ROBUST SEQUENTIAL MONTE-CARLO ESTIMATION METHODS

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ABSTRACT

ROBUST SEQUENTIAL MONTE-CARLO ESTIMATION METHODS

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This thesis addresses the robust system modeling, analysis and state estimation problem for uncertain systems. In the first part of the thesis, polynomial chaos based system representations and some of their important properties such as stability and controllability are studied. A novel relation between the the eigenvalues of the affine uncertain system matrix and the eigenvalues of the polynomial chaos (PC) transformed system is derived. A necessary and sufficient condition that relates stability of the PC transformed system to the original uncertain system is also obtained as a corollary. A necessary condition for the stability of the more general PC transformed systems is obtained in terms of the one-norm matrix measure identity. Furthermore, some necessary conditions for the controllability are obtained. A set-valued estimation problem and its solution for the state estimation of PC transformed system is proposed. The performances of the proposed estimation technique and a technique proposed in literature including an ad-hoc measurement model are evaluated by three framework examples that are used in literature. An observability analysis is also performed for these models. In the second part of the thesis, an extended and robust particle filtering methods are proposed to the solution of the robust nonlinear estimation problem for uncertain systems with cumulative relative entropy constraint. Additionally, robust estimation problem for instantaneous type relative entropy constraint is studied by referring the recent results in literature. Some numerical solutions are proposed for the related problems utilizing particle filtering and unscented Kalman filtering.

Keywords: Polynomial Chaos, Robust Estimation, Robust Stability and Controllability, Relative Entropy

ÖZ

DAYANIKLI SIRALI MONTE-CARLO TABANLI KESTİRİM YÖNTEMLERİ

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Bu tezde dayanıklı sistem modellemesi, analizi ve durum değişkenlerinin kestirim problemi üzerinde durulmuştur. İlk bölümde, polinomsal kaos tabanlı sistem modellemesi, kararlılık ve kontrol edilebilme gibi önemli sistem özellikleri çalışılmıştır. Doğrusal belirsiz sistem matrisleri ile dönüştürülmüş polinomsal kaos (PC) sisteminin ozdeğerlerini ilişkilendiren yeni bir bağıntı bulunmuştur. Ek olarak dönüştürülmüş PC sisteminin kararlılığını esas belirsiz sistemin kararlılı ğına ilişkilendiren yeterli ve gerekli koşul elde edilmiştir. Daha genel doğrusal belirsiz sistemler için elde edilen dönüştürülmüş PC sistemler için bazı gerekli kararlılık koşulları matrix 1-ölçütü kullanılarak elde edilmiştir. Bazı gerekli kontrol edilebilme koşullarıhem tek değikenli hem de birden çok değişkenli belirsiz sistemler için elde edilmiştir. Küme değerli kestirim problemi ve çözümü öne sürülmüştür. Öne sürülen yaklaşımın ve literatürde var olan basit yaklaşımlıölçüm modele sahip yöntemin performansları üç örnek üzerinde değerlendirilmiştir. Bu modellerin gözlenebilirlik analizi gerçekleştirilmiştir. Tezin İkinci kısımında toplamsal göreceli entropi kısıtına sahip doğrusal olmayan belirsiz sistemlerin dayanaklı kestirimi için parçacık filtreleme yöntemi öne sürülmüş ve problem formülasyonu literatürde varolan sonuçlar kullanılarak elde edilmiştir. Anlık göreceli entropi kısıtına sahip doğrusal olmayan belirsiz sistemlerin kestirimi içinde parçacık filtreleme ve kokusuz Kalman filtreleme yöntemleri öne sürülmüştür.

Anahtar Kelimeler: Polinomsal Kaos, Dayanıklı Kestirim, Dayanaklı Stabilite ve Kontrolibilite, Göreceli Entropi

To my family

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TABLE OF CONTENTS

LIST OF TABLES

TABLES

LIST OF FIGURES

FIGURES

LIST OF ABBREVIATIONS

PC	Polynomial Chaos
LQR	Linear Quadratic Regulator
LQG	Linear Quadratic Gaussian
LMI	Linear Matrix Inequality
BMI	Bilinear Matrix Inequality

CHAPTER 1

INTRODUCTION

The invention of the state-space system representations and relevant optimal control techniques was one of the main corner stones in control theory. However, the applicability of the existing optimal control theoretical techniques to the real engineering applications was restrictive due to the lack of robustness to the system uncertainties [52]. Since early 70s, the control community has mainly focused on robust control techniques in order to get satisfactory designs under inevitable system uncertainties

In late 80s and 90s, some probabilistic analysis and synthesis methods were introduced to overcome the cited drawbacks of the worst-case/deterministic approaches. These techniques, namely randomized algorithms, are based on Monte-Carlo realizations of uncertainty and combine the a priori probabilistic information about the uncertainty at the expense of a probabilistic risk [8, 65].

In the last decade, polynomial chaos expansion technique, which has been used for uncertainty quantification of the physical systems, is utilized to get robust control/estimation designs ([53, 46, 64, 23]). The main idea which was introduced by Wiener is that the second order stochastic processes can be expressed in terms of infinite dimensional series of polynomials of the suitable random variables. In this approach, instead of simulating all possible realizations of the uncertainty, the stochastic bases are used to represent the effect of the uncertainty on the system state by transforming the stochastic dynamics into a higher dimensional deterministic ones. In the recent years, the PC theory is utilized in the solutions of the control, estimation and the parameter identification problems of uncertain systems.

Robust estimation is also widely studied subject since it was noticed that performance of the celebrated Kalman filter is vulnerable to system uncertainty [51],[44]. Different approaches have been introduced in parallel to the advancements in robust control. The main purpose of a robust estimation algorithm is to get a good performance (but not the best) under nominal conditions and an acceptable performance for system models other than the nominal model [54],[45]. This is achieved by limiting the effect of the model uncertainty. Different approaches have been proposed in the literature so far to make the estimators more robust against the modeling errors. The common property of those robust filters is that they yield a suboptimal solution to nominal system but they will guarantee an upper bound on the estimation errors in spite of large modeling errors. In other words, robust estimators yield good performance but not best under nominal conditions and acceptable performance for conditions other than nominal. The common criticism to the classical robust estimation techniques is the over conservatism that is induced by minimax problem definitions.

In this work, we study two main robust estimation techniques, namely polynomial chaos based and relative entropy based robust estimation.

In the first part of the thesis, we study the polynomial chaos based uncertainty modeling and investigated the stability and controllability of the polynomial based uncertain systems. Further we studied the robust estimation problem for uncertain systems that is modeled by polynomial chaos expansion. The main motivation behind studying the polynomial chaos based estimation technique is that the polynomial chaos based approach may overcome the over conservatism problem of classical robust estimation techniques. This is mainly due to the fact that it is an stochastic based approach and the polynomial chaos terms conveys the effect of the uncertainty in time thus some average sense optimizations can be performed rather than worst case approaches.

In the recent years, sampling based nonlinear estimation techniques are becoming popular for nonlinear engineering systems due to substantial increase in computational power. Sequential Monte-Carlo estimation techniques such as particle filters are being used in real engineering problems. Even though these algorithms use computational power, they are vulnerable to the system uncertainty since they use nominal system model. The robustness of those methods is a challenging problem as in the linear estimation techniques. In this regard, in the second part of the thesis, we study the robust nonlinear estimation problem for uncertain systems where the uncertainty is modeled by relative entropy constraint.

1.1 Objectives

First part of this thesis addresses the issue of system analysis and robust estimation of uncertain systems where the uncertainty is modeled by the polynomial chaos expansion. Second part of the the thesis addresses the robust estimation problem for nonlinear uncertain stochastic systems where the uncertainty is modeled by relative entropy constraint.

1.2 Thesis Outline

In Chapter 2, we provide some theoretical preliminary information about orthogonal polynomials and polynomial chaos, that is used in the system analysis of the polynomial chaos based uncertain system models derived in Chaper 3.

In Chapter 3, polynomial chaos (PC) based uncertainty modeling techniques are presented and we make a system analysis for the uncertain systems that are modeled by polynomial chaos expansion. First we study the stability of the PC based systems and make some connection between the original uncertain system and the PC transformed system. We derived a direct relationship between the eigenvalues of the uncertain system and the PC transformed system for single uncertainty case. This novel fruitful relationship enabled us to obtain some concrete results about system properties of the PC transformed systems. We derived a necessary and sufficient condition for stability of the PC transformed system for single uncertainty in affine form case. For more general system representations, we have utilized the matrix measure to derive a necessary condition for the stability of the system by exploiting the banded structure of the system matrix. In the final section, we have studied the controllability of the PC based system models. We have provided some necessary conditions for the controllability of the PC transformed systems. In this regard, we propose a Kalman decomposition procedure in order to eliminate the uncontrollable modes of the PC transformed systems.

In Chapter 4, we studied robust estimation problem for polynomial chaos based uncertain systems. Since the polynomial chaos (PC) theory enables the second order stochastic processes to be expressed in terms of the polynomials of random variables. Thus, PC theory allows the transformation of stochastic dynamics into the deterministic dynamics with random coefficients but with higher dimension. The main motivation behind studying polynomial chaos based robust estimation techniques is that in the classical minimax approaches, the decision maker assumes that the worst-case model will act opposed to him thus it gives too much importance to very unlikely cases. The polynomial chaos based robust estimation techniques that we have proposed are expected to reduce this conservatism due to averaging over the uncertainty space rather than seeking the worst-case scenario. The formulation of the corresponding state estimation problem is difficult since it is not in a suitable form that is used in classical estimation algorithms where the polynomial chaos terms appears in the measurement model as unknown disturbance signals. We propose a different approach for the robust state estimation for polynomial chaos based uncertain systems. The state estimation problem is considered as a set estimation problem where the uncertainty and system disturbances satisfy quasi-deterministic enegry constraint. We showed that the set of possible states are actually ellipsoid where the center and shaping matrix of the ellipsoid can be obtained recursively by augmented Kalman filter equations which is advantageous. We also provide a stochastic interpretation of the problem formulation as average maximum a posteriori state estimation problem. In this regard, we propose two other stochastic estimation problems but we can not provide solutions for them. We provide some necessary conditions for observability of the two measurement models which give a better understanding for the differences of these two models. We have evaluated the performance of the considered two approaches by three illustrative examples that are used in robust estimation community as framework examples. The performance of the proposed approaches are compared with the nominal Kalman filter and classical robust estimation algorithms namely regularized robust Kalman filter and H_∞ filter.

In Chapter 5, we study the robust nonlinear estimation problem for uncertain systems where the uncertainty is modeled by relative entropy constraint. In the first problem, the uncertainty is defined on the joint probability measure between the nominal and perturbed measures over a time horizon where the perturbed measure satisfies the relative entropy constraint with respect to the nominal measure. The problem has also been studied by different researches. Using the available results in literature, the optimal state estimation problem is defined as a minimax estimation problem where the nature seeks to select the worst case probability measure that maximizes the estimation error. On the other hand, the minimizer determines the best estimator for the worst case scenario. The constrained minimax optimization problem is converted to the unconstrained optimization by Lagrange multiplier method. The dual optimization problem requires the calculation of the state estimate sequence over a time horizon that minimizes the exponential of the estimation error as worst case scenario. The problem can be considered as a determination of an output feedback controller for an optimal risk-sensitive stochastic control problem. We provide a solution for linear systems. We propose an approximate solution of the problem by an extended robust estimation algorithm where the system is linearized around the current estimate. Since the solution of the original problem has some noted difficulties, a suboptimal problem is also provided. The suboptimal problem converts the original problem to a sequential optimization problem in terms of an information state in forward time. In the solutions of the problem, particle filtering is proposed for the calculations of the recursive probability measure relations. A complete recursive solution of the suboptimal problem cannot be obtained due to the expectation operation over measurements for a time horizon. We provide an framework example that is used in nonlinear estimation community in order to verify the proposed results. In the second problem, we study the nonlinear estimation problem for instantaneous type relative entropy constraint. Two different sub problems are defined for both the time update and measurement update. Then some approximate solutions are proposed by utilizing the available nonlinear estimation techniques such as particle filtering and unscented Kalman filtering.

In Chapter 6, we provide summary and conclusions and some future works.

CHAPTER 2

THEORETICAL BACKGROUND

In this chapter, we have provided some necessary theoretical background for orthogonal polynomials, polynomial chaos and the related sub topics which are the building stones of our dissertation.

2.1 Orthogonal Polynomials

In this section, we briefly introduce the basic literature related with the orthogonal polynomials [63],[25],[15].

Let μ be a Borel measure defined on an interval X (possibly infinite) in $\mathbb{R}$. Assume that the all the moments $m_n = \int_X x_n d\mu(x)$ are finite. Let Π be the space of real polynomials $p(x) = k_n x^n + k_{n-1}x^{n-1} + \ldots + k_0$ and Π_n be the space of polynomials of degree less than n. For any pair $p(x)$ and $q(x)$, let us define the inner product as

$$\langle p, q \rangle = \int_X p(x)q(x)dx \tag{2.1}$$

If there exists a sequence of polynomials $\{p_n(x)\}_{n=0}^{\infty}$ of $p_n(x) \in \Pi$ such that

$$\langle p_m, q_n \rangle_\mu = \int_X p_n(x)p_m(x)d\mu(x) = \delta_{m,n} \tag{2.2}$$

where $\delta_{m,n}$ being Dirac delta function then $\{p_n(x)\}$ is called the set of orthogonal polynomials with respect to the measure μ. If the measure μ is absolutely continuous with respect to the Lebesque measure whereby $d\mu(x) = w(x)dx$ then $w(x)$ which is a nonnegative integrable function on $\mathbb{R}^d$ is called the weight function.

The measure μ is called discrete if its support consists of a finite or countably infinite number of distinct points x_k at which the discrete measure will have jumps w_k. If the number of distinct points have finite value of N then the associated inner product will be denoted as $\langle p, q \rangle_{\mu_N}$ with

$$\langle p, q \rangle_{\mu_N} = \sum_{n=1}^{N} w_k p(x_k)q(x_k). \tag{2.3}$$

$\langle p,q\rangle$ is positive definite for $p \in \Pi_{N-1}$ but not $p \in \Pi_{n>N-1}$. There are only N orthogonal polynomials $p_0(x), p_1(x), \ldots, p_{N-1}(x)$, satisfying the orthogonality relation

$$\sum_{k=1}^{N} w_k p_m(x_k) p_n(x_k) = h_n \delta_{m,n} \tag{2.4}$$

and they are called discrete orthogonal polynomials.

The orthogonal polynomials $\{p_n(x)\}$, satisfy the following three-term recurrence relation

$$x p_n(x) = a_n p_{n+1}(x) + b_n p_n x + c_n p_{n-1}(x) \tag{2.5}$$

for $n = 0, 1, 2$ where $p_{-1}(x) = 0, p_0(x) = 1$ with $a_n = k_n/k_{n+1}$ where k_n represents the leading coefficient of the nth degree polynomial order, $b_n = \frac{\langle x p_n(x), p_n(x)\rangle}{\langle p_n(x), p_n(x)\rangle}$, and $c_n = a_{n-1} \frac{\langle p_n(x), p_n(x)\rangle}{\langle p_{n-1}(x), p_{n-1}(x)\rangle}$ for $n > 1$. If the polynomials $p_n(x)$ are orthonormal, i.e., $h_n = 1$ polynomials, then $c_n = a_{n-1}$

$$x p_n(x) = a_n p_{n+1}(x) + b_n p_n x + a_{n-1} p_{n-1}(x) \tag{2.6}$$

If the recurrence relation is known, then one can introduce the Jacobi matrix as

$$\mathbb{J}_n = \begin{pmatrix} b_0 & a_0 & 0 & \cdots & 0 \\ c_1 & b_1 & a_1 & \cdots & 0 \\ 0 & c_2 & b_2 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & a_{n-2} \\ 0 & 0 & 0 & c_{n-1} & b_{n-1} \end{pmatrix} \tag{2.7}$$

which satisfies the matrix form of the recurrence relation $xP_n(x) = \mathbb{J}_n P_n(x) + a_n p_n(x) e_n$ where $P_n(x) = (p_0(x), p_1(x), \ldots, p_{n-1}(x))^T$ and $e_n = (0, \ldots, 0)^T$ is the last column of the identity matrix of order n. Note here that for any λ that $p(\lambda) = 0$ as being the zero of the nth order orthogonal polynomial, $(\lambda I - \mathbb{J})P_n(\lambda) = \mathbf{0}$. Thus clearly λ is an eigenvalue of $\mathbb{J}_n$ with the corresponding eigenvector $P_n(\lambda)$. Thus a similarity matrix $T = (P_n(\lambda_0), P_n(\lambda_1), \ldots, P_n(\lambda_n))$ diagonalizes the Jacobi matrix such that $\Lambda = T^{-}\mathbb{J}_n T$. For orthonormal polynomials since the corresponding Jacobi matrix is symmetric, the similarity transformation matrix becomes an orthogonal matrix such that $T^{-} = T^T$. Following theorem (Favard) states the inverse of the three term relation [25].

Theorem 2.1.1 ([25])**.** *If the polynomials $p_n(x)$ of degree n (n=0,1,2,...) satisfy $x p_n(x) = a_n p_{n+1}(x) + b_n p_n x + c_n p_{n-1}(x)$ for $(n > 0)$, with a_n, b_n and c_n real constants and $a_n c_{n+1} > 0$ then there exists a (positive) measure μ on $\mathbb{R}$ such that the polynomials $p_n(x)$ are orthogonal with respect to μ.*

Some important properties of the zeros of the orthogonal polynomials can be listed as [25], [63],[70];

1. Zeros of the orthogonal polynomials are real, distinct, simple and located in the support of the measure.

2. The zeros of nth degree polynomial interlaces with the zeros of the (n-1)st degree orthogonal polynomials.

3. Let $\{p_n\}_{n=1}^{\infty}$ be a sequence of orthogonal polynomials on an interval I. Then for any interval $[a,b] \subset I$, it is possible to find an $m \in N$ such that p_m has at least one zero in $[a,b]$. In other words, the set $\bigcup_{n\geq 1}\bigcup_{k=1}^{n}\left\{\lambda_k^n\right\}$ is dense in I.

4. If the orthogonality measure μ is even, i.e. $\mu(x) = \mu(-x)$ then $p_n(-x) = (-1)^n p_n(x)$, hence $b_n = 0$, so $xp_n(x) = a_n p_n(x) + c_n p_{n-1}(x)$.

5. If there is an orthogonality measure μ with bounded support then μ is unique.

6. If μ is unique then Π is dense in $L_2(\mu)$.

2.1.1 Classical Orthogonal Polynomials

Some systems of orthogonal polynomials namely Hermite, Laguerre and Jacobi are widely used and they are named as classical orthogonal polynomials [12],[25],[47].

2.1.1.1 Jacobi Orthogonal Polynomials

The Jacobi polynomials, $P_n^{(\alpha,\beta)}$ are the polynomials that can be defined by the following Rodrigues formula

$$P_n^{(\alpha,\beta)} = (-2)^n (n!)^{-1}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\left[(1-x)^{n+\alpha}(1+x)^{n+\beta}\right] \tag{2.8}$$

on the interval $[-1,1]$ where the parameters α and β are restricted such that $\alpha > -1$ and $\beta > -1$. For some special values of the parameters, the polynomials are named as

The Legendre polynomials ($\alpha = \beta = 0$)

$$L_n(x) = P_n^{(0,0)}(x) \tag{2.9}$$

for which the following three-term relation is satisfied.

$$(n+1)L_{n+1}(x) = (2n+1)xL_{n+1}(x) - nL_{n-1},\ \ L_0(x) = 1,\ L_1(x) = x \tag{2.10}$$

The Tchebichef polynomials of the first kind ($\alpha = \beta = -1/2$)

$$T_n(x) = 2^{2n}\begin{pmatrix} 2n \\ n \end{pmatrix}^{-1} P_n^{(-1/2,-1/2)}(x) \tag{2.11}$$

for which the following three-term relation is satisfied

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)\ \ T_0(x) = 1,\ T_1(x) = x \tag{2.12}$$

The Tchebichef polynomials of the second kind ($\alpha = \beta = 1/2$)

$$U_n(x) = 2^{2n} \begin{pmatrix} 2n+1 \\ n+1 \end{pmatrix}^{-1} P_n^{(1/2,1/2)}(x) \tag{2.13}$$

for which the following three-term relation is satisfied

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x) \;\; U_0(x) = 1,\; U_1(x) = 2x \tag{2.14}$$

The Gegenbauer polynomials of the second kind ($\alpha = \beta$) with $\alpha = \lambda - 1/2 \neq -1/2$ are defined as

$$C_n\lambda(x) = \begin{pmatrix} n+2\alpha \\ \alpha \end{pmatrix} \begin{pmatrix} 2n+1 \\ n+1 \end{pmatrix}^{-1} P_n^{(\alpha,\alpha)}(x) \tag{2.15}$$

for which the following three-term relation is satisfied

$$C_{n+1}(x) = 2xC_n(x) - U_{n-1}(x) \;\; C_0(x) = 1,\; C_1(x) = 2\lambda x \tag{2.16}$$

2.1.1.2 Laguarre Polynomials

The Laguarre polynomial $L_n^\alpha(x)$ is defined by a Rodrigues' type formula as

$$L_n^\alpha(x) = (n!)^{-1} x^{-\alpha} e^x \frac{d^n}{dx^n} [e^{-x} x^{n+\alpha}] \tag{2.17}$$

on the interval $[0, \infty]$ where it is customary to require that $\alpha > -1$. The Laguarre polynomials satsify

$$(n+1)L_{n+1}^\alpha(x) = (2n+\alpha+1-x)L_n^\alpha(x) - (n+\alpha)L_{n-1}^\alpha(x),\;\; L_0^\alpha(x) = 1,\; L_1^\alpha(x) = x \tag{2.18}$$

2.1.1.3 Hermite Orthogonal Polynomials

Hermite polynomials are defined by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \tag{2.19}$$

on the interval $[-\infty, \infty]$ for which the following three-term relation is satisfied.

$$H_{n+1}(x) = xH_n(x) - nH_{n-1},\;\; H_0(x) = 1,\; H_1(x) = x \tag{2.20}$$

2.1.2 Multivariate Orthogonal Polynomials

The basics related with the multivariate orthogonal polynomials are taken from [15]. We will follow the widely used multi-index notation for expressing the multivariate orthogonal polynomials. A multi-index is denoted as $\mathbf{i} = (i_1, i_2, \ldots, i_d) \in \mathbb{N}_0^d$ where $\mathbb{N}_0^d$ is the set of d dimensional vector of nonnegative integers. In this regard, let $|i| = i_1 + i_2 + \ldots + i_d$ and define the monomials in the variables $(x_1, x_2, \ldots, x_d)$ is a product $\mathbf{x}^{\mathbf{i}} = x_1^{i_1} x_2^{i_2} \ldots x_d^{i_d}$. In general, a polynomial $p(x)$ in d variables is a linear combinations of monomials $p(\mathbf{x}) = \sum_{\mathbf{i}} \mathbf{c}_{\mathbf{i}} \mathbf{x}_{\mathbf{i}}$. The space of polynomials of degree at most n with d variables is denoted as $\Pi_n^d = \left\{p : p(x) = \sum_{|\mathbf{i}| \leq n} \mathbf{c}_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}\right\}$. The space of homogeneous (i.e. all monomials in it have the same order) polynomials of degree n in d variables will be denoted as $\mathcal{P}_n^d = \left\{p : p(x) = \sum_{|\mathbf{i}|=n} \mathbf{c}_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}\right\}$. Then every polynomial in Π_n^d can be written as a linear combination of homogenous polynomials.

$$p(\mathbf{x}) = \sum_{k=0}^{n} \sum_{|\mathbf{i}|=k} \mathbf{c}_{\mathbf{i}} \mathbf{x}^{\mathbf{i}} \tag{2.21}$$

If we denote the number of monomials by r_n^d of degree exactly n; it follows that $r_n^d = \frac{(n+d-1)!}{(n)!(d-1)!}$ and the $dim \Pi_n^d = \frac{(n+d)!}{n!d!}$. The ordering of the multi-index $\mathbf{i}$ is not unique, but graded lexicographic order is most widely used one as defining a one-to-one mapping between the multi-index $\mathbf{i}$ and a single index i as $T(\mathbf{i}) = i, T : \mathbb{N}_0^d \to N$. In the degraded lexicographic order, $\mathbf{i} > \mathbf{j}$ if and only if $|\mathbf{i}| > |\mathbf{j}|$ and the first nonzero entry in the difference $\mathbf{i} - \mathbf{j}$ is positive. Thus, the multivariate orthogonal polynomials can also be expressed as $p_i(\mathbf{x})$. In this notation, $p_0(\mathbf{x})$ refers to the zeroth order polynomial and $p_1(\mathbf{x}), \ldots, p_d(\mathbf{x})$ refers to the first order polynomials of $\mathbf{x}$. A linear functional L is called square positive if $L\left((p(\mathbf{x}))^2\right) > 0$ for all $p(x) \in \Pi^d$ and $p(x) \neq 0$. Specifically, we will consider all-linear functionals expressible as integrals against a Borel measure with finite moments. That is

$$L(p(\mathbf{x})) = \int_{\mathbb{R}^d} p(\mathbf{x}) d\mu(\mathbf{x}) \tag{2.22}$$

Thus the square positive linear functional L induces an inner product $< .,. >$. Two multivariate polynomials are said to be orthogonal with respect to L if $L(p(\mathbf{x})q(\mathbf{x})) = 0$. Let $n \in \mathbb{N}_0$ and $x \in \mathbb{R}^d$, denote $\mathbf{x}^n = (\mathbf{x}^{\mathbf{i}})_{|\mathbf{i}|=n}$ as a vector of size r_n^d, where the monomials are arranged according to the graded lexicographical order of $\left\{\mathbf{i} \in \mathbb{N}_0^d : |i| = n\right\}$. The application of Gram-Schmidt orthogonalization process with respect to a linear functional L on the monomials given by $\{\mathbf{x}^n\}$ will yield the sequence of orthogonal polynomials denoted by $\{\varphi_k^n(x)\}_{n=0,k=1}^{\infty, r_n^d}$ such that n refers that $\varphi_k^n(x) \in \Pi_n^d$. In multi-index notation,

$$\int_{\mathbb{R}} \varphi_i(\mathbf{x}) \varphi_j(\mathbf{x}) d\mu(\mathbf{x}) = \gamma_i \delta_{\mathbf{i},\mathbf{j}} \tag{2.23}$$

with $\delta_{\mathbf{i},\mathbf{j}} = \prod_{k=1}^d \delta_{i_k j_k}$ being a multi-index delta operator. The multivariate orthogonal polynomials can also be generated by the tensor product of univariate orthogonal polynomials. That is if an ith order univariate orthogonal polynomial is defined as $\phi_i(x)$ then the multivariate orthogonal polynomials can be built as $\varphi_i(\mathbf{x}) = \prod_{k=1}^d \varphi_{i_k}(x_k)$ for $0 \leq i_k \leq |\mathbf{i}|$. If a column vector representation of the orthogonal polynomials is introduced as

$$P_n(\mathbf{x}) = (\varphi_k^n)_{|\mathbf{i}|=n} = \left[\varphi_1^n(\mathbf{x}), \varphi_2^n(\mathbf{x}), \ldots, \varphi_k^n(\mathbf{x})\right] \tag{2.24}$$

the orthonormality property of the polynomial series $\left\{\varphi_k^n(\mathbf{x})\right\}$ can be expressed as

$$L(P_n(\mathbf{x}) P_m^T(\mathbf{x})) = \left\{ \begin{array}{cc} I_{r_n^d} & \text{if } n = m \\ \mathbf{0}_{n \times m} & \text{if } n \neq m \end{array} \right\} \tag{2.25}$$

We denote by V_n^d the space of d-variate orthogonal polynomials of degree exactly n; $V_n^d = \left\{p \in \Pi_n^d :< p, q >= 0, \forall q \in \Pi_{n-1}^d\right\}$. Thus $\Pi_n^d = \oplus_{k=0}^n V_k^d$ and $\Pi^d = \oplus_{k=0}^\infty V_k^d$. For $n \geq 0$, there exists unique matrices $A_{n,i} : r_n^d \times r_{n+1}^d$, $B_{n,i} : r_n^d \times r_n^d$ and $C_{n,i}^T : r_n^d \times r_{n-1}^d$ such that the following three term relation is satisfied

$$x_i P_n(\mathbf{x}) = A_{n,i} P_{n+1}(\mathbf{x}) + B_{n,i} P_n(\mathbf{x}) + C_{n,i}^T P_{n-1}(\mathbf{x}) \tag{2.26}$$

for $1 \leq i \leq d$ where $P_{-1}(\mathbf{x}) = 0$, $C_{-1,i} = 0$. Here $A_{n,i} H_{n+1} = L(x_i P_n(\mathbf{x}) P_{n+1}^T(\mathbf{x}))$, $B_{n,i} H_n = L(x_i P_n(\mathbf{x}) P_n^T(\mathbf{x}))$ and $A_{n,i} H_{n+1} = H_{n+1} C_{n+1,i}^T$ with $H_n = L(P_n(\mathbf{x}) P_n^T(\mathbf{x})))$. For orthonormal polynomials, $H_n = I$, so

$$x_i P_n(\mathbf{x}) = A_{n,i} P_{n+1}(\mathbf{x}) + B_{n,i} P_n(\mathbf{x}) + A_{n-1,i}^T P_{n-1}(\mathbf{x}) \tag{2.27}$$

for $1 \leq i \leq d$ where $P_{-1}(\mathbf{x}) = 0$, $A_{-1,i} = 0$. Moreover, each $B_{n,i}$ is symmetric. For the associated three term recursions with related matrix coefficients, block tri-diagonal matrices $\mathbb{J}_{n,i} \in \mathbb{R}^{r \times r}$ where $r = dim \Pi_n^d$ for $i = 1, 2, \ldots, d$ can be introduced as

$$\mathbb{J}_{n,i} = \begin{pmatrix} B_{0,i} & A_{0,i} & 0 & \cdots & 0 \\ C_{1,i} & B_{1,i} & A_{1,i} & \cdots & 0 \\ 0 & C_{2,i} & B_{2,i} & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & A_{n-2,i} \\ 0 & 0 & 0 & C_{n-1,i} & B_{n-1,i} \end{pmatrix} \tag{2.28}$$

They are named as block Jacobi matrices. For polynomials in one variable, the nth order orthonormal polynomial has n distinct zeros and they are the eigenvalues of the truncated Jacobi matrix J_n. For multivariable polynomials, the set of zeros can be different algebraic varieties such as a curve, as well as a point [75]. Thus, it is much more convenient to consider the common zeros of the set of orthogonal polynomials. A common zero of $P_n(\mathbf{x})$ is the common zero of the each element of the vector. It can be shown that common zeros of $P_n(\mathbf{x})$ are distinct and simple [75]. $\Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_d)^T \in \mathbb{R}^d$ is a joint eigenvalue of $\mathbb{J}_{n,1}, \mathbb{J}_{n,2}, \cdots, \mathbb{J}_{n,d}$ if there is a $\eta \neq 0, \eta \in R^r$ such that $\mathbb{J}_{n,i}\eta = \lambda_i \eta$ for $i = 1, .., d$ then the vector η is called as a joint eigenvector associated with the joint eigenvalue λ_i.

Theorem 2.1.2 ([75]). *$\Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_d)^T \in R^d$ is a common zero of $P_n(\mathbf{x})$ if and only if it is a joint eigenvalue of $\mathbb{J}_{n,1}, \mathbb{J}_{n,2}, \cdots, \mathbb{J}_{n,d}$; moreover, a joint eigenvector of Λ is $(P_1(\Lambda), \cdots, P_{n-1}(\Lambda))^T$.*

The polynomials in $P_n(\mathbf{x})$ have at most $r = dim \Pi_n^d$ zeros. If the polynomials in $P_n(\mathbf{x})$ have r common zeros, then the Jacobi matrices $\mathbb{J}_{n,1}, \mathbb{J}_{n,2}, \cdots, \mathbb{J}_{n,d}$ have r distinct eigenvalues which implies that the corresponding eigenvectors must be orthogonal. In other words, $\mathbb{J}_{n,1}, \mathbb{J}_{n,2}, \cdots, \mathbb{J}_{n,d}$ can be simultaneously diagonizable, by a non-singular matrix. Since family of matrices are diagonizable if and only if they mutually commute, the family of the Jacobi matrices satisfies $\mathbb{J}_{n,i}\mathbb{J}_{n,j} = \mathbb{J}_{n,j}\mathbb{J}_{n,i}$ for $1 \neq i, j \neq d$ if the polynomials in $P_n(\mathbf{x})$ have $r = dim \Pi_n^d$ common zeros. The commuting property of the block Jacobi matrices necessitates [15]

$$A_{n-2,i}^T A_{n-2,j} + B_{n-1,i}^T B_{n-1,j}^T = A_{n-2,j} A_{n-2,i} + B_{n-1,j}^T B_{n-1,i} \tag{2.29}$$

Unfortunately, commuting property of the block Jacobi matrices is not satisfied for most of the generally used multivariate orthogonal polynomials. For instance, block Jacobi matrices of the multivariable orthogonal polynomials generated by the tensor product of the univariate orthogonal polynomials do not commute. However for discrete orthogonal polynomials of several variables, the corresponding Jacobi matrices commute [76]. As cited before, all zeros of the univariate orthogonal polynomials of support $[a, b]$ are located in the interior of $[a, b]$. However, this is not true for multivariable orthogonal polynomials. A classical counter example is the set of orthogonal polynomials in a region $D = \left\{(x, y)| \ 1 \leq x^2 + y^2 \leq 2\right\}$ with a uniform weight. $(0, 0)$ which is not in D, is a common zero of all the orthogonal polynomials of odd degree. In [43], it is shown that all common zeros are in the closed convex hull of the support D , i.e., $\lambda \in \overline{conv}(D)$. If $D \in \mathbb{R}^d$ is a convex set, then the all common zeros of orthogonal polynomials of degree $n(n \geq 1)$ lie on the closure $\bar{D}$ of the set D [43].

2.2 Polynomial Chaos

There is a close connection between the orthogonal polynomials and the second order stochastic processes. This connection was first established by Wiener as the homogenous chaos in his seminal paper in 1938 [68]. Wiener used Hermite polynomials in terms of the Gaussian random variables to represent Gaussian processes. Subsequently, Cameron and Martin used Hermite polynomials as spectral expansion bases for the square integrable functionals (with respect to Wiener measure) of the continuous functions on the interval [0,1] vanishing at zero [9]. Moreover, they showed that the series expansion converges in mean square sense. The stochastic interpretation of the Cameron-Martin theorem is that every stochastic process with finite second-order moment can be represented by an (infinite) Hermite-chaos series [73], [19].

In this regard, let us consider a Gaussian linear space H, which is a linear space of d dimensional random variables $\xi \in \mathbb{R}^d$, defined on some probability space (Ω, F, μ) such that each element is a vector of zero mean with independent Gaussian components. It is clear that $H \in L_2(\Omega, F, \mu)$ where $L_2(\Omega, F, \mu)$ is the vector space of random vectors $\xi(\omega)$ such that $E_\xi\left\{|\xi(\omega)|^2\right\} = \left\langle|\xi(\omega)|^2\right\rangle_\mu < \infty$. Now consider the space $P_n(H)$ of d-variate, nth order polynomials of random variables in H. Since all the mixed moments of the independent Gaussian random variables are simply the products of the individual moments by using the Holder's inequality [34],[19], it can be stated that the polynomial space $P_n(H)$ and its closure $\bar{P}_n(H)$ are subspaces of the $L_2(\Omega, F, \mu)$. $\bar{P}_n(H)$ can be orthogonally decomposed as $\bar{P}_n(H) = \otimes_{k=0}^n H_k$ where $H_k := \bar{P}_k(H) \cap \bar{P}_{k-1}(H)^\perp$. H_k is will be called as kth order homogenous chaos. The full space can be decomposed as $\otimes_{k=0}^\infty H_k = \overline{\cup_{n=0}^\infty P_n(H)}$. Thus, the statement of Cameron-Martin theorem is expressed as $\otimes_{k=0}^\infty H_k = L_2(\Omega, \sigma(H), \mu)$ where $\sigma(H)$ is the smallest sigma-algebra induced by H. If the multi-index notation is used for the polynomial chaos representation, then the series expansion of any random variable in $x \in L_2(\Omega, F, \mu)$ can be represented as $x(\omega) = \sum_{\mathbf{i} \in \mathbb{N}_0^d} \alpha_\mathbf{i} H_\mathbf{i}(\xi(\omega))$ where $H_\mathbf{i}(\xi(\omega)) = \Pi_{k=1}^d H_{i_k}(\xi_k(\omega))$ and $\alpha_\mathbf{i} = \langle x(\omega), H_\mathbf{i}(\xi(\omega))\rangle$. The expansion can also be represented in terms of the ordered (graded lexicographic) multi-index

Table2.1: Wiener-Askey Polynomial Chaos for Random Variables

Random Variables	**Wiener-Askey Chaos**
Gaussian	Hermite chaos
Gamma	Laguerre-chaos
Beta	Jacobi-chaos
Uniform	Legendre chaos
Poisson	Charlier-Chaos

notation as $x(\omega) = \sum_{i=0}^{\infty} \hat{\alpha}_i H_i(\boldsymbol{\xi}(\omega))$ where there is a one-to-one correspondence between the functionals $H_{\mathbf{i}}(\boldsymbol{\xi}(\omega))$ and $H_i(\boldsymbol{\xi}(\omega))$ and also between the coefficients $\alpha_{\mathbf{i}}$ and $\hat{\alpha}_i$.

