OPTIMAL PORTFOLIO STRATEGIES UNDER VARIOUS RISK MEASURES

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ABSTRACT

OPTIMAL PORTFOLIO STRATEGIES UNDER VARIOUS RISK MEASURES

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In this thesis, we search for optimal portfolio strategies in the presence of various risk measure that are common in financial applications. Particularly, we deal with the static optimization problem with respect to Value at Risk, Expected Loss and Expected Utility Loss measures. To do so, under the Black-Scholes model for the financial market, Martingale method is applied to give closed-form solutions for the optimal terminal wealths, then via representation problem the optimal portfolio strategies are achieved. We compare the performances of these measures on the terminal wealths and optimal strategies of such constrained investors. Finally, we present some numerical results to compare them in several respects to give light to further studies.

Keywords: Portfolio Optimization, Value at Risk, Expected Loss, Expected Utility Loss, Black-Scholes Model, Martingale Method, Risk Constraints.

ÖZ

ÇEŞİTLİ RİSK ÖLÇÜMLERİ ALTINDA EN UYGUN PORTFÖY STRATEJİLERİ

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Bu tezde, finansal uygulamalarda yaygın olan çeşitli risk ölçümleri varlığında en uygun portföy stratejilerini araştırıyoruz. Özellikle, Riskteki Değer, Beklenen Kayıp ve Beklenen Fayda Kaybı ölçümleriyle ilgili olarak statik problem ile ilgileniyoruz. Bunu yapmak için, finansal piyasa için Black-Scholes modeli altında, en uygun nihayi servetlere kapalı form çözümleri vermek için Martingale metodu uygulanır, ardından temsil problemi yoluyla en uygun portföy stratejileri elde edilir. Bu ölçümlerin, böyle sınırlandırılmış yatırımcıların nihayi servetleri ve en uygun stratejileri üzerindeki performanslarını karşılaştırıyoruz. Son olarak, ileriki çalışmalara ışık tutmak adına bu ölçümleri birkaç yönden karşılaştırmak için bazı sayısal sonuçlar sunuyoruz.

Anahtar Kelimeler: Portföy Optimizasyonu, Riskteki Değer, Beklenen Kayıp, Beklenen Fayda Kaybı, Black-Scholes Modeli, Martingale Metodu, Risk Sınırlamaları

To My Family

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CHAPTER 1

Introduction

Harry Markowitz, who is the pioneer of the modern portfolio theory, mentioned about trading off the mean return of a portfolio against its variance in his works (see [18, 19]). In order to solve the portfolio optimization problem, Robert C. Merton presented the concept of Itô calculus with methods of continuous-time stochastic optimal control in two works (see [20, 21]) and when the utility function is a power function or the logarithm, he produced solutions to both finite and infinite-horizon models (see [20]). Harrison and Kreps [10] constituted portfolios from martingale representation theorems and started the modern mathematical approach to portfolio management in complete markets, which were built around the ideas of martingale measures. Harrison and Pliska (see [11, 12]) improved this subject much more in the context of the option pricing. The martingale ideas to utility maximization problems were adapted by Pliska [22], Cox and Huang [4, 5], and Karatzas, Lehoczky and, Shreve [13]. You can further examine about these developments in Karatzas and Shreve [15].

In this thesis, we investigate optimal strategies for portfolios consisting of only one risky stock and one risk-free bond. This study can easily be generalized to the multi-dimensional Black-Scholes model with $d > 1$ risky stocks. We assume that an investor in this economy has some initial wealth at time zero and there is a finite planning horizon $[0, T]$ that is given. The goal of this investor is to maximize the expected utility of the terminal wealth of the portfolio by optimal selection of the proportions of the wealth invested in stock and bond. We assume continuous-time market which allows for permanent trading and re-balancing the portfolio, and we have to find these proportions for every time t to T. Also, we allow the short selling of the stock, which is the selling of a stock that the seller doesn't own, but is promised to be delivered.

Karatzas, Lehoczky, and Shreve [13] and also Cox and Huang [4] solved the utility maximization problem without additional limitations by using martingale approach in the context of the Black-Scholes model of a complete market. Also, the works of Karatzas et al. [14] is an extension of the solution to should be examined for the case of an incomplete market.

We consider shares of a stock and a risk-free bond whose prices follow a geometric Brownian motion in this portfolio. We can obtain the maximum expected utility of the terminal wealth by following the optimal portfolio strategy. However, since the terminal wealth is a random variable with a distribution which is often extremely skew, it shows considerable probability in regions of small values of the terminal wealth. Namely, the optimal terminal wealth may exhibit large shortfall risks. By the term shortfall risk, we indicate the event that the terminal wealth may fall below a given deterministic threshold value, namely, the initial capital or the result of an investment in a pure bond portfolio.

It is necessary to quantify shortfall risks by using appropriate risk measures in order to incorporate such shortfall risks into the optimization. We denote the terminal wealth of the portfolio at time $t = T$ by X_T and let $q > 0$ be threshold value or shortfall level. Then the shortfall risk consists in the random event $\{X_T < q\}$ or $\{Z = X_T - q < 0\}$ and we assign to the random variable (risk) Z the real number $\rho(Z)$ which will be called a *risk measure*.

Therefore, the idea is to restrict the probability of a shortfall:

$$\rho_1(Z) = P(Z < 0) = P(X_T < q).$$

This corresponds to the concept of Value at Risk (VaR) [23], defined by

$$\mathrm{VaR}_\varepsilon(Z) = \inf\{l \in \mathbb{R} : P(Z > l) \leq \varepsilon\},$$

where l can be interpreted such that given $\varepsilon \in (0, 1)$, the VaR of the portfolio at the confidence level $1 - \varepsilon$ is given by the smallest number l such that the probability that the loss Z exceeds l is at most ε. Although it virtually always represents a loss, VaR is conventionally reported as a positive number. A negative VaR would imply that the portfolio may make a profit. VaR describes the loss that can occur over a given period, at a given confidence level, due to exposure to market risk. This risk measure is widely used by banks, securities firms, commodity and energy merchants, and other trading organizations. However, VaR risk managers often optimally choose a larger exposure to risky assets than non-risk managers and consequently incur larger losses when losses occur.

In order to remedy the shortcomings of VaR, an alternative risk-management model is suggested, which is based on the expectation of a loss. This alternative model is called as Expected Loss. This risk management maintains limited expected losses when losses occur. You can see risk management objectives which are embedded into utility maximization problem using Value at Risk (VaR) and Expected Loss (EL), for instance in [8, 9]. The EL risk measure is defined by

$$\rho_2(Z) = \mathrm{EL}(Z) = \mathbb{E}\left[Z^-\right] = \mathbb{E}\left[(X_T - q)^-\right],$$

and it is bounded by a given $\varepsilon > 0$.

As the aim of the portfolio manager is to maximize the expected utility from the terminal wealth, one may also consider the portfolio optimization problem where the portfolio manager is confronted with a risk measured by a constraint of the type

$$\rho_3(Z) = \mathrm{EUL}(Z) = \mathbb{E}\left[Z^-\right] = \mathbb{E}\left[(u(X_T) - u(q))^-\right] \leq \varepsilon,$$

where $\varepsilon > 0$ is a given bound for the Expected Utility Loss (EUL) [7]. This risk constraint causes to more explicit calculations for the optimal strategy we are looking for. Also, it allows to the constrained static problem to be solved for a large class of utility functions.

Alternatively, Artzner et al. (1999) [1] and Delbaen (2002) [6] introduced the concept of coherent measures and you can find further risk measures in the class of coherent measures. These measures have the properties of monotonicity, subadditivity, positive homogeneity and the translation invariance property. However, VaR, EL, EUL risk measures do not belong to this class: VaR is not subadditive, and EL and EUL do not satisfy the translation invariance property.

In this thesis we examine the effects of risk management on optimal terminal wealth choices and on optimal portfolio policies. We consider portfolio managers or investors as expected utility maximizers, who derive utility from wealth at horizon and who must comply with different risk constraints imposed at that horizon.

This thesis is organized as follows. In Chapter 2, in Section 2.1, we introduce basic notations for the Black-Scholes model of the financial market. In Section 2.2, we restrict to the case of a financial market with only one risky stock, and then formulate the portfolio optimization problem. In Section 2.3, we solve the unconstrained portfolio optimization problem using martingale method [4]: the martingale method consists of converting the dynamic optimization problem of finding an admissible strategy that maximizes the expected utility from terminal wealth into a static optimization problem consisting of finding an optimal terminal wealth. Then, via a representation problem, the optimal strategy associated with this optimal terminal wealth is obtained. In Section 2.4, we give a short review of risk measures that will be used in subsequent chapters.

In Chapter 3, we examine the portfolio optimization problem where the shortfall level q is considered as deterministic. Basically, we associate the risk with the random variable $Z = X_T - q$ and adopt three different risk constraints, which are Value at Risk (VaR), Expected Loss (EL) and Expected Utility Loss. In Section 3.1, the shortfall probability, or equivalently the Value at Risk is bounded and added in a form of risk constraint to the optimization. In Section 3.2, the Expected Loss constraint is used, and in Section 3.3 we bound the Expected Utility Loss to the optimization problem.

In Chapter 4, we examine the numerical results of each optimization problem concerning the above risk measures separately, and compare them with the unconstrained problem.

In the light of Chapter 4, Chapter 5 covers a presentation of the advantages and disadvantages of each measure: we try to understand which one might be considered as more suitable for the optimality in the terminal wealth and portfolio strategy according to risk averse investors who dislike risk and prefer more than less wealth. As we assume, throughout the thesis, the investors are risk averse, we choose the power utility function, and try to draw a general conclusion and outlook.

In Appendix A, the proofs of some propositions in this thesis, and in Appendix B, some necessary Matlab algorithms for the risk measures used in this thesis are presented.

CHAPTER 2

A General Overview to the Portfolio Optimization Problem

In this chapter, we describe the economic setting, formulate portfolio optimization problem, solve the unconstrained portfolio optimization problem using martingale method, and give a short review of risk measures used in subsequent chapters.

2.1 The Economic Setting

We consider a continuous-time economy with finite horizon $[0, T]$, which is built on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ on which a 1-dimensional Brownian motion W is defined. Here, T is the end of the holding period of assets. We assume that all stochastic processes are adapted to $(\mathcal{F}_t)$, the augmented filtration generated by W. Through this thesis all inequalities as well as equalities are assumed to hold P-almost surely. Because we deal with characterization problems, all stated processes are assumed to be well defined without giving any regularity conditions to ensure this.

Financial investment opportunities are given by an instantaneously risk-free money market account providing an interest rate r and a risky stock as in the Black-Scholes model [3]. We search for the optimal strategies for portfolios consisting of only one risky stock and one risk-free bond. The stock price S is represented by a geometric Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad S_0 = s_0 \in \mathbb{R}, \tag{2.1}$$

where the stock instantaneous mean return μ and the volatility σ are assumed to be constant. The bond price $S^0 = (S_t^0)_{t\in[0,T]}$ is given by

$$dS_t^0 = rS_t^0 dt, \tag{2.2}$$

where the interest rate r is assumed to be constant.

The dynamic market completeness implies the existence of a unique state price density process H_t, given by the solution of

$$dH_t = -H_t(rdt + \kappa dW_t), \quad H_0 = 1, \tag{2.3}$$

where $\kappa = \frac{\mu - r}{\sigma}$ is the market price of risk in the economy and, it can be considered as a risk premium.

The investor or the portfolio manager should have an initial capital $x > 0$ and a portfolio process $\theta = (\theta_t)_{t\in[0,T]}$, where θ_t indicates the fraction of wealth invested in stock at time t in order to trade in this economy. At any time t, the portfolio manager chooses a trading strategy $(\psi_t^0, \psi_t)_{t\in[0,T]}$, where ψ_t^0 and ψ_t represent the number of shares held by the portfolio manager in the assets, bond and stock, respectively. The $\mathbb{R}^2$-valued process (ψ_t^0, ψ_t) is assumed to be $\mathcal{F}$-measurable such that

$$\int_0^T (\psi_t^0 S_t^0)^2 dt + \int_0^T (\psi_t S_t)^2 dt < \infty.$$

The wealth process X_t of the portfolio manager, on the other hand, is defined by

$$X_t = \psi_t^0 S_t^0 + \psi_t S_t,$$

at time t, in terms of the trading strategy and the movements of bond and stock.

Moreover, we consider that the trading strategy is self-financing in the sense that no other money is going in or out the market except the money generated by the trading strategy, see [16]. Under this assumption, if the wealth $X_t > 0$ the portfolio manager can act in the market using the associated portfolio process $\theta = (\theta_t)_{t\in[0,T]}$:

$$\theta_t = \frac{\psi_t S_t}{X_t^\theta}$$

with $\theta_t^0 = 1 - \theta_t$ is the fraction of wealth invested in the risk-free bond. As a consequence, the wealth process can be formulated in terms of the portfolio process as a linear stochastic differential equation,

$$dX_t^\theta = [r + \theta_t(\mu - r)]X_t^\theta dt + \theta_t \sigma X_t^\theta dW_t, \ X_0^\theta = x. \tag{2.4}$$

At time $t = T$ the portfolio manager reaches the terminal wealth X_T. The portfolio process is assumed to be admissible in the following sense.

Definition 2.1 (Admissible portfolio process [8])**.** Given $x > 0$, we say that a portfolio process θ is admissible at x, and write $\theta \in \mathbf{A}(x)$, where $\mathbf{A}(x)$ is the set of admissible processes, if the wealth process X_t^θ starting at $X_0^\theta = x$ satisfies $X_t^\theta \geq 0$, $0 \leq t \leq T$.

2.2 The Portfolio Optimization Problem

In this section, we consider that the portfolio manager is assumed to derive from the terminal wealth X_T a utility $u(X_T)$, and he wants to maximize the expected utility by choosing an optimal strategy from the set of admissible strategies.

The most frequently used utility function is the power utility function

$$u(z) = \begin{cases} \frac{z^{1-\gamma}}{1-\gamma}, & \gamma \in (0,\infty) \setminus \{1\}, \\ \ln z, & \gamma = 1. \end{cases} \tag{2.5}$$

With positive first derivative and negative second derivative, the power utility function (2.5) meets the requirement of risk averse investor who prefers more than less wealth. The parameter γ of the power utility function can be interpreted as constant relative risk aversion. As we assume that, in this economy, investors are risk averse, we prefer logarithmic utility function in most of the cases, since our applications are done easily with this function. Also, this logarithmic utility meets our requirement regarding the risk aversion concept.

Here, we examine the case of portfolio optimization when the portfolio manager maximizes the expected logarithmic utility of the terminal wealth of one stock with a constant stock mean return μ and a constant volatility σ, and a bond with a constant interest rate r. The portfolio manager begins with initial capital $x > 0$ and follows a portfolio process $\theta = (\theta_t)_{t\in[0,T]}$ which leads to the wealth $X^\theta = (X_t^\theta)_{t\in[0,T]}$, already formulated in (2.4):

$$dX_t^\theta = [r + \theta_t(\mu - r)]X_t^\theta dt + \theta_t \sigma X_t^\theta dW_t, \;\; X_0^\theta = x.$$

We assume that the stochastic integral $(\int_0^t \theta_s \sigma dW_s)_{t\in[0,T]}$ is a martingale, which is the case when the inequality $\mathbb{E}\left[\int_0^t \theta_s^2 ds\right] < \infty$ is fulfilled, or, in particular, when θ is assumed to be a bounded and deterministic. Now, the portfolio optimization problem can be introduced as follows.

Definition 2.2 (Dynamic Problem [8])**.** Find an admissible strategy θ^* in $\mathbf{A}(x)$ that solves

$$\underset{\theta\in A}{\text{maximize}} \; \mathbb{E}\left[u(X_T^\theta)\right], \tag{2.6}$$

where the utility function u satisfies the following conditions:

- u is twice continuously differentiable,
- u is strictly increasing and strictly concave,
- $\lim_{x\to 0} u'(x) = \infty$ and $\lim_{x\to\infty} u'(x) = 0$.

2.3 The Unconstrained Problem

Cox and Huang [4] and Karatzas et al. [13] solved (2.6) in the case of a complete market using martingale method without additional constraints such as risk. The method consists of converting the dynamic optimization problem of finding an admissible strategy that maximizes the expected utility from terminal wealth into a static optimization problem of finding an optimal terminal wealth. Then, the optimal strategy associated with this optimal terminal wealth is found via a representation problem.

Itô's formula implies that the process $H_t X_t^\theta$ is a super-martingale, which implies that the budget constraint

$$\mathbb{E}\left[H_T X_T^\theta\right] \leq x$$

is satisfied for every $\theta \in A(x)$. See for instance [8]. This means that the expected discounted terminal wealth cannot exceed the initial wealth. Here, the state price density H_t serves as a discounting process.

In the present case of a complete market, the following theorem which Karatzas and Shreve (1998) stated in [15] is a basic tool in martingale methods.

Theorem 2.3 ([15])**.** *Let $x > 0$ be given, and let ξ be a non-negative, $\mathcal{F}_T$-measurable random variable such that*

$$\mathbb{E}\left[H_T \xi\right] = x.$$

Then there exists a portfolio process $\theta \in A(x)$ such that $\xi = X_T^\theta$.

In contrast to the dynamic problem, where the investor is required to maximize expected utility from terminal wealth over a set of processes, in the martingale method in the static problem is considered. Here, the investor has the advantage to maximize the expected utility only over a set of random variables:

Definition 2.4 (Static Problem [8])**.** Let $B(x) = \{\xi \geq 0 : \xi \text{ is } \mathcal{F}_T - \text{measurable and } \mathbb{E}\left[H_T \xi\right] \leq x\}$. Find an $\mathcal{F}_T$-measurable random variable ξ^* in $B(x)$ that solves

$$\underset{\xi \in B(x)}{\text{maximize}} \; \mathbb{E}\left[u(\xi)\right]. \tag{2.7}$$

Then, the optimal strategy is found as the solution to the representation problem:

Definition 2.5 (Representation Problem [8])**.** Given $\xi^* \in B$, which solves (2.7), find an admissible strategy $\theta^* \in A(x)$ such that $X_T^{\theta^*} = \xi^*$.

Let, for $y > 0$, the function I be defined as the inverse function of the derivative of the utility function, $I(y) = (u')^{-1}(y)$. The following theorem formulates how to solve the static optimization problem (2.7).

Theorem 2.6 ([17])**.** *Consider the portfolio problem* (2.6)*. Let $x > 0$ and y be a solution of*

$$\mathbb{E}\left[H_T I(y H_T)\right] = x.$$

Then, there exists for $\xi^* = I(yH_T)$, a self-financing portfolio process $\theta_t^* \in \mathbf{A}(x)$ such that

$$X_T^{\theta^*} = \xi^*$$

holds and the portfolio process θ_t^* solves (2.6)*.*

The representation problem can be solved using the fact that the process $H_t X_t$ is a martingale (see [8]). The Markov property of the solution of the stochastic differential equation allows the optimal wealth process before the horizon $X_t^{\theta^*}$ to be written as a function of H_t, for which Itô's formula is applied. By equating coefficients with the wealth process (2.4), we can get the optimal portfolio.

Cox and Huang [4] studied the unconstrained problem where the investor does not manage the risk and has a constant relative risk aversion γ inside the power utility function. According to Theorem 2.6, the static problem (2.7) has the optimal solution

$$\xi^* = I(yH_T),$$

where $I(x) = x^{\frac{-1}{\gamma}}$ is the inverse function of the derivative of the utility function u and $y = \frac{1}{x^\gamma} e^{(1-\gamma)(r+\frac{2}{2\gamma})T}$.

Let X_t^* be the optimal wealth at time $t \in [0, T]$. Then, Itô's lemma applied to equations (2.3) and (2.4) implies that the process $H_t X_t^*$ is an $\mathcal{F}_t$-martingale, namely, $X_t^* = \frac{1}{H_t}\mathbb{E}\left[H_T X_T^* \,\middle|\, \mathcal{F}_t\right]$. See [17, Theorem 12] for details. Hence, the Markov property of H_t is applied to compute this conditional expectation: for the optimal wealth before the horizon, the following form can be derived (see [17]):

$$X_t^* = \frac{e^{\Gamma(t)}}{(yH_t)^{\frac{1}{\gamma}}} \quad \text{with} \quad \Gamma(t) = \frac{1-\gamma}{\gamma}\left(r + \frac{\kappa^2}{2\gamma}\right)(T-t).$$

Thanks to the representation approach, the optimal strategy can be obtained easily: we have $X_t^* = f(H_t)$ with $f(x) = \frac{e^{\Gamma(t)}}{(yx)^{\frac{1}{\gamma}}}$, for which Itô's lemma is applied, and

$$dX_t^* = \left[-rH_t f'(H_t) + \frac{\kappa^2}{2} H_t^2 f''(H_t)\right] dt + \left[-\kappa H_t f'(H_t)\right] dW_t$$

is obtained. Consequently, equating the volatility coefficient of this equation and that of (2.4), the following constant optimal strategy for the unconstrained problem is derived. To be more explicit,

$$\theta_t^* = \theta^* = \frac{\kappa}{\gamma\sigma} = \frac{\mu - r}{\gamma\sigma^2} = \text{constant}$$

is obtained.

In reality, this strategy should not be constant since values of volatility σ, mean return μ and interest rate r may change with time. However, in this thesis we assume that these values are constants in order to understand problems more clearly.

In this section, we have examined the portfolio optimization problem without risk limitations. However, here the optimal terminal wealth may fall below a given deterministic threshold value with high probability. Hence, it is necessary to use risk constraints in the portfolio optimization in order to reduce risk. In

Section 2.4, we present brief definitions of risk measures used in this thesis and then in next chapters we will examine these measures in optimization problems with their examples.

2.4 Risk Measures

Risk is the quantifiable likelihood of loss or less than expected returns, so in general, it is considered as an undesirable outcome. Although we study the portfolio optimization in Section 2.3, we did not impose any risk limitations. However, a portfolio manager would not want risk or strategies that would lead to extreme positions, so it turns out to be necessary to quantify shortfall risks by using appropriate risk measures, to add into the optimization problem. By the term shortfall risk we state the event that the terminal wealth may fall below threshold value or a shortfall level $q > 0$. We can consider the shortfall level as the initial capital or a proportion of the terminal wealth of a pure bond portfolio, which must not be chosen to be larger than xe^{rT}, the result of an investment with the initial capital x in the risk-free bond.

The shortfall risk consists of the random event $\{Z = X_T - q < 0\}$: we assign risk measures to the random variable (risk) Z and denote it by $\rho(Z)$. Consequently, we should incorporate constraints of the type $\rho(Z) \leq \varepsilon$ for some $\varepsilon > 0$ into the formulation of the portfolio optimization problem.

