STATIC AND FREE VIBRATION ANALYSES OF SMALL - SCALE FUNCTIONALLY GRADED BEAMS POSSESSING A VARIABLE LENGTH SCALE PARAMETER USING DIFFERENT BEAM THEORIES

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ABSTRACT

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This study presents static and free vibration analyses of functionally graded (FG) micro - beams on the basis of higher order continuum mechanics used in conjunction with classical and higher order shear deformation beam theories. Unlike conventional ones, higher order elastic theories consider the size effect for the beam. Strain gradient theory (SGT) and modified couple stress theory (MCST) are the two common non-classical continuum approaches capable of capturing the size effect. Shear deformation beam theories consider the effects of shear strain across the thickness. In the base of SGT and generalized beam theories and taking the thermal effects into account, the governing equations and boundary conditions are derived using a variational formulation based on Hamilton's principle. This new model may be reduced to the non-classical Bernoulli-Euler beam model based on the modified couple stress theory (MCST) when two of the material length scale parameters and extra terms of higher order beam theories are taken to be zero. Numerical analyses using differential quadrature method (DQM) are conducted by considering static bending and free vibration problems of a simply supported FG beam.

Keywords: Modified couple stress theory, Strain gradient theory, Higher order shear deformation beam theory, Functionally graded material, Differential quadrature method.

ÖZ

DEĞİŞKEN BOYUT ÖLÇEĞİ PARAMETRESİNE SAHİP FONKSİYONEL DERECELENDİRİLMİŞ KÜÇÜK ÖLÇEKLİ KİRİŞLERİN FARKLI KİRİŞ TEORİLERİ İLE STATİK VE SERBEST TİTREŞİM ANALİZLERİ

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Bu çalışma, yüksek mertebe sürekli ortam mekaniği ve klasik ve yüksek mertebe kiriş kesme deformasyon teorileri temelinde fonksiyonel derecelendirilmiş (FD) kirişlerin statik ve serbest titreşim analizlerini sunmaktadır. Klasik teorilerin aksine, yüksek mertebe elastisite teorileri kiriş için boyut etkisini göz önünde bulundurmaktalar. Gerinim gradyanı teorisi ve modifiye edilmiş kuvvet çifti gerilmesi teorisi boyut etkisini yakalama kapasitesine sahip iki yaygın klasik olmayan sürekli ortam mekaniği yaklaşımlarıdır. Kiriş kesme deformasyon teorileri kalınlık boyunca kesme gerinimi etkilerini dikkate almaktadırlar. Gerinim gradyanı teorisi ve genel kiriş teorisi temelinde ve ısı etkisi göz önünde bulundurularak denklemler ve sınır koşulları Hamilton prensibine dayanarak bir varyasyon yöntemiyle elde edilmiştir. Yüksek mertebe kiriş teorisindeki ekstra terimler ve malzemenin boyut ölçek parametrelerinden ikisi sıfır alınarak modifiye edilmiş kuvvet çifti gerilmesine dayalı klasik olmayan Bernoulli-Euler kiriş modelini elde etmek mümkün olmaktadır. Basit mesnetli bir FD kirişin statik eğilme ve serbest titreşimini göz önünde bulundurularak diferansiyel kare yapma metodu ile sayısal analizler yapılmıştır.

Anahtar kelimeler: Modifiye edilmiş kuvvet çifti gerilmesi teorisi, Gerinim gradyanı teorisi, Yüksek mertebe kiriş kesme deformasyon teorisi, Fonksiyonel derecelendirilmiş malzemeler (FDM), Diferansiyel kare yapma metodu.

To My Family

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TABLE OF CONTENTS

LIST OF TABLES

TABLES

LIST OF FIGURES

FIGURES

CHAPTER 1

INTRODUCTION

1.1 Introduction

Micro - electromechanical systems (MEMS) have attracted researchers' attention because of their growing applications. Micro - beams are important micro - scale structures that have been widely used in micro - and nano - technology industries. The design and optimization of micro - beams are extensively investigated in the literature.

Since classical theories can not predict the size effect for small - scale beams, some researchers have made effort to examine the behavior of small - scale beams using non classical theories. Strain gradient theory (SGT) and modified couple stress theory (MCST) are two prevalent higher order elastic theories that take the small - scale parameters into account. In most of the studies these theories have been combined with a beam deformation theory and governing equations and boundary conditions are obtained.

The main objective in this study on one hand, is to consider the small - scale effect on the static and dynamic behaviors of functionally graded micro - beam; on the other hand, is to investigate thermal effects on deflections and natural frequencies of a beam. Micro - beam is supposed to be made of a functionally graded material (FGM) because in the recent years there has been considerable interest on FGMs. Further, results for homogeneous beams can also be obtained by considering the formulation valid for functionally graded beams.

1.2 Previous Works on Micro - beams: Applications, Higher Order Elasticity and Beam Theories

Since their extensive use in micro - structures and micro - electromechanical systems (MEMS) such as sensors, actuators and atomic force microscopy (AFM) there is need for the methods capable of evaluating mechanical behavior of micro - beams. Further, modeling these systems makes it possible to estimate the size effect.

To make initial predictions about the performance of the micro - switches and design before fabrication, it is necessary to use analytic equations. In a micro - switch, an elastic beam suspends on a rigid substrate and is actuated by electrostatic forces. In a critical voltage which is called pull in voltage the beam deflects toward the substrate and leads to pull in instability. Pull in voltage and static and dynamic behaviors of micro - systems determine the sensitivity and instability of these systems. Coutu et al. [1] modeled a typical micro - switch by a cantilever beam to estimate the pull in voltage, contact force and contact resistance, which are useful to achieve desired performance. In another work, Mojahedi et al. [2] studied the effects of midplane stretching, electrostatic actuation and axial loading on the pull in instability of micro - systems.

Atomic force microscopy (AFM) is another powerful tool in micro - and nano - scale technology. In recent years it has been used in different branches of science from surface characterization in material science to the study of living biological systems and to nanolithography. AFM consists of a micro - cantilever beam and tip interacting with the sample. The deflection of the cantilever is plotted as a function of surface location to give a high resolution image of surfaces. To capture the surface properties of the sample, analytical and numerical models are indispensable to simulate the coupled dynamics of AFM and the sample. Mahdavi et al. [3] presented a micro - cantilever model for AFM considering four major factors which are: rotary inertia and shear deformation of the beam and mass and rotary inertia of the tip. They studied several commercial micro - cantilevers to investigate the effects of these factors on the frequency response of the beam and verified their model. Stan et al. [4] investigated the size dependent elastic properties of zinc oxide nanowires (ZnO NWs). To measure indentation and Young's modulus, they used this fact that when the probe tip of AFM is brought from air into contact with the wire, resonance frequency changes. Using AFM, they also measured the friction which is proportional to the lateral force to determine the tangential shear modulus of the ZnO NW. Non - contact AFM (NC AFM) is a force sensing cantilever which is used to produce atomic resolution images on various surfaces. Wang and Hu [5] carried out a modal response analysis to study the origins and impacts of higher eigenmodes. Fang and Chang [6] improved the surface roughness by AFM based lithography. Their tests are conducted on an aluminum film deposited on a silicon substrate. In AFM lithography method, an AFM tip is used to draw a pattern on a solid surface. Following the work to determine the mechanical and electrical properties of AFM, Cook et al. [7] investigated the two common methods to measure the spring constant of AFM: thermal noise method and Sader method. They compared the results with experimental ones and showed that there is a good agreement between these methods. The spring constant of AFM is essential for determining the force exerted by it.

Micro - cantilever base sensors are capable of detecting extremely small forces and stresses. They can operate in either static or dynamic mode. To detect the blood glucose level, Pei et al. [8] measured mechanical bending induced by the enzyme reaction on the micro - cantilever surface in the presence of glucose. The bending is sensed by reflecting a laser beam from the cantilever surface to a position sensitive detector.

During fabrication, deployment and operation, MEMS devices can be exposed to mechanical shock or impact which may lead to some damage such as cracks. Younis et al. [9] modeled and simulated MEMS devices under the shock loads and electrostatic actuation and showed that the combination of shock load and an electrostatic actuation makes the instability threshold much lower than threshold predicted just by one of them.

In recent years, carbon nanotubes (CNTs) have found versatile applications in nanotechnology as gas storage and nano - pipes to convey fluids. It is a substantial issue to determine the influence of the internal moving fluid on mechanical behavior of CNTs. The effect of fluid flow on structural instability and free vibration of CNTs is studied by Yoon et al. [10].

In most of the applications of micro - scale beams, which are discussed above, experimental results are used to evaluate the desired function. In some cases, an analytical model in the base of classical elastic theory is presented. Experimental results show the size dependency of the deformation behavior of materials and classical elastic theory is incapable of capturing size effect as the dimensions of the beam become smaller. To consider the effects of small - scale parameters, higher order continuum theories are presented.

Yang et al. [11] introduced the moment of couples as an additional equation into the equilibrium equations and presented the modified couple stress theory (MCST). They investigated new modification by analyzing the torsion of a cylindrical bar and bending of a flat plate of infinite width.

Following the studies about the size dependent behavior of small - scale structures, Lam et al. [12] considered the second order deformation gradient in addition to the conventional first order symmetric strain tensor and introduced the strain gradient theory (SGT). They used simple cantilever beam bending to investigate the difference between classical and strain gradient elastic theories and experimentally showed that the higher order theory clearly demonstrates behavior of micro - sized epoxy beams.

Combination of the new continuum theories with appropriate beam theories can give a useful picture of the static and dynamic properties of micro - beams. This issue has been studied by many researchers. In some work, classical Euler Bernoulli beam theory along with MCST approach are used to study homogeneous micro - beams (Şimşek [13], Kong et al. [14], Rafiee et al. [15], Xia [16], Kahrobaiyan et al. [17], Wang [18], Akgöz and Civalek [19], Park and Gao [20]). Buckling, bending and free vibration analyses of micro - beams considering different boundary conditions are investigated in these studies. Ma et al. [21] combined MCST and third - order beam theory (TOBT) and showed the differences between the resulting static bending and natural frequencies of TOBT and Timoshenko beam theory (TBT). Timoshenko beam is also studied in the works related to MCST (Ma et al. [22], Asghari et al. [23], Ke et al. [24], Fu and Zhang[25]). Asghari et al. [23] considered Von Karman nonlinearity for micro - beams and Ke et al. [24] introduced thermal effects into equations.

Kahrobaiyan et al. [26], Kong et al. [27], Zhao et al. [28], Yin et al. [29] and Akgöz and Civalek [30] studied the Euler Bernoulli beam based on SGT and obtained static deflection, post buckling behavior and natural frequencies of the micro - beams. Kahrobaiyan et al. [26] studied the nonlinear effects of the beam and Zhao et al. [28] considered the nonlinearities due to mean axial extension. Timoshenko beam relations in conjunction with SGT were used to depict dynamic behavior of micro - scale beams in the work conducted by Wang et al. [31].

Similar investigations are carried out for the micro - beams made of functionally graded materials (FGMs). FGMs are inhomogeneous composites which are processed by combining the best properties of two distinct phases such as high strength, and high temperature resistance in order to put forward an ideal material. Nowadays FGMs are used extensively in automotive, electronics, biomechanics and aerospace industries. Asghari et al. [32] used Euler Bernoulli beam theory and Ke and Wang [33] and Asghari et al. [34] used TBT to obtain MCST based formulation for FGMs. Reddy [35] and Ke et al. [36] also considered Von Karman nonlinearities. Ansari et al. [37] formulated Timoshenko functionally graded beam based on SGT. In most of the studies the length scale parameter is considered to be constant which is a simplifying assumption. Kahrobaiyan et al. [38] derived the equation of motion and boundary conditions of a micro - beam using strain gradient approach and functionally graded Euler – Bernoulli beam model considering the variation of material length scale parameter through the thickness.

In the studies mentioned above the higher order continuum theories along with a beam theory is used to obtain static deflection, buckling analysis and natural frequencies of micro - beams. The comparisons made in these studies show significant differences between classical and higher order theories of elasticity. Further, different kinds of boundary conditions are studied in the scope of these works. In some cases a comparison is made for different beam theories.

1.3 Motivation and Scope of the Study

In all studies on small - scale functionally graded beams mentioned in the previous section, the length scale parameters used in the formulation are taken as constants. Note that a single length scale parameter is needed in the modified couple stress theory whereas strain gradient elasticity requires the use of three different length scale parameters. In a functionally graded medium, due to the variations in the volume fractions of the constituents, the length scale parameters are also expected to be functions of the spatial coordinates. Thus, a general formulation should take into account the variations in the length scale parameters as well. The only study in the literature that considers the variations in the length scale parameters seems to be that by Kahrobaiyan et al. [38]. In this article, the authors develop analysis methods by using strain gradient elasticity in conjunction with the Euler - Bernoulli beam theory. However, Euler - Bernoulli theory is built on certain restrictive assumptions such as the assumption of zero shear strain; and also as will be shown in this paper, in a number of problems this theory has a tendency to overestimate the normal stresses in small - scale beams. TBT considers shear strain as a constant along beam thickness. Higher order shear deformation beam theories such as TOBT satisfy the shear conditions on the boundaries in the scope of classical continuum theory, so that using them along with higher order elastic theories leads to more proper results.

In Chapter 2, by using the general beam model and strain gradient theory, the governing equations and boundary conditions are obtained for FGM micro - beam using Hamilton’s principle. The relations can be reduced to MCST or classical also to different beam theories for both homogeneous and FGM. Thermal effects have been considered to derive the formulation and the static and dynamic behavior of the micro - beam related to the temperature change is investigated. In chapter 3, differential quadrature method (DQM) is introduced as a means to solve the system of differential equations. The explanation as to how DQM may be used to solve the problem and how boundary conditions are implemented is presented in this chapter. A computer program is developed using MATLAB to implement the developed numerical solution technique. A simply supported micro - beam is considered and static deflections and natural frequencies are obtained for this problem. The results and comparisons are presented in Chapter 4. The discussion of the results and the suggestions for future work are given in Chapter 5.

CHAPTER 2

FORMULATION

2.1 Problem Definition

To investigate the small - scale and shear deformation effects on micro – beams, a beam made of an FGM is considered. Functionally graded materials (FGMs) are inhomogeneous composites which are processed by combining the best properties of two distinct phases such as high strength, and high temperature resistance in order to put forward an ideal material. Nowadays FGMs are used extensively in automotive, electronics, biomechanics and aerospace industries.

Fig. 1 illustrates a small - scale functionally graded (FG) beam with length L and thickness h made from a mixture of ceramics and metals. The bottom surface (x_3 = $-h/2$) is metal rich and the top surface (x_3 = $h/2$) is ceramic rich. The material properties of the FG beam vary continuously in the thickness direction. The effective bulk modulus K_e and shear modulus μ_e are calculated by Mori - Tanaka homogenization method [37]:

$$\frac{K_e - K_m}{K_c - K_m} = \frac{V_c}{1 + V_m (K_C - K_m)/(K_m + \frac{4\mu_m}{3})} \tag{1}$$

$$\frac{\mu_e - \mu_m}{\mu_c - \mu_m} = \frac{V_c}{1 + V_m(\mu_c - \mu_m)/\left(\mu_m + \mu_m\left(9K_m + 8\mu_m\right)/6\left(K_m + 2\mu_m\right)\right)} \tag{2}$$

where V denotes volume fraction of the phase materials. The subscripts m and c denote metal and ceramic phases, respectively. V_c and V_m are defined by using a power law function:

$$V_C + V_m = 1 \tag{3}$$

$$V_c\left(z\right) = \left(0.5 + z/h\right)^n \tag{4}$$

where n is the volume fraction exponent. The effective material properties of the FG micro - beam such as Young's modulus E and Poisson's ratio υ related to K_e and μ_e are expressed as:

$$E\left(z\right) = \frac{9K_e\mu_e}{3K_e + \mu_e} \tag{5}$$

$$\upsilon(z)=\frac{3K_e-2\mu_e}{6K_e+2\mu_e} \tag{6}$$

Using the rule of mixtures the effective mass density ρ and the thermal expansion coefficient α can be given as

$$\rho(z)=\rho_c V_c+\rho_m V_m \tag{7}$$

$$\alpha(z)=\alpha_c V_c+\alpha_m V_m \tag{8}$$

2.2 Shear Deformation Beam Theories

Higher order shear deformation beam theories consider the effects of transverse shear deformation. Assuming that the deformations of the beam are in the x_1 – x_3 plane and denoting the displacement components along the x_1, x_2 and x_3 directions by u_1, u_2 and u_3, based on the general higher - order shear deformation beam theory, the following displacement field is assumed:

$$u_1(x_1,x_3,t)=u(x_1,t)-x_3 w_{,x_1}+f(x_3)\gamma(x_1,t) \tag{9.a}$$

$$u_2(x_1,x_3,t)=0 \tag{9.b}$$

$$u_3(x_1,x_3,t)=w(x_1,t) \tag{9.c}$$

u and w represent middle surface displacement components along the x_1 and x_3 directions, respectively, t is the time, "$()_{,x}$" denotes partial derivative with respect to x_1, γ is the transverse shear strain of any point on the neutral axis:

$$\gamma(x_1,t)=w_{,x_1}(x_1,t)-\phi(x_1,t), \tag{10}$$

where ϕ is the total bending rotation of the cross - section at any point on the neutral axis (Fig. 1)

Shape function f determines the distribution of the transverse shear strain and stress through the thickness. By taking the shape function as zero the Euler – Bernoulli beam theory (EBBT) or classical beam theory is obtained as a particular case. To consider shear effects higher order shear deformation beam theories have been already introduced by researchers. Among these theories third - order beam theory (TOBT) [40, 41], trigonometric shear deformation beam theory [42] and hyperbolic shear deformation beam theory (HSDBT) [43] are mostly used in the literature. Although different shape functions are applicable, only the ones which convert the present theory to the corresponding Euler - Bernoulli beam theory (EBBT), first order shear deformation beam theory or Timoshenko beam theory (TBT) and TOBT are employed in the present study. In more detail, the shape function employed for EBBT, TBT and TOBT are as follows:

EBBT: $f\left(x_3\right)=0$ *(11.a)*

TBT: $f\left(x_3\right)=x_3$ *(11.b)*

TOBT: $f\left(x_3\right)=x_3\left(1-\frac{4x_3^2}{3h^2}\right)$ *(11.c)*

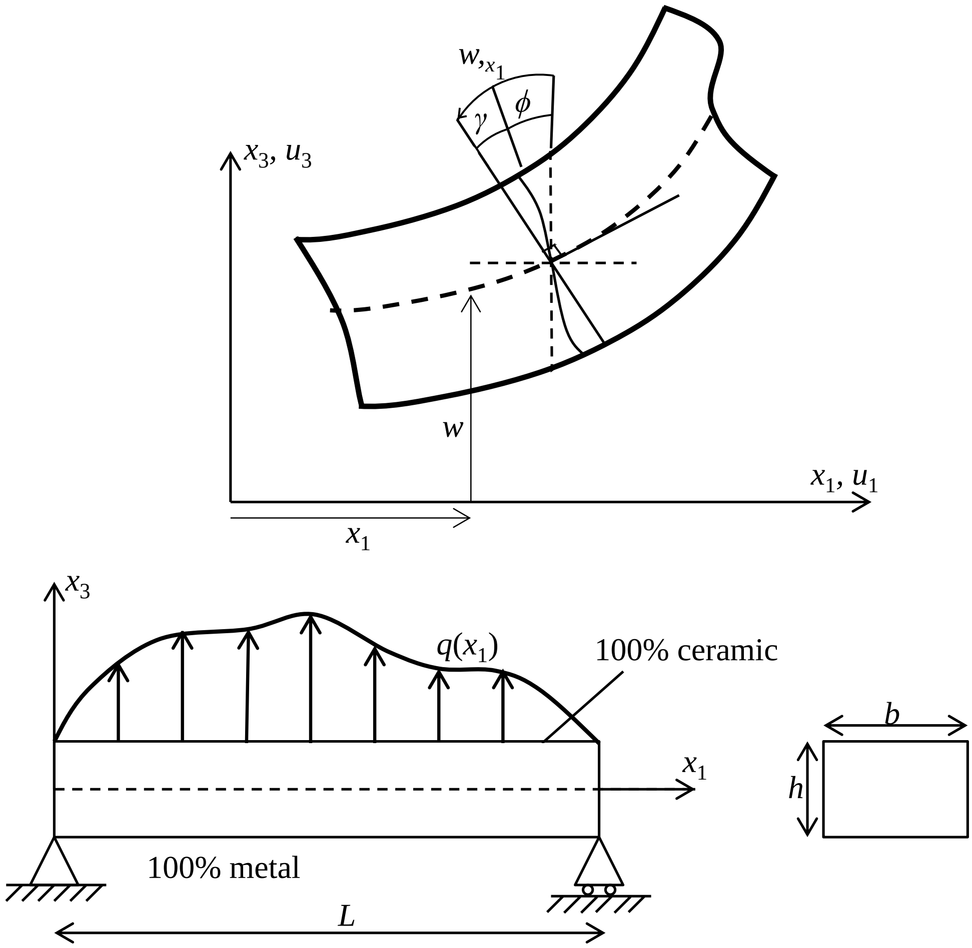

Fig. 1. Functionally graded beam configuration and the shape function schematic sketch for higher order shear deformation beam theories.

2.3 Strain Gradient Theory (SGT)

In the strain gradient theory strain energy density, v, depends on both the conventional strain (the symmetric part of the first order deformation gradient) and on the second order deformation gradient [12]

$$v = v\left(\varepsilon_{ij}, \eta_{ij}\right) \tag{12}$$

where ε_{ij} and η_{ijk} are the strain tensor (first order deformation gradient) and second order deformation gradient tensor, respectively

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{13.a}$$

$$\eta_{ijk} = u_{k,ij} \tag{13.b}$$

where u_i is the displacement vector.

The Cauchy stress tensor, σ_{ij}, and double stress tensor, τ_{ijk} can be defined as:

$$\sigma_{ij} = \frac{\partial v}{\partial \varepsilon_{ij}} \tag{14.a}$$

$$\tau_{ijk} = \frac{\partial v}{\partial \eta_{ijk}} \tag{14.b}$$

The second order deformation gradient, η_{ijk}, can be decomposed into symmetric and anti - symmetric parts, η_{ijk}^{s}, and η_{ijk}^{a}, giving,

$$\eta_{ijk}^{s} = \frac{1}{3}\left(\eta_{ijk} + \eta_{jki} + \eta_{kij}\right) \tag{15.a}$$

$$\eta_{ijk}^{a} = \frac{2}{3}\left(e_{ikl}\chi_{lj} + e_{jkl}\chi_{li}\right) \tag{15.b}$$

where e_{ijk} is the alternating tensor and $\chi_{ij} = 1/2 e_{ipq}\eta_{jpq}$ is the curvature tensor. By splitting the symmetric second order deformation gradient, η_{ijk}^{s}, a trace part, $\eta_{ijk}^{(0)}$, and a traceless part, $\eta_{ijk}^{(1)}$, are obtained

$$\eta_{ijk}^{s} = \eta_{ijk}^{(0)} + \eta_{ijk}^{(1)} \tag{16}$$

where

$$\eta_{ijk}^{(0)} = \frac{1}{5}\left(\delta_{ij}\eta_{mmk}^{s} + \delta_{jk}\eta_{mmi}^{s} + \delta_{ki}\eta_{mmj}^{s}\right) \tag{17.a}$$

$$\eta_{ijk}^{(1)} = \eta_{ijk}^{s} - \eta_{ijk}^{(0)} \tag{17.b}$$

$$\eta_{mmk}^{s} = \frac{1}{3}\left(\eta_{mmk} + 2\eta_{kmm}\right) \tag{17.c}$$

The curvature tensor is decomposed into symmetric and anti - symmetric parts as

$$\chi_{ij} = \chi_{ij}^{s} + \chi_{ij}^{a} \tag{18}$$

where

$$\chi_{ij}^{s} = \frac{1}{2}\left(\chi_{ij} + \chi_{ji}\right) \tag{19.a}$$

$$\chi_{ij}^{a} = \frac{1}{2}\left(\chi_{ij} - \chi_{ji}\right) \tag{19.b}$$

The trace part of the symmetric second order deformation gradient is a function of the dilatation gradient and the anti - symmetric part of the curvature,

$$\eta_{ipp}^{s} = \varepsilon_{,i} + \frac{2}{3}e_{imn}\chi_{mn}^{a} = \varepsilon_{,i} + \frac{2}{3}e_{imn}\chi_{mn} \tag{20}$$

where ε is the dilatation strain,

$$\varepsilon = \varepsilon_{mm} \tag{21}$$

For easy reference, $\varepsilon_{,i}$, $\eta_{ijk}^{(1)}$ and χ_{ij} are named as the dilatation gradient, the deviatoric stretch gradient and the rotational gradient, respectively. The second order virtual work density in terms of the new strain metrics is

$$\delta\hat{v} = \tau_{ijk}^{(0)}\delta\eta_{ijk}^{(0)} + \tau_{ijk}^{(1)}\delta\eta_{ijk}^{(1)} + \tau_{ijk}^{a}\delta\eta_{ijk}^{a} \tag{22}$$

where $\tau_{ijk}^{(0)}$ and $\tau_{ijk}^{(1)}$ are the trace and traceless parts of the symmetric part the double stress tensor (τ_{ijk}^{s}), respectively, and are orthogonal to each other:

$$\tau_{ijk}^{(0)} = \frac{1}{5}\left(\delta_{ij}\tau_{mmk}^{s} + \delta_{jk}\tau_{mmi}^{s} + \delta_{ki}\tau_{mmj}^{s}\right) \tag{23.a}$$

$$\tau_{ijk}^{(1)} = \tau_{ijk}^{s} - \tau_{ijk}^{(0)} \tag{23.b}$$

Using $\varepsilon_{,i}$, $\tau_{ijk}^{(1)}$, χ_{ij} as the second order metrics, equation (22) can be written as

$$\delta\hat{v} = p_i\delta\varepsilon_{,i} + \tau_{ijk}^{(1)}\delta\eta_{ijk}^{(1)} + m_{ij}^{'}\delta\chi_{ij} \tag{24}$$

where

$$p_i = \frac{3}{5}\tau^{s}_{mmi} \tag{25}$$

$$m'_{ij} = \frac{4}{3}\tau^{a}_{ipq}e_{jpq} - \frac{2}{5}e_{ijk}\tau^{s}_{mmk} \tag{26}$$

By neglecting the effects of the dilatation gradient and the deviatoric stretch gradient the *modified couple stress theory* (MCST) is obtained:

$$\delta v = \sigma_{ij}\delta\varepsilon_{ij} + m_{ij}\delta\chi_{ij} \tag{27}$$

In this theory in addition to the classical equations of forces and moments of forces, the equilibrium of moments of couples must be satisfied. To satisfy the higher order equilibrium equation m_{ij} must be symmetric. Finally, considering the effects of conventional strain tensor and using symmetric part of m_{ij}, the total strain energy density for strain gradient theory becomes:

$$v = v\left(\varepsilon_{ij}, \varepsilon_{,i}, \eta^{(1)}_{ijk}, \chi^{s}_{ij}\right) \tag{28}$$

For a deformed linear elastic isotropic material, the strain energy U, occupying region Ω based on the modified strain gradient elasticity theory can be then written as

$$U = \frac{1}{2}\int_{\Omega}\left(\sigma_{ij}\varepsilon_{ij} + p_i\gamma_i + \tau^{(1)}_{ijk}\eta^{(1)}_{ijk} + m^{s}_{ij}\chi^{s}_{ij}\right)dv \tag{29}$$

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{30}$$

$$\gamma_i = \varepsilon_{mm,i} \tag{31}$$

$$\eta^{(1)}_{ijk} = \frac{1}{3}\left(\varepsilon_{jk,i} + \varepsilon_{ki,j} + \varepsilon_{ij,k}\right) - \frac{1}{15}\delta_{ij}\left(\varepsilon_{mm,k} + 2\varepsilon_{mk,m}\right) - \frac{1}{15}\left\{\delta_{jk}\left(\varepsilon_{mm,i} + 2\varepsilon_{mi,m}\right) + \delta_{ki}\left(\varepsilon_{mm,j} + 2\varepsilon_{mj,m}\right)\right\} \tag{32}$$

$$\chi^{s}_{ij} = \frac{1}{2}\left(e_{ipq}\varepsilon_{qj,p} + e_{jpq}\varepsilon_{qi,p}\right) \tag{33}$$

Since the different second order strain metrics are orthogonal to each other and there is no coupling amongst them three different parameters are used to define the constitutive relations which are given by [12]:

$$\sigma_{ij} = \lambda tr\left(\varepsilon\right)\delta_{ij} + 2\mu\varepsilon_{ij} \tag{34}$$

$$p_i = 2\mu l_0^2\gamma_i \tag{35}$$

$$\tau_{ijk}^{(1)} = 2\mu l_1^2 \eta_{ijk}^{(1)} \tag{36}$$

$$m_{ij}^{s} = 2\mu l_2^2 \chi_{ij}^{s} \tag{37}$$

where

$$\lambda = \frac{E\upsilon}{(1+\upsilon)(1-2\upsilon)}, \tag{38}$$

$$\mu = \frac{E}{2(1+\upsilon)}, \tag{39}$$

and l_0, l_1, l_2 are material length - scale parameters which for the FGM micro – beam are taken as follows:

$$l_0(x_3) = l_{0c}V_c + l_{0m}V_m \tag{40.a}$$

$$l_1(x_3) = l_{1c}V_c + l_{1m}V_m \tag{40.b}$$

$$l_2(x_3) = l_{2c}V_c + l_{2m}V_m \tag{40.c}$$

By inserting equations (9.a) - (9.c) into equation (30), the non-vanishing strains are

$$\begin{aligned} \varepsilon_{11} &= \frac{\partial u}{\partial x_1} - x_3 \frac{\partial^2 w}{\partial x_1^2} + f \frac{\partial \gamma}{\partial x_1} \\ \varepsilon_{13} &= \varepsilon_{31} = \frac{1}{2} f' \gamma \end{aligned} \tag{41}$$

where a prime denotes the derivative with respect to x_3.

