MANAGEMENT OF BUNDLED TICKETS IN SPORTS AND ENTERTAINMENT INDUSTRY

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ABSTRACT

MANAGEMENT OF BUNDLED TICKETS IN SPORTS AND ENTERTAINMENT INDUSTRY

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Revenue management techniques can be used in industries where the capacity is limited and perishable, and the market can be segmented. In this work, the sales of event tickets in the sports and entertainment industry is focused, assuming tickets are sold initially as season tickets and as single tickets later in the revenue horizon. Basic decision problems within this context are the determination of optimal time at which the switch from bundled tickets to single tickets should occur and the decision of which event tickets to include into the bundle. Specifically, we study the optimal time to switch between these market segments as a function of the system state where "early switch to a low-demand event" is allowed. Under Poisson demand processes, it is shown that the optimal switching time is a set of time thresholds that depends on the remaining inventory and time. Numerical experiments show that profit enhancement can be obtained by deciding the optimal switch time dynamically over the case when the date of switch is announced in advance. Concerning the latter decision problem, we analyze the optimal selection of tickets for bundling and find increase and decrease patterns in total revenue as the time of bundled events change. We offer a simple heuristic method for finding profitable bundles for a given schedule. We also analyzed the revenue contribution from deciding the ticket bundling and sports league scheduling simultaneously. A heuristic method is offered which utilizes the approximate expected revenue values.

Keywords: Revenue Management, Dynamic Switch, Ticket Bundling, Sports League Scheduling

ÖZ

SPOR VE EĞLENCE SEKTÖRÜNDE GRUP BİLET GELİRİNİN YÖNETİMİ

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Gelir yönetimi teknikleri, sınırlı kapasiteye haiz ve/veya değerini kolay kaybedebilen ürün ya da hizmetlerin satıldığı, segmentlere ayrılabilen pazarların bulunduğu endüstrilerde kullanılabilir. Bu çalışmada, spor ve eğlence sektöründe biletlerin öncelikle sezon bileti olarak, müteakiben de tek tek satıldığı bilet satış sürecine ilişkin karar problemleri incelenmiştir. Bu kapsamda, temel karar problemleri sezon bilet satışından tek tek satışa geçiş zamanının optimizasyonu ile beraber gruplanacak biletlerin optimal seçimleridir. Sistem değişkenlerinin bir fonksiyonu olarak, düşük talepli biletin tekli satışına önceden başlanmasına izin veren (iki geçiş zamanı bulunan) optimal geçiş zamanı problemi incelenmiş, talebin Poisson sürece uyduğu varsayımı altında optimal geçiş zamanı kalan bilet stoğu ile kalan satış süresine bağlı eşik zaman seviyeleri kümesi olarak belirlenmiştir. Geçiş zamanın müşterilere önceden bildirildiği duruma göre bahsedilen dinamik geçiş zamanı uygulamasının kar artırımı sağladığı sayısal örneklerle gösterilmiştir. Diğer karar problemine ilişkin olarak gruplanacak biletlerin seçimi incelenmiş, gruplanan etkinliklerin zamanlarının gişe gelirine etkisi, bu zamanlara bağlı olarak gişe geliri üzerindeki artış ve azalış paternleri tespit edilmiştir. Bu paternlerin yardımı ile, verilen bir etkinlik programı için karlı bilet gruplarının bulunmasını sağlayan sezgisel bir yaklaşım geliştirilmiştir. Ayrıca, bilet gruplama ve spor turnuvasındaki karşılaşma programının eşzamanlı bir şekilde yapılmasının katkısı da analiz edilmiştir. Bilet gruplama kapsamındaki gişe geliri üzerindeki artış ve azalış paternlerinin yardımı ile yaklaşık olarak belirlenebilen gelir seviyelerinin kullanıldığı sezgisel bir metot geliştirilmiş, sayısal örneklerle yapılan testlerde eşzamanlı yaklaşımın belirgin bir katkısı olduğu görülmüştür.

Anahtar Kelimeler: Gelir Yönetimi, Dinamik Zamanlama, Bilet Gruplama, Lig Programlama

to my mother

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CHAPTER 1

INTRODUCTION

Revenue management (RM) aims to maximize profits by allocating the limited resources such as airline seats, hotel rooms, and concert tickets to the right customers at the right price. RM concept first emerged in the airline industry after the deregulation in the early 1980s and since then it has been a vital component in creating strategic and competitive advantages in this highly competitive industry. Today, there exist several successful RM practices in other sectors such as hospitality industry, rental car industry, telecommunications industry and sports and entertainment industry.

Successful RM practices offer significant benefits to the companies. According to Talluri and Van Ryzin (2004) the revenue gains from the use of RM systems are about 4-5%, which corresponds to many airlines' total annual profits. Practitioners agree that implementations of RM techniques in hotel industry can result in 3 to 6 percent revenue increases (Haley & Inge, 2004). Besides increasing revenues, RM is reported to bring additional benefits such as improved forecasting, cohesiveness of information, market identification etc. However, to realize these benefits, a successful implementation of RM is required, which is a quite challenging task. The main reason for this challenge is that RM involves several different aspects of the business processes in the organization, and therefore requires a coordinated effort among diverse groups such as marketing, operations, finance and management. Unfortunately, a seamless integration among these departments is a rare find in most of the companies mainly due to conflicting set of objectives. Hence, an RM system should be designed in a way that suits every stakeholder within the firm and also take into consideration the business characteristics of the specific industry.

Although there is a vast number of successful implementations of RM, some industries still offer a large set of RM-type problems that have not been fully addressed. The sports and entertainment (S&E) industry tackles similar problems like pricing classes of customers , timing of discount offerings, or selection of events to include in a portfolio. The sports business industry is one of the largest and fastest growing industries in the United States and according to the Plunkett Research, which specializes in sports industry statistics, it is estimated at $470 billion in 2013 (Plunkett Research, 2013). The ticket sales alone constitute 6% ($28 billion) of this overall revenue by adopting previous studies to the year of 2013 (SBJ, 2011). The growth and the related statistics in the entertainment industry is similar, however it is quite difficult to present exact figures as what constitutes that sector is not well defined. In S&E industry, RM practices are mostly focused on determining the price of the tickets. Historically, ticket prices change with respect to location of the seat within the venue, however recently, dynamic pricing in regard to factors such as opponent, day of the week, current standing, weather forecast is used in determining ticket prices. For example, San Francisco Giants increased its ticket revenues by 6% using dynamic prices (Render, 2010).

Even though revenue may be increased with dynamic pricing of the tickets of a certain event, the real potential lies within the season ticket sales. In S&E industry, most of the firms offer season tickets first and then start single ticket sale at a later date. This sequential offering of tickets allows the firm to sell the high quality seats to purchasers of season tickets, encouraging commitment to the entire bundle to ensure good seats. In some cases, especially in entertainment industry, firms offer bundled tickets for only a subset of all the events that will take place at their venues. The basic motivation of selling bundled tickets is to increase the revenue of a potentially low demand event by selling it combined with a potentially high demand event that attracts a higher number of customers. In this setting, to maximize the revenue, firms should successfully decide on the content of the bundles, bundled and single ticket prices and finally the time to switch from bundled tickets to single tickets. The challenge in this context is that these decisions all affect each other in a circular sense and making all of these decisions simultaneously is often impossible due to the complexity of the decisions.

While selling bundled tickets has several advantages over traditional ticket sales for the firms (e.g. less variability in demand, higher and earlier revenues), firms should give incentives to the customers to convince them to buy those bundled tickets. Usually, bundled tickets are offered with a discounted price over the total purchasing price of the single tickets included in the bundle. This means that firms actually lose a small portion of their revenues from high demand events, however, this loss is compensated by the increase in revenue from low demand events. Frequently, bundled (season) ticket buyers are offered additional privileges such as premium seats, priority in buying tickets to other events that are not included in the bundle (such as cup game, away game, other events in the venue), which provide further incentives to the customers to buy bundled tickets.

Based on the discussions above, the effectiveness of selling bundled tickets depends on several factors. First of all, the demand patterns of the events and correlation between these patterns are two important factors that determine the excess revenue in selling bundled tickets over traditional single ticket sales. For instance, bundling two events both with potentially high demand may result in negligible increase (or even loss) in revenue over single ticket sales, on the other hand bundling two events, one with potentially high demand and the other with potentially low demand may result in a significant increase in revenue. As for the correlation between demand patterns, bundling one low demand event and one high demand event with negative demand correlation will result in higher gains over single ticket sales compared to bundling two such events with positive demand correlation. Another important factor is pricing. Pricing affects not only the demand patterns of single and bundled tickets, but also the relative valuation of the potential customers for the bundle. The seating capacity of the venue is also important especially when considered with maximum potential demand for the events. For example, if the seating capacity of the venue is smaller than the expected demand for the lowest demand event, there is almost no potential for excess revenue in bundling, and even potential loss due to the discounted bundle price. On the other hand, if the venue capacity is high, bundling has a higher potential in creating extra revenue. Last but not least, timing affects the effectiveness of the bundling. More specifically, the time to switch from selling bundled tickets to selling single tickets, where an early switch increases the revenue from selling single tickets in the expense of the revenue from selling bundled tickets and a late switch vice versa. Another issue related to timing is the schedule of the events. According to the schedule of the events, the extra revenue from bundling may considerably change. For example, if the highest demand event is scheduled as the first event, the firm may consider switching earlier to selling single tickets as otherwise the remaining time until the performance time of the event may not be enough to sell the desired number of tickets. On the other hand, if that particular event is scheduled last, the firm may consider switching at the last possible time to increase the revenue from the bundled tickets as now there is sufficient time for selling the remaining tickets as singles. However, as will be discussed later on, scheduling of the events itself is a complicated task because of the combinatorial nature of the task and several complex feasibility restrictions such as availability of the venues, pattern restrictions (limit on the number of consecutive home/away games), time/event pairing restrictions (scheduling high popularity events on Christmas, Thanksgiving), etc. Hence, the best event schedule for maximizing the revenue might not be feasible for most of the time. Finally, not only these individual factors but their interaction determines the potential gain from bundling. Therefore, it is quite difficult to identify the best strategy for maximizing the potential revenue using ticket bundling.

In our work, we consider the problem of bundling of the event tickets in order to maximize the expected revenue. In this setting, decisions involved for the revenue maximization are determining the optimal time at which the switch from bundled tickets to single tickets should occur and the selection of tickets to be bundled, given the single and bundled ticket prices.

In Chapter 2, firstly we introduce the question of timing the switch from selling bundled tickets to selling single tickets which is discussed in Duran (2007). After introduction of the base problem, we study an extension where "an early switch to a low-demand event" is allowed. We find an optimal policy that is intuitive and easy-to-implement which is defined by a set of threshold pairs of elapsed time and remaining inventory, and determine the switch time. The threshold pairs are computed by an algorithm. Assuming an accurate demand estimate, it is shown that employing two switch times can enhance revenue over employing only one switching time and the alternative of best static decision.

In Chapter 3, we focus on the problem of optimal selection of tickets to be bundled based on the given schedule of the events. We studied the effects of scheduling on the bundle selection decision, assuming predetermined prices, independent and constant demand intensities (customer arrival rates) for individual and bundled tickets. We allow the sale of pure bundles at first and after a switch time only the sale of individual tickets are allowed. We have identified certain properties of potentially high revenue generating bundles, and using these properties we construct a simple yet effective heuristic method to determine good bundles.

In Chapter 4, we analyze scheduling and bundling decisions together for a general double round robin tournament (DRRT). We address the decision problem of scheduling to facilitate the creation of profitable bundles and we attempt to develop a well-performing approach to generate effective schedules. The challenge here is that the bundling and scheduling decisions affect each other in a circular sense and making such a simultaneous decision is quite a daunting task. Nevertheless, we consider the decisions of determining the best schedule of events and determining the best bundle of events and propose an integrated framework to handle these decisions together, which possibly will lead to higher revenue generation compared to sequential decision making process. The developed heuristic method is tested on various experimental cases and it is observed that it determines profitable bundles and corresponding schedule simultaneously in a reasonable amount of time.

CHAPTER 2

DYNAMIC SWITCHING TIMES FROM SEASON TO SINGLE TICKETS

2.1. Introduction

In this chapter, firstly we introduce the question of timing the switch from selling bundled tickets to selling single tickets which is discussed in Duran (2007). This is a problem motivated by the discussions with several large S&E firms, where the timing of the sales, or in some cases timing of the promotion of sales to the public is of key interest.

The availability of the same limited capacity as either bundled or single tickets is key aspect of the problem. In addition, after the switch time, the bundled tickets are split and multiple events start to sell simultaneously as single tickets. These characteristics implied a new research requirement, with the desire to *dynamically* determine the timing decision.

After introduction of the base problem studied in Duran (2007), which is the timing of the single switch from selling bundles to selling single tickets, we study its extension where "an early switch to a low-demand event" is allowed. The problem can be generalized such that timing of different tickets for a two-event performance season (a low- and a high-demand) can be considered to have three switch times: the starting time for the low-demand event ticket sale, τ_1, the starting time for the high-demand event ticket sale, τ_2, and the stopping time for the bundled ticket sale, τ_3. With respect to this generalization, the base problem can be described with a model having equal switch times. In our extension, the season starts with selling bundled tickets, the first switch starts the low-demand event ticket selling, the second switch stops the bundled ticket selling and starts the high-demand game ticket selling. According to generalized problem setting, this model could be considered to have switch times, $\tau_1 < \tau_2 = \tau_3$.

We observe that the problem structure brings an optimal timing policy which can be implemented easily. A set of threshold pairs of remaining inventory and elapsed time defines the resulting optimal policy, and determine the time of optimal switch. After each ticket sale, if the elapsed time is less than the corresponding time threshold, then the switch is exercised. We describe an algorithm for finding the threshold pairs, and we show that the dynamic timing decision enhance the revenue.

In the next section the relevant literature is described. In Section 2.3, the assumptions and the model are introduced, key results for the base case are presented and the model is generalized. We demonstrate numerical experiments in Section 2.5.

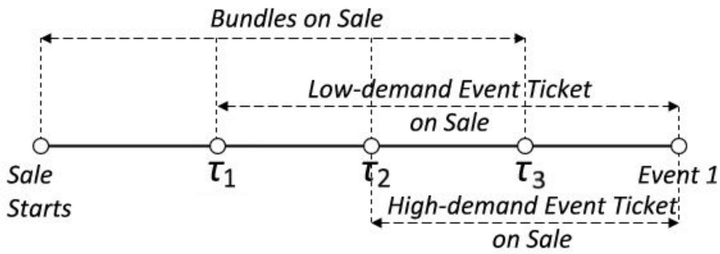

Figure 2.1: The Generalized Multi-Product Sale Timeline Where $\tau_1 \neq \tau_2 \neq \tau_3$.

2.2. Literature Review

RM concept first emerged in the airline industry. The research include how to determine the levels of overbooking (Littlewood, 1972), allocations of seats by class (Belobaba, 1989), or the bid-prices for each flight (Talluri & van Ryzin, 1998); other applications include Bertsimas and Popescu (2003) and Karaesmen and van Ryzin (2004). A network structure matches the application in the airline industry where demand is for a pair of origin and destination, which includes multiple choices of paths for the passenger. Group purchases are considered primarily for groups of passengers purchasing tickets on one plane, rather than a single passenger purchasing many tickets over time, and there is a limited literature on group purchasing as described in Farley (2003). The most similar sale to season tickets is sale of "flexible products", where a passenger buys the option of more than one flights and assigned to one of them later by the carrier (Gallego & Phillips 2004).

Many papers in airline and retail industries RM literature analyzes pricing as a function of time. In the S&E industry, price announcement in the selling period is a common practice, which is described as "price stickiness" (see (Courty, 2000)). Therefore pricing as it is used in retailing industry is less applicable to the S&E industry. In S&E industry, timing of different products is more common than timing of a price change.

A related work is Feng and Gallego (1995), where the authors determine the optimal dynamic time to switch from one price to a second price in order to maximize revenue by selling a given stock of products in a finite time period. A stochastic demand is assumed and rate of demand is higher for the lower price. They show that the optimal policy is a threshold of time which depends on the remaining amount of stock. In Feng and Gallego (2000), the restrictions on the number of price change and constancy of demand intensities are relaxed. They provide an algorithm which finds the optimal value functions and the policy for optimal pricing. Like Feng and Gallego, we use pre-determined prices, however the main difference in our work is that we analyze switching from selling *bundles* of tickets to single tickets. Another important difference is, when the switch occurs, multiple simultaneous processes start. Therefore, the results do not follow directly from their work. Petruzzi and Monahan (2003) study optimal timing of ending a primary selling season to sell in a secondary market, where the first demand process is stochastic.

The timing problem between bundled tickets and single tickets is studied in Drake, Duran, Griffin, and Swann (2008). However, the authors focus on a static timing decision, where the switching date to single tickets is announced in advance to the public. In that paper, a form of the linear Markovian death process is assumed for demand.

Also it is important to mention analysis related to revenue improvement in S&E industry in the fields other than RM. Some researchers have looked at pricing decisions within a venue without considering the ticket bundling, for example, Leslie (2004) and Rosen and Rosenfield (1997) studied ticket prices for different seat qualities in order to maximize revenue. Sainam, Balasubramanian, and Bayus (2010) considers pricing the tickets without knowing which teams play the games. The papers in the marketing and economics literature considering the selling of products which are bundled includes Venkatesh and Kamakura (2003), and McAfee, McMillan, and Whinston (1989) among others. However, these papers analyze product grouping or pricing of the bundles, not the timing of decisions.

2.3. The Dynamic Timing Problem

2.3.1 Assumptions

Let $M \in \mathbb{Z}^+$ be the quantity of seats for sale, and $[0, T]$ be the period of ticket sale where $T \in \mathbb{R}^+$. We have observed that season tickets are rarely purchased after the beginning of performances, and the switch to selling single tickets is made before start of the season. Thus, the time horizon before the beginning of the performance season is focused and the selling period is assumed to end when the first event takes place. This assumption of no single ticket sale after the season starts is relaxed in Section 4.

In the base problem, at the beginning of the selling horizon, bundled tickets are offered at price p_B and after the switch time only individual event tickets at prices p_L and p_H for low- and high-demand events are sold without any assumption about the p values. In the extension of the base problem, ticket selling similarly begins with offering bundled tickets at price p_B, then with the first switch selling low-demand event tickets at p_L starts while bundled tickets continues to sell at p_B, and finally after the second switch only individual event tickets sell at p_L and p_H for low- and high- demand events, respectively. The product prices are assumed to be predetermined before the selling season, which is true for most organizations.

Market segments of bundled tickets and single tickets are assumed to be independent. Discussions with professional sports teams (Depaoli, 2006) supports this assumption, which is common for many models in RM. Constant demand rates are assumed for each of the ticket selling processes.

For each ticket, there is a corresponding Poisson process of demand: $N_B(s)$, $0 \le s \le t$, with given rate λ_B for the bundled events; $N_L(s)$, $0 \le s \le t$, and $N_H(s)$, $0 \le s \le t$, with given rates λ_L and λ_H for the low- and high-demand events, respectively.

In the extended model, additionally, a new Poisson process of $N_{B'}(s)$, $0 \le s \le t$, with given constant rate $\lambda_{B'}(< \lambda_B)$ is introduced for the moderated bundled tickets which is sold along with the low-demand tickets. The extended model represents the expectation that high-demand ticket in the bundle can still motivate the consumers to purchase bundled tickets while low-demand tickets are being sold. The elapsed time t and the remaining inventory at time t, $n(t)$ or simply n if it is observable at time t, defines the state of the system. In the extended model, since they may not be equal to each other, the inventories of individual event tickets are traced separately as $n_L(t)$ and $n_H(t)$.

The revenue rates are defined as $r_L = \lambda_L p_L$ and $r_H = \lambda_H p_H$, for low- and high-demand events respectively. The revenue rate for the bundled tickets before the first switch is $r_B = \lambda_B p_B$. In the extended model, this rate decreases to $r_{B'} = \lambda_{B'} p_B$ after the first switch. The relation between the expected revenue rates: $r_B > r_{B'} + r_L > r_H + r_L$ is assumed to hold. Otherwise, an immediate switch would be optimal. Note that since two classes of tickets (low and high demand) can be grouped according to their demand rates, this model can be applied to larger bundles with more than two tickets.

2.3.2 Models and Results

In this part, the characteristics of the optimal time to switch is analyzed. A function that allows us to determine the optimal switch times is introduced, and with the help of this function, the structure of the optimal switching times is examined. Firstly, the base problem will be addressed and then the extended problem will be discussed.

2.3.2.1 Base Model

The revenue expected to be generated by the ticket sales by implementing the optimal switch times throughout the selling horizon $[t, T]$ is given by $V(t, n)$:

$$V(t,n) = \sup_{\tau \in \mathcal{T}} E[p_B((N_B(\tau) - N_B(t)) \wedge n)] + \Pi(\tau, n(\tau)),$$

where $\tau \in \mathcal{T}$ which is the set of switching times. The relation $t \leq \tau = \tau_1 = \tau_2 = \tau_3 \leq T$ and $n(\tau) = [n - N_B(\tau) + N_B(t)]^+$ is satisfied, where $x^+ = \max\{0, x\}$.

The expected revenue generated by single ticket sales over $[\tau, T]$ is included in $V(t, n)$, and denoted as $\Pi(\tau, n(\tau))$. This function can be described as

$$\Pi(\tau, n(\tau)) = p_L E[(N_L(T) - N_L(\tau)) \wedge n(\tau)] + p_H E[(N_H(T) - N_H(\tau)) \wedge n(\tau)],$$

where the minimum of x and y is indicated by $(x \wedge y)$.

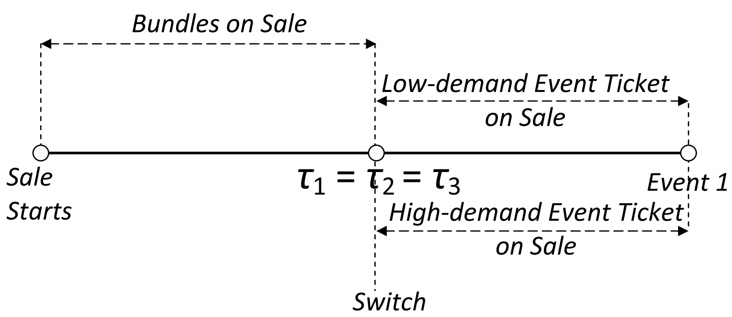

Figure 2.2: Base Model: $\tau_1 = \tau_2 = \tau_3 = \tau$

If the revenue expected over the horizon $[t, T]$ with an immediate switch can be compared to that of the case where the switch is delayed to τ ($t \leq \tau \leq T$), economic decision can easily be made. For these options, the expected values of revenue are given with $\Pi(t, n)$, and $E[p_B((N_B(\tau) - N_B(t)) \wedge n)] + \Pi(\tau, n(\tau))$. For determining the effect of delaying the switch time, a generator function is defined. The infinitesimal generator $\mathcal{G}$ with respect to Poisson process $(t, N_B(t))$ for a uniformly bounded function $g(t,n)$ is described as

$$\mathcal{G}g(t,n) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} E[g(t + \Delta t, n - N_B(\Delta t)) - g(t,n)] = \frac{\partial g(t,n)}{\partial t} + \lambda_B[g(t, n-1) - g(t,n)].$$

The derivation of the $\mathcal{G}$ function is as follows:

$$\begin{aligned}
\mathcal{G}g(t,n) &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} E[g(t + \Delta t, n - N_B(\Delta t)) - g(t,n)] \\
&= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \sum_{k=0}^{\infty} [g(t + \Delta t, (n-k)^+) - g(t,n)] \frac{(\lambda_B \Delta t)^k}{k!} e^{-\lambda_B \Delta t} \\
&= \lim_{\Delta t \to 0} \frac{1}{\Delta t} [(g(t + \Delta t, n) - g(t,n))(1 - \lambda_B \Delta t) + (g(t + \Delta t, n-1) - g(t,n))\lambda_B \Delta t] \\
&= \lim_{\Delta t \to 0} \frac{1}{\Delta t} (g(t + \Delta t, n) - g(t,n)) + \lim_{\Delta t \to 0} \lambda_B (g(t + \Delta t, n-1) - g(t + \Delta t, n)) \\
&= \frac{\partial g(t,n)}{\partial t} + \lambda_B[g(t, n-1) - g(t,n)].
\end{aligned}$$

By application of this generator to $\Pi(t,n)$, we obtain the immediate loss in expected revenue from individually sold tickets when the switch is delayed to a later time. $\mathcal{G}\Pi(t,n) = \frac{\partial \Pi(t,n)}{\partial t} + \lambda_B[\Pi(t,n-1) - \Pi(t,n)]$ can be analyzed in two parts where $\frac{\partial \Pi(t,n)}{\partial t}$ is the expected revenue loss due to loss of selling time, and $\lambda_B[\Pi(t,n-1) - \Pi(t,n)]$ is the expected revenue loss due to loss of inventory which can be sold individually and at a higher price. But during the time when switching is being delayed, the Poisson process for bundles $(t, N_B(t))$ is active, and it generates revenue at the rate of $\mathcal{G}E[p_B((N_B(\tau) - N_B(t)) \wedge n)] = \lambda_B p_B$. Therefore, the net marginal gain (or loss) for delaying the switch from bundles to singles at state (t,n) is given by

$$\mathcal{G}\Pi(t,n) + \lambda_B p_B = \frac{\partial \Pi(t,\ n)}{\partial t} + \lambda_B[\Pi(t,n-1) - \Pi(t,n)] + \lambda_B p_B.$$

The following two expressions are martingales for any s $\geq t$ by Dynkin's Lemma (Rogers & Williams, 1987):

$$\begin{aligned}
\Pi(s, n(s)) - \Pi(t,n) &- \int_t^s \mathcal{G}\Pi(u, n(u))du, \\
p_B((N_B(s) - N_B(t)) \wedge n) &- \int_t^s \lambda_B p_B I_{\{n(u)>0\}} du,
\end{aligned}$$

where $I_{\{n(u)>0\}}$ is the indicator function. Since the expected value of these martingales at any time s is equal to their expected value at the starting time t, we have:

$$\Pi(s, n(s)) - \Pi(t,n) = E \int_t^s \mathcal{G}\Pi(u, n(u))du, \tag{2.1}$$

$$E[p_B((N_B(s) - N_B(t)) \wedge n)] = E \int_t^s \lambda_B p_B I_{\{n(u)>0\}} du. \tag{2.2}$$

We can replace s in (2.1) and (2.2) with any stopping time $\tau \geq t$ by the optional sampling theorem (Karatzas & Shreve, 1988). Therefore, adding Equations (2.1) and (2.2) for a stopping time τ, we get:

$$\begin{aligned}
E[p_B((N_B(\tau) - N_B(t)) \wedge n)] + \Pi(\tau, n(\tau)) &- \Pi(t,n) \\
&= E \int_t^{\tau} [\mathcal{G}\Pi(u, n(u)) + \lambda_B p_B I_{\{n(u)>0\}}]du.
\end{aligned} \tag{2.3}$$

Note that the left-hand side of (2.3) is the expected revenue gained over $\Pi(t,n)$ by delaying the switch from t to τ, and we can quantify it by using $\mathcal{G}$, as shown in the right-hand side. Therefore, we know that delaying the switch to τ from t is beneficial if $E\int_t^\tau [\mathcal{G}\Pi(u,n(u)) + \lambda_B p_B I_{\{n(u)>0\}}]du > 0$.

Taking the supremum of both sides in (2.3) over all stopping times $t \le \tau \le T$, and defining

$$\widetilde{V}(t,n) = \sup_{t\le\tau\le T} E\int_t^\tau [\mathcal{G}\Pi(u,n(u)) + \lambda_B p_B I_{\{n(u)>0\}}]du,$$

we get that $V(t,n) = \Pi(t,n) + \widetilde{V}(t,n)$. Therefore the optimal expected revenue generated in the time horizon $[t,T]$ has two components. They are the expected revenue when the switch is taken immediately and the additional expected revenue obtained by delay of the switch to a later time. Since $\widetilde{V}(t,n)$ is also given by

$$\widetilde{V}(t,n) = \sup_{t\le\tau\le T} E[p_B((N_B(\tau) - N_B(t)) \wedge n) + \Pi(\tau,n(\tau))] - \Pi(t,n), \tag{2.4}$$

it is obvious that $\widetilde{V}(t,n) \ge 0$ for any $0 \le t \le T$ and $0 \le n \le M$. In particular, $\widetilde{V}(t,0) = 0$ for all $0 \le t \le T$ and $\widetilde{V}(T,n) = 0$ for all $0 \le n \le M$. In Equation (2.4), it is shown that delay of the switch is not profitable if $\widetilde{V}(t,n) = 0$, and a revenue enhancement is expected if $\widetilde{V}(t,n) > 0$.

In order to find the numerical values of $\widetilde{V}(t,n)$, a new function $\overline{V}(t,n)$ is defined. Its values for each time and inventory level can be calculated recursively and when the conditions described in Theorem 2.1 hold, the function is identical to $\widetilde{V}(t,n)$.