Basic idea is to restrict the probability of a shortfall,

$$\rho_1(Z) = P(Z < 0) = P(X_T < q) \leq \varepsilon,$$

where, $\varepsilon \in (0, 1)$ is the maximum shortfall probability which is accepted by the portfolio manager. This corresponds to the concept of Value at Risk (VaR) [23], which may be regarded as

$$\mathrm{VaR}_{\varepsilon}(Z) = \inf\{l \in \mathbb{R} : P(Z > l) \leq \varepsilon\},$$

that is, the VaR of the portfolio at the confidence level $1 - \varepsilon$ is given by the smallest number l such that the probability that the loss Z exceeds l is at most ε. Although it virtually always represents a loss, VaR is conventionally reported as a positive number. A negative VaR can imply the portfolio has a high probability of making a profit. In this setting, VaR can be interpreted as the threshold value for which the risk exceeds this value with some given probability ε.

Value at Risk is a widely used risk measure for loss on a specific portfolio of financial assets. It is the worst loss for a given confidence level. For a confidence level of $1 - \varepsilon = 95\%$, one is 95% certain that at the end of a chosen risk horizon, there will be no smaller wealth than the wealth which corresponds to the level of VaR. However, VaR risk managers often optimally choose a larger exposure to risky assets than non-risk managers and consequently incur larger losses when losses occur. Also, the VaR risk measure controls only the probability of loss rather than its magnitude. These shortcomings of VaR risk measure may be remedied.

So, an alternative risk-management model may be regarded as Expected Loss, denoted by EL:

$$\rho_2(Z) = \mathrm{EL}(Z) = \mathbb{E}\left[Z^-\right] = \mathbb{E}\left[(X_T - q)^-\right].$$

As the goal of the portfolio manager is to maximize the expected utility of the terminal wealth X_T, it is also interesting to examine the other risk measure called Expected Utility Loss (EUL). Let u denote a given utility function; the situation of $u(X_T)$ being below the target $u(q)$ may be considered as undesirable. In order to quantify the associated risk, the random variable $Z = u(X_T) - u(q)$ is assigned a real-valued risk measure $\rho_3(Z)$:

$$\rho_3(Z) = \mathrm{EUL}(Z) = \mathbb{E}\left[Z^-\right] = \mathbb{E}\left[(u(X_T) - u(q))^-\right].$$

There are more risk measures, mainly in the class of coherent measures, which have been introduced by Artzner et al. [1] and Delbaen [6]: they have suggested some the properties that must be satisfied by a risk function $\rho(X)$.

Definition 2.7 (Coherent Risk Measure [1])**.** Let M be a set of real-valued random variables and Z_1 and Z_2 be random outcomes in this set. A coherent risk measure is a function $\rho : M \to R$ that satisfies the properties stated below:

(i) *Normalization:* $\rho(0) = 0$. This means when you take no position, you have no risk.

(ii) *Monotonicity:* If $Z_1, Z_2 \in M$ and $Z_1 \leq Z_2$, then $\rho(Z_1) \geq \rho(Z_2)$. That is, if portfolio θ_2 always has better values than portfolio θ_1 under almost all scenarios then the risk of Z_2 should be less than the risk of Z_1.

(iii) *Sub-additivity:* If $Z_1, Z_2, Z_1 + Z_2 \in M$, then $\rho(Z_1 + Z_2) \leq \rho(Z_1) + \rho(Z_2)$. Namely, the risk of two portfolios together cannot be any worse than adding the two risks separately. This is the diversification principle, which is to reduce risk by investing in a variety of assets.

(iv) *Positively homogeneity:* If $Z_1 \in M$ and $h > 0$, then $\rho(hZ_1) = h\rho(Z_1)$. That is, when you double the portfolio, you double the risk.

(v) *Translation invariance:* If $a \in R$ and $Z_1 \in M$, then $\rho(Z_1 + a) = \rho(Z_1) - a$. The value a is just adding cash to the portfolio θ_1, which acts like an insurance. The risk of $Z_1 + a$ is less than the risk of Z_1, and the difference is exactly the added cash a. So a risk-measure is said to be coherent if and only if it has all these properties.

Dalbaen [6] proved that the VaR measure is not a coherent risk measure since it does not satisfy the sub-additivity property. Since the VaR risk measure does not satisfy this property, we can say that diversification, which is commonly considered as a way to reduce risk, can lead to an increase of VaR.

When we examine EL and EUL risk measures, we see that these measures are not coherent risk measures, too, since they both do not satisfy the translation-invariance property.

However, we will only debate the behaviors of a portfolio manager who wants to maximize its expected utility from terminal wealth in the presence of different shortfall risks measured by the VaR, EL and EUL risk measures, but not coherent risk measures, in this thesis.

CHAPTER 3

Portfolio Optimization Under Constraints

In this chapter, we consider the portfolio optimization problem with constraints that are Value at Risk (VaR), Expected Loss (EL), and Expected Utility Loss (EUL) with objective to maximize the expected utility of the terminal wealth. When we discuss these situations, we shall take into account that the terminal wealth X_T may fall below a given deterministic shortfall level q. Also, we will examine the impact of the different risk constraints to the behavior of the portfolio manager.

When there are no additional constraints, the portfolio manager reaches the terminal wealth $X_T^{\theta^*} = \xi_T^*$ with the normal strategy θ_t^* for $t \in [0, T]$, which we see in Theorem 2.6.

3.1 Portfolio Optimization under Value at Risk Constraint

In this section, the portfolio optimization problem is solved by using a Value at Risk constraint, and then the properties of the solution are examined.

The dynamic optimization problem of the VaR investor is solved by using the martingale representation method [4, 13], which allows the problem to be restated as the following static variational problem:

$$\begin{aligned} &\underset{\xi \in B(x)}{\text{maximize}} \ \mathbb{E}\left[u(\xi)\right] \\ &\text{subject to } P(\xi < q) \leq \varepsilon. \end{aligned} \tag{3.1}$$

We recall that the set $B(x)$ contains the budget constraint for the initial capital x. Namely,

$$B(x) = \{\xi \geq 0 : \xi \text{ is } \mathcal{F}_T - \text{measurable and } \mathbb{E}\left[H_T \xi\right] \leq x\}$$

as in Definition 2.4.

The VaR constraint causes to non-concavity for the optimization problem for which the maximization process is more complicated. The following proposition is proved in Basak and Shapiro [2], and also provided in Appendix A.1; it defines the optimal terminal wealth, assuming it exists.

Proposition 3.1 ([2]). *Time-T optimal wealth of the* VaR *investor is*

$$\xi^{\text{VaR}} = \begin{cases} I(yH_T), & \text{if } H_T < \underline{h}, \\ q, & \text{if } \underline{h} \le H_T < \overline{h}, \\ I(yH_T), & \text{if } \overline{h} \le H_T, \end{cases} \tag{3.2}$$

where I is the inverse function of u', $\underline{h} = \frac{u'(q)}{y}$, $\overline{h}$ is such that $P(H_T > \overline{h}) = \varepsilon$, and $y \ge 0$ solves $\mathbb{E}\left[H_T \xi^{\text{VaR}}\right] = x$.

The VaR *constraint ($P(\xi < q) \le \varepsilon$) is binding if, and only if, $\underline{h} < \overline{h}$.*

Basak and Shapiro [2] prove that if a terminal wealth satisfies (3.2) then it is the optimal policy for the VaR portfolio manager. As they note in their proof, to keep the focus, they do not provide general conditions for existence. However, they provide explicit numerical solutions for a variety of parameter values. Their method of proof is applicable to other problems, such as those with non-standard preferences. By the term "non-standard preferences" it means that the optimization problem is not standard because it is non-concave. Also, because the VaR constraint must hold with equality, the definition of $\overline{h}$ is deduced.

We depict in Figure 3.1 the optimal terminal wealth of a VaR portfolio manager with $\varepsilon \in (0, 1)$, a benchmark (as we call, unconstrained) investor with $\varepsilon = 1$ who does not use a risk constraint in the optimization or ignores large losses, and a portfolio insurer with $\varepsilon = 0$ who does not allow large losses but fully insures himself against large losses.

The blue curve, in Figure 3.1, plots the optimal horizon wealth of the VaR risk manager as a function of the horizon state price density H_T, the red curve is for the unconstrained investor and the black curve is for the portfolio insurer investor. For this application, one can find the necessary Matlab codes in Appendix B.1.1. Furthermore, here we note that q_2 is defined by

$$q_2 = \begin{cases} I(y\overline{h}), & \text{if } \underline{h} < \overline{h}, \\ q, & \text{otherwise.} \end{cases} \tag{3.3}$$

The VaR portfolio manager's optimal horizon wealth is divided into three distinct regions, where he displays distinct economic behaviors. In the good states, namely low price of consumption $H_T < \underline{h}$, the VaR portfolio manager behaves like a benchmark (unconstrained) investor. In the intermediate states $[\underline{h} \le H_T < \overline{h}]$, he insures himself against losses by behaving like a portfolio insurer investor, and in the bad states, namely high price of consumption $H_T > \overline{h}$ he is completely uninsured by incurring all losses. Because he is only concerned with the probability (and not the magnitude) of a loss, the VaR portfolio manager chooses to leave the worst states uninsured because they are the most expensive ones to insure against. The measure of these bad states is chosen to comply exactly with the VaR constraint. Consequently, $\overline{h}$ depends solely on ε and the distribution of H_T and is independent of the investor's preferences and initial wealth. The

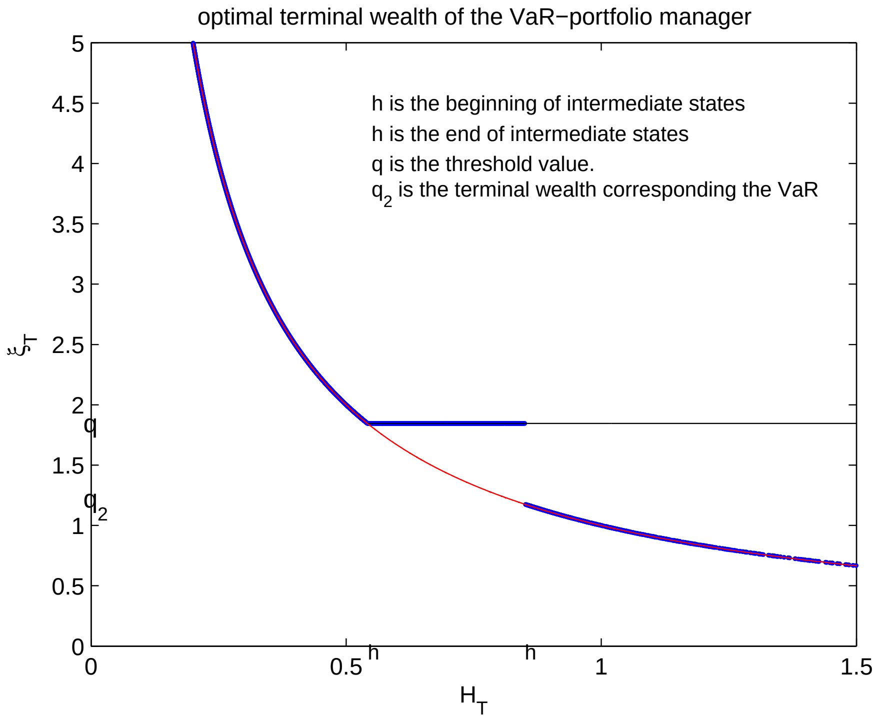

Figure 3.1: Optimal horizon wealth of the VaR risk manager

investor can be considered as one who ignores losses in this upper tail of the H_T distribution, where the consumption is the most costly.

When we take into account Figure 3.1, we can examine the dependence of the solution on the parameters q and ε. If the threshold value q is increased, more states need to be insured against, and the intermediate region grows at the expense of the good states region. Accordingly, the wealth in both good and bad regions must decrease to meet the bigger threshold value q in the intermediate region. When ε increases, namely, when the investor is allowed to make a loss with higher probability, the intermediate, insured region can shrink, and the good and bad regions both can grow. The investor's horizon wealth can increase in both the good and bad states because he is not required to insure against losses in a large state. The solution reveals that when a large loss occurs, it may be an even larger loss under the VaR constraint, and hence more likely to cause to credit problems. Basak and Shapiro show this situation in [2] and presented by the following proposition whose proof can also be found in Appendix A.2

Proposition 3.2 ([2]). *Assume* $u(\xi) = \frac{\xi^{1-\gamma}}{1-\gamma}$, $\gamma > 0$. *For a given terminal wealth* ξ_T, *define the following two measures of loss:* $L_1(\xi) = \mathbb{E}\left[(q_2 - \xi_T)\mathbb{1}_{\{\xi_T \leq q_2\}}\right]$ *and* $L_2(\xi) = \mathbb{E}\left[\frac{H_T}{H_0}(q_2 - \xi_T)\mathbb{1}_{\{\xi_T \leq q_2\}}\right]$. *Then,* *(i)* $L_1(\xi^{\text{VaR}}) \geq L_1(\xi^*)$, *and*

(ii) $L_2(\xi^{\text{VaR}}) \geq L_2(\xi^*)$,

where ξ^* *stands for the solution of the unconstrained (benchmark) problem.*

Proposition 3.2 shows explicitly that under the VaR constraint the expected extreme losses are higher than those which are incurred by an investor who does not use the VaR constraint ($P(\xi < q) \leq \varepsilon$). The bad states, which are the states of large losses, are considered: $L_1(\xi)$ measures the expected future value of a loss, when there is a large loss, while $L_2(\xi)$ measures its present value.

Although the aim of using VaR approach in the optimization is to prevent large and frequent losses that may cause economic investors out of business, under the VaR constraint losses are not frequent, however, the largest losses are more severe than without the VaR constraint.

In his study, Gabih [7] presents explicit expressions for the VaR portfolio manager's optimal wealth and portfolio strategies before the horizon in the following proposition, proof of which can also be found in [7].

Proposition 3.3 ([7])**.** *Let the assumptions of Proposition 3.1 be fulfilled, and let u be the utility function given as in* (2.5)*. Then,*

(i) The VaR*-optimal wealth at time $t < T$ before the horizon is given by*

$$X_t^{\text{VaR}} = F(H_t, t), \tag{3.4}$$

with

$$\begin{aligned} F(z,t) &= \frac{e^{\Gamma(t)}}{(yz)^{\frac{1}{\gamma}}} - \left[\frac{e^{\Gamma(t)}}{(yz)^{\frac{1}{\gamma}}}\Phi(-d_1(\underline{h},z,t)) - qe^{-r(T-t)}\Phi(-d_2(\underline{h},z,t))\right] \\ &+ \left[\frac{e^{\Gamma(t)}}{(yz)^{\frac{1}{\gamma}}}\Phi(-d_1(\overline{h},z,t)) - qe^{-r(T-t)}\Phi(-d_2(\overline{h},z,t))\right], \end{aligned}$$

for $z > 0$. Here, Φ is the standard-normal distribution function, y, $\underline{h}$ and $\overline{h}$ are as in Proposition 3.1. Furthermore,

$$\begin{aligned} \Gamma(t) &= \frac{1-\gamma}{\gamma}\left(r + \frac{\kappa^2}{2\gamma}\right)(T-t), \\ d_1(u,z,t) &= \frac{\ln\frac{u}{z} + \left(r - \frac{\kappa^2}{2}\right)(T-t)}{\kappa\sqrt{T-t}}, \\ d_2(u,z,t) &= d_1(u,z,t) + \frac{1}{\gamma}\kappa\sqrt{T-t}. \end{aligned}$$

(ii) The VaR*-optimal fraction of wealth invested in stock at time $t < T$ before the horizon is*

$$\theta_t^{\mathrm{VaR}} = \theta^N \Theta(H_t, t),$$

where

$$\begin{aligned}\Theta(z,t) &= 1 - \frac{qe^{-r(T-t)}}{F(z,t)}\left[\Phi(-d_2(\underline{h},z,t)) - \Phi(-d_2(\overline{h},z,t))\right]\\ &+ \frac{\gamma}{\kappa\sqrt{T-t}F(z,t)}\frac{e^{\Gamma(t)}}{(yz)^{\frac{1}{\gamma}}}\left[\varphi(d_1(\underline{h},z,t)) - \varphi(d_1(\overline{h},z,t))\right]\\ &- \frac{\gamma q e^{-r(T-t)}}{\kappa\sqrt{T-t}F(z,t)}\left[\varphi(d_2(\overline{h},z,t)) - \varphi(d_2(\underline{h},z,t))\right],\end{aligned}$$

for $z > 0$. Here, $\theta^N = \frac{\kappa}{\gamma\sigma} = \frac{\mu - r}{\gamma\sigma^2}$ denotes the normal strategy, $\Theta(H_t, t)$ is the exposure to risky assets relative to the normal (unconstrained) strategy and φ is the density function of the standard normal distribution.

Proof. ([7])

(i) Using Equations (2.3) and (2.4), Itô's lemma implies that the process $HX^{\mathrm{VaR}} = (H_t X_t^{\mathrm{VaR}})_{t\in[0,T]}$ is an $\mathcal{F}_t$-martingale:

$$\begin{aligned}X_t^{\mathrm{VaR}} &= \mathbb{E}\left[\frac{H_T}{H_t}\xi^{\mathrm{VaR}}\,\middle|\,\mathcal{F}_t\right]\\ &= \mathbb{E}\left[\frac{H_T}{H_t}I(yH_T)\left(\mathbb{1}_{\{H_T<\underline{h}\}} + \mathbb{1}_{\{\overline{h}\le H_T\}}\right)\,\middle|\,\mathcal{F}_t\right]\\ &+ \mathbb{E}\left[\frac{H_T}{H_t}q\mathbb{1}_{\{\underline{h}\le H_T<\overline{h}\}}\,\middle|\,\mathcal{F}_t\right].\end{aligned}$$

These conditional expectations are computed by applying Markov's property of solution stochastic differential equation and using the fact that $\ln H_T$ is normally distributed with mean $\ln H_t - \left(r + \frac{\kappa^2}{2}\right)(T-t)$ and variance $\kappa^2(T-t)$.

(ii) From Equation (3.4) it follows $X_t^{\mathrm{VaR}} = F(H_t, t)$. The process $H = (H_t)_{t\in[0,T]}$ satisfies the SDE (2.3). Applying Itô's lemma to the function $F(H_t, t)$, we find that $X^{\mathrm{VaR}} = (X_t^{\mathrm{VaR}})_{t\in[0,T]}$ satisfies the SDE

$$\begin{aligned}dX_t^{\mathrm{VaR}} &= \left[F_t(H_t,t) - rF_z(H_t,t)H_t + \tfrac{\kappa^2}{2}F_{zz}(H_t,t)H_t^2\right]dt\\ &- F_z(H_t,t)H_t\kappa^T dW_t,\end{aligned}$$

where F_z, F_{zz} and F_t denote the partial derivatives of $F(z,t)$ with respect to z and t, respectively. Equating coefficients of dW_t in the above equation and (2.4) leads to the following equality:

$$\theta_t^{\mathrm{VaR}} = -\sigma^{-1}\kappa\frac{F_z(H_t,t)H_t}{F(H_t,t)} = -\theta^N\gamma\frac{F_z(H_t,t)H_t}{F(H_t,t)}. \tag{3.5}$$

On the other hand, we compute the derivative F_z to get

$$\begin{aligned}
F_z(z,t) &= \frac{1}{\gamma z}\left[-F(z,t) + qe^{-r(T-t)}(\Phi(-d_2(\underline{h},z)) - \Phi(-d_2(\overline{h},z)))\right] \\
&- \frac{e^{\Gamma(t)}}{(yz)^{\frac{1}{\gamma}}\kappa\sqrt{T-t}z}\left[\varphi(d_1(\underline{h},z)) - \varphi(d_1(\overline{h},z))\right] \\
&+ \frac{qe^{-r(T-t)}}{\kappa\sqrt{T-t}z}\left[\varphi(d_2(\overline{h},z)) - \varphi(d_2(\underline{h},z))\right].
\end{aligned}$$

Therefore, using this in (3.5), we get the final form of the optimal strategies before the horizon, and the proof is completed. □

3.2 Portfolio Optimization under Expected Loss Constraint

In this section, we consider the Expected Loss (EL) strategy as an alternative to the Value at Risk (VaR) strategy. We then solve the optimization problem of an EL portfolio manager who wants to limit his expected loss and analyze the properties of the solution.

The portfolio manager who uses Value at Risk (VaR) constraint does not concern with the magnitude of a loss and is just interested in controlling the probability of the loss. However, if one wants to control the magnitude of losses, he should control (all or some of the) moments of the loss distribution. Therefore, we now focus on controlling the first moment and examine how one can remedy the shortcomings of VaR constraint. In this case, the investor defines his strategy as follows:

$$\mathrm{EL}(Z) = \mathbb{E}\left[Z^-\right] = \mathbb{E}\left[(X_T - q)^-\right] \leq \varepsilon, \tag{3.6}$$

where $Z = X_T - q$ and ε is a given bound for the Expected Loss. This strategy will be called EL strategy. Thus, the aim is to solve the optimization problem constrained by (3.6). Using the martingale representation approach the dynamic optimization problem of the EL-portfolio manager can be restated as the following static problem

$$\begin{aligned}
&\underset{\xi\in B(x)}{\text{maximize}}\ \mathbb{E}\left[u(\xi)\right] \\
&\text{subject to } \mathbb{E}\left[(\xi - q)^-\right] \leq \varepsilon.
\end{aligned} \tag{3.7}$$

The EL-constraint (3.6) can be interpreted as a risk measure of time-T losses. This measure satisfies the sub-additivity, positive homogeneity, and monotonicity axioms (but not the translation-invariance axiom) defined by Artzner et al. [1]. Hence EL risk measure can be thought that it has an advantage about this issue according to the VaR measure of risk: because the VaR strategy fails to display sub-additivity when combining the risk of two or more portfolios, the VaR of the whole portfolio may be greater than the sum of the VaRs of the individuals.

A. Gabih, R. Wunderlich [9] characterize the optimal terminal wealth ξ^{EL} in the presence of the EL-constraint (3.6) in the following proposition whose proof is based on the following lemma. The proof of Lemma 3.4 is presented in Appendix A.3.

Lemma 3.4. *Let $z, y_1, y_2, q > 0$. Then the solution of the optimization problem*

$$\max_{x>0}\{u(x) - y_1 zx - y_2(x-q)^-\}$$

is $x^* = \xi^*(z)$.

Now, the following proposition, Proposition 3.5, states the optimal solution of the static variational problem, concerning the EL constraint.