Substitution of equations (41) into equations (31) - (33) yields the nonzero components of $\gamma_i, \eta_{ijk}^{(1)}, \chi_{ij}^{s}$

$$\begin{aligned} \gamma_1 &= \frac{\partial^2 u}{\partial x_1^{\ 2}} - x_3 \frac{\partial^3 w}{\partial x_1^3} + f \frac{\partial^2 \gamma}{\partial x_1^2} \\ \gamma_3 &= -\frac{\partial^2 w}{\partial x_1^2} + f' \frac{\partial \gamma}{\partial x_1} \end{aligned} \tag{42}$$

$$\eta_{111}^{(1)} = \frac{2}{5}\left(\frac{\partial^2 u}{\partial x_1^2} - x_3 \frac{\partial^3 w}{\partial x_1^3} + f \frac{\partial^2 \gamma}{\partial x_1^2} - \frac{1}{2} f'' \gamma \right)$$

$$\eta_{333}^{(1)} = \frac{1}{5}\left(\frac{\partial^2 w}{\partial x_1^2} - 2 f' \frac{\partial \gamma}{\partial x_1} \right)$$

$$\eta_{113}^{(1)} = \eta_{311}^{(1)} = \eta_{131}^{(1)} = -\frac{4}{15}\left(\frac{\partial^2 w}{\partial x_1^2} - 2 f' \frac{\partial \gamma}{\partial x_1} \right) \tag{43}$$

$$\eta_{223}^{(1)} = \eta_{322}^{(1)} = \eta_{232}^{(1)} = \frac{1}{15}\left(\frac{\partial^2 w}{\partial x_1^2} - 2 f' \frac{\partial \gamma}{\partial x_1} \right)$$

$$\eta_{221}^{(1)} = \eta_{122}^{(1)} = \eta_{212}^{(1)} = -\frac{1}{5}\left(\frac{\partial^2 u}{\partial x_1^2} - x_3 \frac{\partial^3 w}{\partial x_1^3} + f \frac{\partial^2 \gamma}{\partial x_1^2} + \frac{1}{3} f'' \gamma \right)$$

$$\eta_{331}^{(1)} = \eta_{133}^{(1)} = \eta_{313}^{(1)} = -\frac{1}{5}\left(\frac{\partial^2 u}{\partial x_1^2} - x_3 \frac{\partial^3 w}{\partial x_1^3} + f \frac{\partial^2 \gamma}{\partial x_1^2} - \frac{4}{3} f'' \gamma \right)$$

$$\chi_{12}^{s} = \chi_{21}^{s} = \frac{1}{4}\left(-2 \frac{\partial^2 w}{\partial x_1^2} + f' \frac{\partial \gamma}{\partial x_1} \right) \tag{44}$$

$$\chi_{23}^{s} = \chi_{32}^{s} = \frac{1}{4} f'' \gamma$$

Consequently, by placing equations (41) - (44) into equations (34) - (37), the nonzero components of the symmetric section of the stress tensor and the higher order stresses in the thermal environment are obtained as follows

$$\sigma_{11} = (\lambda + 2\mu)\left(\frac{\partial u}{\partial x_1} - x_3 \frac{\partial^2 w}{\partial x_1^2} + f \frac{\partial \gamma}{\partial x_1} - \alpha(x_3)\Delta T \right)$$

$$\sigma_{22} = \lambda \left(\frac{\partial u}{\partial x_1} - x_3 \frac{\partial^2 w}{\partial x_1^2} + f \frac{\partial \gamma}{\partial x_1} \right) \tag{45}$$

$$\sigma_{33} = \lambda \left(\frac{\partial u}{\partial x_1} - x_3 \frac{\partial^2 w}{\partial x_1^2} + f \frac{\partial \gamma}{\partial x_1} \right)$$

$$\sigma_{13} = \sigma_{31} = k_s \mu f' \gamma$$

$$p_1 = 2\mu l_0^2 \left(\frac{\partial^2 u}{\partial x_1^2} - x_3 \frac{\partial^3 w}{\partial x_1^3} + f \frac{\partial^2 \gamma}{\partial x_1^2} \right) \tag{46}$$

$$p_3 = 2\mu l_0^2 \left(-\frac{\partial^2 w}{\partial x_1^2} + f' \frac{\partial \gamma}{\partial x_1} \right)$$

$$
\begin{aligned}
&\tau_{111}^{(1)} = \frac{4}{5}\mu l_1^2\left(\frac{\partial^2 u}{\partial x_1^2} - x_3\frac{\partial^3 w}{\partial x_1^3} + f\frac{\partial^2 \gamma}{\partial x_1^2} - \frac{1}{2}f''\gamma\right)\\
&\tau_{333}^{(1)} = \frac{2}{5}\mu l_1^2\left(\frac{\partial^2 w}{\partial x_1^2} - 2f'\frac{\partial \gamma}{\partial x_1}\right)\\
&\tau_{113}^{(1)} = \tau_{311}^{(1)} = \tau_{131}^{(1)} = -\frac{8}{15}\mu l_1^2\left(\frac{\partial^2 w}{\partial x_1^2} - 2f'\frac{\partial \gamma}{\partial x_1}\right)\\
&\tau_{223}^{(1)} = \tau_{322}^{(1)} = \tau_{232}^{(1)} = \frac{2}{15}\mu l_1^2\left(\frac{\partial^2 w}{\partial x_1^2} - 2f'\frac{\partial \gamma}{\partial x_1}\right)\\
&\tau_{221}^{(1)} = \tau_{122}^{(1)} = \tau_{212}^{(1)} = -\frac{2}{5}\mu l_1^2\left(\frac{\partial^2 u}{\partial {x_1}^2} - x_3\frac{\partial^3 w}{\partial {x_1}^3} + f\frac{\partial^2 \gamma}{\partial {x_1}^2} + \frac{1}{3}f''\gamma\right)\\
&\tau_{331}^{(1)} = \tau_{133}^{(1)} = \tau_{313}^{(1)} = -\frac{2}{5}\mu l_1^2\left(\frac{\partial^2 u}{\partial {x_1}^2} - x_3\frac{\partial^3 w}{\partial {x_1}^3} + f\frac{\partial^2 \gamma}{\partial x_1^2} - \frac{4}{3}f''\gamma\right)
\end{aligned}
\tag{47}
$$

$$
\begin{aligned}
&m_{12}^s = m_{21}^s = \frac{\mu l_2^2}{2}\left(-2\frac{\partial^2 w}{\partial x_1^2} + f'\frac{\partial \gamma}{\partial x_1}\right)\\
&m_{23}^s = m_{32}^s = \frac{1}{2}\mu l_2^2 f''\gamma
\end{aligned}
\tag{48}
$$

Note that k_s is the shear correction factor, which is taken as unity in EBBT and TOBT; and specified as $5/6$ in Timoshenko beam theory for rectangular cross - sections. ΔT is the temperature change from a stress free state. Note that the deformed shape of the beam is governed by the total strain while the stress state depends only on the mechanical strains. Although the material properties such as elastic modulus E, Poisson's ration v, density ρ and coefficient of thermal expansion α are temperature dependent, the influence of temperature change on material properties is not considered in this study. Then the strain energy based on modified strain gradient theory (equation (29)) may be rewritten as

$$
\begin{aligned}
U &= \frac{1}{2}\int_0^L\int_A(\sigma_{ij}\varepsilon_{ij} + p_i\gamma_i + \tau_{ijk}^{(1)}\eta_{ijk}^{(1)} + m_{ij}^s\chi_{ij}^s)dAdx\\
&= \frac{1}{2}\int_0^L\int_A\Bigg\{\sigma_{11}\left(\frac{\partial u}{\partial x_1} - x_3\frac{\partial^2 w}{\partial x_1^2} + f\frac{\partial \gamma}{\partial x_1}\right) + \sigma_{13}\left(f'\gamma\right)\\
&+p_1\left(\frac{\partial^2 u}{\partial x_1^2} - x_3\frac{\partial^3 w}{\partial x_1^3} + f\frac{\partial^2 \gamma}{\partial x_1^2}\right) + p_3\left(-\frac{\partial^2 w}{\partial x_1^2} + f'\frac{\partial \gamma}{\partial x_1}\right)\\
&+\tau_{111}^{(1)}\left(\frac{\partial^2 u}{\partial x_1^2} - x_3\frac{\partial^3 w}{\partial x_1^3} + f\frac{\partial^2 \gamma}{\partial x_1^2} - \frac{1}{2}f''\gamma\right) + \frac{4}{3}\tau_{333}^{(1)}\left(\frac{\partial^2 w}{\partial x_1^2} - 2f'\frac{\partial \gamma}{\partial x_1}\right)\\
&+\frac{1}{3}\mu l_1^2\gamma^2 f''^2 + \frac{1}{2}m_{12}^s\left(-2\frac{\partial^2 w}{\partial x_1^2} + f'\frac{\partial \gamma}{\partial x_1}\right) + \frac{1}{2}m_{23}^s\left(f''\gamma\right)\Bigg\}dAdx
\end{aligned}
\tag{49}
$$

The work done by external forces is

$$W=\int_{o}^{L}\left\{q\left(x_1\right)w+\frac{1}{2}\int_{A}\left(\lambda+2\mu\right)\alpha\Delta T\left(\frac{\partial w}{\partial x_1}\right)^2 dA\right\}dx \tag{50}$$

where q is the distributed transverse loading and the second expression is the work done by axial force due to the influence of the temperature change which can be determined from a static thermal bending analysis [24]. The kinetic energy of the micro - beam is given by

$$\begin{aligned}K&=\frac{1}{2}\int_{0}^{L}\int_{A}\rho\left\{\left(\frac{\partial u}{\partial t}-x_3\frac{\partial^2 w}{\partial x_1\partial t}+f\frac{\partial\gamma}{\partial t}\right)^2+\left(\frac{\partial w}{\partial t}\right)^2\right\}dAdx\\&=\frac{1}{2}\int_{0}^{L}\left\{I_1\left(\frac{\partial u}{\partial t}\right)^2+I_1\left(\frac{\partial w}{\partial t}\right)^2-2I_2\left(\frac{\partial u}{\partial t}\right)\left(\frac{\partial^2 w}{\partial x_1\partial t}\right)+I_3\left(\frac{\partial^2 w}{\partial x_1\partial t}\right)^2\right.\\&\left.+2I_4\left(\frac{\partial u}{\partial t}\right)\left(\frac{\partial\gamma}{\partial t}\right)-2I_5\left(\frac{\partial^2 w}{\partial x_1\partial t}\right)\left(\frac{\partial\gamma}{\partial t}\right)+I_6\left(\frac{\partial\gamma}{\partial t}\right)^2\right\}dx\end{aligned} \tag{51}$$

where ρ and A are density and the cross - sectional area of the beam. The terms related to inertia in equation (51) can be defined as

$$\left\{I_1,I_2,I_3,I_4,I_5,I_6\right\}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\rho(x_3)\left\{1,x_3,x_3^2,f,x_3f,f^2\right\}bdx_3 \tag{52}$$

To determine the dynamic governing equations of the beam and all possible boundary conditions the Hamilton's principle is employed which is

$$\delta\int_{t_1}^{t_2}\left(K-\left(U-W\right)\right)dt=0 \tag{53}$$

Substituting equations (49) - (51) into equation (53), taking the variation of u, w and γ and integrating by parts the higher order equations of motion (54.a) - (54.c) and the boundary conditions (55.a) - (55.g) can be obtained by setting the coefficients of δu, δw and $\delta\gamma$ equal to zero. The equations are given as follows:

$$\begin{aligned}&A_{11}\frac{\partial^2 u}{\partial x_1^2}-\left(2A_{550}+\frac{4}{5}A_{551}\right)\frac{\partial^4 u}{\partial x_1^4}-B_{11}\frac{\partial^3 w}{\partial x_1^3}+B_{55}\left(2B_{550}+\frac{4}{5}B_{551}\right)\frac{\partial^5 w}{\partial x_1^5}\\&+\left(F_{11}+\frac{2}{5}F_{671}\right)\frac{\partial^2\gamma}{\partial x_1^2}-\left(2F_{470}+\frac{4}{5}F_{471}\right)\frac{\partial^4\gamma}{\partial x_1^4}=I_1\frac{\partial^2 u}{\partial t^2}-I_2\frac{\partial^3 w}{\partial x_1\partial t^2}+I_4\frac{\partial^2\gamma}{\partial t^2}\end{aligned} \tag{54.a}$$

$$
\begin{aligned}
&B_{11}\frac{\partial^3 u}{\partial x_1^3}-\left(2B_{550}+\frac{4}{5}B_{551}\right)\frac{\partial^5 u}{\partial x_1^5}-H_{11}\Delta T\frac{\partial^2 w}{\partial x_1^2}-\left(D_{11}+2A_{550}+\frac{8}{15}A_{551}+A_{552}\right)\frac{\partial^4 w}{\partial x_1^4}\\
&+\left(2D_{550}+\frac{4}{5}D_{551}\right)\frac{\partial^6 w}{\partial x_1^6}+\left(F_{22}+2F_{570}+\frac{16}{15}F_{571}+\frac{F_{572}}{2}+\frac{2}{5}F_{681}\right)\frac{\partial^3 \gamma}{\partial x_1^3}\\
&-\left(2F_{480}+\frac{4}{5}F_{481}\right)\frac{\partial^5 \gamma}{\partial x_1^5}+q=I_2\frac{\partial^3 u}{\partial x_1\partial t^2}+I_1\frac{\partial^2 w}{\partial t^2}-I_3\frac{\partial^4 w}{\partial x_1^2\partial t^2}+I_5\frac{\partial^3 \gamma}{\partial x_1\partial t^2}
\end{aligned}
\tag{54.b}
$$

$$
\begin{aligned}
&-\left(F_{11}+\frac{2}{5}F_{671}\right)\frac{\partial^2 u}{\partial x_1^2}+\left(2F_{470}+\frac{4}{5}F_{471}\right)\frac{\partial^4 u}{\partial x_1^4}+\\
&\left(F_{22}+2F_{570}+\frac{16}{15}F_{571}+\frac{F_{572}}{2}+\frac{2}{5}F_{681}\right)\frac{\partial^3 w}{\partial x_1^3}-\left(2F_{480}+\frac{4}{5}F_{481}\right)\frac{\partial^5 w}{\partial x_1^5}\\
&+\left(k_sF_{55}+\frac{8}{15}F_{661}+\frac{F_{662}}{4}\right)\gamma-\left(F_{33}+2F_{550}+\frac{32}{15}F_{551}+\frac{F_{552}}{4}+\frac{4}{5}F_{461}\right)\frac{\partial^2 \gamma}{\partial x_1^2}\\
&+\left(2F_{440}+\frac{4}{5}F_{441}\right)\frac{\partial^4 \gamma}{\partial x_1^4}=-I_4\frac{\partial^2 u}{\partial t^2}+I_5\frac{\partial^3 w}{\partial x_1\partial t^2}-I_6\frac{\partial^2 \gamma}{\partial t^2}
\end{aligned}
\tag{54.c}
$$

And, the boundary conditions are:

$$
\begin{aligned}
&\left\{-A_{11}\frac{\partial u}{\partial x_1}+\left(2A_{550}+\frac{4}{5}A_{551}\right)\frac{\partial^3 u}{\partial x_1^3}+B_{11}\frac{\partial^2 w}{\partial x_1^2}-\left(2B_{550}+\frac{4}{5}B_{551}\right)\frac{\partial^4 w}{\partial x_1^4}\right.\\
&\left.-\left(F_{11}+\frac{2}{5}F_{671}\right)\frac{\partial \gamma}{\partial x_1}+\left(2F_{470}+\frac{4}{5}F_{471}\right)\frac{\partial^3 \gamma}{\partial x_1^3}+\frac{1}{2}H_{11}\Delta T\right\}\Big|_{x_1=0,L}=0\quad or\quad \delta u\Big|_{x_1=0,L}=0
\end{aligned}
\tag{55.a}
$$

$$
\begin{aligned}
&\left\{\left(2A_{550}+\frac{4}{5}A_{551}\right)\frac{\partial^2 u}{\partial x_1^2}-\left(2B_{550}+\frac{4}{5}B_{551}\right)\frac{\partial^3 w}{\partial x_1^3}\right.\\
&\left.-\frac{2}{5}F_{671}\gamma+\left(2F_{470}+\frac{4}{5}F_{471}\right)\frac{\partial^2 \gamma}{\partial x_1^2}\right\}\Big|_{x_1=0,L}=0\quad or\quad \delta\left(\frac{\partial u}{\partial x_1}\right)\Big|_{x_1=0,L}=0
\end{aligned}
\tag{55.b}
$$

$$
\begin{aligned}
&\left\{B_{11}\frac{\partial^2 u}{\partial x_1^2}-\left(2B_{550}+\frac{4}{5}B_{551}\right)\frac{\partial^4 u}{\partial x_1^4}-H_{11}\Delta T\frac{\partial w}{\partial x_1}\right.\\
&-\left(D_{11}+2A_{550}+\frac{8}{15}A_{551}+A_{552}\right)\frac{\partial^3 w}{\partial x_1^3}+\left(2D_{550}+\frac{4}{5}D_{551}\right)\frac{\partial^5 w}{\partial x_1^5}\\
&+\left(F_{22}+2F_{570}+\frac{16}{15}F_{571}+\frac{F_{572}}{2}+\frac{2}{5}F_{681}\right)\frac{\partial^2 \gamma}{\partial x_1^2}-\left(2F_{480}+\frac{4}{5}F_{481}\right)\frac{\partial^4 \gamma}{\partial x_1^4}\\
&\left.=I_2\frac{\partial^2 u}{\partial t^2}-I_3\frac{\partial^3 w}{\partial x_1\partial t^2}+I_5\frac{\partial^2 \gamma}{\partial t^2}\right\}\Big|_{x_1=0,L}=0\quad or\quad \delta w\Big|_{x_1=0,L}=0
\end{aligned}
\tag{55.c}
$$

$$\left\{ B_{11}\frac{\partial u}{\partial x_1} - \left(2B_{550} + \frac{4}{5}B_{551}\right)\frac{\partial^3 u}{\partial x_1^3} - \left(D_{11} + 2A_{550} + \frac{8}{15}A_{551} + A_{552}\right)\frac{\partial^2 w}{\partial x_1^2} \right.$$

$$+\left(2D_{550} + \frac{4}{5}D_{551}\right)\frac{\partial^4 w}{\partial x_1^4} + \left(F_{22} + 2F_{570} + \frac{16}{15}F_{571} + \frac{F_{572}}{2} + \frac{2}{5}F_{681}\right)\frac{\partial \gamma}{\partial x_1} \qquad (55.d)$$

$$\left. -\left(2F_{480} + \frac{4}{5}F_{481}\right)\frac{\partial^3 \gamma}{\partial x_1^3} - \frac{1}{2}H_{22}\Delta T \right\}\Big|_{x_1=0,L} = 0 \quad or \quad \delta\left(\frac{\partial w}{\partial x_1}\right)\Big|_{x_1=0,L} = 0$$

$$\left\{ \left(2B_{550} + \frac{4}{5}B_{551}\right)\frac{\partial^2 u}{\partial x_1^2} - \left(2D_{550} + \frac{4}{5}D_{551}\right)\frac{\partial^3 w}{\partial x_1^3} \right.$$

$$\left. -\frac{2}{5}F_{681}\gamma + \left(2F_{480} + \frac{4}{5}F_{481}\right)\frac{\partial^2 \gamma}{\partial x_1^2} \right\}\Big|_{x_1=0,L} = 0 \quad or \quad \delta\left(\frac{\partial^2 w}{\partial x_1^2}\right)\Big|_{x_1=0,L} = 0 \qquad (55.e)$$

$$\left\{ F_{11}\frac{\partial u}{\partial x_1} - \left(2F_{470} + \frac{4}{5}F_{471}\right)\frac{\partial^3 u}{\partial x_1^3} - \left(F_{22} + 2F_{570} + \frac{16}{15}F_{571} + \frac{F_{572}}{2}\right)\frac{\partial^2 w}{\partial x_1^2} \right.$$

$$+\left(2F_{480} + \frac{4}{5}F_{481}\right)\frac{\partial^4 w}{\partial x_1^4} + \left(F_{33} + 2F_{550} + \frac{32}{15}F_{551} + \frac{F_{552}}{4} + \frac{2}{5}F_{461}\right)\frac{\partial \gamma}{\partial x_1} \qquad (55.f)$$

$$\left. -\left(2F_{440} + \frac{4}{5}F_{441}\right)\frac{\partial^3 \gamma}{\partial x_1^3} - \frac{1}{2}H_{33}\Delta T \right\}\Big|_{x_1=0,L} = 0 \quad or \quad \delta\gamma\Big|_{x_1=0,L} = 0$$

$$\left\{ \left(2F_{470} + \frac{4}{5}F_{471}\right)\frac{\partial^2 u}{\partial x_1^2} - \left(2F_{480} + \frac{4}{5}F_{481}\right)\frac{\partial^3 w}{\partial x_1^3} \right.$$

$$\left. -\frac{2}{5}F_{461}\gamma + \left(2F_{440} + \frac{4}{5}F_{441}\right)\frac{\partial^2 \gamma}{\partial x_1^2} \right\}\Big|_{x_1=0,L} = 0 \quad or \quad \delta\left(\frac{\partial \gamma}{\partial x_1}\right)\Big|_{x_1=0,L} = 0 \qquad (55.g)$$

The stiffness components in equations (54.a) - (54.c) and (55.a) - (55.g) are defined as

$$\{A_{11}, B_{11}, D_{11}, F_{11}, F_{22}, F_{33}\} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E(x_3)(1-\upsilon(x_3))}{(1+\upsilon(x_3))(1-2\upsilon(x_3))}\left\{1, x_3, x_3^2, f, x_3 f, f^2\right\} b\,dx_3$$

$$\{H_{11}, H_{22}, H_{33}\} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E(x_3)(1-\upsilon(x_3))}{(1+\upsilon(x_3))(1-2\upsilon(x_3))}\alpha(x_3)\{1, x_3, f\} b\,dx_3 \qquad (56.a)$$

$$\{A_{55}, B_{55}, D_{55}, F_{44}, F_{46}, F_{47}, F_{48}, F_{55}, F_{57}, F_{66}, F_{67}, F_{68}\}$$

$$= \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E(x_3)}{2(1+\upsilon(x_3))}\left\{1, x_3, x_3^2, f^2, ff'', f, x_3 f, f'^2, f', f''^2, f'', x_3 f''\right\} b\,dx_3 \qquad (56.b)$$

$$\{A_{550}, B_{550}, D_{550}, F_{440}, F_{460}, F_{470}, F_{480}, F_{550}, F_{570}, F_{660}, F_{670}, F_{680}\}$$

$$= \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E(x_3).l_0^2}{2(1+\upsilon(x_3))}\left\{1, x_3, x_3^2, f^2, ff'', f, x_3 f, f'^2, f', f''^2, f'', x_3 f''\right\} b\,dx_3 \qquad (56.c)$$

$$\{A_{551},B_{551},D_{551},F_{441},F_{461},F_{471},F_{481},F_{551},F_{571},F_{661},F_{671},F_{681}\}$$
$$=\int_{-\frac{h}{2}}^{\frac{h}{2}}\frac{E(x_3).l_1^2}{2(1+\upsilon(x_3))}\{1,x_3,x_3^2,f^2,ff'',f,x_3f,f'^2,f',f''^2,f'',x_3f''\}bdx_3 \tag{56.d}$$

$$\{A_{552},B_{552},D_{552},F_{442},F_{462},F_{472},F_{482},F_{552},F_{572},F_{662},F_{672},F_{682}\}$$
$$=\int_{-\frac{h}{2}}^{\frac{h}{2}}\frac{E(x_3).l_2^2}{2(1+\upsilon(x_3))}\{1,x_3,x_3^2,f^2,ff'',f,x_3f,f'^2,f',f''^2,f'',x_3f''\}bdx_3 \tag{56.e}$$

Equation (10) is used to get the governing equations and boundary conditions in terms of ϕ instead of γ. Thus, the following equations are obtained:

$$A_{11}\frac{\partial^2 u}{\partial x_1^2}-\left(2A_{550}+\frac{4}{5}A_{551}\right)\frac{\partial^4 u}{\partial x_1^4}+\left(F_{11}+\frac{2}{5}F_{671}-B_{11}\right)\frac{\partial^3 w}{\partial x_1^3}$$
$$+\left(2B_{550}+\frac{4}{5}B_{551}-2F_{470}-\frac{4}{5}F_{471}\right)\frac{\partial^5 w}{\partial x_1^5}-\left(F_{11}+\frac{2}{5}F_{671}\right)\frac{\partial^2\phi}{\partial x_1^2}+\left(2F_{470}+\frac{4}{5}F_{471}\right)\frac{\partial^4\phi}{\partial x_1^4} \tag{57.a}$$
$$=I_1\frac{\partial^2 u}{\partial t^2}+(I_4-I_2)\frac{\partial^3 w}{\partial x_1\partial t^2}-I_4\frac{\partial^2\phi}{\partial t^2}$$

$$\left(B_{11}-F_{11}+\frac{2}{5}F_{671}\right)\frac{\partial^3 u}{\partial x_1^3}+\left(2F_{470}+\frac{4}{5}F_{471}-2B_{550}-\frac{4}{5}B_{551}\right)\frac{\partial^5 u}{\partial x_1^5}$$
$$+\left(k_sF_{55}+\frac{8}{15}F_{661}+\frac{F_{662}}{4}-H_{11}\Delta T\right)\frac{\partial^2 w}{\partial x_1^2}$$
$$+\left(2F_{22}+\frac{4}{5}F_{681}-F_{33}-D_{11}+4F_{570}+\frac{32}{15}F_{571}+F_{572}\right.$$
$$\left.-2F_{550}-\frac{32}{15}F_{551}-\frac{F_{552}}{4}-2A_{550}-\frac{8}{15}A_{551}-A_{552}-\frac{4}{5}F_{461}\right)\frac{\partial^4 w}{\partial x_1^4}$$
$$+\left(2F_{440}+\frac{4}{5}F_{441}+2D_{550}+\frac{4}{5}D_{551}-4F_{480}-\frac{8}{5}F_{481}\right)\frac{\partial^6 w}{\partial x_1^6}$$
$$-\left(k_sF_{55}+\frac{8}{15}F_{661}+\frac{F_{662}}{4}\right)\frac{\partial\phi}{\partial x_1}+\left(F_{33}-F_{22}+\frac{4}{5}F_{461}-\frac{2}{5}F_{681}\right.$$
$$\left.+2F_{550}+\frac{32}{15}F_{551}+\frac{F_{552}}{4}-2F_{570}-\frac{16}{15}F_{571}-\frac{F_{572}}{2}\right)\frac{\partial^3\phi}{\partial x_1^3}$$
$$+\left(2F_{480}+\frac{4}{5}F_{481}-2F_{440}-\frac{4}{5}F_{441}\right)\frac{\partial^5\phi}{\partial x_1^5}+q$$
$$=(I_2-I_4)\frac{\partial^3 u}{\partial x_1\partial t^2}+I_1\frac{\partial^2 w}{\partial t^2}+(2I_5-I_3-I_6)\frac{\partial^4 w}{\partial x_1^2\partial t^2}+(I_6-I_5)\frac{\partial^3\phi}{\partial x_1\partial t^2} \tag{57.b}$$

$$
\begin{aligned}
&-\left(F_{11}+\frac{2}{5}F_{671}\right)\frac{\partial^2 u}{\partial x_1^2}+\left(2F_{470}+\frac{4}{5}F_{471}\right)\frac{\partial^4 u}{\partial x_1^4}\\
&+\left(k_s F_{55}+\frac{8}{15}F_{661}+\frac{F_{662}}{4}\right)\frac{\partial w}{\partial x_1}\\
&+\left(F_{22}-F_{33}+\frac{2}{5}F_{681}-\frac{4}{5}F_{461}+2F_{570}+\frac{16}{15}F_{571}+\frac{F_{572}}{2}-2F_{550}-\frac{32}{15}F_{551}-\frac{F_{552}}{4}\right)\frac{\partial^3 w}{\partial x_1^3}\\
&+\left(2F_{440}+\frac{4}{5}F_{441}-2F_{480}-\frac{4}{5}F_{481}\right)\frac{\partial^5 w}{\partial x_1^5}-\left(k_s F_{55}+\frac{8}{15}F_{661}+\frac{F_{662}}{4}\right)\phi\\
&+\left(F_{33}+2F_{550}+\frac{32}{15}F_{551}+\frac{F_{552}}{4}+\frac{4}{5}F_{461}\right)\frac{\partial^2 \phi}{\partial x_1^2}-\left(2F_{440}+\frac{4}{5}F_{441}\right)\frac{\partial^4 \phi}{\partial x_1^4}\\
&=-I_4\frac{\partial^2 u}{\partial t^2}+\left(I_5-I_6\right)\frac{\partial^3 w}{\partial x_1 \partial t^2}+I_6\frac{\partial^2 \phi}{\partial t^2}
\end{aligned}
\tag{57.c}
$$

