PRICING AMERICAN OPTIONS UNDER DISCRETE AND CONTINUOUS TIME SETTING

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submitted by **SİNEM KOZPINAR** in partial fulfillment of the requirements for the degree of **Master of Science in Department of Financial Mathematics, Middle East Technical University** by,

Prof. Dr. Bülent Karasözen
Director, Graduate School of **Applied Mathematics** __________

Assoc. Prof. Dr. A. Sevtap Kestel
Head of Department, **Financial Mathematics, IAM, METU** __________

Assist. Prof. Dr. Yeliz Yolcu Okur
Supervisor, **Financial Mathematics, IAM, METU** __________

Prof. Dr. Z. Nuray Güner
Co-supervisor, **Department of Business Administration, METU** __________

Examining Committee Members:

Prof. Dr. Gerhard Wilhelm Weber
Financial Mathematics, METU __________

Assist. Prof. Dr. Yeliz Yolcu Okur
Financial Mathematics, METU __________

Assoc. Prof. Dr. Azize Hayfavi
Financial Mathematics, METU __________

Assoc. Prof. Dr. Ömür Uğur
Scientific Computing, METU __________

Assoc. Prof. Dr. A. Sevtap Kestel
Actuarial Sciences, METU __________

Date: __________

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last Name: SİNEM KOZPINAR

Signature :

ABSTRACT

PRICING AMERICAN OPTIONS UNDER DISCRETE AND CONTINUOUS TIME SETTING

Kozpınar, Sinem
M.S., Department of Financial Mathematics
Supervisor : Assist. Prof. Dr. Yeliz Yolcu Okur
Co-Supervisor : Prof. Dr. Z. Nuray Güner

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In this thesis, pricing of American options are analyzed in discrete and continuous time markets. We first discuss the discrete-time valuation of American options assuming that the underlying asset pays no dividend during the life of the option. In this setting, we uniquely price American options by introducing the Snell envelope and optimal stopping time problem. We prove the main results studied in Lamberton and Lapeyre (1996) in details. In addition, we show that the price of an American call option with no dividend is equal to the price of its European counterpart. Then, we extend this results to the continuous-time for both dividend and no dividend case. Following Black-Scholes model, we present two different techniques to price American options: martingale pricing technique and variational inequalities. Under martingale pricing approach, we take the expectation of discounted payoff process and determine the stopping time that maximizes this expected value. Then, we derive a pricing formula for both dividend and no dividend case. We also show that an early exercise is not optimal for American call options without dividend. We observed that this rule is not valid for American call options on a dividend paying underlying asset. Then, we introduce the variational inequalities that an American option satisfies and investigate the regular solutions of this inequalities. However, these approaches generally do not admit a closed-form solution for the price of American options. Therefore, we give a brief introduction to the finite difference and PSOR methods and adapt these methods for American options. Finally, a numerical application is done by comparing the efficiencies of these methods. Moreover, the impact of Black-Scholes parameters K, σ, δ on the price of American options are also investigated. The thesis ends with a conclusion and an outlook to future studies.

Keywords: American option, pricing, optimal stopping time problem, martingale pricing, variational inequalities, finite difference method, PSOR method

ÖZ

AMERİKAN OPSİYONLARININ KESİKLİ VE SÜREKLİ ZAMAN MODELLERİ ALTINDA FİYATLANMASI

Kozpınar, Sinem
Yüksek Lisans, Finansal Matematik Bölümü
Tez Yöneticisi : Yard. Doç. Dr. Yeliz Yolcu Okur
Ortak Tez Yöneticisi : Prof. Dr. Z. Nuray Güner

Eylül 2013, 85 sayfa

Bu tezde, Amerikan tipi opsiyonların kesikli ve sürekli modellerdeki fiyatlaması analiz edilmiştir. İlk olarak dayanak varlığın temettü ödemediği varsayımı altında kesikli fiyatlama modeli incelenmiştir. Bu çerçevede Snell envelope ve optimal durma problemlerinden yararlanarak Amerikan opsiyonlarının teorik fiyatları elde edilmiştir. Ayrıca, bu konuyla ilgili başlıca kaynaklardan olan Lamberton ve Lapeyre'in (1996) temel sonuçları detaylı bir şekilde ispatlanmış ve Amerikan ile Avrupa tipi opsiyonların temettüsüz durumda aynı fiyata sahip olduğu gösterilmiştir. Daha sonrasında kesikli piyasalar için gösterilen bu sonuçlar, temettü durumunda sürekli piyasalar için incelenmiştir. Black-Scholes modeli baz alınarak fiyatlama konusu iki farklı yaklaşım altında incelenmiştir: martingale ile fiyatlama ve varvasyonel eşitsizlik teknikleri. Martingale yaklaşımı altında opsiyonun iskonto edilmiş ödeme değerinin beklenentisinin en yüksek olduğu durma anı belirlenmiştir. Bundan yararlanarak temettülü ve temettüsüz durumlar için iki ayrı fiyatlama yapılmıştır. Ayrıca temettüsüz durumda Amerikan opsiyonlarının erken kullanımının optimal olmadığı ve temettülü durumda ise bu kuralın geçersiz olduğu gösterilmiştir. Daha sonrasında varyasyonel eşitsizlikler takdim edilmiş ve bunların düzenli çözümleri araştırılmıştır. Genel olarak bu yaklaşımlar ile Amerikan opsiyonları için kapalı bir çözüm elde edilemediğinden farklar metodu ile türevler ve PSOR yöntemleri takdim edilmiştir. Son olarak, bu yöntemlerin uygulanabilirlikleri karşılaştırılıp, Black-Scholes modelinin parametreleri K, σ, δ'nın fiyatlamaya etkisi nümerik olarak ortaya konulmuştur. Bu çalışma bir

değerlendirme ve gelecek çalışmalara bir bakış ile sonuçlandırılmıştır.

Anahtar Kelimeler: Amerikan tipi opsiyonlar, fiyatlama, riskten korunma portföyleri, Black-Scholes modeli, Snell envelope, optimal durma problemi, martingale ile fiyatlama, varyasyonel eşitsizlikler, farklar metodu ile türevler, PSOR yöntemi

To My Family

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CHAPTER 1

INTRODUCTION

In an economy, the investors can expose a considerably high risk because of the undesired fluctuations in security prices. Therefore, they need to control that risk with the help of some financial instruments. American options are one of the most popular financial instruments created to meet these needs. Besides being used to reduce the risk, American options also provide their holders, unlike the European options, the flexibility of exercising at any time up to maturity. This structure of American options makes them more valuable for investors. That is why, the valuation of these special contracts is an important research area for investors.

The early exercise possibility turns the valuation of an American option into a difficult problem to solve. One way to solve this problem is the martingale pricing approach. According to this approach, we can price a contingent claim in a complete market by taking the expectation of its discounted payoff with respect to the risk-neutral probability measure (see, e.g. [27] and [23]). Since the completeness of the market allows us to find a replicating portfolio which hedges the option, its price at any time t is indeed equal to the value of this portfolio. Moreover, since we deal with the American options, it can be seen that this valuation approach is closely related to the optimal stopping time problem. The theoretical aspects on this approach were firstly proposed by McKean (1965) [25] and Samuelson (1965) [34]. Since the arbitrage-free pricing of contingent claims was not studied yet, McKean proposed to price the American put options under the natural probability measure. The valuation process was then developed by many authors, such as van Moerbeke (1976) [41], Karatzas (1988) [20], Bensoussan (1984) [2]. Van Moerbeke continued to study on the optimal stopping time problem whereas Karatzas and Bensoussan associated the problem with the replicating portfolios. On the other hand, there is another approach commonly used to price American options. This approach, say pricing with variational inequalities, deals with the solutions of parabolic partial differential inequalities that an American option satisfies. This approach was evolved by Bensoussan and Lions [3]. Then, Jaillet, Lamberton and Lapeyre dealt with the regular solutions of these systems in [18].

These pricing approaches generally do not admit a closed-form price formula for

American options. Therefore, several studies have been conducted on the numerical solutions of these methods. Finite difference method is widely used in financial mathematics for the numerical solutions of partial differential equalities (PDE). One of the pioneering method was proposed by Crank and Nicolson [10] in 1947. Afterwards, Brennan and Schwartz (1977, 1978) introduced explicit and implicit methods in [7, 35]. Then, Courtadon (1982) developed these methods in [9]. Under these numerical approaches, we can approximate the variational inequalities that an American option satisfy with their finite difference quotients. Projected SOR method (PSOR) that was suggested by Cryer (1971) [11] is another numerical approach used to price options numerically. This approach indeed deals with the iterative solutions of a linear system of equations of the form $Ax = b$.

In this thesis, we investigate the valuation of American options in discrete and continuous time markets. We first discuss the discrete-time valuation of American options assuming that the option holder does not take a dividend payment during the life of the option. After prevailing the theoretical aspects, we extend our results to the continuous-time. We follow the well-known Black-Scholes model [5] and analyze the valuation concept for both dividend and non-dividend case. We introduce two fundamental techniques commonly used in the valuation of American options: martingale pricing and pricing with variational inequalities. These approaches generally do not admit a closed-form solution for the price of American options. Hence, we give a brief introduction to the finite difference and PSOR methods. Finally, a numerical application is done by comparing the efficiencies of these methods. We also investigate the impact of Black-Scholes parameters, σ, δ, K, on the price of American options.

The outline of this thesis is as follows: In Chapter 2, we present some mathematical preliminaries including fundamental definitions and results used in discrete and continuous-time markets. In Chapter 3, we discuss the fair price of American options with no dividend under discrete-time settings. Throughout this chapter, the market is assumed to be complete. Under this assumption, we obtain the unique price of an American option with the help of a replicating portfolio that hedges the option. Moreover, we describe the Snell envelope and optimal stopping time problem. We prove the main results studied in [23] in details. We also show that the price of an American call option with no dividend is equal to the price of an European call with the same maturity, same strike price and same underlying. In Chapter 4, we introduce the continuous-time valuation of American options on an underlying asset that does not pay any dividend. We present two different approaches for pricing. The first technique is martingale pricing approach that deals with the expectation of discounted payoff processes under the risk-neutral probability measure [4, 22, 23]. Under this approach, we prove our main theorem (Theorem 4.6) in details. Secondly, we investigate the solutions of variational inequalities that American options admit [3, 18, 28]. Under this approach, we prove our main theorem (Theorem 4.13) for pricing in the case that the inequality system has a regular solution. Moreover, we show that it is not optimal to exercise an American call with no dividend before maturity. In Chapter 5, we extend the valuation process of American options to the dividend case. We closely follow the theory given in Chapter 4. We see that, if the underlying pays any dividend to its holder, American and European options with the same strike and same maturity are differently priced, contrary to the non-dividend case. In Chapter 6, we give a brief introduction about the finite difference and PSOR methods [40] adapting the valuation process for American options. In Chapter 7, we present a numerical application comparing these methods with the Binomial method. Moreover, we analyze the effects of Black-Scholes parameters K, δ, σ to the American option value. In chapter 8, we conclude the thesis and give an outlook to future studies.

CHAPTER 2

MATHEMATICAL PRELIMINARIES

In this chapter, we give some important definitions and results that are commonly used in discrete and continuous-time modeling. For sections, we refer to Lamberton and Lapeyre [23], Shreve [37] and Karatzas and Shreve [21].

2.1 Fundamentals of Discrete-Time Modeling

We consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0,1,\ldots,T}, \mathbb{P})$ on the finite time interval $[0, T]$ and suppose $\mathbb{P}(\{\omega\}) > 0$ for all $\omega \in \Omega$. These are the settings on which our discrete-time model will be constructed.

Now, we introduce the fundamental tools used in discrete-time modeling.

Definition 2.1. A random process $(Y_t)_{t=0,1,\ldots,T}$ is said to be *adapted*, if $Y_t \in \mathcal{F}_t$ for all $t \in \{0, 1, \ldots, T\}$.

Definition 2.2. A random process $(Y_t)_{t=0,1,\ldots,T}$ is said to be *predictable*, if Y_t is measurable with respect to the sigma-algebra $\mathcal{F}_{t-1}$ for all $t \in \{0, 1, \ldots, T\}$.

Definition 2.3. Let us assume that there exist $d + 1$ assets in the market. A *trading strategy* $\varphi = (\varphi_t^0, \varphi_t^1, \ldots, \varphi_t^d)_{t=0,1,\ldots,T}$ at time t is a vector in $\mathbb{R}^{d+1}$ where φ_t^i denotes the number of i'th asset in the portfolio.

In our discrete-time model, we assume that all trading strategies are predictable.

Definition 2.4. The *value of a trading strategy* φ at time t is given by

$$V_t(\varphi) = \varphi_t . S_t = \sum_{j=0}^{d} \varphi_t^j S_t^j .$$

Moreover,

$$\tilde{V}_t(\varphi) = \frac{1}{S_t^0} \varphi_t . S_t = \sum_{j=0}^{d} \varphi_t^j \tilde{S}_t^j$$

refers to the discounted value process of this portfolio at time t.

Definition 2.5. A trading strategy $\varphi = (\varphi_t^0, \varphi_t^1, \ldots, \varphi_t^d)$ is said to be *self-financing* if

$$\varphi_t.\, S_t = \varphi_{t+1}.\, S_t,$$

or, equivalently,

$$V_{t+1}(\varphi) - V_t(\varphi) = \varphi_{t+1}.\, (S_{t+1} - S_t)$$

for all $t = 0, 1, \ldots, T-1$.

Definition 2.6. A trading strategy φ is said to be *admissible* if the followings are satisfied

i) it is self-financing,

ii) $V_t(\varphi) \geq 0$ for all $t \in \{0, 1, \ldots, T\}$.

Definition 2.7. Let us consider an admissible strategy $\varphi = (\varphi_t^0, \varphi_t^1, \ldots, \varphi_t^d)_{t=0,1,\ldots,T}$ traded in the market and let $V_t(\varphi)$ denote its value at time t. Then, it is called an *arbitrage strategy* if the followings are satisfied

i) $V_0(\varphi) = 0$,

ii) $V_T(\varphi) > 0$.

Definition 2.8. Let us consider an $\mathcal{F}_t$-measurable, $\mathbb{R}$-valued random process (Y_t) satisfying

1. $\mathbb{E}\left(Y_{t+1} \mid \mathcal{F}_t\right) = Y_t$, for all $t \leq T-1$. Then, the process (Y_t) is a *martingale*,
2. $\mathbb{E}\left(Y_{t+1} \mid \mathcal{F}_t\right) \leq Y_t$, for all $t \leq T-1$. Then, the process (Y_t) is a *supermartingale*,
3. $\mathbb{E}\left(Y_{t+1} \mid \mathcal{F}_t\right) \geq Y_t$, for all $t \leq T-1$. Then, the process (Y_t) is a *submartingale.*

Proposition 2.1. ***(Constant Expectation Property)***

- *If the process (M_t) is an $\mathcal{F}_t$-martingale, then*

  $$\mathbb{E}(M_t) = \mathbb{E}(M_0)$$

  for all t.

- *If the process (M_t) is an $\mathcal{F}_t$-supermartingale, then*

  $$\mathbb{E}(M_n) \leq \mathbb{E}(M_0)$$

  for all t.

- *If the process (M_t) is an $\mathcal{F}_t$-submartingale, then*

$$\mathbb{E}(M_t) \geq \mathbb{E}(M_0)$$

for all t.

Proposition 2.2. *If the discounted asset price process $(\tilde{S}_t)$ is an $\mathcal{F}_t$-martingale, then the discounted value of a self-financing strategy $\tilde{V}_t(\phi)$ defined by*

$$\tilde{V}_t(\varphi) = V_0 + \sum_{j=1}^{t} \varphi_j . \Delta \tilde{S}_j$$

is also a martingale.

Definition 2.9. A market that does not allow arbitrage is said to be *viable.*

Theorem 2.3. *Let us consider that the market is viable. Then, there is a probability measure $\mathbb{P}^*$ equivalent to $\mathbb{P}$ under which the discounted stock price processes are martingales. Conversely, if we can find a probability measure $\mathbb{P}^*$ equivalent to $\mathbb{P}$ which makes the discounted asset prices martingale, then the market is said to be viable.*

Proof. The proof can be found in [23]. □

Definition 2.10. Let us consider a contingent claim represented by a non-negative stochastic variable c. This contingent claim c is said to be attainable if there is an admissible portfolio φ satisfying $V_T(\varphi) = c$.

Definition 2.11. If every contingent claim is attainable, then the market is said to be *complete.*

The next theorem implies that in a complete market, every contingent claim has a single price due to the uniqueness of the risk neutral probability measure $\mathbb{P}^*$. Hence, it provides us to obtain a unique theoretical price for the options. Moreover, using this theorem, we guarantee the existence of a replicating portfolio that hedges the option.

Theorem 2.4. *Let us suppose that the market is complete. Then, we have a unique probability measure $\mathbb{P}^*$ equivalent to $\mathbb{P}$ such that the discounted stock price processes are martingales. Conversely, if we can find a unique probability measure $\mathbb{P}^*$ equivalent to $\mathbb{P}$ which makes the discounted asset prices martingale, then the market is said to be complete.*

Proof. The proof can be found in [23]. □

Definition 2.12. A random variable ν taking values in $\{0, 1, \ldots, T\}$ is a *stopping time* if, for any $t \in \{0, 1, \ldots, T\}$,

$$\{\nu = t\} \in \mathcal{F}_t.$$

After defining the stopping time, we will proceed to the stopped sequence concept which plays an important role in the discrete time modeling of American options.

Definition 2.13. Let us consider that ν is a stopping time taking values in $\{0, 1, \ldots, T\}$ and let X_t be an $\mathcal{F}_t$-measurable random variable. In the case $\{\nu = k\}$ for $k \in \mathbb{N}$, the stopped sequence $(X_{t\wedge\nu}) = (X_t^\nu)$ is given by

$$X_t^\nu = \begin{cases} X_k \in \mathcal{F}_k (\subseteq \mathcal{F}_t) & \text{if } k \leq t \\ X_t \in \mathcal{F}_t & \text{if } k \geq t, \end{cases}$$

where $t \wedge \nu = \min\{\nu, t\}$.

It is obvious from this definition that the stopped sequence (X_t^ν) is always an $\mathcal{F}_t$-measurable random variable.
The following proposition plays a key role in the discrete-time version of optimal stopping time problem. Indeed, we later see that the optimal stopping time for exercising an American option guarantees the stopped sequence to be a martingale (see Proposition 3.2).

Proposition 2.5. *Let (X_t) be an adapted sequence and ν be a stopping time. Then, the stopped sequence (X_t^ν) is adapted. Moreover, if (X_t) is a martingale (a supermartingale, respectively), then (X_t^ν) is a martingale (a supermartingale, respectively).*

Proof. See Lamberton and Lapeyre [23] for the proof of this proposition. □

2.2 Fundamentals of Continuous-Time Modeling

We consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. We may also work on a smaller time interval $[0, T]$ for the *maturity time* $T < \infty$.
These are the settings on which our continuous-time model will be constructed.

Definition 2.14. A random variable τ taking values in $\mathbb{R}^+ \cup \{\infty\}$ is a *stopping time* with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ if for any $t \geq 0$

$$\{\tau \leq t\} \in \mathcal{F}_t.$$

Definition 2.15. A real-valued *Brownian Motion* is a continuous stochastic process $(B_t)_{t\geq 0}$ with indenpendent and stationary increments. In other words,

- $\mathbb{P}$ a.s., the map $t \mapsto B_t(\omega)$ is continuous.
- for $s \leq t$, $B_t - B_s$ is independent of $\mathcal{F}_s = \sigma(B_v, v \leq s)$.
- for $s \leq t$, $B_t - B_s$ and $B_{t-s} - B_0$ have the same probability law.

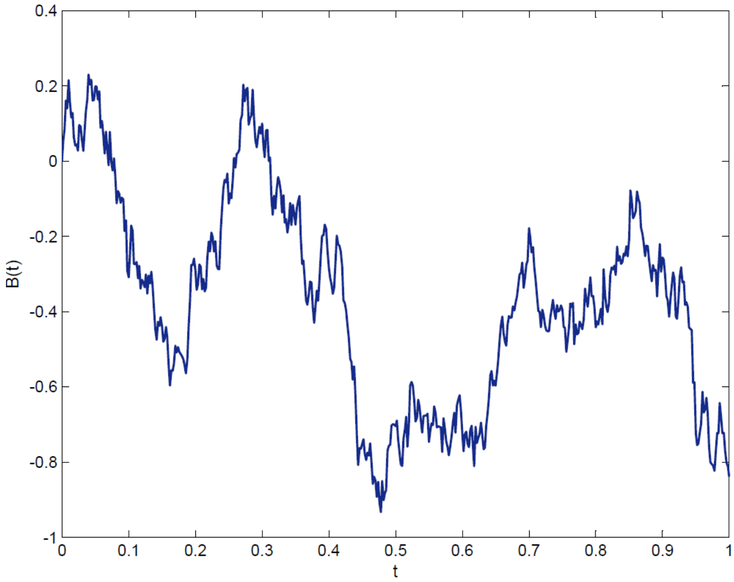


Figure 2.1: A path of a standard Brownian motion.

Definition 2.16. A Brownian motion is said to be standard if

$$B_0 = 0 \quad \mathbb{P}\text{ a.s.}, \qquad \mathbb{E}(B_t) = 0, \qquad \mathbb{E}(B_t^2) = t.$$

A path of a standard Brownian motion is given in Figure 2.1.

Definition 2.17. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space. A process $(X_t)_{t\geq 0}$ is said to be *adapted* with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$, if, for all $t \geq 0$, $(X_t)_{t\geq 0}$ is $\mathcal{F}_t$-measurable.

Definition 2.18. Let us consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $(\mathcal{F}_t)_{t\geq 0}$ on this space. An adapted family $(M_t)_{t\geq 0}$ of integrable random variables, i.e. $\mathbb{E}(|M_t|) < \infty$ for any t is:

- a *martingale* if, for any $s \leq t$, $\mathbb{E}(M_t \mid \mathcal{F}_s) = M_s$,
- a *supermartingale* if, for any $s \leq t$, $\mathbb{E}(M_t \mid \mathcal{F}_s) \leq M_s$,
- a *submartingale* if, for any $s \leq t$, $\mathbb{E}(M_t \mid \mathcal{F}_s) \geq M_s$.

Proposition 2.6. *(**Constant Expectation Property**)*
Given that $(M_t)_{t\geq 0}$ is an $\mathcal{F}_t$-martingale, then it satisfies

$$\mathbb{E}(M_t) = M_0 \quad \textit{for all} \quad t \geq 0.$$

Theorem 2.7. *(**Optional Sampling Theorem**)*
Let $(M_t)_{t\geq 0}$ be a continuous martingale with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$. If τ_1 and τ_2 are two stopping times such that $\tau_1 \leq \tau_2 \leq K$, where K is a finite real number, then M_{τ_2} is integrable and

$$\mathbb{E}(M_{\tau_2} \mid \mathcal{F}_{\tau_1}) = M_{\tau_1} \quad \mathbb{P} \textit{ almost surely.}$$

Proof. The proof can be found in [22]. □

Definition 2.19. Let us define $\mathcal{T}$ as the class of all $(\mathcal{F}_t)_{t\geq 0}$-stopping times τ satisfying $\mathbb{P}(\tau < \infty) = 1$. We set D as the class of right-continuous processes $\{X_t : 0 \leq t < \infty\}$ such that the family $(X_t)_{\tau\in\mathcal{T}}$ is uniformly integrable.

The following theorem plays an important role for constructing hedging portfolios for American options (see Theorem 4.6 and 5.2).

Theorem 2.8. ***(Doob-Meyer Decomposition Theorem for Supermartingales)***
If the right-continuous $\mathcal{F}_t$-supermartingale $(X_t)_{t\geq 0}$ is in class D, then it admits the decomposition

$$X_t = M_t - A_t$$

as the difference of uniformly integrable, RCLL martingale $(M_t)_{t\geq 0}$ and an adapted, non-decreasing, right continuous process $(A_t)_{t\geq 0}$ null at zero.

Theorem 2.9. *Let $(H_t)_{0\leq t\leq T}$ be an adapted stochastic process satisfying $\mathbb{E}(\int_0^T H_s^2 ds) < \infty$. Then the stochastic integral $I(t) = \int_0^t H_s dB_s$ has the following properties:*

- ***Adaptivity**: For all t, $I(t)$ is $\mathcal{F}_t$-measurable.*
- ***Martingale**: $I(t)$ is martingale.*
- ***Itô Isometry**: $\mathbb{E}(I(t)^2) = \mathbb{E}(\int_0^t H_s^2 ds)$.*
- ***Quadratic Variation**: $\langle I, I\rangle_t = \int_0^t H_s^2 ds$.*

Theorem 2.10. ***(Itô Formula)***
Let $(K_t)_{0\leq t\leq T}$ be an Itô process

$$\mathbb{P}\ a.s. \quad \forall t \leq T \qquad X_t = X_0 + \int_0^t K_s ds + \int_0^t H_s dB_s,$$

and f be a twice continuously differentiable function, then

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)dX_s + \frac{1}{2}\int_0^t f''(X_s)d\langle X, X\rangle_s$$

where, by definition

$$\langle X, X\rangle_t = \int_0^t H_s^2 ds,$$

and

$$\int_0^t f'(X_s)dX_s = \int_0^t f'(X_s)K_s ds + \int_0^t f'(X_s)H_s dB_s.$$

Similarly, if $(t, x) \to f(t, x) \in C^{1,2}$ having continuous partial derivatives, then Itô formula turns out to be

$$f(t, X_t) = f(0, X_0) + \int_0^t f'_s(s, X_s)ds + \int_0^t f'_x(s, X_s)dX_s + \frac{1}{2}\int_0^t f''_{xx}(s, X_s)d\langle X, X\rangle_s.$$

Proposition 2.11. ***(Integration by Parts Formula)***
Let X and Y be two Itô processes, i.e.

$$X_t = X_0 + \int_0^t K_s ds + \int_0^t H_s dB_s,$$

and

$$Y_t = Y_0 + \int_0^t M_s ds + \int_0^t N_s dB_s.$$

Then,

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + \langle X, Y \rangle_t \tag{2.1}$$

with the cross-variation

$$\langle X, Y \rangle_t = \int_0^t H_s N_s ds.$$

Definition 2.20. A stochastic differential equation (SDE) with the coefficients b and σ is defined by

$$X_t = X_0 + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dB_s, \tag{2.2}$$

or, equivalently,

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t. \tag{2.3}$$

In order to admit a unique solution to a given SDE (2.3), we need the following theorem.

