PRE-SERVICE ELEMENTARY MATHEMATICS TEACHERS' UNDERSTANDING OF DERIVATIVE THROUGH A MODEL DEVELOPMENT UNIT

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I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

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ABSTRACT

PRE-SERVICE ELEMENTARY MATHEMATICS TEACHERS' UNDERSTANDING OF DERIVATIVE THROUGH A MODEL DEVELOPMENT UNIT

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The purpose of this study was to investigate pre-service mathematics teachers' understanding of 'big ideas' involved in derivative such as covariational reasoning, rate of change, and the graphical connections between a function and its derivative.

In this design-based study, a model development unit was designed, experimented, and evaluated in a real classroom setting as a part of a course offered to pre-service mathematics teachers in two iterations. The data were collected from the 20 pre-service mathematics teachers attending to the course and enrolled in a middle school mathematics teacher education program at a public university in Ankara during the fall semester of 2011-2012. The implementation of the model development unit was continued for 8 weeks. Multiple data collection methods were used in this study, and the data was analyzed by using qualitative and quantitative methods.

The data revealed that, at initial phases, pre-service mathematics teachers had difficulties in covariational reasoning, rate of change, and the graphical connection between a function and its derivative. In the progress of the model development unit, considerable improvements were observed in pre-service teachers' covariational reasoning abilities. Similarly, while they were unaware of the concept of rate of change, they realized it as being a different interpretation of derivative, slope, and difference quotient. However, their confusions between rate (of change) and amount (of change) continued. Furthermore, pre-service teachers’ understanding of the graphical connection between a function and its derivative shifted from thinking only by procedures to thinking by making sense of those procedures. These findings of the study revealed that even university students who already completed calculus courses could not learn the essential ideas involved in derivative. The data of this study also showed the potentials of mathematical modeling activities in promoting students’ contextual understanding of the ideas involved in derivative.

Keywords: Mathematics education, design-based research, derivative, rate of change, mathematical modeling, calculus, covariational reasoning, graphs

ÖZ

İLKÖĞRETİM MATEMATİK ÖĞRETMEN ADAYLARININ BİR MODEL GEİŞTİRME ÜNİTESİ ARACILIĞI İLE TÜREVİ ANLAMALARI

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Bu çalışmanın amacı matematik öğretmen adaylarının kovaryasyonel düşünme, değişim oranı ve bir fonksiyon ile türevi arasındaki grafiksel ilişki gibi türev kavramının temelini oluşturan matematiksel fikirleri nasıl anladıklarını araştırmaktır.

Tasarım-tabanlı bu çalışmada, bir model geliştirme ünitesi tasarlanmış ve matematik öğretmen adayları için açılan bir ders kapsamında gerçek sınıf ortamında iki defa uygulanarak değerlendirilmiştir. Bu çalışmada sunulan veri 2011-2012 sonbahar döneminde derse kayıt yaptıran ve Ankara'da bulunan bir devlet üniversitesinde ilköğretim matematik öğretmenliği programına devam eden 20 öğretmen adayından elde edilmiştir. Model geliştirme ünitesinin uygulanması 8 hafta sürmüştür. Çalışmada birden çok veri toplama yöntemi kullanılmış olup, veri analizinde ise, nitel analiz ağırlıklı olmak üzere, nitel ve nicel yöntemler birlikte kullanılmıştır.

Çalışmadan elde edilen veriler, başlangıç aşamasında, öğretmen adaylarının kovaryasyonel düşünmede zorluk yaşadıklarını, değişim oranı ve bir fonksiyon ile o fonksiyonun türevi arasındaki grafiksel ilişkiye dair bilgilerinin oldukça yetersiz olduğunu ortaya koymuştur. Model geliştirme ünitesinin uygulanması sürecinde, öğretmen adaylarının kovaryasyonel düşünme becerilerinde önemli gelişmeler

gözlemlenmiştir. Aynı şekilde, öğretmen adayları başlangıçta değişim oranı kavramından haberdar değilken, süreçte bunun türev, eğim ve farkların oranı gibi matematiksel kavramların farklı bir yorumu olduğunu fark etmişlerdir. Ancak, öğretmen adaylarının değişim oranı ile değişim miktarını karıştırmaya devam ettikleri de görülmüştür. Ayrıca, öğretmen adaylarının bir fonksiyon ile türevi arasındaki grafiksel ilişkiyi anlamalarında, sadece işlemsel ve prosedürel düşünme yapısından bu işlemler ve prosedürlerin anlamlandırarak düşünmeye doğru bir değişim olmuştur. Bu çalışmanın bulguları, analiz derslerini tamamlayan üniversite öğrencilerinin bile türev ve türev için gerekli temel matematiksel fikirleri öğrenemediklerini göstermektedir. Bu çalışmadan elde edilen veriler ayrıca, öğrencilerin türev ve türev için gerekli temel matematiksel fikirleri anlamalarını geliştirmek için matematiksel modelleme etkinliklerinin etkili olabileceğini göstermiştir.

Anahtar Kelimeler: Matematik eğitimi, tasarım-tabanlı araştırma, türev, değişim oranı, matematiksel modelleme, kalkülüs, kovaryasyonel düşünme, grafikler

To my family

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CHAPTER 1

INTRODUCTION

According to Hughes-Hallett et al. (1994), calculus has been accepted as "*one of the greatest achievements of human intellect*", and it has the power of reducing complicated problems to simple rules in mathematics, physical sciences, and engineering (as cited in Berry & Neyman, 2003, p. 482). The collection of mathematical methods examining change, rate of change, and accumulation of change constitute the content of Calculus. Functions, limit, derivative, and integration are the foundational concepts covered in Calculus. Functions are used for representing how things change in relation to each other, derivative is related to at what rate they change, and integration concerns how they accumulate (Tall, 1996). Although basic calculus concepts such as functions, limit, derivative and integration are initially taught at high school level in some countries, one of which is Turkey, Calculus is generally taught at the university level.

Research regarding learning and teaching of calculus concepts pointed out variety of student and teacher difficulties (e.g., Ferrini-Mundy & Graham, 1991; 1994; Orton, 1983; Tall & Vinner, 1981; Zandieh & Knapp, 2006;White & Mithcelmore, 1996). For instance, Orton (1983) indicated that although students performed successfully on routine derivative and integral problems, they had difficulties in conceptual understanding. Similarly, Selden, Selden, Hauk, and Mason (1999) reported that, in a traditional calculus course, even many good students could not reach the level of conceptual understanding and the ability of solving moderately non-routine problems. Tall (1992) mentioned about a set of general difficulties students encounter in conceptual understanding of calculus concepts that are: (i) difficulties with limit and infinite process, (ii) the difficulty with the Leibniz notation, (iii) weak understanding of functions, (iv) difficulties in translating real-world problem into formal calculus, (v) difficulties in selecting and using appropriate representations; (vi) algebraic manipulation-or lack of it, and (vii) students' preference of procedural methods rather than conceptual understanding.