The boundary conditions are obtained as:

$$
\begin{aligned}
&\left\{-A_{11}\frac{\partial u}{\partial x_1}+\left(2A_{550}+\frac{4}{5}A_{551}\right)\frac{\partial^3 u}{\partial x_1^3}\right.\\
&+\left(B_{11}-F_{11}+\frac{2}{5}F_{671}\right)\frac{\partial^2 w}{\partial x_1^2}+\left(2F_{470}+\frac{4}{5}F_{471}-2B_{550}-\frac{4}{5}B_{551}\right)\frac{\partial^4 w}{\partial x_1^4}\\
&\left.+\left(F_{11}+\frac{2}{5}F_{671}\right)\frac{\partial \phi}{\partial x_1}-\left(2F_{470}+\frac{4}{5}F_{471}\right)\frac{\partial^3 \phi}{\partial x^3}+\frac{1}{2}H_{11}\Delta T\right\}\Big|_{x_1=0,L}=0 \quad or \quad \delta u\,\Big|_{x_1=0,L}=0
\end{aligned}
\tag{58.a}
$$

$$
\begin{aligned}
&\left\{\left(2A_{550}+\frac{4}{5}A_{551}\right)\frac{\partial^2 u}{\partial x_1^2}+\left(2F_{470}+\frac{4}{5}F_{471}-2B_{550}-\frac{4}{5}B_{551}\right)\frac{\partial^3 w}{\partial x_1^3}\right.\\
&\left.-\frac{2}{5}F_{671}\frac{\partial w}{\partial x_1}+\frac{2}{5}F_{671}\phi-\left(2F_{470}+\frac{4}{5}F_{471}\right)\frac{\partial^2 \phi}{\partial x_1^2}\right\}\Big|_{x_1=0,L}=0 \quad or \quad \delta\left(\frac{\partial u}{\partial x_1}\right)\Big|_{x_1=0,L}=0
\end{aligned}
\tag{58.b}
$$

$$
\begin{aligned}
&\left\{B_{11}\frac{\partial^2 u}{\partial x_1^2}-\left(2B_{550}+\frac{4}{5}B_{551}\right)\frac{\partial^4 u}{\partial x_1^4}-H_{11}\Delta T\frac{\partial w}{\partial x_1}\right.\\
&+\left(F_{22}-D_{11}+2F_{570}+\frac{16}{15}F_{571}+\frac{F_{572}}{2}-2A_{550}-\frac{8}{15}A_{551}-A_{552}+\frac{2}{5}F_{681}\right)\frac{\partial^3 w}{\partial x_1^3}\\
&+\left(2D_{550}+\frac{4}{5}D_{551}-2F_{480}-\frac{4}{5}F_{481}\right)\frac{\partial^5 w}{\partial x_1^5}\\
&-\left(F_{22}+2F_{570}+\frac{16}{15}F_{571}+\frac{F_{572}}{2}+\frac{2}{5}F_{681}\right)\frac{\partial^2 \phi}{\partial x_1^2}+\left(2F_{480}+\frac{4}{5}F_{481}\right)\frac{\partial^4 \phi}{\partial x_1^4}\\
&\left.=I_2\frac{\partial^2 u}{\partial t^2}+\left(I_5-I_3\right)\frac{\partial^3 w}{\partial x_1 \partial t^2}-I_5\frac{\partial^2 \phi}{\partial t^2}\right\}\Big|_{x_1=0,L}=0 \quad or \quad \delta w\,\Big|_{x_1=0,L}=0
\end{aligned}
\tag{58.c}
$$

$$
\begin{aligned}
&\Bigg\{\left(B_{11}-F_{11}\right)\frac{\partial u}{\partial x_1}+\left(2F_{470}+\frac{4}{5}F_{471}-2B_{550}-\frac{4}{5}B_{551}\right)\frac{\partial^3 u}{\partial x_1^3}\\
&+\left(2F_{22}-F_{33}-D_{11}+4F_{570}+\frac{32}{15}F_{571}+F_{572}-2A_{550}-\frac{8}{15}A_{551}-A_{552}\right.\\
&\left.-2F_{550}-\frac{32}{15}F_{551}-\frac{F_{552}}{4}-\frac{2}{5}F_{461}+\frac{2}{5}F_{681}\right)\frac{\partial^2 w}{\partial x_1^2}\\
&+\left(2D_{550}+\frac{4}{5}D_{551}+2F_{440}+\frac{4}{5}F_{441}-4F_{480}-\frac{8}{5}F_{481}\right)\frac{\partial^4 w}{\partial x_1^4}\\
&+\left(F_{33}-F_{22}+2F_{550}+\frac{32}{15}F_{551}+\frac{F_{552}}{4}-2F_{570}-\frac{16}{15}F_{571}-\frac{F_{572}}{2}+\frac{2}{5}F_{461}-\frac{2}{5}F_{681}\right)\frac{\partial \phi}{\partial x_1}\\
&+\left(2F_{480}+\frac{4}{5}F_{481}-2F_{440}-\frac{4}{5}F_{441}\right)\frac{\partial^3 \phi}{\partial x_1^3}-\frac{1}{2}\left(H_{22}-H_{33}\right)\Delta T\Bigg\}\Big|_{x_1=0,L}=0\\
&or\quad \delta\left(\frac{\partial w}{\partial x_1}\right)\Big|_{x_1=0,L}=0
\end{aligned}
\tag{58.d}
$$

$$
\begin{aligned}
&\Bigg\{\left(2B_{550}+\frac{4}{5}B_{551}-2F_{470}-\frac{4}{5}F_{471}\right)\frac{\partial^2 u}{\partial x_1^2}\\
&+\left(4F_{480}+\frac{8}{5}F_{481}-2F_{440}-\frac{4}{5}F_{441}-2D_{550}-\frac{4}{5}D_{551}\right)\frac{\partial^3 w}{\partial x_1^3}\\
&+\frac{2}{5}\left(F_{461}-F_{681}\right)\frac{\partial w}{\partial x_1}+\frac{2}{5}\left(F_{681}-F_{461}\right)\phi\\
&+\left(2F_{440}+\frac{4}{5}F_{441}-2F_{480}-\frac{4}{5}F_{481}\right)\frac{\partial^2 \phi}{\partial x_1^2}\Bigg\}\Big|_{x_1=0,L}=0\quad or\quad \delta\left(\frac{\partial^2 w}{\partial x_1^2}\right)\Big|_{x_1=0,L}=0
\end{aligned}
\tag{58.e}
$$

$$
\begin{aligned}
&\Bigg\{F_{11}\frac{\partial u}{\partial x_1}-\left(2F_{470}+\frac{4}{5}F_{471}\right)\frac{\partial^3 u}{\partial x_1^3}\\
&+\left(F_{33}-F_{22}+2F_{550}+\frac{32}{15}F_{551}+\frac{F_{552}}{4}-2F_{570}-\frac{16}{15}F_{571}-\frac{F_{572}}{2}+\frac{2}{5}F_{461}\right)\frac{\partial^2 w}{\partial x_1^2}\\
&+\left(2F_{480}+\frac{4}{5}F_{481}-2F_{440}-\frac{4}{5}F_{441}\right)\frac{\partial^4 w}{\partial x_1^4}-\left(F_{33}+2F_{550}+\frac{32}{15}F_{551}+\frac{F_{552}}{4}+\frac{2}{5}F_{461}\right)\frac{\partial \phi}{\partial x_1}\\
&+\left(2F_{440}+\frac{4}{5}F_{441}\right)\frac{\partial^3 \phi}{\partial x_1^3}-\frac{1}{2}H_{33}\Delta T\Bigg\}\Big|_{x_1=0,L}=0\quad or\quad \delta\phi\Big|_{x_1=0,L}=0
\end{aligned}
\tag{58.f}
$$

$$
\begin{aligned}
&\Bigg\{\left(2F_{470}+\frac{4}{5}F_{471}\right)\frac{\partial^2 u}{\partial x_1^2}+\left(2F_{440}+\frac{4}{5}F_{441}-2F_{480}-\frac{4}{5}F_{481}\right)\frac{\partial^3 w}{\partial x_1^3}-\frac{2}{5}F_{461}\frac{\partial w}{\partial x_1}\\
&+\frac{2}{5}F_{461}\phi-\left(2F_{440}+\frac{4}{5}F_{441}\right)\frac{\partial^2 \phi}{\partial x_1^2}\Bigg\}\Big|_{x_1=0,L}=0\quad or\quad \delta\left(\frac{\partial \phi}{\partial x_1}\right)\Big|_{x_1=0,L}=0
\end{aligned}
\tag{58.g}
$$

Equations (58.a), (58.c), (58.d) and (58.f) represent the classical boundary conditions while the non – classical boundary conditions are expressed in equations (58.b), (58.e) and (58.g). In each of the boundary conditions given in equations (58.a) - (58.g), the first expression is a natural type boundary condition while the second is an essential type.

To obtain the normalized governing equations of motion and boundary conditions the following dimensionless quantities are defined

$$
\begin{aligned}
&\xi=\frac{x_1}{L},\ \eta=\frac{L}{h},\ \{\tilde{u},\ \tilde{w}\}=\frac{\{u,\ w\}}{h},\ \varphi=\phi,\ q_0=qL.\\
&\left\{\bar{I}_1,\bar{I}_2,\bar{I}_3,\bar{I}_4,\bar{I}_5,\bar{I}_6\right\}=\left\{\frac{I_1}{I_{10}},\frac{I_2}{I_{10}h},\frac{I_3}{I_{10}h^2},\frac{I_4}{I_{10}h},\frac{I_5}{I_{10}h^2},\frac{I_6}{I_{10}h^2}\right\},\\
&\{a_{11},\ b_{11},\ d_{11},\ f_{11},\ f_{22},\ f_{33}\}=\left\{\frac{A_{11}}{A_{110}},\frac{B_{11}}{A_{110}h},\frac{D_{11}}{A_{110}h^2},\frac{F_{11}}{A_{110}h},\frac{F_{22}}{A_{110}h^2},\frac{F_{33}}{A_{110}h^2}\right\},\\
&\{h_{11},\ h_{22},\ h_{33}\}=\left\{\frac{H_{11}}{A_{110}},\ \frac{H_{22}}{A_{110}h},\ \frac{H_{33}}{A_{110}h}\right\}\\
&\{a_{55},a_{550},a_{551},a_{552},b_{55},b_{550},b_{551},d_{55},d_{550},d_{551},d_{552},f_{44},f_{440},f_{441},\\
&f_{46},f_{461},f_{47},f_{470},f_{471},f_{48},f_{480},f_{481},f_{55},f_{550},f_{551},f_{552},\\
&f_{57},f_{570},f_{571},f_{572},f_{66},f_{661},f_{662},f_{67},f_{671},f_{68},f_{681}\}\\
&=\left\{\frac{A_{55}}{A_{110}},\frac{A_{550}}{A_{110}h^2},\frac{A_{551}}{A_{110}h^2},\frac{A_{552}}{A_{110}h^2},\frac{B_{55}}{A_{110}h},\frac{B_{550}}{A_{110}h^3},\frac{B_{551}}{A_{110}h^3},\right.\\
&\frac{D_{55}}{A_{110}h^2},\frac{D_{550}}{A_{110}h^4},\frac{D_{551}}{A_{110}h^4},\frac{D_{552}}{A_{110}h^4},\frac{F_{44}}{A_{110}h^2},\frac{F_{440}}{A_{110}h^4},\frac{F_{441}}{A_{110}h^4},\\
&\frac{F_{46}}{A_{110}},\frac{F_{461}}{A_{110}h^2},\frac{F_{47}}{A_{110}h},\frac{F_{470}}{A_{110}h^3},\frac{F_{471}}{A_{110}h^3},\frac{F_{48}}{A_{110}h^2},\frac{F_{480}}{A_{110}h^4},\frac{F_{481}}{A_{110}h^4},\\
&\frac{F_{55}}{A_{110}},\frac{F_{550}}{A_{110}h^2},\frac{F_{551}}{A_{110}h^2},\frac{F_{552}}{A_{110}h^2},\frac{F_{57}}{A_{110}},\frac{F_{570}}{A_{110}h^2},\frac{F_{571}}{A_{110}h^2},\frac{F_{572}}{A_{110}h^2},\\
&\left.\frac{h^2F_{66}}{A_{110}},\frac{h^2F_{661}}{A_{110}h^2},\frac{h^2F_{662}}{A_{110}h^2},\frac{hF_{67}}{A_{110}},\frac{hF_{671}}{A_{110}h^2},\frac{F_{68}}{A_{110}},\frac{F_{681}}{A_{110}h^2}\right\},\\
&\tau=\frac{t}{L}\sqrt{\frac{A_{110}}{I_{10}}}
\end{aligned}
\tag{59}
$$

A_{110} and I_{10} in these equations are respectively reference values of A_{11} and I_1 evaluated by considering a homogeneous beam, whose properties are the same as those of the graded beam computed at $x_3 = -h/2$. By substituting the above relations into equations (57.a) - (57.c) the following normalized equations are obtained

$$
\begin{aligned}
&a_{11}\frac{\partial^2\tilde{u}}{\partial\xi^2}-\frac{1}{\eta^2}\left(2a_{550}+\frac{4}{5}a_{551}\right)\frac{\partial^4\tilde{u}}{\partial\xi^4}+\frac{1}{\eta}\left(f_{11}+\frac{2}{5}f_{671}-b_{11}\right)\frac{\partial^3\tilde{w}}{\partial\xi^3}\\
&+\frac{1}{\eta^3}\left(2b_{550}+\frac{4}{5}b_{551}-2f_{470}-\frac{4}{5}f_{471}\right)\frac{\partial^5\tilde{w}}{\partial\xi^5}-\left(f_{11}+\frac{2}{5}f_{671}\right)\frac{\partial^2\varphi}{\partial\xi^2}\\
&+\frac{1}{\eta^2}\left(2f_{470}+\frac{4}{5}f_{471}\right)\frac{\partial^4\varphi}{\partial\xi^4}=\bar{I}_1\frac{\partial^2\tilde{u}}{\partial\tau^2}+\frac{1}{\eta}\left(\bar{I}_4-\bar{I}_2\right)\frac{\partial^3\tilde{w}}{\partial\xi\partial\tau^2}-\bar{I}_4\frac{\partial^2\varphi}{\partial\tau^2}
\end{aligned}
\qquad (60.a)
$$

$$
\begin{aligned}
&\frac{1}{\eta}\left(b_{11}-f_{11}+\frac{2}{5}f_{671}\right)\frac{\partial^3\tilde{u}}{\partial\xi^3}+\frac{1}{\eta^3}\left(2f_{470}+\frac{4}{5}f_{471}-2b_{550}-\frac{4}{5}b_{551}\right)\frac{\partial^5\tilde{u}}{\partial\xi^5}\\
&+\left(k_s f_{55}+\frac{8}{15}f_{661}+\frac{f_{662}}{4}-h_{11}\Delta T\right)\frac{\partial^2\tilde{w}}{\partial\xi^2}\\
&+\frac{1}{\eta^2}\left(2f_{22}+\frac{4}{5}f_{681}-f_{33}-d_{11}+4f_{570}+\frac{32}{15}f_{571}+f_{572}\right.\\
&\left.-2f_{550}-\frac{32}{15}f_{551}-\frac{f_{552}}{4}-2a_{550}-\frac{8}{15}a_{551}-a_{552}-\frac{4}{5}f_{461}\right)\frac{\partial^4\tilde{w}}{\partial\xi^4}\\
&+\frac{1}{\eta^4}\left(2f_{440}+\frac{4}{5}f_{441}+2d_{550}+\frac{4}{5}d_{551}-4f_{480}-\frac{8}{5}f_{481}\right)\frac{\partial^6\tilde{w}}{\partial\xi^6}\\
&-\eta\left(k_s f_{55}+\frac{8}{15}f_{661}+\frac{f_{662}}{4}\right)\frac{\partial\varphi}{\partial\xi}+\frac{1}{\eta}\left(f_{33}-f_{22}+\frac{4}{5}f_{461}-\frac{2}{5}f_{681}\right.\\
&\left.+2f_{550}+\frac{32}{15}f_{551}+\frac{f_{552}}{4}-2f_{570}-\frac{16}{15}f_{571}-\frac{f_{572}}{2}\right)\frac{\partial^3\varphi}{\partial\xi^3}\\
&+\frac{1}{\eta^3}\left(2f_{480}+\frac{4}{5}f_{481}-2f_{440}-\frac{4}{5}f_{441}\right)\frac{\partial^5\varphi}{\partial\xi^5}+\eta q_0\\
&=\frac{1}{\eta}\left(\bar{I}_2-\bar{I}_4\right)\frac{\partial^3\tilde{u}}{\partial\xi\partial\tau^2}+\bar{I}_1\frac{\partial^2\tilde{w}}{\partial\tau^2}+\frac{1}{\eta^2}\left(2\bar{I}_5-\bar{I}_3-\bar{I}_6\right)\frac{\partial^4\tilde{w}}{\partial\xi^2\partial\tau^2}+\frac{1}{\eta}\left(\bar{I}_6-\bar{I}_5\right)\frac{\partial^3\varphi}{\partial\xi\partial\tau^2}
\end{aligned}
\qquad (60.b)
$$

$$
\begin{aligned}
&-\left(f_{11}+\frac{2}{5}f_{671}\right)\frac{\partial^2\tilde{u}}{\partial\xi^2}+\frac{1}{\eta^2}\left(2f_{470}+\frac{4}{5}f_{471}\right)\frac{\partial^4\tilde{u}}{\partial\xi^4}\\
&+\eta\left(k_s f_{55}+\frac{8}{15}f_{661}+\frac{f_{662}}{4}\right)\frac{\partial\tilde{w}}{\partial\xi}\\
&+\frac{1}{\eta}\left(f_{22}-f_{33}+\frac{2}{5}f_{681}-\frac{4}{5}f_{461}+2f_{570}+\frac{16}{15}f_{571}+\frac{f_{572}}{2}-2f_{550}-\frac{32}{15}f_{551}-\frac{f_{552}}{4}\right)\frac{\partial^3\tilde{w}}{\partial\xi^3}\\
&+\frac{1}{\eta^3}\left(2f_{440}+\frac{4}{5}f_{441}-2f_{480}-\frac{4}{5}f_{481}\right)\frac{\partial^5\tilde{w}}{\partial\xi^5}-\left(k_s f_{55}+\frac{8}{15}f_{661}+\frac{f_{662}}{4}\right)\varphi\\
&+\left(f_{33}+2f_{550}+\frac{32}{15}f_{551}+\frac{f_{552}}{4}+\frac{4}{5}f_{461}\right)\frac{\partial^2\varphi}{\partial\xi^2}-\frac{1}{\eta^2}\left(2f_{440}+\frac{4}{5}f_{441}\right)\frac{\partial^4\varphi}{\partial\xi^4}\\
&=-\bar{I}_4\frac{\partial^2\tilde{u}}{\partial\tau^2}+\frac{1}{\eta}\left(\bar{I}_5-\bar{I}_6\right)\frac{\partial^3\tilde{w}}{\partial\xi\partial\tau^2}+\bar{I}_6\frac{\partial^2\varphi}{\partial\tau^2}
\end{aligned}
\qquad (60.c)
$$

The boundary conditions are derived by placing relations (59) into equations (58.a) - (58.g)

$$\left\{-a_{11}\frac{\partial\tilde{u}}{\partial\xi}+\frac{1}{\eta^2}\left(2a_{550}+\frac{4}{5}a_{551}\right)\frac{\partial^3\tilde{u}}{\partial\xi^3}\right.$$
$$+\frac{1}{\eta}\left(b_{11}-f_{11}+\frac{2}{5}f_{671}\right)\frac{\partial^2\tilde{w}}{\partial\xi^2}+\frac{1}{\eta^3}\left(2f_{470}+\frac{4}{5}f_{471}-2b_{550}-\frac{4}{5}b_{551}\right)\frac{\partial^4\tilde{w}}{\partial\xi^4} \tag{61.a}$$
$$\left.+\left(f_{11}+\frac{2}{5}f_{671}\right)\frac{\partial\varphi}{\partial\xi}-\frac{1}{\eta^2}\left(2f_{470}+\frac{4}{5}f_{471}\right)\frac{\partial^3\varphi}{\partial\xi^3}+\frac{1}{2}\eta h_{11}\Delta T\right\}\Big|_{\xi=0,1}=0 \quad or \quad \delta\tilde{u}\,\Big|_{\xi=0,1}=0$$

$$\left\{\left(2a_{550}+\frac{4}{5}a_{551}\right)\frac{\partial^2\tilde{u}}{\partial\xi^2}+\frac{1}{\eta}\left(2f_{470}+\frac{4}{5}f_{471}-2b_{550}-\frac{4}{5}b_{551}\right)\frac{\partial^3\tilde{w}}{\partial\xi^3}\right.$$
$$\left.-\frac{2}{5}\eta f_{671}\frac{\partial\tilde{w}}{\partial\xi}+\frac{2}{5}\eta^2 f_{671}\varphi-\left(2f_{470}+\frac{4}{5}f_{471}\right)\frac{\partial^2\varphi}{\partial\xi^2}\right\}\Big|_{\xi=0,1}=0 \quad or \quad \delta\left(\frac{\partial\tilde{u}}{\partial\xi}\right)\Big|_{\xi=0,1}=0 \tag{61.b}$$

$$\left\{b_{11}\frac{\partial^2\tilde{u}}{\partial\xi^2}-\frac{1}{\eta^2}\left(2b_{550}+\frac{4}{5}b_{551}\right)\frac{\partial^4\tilde{u}}{\partial\xi^4}-\eta h_{11}\Delta T\frac{\partial\tilde{w}}{\partial\xi}\right.$$
$$+\frac{1}{\eta}\left(f_{22}-d_{11}+2f_{570}+\frac{16}{15}f_{571}+\frac{f_{572}}{2}-2a_{550}-\frac{8}{15}a_{551}-a_{552}+\frac{2}{5}f_{681}\right)\frac{\partial^3\tilde{w}}{\partial\xi^3}$$
$$+\frac{1}{\eta^3}\left(2d_{550}+\frac{4}{5}d_{551}-2f_{480}-\frac{4}{5}f_{481}\right)\frac{\partial^5\tilde{w}}{\partial\xi^5} \tag{61.c}$$
$$-\left(f_{22}+2f_{570}+\frac{16}{15}f_{571}+\frac{f_{572}}{2}+\frac{2}{5}f_{681}\right)\frac{\partial^2\varphi}{\partial\xi^2}+\frac{1}{\eta^2}\left(2f_{480}+\frac{4}{5}f_{481}\right)\frac{\partial^4\varphi}{\partial\xi^4}$$
$$\left.=\bar{I}_2\frac{\partial^2\tilde{u}}{\partial\tau^2}+\frac{1}{\eta}\left(\bar{I}_5-\bar{I}_3\right)\frac{\partial^3\tilde{w}}{\partial\xi\partial\tau^2}-\bar{I}_5\frac{\partial^2\varphi}{\partial\tau^2}\right\}\Big|_{\xi=0,1}=0 \quad or \quad \delta\tilde{w}\,\Big|_{\xi=0,1}=0$$

$$\left\{ \left(b_{11}-f_{11}\right)\frac{\partial\tilde{u}}{\partial\xi}+\frac{1}{\eta^2}\left(2f_{470}+\frac{4}{5}f_{471}-2b_{550}-\frac{4}{5}b_{551}\right)\frac{\partial^3\tilde{u}}{\partial\xi^3}\right.$$
$$+\frac{1}{\eta}\left(2f_{22}-f_{33}-d_{11}+4f_{570}+\frac{32}{15}f_{571}+f_{572}-2a_{550}-\frac{8}{15}a_{551}-a_{552}\right.$$
$$\left.-2f_{550}-\frac{32}{15}f_{551}-\frac{f_{552}}{4}-\frac{2}{5}f_{461}+\frac{2}{5}f_{681}\right)\frac{\partial^2\tilde{w}}{\partial\xi^2}$$
$$+\left(2d_{550}+\frac{4}{5}d_{551}+2f_{440}+\frac{4}{5}f_{441}-4f_{480}-\frac{8}{5}f_{481}\right)\frac{1}{\eta^3}\frac{\partial^4\tilde{w}}{\partial\xi^4} \tag{61.d}$$
$$+\left(f_{33}-f_{22}+2f_{550}+\frac{32}{15}f_{551}+\frac{f_{552}}{4}-2f_{570}-\frac{16}{15}f_{571}-\frac{f_{572}}{2}+\frac{2}{5}f_{461}-\frac{2}{5}f_{681}\right)\frac{\partial\varphi}{\partial\xi}$$
$$\left.+\frac{1}{\eta^2}\left(2f_{480}+\frac{4}{5}f_{481}-2f_{440}-\frac{4}{5}f_{441}\right)\frac{\partial^3\varphi}{\partial\xi^3}-\frac{1}{2}\eta\left(h_{22}-h_{33}\right)\Delta T\right\}\Big|_{\xi=0,1}=0$$
$$or\quad \delta\left(\frac{\partial\tilde{w}}{\partial\xi}\right)\Big|_{\xi=0,1}=0$$

$$\left\{\left(2b_{550}+\frac{4}{5}b_{551}-2f_{470}-\frac{4}{5}f_{471}\right)\frac{\partial^2\tilde{u}}{\partial\xi^2}\right.$$
$$+\frac{1}{\eta}\left(4f_{480}+\frac{8}{5}f_{481}-2f_{440}-\frac{4}{5}f_{441}-2d_{550}-\frac{4}{5}d_{551}\right)\frac{\partial^3\tilde{w}}{\partial\xi^3}$$
$$+\frac{2}{5}\eta\left(f_{461}-f_{681}\right)\frac{\partial\tilde{w}}{\partial\xi}+\frac{2}{5}\eta^2\left(f_{681}-f_{461}\right)\varphi \tag{61.e}$$
$$\left.+\left(2f_{440}+\frac{4}{5}f_{441}-2f_{480}-\frac{4}{5}f_{481}\right)\frac{\partial^2\varphi}{\partial\xi^2}\right\}\Big|_{\xi=0,1}=0\quad or\quad \delta\left(\frac{\partial^2\tilde{w}}{\partial\xi^2}\right)\Big|_{\xi=0,1}=0$$

$$\left\{f_{11}\frac{\partial\tilde{u}}{\partial\xi}-\frac{1}{\eta^2}\left(2f_{470}+\frac{4}{5}f_{471}\right)\frac{\partial^3\tilde{u}}{\partial\xi^3}\right.$$
$$+\frac{1}{\eta}\left(f_{33}-f_{22}+2f_{550}+\frac{32}{15}f_{551}+\frac{f_{552}}{4}-2f_{570}-\frac{16}{15}f_{571}-\frac{f_{572}}{2}+\frac{2}{5}f_{461}\right)\frac{\partial^2\tilde{w}}{\partial\xi^2}$$
$$+\frac{1}{\eta^3}\left(2f_{480}+\frac{4}{5}f_{481}-2f_{440}-\frac{4}{5}f_{441}\right)\frac{\partial^4\tilde{w}}{\partial\xi^4} \tag{61.f}$$
$$-\left(f_{33}+2f_{550}+\frac{32}{15}f_{551}+\frac{f_{552}}{4}+\frac{2}{5}f_{461}\right)\frac{\partial\varphi}{\partial\xi}$$
$$\left.+\frac{1}{\eta^2}\left(2f_{440}+\frac{4}{5}f_{441}\right)\frac{\partial^3\varphi}{\partial\xi^3}-\frac{1}{2}\eta h_{33}\Delta T\right\}\Big|_{\xi=0,1}=0\quad or\quad \delta\varphi\Big|_{\xi=0,1}=0$$

$$\left\{\left(2f_{470}+\frac{4}{5}f_{471}\right)\frac{\partial^2\tilde{u}}{\partial\xi^2}+\frac{1}{\eta}\left(2f_{440}+\frac{4}{5}f_{441}-2f_{480}-\frac{4}{5}f_{481}\right)\frac{\partial^3\tilde{w}}{\partial\xi^3}-\frac{2}{5}\eta f_{461}\frac{\partial\tilde{w}}{\partial\xi}\right.$$
$$\left.+\frac{2}{5}\eta^2 f_{461}\varphi-\left(2f_{440}+\frac{4}{5}f_{441}\right)\frac{\partial^2\varphi}{\partial\xi^2}\right\}\Big|_{\xi=0,1}=0\quad or\quad \delta\left(\frac{\partial\varphi}{\partial\xi}\right)\Big|_{\xi=0,1}=0 \tag{61.g}$$

By using the normalized equations obtained above, dimensionless frequency, $\omega = \Omega L\sqrt{I_{10} / A_{110}}$, will be obtained, where Ω denotes the vibration frequency of the FGM micro - beam. In the case of static analysis, inertia related terms are not taken into account whereas in free vibration analysis the external loading q is equated to zero.