Proposition 3.5 ([9])**.** *The* EL*-optimal terminal wealth is*

$$\xi^{\mathrm{EL}} = \begin{cases} I(y_1 H_T), & \text{if } H_T < \underline{h}, \\ q, & \text{if } \underline{h} \le H_T < \overline{h}, \\ I(y_1 H_T - y_2), & \text{if } \overline{h} \le H_T, \end{cases} \tag{3.8}$$

where $\underline{h} = \underline{h}(y_1) = \frac{u'(q)}{y_1}, \overline{h} = \overline{h}(y_1, y_2) = \frac{u'(q)+y_2}{y_1}$ and $y_1, y_2 > 0$ solve the system of equations,

$$\begin{aligned} \mathbb{E}\left[H_T \xi^{\mathrm{EL}}(T; y_1, y_2)\right] &= x, \\ \mathbb{E}\left[(\xi^{\mathrm{EL}}(T; y_1, y_2) - q)^-\right] &= \varepsilon. \end{aligned}$$

Moreover, the EL*-constraint* (3.6) *is binding, if and only if, $\underline{h} < \overline{h}$.*

Proof. ([9]) In order to solve the optimization problem under EL-constraint, the common convex-duality approach is adapted by introducing the convex-conjugate of the utility function u with an additional term capturing the EL-constraint as it is shown in Lemma 3.4. Thence, applying the lemma point-wise for all $z = H_T$ it follows that $\xi_T^*(H_T)$ is the solution of the maximization problem

$$\max_{\xi>0}\{u(\xi) - y_1 H_T \xi - y_2(\xi - q)^-\}.$$

Obviously, ξ_T^* is $\mathcal{F}_T$-measurable and if y_1, y_2 are chosen as solutions of the system of equations given in the proposition then it follows $\xi_T^* = \xi_T^{\mathrm{EL}} = \xi^{\mathrm{EL}}$.

To complete the proof, let η be any admissible solution satisfying the static budget constraint and the EL-constraint (3.6). We have

$$\begin{aligned} \mathbb{E}\left[u(\xi_T^{\mathrm{EL}})\right] - \mathbb{E}\left[u(\eta)\right] &= \mathbb{E}\left[u(\xi_T^{\mathrm{EL}})\right] - \mathbb{E}\left[u(\eta)\right] - y_1 x + y_1 x - y_2\varepsilon + y_2\varepsilon \\ &\ge \mathbb{E}\left[u(\xi_T^{\mathrm{EL}})\right] - \mathbb{E}\left[y_1 H_T \xi_T^{\mathrm{EL}}\right] - y_2\mathbb{E}\left[(\xi_T^{\mathrm{EL}} - q)^-\right] \\ &\quad - \mathbb{E}\left[u(\eta)\right] + \mathbb{E}\left[y_1 H_T \eta\right] + y_2\mathbb{E}\left[(\eta - q)^-\right] \\ &\ge 0, \end{aligned}$$

where the first inequality follows from the static budget constraint and the constraint for the risk holding with equality for ξ_T^{EL}, while holding with inequality for η. The last inequality is a consequence of the above lemma. Hence, we obtain that ξ_T^{EL} is optimal. □

With the following remark of Gabih (2005) [7], the case of how the EL optimal terminal wealth depends on y_2 is explained:

Remark 3.6. *For $y_2 \downarrow 0$, the situation of $\xi^{\mathrm{EL}} \to I(y_1 H_T)$ is observed. This limit corresponds to $\varepsilon \uparrow \varepsilon_{\max}$ and the results for the unconstrained problem are derived if $y_2 = 0$ and $\xi^{\mathrm{EL}}(y_1, 0) = I(y_1 H_T)$ are set.*

Figure 3.2 depicts the optimal terminal wealth of an EL-portfolio manager [$\varepsilon \in (0, \infty)$], a benchmark (unconstrained) investor ($\varepsilon = \infty$), and a portfolio insurer investor ($\varepsilon = 0$). The blue curve plots the optimal horizon wealth of the EL risk manager as a function of the horizon state price density H_T, the red curve is for the unconstrained investor and the black curve is for the portfolio insurer investor. Implementation in MATLAB is presented in Appendix B.2.1.

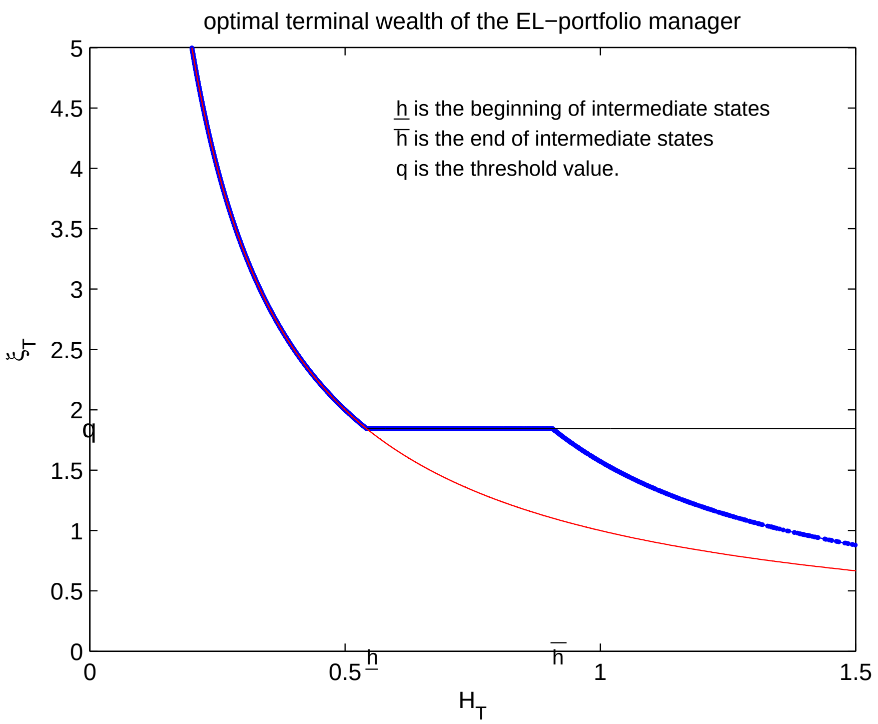

Figure 3.2: Optimal horizon wealth of the EL risk manager

In Figure 3.2, we see that the EL portfolio manager's optimal horizon wealth is divided into three distinct regions, where he exhibits distinct economic behaviors: in the so-called "good states" (for low H_T values), the EL portfolio manager behaves like a benchmark (the unconstrained) investor, while in the "intermediate states" (for $\underline{h} \leq H_T < \overline{h}$) the investor fully insures himself against losses by behaving like a portfolio insurer investor (PI), and in the "bad states" (for high H_T values) the investor partially insures himself by incurring partial losses in contrast to the VaR portfolio manager. Here, we see in the bad-states region, $\xi_T^* <$

$\xi_T^{\mathrm{EL}} < \xi_T^{PI}$, where ξ_T^* stands for the solution of the benchmark (unconstrained) problem. This is constituted in contrast to the findings in the VaR case.

Although in some states he wants to settle for a wealth lower than q, he does so while endogenously choosing a higher ξ_T^{EL} than ξ_T^*. The portfolio manager chooses the bad states in which he maintains a loss, because these are the most expensive states to insure against losses, but maintains some level of insurance. Since insuring a terminal wealth at q level is too costly, he sets for less, but enough to comply with the EL constraint. Unlike $\overline{h}$ for VaR strategy, $\overline{h}$ for EL strategy depends on the investor's preferences and the given initial wealth. Another distinction with VaR strategy is that the terminal wealth policy under EL strategy is continuous across the states of the world.

Gabih (2005) [7] presents the explicit expressions for the EL-optimal wealth and portfolio strategy before the horizon via the following proposition. To make the text self-contained, we also give the proof in Appendix A.4.

Proposition 3.7 ([7]). *Let the assumptions of Proposition 3.5 be fulfilled, and let u be the utility function given in* (2.5). *Then,*

(i) The EL*-optimal wealth at time $t < T$ is given by*

$$X_t^{\mathrm{EL}} = F(H_t, t) \tag{3.9}$$

with

$$\begin{aligned} F(z,t) &= \frac{e^{\Gamma(t)}}{(y_1 z)^{\frac{1}{\gamma}}}\left[1 - \Phi(-d_1(\underline{h}, z))\right] \\ &+ q e^{-r(T-t)}\left[\Phi(-d_2(\underline{h}, z)) - \Phi(-d_2(\overline{h}, z))\right] \\ &+ G(z, \overline{h}), \end{aligned}$$

for $z > 0$, where y_1, y_2 are as defined in Proposition 3.5; $\Gamma(t), d_1, d_2$ are as in Proposition 3.3; and

$$\begin{aligned} \underline{h} &= \frac{1}{y_1 q^\gamma} \text{ and } \overline{h} = \frac{q^{-\gamma} + y_2}{y_1}, \\ G(z, \overline{h}) &= \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_{-\infty}^{c_2(\overline{h}, z)} \frac{e^{-\frac{1}{2}(u-b)^2}}{(y_1 t e^{a+bu} - y_2)^{\frac{1}{\gamma}}} du, \\ c_2(\overline{h}, z) &= \frac{1}{b}\left(\ln(\frac{\overline{h}}{z}\right) - a), \\ a &= -\left(r + \frac{\kappa^2}{2}\right)(T - t) \text{ and} \\ b &= -\kappa\sqrt{T - t}. \end{aligned}$$

(ii) The EL*-optimal fraction of wealth invested in stock at time $t < T$ is*

$$\theta_t^{\mathrm{EL}} = \theta^N \Theta(H_t, t),$$

where

$$
\begin{aligned}
\Theta(z,t) &= \frac{1}{F(z,t)} \frac{e^{\Gamma(t)}}{(y_1 z)^{\frac{1}{\gamma}}} \left[1 - \Phi(-d_1(\underline{h}, z)) + \frac{\gamma}{\kappa\sqrt{T-t}} \varphi(d_1(\underline{h}, z)) \right] \\
&- \frac{q\gamma e^{-r(T-t)}}{F(z,t)\kappa\sqrt{T-t}} \varphi(d_2(\underline{h}, z)) \\
&+ \frac{y_1 z e^{(\kappa^2-2r)(T-t)}}{F(z,t)} \psi_0 \left(c_2(\overline{h}, z), b, y_1 z e^{a}, y_2, 2b, 1, 1+\frac{1}{\gamma} \right),
\end{aligned}
$$

for $z > 0$ *and*

$$
\psi_0(\alpha, \beta, c_1, c_2, m, s, \delta) = \frac{1}{\sqrt{2\pi} s} \int_{-\infty}^{\alpha} \frac{exp(-\frac{(u-m)^2}{2s^2})}{(c_1 e^{\beta u} - c_2)^{\delta}} du.
$$

3.3 Portfolio Optimization under Expected Utility Loss Constraint

In this section, we will be interested in the portfolio optimization problem where the portfolio manager is faced with a risk of loosing expected utility. Here, this risk is measured by a constraint of the type

$$
\mathrm{EUL}(Z) = \mathbb{E}\left[Z^-\right] = \mathbb{E}\left[(u(X_T) - u(q))^-\right] \le \varepsilon, \tag{3.10}
$$

where ε is a given bound for the Expected Utility Loss, and $Z = u(X_T) - u(q)$. This risk constraint leads to more explicit calculations for the optimal strategy we are looking for. Also, it allows to the constrained static problem to be solved for a large class of utility functions. Again, we keep the shortfall level or threshold value q to be constant.

The dynamic optimization problem of the EUL-portfolio manager can be restated as the following static variational problem

$$
\begin{aligned}
&\underset{\xi \in B(x)}{\text{maximize}} \ \mathbb{E}\left[u(\xi)\right] \\
&\text{subject to } \mathbb{E}\left[(u(\xi) - u(q))^-\right] \le \varepsilon.
\end{aligned} \tag{3.11}
$$

Gabih (2005) [7] defines the EUL-optimal terminal wealth which is denoted as ξ_T^{EUL} in the following proposition, the proof of which can also be found in Appendix A.5.

Proposition 3.8 ([7]). *The* EUL-*optimal terminal wealth is*

$$
\xi^{\mathrm{EUL}} = \begin{cases} I(y_1 H_T), & \text{if } H_T < \underline{h}, \\ q, & \text{if } \underline{h} \le H_T < \overline{h}, \\ I(\frac{y_1}{1+y_2} H_T), & \text{if } \overline{h} \le H_T, \end{cases}
$$

for $H_T > 0$, where

$$\begin{aligned}\underline{h} &= \underline{h}(y_1) = \frac{1}{y_1}u'(q),\\ \overline{h} &= \overline{h}(y_1, y_2) = \frac{1+y_2}{y_1}u'(q) = (1+y_2)\underline{h},\end{aligned}$$

and y_1, y_2 satisfy the system of equations

$$\begin{aligned}\mathbb{E}\left[H_T\xi^{\text{EUL}}(T; y_1, y_2)\right] &= x,\\ \mathbb{E}\left[(u(\xi^{\text{EUL}}(T; y_1, y_2)) - u(q))^-\right] &= \varepsilon.\end{aligned}$$

With the following remark, Gabih (2005) [7] explains the case of how the EUL optimal terminal wealth depends on y_2 as follows:

Remark 3.9. *For $y_2 \downarrow 0$, the situation of $\xi^{\text{EUL}} \to I(y_1 H_T)$ is observed. This limit corresponds to $\varepsilon \uparrow \varepsilon_{\max}$ and the results for the unconstrained problem are derived if $y_2 = 0$ and $\xi^{\text{EUL}}(y_1, 0) = I(y_1 H_T)$ are set.*

We depict the optimal terminal wealth of a EUL portfolio manager with $\varepsilon \in (0, \infty)$, a benchmark (the unconstrained) investor ($\varepsilon = \infty$), and a portfolio insurer investor with $\varepsilon = 0$ in Figure 3.3. The blue curve plots the optimal horizon wealth of the EUL risk manager as a function of the horizon state price density H_T, the red curve is for the unconstrained investor and the black curve is for the portfolio insurer investor. MATLAB algorithms presented in Appendix B.3.1 helps us to plot these densities.

The EUL portfolio manager's optimal horizon wealth is divided into three distinct regions, as before, where he shows distinct economic behaviors. In the good states, namely low price of consumption H_T, the EUL portfolio manager behaves like a benchmark investor. In the intermediate states, where $\underline{h} \leq H_T < \overline{h}$, he fully insures himself against utility losses, and in the bad states, namely high price of consumption H_T he partially insures himself against utility losses. That is, EUL portfolio manager behaves like an EL portfolio manager in the case of insurance according to each states. He just considers about utility losses contrary to the EL portfolio manager who is interested in just losses. That is why, the EUL portfolio manager chooses the cases of insurance, like the one above, may be based on the reasons presented for EL portfolio manager. However, here the rules of EUL risk constraint are valid. The measure of bad states is chosen to comply exactly with the EUL constraint. Here $\overline{h}$ for EUL strategy depends on the investor's preferences and initial wealth. As before, another distinction with VaR strategy is that the terminal wealth policy under EUL strategy is continuous across the states of the world.

Gabih (2005) [7] characterizes the explicit expressions for the EUL-optimal wealth and portfolio strategies before the horizon in the following proposition. For the proof we refer to [7] or Appendix A.6.

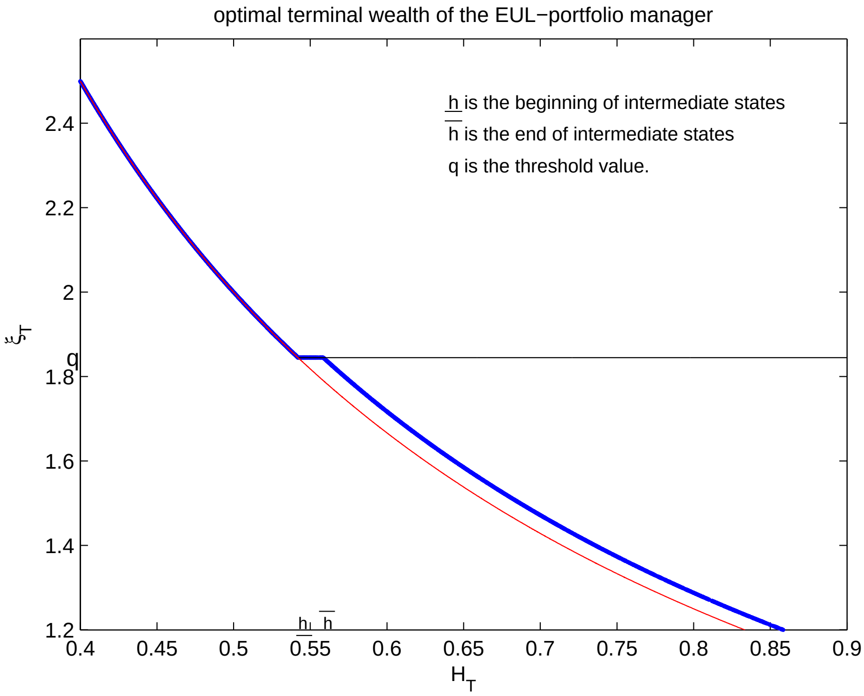

Figure 3.3: Optimal horizon wealth of the EUL risk manager

Proposition 3.10 ([7]). *Let the assumptions of Proposition 3.8 be fulfilled, and let u be the utility function given in* (2.5)*. Then,*

(i) The EUL*-optimal wealth at time $t < T$ before the horizon is given by*

$$X_t^{\mathrm{EUL}} = F(H_t, t), \tag{3.12}$$

where

$$\begin{aligned} F(z,t) &= \frac{e^{\Gamma(t)}}{(y_1 z)^{\frac{1}{\gamma}}} - \left[\frac{e^{\Gamma(t)}}{(y_1 z)^{\frac{1}{\gamma}}}\Phi(-d_1(\underline{h},z,t)) - qe^{-r(T-t)}\Phi(-d_2(\underline{h},z,t))\right] \\ &+ \left[\frac{(1+y_2)^{\frac{1}{\gamma}}e^{\Gamma(t)}}{(y_1 z)^{\frac{1}{\gamma}}}\Phi(-d_1(\overline{h},z,t)) - qe^{-r(T-t)}\Phi(-d_2(\overline{h},z,t))\right], \end{aligned}$$

for $z > 0$, where y_1, y_2 and $\underline{h}, \overline{h}$ are as defined in Proposition 3.8; and

$$\begin{aligned} \Gamma(t) &= \frac{1-\gamma}{\gamma}\left(r + \frac{\kappa^2}{2\gamma}\right)(T-t), \\ d_2(u,z,t) &= \frac{\ln\frac{u}{z} + \left(r - \frac{\kappa^2}{2}\right)(T-t)}{\kappa\sqrt{T-t}}, \\ d_1(u,z,t) &= d_2(u,z,t) + \frac{1}{\gamma}\kappa\sqrt{T-t}. \end{aligned}$$

(ii) The EUL*-optimal fraction of wealth invested in stock at time* $t < T$ *is*

$$\theta_t^{\mathrm{EUL}} = \theta^N \Theta(H_t, t),$$

where

$$\Theta(z,t) = 1 - \frac{qe^{-r(T-t)}}{F(z,t)} \left[\Phi(-d_2(\underline{h}, z, t)) - \Phi(-d_2(\overline{h}, z, t))\right]$$

for $z > 0$.

Gabih [7] also presented the two special properties of the function $\Theta(z,t)$ appearing in the definition of the above representation of the EUL-optimal strategy:

Proposition 3.11 ([7])**.** *Let the assumptions of Proposition 3.8 be fulfilled, and let* u *be the utility function given in* (2.5)*. Then, for the function* $\Theta(z,t)$*, defined in Proposition 3.10, we have,*

(i) $0 < \Theta(z,t) < 1$ *for all* $z > 0$ *and* $t \in [0,T)$*,*

(ii) $\lim_{t \to T} \Theta(z,t) = \begin{cases} 1, & \text{if } z < \underline{h} \text{ or } z > \overline{h}, \\ 0, & \text{if } \underline{h} < z < \overline{h}, \\ \frac{1}{2}, & \text{if } z = \underline{h}, \overline{h} \end{cases}$

Proof. ([7]) Using (3.12) the function $F(z,t)$ can be written as

$$F(z,t) = F_1(z,t) + F_2(z,t),$$

where

$$F_1(z,t) = \frac{e^{\Gamma(t)}}{(y_1 z)^{\frac{1}{\gamma}}} \left[1 - \Phi(-d_1(\underline{h}, z, t)) + (1+y_2)^{\frac{1}{\gamma}} \Phi(-d_1(\overline{h}, z, t))\right]$$

and

$$F_2(z,t) = qe^{-r(T-t)} \left[\Phi(-d_2(\underline{h}, z, t)) - \Phi(-d_2(\overline{h}, z, t))\right] \text{ for } z > 0.$$

On the other hand, from Proposition 3.10, it follows that

$$\Theta(z,t) = 1 - \frac{F_2(z,t)}{F(z,t)} = 1 - \frac{F_2(z,t)}{F_1(z,t) + F_2(z,t)}, \tag{3.13}$$

where the terms $F_1(z,t)$ and $F_2(z,t)$ are strictly positive due to some situations. We can explain them so that $y_2 > 0$ implies $\underline{h} < \overline{h}$ and the functions $d_{\frac{1}{2}}(u,.,.)$,i.e., $(d_1(u,.,.)$ and $d_2(u,.,.))$ are strictly increasing with respect to u. Also, the standard-normal distribution function,Φ, is strictly increasing, too. Therefore we conclude that $0 < \Theta(z,t) < 1$, and hence, the assertion (i) follows.

Table 3.1: The limits as $t \to T$ of the functions appearing in Proposition 3.11: subscript "$\frac{1}{2}$" stands for subscripts "1" and "2".

	$z < \underline{h}$	$z = \underline{h}$	$\underline{h} < z < \overline{h}$	$z = \overline{h}$	$z > \overline{h}$
$d_{\frac{1}{2}}(\underline{h}, z, t)$	$+\infty$	0	$-\infty$	$-\infty$	$-\infty$
$d_{\frac{1}{2}}(\overline{h}, z, t)$	$+\infty$	$+\infty$	$+\infty$	0	$-\infty$
$\Phi(-d_{\frac{1}{2}}(\underline{h}, z, t))$	0	$\frac{1}{2}$	1	1	1
$\Phi(-d_{\frac{1}{2}}(\overline{h}, z, t))$	0	0	0	$\frac{1}{2}$	1
$F_1(z, t)$	$\frac{1}{(y_1 z)^{\frac{1}{\gamma}}}$	$\frac{q}{2}$	0	$\frac{q}{2}$	$\left(\frac{1+y_2}{y_1 z}\right)^{\frac{1}{\gamma}}$
$F_2(z, t)$	0	$\frac{q}{2}$	q	$\frac{q}{2}$	0

For the proof of the second assertion, (ii), we consider the limits of the functions as t tends to T: they are presented in Table 3.1. Here, the relations $\left(\frac{1}{y_1 \underline{h}}\right)^{\frac{1}{\gamma}} = q$ and $\left(\frac{1}{y_1 \overline{h}}\right)^{\frac{1}{\gamma}} = \frac{q}{(1+y_2)^{\frac{1}{\gamma}}}$ have been used. Substituting these limits into (3.13) yields the assertion (ii). $\square$

Based on Proposition 3.11, Gabih [7] makes the following statement about the boundaries of $\Theta(z, t)$:

Remark 3.12. *The second assertion of Proposition 3.11 shows that the lower and upper bounds for $\Theta(z, t)$ given in the first assertion can not be improved. The given bounds are reached (depending on the value of z) asymptotically if time t approaches the horizon T.*

From the proposition we can deduce that the EUL-optimal fraction of wealth θ_T^{EUL} invested in the stock at the horizon is equal to the normal (unconstrained) strategy θ^* in the bad and good states, and equal to zero in the intermediate states of the market, which are described by H_T. Before the horizon T, the optimal EUL strategy, θ_t^{EUL}, is always strictly positive and never exceeds the normal (unconstrained) strategy θ^*.