Theorem 2.1 *Assume that $\overline{V}(t,n)$ is continuous and differentiable with right continuous derivatives for each n in the horizon of $[0,T]$. Additionally, if the following conditions hold $\overline{V}(t,n) = \widetilde{V}(t,n)$.*

(i) $\overline{V}(t,n) \ge 0,\ \ 0 \le t \le T$ *and* $0 \le n \le M$*;*
(ii) $\overline{V}(T,n) = 0$ *for* $0 \le n \le M$ *and* $\overline{V}(t,0) = 0$ *for* $0 \le t \le T$*;*
(iii) $\overline{V}(t,n) = 0 \Rightarrow \mathcal{G}(\overline{V} + \Pi)(t,n) + \lambda_B p_B \le 0,\ 0 \le t \le T$ *and* $0 \le n \le M$*;*
(iv) $\overline{V}(t,n) > 0 \Rightarrow \mathcal{G}(\overline{V} + \Pi)(t,n) + \lambda_B p_B = 0,\ 0 \le t \le T$ *and* $0 \le n \le M$*;*

Non-negativity property and boundary conditions are given in the first two items. The proof of the theorem is along the lines of the proof of Theorem 1 in Feng and Xiao (1999) which is also provided in Duran (2007).

The mirror function allows us to determine if delay of the switch is profitable. The expected revenue enhancement by delay of the switch, which is $\mathcal{G}\Pi(t,n) + \lambda_B p_B$, defines the behavior of $\overline{V}(t,n)$. This term can be written as

$$\begin{aligned}\mathcal{G}\Pi(t,n) + \lambda_B p_B = (r_B - r_L - r_H) \quad &+ \quad p_L(\lambda_L - \lambda_B)P[N_L(T) - N_L(t) \ge n] \\ &+ \quad p_H(\lambda_H - \lambda_B)P[N_H(T) - N_H(t) \ge n].\end{aligned}$$

See (Duran, 2007) for the proof. Note that when $\lambda_B > \lambda_i$ for $i = L, H$, clearly $\mathcal{G}\Pi(t,n) + \lambda_B p_B$ is increasing in t and n.

Now, we will analyze the calculation of $\overline{V}(t,n)$ for all possible states (t,n) of the system. From condition (iv), it is known that $\mathcal{G}\overline{V}(t,n) = -\mathcal{G}\Pi(t,n) - \lambda_B p_B$ when $\overline{V}(t,n) > 0$. Applying the infinitesimal generator $\mathcal{G}$ to $\overline{V}(t,n)$, allows us to obtain the equation

$$\frac{\partial \overline{V}(t,n)}{\partial t} - \lambda_B \overline{V}(t,n) = -[\lambda_B \overline{V}(t,n-1) + \mathcal{G}\Pi(t,n) + \lambda_B p_B],$$

which has the solution $\overline{V}(t,n) = \int_t^T e^{-\lambda_B(s-t)}[\lambda_B \overline{V}(s,n-1) + \mathcal{G}\Pi(s,n) + \lambda_B p_B]ds$ provided that $\overline{V}(t,n-1)$ is known (see Appendix A for the details). Since $\overline{V}(t,0) = 0$, all $\overline{V}(t,n)$ can be solved recursively. The formal procedure is given in Duran (2007).

It is shown that for each level of inventory $n = 1,\ldots,M$, a time threshold x_n exists such that: $\overline{V}(t,n) > 0$ if $t > x_n$, and $\overline{V}(t,n) = 0$ if $t \le x_n$. So, if the inventory decreases to n before x_n, an immediate switch should be taken. Otherwise, it is profitable to delay the switch to a later time.

2.3.2.2 Extended Model with an Early Switch Option

For the extended model with two optimal switching times, the revenue which is expected to be generated in the selling horizon $[t,T]$ is defined as $V_1(t,(n_L,n_H))$:

$$
\begin{aligned}
V_1(t,(n_L,n_H)) &= \sup_{\tau_1\in\mathcal{T}} E[p_B((N_B(\tau_1) - N_B(t)) \wedge n_L)] \\
&+ V_2(\tau_1,(n_L(\tau_1),n_H(\tau_1))), \text{ where} \\
V_2(t,(n_L,n_H)) &= \sup_{\tau_2\in\mathcal{T}} E[p_L((N_{B'L}(\tau_2) - N_{B'L}(t)) \wedge n_L)\frac{\lambda_L}{\lambda_{B'}+\lambda_L} \\
&+ p_B((N_{B'L}(\tau_2) - N_{B'L}(t)) \wedge n_L)\frac{\lambda_{B'}}{\lambda_{B'}+\lambda_L}] \\
&+ V_3(\tau_2,(n_L(\tau_2),n_H(\tau_2))), \text{ and} \\
V_3(t,(n_L,n_H)) &= E\Big[p_L((N_L(T) - N_L(t)) \wedge n_L) + p_H((N_H(T) - N_H(t)) \wedge n_H)\Big],
\end{aligned}
$$

where $\mathcal{T}$ is the set of switching times τ_k satisfying $t \le \tau_1 \le \tau_2 = \tau_3 \le T$. The ticket inventory immediately after the first switch can be described by $n_L(\tau_1) = [n_L - N_B(\tau_1) + N_B(t)]^+$ and $n_H(\tau_1) = [n_H - N_B(\tau_1) + N_B(t)]^+$. In a similar manner, the ticket inventory just after the second switch is $n_L(\tau_2) = [n_L(\tau_1) - N_{B'L}(\tau_2) + N_{B'L}(t)]^+$ and $n_H(\tau_2) = [n_H(\tau_1) - ((N_{B'L}(\tau_2) - N_{B'L}(t)) \wedge n_L(\tau_1))\frac{\lambda_{B'}}{\lambda_{B'}+\lambda_L}]^+$. Note that there is a combined process, $(t, N_{B'L}(t))$, which reflects the capacity sharing of bundled tickets and single tickets of low-demand game, where $\{N_{B'L}(t),\ t \ge 0\} = \{N_{B'}(t) + N_L(t),\ t \ge 0\}$. Because the demand processes $(t, N_{B'}(t))$ and $(t, N_L(t))$ are independent, the demand rate for the combined process $(t, N_{B'L}(t))$ is $\lambda_L + \lambda_{B'}$, with the probability that the first sale from the combined process for a low-demand game is $\frac{\lambda_L}{\lambda_L+\lambda_{B'}}$ The revenue expected to be generated throughout the selling horizon can be analyzed in three parts: the expected revenue from the sale of bundled tickets in the horizon $[0,\tau_1]$ with full inventory; the expected revenue from the sale of bundled tickets and single low-demand tickets in the horizon $[\tau_1,\tau_2]$ with the starting inventory of $n_L(\tau_1)$ and $n_H(\tau_1)$; and the expected revenue from the sale of single tickets in the horizon $[\tau_2,T]$ with the starting inventory of $n_L(\tau_2)$ and $n_H(\tau_2)$.

As in the base model, to determine if delaying the switch is profitable, generator functions are defined regarding to the corresponding Poisson processes. $\mathcal{G}_2$ is the infinitesimal generator for the process $(t, N_{B'L}(t))$ for a uniformly bounded function $g(t,(n_L,n_H))$ when $n_L \ge 1$ and $n_H \ge 1$ described as

$$
\begin{aligned}
\mathcal{G}_2 g(t,(n_L,n_H)) &= \frac{\partial g(t,(n_L,n_H))}{\partial t} + \lambda_{B'}[g(t,(n_L-1,n_H-1)) - g(t,(n_L,n_H))] \\
&+ \lambda_L[g(t,(n_L-1,n_H)) - g(t,(n_L,n_H))],
\end{aligned}
$$

$\mathcal{G}_1$ is described similarly with respect to the process $(t, N_B(t))$ as

$$
\mathcal{G}_1 g(t,(n_L,n_H)) = \frac{\partial g(t,(n_L,n_H))}{\partial t} + \lambda_B[g(t,(n_L-1,n_H-1)) - g(t,(n_L,n_H))].
$$

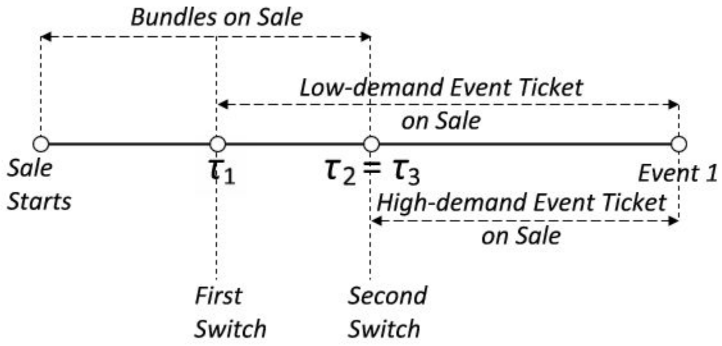

Figure 2.3: Extended Model: $\tau_1 < \tau_2 = \tau_3$

Because of the similarity in the derivations of $\mathcal{G}_1$ and $\mathcal{G}$, here we present only the derivation of $\mathcal{G}_2$. When $n_L \geq 1$ and $n_H \geq 1$:

$$\begin{aligned}
\mathcal{G}_2(t,(n_L,n_H)) &= \lim_{\Delta t\to 0}\frac{1}{\Delta t}E[g(t+\Delta t,(n_L - N_{B'L}(\Delta t), n_H - N_{B'}(\Delta t))) - g(t,(n_L,n_H))] \\
&= \lim_{\Delta t\to 0}\frac{1}{\Delta t}[g(t+\Delta t,(n_L,n_H)) - g(t,(n_L,n_H))](1-(\lambda_L+\lambda_{B'})\Delta t) \\
&+ \lim_{\Delta t\to 0}\frac{1}{\Delta t}[g(t+\Delta t,(n_L-1,n_H)) - g(t,(n_L,n_H))]\frac{\lambda_L}{\lambda_L+\lambda_{B'}}(\lambda_L+\lambda_{B'})\Delta t \\
&+ \lim_{\Delta t\to 0}\frac{1}{\Delta t}[g(t+\Delta t,(n_L-1,n_H-1)) - g(t,(n_L,n_H))]\frac{\lambda_{B'}}{\lambda_L+\lambda_{B'}}(\lambda_L+\lambda_{B'})\Delta t \\
&= \frac{\partial g(t,(n_L,n_H))}{\partial t} + \lim_{\Delta t\to 0}\lambda_{B'}[g(t+\Delta t,(n_L-1,n_H-1)) - g(t+\Delta t,(n_L,n_H))] \\
&+ \lim_{\Delta t\to 0}\lambda_L[g(t+\Delta t,(n_L-1,n_H)) - g(t+\Delta t,(n_L,n_H))] \\
&= \frac{\partial g(t,(n_L,n_H))}{\partial t} + \lambda_{B'}[g(t,(n_L-1,n_H-1)) - g(t,(n_L,n_H))] \\
&+ \lambda_L[g(t,(n_L-1,n_H)) - g(t,(n_L,n_H))].
\end{aligned}$$

The net marginal gain (or loss) for delaying the second switch at state $(t,(n_L,n_H))$ is given by $\mathcal{G}_2 V_3(t,(n_L,n_H))+\lambda_L p_L+\lambda_{B'} p_B$, and similarly the net marginal gain (or loss) for delaying the first switch to selling bundles and low-demand games simultaneously at state $(t,(n_L,n_H))$ is given by $\mathcal{G}_1 V_2(t,(n_L,n_H)) + \lambda_B p_B$.

Using a procedure similar to the case in the base model, by Dynkin's Lemma (Rogers & Williams, 1987) and the optional sampling theorem (Karatzas & Shreve, 1988), it is easy to get that $V_k(t,(n_L,n_H)) = V_{(k+1)}(t,(n_L,n_H)) + \widetilde{V}_k(t,(n_L,n_H))$.

To compute $\widetilde{V}_k(t,(n_L,n_H))$, we introduce the function $\overline{V_k}(t,(n_L,n_H))$, which can be derived recursively and is identical to $\widetilde{V}_k(t,(n_L,n_H))$ when a number of conditions are satisfied in Theorem 2.2.

Theorem 2.2 *Suppose there exists functions $\overline{V_1}(t,(n_L,n_H))$ and $\overline{V_2}(t,(n_L,n_H))$ such that $\overline{V_1}(t,(n_L,n_H))$ and $\overline{V_2}(t,(n_L,n_H))$ are continuous and differentiable with right continuous derivatives in [0,T] for each fixed n_L and n_H. In addition, these functions satisfy:*

(i) $\overline{V_k}(t,(n_L,n_H)) \geq 0$, *for* $k = 1,2$, $0 \leq t \leq T$, $0 \leq n_L \leq M$, *and* $0 \leq n_H \leq M$;
(ii) $\overline{V_k}(T,(n_L,n_H)) = 0$, *for* $k = 1,2$, $0 \leq n_L \leq M$, *and* $0 \leq n_H \leq M$;

(iii) $\overline{V_k}(t,(0,0)) = 0$, *for* $k = 1,2$, $0 \le t \le T$;

(iv) $\overline{V_1}(t,(n_L,n_H)) = 0 \Rightarrow \mathcal{G}_1(\overline{V_1} + V_2)(t,(n_L,n_H)) + \lambda_B p_B \le 0$, *for* $0 \le t \le T$, $0 \le n_L \le M$, *and* $0 \le n_H \le M$;

(v) $\overline{V_1}(t,(n_L,n_H)) > 0 \Rightarrow \mathcal{G}_1(\overline{V_1} + V_2)(t,(n_L,n_H)) + \lambda_B p_B = 0$, *for* $0 \le t \le T$, $0 \le n_L \le M$, *and* $0 \le n_H \le M$;

(vi) $\overline{V_2}(t,(n_L,n_H)) = 0 \Rightarrow \mathcal{G}_2(\overline{V_2} + V_3)(t,(n_L,n_H)) + \lambda_L p_L + \lambda_{B'} p_B \le 0$, *for* $0 \le t \le T$, $0 \le n_L \le M$, *and* $0 \le n_H \le M$;

(vii) $\overline{V_2}(t,(n_L,n_H)) > 0 \Rightarrow \mathcal{G}_2(\overline{V_2} + V_3)(t,(n_L,n_H)) + \lambda_L p_L + \lambda_{B'} p_B = 0$, *for* $0 \le t \le T$, $0 \le n_L \le M$, *and* $0 \le n_H \le M$;

then $\overline{V_k}(t,(n_L,n_H)) = \widetilde{V_k}(t,(n_L,n_H))$ *for* $k = 1,2$.

Proof. First we will show that $\overline{V_1}(t,(n_L,n_H)) = \widetilde{V_1}(t,(n_L,n_H))$. $\widetilde{V_1}$ is defined as:

$$\widetilde{V_1}(t,(n_L,n_H)) = \sup_{t\le\tau_1\le T} E\int_t^{\tau_1} [\mathcal{G}_1 V_2(u,(n_L(u),n_H(u))) + \lambda_B p_B I_{\{n_B(u)>0\}}]du. \tag{2.5}$$

Assume that there exists a function satisfying the conditions in the theorem. For $s \ge t$, let

$$m(s) = \overline{V_1}(s,(n_L(s),n_H(s))) - \overline{V_1}(t,(n_L,n_H)) - \int_t^s \mathcal{G}_1 \overline{V_1}(u,(n_L(u),n_H(u)))du.$$

$m(s)$ is a martingale by Dynkin's Lemma, and since the expected value of this martingale at any time s is equal to its expected value at the starting time t, we have $Em(s) = 0$. Further, by the optional sampling theorem, for any stopping time $\tau_1 \ge t$ we have

$$E[\overline{V_1}(\tau_1,(n_L(\tau_1),n_H(\tau_1)))] - E\int_t^{\tau_1} \mathcal{G}_1 \overline{V_1}(u,(n_L(u),n_H(u)))du = \overline{V_1}(t,(n_L,n_H)) \tag{2.6}$$

$$\begin{aligned} E[\overline{V}(\tau_1,(n_L(\tau_1),n_H(\tau_1)))] - E\int_t^{\tau_1} [\mathcal{G}_1(\overline{V_1} + V_2)(u,(n_L(u),n_H(u))) + \lambda_B p_B I_{\{n_B(u)>0\}}]du \\ = \overline{V_1}(t,(n_L,n_H)) - E\int_t^{\tau_1} [\mathcal{G}_1 V_2(u,(n_L(u),n_H(u))) + \lambda_B p_B I_{\{n_B(u)>0\}}]du. \end{aligned} \tag{2.7}$$

If we subtract $E\int_t^{\tau_1}[\mathcal{G}_1 V_2(u,(n_L(u),n_H(u))) + \lambda_B p_B I_{\{n_B(u)>0\}}]du$ from both sides of (2.6), the left-hand side of the resulting term, given by (2.7), is always positive by conditions (*i*), (*iv*) and (*v*). Therefore,

$$\overline{V_1}(t,(n_L,n_H)) \ge E\int_t^{\tau_1} [\mathcal{G}_1 V_2(u,(n_L(u),n_H(u))) + \lambda_B p_B I_{\{n_B(u)>0\}}]du.$$

Since $\overline{V_1}(t,(n_L,n_H))$ is greater than or equal to each term in the right-hand side of Equation (2.5) for any $\tau_1 \epsilon \mathcal{T}$, it is also greater than or equal to the supremum over all τ_1, which is $\widetilde{V_1}(t,(n_L,n_H))$ in Equation (2.5). Hence, we conclude that $\overline{V_1}(t,(n_L,n_H)) \ge \widetilde{V_1}(t,(n_L,n_H))$ for any stopping time $\tau_1 \ge t$. To prove that $\overline{V_1}(t,(n_L,n_H)) \le \widetilde{V_1}(t,(n_L,n_H))$, we will define a specific stopping time. Let σ be defined as $\sigma = \inf\{t \le s \le T : \overline{V_1}(s,(n_L(s),n_H(s))) = 0\}$. Note that σ is well-defined because $\overline{V_1}(T,\cdot,\cdot) = 0$. Replacing τ_1 in Equation (2.7) with σ, we obtain

$$\begin{aligned} E[\overline{V_1}(\sigma,(n_L(\sigma),n_H(\sigma)))] - E\int_t^{\sigma} [\mathcal{G}_1(\overline{V_1} + V_2)(u,(n_L(u),n_H(u))) + \lambda_B p_B I_{\{n_B(u)>0\}}]du \\ = \overline{V_1}(t,(n_L,n_H)) - E\int_t^{\sigma} [\mathcal{G}_1 V_2(u,(n_L(u),n_H(u))) + \lambda_B p_B I_{\{n_B(u)>0\}}]du. \end{aligned} \tag{2.8}$$

The definition of σ implies that $\overline{V_1}(\sigma,(n_L(\sigma),n_H(\sigma)))=0$, and the definition of σ and condition (v) together imply that $\mathcal{G}_1(\overline{V_1}+V_2)(u,(n_L(u),n_H(u)))+\lambda_B p_B=0$ for all $u\in[t,\sigma]$. Therefore the left-hand side of (2.8) is zero and we have

$$\begin{aligned}\overline{V_1}(t,(n_L,n_H)) &= E\int_t^{\sigma}[\mathcal{G}_1 V_2(u,(n_L(u),n_H(u)))+\lambda_B p_B I_{\{n_B(u)>0\}}]du\\ &\leq \widetilde{V_1}(t,(n_L,n_H)).\end{aligned}$$

The inequality follows from the fact that the left-hand side of the inequality is the right-hand side of Equation (2.5) for a specific stopping time, and $\widetilde{V_1}(t,(n_L,n_H))$ is the supremum over all stopping times τ_1 in that equation. Hence, $\overline{V_1}(t,(n_L,n_H))=\widetilde{V_1}(t,(n_L,n_H))$.

For $\overline{V_2}(t,(n_L,n_H))=\widetilde{V_2}(t,(n_L,n_H))$, we know that $\widetilde{V_2}$ is defined as:

$$\widetilde{V_2}(t,(n_L,n_H))=\sup_{t\leq\tau_2\leq T}E\int_t^{\tau_2}[\mathcal{G}_2 V_3(u,(n_L(u),n_H(u)))+(\lambda_L p_L+\lambda_{B'}p_B)I_{\{n_B(u)>0\}}]du. \tag{2.9}$$

Assume that there exists a function satisfying the conditions in the theorem. We will show that $\overline{V_2}$ is equal to $\widetilde{V_2}$. For $s\geq t$, let

$$m(s)=\overline{V_2}(s,(n_L(s),n_H(s)))-\overline{V_2}(t,(n_L,n_H))-\int_t^s\mathcal{G}_2\overline{V_2}(u,(n_L(u),n_H(u)))du.$$

$m(s)$ is a martingale by Dynkin's Lemma, and since the expected value of this martingale at any time s is equal to its expected value at the starting time t, we have $Em(s)=0$. Further, by the optional sampling theorem, for any stopping time $\tau_2\geq t$ we have

$$E[\overline{V_2}(\tau_2,(n_L(\tau_2),n_H(\tau_2)))]-E\int_t^{\tau_2}\mathcal{G}_2\overline{V_2}(u,(n_L(u),n_H(u)))du=\overline{V_2}(t,(n_L,n_H)) \tag{2.10}$$

$$\begin{aligned}E[\overline{V}(\tau_2,(n_L(\tau_2),n_H(\tau_2)))]-E\int_t^{\tau_2}[\mathcal{G}_2(\overline{V_2}+V_3)(u,(n_L(u),n_H(u)))+(\lambda_L p_L+\lambda_{B'}p_B)I_{\{n_B(u)>0\}}]du \quad (2.11)\\ =\overline{V_2}(t,(n_L,n_H))-E\int_t^{\tau_2}[\mathcal{G}_2 V_3(u,(n_L(u),n_H(u)))+(\lambda_L p_L+\lambda_{B'}p_B)I_{\{n_B(u)>0\}}]du.\end{aligned}$$

If we subtract $E\int_t^{\tau_2}[\mathcal{G}_2 V_3(u,(n_L(u),n_H(u)))+(\lambda_L p_L+\lambda_{B'}p_B)I_{\{n_B(u)>0\}}]du$ from both sides of (2.10), the left-hand side of the resulting term, given by (2.11), is always positive by conditions (i), (vi) and (vii). Therefore,

$$\overline{V_2}(t,(n_L,n_H))\geq E\int_t^{\tau_2}[\mathcal{G}_2 V_3(u,(n_L(u),n_H(u)))+(\lambda_L p_L+\lambda_{B'}p_B)I_{\{n_B(u)>0\}}]du.$$

Since $\overline{V_2}(t,(n_L,n_H))$ is greater than or equal to each term in the right-hand side of Equation (2.9) for any $\tau_2\epsilon\mathcal{T}$, it is also greater than or equal to the supremum over all τ_2, which is $\widetilde{V_2}(t,(n_L,n_H))$ in Equation (2.9). Hence, we conclude that $\overline{V_2}(t,(n_L,n_H))\geq\widetilde{V_2}(t,(n_L,n_H))$ for any stopping time $\tau_2\geq t$. To prove that $\overline{V_2}(t,(n_L,n_H))\leq\widetilde{V_2}(t,(n_L,n_H))$, we will define a specific stopping time. Let σ be defined as $\sigma=\inf\{t\leq s\leq T:\overline{V_2}(s,(n_L(s),n_H(s)))=0\}$. Note that σ is well-defined because $\overline{V_2}(T,\cdot,\cdot)=0$. Replacing τ_2 in Equation (2.11) with σ, we obtain

$$\begin{aligned}E[\overline{V_2}(\sigma,(n_L(\sigma),n_H(\sigma)))]-E\int_t^{\sigma}[\mathcal{G}_2(\overline{V_2}+V_3)(u,(n_L(u),n_H(u)))+(\lambda_L p_L+\lambda_{B'}p_B)I_{\{n_B(u)>0\}}]du \quad (2.12)\\ =\overline{V_2}(t,(n_L,n_H))-E\int_t^{\sigma}[\mathcal{G}_2 V_3(u,(n_L(u),n_H(u)))+(\lambda_L p_L+\lambda_{B'}p_B)I_{\{n_B(u)>0\}}]du.\end{aligned}$$

The definition of σ implies that $\overline{V_2}(\sigma,(n_L(\sigma),n_H(\sigma)))=0$, and the definition of σ and condition (vii) together imply that $\mathcal{G}_2(\overline{V_2}+V_3)(u,(n_L(u),n_H(u)))+\lambda_L p_L+\lambda_{B'}p_B=0$ for all $u\in[t,\sigma]$. Therefore the left-hand side of (2.12) is zero and we have

$$\begin{aligned}\overline{V_2}(t,(n_L,n_H)) &= E\int_t^{\sigma}[\mathcal{G}_2V_3(u,(n_L(u),n_H(u)))+(\lambda_L p_L+\lambda_{B'}p_B)I_{\{n_B(u)>0\}}]du\\ &\leq \widetilde{V_2}(t,(n_L,n_H)).\end{aligned}$$

The inequality follows from the fact that the left-hand side of the inequality is the right-hand side of Equation (2.9) for a specific stopping time, and $\widetilde{V_2}(t,(n_L,n_H))$ is the supremum over all stopping times τ_2 in that equation. Hence, $\overline{V_2}(t,(n_L,n_H))=\widetilde{V_2}(t,(n_L,n_H))$. ■

The first three conditions in the list are the non-negativity property and the boundary conditions of $\overline{V_k}$. Also, since $\widetilde{V_k}(t,(n_L,n_H))$ determines whether it is optimal to switch immediately or not, the conditions for $\overline{V_k}(t,(n_L,n_H))$ to be positive or zero are listed in conditions (iv), (v), (vi), and (vii).

We will first focus on the second switch decision. The properties for it implies a similar policy for the first switch decision. Remember that the net marginal gain (or loss) for delaying the second switch at state $(t,(n_L,n_H))$ is given by $\mathcal{G}_2V_3(t,(n_L,n_H))+\lambda_L p_L+\lambda_{B'}p_B$, and this term will be addressed more closely in the following lemma. First, by noting that $E[(N_i(T)-N_i(t))\wedge n]=\sum_{k=1}^{n}P[N_i(T)-N_i(t)\geq k]$, we can express $V_3(t,(n_L,n_H))$ for $n_L\geq 1$ and $n_H\geq 1$ as

$$V_3(t,(n_L,n_H))=p_L\sum_{k=1}^{n_L}P[N_L(T)-N_L(t)\geq k]+p_H\sum_{k=1}^{n_H}P[N_H(T)-N_H(t)\geq k].$$

Lemma 2.1 *The net marginal gain from delaying for* $0\leq t\leq T$ *can be written as*

$$\begin{aligned}\mathcal{G}_2V_3(t,(n_L,n_H))+\lambda_L p_L+\lambda_{B'}p_B=(r_{B'}-r_H) &+ p_H(\lambda_H-\lambda_{B'})P[N_H(T)-N_H(t)\geq n_H]\\ &- \lambda_{B'}p_L P[N_L(T)-N_L(t)\geq n_L],\end{aligned}$$

and is increasing in t, n_L *and* n_H *when* $\lambda_{B'}>\lambda_H$.

