OSCILLATION CRITERIA FOR FIRST AND SECOND ORDER IMPULSIVE DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT

OSCILLATION CRITERIA FOR FIRST AND SECOND ORDER IMPULSIVE DELAY DIFFERENTIAL EQUATIONS

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There exists a well-developed oscillation theory of delay differential equations. The theory of ordinary differential equations with impulses has also been developed extensively over the past few years. However, oscillation of delay differential equations with impulses seem to have rarely been considered.

In this thesis, we first give a survey on oscillation of solutions of first and second order delay differential equations with impulses having similarities with delay differential equations without impulses. Next, in view of the known results obtained for delay differential equations without impulses, we derived new oscillation and nonoacillation criteria for delay differential equations with impulse effect.

Keywords: Impulsive delay differential equation, Oscillation, Nonoscillation.

ÖZ

BİRİNCİ VE İKİNCİ DERECEDEN İMPULSİVE VE GECİKMELİ DİFERENSİYEL DENKLEMLER İÇİN SALINIM KRİTERLERİ

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Gecikmeli diferensiyel denklemlerin salınımı üzerinde çok sayıda araştırma vardır. İmpulsive diferensiyel denklemler teorisi son zamanlarda hızlı bir şekilde geliştirilmesine rağmen, impulsive ve gecikmeli diferensiyel denklemler alanında yeterli çalışma yoktur. Özellikle, salınım alanında, şimdiye kadar çok az sayıda çalışma yapılmştır.

Bu tezde, amacımız impulsive denklemlerin salınımı konusunda yapılmış olan bazı çalışmalar sunuduktan sonra, impulsive ve gecikmeli diferensiyel denklemlerin salınımı için yeni gerek ve yeter koşullar elde etmektir.

Anahtar kelimeler: İmpulsive ve gecikmeli diferensiyel denklem, Salınımlı çözüm, Salınımsız çözüm.

Dedicated To My Parents FAYZEH and OTHMAN

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CHAPTER 1

INTRODUCTION AND PRELIMINARIES

1.1 Introduction

Many evolution process in nature are charactrized by the fact that at certain moments of time they experience an abrupt change of state. This has been the main reason for the development of the theory of impulsive ordinary differential equations, and now this theory has been elaborated to a considerable extent.

In the last few years the theory of impulsive differential equations and delay differential equations has been studied by many authors[2, 3, 5, 8, 9, 15, 20]. However, not much has been developed in the direction of delay differential equations with impulses[13, 23, 26, 32].

A differential equation of the form

$$x' = f(t, x(t), x(\tau(t))), \qquad \tau(t) < t,\ t \geq t_0 \tag{1.1}$$

in which the right-hand side depends not only on the instantanous position $x(t)$, but also on $x(\tau(t))$, is called a delay differential equation DDE or as in the Russian Literature a differential equation with retarded argument.

The basic initial value function for (1.1) is posed as follows: For $t > t_0$, we seek a continuous function $x(t)$ that satisfies (1.1) and an initial condition

$$x(t) = \phi(t), \quad t \in E_{t_0} \tag{1.2}$$

where t_0 is the initial point and $E_{t_0} = t_0 \cup \{\tau(t) : \tau(t) < t_0, t \geq t_0\}$ is the initial set, the known function $\phi(t)$ on E_{t_0} is called the initial function. Usually, it is assumed that $x(t_0+) = \phi(t_0)$. We always mean a one-sided derivative when we speak of the derivative at an endpoint of an interval. If it is required to determine the solution on the interval $[t_0, T]$, $T \leq +\infty$, then the initial set

$$E_{t_0 T} = \{t_0\} \cup \{\tau(t) : \tau(t) < t, \quad t_0 \leq t \leq T\}$$

is needed.

Example 1.1.1 Consider the equation

$$x'(t) = f(t, x(t), x(t - \cos^2 t)).$$

Here $t_0 = 0, E_0 = [-1, 0]$, therefore the initial function $\phi(t)$ must be given on the interval [-1,0].

If in a real process described by equation (1.1) certain impulses occur at fixed times, the mathematical model of this process could be given by the following initial value function

$$
\begin{aligned}
&x'(t) = f(t, x(t), x(\tau(t))), \qquad t \neq \theta_i, \\
&\Delta x(t)|_{t=\theta_i} = I_i(x(\theta_i)), \\
&x(t) = \phi(t), \quad t \in E_{t_0}, \quad x(t_0+) = x_0, \qquad (1.3)
\end{aligned}
$$

where

$$f : R_+ \times R \times R \to R, \quad I_i : R \to R, \quad \tau : [t_0, \infty) \to [t_0, \infty), \quad \tau(t) < t.$$

The differential equation in (1.3) is called impulsive delay differential equation IDDE. It can be shown that the problem (1.3) is equivalent to

$$
\begin{aligned}
&x(t) = x_0 + \int_{t_0}^{t} f(s, x(s), x(\tau(s)))\, ds + \sum_{t_0 \leq \theta_i < t} I_i(x(\theta_i)), \\
&x(t) = \phi(t), \quad t \in E_{t_0}. \qquad (1.4)
\end{aligned}
$$

The process defined by system (1.3) goes as follows: The point $P_t(t, x(t))$, starting at (t_0, x_0), moves along the curve defined by the solution $x(t) = x(t, t_0, x_0)$ of the equation

$$x'(t) = f(t, x(t), x(\tau(t))) \qquad (1.5)$$

The motion along this curve terminates at time $t = \theta_1$ when the point P_t arrives at the point of disconinuity $t = \theta_1$. At that moment the point P_t

performs a jump $\Delta x|_{t=\theta_1} = I_1(x(\theta_1))$ and proceeds to move along the curve described by the solution $x(t, \theta_1, x(\theta_1+))$ of equation (1.5), until it meets the next point of discontinuity, and so on.

A solution of system (1.3) is such a piecewise continuous function that has discontiuities of the first kind at $t = \theta_i$ such that $\phi'(t) = f(t, \phi(t), \phi(\tau(t)))$ for all $t \neq \theta_i$ and, for $t = \theta_i$, satisfying the jumps condition, i.e,

$$\Delta\phi(t)|_{t=\theta_i} = \phi(\theta_i+) - \phi(\theta_i-) = I_i(\phi(\theta_i-)).$$

where

$$\phi(\theta_i-) = \lim_{t\to\theta_i-} \phi(t),$$

and

$$\phi(\theta_i+) = \lim_{t\to\theta_i+} \phi(t).$$

We assume that $\phi(\theta_i-) = \phi(\theta_i)$ so that ϕ is left continuous.

1.2 Existence and Uniqueness of Solutions of IDDEs

Let $\Omega \subset R^2$ be an open set and let $D = R_+ \times \Omega$. Suppose that for each i=1,2,..., $\theta_i < \theta_{i+1}$ and $\lim_{i\to\infty} \theta_i = \infty$.

Definition 1.2.1 A function $x : (t_0, t_0 + T) \to R, t_0 \geq 0$, $T > 0$, is said to be a *solution* of (1.3) if

(I) $x(t_0+) = x_0$ and $(t, x(t), x(\tau(t))) \in D$ for $t \in [t_0, t_0 + T)$,

(II) $x(t)$ is continuously differentiable and satisfies $x'(t) = f(t, x(t), x(\tau(t)))$ and $x(t) = \phi(t)$, for $t \in [t_0, t_0 + T)$ and $t \neq \theta_i$,

(III) $x(t+) = x(t) + I_i(x(t))$, for $t \in (t_0, t_0 + T]$ and $t = \theta_i$ where $x(t)$ is assumed to be left continuous.

Theorem 1.2.1 *[22] Assume that f, τ and ϕ are continuous in their domains of definition, except possibly at $t = \theta_i$ i=1,2,.... Then, for each $(t_0, x_0, .) \in D$, there exists a solution $x : [t_0, t_0 + \alpha) \to R$ of the initial value problem (1.3) for some $\alpha > 0$.*

1.3 Definition of Oscillation

A non-trivial function $x(t)$ is called *oscillatory* if there exists a sequence $\{t_n\}$ such that $\lim t_n = \infty$ and $x(t_n)x(t_n+) \leq 0$. Otherwise, $x(t)$ is said to be nonoscillatory. A *nonoscillatory* function is either eventually positive or eventually negative, i.e, there exists t_1 such that $x(t) \neq 0$, for all $t > t_1$.

We say that a differential equation is *oscillatory* if every solution of the equation is oscillatory and *nonoscillatory* if it has at least one nonoscillatory solution.

It is clear that in oscillation theory the solutions must exist on an infinite interval of the form $[t_*, \infty)$ for some t_*. The following theorem provides conditions for such solution to exist [22].

Theorem 1.3.1 *[22] Assume that the hypothesis of Theorem 1.2.1 hold. Suppose further that*

$$|f(t,x,y)| \leq g(t,|x|), \quad (t,x,y) \in R_+ \times R \times R,$$

$$|x + I_i(x)| \leq |x|, \quad x \in R,$$

where $g \in C[R_+ \times R_+, R_+]$*, g(t,u) is nondecreasing in u for each* $t \in R_+$*. Let* $r(t) = r(t, t_0, u_0)$ *be the maximal solution of*

$$u' = g(t,u), \quad u(t_0) = u_0 \geq 0,$$

existing on $[t_0, \infty)$*. Then the interval of existence of any solution* $x(t) = x(t, t_0, x_0)$ *of (1.3) such that* $|x_0| \leq u_0$ *is* $[t_0, \infty)$*.*

1.4 Basic Definitions and Fixed Point Theorem

Before starting the main results, we need to mention some basic definitions and theorems that we will rely on later.

Definition 1.4.1 A subset S of a normed linear space X is called *bounded* if there is a number M such that $\|x\| \leq M$ for all $x \in S$.

Definition 1.4.2 A set S in a vector space X is called *convex* if, for any $x, y \in S$, $\lambda x + (1-\lambda)y \in S$ for all $\lambda \in [0,1]$.

Definition 1.4.3 Let N, M be normed linear space, and X be subset of N, An operator $\phi : X \longrightarrow M$ is *continuous at a point* $x \in X$ if and only if for any $\epsilon > 0$ there is a $\delta > 0$ such that $\|\phi x - \phi y\| < \epsilon$ for all $y \in X$ such that $\|x - y\| < \delta$. ϕ is *continuous* on X, or simply *continuous*, if it is continuous at all points of X.

Definition 1.4.4 A subset S of a normed linear space B is *compact* if and only if every infinite sequence of elements of S has a subsequence which converges to an element of S.

Definition 1.4.5 A subset S of a normed linear space N is a *relatively compact* if and only if every sequence in S has a subsequence converging to an element of N.

Definition 1.4.6 A function $f : R \longrightarrow R$ is *bounded* on an interval I if and only if there is a positive real number M such that $|f(x)| < M$ for all $x \in I$. A family F of functions is *uniformly bounded* on I if there is an M such that $|f(x)| \leq M$ for all $x \in I$ and all of $f \in F$.

Definition 1.4.7 A family F of functions is *equicontinuous* on an interval I if and only if for every $\epsilon > 0$ there is δ such that for all $f \in F$, $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta, x, y \in I$.