The general results in the literature have indicated that the symbolic and rule oriented approach do not foster students' conceptual understanding of calculus, and additionally it results in many difficulties (Bezuidenhout, 1998; Orton, 1983). Up to middle of 1990's, standard calculus curricula all around the world were suggesting traditional methods involving mastery of symbolic and algebraic rules and application of these rules for solving problems (Tall, 1996). However, as a result of the dissatisfaction with the students' level of learning, "Calculus Reform Movement" on teaching of calculus took place in USA and followed by some other countries (Garner & Garner, 2001; Habre & Abboud, 2006; Tall, 1992). A parallel process has been observed between the development of computer-based graphing, simulation, and algebra programs and the reform movements in mathematics education. The reformed calculus put more emphasis on the meaningful learning with multiple representations (visuo-spatial, tabular, graphical and algebraic) of concepts within multiple contexts, visualization, and technology usage (Berry & Neyman, 2003; Garner & Garner, 2001; Tall, 1996; Teuscher & Reys, 2012). Studies focusing on meaningful and contextual teaching of calculus concepts were generally conducted within technologically supported and virtually real learning environments. The research studies investigating students' understanding of calculus concepts conducted within technologically supported learning environments have brought innovative pedagogical ideas towards the teaching and learning of calculus (Berry & Nyman, 2003; Stroup, 2002; Tall, 1996). They emphasized the importance of using real life contexts in teaching mathematics. As students worked on real situations, for example animating a moving object with its distance and the duration of time, some hopeful developments in their understanding of qualitative aspects of calculus have been reported (Doerr & O'Neill, 2010; Doorman & Gravemeijer, 2008; Herbert & Pierce, 2008; Hoffkamp, 2011; Stroup, 2002). However, although extensive changes occurred in the mathematics curriculum at secondary level, it is difficult to mention about a reform movement in Turkey with regard to the teaching of Calculus at university level.

1.1 Derivative

Derivative is one of the fundamental topics of Calculus analyzing how things change and at what rate they change (Tall, 1996). However, derivative is not an easy topic to teach and understand, because robust understanding of derivative requires understanding of some essential prior concepts such as slope, function, rate and ratio, limit, and rate of change with their different mathematical representations (Zandieh, 2000). Student difficulties with many of the previous concepts such as; functions and especially lack of dynamic view of functional dependency (Carlson, 1998; Hoffkamp, 2011; Monk, 1992), slope (Stump, 1999), rate and ratio (Herbert & Pierce, 2012a; Thompson & Thompson, 1994), rate of change (Bezuidenhout, 1998; Confrey & Smith, 1994; Orton, 1983; Thompson, 1994a; White & Mitchelmore, 1996; Zandieh, 2000), and limit (Hahkiöniemi, 2006; Tall & Vinner, 1981), have been well documented. The studies specifically focusing on students' understanding of derivative point out the weakness in their conceptual understanding. Although they can perform variety of rule oriented applications, both students and teachers have many difficulties in giving meaning to symbolic expressions used for derivative (Santos & Thomas, 2001; White & Mitchelmore, 1996), in conceptualizing derivative with all its process-object pairs (Habre & Abboud, 2006; Zandieh, 2000), in forming connections between different representations of derivative (Herbert & Pierce, 2012b; Zandieh, 2000; Zandieh & Knapp, 2006), and they have weak understanding of rate of change (Orton, 1983; White & Mitchelmore, 1996).

Restricted images or weak understanding of functions have been determined as one of the possible sources of student difficulties in conceptual understanding of derivative and other calculus concepts (Carlson, 1998; Confrey & Smith, 1994; Monk, 1992; Tall, 1992, 1996). The correspondence approach and formula-based applications are the general orientation in the teaching of function concept. However, correspondence approach fosters the idea of function as a static algebraic rule, used to obtain the output value by substituting the input value, and it does not support the idea of dynamic simultaneous variation of quantities (Confrey & Smith, 1994; Monk, 1992; Tall, 1996; Thompson, 1994b). Dynamic view of function entails the coordination of simultaneous small (and infinitesimal) changes in the input and output variables, and it is the central idea for meaningful understanding of rate of change (Confrey & Smith, 1994; Monk, 1992). Therefore, the dynamic view of function is closely related to the notions of covariation and covariational reasoning. The notions of covariation and covariational reasoning have been identified as being the fundamental idea that students should have in order to understand functions, rate of change, and accumulation of change (Carlson, Jacobs, Coe, Larsen & Hsu, 2002; Cooney, Beckman & Lloyd, 2010; Saldanha & Thomson, 1998; Thompson & Thompson, 1992; Thompson, 1994b; Zandieh, 2000). However, studies on covariational reasoning evidenced that students and teachers were not performed well in covariational reasoning tasks, and the need for further studies focusing on the ways of developing students' covariational reasoning abilities have been offered (Carlson et. al., 2002; Carlson, Larsen & Lesh, 2003; Zeytun, Çetinkaya & Erbaş, 2010).

Weaknesses in students' contextual understanding of derivative have been commonly pointed out in the literature. Many researchers indicated that rate of change was not understood by students or teachers (Bezuidenhout, 1998; Bingolbali, 2008; Coe, 2007; Confrey & Smith, 1994, 1995; Herbert & Pierce, 2012; Orton, 1983; Rowland & Javanoski, 2004; Stroup, 2002, Tall, 1992; Teuscher & Reys, 2012; Thompson, 1994a, 1994b;White& Mitchelmore, 1996; Wilhelm & Confrey, 2004). Students from various grade levels have diverse difficulties and misconceptions related to rate of change. One of them is difficulty in giving meaning to rate of change or interpreting as if it is an arithmetic mean (Bezuidenhout, 1998; Orton, 1983). Confusing rate of change with the amount of change in the dependent variable is another common misconception that students have (Rowland & Javanoski, 2004; Zandieh & Knapp, 2006). In addition, students have a tendency of considering rate of change as the slope only in linear functional situations, and they had difficulties in interpreting it in non-linear situations (Stroup, 2002; Teuscher & Reys, 2012). Additionally, the real life interpretation of derivative is generally introduced within the motion context. However, using only motion context builds an obstacle in front of students for the general idea of rate of change by limiting their understanding with the Physics concepts such as speed, velocity and acceleration (Herbert & Pierce, 2008, 2012; Wilhelm & Confrey, 2003; Gravemeijer & Doorman, 1999; Zandieh & Knapp, 2006; Yoon, Dreyfus & Thomas, 2010). Studies show in common that rate of change is the most problematic dimension of students' conceptual understanding of derivative, and interestingly it is superficially considered in many curricular documents (Bingolbali, 2008).

Researchers also gave importance to the graphical interpretation of derivative and reversing between derivative and antiderivative graphs for conceptual understanding of derivative. Students have difficulties in interpreting cusp points, vertical tangents, discontinuity, inflection point, and second derivative test for deciding the concavity (Asiala, Cottrill, Dubinsky & Schwingendorf, 1997; Baker, Cooley, & Trigueros, 2000; Ubuz, 2007). In addition, students demonstrate weak understanding in reversing between the graphs of derivative and antiderivative function in the absence of algebraic formulas or drawing the antiderivative graph only using the analytical properties (Aspinwall, Shaw & Presmeg, 1997; Haciomeroglu, Aspinwall, Presmeg, 2010). These studies evidence that standardized applications used for reversing between the graph of a function and its derivative do not foster students' understanding of the underlying ideas behind the routines. Moreover, students could not utilize their procedural knowledge for sketching graphs while solving the contextual tasks (Berry & Nyman, 2003; Yoon et al, 2010).

1.2 Purpose of the study

The review of literature indicated that students from various grade levels have difficulties in conceptual understanding of derivative (Bezuidenhout, 1998; Confrey & Smith, 1994; Herbert & Pierce, 2012a; Orton, 1983; Stump, 1999; Tall, 1992, 1996; Thompson, 1994a; Zandieh, 2000). The results of these studies prove the problematic aspect of the way of teaching calculus all around the world and the need for further studies involving new pedagogical approaches as also indicated by many researchers (Berry & Nyman, 2003; Bezuidenhout, 1998; Zandieh, 2000).