Proof. For a Poisson process N_i we know that:

$$P[N_i(T)-N_i(t)=k]=\frac{e^{-(\lambda_i(T)-\lambda_i(t))}(\lambda_i(T)-\lambda_i(t))^k}{k!}. \tag{2.13}$$

Thus derivative of $P[N_i(T)-N_i(t)\geq k]$ with respect to t can be written as:

$$\begin{aligned}\frac{\partial P[N_i(T)-N_i(t)\geq k]}{\partial t} &= \frac{\partial}{\partial t}(1-P[N_i(T)-N_i(t)<k])\\ &= \frac{\partial}{\partial t}(1-\frac{e^{-(\lambda_i(T)-\lambda_i(t))}(\lambda_i(T)-\lambda_i(t))^{k-1}}{(k-1)!}\\ &\quad -\frac{e^{-(\lambda_i(T)-\lambda_i(t))}(\lambda_i(T)-\lambda_i(t))^{k-2}}{(k-2)!}\\ &\quad \vdots\\ &\quad -\frac{e^{-(\lambda_i(T)-\lambda_i(t))}(\lambda_i(T)-\lambda_i(t))^{0}}{(0)!}).\end{aligned}$$

Simply taking derivative of each term we get:

$$\frac{\partial P[N_i(T)-N_i(t)\geq k]}{\partial t} =$$
$$\begin{array}{rcl} -\lambda_i(t)P[N_i(T)-N_i(t)=k-1] & + & \lambda_i(t)P[N_i(T)-N_i(t)=k-2] \\ -\lambda_i(t)P[N_i(T)-N_i(t)=k-2] & + & \lambda_i(t)P[N_i(T)-N_i(t)=k-3] \\ \lambda_i(t)P[N_i(T)-N_i(t)=k-3] & + & (....) \\ (....) & + & (....) \\ (....) & + & \lambda_i(t)P[N_i(T)-N_i(t)=0] \\ \lambda_i(t)P[N_i(T)-N_i(t)=0]. & & \end{array}$$

After eliminations we have:

$$\frac{\partial P[N_i(T)-N_i(t)\geq k]}{\partial t} = -\lambda_i(t)P[N_i(T)-N_i(t)=k-1], \tag{2.14}$$

Therefore, we have $\frac{\partial \sum_{k=1}^{n} P[N_i(T)-N_i(t)\geq k]}{\partial t} = -\lambda_i P[N_i(T)-N_i(t)\leq n-1]$. Using this equality we have

$$\begin{aligned}
\mathcal{G}_2 V_3(t,(n_L,n_H)) &= \frac{\partial V_3(t,(n_L,n_H))}{\partial t} + \lambda_{B'}[V_3(t,(n_L-1,n_H-1)) - V_3(t,(n_L,n_H))] \\
&\quad +\lambda_L[V_3(t,(n_L-1,n_H)) - V_3(t,(n_L,n_H))] \\
&= -\lambda_L p_L P[N_L(T)-N_L(t)\leq n_L-1] - \lambda_H p_H P[N_H(T)-N_H(t)\leq n_H-1] \\
&\quad -\lambda_{B'} p_L P[N_L(T)-N_L(t)\geq n_L] - \lambda_{B'} p_H P[N_H(T)-N_H(t)\geq n_H] \\
&\quad -\lambda_L p_L P[N_L(T)-N_L(t)\geq n_L] \\
&= -\lambda_L p_L(1-P[N_L(T)-N_L(t)\geq n_L]) - \lambda_H p_H(1-P[N_H(T)-N_H(t)\geq n_H]) \\
&\quad -\lambda_L p_L P[N_L(T)-N_L(t)\geq n_L] - \lambda_{B'} p_H P[N_H(T)-N_H(t)\geq n_H] \\
&\quad -\lambda_{B'} p_L P[N_L(T)-N_L(t)\geq n_L] \\
&= -\lambda_L p_L - \lambda_H p_H + p_H(\lambda_H-\lambda_{B'})P[N_H(T)-N_H(t)\geq n_H] \\
&- \lambda_{B'} p_L P[N_L(T)-N_L(t)\geq n_L].
\end{aligned}$$

Note that when $\lambda_{B'} > \lambda_H$, clearly $\mathcal{G}_2 V_3(t,(n_L,n_H)) + \lambda_L p_L + \lambda_{B'} p_B$ is increasing in t, n_L and n_H. ■

Theorem 2.3 *For all $1 \leq n_L \leq n_H \leq M$, and when $\lambda_{B'} > \lambda_H$, $\overline{V_2}(t,(n_L,n_H))$ is recursively determined by*

$$\overline{V_2}(t,(n_L,n_H)) = \begin{cases} \int_t^T L_2(s,(n_L,n_H))e^{-(\lambda_L+\lambda_{B'})(s-t)}ds & if\ t > x^2_{(n_L,n_H)} \\ 0 & otherwise, \end{cases}$$

where

$$\begin{aligned}
x^2_{(n_L,n_H)} &= \inf\{0\leq t\leq T: \int_t^T L_2(s,(n_L,n_H))e^{-(\lambda_L+\lambda_{B'})(s-t)}ds > 0\}, \\
L_2(t,(n_L,n_H)) &= \lambda_{B'}\overline{V_2}(t,(n_L-1,n_H-1)) + \lambda_L\overline{V_2}(t,(n_L-1,n_H)) \\
&+ \mathcal{G}_2 V_3(t,(n_L,n_H)) + \lambda_L p_L + \lambda_{B'} p_B, \qquad 0\leq t\leq T.
\end{aligned}$$

Proof. The proof will be done by induction on n_L. In the theorem, $\overline{V_2}$ is defined as:

$$\overline{V_2}(t,(n_L,n_H)) = \begin{cases} \displaystyle\int_t^T L_2(s,(n_L,n_H))e^{-(\lambda_{B'}+\lambda_L)(s-t)}ds & if\ t > x^2_{(n_L,n_H)} \\ 0 & otherwise. \end{cases} \tag{2.15}$$

Since $\widetilde{V_2}(t,(n_L,n_H)) = V_2(t,(n_L,n_H)) - V_3(t,(n_L,n_H))$, where

$$\begin{aligned} V_2(t,(n_L,n_H)) &= \sup_{\tau_2\in\mathcal{T}} E[p_L((N_{B'L}(\tau_2) - N_{B'L}(t)) \wedge n_L)\frac{\lambda_L}{\lambda_{B'}+\lambda_L} \\ &+ p_B((N_{B'L}(\tau_2) - N_{B'L}(t)) \wedge n_L)\frac{\lambda_{B'}}{\lambda_{B'}+\lambda_L}] \\ &+ V_3(\tau_2,(n_L(\tau_2),n_H(\tau_2))), \end{aligned}$$

when $n_L = 1$, $\tau_2 = t$ which means the switch should be executed immediately. Therefore, $\overline{V_2}(t,(n_L - 1,k)) = 0$, for $1 \le k \le M$, and we have $L_2(t,(1,k)) = \mathcal{G}_2V_3(t,(1,k)) + \lambda_L p_L + \lambda_{B'}p_B$. Therefore, for $1 \le k \le M-1$,

$$\begin{aligned} L_2(t,(1,k+1)) &= \mathcal{G}_2V_3(t,(1,k+1)) + \lambda_L p_L + \lambda_{B'}p_B \\ &\ge \mathcal{G}_2V_3(t,(1,k)) + \lambda_L p_L + \lambda_{B'}p_B = L_2(t,(1,k)), \end{aligned}$$

since $\mathcal{G}_2V_3(t,(1,k))$ is increasing in k. This implies that

$$\int_t^T L_2(s,(1,k+1))e^{-(\lambda_L+\lambda_{B'})(s-t)}ds \ge \int_t^T L_2(s,(1,k))e^{-(\lambda_L+\lambda_{B'})(s-t)}ds.$$

Together with Equation (2.15), this implies $\overline{V_2}(t,(1,k+1)) \ge \overline{V_2}(t,(1,k))$ and $x^2_{(1,k)} \ge x^2_{(1,k+1)}$ for $1 \le k \le M-1$. For any $n_H = k$ ($1 \le k \le M$), we claim that for $t \le x^2_{(1,k)}$ the following holds:

$$\begin{aligned} L_2(t,(1,k)) &= \mathcal{G}_2V_3(t,(1,k)) + \lambda_L p_L + \lambda_{B'}p_B \\ &\le \mathcal{G}_2V_3(x^2_{(1,k)},(1,k)) + \lambda_L p_L + \lambda_{B'}p_B = L(x^2_{(1,k)},(1,k)) \le 0. \end{aligned}$$

The first inequality is a consequence of the increasing property of $\mathcal{G}_2V_3(t,(1,k))$ in t. The second inequality follows from the fact that if $\mathcal{G}_2V_3(x^2_{(1,k)},(1,k)) + \lambda_L p_L + \lambda_{B'}p_B > 0$ then this will contradict definition of $x^2_{(1,k)}$, since $\int_{x^2_{(1,k)}}^T L_2(s,(1,k))e^{-(\lambda_{B'}+\lambda_L)(s-t)}ds > 0$. Hence, for $t \le x^2_{(1,k)}$ (or $\overline{V_2}(t,(1,k)) = 0$ by the definition of $\overline{V_2}$)

$$\begin{aligned} \mathcal{G}_2(\overline{V_2}+V_3)(t,(1,k)) + \lambda_L p_L + \lambda_{B'}p_B &= \frac{\partial\overline{V_2}(t,(1,k))}{\partial t} + \lambda_{B'}[\overline{V_2}(t,(0,k-1)) - \overline{V_2}(t,(1,k))] \\ &+ \lambda_L[\overline{V_2}(t,(0,k)) - \overline{V_2}(t,(1,k))] \\ &+ \mathcal{G}_2V_3(t,(1,k)) + \lambda_L p_L + \lambda_{B'}p_B \\ &= \mathcal{G}_2V_3(t,(1,k)) + \lambda_L p_L + \lambda_{B'}p_B = L_2(t,(1,k)) \le 0. \end{aligned}$$

Thus, condition (*iv*) is satisfied when $(n_L,n_H) = (1,k)$ and $t \le x^2_{(1,k)}$ (or $\overline{V_2}(t,(1,k)) = 0$). When $t > x^2_{(1,k)}$ (or $\overline{V_2}(t,(1,k)) > 0$) we have that

$$\begin{aligned} \mathcal{G}_2(\overline{V_2}+V_3)(t,(1,k)) + \lambda_L p_L + \lambda_{B'}p_B &= \frac{\partial\overline{V_2}(t,(1,k))}{\partial t} + \lambda_{B'}[\overline{V_2}(t,(0,k-1)) - \overline{V_2}(t,(1,k))] \\ &+ \lambda_L[\overline{V_2}(t,(0,k)) - \overline{V_2}(t,(1,k))] \\ &+ \mathcal{G}_2V_3(t,(1,k)) + \lambda_L p_L + \lambda_{B'}p_B \\ &= \frac{\partial\overline{V_2}(t,(1,k))}{\partial t} - (\lambda_L + \lambda_{B'})\overline{V_2}(t,(1,k)) \\ &+ \mathcal{G}_2V_3(t,(1,k)) + \lambda_L p_L + \lambda_{B'}p_B \end{aligned} \tag{2.16}$$

By the definition of $\overline{V_2}(t,(n_L,n_H))$, we have $\overline{V_2}(t,(1,k)) = \int_t^T L_2(s,(1,k))e^{-(\lambda_L+\lambda_{B'})(s-t)}ds$. Taking the derivative with respect to t, we get

$$\begin{aligned} \frac{\partial \overline{V_2}(t,(1,k))}{\partial t} &= (\lambda_L+\lambda_{B'})\int_t^T L_2(s,(1,k))e^{-(\lambda_L+\lambda_{B'})(s-t)}ds - L_2(t,(1,k)) \\ &= (\lambda_L+\lambda_{B'})\overline{V_2}(t,(1,k)) - L_2(t,(1,k). \end{aligned} \tag{2.17}$$

Substituting (2.17) into (2.16), we get $\mathcal{G}_2(\overline{V_2}+V_3)(t,(1,k)) + \lambda_L p_L + \lambda_{B'} p_B = 0$ when $\overline{V_2}(t,(1,k)) > 0$. Therefore, condition ($v$) is satisfied when $(n_L,n_H) = (1,k)$. Moreover, we have $\overline{V_2}(t,(1,k)) \geq \overline{V_2}(t,(0,k)) = 0$ by the definition of $x^2_{(1,k)}$ (there may exist a time t such that $\overline{V_2}(t,(1,k)) > 0$ if $x^2_{(1,k)} > 0$). Now assume that the following statements hold for $1 < n_L \leq \ell < M$: there exist time thresholds $T \geq x^2_{(\ell,1)} \geq x^2_{(\ell,2)} \geq \cdots \geq x^2_{(\ell,M)} \geq 0$ such that $\overline{V_2}(t,(\ell,k))$ is derived from Equation (2.15) and satisfies conditions (i)-(v). Also the inequality $\overline{V_2}(t,(\ell,k)) \geq \overline{V_2}(t,(\ell-1,k))$ holds for $k = 1,\ldots,M$ and $\overline{V_2}(t,(\ell,k+1)) \geq \overline{V_2}(t,(\ell,k)$ holds for $k = 1,\ldots,M-1$. When $n_L = \ell+1$;

$$\begin{aligned} L_2(t,(\ell+1,k)) &= \lambda_{B'}\overline{V_2}(t,(\ell,k-1)) + \lambda_L\overline{V_2}(t,(\ell,k)) \\ &+ \mathcal{G}_2V_3(t,(\ell+1,k)) + \lambda_L p_L + \lambda_{B'} p_B \\ &\geq \lambda_{B'}\overline{V_2}(t,(\ell-1,k-1)) + \lambda_L\overline{V_2}(t,(\ell-1,k)) \\ &+ \mathcal{G}_2V_3(t,(\ell,k)) + \lambda_L p_L + \lambda_{B'} p_B = L_2(t,(\ell,k)), \end{aligned}$$

since $\mathcal{G}_2V_3(t,(\ell,k))$ is increasing in ℓ and by the induction hypothesis. This implies that

$$\int_t^T L_2(s,(\ell+1,k))e^{-(\lambda_L+\lambda_B)(s-t)}ds \geq \int_t^T L_2(s,(\ell,k))e^{-(\lambda_L+\lambda_B)(s-t)}ds.$$

Together with Equation (2.15), this implies $\overline{V_2}(t,(\ell+1,k)) \geq \overline{V_2}(t,(\ell,k))$ and $x^2_{(\ell,k)} \geq x^2_{(\ell+1,k)}$ for $1 \leq k \leq M$. Similarly,

$$\begin{aligned} L_2(t,(\ell+1,k)) &= \lambda_{B'}\overline{V_2}(t,(\ell,k-1)) + \lambda_L\overline{V_2}(t,(\ell,k)) \\ &+ \mathcal{G}_2V_3(t,(\ell+1,k)) + \lambda_L p_L + \lambda_{B'} p_B \\ &\geq \lambda_{B'}\overline{V_2}(t,(\ell,k-2)) + \lambda_L\overline{V_2}(t,(\ell,k-1)) \\ &+ \mathcal{G}_2V_3(t,(\ell+1,k-1)) + \lambda_L p_L + \lambda_{B'} p_B = L_2(t,(\ell+1,k-1)), \end{aligned}$$

since $\mathcal{G}_2V_3(t,(\ell,k))$ is increasing in k and by the induction hypothesis. This implies that

$$\int_t^T L_2(s,(\ell+1,k))e^{-(\lambda_L+\lambda_{B'})(s-t)}ds \geq \int_t^T L_2(s,(\ell+1,k-1))e^{-(\lambda_L+\lambda_{B'})(s-t)}ds.$$

Together with Equation (2.15), this implies $\overline{V_2}(t,(\ell+1,k)) \geq \overline{V_2}(t,(\ell+1,k-1))$ and $x^2_{(\ell+1,k-1)} \geq x^2_{(\ell+1,k)}$ for $1 \leq k \leq M$. For any $n_H = k$ ($1 \leq k \leq M$), when $t \leq x^2_{(\ell+1,k)}$ (or $\overline{V_2}(t,(\ell+1,k)) = 0$),

$$\begin{aligned} \mathcal{G}_2(\overline{V_2}+V_3)(t,(\ell+1,k)) + \lambda_L p_L + \lambda_{B'} p_B &= \frac{\partial \overline{V_2}(t,(\ell+1,k))}{\partial t} + \lambda_{B'}[\overline{V_2}(t,(\ell,k-1)) - \overline{V_2}(t,(\ell+1,k))] \\ &+ \lambda_L[\overline{V_2}(t,(\ell,k)) - \overline{V_2}(t,(\ell+1,k))] \\ &+ \mathcal{G}_2V_3(t,(\ell+1,k)) + \lambda_L p_L + \lambda_{B'} p_B \\ &= \mathcal{G}_2V_3(t,(\ell+1,k)) + \lambda_L p_L + \lambda_{B'} p_B \\ &= L_2(t,(\ell+1,k)) \leq L_2(x^2_{(\ell+1,k)},(\ell+1,k)) \leq 0. \end{aligned}$$

Note that $\overline{V_2}(t,(\ell,k)) = \overline{V_2}(t,(\ell,k-1)) = 0$ since $t \leq x^2_{(\ell+1,k)} \leq x^2_{(\ell,k)} \leq x^2_{(\ell,k-1)}$. The first inequality follows from $\mathcal{G}_2V_3(t,(\ell+1,k))$ being increasing in t, and the second inequality follows from the fact that if $L_2(x^2_{(\ell+1,k)}, (\ell+1,k)) > 0$ then this will contradict the definition of $x^2_{(\ell+1,k)}$. Therefore, condition (iv) is satisfied, when $t \le x^2_{(\ell+1,k)}$ (or $\overline{V_2}(t,(\ell+1,k)) = 0$). When $t > x^2_{(\ell+1,k))}$ (or $\overline{V_2}(t,(\ell+1,k)) > 0$),

$$
\begin{aligned}
\mathcal{G}_2(\overline{V_2}+V_3)(t,(\ell+1,k)) + \lambda_L p_L + \lambda_{B'} p_B &= \frac{\partial \overline{V_2}(t,(\ell+1,k))}{\partial t} + \lambda_{B'}[\overline{V_2}(t,(\ell,k-1)) - \overline{V_2}(t,(\ell+1,k))] \\
&+ \lambda_L[\overline{V_2}(t,(\ell,k)) - \overline{V_2}(t,(\ell+1,k))] \\
&+ \mathcal{G}_2 V_3(t,(\ell+1,k)) + \lambda_L p_L + \lambda_{B'} p_B \\
&= \frac{\partial \overline{V_2}(t,(\ell+1,k))}{\partial t} - (\lambda_L + \lambda_{B'})\overline{V_2}(t,(\ell+1,k)) \\
&+ \lambda_{B'}\overline{V_2}(t,(\ell,k-1)) + \lambda_L \overline{V_2}(t,(\ell,k)) \\
&+ \mathcal{G}_2 V_3(t,(\ell+1,k)) + \lambda_L p_L + \lambda_{B'} p_B = 0,
\end{aligned}
$$

since $\frac{\partial \overline{V_2}(t,(\ell+1,k))}{\partial t} = (\lambda_L + \lambda_{B'})\overline{V_2}(t,(\ell+1,k)) - L(t,(\ell+1,k)$. Therefore condition (v) is satisfied when $t > x^2_{(\ell+1,k)}$ (or $\overline{V_2}(t,(\ell+1,k)) > 0$). For $n_L = \ell+1$, we showed that conditions (i)-(v) hold. Thus the function $\overline{V_2}(t,(\ell,k))$ that is determined by the proposed procedure, is equal to $\widetilde{V_2}(t,(\ell,k))$. Further the switching time thresholds $(x^2_{(n_L,n_H)})$ are monotonically non-increasing in n_L and n_H. ■

We have shown that for any inventory level (n_L,n_H), there exists a time $x^2_{(n_L,n_H)}$ such that: $\overline{V_2}(t,(n_L,n_H)) > 0$ if $t > x^2_{(n_L,n_H)}$, and $\overline{V_2}(t,(n_L,n_H)) = 0$ if $t \le x^2_{(n_L,n_H)}$. Therefore, if the system reaches remaining inventory level (n_L,n_H) at a time $t \le x^2_{(n_L,n_H)}$, then it is optimal for the second switch decision to be implemented immediately. On the other hand if it takes the system longer than $x^2_{(n_L,n_H)}$ time units to reach remaining inventory level (n_L,n_H), then it is optimal to delay the second switch further. Therefore, the $x^2_{(n_L,n_H)}$ values can be interpreted as the latest switching time or the switching-time thresholds for the second switch, when (n_L,n_H) items are unsold.

$\overline{V_1}(t,(n_L,n_H))$ is recursively determined in a similar way to $\overline{V_2}(t,(n_L,n_H))$ as given in Theorem 2.3.

Theorem 2.4 *When $V_2(t,(n_L,n_H)) - V_2(t,(n_L-1,n_H))$ is decreasing in t and n_L, $V_2(t,(n_L,n_H)) - V_2(t,(n_L,n_H-1))$ is decreasing in t and n_H $(1 \le n_L = n_H \le M)$ and $\lambda_B \ge \lambda'_B + \lambda_L$, then $\overline{V_1}(t,(n_L,n_H))$ is recursively determined by*

$$
\overline{V_1}(t,(n_L,n_H)) = \begin{cases} \int_t^T L_1(s,(n_L,n_H))e^{-(\lambda_B)(s-t)}ds & if\ t > x^1_{(n_L,n_H)} \\ 0 & otherwise, \end{cases}
$$

where

$$
\begin{aligned}
x^1_{(n_L,n_H)} &= \inf\{0 \le t \le T : \int_t^T L_1(s,(n_L,n_H))e^{-(\lambda_B)(s-t)}ds > 0\}, \\
T &\ge x^1_{(1,1)} \ge x^1_{(2,2)} \ge \cdots \ge x^1_{(M,M)} \ge 0, \\
L_1(t,(n_L,n_H)) &= \lambda_B \overline{V_1}(t,(n_L-1,n_H-1)) \\
&+ \mathcal{G}_1 V_2(t,(n_L,n_H)) + \lambda_B p_B, \qquad 0 \le t \le T.
\end{aligned}
$$

Proof. When $t > x^2_{(n_L,n_H)}$, from Theorems 2.2 and 2.3 we know that $\mathcal{G}_2(\overline{V_2}+V_3)(t,(n_L,n_H)) + \lambda_L p_L +$

$\lambda_{B'}p_B = \mathcal{G}_2V_2(t,(n_L,n_H)) + \lambda_L p_L + \lambda_{B'}p_B = 0$. Hence,

$$\begin{aligned}
\mathcal{G}_1V_2(t,(n_L,n_H)) &= \frac{\partial V_2(t,(n_L,n_H))}{\partial t} + \lambda_B[V_2(t,(n_L-1,n_H-1)) - V_2(t,(n_L,n_H))] \\
&= -\lambda_L p_L - \lambda_{B'}p_B - \lambda_{B'}[V_2(t,(n_L-1,n_H-1)) - V_2(t,(n_L,n_H))] \\
&- \lambda_L[V_2(t,(n_L-1,n_H)) - V_2(t,(n_L,n_H))] \\
&+ \lambda_B[V_2(t,(n_L-1,n_H-1)) - V_2(t,(n_L,n_H))] \\
&= -\lambda_L p_L - \lambda_{B'}p_B + (\lambda_B - \lambda_{B'} - \lambda_L)[V_2(t,(n_L-1,n_H-1)) - V_2(t,(n_L,n_H))] \\
&+ \lambda_L[V_2(t,(n_L-1,n_H-1)) - [V_2(t,(n_L-1,n_H))] \\
&= -\lambda_L p_L - \lambda_{B'}p_B + (\lambda_B - \lambda_{B'} - \lambda_L)[V_2(t,(n_L-1,n_H-1)) - V_2(t,(n_L-1,n_H))] \\
&+ (\lambda_B - \lambda_{B'} - \lambda_L)[V_2(t,(n_L-1,n_H)) - V_2(t,(n_L,n_H))] \\
&+ \lambda_L[V_2(t,(n_L-1,n_H-1)) - [V_2(t,(n_L-1,n_H))].
\end{aligned}$$

Since $\lambda_B \geq \lambda_{B'} + \lambda_L$, and the assumptions on the behaviour of $V_2(t,(n_L,n_H))$ stated in the theorem, we obtain the result that $\mathcal{G}_1V_2(t,(n_L,n_H))$ is an increasing function of t, n_L and n_H.

When $t \leq x^2_{(n_L,n_H)}$, from Theorem 2.2 and 2.3 we have $\mathcal{G}_2(\overline{V_2} + V_3)(t,(n_L,n_H)) = \mathcal{G}_2V_3(t,(n_L,n_H)) = \mathcal{G}_2V_2(t,(n_L,n_H)$.

$$\begin{aligned}
\mathcal{G}_1V_2(t,(n_L,n_H)) &= \frac{\partial V_2(t,(n_L,n_H))}{\partial t} + \lambda_B[V_2(t,(n_L-1,n_H-1)) - V_2(t,(n_L,n_H))] \\
&= \mathcal{G}_2V_3(t,(n_L,n_H)) - \lambda_{B'}[V_2(t,(n_L-1,n_H-1)) - V_2(t,(n_L,n_H))] \\
&- \lambda_L[V_2(t,(n_L-1,n_H)) - V_2(t,(n_L,n_H))] \\
&+ \lambda_B[V_2(t,(n_L-1,n_H-1)) - V_2(t,(n_L,n_H))] \\
&= \mathcal{G}_2V_3(t,(n_L,n_H)) + (\lambda_B - \lambda_{B'} - \lambda_L)[V_2(t,(n_L-1,n_H-1)) - V_2(t,(n_L,n_H))] \\
&+ \lambda_L[V_2(t,(n_L-1,n_H-1)) - V_2(t,(n_L-1,n_H))] \\
&= \mathcal{G}_2V_3(t,(n_L,n_H)) + (\lambda_B - \lambda_{B'} - \lambda_L)[V_2(t,(n_L-1,n_H-1)) - V_2(t,(n_L-1,n_H))] \\
&+ (\lambda_B - \lambda_{B'} - \lambda_L)[V_2(t,(n_L-1,n_H)) - V_2(t,(n_L,n_H))] \\
&+ \lambda_L[V_2(t,(n_L-1,n_H-1)) - V_2(t,(n_L-1,n_H))].
\end{aligned}$$

Since $\lambda_B \geq \lambda_{B'} + \lambda_L$, the assumptions on the behavior of $V_2(t,(n_L,n_H))$ stated in the theorem and $G_2V_3(t,(n_L,n_H))$'s being increasing in t, n_L and n_H, we obtain the result that $\mathcal{G}_1V_2(t,(n_L,n_H))$ is an increasing function of t, n_L and n_H.

Now, using this result we will prove the theorem. Although we need the proof for the particular case when n_L and n_H are equal to each other, the proof will be done for a general case when n_L and n_H can take any value on the defined domain. In the theorem $\overline{V_1}$ is defined as:

$$\overline{V_1}(t,(n_L,n_H)) = \begin{cases} \int_t^T L_1(s,(n_L,n_H))e^{-\lambda_B(s-t)}ds & if\ t > x^1_{(n_L,n_H)} \\ 0 & otherwise. \end{cases} \tag{2.18}$$

Since $\widetilde{V_1}(t,(n_L,n_H)) = V_1(t,(n_L,n_H)) - V_2(t,(n_L,n_H))$, where

$$\begin{aligned}
V_1(t,(n_L,n_H)) &= \sup_{\tau_1\in\mathcal{T}} E\big[p_B((N_B(\tau_1) - N_B(t)) \wedge n_L)\big] \\
&+ V_2(\tau_1,(n_L(\tau_1),n_H(\tau_1))),\ and \\
V_2(t,(n_L,n_H)) &= \sup_{\tau_2\in\mathcal{T}} E\big[p_L((N_{B'L}(\tau_2) - N_{B'L}(t)) \wedge n_L)\frac{\lambda_L}{\lambda_{B'} + \lambda_L} \\
&+ p_B((N_{B'L}(\tau_2) - N_{B'L}(t)) \wedge n_L)\frac{\lambda_{B'}}{\lambda_{B'} + \lambda_L}\big]
\end{aligned}$$

$$+ \quad V_3(\tau_2, (n_L(\tau_2), n_H(\tau_2))),$$

when $n_L = 1$, $\tau_1 = \tau_2 = t$ which means the switches should be executed immediately. Therefore, $\overline{V_1}(t, (n_L - 1, k)) = 0$, for $1 \le k \le M$, and we have $L_1(t, (1, k)) = \mathcal{G}_1 V_2(t, (1, k)) + \lambda_B p_B$. Therefore, for $1 \le k \le M - 1$,

$$\begin{aligned} L_1(t, (1, k+1)) &= \mathcal{G}_1 V_2(t, (1, k+1)) + \lambda_B p_B \\ &\ge \mathcal{G}_1 V_2(t, (1, k)) + \lambda_B p_B = L_1(t, (1, k)), \end{aligned}$$

since $\mathcal{G}_1 V_2(t, (1, k))$ is increasing in k. This implies that

$$\int_t^T L_1(s, (1, k+1)) e^{-\lambda_B(s-t)} ds \ge \int_t^T L_1(s, (1, k)) e^{-\lambda_B(s-t)} ds.$$

Together with Equation (2.18), this implies $\overline{V_1}(t, (1, k+1)) \ge \overline{V_1}(t, (1, k))$ and $x^1_{(1,k)} \ge x^1_{(1,k+1)}$ for $1 \le k \le M - 1$. For any $n_H = k$ ($1 \le k \le M$), we claim that for $t \le x^1_{(1,k)}$ the following holds:

$$\begin{aligned} L_1(t, (1, k)) &= \mathcal{G}_1 V_2(t, (1, k)) + \lambda_B p_B \\ &\le \mathcal{G}_1 V_2(x_{(1,k)}, (1, k)) + \lambda_B p_B = L_1(x_{(1,k)}, (1, k)) \le 0. \end{aligned}$$

The first inequality is a consequence of the increasing property of $\mathcal{G}_1 V_2(t, (1, k))$ in t. The second inequality follows from the fact that if $\mathcal{G}_1 V_2(x^1_{(1,k)}, (1, k)) + \lambda_B p_B > 0$ then this will contradict definition of $x^1_{(1,k)}$, since $\int_{x^1_{(1,k)}}^T L_1(s, (1, k)) e^{-\lambda_B(s-t)} ds > 0$. Hence, for $t \le x^1_{(1,k)}$ (or $\overline{V_1}(t, (1, k)) = 0$ by the definition of $\overline{V_1}$)

$$\begin{aligned} \mathcal{G}_1(\overline{V_1} + V_2)(t, (1, k)) + \lambda_B p_B &= \frac{\partial \overline{V_1}(t, (1, k))}{\partial t} + \lambda_B[\overline{V_1}(t, (0, k-1)) - \overline{V_1}(t, (1, k))] \\ &+ \mathcal{G}_1 V_2(t, (1, k)) + \lambda_B p_B \\ &= \mathcal{G}_1 V_2(t, (1, k)) + \lambda_B p_B = L_1(t, (1, k)) \le 0. \end{aligned}$$