CHAPTER 4

Numerical Results

In this chapter, we wish to examine the findings of the previous sections with examples of the portfolio optimization under Value at Risk (VaR), Expected Loss (EL), and Expected Utility Loss (EUL) constraints. For the sake of comparison, we also give the corresponding behaviors of the unconstrained investor, and investors who invest in pure stock and pure bond portfolio, separately. First, we examine the probability density functions of the optimal terminal wealth of each of the above investors, and next, the optimal portfolio strategies.

We use Table 4.1 which shows the parameters for the portfolio optimization problem and the underlying Black-Scholes model of the financial market. Our aim is to maximize the expected logarithmic utility ($\gamma = 1$) of the terminal wealth ξ_T of the portfolio with the horizon $T = 15$ years in this example. The shortfall level or threshold value q is chosen to be 75% of the terminal wealth of a pure bond portfolio,namely, $q = 0.75xe^{rT}$, where x is the initial wealth. In the optimization with the VaR constraint, we bound the shortfall probability $P(\xi_T < q)$ by $\varepsilon = 0.06$. In the optimization with the Expected Loss constraint, we bound the expected loss $\mathrm{EL}(\xi_T < q)$ by $\varepsilon = 0.06$ and bound the expected utility loss $\mathrm{EUL}(u(\xi_T) - u(q))$ by $\varepsilon = 0.06$ in the optimization with the Expected Utility Loss.

Table 4.1: Parameters of the optimization problems

stock	$\mu = 9\%, \sigma = 20\%$
bond	$r = 6\%$
horizon	$T = 15$
initial wealth	$x = 1$
utility function	$u(x) = \ln x \ (\gamma = 1)$
shortfall level	$q = 0.75xe^{rT} = 1.8447$
shortfall probability (VaR)	$P(\xi_T < q) < \varepsilon = 0.06$
EL constraint	$\mathrm{EL}(\xi_T - q) \leq \varepsilon = 0.06$
EUL constraint	$\mathrm{EUL}(u(\xi_T) - u(q)) \leq \varepsilon = 0.06$

We consider the solutions of the static problems which leads to the optimal terminal wealths $\xi_T^{\mathrm{VaR}}, \xi_T^{\mathrm{EL}}$ and ξ_T^{EUL}. At first, we show the probability density functions of these random variables, belonging to VaR strategy, EL strategy, EUL strategy, unconstrained strategy, pure stock strategy and pure bond strategy, separately. On the horizontal axes of depicted figures, the expected terminal wealths $\mathbb{E}[\xi_T]$ for the considered portfolios are marked. Next, we examine the solution of the representation problem, that is, we depict the optimal strategy θ_t for each type of investors that we deal with.

For all necessary calculations and plotting the graphics, Matlab software is used. Appendix B consists of some of the necessary Matlab programs for the risk measures used here in this thesis.

4.1 Probability Density Function of VaR Based Optimal Terminal Wealth and The VaR-Optimal Wealth and Strategy at Time $t < T$ before the Horizon

In this section, firstly we examine the probability density function of the optimal terminal wealth which the portfolio manager manages by using Value at Risk (VaR) strategy. Also, for the sake of comparison we give the probability density functions of the terminal wealth of portfolios managed by the pure bond strategy, whose fraction of wealth invested in stock is 0, the pure stock strategy, whose fraction of wealth invested in stock is 1, and the optimal strategy of the unconstrained (benchmark) problem, whose fraction of wealth invested in stock is $\theta_t = \theta^* = \frac{\mu - r}{\gamma\sigma^2} = 0.75$.

Figure 4.1 depicts the shape of the probability density functions of the terminal wealths in the VaR, pure stock, benchmark(unconstrained) and pure bond solutions. The blue curve plots the shape of the probability density function of the VaR portfolio manager's optimal horizon wealth. The black curve is for the pure stock portfolio, the red curve is for the unconstrained portfolio and the line which is found on the "b" mark is for the pure bond portfolio. Also, the expected terminal wealths $\mathbb{E}[\xi_T]$ for the considered portfolios are marked on the horizontal axes. For the implementation that produces the graphs in Figure 4.1, necessary Matlab codes and algorithms are presented in Appendix B.1.2.

In the density plot, in the case of the pure bond portfolio strategy, denoted by $\xi_T^{\theta^0}$, there is a probability mass built up in the single point xe^{rT}. The probability of the terminal wealth of the pure stock portfolio strategy, denoted by $\xi_T^{\theta^1}$, and the probability of the terminal wealth of the unconstrained (benchmark) portfolio strategy $\xi_T^{\theta^*}$ are absolutely continuous. When we compute the expected values of terminal wealth of above strategies and also expected value of terminal wealth of

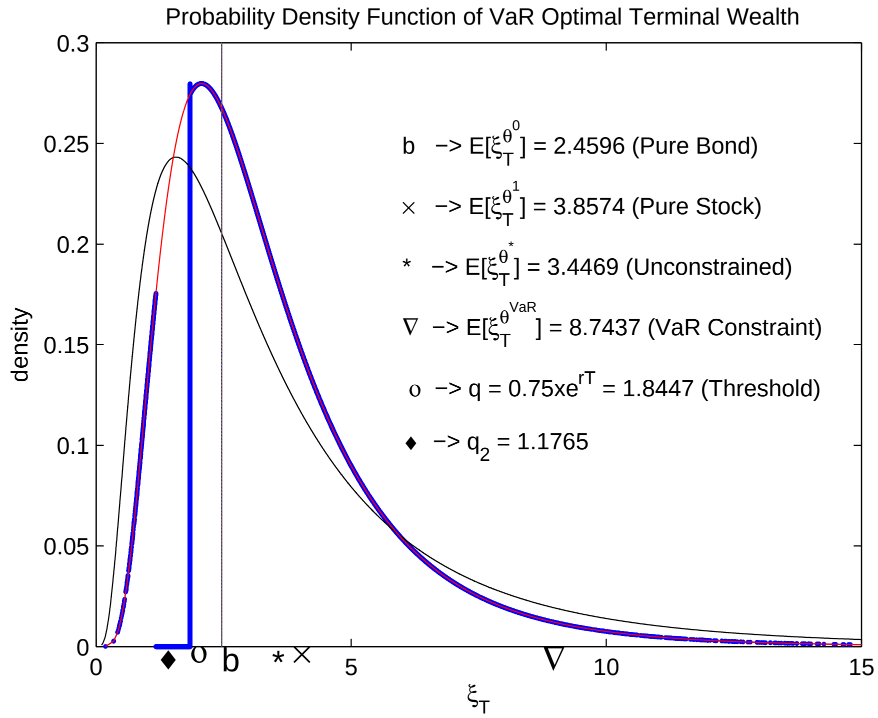

Figure 4.1: Probability density of the optimal horizon wealth belonging to the VaR portfolio manager

VaR strategy $\xi_T^{\theta^{\text{VaR}}}$, we see

$$\begin{aligned} \mathbb{E}\left[\xi_T^{\theta^*}\right] &= 3.4469, \\ \mathbb{E}\left[\xi_T^{\theta^0}\right] = e^{rT} &= 2.4596, \\ \mathbb{E}\left[\xi_T^{\theta^{\text{VaR}}}\right] &= 8.7437 \text{ and} \\ \mathbb{E}\left[\xi_T^{\theta^1}\right] = e^{\mu T} &= 3.8574. \end{aligned}$$

This shows that the following comparison is true:

$$\mathbb{E}\left[\xi_T^{\theta^0}\right] < \mathbb{E}\left[\xi_T^{\theta^*}\right] < \mathbb{E}\left[\xi_T^{\theta^1}\right] < \mathbb{E}\left[\xi_T^{\theta^{\text{VaR}}}\right].$$

Recall that $\xi^* = \xi_T^{\theta^*}$ maximizes the expected utility $\mathbb{E}\left[u(\xi_T^{\theta^*})\right]$, but not the expected terminal wealth $\mathbb{E}\left[\xi_T^{\theta^*}\right]$ itself: thus, the inequalities above is not really a contradiction nor a surprise.

The VaR portfolio manager has a discontinuity, with no states having wealth between the benchmark value of $q = 0.75xe^{rT} = 1.8447$ and $q_2 = 1.1765$. q_2 is the VaR terminal wealth that consists of equation (3.3). However, states with wealth below q_2 have probability $\varepsilon = 6\%$. In these bad states, the VaR portfolio manager has more loss with higher probability than the portfolio manager who does not use any constraint in the portfolio optimization. The VaR portfolio manager allows 6% probability for losses in these bad states, whereas the unconstrained manager allows less probability for these losses. For example, while the probability of VaR optimal terminal wealth whose value is in the interval of (0,1.0807), which is less than $q_2 = 1.1765$, is 6%, the probability of unconstrained terminal wealth whose value is in the interval of (0,1.0807) is 4.56%. The probability mass built up at the shortfall level $q = 1.8447$ is marked by a vertical line at q in Figure 4.1. The gap which we mentioned above is due to an interval $(q_2, q) = (1.1765, 1.8447)$ of values below the shortfall level or threshold value q (small losses) which carries no probability while the interval $(0, q_2] = (0, 1.1765]$ (large losses) carries the maximum allowed probability of $\varepsilon = 6\%$. Due to this situation, we encounter a serious drawback of the VaR constraint, which bounds only the probability of the losses, but does not consider the magnitude of losses.

The solution of the representation problem, in other words, the optimal strategy θ_t^{VaR} performed by the VaR portfolio manager is shown in Figure 4.2. The blue curve plots the shape of the VaR portfolio manager's optimal strategy before the horizon. The red line is for the unconstrained portfolio strategy, the black line is for the pure stock portfolio strategy and the green line is for the pure bond portfolio strategy. For the implementation, refer the readers to the necessary MATLAB algorithms presented in Appendix B.1.3.

For being an example of before the horizon, we take the time to be $t = 5 < T = 15$. Notice also that we allow short selling in the present applications. For the sake of comparison, in Figure 4.2 we depict the strategies of the trivial portfolios, namely, the ones with the pure bond strategy ($\theta^0 \equiv 0$) and the pure stock strategy ($\theta^1 \equiv 1$), as well as and the unconstrained (benchmark) strategy ($\theta^* \equiv \frac{\mu - r}{\gamma \sigma^2} = 0.75$).

As stated before, indeed in Proposition 3.3 (ii), an equivalent representation of θ_t^{VaR} which is a function of time t and, consequently, the state price density H_t. However, on the other hand, because H_t can be expressed in terms t and the stock prices S_t, the optimal strategy θ_t^{VaR} can also be interpreted as a function of time t and the stock prices S_t. Hence, the dependence of θ_t^{VaR} on the stock price S_t for time $t = 5$, before the horizon, is shown in Figure 4.2.

For time $t = 5$ before the horizon $T = 15$, in the case of very small stock prices, that is, in the case of $S_t \in (0, 0.9282)$ computed accordingly by the values of the parameters in Table 4.1, we can see that the investor invests more in risky stock under VaR constraint than without risk management or does short selling the risky stock whose fraction is very close to the investment without risk management. In case of intermediate and large stock prices, the portfolio manager or the investor behaves like an unconstrained investor in terms of fractions of wealth invested in risky stock.

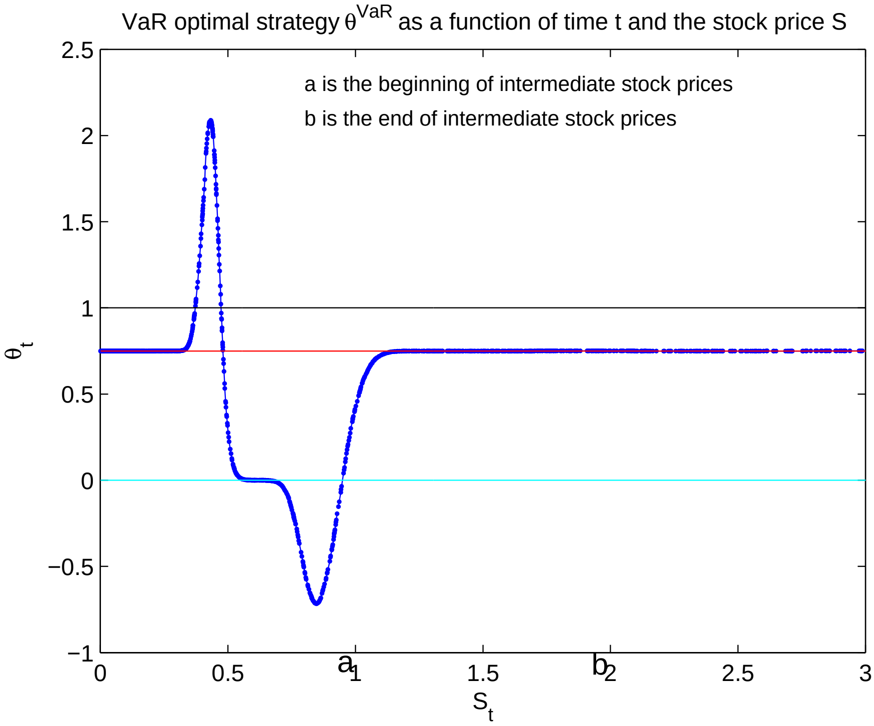

Figure 4.2: The VaR-optimal strategy θ^{VaR} at time $t < T$ before the horizon as a function of time t and the stock price S and the other mentioned strategies

4.2 Probability Density Function of EL Based Optimal Terminal Wealth and The EL-Optimal Wealth and Strategy at Time $t < T$ before the Horizon

In this section, we examine the probability density function of the optimal terminal wealth which the portfolio manager follows the Expected Loss (EL) strategy. Also, for the sake of comparison, we give the probability density functions of the terminal wealth of portfolios which we mentioned in Section 4.1: the trivial portfolios we will use for comparison are the pure bond portfolio ($\theta^0 \equiv 0$), whose fraction of wealth invested in stock is 0, the pure stock portfolio ($\theta^1 \equiv 1$), whose fraction of wealth invested in stock is 1, and the unconstrained (benchmark) portfolio ($\theta^* \equiv \frac{\mu-r}{\gamma\sigma^2} = 0.75$), whose fraction of wealth invested in stock is 0.75.

Again, in this example, the aim is to maximize the expected logarithmic utility ($\gamma = 1$) of terminal wealth ξ_T of the portfolio with the horizon $T = 15$ years. We will use the parameters of Table 4.1 for our applications. Having examined the probability density functions of these above mentioned portfolios, we will try to understand the dynamics of the optimal Expected Loss (EL) strategy at time $t < T$, for instance, by choosing the time to be $t = 5$ before the horizon, as before. Comparison with the pure bond as well as pure stock portfolios, and the unconstrained (benchmark) portfolio will be made.

We consider the solution of the static problem which leads to the optimal terminal wealth ξ^{EL}. Figure 4.3 shows the probability density function of this random variable, and the probability density functions of pure stock, unconstrained (benchmark) and pure bond portfolios. The blue curve plots the shape of the probability density function of the EL portfolio manager's optimal horizon wealth. The black curve is for the pure stock portfolio, the red curve is for the unconstrained portfolio and the line which is found on the "b" mark is for the pure bond portfolio. In addition, the expected terminal wealth $\mathbb{E}[\xi_T]$ for the considered portfolios are marked on the horizontal axes. In Appendix B.2.2, one can find necessary codes and algorithms to plot the graphs depicted in Figure 4.3.

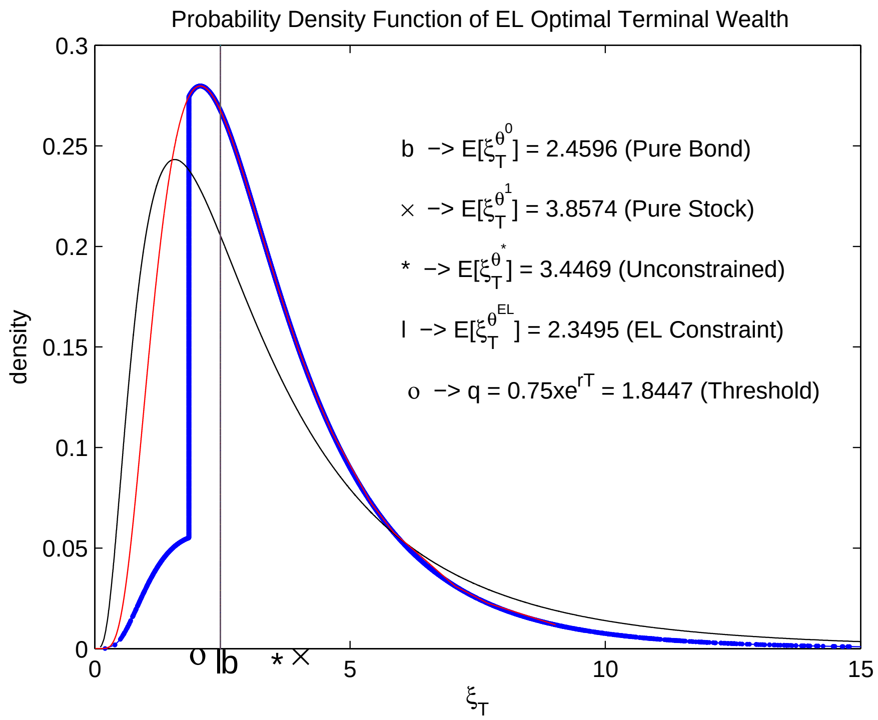

Figure 4.3: Probability density of the optimal horizon wealth belonging to the EL portfolio manager

When Figure 4.3 is closely examined, we see that there is a probability mass build-up in the EL investor's or portfolio manager's horizon wealth, at the floor $q = 0.75xe^{rT} = 1.8447$. However, optimal EL terminal wealth's probability density has no discontinuous across states, unlike that of the optimal VaR terminal wealth. Moreover, contrary to VaR strategy, in the bad states, EL portfolio manager has less loss with higher probability; or we may say that in the bad states EL portfolio manager's probability of large losses is less than the VaR portfolio manager's probability of large losses. For example, while the probability of the EL optimal terminal wealth whose value is in the interval of (0,1.0807), which is less than $q_2 = 1.1765$ and $q = 1.8447$, is 1.14%, the probability of the VaR optimal terminal wealth whose value is in the interval of (0,1.0807) is 6%. Again while in the case of the pure bond portfolio strategy $\xi_T^{\theta^0}$ there is a probability mass built up in the single point xe^{rT}, the probability of the terminal wealth $\xi_T^{\theta^1}$ and the probability of the terminal wealth $\xi_T^{\theta^*}$ are absolutely continuous. That is to say that the probability of the terminal wealth of pure stock portfolio and the probability of the terminal wealth of unconstrained portfolio, respectively, are absolutely continuous.

When the expected terminal wealths are examined, the following equalities are easily deduced:

$$\begin{aligned} \xi_T^{\theta^0} = e^{rT} = \mathbb{E}\left[\xi_T^{\theta^0}\right] &= 2.4596, \\ e^{\mu T} = \mathbb{E}\left[\xi_T^{\theta^1}\right] &= 3.8574, \\ \mathbb{E}\left[\xi_T^{\theta^*}\right] &= 3.4469, \\ \text{and we also obtain } \mathbb{E}\left[\xi_T^{\theta^{\mathrm{EL}}}\right] &= 2.3495. \end{aligned}$$

These equalities ensure

$$\mathbb{E}\left[\xi_T^{\theta^{\mathrm{EL}}}\right] < \mathbb{E}\left[\xi_T^{\theta^0}\right] < \mathbb{E}\left[\xi_T^{\theta^*}\right] < \mathbb{E}\left[\xi_T^{\theta^1}\right].$$

Likewise, as in the VaR strategy of Section 4.1, $\xi^{\mathrm{EL}} = \xi_T^{\theta^{\mathrm{EL}}}$ maximizes the expected utility $\mathbb{E}\left[u(\xi_T^{\theta^{\mathrm{EL}}})\right]$ and not the expected terminal wealth $\mathbb{E}\left[\xi_T^{\theta^{\mathrm{EL}}}\right]$ itself, therefore above inequalities is not at all contradicting the general belief.

On the other hand, solution of the representation problem, namely, the path of the optimal strategy θ_t^{EL} is shown in Figure 4.4 together with the paths of the trivial strategies: The blue curve plots the shape of the EL portfolio manager's optimal strategy before the horizon. The red line is for the unconstrained portfolio strategy, the black line is for the pure stock portfolio strategy and the green line is for the pure bond portfolio strategy. To plot the strategies, MATLAB algorithms are given in Appendix B.2.3.

As for an illustrative example for time t before the horizon T, we take $t = 5 < T = 15$. Also, we allow the short selling in our applications as usual. For the sake of comparison, in Figure 4.4 we present the strategies of the other trivial portfolios considered before and depicted in Figure 4.3: the pure bond strategy ($\theta^0 \equiv 0$), the pure stock strategy ($\theta^1 \equiv 1$) and the unconstrained (benchmark) strategy ($\theta^* \equiv \frac{\mu - r}{\gamma\sigma^2} = 0.75$).

In Proposition 3.7 (ii), on the other hand, we have examined an equivalent representation of θ_t^{EL}, represented in terms of t and the state price density H_t. Thence, as before, one can depict this dependence of θ_t^{EL} on the stock price S_t for time $t = 5$. See Figure 4.4.