The traditional teaching methods of calculus involve the mastery of symbolic and algebraic rules and application of these rules for solving problems (Tall, 1996; Teuscher & Reys, 2012). These methods involve the introduction of concepts with their formal mathematical representations and giving meaning to them in real situations is generally expected from students (Lesh & Doerr, 2003; Teuscher & Reys, 2012). However, formal representations of the mathematical concepts are the mathematical models produced by others in long periods of times. Therefore it is inevitable for students to have difficulties in conceptualizing real life interpretations of them (Gravemeijer, 2002; Lesh & Doerr, 2003; Stroup, 2002). The usage of real situations, modeling activities, or interactive learning environments supported by computer based programs (e.g., simulations, graphing) have been suggested as a possible way for supporting the meaningful learning of mathematical concepts (Gravemeijer & Stephan, 2002; Stroup, 2002). Specifically, mathematical modeling activities have been recommended as pedagogically effective tools for eliciting students' informal ways of thinking, which can later be directed to more meaningful formal understanding (Blum & Niss, 1991; Gravemeijer & Doorman, 1999; Lesh & Doerr, 2003). When students worked on a moving object, they can develop important ideas about the total distance covered in relation to time, the fastness (rate) of change in distance, and graphical representations of them. A few studies have reported worthy developments in students' conceptions of change, rate of change, derivative, antiderivative, and their graphical interpretations as they worked on artificial (interactive computer-based) or authentic real life situations (Doorman & Gravemeijer, 2009; Gravemeijer & Doorman, 1999; Hoffkamp, 2011; Stroup, 2002; Yoon et. al., 2010). These studies have provided me a solid rationale for using authentic problem situations for developing students' understanding of derivative.

In addition, Harel, Selden and Selden (2006) argued that most of the advanced mathematical thinking studies were descriptive or cognitive oriented. The researchers pointed out the need for extending these cognitive oriented studies. In the same vein, the results of many studies also indicated the necessity for creating new pedagogical approaches and curricular materials aiming at effective teaching of derivative (Bezuidenhout, 1998; Carlson et. al., 2002; Tall, 1992). In the current study, a model development unit consisting of four model development sequences (Lesh, Cramer, Doerr, Post, & Zawojewski, 2003; Lesh, 2010) were developed, implemented, and evaluated to develop pre-service teachers' understanding of the big ideas involved in derivative. The big ideas covered within the model development unit were the covariational reasoning, rate of change, and the graphical connections between a function and its derivative. Design-based research approach was adopted in preparing, implementing (or experimenting), and evaluating the model development unit (Cobb, Confrey, diSessa, Lehrer & Schauble, 2003; Hjalmarson & Lesh, 2008; Sloane & Kelly, 2008). The big ideas focused within hypothetical learning trajectory of the model development unit were the concept of covariation and covariational reasoning, rate of change and its connection with other interpretations of derivative, and graphical understanding of derivative.

The main purpose of the current study was investigating pre-service mathematics teachers' understanding of the ideas involved in derivative during the experimentation of the model development unit in a classroom setting. However, because a design study perspective was adopted, the study also involve some other purposes such as (i) determining the critical ideas with regard to teaching and learning of derivative by a comprehensive review of literature, (ii) reaching the principles and conjectures for designing new learning tools aiming at effective way of teaching derivative by refining and reinterpreting the critical ideas obtained from the literature, (iii) designing learning tools and implementing in a real classroom setting during which students' developmental understanding of derivative was investigated, and (iv) reaching theoretical arguments related to pedagogically more effective ways of teaching derivative.

1.3 Research Questions

Covariational reasoning, rate of change, and the graphical connection between a function and its derivative were determined as being the big ideas for conceptual understanding of derivative. In this study, pre-service mathematics teachers' understanding of derivative was investigated focusing on the big ideas involved in it. The following main research questions and sub-questions guided this study:

1) What is the nature of pre-service mathematics teachers' existing conceptions related to the big ideas involved in derivative prior to attending the classroom experimentation of the model development unit?
    - What is the nature of covariational reasoning that pre-service mathematics teachers demonstrated prior to or at initial phases of the model development unit?
    - What is the nature of pre-service teachers' conceptions of rate of change prior to or at initial phases of the model development unit?
    - How did pre-service teachers interpret the graphical connections between a function and its derivative prior to or at initial phases of the model development unit?

2) What conceptions with regard to the big ideas involved in derivative did pre-service mathematics teachers develop as they attended to the classroom experimentation of the model development unit?

    - How did covariational reasoning demonstrated by pre-service teachers change during the process?
    - What is the nature of developments in pre-service mathematics teachers' conceptions of rate of change during the process?
    - How did pre-service teachers' interpretations of the graphical connections between a function and its derivative change during the process?

1.4 Significance of the study

The significance of this study can be stated from various aspects. Foremost, this study is significant in terms of taking three of the big ideas that are covariational reasoning, rate of change, and graphical connections between rate and amount functions into consideration in the teaching of derivative at the same time. In other words, three of the big ideas are equally important for conceptual understanding of derivative. Focusing on only one of them in the teaching may not solve student difficulties for others. Therefore, unlike the other studies, the current study considered three of the big ideas involved in derivative in the hypothetical learning trajectory of the model development unit. Particular student and teacher difficulties have been reported for each of the critical ideas separately. For instance, many studies revealed student and teacher difficulties with the rate of change concept (e.g., Bezuidenhout, 1998; Orton, 1983) and in understanding the graphical connection between rate and amount functions (e.g., Berry & Nyman, 2003; Ubuz, 2007). Similarly, covariational reasoning has been determined as being an essential idea for conceptual understanding of calculus concepts one of which is derivative, but only a few studies considered it in the teaching process (e.g., Hoffkamp, 2011).

Secondly, this study is significant in terms of practicing a hypothetical learning sequence for derivative decided according to the theoretical arguments drawn from the literature. Covariational reasoning has been accepted as being prerequisite for conceptual understanding of rate of change (e.g., Confrey & Smith, 1994;

Thompson, 1994b), and rate of change is the contextual interpretation of derivative. So, covariational reasoning and rate of change are two critical concepts for conceptual understanding of derivative. Furthermore, Stroup (2002) pointed out the central role of contextual graphs in promoting students' qualitative understanding of derivative. Therefore, covariational reasoning, rate of change, and the graphical connections between a function and its derivative formed the learning goals within the hypothetical learning trajectory of the model development unit. According to Simon (1995), the construct of hypothetical learning trajectory consists of a teacher's considerations of the learning goals for a particular topic, designing activities for them, and possible student thinking and learning when engaging in these activities. There are only a few studies using similar learning goals in the teaching of calculus concepts (Doorman & Gravemeijer, 2008; Herbert & Pierce, 2008; Hoffkamp, 2011). Therefore, the results of this study may contribute to the area of research on the teaching of derivative. The theoretical arguments with regard to teaching of derivative, concluded from the literature and revised as a result of this study, may also guide other researchers and educators in designing more effective learning tools.

Thirdly, the number of intervention studies is limited and the need for studies focusing on developing new pedagogical approaches and learning materials for the teaching of derivative has been voiced (Bezuidenhout, 1998; Carlson et al., 2002; Harel et al., 2006). The current study involved an intervention since the designed model development unit was experimented in a classroom setting. Therefore, this study is significant in terms of trying out an intervention and proposing a new pedagogical approach for the teaching of derivative focusing on its contextual and graphical aspects. The findings of this study may contribute to the literature on the teaching of derivative as well as the literature of design-based research studies on calculus.

In addition, the primary rationale behind the design research studies has been indicated as making both practical and scientific contributions, and shortening the gap between theory and practice in education (Hjalmarson & Lesh, 2008; Lesh, 2002; van den Akker, 1999). In this study, a model development unit consisting of a series of model development sequences was designed by a group of researchers. The theoretical conjectures obtained from the literature guided us in designing the model development sequences and so the model development unit. The model development unit was modified passing through a few cycles. Therefore, the study is significant because it provides a practical instructional tool (model development unit) that teachers can directly use in their classrooms in the teaching of derivative.