Theorem 2.12. ***(Existence and Uniqueness Theorem)***
If b and σ are continuous functions, and if there exists a constant $K < \infty$ such that

1. $|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \le K|x-y| \quad$ *(Lipschitz condition)*,
2. $|b(t,x)| + |\sigma(t,x)| \le K(1 + |x|) \quad$ *(polynomial growth)*,
3. $\mathbb{E}(X_0^2) < \infty$,

then (2.3) admits a unique solution in the interval $[0,T]$. Moreover, this solution $(X_t)_{0 \le t \le T}$ satisfies

$$\mathbb{E}(\sup_{0 \le t \le T} |X_t|^2) < \infty.$$

We proceed with the two fundamental theorems of financial mathematics: Girsanov and Martingale Representation Theorem. The proofs can be found in Karatzas and Shreve [21].

In the context of these two theorems, we consider that $(B_t)_{0 \le t \le T}$ is a standard Brownian motion with respect to the probability measure $\mathbb{P}$.

Theorem 2.13. ***(Girsanov Theorem)***
Let $(\theta_t)_{0\leq t\leq T}$ be an adapted process satisfying $\int_0^T \theta_s^2 ds < \infty$ a.s. and such that

$$L_t = \exp\left(-\int_0^t \theta_s dB_s - \frac{1}{2}\int_0^t \theta_s^2 ds\right)$$

is a martingale. Then, under the probability $\mathbb{P}^{(L)}$ given by

$$\mathbb{P}^{(L)}(A) = \int_A Z(\omega)d\mathbb{P}(\omega),\ \forall A \in \mathcal{F}$$

the process $(W_t)_{0\leq t\leq T}$ defined by $W_t = B_t + \int_0^t \theta_s ds$ is an $(F_t)_{0\leq t\leq T}$-standard Brownian motion.

Theorem 2.14. ***(Martingale Representation Theorem)***
Let $(M_t)_{0\leq t\leq T}$ be a square-integrable, $\mathcal{F}_t$-martingale. Then, there exists an adapted process $(K_t)_{0\leq t\leq T}$ satisfying $\mathbb{E}\left(\int_0^T K_s^2 ds\right) < \infty$ and

$$M_t = M_0 + \int_0^T K_s dB_s \text{ a.s.},$$

for all $t \in [0,T]$.
Moreover, in the case the process $(M_t)_{0\leq t\leq T}$ is a local martingale, there exists an $\mathcal{F}_t$-measurable process $(K_t)_{0\leq t\leq T}$ satisfying $\int_0^T K_s^2 ds < \infty$ and

$$M_t = M_0 + \int_0^T K_s dB_s \text{ a.s.},$$

for all $t \in [0,T]$.

In the light of this theorem, we can deduce that, since every martingale (M_t) is also a local martingale, we can always find an adapted process (K_t) satisfying $\int_0^T K_s^2 ds < \infty$ and $M_t = M_0 + \int_0^T K_s dB_s$ for all $t \in [0,T]$.

Definition 2.21. A *partial differential inequality* is an inequality that involves an unknown function of several variables and its partial derivatives. For example, for all $(t,x) \in [0,T] \times \mathbb{R}$,

$$\frac{\partial u}{\partial t}(t,x) + \frac{\sigma^2}{2}(t,x)\frac{\partial^2 u}{\partial x^2}(t,x) + b(t,x)\frac{\partial u}{\partial x}(t,x) - r(t,x)u(t,x) \leq 0$$

is a partial differential inequality with the coefficient functions b, σ and r.

CHAPTER 3

DISCRETE-TIME MODELING FOR AMERICAN OPTIONS ON NON-DIVIDEND PAYING STOCKS

The valuation of American options can be complicated due to the early exercise possibility prior to maturity. Indeed, the holder determines his exercise strategy by comparing the value of the option with its intrinsic value at each time. If the value of the option dominates the money obtained by an immediate exercise, it is beneficial not to use the right to exercise. On the other hand, if the option is not worth as much as its intrinsic value, the holder decides to cancel the contract and takes the money produced by an early exercise. This comparison at each time poses a dilemma for the pricing of American options. In spite of these difficulties, it is possible to simplify the problem with the help of a discrete-time approach.

The discrete-time models are generally built on unrealistic assumptions. Because asset prices are modeled on discrete-time intervals, it is clear that the fluctuations in the market are not efficiently reflected by this approach. However, this deficiency can be eliminated by the practicability of the model. Indeed, under this approach, we do not need to handle with the complicated mathematical tools. Moreover, it provides a better understanding for more complex models owing to the fact that it is constructed on the similar main ideas. Hence, it can be easily extended to other approaches.

In this chapter, we introduce pricing of American options in discrete-time modeling. We will follow mainly the theory provided by Lamberton and Lapeyre (1996) [23].

3.1 The Market Model

We consider a discrete-time interval $[0, T]$, with $T < \infty$. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0,1,\dots,T}, \mathbb{P})$ be a finite, filtered probability space, and let $\mathbb{P}(\{\omega\}) > 0$ for all $\omega \in \Omega$ [14]. Moreover, we assume that there exist $m + 1$ assets in the market: m risky and one riskless asset. Let S_t^i denote the price of the i'th risky asset at time t, and let the riskless asset has a constant return r with the following price process S_t^0

$$S_t^0 = (1 + r)^t, \qquad S_0^0 = 1,$$

for all $t = 0, 1, \ldots, T$.
Under this model, the discount factor at time t is given by

$$\tilde{S}_t^0 = 1/S_t^0.$$

3.2 Pricing and Hedging American Options

In this section, we investigate how to price and hedge American options in the discrete-time setting. The market is assumed to be viable and complete and $\mathbb{P}^*$ is the unique probability measure fulfilling the assumptions of theorem 2.4. We will first show that the discounted price process of an American option appears to be a Snell envelope (see, e.g., [23] and [30]). In order to define such a process, we use backward induction method. After defining the Snell envelope [38], we will proceed to the optimal stopping time problem.
Let U_t denote the price of an American option at time t and Z_t be the intrinsic value of this option. Let S^1 be the underlying asset on which the American option is written. Then, the process Z_t is defined as follows:

$$Z_t = (S_t^1 - K)_+ = \max\{S_t^1 - K, 0\}, \qquad \text{(for call option)},$$

and

$$Z_t = (K - S_t^1)_+ = \max\{K - S_t^1, 0\}, \qquad \text{(for put option)},$$

where K is the strike price of the option.
Note that the price U_T should be equal to the payoff Z_T. Otherwise, an arbitrage opportunity occurs in the market.
In order to price the option for the times $t \neq T$, we first construct a hedging portfolio ψ. Since the market is complete, it is always possible to find a $\mathbb{R}^{m+1}$-valued replicating portfolio ψ such that

$$V_T(\psi) = U_T = Z_T.$$

As we mentioned before, discounted value process $\tilde{V}_t(\psi)$ at the time t is a $\mathcal{F}_t$-martingale under the probability measure $\mathbb{P}^*$ (see Proposition 2.2 in Chapter 2). Then,

$$\mathbb{E}^*(\tilde{V}_T(\psi) \mid \mathcal{F}_t) = \mathbb{E}^*(Z_T/S_T^0 \mid \mathcal{F}_t) = \tilde{V}_t(\psi).$$

Since $\tilde{V}_t(\psi) = V_t(\psi)/S_t^0$, we finally obtain

$$\mathbb{E}^*(Z_T/S_T^0 \mid \mathcal{F}_t)S_t^0 = V_t(\psi).$$

Now, using the backward induction method, we can easily price an American option. At time $T-1$, holder of the option should compare the value of the replicating portfolio $V_{T-1}(\psi)$, whose value is equal to U_N at maturity, with the intrinsic value Z_{T-1}. When $V_{T-1}(\psi) > Z_{T-1}$, an immediate exercise is not profitable, since it does not make sense to sell an option worth $V_{T-1}(\psi)$ for a cheaper price Z_{T-1}. On the other hand, in the case $V_{T-1}(\psi) < Z_{T-1}$, an early exercise is optimal. Then, the price U_{T-1} is given by

$$U_{T-1} = \max\{Z_{T-1}, \mathbb{E}^*(Z_T/S_T^0 \mid \mathcal{F}_{T-1})S_{T-1}^0\},$$

where $\mathbb{E}^*(Z_T/S_T^0 \mid \mathcal{F}_{T-1})S_{T-1}^0 = V_{T-1}(\psi)$.
Continuing backward in time, we get

$$U_t = \max\left\{Z_t, \mathbb{E}^*(U_{t+1}/S_{t+1}^0 \mid \mathcal{F}_t)S_t^0\right\}, \qquad \forall t = 0, \ldots, T-1.$$

Recalling $S_t^0 = (1+r)^t$,

$$U_t = \max\left\{Z_t, \frac{1}{1+r}\mathbb{E}^*(U_{t+1} \mid \mathcal{F}_t)\right\},$$

for all $t = 0, \ldots, T-1$.
Moreover, the discounted price process $(\tilde{U}_t)$ is defined by

$$\tilde{U}_t = \max\left\{\tilde{Z}_t, \mathbb{E}^*(\tilde{U}_{t+1} \mid \mathcal{F}_t)\right\}, \qquad \forall t = 0, \ldots, T-1. \tag{3.1}$$

The following theorem shows that the discounted price process of an American option, contrary to its European counterpart, is a $\mathbb{P}^*$-supermartingale.

Proposition 3.1. *Let Z_t be a non-negative, $\mathcal{F}_t$-measurable random variable interpreting the intrinsic value of an American option, and let $(\tilde{U}_t)$ be its discounted price process given by*

$$\begin{aligned} \tilde{U}_t &= \max\left\{\tilde{Z}_t, \mathbb{E}^*(\tilde{U}_{t+1} \mid \mathcal{F}_t)\right\}, \qquad \forall t = 0, \ldots, T-1, \\ \tilde{U}_T &= \tilde{Z}_T. \end{aligned} \tag{3.2}$$

Then, we say that the process $(\tilde{U}_t)$ is the Snell envelope of the process $(\tilde{Z}_t)$, i.e., it is the smallest supermartingale that dominates $(\tilde{Z}_t)$.

Proof. We first show that the discounted process $(\tilde{U}_t)$ is a supermartingale under the risk neutral probability measure $\mathbb{P}^*$.
Note that $\tilde{U}_t$ is $\mathcal{F}_t$-measurable, since $\tilde{Z}_t$ and $\mathbb{E}^*\left(\tilde{U}_{t+1} \mid \mathcal{F}_t\right)$ are both $\mathcal{F}_t$-measurable random variables. Moreover, it is clear from (3.1) that the adapted process $(\tilde{U}_t)$ dominates the random variables $(\tilde{Z}_t)$ and $\mathbb{E}^*\left(\tilde{U}_{t+1} \mid \mathcal{F}_t\right)$. Hence, $(\tilde{U}_t)$ is a $\mathbb{P}^*$-supermartingale that dominates the process $(\tilde{Z}_t)$.
Now, we will prove that $(\tilde{U}_t)$ is the smallest supermartingale dominating $(\tilde{Z}_t)$. For this reason, let us assume that there exists another supermartingale $\tilde{C}_t$ dominating $\tilde{Z}_t$. Since the price of an American option at maturity must be equal to its payoff Z_T, we have

$$\tilde{C}_T \geq \tilde{U}_T = \tilde{Z}_T.$$

Using backward induction in time, we can derive a similar inequality for an arbitrary time $t = 1, \ldots, T-1$.
Let $\tilde{C}_k \geq \tilde{U}_k$ for all $k = t+1, t+2, \ldots, T-1$. Our aim is to show that the inequality also holds for $k = t$.
Recalling the supermartingale property of $(\tilde{C}_t)$, we obtain

$$\tilde{C}_t \geq \mathbb{E}^*\left(\tilde{C}_{t+1} \mid \mathcal{F}_t\right) \geq \mathbb{E}^*\left(\tilde{U}_{t+1} \mid \mathcal{F}_t\right).$$

Moreover, by definition of $(\tilde{C}_t)$, we have

$$\tilde{C}_t \geq \tilde{Z}_t,$$

for all $t = 0, \ldots, T$.
Hence,

$$\tilde{C}_t \geq \max\left\{\tilde{Z}_t, \mathbb{E}^*\left(\tilde{U}_{t+1} \mid \mathcal{F}_t\right)\right\} = \tilde{U}_t,$$

which completes the proof. □

In this proposition, we have shown that the discounted price process of an American option $(\tilde{U}_t)$ defined by

$$\begin{cases} \tilde{U}_T = \tilde{Z}_T, \\ \tilde{U}_t = \max(\tilde{Z}_t, \mathbb{E}^*(\tilde{U}_{t+1} \mid \mathcal{F}_t)), \qquad \forall t = 0, \ldots, T-1, \end{cases}$$

is a Snell envelope of the discounted intrinsic value process $(\tilde{Z}_t)$.
More generally speaking, all adapted processes $(Y_t)_{t=0,1,\ldots,T}$ in the form of

$$\begin{cases} Y_T = R_T, \\ Y_t = \max(R_t, \mathbb{E}(Y_{t+1} \mid \mathcal{F}_t)), \qquad \forall t = 0, \ldots, T-1, \end{cases}$$

are said to be a *Snell envelope* of an $\mathcal{F}_t$-measurable random process $(R_t)_{t=0,1,\ldots,T}$.

3.2.1 Optimal Stopping Time

For the time being, we have shown that the discounted price process of an American option is the Snell envelope of its discounted intrinsic value. Indeed, it is apparent from (3.2) that this process is obtained by comparing $\tilde{Z}_t$ at each time t with the value $\mathbb{E}^*(\tilde{U}_{t+1} \mid \mathcal{F}_t)$. If $\tilde{Z}_t > \mathbb{E}^*(\tilde{U}_{t+1} \mid \mathcal{F}_t)$, we choose to exercise. In the case $\tilde{Z}_t < \mathbb{E}^*(\tilde{U}_{t+1} \mid \mathcal{F}_t)$, it is more profitable to hold the option. But, with this formulation, we cannot directly determine the optimal stopping times that provide maximum gain to the holder. Instead of making a time-consuming comparison until time 0, we can precisely compute the optimal stopping times with the help of some basic results. For this reason, we now introduce the optimal stopping time problem that plays a key role on pricing of American options.
Let us define an $\mathcal{F}_t$-measurable random variable R_t that represents the reward obtained by quitting a game at time t such that

$$\mathbb{E}(\sup_{t\in\{0,1,\ldots,T\}} |R_t|) < \infty.$$

Since we aim to maximize our gain, we investigate the stopping times satisfying [30]

$$\mathbb{E}(R_{\nu_0}) = \max_{\nu} \mathbb{E}(R_\nu).$$

Here, the stopping time ν_0 is said to be optimal and the optimization problem above is called optimal stopping time problem.
With the following definition, we can generalize the optimal stopping time problem in terms of conditional expectations.

Definition 3.1. Let $\mathcal{T}_{0,T}$ be the family of stopping times taking values in $\{0, 1, \ldots, T\}$. A stopping time $\nu_* \in \mathcal{T}_{0,T}$ is called an *optimal stopping time*, if

$$\mathbb{E}(R_{\nu_*} \mid \mathcal{F}_0) = \sup_{\nu \in \mathcal{T}_{0,T}} \mathbb{E}(R_\nu \mid \mathcal{F}_0).$$

In the case $\nu_* \in \mathcal{T}_{t,T}$, the optimal stopping time satisfies

$$\mathbb{E}(R_{\nu_*} \mid \mathcal{F}_t) = \sup_{\nu \in \tau_{t,N}} \mathbb{E}(R_\nu \mid \mathcal{F}_t).$$

where $\mathcal{T}_{n,T}$ is the family of stopping times taking values in $\{n, \ldots, T\}$.
It is apparent that the holder of an American option always aims to find an exercise strategy that makes his expected gain $\mathbb{E}^*(\tilde{Z}_\nu)$ largest [4, 13]. Therefore, defining $R_t = \tilde{Z}_t$ under $\mathbb{P}^*$, the valuation of an American option turns out to be an optimal stopping time problem.
The following proposition says that the optimal stopping time problem is indeed interested in the explicit solution of Snell envelopes. More precisely, we describe a stopping time ν_0 for a Snell envelope Y_t defined as in (3.2). Then this stopping time appears to be optimal satisfying the expression

$$\mathbb{E}(R_{\nu_0}) = \sup_{\nu \in \mathcal{T}_{0,T}} \mathbb{E}(R_\nu).$$

Moreover, the optimality of ν_0 is closely related to whether the stopped sequence $Y_t^{\nu_0}$ at time ν_0 is a martingale or not.

Proposition 3.2. *Let (Y_t) be Snell envelope of the process (R_t). We consider a stochastic variable $\omega \mapsto \nu_0(\omega)$ such that*

$$\nu_0 = \inf\{t \geq 0 : Y_t = R_t\}. \tag{3.3}$$

Then, we say that ν_0 is an optimal stopping time and the stopped sequence $(Y_t^{\nu_0})$ is a $\mathbb{P}$-martingale.

Proof. In order to prove that ν_0 is a stopping time, we must show

$$\{\omega : \nu_0(\omega) = k\} \in \mathcal{F}_k$$

for all $k = 0, 1, \ldots, T$.
We recall that, from the definition of Snell envelope, we have $Y_T = R_T$. This equality makes the random variable ν_0 well-defined.
Let us consider $k = 0$. Then,

$$\{\nu_0 = 0\} = \{Y_0 = R_0\} \in \mathcal{F}_0,$$

since R_0 is $\mathcal{F}_0$-measurable.
We note that, if $Y_i \neq R_i$ for any $i = 0, 1, \ldots, T-1$, we have

$$Y_i = \mathbb{E}(Y_{i+1} \mid \mathcal{F}_i).$$

Now, let $k \geq 1$. Then,

$$\begin{aligned}
\{\nu_0 = k\} &= \{Y_0 \neq R_0\} \cap \{Y_1 \neq R_1\} \cap \ldots \cap \{Y_{k-1} \neq R_{k-1}\} \cap \{Y_k = R_k\} \\
&= \{Y_0 = \mathbb{E}(Y_1 \mid \mathcal{F}_0)\} \cap \{Y_1 = \mathbb{E}(Y_2 \mid \mathcal{F}_1)\} \cap \ldots \cap \{Y_k = R_k\} \in \mathcal{F}_k.
\end{aligned}$$

Since $\mathbb{E}(Y_{i+1} \mid \mathcal{F}_i)$ is $\mathcal{F}_i$-measurable for all $i = 0, \ldots, T-1$, then we say that ν_0 is a stopping time.
Now, we aim to prove that the stopped sequence $(Y_t^{\nu_0})$ is a $\mathbb{P}^*$-martingale. To do that, we write the process $Y_t^{\nu_0}$ in the following way:

$$Y_t^{\nu_0} = Y_{t\wedge\nu_0} = Y_0 + \sum_{j=1}^{t} \mathbf{1}_{\{\nu_0 \geq j\}} \Delta Y_j.$$

Then, we have

$$Y_{t+1}^{\nu_0} - Y_t^{\nu_0} \quad = \quad \mathbf{1}_{\{t+1 \leq \nu_0\}} (Y_{t+1} - Y_t). \tag{3.4}$$

As it is apparent from the definition of ν_0, on the set $\{\nu_0 \geq t+1\}$

$$Y_t \neq R_t.$$

Moreover, we know

$$Y_t = \max\{R_t, \mathbb{E}(Y_{t+1} \mid \mathcal{F}_t)\},$$

for all $t = 0, 1, \ldots, T-1$.
Thus,

$$Y_t = \mathbb{E}(Y_{t+1} \mid \mathcal{F}_t). \tag{3.5}$$

Plugging (3.5) into (3.4), we obtain

$$Y_{t+1}^{\nu_0} - Y_t^{\nu_0} = \mathbf{1}_{\{t+1 \leq \nu_0\}} (Y_{t+1} - \mathbb{E}(Y_{t+1} \mid \mathcal{F}_t)).$$

Then, taking expectation of both sides

$$\mathbb{E}(Y_{t+1}^{\nu_0} - Y_t^{\nu_0} \mid \mathcal{F}_t) = \mathbb{E}(\mathbf{1}_{\{t+1 \leq \nu_0\}} (Y_{t+1} - \mathbb{E}(Y_{t+1} \mid \mathcal{F}_t) \mid \mathcal{F}_t).$$

Since $\{t+1 \leq \nu_0\} = \{\nu_0 < t\}^C = \{\nu_0 \leq t-1\}^C \in \mathcal{F}_t$, we can take out the expression $\mathbf{1}_{\{t+1 \leq \nu_0\}}$ from the conditional expectation. That is,

$$\mathbb{E}(Y_{t+1}^{\nu_0} - Y_t^{\nu_0} \mid \mathcal{F}_t) = \mathbf{1}_{\{t+1 \leq \nu_0\}} \mathbb{E}(Y_{t+1} - \mathbb{E}(Y_{t+1} \mid \mathcal{F}_t)).$$

Using the linearity property of conditional expectation, we get

$$\begin{aligned}
\mathbb{E}(Y_{t+1}^{\nu_0} - Y_t^{\nu_0} | \mathcal{F}_t) &= \mathbf{1}_{\{t+1 \leq \nu_0\}} (\mathbb{E}(Y_{t+1} \mid \mathcal{F}_t) - \mathbb{E}(\mathbb{E}(Y_{t+1} \mid \mathcal{F}_t) | \mathcal{F}_t)) \\
&= \mathbf{1}_{\{t+1 \leq \nu_0\}} (\mathbb{E}(Y_{t+1} \mid \mathcal{F}_t) - \mathbb{E}(Y_{t+1} \mid \mathcal{F}_t)) \\
&= 0.
\end{aligned}$$

Hence,

$$\mathbb{E}(Y_{t+1}^{\nu_0} | \mathcal{F}_t) = Y_t^{\nu_0}.$$

Now, our aim is to show the optimality of the stopping time ν_0. Because the stopped sequence $Y_T^{\nu_0}$ is a $\mathbb{P}^*$-martingale, we have

$$\mathbb{E}(Y_T^{\nu_0}|\mathcal{F}_0) = Y_0^{\nu_0} = Y_0,$$

and

$$\mathbb{E}(Y_T^{\nu_0}|\mathcal{F}_0) = \mathbb{E}(Y_{\nu_0}|\mathcal{F}_0) = \mathbb{E}(R_{\nu_0}).$$

Now, let us consider a stopping time $\nu \in \mathcal{T}_{0,T}$. By Proposition 2.5, we know that the stopped sequence Y^ν is a $\mathbb{P}$-supermartingale. This property yields

$$Y_0 \geq \mathbb{E}(Y_T^\nu|\mathcal{F}_0) = \mathbb{E}(Y_{T\wedge\nu}|\mathcal{F}_0) = \mathbb{E}(Y_\nu|\mathcal{F}_0) \geq \mathbb{E}(R_\nu|\mathcal{F}_0).$$

Then,

$$Y_0 = \mathbb{E}(Y_T^{\nu_0}|\mathcal{F}_0) = \mathbb{E}(Y_{\nu_0}|\mathcal{F}_0) = \mathbb{E}(R_{\nu_0}|\mathcal{F}_0) \geq \mathbb{E}(R_\nu|\mathcal{F}_0),$$

for all $\nu \in \mathcal{T}_{0,T}$.
Hence,

$$\mathbb{E}(Y_T^{\nu_0}|\mathcal{F}_0) \geq \sup_{\nu\in\mathcal{T}_{0,T}} \mathbb{E}(R_\nu^{\nu_0}|\mathcal{F}_0).$$

Also,we have

$$\mathbb{E}(Y_{\nu_0}|\mathcal{F}_0) \leq \sup_{\nu\in\mathcal{T}_{0,T}} \mathbb{E}(Y_\nu^{\nu_0}|\mathcal{F}_0),$$

since $\nu_0 \in \mathcal{T}_{0,T}$.
Therefore,

$$\mathbb{E}(R_{\nu_0}|\mathcal{F}_0) = \sup_{\nu\in\mathcal{T}_{0,T}} \mathbb{E}(R_\nu^{\nu_0}|\mathcal{F}_0).$$

□

As a result of this proposition, the stopping time for an American option defined by $\nu_0 = \inf\{t \geq 0 : \tilde{U}_t = \tilde{Z}_t\}$ is said to be optimal, and it yields that

$$\mathbb{E}(\tilde{Z}_{\nu_0}) = \max_\nu \mathbb{E}(\tilde{Z}_\nu).$$

With the following result, we can determine all optimal stopping times of an American option.

Proposition 3.3. *Let (R_t) be a non-negative, adapted process, and let (Y_t) be the Snell envelope of (R_t). If a stopping time τ satisfies the followings*

$$\begin{cases} Y_\tau = R_\tau, \\ (Y_{\tau\wedge t})_{t=0,1,\ldots,T} \quad \text{is an } \mathcal{F}_t\text{-martingale,} \end{cases} \tag{3.6}$$

then, τ is an optimal stopping time.

We see that this proposition does not admit a unique solution for the optimal stopping time problem. Indeed, we can find various stopping times that satisfy (3.6). But it is obvious that the stopping time ν_0 mentioned in Proposition 3.2 is the smallest optimal stopping time. More precisely, let $(\tilde{U}_t)$ be the discounted price process of an American option, and let $\tilde{Z}_t$ denote its intrinsic value at time t. We consider $j < \nu_0$. Because ν_0 is the first time that $\tilde{U}_t$ hits the value $\tilde{Z}_t$ and it is not beneficial to exercise the option when $U_j > Z_j$, we say that it is not a good strategy to stop when $j < \nu_0$. That is, the smallest optimal stopping time of an American option is indeed ν_0.

Now, we introduce the well-known Doob decomposition theorem. This theorem was presented by Doob (1953) [12].

Theorem 3.4. ***(Doob Decomposition)***
Let (Y_t) be a supermartingale defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0,1,\ldots,T}, \mathbb{P})$. Then, (Y_t) can be uniquely written in the following way:

$$Y_t = M_t - A_t,$$

where (M_t) is an $\mathcal{F}_t$-martingale and (A_t) is a non-decrasing, $\mathcal{F}_{t-1}$-measurable process with $A_0 = 0$.

For the time being, we discussed the properties of optimal stopping times and decided the smallest optimal date to exercise. With the help of Doob decomposition, we can now determine the largest optimal stopping time of an American option [13, 23].