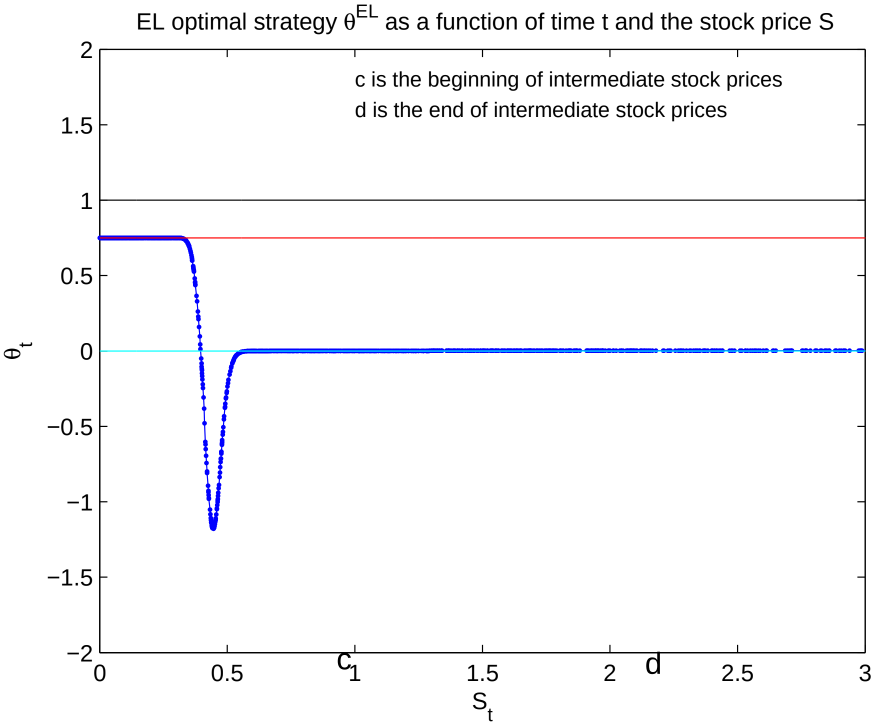

Figure 4.4: The EL-optimal strategy θ^{EL} at time $t < T$ before the horizon as a function of time t and the stock price S and the other mentioned strategies

For time $t = 5$, before the horizon $T = 15$, in the beginning of very small stock prices, $S_t \in (0, 0.9282)$ calculated according to parameters in Table 4.1, the EL portfolio manager behaves like an unconstrained (benchmark) investor by investing 75% of his wealth in risky stock. At the middle of small stock prices, he starts the short selling, whose fraction is larger than the fraction of the unconstrained portfolio manager when the stock price is approximately 0.5. Then, the manager starts to reduce the proportion of short selling, and towards the end of the small stock prices, as the prices increase, investor does not spend on the risky asset by behaving like an investor who only invests in the bond. In the cases of intermediate and large stock prices, that is, in the intervals of $S_t \in (0.9282, 2.1373)$ and $S_t \in (2.1373, \infty)$, respectively, he carries on with this behavior. In these states of stock prices, the optimal strategies θ_t^{EL} and θ^0 of the constrained and pure bond portfolio strategy coincide, which indicates that in these cases the complete capital is invested in the riskless bond, in order to ensure that the terminal wealth exceeds the given threshold value q.

4.3 Probability Density Function of EUL Based Optimal Terminal Wealth and The EUL-Optimal Wealth and Strategy at Time $t < T$ before the Horizon

In this section, we examine the probability density function of the optimal terminal wealth which the portfolio manager manages by using Expected Utility Loss (EUL) strategy. Also, for the sake of comparison, we plot the probability density functions of the terminal wealth of portfolios which were discussed in Section 4.1 and Section 4.2: the portfolios we will use for comparison are the pure bond portfolio ($\theta^0 \equiv 0$), whose fraction of wealth invested in stock is 0, the pure stock portfolio ($\theta^1 \equiv 1$), whose fraction of wealth invested in stock is 1, and the unconstrained (benchmark) portfolio ($\theta^* \equiv \frac{\mu - r}{\gamma\sigma^2} = 0.75$), whose fraction of wealth invested in stock is 0.75.

The aim is again to maximize, in this time, the expected logarithmic utility ($\gamma = 1$) of terminal wealth ξ_T of the portfolio with the horizon $T = 15$ years, and we will be using the values of the parameters of Table 4.1. Having examined the probability density functions of these above mentioned portfolios, we try to extract the Expected Utility Loss (EUL)-optimal strategy at time $t < T$ before the horizon: we choose the time to be $t = 5$, while knowing that our horizon is $T = 15$ years. We will also be considering the pure bond portfolio, pure stock portfolio and the unconstrained (benchmark) portfolio withing the context.

To start with, we consider the solution of the static problem which leads to the optimal terminal wealth ξ^{EUL}. Figure 4.5 shows the probability density function of this random variable, and the probability density functions of pure stock, unconstrained (benchmark) and pure bond portfolios for comparison. The blue curve plots the shape of the probability density function of the EUL portfolio manager's optimal horizon wealth. The black curve is for the pure stock portfolio, the red curve is for the unconstrained portfolio and the line which is found on the "b" mark is for the pure bond portfolio. In addition, the expected terminal wealth $\mathbb{E}[\xi_T]$ for the considered portfolios are marked on the horizontal axes. As usual, for completeness, we present the necessary MATLAB codes in Appendix B.3.2.

When Figure 4.5 is examined, we see immediately that there is a probability mass build-up in the EUL investor's or portfolio manager's horizon wealth, at the floor q. However, this mass is smaller than the mass of that we see in Figure 4.3 due to the definition of EL risk strategy. Similarly, the probability density of the terminal wealth for EUL constrained problem has no discontinuous across states: bad, intermediate, and good ones. In the bad states, EUL portfolio manager has loss with higher probability than EL portfolio manager. However, the probability of that the terminal wealth may fall below the value of $q_2 = 1.1765$ is much more bigger in the VaR strategy than in the EL and EUL strategies. For instance, while the probability of the EUL optimal terminal wealth whose value is in the interval of (0,1.0807), which is less than $q_2 = 1.1765$ and $q = 1.8447$ is 3.93%; the probability of the VaR optimal terminal wealth whose value is in the interval of (0,1.0807) is 6%, and the probability of the EL optimal terminal wealth whose

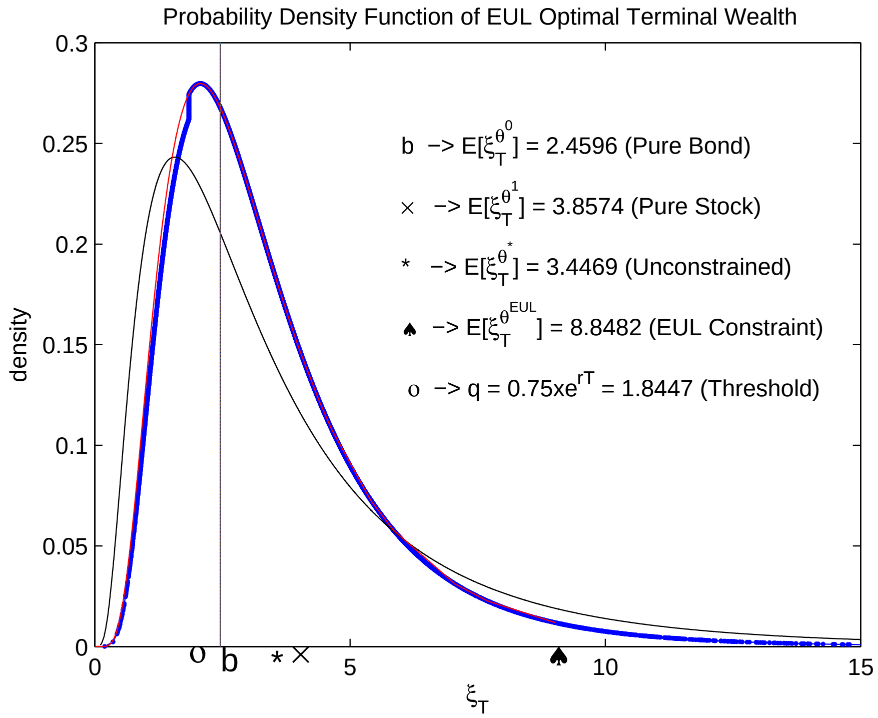

Figure 4.5: Probability density of the optimal horizon wealth belonging to the EUL portfolio manager

value is in the interval of (0,1.0807) is 1.14%. Again while in the case of the pure bond portfolio strategy $\xi_T^{\theta^0}$ there is a probability mass built up in the single point xe^{rT}, the probability of the terminal wealth $\xi_T^{\theta^1}$ and the probability of the terminal wealth $\xi_T^{\theta^*}$ are absolutely continuous. In other words, the probability of the terminal wealth of pure stock portfolio and the probability of the terminal wealth of unconstrained portfolio, respectively, are absolutely continuous.

Calculations of the expected terminal wealths as,

$$\begin{aligned}
\xi_T^{\theta^0} = e^{rT} = \mathbb{E}\left[\xi_T^{\theta^0}\right] &= 2.4596, \\
e^{\mu T} = \mathbb{E}\left[\xi_T^{\theta^1}\right] &= 3.8574, \\
\mathbb{E}\left[\xi_T^{\theta^*}\right] &= 3.4469, \\
\text{and we also obtain } \mathbb{E}\left[\xi_T^{\theta^{\mathrm{EUL}}}\right] &= 8.8482,
\end{aligned}$$

immediately yields the following inequalities:

$$\mathbb{E}\left[\xi_T^{\theta^0}\right] < \mathbb{E}\left[\xi_T^{\theta^*}\right] < \mathbb{E}\left[\xi_T^{\theta^1}\right] < \mathbb{E}\left[\xi_T^{\theta^{\mathrm{EUL}}}\right],$$

which is neither contradicting the previous results, nor surprising.

Accordingly, by the help of the representation problem, the optimal strategy θ_t^{EUL} for the EUL constrained problem is depicted in Figure 4.6 along with the trivial portfolio strategies: The blue curve plots the shape of the EUL portfolio manager's optimal strategy before the horizon. The red line is for the unconstrained portfolio strategy, the black line is for the pure stock portfolio strategy and the green line is for the pure bond portfolio strategy. One may find the necessary MATLAB algorithms used to produce the paths of the strategies in Appendix B.3.3.

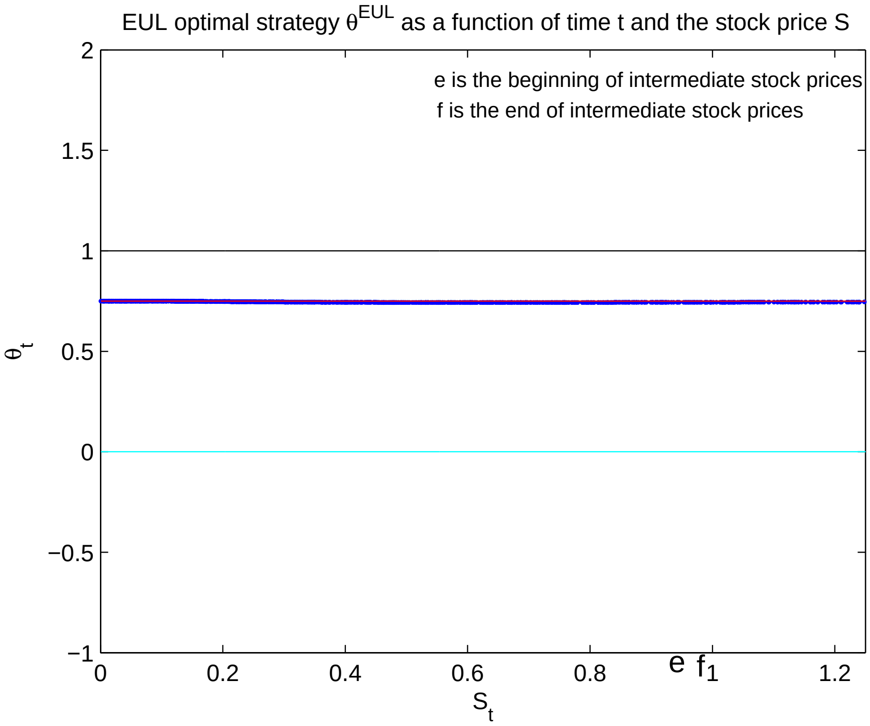

Figure 4.6: The EUL-optimal strategy θ^{EUL} at time $t < T$ before the horizon as a function of time t and the stock price S and the other mentioned strategies

Concerning the case before the horizon, we take the time to be $t = 5 < T = 15$. For the sake of comparison, in Figure 4.6 we present the strategies of the other portfolios considered previously: the pure bond strategy ($\theta^0 \equiv 0$), the pure stock strategy ($\theta^1 \equiv 1$) and the unconstrained (benchmark) strategy ($\theta^* \equiv \frac{\mu - r}{\gamma\sigma^2} = 0.75$). Note that, as before, the optimal strategies are plotted as a function of the stock prices, as the optimal strategies can also be written also as a function of the stock price S_t, and hence, t only. In Figure 4.6, we also show the dependence of θ_t^{EUL} on the stock price S_t for time $t = 5$, before the horizon.

As is clear in Figure 4.6, the fraction of wealth invested in risky stock is very close to the unconstrained fraction, which is 0.75 in this example, in almost every states of the world although there are some little changes in fractions in some states. Thus we can deduce that before the horizon $T = 15$, the EUL-optimal fraction of wealth θ_t^{EUL} is always strictly positive and does not exceed the normal strategy $\theta^* = 0.75$. Refer to Proposition 3.11.

CHAPTER 5

Conclusion and Outlook

Harry Markowitz, who is the pioneer of the modern portfolio theory, considers an investor who would (or should) select one of efficient portfolios which are those with minimum variance for given expected return or more and maximum expected return for given variance or less. However, in Markowitz's model short selling is not allowed, namely the fractions of wealth invested in the securities can not be negative, because necessary portfolios are chosen from inside of the attainable set of portfolios. The attainable set of portfolios consists of all portfolios which satisfy constraints $\sum_{i=0}^{n} \theta_i = 1$ and $\theta_i \geq 0$ for $i = 1, 2, 3, ..., n$. However in this thesis, short selling is allowed. We use the martingale representation approach to solve the optimization problem in continuous time.

Merton presented the method of continuous-time stochastic optimal control when the utility function is a power function or the logarithm [20]. While the static problem is necessary for the martingale approach, in the stochastic optimal control method the dynamic problem is used. However, martingale approach is much easier than the dynamic programming approach. Martingale technique characterizes optimal consumption-portfolio policies simply when there exist non-negativity constraints on consumption and on final wealth [4]. On the other hand, when there is the non-negativity constraint on consumption, the stochastic dynamic programming is more difficult. Also in the dynamic programming, it is in general difficult to construct a solution.

The goal of this thesis is to maximize the expected utility of the terminal wealth of the portfolio by optimal selection of the proportions of the wealth invested in stock and bond, respectively. As we examine in this thesis, when we do not use any risk limitations, the optimal terminal wealth may not exceed the initial capital with a high probability. So we quantify such shortfall risks by using appropriate risk measures and then we add them into the optimization as constraints. Hence, we use Value at Risk (VaR), Expected Loss (EL), and Expected Utility Loss (EUL) risk constraints in order to reduce such shortfall risks. By the term shortfall risk, we mean the event that the terminal wealth may fall below threshold value, namely, the initial capital or the result of an investment in a pure bond portfolio. In this thesis, portfolio optimization under VaR constraint, EL constraint, and EUL constraint are separately examined with their own numerical results. An investor may benefit separately from each strategy by choosing carefully constraint bound ε and the threshold value q for each strategy: ε and q are given and deterministic, and one can choose them in accordance to his risk tolerance for each strategy.

Here, in this thesis, we assume that all investors are risk averse and use the logarithmic utility function for meeting the requirements of these investors. We examine the numerical results of VaR, EL and EUL strategies and, for the sake of comparison, give the results of unconstrained, pure bond and pure stock strategies, and try to understand which is more suitable to risk averse investors and whether these measures are good enough to meet exactly all requirements.

Starting with the portfolio optimization problem under VaR constraint, we choose the shortfall probability as $\varepsilon = 6\%$ and the shortfall level or threshold value as $q = 0.75xe^{rT} = 1.8447$. At the beginning of very small stock prices, before the horizon, the VaR portfolio manager behaves like a benchmark (unconstrained) investor by investing as the fraction of unconstrained strategy. However, towards the middle of very small stock prices he increases the fraction and this fraction exceeds the fraction of unconstrained strategy. In this states, the behavior of VaR agent does not appear as an desirable one because it is risky and not rational. Although in good states unconstrained and VaR agent's optimal fractions which are invested in risky stock result in similar optimal terminal wealth, VaR agent exposures to more risk by investing much more in the risky stock than the unconstrained agent. In the case of intermediate and high stock prices, before the horizon VaR agent's behavior turns to the behavior of the unconstrained agent by investing as the unconstrained fraction of wealth in the risky stock. However, in this case, while the interval (q_2, q) does not carry probability, the interval $(0, q_2)$ carries the maximum allowed probability of ε. That is, while the interval of small losses does not carry probability, the interval of large losses carries the maximum allowed probability of ε. Here, q is the threshold value, and q_2 is the VaR terminal wealth that consists of the equation (3.3) and the maximum allowed probability that the terminal wealth falls below this value (q_2) is ε. This is a serious drawback of the VaR constraint which bounds only the probability of the losses but does not take care of the magnitude of losses. This may cause to credit problems, defeating the purpose of using the VaR constraint in real world applications. A regulatory requirement to manage risk using the VaR approach is designed, in principle, to prevent large and frequent losses that may drive economic investors out of business. It is true that under the VaR constraint losses are not frequent, however, the largest losses are more severe than without the VaR constraint.

In addition to the shortcomings of VaR constraint, we can consider the case of the property of sub-additivity, which is the diversification principle to reduce risk by investing in a variety of assets. Since VaR constraint does not satisfy this property, diversification can lead to an increase of VaR.

In order to remedy the shortcomings of VaR constraint, especially in bad states, as in the case of large losses expected losses are higher in the VaR strategy than those the investor would have incurred if he had not engaged in VaR constraint, Expected Loss (EL) strategy is presented as an alternative risk measure in this thesis. In contrary to the VaR agent who interests in controlling just the probability of the loss, which causes undesirable situations in the bad states as indicated, EL agent concerns with the magnitude of a loss in order to maintain limited expected losses when losses occur. Hence, if one wants to control the magnitude of losses, he should control all moments of the loss distribution, and in this thesis, we focus on controlling the first moment of the loss distribution in the EL strategy. For the EL strategy, in our example concerning this strategy, we choose the bound ε such that $\mathrm{EL}(\xi_T - q) \leq \ \varepsilon = 0.06$. That is, when losses occur, we maintain limited expected losses such that these losses can be at most 6% of our initial capital, and again we choose the threshold value such that $q = 0.75xe^{rT} = 1.8447$.

At the beginning of very small stock prices, before the horizon, the EL portfolio manager behaves like an unconstrained investor by investing of 75% ($\theta^* = \frac{\mu - r}{\gamma\sigma^2} = 0.75$) of wealth in the risky stock of our example in Section 4.2. However, towards the middle of very small stock prices he reduces the fraction and then starts the short selling. When cases of intermediate and high stock prices reached, he stays fixed at the fraction of pure bond strategy, namely $\theta^0 = 0$, in order to ensure that the terminal wealth exceeds the threshold value q. In fact, in the case of small stock prices, the short selling may be considered as a desirable situation since borrowing the low-value stock and selling it when the stock prices increase may lead to the profit for the investor who uses the approximately fraction of unconstrained agent in the short selling case. When the EL optimal terminal wealth is reached, in the bad states EL portfolio manager's probability of large losses becomes less than the VaR portfolio manager's probability of large losses.

Also contrary to the VaR strategy, EL strategy has no discontinuous across states. In the EL strategy, in the bad states, i.e. in the states of large losses, the investor partially insures himself for maintaining limited expected losses, incurring partial losses in contrary to the VaR investor. However, maintaining some level of insurance requires from the investor a cost, too; it is necessary to think well about how much cost is to spent for insurance and whether it is worth leaving bad states completely uninsured.

In addition, contrary to the VaR constraint, EL constraint satisfies the subadditivity property of coherent risk measures. However, it does not satisfy the translation-invariance axiom stated in Section 2.4: for a given $a \in \mathbb{R}$ we should have $\rho(Z_1 + a) = \rho(Z_1) - a$. This might be considered as a disadvantage of EL constraint since when cash which has the value a is added to the portfolio, the risk of $Z_1 + a$ is more than the risk of Z_1 and this risk is as much as the cash which has the value a.

Since one of the goals of a portfolio manager is to maximize the expected utility from the terminal wealth, it is interesting to deal with another risk measure called Expected Utility Loss (EUL), which we investigate in this thesis. EUL risk constraint leads to more explicit calculations for the optimal strategy that we are looking for and allows us to solve the constrained static problem for a large class of utility functions. Thus it might be a convenient risk measure.

In the case of EUL optimal horizon wealth, similar to the EL constraint, in the bad states, namely the high price of consumption H_T, he partially insures himself against losses and therefore in this partially insured states EUL agent may keep the EUL optimal terminal wealth above the optimal terminal wealths of other strategies mentioned. This is achieved by shrinking the insured region in the intermediate states, but by settling for a wealth lower than q so that it is enough to comply with the EUL constraint in the bad states. However, again, since insurance is very costly in these bad states, here EUL agent prefers partially insurance.

For the EUL strategy, in our example, we choose the EUL bound ε such that $\text{EUL}(u(\xi_T) - u(q)) \leq \ \varepsilon = 0.06$. That is, when losses occur, we maintain limited expected utility losses such that those utility losses can be at most 0.06, and again we choose the threshold value such that $q = 0.75xe^{rT} = 1.8447$. As we examine in Section 4.3, before the horizon, in all states of stock prices, the EUL portfolio manager invests in risky stock as a value of fraction that is very close to the fraction of unconstrained strategy. We also infer that the EUL optimal fraction θ_t^{EUL}, before the horizon, is always strictly positive and never exceeds the normal strategy θ^* as is examined in Proposition 3.11. Hence, we understand that if we use the EUL constraint in our optimization problem, when we take drift term μ bigger than r, short selling will not be allowed here, in contrary to the VaR and EL strategies. Finally, to point out that, neither EL nor EUL risk measures not coherent risk measures, unfortunately.

Consequently, each of risk measures in this thesis, which are Value at Risk (VaR), Expected Loss (EL) and Expected Utility Loss (EUL) risk measures, has various advantages and disadvantages separately as mentioned in the above discussions. When a portfolio manager wants to use risk constraints in the optimization problem, it is too significant to choose the bounds and threshold values rationally for each risk constraint and examine in details the advantages and disadvantages of these risk measures before performing an investment in order to be able to achieve the desired results. However, a very serious deficiency of VaR, EL and EUL risk measures is that all of them are not coherent risk measures: the VaR risk measure does not satisfy the sub-additivity property and, the EL and EUL risk measures do not satisfy the translation-invariance property. Sub-additivity property reflects the idea that risk can be reduced by diversification, so non-subadditive measures of risk in portfolio optimization may create portfolios with high risk.

As an outlook, thanks to the translation-invariance property of a risk measure, the risk of a portfolio can be reduced by simply adding a certain amount of riskless money. So, when the shortcomings of these non-coherent risk measures are to be avoided, it appears that, in the constrained portfolio optimization problems, using coherent risk measures may be much more rational and it may be necessary to search coherent risk measures for being alternative to the VaR, EL and EUL risk measures.

APPENDIX A

Proofs of Some Propositions

A.1 Proof of Proposition 3.1

Proof of Proposition 3.1. ([2]) Let $W = \xi_T^{\text{VaR}}$. If $P(W < q) < \varepsilon$, then by their definition, $\overline{h} < \underline{h}$, and $\xi_T^{\text{VaR}} = I(yH_T) = \xi_T^*$, which is optimal following the standard arguments as in the benchmark case. Otherwise, $P(W < q) = \varepsilon$, and $\overline{h} \geq \underline{h}$. The remainder of the proof is for the latter case. We adapt the common convex-duality approach [see, for example [15]] to incorporate the VaR constraint. The expression in Lemma A.1 is the convex conjugate of u with an additional term capturing the VaR constraint.