2.4 Modified Couple Stress Theory (MCST)

By neglecting the effects of the dilatation gradient and the deviatoric stretch gradient, equation (29) reduces to the strain energy based on the modified couple stress theory:

$$U = \frac{1}{2}\int_{\Omega}\left(\sigma_{ij}\varepsilon_{ij} + m_{ij}\chi_{ij}\right)dv \tag{62}$$

By letting l_0 and l_1 to be equal to zero in equations (60.a) - (60.c)and (61.a) - (61.g), the micro - beam formulation for MCST is obtained. Unlike the conventional theories of continuum mechanics, in general higher order theories the material particle is encased with a representative volume element. In conventional theories the material particle is considered as a geometrical point thus only possesses the characteristic translation, but in higher order theories it can undergo rotations and deformations. The equilibrium relations in higher order continuum theories not only involve the conventional force and moment equilibriums but also the equilibrium of moment of couple is considered. In the other words, the couple vector is not a free vector. Fig. 2 shows the equivalence of couple and the moment of couple [11]. The couple vector L_A at point A can be represented by a couple $L_A^{'}$ and a couple of couples $M_A^{'}$ applied to point B.

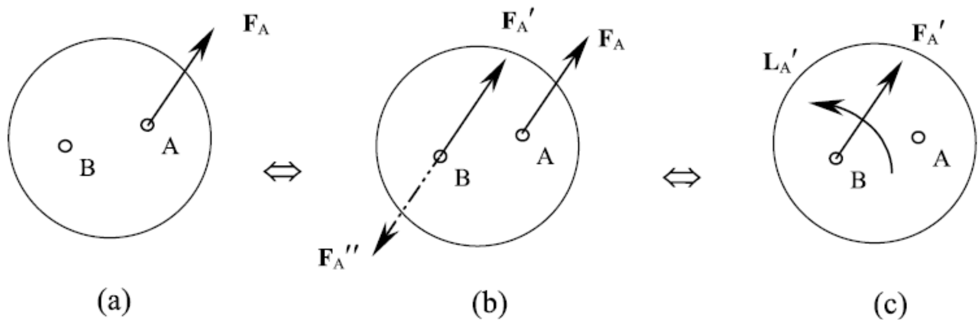

Fig. 2. Equivalence of couple: a couple at point A is equivalent to a couple at point B and a moment of the couple [11].

Yang et al. [11] used the above hypothesizes to obtain the relations in modified couple stress theory. They experimentally showed the importance of small - scale parameter as the sizes become smaller. They also showed how to obtain the value of length scale parameter for a linear isotropic material by twisting slim cylinders of different diameters.

By adding the moment of couple to the equilibrium relations it is observed that the couple stress tensor becomes symmetric and the deformation energy becomes independent of antisymmetric part of the curvature tensor.

In the following chapters a numerical method will be used to solve the equations of higher order continuum in conjunction with newly developed micro - beam theory and classical beam theories. The mechanical characteristics of the micro - beam will be investigated using these theories.

CHAPTER 3

NUMERICAL SOLUTION

In this chapter numerical solution method is described. Differential quadrature method is used to solve the governing differential equations and boundary conditions. A computer program is developed using MATLAB to implement the numerical solution technique.

3.1 Differential Quadrature Method (DQM)

Since analytical solutions are unavailable for complex problems, use of a suitable numerical method in the solution of the system of equations is indispensable. Differential quadrature method (DQM) is a powerful means to solve such problems. This technique is an easy, fast, accurate and applicable method and has some advantages upon other numerical methods. Finite element method (FEM) requires large number of grid points and thus needs more time and memory to solve the problem. In some of the work related to micro - and nano - beams Navier solution is used to solve the simply supported beam problems. The drawback of this technique is that it is restricted to simply supported beams and can not be used for other types of boundary conditions.

Bellman and Casti [44] in 1971 proposed DQM to evaluate the derivatives of a sufficiently smooth function. In this method, weighted sum of function values at nodes are used to approximate the derivatives of the function, i.e. a derivative is written as:

$$u_x^{(m)}(x_i,t)=\sum_{j=1}^{N}c_{ij}^{(m)}u(x_j,t),\quad \text{for } i = 1, 2, ..., N \qquad (63)$$

where $u(x_j,t)$ or u_j is the function value at jth node, $u_x^{(m)}$ is the mth derivative of function u with respect to x, $c_{ij}^{(m)}$ are the weighting coefficients for mth derivative and N is the number of grid points or nodes. To determine the weighting coefficients with no limitation on the choice of grid points, Shu [45] chose Lagrange interpolated polynomial as the set of test functions and obtained the following recursive formula

$$
\begin{aligned}
&c_{ij}^{(1)} = \frac{M^{(1)}\left(x_i\right)}{\left(x_i - x_j\right)M^{(1)}\left(x_j\right)}, \quad \text{for } i \neq j \\
&c_{ii}^{(1)} = \frac{M^{(2)}\left(x_i\right)}{2M^{(1)}\left(x_i\right)}, \quad \text{for } i = j \\
&c_{ij}^{(m)} = m\left(c_{ii}^{(m-1)}c_{ij} - \frac{c_{ij}^{(m-1)}}{x_i - x_j}\right), \quad \text{for } i \neq j, \quad m = 2, 3, ..., N, \quad i,\ j = 1, 2, ..., N \\
&c_{ii}^{(m)} = -\sum_{j=1,\, j\neq i}^{N} c_{ij}^{(m)}, \quad \text{for } i = 1, 2, ..., N
\end{aligned}
\tag{64}
$$

3.2 Numerical Solution for Simply Supported Micro - beam

In this section DQM is used to examine static response and free vibration behavior of simply supported functionally graded micro – beams. The beam geometry is shown in Fig. 1. The bottom surface (x_3 = -$h/2$) is pure metal and the top surface (x_3 = $h/2$) is pure ceramic. In free vibration analysis q is taken as zero.

The Chebyshev nodes which are more stable in comparison with uniform grids [46] are used as the sampling points of DQM as follows

$$
x_j = \frac{1}{2}\left\{1 - cos\left(\frac{\pi\left(j-1\right)}{N-1}\right)\right\}, \text{ for } j = 1, 2, ..., N \tag{65}
$$

For a simply supported micro - beam the boundary conditions of (61.a) - (61.g) can be written as

$$
\begin{aligned}
&\tilde{u}\,|_{\xi=0} = \left\{-a_{11}\frac{\partial\tilde{u}}{\partial\xi} + \frac{1}{\eta^2}\left(2a_{550} + \frac{4}{5}a_{551}\right)\frac{\partial^3\tilde{u}}{\partial\xi^3}\right. \\
&+\frac{1}{\eta}\left(b_{11} - f_{11} + \frac{2}{5}f_{671}\right)\frac{\partial^2\tilde{w}}{\partial\xi^2} + \frac{1}{\eta^3}\left(2f_{470} + \frac{4}{5}f_{471} - 2b_{550} - \frac{4}{5}b_{551}\right)\frac{\partial^4\tilde{w}}{\partial\xi^4} \\
&\left.+\left(f_{11} + \frac{2}{5}f_{671}\right)\frac{\partial\varphi}{\partial\xi} - \frac{1}{\eta^2}\left(2f_{470} + \frac{4}{5}f_{471}\right)\frac{\partial^3\varphi}{\partial\xi^3} + \frac{1}{2}\eta h_{11}\Delta T\right\}|_{\xi=1} = 0
\end{aligned}
\tag{66.a}
$$

$$
\left(\frac{\partial\tilde{u}}{\partial\xi}\right)|_{\xi=0,1} = 0 \tag{66.b}
$$

$$
\tilde{w}\,|_{\xi=0,1} = 0 \tag{66.c}
$$

$$\left\{ \left(b_{11}-f_{11}\right)\frac{\partial \tilde{u}}{\partial \xi}+\frac{1}{\eta^{2}}\left(2f_{470}+\frac{4}{5}f_{471}-2b_{550}-\frac{4}{5}b_{551}\right)\frac{\partial^{3}\tilde{u}}{\partial \xi^{3}} \right.$$
$$+\frac{1}{\eta}\left(2f_{22}-f_{33}-d_{11}+4f_{570}+\frac{32}{15}f_{571}+f_{572}-2a_{550}-\frac{8}{15}a_{551}-a_{552}\right.$$
$$\left.-2f_{550}-\frac{32}{15}f_{551}-\frac{f_{552}}{4}-\frac{2}{5}f_{461}+\frac{2}{5}f_{681}\right)\frac{\partial^{2}\tilde{w}}{\partial \xi^{2}}$$
$$+\frac{1}{\eta^{3}}\left(2d_{550}+\frac{4}{5}d_{551}+2f_{440}+\frac{4}{5}f_{441}-4f_{480}-\frac{8}{5}f_{481}\right)\frac{\partial^{4}\tilde{w}}{\partial \xi^{4}} \tag{66.d}$$
$$+\left(f_{33}-f_{22}+2f_{550}+\frac{32}{15}f_{551}+\frac{f_{552}}{4}-2f_{570}-\frac{16}{15}f_{571}-\frac{f_{572}}{2}+\frac{2}{5}f_{461}-\frac{2}{5}f_{681}\right)\frac{\partial \varphi}{\partial \xi}$$
$$\left.+\frac{1}{\eta^{2}}\left(2f_{480}+\frac{4}{5}f_{481}-2f_{440}-\frac{4}{5}f_{441}\right)\frac{\partial^{3}\varphi}{\partial \xi^{3}}-\frac{1}{2}\eta\left(h_{22}-h_{33}\right)\Delta T\right\}\Big|_{\xi=0,1}=0$$

$$\left(\frac{\partial^{2}\tilde{w}}{\partial \xi^{2}}\right)\Big|_{\xi=0,1}=0 \tag{66.e}$$

$$\left\{ f_{11}\frac{\partial \tilde{u}}{\partial \xi}-\frac{1}{\eta^{2}}\left(2f_{470}+\frac{4}{5}f_{471}\right)\frac{\partial^{3}\tilde{u}}{\partial \xi^{3}} \right.$$
$$+\frac{1}{\eta}\left(f_{33}-f_{22}+2f_{550}+\frac{32}{15}f_{551}+\frac{f_{552}}{4}-2f_{570}-\frac{16}{15}f_{571}-\frac{f_{572}}{2}+\frac{2}{5}f_{461}\right)\frac{\partial^{2}\tilde{w}}{\partial \xi^{2}}$$
$$+\frac{1}{\eta^{3}}\left(2f_{480}+\frac{4}{5}f_{481}-2f_{440}-\frac{4}{5}f_{441}\right)\frac{\partial^{4}\tilde{w}}{\partial \xi^{4}} \tag{66.f}$$
$$-\left(f_{33}+2f_{550}+\frac{32}{15}f_{551}+\frac{f_{552}}{4}+\frac{2}{5}f_{461}\right)\frac{\partial \varphi}{\partial \xi}$$
$$\left.+\frac{1}{\eta^{2}}\left(2f_{440}+\frac{4}{5}f_{441}\right)\frac{\partial^{3}\varphi}{\partial \xi^{3}}-\frac{1}{2}\eta h_{33}\Delta T\right\}\Big|_{\xi=0,1}=0$$

$$\left(\frac{\partial \varphi}{\partial \xi}\right)\Big|_{\xi=0,1}=0 \tag{66.g}$$

By dropping the inertia terms and denoting the static displacements by $\tilde{u}_b, \tilde{w}_b$ and φ_b the nondimensional displacements of the beam under steady state temperature field and static loading are obtained by solving the following static bending problem

$$a_{11}\frac{\partial^{2}\tilde{u}_b}{\partial \xi^{2}}-\frac{1}{\eta^{2}}\left(2a_{550}+\frac{4}{5}a_{551}\right)\frac{\partial^{4}\tilde{u}_b}{\partial \xi^{4}}+\frac{1}{\eta}\left(f_{11}+\frac{2}{5}f_{671}-b_{11}\right)\frac{\partial^{3}\tilde{w}_b}{\partial \xi^{3}}$$
$$+\frac{1}{\eta^{3}}\left(2b_{550}+\frac{4}{5}b_{551}-2f_{470}-\frac{4}{5}f_{471}\right)\frac{\partial^{5}\tilde{w}_b}{\partial \xi^{5}}-\left(f_{11}+\frac{2}{5}f_{671}\right)\frac{\partial^{2}\varphi_b}{\partial \xi^{2}} \tag{67.a}$$
$$+\frac{1}{\eta^{2}}\left(2f_{470}+\frac{4}{5}f_{471}\right)\frac{\partial^{4}\varphi_b}{\partial \xi^{4}}=0$$

$$\frac{1}{\eta}\left(b_{11}-f_{11}+\frac{2}{5}f_{671}\right)\frac{\partial^3\tilde{u}_b}{\partial\xi^3}+\frac{1}{\eta^3}\left(2f_{470}+\frac{4}{5}f_{471}-2b_{550}-\frac{4}{5}b_{551}\right)\frac{\partial^5\tilde{u}_b}{\partial\xi^5}$$
$$+\left(k_s f_{55}+\frac{8}{15}f_{661}+\frac{f_{662}}{4}-h_{11}\Delta T\right)\frac{\partial^2\tilde{w}_b}{\partial\xi^2}$$
$$+\frac{1}{\eta^2}\left(2f_{22}+\frac{4}{5}f_{681}-f_{33}-d_{11}+4f_{570}+\frac{32}{15}f_{571}+f_{572}\right.$$
$$\left.-2f_{550}-\frac{32}{15}f_{551}-\frac{f_{552}}{4}-2a_{550}-\frac{8}{15}a_{551}-a_{552}-\frac{4}{5}f_{461}\right)\frac{\partial^4\tilde{w}_b}{\partial\xi^4}$$
$$+\frac{1}{\eta^4}\left(2f_{440}+\frac{4}{5}f_{441}+2d_{550}+\frac{4}{5}d_{551}-4f_{480}-\frac{8}{5}f_{481}\right)\frac{\partial^6\tilde{w}_b}{\partial\xi^6}$$
$$-\eta\left(k_s f_{55}+\frac{8}{15}f_{661}+\frac{f_{662}}{4}\right)\frac{\partial\varphi_b}{\partial\xi}+\frac{1}{\eta}\left(f_{33}-f_{22}+\frac{4}{5}f_{461}-\frac{2}{5}f_{681}\right.$$
$$\left.+2f_{550}+\frac{32}{15}f_{551}+\frac{f_{552}}{4}-2f_{570}-\frac{16}{15}f_{571}-\frac{f_{572}}{2}\right)\frac{\partial^3\varphi_b}{\partial\xi^3}$$
$$+\frac{1}{\eta^3}\left(2f_{480}+\frac{4}{5}f_{481}-2f_{440}-\frac{4}{5}f_{441}\right)\frac{\partial^5\varphi_b}{\partial\xi^5}+\eta q_0=0 \qquad (67.b)$$

$$-\left(f_{11}+\frac{2}{5}f_{671}\right)\frac{\partial^2\tilde{u}_b}{\partial\xi^2}+\frac{1}{\eta^2}\left(2f_{470}+\frac{4}{5}f_{471}\right)\frac{\partial^4\tilde{u}_b}{\partial\xi^4}$$
$$+\eta\left(k_s f_{55}+\frac{8}{15}f_{661}+\frac{f_{662}}{4}\right)\frac{\partial\tilde{w}_b}{\partial\xi}+\frac{1}{\eta}\left(f_{22}-f_{33}+\frac{2}{5}f_{681}-\frac{4}{5}f_{461}\right.$$
$$\left.+2f_{570}+\frac{16}{15}f_{571}+\frac{f_{572}}{2}-2f_{550}-\frac{32}{15}f_{551}-\frac{f_{552}}{4}\right)\frac{\partial^3\tilde{w}_b}{\partial\xi^3} \qquad (67.c)$$
$$+\frac{1}{\eta^3}\left(2f_{440}+\frac{4}{5}f_{441}-2f_{480}-\frac{4}{5}f_{481}\right)\frac{\partial^5\tilde{w}_b}{\partial\xi^5}-\left(k_s f_{55}+\frac{8}{15}f_{661}+\frac{f_{662}}{4}\right)\varphi_b$$
$$+\left(f_{33}+2f_{550}+\frac{32}{15}f_{551}+\frac{f_{552}}{4}+\frac{4}{5}f_{461}\right)\frac{\partial^2\varphi_b}{\partial\xi^2}-\frac{1}{\eta^2}\left(2f_{440}+\frac{4}{5}f_{441}\right)\frac{\partial^4\varphi_b}{\partial\xi^4}=0$$

The boundary conditions in the static analysis remain the same as in equations (66.a) - (66.g). The total displacements are the sum of static displacements and the additional dynamic displacements which are denoted by $\tilde{u}_d$, $\tilde{w}_d$ and φ_d, that is [47]

$$\tilde{u}=\tilde{u}_b+\tilde{u}_d,\ \tilde{w}=\tilde{w}_b+\tilde{w}_d,\ \varphi=\varphi_b+\varphi_d \qquad (68)$$

By substituting equation (68) into equations (60.a) - (60.c) and boundary conditions in equations (66.a) - (66.g) and noting that $\tilde{u}_b$, $\tilde{w}_b$ and φ_b satisfy the static problem expressed in equations (67.a) - (67.c), the governing equations and boundary conditions in terms of the dynamic displacements $\tilde{u}_d$, $\tilde{w}_d$ and φ_d are obtained. The governing equations are the same as in equations (60.a) - (60.c) and the boundary conditions now become homogeneous for free vibration analysis as follows

$$\tilde{u}_d \rfloor_{\xi=0} = \Bigg\{ -a_{11}\frac{\partial \tilde{u}_d}{\partial \xi} + \frac{1}{\eta^2}\left(2a_{550} + \frac{4}{5}a_{551}\right)\frac{\partial^3 \tilde{u}_d}{\partial \xi^3}$$

$$+\frac{1}{\eta}\left(b_{11} - f_{11} + \frac{2}{5}f_{671}\right)\frac{\partial^2 \tilde{w}_d}{\partial \xi^2} + \frac{1}{\eta^3}\left(2f_{470} + \frac{4}{5}f_{471} - 2b_{550} - \frac{4}{5}b_{551}\right)\frac{\partial^4 \tilde{w}_d}{\partial \xi^4} \qquad (69.a)$$

$$+\left(f_{11} + \frac{2}{5}f_{671}\right)\frac{\partial \varphi_d}{\partial \xi} - \frac{1}{\eta^2}\left(2f_{470} + \frac{4}{5}f_{471}\right)\frac{\partial^3 \varphi_d}{\partial \xi^3}\Bigg\} \rfloor_{\xi=1} = 0$$

$$\left(\frac{\partial \tilde{u}_d}{\partial \xi}\right)\rfloor_{\xi=0,1} = 0 \qquad (69.b)$$

$$\tilde{w}_d \rfloor_{\xi=0,1} = 0 \qquad (69.c)$$

$$\Bigg\{\left(b_{11} - f_{11}\right)\frac{\partial \tilde{u}_d}{\partial \xi} + \frac{1}{\eta^2}\left(2f_{470} + \frac{4}{5}f_{471} - 2b_{550} - \frac{4}{5}b_{551}\right)\frac{\partial^3 \tilde{u}_d}{\partial \xi^3}$$

$$+\frac{1}{\eta}\Bigg(2f_{22} - f_{33} - d_{11} + 4f_{570} + \frac{32}{15}f_{571} + f_{572} - 2a_{550} - \frac{8}{15}a_{551} - a_{552}$$

$$-2f_{550} - \frac{32}{15}f_{551} - \frac{f_{552}}{4} - \frac{2}{5}f_{461} + \frac{2}{5}f_{681}\Bigg)\frac{\partial^2 \tilde{w}_d}{\partial \xi^2}$$

$$+\frac{1}{\eta^3}\left(2d_{550} + \frac{4}{5}d_{551} + 2f_{440} + \frac{4}{5}f_{441} - 4f_{480} - \frac{8}{5}f_{481}\right)\frac{\partial^4 \tilde{w}_d}{\partial \xi^4} \qquad (69.d)$$

$$+\left(f_{33} - f_{22} + 2f_{550} + \frac{32}{15}f_{551} + \frac{f_{552}}{4} - 2f_{570} - \frac{16}{15}f_{571} - \frac{f_{572}}{2} + \frac{2}{5}f_{461} - \frac{2}{5}f_{681}\right)\frac{\partial \varphi_d}{\partial \xi}$$

$$+\frac{1}{\eta^2}\left(2f_{480} + \frac{4}{5}f_{481} - 2f_{440} - \frac{4}{5}f_{441}\right)\frac{\partial^3 \varphi_d}{\partial \xi^3}\Bigg\}\rfloor_{\xi=0,1} = 0$$

$$\left(\frac{\partial^2 \tilde{w}_d}{\partial \xi^2}\right)\rfloor_{\xi=0,1} = 0 \qquad (69.e)$$

$$\Bigg\{f_{11}\frac{\partial \tilde{u}_d}{\partial \xi} - \frac{1}{\eta^2}\left(2f_{470} + \frac{4}{5}f_{471}\right)\frac{\partial^3 \tilde{u}_d}{\partial \xi^3}$$

$$+\frac{1}{\eta}\left(f_{33} - f_{22} + 2f_{550} + \frac{32}{15}f_{551} + \frac{f_{552}}{4} - 2f_{570} - \frac{16}{15}f_{571} - \frac{f_{572}}{2} + \frac{2}{5}f_{461}\right)\frac{\partial^2 \tilde{w}_d}{\partial \xi^2}$$

$$+\frac{1}{\eta^3}\left(2f_{480} + \frac{4}{5}f_{481} - 2f_{440} - \frac{4}{5}f_{441}\right)\frac{\partial^4 \tilde{w}_d}{\partial \xi^4} \qquad (69.f)$$

$$-\left(f_{33} + 2f_{550} + \frac{32}{15}f_{551} + \frac{f_{552}}{4} + \frac{2}{5}f_{461}\right)\frac{\partial \varphi_d}{\partial \xi}$$

$$+\frac{1}{\eta^2}\left(2f_{440} + \frac{4}{5}f_{441}\right)\frac{\partial^3 \varphi_d}{\partial \xi^3}\Bigg\}\rfloor_{\xi=0,1} = 0$$

$$\left(\frac{\partial \varphi_d}{\partial \xi}\right)\Big|_{\xi=0,1} = 0 \qquad (69.g)$$

By substituting finite series representation of the differential operators in equation (64) into equations (60.a) - (60.c) the governing equations of motion of simply supported FGM micro - beam are recast into the following form:

$$\begin{aligned}
&a_{11}\sum_{j=1}^{N} c_{ij}^{(2)}\tilde{u}_j - \frac{1}{\eta^2}\left(2a_{550} + \frac{4}{5}a_{551}\right)\sum_{j=1}^{N} c_{ij}^{(4)}\tilde{u}_j + \frac{1}{\eta}\left(f_{11} + \frac{2}{5}f_{671} - b_{11}\right)\sum_{j=1}^{N} c_{ij}^{(3)}\tilde{w}_j \\
&+\frac{1}{\eta^3}\left(2b_{550} + \frac{4}{5}b_{551} - 2f_{470} - \frac{4}{5}f_{471}\right)\sum_{j=1}^{N} c_{ij}^{(5)}\tilde{w}_j - \left(f_{11} + \frac{2}{5}f_{671}\right)\sum_{j=1}^{N} c_{ij}^{(2)}\varphi_j \qquad (70.a) \\
&+\frac{1}{\eta^2}\left(2f_{470} + \frac{4}{5}f_{471}\right)\sum_{j=1}^{N} c_{ij}^{(4)}\varphi_j = \bar{I}_1\ddot{\tilde{u}}_i + \frac{1}{\eta}\left(\bar{I}_4 - \bar{I}_2\right)\sum_{j=1}^{N} c_{ij}^{(1)}\ddot{\tilde{w}}_j - \bar{I}_4\ddot{\varphi}_i
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{\eta}\left(b_{11} - f_{11} + \frac{2}{5}f_{671}\right)\sum_{j=1}^{N} c_{ij}^{(3)}\tilde{u}_j + \frac{1}{\eta^3}\left(2f_{470} + \frac{4}{5}f_{471} - 2b_{550} - \frac{4}{5}b_{551}\right)\sum_{j=1}^{N} c_{ij}^{(5)}\tilde{u}_j \\
&+\left(k_s f_{55} + \frac{8}{15}f_{661} + \frac{f_{662}}{4} - h_{11}\Delta T\right)\sum_{j=1}^{N} c_{ij}^{(2)}\tilde{w}_j + \frac{1}{\eta^2}\left(2f_{22} + \frac{4}{5}f_{681} - f_{33} - d_{11} + 4f_{570}\right. \\
&\left.+\frac{32}{15}f_{571} + f_{572} - 2f_{550} - \frac{32}{15}f_{551} - \frac{f_{552}}{4} - 2a_{550} - \frac{8}{15}a_{551} - a_{552} - \frac{4}{5}f_{461}\right)\sum_{j=1}^{N} c_{ij}^{(4)}\tilde{w}_j \\
&+\frac{1}{\eta^4}\left(2f_{440} + \frac{4}{5}f_{441} + 2d_{550} + \frac{4}{5}d_{551} - 4f_{480} - \frac{8}{5}f_{481}\right)\sum_{j=1}^{N} c_{ij}^{(6)}\tilde{w}_j \\
&-\eta\left(k_s f_{55} + \frac{8}{15}f_{661} + \frac{f_{662}}{4}\right)\sum_{j=1}^{N} c_{ij}^{(1)}\varphi_j + \frac{1}{\eta}\left(f_{33} - f_{22} + \frac{4}{5}f_{461} - \frac{2}{5}f_{681}\right. \\
&\left.+2f_{550} + \frac{32}{15}f_{551} + \frac{f_{552}}{4} - 2f_{570} - \frac{16}{15}f_{571} - \frac{f_{572}}{2}\right)\sum_{j=1}^{N} c_{ij}^{(3)}\varphi_j \\
&+\frac{1}{\eta^3}\left(2f_{480} + \frac{4}{5}f_{481} - 2f_{440} - \frac{4}{5}f_{441}\right)\sum_{j=1}^{N} c_{ij}^{(5)}\varphi_j + \eta q_{0i} \\
&= \frac{1}{\eta}\left(\bar{I}_2 - \bar{I}_4\right)\sum_{j=1}^{N} c_{ij}^{(1)}\ddot{\tilde{u}}_j + \bar{I}_1\ddot{\tilde{w}}_i + \frac{1}{\eta^2}\left(2\bar{I}_5 - \bar{I}_3 - \bar{I}_6\right)\sum_{j=1}^{N} c_{ij}^{(2)}\ddot{\tilde{w}}_j + \frac{1}{\eta}\left(\bar{I}_6 - \bar{I}_5\right)\sum_{j=1}^{N} c_{ij}^{(1)}\ddot{\varphi}_j \qquad (70.b)
\end{aligned}$$

$$
\begin{aligned}
&-\left(f_{11} + \frac{2}{5} f_{671} \right) \sum_{j=1}^{N} c_{ij}^{(2)} \tilde{u}_j + \frac{1}{\eta^2} \left(2 f_{470} + \frac{4}{5} f_{471} \right) \sum_{j=1}^{N} c_{ij}^{(4)} \tilde{u}_j \\
&+\eta \left(k_s f_{55} + \frac{8}{15} f_{661} + \frac{f_{662}}{4} \right) \sum_{j=1}^{N} c_{ij}^{(1)} \tilde{w}_j + \frac{1}{\eta} \left(f_{22} - f_{33} + \frac{2}{5} f_{681} - \frac{4}{5} f_{461} \right. \\
&\left. +2 f_{570} + \frac{16}{15} f_{571} + \frac{f_{572}}{2} - 2 f_{550} - \frac{32}{15} f_{551} - \frac{f_{552}}{4} \right) \sum_{j=1}^{N} c_{ij}^{(3)} \tilde{w}_j \\
&+ \frac{1}{\eta^3} \left(2 f_{440} + \frac{4}{5} f_{441} - 2 f_{480} - \frac{4}{5} f_{481} \right) \sum_{j=1}^{N} c_{ij}^{(5)} \tilde{w}_j - \left(k_s f_{55} + \frac{8}{15} f_{661} + \frac{f_{662}}{4} \right) \varphi_i \\
&+ \left(f_{33} + 2 f_{550} + \frac{32}{15} f_{551} + \frac{f_{552}}{4} + \frac{4}{5} f_{461} \right) \sum_{j=1}^{N} c_{ij}^{(2)} \varphi_j - \frac{1}{\eta^2} \left(2 f_{440} + \frac{4}{5} f_{441} \right) \sum_{j=1}^{N} c_{ij}^{(4)} \varphi_j \\
&= -\bar{I}_4 \ddot{\tilde{u}}_i + \frac{1}{\eta} \left(\bar{I}_5 - \bar{I}_6 \right) \sum_{j=1}^{N} c_{ij}^{(1)} \ddot{\tilde{w}}_j + \bar{I}_6 \ddot{\varphi}_i
\end{aligned}
\qquad (70.c)
$$

where $i = 1, 2, \ldots, N$ and the over dot denotes partial derivative with respect to dimensionless time τ.