Lemma 1.4.1 ***(Arzela-Ascoli)*** *A set of functions in C([a,b]) with* $\|f\| = sup_{x\in[a,b]}\|f(x)\|$ *is a relatively compact if and only if it is uniformly bounded and equicontinuous in* $[a,b]$.

Remark 1.4.1 If the interval is not finite, one should be very careful in using the above Lemma. To show that a certain family of functions is equicontinuous in an infinite interval one can make use of *Levitan's* result. According to this result, a family of functions is equicontinuous on $[t_0, \infty)$ if for any given $\epsilon > 0$, the interval $[t_0,\infty)$ can be decomposed into a finite number of subintervals in such a way that on each subinterval all functions of the family have variations less than ϵ.

Theorem 1.4.1 ***(Schauder's Fixed Point Theorem)*** *Let S be a closed, convex and non-empty subset of a Banach space X. Let*

$$\phi : S \longrightarrow S$$

be continuous such that ϕS *is a relatively compact subset of X. Then* ϕ *has at least one fixed point in S. That is there exists an* $x \in S$ *such that* $\phi x = x$.

At last we present the statement of the Lebesque dominated convergence theorem.

Theorem 1.4.2 ***(Lebesque's Dominated Convergence Theorem)*** *Let $\{f_n\}$ be a sequence of integrable functions such that*

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \textit{almost everywhere in } A$$

and such that for every n=1,2,3,...

$$|f_n(x)| \leq g(x) \quad \textit{almost everywhere in } A$$

where g is integrable on A. Then

$$\lim_{n\to\infty} \int_A f_n(x)d\mu = \int_A f(x)d\mu.$$

CHAPTER 2

FIRST ORDER IMPULSIVE DELAY EQUATIONS

2.1 Introduction

In this chapter sufficient conditions for oscillation of impulsive delay differential equations of first order are established. Moreover, conditions on existence of at least one nonoscillatory solution are obtained.

First we analyze the oscillatory behavior of solutions of DDEs of the form

$$x'(t) + p(t)x(\tau(t)) = 0 \tag{2.1}$$

and oscillatory and nonoscillarory properties of solutions of the corresponding IDDEs of the type

$$\begin{aligned} &x'(t) + p(t)x(\tau(t)) = 0, \quad t \neq \theta_i, \\ &\Delta x(\theta_i) = b_i x(\theta_i). \end{aligned} \tag{2.2}$$

We will consider equation (2.1) and system (2.2) when $p(t) = p > 0$ and $\tau(t) = t - \tau$. That is, we consider

$$x'(t) + p\,x(t - \tau) = 0, \tag{2.3}$$

and

$$x'(t) + p\,x(t - \tau) = 0, \quad t \neq \theta_i,$$

$$\Delta x(\theta_i) = b_i x(\theta_i). \tag{2.4}$$

It will be shown that the oscillation of a linear impulsive delay differential equation of the type (2.2) is equivalent to the oscillation of a corresponding linear delay differential equation without impulses

$$x'(t) + p(t) \prod_{\tau(t) \leq \theta_i < t} (1 + b_i)^{-1} x(\tau(t)) = 0, \tag{2.5}$$

where it is assumed that the product equals unity if the number of factors is equal to zero.

The characteristic equations of equation (2.3) and system (2.4) are defined respectively as,

$$F(\lambda) = \lambda + p\, e^{-\lambda\tau} = 0,$$

and

$$F(\lambda) = \lambda + p \prod_{t-\tau \leq \theta_i < t} (1 + b_i)\, e^{-\lambda\tau} = 0. \tag{2.6}$$

We will see in Theorem 2.2.6 and Theorem 2.2.8 that equation (2.3) and (2.4) have nonoscillatory solutions if and only if their charactristic equations have a real root.

2.2 Oscillatory Behavior

In this section we shall state and prove some theorems which provide conditions for the oscillation of solutions of DDEs and the corresponding IDDEs. Let us start with the following theorems:

Theorem 2.2.1 *[20] Assume that* $p \in [R_+, R_+]$, $\tau \in [R_+, R_+]$, $\tau(t) < t$, *and* $\lim_{t\to\infty} \tau(t) = \infty$,

If

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)\,ds > \frac{1}{e}, \tag{2.7}$$

then (2.1) is oscillatory.

Proof. Assume, for the sake of contradiction, that (2.1) has an eventually positive solution $x(t)$ such that $x(\tau(t))$ for $t \geq t_1$. Because of (2.7), there exists a $t_2 \geq t_1$ such that

$$\int_{\tau(t)}^{t} p(s)\,ds \geq c > \frac{1}{e} \quad for \quad t \geq t_2. \tag{2.8}$$

Dividing (2.1) by $x(t)$ and integrating from $\tau(t)$ to t, we obtain

$$\ln \frac{x(t)}{x(\tau(t))} + \int_{\tau(t)}^{t} p(s)\,ds = 0, \quad t \geq t_2,$$

and hence

$$\ln \frac{x(\tau(t))}{x(t)} = \int_{\tau(t)}^{t} p(s)\,ds \geq c, \quad t \geq t_2.$$

Since $e^x \geq ex$, for $x \geq 0$, it follows that

$$\frac{x(\tau(t)}{x(t)} \geq ec, \qquad t \geq t_2.$$

Repeating the above procedure, there exists a sequence $\{t_k\}$ such that

$$\frac{x(\tau(t)}{x(t)} \geq (ec)^k, \qquad t \geq t_k. \tag{2.9}$$

From (2.8), there exists a t^* such that

$$\int_{\tau(t)}^{t^*} p(s)\,ds \geq \frac{c}{2} \qquad \textit{and} \qquad \int_{t^*}^{t} p(s)ds \geq \frac{c}{2} \qquad \textit{for} \quad t \geq t_k.$$

Integrating (2.1) from $\tau(t)$ to t^* yields

$$x(t^*) - x(\tau(t)) + \int_{\tau(t)}^{t^*} p(s)x(\tau(s))\,ds \leq 0.$$

This implies that

$$x(\tau(t)) \geq x(\tau(t^*))\frac{c}{2}. \tag{2.10}$$

Similarly, we obtain

$$x(t) - x(t^*) + \int_{t^*}^{t} p(s)x(\tau(s))\,ds \leq 0,$$

and consequently

$$x(t^*) \geq x(\tau(t))\frac{c}{2}. \tag{2.11}$$

Combining (2.10) and (2.11), there results in the inequality

$$x(t^*) \geq x(\tau(t^*))(\frac{c}{2})^2. \tag{2.12}$$

From (2.9) and (2.12), it follows that

$$(\frac{c}{2})^2 \geq \frac{x(\tau(t^*))}{x(t^*)} \geq (ec)^k, \quad for \quad all \quad t \geq t_k. \tag{2.13}$$

Now we choose k sufficiently large such that

$$(ec)^k > (\frac{2}{c})^2,$$

which is possible because $ec > 1$. Therefore (2.13) is a contradiction. A parallel argument holds if $x(t)$ is assumed to be eventually negative solution, therefore we obtain that all solutions of (2.1) are oscillatory.

Consider now system (2.2), that is, the same delay differential equation (2.1) but when it is subject to impulses. In the following theorem we will modify the conditions of Theorem 2.2.1 in order to get an oscillation criterion for (2.2).

Theorem 2.2.2 *[13]Let $\tau(t) = t - \tau$, $\tau \in R$, and assume that*

(H1) for some $T > 0$, $\theta_{i+1} - \theta_i \geq T, i = 1, 2, \ldots$ and $\tau < T$;

(H2) there is an $M > 0$ such that $0 \leq b_i \leq M, \quad i = 1, 2, \ldots$;

(H3) p is continuous on $[0, \infty)$ and $p(t) \geq 0$ for $t \geq 0$.

If

$$\liminf_{t \to \infty} \int_{t-\tau}^{t} p(s)ds > \frac{1+M}{e}, \tag{2.14}$$

then (2.2) is oscillatory .

Proof. Suppose the result is not true. Without loss of generality, there exists an eventually positive solution say $x(t) > 0$ for $t \geq t^*$. Define

$$w(t) = \frac{x(t-\tau)}{x(t)}, \quad for \quad t \geq t^* + \tau.$$

Considering the interval $[t-\tau, t]$ and $\theta_i \in (t-\tau, t)$,

$$x(t-\tau) \geq x(\theta_i) = \frac{1}{1+b_i} x(\theta_i +) \geq \frac{1}{1+b_i} x(t),$$

implying

$$w(t) = \frac{x(t-\tau)}{x(t)} \geq \frac{1}{1+b_i} \geq \frac{1}{1+M}.$$

We shall show that $w(t)$ is bounded above. Let θ_i be a jump point in $[t-2\tau, t-\tau]$. Integrating (2.2) on $[t-\frac{\tau}{2}, t]$,

$$x(t) - x(t-\frac{\tau}{2}) + \int_{t-\frac{\tau}{2}}^{t} p(s)x(s-\tau)ds = 0. \tag{2.15}$$

It follows from (2.15) that

$$\begin{aligned} x(t-\frac{\tau}{2}) \ &\geq \ \int_{t-\frac{\tau}{2}}^{t} p(s)x(s-\tau)\, ds \\ &\geq \int_{t-\frac{\tau}{2}}^{\theta_i+\tau-0} p(s)x(s-\tau)\, ds + \int_{\theta_i+\tau+0}^{t} p(s)x(s-\tau)\, ds \\ &\geq \frac{x(t-\tau)}{1+M} \int_{t-\frac{\tau}{2}}^{t} p(s)\, ds. \end{aligned}$$

On integrating (2.2) over $[\ t-\tau, t-\frac{\tau}{2}]$,

$$x(t-\tau) \geq x(t-\frac{3\tau}{2}) \int_{t-\tau}^{t-\frac{\tau}{2}} p(s)\,ds.$$

Thus

$$x(t-\frac{\tau}{2}) \geq x(t-\frac{3\tau}{2})[\int_{t-\tau}^{t-\frac{\tau}{2}} p(s)\,ds][\int_{t-\frac{\tau}{2}}^{t} p(s)\,ds]\frac{1}{1+M},$$

and, hence

$$\frac{x(t-\frac{3\tau}{2})}{x(t-\frac{\tau}{2})} \leq \frac{1+M}{[\int_{t-\tau}^{t-\frac{\tau}{2}} p(s)\,ds][\int_{t-\frac{\tau}{2}}^{t} p(s)\,ds]} \leq N.$$

We have from (2.2) for large enough t,

$$\int_{t-\tau}^{t} \frac{x'(s)}{x(s)}ds + \int_{t-\tau}^{t} p(s)\frac{x(s-\tau)}{x(s)}\,ds = 0. \qquad (2.16)$$

But

$$\int_{t-\tau}^{t} \frac{x'(s)}{x(s)}\,ds = \int_{t-\tau}^{\theta_i-0} \frac{x'(s)}{x(s)}\,ds + \int_{\theta_i+0}^{t} \frac{x'(s)}{x(s)}\,ds$$
$$= \ln \frac{x(\theta_i-0)}{x(t-\tau)}\frac{x(t)}{x(\theta_i+0)} = \ln \frac{x(t)}{x(t-\tau)}\frac{1}{1+b_k}. \qquad (2.17)$$

From (2.16) and (2.17),

$$\ln \frac{x(t-\tau)}{x(t)}(1+b_k) = \int_{t-\tau}^{t} p(s)\frac{x(s-\tau)}{x(s)}\,ds. \qquad (2.18)$$

If

$$l = \liminf_{t\to\infty} w(t),$$

then l is finite and positive and (2.18) leads to

$$\ln[(1+M)w(t)] \geq l \int_{t-\tau}^{t} p(s)\, ds,$$

which implies that

$$\frac{1+M}{e} \geq \frac{ln[(1+M)l]}{l} \geq \liminf_{t\to\infty} \int_{t-\tau}^{t} p(s)\, ds. \tag{2.19}$$

Since (2.19) contradicts (2.14). The proof is complete.