And finally, covariational reasoning frame developed by Carlson et al. (2002), the theory of quantitative reasoning (Thompson, 1994b; 2011), qualitative calculus (Stroup, 2002), and mathematical modeling (Lesh & Doerr, 2003) were the theoretical perspectives guided me in designing the model development unit and in analyzing the data. Namely, these different perspectives were matched up under a common umbrella. Therefore, this study is significant in terms of showing the common points of these different theoretical perspectives, and how they fulfill each other in explaining more complex situations. The rich and in-depth data collected during the study may provide researchers the opportunity of testing and revising the arguments indicated by these theoretical perspectives. For instance, many studies indicated the weakness of covariational reasoning framework in explaining different ways of thinking (e.g., Carlson et al., 2002; Carlson et al., 2003; Zeytun et al., 2010), and the data of this study may be helpful for identifying the character of covariational reasoning. Additionally, the theory of quantitative reasoning and the notion of qualitative calculus may contribute in characterizing the covariational reasoning.

1.5 Definitions of terms

Covariation

Covariation means the simultaneous variation of two quantities in relation to each other. It is a term used for indicating the dynamic view of functional dependency and involves the coordination of two simultaneously changing quantities (Confrey & Smith, 1994; Saldanha & Thompson, 1998).

Covariational reasoning

Covariational reasoning encapsulates the mental actions used while coordinating the simultaneously changing quantities. Carlson et al. (2002) defined covariational reasoning as "*the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other*" (p.354).

Ratio

Ratio is a quantity obtained as a result of multiplicative comparison of two non-varying quantities (Thompson, 1994b).

Rate

Rate is a quantity obtained as a result of multiplicative comparison of two non-static, varying quantities (Thompson, 1994b).

Rate of change

Rate of change is also a quantity obtained as a result of quantitative operation that is multiplicative comparison of changes in two simultaneously changing quantities, and the idea of rate of change is encapsulated in the mature image of rate (Thompson, 1994b). When the corresponding changes in two simultaneously changing variables remain in constant proportion, it is called *constant rate of change*. *Average rate of change* is obtained as a result of calculating the multiplicative relationship between the change in the dependent variable and the change in the independent variable over an interval, and also this is a kind of constant rate of change in that interval. *Instantaneous rate of change* can be thought as an average rate of change over an infinitesimal interval (Coe, 2007). So, changing rate of change is related to instantaneous rate of change and it means that the instantaneous rate of change of a function takes different values for different input values.

Mathematical modeling

In general, mathematical modeling has been defined as the process of mathematizing, interpreting, verifying, revising, and generalizing real life situations or complex systems (Lingefjard, 2002). Lesh and Doerr (2003) describes mathematical modeling as a process of producing sharable, modifiable, and reusable conceptual tools or mathematical models for describing, predicting, and controlling real life situations. Mathematical modeling applications provide students significant local conceptual developments and meaningful learning of basic mathematical ideas in real situations.

Model development sequence

Instead of using modeling activities as standing alone problem solving applications, a sequence of structurally related modeling and follow up activities aiming at teaching of a particular mathematical concept is called with the notion of *model development sequence* (Lesh et al., 2003). Model development sequences include structurally related modeling and follow-up activities, group discussions, student presentations, and classroom discussions about the structural similarities of mathematical ideas across the activities.

Model development unit

A sequence of model development sequences was called as model development unit (Lesh, 2010). The mathematical topic covered by a model development unit is relatively more extensive.

Hypothetical learning trajectory

Simon (1995) introduced the construct of hypothetical learning trajectory, and it involves determination of the learning goals, designing appropriate learning activities for the learning goals, and consideration of the possible student learning and thinking when engaging the activities. Because design-based studies on teaching a topic involve a kind of hypothetical planning (Cobb & Gravemeijer, 2008), this terminology was used for indicating the learning sequence in the model development unit.

CHAPTER 2

LITERATURE REVIEW

This study adopts a design based research approach during which a model development unit was designed, implemented, and evaluated (Cobb et al., 2003; Hjalmarson & Lesh, 2008; Sloane & Kelly, 2008). The design principles used as a guide in designing the model development unit were determined by a comprehensive literature review. The concept of covariation was determined as being the fundamental idea for understanding of the functions and derivative. Therefore, the research studies related with the concept of derivative were analyzed first. It was followed by the analysis of studies on covariational reasoning which has been identified as the foundational idea for conceptual understanding of derivative. Rate of change and graphical connections between rate and amount functions were determined as being the other critical ideas for conceptual understanding of derivative.

2.1 Derivative

Derivative is one of the important concepts of calculus, but research studies indicate that it is a difficult concept for students. In this section, the review of literature on the concept of derivative was introduced under four subsections focusing on the research studies pointing out (i) student misconceptions and difficulties with the concept of derivative and their theoretical arguments for overcoming those difficulties, and (ii) the way of introducing derivative in curricular documents.

2.1.1 Studies on teaching and learning of derivative

The review of literature have pointed out that general understanding of derivative by students from different grades levels and from different countries were limited to rote application of algebraic rules in artificial and pure-symbolic situations (Berry &

Nyman, 2003; Habre & Abboud, 2006; Orton, 1983; Tall, 1992; White & Mitchelmore, 1996). In an earlier study, Orton (1983) determined students' weak understanding of differentiation. 110 students in England (60 high school students, and 50 of pre-service mathematics teachers) were the participants of the study. Almost all of the students were successful in accurately differentiating the polynomial functions involving the routine aspect of differentiation. However, students had great difficulties in estimating the slope at a point (derivative) of a function represented in graphical form and in interpreting and calculating the rate of change. When they were asked to determine the rate of change of a linear function at a point, among many other irrelevant answers, one-fifth of the students responded with the corresponding y-coordinate. The question related to the rate of change of a quadratic function at a point could not been answered. The general result of this study showed students' difficulties with the concepts of derivative and its rate of change interpretation, and they could not interpret it when involved with the limit and difference quotient. Orton (1983) suggested an informal approach involving numerical and graphical explorations by using real-life data as initial for teaching calculus. This study was a pioneer study putting students' difficulties and weak understanding of derivative into words from various aspects.

In later years, students' limited understanding of the meanings behind symbolic representations of calculus concepts have been mentioned as being the possible source of difficulties (Santos & Thomas, 2001; White & Mitchelmore, 1996). In their study, White and Mitchelmore (1996) evidenced an underdeveloped concept of variable as being the source of students' difficulties in conceptual understanding of derivative. They studied with 40 first-year calculus students during a concept-based calculus course. Students were asked with the different versions of four tasks included in four parallel forms of the tests. The differences between the four versions of the tasks were in their presentation within the continuum of purely verbal expressions to purely symbolic form. Whereas A-version of a task was presented in verbal with only a few symbolic expressions, D-version of the same task was asked within the purely symbolic context. White and Mitchelmore (1996) obtained the result that students were more successful in solving the tasks given in purely symbolic context and they showed important developments in that domain, but they were unable solve and any development was not observed during the teaching period related to the tasks asked in verbal expressions. This study showed students' weak understanding of derivative in real contexts. The researchers concluded that students could not symbolize rate of change in verbally indicated items as derivative where modeling of the situation was required. Additionally, some of the students could not interpret such symbolic expressions as $\frac{dm}{dv}$ because of not seeing the symbols m or v as variables, rather seeing them as symbols standing for particular quantities. According to researchers, this was because of students' dominant conception of variables as symbols to manipulate with and their inadequacy of interpreting derivative as rate of change between two simultaneously changing variables. Likewise, in the study of Santos and Thomas (2001), it has been also evidenced that 13th grade students from a top-performing school who were already taught differentiation and integration attributed different meanings to the $\frac{dy}{dx}$ symbol depending on the context it was used. The results of the study indicated that student could not form relations between different concepts associated with the $\frac{dy}{dx}$ symbol. For instance, while they were interpreting the derivative symbol in the expression $\frac{dy}{dx}=4$ as a gradient, no one interpreted $\frac{dy}{dx}$ as gradient in the equation $2x+\frac{dy}{dx}=1$ .Students generally interpreted the symbol as gradient, but some difficulties observed during the translations between different representations such as words to symbols or symbols to words. The studies of White and Mitchelmore (1996) and Santos and Thomas (2001) both indicated that students had difficulties in interpreting the symbolic representations of derivative by keeping all possible interpretations of them in mind.