By substituting equation (64) into equations (61.a) - (61.g) boundary conditions of simply supported micro - beam can be rewritten as

$$
\begin{aligned}
&\tilde{u}_1 = -a_{11} \sum_{j=1}^{N} c_{Nj}^{(1)} \tilde{u}_j + \frac{1}{\eta^2} \left(2 a_{550} + \frac{4}{5} a_{551} \right) \sum_{j=1}^{N} c_{Nj}^{(3)} \tilde{u}_j \\
&+ \frac{1}{\eta} \left(b_{11} - f_{11} + \frac{2}{5} f_{671} \right) \sum_{j=1}^{N} c_{Nj}^{(2)} \tilde{w}_j + \frac{1}{\eta^3} \left(2 f_{470} + \frac{4}{5} f_{471} - 2 b_{550} - \frac{4}{5} b_{551} \right) \sum_{j=1}^{N} c_{Nj}^{(4)} \tilde{w}_j \\
&+ \left(f_{11} + \frac{2}{5} f_{671} \right) \sum_{j=1}^{N} c_{Nj}^{(1)} \varphi_j - \frac{1}{\eta^2} \left(2 f_{470} + \frac{4}{5} f_{471} \right) \sum_{j=1}^{N} c_{Nj}^{(3)} \varphi_j + \frac{1}{2} \eta h_{11} \Delta T = 0
\end{aligned}
\qquad (71.a)
$$

$$
\sum_{j=1}^{N} c_{1j}^{(1)} \tilde{u}_j = \sum_{j=1}^{N} c_{Nj}^{(1)} \tilde{u}_j = 0 \qquad (71.b)
$$

$$
\tilde{w}_1 = \tilde{w}_N = 0 \qquad (71.c)
$$

$$
\begin{aligned}
&\left(b_{11}-f_{11}\right)\sum_{j=1}^{N}c_{1j}^{(1)}\tilde{u}_j+\frac{1}{\eta^2}\left(2f_{470}+\frac{4}{5}f_{471}-2b_{550}-\frac{4}{5}b_{551}\right)\sum_{j=1}^{N}c_{1j}^{(3)}\tilde{u}_j\\
&+\frac{1}{\eta}\left(2f_{22}-f_{33}-d_{11}+4f_{570}+\frac{32}{15}f_{571}+f_{572}-2a_{550}-\frac{8}{15}a_{551}-a_{552}\right.\\
&\left.-2f_{550}-\frac{32}{15}f_{551}-\frac{f_{552}}{4}-\frac{2}{5}f_{461}+\frac{2}{5}f_{681}\right)\sum_{j=1}^{N}c_{1j}^{(2)}\tilde{w}_j\\
&+\frac{1}{\eta^3}\left(2d_{550}+\frac{4}{5}d_{551}+2f_{440}+\frac{4}{5}f_{441}-4f_{480}-\frac{8}{5}f_{481}\right)\sum_{j=1}^{N}c_{1j}^{(4)}\tilde{w}_j\\
&+\left(f_{33}-f_{22}+2f_{550}+\frac{32}{15}f_{551}+\frac{f_{552}}{4}\right.\\
&\left.-2f_{570}-\frac{16}{15}f_{571}-\frac{f_{572}}{2}+\frac{2}{5}f_{461}-\frac{2}{5}f_{681}\right)\sum_{j=1}^{N}c_{1j}^{(1)}\varphi_j\\
&+\frac{1}{\eta^2}\left(2f_{480}+\frac{4}{5}f_{481}-2f_{440}-\frac{4}{5}f_{441}\right)\sum_{j=1}^{N}c_{1j}^{(3)}\varphi_j-\frac{1}{2}\eta\left(h_{22}-h_{33}\right)\Delta T\\
&=\left(b_{11}-f_{11}\right)\sum_{j=1}^{N}c_{Nj}^{(1)}\tilde{u}_j+\frac{1}{\eta^2}\left(2f_{470}+\frac{4}{5}f_{471}-2b_{550}-\frac{4}{5}b_{551}\right)\sum_{j=1}^{N}c_{Nj}^{(3)}\tilde{u}_j\\
&+\frac{1}{\eta}\left(2f_{22}-f_{33}-d_{11}+4f_{570}+\frac{32}{15}f_{571}+f_{572}-2a_{550}-\frac{32}{15}a_{551}-a_{552}\right.\\
&\left.-2f_{550}-\frac{32}{15}f_{551}-\frac{f_{552}}{4}-\frac{2}{5}f_{461}+\frac{2}{5}f_{681}\right)\sum_{j=1}^{N}c_{Nj}^{(2)}\tilde{w}_j\\
&+\frac{1}{\eta^3}\left(2d_{550}+\frac{4}{5}d_{551}+2f_{440}+\frac{4}{5}f_{441}-4f_{480}-\frac{8}{5}f_{481}\right)\sum_{j=1}^{N}c_{Nj}^{(4)}\tilde{w}_j\\
&+\left(f_{33}-f_{22}+2f_{550}+\frac{32}{15}f_{551}+\frac{f_{552}}{4}\right.\\
&\left.-2f_{570}-\frac{16}{15}f_{571}-\frac{f_{572}}{2}+\frac{2}{5}f_{461}-\frac{2}{5}f_{681}\right)\sum_{j=1}^{N}c_{Nj}^{(1)}\varphi_j\\
&+\frac{1}{\eta^2}\left(2f_{480}+\frac{4}{5}f_{481}-2f_{440}-\frac{4}{5}f_{441}\right)\sum_{j=1}^{N}c_{Nj}^{(3)}\varphi_j-\frac{1}{2}\eta\left(h_{22}-h_{33}\right)\Delta T=0
\end{aligned}
\tag{71.d}
$$

$$
\sum_{j=1}^{N}c_{1j}^{(2)}\tilde{w}_j=\sum_{j=1}^{N}c_{Nj}^{(2)}\tilde{w}_j=0 \tag{71.e}
$$

$$
\begin{aligned}
&f_{11}\sum_{j=1}^{N}c_{1j}^{(1)}\tilde{u}_j-\frac{1}{\eta^2}\left(2f_{470}+\frac{4}{5}f_{471}\right)\sum_{j=1}^{N}c_{1j}^{(3)}\tilde{u}_j\\
&+\frac{1}{\eta}\left(f_{33}-f_{22}+2f_{550}+\frac{32}{15}f_{551}+\frac{f_{552}}{4}-2f_{570}-\frac{16}{15}f_{571}-\frac{f_{572}}{2}+\frac{2}{5}f_{461}\right)\sum_{j=1}^{N}c_{1j}^{(2)}\tilde{w}_j\\
&+\frac{1}{\eta^3}\left(2f_{480}+\frac{4}{5}f_{481}-2f_{440}-\frac{4}{5}f_{441}\right)\sum_{j=1}^{N}c_{1j}^{(4)}\tilde{w}_j\\
&-\left(f_{33}+2f_{550}+\frac{32}{15}f_{551}+\frac{f_{552}}{4}+\frac{2}{5}f_{461}\right)\sum_{j=1}^{N}c_{1j}^{(1)}\varphi_j+\frac{1}{\eta^2}\left(2f_{440}+\frac{4}{5}f_{441}\right)\sum_{j=1}^{N}c_{1j}^{(3)}\varphi_j\\
&-\frac{1}{2}\eta h_{33}\Delta T=f_{11}\sum_{j=1}^{N}c_{Nj}^{(1)}\tilde{u}_j-\frac{1}{\eta^2}\left(2f_{470}+\frac{4}{5}f_{471}\right)\sum_{j=1}^{N}c_{Nj}^{(3)}\tilde{u}_j\\
&+\frac{1}{\eta}\left(f_{33}-f_{22}+2f_{550}+\frac{32}{15}f_{551}+\frac{f_{552}}{4}-2f_{570}-\frac{16}{15}f_{571}-\frac{f_{572}}{2}+\frac{2}{5}f_{461}\right)\sum_{j=1}^{N}c_{Nj}^{(2)}\tilde{w}_j\\
&+\frac{1}{\eta^3}\left(2f_{480}+\frac{4}{5}f_{481}-2f_{440}-\frac{4}{5}f_{441}\right)\sum_{j=1}^{N}c_{Nj}^{(4)}\tilde{w}_j\\
&-\left(f_{33}+2f_{550}+\frac{32}{15}f_{551}+\frac{f_{552}}{4}+\frac{2}{5}f_{461}\right)\sum_{j=1}^{N}c_{Nj}^{(1)}\varphi_j+\frac{1}{\eta^2}\left(2f_{440}+\frac{4}{5}f_{441}\right)\sum_{j=1}^{N}c_{Nj}^{(3)}\varphi_j\\
&-\frac{1}{2}\eta h_{33}\Delta T=0
\end{aligned}
\tag{71.f}
$$

$$
\sum_{j=1}^{N}c_{1j}^{(1)}\varphi_j=\sum_{j=1}^{N}c_{Nj}^{(1)}\varphi_j=0 \tag{71.g}
$$

The governing equations and boundary conditions given above can be used to solve both static and free vibration problems. In the case of static analysis, inertia related terms are not taken into account whereas in free vibration analysis the external loading q and the terms in the boundary conditions resulted from steady state temperature field which effect the static behavior of the beam are equated to zero. In what follows below, we provide the matrix forms of the governing equations and boundary conditions for small - scale beams subjected to static loading and for those undergoing free vibrations.

3.2.1 Small - scale beams subjected to static loading

For a beam subjected to static loading, we define an unknown generalized displacement vector **d**, an unknown static displacement vector $\mathbf{d_b}$ and a dynamic displacement vector $\mathbf{d_d}$ as follows:

$$
\mathbf{d}=\left\{\{\tilde{u}_i\}^{\mathrm{T}},\ \{\tilde{w}_i\}^{\mathrm{T}},\ \{\varphi_i\}^{\mathrm{T}}\right\}^{\mathrm{T}},\quad \text{for } i=1,\ 2,\ ...,\ N \tag{72.a}
$$

$$
\mathbf{d_b}=\left\{\{\tilde{u}_{b_i}\}^{\mathrm{T}},\ \{\tilde{w}_{b_i}\}^{\mathrm{T}},\ \{\varphi_{b_i}\}^{\mathrm{T}}\right\}^{\mathrm{T}},\quad \text{for } i=1,\ 2,\ ...,\ N \tag{72.b}
$$

$$\mathbf{d_d} = \left\{ \left\{ \tilde{u}_{d_i} \right\}^{\mathrm{T}}, \left\{ \tilde{w}_{d_i} \right\}^{\mathrm{T}}, \left\{ \varphi_{d_i} \right\}^{\mathrm{T}} \right\}^{\mathrm{T}}, \quad \text{for } i = 1, 2, ..., N \tag{72.c}$$

where **d = d**$_\mathbf{b}$ **+ d**$_\mathbf{d}$. Eliminating the inertia related terms and substituting equation (72.b) into equations (70.a) - (70.c), we obtain the following matrix form of the governing equations:

$$\mathbf{D_b d_b^e} + \mathbf{D_d d_b^i} + \mathbf{Q} = \mathbf{0}, \tag{73}$$

where $\mathbf{d_b^e}$ and $\mathbf{d_b^i}$ are static displacement vectors for end points and internal points, respectively; $\mathbf{D_b}$ and $\mathbf{D_d}$ are coefficient matrices associated with end and internal points; and $\mathbf{Q}$ is the distributed force matrix. Substitution of equation (72.b) into the boundary conditions expressed by equations (71.a) - (71.g) leads to

$$\mathbf{B_b d_b^e} + \mathbf{B_d d_b^i} + \mathbf{Q_t} = \mathbf{0}, \tag{74}$$

In the last mentioned relation, $\mathbf{B_b}$ and $\mathbf{B_d}$ are coefficient matrices associated with end and internal points and $\mathbf{Q_t}$ is the thermal load vector. The final matrix form of the equations is obtained by substituting equation (74) into equation (73) and is written as

$$\mathbf{K d_b^i} + \mathbf{K_t Q_t} + \mathbf{Q} = \mathbf{0}, \tag{75}$$

where $\mathbf{K_t} = -\mathbf{D_b B_b^{-1}}$ and the stiffness matrix is given by

$$\mathbf{K} = -\mathbf{D_b B_b^{-1} B_d} + \mathbf{D_d}. \tag{76}$$

3.2.2 Small - scale beams undergoing free vibrations

No external forces act on a beam undergoing free vibrations, thus q is taken as zero in the numerical solution of the free vibration problem. In this case, the dynamic displacement vector $\mathbf{d_d}$ is defined in the following form:

$$\mathbf{d_d} = \mathbf{d_d^*} e^{i\omega\tau} \tag{77}$$

where $\omega = \Omega L \sqrt{I_{10} / A_{110}}$ is the dimensionless frequency, Ω denotes the vibration frequency of the FGM micro - beam, $\mathbf{d_d^*} = \left\{ \left\{ \tilde{u}_{d_i}^* \right\}^{\mathbf{T}}, \left\{ \tilde{w}_{d_i}^* \right\}^{\mathbf{T}}, \left\{ \varphi_{d_i}^* \right\}^{T} \right\}^{\mathbf{T}}$ is the vibration mode shape vector. By substituting the equation (77) into equations (70.a) - (70.c), one can derive the governing equations as given below

$$\mathbf{D_b d_d^{*e}} + \mathbf{D_d d_d^{*i}} - \omega^2 \mathbf{M d_d^{*i}} = 0 \tag{78}$$

where $\mathbf{D_b}$ and $\mathbf{D_d}$ are coefficient matrices associated with end and internal points, respectively; $\mathbf{d_d^{*e}}$ and $\mathbf{d_d^{*i}}$ are mode shape vectors for end and internal points; and $\mathbf{M}$ is the mass matrix. Substituting equation (77) into equations (71.a) - (71.g) and eliminating the nonhomogeneous thermal terms one can obtain

$$\mathbf{B_b d_b^{*e}} + \mathbf{B_d d_d^{*i}} = \mathbf{0}, \tag{79}$$

$\mathbf{B_b}$ and $\mathbf{B_d}$ are coefficient matrices associated with end and internal points, respectively. Combining equations (78) and (79), the eigenvalue problem regarding free vibrations of small - scale functionally graded beams is expressed as

$$\left\{\mathbf{K} - \omega^2\mathbf{M}\right\}\ \mathbf{d_d^{*i}} = \mathbf{0}, \tag{80}$$

where $\mathbf{K} = -\mathbf{D_b B_b}^{-1}\mathbf{B_d} + \mathbf{D_d}$ is the stiffness matrix.

Equation (80) is solved to determine the dimensionless frequencies and the corresponding mode shape vectors. A particular frequency obtained via Equation (80) may correspond to any of the three deformation modes, which we identify as transverse deformation, axial deformation, and rotational deformation. These three deformation modes are respectively quantified by the functions *w*, *u* and ϕ. The deformation mode corresponding to a given frequency is determined by examining the mode shape vector.

In order to implement the numerical method developed in this chapter, a computer program is developed using MATLAB [48]. In the following chapter, we give detailed results of simply supported micro - beam.

CHAPTER 4

RESULTS

In the previous chapter the general beam theory in conjunction with SGT is formulated. By substituting the shape function, *f*, the formulation for desired beam theory can be obtained. Letting the small - scale parameters l_0, l_1 and l_2 equal to zero leads to equations of motion and boundary conditions of classical theory of elasticity and by l_0=l_1=0 the relations are reduced to MCST. The formulation is obtained for FGM and the power index *n* determines the type of the variation of the properties along the thickness and by letting *n*=0 homogeneous material formulation is obtained. Setting $\Delta T = 0$ leads to the deflection and natural frequencies of small - scale beam with no thermal effects.

In this chapter the numerical results are presented. To check the accuracy and validity of the computer program, some comparisons with the results already presented in the literature are given. To investigate homogeneous micro - beam it is assumed that the beam is made of epoxy with the material properties *E*=1.44 GPa, υ=0.38 (or μ=521.7 MPa), ρ=1220 kg/m^3 and $\alpha = 54\times10^{-6}$/°C. The FGM micro - beam consists of aluminum (Al) and ceramic (SiC) with the following material properties: $k_s = 5/6$, $\upsilon_m = 0.3$ and $\upsilon_c = 0.17$, $\rho_m = 2702$ kg/m^3 and $\rho_c = 3100$ kg/m^3, $E_m = 70$ GPa and $E_c = 427$ GPa, $\alpha_m = 23\times10^{-6}$/°C and $\alpha_c = 4.76\times10^{-6}$/°C [49]. In classical beam theory it is supposed that the beam is slender and the Poisson effect is neglected ($\upsilon = 0$). For homogeneous epoxy micro - beam the length scale parameter has been experimentally obtained as $l_2 = l = 17.6$ μm and assuming the three length scale parameters in SGT to be equal, they are obtained as $l_0 = l_1 = l_2 = l = 17.6$ μm [12]. There is no available experimental data relevant to the FGM micro - beams in literature. In order to analyze the size effect of the FGM micro - beams, the values of length scale parameters for the metal are approximately assumed to be equal to 15μm ($l_{0m} = l_{1m} = l_{2m}$ $= 15$ μm) in the following numerical evaluations. To investigate the non – constant value of the length scale parameters through the thickness of the FGM micro - beam, distinct sets of length scale parameters of the ceramic ($l_{0c} = l_{1c} = l_{2c}$) are used. For simplicity, in the following analysis material length scale parameter used in MCST is shown by *l* instead of l_2 for both metal and ceramic.

By setting the inertia terms of the governing equations and boundary conditions, the right hand side of the equations (60.a) - (60.c) and (61.a) - (61.g), equal to zero the static formulation is obtained. For static analysis, distributed force of constant value *q* = 1 N/m, is considered to be applied on the upper surface of the micro - beam. The dimensionless deflection (*w/h*) are obtained for the FGM micro - beam with $L/h=10$ and 20 $h/l_{0m}=2$.

In this work the free vibration equations of simply supported beam is derived and numerical solution is used to obtain the natural frequencies. To check the validity of the numerical solution and accuracy of the results, a comparison is made with the results given in previous works.

4.1 Comparisons

In order to check the accuracy of the results obtained using the differential quadrature method and developed computer program, some comparisons are given in section 4.1. Table 1 gives the comparisons between the maximum deflections for both EBBT and TBT which are obtained using MCST. As it is seen, there is a good agreement between the results obtained in this work and those of Reddy [35]. There are small differences between the maximum deflections obtained by TBT because Reddy [35] also considered the Poisson's ratio to be zero for TBT.

Comparisons of natural frequencies for homogeneous and FGM micro - beams predicted by MCST are presented in Table 2 and Table 3. The results generated by our computer program are very close to the results of other researchers. For homogeneous micro - beam, the best agreement is observed between our results and the results obtained using closed form solution by Ma et al. [22] which indicates the accuracy of numerical method used in this work with respect to other methods. In Table 4 the natural frequencies which are obtained using SGT for different FGM power index values are compared. Our results are very close to the results obtained by Navier solution. Although for higher modes small differences are observed, the results are still in good agreement. In the case of taking the thermal effects into account to validate the results of our program, a comparison is made with the natural frequencies obtained by Ke et al. [24] using MCST for different values of ΔT in Table 5. It is observed that the results are also very close in this case.

Table 1. Comparisons of normalized maximum deflection $\bar{w}=w_{max}\times\left(EI/qL^4\right)\times10^2$ for isotropic homogeneous micro - beams, MCST, $L/h=20$, $l=17.6$ μm, $b/h=2$, $q=1.0$ N/m.

l/h	**EBBT**		**TBT**	
	Present	**Reddy [35]**	**Present**	**Reddy [35]**
0.0	1.3021	1.3021	1.3056	1.3103
0.2	1.1092	1.1092	1.1125	1.1162
0.4	0.7679	0.7679	0.7708	0.7731
0.6	0.5076	0.5076	0.5102	0.5116
0.8	0.3442	0.3442	0.3467	0.3475
1.0	0.2435	0.2435	0.2458	0.2464

Table 2. Comparisons of first five natural frequencies (MHz) for isotropic homogeneous micro - beams, MCST, TBT, $L/h=10$, $l=17.6\mu$m.

Mode	h/l = 10		h/l = 5		h/l = 3.33	
	Present	Ma et al. [22]	Present	Ma et al. [22]	Present	Ma et al. [22]
1	0.03765	0.03765	0.07782	0.07782	0.12290	0.12290
2	0.13966	0.13966	0.28874	0.28874	0.45609	0.45609
3	0.28384	0.28384	0.58743	0.58743	0.92894	0.92894
4	0.45136	0.45136	0.93594	0.93594	1.48370	1.48370
5	0.63062	0.63062	1.31143	1.31143	2.08671	2.08671

Table 3. Comparisons of dimensionless natural frequencies of FGM micro - beam for different values of *n*, MCST, TBT, $L/h=10$, $l_m=l_c=15$ µm, $h/l_m=2$.

Mode	Method	FGM volume fraction exponent, *n*				
		Ceramic	*n*=0.6	*n*=1.2	*n*=2	metal
1	**Ke et al. [33]**	0.8336	0.5944	0.5376	0.5048	0.3393
	Ma et al. [22]	0.8538	-	-	-	0.3797
	Ansari et al. [37]	0.8538	0.6084	0.5470	0.5100	0.3863
	Present	0.8538	0.6084	0.5469	0.5099	0.3797
2	**Ke et al. [33]**	3.2081	2.2805	2.0550	1.9230	1.2914
	Ma et al. [22]	3.2551	-	-	-	1.4323
	Present	3.2550	2.3167	2.0772	1.9306	1.4323
3	**Ke et al. [33]**	6.8417	4.8649	4.3725	4.0762	2.7165
	Ma et al. [22]	6.8575	-	-	-	2.9789
	Present	6.8575	4.8744	4.3570	4.0344	2.9789
5	**Ma et al. [22]**	16.4671	-	-	-	6.9886
	Ansari et al. [37]	16.4672	11.6879	10.3919	9.5590	7.0831
	Present	16.4671	11.6880	10.3916	9.5585	6.9886

Table 4. Comparisons of dimensionless natural frequencies of FGM micro - beam for different values of *n*, SGT, TBT, $L/h=10$, $l_{0m}=l_{0c}=15$ µm, $h/l_{0m}=2$.

Mode	Method	FGM volume fraction exponent, *n*				
		Ceramic	*n*=0.6	*n*=1.2	*n*=2	metal
1	**Ansari et al. [37]**	1.2608	0.8976	0.7986	0.7346	0.5430
	Present	1.2608	0.8976	0.7986	0.7346	0.5355
5	**Ansari et al. [37]**	19.2899	13.6798	12.1164	11.0892	8.1240
	Present	19.2938	13.6824	12.1183	11.0908	8.0217

Table 5. Comparisons of dimensionless natural frequencies of homogeneous micro - beam for different values of ΔT, MCST, TBT, $L/h=10$, $l=17.6$ μm, $h/l=2$.

Mode	Method	$\Delta T\,(°C)$					
		0	**20**	**40**	**60**	**80**	**100**
1	**Ke et al. [24]**	0.3478	0.3322	0.3159	0.2986	0.2804	0.2608
	Present	0.3477	0.3322	0.3158	0.2986	0.2803	0.2608
5	**Ke et al. [24]**	5.9628	5.9410	5.9191	5.8971	5.8751	5.8529
	Present	5.9617	5.9399	5.9180	5.8960	5.8740	5.8518

4.2 Static Results

4.2.1 Deflection of the micro - beam under distributed transverse load

To study the static behavior of micro - scale beams, the deflection of the FG beam and its maximum value are obtained for different values of FGM power index, ceramic length scale parameter, temperature change and geometrical parameters of micro - beam, using different higher order continuum approaches and beam deformation theories discussed in this work. Dimensionless deflections of the micro - beam under uniformly distributed load obtained by using MCST and SGT and evaluated by using three different beam theories are depicted in Fig. 3 and Fig. 4, respectively. Fig. 5 is generated by implementing the same parameters of Fig. 4 and letting $h/l_{0m}=10$. By investigating the results one can conclude that the TOBT predicts the smallest midspan deflection compared to other classical beam theories. Although EBBT is seen to result in the largest midspan deflection, once SGT with small values of h/l_{0m} is used the effect of small - scale parameters become more significant than shear considerations which make EBBT predict stiffer beam than TBT.

Fig. 6 and Fig. 7 show normalized normal stress distributions using MCST and SGT, respectively, generated by using the three different beam theories. The curves obtained by the Timoshenko and third - order beam theories are rather close to each other. However, the normal stress estimation by the Euler - Bernoulli is not in agreement with those by the other two beam theories. Since Euler - Bernoulli beam theory does not take into account the in - plane shear deformation, the results calculated by the Timoshenko and the third - order beam theories are deduced to be more reliable. This finding indicates that it may not be appropriate to use the Euler - Bernoulli beam theory in certain problems regarding small - scale FGM beams possessing a variable length scale parameter.

Comparison of the results obtained by different elasticity theories using TOBT are illustrated in Fig. 8. MCST results in higher values of static deflection than SGT, however, it still predicts stiffer beam than the classical elasticity theory.

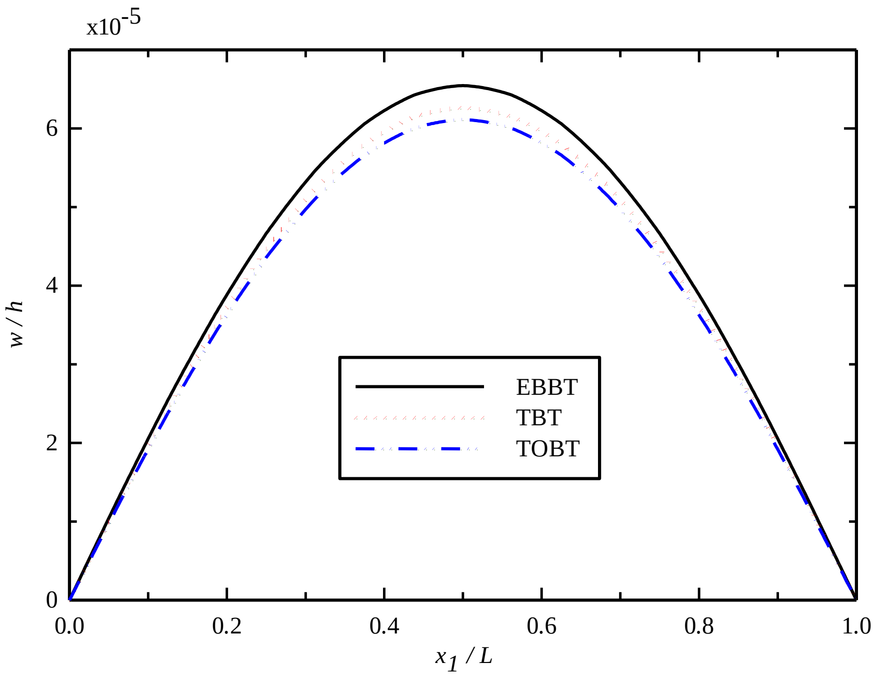

Fig. 3. Dimensionless deflection (*w/h*) of the FGM micro - beam with $L/h=10$, $l_m=15$ μm, $h/l_m=2$, $b/h=2$, $l_c/l_m=3/2$, $n=2$, $q=1.0$ N/m, MCST, considering different beam theories.

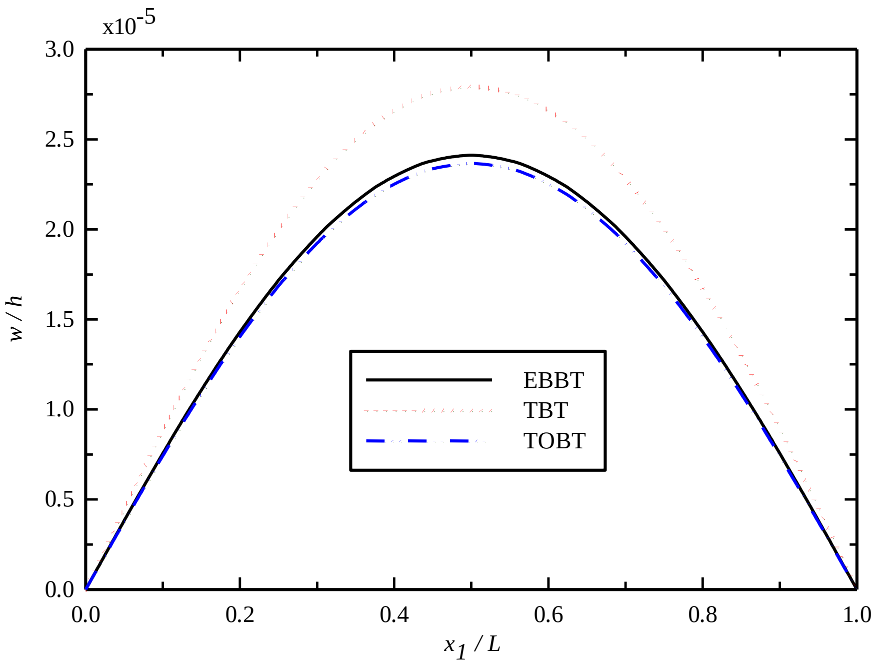

Fig. 4. Dimensionless deflection (*w/h*) of the FGM micro - beam with $L/h=10$, $l_{0m}=15$ μm, $h/l_{0m}=2$, $b/h=2$, $l_{0c}/l_{0m}=3/2$, $n=2$, $q=1.0$ N/m, SGT, considering different beam theories.