Thus, condition (vi) is satisfied when $(n_L, n_H) = (1, k)$ and $t \le x^1_{(1,k)}$ (or $\overline{V_1}(t, (1, k)) = 0$). When $t > x^1_{(1,k)}$ (or $\overline{V_1}(t, (1, k)) > 0$) we have that

$$\begin{aligned} \mathcal{G}_1(\overline{V_1} + V_2)(t, (1, k)) + \lambda_B p_B &= \frac{\partial \overline{V_1}(t, (1, k))}{\partial t} + \lambda_B[\overline{V_1}(t, (0, k-1)) - \overline{V_1}(t, (1, k))] \\ &+ \mathcal{G}_1 V_2(t, (1, k)) + \lambda_B p_B \\ &= \frac{\partial \overline{V_1}(t, (1, k))}{\partial t} - \lambda_B \overline{V_1}(t, (1, k)) + \mathcal{G}_1 V_2(t, (1, k)) + \lambda_B p_B. \end{aligned} \tag{2.19}$$

By the definition of $\overline{V_1}(t, (n_L, n_H))$, we have $\overline{V_1}(t, (1, k)) = \int_t^T L_1(s, (1, k)) e^{-\lambda_B(s-t)} ds$. Taking the derivative with respect to t, we get

$$\begin{aligned} \frac{\partial \overline{V_1}(t, (1, k))}{\partial t} &= \lambda_B \int_t^T L_1(s, (1, k)) e^{-\lambda_B(s-t)} ds - L_1(t, (1, k)) \\ &= \lambda_B \overline{V_1}(t, (1, k)) - L_1(t, (1, k). \end{aligned} \tag{2.20}$$

Substituting (2.20) into (2.19), we get $\mathcal{G}_1(\overline{V_1} + V_2)(t, (1, k)) + \lambda_B p_B = 0$ when $\overline{V_1}(t, (1, k)) > 0$. Therefore, condition (vii) is satisfied when $(n_L, n_H) = (1, k)$. Moreover, we have $\overline{V_1}(t, (1, k)) \ge$

$\overline{V_1}(t,(0,k)) = 0$ by the definition of $x^1_{(1,k)}$ (there exists a time t such that $\overline{V_1}(t,(1,k)) > 0$ if $x^1_{(1,k)} > 0$). Now assume that the following statements hold for $1 < n_L \le \ell < M$: there exist time thresholds $T \ge x^1_{(\ell,1)} \ge x^1_{(\ell,2)} \ge \cdots \ge x^1_{(\ell,M)} \ge 0$ such that $\overline{V_1}(t,(\ell,k))$ is derived from Equation (2.18) and satisfies conditions (i), (ii), (iii), (vii). Also the inequality $\overline{V_1}(t,(\ell,k)) \ge \overline{V_1}(t,(\ell-1,k))$ holds for $k = 1,\dots,M$ and $\overline{V_1}(t,(\ell,k+1)) \ge \overline{V_1}(t,(\ell,k)$ holds for $k = 1,\dots,M-1$. When $n_L = \ell+1$;

$$\begin{aligned} L_1(t,(\ell+1,k)) &= \lambda_B\overline{V_1}(t,(\ell,k-1)) + \mathcal{G}_1V_2(t,(\ell+1,k)) + \lambda_B p_B \\ &\ge \lambda_B\overline{V_1}(t,(\ell-1,k-1)) + \mathcal{G}_1V_2(t,(\ell,k)) + \lambda_B p_B = L_1(t,(\ell,k)), \end{aligned}$$

since $\mathcal{G}_1V_2(t,(\ell,k))$ is increasing in ℓ and by the induction hypothesis. This implies that

$$\int_t^T L_1(s,(\ell+1,k))e^{-\lambda_B(s-t)}ds \ge \int_t^T L_1(s,(\ell,k))e^{-\lambda_B(s-t)}ds.$$

Together with Equation (2.18), this implies $\overline{V_1}(t,(\ell+1,k)) \ge \overline{V_1}(t,(\ell,k))$ and $x^1_{(\ell,k)} \ge x^1_{(\ell+1,k)}$ for $1 \le k \le M$. Similarly,

$$\begin{aligned} L_1(t,(\ell+1,k)) &= \lambda_B\overline{V_1}(t,(\ell,k-1)) + \mathcal{G}_1V_2(t,(\ell+1,k)) + \lambda_B p_B \\ &\ge \lambda_B\overline{V_1}(t,(\ell,k-2)) + \mathcal{G}_1V_2(t,(\ell+1,k-1)) + \lambda_B p_B \\ &= L_1(t,(\ell+1,k-1)), \end{aligned}$$

since $\mathcal{G}_1V_2(t,(\ell,k))$ is increasing in k and by the induction hypothesis. This implies that

$$\int_t^T L_1(s,(\ell+1,k))e^{-\lambda_B(s-t)}ds \ge \int_t^T L_1(s,(\ell+1,k-1))e^{-\lambda_B(s-t)}ds.$$

Together with Equation (2.18), this implies $\overline{V_1}(t,(\ell+1,k)) \ge \overline{V_1}(t,(\ell+1,k-1))$ and $x^1_{(\ell+1,k-1)} \ge x^1_{(\ell+1,k)}$ for $1 \le k \le M$. For any $n_H = k$ $(1 \le k \le M)$, when $t \le x^1_{(\ell+1,k)}$ (or $\overline{V_1}(t,(\ell+1,k)) = 0$),

$$\begin{aligned} \mathcal{G}_1(\overline{V_1}+V_2)(t,(\ell+1,k)) + \lambda_B p_B &= \frac{\partial\overline{V_1}(t,(\ell+1,k))}{\partial t} + \lambda_B[\overline{V_1}(t,(\ell,k-1)) - \overline{V_1}(t,(\ell+1,k))] \\ &+ \mathcal{G}_1V_2(t,(\ell+1,k)) + \lambda_B p_B \\ &= \mathcal{G}_1V_2(t,(\ell+1,k)) + \lambda_B p_B \\ &= L_1(t,(\ell+1,k)) \le L_1(x^1_{(\ell+1,k)},(\ell+1,k)) \le 0. \end{aligned}$$

Note that $\overline{V_1}(t,(\ell,k)) = \overline{V_1}(t,(\ell,k-1)) = 0$ since $t \le x^1_{(\ell+1,k)} \le x^1_{(\ell,k)} \le x^1_{(\ell,k-1)}$. The first inequality follows from $\mathcal{G}_1V_2(t,(\ell+1,k))$ being increasing in t, and the second inequality follows from the fact that if $L_1(x^1_{(\ell+1,k)},(\ell+1,k)) > 0$ then this will contradict the definition of $x^1_{(\ell+1,k)}$. Therefore, condition (vi) is satisfied, when $t \le x^1_{(\ell+1,k)}$ (or $\overline{V_1}(t,(\ell+1,k)) = 0$). When $t > x^1_{(\ell+1,k))}$ (or $\overline{V_1}(t,(\ell+1,k)) > 0$),

$$\begin{aligned} \mathcal{G}_1(\overline{V_1}+V_2)(t,(\ell+1,k)) + \lambda_B p_B &= \frac{\partial\overline{V_1}(t,(\ell+1,k))}{\partial t} + \lambda_B[\overline{V_1}(t,(\ell,k-1)) - \overline{V_1}(t,(\ell+1,k))] \\ &+ \mathcal{G}_1V_2(t,(\ell+1,k)) + \lambda_B p_B \\ &= \frac{\partial\overline{V_1}(t,(\ell+1,k))}{\partial t} - \lambda_B\overline{V_1}(t,(\ell+1,k)) + \lambda_B\overline{V_1}(t,(\ell,k-1)) \\ &+ \mathcal{G}_1V_2(t,(\ell+1,k)) + \lambda_B p_B = 0, \end{aligned}$$

since $\frac{\partial\overline{V_1}(t,(\ell+1,k))}{\partial t} = \lambda_B\overline{V_1}(t,(\ell+1,k)) - L_1(t,(\ell+1,k)$. Therefore condition (vii) is satisfied when $t > x^1_{(\ell+1,k)}$ (or $\overline{V_1}(t,(\ell+1,k)) > 0$). For $n_L = \ell+1$, we showed that conditions (i), (ii), (iii), (vii) hold. Thus the function $\overline{V_1}(t,(\ell,k))$ that is determined by the proposed procedure, is equal to $\widetilde{V_1}(t,(\ell,k))$. Further the switching time thresholds $(x^1_{(n_L,n_H)})$ are monotonically non-increasing in n_L and n_H. ■

For any inventory level (n_L, n_H), there exists a time $x^1_{(n_L,n_H)}$ such that: $\overline{V_1}(t,(n_L,n_H)) > 0$ if $t > x^1_{(n_L,n_H)}$, and $\overline{V_1}(t,(n_L,n_H)) = 0$ if $t \leq x^1_{(n_L,n_H)}$. Moreover, we can easily see that the switching-time thresholds, $x^1_{(n_L,n_H)}$ and $x^2_{(n_L,n_H)}$ are non-increasing in unsold inventory n_L and n_H. So, if the inventory decreases to a certain level of (n_L, n_H) before $x^1_{(n_L,n_H)}$, an immediate first switch should be taken. Otherwise, it is profitable to delay the switch to a later time. After the first switch, if the inventory decreases to a certain level of (n_L, n_H) before $x^2_{(n_L,n_H)}$, an immediate second switch should be taken, if it does not, delaying the switch is profitable.

2.4. Approximation and Calculation of $\overline{V_1}$ and $\overline{V_2}$ Functions

In order to find the optimal switching time thresholds, it is required to calculate the potentials expected from delaying the switches $\overline{V_1}(t,(n_L,n_H))$ and $\overline{V_2}(t,(n_L,n_H))$, respectively. The selling period is divided into many small time intervals of size δ, and then for all inventory levels and elapsed times, the corresponding values of $\overline{V_1}(t,(n_L,n_H))$ and $\overline{V_2}(t,(n_L,n_H))$ are calculated recursively starting with the system states $(T-\delta,(1,1))$ and $(T-\delta,(1,1))$, respectively. We present the approximation and the algorithms in the following subsections.

2.4.1 Approximation of the $\overline{V_2}$ Function

The details of the approximation using discrete time intervals are given below (see also (Feng, 1994)). For any $1 \leq n_L \leq n_H \leq M$, and $x^2(n_L,n_H) < t < T$ with some $\delta > 0$ such that $t+\delta \leq T$, we have

$$
\begin{aligned}
\overline{V_2}(t,(n_L,n_H)) &= \int_t^T L_2(u,(n_L,n_H))e^{-(\lambda_{B'}+\lambda_L)(u-t)}du \\
&= \int_{t+\delta}^T L_2(u,(n_L,n_H))e^{-(\lambda_{B'}+\lambda_L)(u-t)}du + \int_t^{t+\delta} L_2(u,(n_L,n_H))e^{-(\lambda_{B'}+\lambda_L)(u-t)}du \\
&= \int_{t+\delta}^T L_2(u,(n_L,n_H))e^{-(\lambda_{B'}+\lambda_L)(u-(t+\delta))}e^{-(\lambda_{B'}+\lambda_L)\delta}du \\
&+ \int_t^{t+\delta} L_2(u,(n_L,n_H))e^{-(\lambda_{B'}+\lambda_L)(u-t)}du \\
&= \overline{V_2}(t+\delta,(n_L,n_H))e^{-(\lambda_{B'}+\lambda_L)\delta} + \int_t^{t+\delta} L_2(u,(n_L,n_H))e^{-(\lambda_{B'}+\lambda_L)(u-t)}du \\
&= \overline{V_2}(t+\delta,(n_L,n_H))e^{-(\lambda_{B'}+\lambda_L)\delta} \\
&+ \int_t^{t+\delta} [\lambda_{B'}\overline{V_2}(u,(n_L-1,n_H-1)) + \lambda_L\overline{V_2}(u,(n_L-1,n_H)) \\
&+ \mathcal{G}_2 V_3(u,(n_L,n_H)) + \lambda_L p_L + \lambda_{B'} p_B]e^{-(\lambda_{B'}+\lambda_L)(u-t)}du.
\end{aligned}
$$

For a small time interval from t to $t+\delta$, we may take $V_3(t,(n_L,n_H))$ and $\overline{V_2}(t,(n_L,n_H))$ as constants and we get,

$$
\begin{aligned}
\overline{V_2}(t,(n_L,n_H)) &\cong \overline{V_2}(t+\delta,(n_L,n_H))e^{-(\lambda_{B'}+\lambda_L)\delta} \\
&+ (1-e^{-(\lambda_{B'}+\lambda_L)\delta})\overline{V_2}(t,(n_L-1,n_H-1))\frac{\lambda'_B}{\lambda_{B'}+\lambda_L} \\
&+ (1-e^{-(\lambda_{B'}+\lambda_L)\delta})\overline{V_2}(t,(n_L-1,n_H))\frac{\lambda_L}{\lambda_{B'}+\lambda_L} \\
&+ (1-e^{-(\lambda_{B'}+\lambda_L)\delta})[V_3(t,(n_L-1,n_H-1)) - V_3(t,(n_L,n_H))]\frac{\lambda'_B}{\lambda_{B'}+\lambda_L}
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad (1 - e^{-(\lambda_{B'}+\lambda_L)\delta})[V_3(t,(n_L-1,n_H)) - V_3(t,(n_L,n_H))]\frac{\lambda_L}{\lambda_{B'}+\lambda_L} \\
&+ \quad e^{-(\lambda_{B'}+\lambda_L)\delta}[V_3(t+\delta,(n_L,n_H)) - V_3(t,(n_L,n_H))] \\
&+ \quad (1 - e^{-(\lambda_{B'}+\lambda_L)\delta})(\frac{\lambda'_B p_B + \lambda_L p_L}{\lambda_{B'}+\lambda_L}).
\end{aligned}
$$

Therefore $\overline{V_2}(t,(n_L,n_H))$ can be estimated by

$$
\begin{aligned}
\overline{V_2}(t,(n_L,n_H)) \quad &\cong \quad (\overline{V_2}+V_3)(t+\delta,(n_L,n_H))e^{-(\lambda_{B'}+\lambda_L)\delta} \\
&+ \quad \frac{\lambda'_B}{\lambda_{B'}+\lambda_L}(1-e^{-(\lambda_{B'}+\lambda_L)\delta})[p_B + (\overline{V_2}+V_3)(t,(n_L-1,n_H-1))] \\
&+ \quad \frac{\lambda_L}{\lambda_{B'}+\lambda_L}(1-e^{-(\lambda_{B'}+\lambda_L)\delta})[p_L + (\overline{V_2}+V_3)(t,(n_L-1,n_H))] \\
&- \quad V_3(t,(n_L,n_H)).
\end{aligned}
$$

2.4.2 Approximation of the $\overline{V_1}$ Function

The details of the approximation using discrete time intervals are given below. For any $1 \le n_L = n_H \le M$, and $x^1(n_L,n_H) < t < T$ with some $\delta > 0$ such that $t+\delta \le T$, we have

$$
\begin{aligned}
\overline{V_1}(t,(n_L,n_H)) \quad &= \quad \int_t^T L_1(u,(n_L,n_H))e^{-\lambda_B(u-t)}du \\
&= \quad \int_{t+\delta}^T L_1(u,(n_L,n_H))e^{-\lambda_B(u-t)}du + \int_t^{t+\delta} L_1(u,(n_L,n_H))e^{-\lambda_B(u-t)}du \\
&= \quad \int_{t+\delta}^T L_1(u,(n_L,n_H))e^{-\lambda_B(u-(t+\delta))}e^{-\lambda_B\delta}du \\
&+ \quad \int_t^{t+\delta} L_1(u,(n_L,n_H))e^{-\lambda_B(u-t)}du \\
&= \quad \overline{V_1}(t+\delta,(n_L,n_H))e^{-\lambda_B\delta} + \int_t^{t+\delta} L_1(u,(n_L,n_H))e^{-\lambda_B(u-t)}du \\
&= \quad \overline{V_1}(t+\delta,(n_L,n_H))e^{-\lambda_B\delta} \\
&+ \quad \int_t^{t+\delta} [\lambda_B\overline{V_1}(u,(n_L-1,n_H-1)) + \mathcal{G}_1 V_2(u,(n_L,n_H)) + \lambda_B p_B]e^{-\lambda_B(u-t)}du.
\end{aligned}
$$

For a small time interval from t to $t+\delta$, we may take $V_2(t,(n_L,n_H))$ and $\overline{V_1}(t,(n_L,n_H))$ as constants and we get,

$$
\begin{aligned}
\overline{V_1}(t,(n_L,n_H)) \quad &\cong \quad \overline{V_1}(t+\delta,(n_L,n_H))e^{-\lambda_B\delta} \\
&+ \quad (1-e^{-\lambda_B\delta})\overline{V_1}(t,(n_L-1,n_H-1)) \\
&+ \quad (1-e^{-\lambda_B\delta})[V_2(t,(n_L-1,n_H-1)) - V_2(t,(n_L,n_H))] \\
&+ \quad e^{-\lambda_B\delta}[V_2(t+\delta,(n_L,n_H)) - V_2(t,(n_L,n_H))] \\
&+ \quad (1-e^{-\lambda_B\delta})p_B.
\end{aligned}
$$

Therefore $\overline{V_1}(t,(n_L,n_H))$ can be estimated by

$$
\begin{aligned}
\overline{V_1}(t,(n_L,n_H)) \quad &\cong \quad (\overline{V_1}+V_2)(t+\delta,(n_L,n_H))e^{-\lambda_B\delta} \\
&+ \quad (1-e^{-\lambda_B\delta})[p_B + (\overline{V_1}+V_2)(t,(n_L-1,n_H-1))] \\
&- \quad V_2(t,(n_L,n_H)).
\end{aligned}
$$

2.4.3 Algorithm for $\overline{V_2}$ computation

If the selling horizon T is divided into a large number K of intervals of length δ, we obtain

$$\begin{aligned}
\overline{V_2}(k\delta,(n_L,n_H)) &\cong (\overline{V_2}+V_3)((k+1)\delta,(n_L,n_H))e^{-(\lambda_{B'}+\lambda_L)\delta} \\
&+ \frac{\lambda'_B}{\lambda_{B'}+\lambda_L}(1-e^{-(\lambda_{B'}+\lambda_L)\delta})[p_B+(\overline{V_2}+V_3)(k\delta,(n_L-1,n_H-1))] \\
&+ \frac{\lambda_L}{\lambda_{B'}+\lambda_L}(1-e^{-(\lambda_{B'}+\lambda_L)\delta})[p_L+(\overline{V_2}+V_3)(k\delta,(n_L-1,n_H))] \\
&- V_3(k\delta,(n_L,n_H)).
\end{aligned}$$

Starting from the end of the selling horizon T, where $\overline{V_2}(T,(\cdot,\cdot))=0$, the following algorithm guides computations from inventory level $(n_L=1,n_H=1)$ to $(n_L=M,n_H=M)$ (for any inventory level where $1\le n_L\le n_H\le M$).

$$\begin{aligned}
Let,\ \Delta L_2(k\delta,(n_L,n_H)) &= (\overline{V_2}+V_3)((k+1)\delta,(n_L,n_H))e^{-(\lambda_{B'}+\lambda_L)\delta} \\
&+ \frac{\lambda'_B}{\lambda_{B'}+\lambda_L}(1-e^{-(\lambda_{B'}+\lambda_L)\delta})[p_B+(\overline{V_2}+V_3)(k\delta,(n_L-1,n_H-1))] \\
&+ \frac{\lambda_L}{\lambda_{B'}+\lambda_L}(1-e^{-(\lambda_{B'}+\lambda_L)\delta})[p_L+(\overline{V_2}+V_3)(k\delta,(n_L-1,n_H))] \\
&- V_3(k\delta,(n_L,n_H)).
\end{aligned}$$

- **Step 0**: Initialize $\overline{V_2}(T,(\cdot,\cdot))=\overline{V_2}(K\delta,(\cdot,\cdot))=0$ for all inventory levels. Set $n_L=n_H=1$ and $k=(K-1)$.
- **Step 1**: Calculate $\Delta L_2(k\delta,(n_L,n_H))$.
- **Step 2**: Set $\overline{V_2}(k\delta,(n_L,n_H))=(\Delta L_2(k\delta,(n_L,n_H)))^+$ and $k=k-1$. Do the following:
  - if $k\neq -1$ and $\overline{V_2}(k\delta,(n_L,n_H))\ge 0$, go to Step 1;
  - otherwise set $\overline{V_2}(j\delta,(n_L,n_H))=0$ for all $j<k-1$ and do the following:
    - if $n_H\neq M$ set $n_H=n_H+1$, go to Step 1;
    - otherwise do the following:
      - if $n_L\neq M$ set $n_L=n_L+1$ and $n_H=n_L$, go to Step 1;
      - otherwise stop.

2.4.4 Algorithm for $\overline{V_1}$ computation

If the selling horizon T is divided into a large number K of intervals of length δ, we obtain

$$\begin{aligned}
\overline{V_1}(k\delta,(n_L,n_H)) &\cong (\overline{V_1}+V_2)((k+1)\delta,(n_L,n_H))e^{-\lambda_B\delta} \\
&+ (1-e^{-\lambda_B\delta})[p_B+(\overline{V_1}+V_2)(k\delta,(n_L-1,n_H-1))] \\
&- V_2(k\delta,(n_L,n_H)).
\end{aligned}$$

Starting from the end of the selling horizon T, where $\overline{V_1}(T,(\cdot,\cdot))=0$, the following algorithm guides computations from inventory level $(n_L=1,n_H=1)$ to $(n_L=M,n_H=M)$ (for any inventory level where $1\le n_L=n_H\le M$).

$$\begin{aligned}
Let,\ \Delta L_1(k\delta,(n_L,n_H)) &= (\overline{V_1}+V_2)((k+1)\delta,(n_L,n_H))e^{-\lambda_B\delta} \\
&+ (1-e^{-\lambda_B\delta})[p_B+(\overline{V_1}+V_2)(k\delta,(n_L-1,n_H-1))] \\
&- V_2(k\delta,(n_L,n_H)).
\end{aligned}$$

- **Step 0**: Initialize $\overline{V_1}(T,(\cdot,\cdot)) = \overline{V_1}(K\delta,(\cdot,\cdot)) = 0$ for all inventory levels. Set $n_L = n_H = 1$ and $k = (K-1)$.
- **Step 1**: Calculate $\Delta L_1(k\delta,(n_L,n_H))$.
- **Step 2**: Set $\overline{V_1}(k\delta,(n_L,n_H)) = (\Delta L_1(k\delta,(n_L,n_H)))^+$ and $k = k-1$. Do the following:
  - if $k \neq -1$ and $\overline{V_1}(k\delta,(n_L,n_H)) \geq 0$, go to Step 1;
  - otherwise set $\overline{V_1}(j\delta,(n_L,n_H)) = 0$ for all $j < k-1$ and do the following:
    * if $(n_L = n_H) \neq M$ set $(n_L = n_H) = (n_L = n_H) + 1$, go to Step 1;
    * otherwise stop.

2.5. Computational Experiments

Here, the computational analysis on the relation between the parameters of the problem and the optimal switching times is demonstrated. Firstly, features of the extended version of the problem is analyzed, then the alternatives of single switch and two switches is compared.

We consider a stadium with 120 seats where one low-demand game and one high-demand game take place, and let the selling season start 2 months before the first game. The games have monthly demand rates of 50 and 40 tickets and these tickets are priced as \$200 and \$50 for the high- and the low-demand game, respectively. When the tickets are sold as a bundle before switch time, the monthly demand rate is assumed to be 130. If the low-demand game tickets are sold along with the bundled tickets, this monthly rate is decreased to 80. The bundled ticket price is set to \$220. In Table 2.1, the optimal switch times (in months) for the selected inventory levels, where the number of remaining seats for two games are equal to each other, are shown. As expressed in Theorem 2.3 and 2.4, we have non-increasing thresholds for optimal switch times in remaining seat inventory. The graphs of switch time thresholds are presented in Figure 2.4.

Table 2.1: Selected Optimal Switching Times

remained seats (n_L, n_H)	72	73	74	75	76	77	78	79	80	81
first switch time	0.544	0.524	0.508	0.488	0.468	0.452	0.432	0.412	0.396	0.376
second switch time	0.196	0.172	0.148	0.124	0.1	0.076	0.052	0.028	0.004	0

Effect of the model parameters on the switch time thresholds is also a key interest. As the demand rate of the bundled tickets after the first switch is increased, it takes less time to reach the first switch. The reason is the desire of taking advantage of the higher rate after the switch. In a similar manner, if the demand rate of the bundled tickets before the first switch is increased, it takes more time to implement the first switch, because it is advantageous to delay the switch from a higher rate.

Changes in the demand rates of single tickets affect both of the first and second optimal switching time thresholds. An increase in the rate of high-demand ticket leads to increase of both of the time thresholds, because the delay of the switches has less effect on the expected revenue. On the other hand, if the rate of the low-demand tickets is increased, the second switch occurs earlier, since the need for the bundled ticket sale decreases. Note that, since the selling time of high-demand tickets with a higher rate decreases, the first switch time thresholds decrease for taking advantage of highest rate of the bundled tickets. The effects of the changes in single ticket demand rates are illustrated in Figure 2.5.

If it is allowed for the tickets to be sold after the performance season starts, a similar effect is observed. For example, when the second game is low-demand game, it has an effect on the switch time thresholds which is similar to the effect of an increase on the rate of low-demand ticket. The effects of ticket

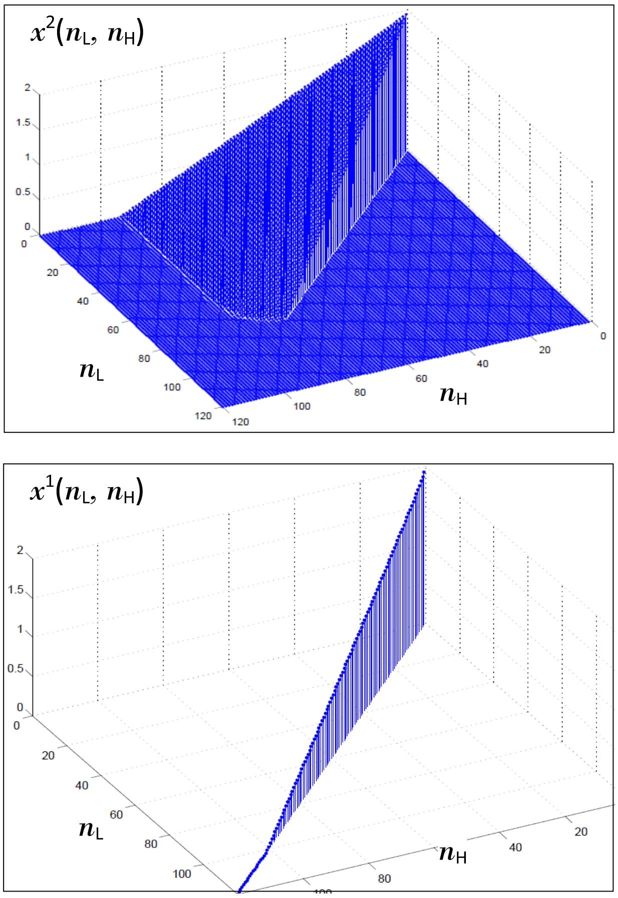

Figure 2.4: Switch Time Thresholds

price changes are similar to those of demand rate changes, therefore it is not explained here to avoid repetition.

We also conduct numerical experiments for examining the potential revenue enhancement with the use of offered dynamic strategy over the strategies with one switch time and static switch time(s). The parameters are set to the same values as they are in the previous experiments. The monthly rates are changed by 10 customer arrivals for analyzing the effect of decrease and increase in the demand rate on revenue enhancement. 100,000 random customer arrival patterns are generated and average expected revenue is calculated. 500 evenly distributed points are taken on the ticket selling period, and all of the combinations for first and second static switch times are determined. The dynamic switch strategy is compared to all of these static switch time combinations and no revenue enhancement less than 0.44% is observed. In Figure 2.6, revenue improvements of only a few selected combinations are presented. The revenue enhancement of the dynamic strategy can increase up to 6.71%. Moreover, with these parameter settings, the strategy with two switch times increases the revenue obtained by single dynamic switch. It is observed that, the revenue enhancement of the model with two switch times increases with the increase in the demand rate of the bundled tickets sold between two switches.

We believe that more research is needed to analyze the model robustness regarding the ticket demand rates. Demand learning is also a key research interest as it is described in the study of Gallego and Talebian (2010).