Below we provide sufficient conditions for the solutions of equation (2.1) and system (2.2) to be oscillatory

Theorem 2.2.3 *[20] Assume that* p, $\tau \in C[R_+, R_+]$, $\tau(t) < t$ *and it is non-decreasing,* $\lim_{t\to\infty} \tau(t) = +\infty$,
If

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)\, ds > 1, \tag{2.20}$$

then (2.1) is oscillatory .

Proof. Without loss of generality, let $x(t) > 0$ be a nonoscillatory solution such that $x(\tau(t)) > 0$, $t \geq t_1$. Integrating (2.1) from $\tau(t)$ to t, we have

$$x(t) - x(\tau(t)) + \int_{\tau(t)}^{t} p(s)x(\tau(s))\, ds = 0,$$

or equivalently

$$x(t) + x(\tau(t))[\int_{\tau(t)}^{t} p(s)\,ds - 1] \leq 0. \tag{2.21}$$

For t is sufficiently large (2.21) is a contradiction. The proof is complete.

Theorem 2.2.4 *Assume that $p \in C[R_+, R_+]$, $\tau(t) = t - \tau < t$ and it is non-decreasing and $\tau < \inf(\theta_{i+1} - \theta_i)$,*
If

$$\limsup_{t\to\infty} \int_{t-\tau}^{t} p(s)\,ds > 1;$$

then (2.2) is oscillatory .

Proof. Without loss of generality, let $x(t) > 0$ be a nonoscillatory solution such that $x(t-\tau) > 0$, $t \geq t_1$. The condition $\tau < \inf(\theta_{i+1} - \theta_i)$ means that there is no discontinuity point between $t - \tau$ and t, i.e, the impulse points have no contribution to this interval. Integrating (2.1) from $t - \tau$ to t, we have

$$x(t) - x(t-\tau) + \int_{t-\tau}^{t} p(s)x(s-\tau)\,ds = 0,$$

or equivalently

$$x(t) + x(t-\tau)[\int_{t-\tau}^{t} p(s)\,ds - 1] \leq 0.$$

For t is sufficiently large, the last inequality is a contradiction. The proof is complete.

Theorem 2.2.5 *[13]Let $\tau(t) = t - \tau$, $\tau \in R$, and assume that*

(H4) p is continuous on $[0, \infty)$ and $p(t) \geq 0$ for $t \geq 0$;

(H5) there is $T > 0$ such that $\theta_{i+1} - \theta_i \geq T$, $i = 1, 2, 3, \ldots$.

If either

$$\limsup_{i \to \infty} (1 + b_i)^{-1} \int_{\theta_i}^{\theta_i + T} p(s)\, ds > 1 \quad if \quad \tau \geq T, \tag{2.22}$$

or

$$\limsup_{i \to \infty} (1 + b_i)^{-1} \int_{\theta_i}^{\theta_i + \tau} p(s)\, ds > 1 \quad if \quad 0 < \tau < T, \tag{2.23}$$

then (2.2) is oscillatory.

Proof. Suppose the conclusion is not true, then there exists a nonoscillatory solution $x(t)$ of (2.2). We shall assume that $x(t) > 0$ for all $t \geq t^*$. Since eventually $x(t) > 0$, $x'(t) \leq 0$ for all large t, x is nonincreasing on intervals of the form $(\theta_j, \theta_{j+1}), j = 1, 2, 3, \ldots$ We shall prove the result in the case of $\tau \geq T$.

It follows from (2.2) by an integration on $(\theta_i, \theta_i + T)$,

$$x(\theta_i + T) - x(\theta_i + 0) + \int_{\theta_i + 0}^{\theta_i + T} p(s) x(s - \tau)\, ds = 0. \tag{2.24}$$

By the nonincreasing nature of x, we have from (2.24),

$$x(\theta_i + T) - x(\theta_i + 0) + [\int_{\theta_i+0}^{\theta_i+T} p(s)\,ds]x(\theta_i + T - \tau) \leq 0,$$

and hence

$$x(\theta_i + T) - x(\theta_i + 0) + x(\theta_i - 0)\int_{\theta_i+0}^{\theta_i+T} p(s)\,ds \leq 0. \qquad (2.25)$$

Now using the jump conditions of (2.2) in (2.25),

$$x(\theta_i + T) + x(\theta_i + 0)[\frac{1}{1+b_i}\int_{\theta_i}^{\theta_i+T} p(s)\,ds - 1] \leq 0. \qquad (2.26)$$

But (2.26) is impossible due to the eventual positivity of x and (2.22). By a similar analysis one can derive a contradiction if (2.23) holds. The proof is complete.

In the following two theorems we will give necessary and sufficient conditions for equation (2.3) and system (2.4) to be oscillatory. The technique is based on the study of the characteristic equations of these equations.

Theorem 2.2.6 *[20] Assume that p and τ are positive numbers.*
If

$$p\,\tau\,e \leq 1,$$

then (2.3) is nonoscillatory.

Proof. Let us look at a solution of (2.3) of the form $x(t) = e^{\lambda t}$. It follows that

$$F(\lambda) = \lambda + pe^{-\lambda\tau} = 0.$$

Observe that $F(0) = p > 0$ and $F(-\frac{1}{\tau}) = -\frac{1}{\tau} + pe = \frac{p\tau e - 1}{\tau} \leq 0$. Hence there exists a negative real number $\lambda \in [-\frac{1}{\tau}, 0]$, such that $e^{\lambda t}$ is a nonoscillatory solution of (2.3).

Corollary 2.2.1 *$p\tau e > 1$ is a necessary and sufficient condition for oscillation of (2.3).*

The following corollary which is a substantial improvement of Theorem (2.3.1) helps in proving the next theorem.

Corollary 2.2.2 *[13, 23] Assume that $b_i \geq 0$ for $i = 1, 2, \ldots,$ and $p\tau e \leq 1$. Then (2.4) is nonoscillatory.*

The next theorem provides a necessary and sufficient condition for (2.4) to have a nonoscillatory solution.

Theorem 2.2.7 *[23] Assume the following:*

(H6) $\theta_{i+1} - \theta_i \geq T$, $T \geq 0$, $i = 1, 2, \ldots$ *and* $\tau < T$;

(H7) $b_i > 0$, $i = 1, 2, \ldots$ *and* $\lim_{i\to\infty} b_i = 0$.

Then (2.4) is nonoscillatory if and only if $p\tau e \leq 1$.

Proof. From Corollary 2.2.2, we know that we only have to show the necessity. If $p\tau e > 1$, then there exists a positive constant M suth that $p\tau e > 1+M$. By $(H7)$, there exists a positive integer K such that $0 < b_i < M$ whenever $i \geq K$. Therefore by Theorem 2.2.2, we know that every solution of (2.4) is oscillatory. Hence, if (2.4) is nonoscillatory, we must have $p\,\tau\,e \leq 1$. The proof is complete.

In the next theorem, we will need the following lemmas.

Lemma 2.2.1 *[32] Assume that* $x : [\sigma - \tau, \infty) \to R$ *is a positive function such that*

$$x'(t) < 0, \quad t > \sigma, \quad t \neq \theta_i$$
$$x(\theta_i+) - x(\theta_i) = b_i x(\theta_i), \quad i = 1, 2, \ldots$$

Then for any $\sigma \leq t_* < t^* < \infty$,

$$x(t^*) < x(t_*) \prod_{t_* \leq \theta_i < t^*} (1 + b_i),$$
$$x(t_*) + \sum_{t_* \leq \theta_i < t^*} b_i x(\theta_i) \leq x(t_*) \prod_{t_* \leq \theta_i < t^*} (1 + b_i^+),$$

and

$$\inf_{t_* \leq t \leq t^*} x(t) \geq x(t^*) \prod_{t_* \leq \theta_i < t^*} (1 + b_i^+)^{-1},$$

where $b_i^+ = \max\{0, b_i\}$.

Lemma 2.2.2 *[32] Assume that* $x : [\sigma - \tau, \infty) \to R$ *is a positive solution of (2.4). Then, for* $t > \sigma + (\frac{3}{2})\tau$,

$$p^2 \tau^2 x(t-\tau) < 4 \prod_{t-\tau \le \theta_i < t} (1 + b_i^+)^2 x(t),$$

where $b_i^+ = \max\{0, b_i\}$.

Theorem 2.2.8 *[32] The following statements are equivalent:*

(a) Equation (2.4) is nonoscillatory.

(b) The characteristic equation (2.6) has a real root.

Proof. To prove (b) $\Rightarrow$ (a), assume that λ_0 is a real root of (2.6) and define

$$x(t) = \prod_{t_1 \le \theta_i < t} (1 + b_i) e^{\lambda_0 t}, \quad for \quad t > t_1.$$

It is obvious that $x(t)$ is left continuous on $[t_1, \infty)$ and is differentiable on $[t_1 + \tau, \infty) - \{\theta_i\}$. Furthermore, for $t > t_1 + \tau$ and $t \neq \theta_i$,

$$\begin{aligned} x'(t) + p\,x(t-\tau) &= \lambda_0 \prod_{t_1 \le \theta_i < t} (1 + b_i) e^{\lambda_0 t} + p \prod_{t_1 \le \theta_i < t-\tau} (1 + b_i) e^{\lambda_0 (t-\tau)} \\ &= \prod_{t_1 \le \theta_i < t} (1 + b_i) e^{\lambda_0 t} [\lambda_0 + p \prod_{t-\tau \le \theta_i < t} (1 + b_i)^{-1} e^{-\lambda_0 \tau}] \\ &= 0 \end{aligned}$$

and for $\theta_i \geq t_1 + \tau$,

$$x(\theta_i+) - x(\theta_i) = b_i x(\theta_i).$$

Thus, $x(t)$ is a positive solution of (2.4) .