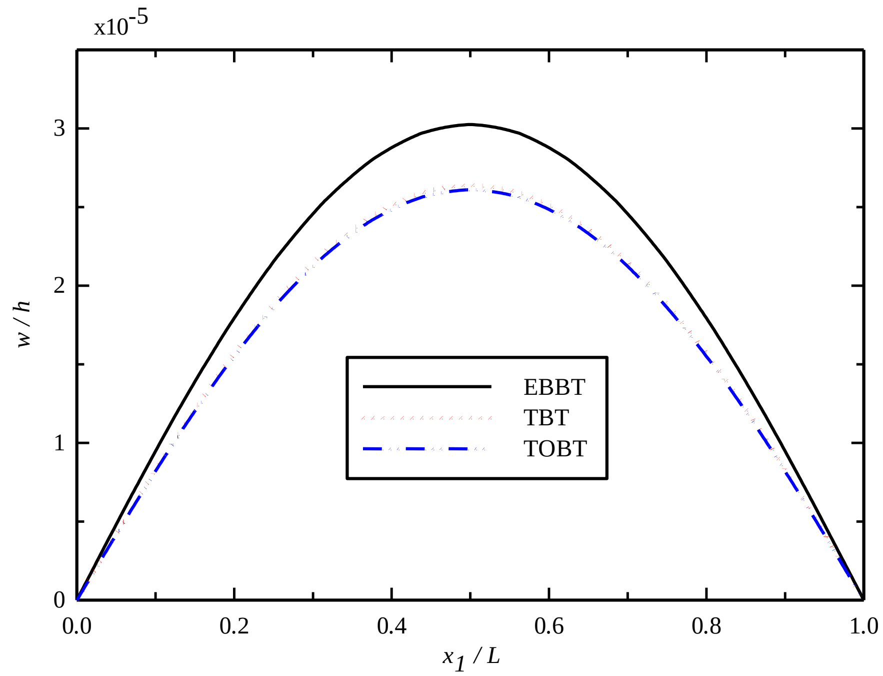

Fig. 5. Dimensionless deflection (*w/h*) of the FGM micro - beam with $L/h=10$, $l_{0m}=15$ µm, $h/l_{0m}=10$, $b/h=2$, $l_{0c}/l_{0m}=3/2$, $n=2$, $q=1.0$ N/m, SGT, considering different beam theories.

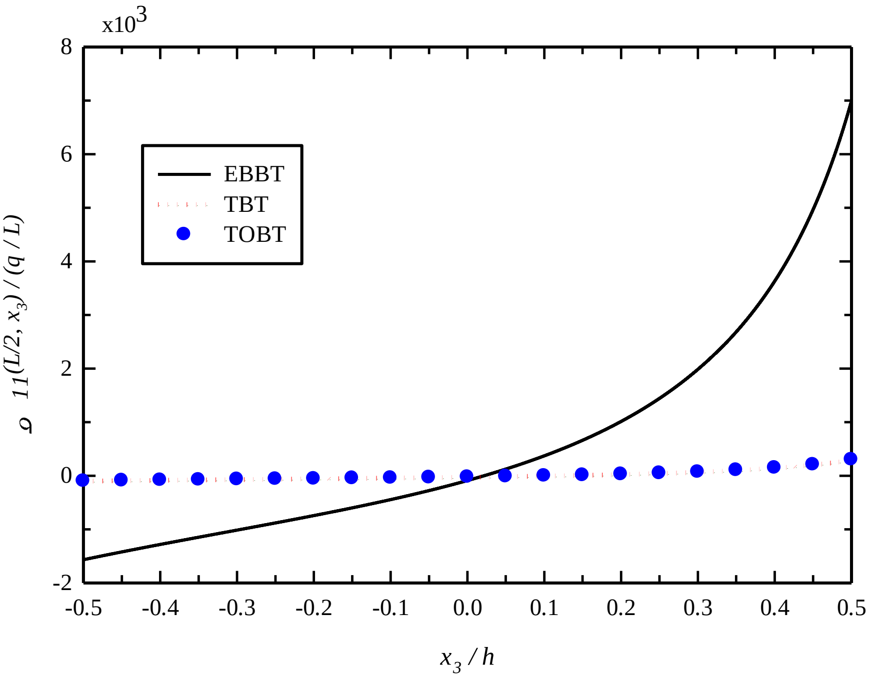

Fig. 6. Normal stress distributions of the FGM micro - beam with $L/h=10$, $l_m=15$ µm, $h/l_m=2$, $b/h=2$, $l_c/l_m=3/2$, $n=2$, $q=1.0$ N/m, MCST, considering different beam theories.

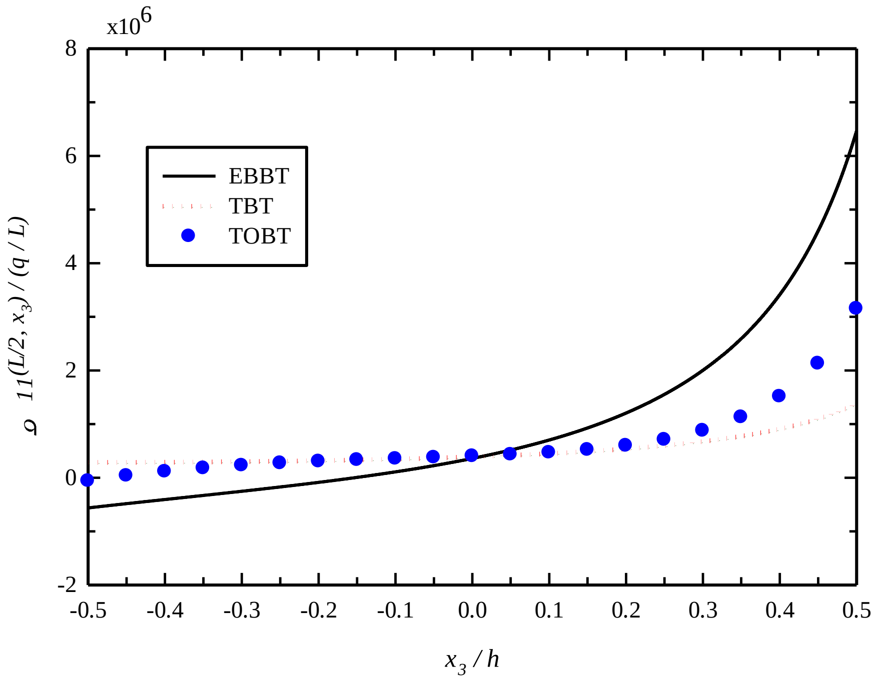

Fig. 7. Normal stress distributions of the FGM micro - beam with $L/h=10$, $l_{0m}=15$ μm, $h/l_{0m}=2$, $b/h=2$, $l_{0c}/l_{0m}=3/2$, $n=2$, $q=1.0$ N/m, SGT, considering different beam theories.

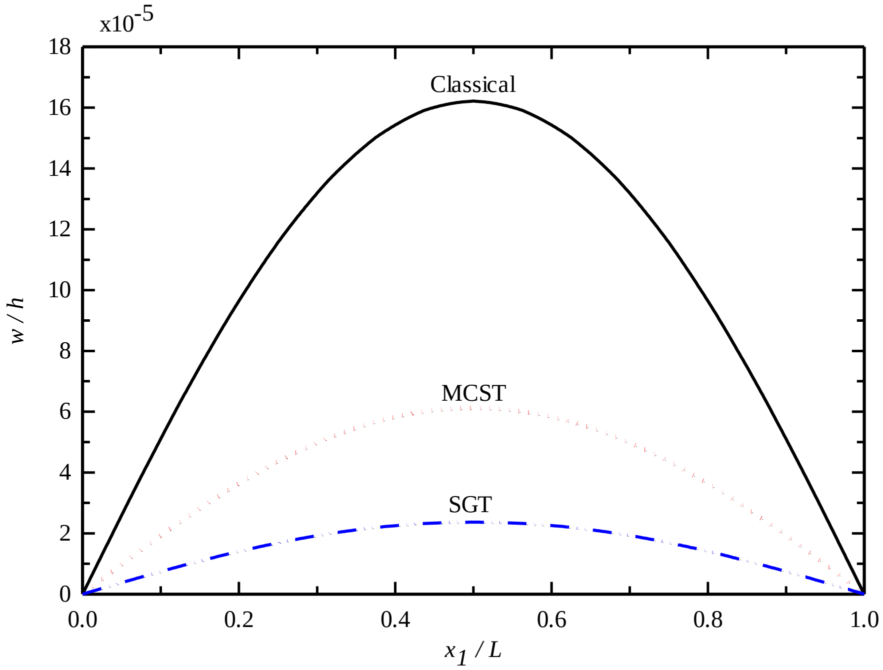

Fig. 8. Dimensionless deflection (*w/h*) of the FGM micro - beam with $L/h=10$, $l_m=l_{0m}=15$ μm, $h/l_m=h/l_{0m}=2$, $b/h=2$, $l_c/l_m=l_{0c}/l_{0m}=3/2$, $n=2$, $q=1.0$ N/m, TOBT, considering different elasticity theories.

Fig. 9 and Fig. 10 depict the influence of the exponent n on the static deflection of the small - scale functionally graded beam predicted by MCST and SGT, respectively. These results are computed by considering the TOBT. Note that distribution profiles of ceramic and metal volume fractions depend on the exponent n. When n is less than unity the beam is ceramic - rich while for an n value greater than one the beam possesses a metal - rich profile. The volume fraction variation is linear for $n = 1$. The results provided in Fig. 9 and Fig. 10 point out that, deflection becomes larger as n is increased from 0.5 to 5, i.e. static deflection computed for a ceramic - rich beam is smaller compared to the deflection evaluated for a metal - rich beam.

Results regarding the impact of the variation of the length scale parameter l upon the static deflection of a small - scale functionally graded beam are presented for MCST in Fig. 11 and for SGT in Fig. 12. These results are also evaluated through the use of the TOBT. Static deflection curves are generated for four different values of l_c / l_m. When this ratio is equal to unity, the beam has a constant length scale parameter. It can be seen that the influence of the variation of the length scale parameter on the static deflection is rather significant. Static deflection decreases as l_c / l_m is increased from $1/3$ to 2. As the length scale parameter of the ceramic component gets larger compared to that of the metallic component, the deflection of the beam becomes smaller considerably. This observation is also a validation of the premise of this study that the variation of the length scale parameter needs to be taken into account in the analysis of small - scale functionally graded beams.

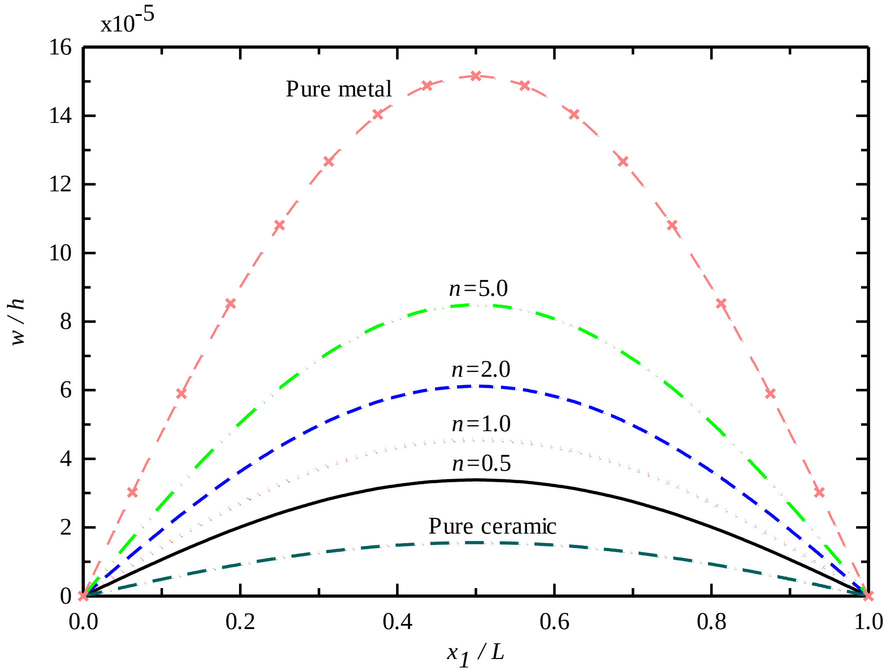

Fig. 9. Dimensionless deflection (*w/h*) of the FGM micro - beam with $L/h = 10$, $l_m = 15$ μm , $h/l_m = 2$, $b/h = 2$, $l_c/l_m = 3/2$, $q = 1.0$ N/m, MCST, TOBT, considering different values of n.

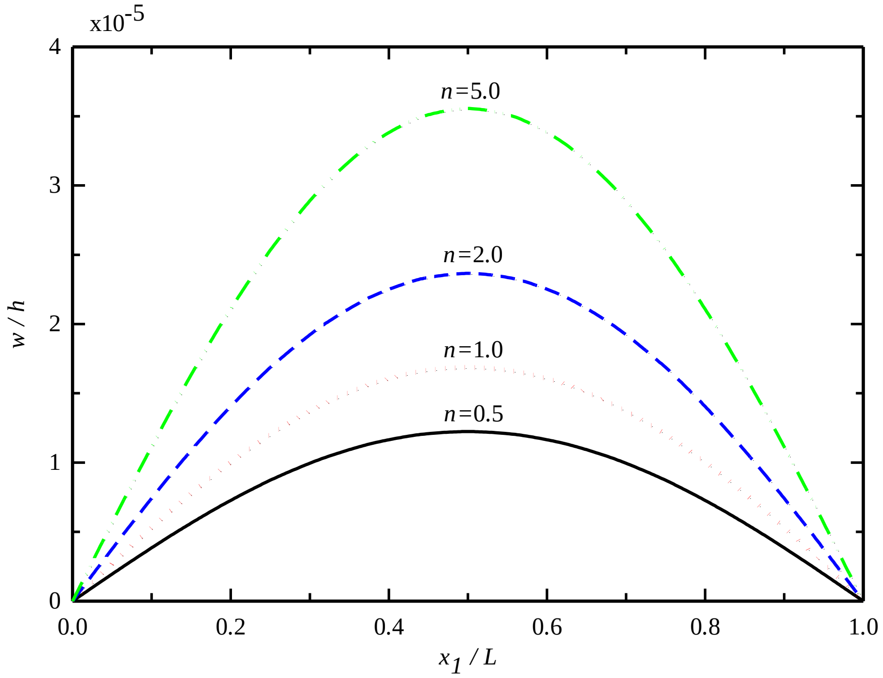

Fig. 10. Dimensionless deflection (*w/h*) of the FGM micro - beam with $L/h=10$, $l_{0m}=15$ μm, $h/l_{0m}=2$, $b/h=2$, $l_{0c}/l_{0m}=3/2$, $q=1.0$ N/m, SGT, TOBT, considering different values of n.

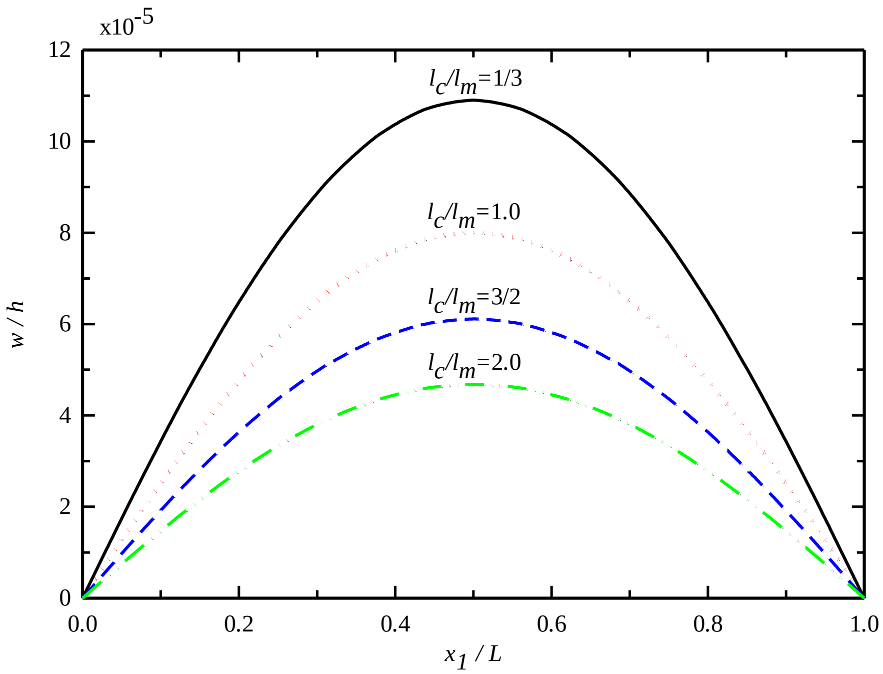

Fig. 11. Dimensionless deflection (*w/h*) of the FGM micro - beam with $L/h=10$, $l_m=15$ μm, $h/l_m=2$, $b/h=2$, $n=2$, $q=1.0$ N/m, MCST, TOBT, considering different values of l_c/l_m.

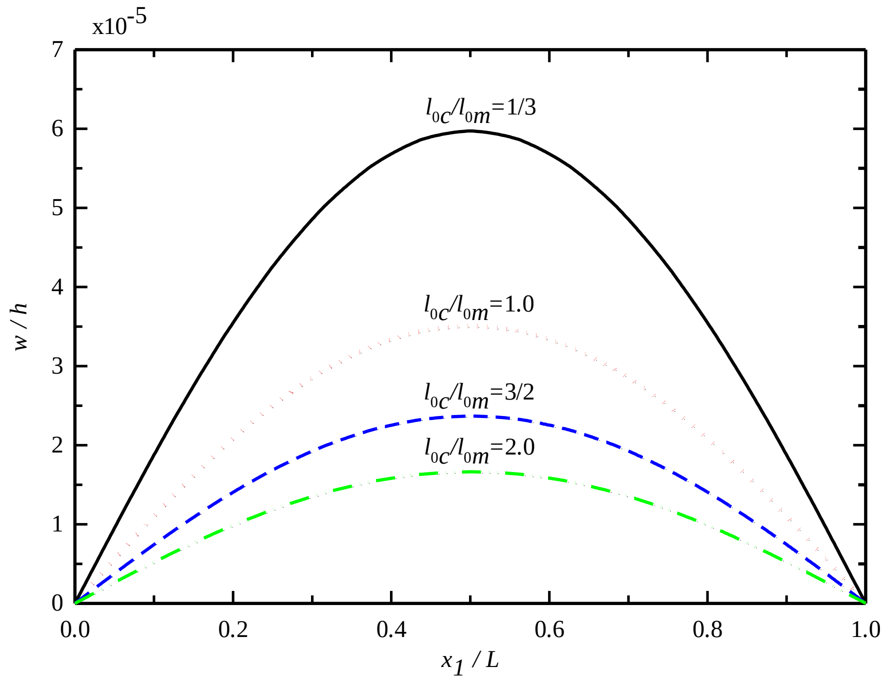

Fig. 12. Dimensionless deflection (*w/h*) of the FGM micro - beam with $L/h=10$, $l_{0m}=15$ μm, $h/l_{0m}=2$, $b/h=2$, $n=2$, $q=1.0$ N/m, SGT, TOBT, considering different values of l_{0c}/l_{0m}.

To have a clear understanding of static analysis, the results of different beam and elasticity theories for different values of n are given for micro - beam with $L/h=20$, $l_m=l_c=l_{0m}=l_{0c}=15$ μm in Table 6 and micro - beam with $L/h=10$, $l_c/l_m=l_{0c}/l_{0m}=3/2$ in Table 7.

Table 6. Maximum deflection $\bar{w}=w_{max}\times 10^3$ of FGM micro - beam with $L/h=20$, $l_m=l_c=l_{0m}=l_{0c}=15$ μm, $h/l_m=h/l_{0m}=2$, $b/h=2$, $q=1.0$ N/m, considering different values of n and different beam and elasticity theories.

Elasticity theory	Beam theory	FGM volume fraction exponent, *n*				
		Ceramic	*n*=0.6	*n*=1.2	*n*=2	metal
Classical	EBBT	0.9758	2.2118	2.7700	3.1277	5.9524
	TBT	0.9133	1.8833	2.2837	2.5354	4.4589
	TOBT	0.9104	1.8778	2.2781	2.5305	4.4446
MCST	EBBT	0.4276	0.9204	1.1835	1.3991	2.7636
	TBT	0.4158	0.8594	1.0865	1.2689	2.3962
	TOBT	0.4146	0.8570	1.0831	1.2644	2.3875
SGT	EBBT	0.1749	0.3680	0.4786	0.5777	1.1620
	TBT	0.1784	0.3692	0.4767	0.5721	1.1273
	TOBT	0.1733	0.3586	0.4624	0.5544	1.0937

Table 7. Maximum deflection $\bar{w} = w_{max} \times 10^5$ of FGM micro - beam with $L/h = 10$, $l_m = l_{0m} = 15$ μm, $h/l_m = h/l_{0m} = 2$, $b/h = 2$, $l_c/l_m = l_{0c}/l_{0m} = 3/2$, $q = 1.0$ N/m, considering different values of n and different beam and elasticity theories.

Elasticity theory	Beam theory	FGM volume fraction exponent, *n*				
		Ceramic	*n*=0.6	*n*=1.2	*n*=2	metal
Classical	EBBT	6.0987	13.8236	17.3123	19.5482	37.2023
	TBT	5.8110	11.9837	14.5523	16.1882	28.5646
	TOBT	5.7928	11.9596	14.5495	16.2169	28.4762
MCST	EBBT	1.5700	3.7936	5.2100	6.5470	10.3450
	TBT	1.5931	3.7288	5.0449	6.2631	9.8398
	TOBT	1.5569	3.6502	4.9327	6.1154	9.5702
SGT	EBBT	0.5389	1.3228	1.8606	2.4126	3.6169
	TBT	0.6695	1.5800	2.1852	2.7937	4.4070
	TOBT	0.5485	1.3262	1.8435	2.3672	3.5905

4.2.2 Deflection of the micro - beam under thermal load

To investigate the thermal effect on the static analysis, the deflection of micro - beam under thermal load is calculated in absence of other external forces. Thermal analysis of static deflection is carried out for different values of ΔT and *n* considering different beam and elasticity theories. The dimensionless static deflections under the load generated by $\Delta T = 40°\text{C}$ predicted by MCST and SGT are shown in Fig. 13 and Fig. 14 respectively. Depicted in Fig. 15 and Fig. 16 are the static deflections for different ΔT values which are calculated by implementing TOBT. As can be seen in these figures, higher ΔT values result in higher static deflections. Investigating the effect of the exponent *n* on the static behavior of the micro - beam undergoing thermal load leads to the conclusion that by increasing the value of *n* and hence increasing the ratio of metal to ceramic phase results in larger static deflection because the coefficient of thermal expansion of metal is large. This fact is illustrated for MCST in Fig. 17 and for SGT in Fig. 18.

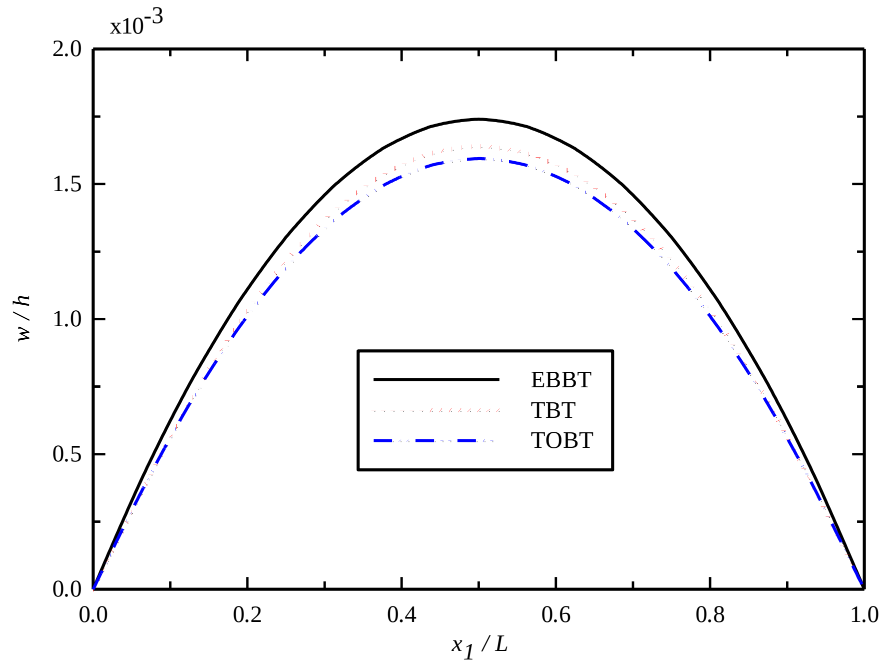

Fig. 13. Dimensionless deflection (*w/h*) of the FGM micro - beam with $L/h=10$, $l_m=15$ µm, $h/l_m=2$, $b/h=2$, $l_c/l_m=3/2$, $n=2$, $\Delta T=40$°C, MCST, considering different beam theories.

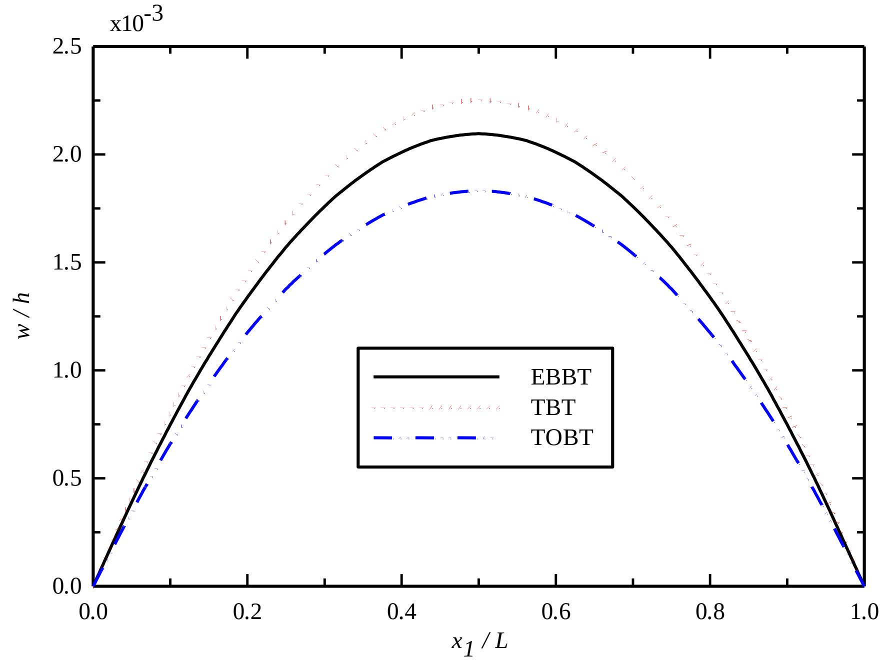

Fig. 14. Dimensionless deflection (*w/h*) of the FGM micro - beam with $L/h=10$, $l_{0m}=15$ µm, $h/l_{0m}=2$, $b/h=2$, $l_{0c}/l_{0m}=3/2$, $n=2$, $\Delta T=40$°C, SGT, considering different beam theories.

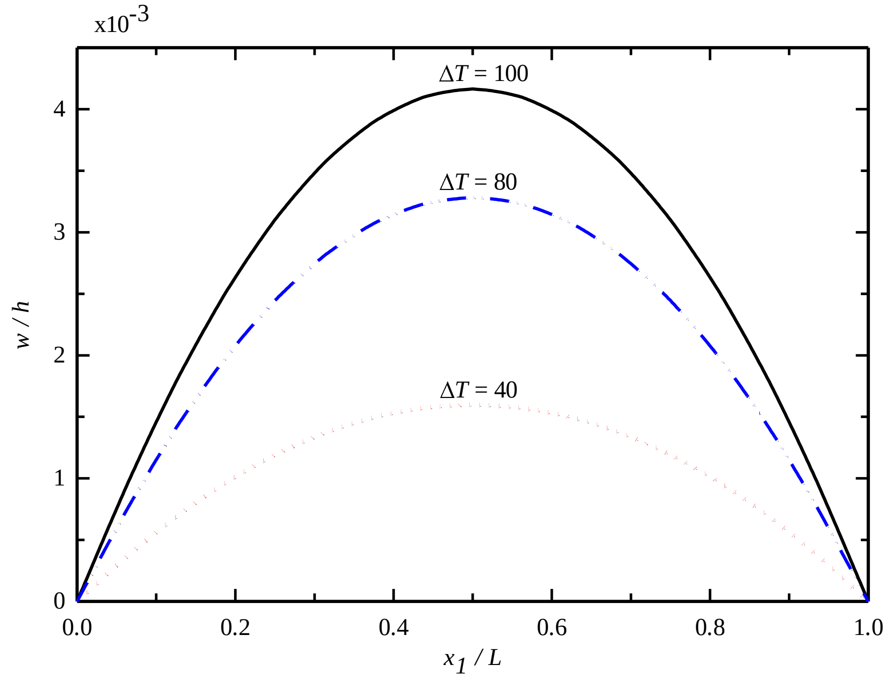

Fig. 15. Dimensionless deflection (*w/h*) of the FGM micro - beam with $L/h = 10$, $l_m = 15$ µm, $h/l_m = 2$, $b/h = 2$, $l_c/l_m = 3/2$, $n = 2$, MCST, TOBT, considering different values of ΔT.