Let us recall that the discounted price process of an American option $(\tilde{U}_t)$ is the Snell envelope of $(\tilde{Z}_t)$, i.e., it is the smallest supermartingale that dominates $(\tilde{Z}_t)$ under the risk-neutral probability measure $\mathbb{P}^*$. Using the Doob Decomposition theorem, we write

$$\tilde{U}_t = \tilde{M}_t - \tilde{A}_t,$$

where $(\tilde{M}_t)$ is an $\mathcal{F}_t$-martingale and $(\tilde{A}_t)$ is a non-decrasing, $\mathcal{F}_{t-1}$-measurable process with $A_0 = 0$.

We note that, since $(\tilde{A}_t)$ is a non-decreasing process with $A_0 = 0$ and $\tilde{U}_t \geq 0$ for all $t = 0, 1, \ldots, T$, we have

$$\tilde{M}_T = \tilde{A}_T + \tilde{U}_T \geq 0.$$

Since the market is complete, we can always find a replicating portfolio ψ for an adapted, non-negative random variable $\tilde{M}_T$ such that

$$\tilde{V}_T(\psi) = \tilde{M}_T.$$

Then, recalling that the discounted value process $(\tilde{V}_t)$ is a martingale under $\mathbb{P}^*$, we get

$$\tilde{V}_t(\psi) = \mathbb{E}^*(\tilde{V}_T(\psi) \mid \mathcal{F}_t) = \mathbb{E}^*(\tilde{M}_T \mid \mathcal{F}_t) = \tilde{M}_t.$$

Hence,

$$\tilde{U}_t = \tilde{V}_t(\psi) - \tilde{A}_t,$$

or

$$U_t = V_t(\psi) - A_t.$$

In the light of foregoing, we can investigate the last time that is optimal to exercise. Let $\tau = \inf\{j : A_{j+1} \neq 0\}$, then an early exercise at time $j > \tau$ yields that

$$\tilde{U}_j = \tilde{V}_j(\psi) - \tilde{A}_j.$$

It is clear that, in that case, the stopped sequence $(\tilde{U}_{j\wedge\tau})$ is not a $\mathbb{P}^*$-martingale that prevents the optimality of the stopping time j (see Proposition 3.3). Therefore, we say that an early exercise is not profitable after time $\tau = \inf\{j, A_{j+1} \neq 0\}$. The following proposition states that fact.

Proposition 3.5. *The largest optimal stopping time for (R_t) is given by:*

$$\nu_{\max} = \begin{cases} T & \text{if } A_T = 0, \\ \inf\{t : A_{t+1} \neq 0\} & \text{if } A_T \neq 0. \end{cases}$$

Proof. From Doob-Decomposition, we know that the process (Y_t) can be written as follows:

$$Y_t = M_t - A_t, \tag{3.7}$$

where (M_t) is a martingale and (A_t) is a non-decreasing, predictable process null at zero.
Using the predictability of the process (A_t), we can easily show that the random variable $\nu_{\max}$ is a stopping time.
Let $A_T \neq 0$. Then,

$$\{\nu_{\max} = t\} = \{A_0 = 0\} \cap \ldots \cap \{A_t = 0\} \cap \{A_{t+1} \neq 0\} \in \mathcal{F}_t.$$

On the other hand, if $A_T = 0$, we have

$$\{\nu_{\max} = T\} = \{A_T = 0\} \in \mathcal{F}_{T-1} \subseteq \mathcal{F}_T.$$

Hence, $\nu_{\max}$ is a stopping time.
For the optimality of $\nu_{\max}$, we need to show that the stopped sequence $(Y_t^{\nu_{\max}})$ is a martingale, with $Y_{\nu_{\max}} = R_{\nu_{\max}}$.
Let us consider $A_j = 0$ for $j \leq \nu_{\max}$. As it is apparent from (3.7), we have $Y_j = M_j$. Then,

$$Y_j^{\nu_{\max}} = Y_j = M_j = M_j^{\nu_{\max}}.$$

When $\nu_{\max} \leq j$, we get

$$Y_j^{\nu_{\max}} = M_{\nu_{\max}} - A_{\nu_{\max}} = M_{\nu_{\max}} = M_j^{\nu_{\max}},$$

since $A_{\nu_{\max}} = 0$.
Hence, $Y^{\nu_{\max}} = M^{\nu_{\max}}$ is a martingale.
In order to show $Y_{\nu_{\max}} = R_{\nu_{\max}}$, we express $Y_{\nu_{\max}}$ as follows:

$$\begin{aligned} Y_{\nu_{\max}} &= \sum_{j=0}^{T-1} \mathbf{1}_{\{\nu_{\max}=j\}} Y_j + \mathbf{1}_{\{\nu_{\max}=T\}} Y_T \\ &= \sum_{j=0}^{T-1} \mathbf{1}_{\{\nu_{\max}=j\}} \max\{R_j, \mathbb{E}(Y_{j+1}|\mathcal{F}_j)\} + \mathbf{1}_{\{\nu_{\max}=T\}} R_T \end{aligned}$$

We note that

$$\begin{aligned} \mathbb{E}(Y_{j+1}|\mathcal{F}_j) &= \mathbb{E}(M_{j+1}|\mathcal{F}_j) - \mathbb{E}(A_{j+1}|\mathcal{F}_j) \\ &= M_j - A_{j+1}, \end{aligned}$$

since (M_j) is martingale and (A_j) is $\mathcal{F}_j$-measurable.
Moreover, $A_j = 0$ and $A_{j+1} > 0$ when $\{\nu_{\max} = j\}$. Therefore,

$$Y_j = M_j,$$

and

$$\mathbb{E}(Y_{j+1}|\mathcal{F}_j) = M_j - A_{j+1} < M_j = Y_j.$$

Thus,

$$Y_j = \max\{R_j, \mathbb{E}(Y_{j+1}|\mathcal{F}_j)\} = R_j.$$

Hence, $Y_{\nu_{\max}} = R_{\nu_{\max}}$.
Now, our aim is to show that $\nu_{\max}$ is the largest optimal stopping time. Let us assume that there exists an optimal stopping time ν such that $\nu \geq \nu_{\max}$ and $\mathbb{P}(\nu > \nu_{\max}) > 0$. As a result of optional sampling theorem, we have
$\mathbb{E}(M_\nu) = \mathbb{E}(M_0)$. Also, we note that $Y_0 = M_0$, since $A_0 = 0$. Therefore,

$$\mathbb{E}(Y_\nu) = \mathbb{E}(M_\nu) - \mathbb{E}(A_\nu) = \mathbb{E}(Y_0) - \mathbb{E}(A_\nu) < \mathbb{E}(Y_0). \tag{3.8}$$

Thus, (Y_t) cannot be a martingale and it contradicts with the assumption that ν is an optimal stopping time. Hence, $\nu_{\max}$ is the largest optimal stopping time. □

3.2.2 Comparison of American and European Options

In this subsection, we make a comparison between American and European options assuming that the underlyings do not distribute dividends.
The following proposition says that an American option is worth at least as its European counterpart. Moreover, we verify that American and European call options are indeed equally priced in the case of no dividend [23].

Proposition 3.6. *Let U_t denote the value of an American option at time t such that*

$$U_t = \max\left\{Z_t, \frac{1}{1+r}\mathbb{E}^*(U_{t+1} \mid \mathcal{F}_t)\right\},$$

where Z_t is the intrinsic value of the option at time t.
Let u_t be the value of an European option interpreted by an adapted, non-negative random variable c such that $c = Z_T$.
Then, we get

$$U_t \geq u_t,$$

for all $t = 0, 1, \ldots, T$.
Also, when $u_t \geq Z_t$ for any t, then we have $u_t = U_t$ for all $t \in \{0, 1, \ldots, T\}$.

Intuitively speaking, $U_t \geq u_t$ holds, since an American option gives its holder an additional right of an early exercise.

Proof. From Proposition 3.1, we know that the discounted value process $(\tilde{U}_t)$ is a supermartingale under the risk-neutral probability measure $\mathbb{P}^*$. Then,

$$\tilde{U}_t \geq \mathbb{E}^*(\tilde{U}_T|\mathcal{F}_t) = \mathbb{E}^*(\tilde{u}_T|\mathcal{F}_t) = \tilde{u}_t,$$

for all $t \in \{0, 1, \ldots, T-1\}$.
Moreover,

$$\tilde{u}_T = \tilde{U}_T = \tilde{Z}_T,$$

since the market is viable.
Hence, $U_t \geq u_t$ for all $t = 0, 1, \ldots, T$.
In the case $u_t \geq Z_t$ for any t, then

$$\tilde{u}_t \geq \tilde{Z}_t.$$

We recall that the discounted value process $(\tilde{u}_t)$ is a $\mathbb{P}^*$-supermartingale due to the fact that every martingale is also a supermartingale. Since $(\tilde{U}_t)$ is the smallest supermartingale dominating $(\tilde{Z}_t)$, we conclude that, for any t

$$\tilde{U}_t \geq \tilde{u}_t.$$

This implies that $U_t = u_t$ for all $t = 0, 1, \ldots, T$. □

With the help of this proposition, we see that the price of an American call option with no-dividend is indeed equal to the price of its European counterpart.
Let us assume that there are only two assets traded in the market: a stock (risky) and a bond (riskless). Let the price of the bond, say S_t^0, defined by

$$\begin{aligned} S_t^0 &= (1+r)^t, \\ S_0^0 &= 1, \end{aligned}$$

where r stands for the risk-free interest rate in the market.
Let S_t be the price of the stock at time t, and let $Z_t = (S_t - K)_+$ be the intrinsic value of an American call with exercise price K.
Recalling that $u_T = Z_T$ and $(\tilde{S}_t)$, $(\tilde{u}_t)$ are $\mathbb{P}^*$-martingales, we have

$$\begin{aligned} \mathbb{E}^*((1+r)^{-T}(S_T - K)_+|\mathcal{F}_t) &= \mathbb{E}^*(\tilde{u}_T|\mathcal{F}_t) \\ &= \tilde{u}_t. \end{aligned}$$

Moreover, we know that

$$(1+r)^{-t}(S_t - K)_+ \geq (1+r)^{-t}(S_t - K),$$

for all $t = 0, 1, \ldots, T$.
Then,

$$\begin{aligned} \tilde{u}_t &\geq \mathbb{E}^*(\tilde{S}_T - K(1+r)^{-T}|\mathcal{F}_t) \\ &= \mathbb{E}^*(\tilde{S}_T|\mathcal{F}_t) - \mathbb{E}^*(K(1+r)^{-T}|\mathcal{F}_t) \\ &= \tilde{S}_t - K(1+r)^{-T}. \end{aligned} \tag{3.9}$$

Multiplying (3.9) with $(1+r)^t$, we get

$$u_t \geq S_t - K(1+r)^{-(T-t)} \geq S_t - K.$$

Because $u_t \geq 0$, then

$$u_t \geq (S_t - K)_+ = Z_t.$$

for all $t = 0, 1, \ldots, T$.
As a result of Proposition 3.6, we conclude that

$$u_t = U_t.$$

CHAPTER 4

CONTINUOUS-TIME MODELING FOR AMERICAN OPTIONS ON NON-DIVIDEND PAYING STOCKS

Continuous-time valuation of options under the Black-Scholes setting [5] can be thought of as an extension of discrete time valuation scheme, since the fundamental tools and ideas we use show similarities with the ones in the discrete case. But, because of the requirement for a strong mathematical background, pricing American options in continuous time is visibly harder and complicated.
In the literature, it can be seen that numerious studies have been conducted on the pricing theory of American options. The pricing scheme was firstly presented by McKean (1965) [25] and Samuelson (1965) [34] associating the valuation concept with the optimal stopping time problem. Then, it was extended by many authors, such as van Moerbeke (1976) [41], Bensoussan (1984) [2] and Karatzas (1988) [20]. Also, the variational inequalities were presented as an another tool to scope out the pricing problem, see, e.g., [3, 18, 22, 28].
In this chapter, we introduce the continuous time valuation of American options written on an underlying asset that does not pay dividend. The main references of this chapter are [4, 22, 23].

4.1 Market Model

In this section, we present the stock price dynamics and trading strategies used in continuous time modeling. Here, stock price processes are modeled under the Black-Scholes setting.

4.1.1 Price Dynamics

We consider a continuous time interval $[0, T]$, with $T < \infty$. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$ be a filtered probability space, and let $(B_t)_{0\leq t\leq T}$ be a standard Brownian motion under the probability measure $\mathbb{P}$. For simplicity, we consider that there exist only two assets in the market: one of them is risky and the other is riskless asset.
Let the price process of the riskless asset $(S_t^0)_{0\leq t\leq T}$ be defined by

$$dS_t^0 = rS_t^0 dt, \qquad S_0^0 = 1,$$

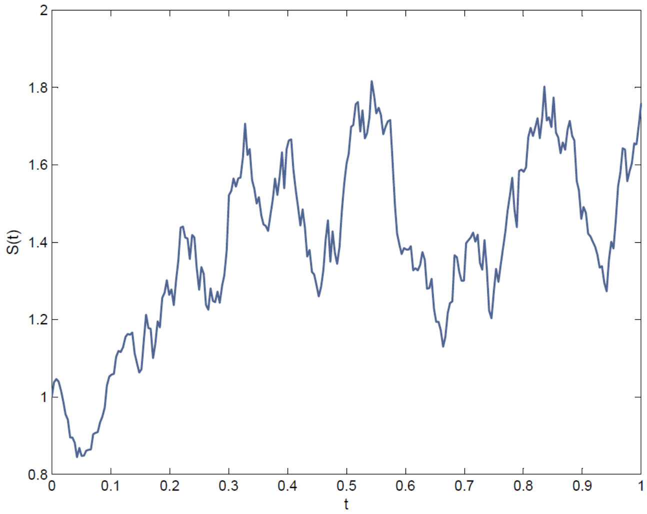


Figure 4.1: A path of geometric Brownian motion

where $r \geq 0$ is the constant interest rate on the the finite time interval $[t - dt, t]$ and $S_t^0 = e^{rt}$.
On the other hand, the price process of the risky asset $(S_t)_{0 \leq t \leq T}$ is supposed to be modeled by the geometric Brownian motion satisfying

$$dS_t = S_t(\mu dt + \sigma dB_t). \tag{4.1}$$

Then, we see that the discounted price process $(\tilde{S}_t)_{0 \leq t \leq T}$ follows

$$d\tilde{S}_t = \tilde{S}_t \left((\mu - r)dt + \sigma dB_t \right). \tag{4.2}$$

Here, the constant coefficients $\mu > 0$ and $\sigma > 0$ are interpreted as the drift and volatility term of the Black-Scholes model [5], respectively.
Because of the deterministic part $(\mu - r)dt$ in (4.2), the discounted asset price process $(\tilde{S}_t)$ turns out to be not a $\mathbb{P}$- martingale. But, using Girsanov theorem, we can define a new probability measure $\mathbb{P}^*$ equivalent to $\mathbb{P}$ such that the discounted asset price processes become a martingale. Then, substituting a $\mathbb{P}^*$-standard Brownian motion $W_t = B_t + \frac{\mu - r}{\sigma} t$ into (4.2), we obtain

$$d\tilde{S}_t = \tilde{S}_t \sigma dW_t.$$

Hence, the discounted price process $(\tilde{S}_t)$ is a $\mathbb{P}^*$-martingale.
Moreover, the SDE (4.1) turns into

$$dS_t = S_t(rdt + \sigma dW_t), \tag{4.3}$$

under the measure $\mathbb{P}^*$. This probability measure is said to be the risk neutral probability measure of the market.
Intuitively speaking, the μ term in the SDE (4.1) reflects the risk choices of investors. The SDES's having smaller drift is more affected by risk averse investors whereas the ones with a larger drift show risk-lover behaviors. Since the expected

stock prices are expressed with μ, with the changing risk attitudes, the expected returns also turn out to differ among investors. But, it is apparent from (4.3) that, under the measure $\mathbb{P}^*$, all investors have the same expectations about the stock returns. Namely, all stocks are supposed to provide the fixed yield r. Then, it can be thought that all investors in the market are risk-neutral. That is the reason why we call to $\mathbb{P}^*$ the risk-neutral probability measure. Because the valuation of options are easier according to this approach, it gains great importance in finance [16].

4.1.2 Trading Strategies

In this section, we will handle the trading strategies constructed in the market and give a brief introduction to their valuation process. The definitions are prepared using Lamberton and Lapeyre in [23].
In the continuous-time modeling, a trading strategy $\phi = (H_t^0, H_t)_{0 \leq t \leq T}$ is supposed to be an $\mathcal{F}_t$-measurable random variable taking values in $\mathbb{R}^2$, where H_t^0 and H_t denote the number of shares of risky and riskless asset respectively, we trade in the market at time t.
Then, we express the value of a trading strategy at time t as follows:

$$V_t(\phi) = H_t^0 S_t^0 + H_t S_t. \tag{4.4}$$

In addition to the valuation of trading strategies, we will now introduce self-financing portfolios under some constraints.

Definition 4.1. An $\mathcal{F}_t$-measurable stochastic process $\phi = (H_t^0, H_t)_{0 \leq t \leq T}$ is said to be a *self-financing strategy* if it satisfies the following properties:

1. $\int_0^T |H_t^0| dt + \int_0^T H_t^2 dt < \infty \quad$ a.s.,
2. $H_t^0 S_t^0 + H_t S_t = H_0^0 S_0^0 + H_0 S_0 + \int_0^t H_u^0 dS_u^0 + \int_0^t H_u dS_u$ for all $t \in [0, T]$ a.s.

As it is apparent from the second condition that the change in the value of a self-financing portfolio arises only from the asset price fluctuations.
Now, we will give the definition of admissible strategy which is closely related to the no-arbitrage principle in the market.

Definition 4.2. We say that an $\mathbb{R}^2$-valued process $\phi = (H_t^0, H_t)_{0 \leq t \leq T}$ is an *admissible strategy* if it is self-financing, its discounted value $\tilde{V}_t(\phi) \geq 0$ for all t, and $\sup_{t \in [0,T]} \tilde{V}_t(\phi)$ is square-integrable under the risk neutral probability measure $\mathbb{P}^*$.

Theorem 4.1. *Let us assume that c is a non-negative, $\mathcal{F}_T$-measurable random variable satisfying the square-integrability condition*

$$\mathbb{E}^*(c^2) < \infty, \quad \forall t \in [0, T],$$

under the risk neutral probability measure $\mathbb{P}^*$. Then, we say that, in the Black-Scholes model, any option whose payoff is equal to c is replicable and the value of any replicating portfolio at time t is defined by

$$V_t = \mathbb{E}^*(e^{-r(T-t)}c \mid \mathcal{F}_t).$$

Proof. The proof can be found in [23]. □

We recall that the payoff function of an European call option with strike price K is defined by $(S_T - K)_+$. Since $\mathbb{E}^*(S_T^2) < \infty$, we deduce that, for any European option with the payoff $(S_T - K)_+$, we can find a replicating portfolio ϕ. Moreover, the price of an European call option $u(t, S_t)$ with strike price K is given by

$$V_t(\phi) = u(t, S_t) = \mathbb{E}^*(e^{-r(T-t)}(S_T - K)_+ \mid \mathcal{F}_t),$$

for all $t \in [0, T]$.

In their famous work [14], Harrison and Pliska show that it is impossible to capture an arbitrage opportunity under the Black-Scholes framework, since there is at least one risk neutral probability measure $\mathbb{P}^*$ under which the discounted stock prices $\tilde{S}_t$ are martingales. To be more precise, we cannot construct such an admissible portfolio $\phi = (H_t^0, H_t)_{0\leq t\leq T}$ satisfying

$$H_0^0 + H_0 S_0 < 0,$$

with

$$H_t^0 S_t^0 + H_t S_t \geq 0.$$

Intuitively speaking, it is not allowed to obtain a riskless gain in the market. As a result of this principle, the contingent claims whose payoffs are equal must have the same prices [4].
We proceed to the concept of trading strategies with consumption which provide a way to hedge American options written on a non-dividend paying asset.

Definition 4.3. An $\mathcal{F}_t$-measurable random variable $\phi = (H_t^0, H_t)_{0\leq t\leq T}$ in $\mathbb{R}^2$ is called a *trading strategy with consumption*, if it satisfies the following properties:

1. $\int_0^T |H_t^0| dt + \int_0^T H_t^2 dt < \infty$ a.s.,
2. $H_t^0 S_t^0 + H_t S_t = H_0^0 S_0^0 + H_0 S_0 + \int_0^t H_u^0 dS_u^0 + \int_0^t H_u dS_u - C_t, \quad \forall t \in [0, T]$,

where $(C_t)_{0\leq t\leq T}$ is an adapted, continuous and non-decreasing process null at zero that is known as the cumulative consumption up to time t .
In the light of this definition, we can easily verify that the discounted value process of a trading strategy with consumption is actually a supermartingale under the risk-neutral probability measure $\mathbb{P}^*$.

Lemma 4.2. *Let $\phi = (H_t^0, H_t)_{0\leq t\leq T}$ be a trading strategy with consumption and let $V_t(\phi) = H_t^0 S_t^0 + H_t S_t$. Then, the discounted value process $\tilde{V}_t(\phi) = e^{-rt} V_t(\phi)$ is a supermartingale under $\mathbb{P}^*$.*

Proof. According to Definition 4.3, a trading strategy with consumption $\phi = (H_t^0, H_t)_{0\leq t\leq T}$ satisfies the following equation:

$$H_t^0 S_t^0 + H_t S_t = H_0^0 S_0^0 + H_0 S_0 + \int_0^t H_u^0 dS_u^0 + \int_0^t H_u dS_u - C_t. \tag{4.5}$$

Then,

$$V_t(\phi) = V_0(\phi) + \int_0^t H_u^0 dS_u^0 + \int_0^t H_u dS_u - C_t.$$

Applying integration by parts formula to the process $e^{-rt}V_t(\phi)$, we get

$$\begin{aligned}
d(e^{-rt}V_t(\phi)) &= -re^{-rt}V_t(\phi)dt + e^{-rt}dV_t(\phi) \\
&= e^{-rt}\left(-r(H_t^0 S_t^0 + H_t S_t)dt + H_t^0 dS_t^0 + H_t dS_t\right) - e^{-rt}dC_t \\
&= e^{-rt}\left(-rH_t S_t dt + H_t dS_t\right) - e^{-rt}dC_t \\
&= H_t d\tilde{S}_t - e^{-rt}dC_t,
\end{aligned}$$

where $d\tilde{S}_t = \sigma\tilde{S}_t dW_t$.
Hence,

$$\tilde{V}_t(\phi) = \tilde{V}_0(\phi) + \int_0^t H_u\sigma\tilde{S}_u dW_u - \int_0^t e^{-ru}dC_u.$$

Note that the stochastic integrals $\int_0^s H_u\sigma\tilde{S}_u dW_u$ and $\int_0^s e^{-ru}dC_u$ are $\mathcal{F}_s$-measurable. Then, for all $s \leq t$,

$$\begin{aligned}
\mathbb{E}^*\left(\tilde{V}_t(\phi) \mid \mathcal{F}_s\right) &= \tilde{V}_0(\phi) + \mathbb{E}^*\left(\int_0^t H_u\sigma\tilde{S}_u dW_u \mid \mathcal{F}_s\right) - \mathbb{E}^*\left(\int_0^t e^{-ru}dC_u \mid \mathcal{F}_s\right) \\
&= \tilde{V}_0(\phi) + \mathbb{E}^*\left(\int_0^s H_u\sigma\tilde{S}_u dW_u + \int_s^t H_u\sigma\tilde{S}_u dW_u \mid \mathcal{F}_s\right) \\
&\quad -\mathbb{E}^*\left(\int_0^s e^{-ru}dC_u + \int_s^t e^{-ru}dC_u \mid \mathcal{F}_s\right) \\
&= \tilde{V}_0(\phi) + \int_0^s H_u\sigma\tilde{S}_u dW_u - \int_0^s e^{-ru}dC_u \\
&\quad +\mathbb{E}^*\left(\int_s^t H_u\sigma\tilde{S}_u dW_u \mid \mathcal{F}_s\right) - \mathbb{E}^*\left(\int_s^t e^{-ru}dC_u \mid \mathcal{F}_s\right) \\
&= \tilde{V}_s(\phi) + \mathbb{E}^*\left(\int_s^t H_u\sigma\tilde{S}_u dW_u \mid \mathcal{F}_s\right) - \mathbb{E}^*\left(\int_s^t e^{-ru}dC_u \mid \mathcal{F}_s\right).
\end{aligned}$$

Since the integral $\int_s^t H_u\sigma\tilde{S}_u dW_u$ is independent of $\mathcal{F}_s$,

$$\mathbb{E}^*\left(\tilde{V}_t(\phi) \mid \mathcal{F}_s\right) = \tilde{V}_s(\phi) + \mathbb{E}^*\left(\int_s^t H_u\sigma\tilde{S}_u dW_u\right) - \mathbb{E}^*\left(\int_s^t e^{-ru}dC_u \mid \mathcal{F}_s\right).$$

Then, from the constant expectation property of the process $\int_s^t H_u\sigma\tilde{S}_u dW_u$, we have

$$\mathbb{E}^*\left(\tilde{V}_t(\phi) \mid \mathcal{F}_s\right) = \tilde{V}_s(\phi) - \mathbb{E}^*\left(\int_s^t e^{-ru}dC_u \mid \mathcal{F}_s\right).$$

Noting that (C_t) is a non-decreasing process null at zero, we obtain

$$\mathbb{E}^*\left(\int_s^t e^{-ru} dC_u \mid \mathcal{F}_s\right) \geq 0,$$

and this implies that

$$\mathbb{E}^*\left(\tilde{V}_t(\phi) \mid \mathcal{F}_s\right) \leq \tilde{V}_s(\phi).$$

It means that the process $(\tilde{V}_t(\phi))$ is a supermartingale under the probability measure $\mathbb{P}^*$. □

4.2 Pricing

In this section, we present two different methods for the valuation of American options. We first illustrate the martingale approach which is based on taking the expectation of discounted payoff process under the risk neutral probability measure. Under this approach, optimal stopping time problem and Snell envelope will be introduced, as in [4] and [28] and then the no-arbitrage pricing formula will be given. Secondly, we mention another method which deals with the solutions of variational inequlities. We point out that this approach has a pivotal role on the numerical solutions of American type options. Our main references for this section are Bjork [4], Lamberton and Lapeyre [23], Karatzas and Shreve [22]. For more details, we also refer to [3], [20] and [21].
Let us assume that the market is arbitrage-free and complete. We recall that the existence of a risk-neutral measure $\mathbb{P}^*$ under which the discounted asset prices $(\tilde{S}_t)_{0\leq t\leq T}$ are martingales prevents an arbitrage in the market. However, the no-arbitrage condition is not enough to gurantee the uniqueness of a price process. Therefore, we need the assumption of completeness for the market.
Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous mapping which satisfies the linear growth property

$$\psi(x) \leq C + Dx,$$

for some $C, D \in \mathbb{R}^+$. For American call and put options with the strike price K, the process ψ is defined by $\psi(x) = \max\{x - K, 0\}$ and $\psi(x) = \max\{K - x, 0\}$, respectively [27].