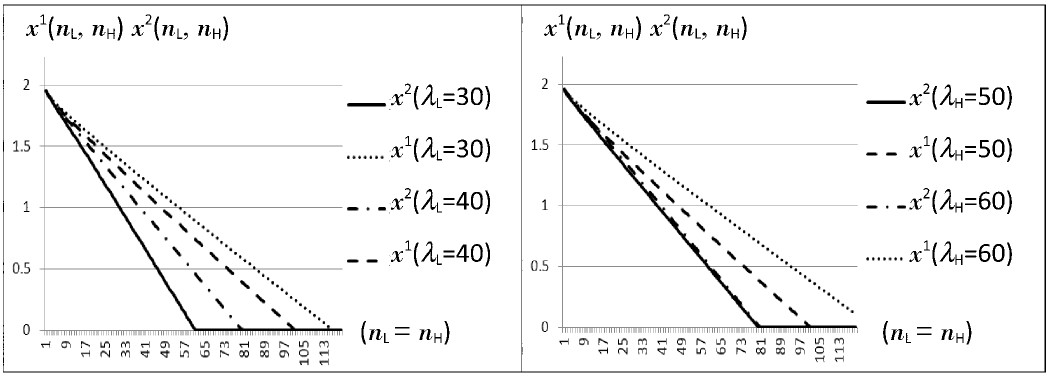

Figure 2.5: Effect of Single Ticket Demand Rates on Switching Time Thresholds

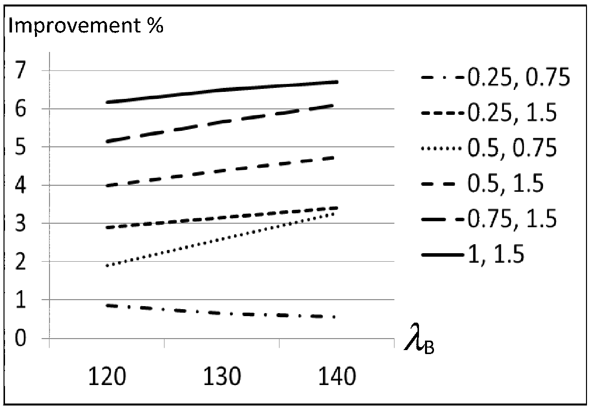

Figure 2.6: Percentage Revenue Improvement of Dynamic Switching Over Static Case.

2.6. Conclusion

In the context of revenue maximization, the problem of determining the optimal dynamic switch times from selling bundled tickets to selling single tickets is studied. An easy and intuitive policy is developed where the optimal switch times are determined with a comparison of elapsed time to the corresponding time threshold of remaining inventory at that moment. This comparison leads to the decision of taking an immediate switch or not. It is observed that, the thresholds for switching times are non-increasing in the remaining ticket inventory which means that when the inventory at hand increases, delaying the switch is profitable because of the advantage of a longer bundle sale time.

If the estimation of the demand rates are correct, the dynamic switch time strategy can enhance the expected revenue over the alternative strategy of static switch time. Moreover, although it is not that significant, the offered model with two switch times can add more revenue on that of the model with only one switch time. It is seen that, the model can be generalized to include multiple events in the performance season, or the ticket demand rates can be allowed to be non-constant.

CHAPTER 3

SELECTION OF EVENT TICKETS FOR BUNDLING

3.1. Introduction

In this chapter, we studied the effects of the schedules of events on the bundle selection decision, assuming predetermined prices, independent and constant demand intensities (customer arrival rates) for individual and bundled tickets. We allow the sale of pure bundles at first and after a switch time only the sale of individual tickets are allowed. In this context, the perishable characteristic of sports or entertainment tickets is of major importance. We observe that the ratio of demand to venue capacity is a determining factor for being selected for the most profitable bundles out of a group of events, and under certain assumptions, the schedule of the events has great influence on the profitability of a selected bundle. We design, implement, and analyze a bundling strategy, which is basically a set of rules to be used in bundling process, and evaluate its performance on instances with different characteristics. We then conduct an extensive computational study to assess the performance on randomly generated instances.

The rest of this chapter is organized as follows. In Section 3.2, we review related literature on RM applications in S&E industry, bundling of commodities and scheduling in S&E. In Section 3.3, we first give a formal statement of our problem and then present some properties of the problem that allow us to state certain rules of thumb in determining the contents of a bundle. In Section 3.4, we demonstrate how our proposed method performs on randomly generated instances. Concluding remarks are provided in Section 3.5.

3.2. Literature Review

The literature on the RM problems in other industries and also on the problems relevant to dynamic ticket pricing and timing is given in the previous chapter. Another stream of literature related to this chapter is on bundling strategies. The initial studies in this area (e.g. (Stigler, 1963) , (Adams & Yellen, 1976)) assume strict demand additivity which is the case when products are not interdependent (neither substitutes nor complements). However, when the products are interdependent, additivity assumption often is not valid. In the literature, bundling of such products is analyzed and concluded that several major factors such as substitutability and complementarity relations, heterogeneity of valuations, correlation between reservation prices, should be considered in determining the optimal bundling strategies (Gürler, Öztop, & Şen 2009).

In bundling of products, there are two main strategies: "price bundling" and "product bundling". According to Stremersch and Tellis (2002), price bundling is "the sale of two or more separate products in a package at a discount without any integration of products" and product bundling is "the integration and sale of two or more separate products or services at any price". Even though there are certain distinctions between the retail industry and S&E industry in terms of bundling the products, the practice of ticket bundling is closer to price bundling as most of the time the motivation of the customers for buying the bundle is discount. In general, the seller's motivation behind the bundling practice may differ; pricing efficiency, market power enhancement or cost savings. But for S&E industry, the seller's motivation in bundling tickets for sports or cultural events is revenue maximization by controlling seat inventory and the price of these perishable products by selling them in different forms and prices.

Venkatesh and Kamakura (2003) studied the effect of complementarity and substitutability on the strategies of no bundling, pure bundling or mixed bundling (i.e., offering bundled and individual items) under monopoly conditions. They suggest mixed bundling for independently valued products and weak substitutes or complements, and no bundling for two moderate or strong substitutes. Ibragimov (2005) observed optimal bundling decisions for complements and substitutes with heavy tail distributions for marketing strategies for the goods with extreme valuations. He discussed that season tickets for entertainment performances that have sufficiently high marginal costs of production might illustrate dual pattern in bundling. The optimal strategy is to offer tickets as a bundle to a small fraction of consumers that have high valuations for performances within the bundle while avoiding separate sale of the most of tickets to consumers with normal valuations. Mixed bundling is typically optimal for most of the cases due to both reduced heterogeneity in consumer valuation (Schmalensee, 1984) and capturing residual demand through separate sale. Considering the relative position of cost and reservation values, Salinger (1995) indicates when positively (or negatively) correlated reservation prices are profitable for bundling.

3.3. Model and Assumptions

We want to maximize the total revenue from the ticket sale of events that will take place periodically (e.g. weekly or bi-weekly) at a venue during a performance season. During the performance season h events, given by the set $H = \{e_{(1)}, e_{(2)}, \ldots, e_{(h)}\}$, take place. The events are numbered within the set H according to their expected demand level from lowest to highest. Dispersion of the events throughout the performance horizon is given by the schedule of the performance season. Schedule of h events, S, is also defined as a set $S = \{T_1^{e_{(\cdot)}}, \ldots, T_h^{e_{(\cdot)}}\}$ where $e_{(\cdot)} \in H$ and $T_1^{e_{(\cdot)}} < \ldots < T_h^{e_{(\cdot)}}$ are the length of selling periods of first, second, ..., last event that take place throughout the performance season. Therefore, the notation of $T_2^{e_{(1)}}$ carries the information that the event with lowest expected demand is located to be the second event in the performance season and the length of the selling period for this event. For a given schedule of h events, the revenue generation from ticket sales occurs within the revenue horizon, starting a certain time period before the first event and ending with the last event's performance time. The revenue horizon for a 3-event performance season when the highest demand event is scheduled to be the first and the lowest demand event is scheduled to be the last is illustrated in Figure 3.1.

Two types of products are on sale throughout the revenue horizon; single and bundled tickets. Selling period of each single ticket that are not included in the bundle ends when the corresponding event takes place. When tickets of k events form a bundle B, a discount is applied to the sum of the prices of these tickets. The bundle schedule, $S^B \subset S$, represents the selling periods of the single tickets within the bundle. The selling period of the bundle, T_B, is set to be the selling period of the first event in the bundle, however a switch from the bundle sale to single ticket sales can occur within this time. Therefore, bundles are on sale until the switch time and afterwards the tickets of the events composing the bundle are on sale individually till their respective performance times. The revenue horizon for a

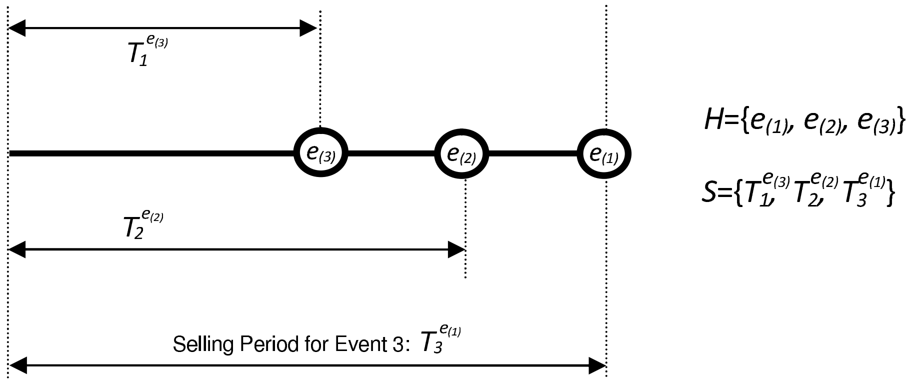

Figure 3.1: Revenue Horizon for a 3-Event Performance Season

4-event performance season with a 2-event bundle is illustrated in Figure 3.2 to clarify the ticket selling mechanism.

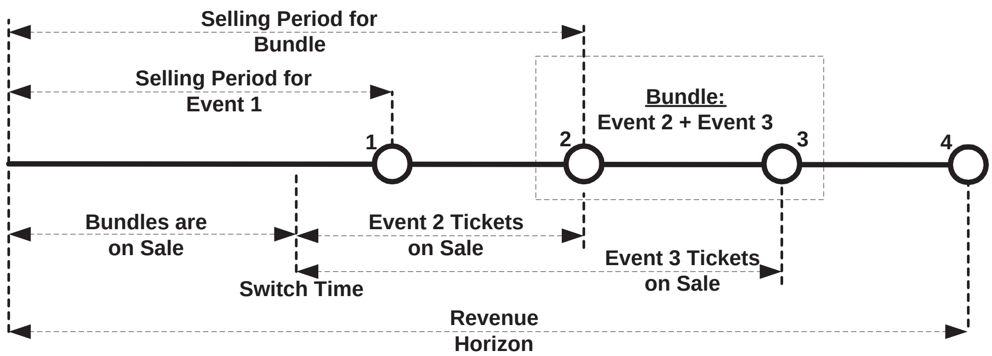

Figure 3.2: Revenue Horizon for a 4-Event Performance Season with a 2-Event Bundle

Demand for tickets arrive according to a Poisson Process for each event. Demand rates are assumed to be constant over the period of sale but this assumption can be relaxed (see (Duran and Pakyardım, 2012)). We have considered that the bundle demand is a certain ratio of the total demand of the bundled events or of the demand of the highest demand event in the bundle. Beyond these demand assumptions, we assumed that the expected demands of the events are fixed and not affected by the schedule.

Other than the discounted price assumption for the bundle, we do not impose any more restrictions on the price and demand rates of the offered products. However, the (instant) revenue rates, multiplication of the price and demand rate of a product, are assumed to satisfy the following relation; bundle revenue rate should be greater than the total of single ticket revenue rates. Otherwise, switching from bundle to single tickets immediately would be optimal for all states and no bundles will be sold.

The expected revenue from a ticket inventory level $I = \{n_1, \ldots, n_h\}$, where n_i corresponds to the number of available tickets for event $e_{(i)} \in H$, over the time horizon starting at a time t before the switch happens and ending with the performance of the last event, $[t, T_h^{(\cdot)}]$ when $T_h^{(\cdot)} \in S$, can be formulated as follows:

$$V(t, I, S) = V_k^B(t, n_B, S^B) + \sum_{e_{(i)} \in H \setminus B} \Pi_{e_{(i)}}(t, n_i, T_{(\cdot)}^{e_{(i)}}),$$

where $B \subset H$, $S^B \subset S$ and $T_{(\cdot)}^{e_{(i)}} \in S$.

The two terms in the right-hand side of the formula represent the revenue from the sale of the bundled tickets (consisting of k events) and the revenue from the events that are sold only as single tickets, respectively. Since the expected demands and the prices are assumed to be fixed throughout the selling period of each event, the time slot of the events which are not bundled does not change the total expected revenue (gate receipt). Therefore, the best bundle selection throughout the revenue horizon for a given schedule is only affected by the demands, the prices and the time slots (or schedule) of the bundled events. The optimal time to switch from bundle sale to single ticket sales of the events in the bundle is crucial to find the overall revenue from the tickets that are bundled (sold in the bundle and as single after the switch) and the method to find that dynamic optimal switch time is developed by Duran, Özener, and Yakıcı (2012).

The ticket inventory level of the bundle, n_B, is equal to the ticket inventory level of any event within the bundle, since they are equal to each other until the switch happens. For the second term, if $t < T_{(\cdot)}^{e_{(i)}}$, then the expected revenue from a single ticket sale of event $e_{(i)}$, $\Pi_{e_{(i)}}(t, n_i, T_{(\cdot)}^{e_{(i)}})$, is equal to $p_i E[(N_i(T_{(\cdot)}^{e_{(i)}}) - N_i(t)) \wedge n_i]$ where N_i is the Poisson Process of demand for the tickets of event $e_{(i)}$, p_i represents the price of the ticket for event $e_{(i)}$ and $(x \wedge y)$ indicates the minimum of x and y. Otherwise, it is equal to zero. The structure and the details of the first term is explained in Subsection 3.1.

When a set of h events is given, the best bundle to select consisting of k tickets to create the greatest marginal revenue contribution when they are bundled, can be found by checking all of the $C(h, k)$ possible alternatives, where $C(h, k)$ is the number of k-size combinations of h events. The problem of determining the optimal bundle is clearly a difficult problem with combinatorial nature. To tackle the problem, we first try to investigate the problem characteristics in detail with a small bundle size, obtain various insights in determining a good/optimal bundle, then we attempt to generalize our findings. Finally, we develop a heuristic method that yields promising results for a real-size problem in reasonable computational time. Hence, we first focus on the two-event ($k = 2$) bundle formation problem to develop structural results. Then, we extend our investigation to larger bundles with $k > 2$.

3.3.1 Bundles of Two Events

Let $M \in \mathbb{Z}^+$ be the number of seats available for sale for each event at a venue. Results that will be obtained in this section are independent of the demand levels of the events. Therefore, we use a simpler notation to denote the schedule of the bundle $S^B = \{T_F, T_S\}$, where $T_F \in \mathbb{R}^+$ is the selling period of the first event and $T_S \in \mathbb{R}^+$ is the selling period of the second event of the bundle. Thus, the selling period of the bundle, $T_B \in \mathbb{R}^+$, is equal to T_F. The selling period begins with first offering tickets as a bundle of two event tickets at price p_B and then switching at the optimal time of τ ($\leq T_B$) to selling individual event tickets at p_F and p_S for individual events of the first and second event, respectively. We assume that the product prices are predetermined at the beginning of the selling season, which is true for most organizations, especially during the time preceding the start of the season.

We assume that for each product, there is a corresponding Poisson Process of demand: $N_B(s)$, $0 \leq s \leq t$, with known constant demand d_B and constant intensity d_B/T_B for the bundled events; $N_F(s)$,

$0 \leq s \leq t$, with known constant demand d_F and constant intensity d_F/T_F, and $N_S(s)$, $0 \leq s \leq t$, with known constant demand d_S and constant intensity d_S/T_S for the single events of the first and second event in revenue horizon, respectively. The revenue horizon with the associated notation of such a 2-event bundle is shown in Figure 3.3.

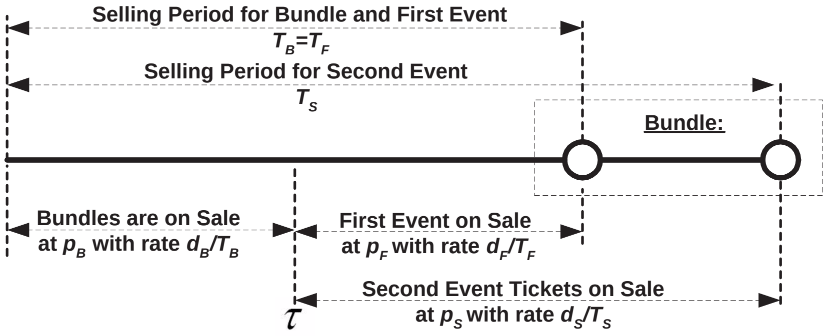

Figure 3.3: Revenue Horizon for a 2-Event Bundle

The state of the system is indicated by the elapsed time t, the remaining inventory level at time t, $n(t)$ or simply n if it is observable at time t and the schedule of *first* and *second* events, $S^B = \{T_F, T_S\}$. The expected total revenue from the bundle composed of two events and single ticket sales over the time horizon $[t, T_S]$ with the optimal switching time is given by $V_2^B(t, n, S^B)$, which is a modified version of the model in Duran et al. (2012) letting the single event tickets to be on sale until those events take place:

$$\begin{aligned}
V_2^B(t, n, S^B) &= \sup_{\tau \in \mathcal{T}} E[p_B((N_B(\tau) - N_B(t)) \wedge n)] + \Pi_F(\tau, n(\tau), T_F) + \Pi_S(\tau, n(\tau), T_S) \\
&= \sup_{\tau \in \mathcal{T}} E[p_B((N_B(\tau) - N_B(t)) \wedge n)] + p_F E[(N_F(T_F) - N_F(\tau)) \wedge n(\tau)] \\
&+ \ p_S E[(N_S(T_S) - N_S(\tau)) \wedge n(\tau)]
\end{aligned}$$

$\mathcal{T}$ is the set of switching times τ satisfying $t \leq \tau \leq T_B$ and $n(\tau) = [n - N_B(\tau) + N_B(t)]^+$, where $x^+ = \max\{0, x\}$. The expected total revenue also includes the expected revenue from single ticket sales after the switch from bundles with $n(\tau)$ items available for sale over $[\tau, T_F]$ and $[\tau, T_S]$, which are denoted as $\Pi_F(\tau, n(\tau), T_F)$ and $\Pi_S(\tau, n(\tau), T_S)$ for the first and the second events, respectively. They are given by

Observation 3.1 *Expected number of tickets to be sold in the time interval t when the starting number of seats is n and customer arrivals are following a Poisson Process ($N(s), 0 \leq s \leq t$) with rate λ;*

$$E[N(t) \wedge n] = \sum_{k=0}^{n} \frac{(\lambda t)^k}{k!} e^{-\lambda t} k + \sum_{k=n+1}^{\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t} n$$

is non-decreasing in λt.

Proof.

$$\begin{aligned}
\frac{\partial E[N(t)\wedge n]}{\partial \lambda t} &= -\sum_{k=1}^{n}\frac{(\lambda t)^k}{(k-1)!}e^{-\lambda t}+\sum_{k=1}^{n}\frac{(\lambda t)^{k-1}}{(k-1)!}ke^{-\lambda t}-n\sum_{k=n+1}^{\infty}\frac{(\lambda t)^k}{k!}e^{-\lambda t}+n\sum_{k=n+1}^{\infty}\frac{(\lambda t)^{k-1}}{k!}ke^{-\lambda t}\\
&= -\sum_{k=1}^{n}\frac{(\lambda t)^k}{k!}ke^{-\lambda t}+\sum_{k=0}^{n-1}\frac{(\lambda t)^k}{k!}(k+1)e^{-\lambda t}-\frac{n}{\lambda t}\sum_{k=n+2}^{\infty}\frac{(\lambda t)^k}{k!}ke^{-\lambda t}+\frac{n}{\lambda t}\sum_{k=n+1}^{\infty}\frac{(\lambda t)^k}{k!}ke^{-\lambda t}\\
&= -\frac{(\lambda t)^n}{n!}ne^{-\lambda t}+\sum_{k=0}^{n-1}\frac{(\lambda t)^k}{k!}e^{-\lambda t}+\frac{n}{\lambda t}\frac{(\lambda t)^{(n+1)}}{(n+1)!}(n+1)e^{-\lambda t}=\sum_{k=0}^{n-1}\frac{(\lambda t)^k}{k!}e^{-\lambda t}\\
&= P[N(t)\le n-1]\ge 0.
\end{aligned}$$

■

Although this observation is simple, it is the foundation of the forthcoming results on revenue and schedule relationship since the demand rate of a product is found as the ratio of its total constant demand and selling period. Next, we propose two lemmas and two following corollaries about the relationship between the revenue of a given schedule and the timing of the events that are included in the bundle. Intuitively, it is obvious that timing of the events in a schedule will affect the revenue generated by bundling. Therefore, the structural properties derived in the following lemmas and corollaries will serve as guidelines in determining which events to bundle in a given schedule.

Lemma 3.1 *For a bundle of two events, keeping the time gap between the bundled events fixed, as the time of the events move further in the revenue horizon, the revenue from these two events decreases.*

Proof. Two different schedules are considered for the bundle as illustrated in Figure 3.4. Before the ticket sale starts, the expected total revenue from bundle and single ticket sales with the optimal switching times are given by $V_2^B(0,n,\{T_{F_1},T_{S_1}\})$ and $V_2^B(0,n,\{T_{F_2},T_{S_2}\})$ for schedule 1 and schedule 2, respectively as:

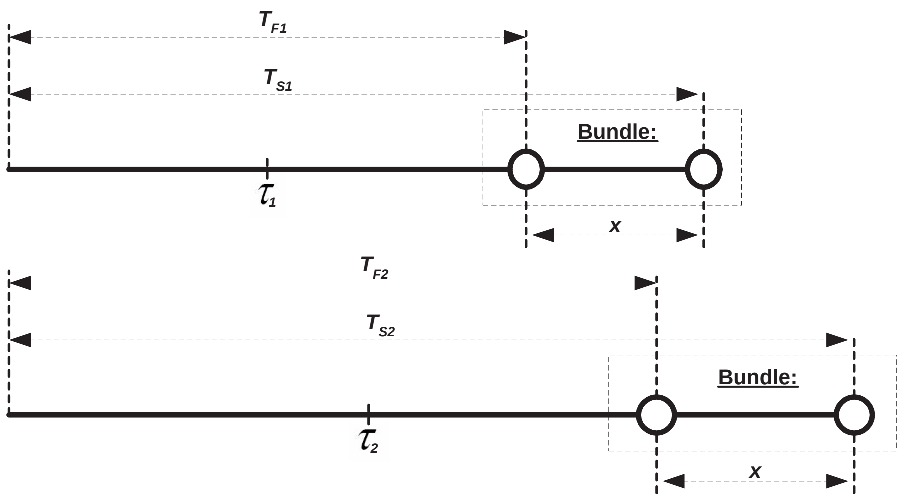

Figure 3.4: Events in the Bundle Move Forward in the Revenue Horizon

$$\begin{aligned}
V_2^B(0,n,\{T_{F_1},T_{S_1}\}) &= \sup_{\tau_1\in\mathcal{T}} E[p_B((N_{B_1}(\tau_1))\wedge n)]+\Pi_{F_1}(\tau_1,n(\tau_1),T_{F_1})+\Pi_{S_1}(\tau_1,n(\tau_1),T_{S_1}),\\
V_2^B(0,n,\{T_{F_2},T_{S_2}\}) &= \sup_{\tau_2\in\mathcal{T}} E[p_B((N_{B_2}(\tau_2))\wedge n)]+\Pi_{F_2}(\tau_2,n(\tau_2),T_{F_2})+\Pi_{S_2}(\tau_2,n(\tau_2),T_{S_2}).
\end{aligned}$$

Let $\tau_2 > 0$ be optimal switch time of schedule 2 and select a switch time τ_1 for schedule 1 such that $\tau_1/T_{F1} = \tau_2/T_{F2}$. Then, $E[N_{B_1}(\tau_1)] = E[N_{B_2}(\tau_2)]$ since $\frac{d_B}{T_{F1}}\tau_1 = \frac{d_B}{T_{F2}}\tau_2$. Moreover, $E[N_{F_1}(T_{F1} - \tau_1)] = E[N_{F_2}(T_{F2} - \tau_2)]$ since $d_F\frac{T_{F1}-\tau_1}{T_{F1}} = d_F\frac{T_{F2}-\tau_2}{T_{F2}}$ and $E[N_{S_1}(T_{S1} - \tau_1)] > E[N_{S_2}(T_{S2} - \tau_2)]$ since $d_S\frac{T_{F1}+x-\tau_1}{T_{F1}+x} > d_S\frac{T_{F2}+x-\tau_2}{T_{F2}+x}$. Therefore,

$$\begin{aligned} V_2^B(0, n, \{T_{F_2}, T_{S_2}\}) &< E[p_B((N_{B_1}(\tau_1)) \wedge n)] + \Pi_{F_1}(\tau_1, n(\tau_1), T_{F_1}) + \Pi_{S_1}(\tau_1, n(\tau_1), T_{S_1}), \\ &\leq V_2^B(0, n, \{T_{F_1}, T_{S_1}\}). \end{aligned}$$

For details see Appendix B. ■

Although under the assumption of a pre-determined schedule, it is not possible to change the schedule of events, the proof above is still of use. Given two different bundles composed of similar events and time gap between events, we prefer to select the earlier bundle than the later, as we expect to generate higher revenue from single ticket sales of the events of the earlier bundle.

Lemma 3.2 *For a bundle of two events, as the time of second event advances in the revenue horizon while the first event time is kept fixed, the revenue from these two events increases.*

Proof. Consider the two schedules of the bundle in Figure 3.5. Before the ticket sale starts, the expected total revenue from bundle and single ticket sales with the optimal switching times are given by $V_2^B(0, n, \{T_{F_1}, T_{S_1}\})$ and $V_2^B(0, n, \{T_{F_2}, T_{S_2}\})$ for schedule 1 and schedule 2, respectively as explained in Lemma 3.1. We have $T_{F_1} = T_{F_2}$ and $T_{S_2} > T_{S_1}$. Let $\tau_1 > 0$ be the optimal switch time of the schedule

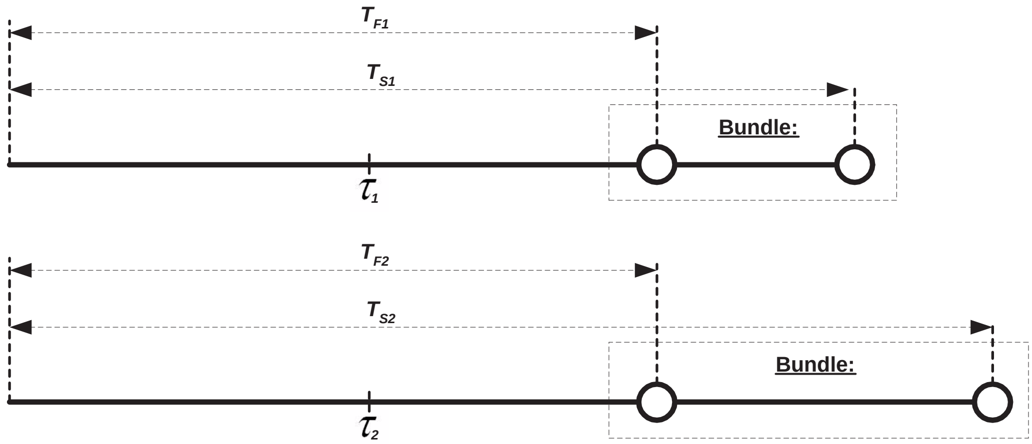

Figure 3.5: Only the Second Event in the Bundle Moves Forward in the Revenue Horizon

1, and select a switch time τ_2 for schedule 2 such that $\tau_1 = \tau_2$. Obviously, $E[N_{B_1}(\tau_1)] = E[N_{B_2}(\tau_2)]$ since $\frac{d_B}{T_{F1}}\tau_1 = \frac{d_B}{T_{F2}}\tau_2$. Moreover, $E[N_{F_1}(T_{F1} - \tau_1)] = E[N_{F_2}(T_{F2} - \tau_2)]$ since $d_F\frac{T_{F1}-\tau_1}{T_{F1}} = d_F\frac{T_{F2}-\tau_2}{T_{F2}}$ and $E[N_{S_1}(T_{S1} - \tau_1)] < E[N_{S_2}(T_{S2} - \tau_2)]$ since $d_S\frac{T_{S1}-\tau_1}{T_{S1}} < d_S\frac{T_{S2}-\tau_2}{T_{S2}}$. Therefore,

$$\begin{aligned} V_2^B(0, n, \{T_{F_1}, T_{S_1}\}) &< E[p_B((N_{B_2}(\tau_2)) \wedge n)] + \Pi_{F_2}(\tau_2, n(\tau_2), T_{F_2}) + \Pi_{S_2}(\tau_2, n(\tau_2), T_{S_2}), \\ &\leq V_2^B(0, n, \{T_{F_2}, T_{S_2}\}). \end{aligned}$$

For details see Appendix B. ■

Similar to Lemma 1, even though we cannot alter the schedule of events, given two different bundles composed of similar events, we prefer to select the one with longer time gap between the events, as we expect to generate a higher revenue from the last event of the bundle with the additional time to sell it as a single ticket.