Hermite-Chaos expansion has been quite effective for solving the differential equations with Gaussian inputs as well as certain types of non-Gaussian inputs such as lognormal distributions [73]. However, for the general non-Gaussian distributed random inputs, the convergence rate is not fast. Hermite-Chaos expansion is generalized by utilizing the orthogonal polynomials in Askey-scheme of hyper-geometric orthogonal polynomials, which are orthogonal with respect to some probability distributions [72, 70]. Then it is named as Wiener-Askey Chaos or Askey-Chaos. Similar to the Hermite-Chaos case, any square integrable random variable can be expressed as $x(\omega) = \sum_{\mathbf{i}\in\mathbb{N}_0^d} \alpha_{\mathbf{i}} \Phi_{\mathbf{i}}(\xi(\omega))$ where $\Phi_{\mathbf{i}}(\boldsymbol{\xi}(\omega))$ denotes the multivariate Askey polynomials or with an ordered (graded lexicographic) multi-index notation $x(\omega) = \sum_{i=0}^{\infty} \hat{\alpha}_i \Phi_i(\boldsymbol{\xi}(\omega))$. The Askey polynomials and their corresponding orthogonality probability distributions are listed in Table 2.1.

Remark 1. The polynomial chaos theory can be applied to the stochastic processes that are functions of both the spatial (s) and the temporal (t) coordinates. That is $x(t, s, \omega) = \sum_{i=0}^{\infty} x_i(t, s)\Phi_i(\boldsymbol{\xi}(\omega))$. Those processes have a general name as random fields.

Recently in [19], it is shown that the polynomial chaos expansion can also be generalized to polynomials of random variables ξ_m, which have finite moments of all order, i.e.,$\left\langle |\xi_m|^k \right\rangle \leq \infty$ for all k and have continuous probability distribution functions $F_{\xi_m}(x) := \mu(\xi_m \leq x)$

Theorem 2.2.1 ([19]). *The sequence of orthogonal polynomials $\{\xi_m\}$, associated with a real random variable ξ which have finite moments of all order, i.e., $< |\xi_m|^k >< \infty$ for all k and having continuous probability distribution functions $F_{\xi_m}(x) := \mu(\xi_m \leq x)$, is dense in $L_2(\mathbb{R}, \mathbb{B}(\mathbb{R}), F_{\xi_m}(dx))$ if and only if the distribution function of ξ is uniquely determined by sequence of its moments.*

2.3 Solution of Stochastic Differential Equations

In the previous section, we have presented the spectral representations of square integrable random variables by orthogonal polynomials in terms of the random basis and with the corresponding deterministic coefficients. This approach can be extended to the stochastic processes by performing the spectral expansion for each time value t [66].

In most of the engineering problems, the propagation of the effect of the system uncertainty on the system outputs is valuable information. In this regard, since the spectral polynomial bases are fixed, if the deterministic coefficients of the spectral bases are determined at each time then the complete response can be obtained in terms of the statistics of the solution. There are two main approaches for determining the time propagation of the spectral bases coefficients namely intrusive and non-intrusive [17]. In the intrusive approach, the truncated polynomial expansions of all dependent variables and uncertain parameters are plugged into the differential equations of the physical system. By taking the projection of the governing equations onto each orthogonal polynomial basis, namely Galerkin projection, the set of deterministic differential (or difference) equations that defines the time propagation of the polynomial bases coefficients is determined. Thus, each element of the governing equation is made orthogonal to the approximation space. In this regard, the stochastic processes $x(t,\omega)$ is defined by the following stochastic differential equation.

$$\dot{x}(t,\omega) = f(x(t,\omega), \Delta(\boldsymbol{\xi}(\omega)), u(t)) \tag{2.30}$$

with a known initial value $x(0,\omega) = x_0 \in \mathbb{R}^n$ and an unknown system parameter $\Delta(\boldsymbol{\xi}(\omega)) \in \mathbb{R}^d$ with a deterministic input $u(t)$. If an approximate solution of the stochastic differential equation is defined as the truncated PC expansion

$$\tilde{x}(t,\omega) = \sum_{j=0}^{r} \tilde{x}_j(t)\Phi_j(\boldsymbol{\xi}(\omega)) \tag{2.31}$$

and

$$\Delta(\boldsymbol{\xi}(\omega)) = \sum_{i=0}^{r} \Delta_i \Phi_i(\boldsymbol{\xi}(\omega)) \tag{2.32}$$

then the residue in the governing equation is obtained by plugging the approximate solution in the state dynamics and the uncertain parameter as

$$e_r(\boldsymbol{\xi}(\omega), t) = f\left(\sum_{j=0}^{r} x_j(t)\Phi_j(\boldsymbol{\xi}(\omega)),\ \sum_{i=0}^{r} \Delta_i\Phi_i(\boldsymbol{\xi}(\omega)), u(t)\right) - \sum_{m=0}^{r} \dot{x}_m(t)\Phi_m(\boldsymbol{\xi}(\omega)) \tag{2.33}$$

If the projection of the residue $e_r(\boldsymbol{\xi}(\omega), t)$ of the governing equation onto the polynomial bases is set to zero, then the dynamic equations of time-varying coefficients is obtained as

$$\dot{x}_m(t) = \frac{1}{\langle \Phi_m(\boldsymbol{\xi}(\omega)), \Phi_m(\boldsymbol{\xi}(\omega))\rangle} \left\langle f\left(\sum_{j=0}^{r} x_j(t)\Phi_j(\boldsymbol{\xi}(\omega)), \sum_{i=0}^{r} \Theta_i\Phi_i(\boldsymbol{\xi}(\omega)), u(t)\right), \Phi_m(\boldsymbol{\xi}(\omega))\right\rangle \tag{2.34}$$

for m=1,2,...,r.

The intrusive method, which creates a simple formulation, can be very difficult to implement for complex problems. Thus, in order to reduce the complexity of the approach, some non-intrusive i.e. sampling based methods can be resorted for more complex problems [17],

[70],[71]. There are two main types of non-intrusive methods namely regression method, based on randomly generated points and pseudo-spectral collocation method, based on deterministically chosen points on the grids [17],[35]. In the regression method via least square, once the uncertainty is randomly sampled from the distribution, the states are propagated in time according to the governing equation. Then, the coefficients of the spectral bases are determined such that the sum of square error between the propagated state and the PC expansion is minimized. That is

$$\{x_j(t)\} \cong argmin_{\{\tilde{x}_j(t)\}} \frac{1}{M} \sum_{z=1}^{Z} \left(x(t, \boldsymbol{\xi}(\omega_z)) - \sum_{j=0}^{r} \tilde{x}_j(t) \Phi_j(\boldsymbol{\xi}(\omega_z)) \right)^2 \tag{2.35}$$

resulting in the following normal equation

$$(A^T A) X(t) = A^T Y(t) \tag{2.36}$$

where $X(t) = (x_0^T(t), \dots, x_r^T(t))$ and $Y(t) = (x(t, \boldsymbol{\xi}(\omega_1)), \dots, x(t, \boldsymbol{\xi}(\omega_r)))$ with $A(i, j) = \Phi_j(\boldsymbol{\xi}(\omega_z))$ $z = 1, 2, ..Z$ and $j = 1, 2, \dots, r$. On the other hand, in the pseudo-spectral collocation approach, the coefficients are evaluated by the numerical integration of the inner product $x_j(t) = \left\langle x(t, \omega), \Phi_j(\boldsymbol{\xi}(\omega)) \right\rangle$ for j=1,2,...,d, where $x(t, \omega)$ is collocated on certain quadrature grids or nodes defined on the support X of the measure μ. That is

$$x_j(t) = \left\langle x(t, \omega), \Phi_j(\boldsymbol{\xi}) \right\rangle = \sum_{q=1}^{Q} x(t, \boldsymbol{\xi}_q) \Phi_j(\boldsymbol{\xi}_q) \tag{2.37}$$

where $\boldsymbol{\xi}_q$ is the set of quadrature nodes in X and w_q are the corresponding quadrature weights. The selection criterion for the quadrature rules is not unique. The most popular one is the Gaussian quadrature method in which the nodes are distributed according to the probability weight of the each random input.

2.3.1 The Divergence of Polynomial Chaos Expansion and Time-Dependent Polynomial Chaos

Polynomial chaos theory assures the mean square convergence of the PC approximations. However, the PC theory does not guarantee the convergence of the higher order moments of the PC approximation with increased number of PC expansion terms [21]. For long-time integrations, divergence of the truncated PC expansion is highly possible due to the nonlinearities in the system dynamics and uncertainty sources making the probability distribution of the solution at a time deviate from the initial distribution of the solution. Thus, the set of orthogonal polynomials that has been chosen initially will not be optimal anymore [27]. More terms are needed to be used in PC expansion in order to get reasonable approximation of the solution as time evolves. In order to prevent the possible divergence of the PC expansion for long-time integrations, a practical method is proposed in [27]. This approach is named as time-dependent polynomial chaos. In their approach, after a certain time of integration before the significant divergence did not begin, PC expansion is stopped and new basis function is constructed from the current solution. The integration is continued with newly defined coordinates [30].

CHAPTER 3

UNCERTAIN SYSTEM MODELLING AND ANALYSIS BY POLYNOMIAL CHAOS

3.1 Introduction

In the last decade, polynomial chaos expansion technique, which has been used for uncertainty quantification of the physical systems, is utilized to get robust control/estimation designs [53, 46, 64, 23]. In this approach, instead of simulating all possible realizations of the uncertainty, the stochastic bases are used to represent the effect of the uncertainty on the system state by transforming the stochastic dynamics into the deterministic systems with a higher dimension. In [46], the uncertainty in the feed-forward controller and system parameters is quantified by the PC simulations. In [31], the stability and the transient response of a class of controlled nonlinear systems is analyzed for uncertain controller gain. They also discussed the tradeoff between the stability and accuracy of the PC expansion. In [23], a Kronecker product form of the system model is presented for linear uncertain systems modeled by PC expansion. They also investigated the stochastic stability conditions for the robust LQR problem. In [22], robust LQR problem is solved by bilinear matrix inequalities (BMI). The proposed method is applied to an F16 longitudinal channel control problem. In the dissertation [64], robust H2 and related LQG control problem with PC uncertainty modeling is investigated.

Stochastic stability of the uncertain systems is analyzed by the truncated PC transformed deterministic systems in [22] by utilizing the Lyapunov stability criteria with linear matrix inequalities (LMI). Due to the truncation, the stability results cannot guarantee the stability of the original uncertain system. Thus in their work, it is assumed that the polynomial chaos expansion order is large enough. Since the analysis relies on some numerical methods such as convex optimization, the interaction between the nominal system matrix and the perturbation matrix is disguised. In [41], the stability of the infinite dimensional polynomial chaos transformed system is studied and a sufficient condition in terms of original system and perturbed matrix is derived for the stability of the infinite dimensional system. The results also imply the moment stability of the original uncertain systems. On the other hand as far as the author's knowledge, there is no detailed analysis on the controllability PC transformed systems.

In this chapter, we study polynomial chaos based system representations and some of their important properties such as stability and controllability.

First we present available system representations induced by polynomial chaos expansion of the system state. Mainly we focus on the affine uncertain systems where the system matrix is an affine function of system uncertainties. In the next section, we studied the stability of the PC transformed systems and tried to make some connections between the original uncertain system and PC transformed system. In this regard for single uncertainty case, we derive a direct relation between the eigenvalues of the uncertain system matrix and the the eigenvalues of nominal system matrix and perturbation matrix for single uncertainty affine uncertain system. We show that this relation can also be extended to the multi-variable uncertainty case for some special cases. This novel relation let the PC truncated system to be block diagonalized. This is the main contribution of this chapter that enables us to obtain some concrete results about the system properties. We derive a necessary and sufficient condition for the PC transformed system. For more general system representations, we utilize the matrix measure to derive a necessary condition for the stability of the system by exploiting the banded structure of the system matrix.

In the final section, we study the controllability of the PC based system models. We have provided some necessary conditions for the controllability of the PC transformed systems. In this regard, we provide a Kalman decomposition procedure in order to eliminate the uncontrollable modes of the PC transformed systems.

All the results derived in this chapter are novel contributions of this study.

3.2 System Modelling

Consider a linear continuous time time-invariant uncertain system

$$\dot{x}(\omega, t) = A(\boldsymbol{\xi}(\omega))x(\omega, t) + Bu(t) \tag{3.1}$$

Here $x(\omega, t) \in \mathbb{R}^n$ is a finite variance stochastic process that represent the system state, $u(t) \in \mathbb{R}^s$ is the system input signal, $A(\boldsymbol{\xi}(\omega)) \in \mathbb{R}^{n\times n}$ is the uncertain system dynamic matrix, $B \in \mathbb{R}^{n\times m}$ is the system input matrix and $\boldsymbol{\xi}(\omega) = (\xi_1(\omega), \ldots, \xi_d(\omega))$ is the uncertainty vector. The state $x(t, \omega)$ of the uncertain system is approximated by the truncated PC expansion as

$$x(\omega, t) \cong \sum_{j=0}^{p} x_j(t)\Phi_j(\boldsymbol{\xi}(\omega)) \tag{3.2}$$

where $\left\{\Phi_j(\boldsymbol{\xi}(\omega))\right\}$ is the set of the orthogonal polynomial bases with the corresponding time-varying coefficients $x_j(t)$. Here the number of PC expansion order p is determined as $p + 1 = \frac{(d+L)!)}{d!L!}$ where d is the dimension of the uncertainty source vector and the L is the order of the orthogonal polynomials. Similarly

$$A(\boldsymbol{\xi}(\omega)) = \sum_{i=0}^{p} A_i\Phi_i(\boldsymbol{\xi}(\omega)) \tag{3.3}$$

with $A_i = \frac{\langle A(\boldsymbol{\xi}(\omega)), \Phi_i(\boldsymbol{\xi}(\omega))\rangle}{\langle \Phi_i(\boldsymbol{\xi}(\omega)), \Phi_i(\boldsymbol{\xi}(\omega))\rangle}$ for $i = 0, 1, \ldots, p$. Substituting the truncated state and system matrix representation into the system dynamics gives

$$\sum_{k=0}^{p} \dot{x}_k(t)\Phi_k(\boldsymbol{\xi}(\omega)) = \sum_{i=0}^{p}\sum_{j=0}^{p} A_i x_j(t)\Phi_i(\boldsymbol{\xi}(\omega))\Phi_j(\boldsymbol{\xi}(\omega)) + Bu(t) \tag{3.4}$$

After taking Galerkin projection,

$$x_k(t) = \sum_{i=0}^{p}\sum_{j=0}^{p} A_i x_j(t)\hat{e}_{kij} + Bu(t)\delta_k \tag{3.5}$$

for $k = 0, 1, 2, \ldots, p$ where $\hat{e}_{kij} = \left\langle \Phi_k, \Phi_i\Phi_j \right\rangle$ the following augmented system dynamics is obtained

$$\bar{x}(t) = \bar{A}\bar{x}(t) + \bar{B}u(t) \tag{3.6}$$

with $\bar{x}(t) = (x_0^T(t), x_1^T(t), \ldots, x_p^T(t))^T \in R^{n(p+1)}$ and $\bar{B} \in R^{n(p+1)\times k}$ with $B = (B^T, 0_{k\times n}, \ldots, 0_{k\times n})^T$. Here $\bar{A} \in R^{n(p+1)\times n(p+1)}$ which can be represented by Kronecker (tensor) product as

$$\bar{A} = \sum_{k=0}^{p} J_{p,k} \otimes A_k \tag{3.7}$$

where

$$J_{p,k} = \begin{pmatrix} \hat{e}_{0k0} & \hat{e}_{0k1} & \cdots & \hat{e}_{0kp} \\ \hat{e}_{1k0} & \hat{e}_{1k1} & \cdots & \hat{e}_{1kp} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{e}_{pk0} & \hat{e}_{pk1} & \cdots & \hat{e}_{pkp} \end{pmatrix} \tag{3.8}$$

with $e_{prs} = \frac{\langle \Phi_p, \Phi_r\Phi_s \rangle}{(\Phi_p)^2}$ where by definition

$$X \otimes Y = \begin{pmatrix} X_{11}Y & X_{12}Y & \cdots & X_{1N}Y \\ X_{21}Y & X_{12}Y & \cdots & X_{2N}Y \\ \vdots & \vdots & \ddots & \vdots \\ X_{N1}Y & X_{N2}Y & \cdots & X_{NN}Y \end{pmatrix} \tag{3.9}$$

for $X \in \mathbb{R}^{N\times N}$.

3.2.1 Multivariable Affine Uncertainty Case

In this section, we consider the case that the system dynamic matrix is an affine function of the system uncertainty vector $\boldsymbol{\xi}(\omega) = (\xi_1(\omega), \xi_2(\omega), \ldots, \xi_d(\omega))$. Thus

$$A(\boldsymbol{\xi}(\omega)) = \sum_{k=0}^{d} A_k\Phi_k(\boldsymbol{\xi}(\omega)) \tag{3.10}$$

and the first order orthogonal polynomials are considered as $\Phi_k(\boldsymbol{\xi}(\omega)) = \xi_k(\omega)$ for $k = 1, 2, \ldots, d$, so

$$A(\boldsymbol{\xi}(\omega)) = A_0 + \sum_{k=1}^{d} A_k\xi_k(\omega) \tag{3.11}$$

In general, different types of constraints can be imposed on the elements of the multivariable uncertainty vector $\boldsymbol{\xi}$ in real engineering problems. The set of possible uncertain parameters can constitute a ball $C_{\mathcal{B}}$ in $\mathbb{R}^d$ such that $C_{\mathcal{B}} = \{\boldsymbol{\xi} : \|\boldsymbol{\xi}(\omega)\|^2 < 1, \forall\omega\}$ or they can constitute a k-dimensional polytope in $\mathbb{R}^d$ as $C_{\mathcal{P}} = \{\boldsymbol{\xi} = (\theta_1\boldsymbol{\xi}_1(\omega) + \theta_2\boldsymbol{\xi}_2(\omega) + \ldots + \theta_k\boldsymbol{\xi}_k(\omega)) \mid \theta_i \geq 0, 0 \leq i \leq k, \sum_{i=0}^{k}\theta_i = 1, \forall\omega\}$. It is possible to construct multivariable orthogonal polynomials with respect to the measures that are defined on these uncertainty constraint sets [15].

In this section, we assume that the uncertainty variables are mutually independent, thus the orthogonal polynomials are constructed as the tensor product of the individual orthogonal polynomials as $\Phi_n(\boldsymbol{\xi}) = \Pi_{s=1}^{d}\varphi_{n_s}^{s}(\xi_s)$ where $\varphi_{n_s}^{s}(\xi_s)$ is the j_sth order univariate orthogonal polynomial of ξ_s. Then the elements $\hat{e}_{nkm}$ of the matrices $J_{p,k}$ is computed as

$$\hat{e}_{nkm} = \left\{ \begin{array}{cc} \delta_{nm} & for\ \ k = 0 \\ \frac{\langle\Phi_n,\xi_k\Phi_m\rangle}{\langle(\Phi_n)^2\rangle} & for\ \ 1 \leq k \leq d \end{array} \right\} \tag{3.12}$$

Note that $\xi_k\Phi_m(\boldsymbol{\xi}) = (\xi_k\varphi_{m_k}^{k}(\xi_k))\Pi_{s=1,m_s\neq m_k}^{d}\varphi_{m_s}(\xi_s)$. Recall that the three term recurrence relation of the univariate orthogonal polynomials is $\xi_k\varphi_{m_k}^{k}(\xi_k) = a_{m_k}\varphi_{m_k+1}^{k}(\xi_k) + b_{m_k}\varphi_{m_k}^{k}(\xi_k) + c_{m_k}\varphi_{m_k-1}^{k}(\xi_k)$. By appropriate substitutions, the inner product in (3.12) takes the following form as shown in [41];

$$\langle\Phi_n,\xi_k\Phi_m\rangle = \left\{ \begin{array}{cc} b_{n_k} & for\ \ \ n = m \\ a_{n_k-1} & for\ \ \ n_k = m_k + 1\ and\ \forall s\ s \neq k\ n_s = m_s \\ c_{n_k+1} & for\ \ \ n_k = m_k - 1\ and\ \forall s\ s \neq k\ n_s = m_s \\ 0 & else \end{array} \right\} \tag{3.13}$$

If the m values that satisfy the conditions of (3.13) are denoted by m_k, m_k^+ and m_k^-, then the inner product can be represented in the following compact form [41];

$$\hat{e}_{nkm} \triangleq \langle\Phi_n,\xi_k\Phi_m\rangle = a_{n_k-1}\delta_{n,m_k^+} + b_{n_k}\delta_{n,m} + c_{n_k+1}\delta_{n,m_k^-} \tag{3.14}$$

Thus, the truncated PC system matrix is obtained as

$$\bar{A} = I_p \otimes A_0 + \sum_{k=1}^{d} J_{p,k} \otimes A_k \tag{3.15}$$

$$J_{p,k} = \begin{pmatrix} \hat{e}_{0k0} & \hat{e}_{0k1} & \cdots & \hat{e}_{0kp} \\ \hat{e}_{1k0} & \hat{e}_{1k1} & \cdots & \hat{e}_{1kp} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{e}_{pk0} & \hat{e}_{pk1} & \cdots & \hat{e}_{pkp} \end{pmatrix} \tag{3.16}$$

3.2.2 Single Variable Affine Uncertainty Case

Consider that the system dynamic matrix is an affine function of a single uncertainty $A(\xi(\omega)) = A_0 + A_1\xi$ then

$$\bar{A} = J_{L,0} \otimes A_0 + J_{L,1} \otimes A_1 \tag{3.17}$$

with

$$J_{L,k} = \begin{pmatrix} \hat{e}_{0k0} & \hat{e}_{0k1} & \cdots & \hat{e}_{0kL} \\ \hat{e}_{1k0} & \hat{e}_{1k1} & \cdots & \hat{e}_{1kL} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{e}_{Lk0} & \hat{e}_{Lk1} & \cdots & \hat{e}_{LkL} \end{pmatrix} \tag{3.18}$$

where

$$\hat{e}_{nkm} = \left\{ \begin{array}{cc} \delta_{n,m} & for\ \ k=0 \\ \frac{1}{\langle(\Phi_n)^2\rangle}(a_m\delta_{n,m+1} + b_m\delta_{n,m} + c_m\delta_{n,m-1}) & for\ \ k=1 \end{array} \right\} \tag{3.19}$$

Thus since $J_{L,0} = I_{L+1}$ and

$$J_{L,1} = \begin{pmatrix} b_0 & c_0 & 0 & \cdots & 0 \\ a_1 & b_1 & c_1 & \cdots & 0 \\ 0 & a_2 & b_2 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & c_{L-1} \\ 0 & 0 & 0 & a_L & b_L \end{pmatrix} \tag{3.20}$$

then

$$\bar{A} = \begin{pmatrix} A_0 + b_0A_1 & c_0A_1 & 0 & \cdots & 0 \\ a_1A_1 & A_0 + b_1A_1 & c_1A_1 & \cdots & 0 \\ 0 & a_2A_1 & A_0 + b_2A_2 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & c_{L-1}A_1 \\ 0 & 0 & 0 & a_LA_1 & A_0 + b_LA_1 \end{pmatrix} \tag{3.21}$$

Note here that the matrix $J_{L,1}$ is the transpose of the Jacobi matrix $\mathbb{J}_{L+1}$ that is expressed in (2.7).

3.3 Stability Analysis

Stability of uncertain systems is one of the main subjects of robust control thus it has been extensively studied by many researchers. Different stability definitions have been introduced and sufficient conditions have been proposed [58, 7, 20].

Here we give three stability definitions for the uncertain system (3.1).

Definition 3.3.1. [58] The uncertain system (3.1) is said to be robustly stable if the system is asymptotically stable for all realizations of $\boldsymbol{\xi}(\omega)$.

Definition 3.3.2. [58] The uncertain system (3.1) is said to be quadratically stable if there exists a single Lyapunov function $V(x(t,\omega)) = (x(t,\omega))^T Px(t,\omega)$ for all possible realizations of $\boldsymbol{\xi}(\omega)$. It means there exists a $P > 0$ such that $A(\boldsymbol{\xi}(\omega))P + PA(\boldsymbol{\xi}(\omega))^T < 0$ for all $\boldsymbol{\xi}(\omega)$.

Note that the quadratic stability implies the robust stability but the reverse is not necessarily true. This is mainly due to the fact that Lyapunov function is independent of the uncertainty [58].

For the uncertainty sources of an unbounded support, the robust stability will not be satisfied almost always, except for a few special systems. In this case, it is much more convenient to study the stability in the average sense rather than the stability of the system for all possible uncertainties. In this regard, definition of the stability in moments is given [41].

Definition 3.3.3. [41] The zero equilibrium point of the system (3.1) is said to be stable in the pth moment if $\forall\epsilon > 0, \exists\delta > 0$ such that $sup_{t\geq 0}E\{|x(t,t_0)|^p\} \leq \epsilon \ \forall x(t_0) : |x(t_0)| \leq \delta$. The uncertain system (3.1) is said to be asymptotically stable in pth moment if it is stable in the pth moment and $\lim_{t\to\infty} E\{|x(t,t_0)|^p\} = 0$

The following theorem is the recent result about the stability relation of the uncertain system and its infinite order PC transform.

Theorem 3.3.1. *[41] The origin of the system* (3.1) *is asymptotically stable in all moments if and only if the PC transformed system for infinite truncation order is asymptotically stable.*

As shown in [24], if the truncated PC transformed system is asymptotically stable for all approximation orders of p, then the response of the state $x_k^{PC} = \sum_{i=0}^{p} x_{k,i}\Phi_i(\boldsymbol{\xi})$ is stable in the mean square sense. As noted in [24], the mean square stability of the PC transformed system does not imply that the uncertain system is stable for all realizations of uncertainty $\boldsymbol{\xi}$.

For practical purposes, the truncation order is finite, thus it is important to analyze the truncated system stability with respect to the original uncertain system.

In the remaining part of this section we concentrate on the relationship between the robust stability and the stability of the truncated PC transformed system and we present our new results.

3.3.1 Scalar Affine Uncertainty Case

The single uncertainty is simpler to handle and so more concrete results can be obtained for this case. Consider the single uncertainty, first order truncated PC transformed linear system is

$$\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}u(t) \tag{3.22}$$

where $\bar{A}$ is defined in (3.17) and (3.18). It is obvious to see that the state dynamics matrix and Jacobi matrix for the related orthogonal polynomials have identically the same form. The following theorem relates the eigenvalues of the truncated system to the eigenvalues of the Jacobi matrix.

Theorem 3.3.2. *The eigenvalues of the truncated system matrix $\bar{A} = I_{L+1}\otimes A_0 + J_{L,1}\otimes A_1$ equal to the collection of the eigenvalues of the matrices $M_j = (A_0 + \lambda_j A_1)$ for $j = 1, 2, \ldots, L+1$ where $\{\lambda_j\}_{j=1}^{L+1}$ are the zeros of the $(L+1)$st degree orthogonal polynomials that satisfy the following three-term recurrence relation $xp_n(x) = a_{n+1}p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x)$ for $n = 0, 1, 2, 3\ldots$ where $p_{-1}(x) = 0, p_0(x) = 1$.*

Proof. Let us consider the related truncated matrix $J_{L,1}$ which is the transpose of the Jacobi matrix $\mathbb{J}_{L+1}$ for the Lth degree orthogonal polynomials. It is known that the eigenvalues of the Jacobi matrix are equal to the zeros of the $(L+1)$st degree orthogonal polynomial thus its transpose. So the eigenvalues of $J_{L,1}$ are real and simple. It can be diagonalized by a similarity transformation matrix such that $\hat{J}_{L,1} = T_L^{-} J_{L,1} T_L$. Now define an augmented similarity transformation matrix where $\bar{T} = T_L \otimes I_n$. Note that $T_L \otimes I_n$ is non-singular since both T_L and I_n are non-singular. If the similarity transformation for the augmented system is performed as $\bar{T}^{-}(I_{L+1} \otimes A_0 + J_{L,1} \otimes A_1)\bar{T} = (T_L \otimes I_n)^{-}(I_{L+1} \otimes A_0)(T_L \otimes I_n) + (T_L \otimes I_n)^{-}(J_{L,1} \otimes A_1)(T_L \otimes I_n)$ Using the identity that $(A \otimes B)(C \otimes D) = AC \otimes BD$ for Kronecker product of matrices with suitable dimensions, the first term of the right hand side is obtained as $(T_L \otimes I_n)^{-}(I_{L+1} \otimes A_0)(T_L \otimes I_n) = (T_L \otimes I_n)^{-}(T_L \otimes A_0) = I_{L+1} \otimes A_0$. On the other hand, the second term is $(T_L \otimes I_n)^{-}(J_{L,1} \otimes A_1)(T_L \otimes I_n) = (T_L \otimes I_n)^{-}(J_{L,1} \otimes A_1) = ((T_L)^{-} J_{L,1} T_L) \otimes A_1 = \hat{J}_{L,1} \otimes A_1$. Here $\hat{J}_{L,1}$ is a diagonal matrix where the diagonal values are the eigenvalues of $J_{L,1}$ which are also equal to the zeros $\{\lambda_j\}_{j=1}^{L+1}$ of the $(L+1)$th orthogonal polynomial. Thus $\bar{T}^{-}\bar{A}\bar{T} = diag((A_0 + \lambda_0 A_1), (A_0 + \lambda_1 A_1), \ldots, (A_0 + \lambda_L A_1))$ where the diagonal values are obviously the eigenvalues of $\bar{A}$ □

Corollary 3.3.3. *The PC transformed system is stable(asymptotically stable) for any truncation order L if and only if the original uncertain system is stable (asymptotically stable) for all possible values of the uncertainty $\xi(\omega)$.*

Proof. The necessity of the stability of the PC truncated system for the stability of the uncertain system is obvious. For the sufficiency, assume that the transformed system is stable for any order L, then it implies $A_0 + \lambda_j A_1$ is stable for all values of $\{\lambda_j\}_{j=1}^{L+1}$. Since the roots of the orthogonal polynomials are located in its support and they are dense, the original system is also stable for any possible value of ξ. □

Note that the above corollary does not say that the truncated system of all orders is unstable when the original system is unstable for some value of uncertainty. To investigate the stability relationship between the truncated PC transformed systems of different orders, matrix measure is utilized. For the sake of completeness we give related information about the matrix measure.

Definition 3.3.4 ([13])**.** The matrix measure $\mu(A)$ of the matrix A induced by a matrix norm $\|.\|_p$ is defined by $\mu_p(A) = lim_{\theta \to 0}\left(\frac{\|I+\theta A\|_p - 1}{\theta}\right)$

By using the corresponding matrix p-norm definitions, following matrix measures are defined [13]

- $\mu_1(A) = max_j([Re(a_{jj}) + \sum_{i=1, i\neq j} |a_{ij}|)$
- $\mu_2(A) = max_{\lambda_i}(\lambda_i(\frac{1}{2}(A + A^T)))$
- $\mu_\infty(A) = max_i([Re(a_{ii}) + \sum_{j=1, i\neq j} |a_{ij}|)$

The matrix measure provides upper and lower bounds for linear differential equations [13].

Theorem 3.3.4. *Let $t \to A(t)$ be a piecewise continuous function from R_+ to $R^{n \times n}$, then the solution of $\dot{x}(t) = A(t)x(t)$ satisfies the following inequality*

$$|x(t_0)|e^{-\int_{t_0}^{t} \mu(-A(\tau)d\tau)} \leq |x(t)| \leq |x(t_0)|e^{\int_{t_0}^{t} \mu(A(\tau)d\tau)} \tag{3.23}$$

Lemma 3.3.5. *The system defined in Theorem 3.3.4 is asymptotically stable if $\mu(A(t)) < 0$ for all t.*

The matrix measure satisfies the following useful properties [13];

1. $\mu(cA) = c\mu(A)\ forall\ c \geq 0$
2. $\mu(A + cI) = \mu(A) + c\ forall\ c\ \in \mathbb{R}$
3. $\mu(A + B) \leq \mu(A) + \mu(B)$
4. $-\|A\| \leq -\mu(-A) \leq Re(\lambda) \leq \mu(A) \leq \|A\|$
5. $\mu(\lambda A + (1 - \lambda)B) \leq \lambda\mu(A) + (1 - \lambda)\mu(B)$

The inequality relation (4) implies that the matrix measure provides a tighter bound for system stability than the matrix norm.

A sufficient condition for stability of a polytope of matrices is given by [20] in terms of matrix measure. For truncated PC expansion of order n, it is easy to show that by using the interlacing property the orthogonal polynomials, the vertices are $(A_0 + A_1\lambda^n_{min})$ and $(A_0 + A_1\lambda^n_{max})$ where λ^n_{min} and λ^n_{max} are the minimum and maximum root of the nth order polynomial which yields the following lemma.

Lemma 3.3.6. *If there exists a matrix measure μ such that $\mu(A_0 + A_1\lambda^n_{max}) < 0$ and $\mu(A_0 + A_1\lambda^n_{min}) < 0$ then all the lower dimensional continuous-time PC truncated systems are stable.*

The above lemma helps to understand the stability of the truncated system. One special case is the first order systems. The system representations for this case is $x_{k+1} = (a_0 + a_1\xi)x_k + bu_k$. The PC expansion of order k has eigenvalues $\lambda_{k,i} = a_0 + a_1\mu_{k,i}$ for $i = 1, 2, \ldots, k$ where $\mu_{k,i}$ is the ith root of the $(k+1)$st degree polynomial. Stability analysis for this case is trivial. Another special case is higher order systems that satisfy $A_0A_1 = A_1A_0$ with diagonizable A_0 and A_1. The commutability reduces the higher order system to a collection of first order systems due to the fact that the diagonizable commuting matrices are mutually diagonizable. The stability result for this case is summarized in the following lemma.

Lemma 3.3.7. *Suppose that $A_0A_1 = A_1A_0$ where A_0 and A_1 are diagonizable matrices. The kth order PC expanded system is stable if and only if all $\lambda_{k,i} = \lambda_{0,i} + d_i\mu_{k,i}$ for $i = 1, 2, \ldots, k+1$ are stable where $\lambda_{0,i}$ is the ith eigenvalue (the ith diagonal element in the diagonal form of A_0 and d_i is the ith diagonal element in the diagonal form of the matrix A_1.*

Proof. Trivial. □

Remark 2. For this special case, it is sufficient to check the stability of the matrices $(A_0 + \xi_{min}A_1)$ and $(A_0 + \xi_{max}A_1)$ for the stability of the uncertain system when the uncertainty has finite support $[\xi_{min}, \xi_{max}]$. This is the due the fact that all roots of the orthogonal polynomials are in the support of the uncertainty and approaching to the boundaries as the order increases.

3.3.2 Multivariate Affine Uncertainty Case

In this section, we study the stability of the multivariate affine uncertain systems. Consider the affine system matrix (3.10) that is derived in the previous section;

$$\bar{A} = I_p \otimes A_0 + \sum_{k=1}^{d} J_{p,k} \otimes A_k \tag{3.24}$$

For the stability of multivariate truncated PC transformed system, we will provide a sufficient condition by exploiting the block tri-diagonal structure of the system matrix. In this regard, we utilized matrix measure concept for the stability analysis. We use the notation S_{ij} to denote the (ij)th entry of a matrix S.