To prove (a) $\Rightarrow$ (b), without loss of generality, asssume that $x(t)$ is an eventually positive solution of (2.4). So there exists $\sigma > t_1 + \tau$ such that $x(t) > 0$ for $t \geq \sigma - \tau_n$. Set

$$\wedge = \{\lambda > 0 : x'(t) + \lambda x(t) < 0 \quad eventually \quad for \quad t \neq \theta_i\}.$$

From (2.4) and Lemma 2.2.1, we have

$$x'(t) + p \prod_{t-\tau \leq \theta_i < t} (1 + b_i)^{-1} x(t) < 0, \quad for \quad t > \sigma + \tau \quad and \quad t \neq \theta_i.$$

Thus, $p \prod_{t-\tau \leq \theta_i < t} (1 + b_i)^{-1} \in \wedge$. Also from Lemma 2.2.2, we get

$$\begin{aligned} 0 &= x'(t) + p\,x(t - \tau) \\ &\leq x'(t) + p\,\Big(\frac{4\prod_{t-\tau \leq \theta_i < t}(1 + b_i^+)^2}{p^2\tau^2}\Big)x(t), \quad for \quad t > \sigma + \frac{3}{2}\tau, \quad t \neq \theta_i. \end{aligned}$$

Therefore, $p\,\Big(\frac{4\prod_{t-\tau \leq \theta_i < t}(1+b_i^+)^2}{p^2\tau^2}\Big)$ is an upper bound for $\wedge$. Since $\wedge$ is nonempty and bounded, we may set $\lambda_0 = sup\wedge$.

Let $\lambda \in \wedge$ be given and define y on $[\sigma - \tau, \infty)$ by $y(t) = x(t)e^{\lambda t}$. Then, there is a suitable $T_\lambda \in (\sigma, \infty)$ such that

$$y'(t) = (x'(t) + \lambda x(t))e^{\lambda t} < 0, \quad for \quad t > T_\lambda \quad and \quad t \neq \theta_i.$$

On the other hand,

$$y(\theta_i+) - y(\theta_i) = b_i y(\theta_i), \quad for \quad \theta_i > T_\lambda.$$

So by Lemma 2.2.1, we know

$$y(t-\tau) > y(t) \prod_{t-\tau\leq\theta_i<t} (1+b_i)^{-1}, \quad for \quad t > T_\lambda + \tau.$$

Hence, for $t > T_\lambda + \tau$ and $t \neq \theta_i$,

$$\begin{aligned}
0 &= x'(t) + p\,x(t-\tau) \\
&= x'(t) + p\,y(t-\tau)e^{-\lambda(t-\tau)} \\
&> x'(t) + p\,y(t) \prod_{t-\tau\leq\theta_i<t} (1+b_i)^{-1} e^{-\lambda(t-\tau)} \\
&= x'(t) + p \prod_{t-\tau\leq\theta_i<t} (1+b_i)^{-1} e^{\lambda\tau} x(t).
\end{aligned}$$

This shows that $p \prod_{t-\tau\leq\theta_i<t} (1+b_i)^{-1} e^{\lambda\tau} \in \Lambda$, and hence $p\prod_{t-\tau\leq\theta_i<t}(1+b_i)^{-1}e^{\lambda\tau} \leq \lambda_0$. Since $\lambda \in \Lambda$ is arbitrary, we conclude that

$$p \prod_{t-\tau\leq\theta_i<t} (1+b_i)^{-1} e^{\lambda_0\tau} \leq \lambda_0.$$

Therefore, $F(-\lambda_0) = -\lambda_0 + p\prod_{t-\tau\leq\theta_i<t}(1+b_i)^{-1}e^{\lambda_0\tau} \leq 0$. Noticing that $F(+\infty) = +\infty$, we know that (2.6) has a real root. The proof of the theorem is complete.

Remark 2.2.1 It follows from Theorem 2.2.8 that a necessary and sufficient condition for oscillation of (2.4) is that the characteristic equation given by

(2.6) has no real roots.

From the Theorem 2.2.8 we can immediately obtain the following results

Corollary 2.2.3 *The following condition is necessary and sufficient for the oscillation of (2.4):*

$$p \prod_{t-\tau \le \theta_i < t} (1+b_i)\,\tau > \frac{1}{e}.$$

Corollary 2.2.4 *Assume that* $p > 0$, $\tau > 0$, $b > -1$, $t_0 > 0$ *and* $\theta_i = t_0 + i\tau$ *for* $i = 1, 2, \ldots$ *. Then a necessary and sufficient condition for the nonoscillation of (2.4) is*

$$p\,\tau\,e \le b + 1.$$

The next two theorems show that the oscillation of a linear impulsive delay differential equation (2.2) is equivalent to the oscillation of a corresponding linear delay differential equation (2.5) without impulses .

These two theorems will enable us to reduce the oscillation and non-oscillation of solutions of (2.2) to the corresponding problem for a delay differential equation without impulses. Before we proceed assume the following conditions:

(A.1) $p, \tau \in C([t_0, \infty), R^+)$, $\tau(t) < t$ and $\tau(t) \to \infty$ as $t \to \infty$;

(A.2) $b_i \in (-\infty, -1) \cup (-1, \infty)$ are constants for i=1,2...;

(A.3) $0 \leq t_0 < \theta_1 < \theta_2 < \ldots < \theta_i < \ldots$, and $\lim_{i\to\infty} \theta_i = \infty$.

Theorem 2.2.9 *[30] Assume that (A.1)-(A.3) hold.*

(H8) If $x(t, \sigma, \phi)$ is a solution of (2.5), then $y(t, \sigma, \phi) = \prod_{\sigma \leq \theta_i < t} (1 + b_i)x(t, \sigma, \phi)$ is a solution of (2.2).

(H9) If $y(t, \sigma, \phi)$ is a solution of (2.2), then $x(t, \sigma, \phi) = \prod_{\sigma \leq \theta_i < t} (1+b_i)^{-1}y(t, \sigma, \phi)$ is a solution of (2.5).

Proof. Let $x(t) = x(t, \sigma, \phi)$ and $y(t) = y(t, \sigma, \phi)$. First, we prove (i). It is easy to see that $y(t) = \prod_{\sigma \leq \theta_i < t} (1 + b_i)x(t)$ is absolutely continuous on each interval $(\theta_i, \theta_{i+1}]$ and for any $t \neq \theta_i$, i=1,2,...

$$y'(t) + p(t)y(\tau(t)) = \prod_{\sigma \leq \theta_i < t} (1 + b_i)x'(t) + p(t) \prod_{\sigma \leq \theta_i < \tau(t)} (1 + b_i)x(\tau(t))$$
$$= \prod_{\sigma \leq \theta_i < t} (1 + b_i)\{x'(t) + p(t) \prod_{\tau(t) \leq \theta_i < t} (1 + b_i)^{-1}x(\tau(t))\} = 0. \qquad (2.27)$$

On the other hand, for every $\theta_i \in \{\theta_i\}$,

$$y(\theta_i+) = \lim_{t\to\theta_i^+} \prod_{\sigma\le\theta_j<t} (1+b_j)x(t) = \prod_{\sigma\le\theta_j<\theta_i} (1+b_j)x(\theta_i)$$

and

$$y(\theta_i) = \prod_{\sigma\le\theta_j<\theta_i} (1+b_j)x(\theta_i).$$

Thus, for every i=1,2,...,

$$y(\theta_i+) = (1+b_i)y(\theta_i). \tag{2.28}$$

It follows from (2.27) and (2.28) that $y(t)$ is the solution of (2.2) corresponding to initial condition (1.2) .

Next we prove (ii). Since $y(t)$ is absolutely continuous on each interval $(\theta_i, \theta_{i+1}]$ and, in view of (2.28), it follows that, for any $i = 1, 2, \ldots$,

$$x(\theta_i+) = \prod_{\sigma\le\theta_j<\theta_i} (1+b_j)^{-1}y(\theta_i+) = \prod_{\sigma\le\theta_j<\theta_i} (1+b_j)^{-1}y(\theta_i) = x(\theta_i)$$

and

$$x(\theta_i^{-1}) = \prod_{\sigma\le\theta_j<\theta_{i-1}} (1+b_j)^{-1}y(\theta_i) = x(\theta_i), \quad i = 1, 2, \ldots,$$

which implies that $x(t)$ is continuous on $[\sigma, \infty)$, It is easy to prove that $x(t)$ is also absolutely continuous in $[\sigma, \infty)$. Now, one can easily check that $x(t) = \prod_{\sigma\le\theta_i<t}(1+b_i)^{-1}y(t)$ is a solution of (2.5) corresponding to initial con-

dition (1.2). The proof is complete.

By applying Theorem 2.2.9 we can prove that the oscillation of (2.2) is equivalent to the oscillation of (2.5).

Theorem 2.2.10 *[30] Assume that* $(A_1)-(A_3)$ *hold and*

$$b_i > -1, \quad i = 1, 2, \ldots . \tag{2.29}$$

Then (2.2) is oscillatory if and only if (2.5) is oscillatory.

Proof. Suppose that $x(t)$ is a solution of (2.5) on $[T, \infty)$, $T \geq t_0$. Let $y(t) = \prod_{T \leq \theta_i < t} (1 + b_i)x(t)$, $t \geq T$, From Theorem 2.2.9, $y(t)$ is a solution of (2.2) on $[T, \infty)$. Since $\prod_{\sigma \leq \theta_i < t}(1 + b_i) > 0, \quad t \geq T$, $y(t)$ is oscillatory if and only if $x(t)$ is oscillatory.

Conversely, suppose that $y(t)$ is a solution of (2.2) on $[T, \infty), T \geq t_0$. Let $x(t) = \prod_{T \leq \theta_i < t} (1 + b_i)^{-1} y(t), \quad t \geq T$. Thus, from Theorem 2.2.9, $x(t)$ is a solution of (2.5) on $[T, \infty)$, $T \geq t_0$, and in view of (2.29), $x(t)$ is oscillatory if and only if $y(t)$ is oscillatory . The proof is complete.

Consider the IDDE

$$x'(t) + p\,x(t - \tau) = 0, \quad t \geq t_0$$

$$\Delta x(\theta_i) = b_i\, x(\theta_i), \quad i = 1, 2, \ldots \tag{2.30}$$

where p is constant and τ is positive constant.

Corollary 2.2.5 *Assume that* $(A_1) - (A_3)$ *and (2.29) hold and for* $t \geq t_0$

$$\prod_{t-\tau \leq \theta_i < t} (1 + b_i) = \alpha.$$

Then (2.30) is oscillatory if and only if the delay differential equation

$$x'(t) + \alpha\, p x(t - \tau) = 0 \tag{2.31}$$

is oscillatory or, equivalently, the characteristic equation

$$\lambda + \alpha\, p\, e^{-\lambda\tau} = 0$$

of (2.31) has no real roots.

Example 2.2.1 *[30]*Let $\tau > 0$ and $b > -1$ be constants and let $\theta_{i+1} - \theta_i = \tau > 0$, $i = 1, 2, \ldots$. Consider the equation

$$x'(t) + p\, x(t - m\tau) = 0 \quad t \geq t_0,$$

$$\Delta x(\theta_i) = b\, x(\theta_i) \quad i = 1, 2, \ldots. \tag{2.32}$$

where m is a positive integer. By Corollary 2.2.5, (2.32) is oscillatory if and only if the delay differential equation

$$x'(t) + p\,(1+b)^m\,x(t - m\tau) = 0$$

is oscillatory or, equivalently, if and only if

$$p\,(1+b)^m\,m\,\tau > \frac{1}{e}$$

2.3 Nonoscillatory Behavior

It is possible that the first order linear homogenous delay differential equations have both nonoscillatory and oscillatory solutions, unlike the first order linear homogenous ordinary differential equations, which have only nonoscillatory solutions. This is one of the reasons why the nonoscillatory and oscillatory theory of delay differential equations has recieved extensive attention. It is not difficult to see that when a delay differential equation is subject to impulse perturbations, its nonoscillatory solutions may or may not continue to persist. Thus the question as how one determines the type of delay differential equation for its nonoscillatory solution to persist under certain impulsive perturbations naturally arises. The following theorem gives an answer to this question.