Corollary 3.1 *For a bundle of two events, as the time of the first event advances in the revenue horizon while the second event time is kept fixed, the revenue from these two events decreases.*

Proof. Consider that first both of the event times are increased by a period of x, and then the second event time is decreased by the same amount x. These operations are applications of Lemma 3.1 and Lemma 3.2, respectively. ■

Corollary 3.2 *For a bundle of two events, as the time of the first event recedes and that of the second event advances in the revenue horizon, the revenue from these two events increases.*

Proof. Consider that first the time of the second event time is being advanced, and then the time of the first event time is receded. These operations are applications of Lemma 3.2 and Corollary 3.1, respectively. ■

These two corollaries are basically implications of the Lemma 1 and Lemma 2, yet they will be quite useful in constructing the heuristic algorithm that will be discussed later on.

3.3.2 Bundles of Three or More Events

After identifying several useful properties for bundles of size two, next we attempt to generalize these results to the situations with bundles of three or more events. This is not a straightforward task due to the complex interactions of multiple events, their demands and of course their relative positions in the given schedules. We merely attempt to come up with "rule of thumbs" in the selection of bundles, which will act as building blocks to the heuristic algorithm.

When the size of the bundle is increased from two to three, we investigate the following situations for determining the relationship between the timing of the events and the revenue generated by bundling these events:

1. the first event or the first two events are fixed, and the other event(s) advance(s)/recede(s) in revenue horizon,
2. the last event or the last two events are fixed, and the other event(s) advance(s)/recede(s) in revenue horizon,
3. the first and the last events are fixed, and the middle event advance(s)/recede(s) between the first and last event performance time.

When the first event's time slot is kept fixed, if the second or third event advance(s) in the revenue horizon (with or without a fixed gap between), the expected revenue from the bundle revenue increases by Lemma 3.2. Lemma 3.2 also explains the schedule where the first and the third event performance times are fixed and the second event is moved between those events. Since the conditions of Lemma 3.1 are independent of the bundle size, it also applies whenever all of the events in the bundle are advanced/receded along the time slots keeping the time gaps between them fixed.

When we generalize these results for bundles of size k, the following corollaries apply.

Corollary 3.3 *When the first event performance time is kept fixed, if any of the other events' performance time advances in the revenue horizon, the expected revenue from the bundle increases.*

Corollary 3.4 *The maximum revenue from a bundle composed of k events is obtained when the first event of the bundle is scheduled at the first time slot and the others are scheduled at the last time slots in the revenue horizon.*

Corollary 3.5 *The minimum revenue from a bundle composed of k events is obtained when all of the events of a bundle are scheduled at the last time slots in the revenue horizon.*

Corollary 3.6 *When the performance time of all events in the bundle advance in the revenue horizon by the same amount of time, the expected revenue from the bundle decreases.*

Corollaries 3.3, 3.4, 3.5, 3.6 immediately follow from Lemmas 3.1 and 3.2. Since we establish the relationships between the timing of events of a bundle and the revenue generated by the bundle, we can construct a heuristic approach to determine a good, hopefully best, bundle out of all possible bundling options.

3.3.3 A Heuristic Approach for Bundle Selection

Next, we develop a heuristic method to choose good/best bundle(s) for a given schedule. Note that there are other factors affecting the revenue generation rate of a bundle besides the timing of the events in a bundle. Hence, we list a set of guidelines that heuristically will result in constructing good bundles.

i) Bundling a high-demand event with low-demand events is better than bundling the events with similar demand levels. Intuitively, the high-demand event(s) will serve as the motivation for the customers to buy bundle, and in turn will increase the revenue generated by the low-demand events in the bundle. This property holds in both cases where the demand rate of the bundle depends on the demand rate of all the events in the bundle and only on the demand rate of the highest-demand event in the bundle. The former situation is usually observed in bundling concert tickets, while the latter situation is usually observed in bundling sports events. Even though, this is an important observation, it is still not clear what low-demand and high-demand event would mean in a given schedule as these concepts are all relative, especially with respect to different venue capacities.

ii) Following the rule above, if an event has sufficient demand such that all seats will be sold out when tickets sold as single tickets, including this event in the bundle would decrease the overall revenue per ticket from the bundle. On the other hand, if an event has low demand with respect to venue capacity, including this event in the bundle would prevent extra revenue opportunity per ticket that can be extracted from (relatively) higher demand events in the bundle. Combined with the guideline (i), in general, we would like to bundle events with low and high demands together, however, based on the venue capacity and the relative values of the demands, we may exclude certain events from consideration (either too high or too low demands).

iii) For the timing of the events, as the time gap between the first and the later events of a bundle increases, the revenue from the bundle increases. Hence, we would like to sell the bundled tickets as fast as possible (based on the demand rate of the bundle), however, we also would like to have enough time to sell the remaining tickets as single tickets.

To summarize, we have two major guidelines in bundling: (1) *time-based*: bundling the events with greater time gaps between the first and the other event(s), and (2) *demand-based*: bundling a high-demand event with low-demand events. If the bundle demand is a function of (or more related to) the demand of the highest-demand event within the bundle, the second strategy becomes even more important. Hence, we should consider this important factor in setting the related parameters in the heuristic for different settings.

3.3.3.1 The Steps of the Heuristic

Our heuristic approach initializes with a pre-processing step of discarding certain events, which generate low marginal revenue, when included in a bundle. In other words, the pre-processing procedure discards the events that have too low or too high demand to venue capacity ratio from being considered for the bundle selection. After determining the candidate events to be considered for bundling, according to the given schedule we create a certain number of bundles from these candidate events which have time gaps as much as possible between the first event and the other events within the bundle. Moreover, we also consider the bundles with a high-demand event and low-demand events as candidates. Hence, we determine a set of candidate bundles using both *time-based* and *demand-based* factors, while the first optimizes the time gap, the second one optimizes the demand gap within the events of a bundle. Note that our objective is to generate only a small and promising subset of the bundles as candidates generating sufficiently good amount of extra revenue without having to consider an exponential number of bundles explicitly.

In the *Event Discard Procedure (EDP)*, we calculate the ratio of demand of an event to the capacity of the venue and we discard events with ratios either too low or too high from consideration. Note that this step has polynomial complexity, $O(n)$, as this ratio is independent of other events within the bundle and their position in the schedule. One important question here is that what is the relative measure of too low or too high. In all our experiments, we set the lower bound as 25% and the upper bound as 80%.

The steps of the Bundle Selection Heuristic (BSH) for creating k-size bundle alternatives are as follows:

Step 1. Determine the events to be discarded from bundling by *EDP* and eliminate these events.
Step 2. For each given schedule:

- **2.a.** Select candidate bundles with time-based guideline: Sort the events according to their performance times; from earliest to latest. Choose the first s_{first} events and last s_{last} events. These events form up E_{first} and E_{last} sets, respectively. Select all k-size bundle alternatives such that one event comes from E_{first}, and other event(s) come(s) from the E_{last}.
- **2.b.** Select candidate bundles with demand-based guideline: Sort the events according to their expected demand levels; from highest to lowest. Choose s_{high} high-demand events and s_{low} low-demand events. These events forms up E_{high} and E_{low} sets, respectively. Select all k-size bundle alternatives such that one event comes from the E_{high} high-demand events set and the other event(s) come(s) from the E_{low} low-demand events set.
- **2.c.** Report the bundles selected at the previous two steps as the profitable bundles.

The parameters s_{first}, s_{last}, s_{high} and s_{low} determine the number of bundles that are labeled as potentially high revenue generating bundles. The tradeoff in setting the values of these parameters is between determining a bundle with higher generated revenue and computational effort in determining the best possible bundle. In any case, these parameters should be set so as to obtain the result in a reasonable amount of time in real-life size instances.

3.4. Numerical Experiments

We perform a numerical experiment in order to evaluate the performance of the BSH discussed in the previous section. We consider a team with a pre-determined schedule. In the first phase of the study, we consider a schedule with seven home games in order to compare our results with optimal bundles found by explicit enumeration and calculate optimality gaps. In the second phase, we consider a real-life instance, a schedule with seventeen home games, where we compare our results to random bundles and also to the bundle with the most demanded game and seven least demanded games which is a simple and yet a smart benchmark. This second benchmark offers a good bundle solution when we ignore the timing of the events as here the potential revenue of the low-demand events increased considerably when bundled with the highest-demand game. The ticket sale for these games is assumed to start a pre-defined time before the first game takes place. The home games are scheduled evenly throughout a season. When a bundle is formed, a discount is applied to the sum of the ticket prices within the bundle. Within all of those discussed parameters; number of games to be played and the time between the games is determined by a decision maker and kept invariant throughout this study. The decision problems the management of a team would encounter are related to the duration of bundle ticket sales, the prices of the tickets and the selection of the games to be bundled. We assume that the start of the selling horizon and the ticket prices are determined according to the market dynamics. Therefore, what we analyze through this experiment is the selection of the games to be bundled.

3.4.1 The Settings of the Experiments

We first investigate the cases where a bundle of two or three games are selected from a schedule of seven home games. The ticket sale is assumed to start two months before the first game. Every week a game is played at the 120-seat home stadium. When two tickets are bundled, a 15% discount and when three tickets are bundled, a 25% discount is applied to the ticket prices within the bundle. The demand of the two-game-bundle is set to 65%, 75% and 85% of the total demand of single tickets included in the bundle, or 120% of the highest-demand observed for the games within the bundle. For three-game bundle, these ratios are set to 46% and 55% of the total demand, or 110% of the highest-demand. The demands of the single tickets for the experimented cases are shown in the Table 3.1. Single ticket price for all of the games are assumed to be \$50.

Table 3.1: Ticket Demands

Opponent Team	Case 1	Case 2	Case 3
A	30	50	60
B	35	55	70
C	40	60	80
D	45	65	90
E	50	70	100
F	55	75	110
G	60	80	120

For the 17-game experiment, we increased the bundle size to eight. A 25% discount is applied to the ticket prices within the bundle, while the demand of the bundle is set to 25% of the total demand of single tickets included in the bundle. The capacity, the ticket price and the period between the start of the ticket sale and the first game are kept same as in the first phase of the experiments. The demands of the single event tickets are assumed to be evenly distributed between 40 and 120.

3.4.2 Evaluating the Performance of the Heuristic

We use "percentage optimality" as the performance measure for the evaluation of our heuristic in the first phase. The percentage optimality of worst possible bundle in a schedule is equal to 0%, while percentage optimality of best bundle in a schedule is equal to 100%. Therefore, this performance measure serves as an indicator of the position of a solution in the range between the worst and the best feasible solutions. The percentage optimality of a bundle B of k events is equal to $100\text{x}\frac{\Delta V_k^B - \Delta V_k^{B_{worst}}}{\Delta V_k^{B^*} - \Delta V_k^{B_{worst}}}$, where $\Delta V_k^{B^*}$, ΔV_k^B and $\Delta V_k^{B_{worst}}$ are the expected marginal revenues of the optimal bundle, bundle B and worst possible bundle, respectively.

3.4.3 Numerical Results

In the first phase of our experiment, the parameters shown in Table 3.2 are chosen such that the solution set includes about one fourth (5 out of 21) of the feasible set for two-game-bundle selection process, and one fifth (7 out of 35) of the feasible set for three-game-bundle selection process. Thus, the solution set provided by the heuristic represents a small part of the original solution space. Therefore, we could also determine the best of the given solution set with a very small extra effort by finding the exact revenues of these solutions when compared to the brute force solution with full enumeration.

Table 3.2: Heuristic Parameters for 7-Game Experiment

Instances	Parameters
1-9	$s_{first} = s_{last} = 2; s_{high} = s_{low} = 1$
10-12	$s_{first} = s_{last} = 1; s_{high} = s_{low} = 2$
13-18	$s_{first} = 2, s_{last} = 3; s_{high} = 1, s_{low} = 2$
19-21	$s_{first} = 1, s_{last} = 2; s_{high} = 2, s_{low} = 3$

The results of the experiments with 7 games are shown in the Table 3 where the first 12 instances represent the experiments with two-game bundles and the instances from number 13 to 21 represent three-game bundles. The second and third columns denote, respectively the demand range of the games and the ratio used in the calculation of the bundle demand, while remaining columns show the average percentage optimalities of the best bundles and the worst bundles that the heuristic finds. Since there are seven games in the considered revenue horizon, there are 5040 possible schedules. Therefore, each percentage optimality denotes an average percentage optimality from 5040 schedules.

The results obtained from the 7-game experiments show us that the BSH provides a set of bundles which includes the optimum or ones very close to optimum. Also the worst bundle of this set consistently provides a guarantee of a fairly good gain over the worst possible bundle.

We also analyze the parameters s_{first}, s_{last}, s_{high} and s_{low} in order to observe how the parameter selection affects the percentage optimality of the heuristic. Since the number of events decreases to 4 in the case where the event demand range is high (between 60 and 120) after the application of *EDP*, those instances are not used. We use the BSH with every possible parameter combination which produces

Table 3.3: Results for 7-Game Experiments

Instance Number[a]	Demand Range	# of Games After EDP	Bundle Demand	Heuristic Results	
				Per.Opt.of Best[b]	Per.Opt.of Worst[c]
1	30-60	7	65%[d] of total	96,02	33,90
2	30-60	7	75% of total	94,72	34,47
3	30-60	7	85% of total	94,78	36,67
4	50-80	7	65% of total	99,60	47,00
5	50-80	7	75% of total	99,74	53,51
6	50-80	7	85% of total	97,70	38,52
7	60-120	4	65% of total	99,51	78,98
8	60-120	4	75% of total	99,17	75,31
9	60-120	4	85% of total	95,68	67,58
10	30-60	7	120% of highest	99,13	53,38
11	50-80	7	120% of highest	99,86	59,45
12	60-120	4	120% of highest	99,28	74,62
13	30-60	7	46%[e] of total	98,65	27,92
14	30-60	7	55% of total	98,24	29,42
15	50-80	7	46% of total	99,96	41,21
16	50-80	7	55% of total	96,58	42,72
17	60-120	4	46% of total	98,45	68,55
18	60-120	4	55% of total	100,00	75,28
19	30-60	7	110% of highest	97,89	43,26
20	50-80	7	110% of highest	98,28	41,74
21	60-120	4	110% of highest	86,31	42,99

[a] The instances from 1 to 12 are with two-game bundles, the remaining ones are with three-game bundles.
[b] Average percentage optimality of best bundle that heuristic finds.
[c] Average percentage optimality of worst bundle that heuristic finds.
[d] 70% is taken when 65% of total demand is less than the highest single ticket demand of the bundle.
[e] 50% is taken when 46% of total demand is less than the highest single ticket demand of the bundle.

from 2 to 5 two-game bundles and from 2 to 7 three-game bundles. The parameter combinations which perform best w.r.t. average *best bundle percentage optimality* and average *worst bundle percentage optimality* are presented in the Tables 3.4 and 3.5. First table shows the results for the case where the bundle demand is a function of total demand of all of the bundled tickets (total demand based case), while the second table shows the results for the other case where the bundle demand is a function of only highest demand ticket within the bundle (highest demand based case). Each combination of four numbers in brackets separated with commas represent a parameter combination s_{first}, s_{last}, s_{high} and s_{low} in the respective order.

We observe a few general characteristics when the best parameters are analyzed. According to the results for the total demand based case in Table 3.4, when only a few alternative bundles are produced, we observe that the the bundles are selected with respect to time-based guideline mostly. However, when the number of alternative bundles are increased, a set of bundles created by a mixture of the two guidelines are preferred. On the other side, when the bundle demand depends on the highest demand ticket within the bundle (see Table 3.5), the bundles created by demand-based guideline becomes more profitable.

Another question is how the heuristic performs when more than one bundle are required. We applied the heuristic to choose two two-game bundles which have no common games and compare the obtained performance with the performance of the case where one bundle is required. Table 3.6 shows the average percentage optimalities of best and worst solutions that the BSH finds for 5040 schedules, along with the set of parameters with which the heuristic is applied. Note that, for the case where two bundles are selected the used benchmarks in finding percentage optimalities are the optimal two bundles and worst two bundles of corresponding game schedule. According to the results, as the number of required bundle is increased from 1 to 2, the average best bundle(s) percentage optimality decreases while the average worst bundle(s) percentage optimality increases. For a schedule with fixed number of games this is an expected result, because when the required number of bundles increases the number of alternative solutions decreases.

An alternative for selecting multiple bundles is to use BSH iteratively and exclude the previously bundled events from consideration. The computational time of the proposed heuristic allows us to implement iteratively for large size problems without any extra effort. We choose not to conduct such a study as the results in later iterations will be similar with a certain diminishing return trend.

For the 17-game experiment, 500 schedules have been chosen randomly. For each schedule, four 8-game bundles are selected by the BSH and four 8-game bundles are selected randomly out of 24,310 alternatives. Also, in order to form another benchmark, the bundle with the most demanded game and seven least demanded games is selected for each of the chosen 500 schedules. For the heuristic; three bundles are created by bundling the first, second and the third game in the schedule one by one with the last seven games and the forth bundle is selected with demand-based strategy which consists of most demanded game and the least demanded seven games. BSH performs very well compared to the randomly selected four bundles. The average of percentage gain among best bundles is 100%, while it is 59% in average. When it is compared to the wisely selected bundle, the corresponding numbers are 6% and -3%. Although it is worse in average, the best of four heuristic bundles is considerably profitable over the selected bundle. For 466 schedules out of 500, the maximum of heuristic bundles is better than the selected bundle. To be on the more profitable side, after the application of heuristic and selection of a very small set from all alternatives (here 4 out of 24,310), the decision maker is advised to calculate the expected revenues of only four bundles.

The proposed heuristic can be easily implemented even by those who are not familiar with details of our

Table 3.4: Best Parameters w.r.t. Average Percentage Optimalities (Total Demand Based Case)

# of Alternative Bundles	Bundle Size:	2-Game Bundle						3-Game Bundle			
	Demand Range:	30-60			50-80			30-60		50-80	
	Bundle Demand:	65%	75%	85%	65%	75%	85%	46%	55%	46%	55%
2		(1,2,0,0)	(1,2,0,0)	(1,2,0,0)	(1,2,0,0)	(1,2,0,0)	(1,2,0,0)	(1,2,1,2)	(1,2,1,2)	(1,2,1,2)	(1,2,1,2)
3		(1,3,0,0)	(1,3,0,0)	(1,3,0,0)	(1,3,0,0)	(1,3,0,0)	(1,3,0,0)	(1,3,0,0)	(1,3,0,0)	(1,3,0,0)	(0,0,1,3)
4		(1,4,0,0)	(1,4,0,0)	(1,4,0,0)	(2,2,0,0)	(2,2,0,0)	(2,2,0,0)	(1,3,1,2)	(1,3,1,2)	(1,3,1,2)	(1,2,1,3)
5		(1,5,0,0)	(1,5,0,0)	(1,5,0,0)	(2,2,1,1)	(2,2,1,1)	(2,2,1,1)	(1,3,2,2)	(1,3,2,2)	(1,3,2,2)	(1,3,2,2)
6		-	-	-	-	-	-	(1,3,3,2)	(1,3,3,2)	(1,3,1,3)	(1,3,1,3)
7		-	-	-	-	-	-	(1,3,4,2)	(1,3,4,2)	(2,3,1,2)	(1,3,4,2)

a. Parameters w.r.t. Best Bundle Average Percentage Optimality

# of Alternative Bundles	Bundle Size:	2-Game Bundle						3-Game Bundle			
	Demand Range:	30-60			50-80			30-60		50-80	
	Bundle Demand:	65%	75%	85%	65%	75%	85%	46%	55%	46%	55%
2		(2,1,0,0)	(2,1,0,0)	(2,1,0,0)	(1,2,0,0)	(1,2,0,0)	(0,0,2,1)	(2,2,0,0)	(2,2,0,0)	(2,2,0,0)	(2,2,0,0)
3		(3,1,0,0)	(3,1,0,0)	(3,1,0,0)	(1,3,0,0)	(1,3,0,0)	(0,0,3,1)	(1,3,0,0)	(1,3,0,0)	(1,3,0,0)	(1,3,0,0)
4		(2,2,0,0)	(2,2,0,0)	(2,2,0,0)	(2,2,0,0)	(1,4,0,0)	(0,0,4,1)	(1,3,1,2)	(1,3,1,2)	(1,3,1,2)	(1,2,1,3)
5		(2,2,1,1)	(2,2,1,1)	(0,0,1,5)	(1,5,0,0)	(1,4,1,1)	(0,0,5,1)	(2,2,1,3)	(2,2,1,3)	(2,2,1,3)	(2,2,1,3)
6		-	-	-	-	-	-	(2,3,0,0)	(2,3,0,0)	(2,3,0,0)	(2,3,0,0)
7		-	-	-	-	-	-	(2,3,1,2)	(2,3,1,2)	(2,3,1,2)	(2,3,1,2)

b. Parameters w.r.t. Worst Bundle Average Percentage Optimality

Table 3.5: Best Parameters w.r.t. Average Percentage Optimalities (Highest Demand Based Case)

# of Alternative Bundles	Bundle Size:	2-Game Bundle		3-Game Bundle	
	Demand Range:	30-60	50-80	30-60	50-80
	Bundle Demand:	120%	120%	110%	110%
2		(0,0,1,2)	(0,0,1,2)	(1,2,1,2)	(1,2,1,2)
3		(0,0,1,3)	(1,1,1,2)	(1,2,2,2)	(1,3,0,0)
4		(1,1,1,3)	(0,0,2,2)	(1,3,1,2)	(1,3,1,2)
5		(1,2,1,3)	(1,1,2,2)	(1,3,2,2)	(1,3,2,2)
6		-	-	(1,3,3,2)	(1,3,1,3)
7		-	-	(1,3,4,2)	(1,3,4,2)

a. Parameters w.r.t. Best Bundle Average Percentage Optimality

# of Alternative Bundles	Bundle Size:	2-Game Bundle		3-Game Bundle	
	Demand Range:	30-60	50-80	30-60	50-80
	Bundle Demand:	120%	120%	110%	110%
2		(0,0,1,2)	(0,0,1,2)	(1,2,1,2)	(1,2,1,2)
3		(0,0,1,3)	(1,1,1,2)	(1,2,2,2)	(1,3,0,0)
4		(0,0,2,2)	(0,0,2,2)	(1,3,1,2)	(1,3,1,2)
5		(1,1,2,2)	(1,1,2,2)	(1,3,2,2)	(1,3,2,2)
6		-	-	(2,3,0,0)	(2,3,0,0)
7		-	-	(2,3,1,2)	(2,3,1,2)

b. Parameters w.r.t. Worst Bundle Average Percentage Optimality

application and RM techniques. Only certain attributes of the events including the expected demand of the events and bundles and the capacity of the venue are sufficient for applying this procedure. The decision maker will set the parameters which determines the number of bundles in the solutions set and find the potentially high revenue yielding set of bundles. The computational study shows that this proposed method is a simple yet an effective way of determining good bundles in real-life size problems.

3.5. Conclusion

In this chapter, we analyze the decision problem of which event tickets to include into the bundle. As a founding result, the method developed by Duran et al. (2012) generating a solution to find the dynamic optimal switch times from bundles to singles is utilized in this chapter. For the problem of event selection for bundling, which has a combinatorial nature, we first analyze the problem to find patterns in the expected revenue of the event tickets when they are bundled at different time slots. We have identified certain properties of potentially high revenue generating bundles, and using these properties we construct a simple yet effective heuristic method to determine good bundles in real-life size problem in reasonable amount of time.

Table 3.6: Comparison of One Bundle and Two Bundles w.r.t. Average Percentage Optimality

	# of Alternative Bundles	Case:	Total Demand Based						Highest Demand Based	
		Demand Range:	30-60			50-80			30-60	50-80
		Bundle Demand:	65%	75%	85%	65%	75%	85%	120%	120%
Best Bundle	5		0,960	0,946	0,947	0,997	0,996	0,977	0,991	0,998
	4		0,957	0,943	0,944	0,996	0,995	0,971	0,987	0,996
Best Two Bundles	5		0,896	0,846	0,822	0,975	0,970	0,868	0,980	0,980
	4		0,882	0,817	0,780	0,972	0,954	0,773	0,969	0,968
Worst Bundle	5		0,339	0,347	0,362	0,469	0,526	0,378	0,540	0,595
	4		0,493	0,441	0,430	0,685	0,621	0,396	0,591	0,626
Worst Two Bundles	5		0,639	0,606	0,580	0,724	0,716	0,566	0,760	0,778
	4		0,851	0,793	0,737	0,889	0,805	0,648	0,844	0,836

a. Average Percentage Optimalities

# of Alternative Bundles	Total Demand Based	Highest Demand Based
4	$s_{first} = s_{last} = 2; s_{high} = s_{low} = 0$	$s_{first} = s_{last} = 0; s_{high} = s_{low} = 2$
5	$s_{first} = s_{last} = 2; s_{high} = s_{low} = 1$	$s_{first} = s_{last} = 1; s_{high} = s_{low} = 2$

b. Heuristic Parameter Settings

CHAPTER 4

LEAGUE SCHEDULING AND GAME BUNDLING IN SPORTS INDUSTRY

4.1. Introduction and Literature Review

In this chapter, we address the decision problem of scheduling to facilitate the creation of profitable bundles and we attempt to develop a well-performing approach to generate effective schedules. The challenge here is that the bundling and scheduling decisions affect each other in a circular sense and making such a simultaneous decision is quite a daunting task. Nevertheless, we consider the decisions of determining the best schedule of events and determining the best bundle of events and propose an integrated framework to handle these decisions together, which possibly will lead to higher revenue generation compared to sequential decision making process.

Apart from bundling strategies mentioned in previous chapter, another relevant literature area is scheduling in S&E industry. In our context, bundling decisions are directly affected by the schedule of the events. For example, if the highest demand event is scheduled as the first event, switching earlier to single tickets sales might be preferable as otherwise the remaining time to the event may not be enough to sell the remaining tickets for a higher price as single tickets. As we show in Chapter 2, the timing of the games in a bundle is one of the most important factor in determining the extra revenue generated. In fact, we proposed a time-efficient bundle selection strategy mostly based on the schedule of events, which performed quite well compared to the our benchmark of optimal bundle strategy. However, scheduling of the events by itself is very difficult due to several complex feasibility restrictions such as availability of the venues, pattern restrictions (limit on the number of consecutive home/away games), time/event pairing restrictions (scheduling high popularity events on Christmas, Thanksgiving), etc. Hence, the best event schedule for maximizing the revenue might not be feasible most of the time. Since each team may have conflicting interests, a central authority prepares the schedule. A welcomed schedule by teams should meet the teams' financial needs and expectations. Among the different structures for league scheduling, the most popular one is round robin tournament (RRT) which has several different forms like single RRT, double RRT, r-RRT, mirrored r-RRT, etc. In all of these RRT variations, each team plays against each other team r times. A complete review on the sports league scheduling is given in Briskorn (2008). There are a number of studies in the literature concerning league scheduling and related areas such as (de Werra, 1988), (de Werra, Descombes, & Masson, 1990), (Dinitz, Froncek, Lamken, & Wallis, 2006), (Drexl & Knust, 2007), (Rasmussen & Trick, 2008) and (Briskorn & Drexl, 2009).

Mostly, researchers focus on constructing feasible schedules and evaluating them with respect to a criterion such as minimization of the number of breaks, travel distances, carryover effects and costs. Our perspective is quite different than the studies listed above. We consider a simple DRRT setting and ignore the other possible side constraints as our focus is to show how schedule affects the bundling decisions and propose a framework to make simultaneous decisions for both. Note that in our context there are several teams each with the objective to maximize their revenues. Hence one schedule might favor some of these teams whereas another schedule might favor some other. Thus, there exist certain tradeoffs between generated revenues by the individual teams, which lead to conflicts in determining the best schedule. We also analyze the schedule decision from the perspective of different teams with different characteristics in our numerical study. We believe that this chapter will serve as a building block to tackle this very challenging, integrated decision problem in S&E industry. If required, one may implement the ideas in this chapter in a more realistic setting where all the complications due to scheduling of events are taken into consideration. However, under such a setting, the computational effort might be significantly higher.

In Section 4.2, we define the problem, list our assumptions and present our proposed solution methodology. In Section 4.3, the performance of the proposed heuristic is discussed through instances with different demand characteristics and the improvement over the method that we have offered in our previous work is reported. Concluding remarks are provided in Section 4.4.

4.2. Model

4.2.1 Basics and Assumptions

Our objective is to maximize the revenue generated throughout a game season of a DRRT. In a general DRRT with $h+1$ teams, each team plays $2h$ games (h home and h away games) during the performance season, and has one home game and one away game with each of the other teams. Our focus is not on determining home-away pattern of a DRRT schedule. Therefore, we partition the performance season to time slots where each time slot has one home and one away game for each of the teams. This assumption guarantees a desired feature that at most two home or two away games are played consecutively. We also assume that the selling times between any two consecutive home games are equal, and therefore revenue enhancement will be observed due to the better bundle creation among home games, not the home-away pattern choice of the games.