Theorem 3.3.8. *The truncated PC transformed system is stable if* $\mu_1(A_0) < -\sum_{k=1}^{d} \beta_k \|A_k\|_1$ *where* β_k *is equal to the maximum column sum of the* $J_{p,k}$*, that is* $\beta_k = \|J_{p,k}\|_1$

Proof. Let us consider the Kronecker product representation of the system dynamics matrix $\bar{A} = I_p \otimes A_0 + \sum_{k=1}^{d} J_{p,k} \otimes A_k$ where $J_{p,k}$ defined in (3.16). Note here that for a fixed $k = k^*$, the elements of the fixed (j^*)th column of the J_{p,k^*} are as follows;

$$\hat{e}_{ik^*j^*} = \left\langle \Phi_i, \xi_{k^*} \Phi_{j^*} \right\rangle = \left\{ \begin{array}{ll} b_{j_k^*} & for \;\; i = j^* \\ a_{j_k^*} & for \;\; i_k = j_k^* + 1 \; and \; \forall s \; s \neq k^* \; i_s = j_s^* \\ c_{j_k^*} & for \;\; i_k = j_k^* - 1 \; and \; \forall s \; s \neq k^* \; i_s = j_s^* \\ 0 & else \end{array} \right\} \tag{3.25}$$

If the indices of i that satisfy the previous conditions are denoted as $j_k^{*,+}$ and $j_k^{*,-}$, then

$$\left\langle \Phi_i, \xi_{k^*} \Phi_{j^*} \right\rangle = a_{j_k^*} \delta_{i, j_k^{*,+}} + b_{j_k^*} \delta_{i,j} + c_{j_k^*} \delta_{i, j_k^{*,-}} \tag{3.26}$$

Thus, since by definition $\mu_1(\bar{A}) = max_j \left[Re(\bar{A}_{jj}) + \sum_{i=1, i\neq j} \left| \bar{A}_{ij} \right| \right]$ then

$$\mu_1(\bar{A}) < max_j \left[\bar{A}_{jj}^0 + \sum_{i=1, i\neq j} \left| A_{ij}^0 \right| + \sum_{k=1}^{d} \sum_i (a_{j_k} + b_{j_k} + c_{j_k}) \left| A_{ij}^k \right| \right] \tag{3.27}$$

By using the (3.25) and since by definition $\mu_1(A_0) = max_j \left[A_{jj}^0 + \sum_{i, i\neq j} \left| A_{ij} \right| \right]$ it is easy to see that

$$\begin{aligned} \mu_1(\bar{A}) &\leq \mu_1(A_0) + max_j \sum_{k=1}^{d} \sum_i (a_{j_k} + b_{j_k} + c_{j_k}) \left| A_{ij}^k \right| \\ &\leq \mu_1(A_0) + \sum_{k=1}^{d} \beta_k max_j \sum_i \left| A_{ij}^k \right| \end{aligned} \tag{3.28}$$

where β_k is the maximum value of the three-term recursion coefficients sum for the univariate orthogonal polynomial corresponding to the kth uncertainty. From (3.25), it can be inferred that it is also equivalent to $||J_{p,k}||_1$ which is the maximum absolute column sum of the matrix $J_{p,k}$. Similarly since $||A_k||_1 = max_j \sum_i \left|A_{ij}^k\right|$ so

$$\mu_1(\bar{A}) \leq \left[\mu_1(A_0) + \sum_{k=1}^{d} ||J_{p,k}||_1 ||A_k||_1\right] \tag{3.29}$$

Since the stability of the augmented system necessiates that $\mu_1(\bar{A}) < 0$ then
$\mu_1(A_0) < -\sum_{k=1}^{d} \beta_k ||A_k||_1$ □

Corollary 3.3.9. *For multi-variable uniformly distributed uncertainty case, the truncated PC transformed system is stable if* $\mu_1(A_0) < -\sum_{k=1}^{d} ||A_k||_1$

Proof. For the uniform distribution case,the corresponding orthogonal polynomiala are Legendre polynomials for which $a_{j_k} = \frac{j_k+1}{2j_k+1}, c_{j_k} = \frac{j_k}{2j_k+1}$ and $b_{j_k} = 0$. Thus the summation of the coefficients on any column of the Jacobi matrix will be equal unity if both $j_k^- \leq p$ and $j_k^+ \leq p$. Otherwise, the summation will be less than unity. Thus $||J_{p,k}|| \leq 1$ which completes the proof. □

Remark 3. In [41], for infinite dimensional multivariate PC truncated system, a sufficient stability condition is derived as $\mu_2(A_0) = max_i \lambda_i \left(\frac{1}{2}(A_0 + A_0^T)\right) \leq -\sum_{k=1}^{d} ||A_k||_2$ by utilizing the Lyapunov stability for row-finite infinite dimensional systems. Their result is the 2-norm version of the our result which is obtained for the truncated PC transformed system.

3.4 Controllability Analysis

In this section, we study the controllability of the PC transformed system representation and relate it to the controllability of the uncertain system. The controllability definition of the uncertain system is given below [61].

Definition 3.4.1. The uncertain system is said to be robustly controllable if it is controllable for every realization of $\boldsymbol{\xi}(\omega)$.

3.4.1 Scalar Affine Uncertainty Case

Consider the truncated PC transformed linear system (3.22). We have shown in the previous section that the system dynamics can be block diagonalized by a similarity transformation matrix $\bar{T}$ such that $\hat{A} = \bar{T}^{-}\bar{A}\bar{T}$ where $\bar{T} = T_L \otimes I_n$ with $\hat{J}_{L,1} = T_L^{-} J_{L,1} T_L$. Note that $\bar{T}^{-} = (T_L \otimes I_n)^{-} = T_L^{-} \otimes I_n$. Corresponding to the state transformation $\hat{\boldsymbol{x}} = \bar{T}^{-}\boldsymbol{x}$ the transformed system takes the following form

$$\dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}u(t) \tag{3.30}$$

Here $\hat{B} = \bar{T}^{-}\bar{B} = \left(T_{11}^{-}B^T, T_{21}^{-}B^T, \ldots, T_{L1}^{-}B^T\right)^T$ and $\hat{A} = diag((A_0{+}\mu_0 A_1), (A_0{+}\mu_1 A_1), \ldots, (A_0{+}\mu_L A_1))$ with T_{ij}^{-} is the $(i,j)th$ element of the matrix T_L^{-}.

For controllability, the block diagonal structure of the transformed system necessitates the diagonal block dynamics $\{A_0 + \mu_i A_1\}$ not to have common modes. This is due to the special structure of the matrix $\hat{B}$, i.e., blocks of it are same except a scalar multiplication. It is obvious that the truncated PC transformed system is not controllable if one of the T_{i1}^{-} values is zero. The following lemma guarantees that these values are not zero.

Lemma 3.4.1. *The first column of the inverse similarity transformation matrix T^{-} that diagonalizes the transpose Jacobi matrix J_L has no zero terms.*

Proof. Consider a non-symmetrical transpose tri-diagonal matrix $J_L = \mathbb{J}_{L+1}^T$. It is known that the symmetric form of a tri-diagonal matrix is obtained by a similarity transformation such that $J_{s,L} = DJ_L D^{-}$ where D is a diagonal matrix. On the other hand, there exists an orthogonal matrix $V^{-} = V^T$ that diagonalizes the symmetric Jacobi matrix such that $\Lambda = V^T J_{s,L} V$ with $V = (p_L(\lambda_0), p_L(\lambda_1), \ldots, p_L(\lambda_L - 1))$ where $p_L(\lambda) = (p_0(\lambda), p_1(\lambda), \ldots, p_0(\lambda))^T$ are composed of orthogonal polynomials $p_i(\lambda)$ and λ_i is the ith root of the Lth degree polynomial. By combining these two transformations it can be obtained that $\Lambda = V^T DJ_L D^{-} V$. Then the similarity transformation $T = D^{-}V$ diagonalizes J_L where $T^{-} = V^{-T}D$. Thus the first column of T^{-} consists of $(d_1 p_0(\lambda_0), d_2 p_1(\lambda_0), \ldots, d_L p_0(\lambda_0))^T$ where $d_1, d_2, \ldots, d_L$ are the diagonal values of the matrix D . Since the zeroth order orthogonal polynomials are positive scalars (generally chosen as one), the first column does not include any zero terms. □

Theorem 3.4.2. *The truncated PC transformed system* (3.22) *and thus* (3.30) *is controllable for any expansion order only if* (A_0, A_1) *is a controllable pair.*

Proof. The similarity transformation $\bar{T}$ makes the PC transformed system block diagonal with block diagonal entries $(A_0 + \mu_i A_1)$ for $i = 1, \ldots, d$ with the corresponding input matrix $\hat{B} = (T_{11}^{-}B^T, T_{21}^{-}B^T, \ldots, T_{L1}^{-}B^T)^T$ where $(T_{11}^{-}, \ldots, T_{L1}^{-})$ are nonzero. Assume that (A_0, A_1) is not controllable then there exists a similarity transformation matrix T_c decomposing the (A_0, A_1) pair as $\hat{A}_0 = T_c^{-}A_0 T_c$ and $\hat{A}_1 = T_c^{-}A_1$ such that

$$\hat{A}_0 = \begin{pmatrix} \hat{A}_0^{11} & \hat{A}_0^{12} \\ 0 & \hat{A}_0^{22} \end{pmatrix} \hat{A}_1 = \begin{pmatrix} \hat{A}_1^{11} \\ \mathbf{0} \end{pmatrix}$$

If the similarity transformation is performed on each $(A_0 + \mu_i A_1)$ then $T_c^{-}(A_0 + \mu_i A_1)T_c = \hat{A}_0 + \mu_i \hat{A}_1 T_c$ where $\hat{A}_1 T_c = \left((\hat{A}_{1i}^{11})^T, 0^T\right)^T$ for $i = 1, 2, \ldots, d$. Thus, the eigenvalues of $\hat{A}_0$ that corresponds to $\hat{A}_0^{22}$ block is same for each i which implies uncontrollability. □

The elimination of the uncontrollable modes may be useful since it reduces the order of the PC transformed system. Define $T_c = (T_1\ T_2)$ as the transformation matrix used in the Kalman decomposition. Define its inverse as $T_c^{-} = (\bar{T}_1^T\ \bar{T}_2^T)^T$ so $T_c^{-}A_0 T_c = \begin{pmatrix} \bar{T}_1 A_0 T_1 & \bar{T}_1 A_0 T_2 \\ \mathbf{0} & \bar{T}_2 A_0 T_2 \end{pmatrix}$,

$T_c^{-}A_1 = \begin{pmatrix} \bar{T}_1 A_1 \\ 0 \end{pmatrix}$ Define the augmented similarity transformation as $T = I_p \otimes T_c$ then it can be shown that after the transformation the PC truncated system can be represented as $\hat{A} = J_{L,0} \otimes \hat{A}_0 + J_{L,1} \otimes \hat{A}_1$ where $\hat{A}_0 = \begin{pmatrix} \bar{T}_1 A_0 T_1 & \bar{T}_1 A_0 T_2 \\ 0 & \bar{T}_2 A_0 T_2 \end{pmatrix}$ and $\hat{A}_1 = \begin{pmatrix} \bar{T}_1 A_1 T_1 & \bar{T}_1 A_1 T_2 \\ 0 & 0 \end{pmatrix}$ since $\bar{T}_2 A_1 = 0$ implies that $\bar{T}_2 A_1 \bar{T}_2 = 0$ and with the PC transformed input matrix being $\bar{B} = \begin{pmatrix} T_c^{-} B \\ 0_{nL\times m} \end{pmatrix}$. The reduced system model is

$$\hat{A}_{red} = \begin{pmatrix} \hat{A}_0 + b_0\hat{A}_1 & c_0\hat{A}_1 & 0 & 0 & \dots & 0 \\ a_1\hat{A}_{1,red} & \hat{A}_0 + b_1\hat{A}_1 & c_1\hat{A}_{1,red} & 0 & \dots & 0 \\ 0 & a_2\hat{A}_{1,red} & \hat{A}_{0,red} + b_2\hat{A}_1 & c_2\hat{A}_{1,red} & \dots & 0 \\ 0 & 0 & a_3\hat{A}_{1,red} & \hat{A}_{0,red} + b_2\hat{A}_1 & \dots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & c_{L-1}\hat{A}_{1,red} \\ 0 & 0 & 0 & \dots & a_L\hat{A}_{1,red} & \hat{A}_{0,red} + b_L\hat{A}_1 \end{pmatrix}$$

with $A_{0,red} = \bar{T}_1 A_0 T_1$ and $\bar{A}_{1,red} = \bar{T}_1 A_1 T_1$.

Note here that the resulting system is almost block tri-diagonal where the structure of the first row block is different than the other row blocks. The following lemma is a direct consequence of the above explanation.

Lemma 3.4.3. *Assume that A_0 has distinct eigenvalues. If (A_0, A_1) pair has $m \leq n$ uncontrollable modes then the truncated PC transformed system (1.45) has at least $(m-1)\times L$ uncontrollable modes.*

Proof. The proof is obvious from above explanations. □

Remark 4. The pair (A_0, B) is not necessarily controllable for the PC transformed system to be controllable as shown by the following example.

Example 3.4.1. Given $A_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $A_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, the pair is (A_0, B) is not controllable but for the second order expansion we have $\bar{A} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ which is controllable.

Lemma 3.4.4. *The PC transformed system* (3.22) *is controllable for any truncation order if and only if the original uncertain system is robustly controllable.*

Proof. Assume that the original uncertain system is not robustly controllable. This means that there is a nonzero left eigenvector $v^T(\lambda_0 I - A_0 - A_1\xi^*) = 0$ and $v^T\ B = 0$ for some ξ^* and λ_0. On the other hand the controllability condition of the PC transformed system is that there does not exist a v such that $v^T(diag((\lambda I-(A_0+\mu_1 A_1)),\ldots,(\lambda I-(A_0+\mu_L A_1))),\hat{B}) = 0$ for any λ where $\{\mu_i\}_{i=1}^L$ is the set of the zeros of the orthogonal polynomials being in the support of the measure. Since the roots of the orthogonal polynomials of increasing order is dense in the support of the orthogonality measure, there will be a polynomial order L^* such that μ_{L^*} is in the ϵ neighborhood of ξ^* for any ϵ for PC truncated system is not controllable. Now consider that the truncated system is not controllable then $v^T(\lambda I - (A_0 + \mu_L, A_1)) = 0$ and $v^T B = 0$ for $\mu_L = \mu_{L^*}$. This implies that the uncertain system is not controllable for a realization of $\xi = \mu_{L^*}$. Thus the proof is complete. □

3.4.2 Multivariate Affine Uncertainty Case

In this section we give a necessary condition for the generalized multivariate orthogonal polynomials. Let us consider the truncated PC transformed linear system

$$\dot{\bar{\boldsymbol{x}}}(t) = \bar{A}\bar{\boldsymbol{x}}(t) + \bar{B}\boldsymbol{u}(t) \tag{3.31}$$

where $\bar{A} = I_p \otimes A_0 + \sum_{i=1}^d J_{p,i} \otimes A_i$ with

$$J_{p,i} = \begin{pmatrix} \hat{e}_{0i0} & \hat{e}_{0i1} & \cdots & \hat{e}_{0ip} \\ \hat{e}_{1i0} & \hat{e}_{1i1} & \cdots & \hat{e}_{1ip} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{e}_{pi0} & \hat{e}_{pi1} & \cdots & \hat{e}_{pip} \end{pmatrix} \quad \bar{B} = \begin{pmatrix} B \\ 0_{np\times m} \end{pmatrix}$$

Theorem 3.4.5. *Truncated PC transformed system which is obtained for generalized multivariate orthogonal polynomials is controllable only if* $(A_0, (A_1, A_2, \ldots, A_d))$ *is a controllable pair.*

Proof. Let us consider the system representation (3.31). Let $M = (\lambda I - \bar{A}, \bar{B})$. M is full row rank for all eigenvalues of $\bar{A}$ if and only if the $(\bar{A}, \bar{B})$ pair is controllable. It can be shown that each row block of $\bar{A}$ contains every A_i for $i = 1, 2, \ldots, d$ at most two times solely. By applying elementary block column operations to the matrix $(\lambda I - \bar{A})$, it is possible to change the matrix to a form that contains exactly one A_i $i = 1, 2 \ldots, d$ in each row and with diagonal element $(\lambda I - A_0 + \sum_{i=1}^d \beta_i A_i)$ for some β_i $i = 1, 2, \ldots, d$. Since all blocks of $\bar{B}$ except the first one is zero, the above operations when applied to M generates rows that contain only $(\lambda I - A_0 + \sum_{i=1}^d \beta_i A_i), A_1, A_2, \ldots, A_d$ nonzero terms which gives the proof. □

3.4.3 Controllability and PC Model Order Reduction for Scalar Uncertainty Case

As previously cited, polynomial chaos system representation is only accurate if the expansion order is high enough. On the other hand, computational complexity increases very rapidly as the expansion order grows especially for multivariable orthogonal polynomials. Thus, the expansion order is an important decision parameter for the PC system representations. In this regard, eliminating the uncontrollable part (of an order t) in the subsystems causes a decrease of $t \times p$ in the overall order of the PC expanded system when the expansion order is p. For the following special case, the PC expansion order is one represents uncertain system dynamics, i.e. higher order expansions do not contribute to the solution.

Fact 1. Assume $A_1^2 = 0$ and $A_1 A_0 = A_0 A_1$. For this special case PC expansion order 1 gives the same solution as higher order expansions.

Proof. The solution of the uncertain system is

$$x(t) = e^{(A_0+A_1\xi)t} x_0 + \int_0^t e^{(A_0+A_1\xi)(t-\tau)} \boldsymbol{Bu}(\tau) d\tau \tag{3.32}$$

Note that

$$e^{(A_0+A_1\xi)t} = \sum_{i=0}^{\infty} (A_0 + A_1\xi)^i \frac{t^i}{i!} \tag{3.33}$$

By the commutativity of the A_0 and A_1 we have $(A_0 + A_1\xi)^i = \sum_{l=0}^{i} \begin{pmatrix} l \\ i \end{pmatrix} A_0^l \xi^{i-l} A_i^{l-i}$. Using $A_1^2 = 0$ we obtain

$$(A_0 + \xi A_1)^i = A_0^i + \xi A_0^{i-1} A_1 \tag{3.34}$$

Inserting (3.34) into (3.33) and assuming that A_0 is asymptotically stable we write

$$e^{(A_0+\xi A_1)t} = \sum_{i=0}^{\infty} (A_0^i + \xi A_0^{i-1} A_1) \frac{t^i}{i!} \tag{3.35}$$

thus $e^{(A_0+\xi A_1)t} = e^{A_0 t} + \xi A_1 A_0^{-} e^{A_0 t}$. The last expression contain only ξ^0 and ξ^1 terms so higher order terms do not contribute to $e^{(A_0+A_1 t)}$. Similarly the second order term contains only zeroth and first order terms of ξ. □

3.5 Conclusion

In this chapter, we study polynomial chaos based system representations and some of their important properties such as stability and controllability.

We studied the stability of the PC transformed systems and tried to make some connections between the original uncertain system and PC transformed system. In this regard for single uncertainty case, we derive a direct relation between the eigenvalues of the uncertain system matrix and the the eigenvalues of nominal system matrix and perturbation matrix for single uncertainty affine uncertain system. We show that this relation can also be extended to the multi-variable uncertainty case for some special cases. This novel relation let the PC truncated system to be block diagonalized. This is the main contribution of this chapter that enables us to obtain some concrete results about the system properties. We derive a necessary and sufficient condition for the PC transformed system. For more general system representations, we utilize the matrix measure to derive a necessary condition for the stability of the system by exploiting the banded structure of the system matrix.

In the final section, we studied the controllability of the PC based system models. We have provided some necessary conditions for the controllability of the PC transformed systems. It is showed that the controllability of the PC transformed system can be analyzed in terms of the controllability of the nominal and perturbed system matrices.

CHAPTER 4

ROBUST STATE ESTIMATION BY POLYNOMIAL CHAOS

4.1 Introduction

Robust estimation is a widely studied subject since it was noticed that performance of the celebrated Kalman filter is vulnerable to system uncertainty [26, 28]. The main purpose of a robust estimation algorithm is to get a good performance (but not the best) under nominal conditions and an acceptable performance for system models other than the nominal model [36, 6, 37]. This is achieved by limiting the effect of the model uncertainty. Various uncertain system modeling methods are proposed so far. The unknown but bounded deterministic disturbances are used as system uncertainty for most of the uncertain systems. In the well-studied estimation approach, the system disturbances are assumed energy bounded and the optimal estimator minimizes the worst-case energy gain from the exogenous signals to the estimation error. Norm-bounded uncertainty in system matrices and uncertainty blocks satisfying an integral (sum) quadratic constraint are other types of uncertain system modeling approaches [37, 45]. State estimation is another branch that the PC theory is applied to. In [59], an observer is designed to estimate the PC expansion modes of a linear uncertain system. In the problem formulation, they considered the measurements as the most likely value of the measured variable. Thus, only the zeroth term of the PC expansion mode is assumed to be measured. Further, the observability of the augmented system is checked by calculating the observability matrix. In [40], generalized PC expansion is combined with Ensemble Kalman filter in order to decrease the sampling error of the Ensemble Kalman filter where samples are taken randomly. In [16], a nonlinear estimation algorithm is proposed which combines the generalized PC theory and higher moment updates. Polynomial chaos theory is used to predict the evolution of uncertainty of the nonlinear random process, and higher order moment updates are used to estimate the posterior non-Gaussian probability density function of the random process. The moments are updated using a linear gain. They stated that the proposed estimator outperforms the linear estimator when measurements are not available very frequently [16]. As given in the literature survey part, the application of the PC theory to the estimation problems for uncertain systems is mainly focused on the estimation of the uncertain parameter and the system state [5],[49]. In this work, we mainly focused on the robust estimation problem where the uncertain parameter is not aimed to be estimated but its average effect on the estimation performance criteria is minimized. Polynomial chaos expansion transforms the system uncertainty from system dynamics to system output. The formulation of the corresponding state estimation problem is difficult since it is not in a suitable form that is used in classical estimation algorithms where the polynomial chaos terms appears in the measurement model as unknown disturbance signals. As noted in the previous paragraph, in [59], the problem is handled by considering the measurement as the most likely value. Thus they take the average of the uncertainty terms in the measurement model. This is the first measurement model that we have considered in this chapter. We propose another approach for the state estimation of the polynomial chaos based uncertain systems by the modifying the set-valued estimation technique that is first introduced by [3] as a deterministic interpretation of Kalman filter. Further this approach was extended to the state estimation problem of uncertain systems where the uncertainty is modeled by sum quadratic constraint [51]. In this regard, we considered the state estimation problem of polynomial chaos based uncertain system as a set estimation problem where the uncertainty and system disturbances satisfy a quasi-deterministic energy constraint. We showed that the set of possible states are actually an ellipsoid where the center and the shaping matrix of the ellipsoid can be obtained recursively by augmented Kalman filter equations which is advantageous. We also provide a stochastic interpretation of the problem formulation as the average maximum a posteriori state estimation problem. In this regard, we propose two other stochastic estimation problems. We provide some necessary conditions for the observability of the two measurement models which give a better understanding of the differences of these two models. We evaluated the performance of the considered two approaches by three illustrative examples that are used in robust estimation community as framework examples. The performance of the proposed approaches are compared with the nominal Kalman filter and classical robust estimation algorithms namely the regularized robust Kalman filter and the H_∞ filter.

4.2 Problem Definition

Let us consider a discrete-time linear uncertain system

$$x_{k+1} = A(\Delta(\xi))x_k + Bw_{k+1} \tag{4.1a}$$

$$y_k = H_k x_k + v_k \tag{4.1b}$$

with an unknown initial value $x_0 \in \mathbb{R}^n$ and an unknown system parameter $\Delta(\boldsymbol{\xi}(\omega)) \in \mathbb{R}^d$ and a disturbance input vector w_k. Here $y_k \in \mathbb{R}^s$ is the system output, $A(\Delta(\xi)) \in \mathbb{R}^{n\times n}$ is the uncertain system matrix and $B \in \mathbb{R}^{n\times m}$ is the input matrix and $H_k \in R^{s\times n}$ is the output matrix, and v_k is the measurement disturbance. Consider that the uncertain system that is approximated by the truncated PC transformed system as

$$\bar{x}_{k+1} = \bar{A}\bar{x}_k + \bar{B}w_{k+1} \tag{4.2a}$$

$$y_k = \sum_{i=0}^{p} H_k x_{i,k}\Phi_i(\boldsymbol{\xi}(\omega)) + v_k \tag{4.2b}$$

where $\bar{A} \in \mathbb{R}^{np \times np}$ is the augmented PC transformed system matrix and $\bar{B} = \begin{pmatrix} B \\ \mathbf{0}_{np \times m} \end{pmatrix}$. Note here that if the random part in the measurement model due to the PC higher order terms is considered as the state dependent measurement disturbances, the measurement model can be expressed as

$$y_k = \bar{H}_{0,k}\bar{x}_k + v_k + \tilde{v}_k(\boldsymbol{\xi}) \tag{4.3}$$

where $\bar{H}_{0,k}$ can be considered as the nominal measurement matrix for the augmented system

$$\bar{H}_{0,k} = (H_k, \mathbf{0}_{s\times n}, \ldots, \mathbf{0}_{s\times n}) \tag{4.4}$$

and $\tilde{v}_k(\xi)$ is the state and the polynomial chaos dependent measurement disturbance

$$\tilde{v}_k(\boldsymbol{\xi}) = \sum_{i=1}^{p} H_k x_{i,k} \Phi_i(\boldsymbol{\xi}(\omega)) \tag{4.5}$$

Since the measurement model includes the random bases in terms of the polynomials of uncertain parameters, the truncated PC transformed system is not in a standard form for the application of the classical state estimation algorithms. We will consider two types of measurement modeling techniques for the state estimation problem of the truncated PC transformed system in the sequel.

Measurement Model I

One possible way of handling the dependence of the measurement disturbance on polynomial chaos is assuming that the sampled measurement is the average (with respect to PC chaos) value [59]. In this regard, the measurement model becomes

$$y_k = \bar{H}_{0,k}\bar{x}_k + v_k \tag{4.6}$$

Measurement Model II

In this technique, we define a quasi-deterministic set-valued state estimation problem and obtain the measurement model as a result of this formulation. Furthermore a stochastic interpretation of the set-valued state estimation problem is done. The two approaches that are explained in the next section give the same measurement equation. Both of these formulation are novel parts of this thesis. The common result of the two approaches induces the following measurement model:

$$\bar{y}_k = \bar{H}_k\bar{x}_k + \bar{v}_k \tag{4.7}$$

where $\bar{H}_k \in \mathbb{R}^{(p+1)s\times(p+1)n}$ with $\bar{H}_k = I_{p+1} \otimes H_k$ and $\bar{v}_k \in \mathbb{R}^{(p+1)s}$ with $E\{(\bar{v}_k\bar{v}_k^T)\} = I_{p+1} \otimes R_k$

4.2.1 Set-Valued Robust State Estimation Problem

Consider the truncated PC transformed system in (4.1) with the modified measurement model (4.2). Let Q_k, R_k be the given positive definite covariance matrices of the process and measurement noises. Assume that the mean and the covariance matrix of the initial state are $\bar{\bar{x}}_0$

and Σ_0 symmetric matrix. For a given fixed measurement sequence $y^s_{0:N}$, assume that the system disturbance sequences w_k,v_k,$\tilde{v}_k(\boldsymbol{\xi})$ and the initial condition of the system $\bar{x}_0$ satisfy the following sum quadratic constraint (energy constraint)

$$\left\|\bar{x}_0 - \tilde{\bar{x}}_0\right\|^2_{\Sigma_0^-} + \sum_{k=0}^{N-1}\left(\|w_k\|^2_{Q_k^-} + \left\|y^s_{k+1} - \bar{H}_{0,k+1}\bar{x}_{k+1}\right\|^2_{R^-_{k+1}}\right) \leq d - E_{\boldsymbol{\xi}}\left[\sum_{k=0}^{N-1}\|\tilde{v}_{k+1}(\boldsymbol{\xi})\|^2_{\bar{R}^-_{k+1}}\right] \quad (4.8)$$

over a finite horizon $[0, N]$ for a given scalar d being sufficiently large that $d > E_{\xi}\left[\sum_{k=0}^{N-1}\|\tilde{v}_{k+1}(\xi)\|^2_{\bar{R}^-_{k+1}}\right]$. Here $E_{\xi}[.]$ denotes the expectation with respect to the probability measure induced by the random variable $\boldsymbol{\xi}$. The left part of the inequality can be considered as the energy of the process disturbance and the measurement disturbance sequences of the the nominal model (nominal model is the model when the uncertainty does not exist). The right hand side of the inequality consists of the energy level d and the possible average energy loss due to the uncertainty in the measurement model. The set of all possible states at time N will be constructed by checking whether a candidate state can be reached by some uncertainty input that satisfies the constraints.

Definition 4.2.1. Let $X_N(\bar{x}_0, y^s_{0:N}, d)$ denote the set of possible states $\bar{x}_N$ at time N for the system (4.1) and the measurement model (4.2) with uncertain inputs w_k,v_k, $\tilde{v}_k(\xi)$ satisfying the constraint (4.8) for a given fixed measurement sequence $y^*_{0:N}$.Then the state estimation problem is defined as finding the set $X_N(\bar{x}_0, y^s_{0:N}, d)$ of all possible states at time N.

Proposition 4.2.1. *The set of possible states for a fixed measurement sequence $y^*_{0:N}$ is an ellipsoid*

$$X_N(\bar{x}_0, y^s_{0:N}, d) = \{\, \varsigma_N \in \mathbb{R}^n : \left\|\varsigma_N - \hat{\bar{x}}_N\right\|^2_{\Sigma^-_{N|N}} \leq \rho(\bar{y}^s_{1:N}) + d\} \quad (4.9)$$

where $\rho(\bar{y}^s_{1:N}) = \sum_{k=1}^{N}\left\|\bar{y}^s_k - \bar{H}_k\bar{A}\hat{\bar{x}}_{k-1}\right\|^2_{(\bar{H}_k\Sigma_{k|k-1}\bar{H}_k^T+\bar{R}_k)^-}$ *and* Σ_N, $\hat{\bar{x}}_N$ *can be computed recursively as*

$$\Sigma_{k+1|k+1} = \left(\Sigma^-_{k+1|k+1} + \bar{H}^T_{k+1}\bar{R}^-_{k+1}\bar{H}_{k+1}\right)^- \quad (4.10a)$$

$$\Sigma_{k+1|k} = \bar{A}\Sigma_{k|k}\bar{A}^T + \bar{B}Q_{k+1}\bar{B}^T, \Sigma_{0|0} = \Sigma_0 \quad (4.10b)$$

$$\hat{\bar{x}}_{k+1} = \bar{A}\hat{\bar{x}}_k + \Sigma_{k+1|k+1}H_k^T\bar{R}_k^-(\bar{y}^s_k - \bar{H}^T_{k+1}\bar{A}\hat{\bar{x}}_k) \;\; \textit{for given } \hat{\bar{x}}_0 \quad (4.10c)$$

$$\bar{R}_{k+1} = diag\left(\frac{1}{\langle\, (\Phi_0(\boldsymbol{\xi}))^2\rangle}, \ldots, \frac{1}{\left\langle\left(\Phi_p(\boldsymbol{\xi})\right)^2\right\rangle}\right) \quad (4.10d)$$

$$\bar{y}^s_k := \left((y^s_k)^T, \mathbf{0}_{ps}\right)^T \quad (4.10e)$$

$$\bar{H}_{k+1} = I_{p+1} \otimes H_{k+1} \quad (4.10f)$$

Proof. Assume that the measurement sequence $y_{0:N} = y^s_{0:N}$ is given. ς_N is an element of $X_N(\bar{x}_0, y^s_{0:N}, d)$ if and only if there exists sequences $\bar{x}_k, w_k, v_k, \tilde{v}_k(\boldsymbol{\xi})$ such that $\bar{x}_N = \varsigma_N$. Since $E_{\boldsymbol{\xi}}\left[\|\tilde{v}_{k+1}(\boldsymbol{\xi})\|^2_{\bar{R}^-_{k+1}}\right] = \sum_{i=1}^{p}\left\|H_{k+1}x_{i,k+1})\right\|^2_{R^-_{k+1}}\left\langle\Phi(\boldsymbol{\xi})^2\right\rangle$ then by defining

$$\bar{y}^s_{k+1} \triangleq \left((y^s_{k+1})^T, \mathbf{0}_{ps}\right)^T \quad (4.11)$$

and

$$\bar{H}_{k+1} = I_{p+1} \otimes H_{k+1} \tag{4.12}$$

it can be stated that $\varsigma_N \in X_N\left(\bar{x}_0, y^s_{0:N}, d\right)$ if and only if there exists a vector $\bar{x}_0$, disturbance sequences $w_{0:N}$ such that

$$J(\bar{x}_0, w_{0:N}, \varsigma_N) = \left\|(\bar{x}_0 - \hat{\bar{x}}_0)\right\|^2_{\Sigma_0^-} + \sum_{k=0}^{N-1}\left(\|w_k\|^2_{Q_k^-} + \left\|\bar{y}^s_{k+1} - \bar{H}_{0,k+1}\bar{x}_{k+1}\right\|^2_{R^-_{k+1}}\right) < d \tag{4.13}$$

subject to the system equation (4.2) and the terminal constraint $x_N = \varsigma_N$. It is obvious that there exist sequences $\bar{x}_0$ and $w_{0:N}$ satisfying the terminal constraint and (4.13) if and only if

$$J^*(\varsigma_N) = \min_{w_{0:N}} J(\bar{x}_0, w_{0:N}, \varsigma_N) < d \tag{4.14}$$

Note here that the initial condition $\bar{x}_0$ is dropped from the argument of the functional $J(\bar{x}_0, w_{0:N}, \varsigma_N)$ since it can be determined by using the system equation and the terminal constraint once the optimal $w_{0:N}$ is determined [3]. The state estimation problem is reduced to the following optimization problem

$$\begin{aligned} J^*(\varsigma_N) &\triangleq \min_{w_0:N} J(\bar{x}_0, w_{0:N}, \varsigma_N) \\ s.t.\ \ \bar{x}_{k+1} &= \bar{A}\bar{x}_k + \bar{B}w_{k+1}, \bar{x}_N = \varsigma_N \end{aligned} \tag{4.15}$$

The optimization problem (4.15) is very similar to the standard linear optimal tracking control problem. There is only one difference. The cost is imposed on the initial state of the system whereas in standard tracking problem the cost is defined on the final state of the problem. This difference can be handled by reversing the time index. The solution of the problem is well known [39], [3]. The optimal cost is obtained as in [3];

$$J^*(\varsigma_N) = \left\|\varsigma_N - \hat{\bar{x}}_N\right\|^2_{\Sigma^-_{N|N}} + \sum_{k=1}^{N}\left\|\bar{y}^s_k - \bar{H}_k\bar{A}\hat{\bar{x}}_{k-1}\right\|^2_{(\bar{H}_k\Sigma_{k|k-1}\bar{H}_k^T + \bar{R}_k)^-} \tag{4.16}$$

where $\Sigma_{k|k}$ and $\hat{\bar{x}}_N$ satisfy the recursions in (4.17). Thus the proof is complete. □

Remark 5. The optimal state estimation recursions are exactly the same as the Kalman filter recursions for the augmented measurement vector $\bar{y}^s_k$ with the corresponding measurement model $\bar{y}^s_k = \bar{H}_k\bar{x}_k + \bar{v}_k$ where $\bar{H}_{k+1} = I_{p+1} \otimes H_{k+1}$. This resemblance can be explained by deterministic least square interpretation of the Kalman filter.