4.2.1 Martingale Pricing Approach

As we know, the thing that distinguishes an American option from its European counterpart is the additional right of an early exercise. Indeed, when the value of the option falls below the intrinsic value, the holder can benefit from this depreciation by taking the immediate payoff derived from an early exercise. But, this early exercise possibility compels us to take the optimal stopping time concept into account. To do that, in addition to the fair price of the option, we also examine the stopping times at which the holder can maximize his expected gain. In this approach, the concept of optimal stopping time and Snell envelope have a great importance.

4.2.1.1 Snell Envelope and Optimal Stopping Time Problem

In this subsubsection, we give a brief explanation on the theory of Snell envelope and optimal stopping time problem avoiding the formal technicalities. For a detailed information, see [22, 36].

Let $(Z_t)_{0\leq t\leq T}$ be a non-negative, $\mathcal{F}_t$-measurable, right-continuous random variable satisfying [22]

$$0 \leq \mathbb{E}(\sup_{0\leq t\leq T} Z(t)) < \infty.$$

The optimal stopping time problem arises from the desire to find a stopping time ϑ taking values between 0 and T such that

$$\mathbb{E}(Z_\vartheta) = \sup_{\tau\in\mathcal{T}_{0,T}} \mathbb{E}(Z_\tau), \tag{4.6}$$

where $\mathcal{T}_{0,T}$ is the set of stopping times taking values in $[0, T]$.

Intuitively, let us consider a game that pays the amount Z_t when stopped at time t. From the perspective of a gambler, we continue playing until we make the largest profit. That is, our problem is to determine the stopping time ϑ at which our expected gain $\mathbb{E}(Z_t)$ is maximized. Here, the process (Z_t) is called the reward process and the random variable ϑ is said to be an optimal stopping time [4]. For American options, the reward process (Z_t) indeed refers to the discounted intrinsic value at time t. Hence, $Z_t = e^{-rt}\max\{S_t - K, 0\}$ for American call options, and in the case of American put options, $Z_t = e^{-rt}\max\{K - S_t, 0\}$, where K is the strike price of the option.

We remark that it is sometimes impossible to find an optimal stopping time satisfying (4.6). To illustrate, the value of a perpetual call option with no dividend converges to the price of the underlying asset as T approaches to infinity. However, we cannot find an optimal stopping time to exercise [22]. But, the following proposition shows that, under some circumstances, we can determine the optimal dates for exercising. It says that we can find an optimal trading strategy in the case the reward process is an $\mathcal{F}_t$-martingale, submartingale or supermartingale.

Proposition 4.3.

- *The optimal stopping time ϑ is 0, if the reward process (Z_t) is an $\mathcal{F}_t$-supermartingale.*
- *The optimal stopping time ϑ is the expiration date T, if the reward process (Z_t) is an $\mathcal{F}_t$-submartingale.*
- *All stopping stopping times ϑ between 0 and T are optimal, if the reward process (Z_t) is an $\mathcal{F}_t$-martingale.*

Proof. This proposition can be found in [4] without proof. Hence, the proof is showed in this thesis. Let us first start with the submartingale case.
Now, let us consider that the reward process (Z_t) is a submartingale. By the optional sampling theorem, we have

$$\mathbb{E}(Z_T \mid \mathcal{F}_\tau) \geq Z_\tau,$$

for all $\tau \in \mathcal{T}_{0,T}$.
Using the tower property, we get

$$\mathbb{E}(Z_T) \geq \mathbb{E}(Z_\tau),$$

for all $\tau \in \mathcal{T}_{0,T}$.
Then,

$$\mathbb{E}(Z_T) \geq \sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}(Z_\tau).$$

Moreover, the following inequality always holds

$$\mathbb{E}(Z_T) \leq \sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}(Z_\tau),$$

since the expiration date T is the last time that an American option can be exercised.
Therefore,

$$\mathbb{E}(Z_T) = \sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}(Z_\tau),$$

which implies that the optimal stopping time for a submartingale is the maturity.
The other parts can be proved similarly. □

Corollary 4.4. *An early exercise is never optimal for an American call written on a stock that does not distribute dividends.*

Let the reward process $(Z_t)_{0 \leq t \leq T}$ be the discounted payoff of an American call with the strike price K such that $Z_t = e^{-rt}(S_t - K)_+$ for all $0 \leq t \leq T$. Recalling that the discounted stock price process $(\tilde{S}_t)_{0 \leq t \leq T}$ is an $\mathcal{F}_t$-martingale under the risk neutral probability measure $\mathbb{P}^*$, we have

$$\begin{aligned}
\mathbb{E}^*(e^{-rt}S_t - e^{-rt}K \mid \mathcal{F}_u) &= \mathbb{E}^*(e^{-rt}S_t \mid \mathcal{F}_u) - e^{-rt}K \\
&= e^{-ru}S_u - e^{-rt}K \\
&\geq e^{-ru}S_u - e^{-ru}K,
\end{aligned}$$

for all $u \leq t$. That is, the discounted payoff of an American call option without dividend is a $\mathbb{P}^*$-submartingale.
Then, we note that the convex and increasing functions preserve the submartingale property. Because the function $(\ x - K\ )_+$ is increasing and convex and the process $e^{-rt}(S_t - K)$ is submartingale under the risk-neutral probability measure $\mathbb{P}^*$ [4], we get that the discounted payoff process (Z_t) is also a $\mathbb{P}^*$-submartingale.
Then, by Proposition 4.3, we conclude that the optimal stopping time for American calls on a non-dividend paying stock is the maturity T.
Now, we define the process Snell envelope which is one of the main tools used in the valuation of American options.

Definition 4.4. A stochastic process $(V_t)_{0\leq t\leq T}$ is said to be a *Snell envelope* of the process $(Z_t)_{0\leq t\leq T}$, if it satisfies the following conditions:

i) (V_t) is a right continuous with left limits (RCLL) supermartingale.

ii) (V_t) dominates the process (Z_t).

iii) If (M_t) is another RCLL supermartingale that dominates the process (Z_t), then $V_t \leq M_t$ for all $t \in [0, T]$.

With the following theorem, we define the Snell envelope of the reward process (Z_t).

Theorem 4.5. *Let us assume that the process $(Z_t)_{0\leq t\leq T}$ is a sequence of non-negative, adapted and right continuous random variables such that*

$$0 \leq \mathbb{E}(\sup_{0\leq t\leq T} Z(t)) < \infty.$$

Then, $\sup_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}(Z_\tau \mid \mathcal{F}_t)$ *is the Snell envelope of Z_t.*

Proof. The proof can be found in [22]. □

As a result of this theorem, we deduce that the process

$$\sup_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}^*(e^{-r\tau}\psi(S_\tau) \mid \mathcal{F}_t)$$

is indeed the Snell envelope of $e^{-rt}\psi(S_t)$.
Now, in the light of this theorem, we can express the fair price of an American option on a stock without dividends.

4.2.1.2 Pricing Under Martingale Approach

As done in discrete-time, we can price American options with the help of hedging portfolios. But, this time, the valuation process will be executed with the help of trading strategies with consumption.

Definition 4.5. A trading strategy with consumption $\phi = (H_t^0, H_t)_{0\leq t\leq T}$ hedges an American option with the intrinsic value $\psi(S_t)$, if the following inequality holds for all $0 \leq t \leq T$

$$V_t(\phi) \geq \psi(S_t). \tag{4.7}$$

We denote Φ^ψ as the set of all trading strategies $\phi = (H_t^0, H_t)$ that hedges the American option on the finite time interval $[0, T]$ and $\mathcal{T}_{t,T}$ as the set of all stopping times taking values between t and T.

The following theorem gives us the unique price of an American option written on a stock without dividends. It shows that the value of any trading strategy with consumption in Φ^{ψ} dominates

$$u(t, S_t) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}^*(e^{-r(\tau-t)}\psi(S_\tau) \mid \mathcal{F}_t).$$

Moreover, it says that it is always possible to find a portfolio in Φ^{ψ} whose value is equal to $u(t, S_t)$. Because the process $(u(t, S_t))$ appears to be the minimum value of a portfolio that hedges the American option, it can be thought of the fair price of this option [23].

Theorem 4.6. *Let us consider a function $u : [0, T] \times \mathbb{R}^+ \to \mathbb{R}$ satisfying*

$$u(t, x) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}^*[e^{-r(\tau-t)}\psi(x \exp((r - (\sigma^2/2))(\tau - t) + \sigma(W_\tau - W_t))],$$

where $\mathcal{T}_{t,T}$ refers to the set of all stopping times taking values in $[t, T]$.
Then, we can find a trading strategy $\bar{\phi} \in \Phi^{\psi}$, such that $V_t(\bar{\phi}) = u(t, S_t)$, for all $t \in [0, T]$. Moreover, all trading strategies with consumption $\phi \in \Phi^{\psi}$ satisfies $V_t(\phi) \geq u(t, S_t)$, for all $t \in [0, T]$.

Proof. The sketch of the proof is given in [23]. Hence, a detailed proof is provided in this thesis. We will first show that there exists a hedging portfolio ϕ in Φ^{ψ} such that

$$V_t(\phi) \geq u(t, S_t),$$

for all $t \in [0, T]$.
Let $\phi = (H_t^0, H_t)_{0 \leq t \leq T}$ be a trading strategy with consumption that hedges an American option such that

$$V_t(\phi) \geq \psi(S_t),$$

for all $t \in [0, T]$.
As a result of Lemma 4.2, the discounted value process of a trading strategy with consumption is an $\mathcal{F}_t$-supermartingale under the probability measure $\mathbb{P}^*$. Also, Theorem 4.5 states that $(e^{-rt}u(t, S_t))$ is the smallest supermartingale that dominates the process $(e^{-rt}\psi(S_t))$. Then, we have

$$e^{-rt}V_t(\phi) \geq e^{-rt}u(t, S_t),$$

for all $\phi \in \Phi^{\psi}$.
Now, we will show that there exists a trading strategy $\bar{\phi} \in \Phi^{\psi}$ such that

$$V_t(\bar{\phi}) = u(t, S_t),$$

for all $t \in [0, T]$.
Because $\tilde{u}(t, S_t)$ is a right-continuous $\mathbb{P}^*$-supermartingale of class D (see Theorem D.13 in [22]), we can use the Doob-Meyer Decomposition as follows:

$$\tilde{u}(t, S_t) = M_t - A_t$$

where (M_t) is a uniformly integrable RCLL martingale under the probability measure $\mathbb{P}^*$ and (A_t) is a non-decreasing, adapted, right-continuous process null at zero.
Note that the filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ is the natural filtration of both $(B_t)_{0\leq t\leq T}$ and $(W_t)_{0\leq t\leq T}$. Using the martingale representation theorem for martingales, we can find an adapted process (K_t) satisfying

$$\int_0^T K_s^2 ds < \infty, \tag{4.8}$$

such that

$$M_t = M_0 + \int_0^t K_s dW_s. \tag{4.9}$$

Now, let us define a strategy $\bar{\phi} = (H_t^0, H_t)$ such that

$$H_t = \frac{K_t}{\sigma \tilde{S}_t}, \quad H_t^0 = \tilde{u}(t, S_t) - H_t\tilde{S}_t, \quad A_t = \int_0^t e^{-rs} dC_s.$$

It is obvious that H_t and H_t^0 are adapted. Also, (4.8) yields that

$$\int_0^T H_s^2 ds = \int_0^T \frac{K_s^2}{\sigma^2 \tilde{S}_s^2} < \infty.$$

Moreover,

$$\int_0^T \left|H_s^0\right| ds = \int_0^T \left|\tilde{u}(s, S_s) - H_s\tilde{S}_s\right| ds < \infty.$$

Hence,

$$\begin{aligned}
\tilde{V}_t(\bar{\phi}) &= H_t^0\tilde{S}_t^0 + H_t\tilde{S}_t = \tilde{u}(t, S_t) \\
&= M_t - A_t \\
&= M_0 + \int_0^t \sigma H_s\tilde{S}_s dW_s - \int_0^t e^{-rs} dC_s,
\end{aligned}$$

where (C_t) is adapted, non-decreasing and continuous process. Indeed, right continuous process (A_t) turns out to be continuous, when ψ has continuous paths (see Theorem D.13 in [22]). This implies the continuity of process C_t. With this results, we conclude that $\bar{\phi}$ is a trading strategy with consumption that indeed perfectly hedges the American option. □

Theorem 4.7. *Under the setting of Theorem 4.6, let $\psi(S_t)$ be defined by $(S_t - K)_+$. Then, the price of an American call option with strike price K is given by*

$$u(t, S_t) = \mathbb{E}^*(e^{-r(T-t)}(S_T - K)_+ \mid \mathcal{F}_t).$$

Proof. Let $t \geq 0$ and τ be any stopping time between t and T.
Since $e^{-r(T-t)}(S_T - K)_+ \geq e^{-r(T-t)}(S_T - K)$ for all $t \in [0, T]$ and the discounted stock price process $(\tilde{S}_t)$ is an $\mathcal{F}_t$-martingale, we have

$$\begin{aligned}
\mathbb{E}^*(e^{-r(T-t)}(S_T - K)_+ \mid \mathcal{F}_\tau) &\geq \mathbb{E}^*(e^{-r(T-t)}(S_T - K) \mid \mathcal{F}_\tau) \\
&= e^{rt}\tilde{S}_\tau - e^{-r(T-t)}K.
\end{aligned} \tag{4.10}$$

Then,

$$\mathbb{E}^*(e^{-r(T-t)}(S_T - K)_+ \mid \mathcal{F}_\tau) \geq e^{rt}\tilde{S}_\tau - e^{-r(\tau-t)}K.$$

Moreover, we know

$$\mathbb{E}^*(e^{-r(T-t)}(S_T - K)_+ \mid \mathcal{F}_\tau) \geq 0.$$

So,

$$\mathbb{E}^*(e^{-r(T-t)}(S_T - K)_+ \mid \mathcal{F}_\tau) \geq (e^{rt}\tilde{S}_\tau - e^{-r(\tau-t)}K)_+.$$

Taking expectation of both sides, we get

$$\mathbb{E}^*(e^{-r(T-t)}(S_T - K)_+) \geq \mathbb{E}^*((e^{rt}\tilde{S}_\tau - e^{-r(\tau-t)}K)_+).$$

for all $\tau \in \mathcal{T}_{t,T}$.
Therefore,

$$\mathbb{E}^*(e^{-r(T-t)}(S_T - K)_+) \geq \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}^*((e^{rt}\tilde{S}_\tau - e^{-r(\tau-t)}K)_+).$$

Furthermore, the following inequality is always satisfied

$$\mathbb{E}^*(e^{-r(T-t)}(S_T - K)_+) \leq \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}^*((e^{rt}\tilde{S}_\tau - e^{-r(\tau-t)}K)_+),$$

since T is the last time that an American option can be exercised.
Hence, we obtain the desired equality

$$\mathbb{E}^*(e^{-r(T-t)}(S_T - K)_+) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}^*((e^{rt}\tilde{S}_\tau - e^{-r(\tau-t)}K)_+).$$

□

4.2.2 Pricing with Variational Inequalities

As seen in the previous section, martingale approach considers maximizing the expectation of discounted payoff process under the risk-adjusted measure $\mathbb{P}^*$. It is apparent from Theorem 4.6 that, under this approach, the maximization problem is indeed closely related to hedging concept. But, instead of directly computing the expectation $\mathbb{E}^*\left(e^{-r(\tau-t)}\psi(S_\tau)\right)$, we can value an American option with the help of parabolic partial differential inequalities.

Note that, throughout the subsection, we omit the subscript t in the stock price process (S_t) for simplicity.

Lemma 4.8. *Let $u(t,S)$ be the value of an American option on a non-dividend paying stock. Then, the value of a put option $u(t,S)$ satisfies the following parabolic partial differential inequality [42]*

$$\frac{\partial u}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 u}{\partial S^2} + rS\frac{\partial u}{\partial S} - ru \leq 0. \tag{4.11}$$

Moreover, the price of an American call without dividends is modeled by the following parabolic partial differential inequality

$$\frac{\partial u}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 u}{\partial S^2} + rS\frac{\partial u}{\partial S} - ru = 0, \tag{4.12}$$

which is the Black-Scholes PDE derived for European options.

Proof. We can derive (4.11) and (4.12) with the help of a delta-hedging portfolio avoiding technicalities. Let $\phi = (1, -\Delta)$ be a portfolio built with an option and -y shares of stock, and let $V(t, S)$ denote the value of this portfolio at time t such that $V = u - \Delta S$. Applying Itô formula (see Theorem 2.10, Chapter 2) to the process $u(t, S)$, we get [42]

$$\begin{aligned} du &= \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}dS + \frac{1}{2}\frac{\partial^2 u}{\partial S^2}d\langle S, S\rangle \\ &= \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial S}S(\mu dt + \sigma dB) + \frac{1}{2}\frac{\partial^2 u}{\partial S^2}\sigma^2 S^2 dt. \end{aligned}$$

Then, by substituting above equation into $dV = du - \Delta dS$, we obtain

$$dV = \left(\frac{\sigma^2}{2}S^2\frac{\partial^2 u}{\partial S^2} + \mu S\frac{\partial u}{\partial S} + \frac{\partial u}{\partial t} - \mu\Delta S\right)dt + \sigma S\left(\frac{\partial u}{\partial S} - \Delta\right)dB. \tag{4.13}$$

We point out that the occurrence of dB term in (4.13) can prevent an efficient hedging, since stock price process can show huge movements due to the unbounded variation of Brownian motion. That is why, we aim to remove the dB term from the above equation. It is apparent that we fulfill this purpose by considering $\frac{\partial u}{\partial S} = \Delta$. Hence,

$$dV = \left(\frac{\sigma^2}{2}S^2\frac{\partial^2 u}{\partial S^2} + \frac{\partial u}{\partial t}\right)dt.$$

It is the reason that why we call this process as delta hedging.
Now, let us consider the case that the portfolio ϕ provides a constant return $rVdt = \left(rS\frac{\partial u}{\partial S} - ru\right)dt$ at each time interval $[t-dt, t]$. By comparing these returns $rVdt$ and dV, we can obtain the desired inequality (4.11). Let $dV > rVdt$, then the holder of an American option can benefit from an arbitrage opportunity [42]. To be more precise, when we receive an amount of V with a constant borrowing rate r and put it into the portfolio ϕ for a future return dV, we can obtain a profit $dV - rVdt$. Hence, the following inequality is always satisfied:

$$\frac{\partial u}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 u}{\partial S^2} + rS\frac{\partial u}{\partial S} - ru \le 0,$$

for an American option written on a non-dividend paying stock.
Specifically, the equality (4.12) derives from the fact that an American call option with no dividend behaves like an European option. Indeed, we know from the Corollary 4.4 that an American call written on a non-dividend paying asset is not exercised until maturity. Therefore, there is no difference between the valuation of an American and European call in the case underlying does not distribute any dividend. □

Lemma 4.9. *Let $\psi(S)$ denote the intrinsic value of an American option with no divided at time t. Then, the followings are always satisfied*

$$
\begin{aligned}
u(t,S) &\geq \psi(S), \qquad (t,S) \in [0,T] \times \mathbb{R}, && (4.14)\\
u(T,S) &= \psi(S). && (4.15)
\end{aligned}
$$

In the case (4.14) and (4.15) do not hold, the option holder can benefit from an arbitrage opportunity, indeed.
In the light of Lemma 4.8 and Lemma 4.9, we can determine an exercise strategy for American options. Actually, since the optimal strategy for American calls is investigated before (see Corollary 4.4), we focus on American put options.
Let us recall that an intrinsic value corresponds to the money that the option holder can get with an early exercise, and let us consider that (4.14) holds. An early exercise is advantageous when $u(t,S) = \psi(S)$. Otherwise, it can cause huge losses, because it means to trade the option less than its worth.
Now, taking the inequality (4.11) into account, we can deduce that it is better to hold the option, if (4.11) turns out to be a PDE in the form of (4.12). Otherwise, an immediate exercise is much more profitable for the holder. To make it more apparent, let us consider an American put option with strike price K and let $u(t,S) = K - S$. When we substitute $K - S$ into the inequality (4.11), we obtain [42]

$$\frac{\partial u}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 u}{\partial S^2} + rS\frac{\partial u}{\partial S} - ru = -rK < 0.$$

On the other hand, when the value of a hedging portfolio is worth less than the risk-free portfolio, it does not make sense to continue holding it. Because holding portfolio ϕ means holding the American option, it is beneficial to take the advantage of an early exercise in this case. Hence, when the inequality $\frac{\partial u}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 u}{\partial S^2} + rS\frac{\partial u}{\partial S} - ru < 0$, then $u(t,S) = K - S$. That is, $u(t,S) = K - S$ is equivalent to

$$\frac{\partial u}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 u}{\partial S^2} + rS\frac{\partial u}{\partial S} - ru < 0.$$

Moreoever,

$$\frac{\partial u}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 u}{\partial S^2} + rS\frac{\partial u}{\partial S} - ru = 0,$$

when $u(t,S) > K - S$.
These equivalences lead us to the following corollary.

Corollary 4.10. *Let $\psi(S)$ be the intrinsic value of an American option with no dividend and let $u(t,S)$ correspond to the value of this option. Then,*

$$(\psi - u)\left(\frac{\partial u}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 u}{\partial S^2} + rS\frac{\partial u}{\partial S} - ru\right) = 0,$$

for all $(t,S) \in [0,T] \times \mathbb{R}$.

In the light of these results, we deduce that the valuation of an American put option is actually an *obstacle problem* [35] such that

$$\frac{\partial u}{\partial t}+\frac{\sigma^2}{2}S^2\frac{\partial^2 u}{\partial S^2}+rS\frac{\partial u}{\partial S}-ru\leq 0, \qquad \forall (t,S)\in[0,T]\times\mathbb{R},$$

$$\left(\frac{\partial u}{\partial t}+\frac{\sigma^2}{2}S^2\frac{\partial^2 u}{\partial S^2}+rS\frac{\partial u}{\partial S}-ru\right)(\psi-u)=0, \qquad \forall (t,S)\in[0,T]\times\mathbb{R}, \tag{4.16}$$

$$u(T,S)=\psi(S), \qquad u\geq\psi \qquad \forall S\in\mathbb{R}.$$

Moreover, the American call option satisfies

$$\frac{\partial u}{\partial t}+\frac{\sigma^2}{2}S^2\frac{\partial^2 u}{\partial S^2}+rS\frac{\partial u}{\partial S}-ru=0, \quad \forall (t,S)\in[0,T]\times\mathbb{R}, \tag{4.17}$$

$$u(T,S)=\psi(S), \quad \forall S\in\mathbb{R}.$$

The following theorems provide us a way to find an elegant solution for the above obstacle problem. Indeed, we generalize the problem into the multidimensional SDE's in the form of

$$\begin{aligned} dX_t^1 &= b^1(t,X_t)dt+\sum_{j=1}^{d}\sigma_{1j}(t,X_t)dB_t^j, \\ &\vdots \\ dX_t^n &= b^n(t,X_t)dt+\sum_{j=1}^{d}\sigma_{nj}(t,X_t)dB_t^j, \end{aligned} \tag{4.18}$$

where $X_t=(X_t^1,\dots,X_t^n)$ is an n-dimensional Itô process, $B_t=(B_t^1,\dots,B_t^d)$ is a d-dimensional Brownian motion, $\sigma(t,x)=(\sigma_{ij}(t,x))_{1\leq i\leq n,1\leq j\leq d}$ is an $n\times d$ matrix and $b(t,x)=(b^1(t,x),\dots,b^n(t,x))$ is an $\mathbb{R}^n$-valued function defined on $\mathbb{R}^+\times\mathbb{R}^n$. In addition, we define an operator $A_t : f\mapsto A_tf$ depending on an $\mathbb{R}$-valued function of class C^2 such that [23]

$$A_tf(x)=\frac{1}{2}\sum_{i,j=1}^{n}a_{i,j}(t,x)\frac{\partial^2 f}{\partial x_i x_j}(x)+\sum_{j=1}^{n}b_j(t,x)\frac{\partial f}{\partial x_j}(x),$$

where $a_{i,j}(t,x)=\sum_{k=1}^{d}\sigma_{i,k}(t,x)\sigma_{j,k}(t,x)$.
This operator is called *infinitesimal generator of the process* (X_t).
In order to solve this obstacle problem, we need some postulations about the coefficients b, σ and the operator A_t. To be more precise, let us assume that b and σ are bounded and Hölder continuous [18, 29], and let the infinitesimal generator A_t fulfill ellipticity property

$$\exists M\in\mathbb{R}^+, \quad \sum_{ij}a_{ij}(t,x)\varepsilon_i\varepsilon_j\geq M\left(\sum_{i=1}^{n}\varepsilon_i^2\right),$$

for all $(\varepsilon_1, \ldots, \varepsilon_n) \in \mathbb{R}^n$ [23].
Under these settings, we can now prove our fundamental theorems used for the pricing of American options with no dividend.

Theorem 4.11. *Let $r : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a bounded, continuous function interpreting the risk-free interest rate in the market and let u be a regular function satisfying the following system:*

$$\begin{cases} \frac{\partial u}{\partial t} + A_t u - ru \leq 0, \quad u \geq f \qquad in\ [0,T] \times \mathbb{R}^n, \\ \left(\frac{\partial u}{\partial t} + A_t u - ru\right)(f-u) = 0 \qquad in\ [0,T] \times \mathbb{R}^n, \\ u(T,x) = f(x) \qquad in\ \mathbb{R}^n. \end{cases} \tag{4.19}$$

Then,

$$u(t,x) = \Phi(t,x) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}\left(e^{\int_t^\tau r(s,X_s^{t,x})ds} f(X_\tau^{t,x})\right).$$

These parabolic partial differential equations given in system (4.30) are called variational inequalities.