Lemma A.1 ([2]). *Expression* (3.2) *solves the following point-wise problem for every* ξ_T*:*

$$u(W) - yH_TW + y_2\mathbb{1}_{\{W\geq q\}} = \max_{\xi}\left\{u(\xi) - yH_T\xi + y_2\mathbb{1}_{\{\xi\geq q\}}\right\},$$

where

$$y_2 \equiv u(I(y\overline{h})) - y\overline{h}I(y\overline{h}) - u(q) + y\overline{h}q \geq 0.$$

Proof. ([2]) The function on which $\max\{\cdot\}$ operates is not concave in ξ, but can only exhibit local maxima at $\xi = I(yH_T)$ and/or $\xi = q$. To find the global maximum, we need to compare the value of these two local maxima. When $H_T < \underline{h}$, we have $I(yH_T) > q$ and

$$u(I(yH_T)) - yH_TI(yH_T) + y_2 > u(q) - yH_Tq + y_2,$$

so $I(yH_T)$ is the global maximum. When $\underline{h} \leq H_T < \overline{h}$, we have $I(yH_T) \leq q$ and

$$\begin{aligned} u(q) - yH_Tq + y_2 &= u(I(y\overline{h})) - y\overline{h}I(y\overline{h}) + yq(\overline{h} - H_T) \\ &> u(I(yH_T)) - yH_TI(yH_T), \end{aligned} \tag{A.1}$$

where the inequality follows from $H_T < \overline{h}$ and $\frac{\partial[u(I(yH)) - yHI(yH) + yqH]}{\partial H} = -yI(y\xi) + yq \geq 0$ whenever $H \geq \underline{h}$. So, q is the global maximum. When $H_T \geq \overline{h}$, the inequality in (A.1) is reversed and so $I(yH_T)$ is the global maximum.

Finally, to show $y_2 \geq 0$ note that

$$y_2 = \left[u(I(y\overline{h}) - y\overline{h}I(y\overline{h})\right] - [u(I(y\underline{h})) - y\underline{h}I(y\underline{h}) + yq\underline{h}] \geq 0,$$

again from $\frac{\partial[u(I(yH))-yHI(yH)+yqH]}{\partial H} \geq 0$ and $\overline{h} \geq \underline{h}$. $\square$

Now, let ξ_T be any candidate optimal solution, which satisfies the VaR constraint $P(\xi < q) \leq \varepsilon$ and the static budget constraint $\mathbb{E}[H_T\xi] \leq x$ in (3.1). We have

$$\begin{aligned}
& \mathbb{E}[u(W)] - \mathbb{E}[u(\xi_T)] \\
= \;& \mathbb{E}[u(W)] - \mathbb{E}[u(\xi_T)] - yH_0\xi_0 + yH_0\xi_0 + y_2(1-\varepsilon) - y_2(1-\varepsilon) \\
\geq \;& \mathbb{E}[u(W)] - \mathbb{E}[u(\xi_T)] - \mathbb{E}[yH_TW] + \mathbb{E}[yH_T\xi_T] \\
+ \;& \mathbb{E}\left[y_2\mathbb{1}_{\{W\geq q\}}\right] - \mathbb{E}\left[y_2\mathbb{1}_{\{\xi_T\geq q\}}\right] \geq 0,
\end{aligned}$$

where the former inequality follows from the static budget constraint and the VaR constraint holding with equality for W, while holding with inequality for ξ_T. The latter inequality follows from Lemma A.1. Hence W is optimal. The optimization problem is not standard because it is non-concave; to gain insight into its structure, it is found satisfactory to provide a general proof of sufficiency for optimality. To prove existence, one has to follow the standard path of stating and verifying conditions for integrability of wealth, prices, and portfolio holdings. In addition, one has to present the appropriate growth conditions on u and moment conditions on H, followed by an elaborate analysis to verify that the expectations in the objective function and in the budget constraint are well defined (e.g., as in [5]). To prevent diverting the focus with a series of technical conditions and to not unnecessarily lengthen the article, Basak and Shapiro [2] chose to solve explicit examples of interest, instead of providing existence in general. Finally, because the VaR constraint must hold with equality, the definition of $\overline{h}$ is deduced. $\square$

A.2 Proof of Proposition 3.2

Proof of Proposition 3.2. ([2]) We prove the claim step by step as follows:

(i) It is easy to verify that $L_1(\xi^*) = G_1(a_*, y^*)$ and $L_1(\xi^{\text{VaR}}) = G_1(a_v, y)$ are satisfied, where

$$\begin{aligned}
G_1(a,x) &= q_2N(a) - x^{-\frac{1}{\gamma}}e^{\left(\frac{m}{\gamma}+\frac{s^2}{2\gamma^2}\right)}N\left(a - \frac{s}{\gamma}\right), \\
m &= \mathbb{E}[-\ln H_T], \\
s^2 &= \text{VaR}[-\ln H_T], \\
a_* &= \frac{(\ln(q_2^\gamma)y^* - m)}{s}, \\
a_v &= \frac{(\ln(q_2^\gamma)y - m)}{s},
\end{aligned}$$

and y solves $\mathbb{E}[H_T I(yH_T)] = H_0\xi_0$. Next, it is also straightforward to show that, for $x > 0$, $\frac{\partial G_1(a,x)}{\partial a} \geq 0$ if, and only if, $a \leq a_v$. Hence, because $a_* \leq a_v, G_1(a_*, y) \leq G_1(a_v, y)$. Also, as $\frac{\partial G_1(a,x)}{x} \geq 0$ and $y \geq y^*, G_1(a, y^*) \leq G_1(a, y)$. Then,

$$\begin{aligned} L_1(\xi^{\mathrm{VaR}}) - L_1(\xi^*) &= G_1(a_v, y) - G_1(a_*, y^*) \\ &\geq G_1(a_v, y) - G_1(a_*, y) \\ &\geq 0. \end{aligned}$$

(ii) It is straightforward to verify that $L_2(\xi^*) = G_2(a_*, y^*), L_2(\xi^{\mathrm{VaR}}) = G_2(a_v, y)$, where

$$\begin{aligned} G_2(a, x) &= \left(q_2 e^{-m+\frac{s^2}{2}} N(a+s) - x^{-\frac{1}{\gamma}} e^{\Gamma} N\left(a - \frac{1-\gamma}{\gamma} s)\right)\right) / H_0, \\ \Gamma &= \frac{1-\gamma}{\gamma} m + \left(\frac{1-\gamma}{\gamma}\right)^2 \frac{s^2}{2}, \end{aligned}$$

a_*, a_v, as in part (i). Also, for $x > 0$, $\frac{\partial G_2(a,x)}{\partial a} \geq 0$ if, and only if, $a \leq a_v$, and since $\frac{\partial G_2(a,x)}{\partial x} \geq 0, G_2(a, y^*) \leq G_2(a, y)$. Therefore,

$$\begin{aligned} L_2(\xi^{\mathrm{VaR}}) - L_2(\xi^*) &= G_2(a_v, y) - G_2(a_*, y^*) \\ &\geq G_2(a_v, y) - G_2(a_*, y) \\ &\geq 0. \end{aligned}$$

Collecting these completes the proof. □

A.3 Proof of Lemma 3.4

Proof of Lemma 3.4. ([9]) Let $z > 0$ and consider the function $h(x) = u(x) - y_1 z x - y_2 (x-q)^-$. Defining the two functions

$$\begin{aligned} h_1(x) &= u(x) - y_1 z x \\ h_2(x) &= u(x) - y_1 z x + y_2(x-q) = u(x) - (y_1 z - y_2)x - y_2 q, \end{aligned}$$

the function h can be written as

$$h(x) = \begin{cases} h_1(x), & \text{for } x \geq q, \\ h_2(x), & \text{for } x < q. \end{cases} \tag{A.2}$$

Since h_1 and h_2 are strictly concave and continuously differentiable, the function h is a continuous and strictly concave function which is differentiable in $(0, q)$ and (q, ∞) and possesses different one-sided derivatives in the point $x = q$ which are $h'(q-0) = h_2'(q)$ and $h'(q+0) = h_1'(q)$.

The functions h_1 and h_2 attain their maximum values at $x_1 = I(y_1 z)$ and $x_2 = I(y_1 z - y_2)$, respectively. Since the function I is strictly decreasing and $y_2 > 0$ it follows $x_1 < x_2$. To find the maximum of h one has to consider the following three cases.

(i) $q < x_1$:
Since u' is strictly decreasing we have $u'(q) > u'(x_1) = u'(I(y_1 z)) = y_1 z$, hence $z < \frac{u'(q)}{y_1} = \underline{h}$. Considering the one-sided derivatives at $x = q$ one obtains

$$h'(q-0) = h_2'(q) = u'(q) - (y_1 z - y_2) > u'(q) - y_1 \frac{u'(q)}{y_1} + y_2 > 0$$

and

$$h'(q+0) = h_1'(q) = u'(q) - y_1 z > u'(q) - y_1 \frac{u'(q)}{y_1} = 0,$$

that is, the function h is increasing at $x = q$. It follows that the function h attains its maximum on (q, ∞) where $h(x) = h_1(x)$, i.e., the maximum is at $x^* = x_1 = I(y_1 z)$.

(ii) $x_1 \le q < x_2$:
Now the relation $q \ge x_1$ implies $z \ge \underline{h}$ while $q > x_2$ leads to

$$u'(q) < u'(x_2) = u'(I(y_1 z - y_2)) = y_1 z - y_2,$$

that is, $z < \frac{u'(q)+y_2}{y_1} = \overline{h}$, which gives $\underline{h} \le z < \overline{h}$. It follows that

$$h'(q-0) = h_2'(q) = u'(q) - (y_1 z - y_2) > u'(q) - y_1 \frac{u'(q)+y_2}{y_1} + y_2 = 0$$

and

$$h'(q+0) = h_1'(q) = u'(q) - y_1 z \le u'(q) - y_1 \frac{u'(q)}{y_1} = 0.$$

From the strict concavity of h we deduce that

$$\begin{aligned} h'(x) &= h_2'(x) > h_2'(q) > 0 \text{ for } x < q, \\ h'(x) &= h_1'(x) < h_1'(q) \le 0 \text{ for } x > q. \end{aligned}$$

Thus the function h is strictly increasing for $x < q$ and strictly decreasing for $x > q$, hence h attains its maximum at $x^* = q$.

The relations

$$\frac{y_1}{1+y_2} z < u'(q) \le y_1 z$$

imply

$$\underline{h} \le z < \overline{h} = \frac{1+y_2}{y_1} u'(q). \tag{A.3}$$

(iii) $q \geq x_2$:
This case is equivalent to $z \geq \overline{h} = \frac{u'(q)+y_2}{y_1}$. For the one-sided derivatives at $x = q$ one obtains

$$h'(q-0) = h_2'(q) = u'(q) - (y_1 z - y_2) \leq u'(q) - y_1 \frac{u'(q) + y_2}{y_1} + y_2 = 0$$

and

$$h'(q+0) = h_1'(q) = u'(q) - y_1 z \leq u'(q) - y_1 \frac{u'(q) + y_2}{y_1} = -y_2 < 0.$$

It follows that the function h is decreasing at $x = q$ attains its maximum on $(0, q)$ where $h(x) = h_2(x)$ and, hence, the maximum is at $x^* = x_2 = I(y_1 z - y_2)$.

The proof is completed. □

A.4 Proof of Proposition 3.7

Proof of Proposition 3.7. ([7]) We proceed as follows:

(i) The computations and the arguments of this proof are the same as in Proposition 3.3 part (i), except that we can not compute the conditional expectation

$$J_2 = G(H_t, \overline{h}) = \mathbb{E}\left[\frac{H_T}{H_t} I(y_1 H_T - y_2) \mathbb{1}_{\{\overline{h} \leq H_T\}} \,\middle|\, \mathcal{F}_t\right]$$

explicitly, but give it in terms of the integral $G(z, \overline{h})$.

(ii) From (3.9) it follows $X_t^{\text{EL}} = F(H_t, t)$. The process $H = (H_t)_{t \in [0,T]}$ satisfies the SDE (2.3). Applying Itô's lemma to the function $F(H_t, t)$ we find that the process $X^{\text{EL}} = (X_t^{\text{EL}})_{t \in [0,T]}$ satisfies the SDE

$$dX_t^{\text{EL}} = \left[F_t(H_t, t) - rF_z(H_t, t)H_t + \frac{\kappa^2}{2} F_{zz}(H_t, t)H_t^2\right] dt - F_z(H_t, t)H_t \kappa^T dW_t,$$

where F_z, F_{zz} and F_t denote the partial derivatives of $F(z, t)$ with respect to z and t, respectively. Equating coefficients in front of dW_t in the above equation and Equation (2.4) leads to the following equality:

$$\theta_t^{\text{EL}} = -\sigma^{-1} \kappa \frac{F_z(H_t, t)H_t}{F(H_t, t)} = -\theta^N \gamma \frac{F_z(H_t, t)H_t}{F(H_t, t)} \tag{A.4}$$

Computing the derivative F_z we get

$$\begin{aligned} F_z(z, t) = &- \frac{e^{\Gamma(t)}}{z\gamma (y_1 z)^{\frac{1}{\gamma}}} \left[1 - \Phi(-d_1(\underline{h}, z)) + \frac{\gamma}{\kappa\sqrt{T-t}} \varphi(d_1(\underline{h}, z))\right] \\ &+ \frac{q e^{-r(T-t)}}{\kappa\sqrt{T-t}z} \left[\varphi(d_2(\underline{h}, z)) - \varphi(d_2(\overline{h}, z))\right] + \frac{\partial G(z, \underline{h})}{\partial z}. \end{aligned}$$

For the last term we have

$$\begin{aligned}\frac{\partial G(z,\overline{h})}{\partial z} &= \frac{e^{-r(T-t)}}{\sqrt{2\pi}}\frac{\partial \int_{-\infty}^{c_2(\overline{h},z)} \iota(z,u)du}{\partial z} \\ &= \frac{e^{-r(T-t)}}{\sqrt{2\pi}}\left[\int_{-\infty}^{c_2(\overline{h},z)} \frac{\partial \iota(z,u)}{\partial z}du + \frac{\partial c_2(\overline{h},z)\iota(z,c_2(\overline{h},z))}{\partial z}\right],\end{aligned}$$

where

$$\iota(z,u) = \frac{e^{-\frac{1}{2}(u-b)^2}}{(y_1 z e^{a+bu} - y_2)^{\frac{1}{\gamma}}}$$

Finally, we get

$$\begin{aligned}\frac{\partial G(z,\overline{h})}{\partial z} &= \frac{-y_1}{\gamma}e^{(\kappa^2-2r)(T-t)}\psi_0\left(c_2(\overline{h},z), b, y_1 z e^a, y_2, 2b, 1, 1+\frac{1}{\gamma}\right) \\ &+ \frac{qe^{r(T-t)}}{\kappa\sqrt{T-t}z}\varphi(-d_2(\overline{h},z)).\end{aligned}$$

Consequently. plugging the last equality in (A.4), we get the final form of the optimal strategies before the horizon, and this completes the proof. □

A.5 Proof of Proposition 3.8

Proof of Proposition 3.8. ([7]) The assumption on the existence of solutions $y_1, y_2 > 0$ of the system of equations given in the proposition implies that ξ_T^{EUL} fulfills the risk constraint with equality. In order to solve the optimization problem under the risk constraint, we adopt the common convex-duality approach by introducing the convex conjugate of the utility function u with an additional term capturing the risk constraint as it is shown in the following lemma.

Lemma A.2 ([7]). *Let $z, y_1, y_2, q > 0$. Then the solution of the optimization problem*

$$\max_{x>0}\left\{u(x) - y_1 z x - y_2(u(x) - u(q))^-\right\}$$

is $x^* = \xi^*(z)$.

Applying the above lemma point-wise for all $z = H_T$ it follows that $\xi^* = \xi_T^*(H_T)$ is the solution of the maximization problem

$$\max_{\xi>0}\left\{u(\xi) - y_1 H_T \xi - y_2(u(\xi) - u(q))^-\right\}.$$

Obviously, ξ^* is $\mathcal{F}_T$-measurable and if y_1, y_2 are chosen as solutions of the system of equations given in the proposition then it follows $\xi^* = \xi_T^{\mathrm{EUL}} = \xi^{\mathrm{EUL}}$. To complete the proof, let η be any admissible solution satisfying the static budget constraint and the EUL-constraint (3.10). We have

$$\begin{aligned}
\mathbb{E}\left[u(\xi_T^{\mathrm{EUL}})\right] - \mathbb{E}\left[u(\eta)\right] &= \mathbb{E}\left[u(\xi_T^{\mathrm{EUL}})\right] - \mathbb{E}\left[u(\eta)\right] - y_1 x + y_1 x - y_2\varepsilon + y_2\varepsilon \\
&\geq \mathbb{E}\left[u(\xi_T^{\mathrm{EUL}})\right] - y_1\mathbb{E}\left[H_T\xi_T^{\mathrm{EUL}}\right] - y_2\mathbb{E}\left[(u(\xi_T^{\mathrm{EUL}}) - u(q))^-\right] \\
&\quad -\mathbb{E}\left[u(\eta)\right] + y_1\mathbb{E}\left[H_T\eta\right] + y_2\mathbb{E}\left[(u(\eta) - u(q))^-\right] \\
&\geq 0,
\end{aligned}$$

where the first inequality follows from the static budget constraint and the constraint for the risk holding with equality for ξ_T^{EUL}, while holding with inequality for η. The last inequality is a consequence of the above lemma. Hence we obtain that ξ_T^{EUL} is optimal; completing the proof. □

Proof of Lemma A.2. ([7]) Consider the function $h(x) = u(x) - y_1 z x - y_2(u(x) - (q))^-$. Defining the two functions

$$\begin{aligned}
h_1(x) &= u(x) - y_1 z x \\
h_2(x) &= u(x) - y_1 z x + y_2(u(x) - u(q)) = (1 + y_2)u(x) - y_1 z x - y_2 u(q),
\end{aligned}$$

the function h can be written as

$$h(x) = \begin{cases} h_1(x), & \text{for } x \geq q, \\ h_2(x), & \text{for } x < q. \end{cases} \tag{A.5}$$

Since h_1 and h_2 are strictly concave and continuously differentiable, the function h is a continuous and strictly concave function which is differentiable in $[0, q)$ and (q, ∞) and possesses different one-sided derivatives in the point $x = q$ which are $h'(q-0) = h_2'(q)$ and $h'(q+0) = h_1'(q)$.

The functions h_1 and h_2 attain their maximum values at $x_1 = I\left(y_1 z\right)$ and $x_2 = I\left(\frac{y_1}{1+y_2} z\right)$, respectively. Since the function I is strictly decreasing and $y_2 > 0$ it follows $x_1 < x_2$. To find the maximum of h one has to consider the following three cases.

(i) $q < x_1$:
Since u' is strictly decreasing we have $u'(q) > u'(x_1) = u'\left(I\left(y_1 z\right)\right) = y_1 z$. Considering the one-sided derivatives at $x = q$ one obtains

$$h'(q-0) = h_2'(q) = (1 + y_2)u'(q) - y_1 z > (1 + y_2)y_1 z - y_1 z = y_1 y_2 z > 0$$

and

$$h'(q+0) = h_1'(q) = u'(q) - y_1 z > y_1 z - y_1 z = 0,$$

that is, the function h is increasing at $x = q$. It follows that the function h attains its maximum on (q, ∞) where $h(x) = h_1(x)$, namely., the maximum is at $x^* = x_1 = I\left(y_1 z\right)$. Solving the inequality $u'(q) > y_1 z$ for z it yields

$$z < \frac{u'(q)}{y_1} = \underline{h} \tag{A.6}$$

(i) $x_1 \le q < x_2$:
Now the relation $q \ge x_1$ implies $u'(q) \le y_1 z$ while $q < x_2$ leads to

$$u'(q) > u'(x_2) = u'\left(I\left(\frac{y_1}{1+y_2}z\right)\right) = \frac{y_1}{1+y_2}z.$$

For the one-sided derivatives at $x = q$ we find

$$h'(q-0) = h_2'(q) = (1+y_2)u'(q) - y_1 z > (1+y_2)\frac{y_1}{1+y_2}z - y_1 z = 0$$

and

$$h'(q+0) = h_1'(q) = u'(q) - y_1 z \le y_1 z - y_1 z = 0.$$

From the strict concavity of h we deduce that $h'(x) = h_1'(x) < h_1'(q) < 0$ for $x > q$. Thus the function h is strictly increasing for $x < q$ and strictly decreasing for $x > q$, hence h attains its maximum at $x^* = q$. The relations

$$\frac{y_1}{1+y_2}z < u'(q) \le y_1 z$$

imply

$$\underline{h} \le z < \overline{h} = \frac{1+y_2}{y_1}u'(q) \tag{A.7}$$

(iii) $q \ge x_2$:
In this case we have $u'(q) \le u'(x_2) = \frac{y_1}{1+y_2}z$. For the one-sided derivatives at $x = q$ one obtains

$$h'(q-0) = h_2'(q) = (1+y_2)u'(q) - y_1 z \le y_1 z - y_1 z = 0$$

and

$$h'(q+0) = h_1'(q) = u'(q) - y_1 z \le \frac{y_1 z}{1+y_2} - y_1 z < 0.$$

It follows that the function h is decreasing at $x = q$ attains its maximum on $(0, q)$ where $h(x) = h_2(x)$ and hence the maximum is at $x^* = x_2 = I\left(\frac{y_1}{1+y_2}z\right)$. Solving the inequality $u'(q) \le u'(x_2) = \frac{y_1}{1+y_2}z$ for z it follows

$$z \ge \frac{1+y_2}{y_1}u'(q) = \overline{h}. \tag{A.8}$$

The proof is completed. □

A.6 Proof of Proposition 3.10

Proof of Proposition 3.10. ([7]) the proof of this proposition is rather similar to that of those proved earlier:

(i) The process HX^{EUL} is an $\mathcal{F}$-martingale and the proof is as in part (i) of Proposition 3.3, if $\underline{h}$ and $\overline{h}$ are replaced appropriately.