4.2.2 Stochastic Polynomial Chaos Based Estimation

In the previous section, the measurement and process disturbance sequences are assumed to be unknown but deterministic sequences satisfying a semi-deterministic energy constraint. In this section the disturbances are modeled as the stochastic processes. In this regard, we consider the following quite general discrete-time stochastic linear uncertain system defined on a probability space $(\Omega, \mathfrak{F}, \mu)$:

$$x_{k+1} = A(\Delta(\boldsymbol{\xi}))x_k + Bw_{k+1} \tag{4.17a}$$

$$y_k = H_k x_k + v_k \tag{4.17b}$$

Here $y_k \in \mathbb{R}^m$ is the system output, $A(\Delta(\boldsymbol{\xi}))\ \mathbb{R}^{n\times n}$ is the uncertain system matrix, $B^{n\times m}$ is the input matrix and $H_k \in \mathbb{R}^{s\times n}$ is the output matrix. The uncertainty vector $\Delta(\xi)$ is defined on a probability space $(\Omega_\xi, \mathfrak{F}_\xi, \mu_\xi)$ that is called the uncertainty space. The process noise $w_k \in \mathbb{R}$, and measurement noise $w_k \in R^m$ and the initial state $x_0 \in \mathbb{R}^n$ are random variables defined on a different probability space denoted by $(\Omega_\eta, \mathfrak{F}_\eta, \mu_\eta)$. This space is called the noise space. It is assumed that the random variables w_k, v_k, x_0 are independent zero mean Gaussian distributed random variables with the corresponding covariance matrices Q_k, R_k, Σ_0. The whole probability space $(\Omega, \mathfrak{F}, \mu)$ can be considered as the Cartesian product space of the uncertainty and noise spaces $(\Omega_\xi \times \Omega_\eta, \mathfrak{F}_\xi \times \mathfrak{F}_\eta, \mu_\xi \times \mu_\eta)$. Then $\{\omega = (\omega_\xi, \omega_\eta) : (\omega_\xi \in \Omega_\xi, \omega_\eta \in \Omega_\eta\}$ and $\mathfrak{F} = \mathfrak{F}_\xi \times \mathfrak{F}_\eta$ is the σ-algebra generated by the collection of all measurable rectangles $\mathfrak{B}_\xi \in \mathfrak{F}_\xi$ and $\mathfrak{B}_\eta \in F_\eta$ i.e, $\mathfrak{F}_\xi \times \mathfrak{F}_\eta = \sigma\{\mathfrak{B}_\xi \times \mathfrak{B}_\eta\}$. Additionally, the random variables $\Theta(\xi), w_k, v_k, x_0$ are assumed to be in $L^2(\Omega, \mathfrak{F}, \mu)$, which is a vector space of the random vectors $x(\omega)$ such that $E_\mu\{x^T(\omega)x(\omega)\} < \infty$ After applying the Galerkin projection, stochastic version of the truncated PC transformed system model is obtained

$$\bar{x}_{k+1} = \bar{A}x_k + \bar{B}w_{k+1} \tag{4.18a}$$

$$y_k = \sum_{i=0}^{p} H_k x_{k,i} \Phi_i(\boldsymbol{\xi}(\omega)) + v_{k+1} \tag{4.18b}$$

As mentioned in the previous section, the linear uncertain system model (4.18) is not in the form of classical state estimation problems. Since the measurement model includes the uncertainty parameters, which is the main role of the polynomial chaos expansion system as transferring the internal system uncertainties to the system output. In order to handle this problem, we have proposed the optimization of cost functions for the state estimation problem. In the problem formulations, the uncertainty vector is assumed to be independent from the given measurements even though the observed measurement gives information about the uncertainty space. In other words, the estimation of the uncertain parameter is not considered but its effect on the estimation error is considered. In this regard, we will define three possible formulations for the state estimation problem. Each of the formulations corresponds to a minimization problem with a different objective function that should be minimized. In the first formulation we assume a linear Gaussian uncertain system structure, and convert the problem to the one that can be solved by Kalman filtering. The other two formulations are given for the general case with no proposed solution.

4.2.3 Expected Maximum a Posteriori Probability Estimation for Linear Gaussian Systems

In this problem, the expected value of the logarithm of the conditional probability density function is minimized. That is

$$J(\hat{\bar{x}}_{0:k}) = \underset{\bar{x}_{0:k}}{max} E_{\mu_\xi}[log(p(\bar{x}_{0:k}|y_{0:k}, \boldsymbol{\xi}))] \tag{4.19}$$

Due to the Bayesian rule

$$p(\bar{x}_{0:k}|y_{0:k}, \boldsymbol{\xi}) = \frac{p(y_{0:k}|\bar{x}_{0:k}, \boldsymbol{\xi})p(\bar{x}_{0:k}|\boldsymbol{\xi})}{p(y_{0:k}|\boldsymbol{\xi})} \tag{4.20}$$

Since the system's time propagation is independent of the uncertain parameter and the process and the measurement noises are independent we can write

$$p(\bar{x}_{0:k}|\boldsymbol{\xi}) = p(\bar{x}_{0:k}) = \prod_{m=1}^{k} p(\bar{x}_m|\bar{x}_{m-1})p(\bar{x}_0) \tag{4.21}$$

and

$$p(y_{0:k}|\bar{x}_{0:k}, \boldsymbol{\xi})\prod_{m=1}^{k} p(y_m|\bar{x}_m, \boldsymbol{\xi}) \tag{4.22}$$

Thus, the optimal state estimation problem (4.19) takes the following form

$$J(\hat{\bar{x}}_{0:k}) = \underset{\bar{x}_{0:k}}{max} E_{\mu_\xi}\left[log\left(\prod_{m=1}^{k} p(\bar{x}_m|\bar{x}_{m-1})p(\bar{x}_0)\prod_{m=1}^{k} p(y_m|\bar{x}_m, \boldsymbol{\xi})\right)\right] - log(p(y_{0:k}|\boldsymbol{\xi})) \tag{4.23}$$

Since the term $log(p(y_{0:k}|\xi))$ does not include the decision variable, it can be eliminated from the optimization problem. Then the optimal state estimation problem is reduced to the following quadratic optimization problem by assuming that the noise sequences are jointly Gaussian distributed

$$J(\hat{\bar{x}}_{0:k}) = \underset{\bar{x}_{0:k}}{min} E_{\mu_\xi}\left[\frac{1}{2}\left\|\bar{x}_0 - \hat{\bar{x}}_0\right\|^2_{\Sigma_0^-} + \frac{1}{2}\sum_{m=1}^{k}\left\|\bar{x}_m - \bar{A}_m\bar{x}_{m-1}\right\|^2_{\bar{Q}_m^-} + \|y_m - H_m(\boldsymbol{\xi})\|^2_{\bar{R}_m^-}\right] \tag{4.24}$$

where $H_k(\boldsymbol{\xi}) = (H_k\Phi_0(\boldsymbol{\xi}(\omega)), H_k\Phi_1(\boldsymbol{\xi}(\omega)), \ldots, H_k\Phi_p(\boldsymbol{\xi}(\omega)))$. Note here that due to zero rows of the input matrix B in system model, the covariance of the input noise Bw_{k+1} is singular. Thus the inverse of $\bar{Q}_m$ does not exist. However due to the PC truncation this model does not actually represent the true system model. Thus we tacitly assume that that input matrix is non singular by assuming that very small fictitious noise as input to the system. By taking the expectation and by using the orthogonality of the polynomial bases the optimal state estimation problem can be written as:

$$J(\hat{\bar{x}}_{0:k}) = \underset{\bar{x}_{0:k}\in\mathbb{R}^n}{min} E_{\mu_\xi}\left[\frac{1}{2}\left\|\bar{x}_0 - \hat{\bar{x}}_0\right\|^2_{\Sigma_0^-} + \frac{1}{2}\sum_{m=1}^{k}\left\|\bar{x}_m - \bar{A}_m\bar{x}_{m-1}\right\|^2_{\bar{Q}_m^-} + \left\|y_m - \bar{H}_m(\boldsymbol{\xi})\right\|^2_{\bar{R}_m^-}\right] \tag{4.25}$$

where

$$\bar{H}_m \triangleq I_{p+1} \otimes H_m \tag{4.26}$$

and

$$\bar{R}_{k+1} = diag1/\left\langle\Phi_0(\boldsymbol{\xi})\right\rangle, \ldots, 1/\left\langle\Phi_p(\boldsymbol{\xi})\right\rangle \otimes R_{k+1} \tag{4.27}$$

Note that the obtained problem is the augmented version of the deterministic formulation of the celebrated Kalman filter . Thus, the following augmented Kalman filter equations can be used for the solution of the problem.

Table4.1: Kalman Filter Recursive Equations

Time Update
$\bar{x}_{k\|k-1} = \bar{A}_{k\|k-1}\bar{x}_{k-1\|k-1}$
$\Sigma_{k\|k-1} = \bar{A}_{k\|k-1}\Sigma_{k-1\|k-1}\bar{A}^T_{k\|k-1} + \bar{Q}_k$
Measurement Update
$K_k = \Sigma_{k\|k-1}\bar{H}^T_k(\bar{H}\Sigma_{k\|k-1}\bar{H}^T + \bar{R}_k)^-$
$\bar{x}_{k\|k} = \bar{x}_{k\|k-1} + K_k(\bar{y}_k - \bar{H}_k\bar{x}_{k\|k-1})$
$\Sigma_{k\|k} = (I - K_k * \bar{H}_k)\Sigma_{k\|k-1}$

4.2.4 Other Possible Probabilistic Estimation Problems for PC Based Uncertain Systems

4.2.4.1 Average Mean Square Estimation (MSE)

In this formulation, the expectation of the conditional mean square estimation error with respect to the uncertainty random variable becomes the objective function that must be minimized. That is

$$\hat{\bar{x}}_k = \underset{u\in\mathbb{R}^n}{argmin} E_{\mu_\xi}\left[E_{\mu_\eta}\left[\frac{1}{2}\left\|\hat{\bar{x}}_k - u\right\|^2 \middle| y_{0:k}, \boldsymbol{\xi}\right]\right] \tag{4.28}$$

4.2.4.2 Average Risk-Sensitive Estimation

In this problem, the expectation of the exponential function of the cumulative mean square error with respect to the uncertainty random variable is minimized under the condition that the a priori estimates are available. That is

$$\hat{\bar{x}}_k = \underset{u\in\mathbb{R}^n}{argmin} E_{\mu_\xi}\left[E_{\mu_\eta}\left[exp\left(\frac{\theta}{2}\sum_{m=1}^{k-1}\left\|\hat{\bar{x}}_k - \bar{x}_m\right\|^2 + \left\|\hat{\bar{x}}_k - u\right\|^2\right)|y_{0:k}, \boldsymbol{\xi}\right]\right] \tag{4.29}$$

where $\hat{\bar{x}}_{0:k-1}$ is the set of the previous state estimates. If the outer expectation is disregarded, the problem is a classical risk-sensitive estimation where all the moments of the estimation error is minimized where the θ parameter weights the moments of the error.

The proposed problems are difficult to solve since the cost function and thus the solution include integral expressions that are not analytically tractable in most cases.

4.2.5 Observability of the Equivalent Measurement Models

In the state estimation problem observability plays an important role. The unobservable part of the state is estimated by means of the system dynamics only and the measurements have no role in its estimation.The observability of the truncated PC expanded system is worth to study due to this fact. In this section, we have analyzed the observability of both the classical measurement model (Model I) $y_k = \bar{H}_{0,k}\bar{x}_k + v_k$ where the sampled measurement is considered as the average of the realizations of all possible uncertainty vector $\boldsymbol{\xi}$ and the proposed estimation model (Model II) $\bar{y}_k = \bar{H}_k\bar{x}_k + \bar{v}_k$ that is induced by the solution of the our novel set-valued state estimation problem (or the equivalent expected maximum a posteriori probability estimation problem).

From now on we assume the uncertain system matrix is an affine function of the system uncertainty vector $\boldsymbol{\xi}$. Thus the system model is as defined in (4.2) where the system matrix is $\bar{A} = I_{p+1} \otimes A_0 + \sum\limits_{i=1}^{d} J_{p,i} \otimes A_i$. The common system model (4.30a) and the considered two measurement models (4.30b),(4.30c) are presented as

$$\bar{x}_{k+1} = \bar{A}x_k + \bar{B}w_{k+1} \tag{4.30a}$$

$$Model\ I\ :\ y_k = \bar{H}_{0,k}\bar{x}_k + v_k \tag{4.30b}$$

$$Model\ II\ :\ \bar{y}_k = \bar{H}_k\bar{x}_k + \bar{v}_k \tag{4.30c}$$

4.2.5.1 Observability of Model I

We give a necessary condition for the observability of the system model (4.2) with the measurement model I by the following lemma

Lemma 4.2.2. *Consider the PC transformed system* (4.2) *for single uncertainty case where the measurement matrix is constant $H_k = H$. Then the equivalent system* (4.30a) *with the measurement model* (4.30b) *is not observable if* (A_1, A_0) *pair is not observable.*

Proof. According to PBH eigenvector test, if (A_1, A_0) pair is not observable, then there exists a right eigenvector v_0 such that $(\lambda_0 I - A_0)v_0 = 0$, and $A_1 v_0 = 0$. Consider now the PC transformed system which is in the following block tri-diagonal form, then

$$\lambda I - \bar{A} = \begin{pmatrix} \lambda I - A_0 - b_0 A_1 & c_0 A_1 & 0 & \dots & 0 \\ a_1 A_1 & \lambda I - A_0 - b_1 A_1 & c_1 A_1 & \dots & 0 \\ 0 & a_2 A_1 & \lambda I - A_0 - b_2 A_1 & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & c_{p-1}A_1 \\ 0 & 0 & \dots & a_p A_1 & \lambda I - A_0 - b_p A_1 \end{pmatrix} \tag{4.31}$$

with the corresponding measurement matrix $\bar{H}_0 = [H, \mathbf{0}, \dots, \mathbf{0}]$. It is clear that $(\lambda_0 I - \bar{A})\bar{v} = \mathbf{0}$ and $\bar{H}_0\bar{v} = 0$ for a nonzero vector $\bar{v} = \left(\mathbf{0}^T, v_0^T, \dots, v_0^T\right)$ showing that the the pair $\left(\bar{H}_0, \bar{A}\right)$ is not observable. □

4.2.5.2 Observability of Model II

Following lemma provides a necessary condition for the observability of the system model (4.2) with the measurement model II.

Lemma 4.2.3. *Consider the PC transformed system* (4.2) *for single uncertainty case where the measurement matrix is constant* $H_k = H$*. Then the equivalent system* (4.30a) *with the measurement model* (4.30c) *is not observable if* $\left(\begin{pmatrix} H \\ A_1 \end{pmatrix}, A_0\right)$ *pair is not observable.*

Proof. According to PBH eigenvector test, if $\left(\begin{pmatrix} H \\ A_1 \end{pmatrix}, A_0\right)$ pair is not observable, then there exists a right eigenvector v_0 such that $(\lambda_0 I - A_0)v_0 = 0, Hv_0 = 0$ and $A_1 v_0 = 0$. Consider now the PC transformed system which is in the following block tri-diagonal form. By exploiting the special structure of the PC transformed system matrix (4.31) it is clear that $(\lambda_0 I - \bar{A})\bar{v} = \mathbf{0}$ and $\bar{H}\bar{v} = 0$ for $\bar{v} = \left(v_0^T, v_0^T, \ldots, v_0^T\right)$ showing that the the pair $\left(\bar{H}, \bar{A}\right)$ is not observable. □

4.2.6 The Effect of Polynomial Truncation Order for the Polynomial Chaos Based Estimation

In this section, we will study the effect of the polynomial truncation order on the proposed polynomial chaos based estimation method. The effect of the proposed estimation algorithm will be studied in two parts as the time update and the measurement update. First we consider the case that there is no measurement available in the time interval of interest. For this case consider the time update of the PC transformed system covariance matrix (or equivalently weight matrix that defines the ellipsoid in the set-valued estimation)

$$\Sigma_{k|k-1} = \bar{A}\Sigma_{k|k-1}\bar{A}^T + \bar{Q}_k \tag{4.32}$$

The covariance matrix of the augmented state vector can be related to the initial covariance matrix as

$$\Sigma_{k|k-1} = \bar{A}^k \Sigma_{0|0} (\bar{A}^k)^T + \sum_{i=1}^{k} \bar{A}^{k-i} \bar{Q}_i (\bar{A}^{k-i})^T \tag{4.33}$$

where $\bar{Q}_k = \begin{pmatrix} Q_k & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$. It is suitable for the initial covariance matrix to be set as $\bar{\Sigma}_{0|0} = \begin{pmatrix} \Sigma_{0|0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$ since the initial value of the higher order PC coefficients are exactly zero. Let us partition the state transition matrix

$$\bar{A_p}^k = \begin{bmatrix} \left[(\bar{A}_p)^k\right]_{0,0} & \left[(\bar{A}_p)^k\right]_{0,1} & \cdots & \left[(\bar{A}_p)^k\right]_{0,p} \\ \left[(\bar{A}_p)^k\right]_{1,0} & \left[(\bar{A}_p)^k\right]_{1,1} & \cdots & \left[(\bar{A}_p)^k\right]_{1,p} \\ \vdots & \vdots & \ddots & \vdots \\ \left[(\bar{A}_p)^k\right]_{p,0} & \left[(\bar{A}_p)^k\right]_{p,1} & \cdots & \left[(\bar{A}_p)^k\right]_{p,p} \end{bmatrix} \tag{4.34}$$

Then the covariance matrix of the zeroth order coefficient vector $x_{0,k}$ at time k can be determined as

$$\Sigma^{(0,0)}_{k|k-1} = \left[(\bar{A}_p)^k\right]_{(0,0)} \Sigma^{(0,0)}_{0|-1} \left(\left[(\bar{A}_p)^k\right]_{(0,0)}\right)^T + \sum_{i=1}^{k} \left[(\bar{A}_p)^{k-i}\right]_{(0,0)} Q_i \left(\left[(\bar{A}_p)^{k-i}\right]_{(0,0)}\right)^T \tag{4.35}$$

Lemma 4.2.4. *For a fixed value of k, the sub-matrices $\left[(\bar{A}_n)^k\right]$are same for any PC truncation order n>1.*

Proof. Since the state transition matrix of truncated PC transformed system at order n is

$$\bar{A}_n = \begin{bmatrix} A_0 & c_0A_1 & 0 & \dots & 0 \\ a_1A_1 & A_0 & c_1A_1 & \dots & 0 \\ 0 & a_2A_1 & A_0 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & a_nA_1 & A_0 \end{bmatrix} \tag{4.36}$$

then $\bar{A}_1 = \begin{bmatrix} A_0 & c_0A_1 \\ a_1A_1 & A_0 \end{bmatrix}$ where

$$(\bar{A}_1)^2 = \begin{bmatrix} A_0^2 + a_1c_0A_1^2 & c_0(A_1A_0 + A_0A_1) \\ a_1(A_1A_0 + A_0A_1) & A_0^2 + a_1c_0A_1^2 \end{bmatrix}$$

On the other hand $\bar{A}_2 = \begin{bmatrix} A_0 & c_0A_1 & 0 \\ a_1A_1 & A_0 & c_1A_1 \\ 0 & a_2A_1 & A_0 \end{bmatrix}$ where

$$\bar{A}_2^2 = \begin{bmatrix} A_0^2 + a_1c_0A_1^2 & c_0(A_1A_0 + A_0A_1) & c_0c_1A_1^2 \\ a_1(A_1A_0 + A_0A_1) & A_0^2 + (a_1c_0 + a_2c_1)A_1^2 & c_1(A_1A_0 + A_0A_1) \\ a_1a_2A_1^2 & a_2(A_1A_0 + A_0A_1) & [(\bar{A}_2)^2]_{(2,2)} \end{bmatrix}$$

Note here that (1,1),(1,2) and (2,1) blocks of the matrices $(\bar{A}_1)^2$ and $(\bar{A}_2)^2$ are same. Due to the tridiagonal structure of the $\bar{A}_n$ for any order n, this fact yields that these three elements will remain same for each power k. Thus $\left[(\bar{A}_n)^k\right]_{(0,0)} = \left[(\bar{A}_m)^k\right]_{(0,0)}$ for any n,m>1. □

Theorem 4.2.5. *The time propagated covariance $\Sigma^{(0,0)}_{k|k-1}$ of the sub set of the state vector of PC transformed system corresponding to the zeroth order coefficients of the PC expansion will not change by increasing the PC expansion order n for n>1.*

Proof. By using the relation (4.35) and the lemma 4.2.4, it is obvious that the covariance matrix will not change by increasing the PC order. □

Now let us consider the measurement update part. Measurement update is done by using the following equations of Kalman filtering.

$$K_k = \Sigma_{k|k-1}\bar{H}_k^T \left(\bar{H}_k\Sigma_{k|k-1}\bar{H}_k^T + \bar{R}_k\right)^{-} \quad \Sigma_{k|k} = \left(I - K_k\bar{H}_k\right)\Sigma_{k|k-1}$$

Table4.2: Legendre Polynomials

Polynomial Order	Legendre Polynomial
1	$\Phi_0(\xi) = 1$
2	$\Phi_1(\xi) = \xi$
3	$\Phi_2(\xi) = \frac{1}{2}(3\xi^2 - 1)$
4	$\Phi_3(\xi) = \frac{1}{2}(5\xi^3 - 3\xi)$
5	$\Phi_4(\xi) = \frac{1}{8}(35\xi^4 - 30\xi + 3)$
6	$\Phi_5(\xi) = \frac{1}{8}(63\xi^4 - 70\xi^3 + 15\xi)$

In the above equations the matrices $\Sigma_{k|k-1}\bar{H}_k^T$ and $\left(\bar{H}_k\Sigma_{k|k-1}\bar{H}_k^T + R_k^T\right)$ contain sub-matrices that are independent of the expansion order. However due to the inverse operator, expansion order effects the Kalman gain so the posterior covariance of the state estimate. However, the process noise covariance matrix of the truncated PC expansion has zero blocks in the B matrix that reduces the values of the blocks of $\Sigma_{k|k}$ other than the first few as time increases. This conclusion is certainly true when we consider the steady state value of the state covariance matrix when process noise is zero.

4.3 Illustrative Examples

In this section, we evaluate the performance of the proposed PC based robust filter by some illustrative examples. Since in classical robust estimation applications, the uncertainties are considered as unknown but bounded quantities, the uncertainty in this performance analysis is modeled as a uniformly distributed random variable. Thus Legendre polynomials are used for the polynomial chaos expansion. Legendre polynomials satisfy the following three-term recurrence relation

$$xp_n(x) = \frac{n+1}{2n+1}p_{n+1}x + \frac{n}{2n+1}p_{n-1}(x) \;\; n = 0, 1, 2\ldots \tag{4.37}$$

Thus the first few orthogonal Legendre polynomials are listed as in Table 4.2

4.3.1 System Dynamics

For single uncertainty case, if the state is approximated as $\prod_{m=1}^{k} x_{i,k}\Phi_i(\xi)$ then the following 3th order truncated PC transformed system can be obtained

$$\bar{x}_{k+1} = \begin{bmatrix} A_0 & \frac{1}{3}A_1 & \mathbf{0} & \mathbf{0} \\ A_1 & A_0 & \frac{2}{5}A_1 & \mathbf{0} \\ \mathbf{0} & \frac{2}{3}A_1 & A_0 & \frac{3}{7}A_1 \\ \mathbf{0} & \mathbf{0} & \frac{3}{5}A_1 & A_0 \end{bmatrix} \bar{x}_k + \begin{bmatrix} B \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} w_{k+1} \tag{4.38}$$

4.3.2 Measurement Models

In the analysis we have evaluated two types of PC based measurement models. In the first one the classical measurement model in which the sampled measurement is considered as the average measurement.

$$y_{k+1} = [H\ 0\ 0\ 0]\begin{bmatrix} x_{0,k+1} \\ x_{1,k+1} \\ x_{2,k+1} \\ x_{3,k+1} \end{bmatrix} + v_{k+1} \tag{4.39}$$

In the second measurement model, we have considered the proposed average maximum a priori estimation problem or equivalently the proposed set-valued estimation problem. In this regard, the measurement model takes the following form;

$$y_{k+1} = \begin{bmatrix} H & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & H & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & H & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & H \end{bmatrix}\begin{bmatrix} x_{0,k+1} \\ x_{1,k+1} \\ x_{2,k+1} \\ x_{3,k+1} \end{bmatrix} + \bar{v}_{k+1} \tag{4.40}$$

where $E\{v_{n,k}^2\} = \frac{1}{2n+1}$

4.3.3 The Filter Performance and Sensitivity

We have studied three framework example that is widely used in robust community [54],[74]. In order to evaluate the performance, the empirical average error variance is used [74]. For this purpose, 500 Monte Carlo simulations are performed each with a time span of N samples. The uncertainty ξ is re-generated and is fixed for each run. For the j^{th} trajectory, the observation series y_i^j is then filtered by five different algorithms; Kalman filter (Kal) which has the nominal model, the proposed robust filter (Pct1, Pct2, Pct3) and the optimal Kalman filter (KalTrue) which has the true system parameter for each run, regularized robust Kalman filter (Reg) [54] and H-infinity filter [56] which is one of the traditional robust estimation algorithms. The mean square error for each sample run is calculated as follows

$$E\{\|x_k - \hat{x}_k\|^2\} = \frac{1}{M}\sum_{j=1}^{M}\left\|\bar{x}_k^{(j)} - \hat{\bar{x}}_k^{(j)}\right\|^2 \tag{4.41}$$

for $k = 1, 2, \ldots, N$ where N is the time horizon and where $\hat{x}_k^{(j)}$is the kth value of the estimated j-th state trajectory and $\bar{x}_k^{(j)}$ is the i^{th} value of the true j^{th} state trajectory. We have also analyzed the sensitivity of the filter to the system uncertainty. By sensitivity we mean the change in the performance as the uncertain system deviates from the nominal one. The analysis on sensitivity is done for measurement type I.

4.3.4 Example I

Consider a time-invariant discrete-time uncertain system

$$x_{k+1} = (A_0 + A_1\xi)x_k + w_{k+1} \tag{4.42a}$$

$$y_k = H_k x_k + v_k \tag{4.42b}$$

where $A_0 = \begin{bmatrix} 0.9802 & 0.0196 \\ 0 & 0.9802 \end{bmatrix}$, $A_1 = \begin{bmatrix} 0 & 0.0099 \\ 0 & 0 \end{bmatrix}$, $H_k = \begin{bmatrix} 1 & -1 \end{bmatrix}$, $Q_k = E\{w_{k+1}w_{k+1}^T\} = \begin{bmatrix} 1.9608 & 0.0195 \\ 0.0195 & 1.9608 \end{bmatrix}$ and $R_k = 1$. Here the uncertain parameter ξ is uniformly distributed in the range [-1, 1].

4.3.4.1 The System Properties

The eigenvalues of the truncated PC transformed systems can be determined using the relationship between the nominal matrix A_0, and the perturbation matrix A_1 and the roots of the Legendre polynomials as stated in Theorem 3.1. That is $eig\left(\bar{A}\right) = \bigcup_{i=1}^{n} eig\left(A_0 + \mu_i A_1\right)$ where $\{\mu_i\}$ are the roots of the nth order Legendre polynomials. Since none of the modes of A_0 are controllable, the eigenvalues of the augmented PC transformed system at any order is equal to the single eigenvalue of the nominal system matrix A_0 which is 0.9802. Other system properties are summarized in Table 4.3. As seen in the table the truncated PC transformed system is not fully controllable. This will yield the higher order modes to be ineffective in the filter performances since these modes will not be excited by the process noise and the initial condition which is zero for them.

Table4.3: The System Properties (Example 1)

	System Order	Obs. States Model I	Obs. States Model II	Contr. States	Stability
First Order	4	4	4	3	Stable
Second Order	6	6	6	3	Stable
Third Order	8	8	8	3	Stable

4.3.4.2 The Filter Performance

The results for the Model I and Model II are presented in Figure 4.1 and Figure 4.2 respectively. For this system, A_0 and A_1 commute, furthermore $A_1^2 = \mathbf{0}$. This indicates that it is useless to increase the expansion order to a value greater than 1 for type I measurement model since additional modes are unobservable. This result is observed in the figures. The interesting result is that increasing the expansion order does increase the performance of the estimation also for type II measurements (see Figure 4.2). As expected the nominal Kalman

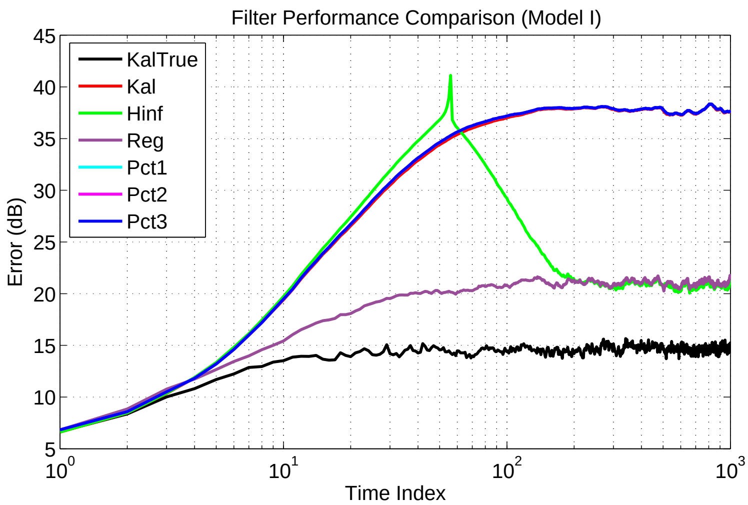

Figure 4.1: Polynomial Chaos filter performance Comparison for Example 1 (Model I). Kal: Kalman filter for the nominal model; Pct1, Pct2, Pct3: the proposed robust filter of expansion order 1, 2, 3; KalTrue: optimal Kalman filter which uses the true system parameter for each run; Reg: regularized robust Kalman filter; Hinf: H-infinity filter

filter which is optimal for the nominal system model is most sensitive to the system uncertainty (see Figure 4.3). The proposed filter sensitivity performance is comparable with the regularized robust filter and much better compared to nominal one. Again as seen in the performance figures, the PC order is almost non-effective.

4.3.5 Example 2

Consider a time-invariant discrete-time uncertain system

$$x_{k+1} = (A_0 + A_1\xi)x_k + Bw_{k+1} \tag{4.43a}$$

$$y_k = H_k x_k + v_k \tag{4.43b}$$

where $A_0 = \begin{bmatrix} 0 & -0.5 \\ 1 & 1 \end{bmatrix}$, $A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0.3 \end{bmatrix}$, $H_k = \begin{bmatrix} -100 & 10 \end{bmatrix}$, $B = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$ $Q_k = E\{w_{k+1}w_{k+1}^T\} = 1$ and $R_k = 1$. Here the uncertain parameter ξ is uniformly distributed in the range $[-1, 1]$.

4.3.5.1 The System Properties

The eigenvalues of the PC transformed system are plotted in Figure 4.4 for different expansion orders. Note that since the poles of different expansion orders are obtained as the union of the eigenvalues of the matrices $(A_0 + A_1\mu_i)$ for $i = 1, 2, \ldots, M$ where M is the expansion order and μ_i is the i^{th} root of the Legendre polynomial of order M, we can find the places

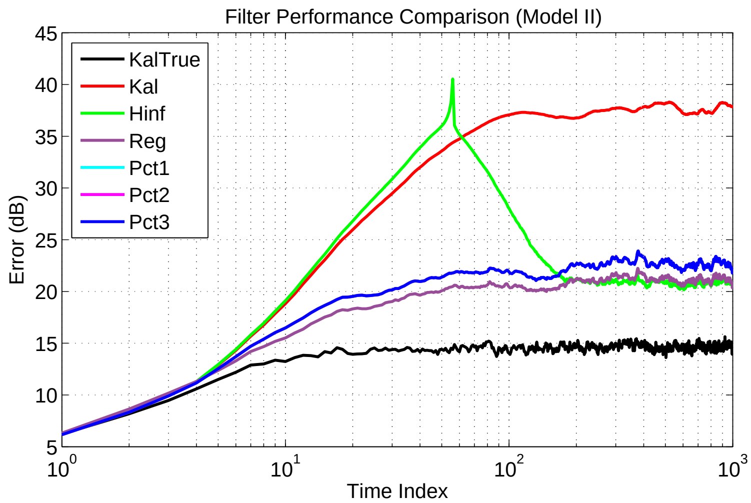

Figure 4.2: Polynomial Chaos filter performance Comparison for Example 1 (Model II). Kal: Kalman filter for the nominal model; Pct1, Pct2, Pct3: the proposed robust filter of expansion order 1, 2, 3; KalTrue: optimal Kalman filter which uses the true system parameter for each run; Reg: regularized robust Kalman filter; Hinf: H-infinity filter

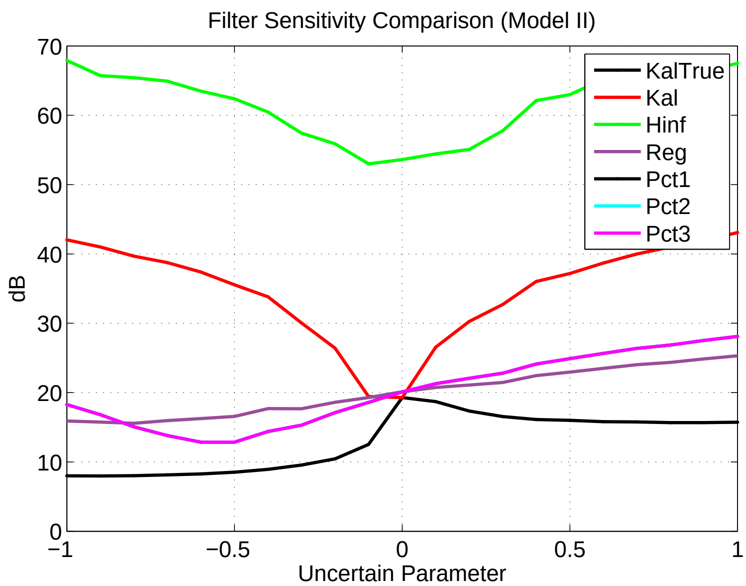

Figure 4.3: Sensitivity Results for the Proposed PC Robust Filter: Kal: Kalman filter for the nominal model; Pct1, Pct2, Pct3: the proposed robust filter of expansion order 1, 2, 3; KalTrue: optimal Kalman filter which uses the true system parameter for each run; Reg: regularized robust Kalman filter; Hinf: H-infinity filter

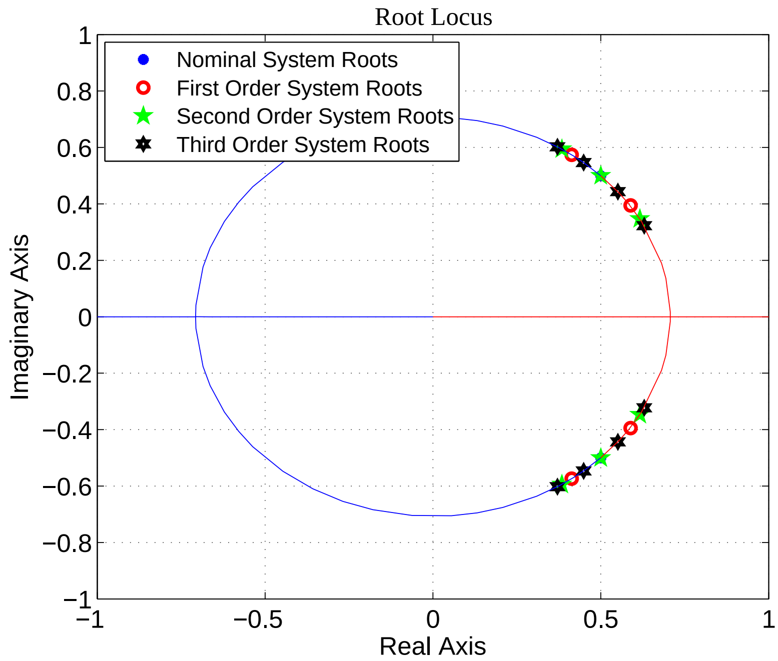

Figure 4.4: Locus of the eigenvalues of the truncated PC expansion of any order and for all possible values of $-\infty < \xi < \infty$ of Example 2

of all possible eigenvalues of any order by equating $|\lambda I - A_0 - A_1\xi| = 0$. For this example $|\lambda I - A_0 - A_1\xi| = \lambda^2 - \lambda + 0.5 - \lambda\xi = 0$. Root locus is helpful at this step to observe the location of the roots with respect to the changes in the uncertainty parameter ξ. Figure 4.4 shows the root locus for all possible ξ values and the actual roots of different orders. The other system properties are summarized in Table 4.4.