Theorem 2.3.1 *[13] Suppose that the functions $p(t)$ and $\tau(t)$ satisfy the following:*

(H10) there exists a positive number c such that

$$p\tau e \leq 1 - c,$$

(H11) $b_i > 0, i = 1, 2, \ldots$ *and* $\sum_{i=1}^{\infty} b_i < \infty$.

Then (2.2) is nonoscillatory.

Proof. We will prove this theorem for the particular case $\tau(t) = t - \tau$. Let t_0 be a real number and let $L_1[t_0 - \tau, \infty)$ denote the space of all equivalence classes of real valued functions defined on $[t_0 - \tau, \infty)$ such that

$$L_1[t_0 - \tau, \infty) = \{f : [t_0 - \tau, \infty) \to R | \int_{t_0-\tau}^{\infty} |f(t)| dt < \infty\}.$$

It is known that L_1 is a complete metric space with the metric ρ defined by

$$\rho(f, g) = \int_{t_0-\tau}^{\infty} |f(t) - g(t)|\, dt.$$

Consider a set $A \subset L_1$ defined as

$$A = \{f \in L_1[t_0 - \tau, \infty) | e^{-\mu_1 t} \leq f(t) \leq e^{-\mu_2 t}; \quad \mu_1 > \mu_2 > 0\},$$

where μ_2 satisfies $pe^{\mu_2 \tau} \leq (1 - c)\mu_2$. Define a map ϕ,

$$\phi : A \to L_1[t_0 - \tau, \infty),$$

where

$$\phi(x)(t) = \int_t^{\infty} [px(s - \tau) - \sum_{j=1}^{\infty} b_j x(\theta_j-) \delta(s - \theta_j)]\, ds, \quad t \geq t_0$$

$$\phi(x)(t) = \int_{t_0}^{\infty} [px(s-\tau) - \sum_{j=1}^{\infty} b_j x(\theta_j -)\delta(s-\theta_j)]\, ds, \quad t_0 - \tau \le t \le t_0.$$

It is easy to see that $\phi(A) \subset A$; for instance,

$$\phi(x)(t) \le \frac{pe^{\mu_2 \tau}}{\mu_2} e^{-\mu_2 t} + \sum_{n(t)}^{\infty} b_i e^{-\mu_2 t} \le e^{\mu_2 t},$$

provided $\frac{pe^{\mu_2\tau}}{\mu_2} \le 1 - c$ and $\sum_{n(t)} b_i < c$. This is possible since we can choose t_0 sufficiently large so that $n(t)$ will be a large enough positive integer. There exists a $\mu_1 > 0$ such that

$$\phi(x)(t) \ge \frac{pe^{\mu_1 \tau}}{\mu_1} e^{-\mu_1 t} \ge e^{-\mu_1 t}.$$

For instance, if we let $\mu_2 = \frac{1}{\tau}$, we get $p\tau e \le 1 - c$ from $\frac{pe^{\mu_2\tau}}{\mu_2} \le 1 - c$. It is easy to see that $\phi(x) \in L_1$ for $x \in A$. Thus $\phi(A) \subset A$. Since ϕ maps the bounded closed subset A of L_1 into itself, ϕ is a compact map. By the *Schauder's fixed point theorem*, ϕ has a fixed point x^* satisfying $\phi x^* = x^*$ and implies that x^* is a nonoscillatory solution.

2.4 New Oscillation Criteria

In this section we obtain new oscillation criteria for solution of

$$\begin{aligned} &x'(t) + p(t)x(\tau(t)) = 0, \quad t \neq \theta_i, \\ &\Delta x(\theta_i) + b_i x(\theta_i) = 0. \end{aligned} \tag{2.33}$$

Theorem 2.4.1 *Assume that $p(t)$, $\tau(t) \in C([0,\infty))$, $p(t) \geq 0$, for $t \geq 0$, and $\tau(t) < t$, $\lim_{t\to\infty} \tau(t) = \infty$.*

If

$$\limsup_{t\to\infty}[\int_{\tau(t)}^{t} p(s)\,ds + \sum_{\tau(t)\leq\theta_i<t} b_i] > 1, \tag{2.34}$$

then (2.33) is oscillatory.

Proof. Without loss of generality, let $x(t) > 0$ be a nonoscillatory solution such that $x(\tau(t)) > 0$, $t \geq t_1$. Integrating (2.33) from $\tau(t)$ to t we have

$$\int_{\tau(t)}^{t} x'(s)\,ds + \int_{\tau(t)}^{t} p(s)x(\tau(s))\,ds = 0. \tag{2.35}$$

We can write

$$\int_{\tau(t)}^{t} x'(s)\,ds = x(t) - x(\tau(t)) + \sum_{i=1}^{n} b_i x(\theta_i). \tag{2.36}$$

Substitute (2.36) in (2.35) we get

$$x(t) - x(\tau(t)) + \sum_{\tau(t)\leq\theta_i<t} b_i x(\theta_i) + \int_{\tau(t)}^{t} p(s)x(\tau(s))\,ds = 0.$$

It follows from (2.33) that $x(t)$ is nonincreasing, so

$$x(t) + x(\tau(t))[\int_{\tau(t)}^{t} p(s)\,ds + \sum_{\tau(t)\leq\theta_i<t} b_i - 1] \leq 0. \tag{2.37}$$

For t is sufficiently large (2.37) is a contradiction. The proof is complete.

Example 2.4.1 *Assume that $p(t) = \frac{1}{e}$ and $\tau(t) = t - 1$ in equation (2.1), that is, we consider*

$$x'(t) + \frac{1}{e}x(t-1) = 0 \tag{2.38}$$

Equation (2.38) does not satisfy condition (2.20) of Theorem 2.2.3. Therefore (2.38) is nonoscillatory.

Example 2.4.2 Consider the equation (2.33) where $p(t) = \frac{1}{e}$, $\tau(t) = t - 1$, $\theta_i = i$ and $b_i = 2 - \frac{1}{e}$, that is,

$$\begin{aligned} & x'(t) + \frac{1}{e}x(t-1) = 0, \quad t \neq i, \\ & \Delta x(i) + (2 - \frac{1}{e})x(i) = 0. \end{aligned} \tag{2.39}$$

The condition (2.34) of Theorem 2.4.1 becomes

$$\frac{1}{e} + \sum_{t-1 \leq i < t} (2 - \frac{1}{e}) = \frac{1}{e} + 2 - \frac{1}{e} = 2 > 1.$$

Therefore (2.39) is oscillatory.

Remark 2.4.1 In the Example 2.4.1 we find that (2.38) is nonoscillatory. However, if this equation is subject to impulsive effect as in Example 2.4.2 then every solution of (2.39) becomes oscillatory. That is, impulse causes solutions to oscillate.

Lastly, we consider

$$x'(t) + p(t)x(t-\tau) = 0, \qquad t \neq \theta_i$$
$$\Delta x(\theta_i) + b_i x(\theta_i - \sigma) = 0, \tag{2.40}$$

where the impulse condition involves a delay as well.

Theorem 2.4.2 *Assume that $p(t) \geq 0$, for $t \geq 0$, and $\tau(t) = t - \tau < t$, $\lim_{t\to\infty} \tau(t) = \infty$.*
If

$$\limsup_{i\to\infty} [\int_{\theta_i-\sigma}^{\theta_i-\sigma+\tau} p(s)\,ds + b_i] > 1, \tag{2.41}$$

then (2.40) is oscillatory .

Proof. Without loss of generality, let $x(t) > 0$ be a nonoscillatory solution such that $x(\tau(t)) > 0$, $t \geq t_1$. Let i be fixed and integrate (2.40) from α to β where $\alpha = \theta_i - \sigma$ and $\beta = \theta_i - \sigma + \tau$, we have

$$x(\theta_i - \sigma + \tau) - x(\theta_i - \sigma) + \sum_{\alpha \leq \theta_j < \beta} b_i x(\theta_j - \sigma) + \int_{\theta_i-\sigma}^{\theta_i-\sigma+\tau} p(s)x(s-\tau)ds = 0.$$

Since $x(t)$ is nonincreasing,

$$-x(t_i - \sigma) + \sum_{\alpha<\theta_j<\beta} b_i x(\theta_j - \sigma) + x(\theta_i - \sigma) \int_{\theta_i-\sigma}^{\theta_i-\sigma+\tau} p(s)ds \leq 0.$$

If $j = i$ then we will have only one impulse point, thus no summation is involved, and so

$$-x(\theta_i - \sigma) + b_i x(\theta_j - \sigma) + x(\theta_i - \sigma) \int_{\theta_i - \sigma}^{\theta_i - \sigma + \tau} p(s)\, ds \leq 0,$$

or

$$x(\theta_i - \sigma)\Big[\int_{\theta_i - \sigma}^{\theta_i - \sigma + \tau} p(s)\, ds + b_i - 1\Big] \leq 0. \qquad (2.42)$$

When i is sufficiently large, (2.42) is a contradiction. The proof is complete.

CHAPTER 3

SECOND ORDER IMPULSIVE DELAY EQUATIONS

3.1 Introduction

In this chapter we will introduce some oscillation criteria for bounded solutions of second order DDEs of the type

$$(r(t)x'(t))' = p(t)x(\tau(t)) \tag{3.1}$$

and for the corresponding IDDEs of the form

$$\begin{aligned}
&(r(t)x'(t))' = p(t)x(\tau(t)), \quad t \neq \theta_i, \quad i \in N \\
&\Delta(r(\theta_i)x'(\theta_i)) = b_i x(\tau(\theta_i)), \\
&\Delta x(\theta_i) = 0.
\end{aligned} \tag{3.2}$$

We will also give new necessary and sufficient conditions for oscillation of all bounded solutions of the following nonlinear impulsive system

$$\begin{aligned}
&(r(t)x'(t))' + p(t)f(x(\tau(t))) = 0, \quad t \neq \theta_i, \quad i \in N \\
&\Delta(r(\theta_i)x'(\theta_i)) + b_i g(x(\theta_i)) = 0, \\
&\Delta x(\theta_i) = 0.
\end{aligned} \tag{3.3}$$

The following lemma proves very helpful

We will start with the following two theorems which provide sufficient conditions for oscillation of all bounded solutions of (3.1) and (3.2), respectively. Before we proceed, we suppose that

$$x(\theta_i+) = x(\theta_i-); \quad x'(\theta_i-) = x'(\theta_i); \quad r(\theta_i-) = r(\theta_i),$$

and

$$0 < \theta_1 < \theta_2 < \ldots, \qquad \lim_{i\to\infty} \theta_i = +\infty.$$

Theorem 3.2.1 *[20] Assume the following:*

(H12) $p \geq 0, \quad r > 0$ *are continuous;*

(H13) $\tau \in C[R_+, R_+], \quad \tau(t)$ *is nondecreasing,* $\tau(t) \leq t,$ *and* $\lim_{t\to\infty} \tau(t) = \infty;$

(H14) $\lim_{t\to\infty} \int_{t_0}^{t} \frac{ds}{r(s)} = \infty.$

If

$$\limsup_{t\to\infty} \frac{1}{r(t)} \int_{\tau(t)}^{t} (u-\tau(t))p(u)\,du > 1, \tag{3.4}$$

then all bounded solutions of (3.1) are oscillatory.