(ii) The same arguments as in the proof of Proposition 3.7 (ii), lead to following equality for the optimal trading strategy

$$\theta_t^{\mathrm{EUL}} = -\sigma^{-1}\kappa \frac{F_z(H_t,t)H_t}{F(H_t,t)} = -\theta^N \gamma \frac{F_z(H_t,t)H_t}{F(H_t,t)}, \tag{A.9}$$

where $F(z,t)$ is defined in Proposition 3.10. Further, formal evaluation of the derivative F_z yields

$$\begin{aligned} F_z(z,t) &= \frac{1}{\gamma z}\left[-F(z,t) + qe^{-r(T-t)}\left(\Phi(-d_2(\underline{h},z,t) - \Phi(-d_2(\overline{h},z,t))\right)\right] \\ &- \frac{e^{\Gamma(t)}}{(y_1 z)^{\frac{1}{\gamma}}\kappa\sqrt{T-t}z}\left[\varphi(d_1(\underline{h},z,t)) - (1+y_2)^{\frac{1}{\gamma}}\varphi(d_1(\overline{h},z,t))\right] \\ &+ \frac{qe^{-r(T-t)}}{\kappa\sqrt{T-t}z}\left[\varphi(d_2(\underline{h},z,t)) - \varphi(d_2(\overline{h},z,t))\right]. \end{aligned}$$

Here, φ denotes the standard-normal probability density function. Substituting these into (A.9), we get the final form of the optimal strategy before the horizon. This completes the proof. □

APPENDIX B

Implementation in Matlab

B.1 Matlab Algorithms Related to VaR Risk Measure

B.1.1 Optimal Horizon Wealth of the VaR Risk Manager

```
% The optimal horizon wealth of the VaR risk manager as a function
% of the horizon state price density H(T).
clear all, close all, clc

randn('state', 100); rand('state', 100)
y = 1; h1 = 0.5421; h2 = 0.85;
X = lognrnd(0,sqrt(0.3375),1,15000);
HT=exp(-1.0695)*X; s=sort(HT); ns = length(s);
for i=1:ns;
    if s(i)<h1
        I1(i) = 1/(y*s(i));
        s1(i) = s(i);
    elseif ((h1<=s(i)) && (s(i)<h2))
        I2(i) = 1.8447 ;
        s2(i) = s(i);
    elseif h2<=s(i)
        I3(i) = 1/(y*s(i));
        s3(i) = s(i);
    end
end
I2(I2==0)=[]; I3(I3==0)=[]; s2(s2==0)=[]; s3(s3==0)=[];
plot(s1,I1, 'b.-'), hold on
plot(s2,I2, 'b.-'), hold on
plot(s3,I3, 'b.-'), hold on
title('optimal terminal wealth of the VaR-portfolio manager')
xlabel('H_{T}'), ylabel('\xi_{T}'), axis([0,2,0,15])

% Trml as a function of the horizon state price density H(T) for
% the portfolio insurer
for i=1:ns;
```

```
    if s(i)<h1
        K1(i) = 1/(y*s(i));
        St1(i)=s(i);
    elseif h1<=s(i)
        K2(i) = 1.8447; % Threshold value.
        St2(i) = s(i);
    end
end
K2(K2==0)=[]; St2(St2==0)=[];
plot(St1,K1,'k-'), hold on
plot(St2,K2,'k-'), hold on
axis([0,2,0,15])

% Trml as a function of the horizon state price density H(T) for
% the unconstrained benchmark
I = 1./s;
plot(s,I,'r')
axis([0,1.5,0,5])
text(-0.015,1.8447,'q','FontSize',10.5);
text(-0.015,1.1765,'q_{2}','FontSize',10.5);
text(0.5421,-0.05,'h','FontSize',9);
text(0.85,-0.05,'h','FontSize',9);
text(0.55,4,'q is the threshold value.','FontSize',9);
text(0.55,3.75,...
    'q_{2} is the terminal wealth corresponding the VaR',...
    'FontSize',9);
text(0.55,4.5,'h is the beginning of intermediate states',...
    'FontSize',9);
text(0.55,4.25,'h is the end of intermediate states',...
    'FontSize',9);
```

B.1.2 Probability Density of the Optimal Horizon Wealth Belonging to the VaR Portfolio Manager

```
% The probability density function of the optimal terminal wealth
% concerning VaR
clear all, clc, close all

randn('state', 100); rand('state', 100)
y = 1; h1 = 0.5421; h2 = 0.85;
X = lognrnd(0,sqrt(0.3375),1,15000);
s = sort(X); HT = exp(-1.0695)*s; ns = length(HT); q2 = 1/(y*h2);
for i=1:ns;
    if ( HT(i)<h1 )
        I1(i) = 1./(y*HT(i));
        PDI1(i) = (0.6867*exp(-1.4569*...
```

```
            (log(0.3432*I1(i))).^2))./I1(i);
   elseif (h1<=HT(i) && (HT(i)<h2))
        I2(i) = 1.8447 ;
   elseif ( h2<=HT(i))
        I3(i) = 1./(y*HT(i));
        PDI3(i) = (0.6867*exp(-1.4569*...
            (log(0.3432*I3(i))).^2))./I3(i);
   end
end
I2(I2==0)=[]; I3(I3==0)=[]; PDI3(PDI3==0)=[];
plot(I1,PDI1,'b.-'), hold on
j = 0:0.0001:max(PDI1);
plot(max(I2),j,'b.-'), hold on
k = q2:0.0001:max(I2) ;
plot(k,0,'b.-'), hold on
plot(I3,PDI3,'b.-'), hold on
xlabel('\xi_{T}'), ylabel('density'), axis([0,15,0,0.3])
text(3.4469,-0.0075,'*','FontSize',14);
text(2.4596,-0.0065,'b','FontSize',14);
text(8.7437,-0.006,'\nabla','FontSize',14);
text(3.8574,-0.0025,'\times','FontSize',14);
text(1.1765,-0.005,'\diamondsuit','FontSize',14);
text(1.8447,-0.002,'\o','FontSize',14);
text(6,0.25,...
    'b   -> E[\xi_{T}^{\theta^{0}}] = 2.4596 (Pure Bond)',...
    'FontSize',10);
text(6,0.22,...
    '\times   -> E[\xi_{T}^{\theta^{1}}] = 3.8574 (Pure Stock)',...
    'FontSize',10);
text(6,0.19,...
    '*   -> E[\xi_{T}^{\theta^{*}}] = 3.4469 (Unconstrained)',...
    'FontSize',10);
text(6,0.16,...
    '\nabla -> E[\xi_{T}^{\theta^{VaR}}] = 8.7437 (VaR Constraint)',...
    'FontSize',10);
text(6,0.13,...
    ' \o  -> q = 0.75xe^{rT} = 1.8447 (Threshold)',...
    'FontSize',10);
text(6,0.10,'\diamondsuit  -> q_2 = 1.1765','FontSize',10);

% The probability density function of the terminal wealth of
% pure stock portfolio
randn('state', 100); rand('state', 100)
xLN = lognrnd(0,sqrt(0.6),1,15000); XT1 = 2.8577*xLN;
s = sort(XT1);
PDXT1 = (0.5150*exp(-0.8333*((log(0.3499*s)).^2)))./s ;
plot(s,PDXT1,'k-'), hold on
```

```
xlabel('\xi_{T}'), ylabel('density'), axis([0,15,0,0.3])

% The probability density function of the terminal wealth of
% pure bond portfolio.
XT0 = 2.4596 ; % The terminal wealth if I only invest in bond.
P = 0:0.0001:1;
plot(XT0,P), hold on
xlabel('\xi_{T}'), ylabel('density')

% The probability density function of the terminal wealth of
% unconstrained portfolio
randn('state', 100); rand('state', 100)
XTB = 1./HT ;
PDFXTB = (0.6867*exp(-1.4569*(log(0.3432*XTB)).^2))./XTB ;
plot(XTB,PDFXTB,'r')
title(strcat('Probability Density Function of',...
    ' VaR Optimal Terminal Wealth'))
xlabel('\xi_{T}'), ylabel('density'), axis([0,15,0,0.3])
```

B.1.3 The VaR Optimal Strategy at Time $t < T$ Before the Horizon as a Function of Time t and the Stock Price S and the other Mentioned Strategies

```
% The VaR-optimal wealth and the fraction of wealth invested in
% stock before horizon
clear all, close all, clc

randn('state', 100); rand('state', 100)
t = 5; StockRN = lognrnd(0,sqrt(0.04*t),1,2000);
S = exp(0.07*t).* StockRN; Sr = sort(S);
statePriceRN = lognrnd(0,sqrt( 0.1225*t),1,2000);
SP = sort(statePriceRN); H = exp(-0.1413*t)*Sr.*SP ;
d11 = (log(0.5421./H)-0.0488*(15-t))/0.15*sqrt(15-t);
d21 = d11+0.15*sqrt(15-t);
d12 = (log(0.85./H)-0.0488*(15-t))/0.15*sqrt(15-t);
d22 = d12+0.15*sqrt(15-t);

kp = exp(0.1413*t)*SP ; % A part of stock price for writing in
                        % terms of the state price density function
STP = H.*kp ; % Stock price
m = -Inf;
DsV11 = @(u) NewDst11V(u);
DsV12 = @(u) NewDst12V(u);
DsV21 = @(u) NewDst21V(u);
DsV22 = @(u) NewDst22V(u);
h1 = 0.5421; h2 = 0.85; y = 1;
```

```
FrB = (0.09-0.06)/0.04 ; % The constant benchmark value (fraction)
for jj = 1:length(Sr)
        intDsV11(jj) = quadgk(DsV11,m,-d11(jj));
        intDsV12(jj) = quadgk(DsV12,m,-d12(jj));
        intDsV21(jj) = quadgk(DsV21,m,-d21(jj));
        intDsV22(jj) = quadgk(DsV22,m,-d22(jj));
        Pdfs11(jj) = (1/sqrt(2*pi))*exp((-1/2)*(d11(jj).^2));
        Pdfs12(jj) = (1/sqrt(2*pi))*exp((-1/2)*(d12(jj).^2));
        Pdfs21(jj) = (1/sqrt(2*pi))*exp((-1/2)*(d21(jj).^2));
        Pdfs22(jj) = (1/sqrt(2*pi))*exp((-1/2)*(d22(jj).^2));
    if ( H(jj)<h1 )
        Xt1(jj) = 1./(H(jj))-((1./(H(jj))).*intDsV11(jj)-...
            1.8447*exp(-0.06*(15-t))*intDsV21(jj))+...
            ((1./(H(jj))).*intDsV12(jj)-1.8447*...
            exp(-0.06*(15-t))*intDsV22(jj));
        ptVaR1(jj) = 1-((1.8447*exp(-0.06*(15-t)))./...
            Xt1(jj)).*(intDsV21(jj)-intDsV22(jj))+...
            (1./(0.15*sqrt(15-t)*Xt1(jj).*H(jj))).*...
            ( Pdfs11(jj)-Pdfs12(jj))-((1.8447*...
            exp(-0.06*(15-t)))./(0.15*sqrt(15-t)*...
            Xt1(jj))).*(Pdfs22(jj)-Pdfs21(jj));
        FrVaR1(jj) = FrB*ptVaR1(jj);
        kp1(jj) = H(jj).*kp(jj); % The stock price
    elseif (h1<=H(jj) && (H(jj)<h2))
        Xt2(jj) = 1./(H(jj))-((1./(H(jj))).*intDsV11(jj)-...
            1.8447*exp(-0.06*(15-t))*intDsV21(jj))+...
            ((1./(H(jj))).*intDsV12(jj)-1.8447*...
            exp(-0.06*(15-t))*intDsV22(jj)) ;
        ptVaR2(jj) = 1-((1.8447*exp(-0.06*(15-t)))./...
            Xt2(jj)).*(intDsV21(jj)-intDsV22(jj))+...
            (1./(0.15*sqrt(15-t)*Xt2(jj).*H(jj))).*...
            ( Pdfs11(jj)-Pdfs12(jj))-((1.8447*...
            exp(-0.06*(15-t)))./(0.15*sqrt(15-t)*...
            Xt2(jj))).*(Pdfs22(jj)-Pdfs21(jj));
        FrVaR2(jj) = FrB*ptVaR2(jj);
        kp2(jj) = H(jj).*kp(jj); % The stock price
    elseif ( h2<=H(jj))
        Xt3(jj) = 1./(H(jj))-((1./(H(jj))).*intDsV11(jj)-...
            1.8447*exp(-0.06*(15-t))*intDsV21(jj))+...
            ((1./(H(jj))).*intDsV12(jj)-1.8447*...
            exp(-0.06*(15-t))*intDsV22(jj)) ;
        ptVaR3(jj) = 1-((1.8447*exp(-0.06*(15-t)))./...
            Xt3(jj)).*(intDsV21(jj)-intDsV22(jj))+...
            (1./(0.15*sqrt(15-t)*Xt3(jj).*H(jj))).*...
            ( Pdfs11(jj)-Pdfs12(jj))-((1.8447*...
            exp(-0.06*(15-t)))./(0.15*sqrt(15-t)*...
            Xt3(jj))).*(Pdfs22(jj)-Pdfs21(jj));
```

```
        FrVaR3(jj) = FrB*ptVaR3(jj);
        kp3(jj) = H(jj).*kp(jj); % The stock price
    end
end
Xt2(Xt2==0)=[]; Xt3(Xt3==0)=[];
ptVaR2(ptVaR2==0)=[]; ptVaR3(ptVaR3==0)=[];
FrVaR2(FrVaR2==0)=[]; FrVaR3(FrVaR3==0)=[];
kp2(kp2==0)=[]; kp3(kp3==0)=[];
plot(kp1,FrVaR1,'b.-'), hold on
plot(kp2,FrVaR2,'b.-'),hold on
plot(kp3,FrVaR3,'b.-'), hold on
xlabel('S_{t}'), ylabel('\theta_{t}'), axis([0,3,-1,2.5])
text(0.9282,-1.05,'a','FontSize',13);
text(1.9254,-1.065,'b','FontSize',13);
text(0.8,2.3,...
    'a is the beginning of intermediate stock prices',...
    'FontSize',8.5);
text(0.8,2.1,...
    'b is the end of intermediate stock prices',...
    'FontSize',8.5);

% The fraction of unconstrained portfolio
randn('state', 100); rand('state', 100)
ns = length(STP);
for i=1:ns;
   Tetaunc(i) = 0.75 ;
end
plot(STP,Tetaunc,'r-'), hold on
xlabel('S_{t}'), ylabel('\theta_{t}')

% The fraction of pure stock portfolio
randn('state', 100); rand('state', 100)
for i=1:ns;
   Tetastock(i) = 1;
end
plot(STP,Tetastock,'k-'), hold on
xlabel('S_{t}'), ylabel('\theta_{t}')

% The fraction of pure bond portfolio
randn('state', 100); rand('state', 100)
for i=1:ns;
   Tetabond(i) = 0;
end
plot(STP,Tetabond,'c')
title(strcat('VaR optimal strategy \theta^{VaR} as',...
    ' a function of time t and the stock price S'))
xlabel('S_{t}'), ylabel('\theta_{t}')
```

B.1.4 Necessary m.files which is in the above VaR Strategy Algorithms (Distribution Function Files)

```
function D11 = NewDst11V(u)
D11 = (1/sqrt(2*pi))*exp((-1/2)*u.^2) ;
end
```

```
function D22 = NewDst12V(u)
D22 = (1/sqrt(2*pi))*exp((-1/2)*u.^2);
end
```

```
function D21 = NewDst21V(u)
D21 = (1/sqrt(2*pi))*exp((-1/2)*u.^2);
end
```

```
function D22 = NewDst22V(u)
D22 = (1/sqrt(2*pi))*exp((-1/2)*u.^2);
end
```

B.2 Matlab Algorithms Related to EL Risk Measure

B.2.1 Optimal Horizon Wealth of the EL Risk Manager

```
% The optimal horizon wealth of the EL risk manager as a function
% of the horizon state price density H(T).
clear all, close all, clc

randn('state', 100); rand('state', 100)
y = 1; y2 = 0.364; h1 = 0.5421; h2 = 0.9061;
HT=exp(-1.0695)*lognrnd(0,sqrt(0.3375),1,15000);
s=sort(HT); ns = length(s);
for i=1:ns;
    if s(i)<h1
        I1(i) = 1/(y*s(i));
        s1(i) = s(i);
    elseif ((h1<=s(i)) && (s(i)<h2))
        I2(i) = 1.8447 ;
        s2(i) = s(i);
    elseif h2<=s(i)
        I3(i) = 1/(y*s(i)-y2);
        s3(i) = s(i);
    end
```

```
end
I2(I2==0)=[]; I3(I3==0)=[]; s2(s2==0)=[]; s3(s3==0)=[];
plot(s1,I1,'b.-'), hold on
plot(s2,I2,'b.-'), hold on
plot(s3,I3,'b.-'), hold on
title('optimal terminal wealth of the EL-portfolio manager')
xlabel('H_{T}'), ylabel('\xi_{T}'), axis([0,2,0,15])
text(-0.015,1.8447,'q','FontSize',10.5);
text(0.5421,-0.05,'h','FontSize',9);
text(0.9061,-0.05,'h','FontSize',9);
text(0.6,4,'q is the threshold value.','FontSize',9);
text(0.6,4.5,'h is the beginning of intermediate states',...
    'FontSize',9);
text(0.6,4.25,'h is the end of intermediate states',...
    'FontSize',9);

% Trml as a function of the horizon state price density H(T)
% for the portfolio insurer
for i=1:ns;
    if s(i)<h1
        K1(i) = 1/(y*s(i));
        St1(i)=s(i);
    elseif h1<=s(i)
        K2(i) = 1.8447; % Threshold value.
        St2(i) = s(i);
    end
end
K2(K2==0)=[]; St2(St2==0)=[];
plot(St1,K1,'k-'), hold on
plot(St2,K2,'k-'), hold on
axis([0,2,0,15])

% Trml as a function of the horizon state price density H(T)
% for the unconstrained benchmark
I = 1./s;
plot(s,I,'r')
axis([0,1.5,0,5])
```

B.2.2 Probability Density of the Optimal Horizon Wealth Belonging to the EL Portfolio Manager

```
% The probability density function of the optimal terminal wealth
% concerning EL risk measure
clear all, close all, clc

randn('state', 100); rand('state', 100)
```

```
y1 = 1; y2 = 0.364; h1 = 0.5421; h2 = 0.9061;
X = lognrnd(0,sqrt(0.3375),1,15000);
HT = exp(-1.0695)*X; s = sort(HT); ns = length(s);
for i=1:ns;
    if ( s(i)<h1 )
        I1(i) = 1/(y1*s(i));
        PDI1(i) = (0.6867*exp(-1.4569*...
            (log(0.3432*I1(i))).^2))./I1(i);
    elseif (h1<=s(i) && (s(i)<h2))
        I2(i) = 1.8447 ;
    elseif ( h2<=s(i))
        I3(i) = 1./(y1*s(i)-y2);
        PDI3(i) = (0.6867*exp(-( 1.4815*((log(0.3432*I3(i)./...
            (1+0.364*I3(i)))).^2))))./(I3(i).*(1+0.364*I3(i)));
    end
end
I2(I2==0)=[]; I3(I3==0)=[]; PDI3(PDI3==0)=[];
plot(I1,PDI1,'b.-'), hold on
j = PDI3(1,1):0.0001:PDI1(1,11779);
plot(max(I2),j,'b.-'), hold on
plot(I3,PDI3,'b.-'), hold on
xlabel('\xi_{T}'), ylabel('density'), axis([0,15,0,0.3])
text(3.4469,-0.0075,'*','FontSize',14);
text(2.4596,-0.0065,'b','FontSize',14);
text(2.3495,-0.0065,'l','FontSize',14);
text(3.8574,-0.0025,'\times','FontSize',14);
text(1.8447,-0.002,'\o','FontSize',14);
text(6,0.25,...
    'b  -> E[\xi_{T}^{\theta^{0}}] = 2.4596 (Pure Bond)',...
    'FontSize',10);
text(6,0.22,...
    '\times   -> E[\xi_{T}^{\theta^{1}}] = 3.8574 (Pure Stock)',...
    'FontSize',10);
text(6,0.19,...
    '*   -> E[\xi_{T}^{\theta^{*}}] = 3.4469 (Unconstrained)',...
    'FontSize',10);
text(6,0.16,...
    'l  -> E[\xi_{T}^{\theta^{EL}}] = 2.3495 (EL Constraint)',...
    'FontSize',10);
text(6,0.13,...
    ' \o  -> q = 0.75xe^{rT} = 1.8447 (Threshold)',...
    'FontSize',10);

% The probability density function of the terminal wealth of
% pure stock portfolio
randn('state', 100); rand('state', 100)
xLN = lognrnd(0,sqrt(0.6),1,15000); XT1 = 2.8577*xLN; s = sort(XT1);
```

```
PDXT1 = (0.5150*exp(-0.8333*((log(0.3499*s)).^2)))./s;
plot(s,PDXT1,'k-'), hold on
xlabel('\xi_{T}'), ylabel('density'), axis([0,15,0,0.3])

% The probability density function of the terminal wealth of
% pure bond portfolio.
XT0 = 2.4596 ; % the terminal wealth if I only invest in bond.
P = 0:0.0001:1;
plot(XT0,P), hold on
xlabel('\xi_{T}'), ylabel('density')

% The probability density function of the terminal wealth of
% unconstrained portfolio
randn('state', 100); rand('state', 100)
XTB = 1./s ;
PDFXTB = (0.6867*exp(-1.4569*(log(0.3432*XTB)).^2))./XTB ;
plot(XTB,PDFXTB,'r-')
title(strcat('Probability Density Function of',...
    ' EL Optimal Terminal Wealth'))
xlabel('\xi_{T}'), ylabel('density'), axis([0,15,0,0.3])
```