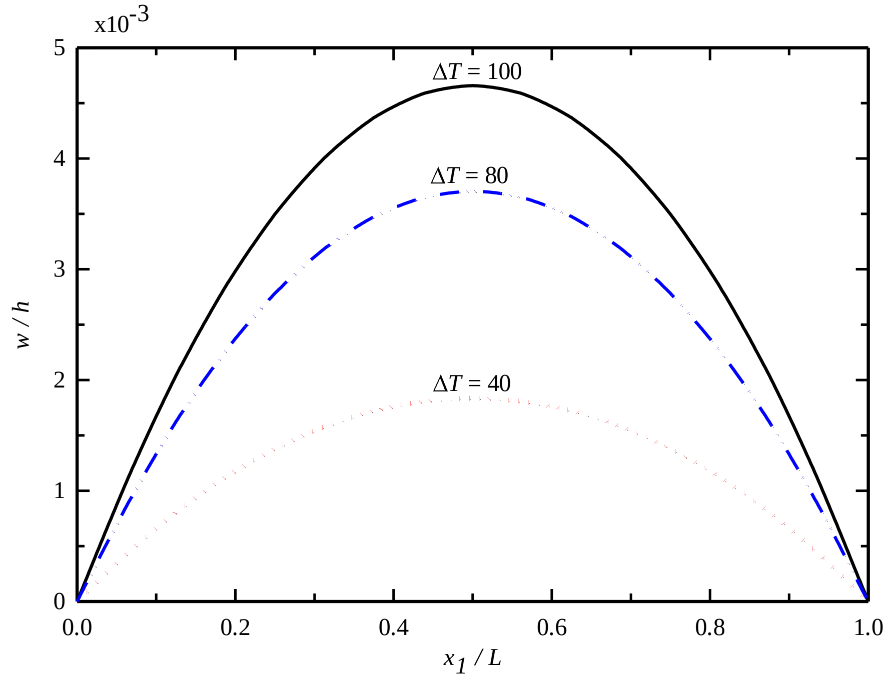

Fig. 16. Dimensionless deflection (*w/h*) of the FGM micro - beam with $L/h = 10$, $l_{0m} = 15$ µm, $h/l_{0m} = 2$, $b/h = 2$, $l_{0c}/l_{0m} = 3/2$, $n = 2$, SGT, TOBT, considering different values of ΔT.

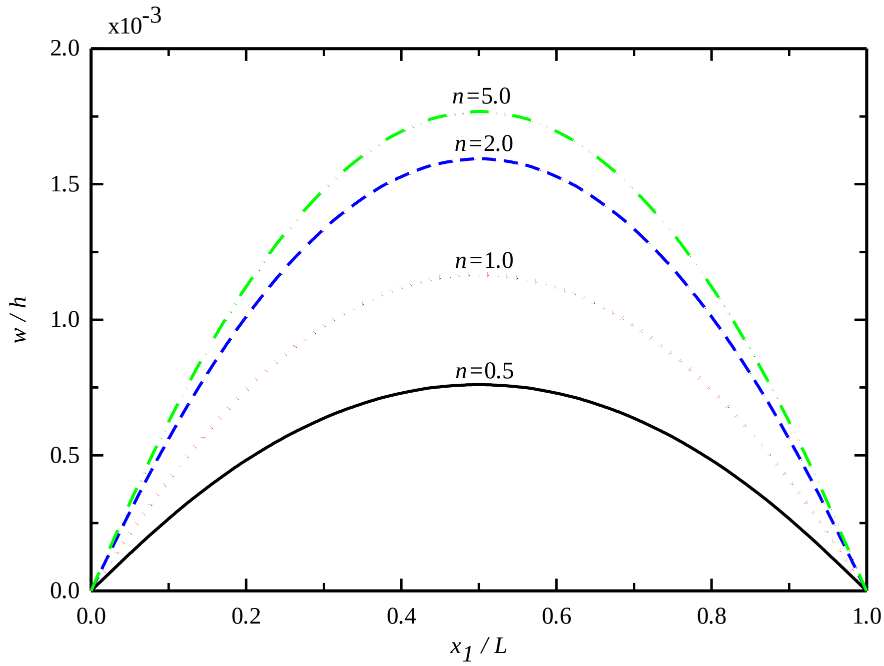

Fig. 17. Dimensionless deflection (*w/h*) of the FGM micro - beam with $L/h=10$, $l_m=15$ μm, $h/l_m=2$, $b/h=2$, $l_c/l_m=3/2$, $\Delta T=40$°C, MCST, TOBT, considering different values of n.

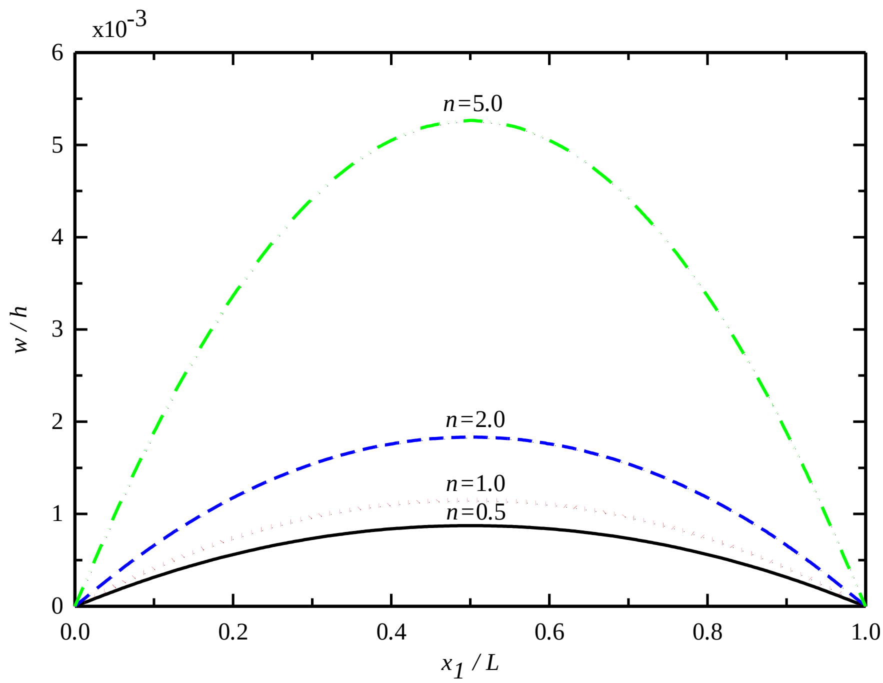

Fig. 18. Dimensionless deflection (*w/h*) of the FGM micro - beam with $L/h=10$, $l_{0m}=15$ μm, $h/l_{0m}=2$, $b/h=2$, $l_{0c}/l_{0m}=3/2$, $\Delta T=40$°C, SGT, TOBT, considering different values of n.

4.3 Free Vibration Results

4.3.1 Natural frequencies without thermal effects

The free vibration analysis of the simply supported micro - beam is carried out using the numerical method and the natural frequencies are obtained for different values of material and geometrical parameters.

Presented in Fig. 19 are the variations of the first nondimensional natural frequency ω_1 with respect to the ratio of height to length scale parameter of MCST, h/l_m which are generated by considering the three different beam theories. These results for SGT are shown with respect to h/l_{0m} in Fig. 20. Note that in the examined problems the first natural frequency always corresponds to the transverse deformation mode. The results obtained by the use of Timoshenko beam theory and the third - order beam theory are again almost identical whereas Euler - Bernoulli beam theory leads to slightly smaller results especially for relatively larger values of h/l_m The sensitivity of the nondimensional frequency ω_1 to the variations in h/l_m becomes rather pronounced as this ratio gets smaller.

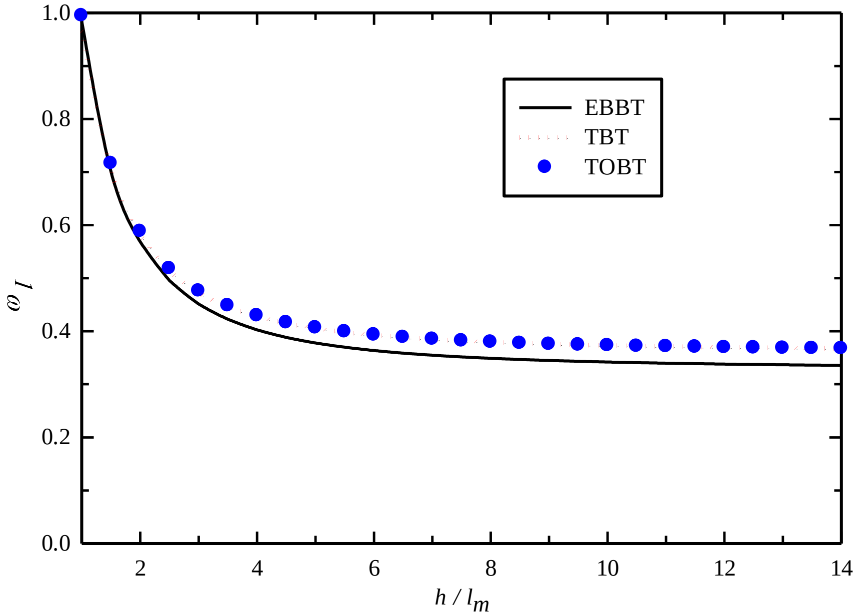

Fig. 19. Variations of the first dimensionless natural frequency with respect to h/l_m for FGM micro - beam with $L/h=10$, $l_m=15$ µm, $b/h=2$, $l_c/l_m=3/2$, $n=2$, MCST, considering different beam theories.

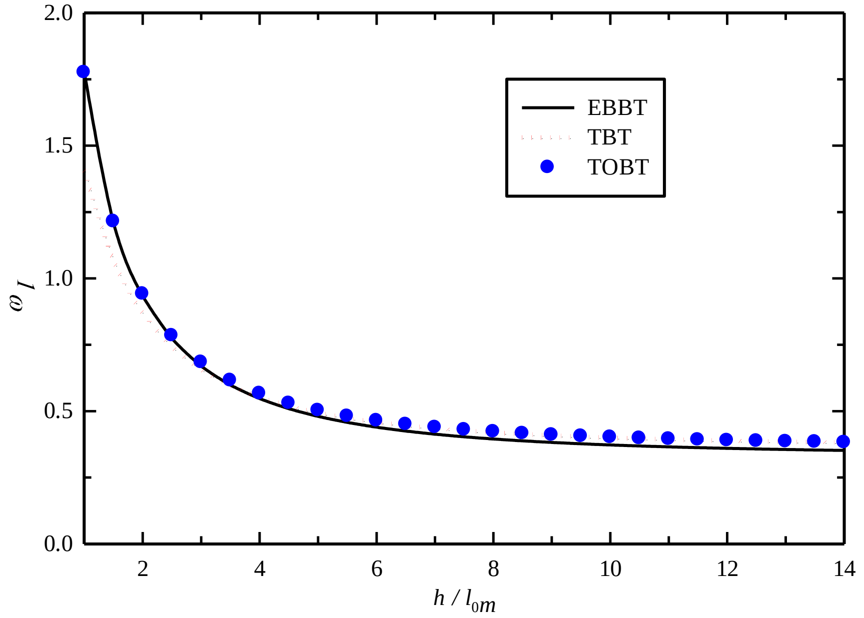

Fig. 20. Variations of the first dimensionless natural frequency with respect to h/l_{0m} for FGM micro - beam with $L/h=10$, $l_{0m}=15$ μm, $b/h=2$, $l_{0c}/l_{0m}=3/2$, $n=2$, SGT, considering different beam theories.

Fig. 21 illustrates the variation of dimensionless natural frequencies of FGM micro - beam with the height to metal length scale parameter ratio using three different elasticity theories. These results are obtained by using TOBT. Similar to the static case, SGT predicts the stiffest beam. In classical elasticity theory in which the small - scale effect is neglected, the smallest dimensionless natural frequencies are obtained. In this theory, by keeping the length to height ratio constant, the variation of the height to material length scale parameter has no effect on the value of dimensionless natural frequency. Increase in the value of height to length scale parameter ratio leads to smaller dimensionless natural frequencies. The effect of material length scale parameter change is more considerable for smaller values of *h* and it becomes ineffective as the height increases.

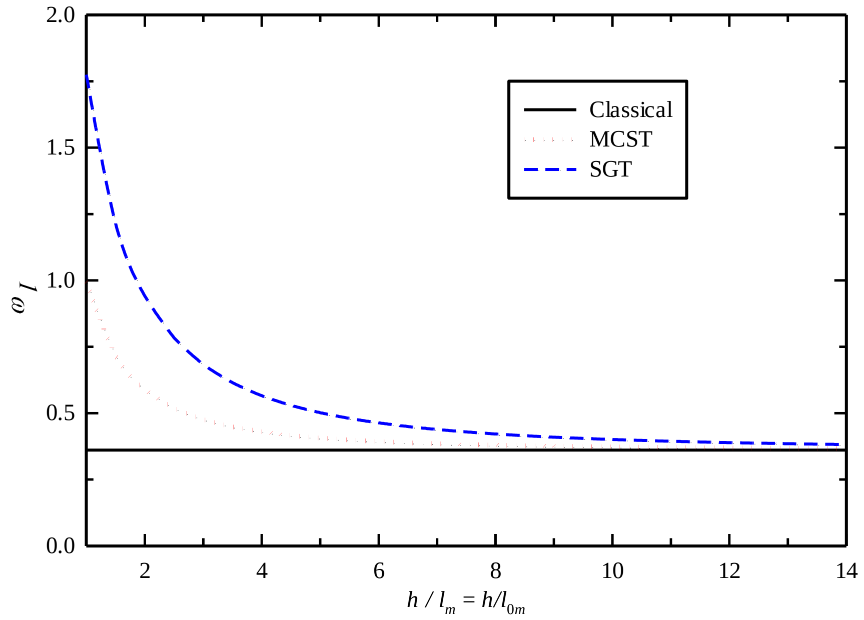

Fig. 21. Variations of the first dimensionless natural frequency with respect to $h/l_m = h/l_{0m}$ for FGM micro - beam with $L/h = 10$, $l_m = l_{0m} = 15$ μm, $b/h = 2$, $l_c/l_m = l_{0c}/l_{0m} = 3/2$, $n = 2$, TOBT, considering different elasticity theories.

The results provided in Fig. 22 and Fig. 23 are calculated by using the third - order beam theory. Fig. 22 depicts ω_1 as a function of the ratio h/l_m and Fig. 23 presents ω_1 as a function of the ratio h/l_{0m} and the volume fraction exponent *n*, by utilizing MCST and SGT, respectively. The increase in the exponent n is seen to result in a corresponding decrease in ω_1, which implies that metal - rich small - scale beams exhibit smaller nondimensional natural frequencies. By using MCST and SGT, respectively in Fig. 24 and Fig. 25 variations of ω_1 with respect to h/l_m and h/l_{0m} are shown for four different values of the length scale parameter ratio l_c/l_m and l_{0c}/l_{0m}. Nondimensional frequency increases as the ratio l_c/l_m is increased from $1/3$ to 2. The increase is much more significant when h/l_m is relatively smaller. This observation is another verification of the fact that through - the - thickness variation of the length scale parameter l has to be taken into account in the analysis of small - scale functionally graded beams.

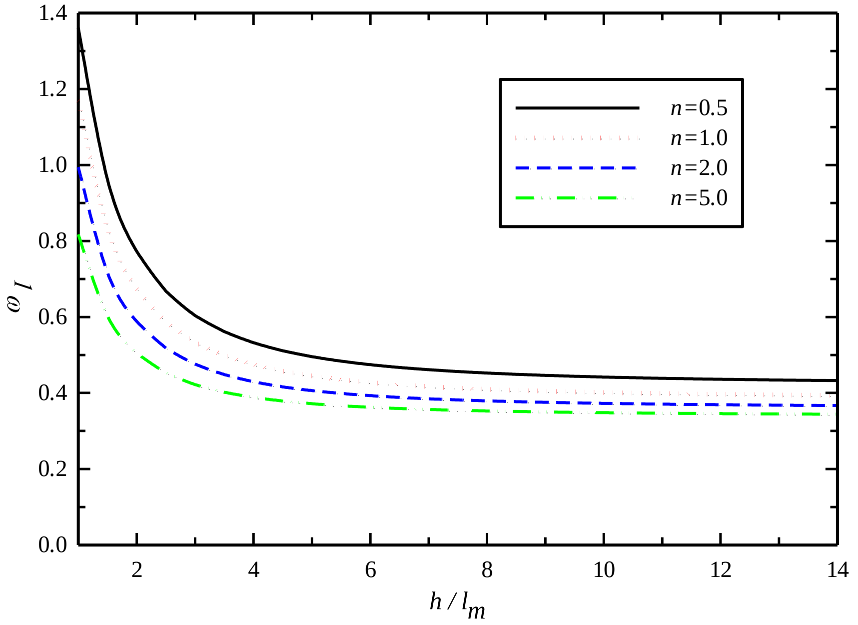

Fig. 22. Variations of the first dimensionless natural frequency with respect to h/l_m and n for FGM micro - beam with $L/h = 10$, $l_m = 15$ μm, $b/h = 2$, $l_c/l_m = 3/2$, MCST, TOBT.

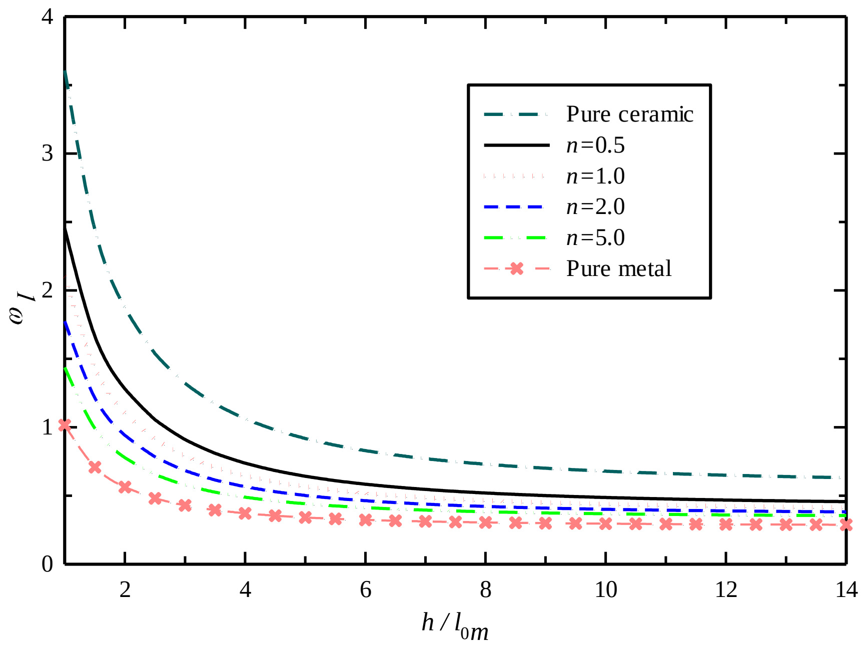

Fig. 23. Variations of the first dimensionless natural frequency with respect to h/l_{0m} and n for FGM micro - beam with $L/h = 10$, $l_{0m} = 15$ μm, $b/h = 2$, $l_{0c}/l_{0m} = 3/2$, SGT, TOBT.

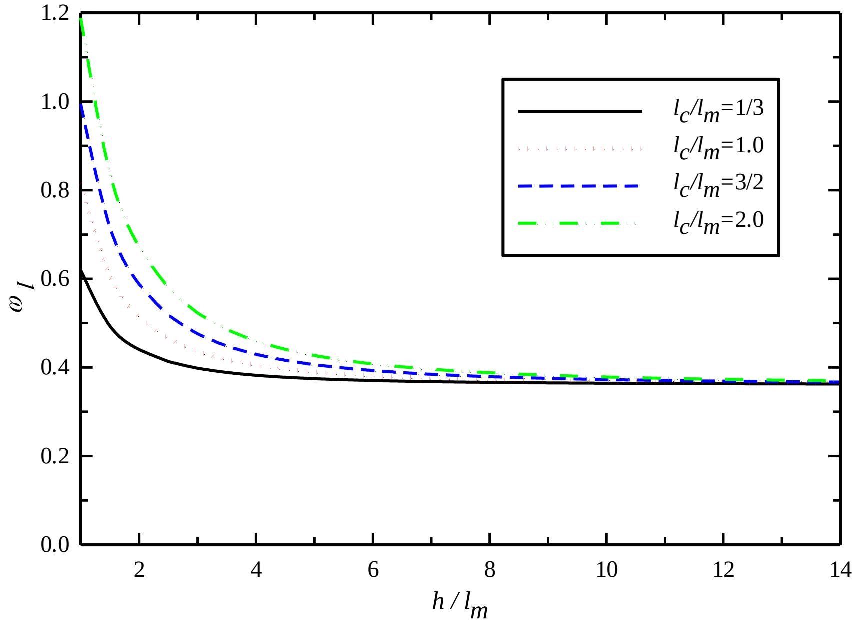

Fig. 24. Variations of the first dimensionless natural frequency with respect to h/l_m and l_c/l_m for FGM micro - beam with $L/h=10$, $l_m=15$ μm, $b/h=2$, $n=2$, MCST, TOBT.

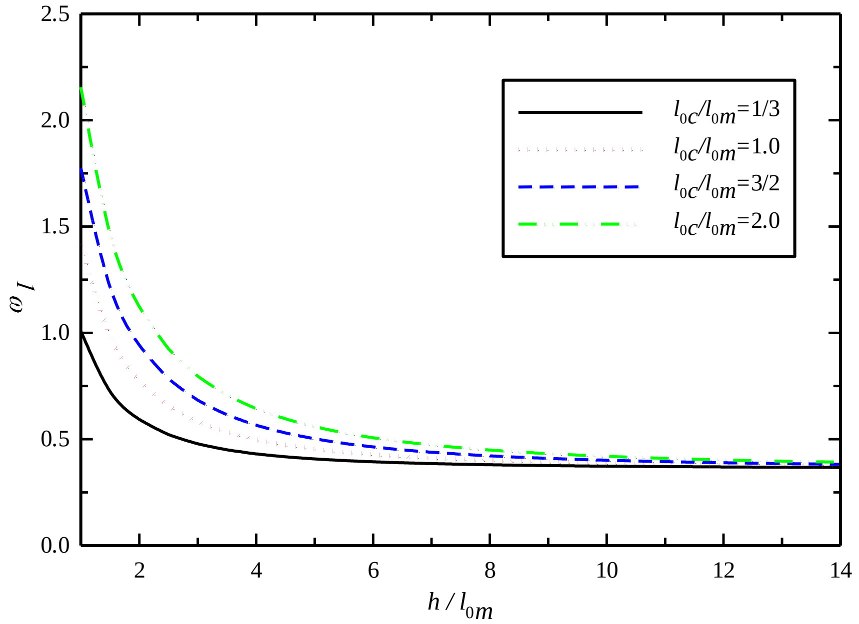

Fig. 25. Variations of the first dimensionless natural frequency with respect to h/l_{0m} and l_{0c}/l_{0m} for FGM micro - beam with $L/h=10$, $l_{0m}=15$ μm, $b/h=2$, $n=2$, SGT, TOBT.

The variations of dimensionless natural frequency of FGM micro - beam with the length to thickness ratio predicted by different beam and elasticity theories are depicted in Fig. 26 - Fig. 28. In Fig. 26 and Fig. 27 it is observed that higher values of dimensionless natural frequency can be obtained for smaller values of slenderness ratio L / h. Also, it can be inferred that, like the static case, TOBT predicts stiffer beam than TBT and EBBT. In the free vibration problem, for all three elasticity theories considered in this study, similar trends are observed, where it is seen that the natural frequency predicted by MCST is higher than that by the classical elasticity theory and smaller than that of SGT. This behavior is observed in Fig. 28 which depicts the natural frequency versus slenderness ratio curve and obtained by using TOBT.

Table 8 tabulates the first three dimensionless natural frequencies corresponding to the transverse deformation mode computed by implementing MCST for various values of the exponent n and the ratio l_c / l_m. Similar results for SGT are given in Table 9. The results are generated by utilizing the third - order beam theory. The effects of both n and l_c / l_m (or l_{0c} / l_{0m}) are seen to be important. For each value of n, dimensionless frequency increases significantly as l_c / l_m (or l_{0c} / l_{0m}) is increased. On the other hand, the increase in the exponent n leads to a drop in the dimensionless frequency ω.

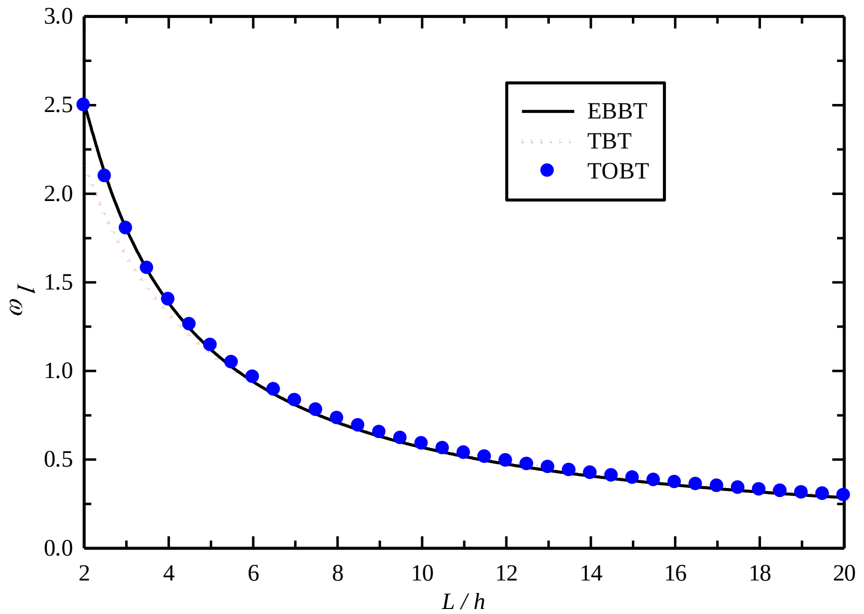

Fig. 26. Variations of the first dimensionless natural frequency with respect to L / h for FGM micro - beam with, $l_m = 15$ µm, $h / l_m = 2$, $b / h = 2$, $l_c / l_m = 3 / 2$, $n = 2$, MCST, considering different beam theories.

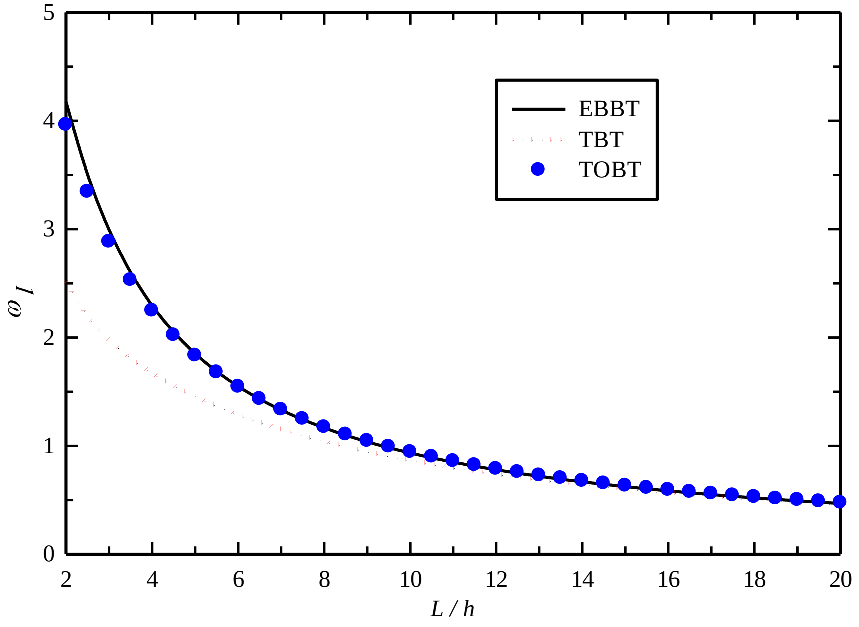

Fig. 27. Variations of the first dimensionless natural frequency with respect to L/h for FGM micro - beam with, $l_{0m} = 15$ µm, $h/l_{0m} = 2$, $b/h = 2$, $l_{0c}/l_{0m} = 3/2$, $n = 2$, SGT, considering different beam theories.