Proof. Suppose that the conclusion is not true. Without loss of generality, let $x(t) > 0$ be a bounded nonoscillatory solution of (3.1) for all $t > T$. Hence $(r(t)x'(t))' \geq 0$, i.e., $r(t)x'(t)$ is nondecreasing .

Case 1. $r(t)x'(t) \geq c > 0$ for $t \geq T_1 \geq T$. Divide this inequality by $r(t)$ and then integrate from T_1 to t we see in view of $(H14)$ that $x(t)$ is unbounded, a contradiction.

Case 2. $r(t)x'(t) \leq 0$. Then $x'(t) \leq 0$. Integrating (3.1) from s to t, we have

$$r(t)x'(t) = r(s)x'(s) + \int_{s}^{t} p(u)x(\tau(u))\,du. \tag{3.5}$$

Integrating (3.5) from $\tau(t)$ to t we see that

$$\int_{\tau(t)}^{t} r(s)x'(s)ds + \int_{\tau(t)}^{t} [\int_{s}^{t} p(u)x(\tau(u))\,du]ds \leq 0.$$

Changing the order of integration we get

$$\int_{\tau(t)}^{t} r(s)x'(s)ds + \int_{\tau(t)}^{t} (u-\tau(t))p(u)x(\tau(u))\,du \leq 0.$$

Now, using integration by parts and that $x(t)$ is nonoincreasing function, it follows

$$r(t)x(t) - r(t)x(\tau(t)) + x(\tau(t)) \int_{\tau(t)}^{t} (u-\tau(t))p(u)\,du \leq 0,$$

which means

$$x(t) - x(\tau(t)) + \frac{x(\tau(t))}{r(t)} \int_{\tau(t)}^{t} (u - \tau(t))p(u)\, du \leq 0,$$

i.e.,

$$\frac{x(t)}{x(\tau(t))} + [\frac{1}{r(t)} \int_{\tau(t)}^{t} (u - \tau(t))p(u)\, du - 1] \leq 0$$

But this inequality contradicts the condition (3.4) and hence the proof is complete.

Remark 3.2.1 If we do not require $\int^{\infty} \frac{ds}{r(s)} = \infty$, but assume that $r(t)$ is nondecreasing and (3.4) is satisfied, then the conclusion of Theorem 3.2.1 remains valid.

Theorem 3.2.2 *[5] Let the following conditions be fulfilled:*

(H15) $p \geq 0, r > 0$ *are continuous;*

(H16) $\tau \in C[R_+, R_+]$, $\tau(t)$ *is nondecreasing,* $\tau(t) \leq t$, *and* $\lim_{t\to\infty} \tau(t) = \infty$*;*

(H17) $\lim_{t\to\infty} \int_{t_0}^{t} \frac{ds}{r(s)} = +\infty$ *and* $r'(t) \geq 0$ *for* $t \in R_+$*;*

(H18) $b_i \geq 0$, $i \in N$.

If

$$\limsup_{t\to\infty} \frac{1}{r(t)} [\int_{\tau(t)}^{t} (u - \tau(t))p(u)\, du + \sum_{\tau(t)\leq\theta_i<t} (\theta_i - \tau(t))b_i] > 1, \qquad (3.6)$$

then all bounded solution of the (3.2) are oscillatory.

Proof. Let $x(t)$ be a nonoscillatory bounded solution of the equation (3.2). Without loss of generality, we may suppose that $x(t) > 0$ for $t \geq t_1 \geq 0$ and $x(t) \leq M$, $M = const > 0$. It is clear that $x(\tau(t)) > 0$, hence $(r(t)x'(t))' > 0$, i.e., $r(t)x'(t)$ is nondecreasing function, then two cases are possible

Case 1. $r(t)x'(t) \geq c > 0$ for $t \geq t_2$. It follows that $\lim_{t\to\infty} x(t) = \infty$, which contradicts the boundedness of the solution.

Case 2. $r(t)x'(t) \leq 0$. Clearly, $x(t)$ is nonincreasing. Integrating (3.2) from s to t, we obtain

$$r(t)x'(t) = r(s)x'(s) + \int_s^t p(u)x(\tau(u))\,du + \sum_{s\leq\theta_i<t} b_i x(\tau(\theta_i)) \leq 0.$$

We integrate again the above inequality from $\tau(t)$ to t and arrive at the inequality

$$\int_{\tau(t)}^t r(s)x'(s)\,ds + \int_{\tau(t)}^t [\int_s^t p(u)x(\tau(u))\,du + \sum_{s\leq\theta_i<t} b_i x(\tau(\theta_i))]\,ds \leq 0.$$

Changing the order of integration and using Lemma 3.1.1 we get

$$\int_{\tau(t)}^t r(s)x'(s)\,ds + \int_{\tau(t)}^t (u-\tau(t))p(u)x(\tau(u))\,du + \sum_{\tau(t)\leq\theta_i<t} (\theta_i-\tau(t))b_i x(\tau(\theta_i)) \leq 0.$$

Now, from the fact that $r(t)$ is nondecreasing function and $x(t)$ is nonincreasing function, it follows

$$r(\tau(t))[x(t)-x(\tau(t))] + x(\tau(t))[\int_{\tau(t)}^t (u-\tau(t))p(u)du + \sum_{\tau(t)\leq\theta_i<t} (\theta_i-\tau(t))b_i] \leq 0,$$

which means

$$x(\tau(t))[\int_{\tau(t)}^{t}(u-\tau(t))p(u)\,du+\sum_{\tau(t)\le\theta_i<t}(\theta_i-\tau(t))b_i] \le r(\tau(t))x(\tau(t)) \le r(t)x(\tau(t)),$$

i.e.,

$$\frac{1}{r(t)}[\int_{\tau(t)}^{t}(u-\tau(t))p(u)\,du+\sum_{\tau(t)\le\theta_i<t}(\theta_i-\tau(t))b_i] \le 1.$$

The last inequality is a contradiction. The proof is complete.

Consider the delay diferential equation

$$x''(t)+p(t)x(\tau(t))=0. \tag{3.7}$$

Theorem 3.2.3 *[20] Assume the following:*

(H19) $\tau(t)\in C[R_+,R_+]$ *and* $\lim_{t\to\infty}\tau(t)=\infty$;

(H20) $p\in C[R_+,R_+]$ *and* $p(t)\ge 0$ *for* $t\ge 0$.

Then

$$\int^{\infty} t\,p(t)\,dt=\infty \tag{3.8}$$

is a necessary and sufficient condition for oscillation of all bounded solutions of (3.7).

Proof.(*Necessity*) Suppose that (3.8) fails. Then there exists T such that

$$\int_T^\infty t\,p(t)\,dt < \frac{1}{4}. \tag{3.9}$$

Set $T_0 = inf_{t\geq T}(\tau(t))$, and let X be the space of bounded continuous functions on $[T_0,\infty)$ with the *sup* norm $\|x\| = sup\{|x(t)| : t \geq t_0\}$. Let $S \subset X$ be defined by $S = \{x \in X : a \leq \|x\| \leq 2a\}$. Then S is a bounded convex closed subset of X. Define an operator ϕ on S by

$$(\phi x)(t) = \frac{3a}{2} + \int_t^\infty (t-s)\,p(s)\,x(\tau(s))\,ds, \quad t \geq T$$
$$(\phi x)(t) = \frac{3a}{2} + \int_T^\infty (t-s)\,p(s)\,x(\tau(s))\,ds, \quad T_0 \leq t < T$$

We will show that

$$\phi : S \longrightarrow S$$

is continuous such that ϕS is a relatively compact subset of X, So we can apply *Shauder fixed point theorem* in order to get nonoscillatory solution.

(i) *ϕ maps S into S* : In fact, knowing that $t \geq T$ and for $s \geq t \geq 0$ $t - s \leq s + s = 2s$ we have

$$|\phi x(t)| = |\frac{3a}{2} + \int_t^\infty (t-s)\,p(s)\,x(\tau(s))\,ds| \leq \frac{3a}{2} + 4a\int_T^\infty |\,s\,p(s)|\,ds = 2a,$$

and

$$|\phi x(t)| = |\frac{3a}{2} + \int_t^\infty (t-s)\,p(s)\,x(\tau(s))\,ds| \geq \frac{3a}{2} - 4a\int_T^\infty |\,s\,p(s)|\,ds = a.$$

Hence $\phi x \in S$

(ii) *ϕ is continuous.* To prove this, let $\{x_n\}$ be a Cauchy sequence in S, and let $\lim_{n\to\infty} \|x_n - x\| = 0$. Because S is closed, $x \in S$. To prove the continuity of ϕ, we see that

$$|\phi x_n - \phi x| \leq \int_t^\infty (t-s)\, p(s)|x_n(\tau(s)) - x(\tau(s))|\, ds \leq 2\int_t^\infty s\, p(s)|x_n - x|\, ds.$$

Set

$$G_n(s) = 2\, s\, p(s)|x_n - x|.$$

Then the above inequality reduces to

$$|\phi x_n - \phi x| \leq \int_T^\infty G_n(s) ds. \tag{3.10}$$

Noting the fact that $G_n(s) = s\, p(s)|x_n - x| \leq 4asp(s)$. It is obvious that $\lim_{n\to\infty} G_n(s) = 0$. Since $tp(t)$ is integrable on $[T, \infty)$, by *Lebesgue convergence theorem* we have

$$\lim_{n\to\infty} \|\phi x_n - \phi x\| = 0,$$

which means that ϕ is continuous .