In order to prove Theorem 4.13, we need the following proposition [23].

Proposition 4.12. *Let $X_t = (X_t^1, \ldots, X_t^n)$ be an n-dimensional Itô process satisfying (4.18). If $r : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is a bounded, continuous function and $u : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a function of class $C^{1,2}$ having bounded partial derivatives in x, then the process $(M_t)_{t\geq 0}$ defined by*

$$M_t = e^{-\int_0^t r(s,X_s)ds} u(t,X_t) - \int_0^t e^{-\int_0^s r(v,X_v)dv} \left(\frac{\partial u}{\partial t} + A_s u - ru\right)(s,X_s)ds$$

is a martingale.

Proof of Theorem 4.13. The sketch of the proof is given in [23]. Hence, a detailed proof is provided in this thesis. We emphasize that the solution $u(t,x)$ of the system (4.30) is generally not a function of class $C^{1,2}$ [23]. That is why, Itô formula for the process $u(t,x)$ is not always applicable. Therefore, we prove the theorem for the functions $u(t,x) \in C^{1,2}$.
We will first prove the theorem for $t = 0$. For $t \neq 0$, it can be shown in a similar way.
Let $t = 0$, and let us consider a stochastic process (X_t) satisfying (4.18), with $X_0 = x$. From Proposition 4.12, we know that the process (M_t) defined by

$$M_t = e^{-\int_0^t r(s,X_s^x)ds} u(t,X_t^x) - \int_0^t e^{-\int_0^s r(v,X_v^x)dv} \left(\frac{\partial u}{\partial t} + A_s u - ru\right)(s,X_s^x)ds$$

is an $\mathcal{F}_t$-martingale.
Using the optional sampling theorem (see Proposition 2.7, Chapter 2), we get

$$\mathbb{E}(M_\tau) = \mathbb{E}(M_0),$$

for all stopping times $\tau \in \mathcal{T}_{0,T}$.
Then,

$$\begin{aligned} u(0,x) &= \mathbb{E}\left(e^{-\int_0^\tau r(s,X_s^x)ds} u(\tau, X_\tau^x)\right) \\ &\quad - \mathbb{E}\left(\int_0^\tau e^{-\int_0^s r(v,X_v^x)dv}\left(\frac{\partial u}{\partial t} + A_s u - ru\right)(s, X_s^x)ds\right), \end{aligned}$$

where $\mathbb{E}(M_0) = u(0,x)$.
Since $\frac{\partial u}{\partial t} + A_t u - ru \leq 0$ and $u(t,x) \geq f(x)$, we obtain

$$\begin{aligned} u(0,x) &\geq \mathbb{E}\left(e^{-\int_0^\tau r(s,X_s^x)ds} u(\tau, X_\tau^x)\right) \\ &\geq \mathbb{E}\left(e^{-\int_0^\tau r(s,X_s^x)ds} f(X_\tau^x)\right), \end{aligned}$$

for all $\tau \in \mathcal{T}_{0,T}$.
Thus,

$$u(0,x) \geq \sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}\left(e^{\int_0^\tau r(s,X_s^x)ds} f(X_\tau^x)\right). \tag{4.20}$$

In order to complete the proof for $t = 0$, we also need to show

$$u(0,x) \leq \sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}\left(e^{\int_0^\tau r(s,X_s^x)ds} f(X_\tau^x)\right).$$

Therefore, we will define a new random variable τ_{opt} such that

$$\tau_{opt} := \inf\left\{0 \leq s \leq T \mid u(s, X_s^x) = f(X_s^x)\right\}.$$

We note that the random variable τ_{opt} is indeed a stopping time. Moreover, we have

$$u(s, X_s^x) \neq f(X_s^x),$$

for all $0 \leq s < \tau_{opt}$.
Then, $\left(\frac{\partial u}{\partial t} + A_t u - ru\right)(f - u) = 0$ yields that

$$\left(\frac{\partial u}{\partial t} + A_t u - ru\right) = 0,$$

for all $0 \leq s < \tau_{opt}$.
Hence, the process $M_{\tau_{opt}}$ turns out to be

$$\begin{aligned} M_{\tau_{opt}} &= e^{-\int_0^{\tau_{opt}} r(s,X_s^x)ds} u(\tau_{opt}, X_{\tau_{opt}}^x) \\ &\quad - \int_0^{\tau_{opt}} e^{-\int_0^s r(v,X_v^x)dv}\left(\frac{\partial u}{\partial t} + A_s u - ru\right)(s, X_s^x)ds \\ &= e^{-\int_0^{\tau_{opt}} r(s,X_s^x)ds} u(\tau_{opt}, X_{\tau_{opt}}^x). \end{aligned}$$

As a result of optional sampling theorem, we get

$$\mathbb{E}(M_{\tau_{opt}}) = \mathbb{E}(M_0).$$

Then,

$$u(0,x) = \mathbb{E}(e^{-\int_0^{\tau_{opt}} r(s,X_s^x)ds} u(\tau_{opt}, X_{\tau_{opt}}^x)), \tag{4.21}$$

where $\mathbb{E}(M_0) = u(0,x)$.
Since $u(\tau_{opt}, X_{\tau_{opt}}^x) = f(X_{\tau_{opt}}^x)$,

$$u(0,x) = \mathbb{E}(e^{-\int_0^{\tau_{opt}} r(s,X_s^x)ds} f(X_{\tau_{opt}}^x)).$$

That is,

$$u(0,x) \leq \sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}\left(e^{\int_0^\tau r(s,X_s^x)ds} f(X_\tau^x)\right), \tag{4.22}$$

since τ_{opt} is a stopping time taking values in $[0,T]$.
From (4.20) and (4.22), we obtain the desired equality

$$u(0,x) = \Phi(0,x) = \sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}\left(e^{\int_0^\tau r(s,X_s^x)ds} f(X_\tau^x)\right).$$

Now, let $t > 0$, and let the n-dimensional process $X_t = (X_t^1, \ldots, X_t^n)$ be the solution of (4.18). Applying integration by parts formula to the process

$$e^{-\int_0^\tau r(v,X_v)dv} u(\tau, X_\tau),$$

we get

$$\begin{aligned} e^{-\int_0^\tau r(v,X_v)dv} u(\tau, X_\tau) &= u(0,X_0) + \int_0^\tau e^{-\int_0^s r(v,X_v)dv} du(s,X_s) \\ &+ \int_0^\tau u(s,X_s) de^{-\int_0^s r(v,X_v)dv}, \end{aligned} \tag{4.23}$$

for all $\tau \in \mathcal{T}_{0,T}$.
We note that, by applying Itô's formula to $u(t,X_t)$, we get

$$\begin{aligned} du(t,X_t) &= \frac{\partial u}{\partial t}(t,X_t)dt + \sum_{i=1}^n \frac{\partial u}{\partial x_i}(t,X_t)dX_t^i \\ &+ \frac{1}{2}\sum_{i,m=1}^n \frac{\partial^2 u}{\partial x_i x_m}(t,X_t) d\langle X^i, X^m\rangle_t \\ &= \frac{\partial u}{\partial t}(t,X_t) + \sum_{i=1}^n \frac{\partial u}{\partial x_i} b_t^i(t,X_t)dt + \sum_{i=1}^n \frac{\partial u}{\partial x_i}\left(\sum_{j=1}^d \sigma_{i,j} dB_t^j\right)(t,X_t) \\ &+ \frac{1}{2}\sum_{i,m=1}^n \frac{\partial^2 u}{\partial x_i x_m}\left(\sum_{k=1}^d \sigma_{i,k}\sigma_{m,k}\right)(t,X_t)dt. \end{aligned} \tag{4.24}$$

Moreover, we have

$$de^{-\int_0^t r(v,X_v)dv} = -e^{-\int_0^t r(v,X_v)} r(t,X_t)dt. \tag{4.25}$$

Plugging (4.24) and (4.25) into (4.23),

$$\begin{aligned} e^{-\int_0^\tau r(s,X_s)ds}u(\tau,X_\tau) &= u(0,X_0) \\ &\quad + \int_0^\tau e^{-\int_0^s r(v,X_v)dv}\left(\frac{\partial u}{\partial t} + A_s u - ru\right)(s,X_s)ds \\ &\quad + \sum_{i=1}^n \sum_{j=1}^d \int_0^\tau e^{-\int_0^s r(v,X_v)dv}\frac{\partial u}{\partial x_i}\sigma_{i,j}(s,X_s)dB_s^j. \end{aligned}$$

From the Markov property of $(X_t)_{0\le t\le T}$, we have

$$e^{-\int_0^\tau r(v,X_v^{0,y})dv}u(\tau,X_\tau^{0,y}) = e^{-\int_t^\tau r(v,X_v^{t,X_t^y})dv}u(\tau,X_\tau^{t,X_t^y}),$$

where X_τ^{t,X_t^y} is the solution of (4.18) starting from X_t^y at time $t \le \tau$.
Considering $X_t^y = x$, we get

$$\begin{aligned} e^{-\int_t^\tau r(v,X_v^{t,x})dv}u(\tau,X_\tau^{t,x}) &= u(t,x) \\ &\quad + \int_t^\tau e^{-\int_t^s r(v,X_v^{t,x})dv}\left(\frac{\partial u}{\partial t} + A_s u - ru\right)(s,X_s^{t,x})ds \\ &\quad + \sum_{i=1}^n \sum_{j=1}^d \int_t^\tau e^{-\int_t^s r(v,X_v^{t,x})dv}\frac{\partial u}{\partial x_i}\sigma_{i,j}(s,X_s^{t,x})dB_s^j. \end{aligned}$$

for all $\tau \in \mathcal{T}_{t,T}$.
Thus,

$$\begin{aligned} u(t,x) &= e^{-\int_t^\tau r(v,X_v^{t,x})dv}u(\tau,X_\tau^{t,x}) \\ &\quad - \int_t^\tau e^{-\int_t^s r(v,X_v^{t,x})dv}\left(\frac{\partial u}{\partial t} + A_s u - ru\right)(s,X_s^{t,x})ds \\ &\quad - \sum_{i=1}^n \sum_{j=1}^d \int_t^\tau e^{-\int_t^s r(v,X_v^{t,x})dv}\sigma_{i,j}(s,X_s^{t,x})\frac{\partial u}{\partial x_i}(s,X_s^{t,x})dB_s^j. \end{aligned}$$

Taking expectation of both sides,

$$\begin{aligned} u(t,x) &= \mathbb{E}\Bigg(e^{-\int_t^\tau r(v,X_v^{t,x})dv}u(\tau,X_\tau^{t,x}) \\ &\quad - \int_t^\tau e^{-\int_t^s r(v,X_v^{t,x})dv}\left(\frac{\partial u}{\partial t} + A_s u - ru\right)(s,X_s^{t,x})ds \\ &\quad - \sum_{i=1}^n \sum_{j=1}^d \int_t^\tau e^{-\int_t^s r(v,X_v^{t,x})dv}\frac{\partial u}{\partial x_i}\sigma_{i,j}(s,X_s^{t,x})dB_s^j\Bigg). \end{aligned}$$

Note that the stochastic integral

$$\int_t^\tau e^{-\int_t^s r(v,X_v^{t,x})dv}\frac{\partial u}{\partial x_i}\sigma_{i,j}(s,X_s^{t,x})dB_s^j$$

is a martingale, since r, σ and $\frac{\partial u}{\partial x}$ are bounded functions.
Then, by the constant expectation property of martingales, we have

$$\mathbb{E}\left(\int_t^\tau e^{-\int_t^s r(v,X_v^{t,x})dv}\frac{\partial u}{\partial x_i}(s,X_s^{t,x})\sigma_{i,j}(s,X_s^{t,x})dB_s^j\right)=0.$$

This implies that

$$\begin{aligned} u(t,x) &= \mathbb{E}\left(e^{-\int_t^\tau r(v,X_v^{t,x})dv}u(\tau,X_\tau^{t,x})\right. \\ &\quad \left.-\int_t^\tau e^{-\int_t^s r(v,X_v^{t,x})dv}\left(\frac{\partial u}{\partial t}+A_s u-ru\right)(s,X_s^{t,x})ds\right). \end{aligned} \tag{4.26}$$

Moreover, we know $\frac{\partial u}{\partial t}+A_s u-ru\leq 0$ and $u(t,x)\geq f(x)$. Then,

$$\begin{aligned} u(t,x) &= \mathbb{E}\left(e^{-\int_t^\tau r(v,X_v^{t,x})dv}u(\tau,X_\tau^{t,x})\right) \\ &\geq \mathbb{E}\left(e^{-\int_t^\tau r(v,X_v^{t,x})dv}f(X_\tau^{t,x})\right). \end{aligned}$$

Hence,

$$u(t,x)\geq \sup_{\tau\in\mathcal{T}_{t,T}}\mathbb{E}\left(e^{-\int_t^\tau r(v,X_v^{t,x})dv}f(X_\tau^{t,x})\right). \tag{4.27}$$

Now, let us define a stopping time $\hat{\tau}_{opt}$ as follows:

$$\hat{\tau}_{opt}:=\inf\left\{t\leq s\leq T \mid\ u(s,X_s^{t,x})=f(X_s^{t,x})\right\}.$$

By definition of $\hat{\tau}_{opt}$, it is clear that

$$u(s,X_s^{t,x})\neq f(X_s^{t,x}),$$

for all $t\leq s<\hat{\tau}_{opt}$.
Moreover,

$$u(\hat{\tau}_{opt},X_{\hat{\tau}_{opt}}^{t,x})=f(X_{\hat{\tau}_{opt}}^{t,x}).$$

Since $\left(\frac{\partial u}{\partial t}+A_s u-ru\right)(f-u)=0$, we have

$$\left(\frac{\partial u}{\partial t}+A_s u-ru\right)=0,\qquad \forall t\leq s<\hat{\tau}_{opt}. \tag{4.28}$$

Substituting $\hat{\tau}_{opt}=\tau$ into (4.26), the process $(u(t,x))$ becomes

$$\begin{aligned} u(t,x) &= \mathbb{E}\left(e^{-\int_t^{\hat{\tau}_{opt}} r(v,X_v^{t,x})dv}u(\hat{\tau}_{opt},X_{\hat{\tau}_{opt}}^{t,x})\right. \\ &\quad \left.-\int_t^{\hat{\tau}_{opt}} e^{-\int_t^s r(v,X_v^{t,x})dv}\left(\frac{\partial u}{\partial t}+A_s u-ru\right)(s,X_s^{t,x})ds\right). \end{aligned}$$

Then, (4.28) yields that

$$u(t,x)=\mathbb{E}\left(e^{-\int_t^{\hat{\tau}_{opt}} r(v,X_v^{t,x})dv}u(\hat{\tau}_{opt},X_{\hat{\tau}_{opt}}^{t,x})\right).$$

Recalling $u(\hat{\tau}_{opt}, X^{t,x}_{\hat{\tau}_{opt}}) = f(X^{t,x}_{\hat{\tau}_{opt}})$, we obtain

$$u(t,x) = \mathbb{E}\left(e^{-\int_t^{\hat{\tau}_{opt}} r(v,X_v^{t,x})dv} f(X^{t,x}_{\hat{\tau}_{opt}})\right).$$

Hence,

$$u(t,x) \leq \sup_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}\left(e^{-\int_t^{\hat{\tau}_{opt}} r(v,X_v^{t,x})dv} f(X^{t,x}_{\hat{\tau}_{opt}})\right). \tag{4.29}$$

From (4.27) and (4.29), we get

$$u(t,x) = \sup_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}\left(e^{-\int_t^{\hat{\tau}_{opt}} r(v,X_v^{t,x})dv} f(X^{t,x}_{\hat{\tau}_{opt}})\right).$$

□

The following theorem plays an important role for pricing American call options without dividends.

Theorem 4.13. *Let $r : [0,T]\times\mathbb{R} \to \mathbb{R}$ be a bounded, continuous function interpreting the risk-free interest rate in the market and let u be a regular function satisfying the following system:*

$$\begin{cases} \frac{\partial u}{\partial t} + A_t u - ru = 0 & \text{in } [0,T]\times\mathbb{R}^n, \\ u(T,x) = f(x) & \text{in } \mathbb{R}^n. \end{cases} \tag{4.30}$$

Then,

$$u(t,x) = \Phi(t,x) = \mathbb{E}\left(e^{\int_t^T r(s,X_s^{t,x})ds} f(X_T^{t,x})\right).$$

Proof. The proof can be easily shown by using the proposition Proposition 4.12. It can also be found in [23]. □

As we mentioned before, it is not always possible to find a solution of class $C^{1,2}$ for the system (4.30). Therefore, we need an additional theorem that makes the valuation process more meaningful. With the following theorem, we achieve our goal by defining an infinitesimal generator $A_t^{\log}$ associated with the Black-Scholes model. As a result, we will find the unique price of American options with no dividend.
Now, let us recall that the stock price process in the Black-Scholes model is actually the solution of the following SDE

$$dS_t = S_t r dt + S_t \sigma dW_t,$$

under the risk-neutral probability measure $\mathbb{P}^*$. Then, we can easily verify that the infinitesimal generator of the Black-Scholes model is actually in the form

$$A_t f(x) = \frac{\sigma^2}{2} x^2 \frac{\partial^2 f}{\partial x^2} + rx\frac{\partial f}{\partial x}.$$

Since this generator is not elliptic [23], we need to find another operator that fulfills the ellipticity condition.
Making a little adjustment, it is possible to obtain an elliptic infinitesimal generator. Here, we focus on American put options, because the price of an American call without dividends is already given by

$$u(t, S_t) = \mathbb{E}\left(e^{\int_t^T r(s, X_s^{t,x})ds} f(X_T^{t,x})\right).$$

Let $X_t = \log(S_t)$ satisfying the following SDE

$$dX_t = \left(r - \frac{\sigma^2}{2}\right) dt + \sigma dW_t.$$

Then, its infinitesimal generator is in the form

$$A_t^{\log} = \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + \left(r - \frac{\sigma^2}{2}\right)\frac{\partial}{\partial x},$$

which satisfies the ellipticity condition [23].
That is, if we set $S = e^x$, we obtain an elliptic infinitesimal generator under Black-Scholes model. After defining this operator $A_t^{\log}$, we can now investigate the variational inequalities associated with the American options.
Let $u(t, S)$ denote the price of an American put option with strike price K and $\psi(S) = (K - S)_+$ be the intrinsic value of the option. We remark that, for simplicity we use S instead of S_t. Now, let us set $S = e^x$ and $v(t, x) = u(t, e^x)$, with $x \in \mathbb{R}$. With the help of multivariable chain rule, the partial derivatives $\frac{\partial v}{\partial x}, \frac{\partial v}{\partial t}, \frac{\partial^2 v}{\partial x^2}$ are defined as follows:

$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial S}S, \quad \frac{\partial v}{\partial t} = \frac{\partial u}{\partial t}, \quad \frac{\partial^2 v}{\partial x^2} = \frac{\partial u}{\partial S}S + S^2\frac{\partial^2 u}{\partial S^2}.$$

From the system (4.16), we know $\frac{\partial u}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 u}{\partial S^2} + rS\frac{\partial u}{\partial S} - ru \leq 0$. Therefore,

$$\frac{\partial v}{\partial t} + A_t^{\log}u - rv = \frac{\partial u}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 u}{\partial S^2} + rS\frac{\partial u}{\partial S} - ru \leq 0,$$

for all $(t, x) \in [0, T] \times \mathbb{R}$.
Moreover, when we define $\phi(x)$ as $(K - e^x)_+$, we have

$$v(T, x) = u(T, e^x) = \psi(e^x) = \phi(x), \qquad \forall x \in \mathbb{R},$$

and

$$v(t, x) = u(t, e^x) \geq \psi(e^x) = \phi(x), \qquad \forall (t, x) \in [0, T] \times \mathbb{R}.$$

Hence, in the case $S = e^x$, an American put option with strike price K satisfies the following variational inequalities

$$\frac{\partial v}{\partial t}(t, x) + A_t^{\log}v(t, x) - rv(t, x) \leq 0, \qquad \forall (t, x) \in [0, T] \times \mathbb{R},$$

$$\left(\frac{\partial v}{\partial t}(t, x) + A_t^{\log}v(t, x) - rv(t, x)\right)(\phi - v) = 0, \qquad \forall (t, x) \in [0, T] \times \mathbb{R}, \quad (4.31)$$

$$v(T, x) = \phi(x), \qquad v \geq \phi \qquad \forall x \in \mathbb{R}.$$

Now, we introduce the fundamental theorem used to price American put options without dividend. The theorem says that the system defined above admits a unique solution (not necessarily regular). Moreover, this result presents a way to price American put options with the help of inequality system (4.31).

Theorem 4.14. *The variational inequality system (4.31) admits a unique solution $v(t,x)$ such that this solution is continuous and bounded and has the locally bounded partial derivatives $\frac{\partial v}{\partial x}, \frac{\partial v}{\partial t}, \frac{\partial^2 v}{\partial x^2}$.*

Also, the following equality always holds

$$v(t, \log(x)) = \Phi(t,x) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}^*[e^{-r(\tau-t)}\psi(x \exp((r-(\sigma^2/2))(\tau-t)+\sigma(W_\tau - W_t))].$$

For the proof, we refer to Jaillet, Lamberton and Lapeyre (1990) [18].
We point out that the variational inequality approach indeed gives the same solution as the martingale approach for American put options.

CHAPTER 5

CONTINUOUS-TIME MODELING FOR AMERICAN OPTIONS ON DIVIDEND PAYING STOCKS

In the previous chapters, we have considered at length pricing of American options written on an underlying asset that does not distribute dividends. Now, we will investigate the valuation of American options in the case the underlying pays dividends with a constant continuous yield.
In this chapter, we will closely follow the theory given in Chapter 4. Since the fundamental results show similarities with the previous chapter, we will only discuss the main theorems avoiding technicalities.

5.1 Description of the Model with Dividends

We consider a continuous time interval $[0, T]$, with $T < \infty$. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t \le T}, \mathbb{P})$ be a filtered probability space, and let B_t^δ be the standard Brownian motion under the probability measure $\mathbb{P}$. We suppose that the market, similar to the non-dividend case, includes only 2 securities: a stock with a constant continuous dividend yield $\delta \ge 0$ and a bond with a constant interest rate $r > 0$. Then, at time t,

- the bond price $S_t^0 = e^{rt}$ is modeled by the following SDE

$$dS_t^0 = rS_t^0 dt, \qquad S_0^0 = 1.$$

- the stock price process (S_t) satisfies

$$dS_t = S_t(\mu dt + \sigma dB_t^\delta). \tag{5.1}$$

- the discounted price process $(\tilde{S}_t)$ is the solution of the following SDE

$$d\tilde{S}_t = \tilde{S}_t \left((\mu - r)dt + \sigma dB_t^\delta\right),$$

where the constant coefficients $\mu > 0$ and $\sigma > 0$ represent the drift and volatility terms of the Black-Scholes model [5], respectively.

- the total change in the value of the stock is given by

$$dS_t + \delta S_t dt.$$

Under these settings, we can easily show that the process $(e^{\delta t}\tilde{S}_t)_{0\le t\le T}$ satisfies the following stochastic differential equation

$$d(e^{\delta t}\tilde{S}_t) = e^{\delta t}\tilde{S}_t\left((\delta+\mu-r)dt + \sigma dB_t^{\delta}\right). \tag{5.2}$$

But, the deterministic term $(\delta+\mu-r)dt$ in (5.2) prevents the process $(e^{\delta t}\tilde{S}_t)$ to be a martingale under the measure $\mathbb{P}$. However, Girsanov theorem gives us a way to make the process $(e^{\delta t}\tilde{S}_t)$ a martingale under an equivalent measure $\mathbb{P}^{\delta}$. Under this measure, we define a standard Brownian motion $W_t^{\delta} = B_t^{\delta} + \frac{(\mu+\delta-r)}{\sigma}t$ such that the SDE (5.2) turns into

$$d(e^{\delta t}\tilde{S}_t) = \tilde{S}_t e^{\delta t}\sigma dW_t^{\delta}. \tag{5.3}$$

Hence, the process $(e^{(\delta-r)t}S_t)$ is a martingale under the probability measure $\mathbb{P}^{\delta}$. Moreover, we have

$$\begin{aligned} dS_t &= S_t(\mu dt + \sigma dB_t^{\delta}) \\ &= S_t(\mu dt + \sigma dW_t^{\delta} - (\mu+\delta-r)dt) \\ &= S_t\left((r-\delta)dt + \sigma dW_t^{\delta}\right), \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} d\tilde{S}_t &= \tilde{S}_t\left((\mu-r)dt + \sigma dB_t^{\delta}\right) \\ &= \tilde{S}_t((\mu-r)dt + \sigma dW_t^{\delta} - (\mu+\delta-r)dt) \\ &= \tilde{S}_t\left(-\delta dt + \sigma dW_t^{\delta}\right), \end{aligned} \tag{5.5}$$

under the new measure $\mathbb{P}^{\delta}$. We say that this measure $\mathbb{P}^{\delta}$ is a risk-neutral probability measure that makes the process $(e^{(\delta-r)t}S_t)$ a martingale.
Afterwards, (5.4) and (5.5) have the following closed-form solutions

$$S_t = S_0 e^{(r-\delta-\frac{\sigma^2}{2})t+\sigma W_t^{\delta}},$$

and

$$\tilde{S}_t = \tilde{S}_0 e^{\sigma W_t^{\delta}-(\delta+\frac{\sigma^2}{2})t}.$$

5.2 Trading Strategies

In this subsection, we give a brief introduction about the trading strategies that can be constructed in the market. The definitions show similarities with the ones defined in Chapter 4.
A trading strategy $\phi = (H_t^0, H_t)$ with dividend is an $\mathcal{F}_t$-measurable, $\mathbb{R}^2$-valued random variable such that the components H_t^0 and H_t denote the number of

shares of bond and stock, respectively, we trade in the market at time t [23]. Then, the value of the portfolio at time t becomes

$$V_t(\phi) = H_t^0 S_t^0 + H_t S_t. \tag{5.6}$$

Now, we continue with the definition of self-financing strategies.