B.2.3 The EL Optimal Strategy at Time $t < T$ Before the Horizon as a Function of Time t and the Stock Price S and the other Mentioned Strategies

```
% The EL-optimal wealth and the fraction of wealth invested
% in stock before horizon
clear all, close all, clc

randn('state', 100); rand('state', 100)
y1 = 1; y2 = 0.364; h1 = 0.5421; h2 = 0.9061; t = 5;
b = -0.15*sqrt(15-t); a = -(0.06+(0.15^2)/2)*(15-t);
StockRN = lognrnd(0,sqrt(0.04*t),1,2000);
S = exp(0.07*t).* StockRN; Sr = sort(S);
statePriceRN = lognrnd(0,sqrt( 0.1225*t),1,2000);
SP = sort(statePriceRN);
H = exp(-0.1413*t)*Sr.*SP ;  % the state price density function.
c2 = (1/b)*(log(h2./H)-a);
d11 = (log(h1./H)-(0.06-(0.15^2)/2)*(15-t))./0.15*sqrt(15-t);
d21 = d11+0.15*sqrt(15-t);
d12 = (log(h2./H)-(0.06-(0.15^2)/2)*(15-t))./0.15*sqrt(15-t);
d22 = d12+0.15*sqrt(15-t);
kp = exp(0.1413*t)*SP ; % A part of stock price for writing in
                        % terms of the state price density function
STP = H.*kp; m = -Inf; fn = @(u) myNewfn(u);
Ds11 = @(u) DstEL11(u); Ds21 = @(u) DstEL21(u);
```

```
Ds22 = @(u) DstEL22(u); fo = @(u) NEWfio(u);
FrB = (0.09-0.06)/0.04; % the benchmark value
q = 1.8447 ; % threshold value.
for jj = 1:length(Sr)
    intg(jj) = (exp(-0.06*(15-t))/sqrt(2*pi))*quadgk(fn,m,c2(jj));
    intDs11(jj) = quadgk(Ds11,m,-d11(jj));
    intDs21(jj) = quadgk(Ds21,m,-d21(jj));
    intDs22(jj) = quadgk(Ds22,m,-d22(jj));
    Pdfs11(jj) = (1/sqrt(2*pi))*exp((-1/2)*(d11(jj).^2));
    Pdfs21(jj) = (1/sqrt(2*pi))*exp((-1/2)*(d21(jj).^2));
    intfio(jj) = (1/sqrt(2*pi))*quadgk(fo,m,c2(jj));
    if ( H(jj)<h1 )
       XtEL1(jj) = (1./(y1*H(jj))).*(1-intDs11(jj))+...
           q*exp(-0.06*(15-t))*(intDs21(jj)-intDs22(jj))+intg(jj);
       ptEL1(jj) = 1./(XtEL1(jj).*y1.*H(jj))-...
           (1./(XtEL1(jj).*y1.*H(jj))).*...
           (intDs11(jj)+Pdfs11(jj)./(0.15*sqrt(15-t)))-...
           (q*exp(-0.06*(15-t))*Pdfs21(jj))./(0.15*sqrt(15-t)*...
           XtEL1(jj))+(y1*H(jj)*exp((0.15^2-2*0.06)*...
           (15-t)).*intfio(jj))./XtEL1(jj);
       FrEL1(jj) = FrB*ptEL1(jj);
       kp1(jj) = H(jj).*kp(jj); % The stock price
     elseif (h1<=H(jj) && (H(jj)<h2))
       XtEL2(jj) = (1./(y1*H(jj))).*(1-intDs11(jj))+...
           q*exp(-0.06*(15-t))*(intDs21(jj)-intDs22(jj))+intg(jj);
       ptEL2(jj) = 1./(XtEL2(jj).*y1.*H(jj))-...
           (1./(XtEL2(jj).*y1.*H(jj))).*...
           (intDs11(jj)+Pdfs11(jj)./(0.15*sqrt(15-t)))-...
           (q*exp(-0.06*(15-t))*Pdfs21(jj))./(0.15*sqrt(15-t)*...
           XtEL2(jj))+(y1*H(jj)*exp((0.15^2-2*0.06)*...
           (15-t)).*intfio(jj))./XtEL2(jj);
       FrEL2(jj) = FrB*ptEL2(jj);
       kp2(jj) = H(jj).*kp(jj); % The stock price
    elseif ( h2<=H(jj))
       XtEL3(jj) = (1./(y1*H(jj))).*(1-intDs11(jj))+...
           q*exp(-0.06*(15-t))*(intDs21(jj)-intDs22(jj))+intg(jj);
       ptEL3(jj) = 1./(XtEL3(jj).*y1.*H(jj))-...
           (1./(XtEL3(jj).*y1.*H(jj))).*...
           (intDs11(jj)+Pdfs11(jj)./(0.15*sqrt(15-t)))-...
           (q*exp(-0.06*(15-t))*Pdfs21(jj))./(0.15*sqrt(15-t)*...
           XtEL3(jj))+(y1*H(jj)*exp((0.15^2-2*0.06)*...
           (15-t)).*intfio(jj))./XtEL3(jj);
       FrEL3(jj) = FrB*ptEL3(jj);
       kp3(jj) = H(jj).*kp(jj); % The stock price
    end
end
XtEL2(XtEL2==0)=[]; XtEL3(XtEL3==0)=[];
```

```
ptEL2(ptEL2==0)=[]; ptEL3(ptEL3==0)=[];
FrEL2(FrEL2==0)=[]; FrEL3(FrEL3==0)=[];
kp2(kp2==0)=[]; kp3(kp3==0)=[];
plot(kp1,FrEL1,'b.-'), hold on
plot(kp2,FrEL2,'b.-'), hold on
plot(kp3,FrEL3,'b.-'), hold on
xlabel('S_{t}'), ylabel('\theta_{t}'), axis([0,3,-2,2])
text(0.9282,-2.04,'c','FontSize',13);
text(2.1373,-2.07,'d','FontSize',13);
text(1,1.8,'c is the beginning of intermediate stock prices',...
    'FontSize',8.5);
text(1,1.6,'d is the end of intermediate stock prices',...
    'FontSize',8.5);

% The fraction of unconstrained portfolio
randn('state', 100); rand('state', 100)
ns = length(STP);
for i=1:ns;
   Tetaunc(i) = 0.75 ;
end
plot(STP,Tetaunc,'r-'), hold on
xlabel('S_{t}'), ylabel('\theta_{t}')

% The fraction of pure stock portfolio
randn('state', 100); rand('state', 100)
for i=1:ns;
   Tetastock(i) = 1;
end
plot(STP,Tetastock,'k-'), hold on
xlabel('S_{t}'), ylabel('\theta_{t}')

% The fraction of pure bond portfolio
randn('state', 100); rand('state', 100)
for i=1:ns;
   Tetabond(i) = 0;
end
plot(STP,Tetabond,'c')
title(strcat('EL optimal strategy \theta^{EL} as',...
    ' a function of time t and the stock price S'))
xlabel('S_{t}'), ylabel('\theta_{t}')
```

B.2.4 Necessary m.files which is in the above EL Strategy Algorithms

```
function G = myNewfn(u)
y1 = 1; y2 = 0.364; t = 5;
b = -0.15*sqrt(15-t); a = -(0.06+(0.15^2)/2)*(15-t);
```

```
G = exp((-1/2)*(u-b).^2)./(y1*t*exp(a+b*u)-y2);
end
```

```
function fi = NEWfio(u)
randn('state', 100); rand('state', 100)
y1 = 1; y2 = 0.364; h1 = 0.5421; h2 = 0.9061; t = 5;
b = -0.15*sqrt(15-t); a = -(0.06+(0.15^2)/2)*(15-t);
StockRN = lognrnd(0,sqrt(0.04*t),1,2000);
S = exp(0.07*t).* StockRN; Sr = sort(S);
statePriceRN = lognrnd(0,sqrt( 0.1225*t),1,2000);
SP = sort(statePriceRN);
H = exp(-0.1413*t)*Sr.*SP ;  % the state price density function.
fi = exp(-((u-2*b).^2)/2)./((y1*exp(a)*H(2000).*exp(b*u)-y2).^2);
end
```

```
function D11 = DstEL11(u)
% Standard normal distribution function
D11 = (1/sqrt(2*pi))*exp((-1/2)*u.^2) ;
end
```

```
function D21 = DstEL21(u)
% Standard normal distribution function
D21 = (1/sqrt(2*pi))*exp((-1/2)*u.^2);
end
```

```
function D22 = DstEL22(u)
D22 = (1/sqrt(2*pi))*exp((-1/2)*u.^2);
end
```

B.3 Matlab Algorithms Related to EUL Risk Measure

B.3.1 Optimal Horizon Wealth of the EUL Risk Manager

```
% The optimal horizon wealth of the EUL risk manager as a function
% of the horizon state price density H(T).
clear all, close all, clc

randn('state', 100); rand('state', 100)
y = 1; y2 = 0.0303; h1 = 0.5421; h2 = 0.5585;
HT=exp(-1.0695)*lognrnd(0,sqrt(0.3375),1,15000);
s=sort(HT); ns = length(s);
```

```
for i=1:ns;
    if s(i)<h1
        F1(i) = 1./(y*s(i)); % the optimal terminal wealth.
        s1(i) = s(i); %the state price density function.
    elseif ((h1<=s(i)) && (s(i)<h2))
        F2(i) = 1.8447 ; % threshold value.
        s2(i) = s(i); %the state price density function.
    elseif h2<=s(i)
        F3(i) = (1+y2)./(y*s(i)); % the optimal terminal wealth.
        s3(i) = s(i); %the state price density function.
    end
end
F2(F2==0)=[]; F3(F3==0)=[];
s2(s2==0)=[]; s3(s3==0)=[];
plot(s1,F1,'b.-'), hold on
plot(s2,F2,'b.-'), hold on
plot(s3,F3,'b.-'),hold on
title('optimal terminal wealth of the EUL-portfolio manager')
xlabel('H_{T}'), ylabel('\xi_{T}'), axis([0,1,0,15])
text(0.391,1.8447,'q','FontSize',10.5);
text(0.5421,1.216,'h','FontSize',7.5);
text(0.5585,1.216,'h','FontSize',7.5);
text(0.64,2.3,'q is the threshold value.','FontSize',9);
text(0.64,2.450,'h is the beginning of intermediate states',...
    'FontSize',9);
text(0.64,2.375,'h is the end of intermediate states',...
    'FontSize',9);

% Trml as a function of the horizon state price density H(T)
% for the portfolio insurer
for i=1:ns;
    if s(i)<h1
        K1(i) = 1/(y*s(i));
        St1(i)=s(i);
    elseif h1<=s(i)
        K2(i) = 1.8447; % Threshold value.
        St2(i) = s(i);
    end
end
K2(K2==0)=[]; St2(St2==0)=[];
plot(St1,K1,'k-'), hold on
plot(St2,K2,'k-'), hold on
axis([0,2,0,15])

% Trml as a function of the horizon state price density H(T)
% for the unconstrained benchmark
I = 1./s;
```

```
plot(s,I,'r')
axis([0.4,0.9,1.2,2.6])
```

B.3.2 Probability Density of the Optimal Horizon Wealth Belonging to the EUL Portfolio Manager

```
% The probability density function of the optimal terminal wealth
% concerning EUL risk measure
clear all, close all, clc

randn('state', 100); rand('state', 100)
y1 = 1; y2 = 0.0303; h1 = 0.5421; h2 = 0.5585;
X = lognrnd(0,sqrt(0.3375),1,15000); HT = exp(-1.0695)*X;
s = sort(HT); ns = length(s);
for i=1:ns;
    if ( s(i)<h1 )
        I1(i) = 1./(y1*s(i));
        PDI1(i) = (0.6867*exp(-1.4569*...
            (log(0.3432*I1(i))).^2))./I1(i);
    elseif (h1<=s(i) && (s(i)<h2))
        I2(i) = 1.8447 ;
    elseif ( h2<=s(i))
        I3(i) = (1+y2)./(y1*s(i));
        PDI3(i) =(0.6867*exp(-1.4815*...
            ((log(0.3331*I3(i))).^2)))./I3(i);
    end
end
I2(I2==0)=[]; I3(I3==0)=[]; PDI3(PDI3==0)=[];
plot(I1,PDI1,'b.-'), hold on
j = PDI3(1,1):0.0001:PDI1(1,11779);
plot(max(I2),j,'b.-'), hold on
plot(I3,PDI3,'b.-'), hold on
xlabel('\xi_{T}'), ylabel('density'), axis([0,15,0,0.3])
text(3.4469,-0.0075,'*','FontSize',14);
text(2.4596,-0.0065,'b','FontSize',14);
text(8.8482,-0.003,'\spadesuit','FontSize',14);
text(3.8574,-0.0025,'\times','FontSize',14);
text(1.8447,-0.002,'\o','FontSize',14);
text(6,0.25,...
    'b  -> E[\xi_{T}^{\theta^{0}}] = 2.4596 (Pure Bond)',...
    'FontSize',10);
text(6,0.22,...
    '\times   -> E[\xi_{T}^{\theta^{1}}] = 3.8574 (Pure Stock)',...
    'FontSize',10);
text(6,0.19,...
    '*   -> E[\xi_{T}^{\theta^{*}}] = 3.4469 (Unconstrained)',...
```

```
    'FontSize',10);
text(6,0.16,...
    '\spadesuit  -> E[\xi_{T}^{\theta^{EUL}}] = 8.8482 (EUL Constraint)',...
    'FontSize',10);
text(6,0.13,...
    ' \o  -> q = 0.75xe^{rT} = 1.8447 (Threshold)',...
    'FontSize',10);

% The probability density function of the terminal wealth of
% pure stock portfolio
randn('state', 100); rand('state', 100)
xLN = lognrnd(0,sqrt(0.6),1,15000);
XT1 = 2.8577*xLN; s = sort(XT1);
PDXT1 = (0.5150*exp(-0.8333*((log(0.3499*s)).^2)))./s ;
plot(s,PDXT1,'k-'), hold on
xlabel('\xi_{T}'), ylabel('density'), axis([0,15,0,0.3])

% The probability density function of the terminal wealth of
% pure bond portfolio.
XT0 = 2.4596 ; % the terminal wealth if I only invest in bond.
P = 0:0.0001:1;
plot(XT0,P), hold on
xlabel('\xi_{T}'), ylabel('density')

% The probability density function of the terminal wealth of
% unconstrained portfolio
randn('state', 100); rand('state', 100)
XTB = 1./s ;
PDFXTB = (0.6867*exp(-1.4569*(log(0.3432*XTB)).^2))./XTB ;
plot(XTB,PDFXTB,'r-')
title(strcat('Probability Density Function of',...
    ' EUL Optimal Terminal Wealth'))
xlabel('\xi_{T}'), ylabel('density'), axis([0,15,0,0.3])
```

B.3.3 The EUL Optimal Strategy at Time $t < T$ Before the Horizon as a Function of Time t and the Stock Price S and the other Mentioned Strategies

```
% The EUL-optimal wealth and the fraction of wealth invested
% in stock before horizon
clear all, close all, clc

randn('state', 100); rand('state', 100)
y1 = 1; y2 = 0.0303; h1 = 0.5421; h2 = 0.5585; t = 5;
StockRN = lognrnd(0,sqrt(0.04*t),1,2000);
S = exp(0.07*t).* StockRN; Sr = sort(S);
```

```
statePriceRN = lognrnd(0,sqrt( 0.1225*t),1,2000);
SP = sort(statePriceRN);
H = exp(-0.1413*t)*Sr.*SP ;  % the state price density function.
d21 = (log(h1./H)-(0.06-(0.15^2)/2)*(15-t))./(0.15*sqrt(15-t));
d11 = d21+0.15*sqrt(15-t);
d22 = (log(h2./H)-(0.06-(0.15^2)/2)*(15-t))./(0.15*sqrt(15-t));
d12 = d22+0.15*sqrt(15-t);
kp = exp(0.1413*t)*SP; STP = H.*kp; m = -Inf;
DsE11 = @(u) NewDst11EUL(u); DsE21 = @(u) NewDst21EUL(u);
DsE22 = @(u) NewDst22EUL(u); DsE12 = @(u) NewDst12EUL(u);
FrB = (0.09-0.06)/0.04; q = 1.8447;
for jj = 1:length(Sr)
    intDsE11(jj) = quadgk(DsE11,m,-d11(jj));
    intDsE21(jj) = quadgk(DsE21,m,-d21(jj));
    intDsE22(jj) = quadgk(DsE22,m,-d22(jj));
    intDsE12(jj) = quadgk(DsE12,m,-d12(jj));
    if ( H(jj)<h1 )
       XtEUL1(jj) = (1./(y1*H(jj)))-((1./(y1*H(jj))).*...
           intDsE11(jj)-q*exp(-0.06*(15-t))*...
           intDsE21(jj))+(((1+y2)./(y1*H(jj))).*...
           intDsE12(jj)-q*exp(-0.06*(15-t))*intDsE22(jj));
       ptEUL1(jj) = (1-((q*exp(-0.06*(15-t)))./...
           XtEUL1(jj)).*(intDsE21(jj)-intDsE22(jj)));
       FrEUL1(jj) = FrB*ptEUL1(jj);
       kp1(jj) = H(jj).*kp(jj);
    elseif (h1<=H(jj) && (H(jj)<h2))
       XtEUL2(jj) = (1./(y1*H(jj)))-((1./(y1*H(jj))).*...
           intDsE11(jj)-q*exp(-0.06*(15-t))*...
           intDsE21(jj))+(((1+y2)./(y1*H(jj))).*...
           intDsE12(jj)-q*exp(-0.06*(15-t))*intDsE22(jj));
       ptEUL2(jj) = (1-((q*exp(-0.06*(15-t)))./...
           XtEUL2(jj)).*(intDsE21(jj)-intDsE22(jj)));
       FrEUL2(jj) = FrB*ptEUL2(jj);
       kp2(jj) = H(jj).*kp(jj);
    elseif ( h2<=H(jj))
       XtEUL3(jj) = (1./(y1*H(jj)))-((1./(y1*H(jj))).*...
           intDsE11(jj)-q*exp(-0.06*(15-t))*...
           intDsE21(jj))+(((1+y2)./(y1*H(jj))).*...
           intDsE12(jj)-q*exp(-0.06*(15-t))*intDsE22(jj));
       ptEUL3(jj) = (1-((q*exp(-0.06*(15-t)))./...
           XtEUL3(jj)).*(intDsE21(jj)-intDsE22(jj)));
       FrEUL3(jj) = FrB*ptEUL3(jj);
       kp3(jj) = H(jj).*kp(jj);
    end
end
XtEUL2(XtEUL2==0)=[]; XtEUL3(XtEUL3==0)=[];
ptEUL2(ptEUL2==0)=[]; ptEUL3(ptEUL3==0)=[];
```

```
FrEUL2(FrEUL2==0)=[]; FrEUL3(FrEUL3==0)=[];
kp2(kp2==0)=[]; kp3(kp3==0)=[];
plot(kp1,FrEUL1,'b.-'), hold on
plot(kp2,FrEUL2,'b.-'), hold on
plot(kp3,FrEUL3,'b.-'), hold on
xlabel('S_{t}'), ylabel('\theta_{t}'), axis([0,1.25,-1,2])
text(0.9282,-1.0435,'e','FontSize',13);
text(0.9739,-1.065,'f','FontSize',13);
text(0.52,1.85,...
    'e is the beginning of intermediate stock prices',...
    'FontSize',8.5);
text(0.52,1.7,...
    'f is the end of intermediate stock prices',...
    'FontSize',8.5);

% The fraction of unconstrained portfolio
randn('state', 100); rand('state', 100)
ns = length(STP);
for i=1:ns;
   Tetaunc(i) = 0.75 ;
end
plot(STP,Tetaunc,'r-'), hold on
xlabel('S_{t}'), ylabel('\theta_{t}')

% The fraction of pure stock portfolio
randn('state', 100); rand('state', 100)
for i=1:ns;
   Tetastock(i) = 1;
end
plot(STP,Tetastock,'k-'), hold on
xlabel('S_{t}'), ylabel('\theta_{t}')

% The fraction of pure bond portfolio
randn('state', 100); rand('state', 100)
for i=1:ns;
   Tetabond(i) = 0;
end
plot(STP,Tetabond,'c')
title(strcat('EUL optimal strategy \theta^{EUL} as',...
    ' a function of time t and the stock price S'))
xlabel('S_{t}'), ylabel('\theta_{t}')
```

B.3.4 Necessary m.files which is in the above EUL Strategy Algorithms

```
function D11 = NewDst11EUL(u)
```

```
% Standard normal distribution function
D11 = (1/sqrt(2*pi))*exp((-1/2)*u.^2) ;
end
```

```
function D22 = NewDst12EUL(u)
% Standard normal distribution function
D22 = (1/sqrt(2*pi))*exp((-1/2)*u.^2);
end
```

```
function D21 = NewDst21EUL(u)
% Standard normal distribution function
D21 = (1/sqrt(2*pi))*exp((-1/2)*u.^2);
end
```

```
function D22 = NewDst22EUL(u)
% Standard normal distribution function
D22 = (1/sqrt(2*pi))*exp((-1/2)*u.^2);
end
```

REFERENCES

[1] P. Artzner, F. Delbaen, J.-M. Eber, and D. Heath, Coherent measures of risk, Mathematical Finance, 9, pp. 203–228, 1999.

[2] S. Basak and A. Shapiro, Value-at-risk-based risk management: Optimal policies and asset prices, The Review of Financial Studies, 14(2), pp. 371–405, 2001.

[3] F. Black and M. Scholes, The pricing of options and corporate liabilities, The Journal of Political Economy, 81(3), pp. 637–654, 1973.

[4] J. C. Cox and C.-f. Huang, Optimal consumption and portfolio policies when asset prices follow a diffusion process, Journal of Economic Theory, 49, pp. 33–83, 1989.

[5] J. C. Cox and C.-f. Huang, A variational problem arising in financial economics, Journal of Mathematical Economics, 20, pp. 465–487, 1991.

[6] F. Delbaen, Coherent risk measures on general probability spaces, Advances in Finance and Stochastics, pp. 1–37, 2002.

[7] A. Gabih, *Portfolio optimization with bounded shortfall risks*, Ph.D. thesis, Martin-Luther-Universität, 2005.

[8] A. Gabih, W. Grecksch, and R. Wunderlich, Dynamic portfolio optimization with bounded shortfall risks., Stochastic Analysis and Applications, 23(3), pp. 579–594, 2005.

[9] A. Gabih and R. Wunderlich, Optimal portfolios with bounded shortfall risks, Stochastische Analysis, pp. 21–42, 2004.

[10] J. M. Harrison and D. M. Kreps, Martingales and arbitrage in multiperiod security markets, Journal of Economics Theory, 20, pp. 381–408, 1979.

[11] J. M. Harrison and S. R. Pliska, Martingales and stochastic integrales in the theory of continuous trading, Stochastic Processes and Applications, 11, pp. 215–260, 1981.

[12] J. M. Harrison and S. R. Pliska, A stochastic calculus model of continuous trading: complete markets, Stochastic Processes and Applications, 15, pp. 313–316, 1983.

[13] I. Karatzas, J. P. Lehoczky, and S. E. Shreve, Optimal portfolio and consumption decisions for a small investor on a finite horizon, SIAM Journal of Control and Optimization, 25, pp. 1557–1586, 1987.

[14] I. Karatzas, J. P. Lehoczky, S. E. Shreve, and G. L. Xu, Martingale and duality methods for utility maximization in an incomplete market, SIAM Journal of Control and Optimization, 29, pp. 702–730, 1991.

[15] I. Karatzas and S. E. Shreve, *Methods of Mathematical Finance*, Springer-Verlag, New York, 1998.

[16] E. Korn and R. Korn, *Option Pricing and Portfolio Optimization*, AMS. Providence, 2000.

[17] R. Korn, *Optimal Portfolios*, World Scientific, Singapore, 1998.

[18] H. Markowitz, Portfolio selection, The Journal of Finance, 8, pp. 77–91, 1952.

[19] H. Markowitz, *Portfolio Selection: Efficient Diversification of Investment*, John Wiley & Sons, Inc., New York, 2nd edition, 1959, 2nd ed. Basil Blackwell, 1991.

[20] R. C. Merton, Life time portfolio selection under uncertainty: The continuous case, The Review of Economics and Statistics, 51, pp. 247–257, 1969.

[21] R. C. Merton, Optimum consumption and portfolio rules in a continuous-time model, The Journal of Economic Theory, 3, pp. 373–413, 1971.

[22] S. R. Pliska, A stochastic calculus model of continuous trading: optimal portfolios, Mathematical Methods of Operations Research, 11, pp. 371–382, 1986.

[23] Wikipedia, Value at risk — Wikipedia, the Free Encyclopedia, http://en.wikipedia.org/wiki/Value_at_Risk, 2013, [Online; accessed 20-July-2013].