As the previous studies indicated, performing procedural aspects of derivative does not mean understanding of the underlying ideas behind the symbolic expressions. For example, in the study of Habre and Abboud (2006), 61% of the 56 calculus students could not interpret geometrically or analytically the question that "*Why the derivative of* $g(x)=x^2+1$ *should also be* $2x$" (p.67) asked after providing the geometric explanation of derivative of $f(x)=x^2$ as $f'(x)=2x$.

A comprehensive framework was developed by Zandieh (2000) for explaining what it means to understand derivative. The framework had two main components that are multiple representations or contexts of the derivative (slope, rate, velocity, and difference quotient) and layers of process-object pairs (ratio, limit, function). According to Zandieh (2000), derivative can be represented graphically (slope, tangent line), verbally (instantaneous rate of change), physically (speed or velocity) and symbolically (limit of the difference quotient). These are the dimensions of the multiple representations (or contexts) component. On the other hand, the concept of derivative mathematically involves a ratio, a limit and a function which were accepted as forming the dimensions of the process-object pair component. The following expressions show understanding of difference quotient representation of derivative at ratio, limit, and function layers:

1) $\dfrac{f(x)-f(x_0)}{x-x_0}$ or $\dfrac{f(x_0+h)-f(x_0)}{h}$ (ratio-layer)

2) $f^{'}(x_0)=\lim\limits_{x\to x_0}\dfrac{f(x)-f(x_0)}{x-x_0}$ or $f^{'}(x_0)=\lim\limits_{h\to 0}\dfrac{f(x_0+h)-f(x_0)}{h}$ (limit-layer)

3) $f^{'}(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$ (function-layer)

In the first expression, the ratio between the difference in dependent variable and difference in the independent variable is obtained without a limiting process. In the second expression, the limiting process involves and the expression gives the value of derivative function at a particular point (x_0). In the function layer, the idea is possibility of applying the formula for infinitely many values of x. While only the derivative at a particular point is obtained in limit-layer understanding, the derivative as function is the central idea in function-layer understanding. Zandieh (2000) explained the meaning of understanding the other representations of derivative at ratio, limit, or function layers in detail. By using the framework, she investigated nine high school students' understanding of derivative who attends to the Advanced Placement Calculus class. The results of the study showed that the slope and rate interpretations were the most preferred representations that students used to explain derivative. In contrast, formal definition of derivative was less preferred and less integrated students' understanding. Although velocity was not preferred for explaining derivative, it was used by students as a reference representation for explaining contextual meaning of derivative. In addition, the results of the study showed that a student may explain derivative by demonstrating all three process-object layers in a particular representation (e.g., slope, velocity), but the same student cannot explain derivative again by demonstrating all process-object layers in another representation. According to Zandieh, it is difficult to mention about a complete understanding of derivative for a student, if he or she cannot think with each of ratio, limit and function processes at any representation of derivative. Also, understanding the derivative at a particular process level in one context does not mean understanding it at the same process level in another context. For conceptual understanding of derivative, Zandieh emphasized the necessity of understanding different representations of derivative by conceptualizing them at each of process-object layers.

At that point, limit seems one of the critical ideas in conceptual understanding of derivative (Orton, 1983; Tall, 1992; Hahkiöniemi, 2006). Orton (1983) and Zandieh (2000) pointed out that although students procedurally perform limit of difference quotient, they have many difficulties in conceptual understanding of limiting process in the derivative. Even if a student could define derivative, for example as the slope of tangent line, his or her understanding can be in the form of pseudo-structural conception which means lacking of the underlying reasoning processes behind it such as the limit of slopes of secant lines. Hahkiöniemi (2006), on the other hand, evidenced that students have the necessary ideas of limiting process, but they have difficulty in connecting their ideas with the formal mathematical representations.

Forming connections between different representations and different process-object layers have been indicated as being crucial for the conceptual understanding of derivative (Herbert & Pierce, 2012; Santos & Thomas, 2001; White & Mithcelmore, 1996; Zandieh, 2000). However, it involves many difficulties stemming either from the individuals or from the nature of concepts. This was indicated by Herbert and Pierce (2012) as "*Understandings of rate in one representation or context are not necessarily transferred to another representation or context*" (p. 455). Students' weaknesses in forming connections between different representations of derivative were observed in the study by Zandieh and Knapp (2006). The results showed that students may prefer contexts such as slope on graph or velocity for explaining derivative, but using a particular representation does not show their understanding in another representation. Students' use of average rate of change (ratio-layer) or expressions as "*derivative is instantaneous velocity*" does not represent their understanding of derivative concept as a whole.

Student difficulties in forming connections between different interpretations of derivative are apparent (Herbert & Pierce, 2012; Orton, 1983; White & Mithcelmore, 1996; Zandieh, 2000; Zandieh & Knapp, 2006). However, their preferences for thinking with a particular representation or their inability to form connections between other representations can be a result of the teaching orientations. In the study by Bingolbali, Monaghan and Roper (2007), it was observed that undergraduate students' understanding of derivative from different departments have different orientations, that is engineering students were thinking in terms of *rate of change* and mathematics students were thinking in terms of *tangents*. This was interpreted as the contextual aspect of concept image and concept definition by the researchers (Bingolbali & Monaghan, 2008). This study shows how different instructional orientations may result in an underdeveloped understanding of derivative. Different instructional orientations may be related with the textbooks or curriculum. In the following section, the literature about the way of introducing derivative in curricular documents was examined.

2.1.2 **Derivative in curricular documents**

The difficulties that many students have related to derivative concept may be because of its way of teaching (Bingolbali, 2008; Bingolbali et al., 2007). After examining the seven calculus textbooks, Berry and Neyman (2003) determined that derivative is generally introduced with the geometric interpretation of it, which is the slope of a tangent line obtained by the approximation of the secant lines. With the idea that "*secant lines approximate to a tangent line*", the formal definition of derivative with difference quotient and limiting process is introduced. Then, it is followed by the algorithmic rules of differentiation and applications of them in some real situations. Only a few textbooks started with the average and instantaneous rate of change by using the motion context (Berry & Neyman, 2003). Furthermore, Teuscher and Reys (2012) and Stroup (2002) pointed out that textbooks introduces the concepts of slope and rate of change in linear situations, and slope is not discussed during the introduction of other functions such as quadratics, polynomials and exponentials. According to Stroup (2002), this may be the reason of students' difficulties in understanding of derivative and rate of change in non-linear functions.