Definition 5.1. An $\mathcal{F}_t$-measurable stochastic variable $\phi = (H_t^0, H_t)_{0\leq t\leq T}$ is said to be a self-financing strategy with dividend if it satisfies the following properties:

1. $\int_0^T |H_t^0|\, dt + \int_0^T H_t^2 dt < \infty$ a.s.,
2. $H_t^0 S_t^0 + H_t S_t = H_0^0 S_0^0 + H_0 S_0 + \int_0^t H_u^0 dS_u^0 + \int_0^t H_u(dS_u + \delta S_u du)$,

for all $t \in [0, T]$.

We see that the value of a self-financing portfolio is only affected by the price movements and the dividend payments of the stock.

We proceed to the concept of trading strategies with dividend and consumption which provides a way to hedge American options written on a dividend paying stocks.

Definition 5.2. An $\mathcal{F}_t$-measurable random variable $\phi = (H_t^0, H_t, C_t)_{0\leq t\leq T}$ is called a trading strategy with dividend and consumption , if it satisfies the following properties:

1. $\int_0^T |H_t^0|\, dt + \int_0^T |H_t|^2\, dt < \infty$ a.s.,
2. $H_t^0 S_t^0 + H_t S_t = H_0^0 S_0^0 + H_0 S_0 + \int_0^T H_u^0 dS_u^0 + \int_0^T H_u(dS_u + \delta S_u du) - C_t$,

for all $t \in [0, T]$, where $(C_t)_{0\leq t\leq T}$ is an adapted, continuous, non-decreasing process null at $t = 0$.

With the help of this definition, we can show that the discounted value process of a trading strategy with dividend and consumption is a supermartingale under the risk-neutral probability measure $\mathbb{P}^\delta$.

Lemma 5.1. *Let $\phi = (H_t^0, H_t, C_t)_{0\leq t\leq T}$ be a trading strategy with dividend and consumption and let $V_t(\phi) = H_t^0 S_t^0 + H_t S_t$. We consider that the stock pays dividend with a continuous yield δ. Then, we say that the discounted value process $\tilde{V}_t(\phi) = e^{-rt} V_t(\phi)$ is a supermartingale under $\mathbb{P}^\delta$.*

Proof. From Definition 5.2, we know that a trading strategy with dividend and consumption satisfies the following equation:

$$H_t^0 S_t^0 + H_t S_t = H_0^0 S_0^0 + H_0 S_0 + \int_0^t H_u^0 dS_u^0 + \int_0^t H_u(dS_u + \delta S_u du) - C_t. \tag{5.7}$$

That is,

$$V_t(\phi) = V_0(\phi) + \int_0^t H_u^0 dS_u^0 + \int_0^t H_u(dS_u + \delta S_u du) - C_t.$$

Applying integration by parts formula to the process $(e^{-rt}V_t(\phi))$, we get

$$\begin{aligned}
d(e^{-rt}V_t(\phi)) &= -re^{-rt}V_t(\phi)dt + e^{-rt}dV_t(\phi) \\
&= e^{-rt}\left(-r(H_t^0 S_t^0 + H_t S_t)dt + H_t^0 dS_t^0 + H_t(dS_t + \delta S_t dt)\right) - e^{-rt}dC_t \\
&= e^{-rt}\left(-rH_t S_t dt + H_t(dS_t + \delta S_t dt)\right) - e^{-rt}dC_t \\
&= H_t(d\tilde{S}_t + \delta \tilde{S}_t dt) - e^{-rt}dC_t,
\end{aligned}$$

where $d\tilde{S}_t + \delta \tilde{S}_t dt = \sigma \tilde{S}_t dW_t^\delta$.
Hence,

$$\tilde{V}_t(\phi) = \tilde{V}_0(\phi) + \int_0^t H_u \sigma \tilde{S}_u dW_u^\delta + \int_0^t e^{-ru} dC_u.$$

We note that the stochastic integrals $\int_0^s H_u \sigma \tilde{S}_u dW_u^\delta$ and $\int_0^s e^{-ru} dC_u$ are $\mathcal{F}_s$-measurable. Then, we have

$$\begin{aligned}
\mathbb{E}^\delta\left(\tilde{V}_t(\phi) \mid \mathcal{F}_s\right) &= \tilde{V}_0(\phi) + \mathbb{E}^\delta\left(\int_0^t H_u \sigma \tilde{S}_u dW_u^\delta \mid \mathcal{F}_s\right) - \mathbb{E}^\delta\left(\int_0^t e^{-ru} dC_u \mid \mathcal{F}_s\right) \\
&= \tilde{V}_0(\phi) + \mathbb{E}^\delta\left(\int_0^s H_u \sigma \tilde{S}_u dW_u^\delta + \int_s^t H_u \sigma \tilde{S}_u dW_u^\delta \mid \mathcal{F}_s\right) \\
&\quad -\mathbb{E}^\delta\left(\int_0^s e^{-ru} dC_u + \int_s^t e^{-ru} dC_u \mid \mathcal{F}_s\right) \\
&= \tilde{V}_0(\phi) + \int_0^s H_u \sigma \tilde{S}_u dW_u^\delta - \int_0^s e^{-ru} dC_u \\
&\quad +\mathbb{E}^\delta\left(\int_s^t H_u \sigma \tilde{S}_u dW_u^\delta \mid \mathcal{F}_s\right) - \mathbb{E}^\delta\left(\int_s^t e^{-ru} dC_u \mid \mathcal{F}_s\right) \\
&= \tilde{V}_s(\phi) + \mathbb{E}^\delta\left(\int_s^t H_u \sigma \tilde{S}_u dW_u^\delta \mid \mathcal{F}_s\right) - \mathbb{E}^\delta\left(\int_s^t e^{-ru} dC_u \mid \mathcal{F}_s\right).
\end{aligned}$$

Since the integral $\int_s^t H_u \sigma \tilde{S}_u dW_u^\delta$ is independent of the sigma-algebra $\mathcal{F}_s$,

$$\mathbb{E}^\delta\left(\tilde{V}_t(\phi) \mid \mathcal{F}_s\right) = \tilde{V}_s(\phi) + \mathbb{E}^\delta\left(\int_s^t H_u \sigma \tilde{S}_u dW_u^\delta\right) - \mathbb{E}^\delta\left(\int_s^t e^{-ru} dC_u \mid \mathcal{F}_s\right).$$

Recalling that $\int_0^t H_u \sigma \tilde{S} f_u dW_u^\delta$ is an $\mathcal{F}_t$-martingale, we have

$$\mathbb{E}^\delta\left(\int_s^t H_u \sigma \tilde{S}_u dW_u^\delta\right) = 0.$$

Thus,

$$\mathbb{E}^\delta\left(\tilde{V}_t(\phi) \mid \mathcal{F}_s\right) = \tilde{V}_s(\phi) - \mathbb{E}^\delta\left(\int_s^t e^{-ru} dC_u \mid \mathcal{F}_s\right).$$

Because (C_t) is a non-decreasing process null at zero, we obtain

$$\mathbb{E}^{\delta}\left(\int_s^t e^{-ru}dC_u \mid \mathcal{F}_s\right) \geq 0.$$

Hence,

$$\mathbb{E}^{\delta}\left(\tilde{V}_t(\phi) \mid \mathcal{F}_s\right) \leq \tilde{V}_s(\phi),$$

which means that the process $(\tilde{V}_t(\phi))$ is a supermartingale under the probability measure $\mathbb{P}^{\delta}$. $\square$

5.3 Pricing

In this section, we will closely follow the theory given in Chapter 4. We will first introduce the martingale pricing approach and then discuss the solutions of variational inequalities.

Let us assume that the market is viable and complete. We recall that the viability of the market guarantees the existence of a risk-neutral probability measure $\mathbb{P}^{\delta}$ that makes the process $(e^{\delta t}\tilde{S}_t)_{0\leq t\leq T}$ an $\mathcal{F}_t$-martingale. Moreover, if the market fulfills the completeness assumption, then this risk neutral probability measure is said to be unique.

Under the same setting of Chapter 4, let us define a continuous function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ which satisfies the linear growth property

$$\psi(x) \leq C + Dx,$$

for some $C, D \in \mathbb{R}^+$. In the case of an American call option with strike price K, $\psi(S_t)$ is given by $\psi(S_t) = \max\{S_t - K, 0\}$. Moreover, if we have an American put with strike K, then $\psi(S_t) = \max\{K - S_t, 0\}$ [27].

We recall that, since the underlying pays dividends with a constant continuous yield δ, the stock price process is defined as follows:

$$S_t = S_0 e^{(r-\delta-(\sigma^2/2))t+\sigma W_t^{\delta}}.$$

5.3.1 Pricing Under Martingale Approach

In this subsubsection, we do not give detailed information about the theoretical aspects, since the overall theory of this approach can be found in Subsection 4.2.1. We price American options written on an underlying asset that pays dividends with the help of hedging portfolios. But, this time, the valuation process will be executed with the trading strategies with dividend and consumption.

Definition 5.3. A trading strategy $\phi = (H_t^0, H_t, C_t)_{0\leq t\leq T}$ with dividend and consumption hedges an American option, if the following inequality holds for all $0 \leq t \leq T$:

$$V_t(\phi) \geq \psi(S_t). \tag{5.8}$$

We denote Φ^{ψ} as the set of all trading strategies $\phi = (H_t^0, H_t, C_t)$ with dividend and consumption that hedges the American option on the finite time interval $[0, T]$.

Since the discounted value process of a trading strategy with dividend and consumption is supermartingale, the definition above indeed states that the process $(\tilde{V}_t(\phi))$ is an $\mathcal{F}_t$-supermartingale dominating $(\tilde{\psi}(S_t))$. Moreover, the process

$$\tilde{V}_t(\phi) = \tilde{V}_0(\phi) + \int_0^t H_u \sigma \tilde{S}_u dW_u^{\delta} + \int_0^t e^{-ru} dC_u$$

is right-continuous with left limits, since the stochastic integrals $\int_0^t H_u \sigma \tilde{S}_u dW_u^{\delta}$ and $\int_0^t e^{-ru} dC_u$ are continuous.
We know that the process $(\tilde{u}(t, S_t))$ is the smallest RCLL supermartingale dominating $(\tilde{\psi}(S_t))$ (see Theorem 4.5). Then, we have

$$\tilde{u}(t, S_t) \leq \tilde{V}_t(\phi).$$

Hence,

$$u(t, S_t) \leq V_t(\phi).$$

That is, all trading strategies with dividend and consumption that hedges the American option dominates the process $(u(t, S_t))$.
With the following theorem, we can uniquely price an American option written on a stock paying dividends. It states that the value of any hedging portfolio in Φ^{ψ} is worth at least as the process

$$u(t, S_t) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}^*(e^{-r\tau} \psi(S_0 e^{(r-\delta-(\sigma^2/2))\tau + \sigma W_\tau^{\delta}}) \mid \mathcal{F}_t).$$

Moreover, it is always possible to find a portfolio in Φ^{ψ} whose value is equal to $u(t, S_t)$. Because the process $(u(t, S_t))$ appears to be the minimum value of a portfolio that hedges the American option, it can be thought of the fair price of this option [23].

Theorem 5.2. *Let us consider a function $u : [0, T] \times \mathbb{R}^+ \to \mathbb{R}$ satisfying*

$$u(t, x) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}^{\delta}[e^{-r(\tau - t)} \psi(x \exp((r - \delta - (\sigma^2/2))(\tau - t) + \sigma(W_\tau^{\delta} - W_t^{\delta}))],$$

where $\mathcal{T}_{t,T}$ refers to the set of all stopping times taking values in $[t, T]$. Then, we can find a trading strategy $\bar{\phi} \in \Phi^{\psi}$ such that $V_t(\bar{\phi}) = u(t, S_t)$ for all $t \in [0, T]$. Moreover, all trading strategies with dividend and consumption $\phi \in \Phi^{\psi}$ satisfies $V_t(\phi) \geq u(t, S_t)$ for all $t \in [0, T]$.

Proof. As shown above, the value of all trading strategies with dividend and consumption in $\mathcal{T}_{t,T}$ dominate the American option price.
Since $\tilde{u}(t, S_t)$ is a right-continuous $\mathbb{P}^{\delta}$-supermartingale of class D (see Theorem D.13 in [22]), we can use the Doob-Meyer decomposition such that

$$\tilde{u}(t, S_t) = M_t - A_t, \tag{5.9}$$

where (M_t) is a uniformly integrable RCLL martingale under the probability measure $\mathbb{P}^\delta$ and (A_t) is a non-decreasing, adapted, right-continuous process null at zero.
Note that the filtration $(\mathcal{F}_t)_{0\le t\le T}$ is the natural filtration of both $(B_t^\delta)_{0\le t\le T}$ and $(W_t^\delta)_{0\le t\le T}$. Then, using the martingale representation theorem for martingales, we can find an adapted process (K_t) satisfying

$$\int_0^T K_s^2 ds < \infty \tag{5.10}$$

such that

$$M_t = M_0 + \int_0^t K_s dW_s^\delta. \tag{5.11}$$

Now, let us define a strategy $\bar{\phi} = (H_t^0, H_t, C_t)$ such that

$$H_t = \frac{K_t}{\sigma \tilde{S}_t}, \quad H_t^0 = \tilde{u}(t, S_t) - H_t \tilde{S}_t, \quad A_t = \int_0^t e^{-rs} dC_s.$$

It is obvious that the components H_t and H_t^0 are adapted. Also, from the condition (5.10), we deduce

$$\int_0^T H_s^2 ds = \int_0^T \frac{K_s^2}{\sigma^2 \tilde{S}_s^2} < \infty.$$

Moreover,

$$\int_0^T \left|H_s^0\right| ds = \int_0^T \left|\tilde{u}(s, S_s) - H_s \tilde{S}_s\right| ds < \infty.$$

Then,

$$\begin{aligned} \tilde{V}_t(\phi) &= H_t^0 \tilde{S}_t^0 + H_t \tilde{S}_t = \tilde{u}(t, S_t) \\ &= M_t - A_t \\ &= M_0 + \int_0^t \sigma H_s \tilde{S}_s dW_s^\delta - \int_0^t e^{-rs} dC_s, \end{aligned}$$

where C_t is adapted, non-decreasing and continuous process. Indeed, the right-continuous process (A_t) turns into a continuous process when $\psi(S_t)$ has continuous paths (see Theorem D.13 in [22]). Therefore, the continuity of A_t yields to the continuity of C_t. Hence, we have found a trading strategy with dividend and consumption that perfectly hedges the American option. □

As a result of this theorem, wee see that the fair price of an American put option with dividend is given by

$$u(t, S_t) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}^\delta \left(e^{-r(\tau - t)} (K - S_\tau)_+ \right),$$

whereas the unique price of an American call on a dividend paying stock becomes

$$u(t, S_t) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}^\delta \left(e^{-r(\tau - t)} (S_\tau - K)_+ \right).$$

5.3.2 Pricing with Variational Inequalities

In this subsubsection, we will discuss the solutions of variational inequalities for American options avoiding the technicalities. The overall theory can be found in Subsection 4.2.2. For more details, we refer to [29].
Variational inequalities give us a way to price American options without explicitly computing the expectation

$$\mathbb{E}^{\delta}[e^{-r(\tau-t)}\psi(x\exp((r-\delta-(\sigma^2/2))(\tau-t)+\sigma(W_\tau^\delta-W_t^\delta))].$$

Indeed, we will deal with the parabolic partial differential inequality systems under some regularity assumptions. These assumptions will guarantee the existence and uniqueness of the solution to the parabolic system.
Now, let us define the partial differential inequality that the American options satisfy.

Lemma 5.3. *Let $u(t, S_t)$ be the value of an American option on a dividend paying stock. Under the Black-Scholes setting, the value $u(t, S_t)$ satisfies the following partial differential inequality [42]*

$$\frac{\partial u}{\partial t}+\frac{\sigma^2}{2}S_t^2\frac{\partial^2 u}{\partial S^2}+(r-\delta)S_t\frac{\partial u}{\partial S}-ru\le 0. \tag{5.12}$$

Proof. As being different from the previous chapter, we derive the Black-Scholes inequality with the help of supermartingale property of $\tilde{u}(t, S_t)$ under the risk-neutral probability measure $\mathbb{P}^\delta$.
Let us apply Itô formula to the processes $(u(t, S_t))$ and $(\tilde{u}(t, S_t))$. Then, we have

$$u(t,S_t)=\int_0^t\left(\frac{\partial u}{\partial t}+\frac{\sigma^2}{2}S_v^2\frac{\partial^2 u}{\partial S^2}+(r-\delta)S_v\frac{\partial u}{\partial S}\right)dv+\int_0^t\sigma S_v\frac{\partial u}{\partial S}dW_v^\delta,$$

and

$$\tilde{u}(t,S_t)=\int_0^t e^{-rv}\left(\frac{\partial u}{\partial t}+\frac{\sigma^2}{2}S_v^2\frac{\partial^2 u}{\partial S^2}+(r-\delta)S_v\frac{\partial u}{\partial S}-ru\right)dv+\int_0^t e^{-rv}\sigma S_v\frac{\partial u}{\partial S}dW_v^\delta. \tag{5.13}$$

Also, we know from the previous section that the discounted price of an American option is supermartingale under the probability measure $\mathbb{P}^\delta$. Using this fact, we can easily derive the desired inequality (5.12). Taking the conditional expectation of both sides in (5.13),

$$\begin{aligned}\mathbb{E}^\delta(\tilde{u}(t,S_t)\mid\mathcal{F}_k) &= \mathbb{E}^\delta\left(\int_0^t e^{-rv}\left[\frac{\partial u}{\partial t}+\frac{\sigma^2}{2}S_v^2\frac{\partial^2 u}{\partial S^2}+(r-\delta)S_v\frac{\partial u}{\partial S}-ru\right]dv\mid\mathcal{F}_k\right)\\ &\quad+\mathbb{E}^\delta\left(\int_0^t e^{-rv}\sigma S_v\frac{\partial u}{\partial S}dW_v^\delta\mid\mathcal{F}_k\right)\\ &= \tilde{u}(k,S_k)+\mathbb{E}^\delta\left(\int_k^t e^{-rv}\sigma S_v\frac{\partial u}{\partial S}dW_v^\delta\mid\mathcal{F}_k\right)\\ &\quad+\mathbb{E}^\delta\left(\int_k^t e^{-rv}\left[\frac{\partial u}{\partial t}+\frac{\sigma^2}{2}S_v^2\frac{\partial^2 u}{\partial S^2}+(r-\delta)S_v\frac{\partial u}{\partial S}-ru\right]dv\mid\mathcal{F}_k\right).\end{aligned}$$

Because the stochastic integral $\int_0^t e^{-rv}\sigma S_v \frac{\partial u}{\partial S} dW_v^\delta$ is a martingale with the independent and stationary increments property, we get

$$\begin{aligned}\mathbb{E}^\delta\left(\int_k^t e^{-rv}\sigma S_v \frac{\partial u}{\partial S} dW_v^\delta \mid \mathcal{F}_k\right) &= \mathbb{E}^\delta\left(\int_k^t e^{-rv}\sigma S_v \frac{\partial u}{\partial S} dW_v^\delta\right)\\ &= \mathbb{E}^\delta\left(\int_0^{t-k} e^{-rv}\sigma S_v \frac{\partial u}{\partial S} dW_v^\delta\right)\\ &= 0.\end{aligned}$$

Hence,

$$\begin{aligned}\mathbb{E}^\delta(\tilde{u}(t,S_t) \mid \mathcal{F}_k) &= \tilde{u}(k,S_k) + \mathbb{E}^\delta\left(\int_k^t e^{-rv}\left(\frac{\partial u}{\partial t} + \frac{\sigma^2}{2}S_v^2\frac{\partial^2 u}{\partial S^2}\right.\right.\\ &\qquad \left.\left. +(r-\delta)S_v\frac{\partial u}{\partial S} - ru\right) dv \mid \mathcal{F}_k\right).\end{aligned}$$

Because $\tilde{u}(t,S_t)$ is supermartingale, then

$$\frac{\partial u}{\partial t} + \frac{\sigma^2}{2}S_t^2\frac{\partial^2 u}{\partial S^2} + (r-\delta)S_t\frac{\partial u}{\partial S} - ru \le 0,$$

for all $t \ge 0$. □

Lemma 5.4. *Let $\psi(S_t)$ denote the intrinsic value of an American option with dividend at time t. Then, the followings are always satisfied*

$$\begin{aligned}u(t,S_t) &\ge \psi(S_t), \qquad (t,S_t) \in [0,T]\times\mathbb{R}, &(5.14)\\ u(T,S_T) &= \psi(S_T). &(5.15)\end{aligned}$$

In the case (5.14) and (5.15) does not hold, it is obvious that an arbitrage opportunity exists in the market.
Note that if $u(t,S_t) = \psi(S_t)$ or $\frac{\partial u}{\partial t} + \frac{\sigma^2}{2}St^2\frac{\partial^2 u}{\partial S^2} + (r-\delta)S_t\frac{\partial u}{\partial S} - ru < 0$, then it is optimal to exercise the option.
Moreover, if $u(t,S_t) > \psi(S_t)$ or $\frac{\partial u}{\partial t} + \frac{\sigma^2}{2}S_t^2\frac{\partial^2 u}{\partial S^2} + (r-\delta)S_t\frac{\partial u}{\partial S} - ru = 0$, it is beneficial not to use the early exercise right of the option.
This results lead us to the following lemma.

Lemma 5.5. *Let $\psi(S_t)$ be the intrinsic value of an American option with dividend yield δ and let $u(t,S_t)$ corresponds to the value of this option depending on time and stock price S_t. Then,*

$$(\psi - u)\left(\frac{\partial u}{\partial t} + \frac{\sigma^2}{2}S_t^2\frac{\partial^2 u}{\partial S^2} + (r-\delta)S_t\frac{\partial u}{\partial S} - ru\right) = 0, \qquad (5.16)$$

for all $(t,S_t) \in [0,T]\times\mathbb{R}$.

In the light of foregoings, we deduce that the price of an American option is actually the solution of the following inequality system

$$\frac{\partial u}{\partial t} + \frac{\sigma^2}{2} S_t^2 \frac{\partial^2 u}{\partial S^2} + (r-\delta) S_t \frac{\partial u}{\partial S} - ru \leq 0, \qquad \forall (t, S_t) \in [0,T] \times \mathbb{R},$$

$$\left(\frac{\partial u}{\partial t} + \frac{\sigma^2}{2} S_t^2 \frac{\partial^2 u}{\partial S^2} + (r-\delta) S_t \frac{\partial u}{\partial S} - ru \right) (\psi - u) = 0, \qquad \forall (t, S_t) \in [0,T] \times \mathbb{R}, \tag{5.17}$$

$$u(T, S_T) = \psi(S_T), \qquad u \geq \psi \qquad \forall S_t \in \mathbb{R}.$$

As we did in the previous chapter, in order to find an elegant solution for the above inequality system, we generalize the problem into the multidimensional SDE's in the form of [23]

$$\begin{aligned} dX_t^1 &= b^1(t, X_t)dt + \sum_{j=1}^{d} \sigma_{1j}(t, X_t) dB_t^j, \\ &\vdots \\ dX_t^n &= b^n(t, X_t)dt + \sum_{j=1}^{d} \sigma_{nj}(t, X_t) dB_t^j, \end{aligned} \tag{5.18}$$

where $X_t = (X_t^1, \ldots, X_t^n)$ is an n-dimensional Itô process, $B_t = (B_t^1, \ldots, B_t^d)$ is a d-dimensional Brownian motion, $\sigma(t,x) = (\sigma_{ij}(t,x))_{1 \leq i \leq n, 1 \leq j \leq d}$ is an $n \times d$ matrix and $b(t,x) = (b^1(t,x), \ldots, b^n(t,x))$ is an $\mathbb{R}^n$-valued function defined on $\mathbb{R}^+ \times \mathbb{R}^n$. Also, the infinitesimal generator $A_t : f \mapsto A_t f$ depending on an $\mathbb{R}$-valued function of class C^2 is defined by [23]

$$A_t f(x) = \frac{1}{2} \sum_{i,j=1}^{n} a_{i,j}(t,x) \frac{\partial^2 f}{\partial x_i x_j}(x) + \sum_{j=1}^{n} b_j(t,x) \frac{\partial f}{\partial x_j}(x),$$

where $a_{i,j}(t,x) = \sum_{k=1}^{d} \sigma_{i,k}(t,x) \sigma_{j,k}(t,x)$.
Recalling the generalized version of the partial inequality system (5.17)

$$\begin{cases} \frac{\partial u}{\partial t} + A_t u - ru \leq 0, \quad u \geq f \qquad \text{in } [0,T] \times \mathbb{R}^n, \\ \left(\frac{\partial u}{\partial t} + A_t u - ru \right) (f - u) = 0 \qquad \text{in } [0,T] \times \mathbb{R}^n, \\ u(T,x) = f(x) \qquad \text{in } \mathbb{R}^n, \end{cases}$$

we consider that the coefficients b, σ are the bounded and Hölder continuous [35, 18], and the infinitesimal generator A_t satisfies the ellipticity property [23, 18]

$$\exists M \in \mathbb{R}^+, \quad \sum_{ij} a_{ij}(t,x) \varepsilon_i \varepsilon_j \geq M \left(\sum_{i=1}^{n} \varepsilon_i^2 \right),$$

for all $(\varepsilon_1, \ldots, \varepsilon_n) \in \mathbb{R}^n$.
As we mentioned before, this inequality system does not admit a regular solution. But, in the case the solution is regular, the process $(u(t,x))$ is given by (see Theorem 4.13)

$$u(t,x) = \Phi(t,x) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}\left(e^{\int_t^\tau r(s,X_s^{t,x})ds} f(X_\tau^{t,x})\right).$$

In order to extend this result to the inequality systems having non-regular solutions, we need some adjustments.
Now, let us recall that, in the Black-Scholes model, the price process of a dividend paying stock is actually the solution of the following SDE

$$dS = S(r-\delta)dt + S\sigma dW_t^\delta,$$

under $\mathbb{P}^*$. Then, we can easily verify that the infinitesimal generator of the Black-Scholes model is actually in the form

$$A_t f(x) = \frac{\sigma^2}{2} x^2 \frac{\partial^2 f}{\partial x^2} + (r-\delta)x\frac{\partial f}{\partial x}.$$

Because this generator is not elliptic [23], we need to find another operator that fulfills the ellipticity condition. Letting $X_t = \log(S_t)$, we have

$$dX_t = \left(r - \frac{\sigma^2}{2} - \delta\right)dt + \sigma dW_t^\delta.$$

Then, its infinitesimal generator is in the form [29]

$$A_t^\delta = \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + \left(r - \frac{\sigma^2}{2} - \delta\right)\frac{\partial}{\partial x},$$

satisfying the ellipticity condition.
That is, if we set $S = e^x$, we obtain an elliptic infinitesimal generator under Black-Scholes model. After defining this operator A_t^δ, we can now investigate the variational inequalities associated with the American options.
Let $u(t,S_t)$ denote the price of an American option with strike price K and δ be the constant dividend yield that underlying pays. Also, we consider $\psi(S_t) = (K - S_t)_+$ and $\psi(S_t) = (S_t - K)_+$ are the intrinsic value of the put and call option, respectively. Now, let us set $S = e^x$ and $v(t,x) = u(t,e^x)$. Following the same steps mentioned in Chapter 4, we conclude that, when $S = e^x$, an American option with strike price K satisfies the following variational inequalities

$$\begin{aligned} &\frac{\partial v}{\partial t}(t,x) + A_t^\delta v(t,x) - rv(t,x) \le 0, \qquad \forall (t,x) \in [0,T]\times\mathbb{R} \\ &\left(\frac{\partial v}{\partial t}(t,x) + A_t^\delta v(t,x) - rv(t,x)\right)(\phi - v) = 0, \qquad \forall (t,x) \in [0,T]\times\mathbb{R} \\ &v(T,x) = \phi(x), \qquad v \ge \phi \qquad \forall x \in \mathbb{R}. \end{aligned} \tag{5.19}$$

Now, we introduce the fundamental theorem used to price American options on a dividend paying stock. The theorem says that the system defined in (5.19) admits a unique solution. Moreover, this result presents a way for pricing American options with the help of this inequality system.