Table4.4: The System Properties (Example 2)

	System Order	Obs. States Model I	Obs. States Model II	Contr. States	Stability
First Order	4	4	4	4	Stable
Second Order	6	6	6	6	Stable
Third Order	8	8	8	8	Stable

4.3.5.2 The Filter Performance

The performances for Model I and Model II are presented in Figure 4.5 and Figure 4.6. One observation is that, the performances of both of the proposed filters are better than the other robust estimation methods. This can be seen from the sensitivity graphs as well. (see Figure 4.7) Increase in the expansion order cannot increase the performance of the filter for the

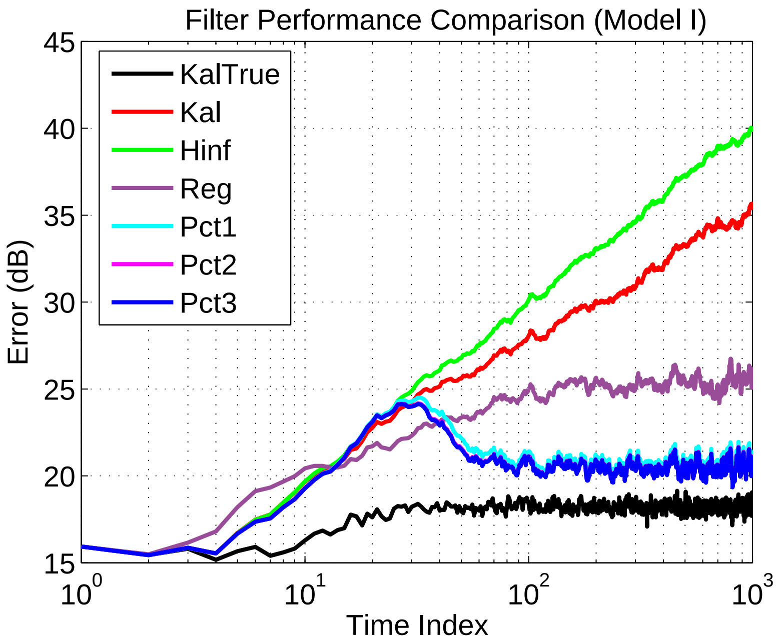

Figure 4.5: Polynomial Chaos filter performance Comparison for Example 2 (Model I). Kal: Kalman filter for the nominal model; Pct1, Pct2, Pct3: the proposed robust filter of expansion order 1, 2, 3; KalTrue: optimal Kalman filter which uses the true system parameter for each run; Reg: regularized robust Kalman filter; Hinf: H-infinity filter

observation model II.

4.3.6 Example 3

In this example, state estimation of the following uncertain oscillator system is studied.

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 + 0.1\xi & 0 \end{bmatrix} x(t) + w_(t) \tag{4.44a}$$

$$y(t) = Hx(t) + v(t) \tag{4.44b}$$

The corresponding 50 Hz sampled discrete-time equivalent system model found by a first-order hold method takes the following form

$$x_{k+1} = (A_0 + A_1\xi)x_k + w_{k+1} \tag{4.45a}$$

$$y_k = H_k x_k + v_k \tag{4.45b}$$

where $A_0 = \begin{bmatrix} 0.9996 & 0.02 \\ -0.04 & 0.9996 \end{bmatrix}$, $A_1 = \begin{bmatrix} 0 & 0 \\ 0.002 & 0 \end{bmatrix}$, $H_k = \begin{bmatrix} 1 & -1 \end{bmatrix}$, $Q_k = E\{w_{k+1}w_{k+1}^T\} = 10^{-6}I_2$ and $R_k = 10^{-2}$. Here the uncertain parameter ξ is uniformly distributed in the range [-1, 1].

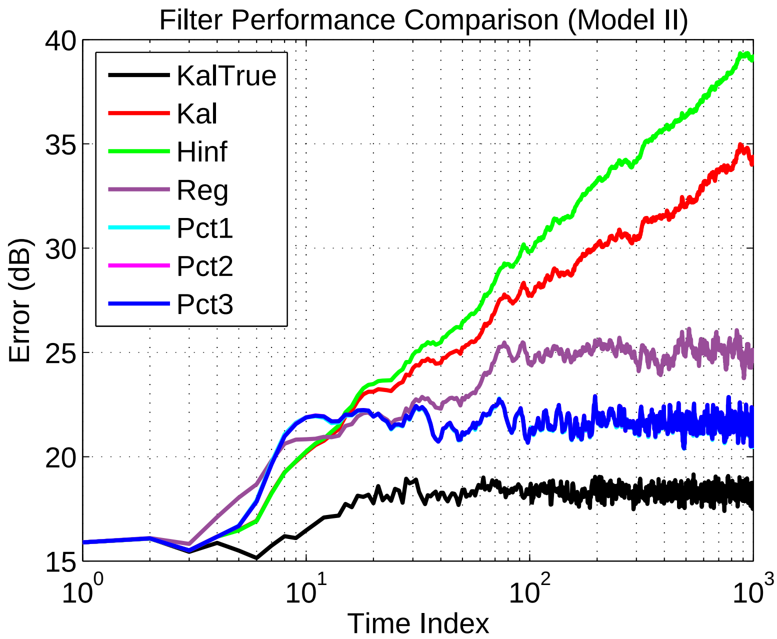

Figure 4.6: PC filter performances for Example 2 (Model I). Kal: Kalman filter for the nominal model; Pct1, Pct2, Pct3: the proposed robust filter of expansion order 1, 2, 3; KalTrue: optimal Kalman filter which uses the true system parameter for each run; Reg: regularized robust Kalman filter; Hinf: H-infinity filter

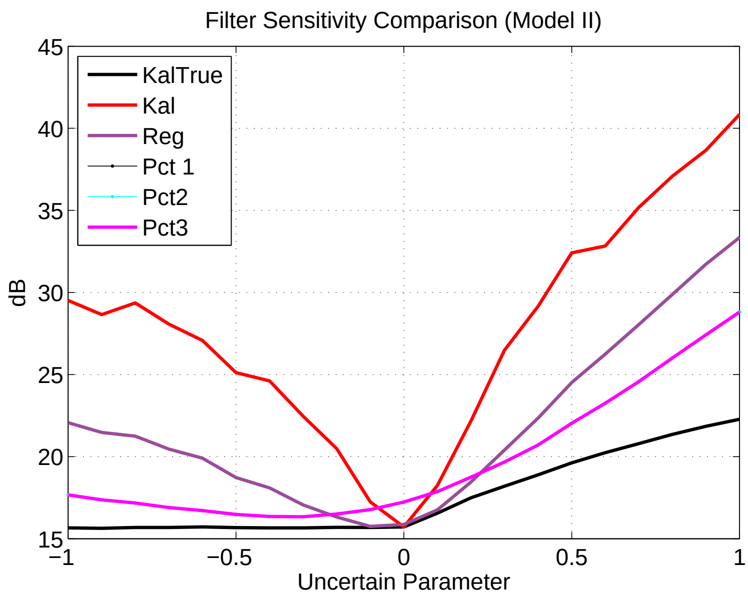

Figure 4.7: Sensitivity Results for the Proposed PC Robust Filter for Example 2 (Model II): Kal: Kalman filter for the nominal model; Pct1, Pct2, Pct3: the proposed robust filter of expansion order 1, 2, 3; KalTrue: optimal Kalman filter which uses the true system parameter for each run; Reg: regularized robust Kalman filter

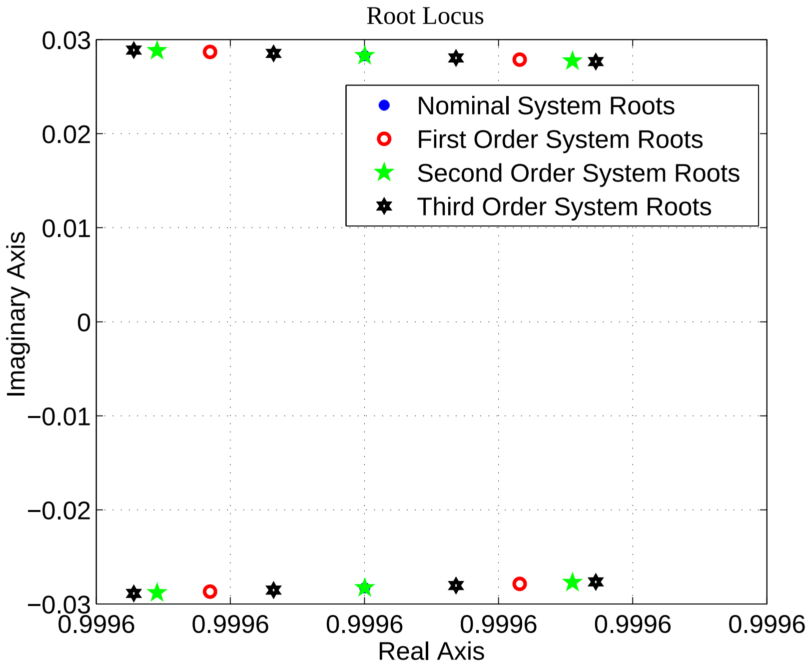

Figure 4.8: Locus of the eigenvalues of the truncated PC expansion of the first three order

4.3.6.1 System Properties

The eigenvalues of the PC transformed system is on the unit circle for all orders with the corresponding eigenvalues(see Figure 4.8). The summary of the system properties is given in Table 4.5.

Table4.5: The System Properties (Example 3)

	System Order	Obs. States Model I	Obs. States Model II	Contr. States	Stability
First Order	4	4	4	4	Stable
Second Order	6	6	6	6	Stable
Third Order	8	8	6	7	Stable

4.3.6.2 The Filter Performance

The performances for Model I and Model II are presented in Figure 4.9 and Figure 4.10. It is seen that the both models perform poorly for this case. It is interesting to see that the Model I diverges with time for all orders but the PC order increase prevents the divergence to some extent. This observation is well known for the truncated PC model approximations of differential equations One of the Our explanation for the poor performance under oscillatory case

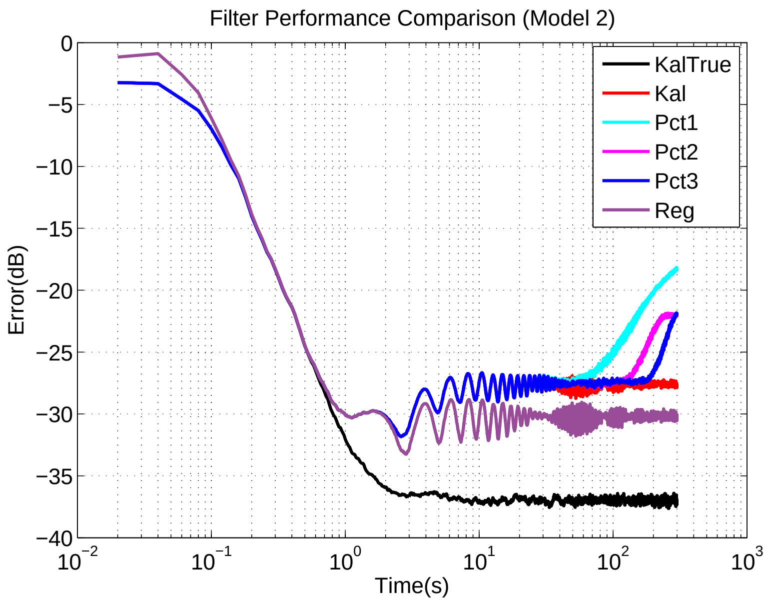

Figure 4.9: PC filter performances for Example 3 (Model I). Kal: Kalman filter for the nominal model; Pct1, Pct2, Pct3: the proposed robust filter of expansion order 1, 2, 3; KalTrue: optimal Kalman filter which uses the true system parameter for each run; Reg: regularized robust Kalman filter

is as follows. Actually, PC based system approximates the system state as a weighted some of the modes of the PC transformed system which are pure sinusoids in oscillatory case since the all the eigenvalues are on unit circle. It is known that the summation of the two sinusoids with very close frequencies yields a beating phenomenon which is perceived as a periodic variations in amplitude whose frequency is the difference between the two frequencies. Thus in actual system realization system output is a pure sinusoid however PC transformed system yields an amplitude varying output.

4.4 Conclusion

In this section, we study robust estimation problem of uncertain systems that are modeled by polynomial chaos. We propose a set-valued estimation approach for which the disturbance signals satisfy average energy constraint. We also provide the stochastic interpretation of the proposed estimation technique as average maximum likelihood estimation problem. The solution actually the augmented version of the Kalman filter. This is quite advantageous since Kalman filter is a well-known algorithm and it has been implemented in various systems over the world. We also compare the proposed measurement model with the existing empirical model in literature. In this regard, we provide some necessary conditions for the observability of the two measurement models which give a better understanding of the differences of these two models. We have evaluated the performance of the considered two approaches by three illustrative examples that are used in robust estimation community as framework examples.

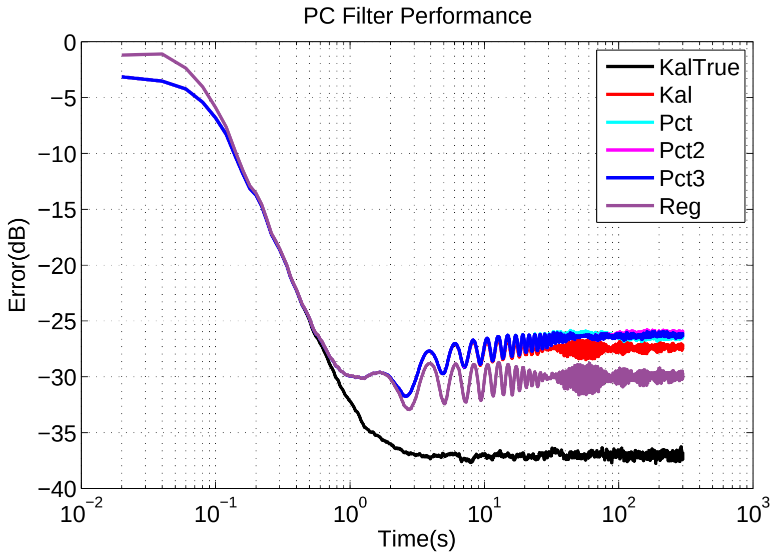

Figure 4.10: PC filter performances for Example 3 (Model II). Kal: Kalman filter for the nominal model; Pct1, Pct2, Pct3: the proposed robust filter of expansion order 1, 2, 3; KalTrue: optimal Kalman filter which uses the true system parameter for each run; Reg: regularized robust Kalman filter

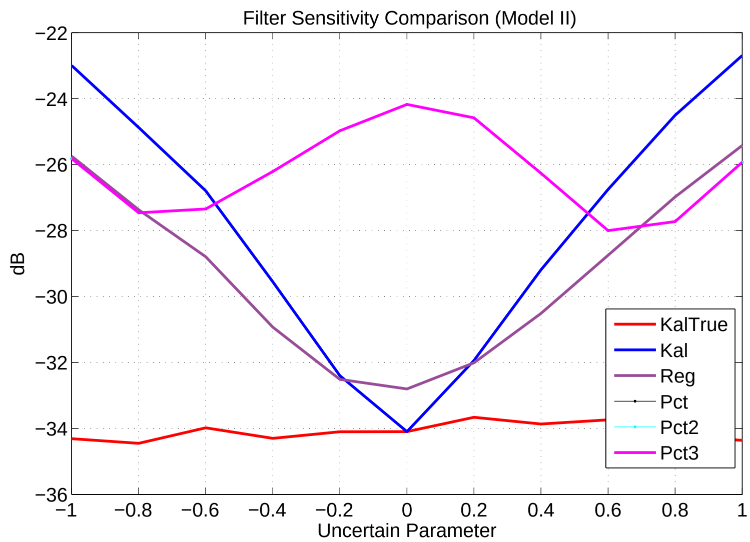

Figure 4.11: PC filter performances for Example 3 (Model II). Kal: Kalman filter for the nominal model; Pct1, Pct2, Pct3: the proposed robust filter of expansion order 1, 2, 3; KalTrue: optimal Kalman filter which uses the true system parameter for each run; Reg: regularized robust Kalman filter

The performance of the proposed approaches were compared with the nominal Kalman filter and classical robust estimation algorithms namely the regularized robust Kalman filter and the H_∞ filter. It can be concluded that the proposed approach performs as classical robust estimation algorithm in that it is less sensitive to the model uncertainty. It has been observed by the examples that for moderately damped systems, it performs better than the other classical robust estimation techniques such as H_∞ and regularized robust Kalman filters. However, the proposed approach does not perform as good as the other techniques for oscillatory systems. We give some explanations to this lack of performance for oscillatory systems. But further study is necessary in order to get some concrete results. Furthermore it is observed that the performance of the proposed approach is not sensitive to the polynomial chaos order. We provide some reasoning to this characteristics.

CHAPTER 5

NONLINEAR ROBUST ESTIMATION WITH RELATIVE ENTROPY CONSTRAINT

5.1 Introduction

In the previous chapters, we formulate the robust estimation problem and give a solution that is based on the polynomial chaos expansion and analyze the proposed solution from different points of view. The systems handled are linear except for a special class of nonlinearity. In this chapter, we study the robust estimation problem for uncertain nonlinear systems where the uncertainty is modeled by the relative entropy constraint.

Initially, robust estimation techniques were based on worst-case considerations of the disturbances in parallel to the robust control theory where the disturbance signals are modeled as deterministic signals. The risk-sensitive approach in which the average exponential of the square of the estimation error is imposed as error criteria [44] is a stochastic interpretation of the deterministic worst case approach (minimax games). Due to the exponential in the cost function, the error criterion includes all the higher order moments of the estimation error which yields a robust approach for unmodeled plant dynamics [1].

The risk-sensitive criterion for optimal control problems was first introduced by Jacobson in [32] and for linear systems a solution was provided for fully observed case. In the following years, Whittle in [67], provided the the solution of the partially observed risk-sensitive control problem for discrete-time systems and the continuous-time version of the solution was presented in [2].

In parallel to advances in risk-sensitive control techniques, optimal stochastic estimation problem with exponential cost criteria is solved in [62]. The optimal estimator is linear in structure but is not the conditional mean. In [14], the risk sensitive estimation problem is expressed in terms of an information state, which is a combination of the system state and the risk-sensitive cost function to be minimized. A recursive calculation of the information state is obtained by a measure change process where the measurement sequence becomes an independent identically distributed sequence and independent from the state process. The same recursions are obtained in [4], without utilizing the measure change process. The risk-sensitive estimation problem for nonlinear systems does not yield a finite-dimensional solution except for few specific cases.

Relative entropy (a.k.a. Kullback–Leibler divergence) which quantifies the difference between two probability measures is a widely studied measure especially in information theory. The researchers used relative entropy as a measure for defining the uncertainty on system models. Since it is a convex function, it is advantageous in the related optimal control problems. In [50], a stochastic uncertain system modeling is proposed, where the uncertain system is modeled by a relative entropy constraint on the driving stochastic disturbance signals (process and measurement noise sequences and initial condition uncertainty). This generalized stochastic uncertainty modeling allows the stochastic version of the sum quadratic constraint uncertain system modeling under Gaussian distributed noise case. The state estimation problem is considered as a minimax estimation problem where the designer seeks an optimal estimator that minimizes the worst case functional for the admissible sets of uncertain systems satisfying the relative entropy constraint. This unconstrained minimax problem is converted to a parameterized risk-sensitive problem by utilizing the duality between the free energy and the relative entropy [50]. In [77], the solution of the robust estimation is provided for linear uncertain systems. The optimal state estimation problem is solved by dynamic programming method once the problem is converted to parameterized risk-sensitive estimation. In [69], conditional relative entropy constrained is considered for the uncertainty modeling. This approach is applied to the finite horizon robust estimation for uncertain Finite-Alphabet Hidden Markov models. Relative entropy constrained stochastic uncertain systems and related minimax control and estimation problems are also studied in [10]. In [11], robust nonlinear estimation problem is studied in Banach space for uncertain signal models which are described by relative entropy constraint. The problems are defined for class of uncertain models where the uncertainty is on the joint probability measure and conditional probability measure. Recently, the instead of cumulative relative entropy constrained, an instantaneous relative entropy constrained is proposed in order to prevent the over conservatism of robust estimators [**?**].

Sequential Monte-Carlo estimation methods (particle filters) are becoming a popular and a practical estimation method for nonlinear non Gaussian estimation problems with the help of the increasing computing power. This solution technique is also applied to risk sensitive estimation of non-linear systems [48]. In [48], infinite-dimensional information state recursions are obtained for the proposed general risk functions that are in the product form. Then the information state recursions are handled by particles in a standard particle filtering algorithm. In their work, risk-sensitive particle filters are proposed as an alternative solution to sample impoverishment problem and robustness is not an explicit aim.

In this chapter, we study the robust nonlinear estimation problem for uncertain systems where the uncertainty is modeled by relative entropy constraint.

In the first problem, the uncertainty is defined on the joint probability measure between the nominal and perturbed measures over a time horizon where the perturbed measure satisfies the relative entropy constraint with respect to the nominal measure. Using the available results in literature, the optimal state estimation problem is defined as a minimax estimation problem. The constrained minimax optimization problem is converted to the unconstrained optimization by Lagrange multiplier method. The dual optimization problem requires the calculation of the state estimate sequence over a time horizon that minimizes the exponential of the estimation error as worst case scenario. The problem can be considered as a determination of an output feedback controller for an optimal risk-sensitive stochastic control problem. The solution of this problem can be obtained via information state dynamic programming where the output feedback problem is converted to information state feedback problem. We provide the solution of the problem for linear systems. We utilize this solution in order to solve the robust estimation problems nonlinear systems by extended approach where the system is linearized around the current estimate. In addition to extended approach, we define a suboptimal problem. The suboptimal problem converts to the original problem to a sequential optimization problem in terms of an information state in forward time. In the solutions of the problem, particle filtering is proposed for the calculations of the recursive probability measure relations. But a complete recursive solution of the suboptimal problem cannot be obtained due to expectation operation over measurements over a time horizon. But the optimal solution can be obtained by Monte-Carlo simulations. In the final part of the subsection a non-analytical method which determines the optimal Lagrange multiplier by trial and error is applied to a framework example.

In the second problem, we study the nonlinear estimation problem for instantaneous type relative entropy constraint. Two different sub problems are defined for both the time update and measurement update. Then some numerical solutions are proposed for the problems. First proposed approach is particle filtering method. An approximate but less complex solution method is proposed for the problems by utilizing the unscented transformation technique.

5.2 Robust Nonlinear Estimation with a Cumulative Relative Entropy Constraint

In this section, robust nonlinear estimation methods for uncertain systems where the uncertainty is defined by the cumulative relative entropy constrained are studied. Approximate solution methods are provided.

5.2.1 Uncertain Model

Let $(\Omega, \mathfrak{F}, P)$ be a probability space on which the unobserved (system state) process $\{x_k\}$ and observed process $\{y_k\}$ are defined. Assume that these processes satisfy the following recursions

$$x_{k+1} = f_k(x_k) + B_{k+1}w_{k+1}, \ x_0 \in \mathbb{R}^n \tag{5.1a}$$

$$y_k = h_k(x_k) + D_k v_k \tag{5.1b}$$

for $0 < k < N$. Here $x_0 : \Omega \to \mathbb{R}^n$ is the initial state and $w_k : \Omega \to \mathbb{R}^n$, $v_k : \Omega \to \mathbb{R}^m$ are the random vectors that define stochastic noise inputs. We further assume that the noise sequences (x_0, w_k, v_k) are independent. The functions $f_k : \Omega \to \mathbb{R}^n$ and $h_k : \Omega \to \mathbb{R}^n$ are measurable nonlinear functions of the system state x_k. Let X_k be a complete filtration, i.e., the σ field that is generated by the state sequence $x_{0:k} = \{x_0, \ldots, x_k\}$ and Y_k be a complete filtration generated by the measurement sequence $y_{0:k} = \{y_0, \ldots, y_k\}$ for $0 \le k \le N$. In this probability space, E_P denotes the expectation operator with respect to the probability measure P. The nominal joint probability measure of the state and the measurement sequences over a time horizon $[0, N]$ is denoted as $P_{x_{0:N}, y_{0:N}}$. However, the actual joint probability measure is $Q_{x_{0:N}, y_{0:N}}$ and it is unknown. The unknown measure is constrained to be in a class of admissible measures $C\left(P_{x_{0:N}, y_{0:N}}\right)$. The set $C\left(P_{x_{0:N}, y_{0:N}}\right)$ is defined by the following relative entropy constraint

$$C\left(P_{x_{0:N}, y_{0:N}}\right) = \left\{Q_{x_{0:N}, y_{0:N}} : R\left(Q_{x_{0:N}, y_{0:N}} \| P_{x_{0:N}, y_{0:N}}\right) \le d\right\} \tag{5.2}$$

where

$$R\left(Q_{x_{0:N}, y_{0:N}} \| P_{x_{0:N}, y_{0:N}}\right) = E_{Q_{x_{0:N}, y_{0:N}}}\left(log\left(\frac{dQ_{x_{0:N}, y_{0:N}}}{dP_{x_{0:N}, y_{0:N}}}\right)\right) = \int log\left(\frac{dQ_{x_{0:N}, y_{0:N}}}{dP_{x_{0:N}, y_{0:N}}}\right) dP_{x_{0:N}, y_{0:N}} \tag{5.3}$$

provided that $Q_{x_{0:N}, y_{0:N}}$ is absolutely continuous with respect to $P_{x_{0:N}, y_{0:N}}$ and

$$\int log\left(\frac{dQ_{x_{0:N}, y_{0:N}}}{dP_{x_{0:N}, y_{0:N}}}\right) dP_{x_{0:N}, y_{0:N}} < \infty \tag{5.4}$$

Here $d > 0$ is a scalar that determines the size of the constraint set. Using the additive noise assumption and independence property of the noise sequences, the nominal joint probability measure can be decomposed as

$$P_{x_{0:N}, y_{0:N}} = \prod_{k=1}^{N} P_{x_k|x_{k-1}} \prod_{k=1}^{N} P_{y_k|x_k} P_{x_0} \tag{5.5}$$

where $P_{x_k|x_{k-1}}$ is the (regular) conditional probability measure of the state x_k given the previous state x_{k-1} and $P_{y_k|x_k}$ is the (regular) conditional probability measure of the measurement y_k given the state x_k. On the other hand, if the unknown joint probability measure is assumed to satisfy

$$Q_{x_{0:N}, y_{0:N}} = \prod_{k=1}^{N} Q_{x_k|x_{k-1}} \prod_{k=1}^{N} Q_{y_k|x_k} Q_{x_0} \tag{5.6}$$

then the relative entropy between the two measures becomes

$$R\left(Q_{x_{0:N}, y_{0:N}} \| P_{x_{0:N}, y_{0:N}}\right) = E_{Q_{x_{0:N}, y_{0:N}}}\left[R\left(Q_{x_0} \| P_{x_0}\right) + \sum_{k=1}^{N}\left(R\left(Q_{x_k|x_{k-1}} \| P_{x_k|x_{k-1}}\right) + R\left(Q_{y_k|x_k} \| P_{y_k|x_k}\right)\right)\right] \tag{5.7}$$

The relative entropy constraint allows the perturbations in the mean of the nominal measure [50]. The perturbations in the mean can be generated by the additive uncertain system dynamics $\Delta(x_k)$. Note that the relative entropy between the two Gaussian measures (P and Q) with same covariance Σ but different means (m_P and m_Q) is $\frac{1}{2}\|m_p - m_Q\|^2_{\Sigma^-}$. Thus, the relative entropy constraint allows the following stochastic energy constraint over the time horizon

$[0, N]$ for a Gaussian nominal measure and additive uncertain system dynamics

$$E_Q\left[(x_0 - \tilde{x}_0)^T \Sigma_0^- (x_0 - \tilde{x}_0)\right] + E_Q\left[\sum_{k=1}^{N} ||\Delta^T(x_k) Q_k^- \Delta(x_k)||^2\right] < d \tag{5.8}$$

where $\tilde{x}_0$ x_0 are the initial perturbed and nominal measures' means.

5.2.2 The Optimal State Estimation Problem

The pay-off function for the state estimation problem is defined as the expected value of the cumulative error function over a time horizon [0,N],

$$\Psi_{0:N}(\hat{x}_{0:N}) = \sum_{k=0}^{N} \ell(x_k, \hat{x}_k) \tag{5.9}$$

where $\hat{x}_{0:N}$ is the sequence of estimated states which belongs to the set of possible state estimate sequences $\aleph_{0:N}$. Here

$$\aleph_{0:N} \triangleq \left\{\hat{x}_{0:N} \middle| \, \hat{x}_k = \Omega \rightarrow \mathbb{R}^n \; for \; 0 \leq k \leq N; \hat{x}_k \; is \; adapted \; to \; Y_k\right\} \tag{5.10}$$

Thus the state estimate at time k ($\hat{x}_k$) is a causal function of the data set $\{y_{0:k}\}$. Here $\ell(x_k, \hat{x}_k)$ is the instantaneous error that is continuous both in x_k and $\hat{x}_k$ and it is bounded from below. Obviously one possible selection of the incremental cost is the widely used square of the norm of the estimation error $\ell(x_k, \hat{x}_k) = ||x_k - \hat{x}_k||^2$. Then the state estimation problem is defined as

$$J(\hat{x}^*_{0:N}, Q^*_{x_{0:N}, y_{0:N}}) = \inf_{\hat{x}_{0:N} \in \aleph_{0:N}} \; \sup_{Q_{x_{0:N}, y_{0:N}} \in C\left(Q_{x_{0:N}, y_{0:N}}\right)} E_Q\left[\Psi_{0:N}(\hat{x}_{0:N})\right] \tag{5.11}$$

5.2.2.1 Existence of the Solution

The existence of the solution of the minimax problems with relative entropy constraint is investigated in the literature [50] and [10]. Thus in this section, we provide the main facts that has been presented before. The existence of the maximizing measure is proven by the Wierstrass theorem.

Theorem 5.2.1 (Wierstrass Theorem). *An upper semi-continuous functional on a compact subset K of a normed linear space S, achieves a maximum on K.*

For the state estimation problem (5.11), as shown in [10], the relative entropy constraint set $C(P)$ is compact and the functional $E_{Q_{x_{0:N}, y_{0:N}}}\left[\Psi^c_{0:N}(\hat{x}_{0:N}\right]$ is upper semi-continuous function provided the cost function $\Psi^c_{0:N}(\hat{x}_{0:N})$ is a continuous function of $x_{0:N}$, which is satisfied by definition. The existence of saddle point solution of the minimax problem is shown by the generalization of the Von-Neumann's minimax theorem to infinite dimensional case [10].

Theorem 5.2.2 ([57]). *Let M and N be convex sets one of which is compact. Let $f(m,n)$ be a functional defined on $M \times N$, quasi convex/concave and upper semi-continuous/lower semi-continuous. Then f has a saddle point, that is* $\underset{m\in M}{sup}\underset{n\in N}{inf} f(m,n) = \underset{n\in N}{inf}\underset{m\in M}{sup} f(m,n)$

The saddle point property implies that

$$\begin{aligned} J(\hat{x}^*_{0:N}, Q^*_{x_{0:N},y_{0:N}}) &= \inf_{\hat{x}_{0:N}\in\aleph_{0:N}} \sup_{Q_{x_{0:N},y_{0:N}}\in C\left(Q_{x_{0:N},y_{0:N}}\right)} E_Q\left[\Psi_{0:N}(\hat{x}_{0:N})\right] \\ &= \sup_{Q_{x_{0:N},y_{0:N}}\in C\left(Q_{x_{0:N},y_{0:N}}\right)} \inf_{\hat{x}_{0:N}\in\aleph_{0:N}} E_Q\left[\Psi_{0:N}(\hat{x}_{0:N})\right] \end{aligned} \tag{5.12}$$

We know that the set of admissible measures C is a convex set due to the convexity property of the relative entropy constraint; it is compact as stated previously. On the other hand, by definition, the set of admissible estimators is convex. The functional $J(\hat{x}_{0:N}, Q_{x_{0:N},y_{0:N}})$ is a convex function of any admissible estimate sequence and it is continuous thus a lower semi-continuous function for each admissible measure. Additionally, the functional is a concave function of any admissible measure due to linearity and it is an upper semi-continuous function of the admissible measure for any estimator. Thus, the above theorem proves the existence of the saddle point solution.

5.2.2.2 The Maximizing Measure

The maximization part of the minimax problem is solved by Lagrange multiplier method [50]. Let us first define the maximization problem as

$$J(\hat{x}_{0:N}, Q^*_{x_{0:N},y_{0:N}}) = \underset{Q_{x_{0:N},y_{0:N}}\in C\left(Q_{x_{0:N},y_{0:N}}\right)}{sup} E_{Q_{x_{0:N},y_{0:N}}}\left[\Psi_{0:N}(\hat{x}_{0:N})\right] \tag{5.13}$$

Then the Lagrangian is defined as

$$\begin{aligned} L\left(\hat{x}_{0:N}, Q_{x_{0:N},y_{0:N}}, s, \lambda\right) \triangleq & \int \Psi_{0:k}\left(\hat{x}_{0:N}\right) dQ_{x_{0:N},y_{0:N}} - \lambda\left(R\left(Q_{x_{0:N},y_{0:N}} \middle\| P_{x_{0:N},y_{0:N}}\right) - d\right) \\ & - s\left(\int dQ_{x_{0:N},y_{0:N}} - 1\right) \end{aligned} \tag{5.14}$$

where $s \geq 0$ and $\lambda \geq 0$ are the Lagrange multipliers of the corresponding constraints. Then the Lagrange dual function is defined as

$$L\left(\hat{x}_{0:N}, Q^*_{x_{0:N},y_{0:N}}, s, \lambda\right) = \sup_{Q_{x_{0:N},y_{0:N}}\in\Gamma}\left(L\left(\hat{x}_{0:N}, Q_{x_{0:N},y_{0:N}}, s, \lambda\right)\right) \tag{5.15}$$

where Γ represents the set of all possible probability measures. Then the dual problem becomes

$$L\left(\hat{x}_{0:N}, Q^*_{x_{0:N},y_{0:N}}, s^*, \lambda^*\right) = \inf_{s\geq 0}\inf_{\lambda\geq 0}\sup_{Q_{x_{0:N},y_{0:N}}\in\Gamma}\left(L\left(\hat{x}_{0:N}, Q_{x_{0:N},y_{0:N}}, s, \lambda\right)\right) \tag{5.16}$$

The Equivalence between the Primal and the Dual Problems

The equivalence between the primal and the dual optimization problems is established by the Lagrange-Duality theorem [10],[50], which is named as the Strong Duality Theorem in the optimization literature.

Theorem 5.2.3 (Lagrange Duality)**.** *Let f be a real-valued convex functional defined on a convex subset Ω of a vector space S, and let G be a convex mapping of S into a normed space Z. Suppose there exists an s_1 such that $G(s_1) < 0$ and that $\mu_0 = \inf\{f(s) : G(s) \le 0, s \in \Omega\}$ is finite. Then*

$$\inf_{G(s)\le 0, s\in\Omega} f(s) = \max_{\lambda^*\ge 0}\inf_{s\in\Omega}\left(f(s) + \langle G(s), \lambda^*\rangle\right) \tag{5.17}$$

and the maximum on the right is achieved by some $\lambda^* \ge 0$. If the infinimum is achieved by some $s_0 \in \Omega$, then $\left\langle G(s_0), \lambda_0^*\right\rangle = 0$ and s_0 minimizes $f(s) + \left\langle G(s), \lambda_0^*\right\rangle,\ s \in \Omega$.

The Lagrange-Duality theorem can be applied to robust state estimation problem after some modifications [10]. Note that the constraint qualification condition (existence of a strictly feasible point for the inequality constraint) for the strong duality is satisfied for $Q_{x_{0:N},y_{0:N}} = P_{x_{0:N},y_{0:N}} \in \Gamma$ then $R\left(Q_{x_{0:N},y_{0:N}} \middle\| P_{x_{0:N},y_{0:N}}\right) = 0$ thus, $-d < 0$. Further, the theorem implies that

$$\lambda^* \left(R\left(Q_{x_{0:N},y_{0:N}} \middle\| P_{x_{0:N},y_{0:N}}\right) - d\right) = 0 \tag{5.18}$$

This means that for nonzero values of $\lambda^*>0$, the solution is at the boundary. Once the equivalence of the primal and dual problems is shown, the maximizing measure is found by the solution of the dual problem.