Consider any one of the $h + 1$ teams within the DRRT. This team plays h home games with visiting teams during the tournament, given by the set $V = \{v_{(1)}, v_{(2)}, \ldots, v_{(h)}\}$. The games are numbered within the set V according to their expected demand level from lowest to highest when the game played at home stadium with the visitor teams. In other words, $v_{(1)}$ is the lowest demand generating visiting opponent for the home team. Timing of the games throughout the performance season is given by the team schedule. Team schedule of h games, placement of the games to be played with visiting teams among h time slots, can be shown as a set $S = \{T_1^{v_{(\cdot)}}, \ldots, T_h^{v_{(\cdot)}}\}$ where $v_{(\cdot)} \in V$ and $T_1^{v_{(\cdot)}} < \ldots < T_h^{v_{(\cdot)}}$ are the length of ticket selling periods of first, second, ..., last home game to be played throughout the tournament. For a given team schedule of h home games, the revenue generation from ticket sales occurs within the revenue horizon, starting a certain time before the first home game and ending with the last home game's performance time.

When tickets of k games form a bundle B, a discount is applied to the sum of the prices of these single game tickets. The bundle schedule, $S^B \subset S$, represents the ticket selling periods of the single tickets (or time slots of the corresponding games) within the bundle. The selling period of the bundle, $T_B \in \mathbb{R}^+$, is equal to the ticket selling period of the first game in the bundle, $T_{B_1} \in \mathbb{R}^+$, however a switch from the bundle sale to single ticket sales can occur within T_B. The optimal time to switch is found by the dynamic optimal switch time method developed by Duran et al. (2012). Therefore, bundles are on sale until the switch time and afterwards the tickets of the games composing the bundle are on sale individually till their respective game times. We assume that the bundle demand is a certain ratio of the total demand of the bundled games and the expected demands of the games are fixed and not affected by the schedule. Therefore, for a bundle with k games, there is a corresponding Poisson process of demand: $N_B(s)$, $0 \leq s \leq t$, with known constant demand d_B and constant intensity d_B/T_B for the bundled games and $N_{B_i}(s)$, $0 \leq s \leq t$, with known constant demand d_{B_i} and constant intensity d_{B_i}/T_{B_i} for the single games in revenue horizon, where $i = \{1, \ldots, k\}$. To illustrate the schedule and bundle relation graphically, suppose that for the selected team, the home game schedule is $S = \{T_1^{v(1)}, \ldots, T_h^{v(h)}\}$. Then, the time line with the associated notation of a game bundle, where the first k games in the team schedule are bundled, is shown in Figure 4.1. In Figure 4.1 p_B and p_{B_i} are the respective ticket prices of the bundled and the single tickets of ith game in the bundle. Note that even for a given schedule, $S = \{T_1^{v(1)}, \ldots, T_h^{v(h)}\}$ for one team in the league, there are $C(h, k)$ different ways to choose a k-game bundle. The bundle selection affects the bundle selling period T_B and bundle demand directly, thus we need to decide the schedule and bundle together.

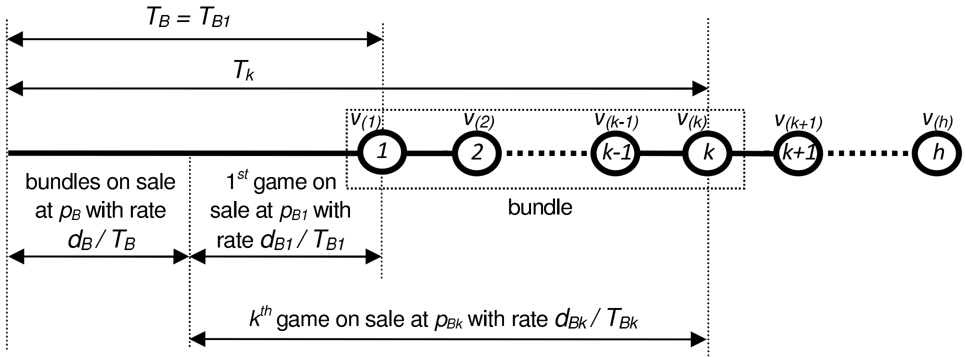

Figure 4.1: Time Line for a k-Game Bundle

The optimal expected revenue from a bundle can be modeled by dynamic programming where the states of the system are the elapsed time (t), the remaining inventory level at time t (n) and the schedule of games in the bundle (S^B). Let the expected total revenue from the bundle composed of k home games and those k single home game ticket sales over the time horizon $[t, T_k]$ with the optimal switching time be denoted by $V^B(t, n, S^B)$ as it is given in the previous chapter. Then the marginal expected revenue from bundled tickets at $t = 0$ is given by $\Delta V^B(0, n, S^B) = V^B(0, n, S^B) - \Pi_1(0, n, T_{B_1}) - \ldots - \Pi_k(0, n, T_{B_k})$, where $\Pi_i(0, n, T_{B_i})$ represents the expected revenue from single home game ticket sales when it is sold as single tickets throughout its respective ticket selling horizon.

Note that our objective is to maximize the total revenue generated from the ticket sales of all participating teams in a DRRT, and the schedules of the single-sold tickets not included in the bundles do not affect the total revenue since their demands are independent of their game time. Therefore, our problem is a DRRTP (Double Round Robin Tournament Problem) with the objective of maximizing only the expected marginal revenue from home games in the bundles. Let L be the set of teams and P be the set of all possible 3-tuples (i, j, k) where $i \in$ L, $j \in$ L$\backslash\{i\}$ and $k \in$ L$\backslash\{i, j\}$. The Bundle and Schedule Creation Problem (B&SCP) with one 2-ticket bundle for each team is formulated as follows:

$$\textbf{(B\&SCP)} \quad z_{B\&SCP} = \max \sum_{i \in L} \sum_{j \in L \backslash \{i,k\}} \sum_{k \in L \backslash \{i,j\}} \sum_{p \in P} \sum_{q \in P, p<q} y_{ijkpq} r_{ijkpq} \tag{4.1}$$

subject to

$$\sum_{i \in L \setminus \{j\}} x_{ijp} = 1 \quad \forall j \in L, \forall p : T_p \in S \tag{4.2}$$

$$\sum_{j \in L \setminus \{i\}} x_{ijp} = 1 \quad \forall i \in L, \forall p : T_p \in S \tag{4.3}$$

$$\sum_{p : T_p \in S} x_{ijp} = 1 \quad \forall i \in L, \forall j \in L, i \neq j \tag{4.4}$$

$$x_{ijp} + x_{ikq} - 1 \geq y_{ijkpq} - y'_{ijkpq} \quad \forall \{i, j, k\} \in P, \forall p : T_p \in S, \forall q : T_p \in S, p < q \tag{4.5}$$

$$y_{ijkpq} + y'_{ijkpq} \leq 1 \quad \forall \{i, j, k\} \in P, \forall p : T_p \in S, \forall q : T_p \in S, p < q \tag{4.6}$$

$$\sum_{j \in L} \sum_{k \in L \setminus \{j\}} \sum_{p : T_p \in S} \sum_{q : T_p \in S} y_{ijkpq} = 1 \quad \forall i \in L \setminus \{j, k\}, p < q \tag{4.7}$$

$$x_{ijp} \in \{0, 1\} \quad \forall i \in L, \forall j \in L \setminus \{i\}, \forall p : T_p \in S \tag{4.8}$$

$$y_{ijkpq} \in \{0, 1\} \quad \forall \{i, j, k\} \in P, \forall p : T_p \in S, \forall q : T_p \in S, p < q \tag{4.9}$$

$$y'_{ijkpq} \geq 0 \quad \forall \{i, j, k\} \in P, \forall p : T_p \in S, \forall q : T_p \in S, p < q \tag{4.10}$$

Let a game be denoted with a pair of indices $(i - j)$, where the indices represent the home team and the visiting team, respectively. The decision variable x_{ijp} indicates whether the ticket selling horizon of game $(i - j)$ is T_p, or not; while the other decision variable y_{ijkpq} indicates whether the games $(i - j)$ and $(i - k)$ with respective ticket selling horizons T_p and T_q are bundled, or not. The parameter r_{ijkpq} indicates the expected marginal revenue from the bundle of the games $(i - j)$ and $(i - k)$ with respective ticket selling horizons T_p and T_q. Formally, r_{ijkpq} is equal to $\Delta V^B(0, n, S^B)$, the expected marginal revenue of the bundle B having schedule $S^B = \{T_{B_1}, T_{B_2}\}$, where T_{B_1} and T_{B_2} are equal to T_p and T_q, respectively. The dummy variable y'_{ijkpq} is required for the cases when the term $(x_{ijp} + x_{ikq})$ in Constraint (4) is equal to 0. Constraints (2), (3) and (4) assure a feasible schedule; Constraints (5) and (6) assure that the games $(i - j)$ and $(i - k)$ with respective selling periods T_p and T_q can be bundled only if they are scheduled at corresponding time slots; Constraints (7) limit the number of bundles formed for each team to one. The other constraints define the decision variable domains.

The B&SCP is $\mathcal{NP}$-hard and explodes combinatorially as the number of teams and bundle size increases. Moreover, especially for a large instance like an 18-team soccer league, generating expected revenue data (r_{ijkpq}) for all alternative bundles for each team takes long time. Therefore, we should find an approach to solve the problem while overcoming those difficulties. At this point, we utilize the lemmas and corollaries from the previous chapter. These lemmas and corollaries help us in calculating approximate values for bundle expected revenue data (r_{ijkpq}) and also constructing a heuristic to find profitable bundles along with a corresponding league schedule within reasonable time.

Consider selected two visiting opponents for a team. Table 4.1 shows all possible bundled games' schedules S^B of these two visiting opponents. According to lemmas and corollaries in the previous chapter, expected bundle revenue

- increases from left to right within the same row,
- decreases from top to down within the same column,
- decreases diagonally from upper left corner to lower right corner,
- increases in the perpendicular direction to the diagonal.

These bundle schedule properties allow us to obtain an expected revenue range for a selected bundle (opponent visitor teams to be included in the bundle fixed) of two games independent of their exact schedule. If the first game of a 2-game bundle is scheduled at the first time slot and the second event is scheduled at the last in the revenue horizon, the maximum marginal revenue is obtained for the

Table 4.1: Possible Time Slots of the Games Within a 2-Game Bundle

		Second Game Time Slot				
		2	3	4	...	h
First Game Time Slot	1	$T_1 - T_2$	$T_1 - T_3$	$T_1 - T_4$	...	$T_1 - T_h$
	2		$T_2 - T_3$	$T_2 - T_4$	...	$T_2 - T_h$
	3			$T_3 - T_4$	...	$T_3 - T_h$
	...				...	...
	$(h-1)$					$T_{(h-1)} - T_h$

considered bundle. Similarly, if the events are scheduled at the last two time slots, then the minimum revenue is obtained.

Calculating only three possible bundle schedules, at three corners of the triangle in Table 4.1 allows us to approximately find the expected revenues for all other possible schedules of the bundle. Consider the case of an 8-team DRRT where home game schedule, (ticket selling horizons throughout the tournament) is given as $S = \{T_1^{v_{(\cdot)}}, \ldots, T_8^{v_{(\cdot)}}\}$ for a selected team. For this selected team, also two visiting opponent teams are selected to be bundled and those two home games can be scheduled at any pair of those home game time slots, i.e. $T_i - T_j$ where $T_i, T_j \in S$. With respect to the Table 4.1 representation of the possible schedules of the games in the bundle, the corner time slot cells would be $T_1 - T_2$, $T_7 - T_8$ and $T_1 - T_8$. Let R_1, R_2 and R_3 represent the corresponding expected revenues by those three bundle schedules to be used in the approximation, respectively. Then the approximate expected revenues from the bundle, if the games are scheduled at other time slots, can be linearly interpolated. As an example, the approximate expected revenue value of the bundle with $S^B = \{T_1, T_3\}$ is equal to $R_1 + (R_3 - R_1)/6$, or with $S^B = \{T_1, T_4\}$ is equal to $R_1 + 2(R_3 - R_1)/6$, or with $S^B = \{T_1, T_5\}$ is equal to $R_1 + 3(R_3 - R_1)/6$.

Until now, we have discussed the case where only two of the games are chosen to be bundled. Now, we analyze the case where three or more games are considered for bundling. Since it is quite straight forward to modify the formulation of B&SCP for the bundles with three or more games, we do not present the reformulation here.

The rules discussed previously for 2-game bundles are also generalized for the cases with bundles of three or more games in the previous chapter. According to this generalization:

- When the first home game time slot in the bundle is kept fixed, if any of the other home games' time slot in the bundle advances in the revenue horizon, the expected revenue from the bundle increases.
- The maximum revenue from a bundle composed of k home games is obtained when the first game of the bundle is scheduled at the first time slot and the others are scheduled at the last time slots in the revenue horizon.
- The minimum revenue from a bundle composed of k home games is obtained when all of the games of a bundle are scheduled at the last time slots in the revenue horizon.
- When the performance time of all games in the bundle advance in the revenue horizon by the same amount of time, the expected revenue from the bundle decreases.

According to these rules; we can apply the interpolation method with a similar approach as it is applied to the 2-game bundle case. Again, using the three corner time slot cells, it is straightforward when second and the latter games are scheduled to consecutive time slots. Note that for a 3-game bundle in an 8-team DRRT, these points are $T_1 - T_2 - T_3$, $T_1 - T_7 - T_8$ and $T_6 - T_7 - T_8$. When second and the latter games within the bundle are scheduled to the time slots that are not consecutive, an imaginary game time, which is the average of the game times of the second and the latter games, is taken and the games (except the first game) are distributed consecutively around this average time slot. Note that there are other ways of obtaining approximate revenue values by interpolation. Moreover, if we calculate the expected bundle revenues of more points in the event time line, we can utilize those calculated revenue values to make more precise estimations, of course in exchange of more computational time.

4.2.2 Schedule and Bundle Selection Heuristic

Since B&SCP is $\mathcal{NP}$-hard, we develop the *Schedule and Bundle (S&B) Selection Heuristic* to generate a solution providing preferable bundles and a corresponding feasible DRRT schedule. There are several factors affecting the revenue enhancement potential from a bundle besides the timing of the games in the bundle. Hence, for the heuristic we list a set of guidelines that will result in constructing good bundles.

i) Bundling a high-demand game with low-demand games is better than bundling the games with similar demand levels. High-demand game(s) will serve as the motivator to buy the bundle, therefore the revenue generated by the low-demand games in the bundle will increase.

ii) Although guideline i) is valid in general, if a game has sufficient demand to sell out all seats in the venue when sold as single tickets, including this game in the bundle would decrease the overall revenue per ticket from the bundle. Similarly, if a game has significantly low demand rate w.r.t. venue capacity, including this game in the bundle would result in forgone revenue opportunity per ticket that can be extracted from higher demand games in the bundle. Therefore, guideline i) is valid only when either too high or too low demand games w.r.t. the venue capacity are excluded from consideration for bundling.

iii) For time slots of the bundled games, as the time gap between the first and the other games within a bundle increases, the revenue from that bundle increases.

4.2.2.1 The Steps of the Heuristic

For each participating team, our heuristic approach initializes with a pre-processing step called *Game Discard Procedure* (*GDP*) where certain opponent visiting teams are discarded from consideration for bundling, that will potentially generate low marginal revenue if included in bundles. In other words, the pre-processing procedure discards the opponent visiting teams for each home team that have too low or too high demand when played with, considering the home venue capacity. Note that this step has polynomial complexity, $O(n)$, as this ratio is independent of other games within the bundle and their position in the schedule. After the candidate opponent teams to be considered for bundling are determined, these visiting teams are separated as basic and nonbasic for each team in the league. The opponent visiting teams that will create low demand when played against are selected to be included in the basic set of that home team such that the cardinality of the basic set should be at least 1 less than the bundle size. Each bundle is constructed such that it should have one nonbasic visiting team and as many basic visiting teams as the bundle size allows. Bundle construction process is an iterative process where at each iteration, one nonbasic visiting team is selected for generating all possible bundles with the basic visiting teams. At each iteration, after creation of the new bundles and calculation of their approximate expected marginal revenues with the method discussed in section 4.2.1, these new bundles are added to B&SCP before it is solved. The iterations are ceased when the revenue improvement between two consecutive solutions of B&SCP gets insignificant. After last iteration, a final improvement procedure (*ImpProcedure*) is applied. The *ImpProcedure* checks if there is any further revenue improvement possibility over the final schedule by generating a small number of promising bundles which cannot be generated by the basic-nonbasic set matings. The steps of the S&B Selection is given at Table 4.2 in detail.

Table 4.2: S&B Selection Heuristic

Step 0.	For each home team *GDP* is utilized; we discard very low and very high demand visiting opponents w.r.t. home venue capacities.
Step 1.	For each home team define the basic and nonbasic visiting team sets B and N; we choose b visiting opponent(s) which generate(s) lowest demand when played against as basic and assign the remaining *nonb* visiting opponents as nonbasic.
Step 2.	Set iteration counter t to 1, optimal objective function value $z(t)$ to 0 and the bundle set S to be empty set.
Step 3.	For each team with nonempty set N, choose the highest-demand generating visiting opponent g from set N, generate all alternative bundles including the games with the chosen opponent g and the basic visiting opponents, and estimate the marginal expected revenues of these new bundles independent of the bundle schedule using approximation method in Section 4.2. Add this new set of bundles to set S. Solve B&SCP with the current set of bundles S. Assign the optimal objective function value to $z(t)$. Increase t by 1. Set N to $N \backslash \{g\}$.
Step 4.	If $[z(t-1)-z(t-2)]/z(t-2) < \epsilon$ (where ϵ is a predefined small number) or N is empty for all teams, STOP, apply the *ImpProcedure* to the solution of iteration $(t-1)$ and report the final solution after the process of *ImpProcedure*. Otherwise, go to Step 3.

Before proceeding with the steps of *ImpProcedure*, we should define the terms *preferable slot*, *totally preferable* and *slot deviation*. Consider a bundle of k games. The preferable slot of the first game in the bundle is the first time slot and the preferable slots of the remaining $k-1$ games in the bundle are the last $k-1$ time slots, with the respective order. Note that if the games are scheduled at their preferable slots, the bundle will create the maximum revenue. If all of the games within a bundle are at their *preferable slots*, then the bundle is called *totally preferable*. *Slot deviation* of a bundle is equal to the required number of slot shift(s) of the games within a bundle to make the bundle totally preferable.

The detailed steps of the *ImpProcedure* is given at Table 4.3. Moreover, to make the *ImpProcedure* more clearly understandable, we explain the details through an example in the following section.

Note that in the *ImpProcedure* we change only one of the basic visiting teams which is not at its preferable slot. This procedure can be applied more than once or until a determined termination criteria, in order to consider more basic visiting teams in expense of more computational burden to calculate the approximate expected marginal bundle revenues. Moreover, with slight changes, one can modify the algorithm in order to apply to a multiple bundle selection problem. Also, with required changes in the MIP model (B&SCP), it is possible to apply the heuristic to the problems with more scheduling restrictions, or with forms of tournaments other than DRRT.

Table 4.3: Improvement Procedure (*ImpProcedure*)

Step 0.	For each home team, check if the solution offers a *totally preferable* bundle or not. If there are one or more games played with basic visiting opponents, not at their *preferable slots*, mark the game which has the maximum *deviation* from its *preferable slot* as "candidate to leave (the bundle)". (If there is a tie, choose the earliest one, since it is expected to be more profitable.)
Step 1.	For each bundle that is not *totally preferable* with a game marked as "candidate to leave", determine the time slot for the game "candidate to enter" (which may change the game "candidate to leave") in order to minimize the *slot deviation* of the bundle.
Step 2.	For each bundle under consideration, generate the new alternative bundles where the game "candidate to leave" is changed with the games "candidate to enter" that are played with nonbasic opponents, and calculate the expected marginal revenues of these newly generated bundles using the approximation explained in Section 4.2.
Step 3.	Solve B&SCP, after adding the generated set of bundles to S. Report the objective function value $z(ImpProcedure)$ as the final solution.

4.3. Numerical Experiments

We perform numerical experiments in order to evaluate the performance of the S&B selection heuristic approach discussed in the previous section. We consider a small league setting with eight teams and a greater league setting with eighteen teams. To compare the performance of the S&B selection heuristic with the optimal solution, we use the small league setting. With the greater league setting, obtaining optimal solutions within reasonable computational time is not possible, therefore S&B selection heuristic's performance is compared to that of Bundle Selection Heuristic developed in the previous chapter. In both of the league size settings, the Bundle Selection Heuristic is applied to randomly generated schedules of the league since this method requires predetermined schedules. Table 4.4 indicates what type of performance evaluation experiments are held on the combinations of league size and bundle size.

Table 4.4: Performance Evaluation Experiments

League Size & Bundle Size	Comparison	
	to Optimal	to *Bundle Selection Heuristic*
8-team; 2-game	+	+
8-team; 3-game	+	+
18-team; 3-game	-	+

In addition to the experiments executed for performance evaluation of the offered S&B selection heuristic, we also perform an experimental study to investigate the contribution of this heuristic to the revenues of popular and unpopular teams of the league.

4.3.1 The Cases and Assumptions

General assumptions throughout the experiments are as follows. The ticket sale is assumed to start two months before the first game. Every week one home game is played at each one of the home stadiums of participating teams. When two tickets are bundled, a 15% discount and when three tickets are bundled, a 25% discount is applied to the ticket prices within the bundle. Single ticket price for all of the games are assumed to be $50. For the small league setting, the home stadium capacities and demands of the single game tickets are shown in the Table 4.5.

Table 4.5: Stadium Capacities and Demands for Single Game Tickets

Home Team	Stadium Cap.	Visiting Teams							
		1	2	3	4	5	6	7	8
1	120	-	120	100	90	80	70	60	30
2	120	120	-	110	100	90	80	70	40
3	100	90	95	-	70	60	50	40	40
4	80	80	85	60	-	50	45	35	35
5	80	70	80	55	50	-	40	35	35
6	70	60	70	50	45	35	-	40	35
7	70	50	55	40	30	20	35	-	30
8	50	55	60	45	40	30	25	25	-

In the small league setting, for each home game, we discard the visiting team with the most demanded game ticket by *GDP*. For each home team, the number of basic visiting teams, *b*, is set to 2, while the remaining teams are labeled as nonbasic, making *nonb* parameter equal to 4. The same basic visiting teams are chosen for both the 2-game bundle and 3-game bundle cases. Basic and nonbasic opponent visiting teams for each home team are presented in Table 4.6.

Table 4.6: Basic and Nonbasic Visiting Team Selection

Home Team	Visiting Teams					
	Basic		Nonbasic			
1	7	8	3	4	5	6
2	7	8	3	4	5	6
3	7	8	1	4	5	6
4	7	8	1	3	5	6
5	7	8	1	3	4	6
6	5	8	1	3	4	7
7	5	8	1	3	4	6
8	6	7	1	3	4	5

The stadium capacities and demands of single game tickets for the 18-team league are given in Table 4.7. For bundling, we have discarded the games with demand to capacity ratio over 90%, while the least demanded two games are selected as basic.

Assumptions on the bundle demand creates cases in our numerical experiments. The demand of the 2-game bundle is set to 85% of the total demand of single game tickets included in the bundle, or 120% of the highest demand observed for the games within the bundle. For 3-game bundle, these ratios are set to 55% of the total demand, or 110% of the highest-demand game. The parameters used in Bundle Selection Heuristic along with the cases are shown in Table 4.8.

According to the selected parameters, the Bundle Selection heuristic selects a group of promising bundles for a given team schedule. In our experiments, we compare only the best (most profitable among the promising bundles selected) bundle created by the Bundle Selection heuristic to the bundle

Table 4.7: Stadium Capacities and Demands of Single Tickets for 18-Team League

Home Team	St.Cap.	Visiting Teams																	
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	120	-	130	125	120	115	110	105	100	95	90	85	80	75	70	65	60	55	50
2	115	130	-	125	120	115	110	105	100	95	90	85	80	75	70	65	60	55	50
3	110	130	125	-	120	115	110	105	100	95	90	85	80	75	70	65	60	55	50
4	110	125	120	115	-	110	105	100	95	90	85	80	75	70	65	60	55	50	45
5	100	125	120	115	110	-	105	100	95	90	85	80	75	70	65	60	55	50	45
6	100	120	115	110	105	100	-	95	90	85	80	75	70	65	60	55	50	45	40
7	90	120	115	110	105	100	95	-	90	85	80	75	70	65	60	55	50	45	40
8	90	115	110	105	100	95	90	85	-	80	75	70	65	60	55	50	45	40	35
9	80	115	110	105	100	95	90	85	80	-	75	70	65	60	55	50	45	40	35
10	80	110	105	100	95	90	85	80	75	70	-	65	60	55	50	45	40	35	30
11	80	105	100	95	90	85	80	75	70	65	60	-	55	50	45	40	35	30	28
12	80	100	95	90	85	80	75	70	65	60	55	50	-	45	40	35	33	30	28
13	75	100	95	90	85	80	75	70	65	60	55	50	45	-	40	35	33	30	25
14	70	100	95	90	85	80	75	70	65	60	55	50	45	40	-	35	30	28	23
15	65	95	90	85	80	75	70	65	60	55	50	45	40	35	30	-	28	25	23
16	60	95	90	85	80	75	70	65	60	55	50	45	40	35	30	28	-	25	20
17	55	95	90	85	80	75	70	65	60	55	50	45	40	35	30	25	20	-	18
18	50	95	90	85	80	75	70	65	60	55	50	45	40	35	30	25	20	15	-

Table 4.8: Cases and Parameters for Bundle Selection Heuristic

League Size	Cases	Parameters
8-team	2-game bundle w.r.t. total demand	$s_{first} = s_{last} = 2; s_{high} = s_{low} = 1$
	2-game bundle w.r.t. high demand	$s_{first} = s_{last} = 1; s_{high} = s_{low} = 2$
	3-game bundle w.r.t. total demand	$s_{first} = 2, s_{last} = 3; s_{high} = 1, s_{low} = 2$
	3-game bundle w.r.t. high demand	$s_{first} = 1, s_{last} = 2; s_{high} = 2, s_{low} = 3$
18-team	3-game bundle w.r.t. total demand	$s_{first} = 6, s_{last} = 3; s_{high} = 1, s_{low} = 2$
	3-game bundle w.r.t. high demand	$s_{first} = 1, s_{last} = 2; s_{high} = 6, s_{low} = 3$

created by the S&B Selection heuristic.

4.3.2 Numerical Results

For the small league setting, the proximity of the values found by offered heuristic to corresponding optimal marginal values are presented in Table 4.9. Note that we perform all possible iterations since the problem size is small.

Table 4.9: Proximity to Optimal Solution for the 8-Team League Setting

Cases	Heuristic Iterations				
	t=1	t=2	t=3	t=4	Impr.
1. 2-game bundle w.r.t. total demand	91.75%	98.19%	99.05%	99.83%	-
2. 2-game bundle w.r.t. high demand	86.75%	97.97%	99.90%	99.98%	-
3. 3-game bundle w.r.t. total demand	84.24%	92.39%	95.00%	97.86%	98.76%
4. 3-game bundle w.r.t. high demand	95.08%	97.90%	98.82%	98.30%	99.00%

In order to illustrate the bundles created and their schedule, we choose the third case in Table 4.9 where we can observe the improvement procedure also. Table 4.10 shows the bundled games at each of the heuristic iterations. In the first iteration, for the home team indexed with 1, the selected bundle is composed of the games to be played against opponent visiting teams indexed with 3, 7 and 8 with schedule $S^B = \{T_2, T_6, T_7\}$. As iterations proceed, opponent visiting team 3 is replaced by team 4 and then team 5. Finally the selected bundle is composed of the games played against teams 6, 7 and 8 with the respective schedule $S^B = \{T_1, T_5, T_7\}$.

After the forth iteration, the improvement procedure is applied. In the left hand side of Table 4.11, the candidate nonbasic visiting teams “to enter” the bundle and the basic visiting team “to be exchanged” (which may leave the bundle) are shown in italic and bold styles, respectively. The final solution is given in the right hand side of Table 4.11, and the bundle created for Team 7 has a new nonbasic visiting team. Note that this change in one bundle affects the bundles of Team 2 and Team 6 such that the time gaps between the first and the second games within the bundles for these home teams are increased.

In order to compare the solution of the offered heuristic with the the Bundle Selection Heuristic presented in the previous chapter where league schedules are given as inputs, we generated 15 random schedules and applied the Bundle Selection Heuristic to these random schedules using the parameters given in Table 4.8.

Table 4.10: Bundle Schedules by the S&B Selection Heuristic for Case 3 (3-game bundle w.r.t. total demand)

Home Team	Periods						
	1	2	3	4	5	6	7
1		3				7	8
2	3			7	8		
3	7		8		1		
4		8	7	1			
5	8					1	7
6	1			8			5
7	5					8	1
8		1			7		6

a. First Iteration

Home Team	Periods						
	1	2	3	4	5	6	7
1	4			7	8		
2		7	8				4
3	8					4	7
4		8				7	3
5	7					3	8
6	3			8			5
7	5					8	1
8		3			7		6

b. Second Iteration

Home Team	Periods						
	1	2	3	4	5	6	7
1		5			7		8
2		7	8		5		
3	8					5	7
4		8				7	3
5	7					8	4
6	4			8			5
7	5				8		1
8	6			7		4	

c. Third Iteration

Home Team	Periods						
	1	2	3	4	5	6	7
1	6				7		8
2		7		8			6
3	8					5	7
4		8				7	3
5	7					8	4
6	4		8				5
7	5				8		1
8		6		7	5		

d. Forth Iteration

For the small league setting, to evaluate the performances of both of the heuristics under same conditions, we discard the most demanded visiting team by *GDP* before application of the Bundle Selection Heuristic just as we do in the currently offered S&B Selection Heuristic. The figures showing the improvement provided by the S&B Selection Heuristic over the Bundle Selection Heuristic are presented in the Tables 4.12 and 4.13 for total demand based and high-demand based cases, respectively. The results show that, the offered method, which integrates scheduling and bundling of the games, significantly contributes to the revenue enhancement. By schedule consideration, on average around 10% revenue enhancement is realized over Bundle Selection Heuristic.