When looking at the textbooks and curriculum in Turkey, Bingolbali (2008) examined several high school and university level textbooks covering the derivative topic, and he determined that many of the textbooks start with the formal definition of derivative involving the difference quotient with limiting process. After introducing the formal definition of derivative and the general differentiation rules, the slope-derivative relationship was introduced under the part of geometrical interpretation of derivative. In the textbooks examined by Bingolbali (2008), rate of change interpretation of derivative was never or superficially mentioned. The speed and acceleration concepts were superficially mentioned as being the contextual interpretation of derivative. When I examined the current textbook used by Ministry of Education in Turkey (MEB), I observed the same path. In the "*Mathematics for Grade 12*" textbook written by Kaplan (2012), which is currently being used, the derivative concept is started with the example of finding the average speed. The instantaneous speed is mentioned as the physical interpretation of derivative. It is followed by the formal definition and geometric interpretation of derivative. After the introduction of the differentiation rules for taking the derivative of algebraic functions, applications of derivative are introduced with the max-min problems. The term "rate of change" is newer mentioned in the book, except the speed concept. Bingolbali (2008) also examined the objectives stated for the teaching of derivative in the national curriculum of Turkey (TTKB, 2005). He determined that the ideas of average speed and instantaneous speed from motion context have been used as an introduction of derivative, and it was followed by the slope interpretation. Only motion context was used for the idea of rate of change. However, it has been known from many studies that using only motion context did not support more general understanding of rate of change (Gravemeijer & Doorman, 1999; Yoon, Dreyfus & Thomas, 2010; Zandieh & Knapp, 2006). Bingolbali (2008) also criticized the curriculum in terms of using only the motion context for clarifying the rate of change of interpretation of derivative. According to him, rate of change is more inclusive concept and speed is only one interpretation of it. Bingolbali (2008) concluded that rate of change interpretation of derivative is the "lost ring" in national and international textbooks. When we look at the renewed curriculum of Turkey (TTKB, 2013), more emphasis on the concept of rate of change is observable. The derivative concept is first introduced with the learning objective that "*the rate of change concept is explained benefiting from physical and geometric models*" (TTKB, 2013, p.46). In addition, rate of change is not limited to speed and acceleration concepts. According to new curriculum (TTKB, 2013), after the rate of change interpretation of derivative is given, it is followed by the slope of tangent line and the formal definition of derivative.

In summary, derivative is generally introduced by its formal definition and geometric interpretation of it and followed by the rules of differentiation (Berry & Nyman, 2003; Habre & Abboud, 2006). In the textbooks, rate of change interpretation of derivative is generally ignored or it is only given with the Physics concepts (Bingolbali, 2008). In the following section, the research studies specifically focusing on the big ideas involved in derivative, which are covariational reasoning, rate of change, and graphical connections between a function and its derivative were examined.

2.2 The Concept of Covariation

The concept of covariation has been introduced as being a foundational idea for understanding of functions, rate of change and derivative. The origins of the concept of covariation in mathematics education literature emerged within the studies on functions and rate (of change) concepts (Confrey & Smith, 1994; Confrey & Smith, 1995; Monk, 1992; Thompson, 1994a; Zandieh, 2000). The concept of covariation means the coordination of changes in two functionally related variables. In earlier studies, different terminologies were used in place of covariation such as across-time view of function (Monk, 1992), co-variation (Confrey & Smith; 1994, 1995; Thompson & Thompson; 1992), and function as process (Zandieh, 2000). In later years, the concept of covariation appeared as an independent notion and it has been introduced as an effective way of reasoning for comprehending functions and calculus concepts (Carlson et al., 2002; Cooney, Beckman & Lloyd, 2010; Saldanha & Thomson, 1998; Zandieh, 2000).

While studying on a group of calculus students' understanding of functions given with a dynamical physical model (i.e., Sliding Ladder problem), Monk (1992) determined two ways of thinking that are pointwise and across-time views of functions. While the pointwise view of function indicates focusing on the particular input and output values as pairs, the *across-time* view indicates concentrating on the patterns of change in the value of a function that result from a pattern of change in the values of the input variable. It is important to notice here that *across-time* view of function indicates being able to focus on the covariation between dependent and independent variables. Monk (1992) evidenced in his study that, students who thought with pointwise had difficulties in successfully preforming the given task and in translating the information obtained within the physical model to the graphical representation. He also evidenced that students having across-time view of function were more successful in performing the sliding ladder task and in translating their ideas to the graphical form. Monk pointed out the need for developing students' across-time view of function and offered the usage of dynamic and physical models as an effective way for this.

Thompson and Thompson (1992) first mentioned about *covariation* as a mental operation while explaining the four levels of students' conceptions of rate. In explaining the *internalized ratio* level of rate scheme, they mentioned the mental operation that "*construction of co-varying accumulations of quantities where the accrual of quantities occurs additively* (p.7)". The term covariation has been used by Thompson in the same way in the following years without providing a formal definition (Thompson, 1994a; 1994b; 1995). Concurrently with the studies of Thompson (1994; 1995), Confrey and Smith (1994; 1995) also mentioned and explicitly used the term *covariation* as a way of approaching the concept of function. In their studies on exponential functions, Confrey and Smith (1994; 1995) realized the coordination of the increments in the changes of two functionally related variables as being more powerful way for describing the functional relationship and called it as covariation approach. According to Confrey and Smith (1995), the correspondence approach by which a function is described via a rule generally leads to an overreliance on algebraic representations. Instead, they offered the covariation approach as an alternative way for conceptualizing the function concept. The covariation approach for a function was described by Confrey and Smith (1994) as "*moving operationally from* y_m *to* y_{m+1} *coordinating with movement from* x_m *to* x_{m+1}(p.33)" by which a function has been understood as the juxtaposition of two data sequences. The covariation approach emphasizes the coordination of the changes in the independent and dependent variables which also promote the idea of rate of change. Therefore, Confrey and Smith (1994) see covariation approach as being central for conceptual understanding of the rate of change concept. According to them, the power of covariation approach lies in its emphasis for the usage of rate of change as an entry, because moving between successive values of one variable and coordinating this with moving between corresponding successive values of another variable entails consideration of rate of change.

By specifically focusing on the concept of covariation, Saldanha and Thompson (1998) examined it from quantitative reasoning perspective and provided an explanation that "*Covariation is of someone holding in mind a sustained image of two quantities' values simultaneously. It entails the coupling of two quantities, so that a multiplicative object is formed of the two*" (p.298). Saldanha and Thompson (1998) also emphasized the developmental nature of thinking on covariation evolving from coordination of two quantities for some discrete values to images of continuous coordination of two quantities. By the continuous covariation, it is indicated that "*one understands that if either quantity has different values at different times, it changed from one to another by assuming all intermediate values*" (Saldanha & Thompson, 1998, p.299). Thompson modified his explanation of the covariation by adding the phrase of "conceptual time" and by emphasizing the dynamic nature of covariation (Thompson, 2008; 2011). He emphasized that any conception of variation also entails covariation, because when the magnitude of a quantity vary, there is always a tacit variable whose values varies in tandem. According to Thompson (2011), when a quantity's magnitude vary, the variation always happens over an interval of conceptual time that can be represented as $x_\varepsilon = x(t_\varepsilon)$ where t_ε represents the interval $[t, t+\varepsilon)$. By the conceptual time, not the time on clock, but smoothly changing, the variation of a quantity with infinitesimal amounts, and variation within those amounts as well are indicated. Therefore, he represented the covariation symbolically as $(x_\varepsilon, y_\varepsilon) = (x(t_\varepsilon), y(t_\varepsilon))$ for indicating the simultaneous variation in two quantities. In both studies (Saldanha & Thompson, 1998; Thompson, 2008), simultaneous continuous nature of covariation has been emphasized in response to the discrete nature of explanation provided by Confrey and Smith (1994). Comparing the explanations for covariation, whereas Confrey and Smith (1994) consider the covariation discretely as the coordination of the successive changes in two variables by benefiting from the tabular data, Saldanha and Thompson (1998) emphasizes the simultaneous continuous variation.

Zandieh's (2000) consideration of the concept of covariation appears in the explanation of the theoretical framework that she developed for the concept of derivative using the process-objet views of functions explained by Sfard (1991). According to Zandieh (2000), while the action conception of function emphasizes the correspondence between input and output variables, the process conception of function entails consideration of the covariation. In other words, having a process view of function indicates the ability of coordinating the patterns of changes in two dynamically changing quantities which has been called as covariational reasoning ability (Carlson et. al., 2002). It can be said that process view of function entails the ability of covariational reasoning. For developing the covariational reasoning ability and building a process view of function, the usage of learning tools involving dynamically changing real life situations have been proposed by many researchers (Carlson et al., 2002; Monk, 1992; Oehrtman, Carlson & Thompson, 2008).