A necessary condition for $L\left(\hat{x}_{0:N}, Q_{x_{0:N},y_{0:N}}, s, \lambda\right)$ to have an extremum at $Q^*_{x_{0:N},y_{0:N}}$ is that the Gateaux derivative of $L\left(\hat{x}_{0:N}, Q_{x_{0:N},y_{0:N}}, s, \lambda\right)$ is zero at $Q^*_{x_{0:N},y_{0:N}}$ in any direction $\Delta Q_{x_{0:N},y_{0:N}} = Q_{x_{0:N},y_{0:N}} - Q^*_{x_{0:N},y_{0:N}}$ where Gateaux derivative is defined as [42],

$$\delta L\left(Q^*_{x_{0:N},y_{0:N}}; Q_{x_{0:N},y_{0:N}} - Q^*_{x_{0:N},y_{0:N}}\right) = \frac{d}{dh} L\left(\hat{x}_{0:N}, Q_{x_{0:N},y_{0:N}} + h\left(Q_{x_{0:N},y_{0:N}} - Q^*_{x_{0:N},y_{0:N}}\right), s, \lambda\right)\Big|_{h=0} \tag{5.19}$$

Then

$$\begin{aligned}
&\delta L\left(Q^*_{x_{0:N},y_{0:N}}; Q_{x_{0:N},y_{0:N}} - Q^*_{x_{0:N},y_{0:N}}\right)\\
&= \int \left(\Psi_{0:N}(\hat{x}_{0:N}) - \lambda \log\left(\tfrac{dQ_{x_{0:N},y_{0:N}}}{dP_{x_{0:N},y_{0:N}}}\right) - (\lambda + s)\right)\left(dQ_{x_{0:N},y_{0:N}} - dQ^*_{x_{0:N},y_{0:N}}\right)\\
&= \int \log\left(\exp\left(\Psi_{0:N}(\hat{x}_{0:N}) - (\lambda + s)\right) + \left(\tfrac{dQ_{x_{0:N},y_{0:N}}}{dP_{x_{0:N},y_{0:N}}}\right)^{-\lambda}\right)\left(dQ_{x_{0:N},y_{0:N}} - dQ^*_{x_{0:N},y_{0:N}}\right)
\end{aligned}$$

Then for any direction of $\left(dQ_{x_{0:N},y_{0:N}} - dQ^*_{x_{0:N},y_{0:N}}\right)$

$$\frac{dQ^*_{x_{0:N},y_{0:N}}}{dP_{x_{0:N},y_{0:N}}} = \exp\left(\frac{\Psi_{0:N}(\hat{x}_{0:N}) - (\lambda + s)}{\lambda}\right) \tag{5.20}$$

Since $Q^*_{x_{0:N},y_{0:N}}$ is required to be a probability measure

$$\int dQ^*_{x_{0:N},y_{0:N}} = \int \exp\left(\frac{\Psi_{0:N}(\hat{x}_{0:N}) - (\lambda + s)}{\lambda}\right) dP_{x_{0:N},y_{0:N}} = 1 \tag{5.21}$$

Then

$$\exp\left(-\frac{\lambda + s}{\lambda}\right) = \left(\int \exp\left(\lambda^{-1}\Psi_{0:N}(\hat{x}_{0:N})\right) dP_{x_{0:N},y_{0:N}}\right)^{-1} \tag{5.22}$$

Thus the optimal (worst-case) measure satisfy the following relation [11],

$$dQ^{*}_{x_{0:N},y_{0:N}} = \frac{\exp\left(\lambda^{-1}\Psi_{0:N}\left(\hat{x}_{0:N}\right)\right)}{\int \exp\left(\lambda^{-1}\Psi_{0:N}\left(\hat{x}_{0:N}\right)\right) dP_{x_{0:N},y_{0:N}}} dP_{x_{0:N},y_{0:N}} \tag{5.23}$$

The relation implies that the worst-case measure is the exponentially tilted version of the nominal measure. This is a classical result existing in Large Deviations theory []. The following duality relation between the free energy and relative entropy establishes it.

$$\log \int \exp(\psi)\, dP = \sup_{Q} \left\{ \int \psi dQ - R\left(Q \| P\right) \ ; Q \leq P, \psi \in L_1\left(Q\right) \right\} \tag{5.24}$$

Further, it can be easily shown that the worst-case measure satisfies the boundary condition by substituting it into the relative entropy constraint [10],

$$R\left(Q^{*}_{x_{0:N},y_{0:N}} \,\|\, P_{x_{0:N},y_{0:N}}\right) = d \tag{5.25}$$

This implies that the Lagrange multiplier λ is greater than zero by the complementary slackness condition.

5.2.3 The Unconstrained Optimization Problem

By substituting the worst-case measure on the cost function, the Lagrangian becomes

$$L\left(\hat{x}_{0:N}, Q^{*}_{x_{0:N},y_{0:N}}, \lambda\right) = \lambda \log E_P \left[\exp\left(\lambda^{-1}\Psi_{0:N}\left(\hat{x}_{0:N}\right)\right)\right] + \lambda d \tag{5.26}$$

Note here that the expectation is defined in terms of the nominal joint probability measure $P_{x_{0:N},y_{0:N}}$. Thus, the state estimation problem is converted to following form [50], [77],

$$J\left(\hat{x}^{*}_{0:N}, \lambda^{*}\right) = \inf_{\lambda>0} \inf_{\hat{x}_{0:N}\in\aleph_{0:N}} \lambda \log\left(E_P\left[\exp\left(\lambda^{-1}\Psi_{0:N}\left(\hat{x}_{0:N}\right)\right)\right]\right) + \lambda d \tag{5.27}$$

The inner minimization problem

$$J\left(\hat{x}^{*}_{0:N}, \lambda\right) = \inf_{\hat{x}_{0:N}\in\aleph_{0:N}} E_P\left[\exp\left(\lambda^{-1}\sum_{k=0}^{N}\ell\left(x_k, \hat{x}_k\right)\right)\right] \tag{5.28}$$

can be considered as a finite-horizon risk-sensitive estimation problem that considers $\lambda > 0$ as a fixed parameter under the nominal system dynamics. The inner minimization can be achieved by interpreting the state estimate sequence as an output feedback controller for an optimal risk-sensitive stochastic control problem [6]. The solution of this problem can be obtained via information state dynamic programming where the output feedback problem is converted to an information state feedback problem [33]. There are two main difficulties of this approach as stated in [6]. The solution is dependent on the finite-time interval. Thus, a recursive solution cannot be obtained by incrementing the finite time interval. That is, the problem is needed to be solved from the beginning for the new time interval. Secondly, the solution method results in an infinite-dimensional nonlinear dynamic programming equation whose solution is not analytically possible in most cases. In [60], the inner minimization problem is considered as a batch optimization problem rather than sequential minimization problem like dynamic programming for constant dynamics systems.

In the following section, we will provide the partially observed risk-sensitive nonlinear control approach for the solution of robust state estimation problem. The problem formulation is referenced by [33].

5.2.4 Partially Observed Risk-sensitive Control Problem Approach

The inner minimization part of the unconstrained optimization problem in (5.28) is actually risk-sensitive estimation problem for the nominal nonlinear system model which can be formulized as the partially observed risk-sensitive control problem. In this regard, recall the system model

$$x_{k+1} = f_k(x_k) + B_{k+1} w_{k+1},\ x_0 \in \mathbb{R}^n \tag{5.29a}$$

$$y_k = h_k(x_k) + D_k v_k \tag{5.29b}$$

such that the optimal state sequence is defined as

$$\bar{J}\left(\hat{x}^*_{0:N}, \lambda\right) = \inf_{\hat{x}_{0:N} \in \aleph_{0:N}} E_P \left[\exp\left(\lambda^{-1} \left(\Psi_{0:N} + \Psi_{N+1}(x_{N+1})\right)\right)\right] \tag{5.30}$$

where $\Psi_{0:N} = \sum_{k=0}^{N} \ell\left(x_k, \hat{x}_k\right)$

Remark 6. Note here that a fictitious terminal state cost Ψ_{N+1} is imposed to (5.30) in order to interpret the optimal state estimation problem as optimal control problem. Actually $\Psi_{N+1}(x_{N+1}) = 0$.

5.2.4.1 Measure Change

In the solution of the equivalent optimal control problem, the measure change technique will be utilized [18]. The measure change technique, which is known as reference probability method, is a widely used technique in risk-sensitive type optimization problems where an ideal reference probability measure is used to formulize the problem in order to ease the solution of the problem [18]. In this regard, we work under a new probability measure $\bar{P}_{x_{0:N}, y_{0:N}}$ where the state and measurement sequences are independent and identically distributed (i.i.d.). For this ideal probability measure, the distribution of the state is $x_k \sim p_{w_k}(.)$ and the distribution of the measurement is $y_k \sim p_{v_k}(.)$.

Let us define the following Radon-Nikodym derivative under the complete sigma-field $Z_{N+1} = \sigma\left(x_{0:N+1}, y_{0:N+1}\right)$ generated by the random variables $\{x_0, x_1, ..., x_{N+1}, y_0, y_1, ..., y_{N+1}\}$

$$\bar{\Gamma}_{0:N+1} \triangleq \left.\frac{dP}{d\bar{P}}\right|_{Z_{N+1}} = \prod_{k=0}^{N} \bar{\gamma}_k \tag{5.31}$$

where

$$\bar{\gamma}_k = \left\{ \begin{array}{ll} \frac{p_{v_k}(D_k^{-1}(y_k - h_k(x_k)))}{p_{v_k}(y_k)} \frac{p_{w_k}(B_k^{-1}(x_k - \bar{x}_0))}{p_{w_k}(x_k)} & for\ \ k = 0 \\ \frac{p_{v_k}(D_k^{-1}(y_k - h_k(x_k)))}{|D_k| p_{v_k}(y_k)} \frac{p_{w_k}(B_k^{-1}(x_k - f_k(x_{k-1})))}{|B_k| p_{w_k}(x_k)} & for\ k > 0 \end{array} \right\} \tag{5.32}$$

Using Bayes theorem, the optimization problem can be redefined under the new probability measure as follows;

$$\begin{aligned} \bar{J}_N(x^*_{0:N}, \lambda) &= \inf_{\hat{x}_{0:N} \in \aleph_{0:N}} E_P \left[\exp(\lambda^{-1} (\Psi_{0:N} + \Psi_{N+1}(x_{N+1}))) \right] \\ &= \inf_{\hat{x}_{0:N} \in \aleph_{0:N}} E_{\bar{P}} \left[\bar{\Gamma}_{0:N+1} \exp\left(\lambda^{-1} (\Psi_{0:N} + \Psi_{N+1}(x_{N+1})) \right) \right] \end{aligned} \tag{5.33}$$

since $E_{\bar{P}}\left[\bar{\Gamma}_{0:N+1}\right] = 1$. Using the smoothing property of the conditional expectation

$$\begin{aligned} & E_{\bar{P}} \left[\bar{\Gamma}_{0:N+1} \exp\left(\lambda^{-1} (\Psi_{0:N} + \Psi_{N+1}(x_{N+1})) \right) \right] \\ &= E_{\bar{P}} \left[E_{\bar{P}} \left[\bar{\Gamma}_{0:N+1} \exp\left(\lambda^{-1} (\Psi_{0:N} + \Psi_{N+1}(x_{N+1})) \right) \middle| Y_{N+1} \right] \right] \\ &= \int E_{\bar{P}} \left[\bar{\Gamma}_{0:N+1} \exp\left(\lambda^{-1} (\Psi_{0:N} + \Psi_{N+1}(x_{N+1})) \right) \middle| Y_{N+1} \right] d\bar{P}_{Y_{N+1}} \end{aligned} \tag{5.34}$$

Thus

$$\bar{J}_N\left(x^*_{0:N}, \lambda\right) = \inf_{\hat{x}_{0:N} \in \aleph_{0:N}} \int E_{\bar{P}} \left[\bar{\Gamma}_{0:N+1} \exp\left(\lambda^{-1} (\Psi_{0:N} + \Psi_{N+1}(x_{N+1})) \right) \middle| Y_{N+1} \right] d\bar{P}_{Y_{N+1}} \tag{5.35}$$

5.2.4.2 The Information State

For a given estimated state sequence $\hat{x}_{0:N}$, we will define the following measure-valued process

$$\sigma_k^\lambda(B) \triangleq \int_{\mathbb{R}^n} I_B(x) d\sigma_k^\lambda(x) = E_{\bar{P}} \left[I_B(x) \bar{\Gamma}_k \exp\left(\lambda^{-1} \Psi_{0,k-1} \right) | Y_k \right] \tag{5.36}$$

where $I_B(.)$ is the indicator function of the Borel set B. Furthermore let α_k^λ be the density of the measure-valued process such that

$$\sigma_k^\lambda(B) = \int_B \alpha_k^\lambda(x) dx \tag{5.37}$$

Then following inner product can be defined

$$\langle \alpha_k^\lambda, g \rangle = \int \alpha_k(x) g(x) dx = E_{\bar{P}} \left[g(x_k) \bar{\Gamma}_{0:k} \exp\left(\lambda^{-1} \Psi_{0,k-1} \right) | Y_k \right] \tag{5.38}$$

$\alpha_k^\lambda(x)$ can be considered as an information state for the optimal control problem in order to convert the partially observed risk-sensitive control problem to information state observed control problem [33].

Lemma 5.2.4. *The information state density $\alpha(x)$ satisfies the following recursion*

$$\alpha_k^\lambda(x_k) = \Xi_k^\lambda(\hat{x}_{k-1}, y_k) \alpha_k^\lambda(x_{k-1}) \tag{5.39a}$$

$$\Xi_k^\lambda(u,y)\alpha_k^\lambda(z) = \frac{p_{v_k}\left(D_k^{-1}\left(y - h_k\left(z\right)\right)\right)}{|B_k|\,|D_k|\,p_{v_k}\left(y\right)} \int \exp\left(\lambda^{-1}\ell\left(\xi,u\right)\right) p_{w_k}\left(B_k^{-1}\left(z - f_k\left(\xi\right)\right)\right)\alpha_{k-1}^\lambda\left(\xi\right) d\xi \tag{5.39b}$$

$$\alpha_0^\lambda\left(x_0\right) = \frac{p_{v_0}\left(D_0^{-1}\left(y_0 - h_0\left(x_0\right)\right)\right)}{|D_0|\,\left|\sqrt{\Sigma_0}\right| p_{v_0}\left(y_0\right)} p_{w_0}\left(\sqrt{\Sigma_0}^{-1}\left(x_0 - \bar{x}_0\right)\right) \tag{5.39c}$$

where $\Xi_k^\lambda(u,y)$ is considered as a linear transformation (infinite dimensional) that propogates the information state.

Proof. The problem formulation is very similar to risk-sensitive estimation problem solution by reference probability method. The proof of the theorem is available in [48, 18]. For the sake of completeness, we provide the derivation in the sequel.

Let $g : \mathbb{R}^n \to \mathbb{R}$ be any test function. Then

$$\begin{aligned}
\int g(x) d\alpha_k^\lambda(x) &= E_{\bar{P}}\left[\bar{\Gamma}_{0:k}\exp(\lambda^{-1}\Psi_{0:k-1})f(x_k)|Y_k\right] \\
&= E_{\bar{P}}\left[\bar{\Gamma}_{0:k-1}\exp\left(\lambda^{-1}\ell(x_{k-1},\hat{x}_{k-1})\right)\exp(\lambda\Psi_{0:k-2})g(x_k)\right. \\
&\quad \left.\times \frac{p_{v_k}\left(D_k^{-1}\left(y_k - h_k\left(x_k\right)\right)\right)}{|D_k| p_{v_k}\left(y_k\right)} \frac{p_{w_k}\left(B_k^{-1}\left(x_k - f_k\left(x_{k-1}\right)\right)\right)}{|B_k|\,p_{w_k}\left(x_k\right)} \right| Y_k\Bigg] \\
&= E_{\bar{P}}\Bigg[\bar{\Gamma}_{0:k-1}\exp\left(\lambda^{-1}\ell(x_{k-1},\hat{x}_{k-1})\right)\exp(\lambda\Psi_{0:k-2})g(x_k) \\
&\quad \times E_{\bar{P}}\left[\frac{p_{v_k}\left(D_k^{-1}\left(y_k - h_k\left(x_k\right)\right)\right)}{|D_k| p_{v_k}\left(y_k\right)} \frac{p_{w_k}\left(B_k^{-1}\left(x_k - f_k\left(x_{k-1}\right)\right)\right)}{|B_k|\,p_{w_k}\left(x_k\right)} \,\middle|\, x_{k-1}, Y_k\right]\Bigg|Y_k\Bigg]
\end{aligned}$$

Since under $\bar{P}$, state and measurement sequences are independent

$$\begin{aligned}
&E_{\bar{P}}\left[\frac{p_{v_k}\left(D_k^{-1}\left(y_k - h_k\left(x_k\right)\right)\right)}{|D_k|\,p_{v_k}\left(y_k\right)} \frac{p_{w_k}\left(B_k^{-1}\left(x_k - f_k\left(x_{k-1}\right)\right)\right)}{|B_k|\,p_{w_k}\left(x_k\right)} g(x_k) \,\middle|\, x_{k-1}, Y_k\right] \\
&= \int \frac{p_{v_k}\left(D_k^{-1}\left(y_k - h_k\left(x\right)\right)\right)}{|D_k|\,p_{v_k}\left(y_k\right)} \frac{p_{w_k}\left(B_k^{-1}\left(x - f_k\left(x_{k-1}\right)\right)\right)}{|B_k|\,p_{w_k}\left(x\right)} g(x) p_{w_k}(x) dx
\end{aligned}$$

Then

$$\begin{aligned}
\int g(x)\hat{\alpha}_k^\lambda(x) dx = \int g(x)\Bigg[&\frac{p_{v_k}\left(D_k^{-1}\left(y_k - h_k\left(x\right)\right)\right)}{|B_k||D_k| p_{v_k}\left(y_k\right)} \\
&\times \int exp\left(\lambda^{-1}\ell(z,\hat{x}_{k-1})\right) p_{w_k}\left(B_k^{-1}(x - f_k\left(z\right))\right)\alpha_k^\lambda(z) dz\Bigg] dx
\end{aligned}$$

On the other hand, the initial density can be evaluated as follows

$$\begin{aligned}
E_{\bar{P}}\left[\bar{\Gamma}_0\exp\left(\lambda^{-1}\Psi_{0,-1}\right) g(x_0)|Y_0\right] &= E_{\bar{P}}\left[\bar{\Gamma}_0 g(x_0)|Y_0\right] = E_{\bar{P}}\left[\bar{\gamma}_0 g(x_0)|Y_0\right] \\
&= E_{\bar{P}}\left[\frac{p_{v_0}\left(D_0^{-1}\left(y_0 - h_0\left(x_0\right)\right)\right)}{p_{v_0}\left(y_0\right)|D_0|} \frac{p_{w_0}\left(\Sigma_0^{-1}\left(x_0 - \bar{x}_0\right)\right)}{p_{w_0}\left(x_0\right)|\Sigma_0|} g(x_0)\middle| Y_0\right] \\
&= \int \frac{p_{v_0}\left(D_0^{-1}\left(y_0 - h_0\left(x\right)\right)\right)}{p_{v_0}\left(y_0\right)|D_0|} \frac{p_{w_0}\left(\Sigma_0^{-1}\left(x - \bar{x}_0\right)\right)}{p_{w_0}\left(x\right)|\Sigma_0|} g(x) p_{w_0}(x) dx
\end{aligned}$$

Thus

$$\alpha_0^\lambda(x) = \frac{p_{v_0}\left(D_0^{-1}(y_0 - h_0(x))\right)}{p_{v_0}(y_0)\,|D_0|\left|\sqrt{\Sigma_0}\right|} p_{w_0}\left(\sqrt{\Sigma_0}^{-1}(x - \bar{x}_0)\right)$$

Thus the proof is complete. □

5.2.4.3 Dynamic Programming

In this section, the cost function will be represented in terms of the information state thus the partially observed control problem is converted to information-state observable optimization problem. In this regard, the optimization problem is defined as

$$\bar{J}_N\left(x^*_{0:N}, \lambda\right) = \inf_{\hat{x}_{0:N} \in \aleph_{0:N}} \int\int \exp\left(\lambda^{-1}\Psi_{N+1}\right)\alpha^\lambda_{N+1} dx_{N+1} d\bar{P}_{Y_{N+1}} \tag{5.40}$$

Let us define

$$\beta_{N+1} = \exp(\lambda^{-1}\Psi_{N+1}) \tag{5.41}$$

Then the cost function can be represented in terms of the information state as

$$\bar{J}_N(\hat{x}_{0:N}, \lambda) = E_{\bar{P}}\left[\langle \alpha^\lambda_{N+1}, \beta_{N+1}\rangle\right] \tag{5.42}$$

where by definition $\langle q, \eta\rangle = \int \eta(x)q(x)dx$ for any integrable function $q(x)$ and any bounded function η. Since

$$\begin{aligned}\bar{J}_N(x_{0:N}, \lambda) &= E_{\bar{P}}\left[\Gamma_{0:k}\Gamma_{k+1:N+1}\Psi_{0:k-1}\Psi_{k:N}\Psi_{N+1}\right] \\ &= E_{\bar{P}}\left[\Gamma_{0:k}\Psi_{0:k-1}E_{\bar{P}}\left[\Gamma_{k+1:N+1}\Psi_{k:N}\Psi_{N+1}\middle|x_{0:k}, Y_{N+1}\right]\right]\end{aligned} \tag{5.43}$$

then define the adjoint process as

$$\beta^\lambda_k(x) \triangleq E_{\bar{P}}\left[\Gamma_{k:N+1}\Psi_{k:N}\Psi_{N+1}\middle|x_{0:k}, Y_{N+1}\right] = E_{\bar{P}}\left[\Gamma_{k:N+1}\Psi_{k:N}\Psi_{N+1}\middle|x_k, Y_{N+1}\right] \tag{5.44}$$

Thus the cost function is expressed as

$$\bar{J}_N(x_{0:N}, \lambda) = E_{\bar{P}}\left[\langle\alpha^\lambda_k, \beta^\lambda_k\rangle\middle|Y_{0:N+1}\right] \tag{5.45}$$

The value function (cost-to-go) of this optimization problem can be defined as

$$V_k(\alpha) = \inf_{x_{k:N}\in\aleph_{k:N}} E_{\bar{P}}\left[\langle\alpha^\lambda_k, \beta^\lambda_k\rangle\middle|\alpha^\lambda_k = \alpha\right] \tag{5.46}$$

It is possible to show that the following backward dynamic programming recursion is satisfied [18].

$$\begin{aligned}V_k(\alpha) &= \inf_{\hat{x}_k\in\aleph_k} E_{\bar{P}}\left[V_{k+1}\left(\Xi^\lambda_k(y_{k+1}, \hat{x}_k)\,\alpha_k\right)\middle|\alpha^\lambda_k = \alpha\right] \\ V_{N+1}(\alpha) &= \langle\alpha, \beta_{N+1})\rangle = 1\end{aligned} \tag{5.47}$$

As previously noted, the solution method resulted in an infinite-dimensional nonlinear dynamic programming equation whose solution is not analytically possible in most cases. In the next section, we will provide solution of the optimization problem for linear Gaussian systems with quadratic cost function for which a finite-dimensional solution can be obtained.

5.2.4.4 Linear Gaussian Case

Consider linear version of the system model (5.3)

$$x_{k+1} = A_k x_k + w_k \tag{5.48a}$$

$$y_k = H_k x_k + v_k \tag{5.48b}$$

where w_k and v_k are zero mean Gaussian distributed process and measurement noises with corresponding covariances matrices Q_k and R_k. Consider the unconstrained optimization problem for the relative entropy constrained robust estimation problem

$$J\left(\hat{x}^*_{0:N}, \lambda^*\right) = \inf_{\lambda>0} \inf_{\hat{x}_{0:N}\in\aleph_{0:N}} \lambda \log\left(E_P\left[\exp\left(\lambda^{-1}\Psi_{0:N}\left(\hat{x}_{0:N}\right)\right)\right]\right) + \lambda d \tag{5.49}$$

where inner minimization problem is modified to utilize the optimal control techniques by introducing fictitious terminal state cost

$$J\left(\hat{x}^*_{0:N}, \lambda\right) = \inf_{\hat{x}_{0:N}\in\aleph} E_P\left[\exp\left(\lambda^{-1}\sum_{k=0}^{N} \ell\left(x_k, \hat{x}_k\right) + \Psi_{N+1}(x_{N+1})\right)\right] \tag{5.50}$$

Here

$$\ell\left(x_k, \hat{x}_k\right) = \frac{1}{2}\|x_k - \hat{x}_k\|^2 = \frac{1}{2}\left(x_k^T x_k - 2x_k^T\hat{x}_k + \hat{x}_k^T\hat{x}_k\right) \tag{5.51}$$

and $\Psi_{N+1}(x_{N+1}) = 0$. Using the linearity of the system dynamics, it can be concluded that the information state density is an unnormalized Gaussian density

$$\alpha_k(x) = Z_k \exp\left(\frac{1}{2}(x-\mu_k)^T\Sigma_k^{-1}(x-\mu_k)\right),\quad Z_k = (2\pi)^{-n/2}|\Sigma_k|^{-1/2} \tag{5.52}$$

Lemma 5.2.5. *The finite parameters ($\Upsilon_k \triangleq (Z_k, \mu_k, \Sigma_k)$) of the information state $\alpha_k(x)$ satisfy the following recursions*

$$\tilde{\mu}_k = \mu_k + \Sigma_k\left(\Sigma_k - \lambda I\right)^{-1}\left(\hat{x}_k - \mu_k\right) \tag{5.53a}$$

$$S_{k+1} = A_k\left(\Sigma_k^{-1} - (\lambda^{-1}I)\right)^{-1}A_k^T + Q_{k+1} \tag{5.53b}$$

$$\Sigma_{k+1}^{-1} = S_{k+1}^{-1} + H_{k+1}^T R_{k+1}^{-1} H_{k+1} \tag{5.53c}$$

$$\mu_{k+1} = A_k\tilde{\mu}_k + S_{k+1}H_{k+1}^T\left(H_{k+1}S_{k+1}H_{k+1}^T + R_{k+1}\right)^{-1}\left(y_{k+1} - H_{k+1}A_{k+1}\tilde{\mu}_k\right) \tag{5.53d}$$

$$\begin{aligned} Z_{k+1} = Z_k\,|Q_{k+1}|^{-1/2}|N_{k+1}|^{-1/2}\exp\left(\frac{1}{2}\|y_{k+1}\|^2_{R_{k+1}^{-1}}\right)\exp\left(-\frac{1}{2}\|\hat{x}_k - \mu_k\|^2_{\left(\Sigma_k - (\lambda^{-1}I)\right)^{-1}}\right) \\ \times\exp\left(-\frac{1}{2}\|y_{k+1} - H_{k+1}A_k\tilde{\mu}_k\|^2_{\left(H_{k+1}S_{k+1}H_{k+1}^T + R_{k+1}\right)^{-1}}\right) \end{aligned} \tag{5.53e}$$

$$N_{k+1} = \left(\Sigma_k^{-1} - \lambda^{-1}I + A_k^T Q_{k+1}^{-1} A_k\right) \tag{5.53f}$$

Proof. The recursions can be obtained as a special case (estimation problem) of the discrete-time linear exponential quadratic Gaussian control(LEQG) problem [55]. □

The Optimal Estimator :
Since the system dynamics is linear and the information state is unnormalized Gaussian, the value function can be assumed to be in the following generalized exponential form;

$$V_{k+1}\left(\Upsilon_{k+1}\right) \triangleq Z_{k+1} \exp\left(\frac{\lambda^{-1}}{2}\left(\mu_{k+1}^T M_{k+1}\mu_{k+1} + m_{k+1}\right)\right) \tag{5.54}$$

For this optimal state estimation problem, another possibility for the objective function is the following simpler form;

$$V_{k+1}\left(\Upsilon_{k+1}\right) \triangleq Z_{k+1} \exp\left(\frac{\lambda^{-1}}{2} m_{k+1}\right) \tag{5.55}$$

then the dynamic programming recursion takes the following form;

$$V_k(\Upsilon_k) = \inf_{\hat{x}_k} E_{\bar{P}}\left[Z_{k+1} \exp\left(\frac{\lambda^{-1}}{2} m_{k+1}\right)\middle|\Upsilon_k = \Upsilon\right] \tag{5.56}$$

By substituting the Z_{k+1} expression into (5.53e) into (5.56) yields

$$\begin{aligned} V_k(\Upsilon_k) = \inf_{\hat{x}_k} \int_{\mathbb{R}^m} & Z_k\, |Q_{k+1}|^{-1/2}|N_{k+1}|^{-1/2}(2\pi)^{-m/2}|R_{k+1}|^{-1/2} \exp\left(-\frac{1}{2}\|\hat{x}_k - \mu_k\|^2_{(\Sigma_k-(\lambda^{-1}I))^{-1}}\right) \\ & \times \exp\left(\frac{\lambda^{-1}}{2} m_{k+1}\right) \exp\left(-\frac{1}{2}\|y_{k+1} - H_{k+1}A_k\tilde{\mu}_k\|^2_{(H_{k+1}S_{k+1}H_{k+1}^T+R_{k+1})^{-1}}\right) dy_{k+1} \end{aligned} \tag{5.57}$$

Since

$$\begin{aligned} &\int_{\mathbb{R}^m} \exp\left(-\frac{1}{2}\|y_{k+1} - H_{k+1}A_k\tilde{\mu}_k\|^2_{(H_{k+1}S_{k+1}H_{k+1}^T+R_{k+1})^{-1}}\right) dy_{k+1} \\ &\quad = (2\pi)^{m/2}\left|\left(H_{k+1}S_{k+1}H_{k+1}^T + R_{k+1}\right)\right|^{1/2} \end{aligned} \tag{5.58}$$

by noting that the optimal state estimate is equal to the information state mean $\hat{x}_k = \mu_k$, it is easy to see that

$$V_k(\Upsilon_k) = Z_k \left|\left(H_{k+1}S_{k+1}H_{k+1}^T + R_{k+1}\right)\right|^{1/2} |Q_{k+1}|^{-1/2}|R_{k+1}|^{-1/2}|N_{k+1}|^{-1/2} \exp\left(\frac{\lambda^{-1}}{2} m_{k+1}\right) \tag{5.59}$$

then since by definition $V_k\left(\Upsilon_k\right) \triangleq Z_k \exp\left(\frac{\lambda^{-1}}{2} m_k\right)$ then

$$\exp\left(\frac{\lambda^{-1}}{2} m_k\right) = \left|\left(H_{k+1}S_{k+1}H_{k+1}^T + R_{k+1}\right)\right|^{1/2} |Q_{k+1}|^{-1/2}|R_{k+1}|^{-1/2}|N_{k+1}|^{-1/2} \exp\left(\frac{\lambda^{-1}}{2} m_{k+1}\right) \tag{5.60}$$

Since $V_{N+1}(\Upsilon_{N+1}) = \langle \alpha_{N+1}, \beta_{N+1})\rangle = 1$ with $\exp\left(\frac{\lambda^{-1}}{2} m_{N+1}\right) = |\Sigma_{N+1}|^{1/2}(2\pi)^{n/2}$, then the optimal cost takes the following form;

$$\begin{aligned} V_0(\Upsilon_0) &= |\Sigma_0|^{-1/2}(2\pi)^{-n/2} \exp\left(\lambda^{-1} m_0\right) \\ &= |\Sigma_0|^{-1/2}|\Sigma_{N+1}|^{1/2} \prod_{k=0}^{N} \left|\left(H_{k+1}S_{k+1}H_{k+1}^T + R_{k+1}\right)\right|^{1/2} |Q_{k+1}|^{-1/2}|R_{k+1}|^{-1/2}|N_{k+1}|^{-1/2} \end{aligned} \tag{5.61}$$

where $N_{k+1} = \left(\Sigma_k^{-1} - \lambda^{-1}I + A_k^T Q_{k+1}^{-1} A_k\right)$.

Lemma 5.2.6. *Consider a nonsingular matrix A in partitioned form as* $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$. *Then consider the Schur complement of invertible square matrix* A_{11} *in A as* $A_{11}^c = A_{22} - A_{21}A_{11}^{-}A_{12}$ *and similarly the Schur complement of invertible square matrix* A_{22} *in A as* $A_{22}^c = A_{11} - A_{12}A_{22}^{-}A_{21}$. *Then*

$$\det|A| = \det|A_{11}|\,|A_{11}^c| = \det|A_{22}|\ \det|A_{22}^c| \tag{5.62}$$

By using the lemma 5.2.6,

$$|Q_{k+1}|\left|\left(\Sigma_k^- - \lambda^- I + A_k^T Q_{k+1}^- A_k\right)\right| = \left|\Sigma_k^- - \lambda^- I\right| \left|A_k\left(\Sigma_k^- - \lambda^- I\right)^- A_k^T + Q_{k+1}\right| \tag{5.63}$$

and

$$\left|H_{k+1}S_{k+1}H_{k+1}^T + R_{k+1}\right|\left|S_{k+1}^-\right| = \left|H_{k+1}R_{k+1}^- H_{k+1}^T + S_{k+1}^-\right|\left|R_{k+1}\right| \tag{5.64}$$

Thus

$$\exp\left(\frac{\lambda^-}{2}s_k^c\right) = \left|\Sigma_k^- - \lambda^- I\right|^{-1/2}\left|\Sigma_{k+1}\right|^{-1/2}\exp\left(\frac{\lambda^-}{2}s_{k+1}^c\right) \tag{5.65}$$

By noting that $\exp\left(\frac{\lambda^-}{2}s_{N+1}^c\right) = (2\pi)^{n/2}\left|P_{N+1}\right|^{1/2}$,

$$V_0 = Z_0\exp\left(\frac{\lambda^-}{2}s_0^c\right) = \left|\Sigma_0\right|^{-1/2}\prod_{k=1}^{N}\left|\Sigma_{k-1}^- - \lambda^- I\right|^{-1/2}\left|\Sigma_k\right|^{-1/2} \tag{5.66}$$

Thus the optimal value of the Lagrange multiplier is determined by the following scalar optimization problem

$$J\left(\hat{x}_{0:N}^*, \lambda^*\right) = \inf_{\lambda>0}\lambda\left(\log(V_0) + d\right) \tag{5.67}$$

where

$$\log(V_0) = -\frac{1}{2}\log\left|\Sigma_0\right| - \frac{1}{2}\sum_{k=1}^{N}\left(\log\left|\Sigma_{k-1}^- - \lambda^- I\right| + \log\left|\Sigma_k\right|\right) \tag{5.68}$$

Example 5.2.1. Let us consider the following time-invariant linear system

$$x_{k+1} = Ax_k + w_{k+1}y_k = Hx_k + v_k$$

where $A = \begin{bmatrix} 0.6 & 0 \\ 0 & -0.6 \end{bmatrix}$ and $H = \begin{bmatrix} 1 & 1 \end{bmatrix}$ with $Q_k = E[w_k w_k^T] = I_{2\times2}$ and $R_k = E\left[v_k v_k^T\right] = 1.0$ and the initial covariance matrix of the state vector $E[x_0 x_0^T] = I_{2\times2}$. Consider the robust state estimation problem over a finite horizon $[0, 100]$ where the relative entropy between the nominal and perturbed joint probability measures is less than $d = 0.1$. The cost function as a function of the Lagrange multiplier can be obtained as in Figure 5.1. Thus the optimal value of the Lagrange multiplier is 4.85.