The same experiment, which compares the Bundle Selection Heuristic with the S&B Selection Heuristic, is applied to the problem of larger size with 18 teams and bundle size of three games. 10 iterations, where all of the nonbasic visiting teams are introduced into the problem, are executed in application of the S&B Selection Heuristic. Since the problem size is large, by the first guideline, a number of the time slots which are in the middle of the event time line are disregarded in creation of the candidate bundles in order to decrease the size of input data for the B&SCP. Therefore, only certain bundles are selected such that the first game within the bundle is played at one of the first time slots and the other games of the bundle are played at the last time slots. In the final iteration, the input data for B&SCP have reached to a size of approximately 132,000 candidate bundles. The improvement pro-

Table 4.11: Improvement Procedure for 8-Team League Setting

Home Team	Periods						
	1	2	3	4	5	6	7
1	6				**7**	*3,4,5*	8
2		7		**8**		*3,4,5*	6
3	8					5	7
4	*1,5,6*	**8**				7	3
5	7					8	4
6	4		**8**			*1,3,7*	5
7	5				**8**	*3,4,6*	1
8		6		**7**	5		*1,3,4*

a. Candidates

Home Team	Periods						
	1	2	3	4	5	6	7
1	6				7		8
2		7			8		6
3	8					5	7
4		8				7	3
5	7					8	4
6	4			8			5
7	5					4	1
8		6		7	5		

b. Final Solution

vided over the Bundle Selection Heuristic is reported in Table 4.14. It is observed that, the percentage improvement over the Bundle Selection Heuristic in the 18-team league setting is even greater than the percentage improvement obtained in the 8-team league setting.

4.3.3 Distribution of Revenue Enhancement Among Teams

Using S&B Selection Heuristic, we show numerically that significantly better schedule and bundle combinations can be created. However, how the revenue enhancement obtained from bundling is distributed among teams is still an open question. When we elaborate the optimal (or heuristic) solutions, it is easily observed that the share of the teams in the total marginal revenue are not distributed evenly, as expected.

When an average or below-average team plays a home game with one of the big clubs of the league, they obtain a higher revenue compared to their average revenue throughout the season. Therefore, one can think that the big clubs should be given priority when scheduling and bundling decisions are given simultaneously. Or other more fair options may also be used like taking a ballot for determining the groups to be favored, or applying a rotation to the teams to be favored.

In order to favor a team or a group of teams, we can solve the problem iteratively. We begin with

Table 4.12: Improvement over the Bundle Selection Heuristic for 8-Team League

Total Demand Based (Cases 1&3)									
Instance	2-Game Bundle				3-Game Bundle				
	Iter1	Iter2	Iter3	Iter4	Iter1	Iter2	Iter3	Iter4	Imp
1	0%	6%	7%	8%	-4%	6%	9%	12%	13%
2	4%	11%	12%	13%	-4%	5%	8%	11%	12%
3	5%	12%	13%	14%	-3%	7%	10%	13%	14%
4	-2%	5%	6%	7%	-4%	5%	8%	11%	12%
5	-3%	4%	5%	6%	-9%	0%	3%	6%	7%
6	-2%	5%	6%	7%	-8%	1%	4%	7%	8%
7	1%	8%	9%	10%	-5%	5%	7%	11%	12%
8	4%	12%	13%	14%	-7%	2%	5%	8%	9%
9	-4%	3%	4%	5%	-2%	8%	11%	14%	15%
10	4%	12%	13%	14%	-7%	2%	5%	8%	9%
11	3%	10%	11%	12%	-8%	1%	4%	7%	8%
12	-1%	6%	7%	8%	-3%	6%	9%	12%	13%
13	4%	11%	12%	13%	-5%	4%	7%	10%	11%
14	1%	8%	9%	10%	-5%	4%	7%	11%	12%
15	-2%	5%	5%	6%	-9%	0%	3%	6%	7%
min	-4%	3%	4%	**5%**	-9%	0%	3%	6%	**7%**
average	1%	8%	9%	**10%**	-5%	4%	7%	10%	**11%**
max	5%	12%	13%	**14%**	-2%	8%	11%	14%	**15%**

the objective of maximizing the chosen team's/group's profit and the schedule of the bundled games created are used as additional constraint set in the following iteration. Of course, two or more ordered teams or group of teams can be chosen to be favored. In that case, the same method can be applied iteratively.

Accordingly, we perform an experimental study in order to investigate the contribution of the S&B Selection Heuristic to the revenues of the teams. The league with eight teams and the ticket demand setting given in Table 4.5 is taken as original league and demand setting. We create two more demand settings (see Table 4.15 and Table 4.16) by decreasing the demands of the last five teams gradually, therefore we have three different 8-team league demand settings including the original one.

Within these demand settings, the teams are grouped as "popular" and "unpopular" with respect to their ticket demand levels. Therefore, the first three teams are grouped as popular teams while the others are grouped as unpopular.

The expected total revenue of popular and unpopular team tickets when they are sold as singles (without any bundling) are assumed to be the benchmarks of respective groups. Since the most demanded ticket for each home team is not considered for bundling, the single ticket revenue generated by the most demanded ticket is not included in this benchmark revenue. We have inquired the percentage increase over the group benchmarks of popular and unpopular teams when three of the six considered tickets for each home team are bundled. We have also analyzed how the results change when the popular or unpopular teams are favored. The graph in Figure 4.2 shows the percentage increase over benchmark revenue for popular/unpopular teams, for the three different demand settings and for the total demand and high demand based cases. Moreover, Figure 4.3 shows the marginal percentage

Table 4.13: Improvement over the Bundle Selection Heuristic for 8-Team League

High Demand Based (Cases 2&4)									
Instance	2-Game Bundle				3-Game Bundle				
	Iter1	Iter2	Iter3	Iter4	Iter1	Iter2	Iter3	Iter4	Imp
1	-8%	4%	6%	6%	7%	11%	12%	11%	12%
2	-5%	7%	9%	9%	6%	10%	11%	10%	11%
3	-9%	3%	5%	5%	4%	8%	9%	8%	9%
4	-8%	4%	6%	7%	6%	10%	11%	10%	11%
5	-6%	6%	8%	8%	6%	9%	10%	9%	10%
6	-6%	6%	8%	8%	6%	9%	10%	10%	10%
7	-6%	6%	8%	8%	7%	10%	11%	10%	11%
8	-8%	4%	6%	6%	2%	5%	6%	6%	7%
9	-8%	4%	6%	7%	5%	8%	9%	9%	10%
10	-6%	6%	8%	8%	7%	10%	11%	10%	11%
11	-8%	4%	6%	6%	8%	11%	12%	11%	12%
12	-6%	7%	9%	9%	2%	5%	6%	5%	6%
13	-6%	7%	9%	9%	5%	8%	9%	9%	10%
14	-6%	6%	8%	8%	5%	8%	9%	8%	9%
15	-5%	7%	9%	9%	9%	12%	13%	13%	13%
min	-9%	3%	5%	**5%**	2%	5%	6%	5%	**6%**
average	-7%	5%	8%	**8%**	6%	9%	10%	9%	**10%**
max	-5%	7%	9%	**9%**	9%	12%	13%	13%	**13%**

Table 4.14: Improvement Provided over Bundle Selection Heuristic for 18-Team League

Instance	Case 3	Case 4
1	19%	13%
2	16%	11%
3	15%	13%
4	13%	13%
5	16%	12%
6	18%	14%
7	16%	10%
8	15%	13%
9	15%	10%
10	18%	16%
11	16%	13%
12	13%	11%
13	13%	12%
14	19%	12%
15	15%	13%
min	**13%**	**10%**
average	**16%**	**12%**
max	**19%**	**16%**

Table 4.15: Stadium Capacities and Demands of Single Tickets (Demand Setting 2)

Team	Stadium Cap.	Rival Teams							
		1	2	3	4	5	6	7	8
1	120	-	120	110	100	90	80	70	40
2	120	120	-	100	90	80	70	60	30
3	100	95	90	-	70	60	50	40	40
4	80	85	60	40	-	35	30	25	25
5	80	80	50	45	40	-	35	30	25
6	70	70	45	40	35	30	-	25	20
7	70	70	40	35	30	25	20	-	15
8	50	50	35	30	25	20	15	15	-

Table 4.16: Stadium Capacities and Demands of Single Tickets (Demand Setting 3)

Team	Stadium Cap.	Rival Teams							
		1	2	3	4	5	6	7	8
1	120	-	120	110	100	90	80	70	40
2	120	120	-	100	90	80	70	60	30
3	100	95	90	-	70	60	50	40	40
4	80	85	50	30	-	25	23	22	20
5	80	80	40	35	30	-	25	20	15
6	70	70	35	30	25	20	-	15	14
7	70	70	30	25	20	15	14	-	12
8	50	50	30	25	20	15	13	12	-

increase when the groups are favored.

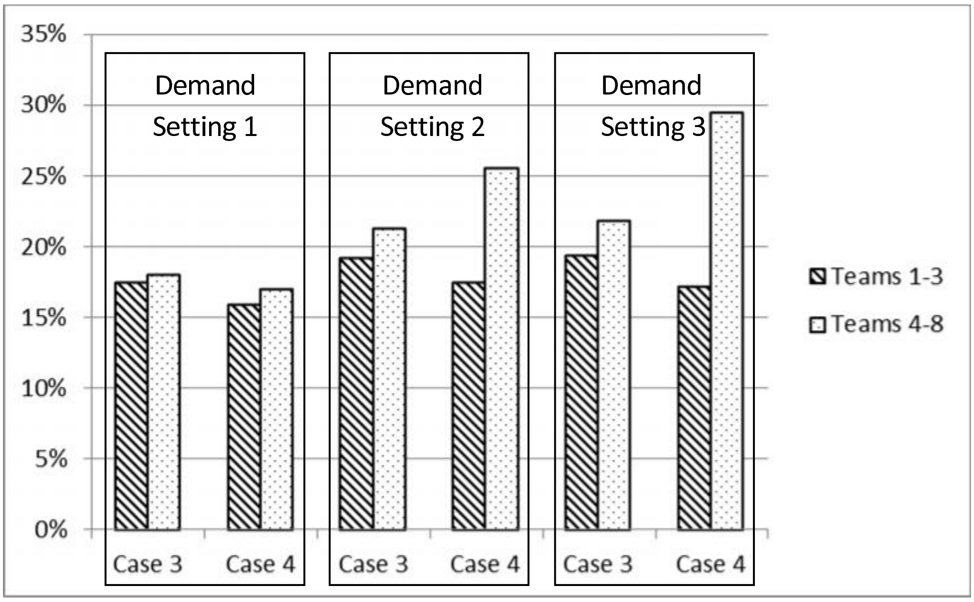

Figure 4.2: Percentage Increase for Popular/Unpopular Teams

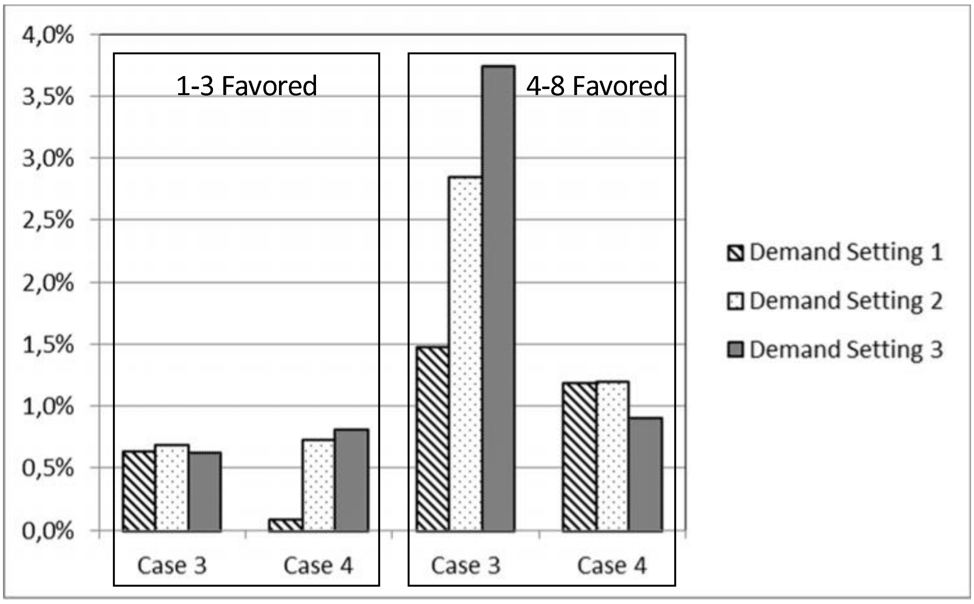

Figure 4.3: Marginal Percentage Increase When the Popular/Unpopular Teams are Favored

As we expect, the percentage revenue enhancement for popular teams (teams 1, 2 and 3) are less than the one for unpopular teams (teams 4, 5, 6, 7 and 8). On the other hand, revenue enhancement of the popular teams are affected slightly with the change in demand (as we move from one demand setting to another), while there is a significant increase in the percentage revenue enhancement of unpopular games. This result is also expected since the benchmark revenues of unpopular teams are decreased gradually in the last two settings. Another prominent result is, the increase in the percentage revenue enhancement of unpopular teams being stronger in high demand based case than the one observed in total demand based case. The total demand based case is more influential since all of the three tickets within the bundle, of which demands are decreased, affects the bundled ticket demand. On the other hand, in the high demand based case, decreased demand of only one ticket affects the bundled ticket demand. Therefore, the unpopular group in high demand based case exploits more, compared to the one in total demand based case.

In Figure 4.3, it is seen that, the marginal contribution of favoring groups is very low except one combination where the group is unpopular and the case is total demand based. The reason is hidden in the rivalry between the popular and unpopular groups. When no group is favored, for the total demand based case, the contest to win the highest demand games between the groups are more severe than the one for the high based demand case, since the bundle demand is affected by all of the games within the bundle and the popular teams challenge to take highly demanded games into their own bundle. Therefore, when the unpopular group is favored in total based case, it exploits more. Note that this contest is highly dependent on the games' demand to capacity ratios. If all of the games' ratios are high for popular teams, since the single ticket revenues increase and marginal revenues decrease for them, the unpopular teams exploit without being favored.

4.4. Conclusion

In this chapter, with a holistic perspective, we consider an integrated approach which generates schedule and bundle combinations to maximize the overall revenue of the teams in a league. We have constructed an effective heuristic method to determine profitable bundles and corresponding schedule simultaneously in a reasonable amount of time. The experiments show that, the *S&B Selection Heuristic* contributes significantly over the result obtained from the *Bundle Selection Heuristic* developed in Chapter 3, when it is applied on a randomly generated feasible schedule. Thus, increasing the revenues from the bundling practise requires creating schedules while also considering the dynamics of profitable bundle selection. We analyze experimentally how the popular and unpopular teams share the gain from bundling with and without preferential treatment.

CHAPTER 5

CONCLUSION AND FUTURE RESEARCH

In this thesis we focus on the sales of event tickets in the sports and entertainment industry, where tickets are sold exclusively as season tickets initially and as single events later in the selling horizon. Selling bundled tickets presents additional challenges over selling single tickets. In this context, we have analyzed three decision problems. According to the order of their presentation in this thesis, the first decision is determining the optimal switch time from selling bundled tickets to single event tickets. Both early and late switch might result in decrease in total revenue as either bundled ticket revenue or single ticket revenue might decrease more than the increase in the other. We have studied the dynamic switching between selling bundles of tickets to selling individual tickets in order to maximize the revenue. The basic problem studied in Duran (2007), which is the timing of a single switch from selling bundles to selling single tickets, is extended to a model which allows an early switch to a low-demand event. The optimal time to switch consists of a set of threshold pairs defined by the remaining seat inventory and the time left in the horizon. While the dynamic decisions can improve revenues over the best static decision, the revenue improvement of employing two switching times over the alternative of employing only one switching time is not that significant. We also observe that our results can be generalized to multiple events in the horizon or to a demand rate that depends on time.

The second decision is to determine the contents of a bundle. Generally speaking, bundling high demand events and low demand events will result in higher revenue generation compared to bundling two events with similar demand characteristics. The reason is that high demand events attract a higher demand from the customers and this will increase the utilization of the venue capacity for low demand events. However, this is just a generalization in the bundling process as the best bundle depends on demand distributions, correlation between demand of different events, single and bundled ticket prices, capacity of the venue, timing of the sales and schedule of the events. Therefore, we analyze the problem of optimal bundling to find patterns in the expected revenue of the event tickets when they are bundled at different time slots. The properties of potentially high revenue generating bundles are used to construct a simple and effective method, *Bundle Selection Heuristic*, to determine profitable bundles in reasonable amount of time.

The third decision actually precedes the first two decision, which is creating the best schedule of the events so as to maximize the overall revenue. Within this context, an integrated approach is developed to generate schedule and bundle combinations to maximize the overall revenue of the participating teams in a league. A heuristic method is constructed to find profitable bundles and corresponding schedule simultaneously in a reasonable amount of time. The experiments show that, the *S&B Selection Heuristic* contributes significantly over the result obtained from the *Bundle Selection Heuristic* when it is applied on a randomly generated feasible schedule. Thus, increasing the revenues from the bundling practise requires creating schedules while also considering the dynamics of profitable bundle selection. Also, the share of popular and unpopular teams on the revenue gain that comes from bundling is analyzed under different assumptions.

There are several directions of research that could further the study in this thesis. Relaxing some restrictive assumptions about the relation of the demand rates before and after a switch would enhance the applicability of our model. Extending the model towards dependent and non-constant demand certainly will make it more practical. Tickets may follow a non-constant demand pattern due to remained seats that are not preferred, timing of the events, advertisement policy or possible impacts of other substitutable events, etc. Therefore, time-invariance of the demand behavior may not hold in many real instances. Similarly, full information assumption in which the seller knows the underlying distribution of demand and price sensitivity is not a realistic assumption. Having a demand learning model may result in a more accurate estimation of ticket demand processes.

A possible extension of the model can be offering bundles and singles simultaneously but with limited seat selection for the singles and considering a menu of bundles (adding mini bundles). Incorporating the dynamic price switches along with the switches from bundled tickets to single tickets would also enhance the revenue.

Moreover, there are many untouched problems still in the overall area of revenue management for sports and entertainment, as RM research has really just begun in this industry segment, waiting to be studied.

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APPENDIX A

Obtaining $\overline{V}$ from the Differential Equation

The following theorem is used for the general solution of a first order linear inhomogeneous differential equation.

Theorem A.1. If $g(t)$ is the solution of the differential equation

$$g'(t) + \lambda g(t) = h(t), \quad 0 \le t \le T$$

where λ is a real number, $h(t)$ is a continuous function, then

$$g(t) = g(T)e^{\lambda(T-t)} - \int_t^T e^{\lambda(s-t)} h(s) ds$$

Proof. Define $G(t) = e^{\lambda t} g(t)$. Then $g(t) = G(t)e^{-\lambda t}$ and $g'(t) = G'(t)e^{-\lambda t} - \lambda G(t)e^{-\lambda t}$. Consequently, $g'(t) + \lambda g(t) = G'(t)e^{-\lambda t}$. Then $h(t) = G'(t)e^{-\lambda t}$ $(*)$, and $(*)$ has solution

$$G(t) = G(T) - \int_t^T e^{\lambda s} h(s) ds$$

then, $g(t) = g(T)e^{\lambda(T-t)} - \int_t^T e^{\lambda(s-t)} h(s) ds$. ■

Therefore the solution of the differential equation:

$$\frac{\partial \overline{V}(t,n)}{\partial t} - \lambda_B \overline{V}(t,n) = -[\lambda_B \overline{V}(t,n-1) + \mathcal{G}\Pi(t,n) + \lambda_B p_B],$$

is $\overline{V}(t,n) = \int_t^T e^{-\lambda_B(s-t)} [\lambda_B \overline{V}(s,n-1) + \mathcal{G}\Pi(s,n) + \lambda_B p_B] ds = \int_t^T e^{-\lambda_B(s-t)} L(s,n) ds$, since $\overline{V}(T,\cdot) = 0$.

APPENDIX B

Details of the proofs of Lemma 3.1 and Lemma 3.2

Proof.[Details of the proof of Lemma 3.1]

$$
\begin{aligned}
V_2(0,n) &= E[p_B((N_{B_2}(\tau_2)) \wedge n)] + \Pi_{F_2}(\tau_2, n(\tau_2)) + \Pi_{S_2}(\tau_2, n(\tau_2)) \\
&= p_B\Big[\sum_{k=0}^{n-1} \frac{((d_B/T_{F2})\tau_2)^k}{k!} e^{-((d_B/T_{F2})\tau_2)} k + \sum_{k=n}^{\infty} \frac{((d_B/T_{F2})\tau_2)^k}{k!} e^{-((d_B/T_{F2})\tau_2)} n\Big] \\
&+ p_F\Big[\sum_{k=0}^{n-1} \frac{((d_F/T_{F2})(T_{F2}-\tau_2))^k}{k!} e^{-((d_B/T_{F2})(T_{F2}-\tau_2))} k \\
&+ \sum_{k=n}^{\infty} \frac{((d_F/T_{F2})(T_{F2}-\tau_2))^k}{k!} e^{-((d_B/T_{F2})(T_{F2}-\tau_2))} n\Big] \\
&+ p_S\Big[\sum_{k=0}^{n-1} \frac{((d_S/T_{S2})(T_{S2}-\tau_2))^k}{k!} e^{-((d_B/T_{S2})(T_{S2}-\tau_2))} k \\
&+ \sum_{k=n}^{\infty} \frac{((d_S/T_{S2})(T_{S2}-\tau_2))^k}{k!} e^{-((d_B/T_{S2})(T_{S2}-\tau_2))} n\Big] \\
&< p_B\Big[\sum_{k=0}^{n-1} \frac{((d_B/T_{F1})\tau_1)^k}{k!} e^{-((d_B/T_{F1})\tau_1)} k + \sum_{k=n}^{\infty} \frac{((d_B/T_{F1})\tau_1)^k}{k!} e^{-((d_B/T_{F1})\tau_1)} n\Big] \\
&+ p_F\Big[\sum_{k=0}^{n-1} \frac{((d_F/T_{F1})(T_{F1}-\tau_1))^k}{k!} e^{-((d_B/T_{F1})(T_{F1}-\tau_1))} k \\
&+ \sum_{k=n}^{\infty} \frac{((d_F/T_{F1})(T_{F1}-\tau_1))^k}{k!} e^{-((d_B/T_{F1})(T_{F1}-\tau_1))} n\Big] \\
&+ p_S\Big[\sum_{k=0}^{n-1} \frac{((d_S/T_{S1})(T_{S1}-\tau_1))^k}{k!} e^{-((d_B/T_{S1})(T_{S1}-\tau_1))} k \\
&+ \sum_{k=n}^{\infty} \frac{((d_S/T_{S1})(T_{S1}-\tau_1))^k}{k!} e^{-((d_B/T_{S1})(T_{S1}-\tau_1))} n\Big] \\
&= E[p_B((N_{B_1}(\tau_1)) \wedge n)] + \Pi_{F_1}(\tau_1, n(\tau_1)) + \Pi_{S_1}(\tau_1, n(\tau_1)) \le V_1(0,n).
\end{aligned}
$$

The inequality above follows from the obvious fact that

$$E[p_B((N_{B_2}(\tau_2)) \wedge n)] + \Pi_{F_2}(\tau_2, n(\tau_2)) = E[p_B((N_{B_1}(\tau_1)) \wedge n)] + \Pi_{F_1}(\tau_1, n(\tau_1))$$

and $\Pi_{S_2}(\tau_2, n(\tau_2)) < \Pi_{S_1}(\tau_1, n(\tau_1))$. We know from Observation 3.1 that the expected number of sales term $\sum_{k=0}^{n-1} \frac{((d_S/T_S)(T_S-\tau))^k}{k!} e^{-((d_B/T_S)(T_S-\tau))} k + \sum_{k=n}^{\infty} \frac{((d_S/T_S)(T_S-\tau))^k}{k!} e^{-((d_B/T_S)(T_S-\tau))} n$ decreases as the rate term $((d_S/T_S)(T_S-\tau))$ decreases. Let $T_S = T_F + x$ where $x > 0$, then the latter term is equal to $d_S(1 - \frac{\tau}{T_F+x})$, which is decreasing in T_F. ■

Proof.[Details of the proof of Lemma 3.2]

$$\begin{aligned}
V_1(0,n) &= E[p_B((N_{B_1}(\tau_1)) \wedge n)] + \Pi_{F_1}(\tau_1, n(\tau_1)) + \Pi_{S_1}(\tau_1, n(\tau_1)) \\
&= p_B\Big[\sum_{k=0}^{n-1} \frac{((d_B/T_{F1})\tau_1)^k}{k!} e^{-((d_B/T_{F1})\tau_1)} k + \sum_{k=n}^{\infty} \frac{((d_B/T_{F1})\tau_1)^k}{k!} e^{-((d_B/T_{F1})\tau_1)} n\Big] \\
&+ p_F\Big[\sum_{k=0}^{n-1} \frac{((d_F/T_{F1})(T_{F1}-\tau_1))^k}{k!} e^{-((d_F/T_{F1})(T_{F1}-\tau_1))} k \\
&+ \sum_{k=n}^{\infty} \frac{((d_F/T_{F1})(T_{F1}-\tau_1))^k}{k!} e^{-((d_F/T_{F1})(T_{F1}-\tau_1))} n\Big] \\
&+ p_S\Big[\sum_{k=0}^{n-1} \frac{((d_S/T_{S1})(T_{S1}-\tau_1))^k}{k!} e^{-((d_S/T_{S1})(T_{S1}-\tau_1))} \\
&+ \sum_{k=n}^{\infty} \frac{((d_S/T_{S1})(T_{S1}-\tau_1))^k}{k!} e^{-((d_S/T_{S1})(T_{S1}-\tau_1))} n\Big] \\
&< p_B\Big[\sum_{k=0}^{n-1} \frac{((d_B/T_{F2})\tau_2)^k}{k!} e^{-((d_B/T_{F2})\tau_2)} k + \sum_{k=n}^{\infty} \frac{((d_B/T_{F2})\tau_2)^k}{k!} e^{-((d_B/T_{F2})\tau_2)} n\Big] \\
&+ p_F\Big[\sum_{k=0}^{n-1} \frac{((d_F/T_{F2})(T_{F2}-\tau_2))^k}{k!} e^{-((d_F/T_{F2})(T_{F_2}-\tau_2))} k \\
&+ \sum_{k=n}^{\infty} \frac{((d_F/T_{F2})(T_{F2}-\tau_2))^k}{k!} e^{-((d_F/T_{F2})(T_{F2}-\tau_2))} n\Big] \\
&+ p_S\Big[\sum_{k=0}^{n-1} \frac{((d_S/T_{S2})(T_{S2}-\tau_2))^k}{k!} e^{-((d_S/T_{S2})(T_{S2}-\tau_2))} k \\
&+ \sum_{k=n}^{\infty} \frac{((d_S/T_{S2})(T_{S2}-\tau_2))^k}{k!} e^{-((d_S/T_{S2})(T_{S2}-\tau_2))} n\Big] \\
&= E[p_B((N_{B_2}(\tau_2)) \wedge n)] + \Pi_{F_2}(\tau_2, n(\tau_2)) + \Pi_{S_2}(\tau_2, n(\tau_2)) \leq V_2(0,n).
\end{aligned}$$

The inequality above follows from the fact that

$$E[p_B((N_{B_1}(\tau_1)) \wedge n)] + \Pi_{F_1}(\tau_1, n(\tau_1)) = E[p_B((N_{B_2}(\tau_2)) \wedge n)] + \Pi_{F_2}(\tau_2, n(\tau_2))$$

and $\Pi_{S_1}(\tau_1, n(\tau_1)) < \Pi_{S_2}(\tau_2, n(\tau_2))$. The equalities are obvious and the inequality follows from $\sum_{k=0}^{n-1} \frac{((d_S/T_S)(T_S-\tau))^k}{k!} e^{-((d_B/T_S)(T_S-\tau))} k + \sum_{k=n}^{\infty} \frac{((d_S/T_S)(T_S-\tau))^k}{k!} e^{-((d_S/T_S)(T_S-\tau))} n$ being increasing in the rate term $((d_S/T_S)(T_S - \tau))$ by Observation 3.1, which is increasing in T_S. ■

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