(iii) To show that *ϕS is precompact*, we see that $(\phi(x))$, $x \in S$, is uniformaly bounded. Now we will prove that ϕS is an equicontinuous family of functions on $[T_0, \infty)$. Let $x \in S$ and $t_2 > t_1$. Since

$$
\begin{aligned}
|\phi x(t_2) - \phi x(t_1)| &= |\int_{t_2}^{\infty} (t_2 - s)\, p(s)\, x(\tau(s))\, ds - \int_{t_1}^{\infty} (t_1 - s)\, p(s)\, x(\tau(s))\, ds| \\
&\leq |\int_{t_2}^{\infty} (t_2 - s)\, p(s)\, x(\tau(s))\, ds| + |\int_{t_1}^{\infty} (t_1 - s)\, p(s)\, x(\tau(s))\, ds| \\
&\leq |\int_{t_1}^{\infty} (t_1 - s)\, p(s)\, x(\tau(s))\, ds| + |\int_{t_1}^{\infty} (t_1 - s)\, p(s)\, x(\tau(s))\, ds| \\
&\leq 8a \int_{t_1}^{\infty} |\, s\, p(s)|\, ds.
\end{aligned}
$$

For a given $\epsilon > 0$, there exists $T^* > T$ such that

$$\int_{T^*}^{\infty} s\, |p(s)|\, ds < \epsilon.$$

So if $t_2 > t_1 > T^*$, we have

$$|\phi x(t_2) - \phi x(t_1)| < \epsilon \quad for \quad all \quad x \in S.$$

Now suppose that $T \leq t_1 < t_2 \leq T^*$. Then

$$
\begin{aligned}
|\phi x(t_2) - \phi x(t_1)| &= |\int_{t_2}^{\infty} (t_2 - s)\, p(s)\, x(\tau(s))\, ds - \int_{t_1}^{\infty} (t_1 - s)\, p(s)\, x(\tau(s))\, ds| \\
&\leq 2a|t_2 - t_1| \int_{T}^{\infty} p(s)\, ds + 4a \int_{t_1}^{t_2} |\, s\, p(s)|\, ds.
\end{aligned}
$$

Hence, for any given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|\phi x(t_2) - \phi x(t_1)| < \epsilon, \quad |t_2 - t_1| < \delta, \quad for \quad all \quad x \in S.$$

That is, the interval $[T_0, \infty)$ can be divided into two subintervals $[T_0, T^*]$ and $[T^*, \infty)$ on which every $(\phi x)(t)$, $x \in S$, has variation less than ϵ.

Therefore, ϕS is equicontinuous family on $[T_0, \infty)$. Hence ϕS is a compact subset of S. According to the *Shauder fixed point theorem*, there exists an $x \in S$ such that $\phi x = x$. This x is a bounded nonoscillatory solution.

(*Sufficiency*) Suppose that (3.8) holds and there is a nonoscillatory solution of (3.7). Without loss of generality, we may assume that $x(t) > 0$ and $x(\tau(t)) > 0$ for $t \geq t_0$, hence $x''(t) < 0$, i.e., $x'(t)$ is nonincreasing. The following cases are possible

Case 1. $x'(t) < 0$. Since $x(t)$ is nonincreasing and concave down, we will have a contradiction with the positivity of $x(t)$.

Case 2. $x'(t) > 0$. Since $x(t)$ is nondecreasing and positive there exists c and T such that $x(t) > c$ and $x(\tau(t)) > c$ for all $t \geq T$. Integrating the above equation from s to t, we have

$$x'(t) - x'(s) + \int_s^t p(u)x(\tau(u))du = 0,$$

or

$$x'(s) > \int_s^t p(u)x(\tau(u))du.$$

Integrate again from T to t, we obtain

$$x(t) - x(T) > \int_T^t [\int_s^t p(u)x(\tau(u))du]ds.$$

Changing the order of integration, we get

$$x(t) - x(T) > \int_T^t (u - T)p(u)x(\tau(u))du.$$

Using the nondecreasing nature of $x(t)$ we have

$$x(t) - x(T) > c\int_T^t (u-T)p(u)du.$$

Hence,

$$\int_T^t (u-T)p(u)du < \frac{x(t)-x(T)}{c}.$$

Since

$$\lim_{u\to\infty} \frac{u-T}{u} = 1,$$

it follows that $u - T > \frac{1}{2}u$ if $u > 2T$. Therefore, we have

$$\frac{1}{2}\int_{2T}^t u\,p(u)\,du \le \int_{2T}^t (u-T)\,p(u)\,du \le \int_T^t (u-T)\,p(u)\,du < \frac{x(t)-x(T)}{c},$$

which contradicts (3.8). Then all bounded solutions of (3.7) are oscillatory.

3.3 New Oscillation Criteria

The following is a new contribution that gives a necessary and sufficient condition for oscillation of all solutions of (3.3).

Theorem 3.3.1 *Assume the following:*

(H21) $\tau(t) \in C[R_+, R_+]$ *and* $\lim_{t\to\infty} \tau(t) = \infty$*;*

(H22) $p(t) \in C[R_+, R_+]$ *for* $t \ge 0$*;*

(H23) f *and* g *are nondecresing and continuous functions;*

(H24) $r(t) \in C[R_+, R_+]$, $r(t) > 0$ *for* $t \geq t_0, t_0 \in R_+$ *and* $\lim_{t\to\infty} R(t,u) = \infty$ *where* $R(t,u) = \int_u^t \frac{1}{r(s)} ds$;

(H25) $b_i \geq 0,\ i \in N$;

(H26) $xf(x) > 0$ *and* $xg(x) > 0$ *for* $x \neq 0$.

Then

$$\int^{\infty} R(u,0)p(u)\,du + \sum^{\infty} R(\theta_i, 0)b_i = \infty \tag{3.11}$$

is necessary and sufficient condition for oscillation of all bounded solutions of (3.3).

Proof.(*Necessity*) Suppose that (3.11) fails. Then one can find a sufficiently large T such that

$$\int_T^{\infty} R(u,0)p(u)du + \sum_{T\leq\theta_i<\infty} R(\theta_i,0)b_i < \frac{a}{2\max\{f(2a), g(2a)\}}$$

Set $T_0 = inf_{t\geq T}\tau(t)$ and let X be the space of bounded continuous functions on $[T_0, \infty)$ with the *sup* norm $\|x\| = sup\{|x(t)| : t \geq t_0\}$. Le t $S \subset X$ be defined by $S = \{x \in X : a \leq \|x\| \leq 2a\}$. Clearly, S is bounded convex closed subset of X. Define an operator ϕ on S by

$$\begin{aligned}(\phi x)(t) &= \frac{3a}{2} + \int_t^{\infty} R(t,u)p(u)f(x(\tau(u)))\,du \\ &\quad + \sum_{t\leq\theta_i<\infty} R(t,\theta_i)b_i g(x(\theta_i)), \quad t \geq T\end{aligned}$$

$$(\phi x)(t) = \frac{3a}{2} + \int_T^\infty R(T,u)p(u)f(x(\tau(u)))\,du + \sum_{T\leq\theta_i<\infty} R(T,\theta_i)b_i g(x(\theta_i)), \quad t\in[T_0,T].$$

We will show that

$$\phi : S \longrightarrow S$$

is continuous such that ϕS is relatively compact subset of X so we can apply *Shauder fixed point theorem* in order to get nonoscillatory solution.

(i) *ϕ maps S into itslef*: In fact

$$|\phi x(t)| \leq \frac{3a}{2} + |\int_t^\infty R(t,u)p(u)f(x(\tau(u)))\,du + \sum_{t\leq\theta_i<\infty} R(t,\theta_i)b_i g(x(\theta_i))|.$$

From $(H23)$, for $t \geq T$,

$$|\phi x(t)| \leq \frac{3a}{2} + \max\{f(2a), g(2a)\}\Big[\int_T^\infty |R(t,u)p(u)|du + \sum_{T\leq\theta_i<\infty} |R(t,\theta_i)b_i|\Big].$$

Since

$$|R(t,u)| \leq 2|R(u,0)| \quad \text{for all } u \geq t,$$

we have

$$|\phi x(t)| \leq \frac{3a}{2} + 2\max\{f(2a), g(2a)\}\Big[\int_T^\infty R(u,0)p(u)\,du + \sum_{T\leq\theta_i<\infty} R(\theta_i,0)b_i\Big] = 2a,$$

and

$$|\phi x(t)| \geq \frac{3a}{2} - |\int_t^\infty R(t,u)p(u)f(x(\tau(u)))\,du + \sum_{t\leq\theta_i<\infty} R(t,\theta_i)b_i g(x(\theta_i))|$$

$$\geq \frac{3a}{2} - 2\max\{f(2a), g(2a)\}\Big[\int_T^\infty |R(u,0)p(u)|\,du + \sum_{T\leq\theta_i<\infty} |R(\theta_i,0)b_i|\Big]$$
$$= a.$$

Therefore, $\phi x \in S$.

(ii) *ϕ is continuous.* To prove this, let $\{x_n\}$ be Cauchy sequence in S, and let $\lim_{n\to\infty} \|x_n - x\| = 0$. Because S is closed, $x \in S$. To prove the continuity of ϕ, we first see that

$$\begin{aligned} |\phi x_n - \phi x| &\leq \int_t^\infty R(t,u)p(u)|f(x_n) - f(x)|\,du \\ &\quad + \sum_{t\leq\theta_i<\infty} R(t,\theta_i)b_i|g(x_n) - g(x)| \\ &\leq 2\int_t^\infty R(u,0)p(u)|f(x_n) - f(x)|du \\ &\quad + 2\sum_{t\leq\theta_i<\infty} R(\theta_i,0)b_i|g(x_n - g(x))|. \end{aligned}$$

Set

$$G_n(u) = R(u,0)p(u)|f(x_n) - f(x)|,$$

and

$$H_n(\theta_i) = R(\theta_i,0)b_i|g(x_n) - g(x)|.$$

Then the above inequality reduces to

$$|\phi x_n - \phi x| \leq 2\int_T^\infty G_n(u)du + 2\sum_{T\leq\theta_i<\infty} H_n(\theta_i) \qquad (3.12)$$

Notice that $G_n(u) = R(u,0)p(u)|f(x_n) - f(x)| \leq 2f(2a)R(u,0)p(u)$ and $H_n(\theta_i) = R(\theta_i,0)b_i|g(x_n)-g(x)| \leq 2g(2a)R(\theta_i,0)b_i$, and that $\lim_{n\to\infty} G_n(u) = 0$ and $\lim_{n\to\infty} H_n(\theta_i) = 0$. Applying *Lebesque Convergence theorem* on integral and the uniform convergence on the sum, we have

$$\lim_{n\to\infty} \|\phi x_n - \phi x\| = 0.$$

Hence ϕ is continuous.