5.2.4.5 Extended Relative Entropy Constrained Robust Estimation

In the previous section, we have provided the solution of the robust state estimation problem for the linear Gaussian systems with quadratic cost function. This solution approach can be

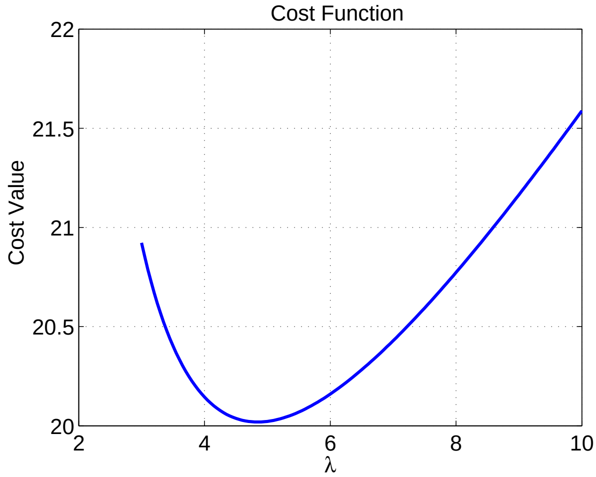

Figure 5.1: The Cost Function for the Example 5.2.1

applied to linearized models of the nonlinear systems as an approximate method. The solution approach quite similar to the extended Kalman filter. In this regard, let us consider the first order Taylor series approximation of the nonlinear system expressed in (5.2.1) around the mean (μ_k) of the information state density (un-normalized Gaussian) that is introduced in the previous section

$$x_{k+1} \approx A_k(\mu_k)x_k + (f(\mu_k) - A_k(\mu_k)\mu_k) + w_{k+1} \tag{5.69a}$$

$$y_k \approx H_k(\mu_k)x_k + (h_k(\mu_k) - H_k(\mu_k)\mu_k) + v_k \tag{5.69b}$$

where $f(x_k) \approx f(\mu_k)+A_k(\mu_k)(x_k-\mu_k)$ with $A_k \triangleq \frac{\partial f(x_k)}{\partial x_k}\Big|_{x_k=\mu_k}$ and $h(x_k) \approx h(\mu_k)+H_k(\mu_k)(x_k-\mu_k)$ with $H_k \triangleq \frac{\partial H(x_k)}{\partial x_k}\Big|_{x_k=\mu_k}$. Thus we have an approximate linear system for which the robust optimal state estimation problem solution that has been obtained in the previous section can be applied.

Lemma 5.2.7. *The information state $\alpha_k(x)$ parameters $\Upsilon_k \triangleq (Z_k, \mu_k, \Sigma_k)$ satisfy the following recursions*

$$\tilde{\mu}_k = \mu_k + \Sigma_k^{\mu}\left(\Sigma_k^{\mu} - \lambda I\right)^{-}(\hat{x}_k - \mu_k) \tag{5.70a}$$

$$S_{k+1} = A_k(\mu_k)\left(\Sigma_k^{\mu,-} - (\lambda^{-} I)\right)^{-} A_k(\mu_k)^T + Q_{k+1} \tag{5.70b}$$

$$\Sigma_{k+1}^{-} = S_{k+1}^{-} + H_{k+1}^T(\mu_k)R_{k+1}^{-}H_{k+1}(\mu_k) \tag{5.70c}$$

$$\mu_{k+1} = f_k(\tilde{\mu}_k) + S_{k+1}H_{k+1}^T(\mu_k)\left(H_{k+1}(\mu_k)S_{k+1}H_{k+1}^T(\mu_k) + R_{k+1}\right)^{-}(y_{k+1} - h_{k+1}(\tilde{\mu}_k)) \tag{5.70d}$$

$$\begin{aligned} Z_{k+1} = {} & Z_k\,|Q_{k+1}|^{-1/2}|N_{k+1}|^{-1/2}\exp\left(\frac{1}{2}\|y_{k+1}\|^2_{R_{k+1}^{-}}\right)\exp\left(-\frac{1}{2}\|\hat{x}_k - \mu_k\|^2_{(\Sigma_k^{\mu}-(\lambda^{-}I))^{-}}\right) \\ & \times\exp\left(-\frac{1}{2}\|y_{k+1} - h_{k+1}(f_k(\tilde{\mu}_k))\|^2_{(H_{k+1}(\mu_k)S_{k+1}H_{k+1}^T(\mu_k)+R_{k+1})^{-}}\right) \end{aligned} \tag{5.70e}$$

$$N_{k+1}(\mu_k) = \left(\Sigma_k^{\mu,-} - \lambda^{-}I + A_k^T(\mu_k)Q_{k+1}^{-}A_k(\mu_k)\right) \tag{5.70f}$$

Proof. By referring the classical extended Kalman filter derivation in [18] it is straight forward to achieve the result. □

Now consider the dynamic programming recursion relation that is provided in the previous section.

$$V_k(\Upsilon_k) = \inf_{\hat{x}_k} \int_{\mathbb{R}^m} Z_k |Q_{k+1}|^{-1/2} |N_{k+1}(\mu_k)|^{-1/2} |R_{k+1}|^{-1/2} \exp\left(-\frac{1}{2}\|\hat{x}_k - \mu_k\|^2_{(\Sigma_k^{\mu} - (\lambda^{-1} I))^{-1}}\right)$$
$$\times \exp\left(\frac{\lambda^{-1}}{2} m_{k+1}\right) \exp\left(-\frac{1}{2}\|y_{k+1} - h_k(f_k(\mu_k))\|^2_{(H_{k+1}(\mu_k) S_{k+1} H_{k+1}^T(\mu_k) + R_{k+1})^{-1}}\right) dy_{k+1} \tag{5.71}$$

as in the linear Gaussian case,

$$\log(V_0^{\mu}) = -\frac{1}{2} log\left|\Sigma_0^{\mu}\right| - \frac{1}{2} \sum_{k=1}^{N} \left(\log\left|\Sigma_{k-1}^{\mu,-} - \lambda^{-} I\right| + \log\left|\Sigma_k^{\mu}\right|\right) \tag{5.72}$$

Note here that the covariance matrix Σ_k^{μ} is information state dependent thus it is dependent on the measurement sequence realization. For online batch processing applications, the optimal value of the Lagrange multiplier can be determined by the following scalar optimization problem for a single realization of measurement sequence.

$$J\left(\hat{x}_{0:N}^*, \lambda^*\right) = \inf_{\lambda > 0} \lambda \left(\log(V_0^{\mu}) + d\right) \tag{5.73}$$

For offline design cases, the optimal Lagrange multiplier can be determined by the solution of 5.73 for different realizations of the measurement sequence. Ensemble mean can be used for determination of the optimal Lagrange multiplier.

Example 5.2.2. Let us consider the following FM demodulation problem that is represented by the following continous-time nonlinear system

$$\dot{x}(t) = Ax(t) + Bw(t)$$
$$y(t) = h(x(t)) + v(t)$$

where $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, $A = \begin{bmatrix} 0 & 1 \\ 0 & -\beta \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ \sqrt{(2\sigma^2\beta)} \end{bmatrix}$ and $h(x(t)) = \sqrt{2}\sin(t + x_2(t))$.

The discrete-time version of the system 5.2.2 can be obtained for a sampling rate $dt = 0.05$, $\beta = 1$ and $\sigma = 0.1$ as

$$x_{k+1} = A_k x_k + w_{k+1}$$
$$y_k = h_k(x_k) + v_k$$

where $A_k = \begin{bmatrix} 1.0 & 0.0488 \\ 0 & 0.9512 \end{bmatrix}$, $Q_k = E[w_k w_k^T] = \begin{bmatrix} 8 \times 10^{-7} & 2.38 \times 10^{-5} \\ 2.38 \times 10^{-5} & 0.95 \times 10^{-5} \end{bmatrix}$, $R_k = E\left[v_k v_k^T\right] = 0.01$ and the initial covariance matrix of the state vector $E[x_0 x_0^T] = 0.01 \times I_{2\times2}$.

Consider the robust state estimation problem over a finite horizon $[0, 10]$ where the relative entropy between the nominal and perturbed joint probability measures is less than $d = 1$. The

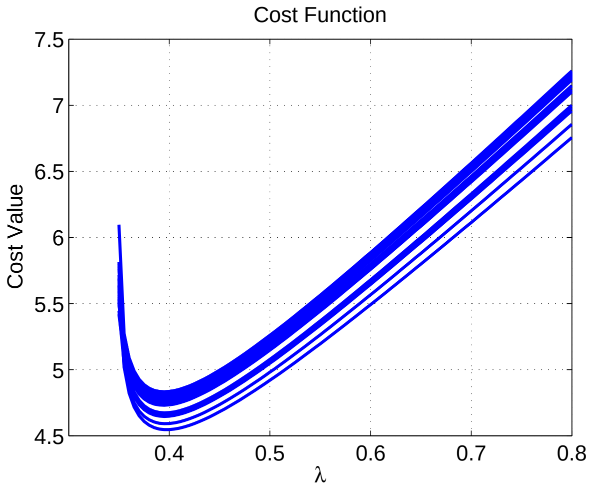

Figure 5.2: The Cost Function for the Example 5.2.2 for Different Realizations of the Measurement Sequence

dual function as a function of the Lagrange multiplier for different realizations is provided in Figure 5.2. The optimal value for the Lagrange multiplier can be obtained by determining the ensemble mean of the optimal Lagrange multipliers for each sequence. It is found that the optimal value is 4.76.

5.2.5 A Suboptimal Nonlinear Estimation Solution

In this section, due to the cited difficulties of the optimal robust state estimation problems given in the previous section, a suboptimal version of the solution of problem (5.28) is proposed in order to get a more tractable solutions. In this approach, in the minimax context, the maximizer is allowed to select the worst case measure over the entire finite time interval [0, N]. However, the minimizer is restricted to make a decision at each k and it is assumed the decision has been made for all times [0, k-1]. This is a common problem definition in classical risk-sensitive estimation [14],[6],[1] where the λ is considered as a design parameter. In this regard, the modified state estimation problem for a fixed Lagrange multiplier can be defined as sequential minimization problems in forward time as follows.

$$V_k^s\left(x_k^*, \lambda\right) = \inf_{\hat{x}_k \in \aleph_k} E_P\left(\exp\left(\lambda^{-1}\tilde{\Psi}_{0:k}\left(\hat{x}_k\right)\right)\right) \tag{5.74}$$

for k=1,...,N where

$$\tilde{\Psi}_{0:k}\left(\hat{x}_k\right) = \tilde{\Psi}^*_{0:k-1} + \ell\left(x_k, \hat{x}_k\right) \tag{5.75}$$

with $\tilde{\Psi}^*_{0:k-1} = \sum_{i=0}^{k-1} \ell\left(x_i, \hat{x}_i^*\right)$ and $\hat{x}_i^*$ is the suboptimal state estimate at time i. Here $\hat{x}_k$ is the state estimate at time k, which belongs to following set of possible state estimates

$$\aleph_k \triangleq \{\hat{x}_k : \ \Omega \rightarrow \mathbb{R}^n \ \text{for } 0 \leq k \leq N \,; \hat{x}_k \ \text{ is adapted } \ \text{to } Y_k\} \tag{5.76}$$

The inner minimization problem is expressed as the final step of the sequential state estimation problem

$$J^s\left(\hat{x}^*_{0:N}, \lambda\right) = \inf_{\hat{x}_N \in \aleph_N} V^s_N\left(x^*_N, \lambda\right) \tag{5.77}$$

Then the optimal Lagrange multiplier is determined at the final stage of the finite time interval as

$$J^s\left(\hat{x}^*_{0:N}, \lambda^*\right) = \inf_{\lambda>0}\left(\lambda \log\ V^s_N\left(x^*_N, \lambda\right) + \lambda d\right) \tag{5.78}$$

The Information State and Measure Change :

In the solution of the suboptimal state estimation problem, we utilize the measure change technique which is also applied in the previous section. In this regard, we work under a new probability measure $\bar{P}$ where the state and measurement sequences are independent and identically distributed (i.i.d.). Let us define the following Radon-Nikodym derivative under the complete sigma-field $Z_k = \sigma\left(x_{0:k}, y_{0:k}\right)$ generated by the random variables $\{x_0, x_1, ..., x_k, y_0, y_1, \ldots, y_k\}$

$$\bar{\Gamma}_{0:k} \triangleq \left.\frac{dP}{d\bar{P}}\right|_{Z_k} = \prod_{i=0}^{k} \bar{\gamma}_i \tag{5.79}$$

where

$$\bar{\gamma}_i = \left\{ \begin{array}{ll} \frac{p_{v_i}\left(D_k^{-1}(y_i-h_i(x_i))\right)}{p_{v_i}(y_i)} \frac{p_{w_i}\left(B_i^{-1}(x_i-\bar{x}_0)\right)}{p_{w_i}(x_i)} & \text{for } i = 0 \\ \frac{p_{v_i}\left(D_i^{-1}(y_i-h_i(x_i))\right)}{|D_i| p_{v_i}(y_i)} \frac{p_{w_i}\left(B_i^{-1}(x_i-f_i(x_{i-1}))\right)}{|B_i| p_{w_i}(x_i)} & \text{for } i > 0 \end{array} \right\} \tag{5.80}$$

Using Bayes theorem, the recursive optimization problem can be redefined under the new probability measure as follows;

$$\begin{aligned} V_k\left(x^*_k, \lambda\right) &= \inf_{\hat{x}_k \in \aleph_k} E_P\left(\exp\left(\lambda^{-1}\tilde{\Psi}^c_{0:k}\left(\hat{x}_k\right)\right)\right) \\ &= \inf_{\hat{x}_k \in \aleph_k} E_{\bar{P}}\left[\bar{\Gamma}_k \exp\left(\lambda_k^{-1}\tilde{\Psi}_{0:k}\left(\hat{x}_k\right)\right)\right] \end{aligned} \tag{5.81}$$

since $E_{\bar{P}}\left[\bar{\Lambda}_{0:k}\right] = 1$. Using the smoothing property of the conditional expectation

$$\begin{aligned} E_{\bar{P}}\left[\bar{\Gamma}_{0:k} \exp\left(\lambda_k^{-1}\tilde{\Psi}_{0:k}\left(\hat{x}_k\right)\right)\right] &= E_{\bar{P}}\left[\left. E_{\bar{P}}\left[\bar{\Gamma}_{0:k} \exp\left(\lambda_k^{-1}\tilde{\Psi}_{0:k}\left(\hat{x}_k\right)\right)\right| Y_k\right]\right] \\ &= \int E_{\bar{P}}\left[\bar{\Gamma}_{0:k} \exp\left(\lambda_k^{-1}\tilde{\Psi}_{0:k}\left(\hat{x}_k\right)\right)\Big| Y_k\right] d\bar{P}_{Y_k} \end{aligned} \tag{5.82}$$

Thus

$$V^s_k\left(x^*_k, \lambda\right) = \inf_{\hat{x}_k \in \aleph_k} \int E_{\bar{P}}\left[\bar{\Gamma}_{0:k} \exp\left(\lambda^{-1}\tilde{\Psi}_{0:k}\left(\hat{x}_k\right)\right)\Big| Y_k\right] d\bar{P}_{Y_k} \tag{5.83}$$

In order get a recursive solution for the inner conditional expectation term, the following Lagrange multiplier dependent information state can be introduced

$$\sigma^\lambda_k(B) \triangleq \int_{\mathbb{R}^n} I_B(x) d\sigma^\lambda_k(x) = E_{\bar{P}}\left[I_B(x)\bar{\Gamma}_{0:k} \exp\left(\lambda^{-1}\tilde{\Psi}^*_{0,k-1}\right)\Big| Y_k\right] \tag{5.84}$$

where I_B is the indicator function of the Borel set B. Furthermore let α^λ_k be the density of the measure-valued process such that

$$\sigma^\lambda_k(B) = \int_B \alpha^\lambda_k(x) dx \tag{5.85}$$

Thus, the recursive optimization problem is defined as

$$V_k^s\left(x_k^*, \lambda\right) = \inf_{\hat{x}_k \in \aleph_k} \int\int \exp\left(\lambda^{-1}\ell\left(x_k', \hat{x}_k\right)\right) \alpha_k^\lambda\left(x_k'\right) dx_k' d\bar{P}_{Y_k} \tag{5.86}$$

Theorem 5.2.8. *The information state α_k^λ satisfies the following recursion*

$$\begin{aligned}\alpha_k^\lambda(x) = &\frac{1}{|B_k|\,|D_k|} \frac{p_{v_k}\left(D_k^{-1}\left(y_k - h_k\left(x_k'\right)\right)\right)}{p_{v_k}\left(y_k\right)} \\ &\int \exp\left(\lambda^{-1}\ell\left(x_{k-1}', \hat{x}_{k-1}\right)\right) p_{w_k}\left(B_k^{-1}\left(x_k' - f_k\left(x_{k-1}'\right)\right)\right) \alpha_{k-1}^\lambda\left(x_{k-1}'\right) dx_{k-1}'\end{aligned} \tag{5.87}$$

with an initial condition

$$\alpha_0^\lambda\left(x_0\right) = \frac{p_{v_0}\left(D_0^{-1}\left(y_0 - h_0\left(x_0\right)\right)\right)}{|D_0|\,\left|\sqrt{\Sigma_0}\right| p_{v_0}\left(y_0\right)} p_{w_0}\left(\sqrt{\Sigma_0}^{-1}\left(x_0 - \bar{x}_0\right)\right) \tag{5.88}$$

Proof. The lemma is almost the same as in 5.2.4 whose derivation is provided previously. □

5.2.5.1 Determination of the Suboptimal State Estimate: Particle Filter Approach

Let us reconsider the suboptimal state estimation problem

$$V_k^s\left(x_k^*, \lambda\right) = \inf_{\hat{x}_k \in \aleph_k} \int\int \exp\left(\lambda^{-1}\ell\left(x_k', \hat{x}_k\right)\right) \alpha_k^\lambda\left(x_k'\right) dx_k'\, d\bar{P}_{Y_k} \tag{5.89}$$

Using the fact that the outer integral is a positive weighted integral of the inner integral, the optimal state estimate for a given λ value and for any given measurement sequence $\{y_{0:k}\}$

$$\hat{x}_k^* = \arg\inf_{\hat{x}_k \in \aleph_k} \int \exp\left(\lambda^{-1}\ell\left(x_k', \hat{x}_k\right)\right) \alpha_k^\lambda\left(x_k'\right) dx_k' \tag{5.90}$$

Thus, it is possible to determine the suboptimal state estimate sequence in forward recursion by solving the previous equation at each time k for a fixed value of λ and for a given measurement sequence. We propose an approximate solution for nonlinear systems by particle filtering. In this regard, if the information state is approximated by M samples

$$\alpha_{k-1}^\lambda\left(x_{k-1}\right) \approx \sum_{j=1}^{M} \pi_{k-1}^{\lambda,(j)} \delta\left(x_{k-1} - x_{k-1}^{\lambda,(j)}\right) \tag{5.91}$$

where $\pi_{k-1}^{\lambda,(j)}$ is the normalized weight of the jth particle. By substituting this relation into (5.90), the optimal state estimate can be found as

$$\hat{x}_k^* = \arg\min_{\hat{x}_k} \left(\sum_{j=1}^{M} \pi_k^{\lambda,(j)} \exp\left(\lambda^{-1}\ell\left(x_k^{\lambda,(j)}, \hat{x}_k\right)\right)\right) \tag{5.92}$$

Note that for a quadratic cost function $\ell\left(x_k^{\lambda,(j)}, \hat{x}\right) = \left\|\hat{x} - x_k^{\lambda,(j)}\right\|^2$, the optimal state estimate satisfies the following fixed-point relation;

$$\hat{x}_k^* = \frac{1}{\sum_{j=1}^{M} \pi_k^{\lambda,(j)} \exp\left(\lambda^{-1}\left\|\hat{x}_k^* - x_k^{\lambda,(j)}\right\|^2\right)} \sum_{j=1}^{M} \pi_k^{\lambda,(j)} x_k^{\lambda,(j)} \exp\left(\lambda^{-1}\left\|\hat{x}_k^* - x_k^{\lambda,(j)}\right\|^2\right) \tag{5.93}$$

On the other hand, substituting the approximation (5.91) into the information state recursion relation yields

$$\alpha_k^{\lambda}(x_k) \propto \; p(y_k|\, x_k) \sum_{j=1}^{M} \pi_{k-1}^{\lambda,(j)} p\left(x_k|\, x_{k-1}^{\lambda,(j)}\right) \exp\left(\lambda^{-1}\ell\left(x_{k-1}^{\lambda,(j)}, \hat{x}_{k-1|k-1}\right)\right) \tag{5.94}$$

if the updated information state is approximated by sampled x_k values from a priori density $\left\{x_k^{(j)}\right\} \sim p\left(x_k \middle| x_{k-1}^{(j)}\right)$;

$$\alpha_k^{\lambda}(x_k) \propto \sum_{j=1}^{M} \pi_k^{\lambda,(j)} \delta\left(x_k - x_k^{\lambda,(j)}\right) \tag{5.95}$$

where

$$\pi_k^{\lambda,(j)} = p\left(y_k \middle| x_k^{\lambda,(j)}\right) \pi_{k-1}^{(j)} \exp\left(\lambda^{-1}\ell\left(x_{k-1}^{\lambda,(j)}, \hat{x}_{k-1|k-1}\right)\right) \tag{5.96}$$

Determination of the Lagrange Multiplier :

Now reconsider final stage optimization problem

$$J^s\left(\hat{x}_{0:N}^*, \lambda^*\right) = \inf_{\lambda>0}\left(\lambda \log\, V_N^s\left(x_N^*, \lambda\right) + \lambda d\right) \tag{5.97}$$

with

$$V_N^s\left(x_N^*, \lambda\right) = \inf_{\hat{x}_N \in \aleph_N} \int\int \exp\left(\lambda^{-1}\ell\left(x_N', \hat{x}_N\right)\right)\, \alpha_k^{\lambda}\left(x_N'\right) dx_N'\, d\bar{P}_{Y_N} \tag{5.98}$$

The determination of the suboptimal Lagrange multiplier is not a tractable problem since the optimization with respect to the Lagrange multiplier is dependent on the expectation operation over the measurement space on the time horizon [0, N]. Unfortunately, this fact impedes us to get a complete recursive solution. At this stage several approximations of the objective function is possible to obtain a recursive solution.

One possible way for evaluating the expectation for the measurement sequence over the time horizon [0,N] is off-line Monte-Carlo simulations. In this regard, for each possible measurement sequence that is consistent with nominal system dynamics, the expectation is performed for possible values of Lagrange multipliers.

Another possible way to handle the computational complexity is to change the original optimal estimation problem such that the optimal state estimation cost are conditional to the given measurement sequence. This is possible if the relative entropy constraint is defined on nominal and perturbed conditional probability measures. In this regard, we propose the following function as the cost function of the optimal state estimation problem

$$J\left(\hat{x}_{0:N}^*, Q_{x_{0:N}|y_{0:N}}^*\right) = \inf_{\hat{x}_{0:N} \in \aleph_{0:N}} \; \sup_{Q_{x_{0:N}|y_{0:N}} \in \bar{C}\left(P_{x_{0:N}|y_{0:N}}\right)} E_{Q_{x_{0:N}|y_{0:N}}}\left[\Psi_{0:N}\left(\hat{x}_{0:N}\right)|\, y_{0:N}\right] \tag{5.99}$$

Although this approach seems to be promising, we left the remaining analysis as a future work.

We conclude this section by an application of the method that we have proposed to an example. The example is widely used in the nonlinear estimation literature and is reported as the one that the exponentials risk function that we have used performs worse than the other proposed functions [48]. So our expectation is not to have a good robust estimation result but the proof of the concept of what we have proposed.

Example 5.2.3. As a case study, we have selected a first order nonlinear system that has been widely used in related literature [48]. The uncertain system is represented by the following equations.

$$x_{k+1} = \frac{1}{2}x_k + (\bar{\mu}_k + \Delta_k)\frac{x_k}{1+x_k^2} + 8\cos(1.2k) + w_{k+1} \tag{5.100a}$$

$$y_k = \frac{1}{20}x_k^2 + v_k \tag{5.100b}$$

where x_k is the state and y_k is measurement with $\bar{\mu} = 25$ and δ_k is the unknown but bounded parameter satisfying $|\delta_k| \leq 25$. Here w_k and v_k are the process and the measurement noises that are assumed to be white Gaussian with variance 10.0 and 1.0 respectively. The block diagram of the system is given in Figure 5.3

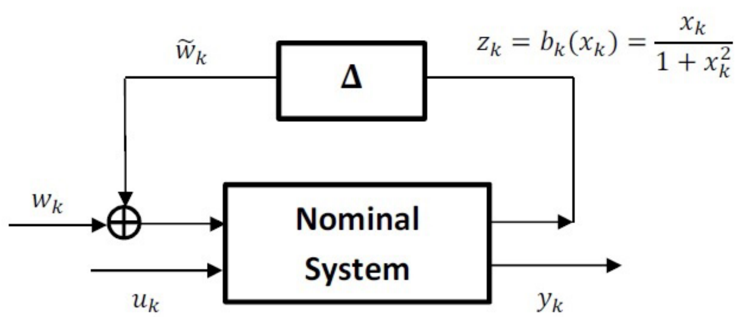

Figure 5.3: The Uncertain Nonlinear System

200 Monte-Carlo runs are performed in order to evaluate the performance of the robust particle filter algorithm. Performance comparison of the proposed approach is done with the classical particle filter. 100 particles are generated for each algorithm. The uncertain parameter which has a nominal value of μ 25 is varied from 0 to 80. The value of is chosen as 65.8 after few trials without applying an explicit optimization procedure. The results are summarized in Figure 5.4. From the figure, it can be concluded that the proposed robust particle filter behaves similar to a classical robust filtering method although it is not very effective. The classical particle filter algorithm performs better than the robust algorithm for uncertain parameter values that are near the nominal value; however the performance of the robust algorithm is acceptable. On the other hand, for the other parameter values, the robust algorithm performance is slightly better than the classical one. The same example is studied in [48] for different risk functions (polynomials and exponentials) in product form. They have concluded

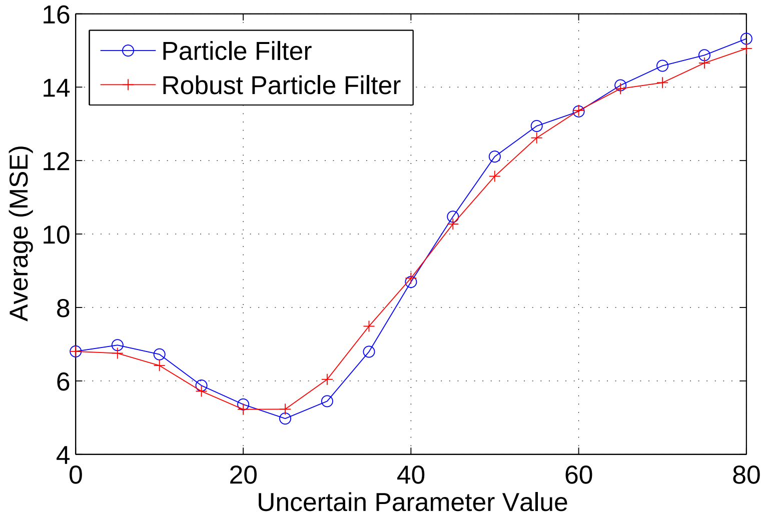

Figure 5.4: Nonlinear Example Performance Results

that the exponential risk function performs worse than the other proposed functions. However, it will be cumbersome to determine the best suitable risk function for different applications.

5.3 Robust Nonlinear Estimation with an Instantaneous Relative Entropy Constraint

In the previous section, the system uncertainty is assumed to satisfy the relative entropy constraint over a finite horizon.

In [38], instead of a single relative entropy constraint defined on a finite time interval, conditional instantaneous relative entropy constraint is proposed. They derived a robust filter for linear Gaussian state space systems as a solution of the minimax estimation problem. The form of the filter is actually risk-sensitive filter with a time-varying risk sensitivity parameter depending on the tolerance bound. They state that single relative entropy constraint yields conservative results since it allows the maximizer to identify the moment when the system most susceptible to distortions".

In this section, we applied the results of the recent paper to the nonlinear systems by utilizing existing nonlinear estimation methods such as particle filter and unscented Kalman filter.

Note here that these two stochastic uncertainty models actually correspond to the deterministic energy constraint and norm bounded uncertainty cases. In the deterministic energy constrained case, the system uncertainty satisfies integral quadratic constraint (or in discrete-time case sum quadratic constraint) and in the norm bounded system uncertainty case, the uncertainty is modeled as unknown but norm-bounded system dynamics.

Let us consider a nonlinear uncertain model

$$x_{k+1} = f(x_k) + w_{k+1} \tag{5.101a}$$

$$y_k = h(x_k) + v_k \tag{5.101b}$$

$$z_k^s = b^s(x_k) \tag{5.101c}$$

$$z_k^m = b^m(x_k) \tag{5.101d}$$

where $x_k \in \mathbb{R}^n$ is the system state, $y_k \in \mathbb{R}^m$ is the system measurement and z_k^s and z_k^m are the system uncertainty outputs that will be used in the sequel to define the uncertainty sets.

Here w_{k+1} and v_k are Gaussian distributed white noise sequences with appropriate dimensions. The true system is assumed to be in a set of admissible systems. It is assumed that the set of admissible uncertain systems are defined in terms of the perturbations or distortions on the system transition and observation densities. Relative entropy constraint is used to evaluate the discrepancy between the nominal densities and the perturbed ones. Then the robust state estimation is defined as the minimization of the worst case estimation error.

Different from the recent results of [38], in this work, two independent distortion sets, (Θ_k^s, Θ_k^m), for system transition and observation densities are defined as follows.

$$\Theta_k^s = \left\{ \bar{\rho}(y_k|\, x_k) : E_{\bar{P}}\left[R(\bar{\rho}(x_k|\, x_{k-1}) \,\|\, \rho(x_k|x_{k-1}))\,|\, y_{0:k-1}\right] \le d_1 + \frac{1}{2} E_{\bar{P}}\left[\|z_{k-1}^s\|^2 \middle| y_{0:k-1} \right] \right\} \tag{5.102a}$$

$$\Theta_k^m = \{\bar{\rho}(y_k|\, x_k) : E_{\bar{P}}\left[R(\bar{\rho}(y_k|\, x_k) \,\|\, \rho(y_k|\, x_k))\,|\, y_{0:k-1}\right] \le d_2 + \frac{1}{2} E_{\bar{P}}\left[\left\|z_k^m\right\|^2 \middle| y_{0:k-1} \right] \tag{5.102b}$$

where $\bar{\rho}(x_k|\, x_{k-1})$ and $\bar{\rho}(y_k|\, x_k)$ are the perturbed probability density functions and $\rho(x_k|\, x_{k-1})$ and $\rho(y_k|\, x_k)$ are the nominal probability density functions. In a more explicit form,

$$\begin{aligned} & E_{\bar{P}}\left[R(\bar{\rho}(x_k|x_{k-1}) \| \rho(x_k|x_{k-1}))\, \middle| y_{0:k-1} \right] \\ & = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \log\left(\frac{\bar{\rho}(x_k|x_{k-1})}{\rho(x_k|x_{k-1})} \right) \bar{\rho}(x_k|x_{k-1})\, p(x_{k-1}|y_{0:k-1})\, dx_k dx_{k-1} \\ & < d_1 + \int_{-\infty}^{\infty} \|z_{k-1}^s\|^2 p(x_{k-1}|y_{0:k-1})\, dx_{k-1} \end{aligned} \tag{5.103}$$

and

$$\begin{aligned} & E_{\bar{P}}\left[R(\bar{\rho}(y_k|\, x_k) \,\|\, \rho(y_k|\, x_k))\,|\, y_{0:k-1}\right] \\ & = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \log\left(\frac{\bar{\rho}(y_k|\, x_k)}{\rho(y_k|\, x_k)} \right) \bar{\rho}(y_k|\, x_k)\, p(x_k|\, y_{0:k-1})\, dy_k dx_k < d_2 + \int_{-\infty}^{\infty} \left\|z_k^m\right\|^2 p(x_k|\, y_{0:k-1}) dx_k \end{aligned} \tag{5.104}$$

Here $p(x_k|y_{0:k-1})$ and $p(x_{k-1}|y_{0:k-1})$ are prior conditional probability density functions. Note that, we made the size of the relative entropy constraint to be state dependent that is defined by the uncertainty output signals (z_k^s, z_k^m). This allows the relative entropy constrained formulation to be generalized the norm bounded uncertainty description. Under the Gaussian assumption on the system noise sequence and additive uncertainty term $\Delta f(x_{k-1})$, the relative entropy constraint reduces to the following conditional instantaneous energy constraint.

$$\bar{\rho}(x_k|x_{k-1}) \propto \exp\left(-\frac{1}{2}\|x_k - f(x_{k-1})\|^2_{Q_k^{-1}}\right) \tag{5.105}$$

$$\bar{\rho}(x_k|x_{k-1}) \propto \exp\left(-\frac{1}{2}\|x_k - f(x_{k-1}) - \Delta f(x_{k-1})\|^2_{Q_k^{-1}}\right) \tag{5.106}$$

Then the relative entropy constraint results in

$$E_{\bar{P}}\left[\frac{1}{2}\|\Delta f(x_{k-1})\|^2_{Q_k^{-1}}\middle| y_{0:k-1}\right] \leq E_{\bar{P}}\left[\left\|z_k^m\right\|^2 \middle| y_{0:k-1}\right] \tag{5.107}$$

where $\Delta f(x_{k-1})$ is the additive system uncertainty term. For the uncertain system model with $z_{k-1}^s = b^s(x_{k-1})$, if the uncertainty signal is defined as $\Delta f(x_{k-1}) = \Delta_{k-1} b^s(x_{k-1})$ then the relative entropy constraint implies:

$$\frac{1}{2}\int_{-\infty}^{\infty}\left(\left\|\Delta_{k-1} b^s(x_{k-1})\right\|^2_{Q_k^{-1}} - \left\|b^s(x_{k-1})\right\|^2\right) p(x_{k-1}|y_{0:k-1})\, dx_{k-1} \leq 0 \tag{5.108}$$

or $\|\Delta_k b^s(x_{k-1})\|^2_{Q_k^{-1}} - \|b^s(x_{k-1})\|^2 \leq 0$ for all possible values of x_{k-1}. Thus $\|\Delta_{k-1}\| \leq 1$.

Following two sequential optimization problems are defined for the robust time and measurement update of the state estimation problems.

PROBLEM I (Time Update):

$$\hat{x}_{k|k-1} = \arg\min_{\xi\in\Im}\max_{\bar{\rho}(x_k|x_{k-1})\in\Phi^s} J_k(\xi, \bar{\rho}(x_k|x_{k-1})) = \frac{1}{2}E_{\bar{P}}\left[\|x_k - \xi\|^2 \middle| y_{0:k-1}\right] \tag{5.109a}$$

$$\text{s.t.}\quad E_{\bar{P}}\left[R(\bar{\rho}(x_k|x_{k-1}) \,\|\, \rho(x_k|x_{k-1})) \,|\, y_{0:k-1}\right] \leq d + \frac{1}{2}\bar{E}\left[\left\|z_{k-1}^s\right\|^2 \middle| y_{0:k-1}\right] \tag{5.109b}$$

$$\text{and}\quad \int_{-\infty}^{\infty}\bar{\rho}(x_k|x_{k-1})\, dx_k = 1 \tag{5.109c}$$

where $\xi \in \Im$ is the state estimate and $\Im$ is the class of state estimators that have finite energy under all distorted probability density functions $\bar{\rho}(x_k|x_{k-1}) \in \Phi^s$.

PROBLEM II (Measurement Update):

$$\hat{x}_{k|k} = \arg\min_{\xi(y_k)\in\Im}\max_{\bar{\rho}(y_k|x_k)\in\Phi^m} J_k(\xi(y_k), \bar{\rho}_k) = \frac{1}{2}E_{\bar{P}}\left[\|x_k - \xi(y_k)\|^2 \middle| y_{0:k}\right] \tag{5.110a}$$

$$\text{s.t.}\quad E_{\bar{P}}\left[R(\bar{\rho}(y_k|x_k) \,\|\, \bar{\rho}(y_k|x_k)) \,|\, y_{0:k-1}\right] \leq d + \frac{1}{2}E_{\bar{P}}\left[\left\|z_k^m\right\|^2 \middle| y_{0:k}\right] \tag{5.110b}$$

$$\text{and}\quad \int_{-\infty}^{\infty}\bar{\rho}(y_k|x_k)\, dy_k = 1 \tag{5.110c}$$

5.3.1 Unconstrained Optimization Problems

The constrained optimization problems defined in the previous section will be converted to unconstrained one by utilizing the Lagrange multipliers theorem.