Theorem 5.6. *The variational inequality system (5.19) admits a unique solution $v(t,x)$ such that this solution is continuous, bounded and has the locally bounded partial derivatives $\frac{\partial v}{\partial x}, \frac{\partial v}{\partial t}, \frac{\partial^2 v}{\partial x^2}$.*

Also, the following equality always holds

$$v(t, \log(x)) = \Phi(t,x) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}^{\delta}(e^{-r(\tau-t)}\psi(xe^{(r-(\sigma^2/2)-\delta)(\tau-t)+\sigma(W_\tau - W_t)})).$$

With the help of this theorem, we see that the price of an American call option with dividend is given by

$$u(t, S_t) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}^{\delta}\left(e^{-r(\tau-t)}(S_t e^{(r-(\sigma^2/2)-\delta)(\tau-t)+\sigma(W_\tau - W_t)} - K)_+\right),$$

whereas an American put price is expressed as follows:

$$u(t, S_t) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}^{\delta}\left(e^{-r(\tau-t)}(K - S_t e^{(r-(\sigma^2/2)-\delta)(\tau-t)+\sigma(W_\tau - W_t})_+\right).$$

We emphasize that, although martingale and variational inequality approach deal with the different technical aspects, they give the same formula for the price of American options.

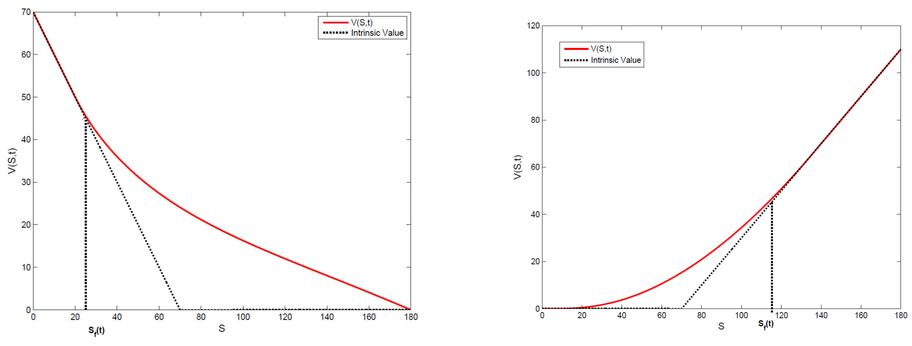
CHAPTER 6

NUMERICAL METHODS

American options provide its holder the flexibility of exercising at any time up to maturity. By this flexibility, the holder of the option can terminate the contract in case the option value is less than its intrinsic value, which is actually the revenue generated by an immediate exercise. That is why, the value of an American option is said to be worth at least as much as its intrinsic value. Hence, for all (S, t),

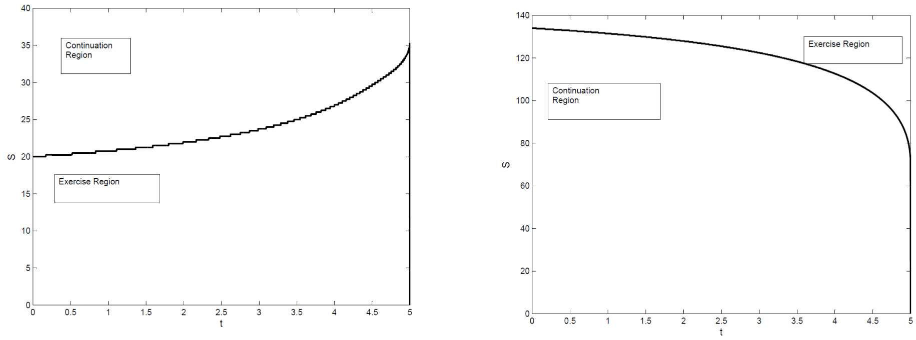
$$V(S, t) \geq \psi(S, t), \tag{6.1}$$

where $V(S, t)$ is the value of the option at time t and $\psi(S, t)$ is the intrinsic value. Here, we recall that $\psi(S, t) = (S - K)_+$ (respectively $(K - S)_+$) for a call option with a strike price K (respectively for a put option with the same strike price).
From this inequality, it is clear that an early exercise is profitable only when the value of the option is dominated by its intrinsic value. In case the inequality (6.1) holds, an early exercise means to trade the option for less than worth. Therefore, it can cause a huge loss for the option holder.
In Figure 6.1a and Figure 6.1b, we figure out these results. It is clear that it is optimal to hold the put for the points $S > S_f(t)$. Indeed, since the value function of a put option is continuous and non-increasing with respect to S and $V(S, t) \geq \psi(S, t)$ for all (S, t), there is an at least one intersection point (S^1, t) such that $V(S^1, t) = \psi(S^1, t)$. Here, $S_f(t)$ is the last contact point of the curves



(a) (b)

Figure 6.1: Value curve for (a) an American call option, (b) an American put option.



(a) (b)

Figure 6.2: Free boundary curve for (a) an American put option, (b) an American call option.

$V(S,t)$ and $\psi(S,t)$. To be more precise, all points $S > S_f(t)$ determine the continuation region of the option for a fixed time t, whereas the the points $S \leq S_f(t)$ belong to the exercise region of the option. In the case of an American call option, the buyer can experience a huge loss when exercising at the points $S \leq S_f(t)$. The existence of such a contact point $S_f(t)$ derives from the fact that the value function of a call option is continuous and non-decreasing with respect to S and $V(S,t) \geq \psi(S,t)$ for all (S,t). The parameters we use in Figure 6.1 are: $S_0 = 70$, $K = 70$, $D = 0.1$, $\sigma = 0.4$, $r = 0.05$, $S_{\min} = 0$, $S_{\max} = 180$, $T = 5$, $dS = 0.25$, $dt = 0.002$.
The intersection points $S_f(t)$ relocate by the passage of time, as shown in Figure 6.2. Because of this relocation, the boundaries of exercise and continuation regions change at each time instant. Therefore, the option holder is obliged to find these boundaries concurrently with the unknown value $V(S,t)$. In finance, this problem is said to be a free boundary problem. The parameters we use in Figure 6.2 are: $S_0 = 70$, $K = 70$, $D = 0.1$, $\sigma = 0.4$, $r = 0.05$, $S_{\min} = 0$, $S_{\max} = 180$, $T = 5$, $dS = 0.25$, $dt = 0.002$.
Since it is required to regulate the boundaries along with the price $V(S,t)$, the free boundary problem can entail an additional difficulty to the valuation of an American option. In order to cope with this difficulty, we investigate the numerical solutions of the following PDE

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-\delta)S\frac{\partial V}{\partial S} - rV \quad \leq \quad 0, \tag{6.2}$$

with the terminal condition

$$V(S,T) = \psi(S,T), \tag{6.3}$$

and with the constraint (6.1).
We recall that, in the case (6.2) is equal to zero, we have $V(S,t) > \psi(S,t)$ for all (S,t). If we get rid of the equal sign, then it is possible to maximize our gain only by holding the option. To put it another way, the free boundary problem is not taken into account at that instant. Since a numerical valuation does not make sense at the time of an early exercise, we try to solve the pricing problem in the

holding region. That is, a numerical method used to price an American option considers first the equality of (6.2) and then examines whether the intrinsic value is greater than the approximate price or not (see, e.g., [6, 40]). It makes the numerical pricing of an American option a two-step valuation process such that

$$V(S,t) = \max\left\{\bar{V}(S,t), \psi(S,t)\right\}, \tag{6.4}$$

where $\bar{V}(S,t)$ is the approximate price of the American option derived with no free-boundary.
In this chapter, we introduce some numerical methods used to price American options.

6.1 Finite-difference Methods

Finite-difference methods are widely used in finance to price derivative securities. Because it can be a hard work, sometimes impossible, to find a closed-form solution for a price function, many studies conducted on the numerical approaches. Therefore, in the literature, there are several publications and books written on this topic, see, e.g., [6, 7, 39, 40].
To obtain a numerical result from (6.2), we will work with the finite difference quotients of partial derivatives derived from the Taylor series expansions. To do that, we will first start with the discretization of domains of S and t such that [40]

$$S_{\min} \leq S \leq S_{\max} \quad \text{and} \quad t_0 \leq t \leq T.$$

Dividing the intervals $[t_0, T]$ and $[S_{\min}, S_{\max}]$ into M and N parts, respectively, we have

$$\Delta t = \frac{T - t_0}{M},$$

and

$$\Delta S = \frac{S_{\max} - S_{\min}}{N}.$$

Namely,

$$t_i = t_0 + i\Delta t, \qquad i = 0, 1, \ldots, M,$$

and

$$S_k = S_{\min} + k\Delta S, \qquad k = 0, 1, \ldots, N.$$

After explaining the discretization of the domain of the PDE, let us denote the approximation of the value $V(S_k, t_i)$ with

$$V(S_k, t_i) \approx w_{ki}, \tag{6.5}$$

for all grid points (S_k, t_i).

6.1.1 Explicit Method

Explicit method is a well-known finite difference method which is widely used for the numerical solutions of PDE's. It was first introduced by Brennan and Schwartz (1977, 1978) [7, 35] and then developed by Courtadon (1982) [9].
According to this method, the partial derivative $\frac{\partial V}{\partial t_i}$ is approximated by the backward finite difference scheme whereas $\frac{\partial V}{\partial S_k}$ and $\frac{\partial^2 V}{\partial S_k^2}$ are approximately valued by the central difference approach. That is,

$$\begin{aligned}\frac{\partial V}{\partial t_i} &\approx \frac{w_{ki} - w_{k,i-1}}{\Delta t}, &(6.6)\\ \frac{\partial V}{\partial S_k} &\approx \frac{w_{k+1,i} - w_{k-1,i}}{2\Delta S}, &(6.7)\\ \frac{\partial^2 V}{\partial S_k^2} &\approx \frac{w_{k+1,i} - 2w_{k,i} + w_{k-1,i}}{(\Delta S)^2}. &(6.8)\end{aligned}$$

Substituting (6.6), (6.7) and (6.8) into the Black-Scholes PDE, we get

$$\begin{aligned}\frac{w_{ki} - w_{k,i-1}}{\Delta t} &= r w_{ki} - (r-\delta) S_k \frac{w_{k+1,i} - w_{k-1,i}}{2\Delta S}\\ &\quad -\frac{1}{2}\sigma^2 S_k^2 \frac{w_{k+1,i} - 2w_{k,i} + w_{k-1,i}}{(\Delta S^2)}. \qquad (6.9)\end{aligned}$$

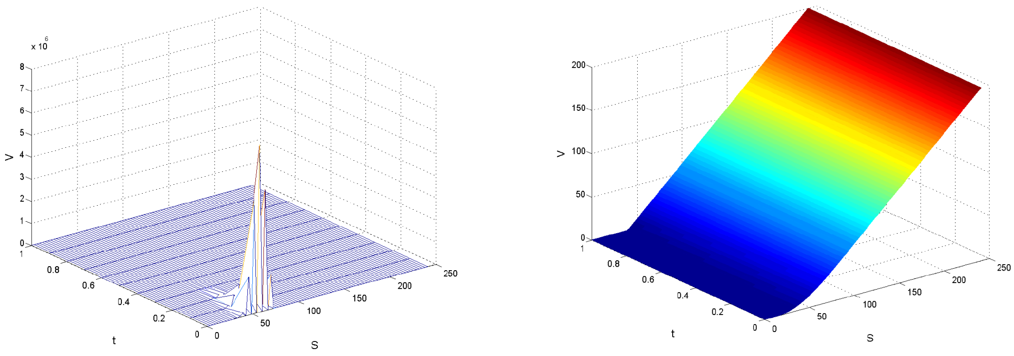
Then,

$$w_{k,i-1} = x_k w_{k-1,i} + y_k w_{k,i} + z_k w_{k+1,i}, \tag{6.10}$$

where

$$\begin{aligned}x_k &= \frac{1}{2}\Delta t \left\{ \sigma^2 (\frac{S_k}{\Delta S})^2 - (r-\delta)\frac{S_k}{\Delta S} \right\},\\ y_k &= 1 - \Delta t \left\{ \sigma^2 (\frac{S_k}{\Delta S})^2 + r \right\},\\ z_k &= \frac{1}{2}\Delta t \left\{ (r-\delta)\frac{S_k}{\Delta S} + \sigma^2 (\frac{S_k}{\Delta S})^2 + \right\}.\end{aligned}$$

It is apparent from (6.10) that explicit method is easy to implement, since we do not need to take the inverse of any matrix at the time of the implementation and the computation of some boundaries is not necessary for the solution (Hull and White (1990) [15]). However, it can sometimes show conditional stability depending on the choice of Δt and ΔS. To be more precise, the errors occurred in the previous steps of the implementation can be increasingly carried to the next steps because of the wrong choice of Δt and ΔS. Therefore, the conditional stability can cause the efficiency of the method to decrease.
In the Figure 6.3a, we plot the instability of explicit method for an American call option. It shows that, with the wrong choice of parameters, the method can cause misleading results. With the Figure 6.3b, we see that the instability of the method is improved by choosing the suitable parameters for the valuation. The parameters we use are: $S_0 = 40$, $S_{\min} = 0$, $S_{\max} = 240$, $T = 1$, $dS = 3$, , $r = 0.1$, $D = 0.1$, $\sigma = 0.5$, $K = 40$.
The MATLAB code can be found in [40].



(a) (b)

Figure 6.3: The conditional stability of explicit method with (a) $dt = 0.1$, (b) $dt = 0.001$.

6.1.2 Implicit Method

This scheme, too, was firstly introduced by Schwartz (1977) [35] and Brennan and Schwartz (1978) [7]. Then, it was developed by Courtadon (1982) [9].
In the implicit method, contrary to explicit scheme, we are not restricted with the choice of step lengths Δt and ΔS. Moreover, it also differs from the explicit method with the use of forward difference for the approximation of partial derivative $\frac{\partial V}{\partial t}$.
According to this scheme, the partial derivatives $\frac{\partial V}{\partial t_i}$, $\frac{\partial V}{\partial S_k}$ and $\frac{\partial^2 V}{\partial S_k^2}$ are approximated as follows:

$$\frac{\partial V}{\partial t_i} \approx \frac{w_{k,i+1} - w_{k,i}}{\Delta t}, \tag{6.11}$$

$$\frac{\partial V}{\partial S_k} \approx \frac{w_{k+1,i} - w_{k-1,i}}{2\Delta S}, \tag{6.12}$$

$$\frac{\partial^2 V}{\partial S_k^2} \approx \frac{w_{k+1,i} - 2w_{k,i} + w_{k-1,i}}{(\Delta S)^2}. \tag{6.13}$$

Plugging (6.11), (6.12) and (6.13) into the Black-Scholes PDE, we obtain

$$\begin{aligned} \frac{w_{k,i+1} - w_{k,i}}{\Delta t} &= rw_{ki} - (r - \delta)S_k \frac{w_{k+1,i} - w_{k-1,i}}{2\Delta S} \\ &\quad -\frac{1}{2}\sigma^2 S_k^2 \frac{w_{k+1,i} - 2w_{k,i} + w_{k-1,i}}{(\Delta S^2)}. \end{aligned} \tag{6.14}$$

Then,

$$w_{k,i+1} = x_k w_{k-1,i} + y_k w_{k,i} + z_k w_{k+1,i}, \tag{6.15}$$

where

$$\begin{aligned}
x_k &= \frac{1}{2}\Delta t\left\{(r-\delta)\frac{S_k}{\Delta S} - \sigma^2(\frac{S_k}{\Delta S})^2\right\}, \\
y_k &= 1+\Delta t\left\{\sigma^2(\frac{S_k}{\Delta S})^2 + r\right\}, \\
z_k &= -\frac{1}{2}\Delta t\left\{(r-\delta)\frac{S_k}{\Delta S} + \sigma^2(\frac{S_k}{\Delta S})^2 +\right\}.
\end{aligned}$$

By rewriting (6.15) in matrix form, we get

$$Aw^{(i)} = Bw^{(i+1)} + y^{(i+1)},$$

for all $i = M-1, M-2, \ldots, 1$.
Here,

$$w^{(i)} = \begin{pmatrix} w_{1,i} \\ w_{2,i} \\ \vdots \\ w_{N-1,i} \end{pmatrix},$$

$$y^{(i+1)} = \begin{pmatrix} -x_1 w_{0,i} \\ 0 \\ \vdots \\ 0 \\ -z_{N-1} w_{N,i} \end{pmatrix},$$

$$A = \begin{pmatrix} y_1 & z_1 & 0 & \ldots & 0 & 0 \\ x_2 & y_2 & z_2 & \ldots & 0 & 0 \\ 0 & -x_3 & 1-y_3 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & x_{N-1} & y_{N-1} \end{pmatrix}.$$

Now, our aim is to find the price of an American option with respect to this scheme. In order to do that, we compare each approximate value $w_{k,i}^{\mathrm{IM}}$, which is computed under the assumption of no free-boundary, with the intrinsic value $P_{k,i}$. That is, the value of an American option at the grid point (S_k, t_i) is defined by

$$w_{k,i}^{Am} = \max\left\{w_{k,i}^{\mathrm{IM}}, \psi_{k,i}\right\},$$

where $i = 1, \ldots, N-1$ and $k = M, M-1, \ldots, 1$.

6.1.3 Crank-Nicolson method

One of the simplest numerical methods used to price options is the Crank-Nicolson method. This approach was introduced by Crank and Nicolson (1947) [10].

According to this method, the discretized partial derivative $\frac{\partial V_k}{\partial t_i}$ is approximated by the backward finite difference such that

$$\frac{\partial V_k}{\partial t_i} \approx \frac{w_{ki} - w_{k,i-1}}{\Delta t}.$$

Then, by multiplying the right hand side of (6.9) and (6.14) with 1/2, we have

$$\begin{aligned}
\frac{w_{ki} - w_{k,i-1}}{\Delta t} \quad = \quad & \frac{1}{2}\left(r w_{ki} - (r-\delta) S_k \frac{w_{k+1,i} - w_{k-1,i}}{2\Delta S} \right) \\
& -\frac{1}{4}\left(\sigma^2 S_k^2 \frac{w_{k+1,i} - 2w_{k,i} + w_{k-1,i}}{(\Delta S^2)} \right) \\
& +\frac{1}{2}\left(r w_{k,i-1} - (r-\delta) S_k \frac{w_{k+1,i-1} - w_{k-1,i-1}}{2\Delta S} \right) \\
& -\frac{1}{4}\left(\sigma^2 S_k^2 \frac{w_{k+1,i-1} - 2w_{k,i-1} + w_{k-1,i-1}}{(\Delta S^2)} \right).
\end{aligned} \tag{6.16}$$

Hence, arranging (6.16), we get

$$\begin{aligned}
& - x_k w_{k-1,i-1} + (1 - y_k) w_{k,i-1} - z_k w_{k+1,i-1} \\
& \qquad\qquad\qquad\qquad = x_k w_{k-1,i} + (1 + y_k) w_{k,i} - z_k w_{k+1,i},
\end{aligned}$$

where

$$\begin{aligned}
x_k &= \frac{1}{4}\Delta t \left\{ \sigma^2 (\frac{S_k}{\Delta S})^2 - (r-\delta)\frac{S_k}{\Delta S} \right\}, \\
x_k &= -\frac{1}{2}\Delta t \left\{ \sigma^2 (\frac{S_k}{\Delta S})^2 + r \right\}, \\
z_k &= \frac{1}{4}\Delta t \left\{ \sigma^2 (\frac{S_k}{\Delta S})^2 + (r-\delta)\frac{S_k}{\Delta S} \right\}.
\end{aligned}$$

Moreover, we can rewrite (6.16) in matrix form as follows:

$$A w^{(i-1)} = B w^{(i)} + b^{(i)}, \tag{6.17}$$

for all $i = M-1, M-2, \ldots, 1$.
Here,

$$w^{(i)} = \begin{pmatrix} w_{1,i} \\ w_{2,i} \\ \vdots \\ w_{N-1,i} \end{pmatrix},$$

$$b^{(i)} = \begin{pmatrix} x_1(w_{0,i-1} + w_{0,i}) \\ 0 \\ \vdots \\ 0 \\ z_{N-1}(w_{N,i-1} + w_{N,i}) \end{pmatrix},$$

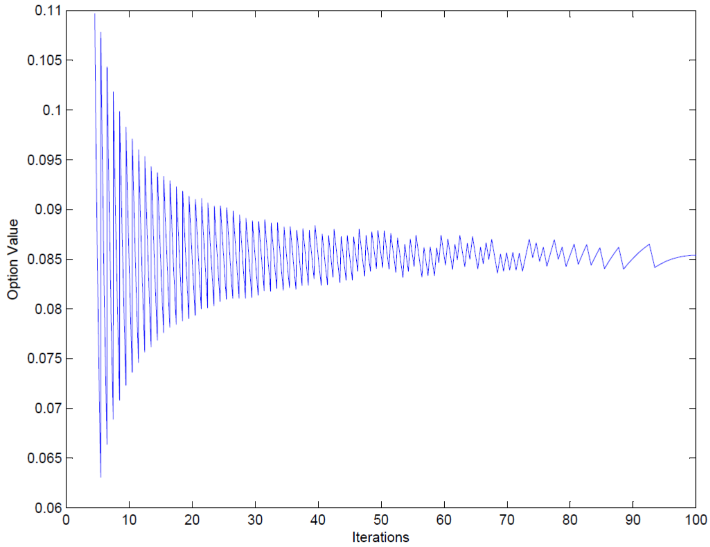


Figure 6.4: The price of an American put option for different number of time steps.

$$A = \begin{pmatrix} 1-y_1 & -z_1 & 0 & \dots & 0 & 0 \\ -x_2 & 1-y_2 & -z_2 & \dots & 0 & 0 \\ 0 & -x_3 & 1-y_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -x_{N-1} & 1-y_{N-1} \end{pmatrix},$$

$$B = \begin{pmatrix} 1+y_1 & z_1 & 0 & \dots & 0 & 0 \\ x_2 & 1+y_2 & z_2 & \dots & 0 & 0 \\ 0 & -x_3 & 1-y_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & x_{N-1} & 1+y_{N-1} \end{pmatrix}.$$

We note that the Crank-Nicolson method, too, is unconditionally stable, because the implicit method protects its stability at the time of the implementation.
In order to adjust this method for American options, we compare the approximate value $w_{k,i}^{\text{CN}}$, which is computed under the assumption of no-free boundary, with the intrinsic value $P_{k,i}$. Hence, the value of an American option at the grid point (S_k, t_i) is given by

$$w_{k,i}^{Am} = \max\left\{w_{k,i}^{\text{CN}}, \psi_{k,i}\right\},$$

where $i = 1, 2, \dots, N-1$ and $k = M, M-1, \dots, 1$.
Figure 6.4 shows how the price of an American put option under CN method is affected by the number of time steps. It is clear that, when we increase the number of time steps, the oscillations in the option value decrease. The parameters we use in the graph are: $S0 = 40$, $K = 40$, $r = 0.1$, $T = 1$, $\sigma = 0.50$, $dS = 3$, $S_{\min} = 0$, $S_{\max} = 240$. The time step length changes from 0.01 to 0.22 with an increment of 0.0001.

Table 6.1: The errors realized at the implementation of Crank-Nicolson and Implicit Euler method [17].

	Implicit-Euler		Crank-Nicolson	
n	error	ratio	error	ratio
2	1.905E-1		2.739E-1	
4	1.015E-1	1.88	1.403E-1	1.95
8	5.305E-2	1.91	6.929E-2	2.02
16	2.732E-2	1.94	3.265E-2	2.12
32	1.392E-2	1.96	1.422E-2	2.30
64	7.044E-3	1.98	5.237E-3	2.71
128	3.547E-3	1.99	1.271E-3	4.12
256	1.780E-3	1.99	9.369E-5	13.57
512	8.920E-4	1.99	2.182E-6	42.93

The MATLAB code of Crank-Nicolson can be found in [40].

Although Crank Nicolson method is unconditionally stable, remarkable errors can be occurred at the time of the implementation. Therefore, it can cause poor numerical results for the valuation of the option [17, 24, 31].
To handle this problem, Rannacher [32] proposed to use Implicit Euler method for a few time steps and then to apply the Crank-Nicolson method. With this adjustment, he showed that large errors occurred in the implementation have been reduced. As a result, better numerical approximations were obtained for the option prices. The Table 6.1 illustrates this fact. As can be seen, Crank-Nicolson method shows more oscillating behavior than the Implicit Euler method in the valuation of an American call option. The table is prepared for different number of time steps n. The parameters we use are $K = 10$, $\sigma = 0.6$, $D = 0.2$, $T = 1$, $r = 0.25$, $S_{\min} = 0$, $S_{\max} = 50$ [17].