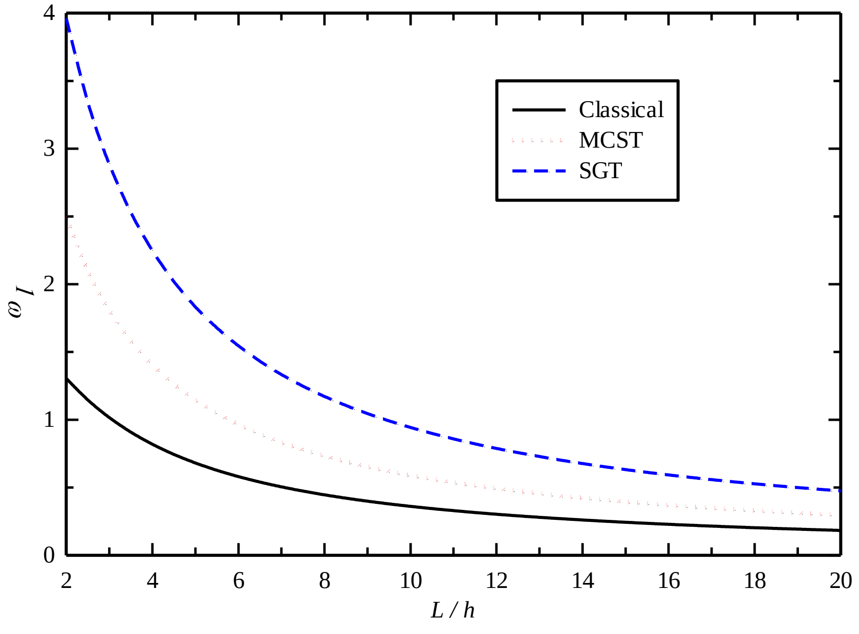

Fig. 28. Variations of the first dimensionless natural frequency with respect to L/h for FGM micro - beam with $l_m = l_{0m} = 15$ µm, $h/l_m = h/l_{0m} = 2$, $b/h = 2$, $l_c/l_m = l_{0c}/l_{0m} = 3/2$, $n = 2$, TOBT, considering different elasticity theories.

Table 8. Dimensionless natural frequencies corresponding to the transverse deformation mode computed for various values of n and l_c / l_m, for FGM micro - beam with $L/h = 10$, $l_m = 15$ μm, $h / l_m = 2$, $b / h = 2$, MCST, TOBT.

Mode	n	l_c / l_m			
		1 / 3	**1.0**	**3 / 2**	**2.0**
First	**0.5**	0.4847	0.6298	0.7730	0.9308
	1.0	0.4572	0.5660	0.6737	0.7937
	2.0	0.4403	0.5140	0.5881	0.6725
	5.0	0.4268	0.4652	0.5050	0.5519
Second	**0.5**	1.8618	2.4387	3.0055	3.6283
	1.0	1.7524	2.1901	2.6201	3.0968
	2.0	1.6822	1.9859	2.2867	2.6252
	5.0	1.6267	1.7928	1.9599	2.1525
Third	**0.5**	3.9540	5.2405	6.4954	7.8666
	1.0	3.7137	4.7023	5.6642	6.7214
	2.0	3.5517	4.2558	4.9415	5.7019
	5.0	3.4261	3.8302	4.2259	4.6716

Table 9. Dimensionless natural frequencies corresponding to the transverse deformation mode computed for various values of n and l_{0c} / l_{0m}, for FGM micro - beam with $L/h = 10$, $l_{0m} = 15$ μm, $h / l_{0m} = 2$, $b / h = 2$, SGT, TOBT.

Mode	n	l_{0c} / l_{0m}			
		1 / 3	**1.0**	**3 / 2**	**2.0**
First	**0.5**	0.6130	0.9690	1.2804	1.6057
	1.0	0.5993	0.8651	1.1029	1.3546
	2.0	0.5928	0.7737	0.9415	1.1233
	5.0	0.5882	0.6839	0.7775	0.8832
Second	**0.5**	2.3531	3.7247	4.9260	6.1760
	1.0	2.2937	3.3353	4.2667	5.2457
	2.0	2.2602	2.9934	3.6654	4.3843
	5.0	2.2382	2.6529	3.0444	3.4760
Third	**0.5**	5.0030	7.9395	10.4969	13.1237
	1.0	4.8619	7.1331	9.1425	11.2120
	2.0	4.7717	6.4292	7.9132	9.4551
	5.0	4.7159	5.7190	6.6300	7.5968

4.3.2 Natural frequencies with thermal effects

The relationships between dimensionless natural frequency and height to material length scale parameter and length to height of the FG micro - beam for different values of temperature change ΔT are illustrated in Fig. 29 - Fig. 36 and Table 10 and Table 11. Similar to the static case, increasing the value of ΔT decreases the stiffness of the beam resulting in small values of dimensionless natural frequency. The thermal influence is more sensible in higher values of height to material length scale parameter and slenderness ratio. The effect of temperature change on the first three dimensionless natural frequencies for different values of exponent n are numerically presented in Table 10 for MCST and Table 11 for SGT.

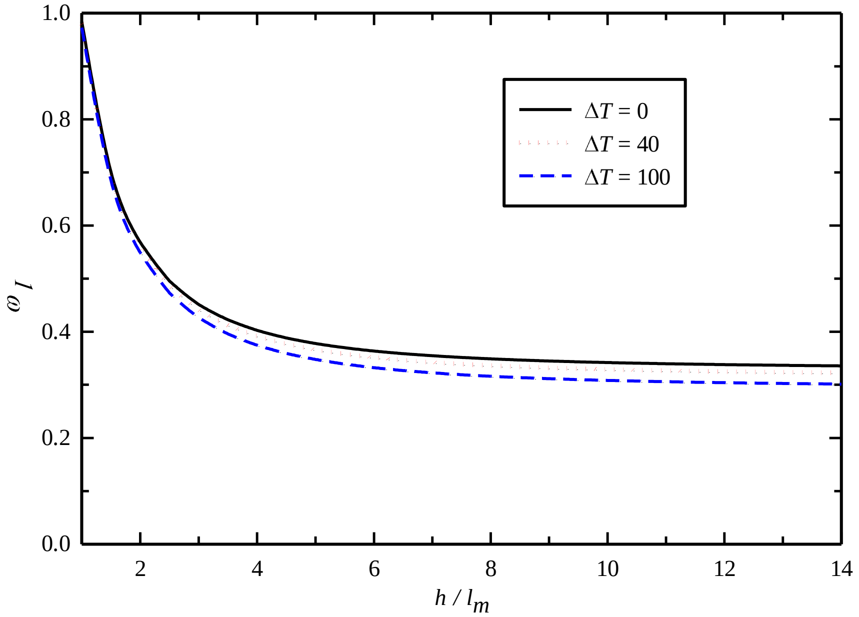

Fig. 29. Variations of the first dimensionless natural frequency with respect to h / l_m and ΔT for FGM micro - beam with $L / h = 10$, $l_m = 15$ μm, $b / h = 2$, $l_c / l_m = 3/2$, $n = 2$, MCST, EBBT.

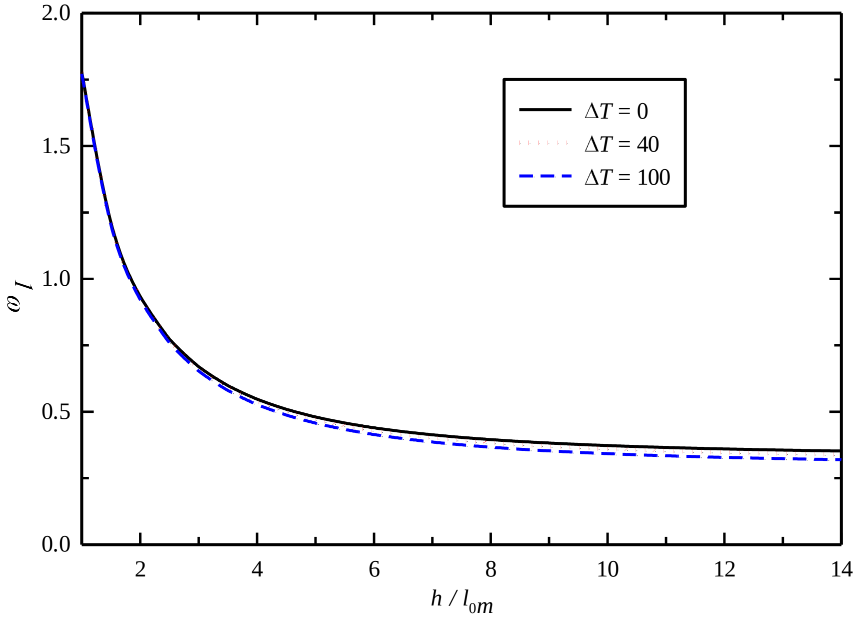

Fig. 30. Variations of the first dimensionless natural frequency with respect to h / l_{0m} and ΔT for FGM micro - beam with $L / h = 10$, $l_{0m} = 15$ μm, $b / h = 2$, $l_{0c} / l_{0m} = 3/2$, $n = 2$, SGT, EBBT.

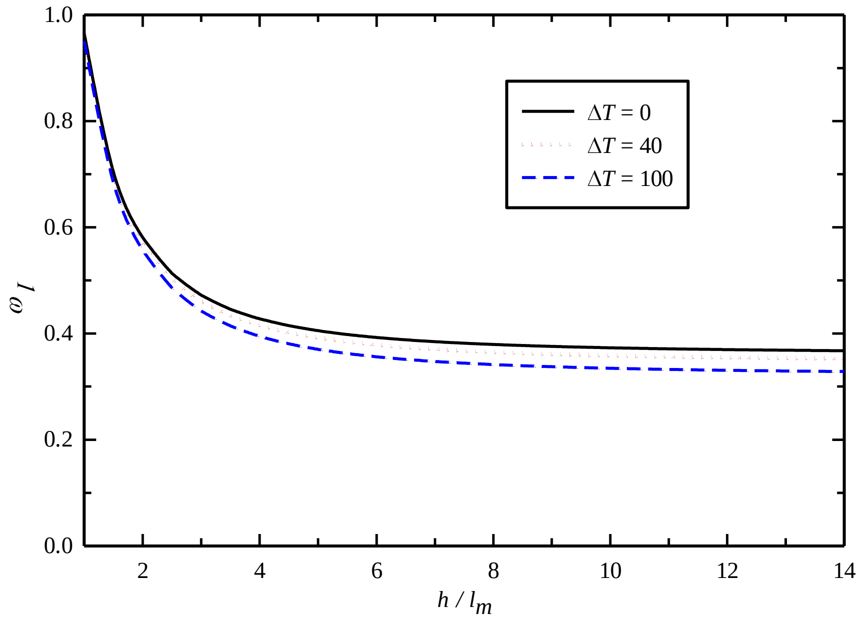

Fig. 31. Variations of the first dimensionless natural frequency with respect to h/l_m and ΔT for FGM micro - beam with $L/h = 10$, $l_m = 15$ μm, $b/h = 2$, $l_c / l_m = 3/2$, $n = 2$, MCST, TBT.

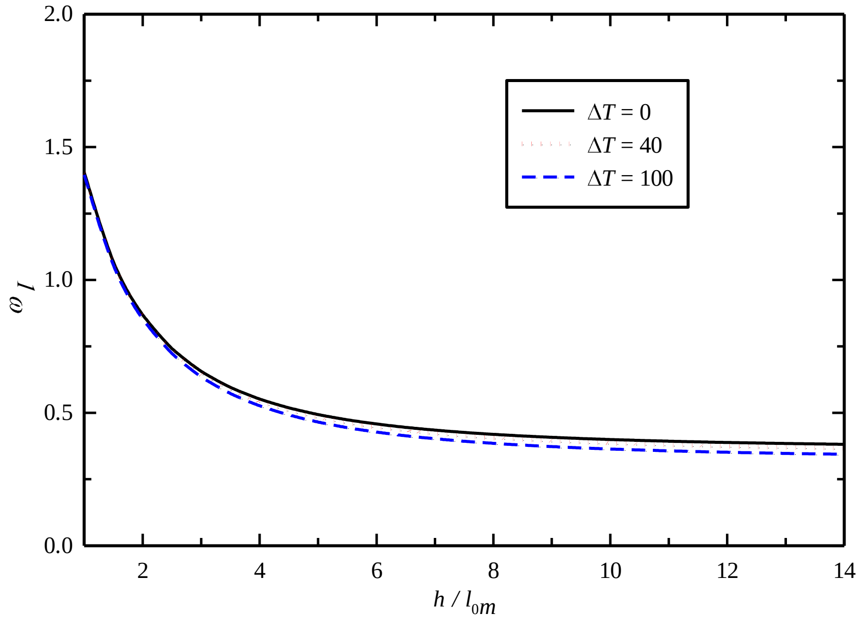

Fig. 32. Variations of the first dimensionless natural frequency with respect to h/l_{0m} and ΔT for FGM micro - beam with $L/h = 10$, $l_{0m} = 15$ μm, $b/h = 2$, $l_{0c} / l_{0m} = 3/2$, $n = 2$, SGT, TBT.

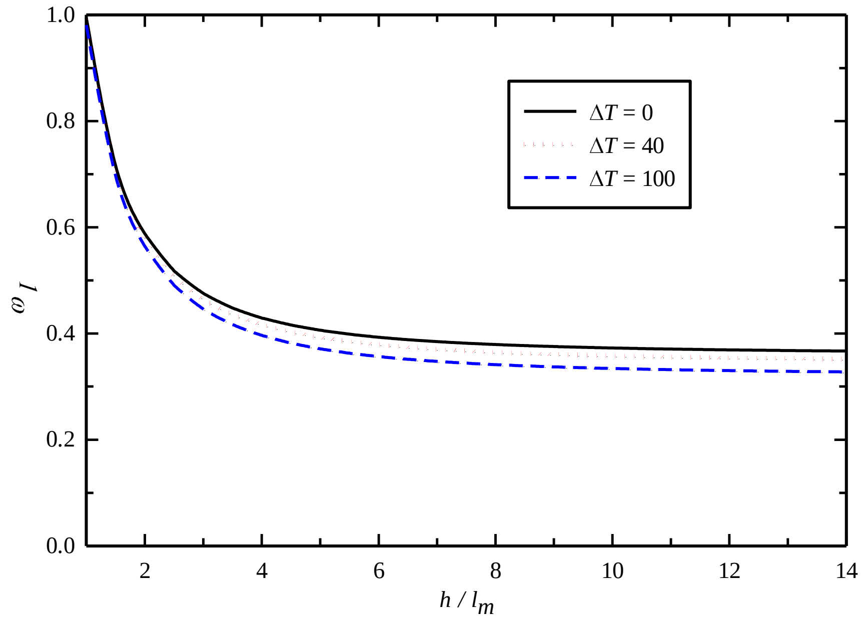

Fig. 33. Variations of the first dimensionless natural frequency with respect to h / l_m and ΔT for FGM micro - beam with $L / h = 10$, $l_m = 15$ μm, $b / h = 2$, $l_c / l_m = 3/2$, $n = 2$, MCST, TOBT.

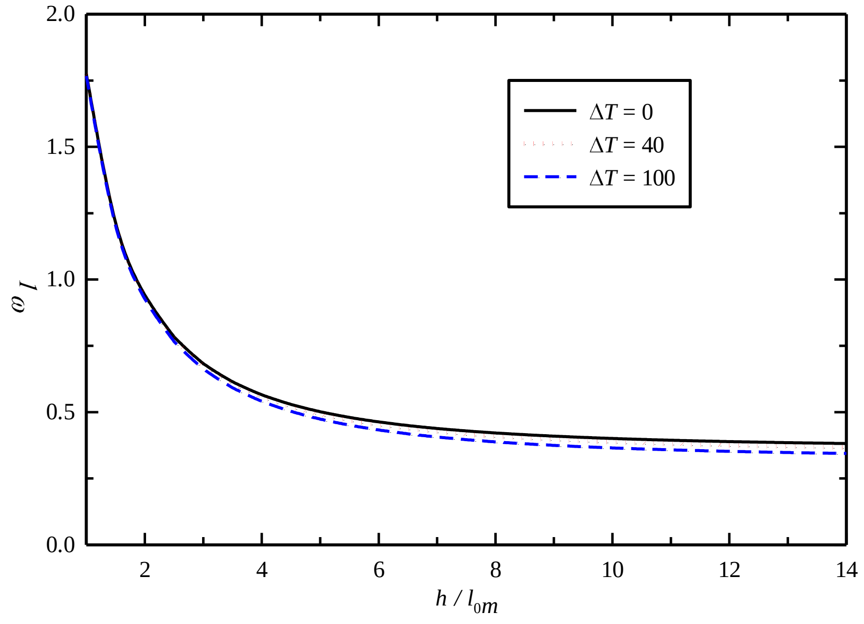

Fig. 34 Variations of the first dimensionless natural frequency with respect to h / l_{0m} and ΔT for FGM micro - beam with $L / h = 10$, $l_{0m} = 15$ μm, $b / h = 2$, $l_{0c} / l_{0m} = 3/2$, $n = 2$, SGT, TOBT.

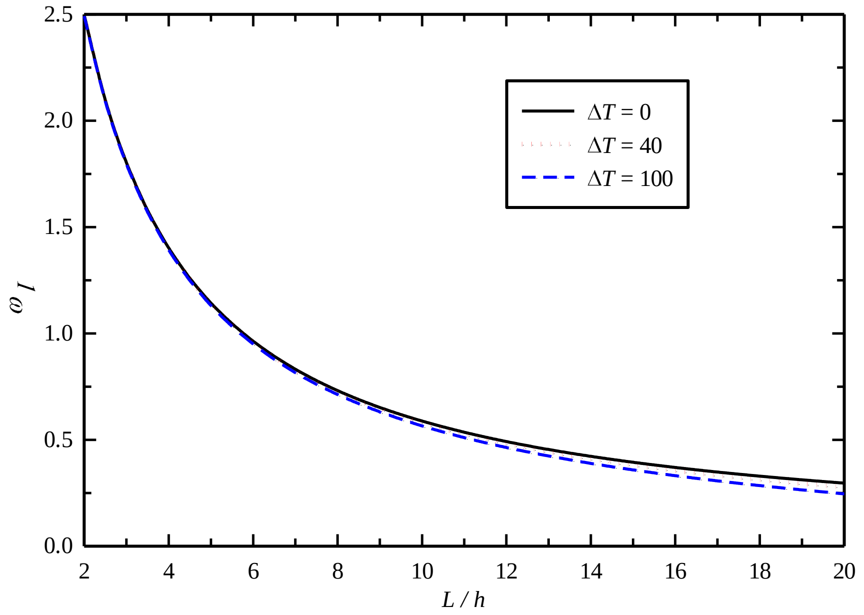

Fig. 35. Variations of the first dimensionless natural frequency with respect to L/h and ΔT for FGM micro - beam with, $l_m = 15$ µm , $h/l_m = 2$, $b/h = 2$, $l_c / l_m = 3/2$, $n = 2$, MCST, TOBT.

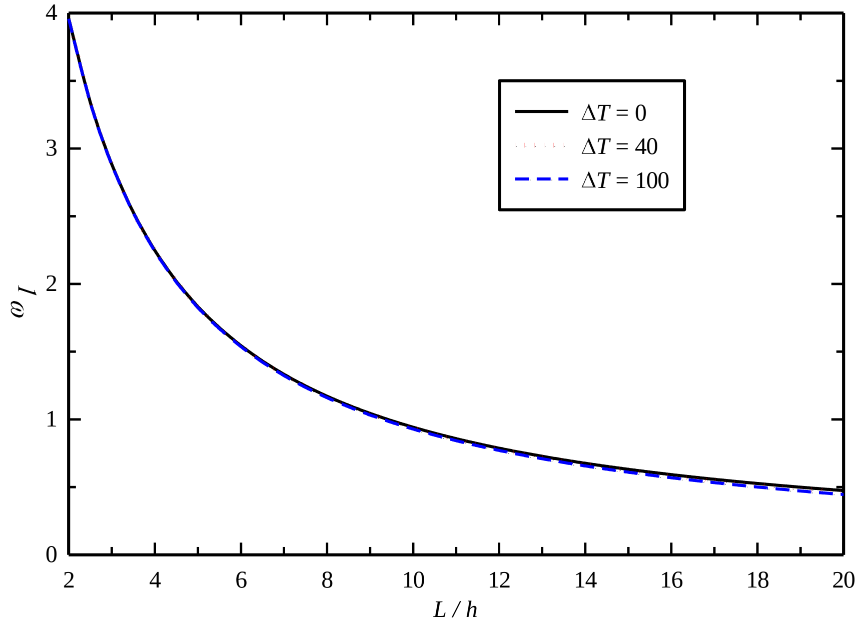

Fig. 36. Variations of the first dimensionless natural frequency with respect to L/h and ΔT for FGM micro - beam with, $l_{0m} = 15$ µm , $h/l_{0m} = 2$, $b/h = 2$, $l_{0c} / l_{0m} = 3/2$, $n = 2$, SGT, TOBT.

Table 10. Dimensionless natural frequencies corresponding to the transverse deformation mode computed for various values of n and ΔT, for FGM micro - beam with $L/h=10$, $l_m = 15$ μm, $h/l_m = 2$, $b/h=2$, $l_c/l_m = 3/2$, MCST, TOBT.

Mode	n	$\Delta T\,(°C)$		
		0	40	100
First	0.5	0.7730	0.7670	0.7579
	1.0	0.6737	0.6667	0.6561
	2.0	0.5881	0.5802	0.5680
	5.0	0.5050	0.4958	0.4818
Second	0.5	3.0055	2.9995	2.9904
	1.0	2.6201	2.6131	2.6027
	2.0	2.2867	2.2788	2.2668
	5.0	1.9599	1.9507	1.9369
Third	0.5	6.4954	6.4893	6.4802
	1.0	5.6642	5.6572	5.6467
	2.0	4.9415	4.9335	4.9215
	5.0	4.2259	4.2166	4.2026

Table 11. Dimensionless natural frequencies corresponding to the transverse deformation mode computed for various values of n and ΔT, for FGM micro - beam with $L/h=10$, $l_{0m} = 15$ μm, $h/l_{0m} = 2$, $b/h=2$, $l_{0c}/l_{0m} = 3/2$, SGT, TOBT.

Mode	n	$\Delta T\,(°C)$		
		0	40	100
First	0.5	1.2804	1.2768	1.2714
	1.0	1.1029	1.0987	1.0923
	2.0	0.9415	0.9366	0.9291
	5.0	0.7775	0.7715	0.7626
Second	0.5	4.9260	4.9224	4.9169
	1.0	4.2667	4.2624	4.2560
	2.0	3.6654	3.6605	3.6531
	5.0	3.0444	3.0385	3.0296
Third	0.5	10.4969	10.4932	10.4875
	1.0	9.1425	9.1383	9.1318
	2.0	7.9132	7.9083	7.9009
	5.0	6.6300	6.6241	6.6154

In most of the works on the free vibration analysis of micro - beams, only the transverse vibration is considered and the axial and rotational mode shapes are neglected. Unless very small values of length to scale ratio are used, the dominant mode shapes of first natural frequencies belong to transverse vibration. However, the axial natural frequencies may occur after the first one. Table 12 gives the first eight dimensionless natural frequencies predicted by TBT and SGT. For each mode, the mode shapes can be drawn for axial, transverse and rotational vibrations separately as shown in Fig. 37. By inspecting the order of the mode shapes, the dominant one is detected at that natural frequency. As it can be seen in Table 12 in the first two natural frequencies the transverse vibration is dominant and the third natural frequency belongs to axial mode shape.

Table 12. First eight dimensionless natural frequencies of FGM micro - beam with $L/h=10$, $h/l_{0m}=2$, $l_{0c}/l_{0m}=1$, $n=2$, SGT, TBT.

Mode	Dimensionless natural frequency	Dominant mode shape
1	0.7346	Transverse
2	2.5532	Transverse
3	4.1003	Axial
4	4.9649	Transverse
5	7.8083	Transverse
6	8.4369	Axial
7	11.0908	Transverse
8	13.2346	Axial

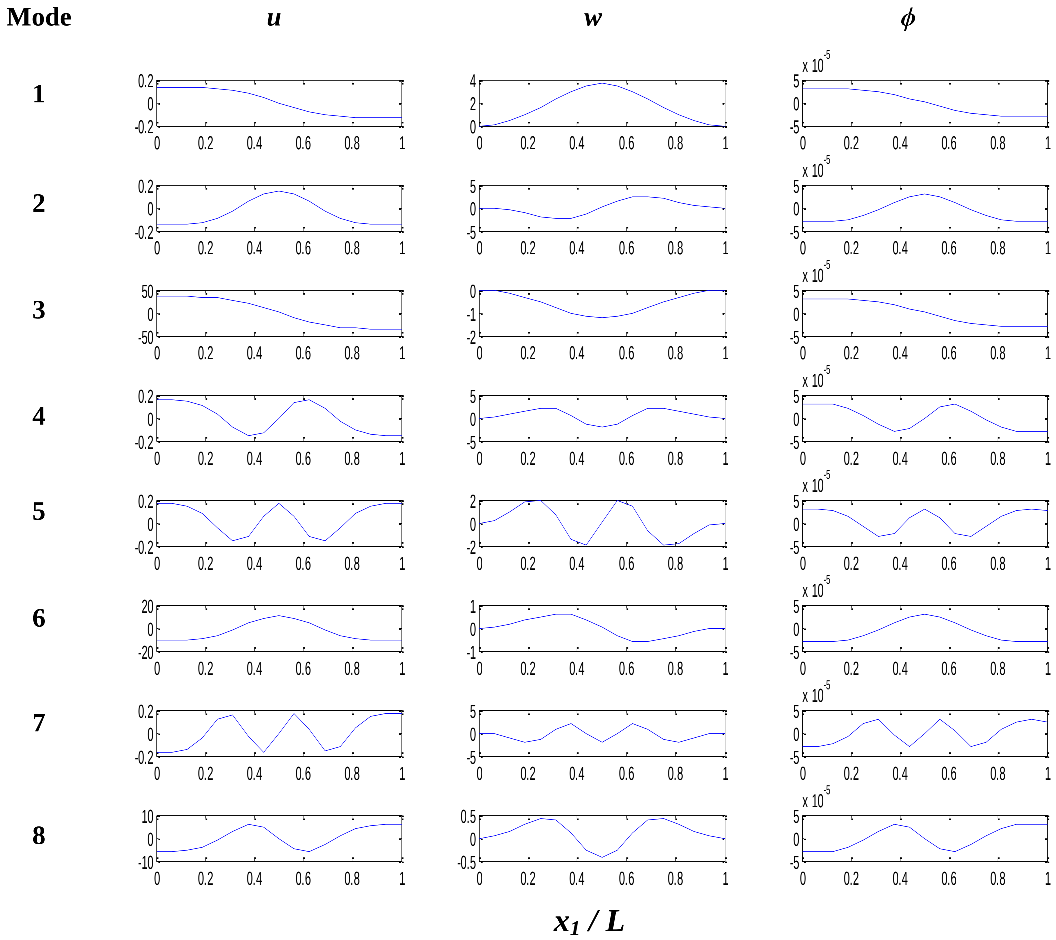

Fig. 37. Axial (u), transverse (w) and rotational (ϕ) mode shapes for first eight dimensionless natural frequencies of FGM micro - beam with $L/h=10$, $h/l_{0m}=2$, $l_{0c}/l_{0m}=1$, $n=2$, SGT, TBT.

CHAPTER 5

CONCLUSION AND FUTURE WORK

In this work a method of analysis based on the higher order continuum theories for small - scale functionally graded beams, that possess a variable length scale parameter, is presented. The formulation is carried out in such a way that, proposed procedures make it possible to generate results regarding three different beam theories, which are Euler - Bernoulli beam theory, Timoshenko beam theory, and third - order beam theory. Both, beams that are statically loaded and those undergoing free vibrations are considered in the developments. Governing partial differential equations are derived by employing Hamilton's principle. These equations are solved numerically by means of the differential quadrature method.

Detailed numerical analyses for the static deflection and free vibration of the FGM micro - beam are given in Chapter 4. Comparisons of our results to those of other researchers show the accuracy of numerical method used in this work. Further results presented illustrate the influences of geometric and material parameters upon the static and the free vibration responses of small - scale FGM beams.

Two main findings of this study justify the development of a general approach for the analysis of small - scale functionally graded beams possessing a variable length scale parameter. It is seen that for such a small - scale beam normal stresses predicted by the use of the Euler - Bernoulli beam theory deviate significantly from those calculated by using either of the Timoshenko theory or the third - order theory. Hence, in the development of an analysis technique, the in - plane shear deformation needs to be incorporated into the formulation. Furthermore, the variation in the length scale parameter is shown to strongly influence both the static and the free vibration responses of a small - scale FGM beam. As a result, in the formulation of the small - scale beam problems, this variation needs to be taken into account. The method presented in this article is general in the sense that it allows the consideration of the in - plane shear deformation as well as the spatial variation of the length scale parameter. Thus, it could prove useful in the analysis, design, and optimization of small - scale functionally graded beams.

The results of static deflection and free vibration analyses of FG micro - beam for different values of temperature change ΔT show that, increasing the value of ΔT leads to decrease the stiffness of the beam.

In higher order elasticity approach traction boundary conditions are changed and to improve the results of analyses it is better to derive new beam model which satisfies these conditions.

Except for homogeneous epoxy beam, there are no experimental data on the small - scale parameters of other materials. To present more accurate static and dynamic characteristics of the micro - beam, it is essential to make efforts to find the value of small - scale parameters for other materials; consequently the small - scale parameter of FGM can be evaluated.

The developed models in this study can be used accurately for static and free vibration analyses of sensors, actuators, atomic force microscopy (AFM) and other MEMS made of homogeneous material or FGM with different types of boundary conditions. The models can be used to validate and improve the approach to predict the static deflection and natural frequencies of the micro - beam.

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