5.3.1.1 Time Update

In a more detailed form, the optimization problem can be stated as

$$\hat{x}_{k|k-1} = \arg\min_{\xi\in\Im}\max_{\bar{\rho}(x_k|x_{k-1})\in\Phi^s} J_k(\xi,\bar{\rho}_k) = \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\|x_k-\xi\|^2\bar{\rho}(x_k|\,x_{k-1})\,p(x_{k-1}|\,y_{0:k-1})\,dx_k dx_{k-1} \tag{5.111a}$$

$$\text{s.t.}\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\log\left(\frac{\bar{\rho}(x_k|\,x_{k-1})}{\rho(x_k|\,x_{k-1})}\right)\bar{\rho}(x_k|\,x_{k-1})\,p(x_{k-1}|\,y_{0:k-1})\,dx_k dx_{k-1} < d+\int_{-\infty}^{\infty}\left\|z^s_{k-1}\right\|^2 p(x_{k-1}|\,y_{0:k-1})dx_{k-1} \tag{5.111b}$$

$$\text{and}\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\bar{\rho}(x_k|\,x_{k-1})\,p(x_{k-1}|\,y_{0:k-1})\,dx_k dx_{k-1} = 1 \tag{5.111c}$$

The existence of the minimax problem is shown in [38] for a very similar problem. The cost function is quadratic thus convex function of $\xi\in\Im$ and it is linear, thus concave, function of $\bar{\rho}(x_k|\,x_{k-1})$ and the set Φ is a convex set of the probability density functions $\bar{\rho}(x_k|\,x_{k-1})$ further it is compact since relative entropy constrained is weakly sequentially compact [10]. The set of estimators is $\Im$ and is convex. It is also compact if the second moment of the estimators have large but fixed upper bound. Thus according to minimax theorem [10] the saddle point exists. Using the Lagrange multiplier method, the constrained optimization problem can be converted to the following unconstrained optimization one.

$$\hat{x}_{k|k-1} = \arg\min_{\xi\in\Im}\min_{\lambda^s_k\geq 0}\max_{\bar{\rho}(x_k|x_{k-1})\in\Phi^s} L_k\left(\xi,\bar{\rho}(x_k|\,x_{k-1}),\lambda^s_k,\mu^s_k\right) \tag{5.112}$$

where

$$\begin{aligned}L_k(\xi,\bar{\rho}(x_k|x_{k-1}),\lambda^s_k,\mu^s_k = {}&\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\|x_k-\xi\|^2\bar{\rho}(x_k|\,x_{k-1})\;p(x_{k-1}|\,y_{0:k-1})\,dx_k dx_{k-1}\\&+\lambda^s_k\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\log\left(\frac{\bar{\rho}(x_k|\,x_{k-1})}{\rho(x_k|\,x_{k-1})}\right)\bar{\rho}(x_k|\,x_{k-1})\,p(x_{k-1}|\,y_{0:k-1})\;dx_k dx_{k-1}-d^s_k\right)\\&+\mu^s_k\left(\int_{-\infty}^{\infty}\bar{\rho}(x_k|\,x_{k-1})\,dx_k-1\right)\end{aligned} \tag{5.113}$$

with $d^s_k = d+\int_{-\infty}^{\infty}\left\|z^s_{k-1}\right\|^2 p(x_{k-1}|\,y_{0:k-1})\,dx_{k-1}$ and λ^s_k and μ^s_k are the Lagrange multipliers.

The maximization part can be handled by equating the Gateaux derivative of the objective function to zero, thus it can be obtained that the least favorable density satisfies the following relation.

$$\bar{\rho}^*\left(x_k | x_{k-1}\right) = \frac{\exp\left(\frac{1}{2\lambda_k^s}\left\|x_k - \xi\right\|^2\right)}{\int\limits_{-\infty}^{\infty} \exp\left(\frac{1}{2\lambda_k^s}\left\|x_k - \xi\right\|^2\right)\rho\left(x_k | x_{k-1}\right) dx_k}\rho\left(x_k | x_{k-1}\right) \tag{5.114}$$

By substituting the least favorable density into the Lagrangian yields

$$\hat{x}_{k|k-1} = \arg\min_{\xi_k \in \Im}\min_{\lambda_k^s > 0} \lambda_k^s\left(V^s\left(\lambda_k^s, \xi_k\right) + d_k^s\right) \tag{5.115}$$

where

$$V^s\left(\lambda_k^s, \xi_k\right) \triangleq \log \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(\frac{1}{2\lambda_k^s}\left\|x_k - \xi_k\right\|^2\right)\rho_k\left(x_k | x_{k-1}\right) p\left(x_k | y_{0:k-1}\right) dx_k dx_{k-1} \tag{5.116}$$

5.3.1.2 Measurement Update

Since

$$\begin{aligned} J_k\left(\xi\left(y_k\right), \bar{\rho}_k\right) &= \frac{1}{2}\bar{E}\left[\left\|x_k - \xi\left(y_k\right)\right\|^2 \middle| y_{0:k-1}\right] \\ &= \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \left\|x_k - \xi\left(y_k\right)\right\|^2 \rho_k\left(x_k, y_k | y_{0.k-1}\right) dx_k dy_k \\ &= \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \left\|x_k - \xi\left(y_k\right)\right\|^2 \rho_k\left(y_k | x_k\right) p_k\left(x_k | y_{0.k-1}\right) dx_k dy_k \end{aligned} \tag{5.117}$$

The constrained optimization problem can be converted to the following unconstrained optimization problem using the similar arguments presented for time update part;

$$\hat{x}_{k|k} = \arg\min_{\xi_k \in \Im}\min_{\lambda_k^m > 0} \lambda_k^m\left(V\left(\lambda_k^m, \xi(y_k)\right) + d_k^s\right) \tag{5.118}$$

where

$$V^m\left(\lambda_k^m, \xi(y_k)\right) \triangleq \log \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \exp\left(\frac{1}{2\lambda_k^m}\left\|x_k - \xi_k\right\|^2\right)\rho_k\left(y_k | x_k\right) p\left(x_k | y_{0:k-1}\right) dx_k dy_k \tag{5.119}$$

and λ_k^m is the Lagrange multiplier of the measurement update part.

In the following sections, we will utilize the some approximate nonlinear estimation techniques in order to solve the optimal estimation problems.

5.3.2 The Unscented Approach

Let us approximate the following integral

$$p\left(x_k | y_{0:k-1}\right) = \int \rho_k\left(x_k | x_{k-1}\right) p\left(x_{k-1} | y_{0:k-1}\right) dx_{k-1} \tag{5.120}$$

with a Gaussian density by unscented transformation of the nonlinear nominal system model $f(x_k)$. By assuming that the posterior density $p(x_{k-1}|\, y_{0:k-1})$ is Gaussian

$$p(x_{k-1}|\, y_{0:k-1}) = N(x_{k-1}; x_{k-1|k-1}, P_{k-1|k-1}) \tag{5.121}$$

we construct the sigma points

$$\aleph^{(i)}_{k-1|k-1} = \begin{pmatrix} x_{k-1|k-1} for \;\; i=0 \\ x_{k-1|k-1} + \left(\sqrt{(n+\lambda)P_{k-1|k-1}}\right)_i for \;\; 1 \le i \le n \\ x_{k-1|k-1} - \left(\sqrt{(n+\lambda)P_{k-1|k-1}}\right)_i for \;\; n < i \le 2n \end{pmatrix}$$

Here $\left(\sqrt{A}\right)_i$ denotes the ith column of the matrix square root of A. Then by propagating the sigma points through the nominal nonlinear system model

$$\aleph^{(i)}_{k|k-1} = f_k\left(\aleph^{(i)}_{k|k-1}\right) \tag{5.122}$$

$$x_{k|k-1} = \sum_{i=0}^{2N} W_i^{(m)} \aleph_{k|k-1}(i) \tag{5.123}$$

$$P_{k|k-1} = \sum_{i=0}^{2N} W_i^{(c)} (\aleph_{k|k-1}(i) - x_{k|k-1})(\aleph_{k|k-1}(i) - x_{k|k-1})^T + Q_k \tag{5.124}$$

with $W_i^{(m)}$and $W_i^{(c)}$ are the weights of the sigma points. Then the optimization problem

$$\hat{x}_{k|k-1} = \arg\min_{\xi\in\Im}\min_{\lambda_k^s>0} \lambda_k^s \left(\log \int_{-\infty}^{\infty} \exp\left(\frac{\|x_k - \xi\|^2}{2\lambda_k^s}\right) N(x_k; x_{k|k-1}, P_{k|k-1})\, dx_k + d_k^s \right) \tag{5.125}$$

can be converted to the following form

$$\hat{x}_{k|k-1} = \arg\min_{\xi\in\Im}\min_{\lambda_k^s\ge 0} \lambda_k^s \left(\log \frac{\sqrt{\det M_k^1\left(\lambda_k^s\right)}}{\sqrt{\det P_{k|k-1}}} + \left(\frac{1}{2}(\xi - x_{k|k-1})^T S_k^1\left(\lambda_k^s\right)(\xi - x_{k|k-1})\right)\right) + \lambda_k^s d_k^s \tag{5.126}$$

where $S_k^1\left(\lambda_k^s\right) = \left(\lambda_k^s I_{nxn} - P_{k|k-1}\right)^{-1} > 0$ and $M_k^1\left(\lambda_k^s\right) = \left(P_{k|k-1}^{-1} - \left(\lambda_k^s\right)^{-1} I_{nxn}\right)^{-1} > 0$.

Thus it can be easily seen that the optimal state estimate is the mean of the nominal prediction density $p(x_k|y_{0:k-1}) = N(x_k; x_{k|k-1}, P_{k|k-1})$. Thus $\hat{x}_{k|k-1} = x_{k|k-1}$. This is an important fact that has been noted in [29],[38] which is derived for the linear system. Least-squares estimator is robust with respect to the relative entropy constraint. Thus, any non-causal estimators (Kalman or Wiener smoothers) over a finite horizon are robust.

The worst case covariance becomes

$$P^*_{k|k-1} = \left(P_{k|k-1}^{-1} - \left(\lambda_k^s\right)^{-1} I_{nxn}\right)^{-1} \tag{5.127}$$

Using the fact that the worst-case probability density is on the boundary of the relative entropy constraint [38], we can say that the Lagrange multiplier λ_k^s satisfies the following relation

$$\lambda_k^{s,*} = \arg\min_{\lambda_k^s} \lambda_k^s \log \frac{\sqrt{\det M_k^1\left(\lambda_k^s\right)}}{\sqrt{\det P_{k|k-1}}} + \lambda_k^s d_k^s \quad . \tag{5.128}$$

Then it can be obtained that

$$\begin{aligned} c\left(\lambda_k^s\right) &\triangleq R\left(\bar{p}^*\left(x_k | x_{k-1}\right) \| \, p\left(x_k | x_{k-1}\right)\right) \\ &= \frac{1}{2}\log\det\left(I_{n\times n} - \left(\lambda_k^s\right)^{-1} P_{k|k-1}\right) + \frac{1}{2} trace\left(\left(P_{k|k-1}^{-1} - \left(\lambda_k^s\right)^{-1} I_{n\times n}\right)^{-1}\right) = d_k^s \end{aligned} \tag{5.129}$$

It is shown in [38] that the function $c(\lambda_k)$ is a monotone decreasing function. As λ_k goes to the infinity then $c\left(\lambda_k^s\right)$ becomes zero. Thus, a simple numerical root finding algorithm such as bisection algorithm can be utilized to calculate the optimal λ_k value.

Similarly the measurement update part can be solved using the predicted a priori worst case (least favorable) Gaussian density $p\left(x_k | y_{0:k-1}\right)$ as follows;

$$\hat{x}_{k|k} = \arg\min_{\xi\in\Im}\min_{\lambda_k^m \geq 0} \lambda_k^m \left(E\left[\exp\left(\frac{1}{2\lambda_k^m}\left\|x_k - \xi\left(y_k\right)\right\|^2\right)\middle| y_{0:k-1}\right] + d_k^m\right) \tag{5.130}$$

Using the law of iterated expectations

$$E\left[\exp\left(\frac{1}{2\lambda_k^m}\left\|x_k - \xi\left(y_k\right)\right\|^2\right)\middle| y_{0:k-1}\right] = E\left[E\left[\frac{1}{2\lambda_k^m}\left\|x_k - \xi\left(y_k\right)\right\|^2\middle| y_{0:k}\right]\middle| y_{0:k-1}\right] \tag{5.131}$$

where

$$E\left[\frac{1}{2\lambda_k^m}\left\|x_k - \xi\left(y_k\right)\right\|^2\middle| y_{0:k}\right] = \int_{-\infty}^{\infty} \exp\left(\frac{\left\|x_k - \xi\left(y_k\right)\right\|^2}{2\lambda_k^m}\right) p\left(x_k | y_{0:k}\right) dx_k \tag{5.132}$$

The conditional density function $p\left(x_k | y_{0:k}\right)$ can be approximated by an unscented transformation of the nominal nonlinear measurement model as

$$p\left(x_k | y_{0:k}\right) = N\left(x_k; x_{k|k}, P_{k|k}\right) = \frac{1}{(2\pi)^{n/2}\left|P_{k|k}\right|^{1/2}} \exp\left(\frac{-1}{2}\left\|x_k - x_{k|k}\right\|_{P_{k|k}^{-1}}^2\right) \tag{5.133}$$

where

$$x_k = x_k + K_k\left(z_k - z_{k|k-1}\right) \qquad P_{k|k} = P_{k|k-1} - K_k P_{zz,k} K_k^T \tag{5.134}$$

with

$$\aleph_{k-1|k-1}^{(i)} = \begin{pmatrix} x_{k-1|k-1} & for \;\; i = 0 \\ x_{k-1|k-1} + \left(\sqrt{(n+\lambda) P_{k-1|k-1}}\right)_i & for \;\; 1 \leq i \leq n \\ x_{k-1|k-1} - \left(\sqrt{(n+\lambda) P_{k-1|k-1}}\right)_i & for \;\; n < i \leq 2n \end{pmatrix}$$

$$Z_{k|k-1} = h_k\left(\aleph_{k|k-1}\right), \quad z_{k|k-1} = \sum_{i=0}^{2N} W_i^{(m)} Z_{k|k-1}^{(i)} \tag{5.135}$$

$$P_{zz,k} = \sum_{i=0}^{2N} W_i^{(c)} \left(Z_{k|k-1}^{(i)} - z_{k|k-1}\right)\left(Z_{k|k-1}^{(i)} - z_{k|k-1}\right)^T + R_k \tag{5.136}$$

$$P_{xz,k} = \sum_{i=0}^{2N} W_i^{(c)} \left(X_{k|k-1}^{(i)} - x_{k|k-1}\right)\left(Z_{k|k-1}^{(i)} - z_{k|k-1}\right)^T \tag{5.137}$$

$$K_k = P_{xz,k} \cdot (P_{zz,k})^{-1} \tag{5.138}$$

then

$$\int_{-\infty}^{\infty} \exp\left(\frac{\|x_k - \xi(y_k)\|^2}{2\lambda_k^s}\right) N(x_k; x_{k|k}, P_{k|k}) dx_k = \frac{\sqrt{\left|M_k^2(\lambda_k)\right|}}{\sqrt{\left|P_{k|k}\right|}} \exp\left(\frac{1}{2}\left\|\xi(y_k) - x_{k|k}(y_k)\right\|^2_{S_k^s(\lambda_k^s)}\right) \tag{5.139}$$

where $S_k^m\left(\lambda_k^m\right) = \left(\lambda_k^m I - P_{k|k}\right)^{-1} > 0$ and $M_k^m\left(\lambda_k^m\right) = \left(P_{k|k}^{-1} - \left(\lambda_k^m\right)^{-1} I_{nxn}\right)^{-1} > 0$. then the optimization problem reduces to

$$\hat{x}_{k|k} = \arg\min_{\xi \in \Im} \min_{\lambda_k^m \geq 0} \lambda_k^m \left(\log \int_{-\infty}^{\infty} \frac{\sqrt{\left|M_k^2\left(\lambda_k^m\right)\right|}}{\sqrt{\left|P_{k|k}\right|}} \exp\left(\frac{1}{2}\left\|\xi(y_k) - x_{k|k}(y_k)\right\|^2_{S_k^m(\lambda_k^m)}\right) dy_k + d_k^m\right) \tag{5.140}$$

Thus it can be seen easily that the mean of the robust estimator is the same as the nominal one. That is $\hat{x}_{k|k} = x_{k|k}(y_k)$ and then

$$\lambda_k^{m,*} = \arg\min_{\lambda_k^m} \lambda_k^m \log \frac{\sqrt{\det M_k\left(\lambda_k^m\right)}}{\sqrt{\det P_{k|k-1}}} + \lambda_k^m d_k^m \tag{5.141}$$

It can be obtained that

$$c\left(\lambda_k^m\right) = \frac{1}{2} \log\det\left(I_{nxn} - \left(\lambda_k^m\right)^{-1} P_{k|k}\right) + \frac{1}{2} trace\left(\left(P_{k|k}^{-1} - \left(\lambda_k^m\right)^{-1} I_{nxn}\right)^{-1}\right) = d_k^m \tag{5.142}$$

Note here that the left side of the equation is actually the relative entropy between the worst case measure and the nominal measure. This shows that the worst-case measure is at the boundary. Using this relation optimal Lagrange multiplier can be found by a simple bisection algorithm.

5.3.3 Particle Filter Approach

In this section, particle filtering approach is utilized for the robust nonlinear estimation problem.

5.3.3.1 Time Update

For the time update part, the optimization problem can be handled by approximating the posterior conditional density in terms of particles as

$$p(x_{k-1} | y_{0:k-1}) = \sum_{i=1}^{M} W_{k-1|k-1}^{(i)} \delta\left(x_{k-1} - x_{k-1|k-1}^{(i)}\right) \tag{5.143}$$

Table5.1: Relative Entropy Unscented Kalman Filter

Time Update
Calculate Sigma Points
$\aleph_{k-1\mid k-1} = \left[\begin{array}{ccc} x_{k-1\mid k-1} & x_{k-1\mid k-1} + \sqrt{(n+\lambda) P_{k-1\mid k-1}} & \hat{x}_{k-1\mid k-1} - \sqrt{(n+\lambda) P_{k-1\mid k-1}} \end{array}\right]$
$\aleph_{k\mid k-1} = f_k\left(\aleph_{k-1\mid k-1}\right)$
$x_{k\mid k-1} = \sum_{i=0}^{2N} W_i^{(m)} \aleph_{k\mid k-1}(i)$
$P_{k\mid k-1} = \sum_{i=0}^{2N} W_i^{(c)} \left(\aleph_{k\mid k-1}(i) - x_{k\mid k-1}\right)\left(\aleph_{k\mid k-1}(i) - x_{k\mid k-1}\right)^T + Q_k$
Find λ_k^s such that $c\left(\lambda_k^s\right) = d_k^s$
$P_{k\mid k-1}^s = \left(\left(P_{k\mid k-1}\right)^{-1} - \left(\lambda_k^s\right)^{-1} I_{nxn}\right)^{-1}$
Measurement Update
Recalculate Sigma Points
$\aleph_{k-1\mid k-1} = \left[\begin{array}{ccc} x_{k\mid k-1} & x_{k\mid k-1} + \sqrt{(n+\lambda) P_{k\mid k-1}^s} & \hat{x}_{k-1\mid k-1} - \sqrt{(n+\lambda) P_{k\mid k-1}^s} \end{array}\right]$
$z_{k\mid k-1} = \sum_{i=0}^{2N} W_i^{(m)} Z_{k\mid k-1}(i)$
$Z_{k\mid k-1} = h_k\left(x_{k\mid k-1}\right)$
$P_{zz,k} = \sum_{i=0}^{2N} W_i^{(c)} \left(Z_{k\mid k-1} - z_{k\mid k-1}\right)\left(Z_{k\mid k-1} - z_{k\mid k-1}\right)^T + R_k$
$P_{xz,k} = \sum_{i=0}^{2N} W_i^{(c)} \left(X_{k\mid k-1} - x_{k\mid k-1}\right)\left(Z_{k\mid k-1} - z_{k\mid k-1}\right)^T$
$K_k = P_{xz,k} \cdot \left(P_{zz,k}\right)^{-1}$
$x_k = x_k + K_k\left(z_k - z_{k\mid k-1}\right) \qquad P_{k\mid k} = P_{k\mid k-1} - K_k P_{zz,k} K_k^T$
$c\left(\lambda_k^m\right) = d_k$
Find λ_k^m such that $c\left(\lambda_k^m\right) = d_k$

then since

$$
\begin{aligned}
p\left(x_k | y_{0:k-1}\right) &= \int_{-\infty}^{\infty} \rho_k\left(x_k | x_{k-1}\right) \, p\left(x_{k-1} | y_{0:k-1}\right) dx_{k-1} \\
&= \int_{-\infty}^{\infty} \rho_k\left(x_k | x_{k-1}\right) \sum_{i=1}^{M} W_{k-1|k-1}^{(i)} \delta\left(x_{k-1} - x_{k-1|k-1}^{(i)}\right) dx_{k-1} \\
&= \sum_{i=1}^{M} W_{k-1|k-1}^{(i)} \rho_k\left(x_k | x_{k-1|k-1}^{(i)}\right)
\end{aligned}
\tag{5.144}
$$

the predicted conditional density can be approximated as

$$
p_k\left(x_k | y_{0:k-1}\right) \approx \sum_{k=1}^{N} W_{k-1|k-1}^{(i)} \delta\left(x_k - x_{k|k-1}^{(i)}\right) \tag{5.145}
$$

where $x_{k|k-1}^{(i)}$ are sampled from $\rho_k\left(x_k | x_{k-1|k-1}^{(i)}\right)$ as $x_{k|k-1}^{(i)} = f\left(x_{k|k-1}^{(i)}\right) + w_{k+1}^{(i)}$. Thus, the time update part of the problem can be approximated as

$$
\hat{x}_{k|k-1} = \arg\min_{\xi_k \in \mathfrak{I}} \min_{\lambda_k^s > 0} F\left(\lambda_k^s, \xi_k\right) \tag{5.146}
$$

where

$$
F\left(\lambda_k^s, \xi_k\right) = \lambda_k^s \left(\log \sum_{i=1}^{M} W_{k-1}^{(i)} \exp\left(\frac{1}{2\lambda_k^s} \left\| x_{k|k-1}^{(i)} - \xi_k \right\|^2 \right) + d_k^s \right). \tag{5.147}
$$

This problem can be solved by an alternating minimization algorithm as shown in Table 5.2

Table5.2: Alternating Minimization Algorithm for Robust Particle Filter (Time Update)

$\xi_{k,(0)} \leftarrow \hat{x}_{k-1\|k-1}$
while $\left\| \xi_{k,(j)} - \xi_{k,(j-1)} \right\| > \varepsilon$
$\lambda_{k,(j+1)}^s \leftarrow \arg\min_{\lambda_k^s} F\left(\lambda_k^s, \xi_{k,(j)}\right)$
$\xi_{k,(j+1)} \leftarrow \arg\min_{\xi} F\left(\lambda_{k,(j+1)}^s, \xi_k\right)$
$j \leftarrow j+1$
end

The algorithm consists of two sub optimization problems. The solution of the problems will be cited in the following sections.

Minimization with Respect to the Lagrange Multipliers:
The minimization with respect to the Lagrange multiplier λ_k^s is an alternating minimization problem will be accomplished by a fixed point algorithm by taking the derivative of objective function with respect to λ_k^s and equating it to zero. We assume that fixed point can be applied to the minimization problem.

$$
\nabla_{\xi_k} F\left(\lambda_k^s, \xi_k\right) = \log \Sigma_1\left(\lambda_k^s\right) + d_k - \left(\lambda_k^s\right)^{-1} \frac{\Sigma_2\left(\lambda_k^s\right)}{\Sigma_1\left(\lambda_k^s\right)} = 0 \tag{5.148}
$$

where

$$\Sigma_1\left(\lambda_k^s\right) = \sum_i W_{k-1}^{(i)} \exp\left(\frac{\left\|x_{k|k-1}^{(i)} - \xi_k\right\|^2}{2\lambda_k^s}\right) \tag{5.149}$$

and

$$\Sigma_2\left(\lambda_k^s\right) = \sum_i W_{k-1}^{(i)} \left\|x_{k|k-1}^{(i)} - \xi_k\right\|^2 \exp\left(\frac{\left\|x_{k|k-1}^{(i)} - \xi_k\right\|^2}{2\lambda_k^s}\right) \tag{5.150}$$

Thus a fixed point algorithm can be devised for

$$\lambda_k^s = \frac{\Sigma_1\left(\lambda_k\right)\log\left(\Sigma_1\left(\lambda_k^s\right)\right) + d_k^s\Sigma_1\left(\lambda_k^s\right)}{\Sigma_2\left(\lambda_k^s\right)} \tag{5.151}$$

The algorithm is summarized in Table 5.3

Table5.3: Fixed Point Algorithm for Lagrange Multiplier Optimization (Time Update)

$$\begin{array}{l}
\lambda_{k,(j)}^s \leftarrow 10^n \ for\ n > 3 \\
while\ \left\|\lambda_{k,(j)} - \lambda_{k,(j-1)}\right\| > \varepsilon \\
\lambda_{k,(j+1)}^s \leftarrow \frac{\Sigma_1\left(\lambda_{k,(j)}\right)\log\left(\Sigma_1\left(\lambda_{k,(j)}^s\right)\right) + d_k^s\Sigma_1\left(\lambda_{k,(j)}^s\right)}{\Sigma_2\left(\lambda_{k,(j)}^s\right)} \\
j \leftarrow j+1 \\
end
\end{array}$$

Since the gradient of the objective function with respect to state estimate is

$$\begin{aligned}\nabla_{\xi_k} F\left(\lambda_k^s, \xi_k\right) = &\left(\sum_{i=1}^{M} W_{k|k-1} \exp\left(\tfrac{1}{2\left(\lambda_k^s\right)}\left\|x_{k|k-1}^{(i)} - \xi_k\right\|^2\right)\right)^{-1} \\ &\times \sum_{i=1}^{M} \tfrac{1}{\lambda_k^s} W_{k|k-1}\left(x_{k|k-1}^{(i)} - \xi_k\right) \exp\left(\tfrac{1}{2\left(\lambda_k^s\right)}\left\|x_{k|k-1}^{(i)} - \xi_k\right\|^2\right)\end{aligned} \tag{5.152}$$

a gradient descent algorithm can be constructed as presented in the following table.

Table5.4: Gradient Descent Algorithm for the Optimal State Determination

$$\begin{array}{l}
\xi_{k,(0)} \leftarrow \hat{x}_{k-1|k-1} \\
while\ \left\|\xi_{k,(j)} - \xi_{k,(j-1)}\right\| > \varepsilon \\
j \leftarrow j+1 \\
\quad if\ \ j < MaxN\, or\ \ |Grad| > MinGrad \\
\quad\quad Grad_{(j)} \leftarrow Equation, (5.152) \\
\quad\quad \xi_{k,(j)} \leftarrow \xi_{k,(j)} - \mu\, Grad_{(j)} \\
\quad end \\
end
\end{array}$$

Here MaxN is the maximum allowed iteration number and μ is the Gradient step size, which can be optimized by any line search algorithm and MinGrad is the stopping criteria for the minimum gradient value of the objective function.

5.3.3.2 Measurement Update

Let us consider the measurement update part of the robust state estimation. That is

$$\hat{x}_{k|k} = \arg\min_{\xi(.)\in\Im}\min_{\lambda_k^m>0}\lambda_k^m\left(\log V^m(\lambda_k^m,\xi(y_k)) + d_k\right) \tag{5.153}$$

where

$$V^m(\lambda_k,\xi(.)) \triangleq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(\frac{1}{2\lambda_k^m}\|x_k - \xi(y_k)\|^2\right)\rho_k(y_k|x_k)\,p(x_k|y_{0:k-1})\,dx_k dy_k \tag{5.154}$$

If the predicted probability density is approximated as

$$p(x_k|y_{0:k-1}) \approx \sum_{i=1}^{M} W_{k|k-1}^{(i)}\delta\left(x_k - x_{k|k-1}^{(i)}\right) \tag{5.155}$$

then

$$V^m(\lambda_k,\xi(.)) \triangleq= \int_{-\infty}^{\infty}\sum_{i=1}^{M}\left(W_{k|k-1}^{i}\exp\left(\frac{1}{2\lambda_k^m}\left\|x_k^{(i)} - \xi(y_k)\right\|^2\right)\rho_k\left(y_k|x_k^{(i)}\right)\right)p(y_k)\,dy_k \tag{5.156}$$

Note that since the integrand is positive, for a fixed λ_k^m , the minimization with respect to state estimate is equal to the

$$\hat{x}_{k|k} = \arg\min_{\xi(.)\in\Im}\sum_{i=1}^{M} W_{k|k}^{i}\exp\left(\frac{1}{2\lambda_k^m}\left\|x_k^{(i)} - \xi_k\right\|^2\right) \tag{5.157}$$

where $W_{k|k}^{i} = W_{k|k-1}^{i}\rho_k\left(y_k|x_k^{(i)}\right)$ for a given y_k value. On the other hand, the optimization with respect to λ_k^m necessitates a functional form of the $\xi(y_k)$ due integral operation with respect to y_k. Thus it is not possible to solve the problem without defining a structure for the estimator of the state. For instance, a linear structure can be imposed to the problem. Thus, the best linear estimator can be sought. In this regard, let us define

$$\xi(y_k) = \bar{x}_{k|k-1} + K_k\left(y_k - \bar{h}_k\right) \tag{5.158}$$

where $\bar{x}_{k|k-1} = \sum_{i=1}^{M} W_{k|k-1}^{(i)}x_{k|k-1}^{(i)}$ and $\bar{h}_k = \sum W_{k|k-1}^{(i)}h\left(x_{k|k-1}^{(i)}\right)$. Then if previous quantities are plugged into the optimization problem,

$$\hat{x}_{k|k} = \arg\min_{K_k\in\Im}\min_{\lambda_k^m>0}\ \lambda_k^m\left(\log V^m(\lambda_k^m,K_k) + d_k\right) \tag{5.159}$$

where

$$V^m(\lambda_k^m,K_k) \triangleq \sum_{i=1}^{M}\int_{-\infty}^{\infty}\exp\left(\frac{\left\|x_{k|k-1}^{(i)} - \bar{x}_{k|k-1} + K_k\left(y_k - \bar{h}_k\right)\right\|^2}{2\lambda_k^m}\right)\rho_k\left(y_k|x_{k|k-1}^{(i)}\right)dy_k \tag{5.160}$$

Since $\rho_k\left(y_k|\,x^{(i)}_{k|k-1}\right) = N\left(y_k; x^{(i)}_{k|k-1}, R_k\right)$ then

$$V^m(\lambda_k^m, K_k) = \sum_{i=1}^{M} W^{(i)}_{k|k-1} \int_{-\infty}^{\infty} \exp\left(\frac{\left\|x^{(i)}_{k|k-1} - \bar{x}_{k|k-1} - K_k\left(y_k - \bar{h}_k\right)\right\|^2}{2\lambda_k^m}\right) N\left(y_k; x^{(i)}_{k|k-1}, R_k\right) dy_k \tag{5.161}$$

It can be obtained that

$$\begin{aligned}\hat{x}_{k|k} = {} & \arg\min_{K_k\in\Im}\min_{\lambda_k^m>0} \\ & \lambda_k^m \log\sum_{i=1}^{M} W^{(i)}_{k|k-1}\begin{pmatrix}(2\pi)^{-m/2}\,|M_k|^{-1/2} \\ \exp\left(\frac{1}{2}\left\|x^{(i)}_{k|k-1} - \bar{x}_{k|k-1} + K_k\bar{h}_k - K_k h\left(x^{(i)}_{k|k-1}\right)\right\|^2_{S_k^{-1}(\lambda_k^m)}\right)\end{pmatrix} \\ & + \lambda_k^m d_k)\end{aligned} \tag{5.162}$$

where $S_k^{(i)} = \left[\lambda_k - K_k R_k^{-1}\left(K_k\right)^T\right]^{-1}$ and $M_k^{(i)} = \left[R_k^{-1} - \left(K_k\right)^T\left(\lambda_k^s\right)^{-1}\left(K_k\right)\right]^{-1}$. A numerical algorithm that has been proposed for the time update part can be utilized for this part also. Details of the numerical algorithm is considered as a future work.

5.4 Conclusion

In this chapter, we study nonlinear robust estimation problems for uncertain systems where the uncertainty is defined as relative entropy constraint. Two different relative entropy constrained uncertain systems are studied namely relative entropy constrained over a time horizon and instantaneous relative entropy constrained.

For the former case, we provide the problem formulation using the available results in literature. Then by using the available results for linear systems, we propose the extended relative entropy estimation technique by linearizing the nonlinear system around the information state mean. A particle filter solution is also proposed which necessitates Monte-Carlo simulations in order to evaluate the optimal Lagrange multiplier. As a future work, conditional relative entropy constrained problem is worth to study in order to eliminate the expectations operation over measurement sequence. Thus, a more tractable solution can be obtained.

The second problem that we study is the instantaneous relative entropy constrained robust nonlinear estimation problem. We propose two solution methods for this problem. The first is unscented approach where sigma-points are utilized to approximate the information state density. We gave an algorithm for the solution of the problem. The second solution method that we propose is the particle filtering approach. Some numerical algorithms are proposed the time update part of the state estimation problem. For the measurement update part, we propose to give a linear structure to the estimator in order to get finite-dimensional solution. But the details of the algorithm is considered as a future work.

CHAPTER 6

MAIN CONTRIBUTIONS AND CONCLUSIONS

The main contributions that have provided in this thesis are summarized as follows;

In Chapter 3

1) We derived a novel relation between the the eigenvalues of the uncertain system matrix that is in an affine function of the system uncertainty and the eigenvalues of the polynomial chaos (PC) transformed system in terms of the nominal system matrix, perturbation matrix and the roots of the polynomials that corresponds to the polynomial chaos expansion. By using this novel relationship, we showed that the PC transformed system is block diagonalized by a similarity transformation which is used in the stability and controllability analysis.

2) By using the our derived direct relation, we obtain a necessary and sufficient stability condition that relates the stability of the PC transformed system to the original uncertain system as a corollary.

3) We obtain a necessary condition for more general truncated PC transformed systems in terms of the one-norm matrix measure identity by exploiting the its banded structure. It is interesting that our derived result is the one-norm version of the recently obtained result (which is derived in terms of two-norm) in the literature that is derived for infinite-dimensional PC transformed systems by utilizing the infinite dimensional Lyapunov stability.

4) We provide some necessary conditions for the controllability of the PC transformed systems both for single uncertainty and multivariable uncertainty case. As far as our knowledge, the obtained results are new and quite informative.

5) We provide a Kalman decomposition procedure to eliminate the uncontrollable modes of the PC transformed systems.

In Chapter 4

1) We propose a set-valued estimation problem (by modifying the existing results in literature) and its solution for the robust state estimation of uncertain system that is modeled by polynomial chaos expansion. We also showed the probabilistic analogue of the quasi-deterministic approach by a slight modification to the model. We showed that the solution is appeared as

an augmented Kalman filter equations which is advantageous since there are well grown literature for Kalman filter and its solutions techniques. The performances of the our proposed estimation technique and a technique proposed in literature including an ad-hoc measurement model are evaluated by three framework examples that are used in literature. We showed that for stable uncertain systems the proposed technique better than the regularized Kalman filter. For marginally stable system (where eigenvalues are on the unit circle), the filter performed is not satisfactorily. Since PC filter gives a combination of the oscillatory responses due to augmented structure of the system matrix, a beating phenomenon occurs on the state estimation which degrades the performance.

2) An observability analysis is performed both for the proposed estimation technique and the available ad-hoc measurement model. Different necessary conditions are derived for two estimation approaches.

In Chapter 5

1) We propose extended robust filter and a particle filtering method for the solution of the robust nonlinear estimation of uncertain systems with cumulative relative entropy constraint. By particle filtering method, a complete recursive solution could not be obtained for the cumulative relative entropy constraint problem since the determination of the Lagrange multiplier necessitates an averaging process over measurement. Monte-Carlo simulations are proposed for determination of the Lagrange multiplier.

2) Nonlinear estimation problem for instantaneous type relative entropy constraint is studied referring the recent results in the literature. Different form the available results, we define two sub problems both for the time update and the measurement update. Then numerical solutions are proposed for the problems utilizing particle filtering and unscented Kalman filtering.

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PUBLICATIONS

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3. S. Sırtkaya, B. Seymen, "Practical Considerations for Integration of Inertial Navigation and Visual Odometry", presented at NATO SET-168 Symposium on "Navigation Sensors and Systems in GNSS Denied Environments", İzmir, Turkey, September 2012

4. S. Sırtkaya, B. Seymen, A. Alatan, "Loosely Coupled Kalman Filtering for Fusion of

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5. Seymen N.B., Demirekler, M., "Stability and Controllability Analysis on Polynomial Chaos Based Uncertain Systems", is submitted to International Journal of Systems Science, September 2013