2.2.1 **Covariational Reasoning**

As an extension of the earlier studies on the concept of covariation, Carlson et al. (2002) developed a framework for describing and characterizing students' ways of reasoning about the situations involving dynamically changing quantities. They defined covariational reasoning as "*the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other*" (p.354). A student's covariational reasoning ability relative to a particular task has been decided by looking at the collection of mental actions appeared in his/her responses. Five mental actions, their descriptions, and the related behaviors are represented and explained on the Table 1. Regarding to the appearance of these mental actions in one's reasoning, Carlson et al. (2002, p.357) determined five distinct levels of covariational reasoning which are: (i) Coordination (MA1 is supported), (ii) Direction (MA1 and MA2 are supported), (iii) Quantitative Coordination (MA1, MA2, and MA3 are supported), (iv) Average Rate (MA1 through MA4 are supported), and (v) Instantaneous Rate (MA1 through MA5 are supported).

Table 1: Mental Actions of the Covariation Framework (Carlson et. al., 2002, p.357)

Mental Action	Description of Mental Action	Behaviors
Mental Action 1 (MA1)	Coordinating the dependence of one variable on another variable	• Labeling the axes with verbal indications of coordinating the two variables (e.g., y changes with changes in x)
Mental Action 2 (MA2)	Coordinating the direction of change of one variable with changes in the other variable	• Constructing a monotonic straight line • Verbalizing an awareness of the direction of change of the output while considering changes in the input
Mental Action 3 (MA3)	Coordinating the amount of change of one variable with changes in the other variable	• Plotting points/constructing secant lines • Verbalizing an awareness of the amount of change of the output while considering changes in the input
Mental Action 4 (MA4)	Coordinating the average rate of change of the function with uniform increments of change in the input variable	• Constructing secant lines for contiguous intervals in the domain • Verbalizing an awareness of the rate of change of the output (with respect to the input) while considering uniform increments of the input
Mental Action 5 (MA5)	Coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function	• Constructing a smooth curve with clear indications of concavity changes • Verbalizing an awareness of the instantaneous changes in the rate-of change for the entire domain of the function (direction of concavities and inflection points are correct)

When a student's covariational reasoning ability is determined at a particular level, it means that the student demonstrated the behaviors supporting the mental actions associated with that level and the actions associated with all lower levels. Carlson et al.'s (2002) framework points out the evolving nature of covariational reasoning in a way that whereas a person initially focuses on only discrete changes, later he/she becomes aware of continuity of changes. However, the framework does not explain clearly how to prescribe students' images of continuously changing rate over the entire domain. Castillo-Garsow (2010, 2012) provided an extension for covariational reasoning by introducing the notion of continuous quantitative reasoning. Castillo-Garsow (2012) tried to explain the mechanics of covariational reasoning in a continuously changing functional situation. She determined two types in students' thinking about the change which are "chunky" and "smooth" thinking. While chunky thinking is discrete in nature, smooth thinking is continuous. For example, let us consider a moving car with a speed of 80 kilometer per hour for a period of time. Thinking the change in distance at only exact times (e.g., after two hours the distance is 160 km) without considering the bits of time period and the distance at every moment is an example of chunky thinking. Thinking by imagining the every moment of time period and being aware of the corresponding changes of distance with a rate of 80 kilometer per hour may be an example of smooth thinking. Castillo-Garsow (2012) determined that smooth thinking is more critical and important for students to comprehend non-linear and exponential functional situations and ideas of calculus.

2.2.2 Research on covariational reasoning

By using the aforementioned framework, Carlson et al. (2002) investigated twenty high-performing 2nd-semester calculus students' ability to reason about covarying quantities. Although all the participants were high-performing in calculus course, they had difficulty in interpreting and representing images of continuously changing rate while analyzing the dynamically changing situations. They were able to apply quantitative coordination with MA3 consistently. However, they were unable to coordinate the average rate of change with fixed changes in the independent variable (MA4) and the instantaneous rate of change with continuous change in the independent variable (MA5). Only small percent of the students constructed the true graph for showing the height as a function of the volume. The researchers offered development of curricular materials involving dynamically and simultaneously changing situations for developing students' covariational reasoning abilities. They also mentioned about some weaknesses of the framework that they developed. The weaknesses of the frame were inadequacy for considering the implicit time variable and the need for a more finely grained way of analysis of the reasoning on instantaneous rate of change. They call for further studies on the revision and extension of the framework.

In a following study, Carlson, Larsen and Lesh (2003) also investigated the covariational reasoning of pre-service elementary teachers by converting the filling bottle problem to a model-eliciting activity. Covariational reasoning framework was used for analyzing students' reasoning. Some developments in students' covariational reasoning abilities were observed. However, students thought with the cross-sectional area as the independent variable and they used some other justifications during the group discussion that could not be analyzed by the framework. The general results obtained in this study were; (i) treating height as the input variable, (ii) considering time as the input variable, and (iii) students did not describe their way of drawing a smooth curve by plotting points or by connecting line segments. These findings also appeared in a recent study conducted by Zeytun, Çetinkaya and Erbaş (2010). Zeytun et al. (2010), in their study, investigated five in-service secondary mathematics teachers' covariational reasoning levels and their predictions about students' abilities. Teachers studied on a modeling activity asking them to draw a volume-height graph for a filling-bottle problem. The results showed that teachers have many difficulties related to covariational reasoning. Some of difficulties appeared in teachers' ways of reasoning were considering "time" as input variable, thinking as if the flow rate of water will affect the shape of the curve, thinking input and output variables in reverse order, and the difficulty in interpreting the relationship between varying rate of change and the concavity of graphs. Their predictions about students' ways of reasoning were concurrent with their own ways of thinking. This study evidenced that not only students have difficulties with covariational reasoning but also teachers demonstrate the same difficulties. In addition, the findings of both studies (Carlson et al., 2003; Zeytun et al., 2010) pointed out the weakness of the Covariational Reasoning framework in terms of comprehensively characterizing different ways of reasoning. Carlson et al. (2003) realized students' expressions of mental actions such as considering time as an input variable and thinking with the cross-sections differed qualitatively from the descriptions provided within the framework, and they indicated the need for a refinement. In a study on the role of covariational reasoning in learning and understanding exponential functions, Strom (2006) also determined the necessity for a refinement in this frame.

Putting all together, student difficulties in conceptual understanding of derivative are rooted in their weak understandings of prior concepts such as functional relationship. Students' weak understanding of the function concept is a common finding in many studies (Carlson, 1998; Confrey & Smith, 1994; Monk, 1992; Tall & Vinner, 1981; Tall, 1996; Thompson, 1994a). Even successful students have static and undeveloped images of continuous and dynamic variation (Monk, 1992). This was attributed to overreliance on the use of correspondence approach in the teaching functions (Confrey & Smith, 1994; Hoffkamp, 2011; Tall, 1996; Thompson, 1994a). Using the covariation approach in teaching functions and developing students' covariational reasoning abilities have been suggested for overcoming student difficulties (Confrey & Smith, 1995; Monk, 1992; Carlson et. al., 2002; Oehrtman, Carlson & Thompson, 2008; Thompson & Saldanha, 1998; Zandieh, 2000). The covariation approach has been expected to develop students' reasoning abilities from two aspects: (i) fostering to focus on the coordination of changes in simultaneously changing quantities, and (ii) supporting the idea of rate of change as an entry (Confrey & Smith, 1994). Additionally, covariational reasoning ability have been shown to be essential for comprehending many of calculus concepts such as functions, rate of change, and derivative and (Carlson et. al., 2002; Castillo-Garsow, 2011; Confrey & Smith, 1994; Monk, 1992; Saldanha & Thompson, 1998). Recently, the covariation approach in the teaching functions started to find place in curricular documents as well. In a book prepared by NCTM for the essential understandings for the concept of function, the covariation approach is highly emphasized not only for teaching the concept of function, but also increasing students' and teachers' awareness for families of functions as they engaged in thinking with rate of change (Cooney, Beckman & Lloyd, 2010). Therefore, the first theoretical principle that should be considered in the teaching of derivative was the following:

Principle 1: Covariational reasoning ability is a critical prerequisite in conceptualizing functional dependency, rate, rate of change, and so derivative. For developing covariational reasoning ability, dynamically changing real situations involving two simultaneously changing quantities can be utilized. When students worked on such situations, they can develop important ideas such as dynamic image of functional dependency, and intuitive understanding of rate of change and derivative (Carlson et al., 2002; Confrey & Smith, 1994; Monk, 1992).