6.1.4 θ-Averaged Method

θ-Averaged method is a generalized version of Crank-Nicolson which associates the implicit and explicit finite differences with the weights θ and $1-\theta$, respectively, [26].
Given $0 \leq \theta \leq 1$, the method is defined as

$$\begin{aligned}
\frac{w_{k,i} - w_{k,i-1}}{\Delta t} &= \theta \left(r w_{k,i} - (r-\delta) S_k \frac{w_{k+1,i} - w_{k-1,i}}{2\Delta S} \right) \\
&- \frac{1}{2}\theta \left(\sigma^2 S_k^2 \frac{w_{k+1,i} - 2w_{k,i} + w_{k-1,i}}{(\Delta S^2)} \right) \\
&+ (1-\theta) \left(r w_{k,i-1} - (r-\delta) S_k \frac{w_{k+1,i-1} - w_{k-1,i-1}}{2\Delta S} \right) \\
&- \frac{1}{2}(1-\theta) \left(\sigma^2 S_k^2 \frac{w_{k+1,i-1} - 2w_{k,i-1} + w_{k-1,i-1}}{(\Delta S^2)} \right).
\end{aligned} \tag{6.18}$$

Note that

- if $\theta = 1$, then the method turns out to be explicit,
- if $\theta = \frac{1}{2}$, then the method turns out to be Crank-Nicolson,
- if $\theta = 0$, then the method turns out to be implicit.

Rearranging (6.18), we obtain

$$- x_k w_{k-1,i-1} + (1 - y_k) w_{k,i-1} - z_k w_{k+1,i-1} \\ = x_k^1 w_{k-1,i} + (1 + y_k^1) w_{k,i} - z_k^1 w_{k+1,i},$$

where

$$\begin{aligned}
x_k &= \frac{1}{2}(1-\theta)\Delta t \left\{ \sigma^2 (\frac{S_k}{\Delta S})^2 - (r - \delta)\frac{S_k}{\Delta S} \right\}, \\
y_k &= -(1-\theta)\Delta t \left\{ \sigma^2 (\frac{S_k}{\Delta S})^2 + r \right\}, \\
z_k &= \frac{1}{2}(1-\theta)\Delta t \left\{ \sigma^2 (\frac{S_k}{\Delta S})^2 + (r - \delta)\frac{S_k}{\Delta S} \right\}, \\
x_k^1 &= \frac{1}{2}\theta\Delta t \left\{ \sigma^2 (\frac{S_k}{\Delta S})^2 - (r - \delta)\frac{S_k}{\Delta S} \right\}, \\
y_k^1 &= -\theta\Delta t \left\{ \sigma^2 (\frac{S_k}{\Delta S})^2 + r \right\}, \\
z_k^1 &= \frac{1}{2}\theta\Delta t \left\{ \sigma^2 (\frac{S_k}{\Delta S})^2 + (r - \delta)\frac{S_k}{\Delta S} \right\}.
\end{aligned}$$

In matrix notation,

$$Aw^{(i)} = Bw^{(i+1)} + y^{(i+1)},$$

for all $i = M-1, M-2, \ldots, 1$.
Here,

$$w^{(i)} = \begin{pmatrix} w_{1,i} \\ w_{2,i} \\ \vdots \\ w_{N-1,i} \end{pmatrix},$$

$$b^{(i+1)} = \begin{pmatrix} -x_1 w_{0,i} \\ 0 \\ \vdots \\ 0 \\ -z_{N-1} w_{N,i}) \end{pmatrix},$$

$$A = \begin{pmatrix} y_1 & z_1 & 0 & \ldots & 0 & 0 \\ x_2 & y_2 & z_2 & \ldots & 0 & 0 \\ 0 & -x_3 & 1 - y_3 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & x_{N-1} & y_{N-1} \end{pmatrix},$$

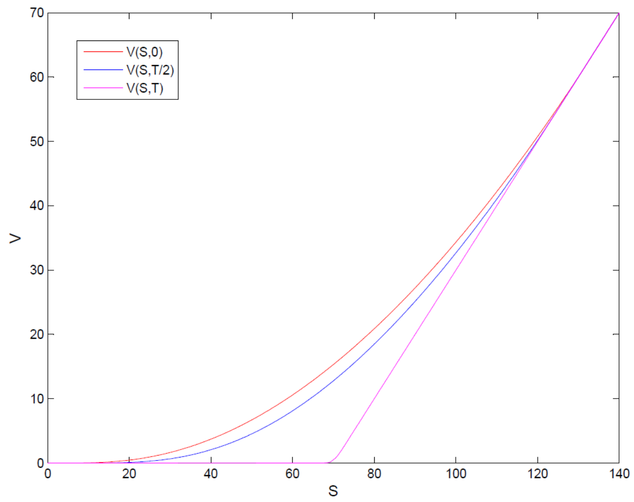


Figure 6.5: The price of an American call option for different time to maturities.

$$B = \begin{pmatrix} 1+y_1^1 & z_1^1 & 0 & \dots & 0 & 0 \\ x_2^1 & 1+y_2^1 & z_2^1 & \dots & 0 & 0 \\ 0 & -x_3^1 & 1-y_3^1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & x_{N-1}^1 & 1+y_{N-1}^1 \end{pmatrix}.$$

In order to price an American option, we take the following equality into account

$$w_{k,i}^{Am} = \max\left\{w_{k,i}^{\theta}, \psi_{k,i}\right\},$$

where $i = 1, \dots, N-1$ and $j = M, M-1, \dots, 1$. Here, $w_{k,i}^{\theta}$ is the approximate solution of the Black-Scholes PDE computed with the assumption of no free boundary.

In Figure 6.5, we show the valuation of an American put option for T=0, T=1, T=2. The parameters we use are: $S0 = 70$, $K = 70$, $D = 0.1$, $\sigma = 0.4$, $r = 0.05$, $T = 5$, $dS = 0.25$, $dt = 0.002$, $S_{\min} = 0$, $S_{\max} = 140$, $\theta = 0.4$.

6.2 Projected Sor Method (PSOR)

Projected SOR method (PSOR) which is used to solve the linear system of equations iteratively was first suggested by Cryer (1971) [11]. In this section, this method is applied to the system (6.17)

$$Aw^{(i)} = Bw^{(i+1)} + b^{(i+1)} = y^{(i+1)}, \tag{6.19}$$

where $i = M, M-1, \dots, 1$.

To be more precise, instead of using the LU decomposition of the matrix

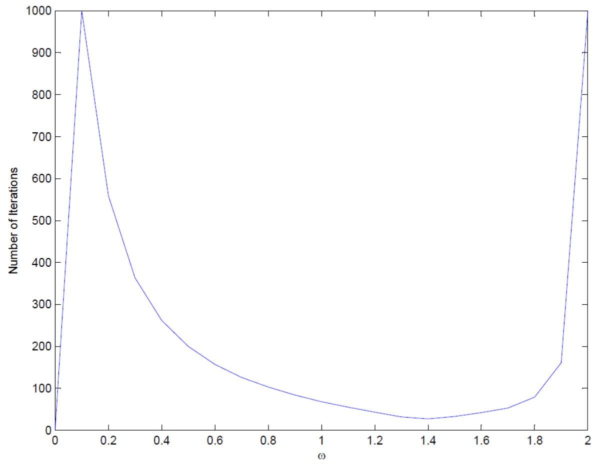


Figure 6.6: The number of iterations for increasing number of ω's.

$A = (a_{ji})$, we work with the sequences that converge to our desired solution $w^{(i)} = (w_{1,i}, \ldots, w_{N-1,i})^T$.
Let us start with an initial guess $w_i^{(0)} = (w_{1,i}^0, \ldots, w_{N-1,i}^0)^T$ and let ω be a constant in $\mathbb{R}$. For each component $(w_{k,i})_{1\leq k \leq N-1}$, we define a sequence $(\tilde{w}_{k,i}^{(l)})_{l\geq 1}$ such that

$$\tilde{w}_{k,i}^{(l)} = (1-\omega)\tilde{w}_{k,i}^{(l-1)} + \frac{\omega}{a_{kk}}\left\{ y_{k,i} - \sum_{t=1}^{k-1} a_{k,t} w_{t,i}^{(l)} - \sum_{t=k+1}^{N-1} a_{k,t} w_{t,i}^{(l-1)} \right\}. \tag{6.20}$$

Here, the constant ω is called relaxation parameter. Kahan (1958) [19] proved that the iterate $\tilde{w}_i^{(l+1)}$ converges to the desired solution $w^{(i)}$ iff $\omega \in (0,2)$. Moreover according to Young (1971) [43], if $\omega > 1$, we can maximize the convergence rate of the iteration process. This method that causes the convergence to rise is known as successive overrelaxation method (SOR). For more details, we refer to [1] and [33].
We note that the continuity of the iteration is fulfilled by the condition

$$\|\tilde{w}_{k,i}^{(l+1)} - \tilde{w}_{k,i}^{(l)}\| > \xi, \tag{6.21}$$

where ξ is a predetermined constant.
In order to construct an algorithm for an American type option, we just need to compare the SOR iterate $\tilde{w}_{k,i}^{(l)}$ with the intrinsic value $P_{k,i}$. This variant of the SOR method is known as Projected SOR method (PSOR). Hence, according to this method, the approximate value of an American option is given by

$$\tilde{w}_{k,i}^{PSOR} = \max\left\{\tilde{w}_{k,i}^{SOR}, \psi_{k,i}\right\},$$

where $\tilde{w}_{k,i}^{SOR}$ is the SOR iterate computed under the assumption of no free boundary and $P_{k,i}$ is the intrinsic value of the American option at the grid point (S_k, t_i). The MATLAB code of PSOR method can be found in [40].
Figure 6.6 plots the number of iterations used to price an American put option

for different value of ω's. The parameters we use are: $S0 = 40$, $K = 40$, $D = 0.1$, $\sigma = 0.5$, $r = 0.1$, $S_{\min} = 0$, $S_{\max} = 240$, $T = 1$, $dS = 2$, $dt = 0.001$, $tol = 1e - 8$. We see that convergence rate is maximized, when $\omega \approx 1.4$.

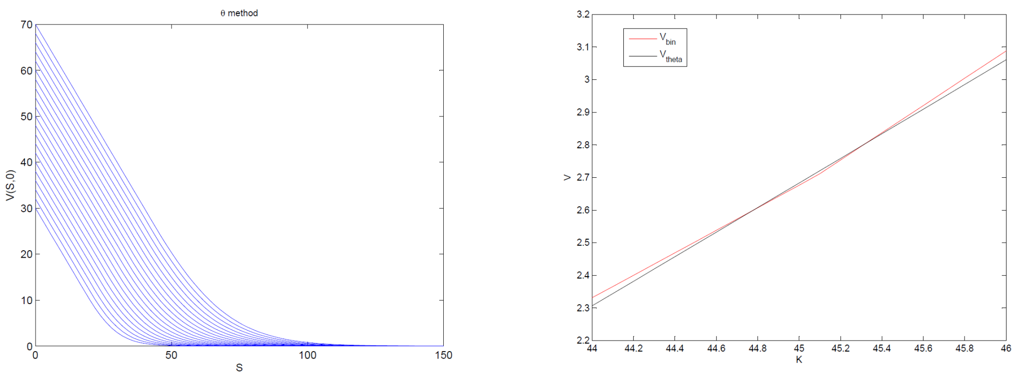
CHAPTER 7

APPLICATION

For the time being, we analyzed the fair price of American options through the different type of markets. We first gave a brief introduction about the discrete-time pricing of American options in case the holder does not receive a dividend payment. After gaining a deeper insight on pricing, we extended our results to the continuous-time. In this setting, we followed the well-known Black-Scholes model and we discussed the pricing theory for both dividend and non-dividend case. Indeed, assuming that the stock prices are modeled with the geometric Brownian motion, we mentioned the fundamental techniques for the valuation processes: martingale pricing and variational inequalities. But, these approaches we mentioned generally do not admit a closed-form pricing formula for the American options. For this reason, we will get the assistance of numerical methods in order to cope with this problem.
In this chapter, we focus on the numerical solutions of the price of American options for both discrete and continuous-time setting.
For continuous time, we employ the finite-difference methods described in Sections 6.1-6.2 assuming that stock prices follow geometric Brownian motion. Moreover, we suppose that the underlying asset pays a constant dividend yield $D = \delta$. We note that, for these implementations, we use the Black-Scholes inequalities with variable coefficients. In addition, it is possible to deal with the ones with constant coefficients by using the log prices.
As for the discrete-time setting, we will apply Binomial method to value American options. According to this method, all stock prices rise with an up factor u or fall with a down factor d in each time period. In order to reflect the stock price fluctuations efficiently, we work with the large number of time steps. Therefore, we use the continuously compounding discount factor e^{-rdt} for each time interval $[t - dt, t]$ to be more realistic. More precisely, letting (S_t) denote the stock price process at time t, we have $S_t = S_{t-dt}u$ or $S_t = S_{t-dt}d$, where dt corresponds to the size of the time steps. Let V_t^u be the value of the American option when $S_t = S_{t-dt}u$ and let V_t^d be the value of the option in the case $S_t = S_{t-dt}d$. In order to find the price of an American option at time t, we take the maximum of the intrinsic value and the expression $e^{-rdt}(qV_{t+dt}^u + (1-q)V_{t+dt}^d)$, as done similarly in Chapter 3. Here, q is the risk-neutral probability measure under which the discounted asset prices are martingales.
With the help of these methods, we will see the impact of Black-Scholes param-

Table 7.1: The value of an American put option for different strike prices

K	Binomial	θ-Averaged	CN	PSOR
30	0.0843	0.0865	0.0866	0.0866
35	0.3922	0.4016	0.4016	0.4016
40	1.1920	1.1783	1.1782	1.1782
45	2.6764	2.6837	2.6834	2.6835
50	4.9629	4.9558	4.9555	4.9556
55	8.0147	8.0087	8.0084	8.0085
60	11.6884	11.6668	11.6666	11.6668
65	15.8208	15.8204	15.8202	15.8205
70	20.3220	20.3132	20.3131	20.3135



(a) (b)

Figure 7.1: The valuation of an American put (a) for different strike prices. (b) with θ-averaged and Binomial method depending on K.

eters K, σ and δ on the valuation of American options. Also, we will compare Crank-Nicolson, θ-Averaged, PSOR and Binomial methods to analyze their sensitivities against to the changes in the parameters K, σ and D given above.
In the implementation, we will use these parameters: $S_0 = 50$, $K = 45$, $D = 0.1$, $\sigma = 0.4$, $r = 0.1$, $S_{\min} = 0$, $S_{\max} = 150$, $T = 5/12$, $dS = 2$, $dt = 1/1200$, $\omega = 1$, $tol = 1e-8$, $\theta = 0.4$, $M = 100$ [40].
We recall that the parameter $S0$ is the initial stock price, V is the value of the option, K is the strike price of the option, D is constant dividend yield, σ is volatility, r is risk-free interest rate, T is the maturity of the option, ω is the relaxation parameter, M is the number of points in Binomial method and tol is the lower bound providing the continuity of iterations in PSOR method.
In Table 7.1, we compare the numerical solutions of the price of an American put option for different strike prices. It is apparent that strike price has a positive impact on the price of the American put. Also, it can be seen that the performance of finite difference methods are almost the same. On the other hand, when $K \geq 50$, binomial method seems to reflect the change in the strike price to the option value much more than the others.

In Figure 7.1, we illustrate these results. In Figure 7.1a, when K increases from

Table 7.2: The value of an American call option for different strike prices

K	Binomial	θ-Averaged	CN	PSOR
30	20.0000	20.0000	20.0000	20.0000
35	15.1558	15.1576	15.1575	15.1578
40	10.9648	10.9490	10.9489	10.9491
45	7.5473	7.5517	7.5514	7.5516
50	4.9629	4.9562	4.9558	4.9559
55	3.1460	3.1417	3.1415	3.1415
60	1.9305	1.9108	1.9106	1.9107
65	1.1347	1.1402	1.1402	1.1402
70	0.6652	0.6587	0.6587	0.6587

30 to 70 with an increment of two, the price of the option experiences a rise. In Figure 7.1b, we simulate the price of the option $V(S, 0)$ with θ-averaged and Binomial method depending on the strike price K. Since the other finite differences give almost the same results with θ-averaged method, we compare the Binomial method only with θ scheme. Here, K changes from 44 to 46 with an increment of 0.1. It is clear that, as getting closer to the strike price 45, the methods give closer numerical solutions.

In Table 7.2, we analyze this time pricing of an American call option. We use the same parameters we mentioned above. It is clear that strike price K is inversely correlated with the price of an American call option, as also shown in Figure 7.2. In the figure, we simulate the option value depending on K and S_0 by Binomial method. As can be seen, when we fix the initial stock price, the option value decreases with rising exercise price. As mentioned in the previous chapter,

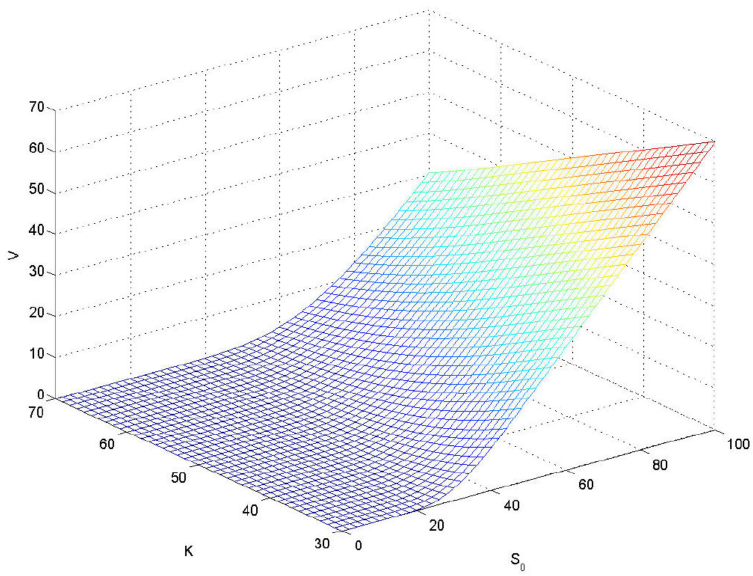
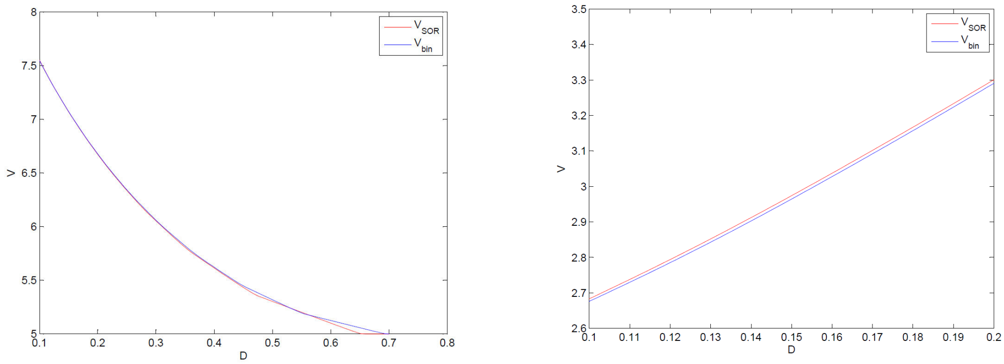


Figure 7.2: Binomial method; the value of an American call option for different strike prices.

Table 7.3: The value of an American put option for different dividend yields

D	Binomial	θ-Averaged	CN	PSOR
0.1	2.6764	2.6832	2.6834	2.6835
0.2	3.2910	3.3008	3.3010	3.3010
0.3	4.0103	4.0209	4.0211	4.0211
0.4	4.8129	4.8239	4.8241	4.8241
0.5	5.6933	5.7041	5.7043	5.7043
0.6	6.6435	6.6537	6.6540	6.6540
0.7	7.6537	7.6627	7.6631	7.6631



(a) (b)

Figure 7.3: SOR, Binomial Method; the valuation for different dividend yields (a) call option (b) put option.

the dividend yields play a key role on pricing of American options. Under the Black-Scholes setting, we assume that the price of the underlying asset goes down with the continuously paid dividends. When the stock price does not decrease at those dates, we can obtain an arbitrage profit in the market. Indeed, by taking the non-negative dividend payment, we can sell the asset to its new holder as soon as we buy it. This causes to occur a riskless profit in the market. Therefore, we say that that the stock price goes down at those days to prevent an arbitrage opportunity [8, 40].

We recall that an American call option increases in value when the stock price goes up. This increase in the stock price makes the American call option more valuable. On the other hand, the put holder suffers from a decrease in the option value, if the stock price continues to rise. In light of the foregoing, it can be easily verified that an increasing dividend yield causes the put option to worth more than before. Table 7.3 figures out these facts. However, as seen in Table 7.4, call option suffers from a depreciation in value for increasing dividend yields. Moreover, the discrete-time valuation technique, Binomial method, seems not so successful in capturing the dividend gains for the American put options. On the other hand, the call option does not experience a strict fall as much as the ones in continuous-time. With the Figures 7.3a and 7.3b, we point out these facts. Since the finite difference methods give similar results, we compare the

Table 7.4: The value of an American call option for different dividend yields

D	Binomial	θ-Averaged	CN	PSOR
0.1	7.5473	7.5511	7.5514	7.5516
0.2	6.6778	6.6720	6.6725	6.6732
0.3	6.0598	6.0520	6.0525	6.0538
0.4	5.6220	5.6109	5.6115	5.6138
0.5	5.3185	5.3010	5.3014	5.3031
0.6	5.1254	5.0933	5.0947	5.1009
0.7	5.0000	5.0000	5.0000	5.0000

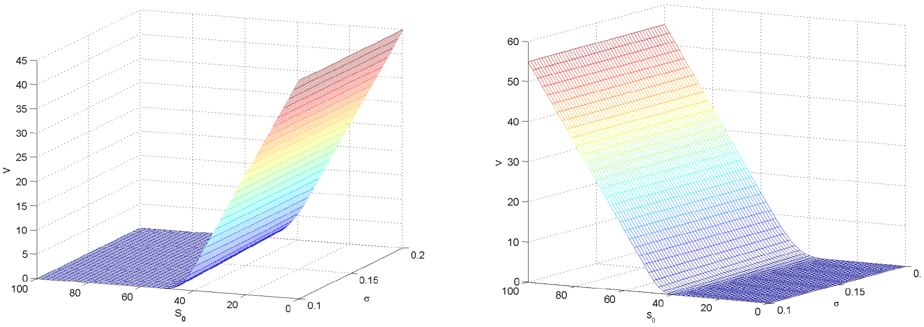
Binomial method only with PSOR method.
It is clear from Table 7.5 and Table 7.6 such that the volatility has a positive effect on both call and put options. It is actually to the fact that options are the financial instruments commonly used to hedge the risks of the investors. Indeed, an option takes its value from the underlying asset that has written on it. Because the volatility means uncertainty in the market, an increasing uncertainty makes the stocks more riskier. That is why, the investors aim to guarantee themselves against an unexpected movements in the market. In that case, the options used to eliminate that risk becomes more valuable and attractive for investors. Therefore, an increasing volatility causes the options to increase in value.

Table 7.5: The value of an American put option for different volatilities

σ	Binomial	θ-Averaged	CN	PSOR
0.1	0.0634	0.0789	0.0788	0.0788
0.2	0.6874	0.6959	0.6960	0.6960
0.3	1.6378	1.6347	1.6348	1.6349
0.4	2.6764	2.6832	2.6834	2.6835
0.5	3.7902	3.7760	3.7763	3.7764
0.6	4.9055	4.8882	4.8886	4.8887
0.7	6.0151	6.0080	6.0085	6.0086

Table 7.6: The value of an American call option for different volatilities

σ	Binomial	θ-Averaged	CN	PSOR
0.1	5.0070	5.0012	5.0012	5.0012
0.2	5.5720	5.5737	5.5738	5.5739
0.3	6.5108	6.5050	6.5052	6.5053
0.4	7.5473	7.5511	7.5514	7.5516
0.5	8.6579	8.6430	8.6434	8.6436
0.6	9.7733	9.7548	9.7553	9.7555
0.7	10.8835	10.8744	10.8750	10.8752



(a) (b)

Figure 7.4: Binomial Method; The valuation for different volatilities (a) American call option (b) American put option.

Figure 7.4a and Figure 7.4b illustrates these results. It is clear that the call and put option prices are positively affected from the increase of volatility in the market.

CHAPTER 8

CONCLUSION AND OUTLOOK

In this thesis, we studied the valuation of American options in discrete and continuous time models. We first examined the discrete-time valuation of American options with no dividend. In this model, American options were uniquely priced with the help of replicating portfolios. After introducing Snell envelope and optimal stopping time problem, it was also shown that American call options with no dividend were equally priced with their European counterparts.
Afterwards, we extended our results to the continuous-time market. Following the well-known Black-Scholes model, we investigated the valuation concept for both dividend and non-dividend case. Under this setting, two fundamental valuation techniques commonly used to price American options were introduced: martingale pricing and variational inequalities. Martingale pricing of American options deals with maximizing the expected value of discounted payoff process under the risk-neutral probability measure. With this approach, it was shown that the unique price of an American option was described by a hedging portfolio that replicates it. Moreover, it was shown that an early exercise is not optimal for an American call option with no dividend. That is, the price of an American call option is equal to the price of its European counterpart in the case the underlying does not pays dividend. On the other hand, this result is not valid for the American calls on a dividend paying stock. Secondly, the variational inequalities were introduced. Under this approach, we gave the parabolic partial differential inequalities that the American options satisfy. Then, we proved the main theorem for pricing in detail when the inequality system has a regular solution.
Because these approaches generally do not offer a closed-form pricing formula for the American options, we gave a brief introduction to the finite difference and PSOR methods. Finally, a numerical application was done by comparing the efficiencies of these methods. We also investigated the impact of Black-Scholes parameters, σ, δ, K, on the price of American options. We showed that strike price, volatility and dividend yield are positively correlated with the put option price, whereas the value of a call option tends to decrease for increasing values of K and δ.
As for future work, Monte-Carlo method can be used to price American options numerically. This scheme is based on approximating the expected discounted payoff process with the help of dynamic programming. Moreover, these contracts may be valued with the help of trinomial tree method working on the discrete-

time setting. Indeed, this scheme is equivalent to the explicit finite difference approach. Instead of dealing with the numerical methods, we may also use analytic techniques that generally provide a lower cost. Interpolation, quadratic approximation and methods of lines can be preferable to fulfill this purpose.

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