(iii) To show ϕ *is precompact*, we see that $(\phi(x))$, $x \in S$ is uniformaly bounded. Now we will prove that ϕS is equicontinous family of functions on $[T_0, \infty)$. For $x \in S$, and $t_2 > t_1$, we have

$$\begin{aligned}
|\phi x(t_2) - \phi x(t_1)| &= |\int_{t_2}^{\infty} R(t_2,u)p(u)f(x(\tau(u)))\,du \\
&\quad + \sum_{t_2\leq\theta_i<\infty} R(t_2,\theta_i)b_i g(x(\theta_i)) \\
&\quad - \int_{t_1}^{\infty} R(t_1,u)p(u)f(x(\tau(u)))\,du \\
&\quad - \sum_{t_1\leq\theta_i<\infty} R(t_1,\theta_i)b_i g(x(\theta_i))| \\
&\leq 2f(2a)\int_{t_1}^{\infty} |R(t,u)p(u)|\,du + 2g(2a)\sum_{t_1\leq\theta_i<\infty} |R(t,\theta_i)b_i| \\
&\leq 4f(2a)\int_{t_1}^{\infty} |R(u,0)p(u)|\,du + 4g(2a)\sum_{t_1\leq\theta_i<\infty} |R(\theta_i,0)b_i|.
\end{aligned}$$

For a given $\epsilon > 0$, there exists $T^* > T$ such that

$$\int_{T^*}^{\infty} R(u,0)p(u)du + \sum_{T^*\leq\theta_i<\infty} R(\theta_i,0)b_i < \epsilon.$$

So if $T^* \leq t_1 < t_2$ we have

$$|\phi x(t_2) - \phi(x_1)| < \epsilon \quad for \quad all \quad x \in S.$$

Now, suppose that $T \leq t_1 < t_2 \leq T^*$. Clearly,

$$\begin{aligned}
|\phi x(t_2) - \phi x(t_1)| &= |\int_{t_2}^{\infty} (R(t_2, u) - R(t_1, u)) p(u) f(x(\tau(u)))\, du \\
&\quad + \sum_{t_2 \leq \theta_i < \infty} (R(t_2, \theta_i) - R(t_1, \theta_i)) b_i g(x(\theta_i)) \\
&\quad - \int_{t_1}^{t_2} R(t_1, u) p(u) f(x(\tau(u)))\, du \\
&\quad - \sum_{t_1 \leq \theta_i < t_2} R(t_1, \theta_i) b_i g(x(\theta_i))|.
\end{aligned}$$

From $(H23)$ and

$$R(t_2, u) - R(t_1, u) = \int_u^{t_2} \frac{1}{r(s)} ds - \int_u^{t_1} \frac{1}{r(s)} ds = \int_{t_1}^{t_2} \frac{1}{r(s)} ds = R(t_2, t_1),$$

we have

$$\begin{aligned}
|\phi x(t_2) - \phi x(t_1)| &\leq f(2a) R(t_2, t_1) \int_{t_2}^{\infty} |p(s)|\, ds \\
&\quad + g(2a) R(t_2, t_1) \sum_{t_2 \leq \theta_i < \infty} |b_i| \\
&\quad + f(2a) \int_{t_1}^{t_2} |R(t_1, u) p(u)|\, du \\
&\quad + g(2a) \sum_{t_1 \leq \theta_i < t_2} |R(t_1, \theta_i) b_i|,
\end{aligned}$$

or

$$
\begin{aligned}
|\phi x(t_2)-\phi x(t_1)| \;\leq\; & \max\{f(2a),g(2a)\}|R(t_2,t_1)|[\int_T^{\infty} p(u)\,du + \sum_{T\leq\theta_i<\infty} b_i] \\
& + 2f(2a)\int_{t_1}^{t_2} |R(u,0)p(u)|\,du \\
& + 2g(2a)\sum_{t_1\leq\theta_i<t_2} |R(\theta_i,0)b_i|.
\end{aligned}
$$

Clearly,

$$
\begin{aligned}
\int_T^{\infty} p(u)\,du + \sum_{T\leq\theta_i<\infty} b_i \;&<\; \int_T^{\infty} R(u,0)p(u)du + \sum_{T\leq\theta_i<\infty} R(\theta_i,0)b_i \\
&<\; \frac{a}{2\max\{f(2a),g(2a)\}}
\end{aligned}
$$

Hence, for any given $\epsilon>0$, there exists a δ such that

$$
|\phi x(t_2)-\phi x(t_1)|<\epsilon, \quad |t_2-t_1|<\delta \quad for \quad all \quad x\in S
$$

That is, the interval $[T_0,\infty)$ can be divided into two subintervals $[T_0,T^*]$ and $[T^*,\infty)$ on which every $(\phi x)(t)$, $x\in S$, has variation less than ϵ.

Therefore ϕS is an equicontinuous family on $[T_0,\infty)$, Hence ϕS is a compact subset of S. According to the *Schauder fixed point theorem* there exists a $x \in S$ such that $x=\phi x$. This x is bounded nonoscillatory solution.

(*Sufficiency*) Suppose that (3.11) holds and there exists a nonoscillatory solution of (3.3). We may without loss of generality assume that $x(t)>0$ and $x(\tau(t))>0$ for $t\geq t_0$, hence $(r(t)x'(t))'<0$, i.e., $r(t)x'(t)$ is nonincreasing. The following cases are possible

Case 1. $r(t)x'(t) < 0$. Since $(r(t)x'(t))' < 0$, $r(t)x'(t)$ is nonincreasing. Integrating from t_0 to t we get

$$x'(t) < r(t_0)x'(t_0)\frac{1}{r(t)}.$$

Integrating again from T to t we have

$$x(t) - x(T) < r(t_0)x'(t_0)\int_T^t \frac{1}{r(s)}ds.$$

But $r(t_0)x'(t_0) < 0$ and $\int_T^t \frac{1}{r(s)}ds \to \infty$ as $t \to \infty$. Therefore we have a contradiction with the positivity of x .

Case 2. $r(t)x'(t) > 0$. Integrating (3.3) from s to t,

$$r(t)x'(t) - r(s)x'(s) + \int_s^t p(u)f(x(\tau(u)))\,du + \sum_{s\leq\theta_i<t} b_i g(x(\theta_i)) = 0,$$

then

$$r(s)x'(s) > \int_s^t p(u)f(x(\tau(u)))\,du + \sum_{s\leq\theta_i<t} b_i g(x(\theta_i)).$$

Divide the above inequality by $r(s)$ and Integrate again from T to t,

$$x(t) - x(T) > \int_T^t [\frac{1}{r(s)}\int_s^t p(u)f(x(\tau(u))\,du + \frac{1}{r(s)}\sum_{s\leq\theta_i<t} b_i g(x(\theta_i))]ds.$$

Changing the order of integration and using Lemma 3.1.1 we get

$$x(t) - x(T) > \int_T^t R(u,T)p(u)f(x(\tau(u))\,du + \sum_{T\leq\theta_i<t} R(\theta_i,T)b_i g(x(\theta_i)).$$

We know that x is positive and increasing so it is bounded from below by a positive real number. This means that by increasing the size of T if necessary there exists $c > 0$ such that $x(\tau(t)) > c$ and $x(\theta_i) > c$ for all $t \geq T$. Since f and g are nondecreasing then $f(x(\tau(t))) > f(c)$ and $g(x(\theta_i)) > g(c)$, hence

$$f(c)\int_T^t R(u,T)p(u)\,du + g(c)\sum_{T\leq\theta_i<t} R(\theta_i,T)b_i < x(t) - x(T),$$

$$\int_T^t R(u,T)p(u)\,du + \sum_{T\leq\theta_i<t} R(\theta_i,T)b_i < \frac{x(t)}{\min\{f(c),g(c)\}}. \tag{3.13}$$

Since

$$\lim_{u\to\infty}\frac{R(u,T)}{R(u,0)} = 1,$$

it follows that $R(u,T) > k\,R(u,0)$ for some $k \in (0,1)$ if u is sufficiently large, say $u \geq T_* \geq T$. Thus

$$k\Big[\int_{T_*}^t R(u,0)p(u)\,du + \sum_{T_*\leq\theta_i<t} R(\theta_i,0)b_i\Big] < \int_{T_*}^t R(u,T_*)p(u)\,du + \sum_{T_*\leq\theta_i<t} R(\theta_i,T_*)b_i.$$

In view of (3.11), we see that

$$\int_{T_*}^t R(u,T_*)p(u)\,du + \sum_{T_*\leq\theta_i<t} R(\theta_i,T_*)b_i = \infty,$$

which, on replacing T by T_*, contradicts (3.13). Therefore all bounded solutions must be oscillatory.

Consider the following system which is a particular varient of system (3.3)

$$\begin{aligned}
&x''(t) + p(t)f(x(\tau(t))) = 0, \quad t \neq \theta_i, \quad i \in N\\
&\Delta x'(\theta_i) + b_i g(x(\theta_i)) = 0,\\
&\Delta x(\theta_i) = 0.
\end{aligned} \tag{3.14}$$

The following theorem is a particular case of Theorem 3.3.1 and it is the corresponding result to the Theorem 3.2.3.

Theorem 3.3.2 *Assume the following:*

(H27) $\tau(t) \in C[R_+, R_+]$ *and* $\lim_{t\to\infty} \tau(t) = \infty$;

(H28) $p \in C[R_+, R_+]$ *and* $p(t) \geq 0$ *for* $t \geq 0$;

(H29) f *and* g *are nondecresing and continuous functions;*

(H30) $b_i \geq 0$ *for* $i \in N$;

(H31) $xf(x) > 0$ *and* $xg(x) > 0$ *for* $x \neq 0$.

Then

$$\int^{\infty} up(u)\,du + \sum^{\infty} \theta_i b_i = \infty \tag{3.15}$$

is necessary and sufficient condition for oscillation of all bounded solutions of (3.14)

Proof. It is similar to the proof of Theorem 3.3.1, and hence omitted.

Example 3.3.1 Assume that $p(t) = \frac{1}{t^3}$, $\tau(t) = t - \tau$, $\theta_i = i$, $b_i = \frac{1}{i^{\frac{3}{2}}}$ and $r(t) = 1$, that is, consider

$$\begin{aligned}
&x''(t) + \frac{1}{t^3}x(t-\tau) = 0, \quad t \neq i,\\
&\Delta x'(i) + \frac{1}{i^{\frac{3}{2}}}x(i) = 0,\\
&\Delta x(i) = 0.
\end{aligned} \tag{3.16}$$

Then condition (3.15) is satisfied, since

$$\int^{\infty} \frac{1}{u^2}\, du + \sum^{\infty} \frac{1}{i^{\frac{1}{2}}} = \infty$$

Therefore, (3.16) is oscillatory.

We note that if the equation is not subject to impulse condition, then

$$\int^{\infty} \frac{1}{u^2}\, du < \infty$$

and so,

$$x''(t) + \frac{1}{t^3}x(t-\tau) = 0$$

is nonoscillatory by Theorem 3.2.3.

CHAPTER 4
CONCLUSIONS

There is a well-developed theory of delay differential equations without impulses. The theory of ordinary impulsive equations with impulses has also been developed over the past few years. However, the contributions in the direction of delay differential equations with impulses are limited.

In view of the known results obtained for delay differential equations without impulses, we derived new oscillation and nonoscillation criteria for delay differential equations with impulses. The first result we obtaind is the corresponding theorem to Theorem 2.2.3. The beauty of this theorem can be seen in examples 2.4.1 and 2.4.2. We deduced that for a delay differential equation which is nonoscillatory we can obtain oscillatory solution by imposing impulse condition, that is, impulses causes oscillatory solution. The second result we established is Theorem 2.4.2 in which we give sufficient condition for oscillation of system (2.40) where the impulse condition involves a delay as well.

For second order, we derived a new oscillation criterion that provides necessary and sufficient condition for oscillation of all bounded solutions of system (3.3). Similar result of this theorem can be obtained for linear impulsive system. Example 3.3.1 shows the importance of this result, that is, impulses provides oscillatory solution.

The oscillation theory of the impulsive differential equations is not yet elaborated in contrast to the oscillation theory of ordinary differential equations with deviating arguments. The first paper published in this area is in 1987 [13]. Since that time, this subject has been developed extensively. The problems related to oscillation and stability seem to be chalenging and interesting.

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