2.3 Rate of Change

Rate of change is a more inclusive interpretation of derivative and it is fundamental for conceptual understanding of foundational calculus concepts (Confrey & Smith, 1994; Thompson, 1994a). As pointed out in many studies, students from various grade levels have difficulties with interpreting the derivative in real contexts (Bezuidenhout, 1998; Orton, 1983; Santos & Thomas, 2001, 2003; Tall, 1992; White & Mitchelmore, 1996; Zandieh & Knapp, 2006). In other words, rate of change is the least understood interpretation of derivative. Additionally, it is less stressed in curricular documents (Bingolbali, 2008; Herbert & Pierce, 2012b; Teuscher & Reys, 2012). More comprehensive research studies have been conducted specifically focusing on student understanding of rate of change (Bezuidenhout, 1998; Coe, 2007; Confrey & Smith, 1994, 1995; Herbert & Pierce, 2008; Teuscher & Reys, 2012; Thompson, 1994a, 1994b; Wilhelm & Confrey, 2004). In this section, the concept of rate of change was explained first and then an overview of the research studies were provided.

2.3.1 **Conceptualizing rate of change**

The conceptual explanations for the rate of change concept that appears in the literature were explained here. The studies by Thompson (1994a; 1994b), Confrey and Smith (1994; 1995), and Stroup (2002) have shed light on the meaning of rate of change by discussing conventional and some possible different understandings of it from a pedagogical point of view.

To begin with, Thompson and Thompson (1992), and Thompson (1994a) described the concept of rate of change as a quantitative multiplicative relationship comparing the changes in two functionally related quantities and according to them, it is closely related with the concepts of ratio and proportion. A mature image of rate is required in order to be able to understand the rate of change. Thompson and Thompson (1992) identified four levels for describing the developments of someone's ratio/rate schemes which are ratio, internalized ratio, interiorized ratio, and rate. They described each level as follows:

> The first level, ratio, is characterized by children's comparison of two taken-as-unchanging quantities according to the criterion "as many times as". The second level, internalized ratio, is characterized by children's construction of co-varying accumulations of quantities, where the accrual of the quantities occurs additively, but there is no conceptual relationship between ratio of accumulated quantities at iteration x and the ratio of accumulated quantities at iteration x+1. The third level, interiorized ratio, is characterized by children's construction of co-varying amounts of quantities, where the amounts vary additively but with the anticipation that the ratio of the accumulations does not change. The fourth level, rate, is characterized by children's conception of constant ratio variation as being a single quantity–the multiplicative variation of a pair of quantities as a measure of a single attribute. A rate is a reflectively-abstracted conception of constant ratio variation. (p.7, [electronic])

The rate and ratio concepts are both characterized as a quantity obtained as the result of multiplicative comparison of two quantities (Thompson, 1994b). However, the difference between rate and ratio according to Thompson and Thompson (1992) is that while ratio is characterized as a quantity obtained as the result of multiplicative comparison of two non-varying quantities; rate involves the multiplicative comparison of two dynamically varying quantities. The mental schemes staying in between these two concepts are indicated as internalized ratio and interiorized ratio. Rate is defined as "*reflectively abstracted constant ratio*" (p.18). Consideration of two simultaneously and dynamically covarying quantities, so that their measures remain in constant ratio, was identified as the mature image of rate (Thompson, 1994b). In addition, for mature image of rate, it is required a "*schematic coordination of relationships among accumulations of two quantities and accruals by which the accumulations are constructed*" (Thompson, 1994a, p.232). The mature image of rate also encapsulates the rate of change concept. The expression "*constant ratio*" indicates linearity of the rate, however for interpreting the varying rate of change in nonlinear situations; this linearity or constancy of ratio should be understood within infinitesimal intervals. According to Thompson (1994b), rate of change is an intensive quantity and it can be quantified in two different ways that are gross quantification and extensive quantification. Gross quantification of rate of change in different situations involves some perceptual and experiential criteria, whereas the extensive quantification involves a ratio-based conception of it as a multiplicative relationship between two simultaneously changing quantities. While perceiving speed directly as "seems faster" is an example of gross quantification, perceiving it as a new quantity resulted from the ratio between distance and time and stating it with the measured values as 50 kilometers per hour involves an extensive quantification.

On the other hand, Confrey and Smith (1994) pointed out the changing nature of rate of change concept in different functional situations. They investigated students' way of thinking on rate of change in exponential situations. Students were worked on a lab experiment analyzing the number of growing cells recorded for every hour. A two-column tabular data was provided to the students, the first column was showing the time values changing with 1 unit increments, and the second column was showing the number of cells increasing by the powers of 9. They determined three possible ways of interpreting rate of change that were (i) *additive rate of change*, (ii) *multiplicative rate of change*, (iii) and *proportion new to old rate of change*. Additive rate of change was obtained as the ratio between differences in succeeding values of cell population and difference in time. This was the conventional understanding of rate of change. In this lab context, the additive rate of change varies and it gradually increases which make it difficult to interpret. Multiplicative rate of change is obtained by taking the ratios between succeeding cell values for every unit change in time values. By this interpretation, rate of change of cell grow is constant and it is increasing by a constant rate of nine. Most of the students have preferred multiplicative interpretation of rate in this context. Proportion new to old rate of change is demonstrated with the explanation that the pattern on the table "*increases by eight times*" by which the initial amount and the amount being added on each time is considered. The last interpretation of rate of change involves both additive and multiplicative interpretations of rate.

By observing these three ways of thinking, Confrey and Smith (1994) realized the weakness of conventional understanding of rate of change. They described rate of change as "*a unit per unit comparison*" of simultaneously changing quantities where unit is defined as an "*invariant relationship between a successor and its predecessor*" (p.135). For instance, taking the difference between succeeding values of time yield 1 and this is an invariant additive unit. In the same way, taking the difference between succeeding values of cells is varying and it is also an additive unit. Taking the ratio between two differences is the additive rate of change. On the other hand, the ratio between succeeding values of cells is 9 and this ratio is an invariant multiplicative unit. Namely, unit can be additive or multiplicative and one can prefer to use one of them according to the situation. Therefore, "*unit per unit comparison*" definition encapsulates both of additive and multiplicative interpretations of rate of change.

"*Unit per unit comparison*" is the analytical understanding of rate of change where a ratio based reasoning involved. But according to Confrey and Smith (1994), there are some other interpretations of derivative and rate of change as powerful as the analytical one. The first one is named as *comparative dimension* by which experiential understanding of rate of change involves. One can intuitively realize a changing speed or compare the speeds of two moving cars non-numerically without needing a ratio or unit per unit approach. Another understanding of rate of change is the slope on graph. Slope on graph supports students' understanding of varying rate that can be also used in earlier grades. According to Confrey and Smith (1994), being able to coordinate between unit per unit, slope, and comparative dimension is necessary for robust conception of rate of change as represented in the figure below.

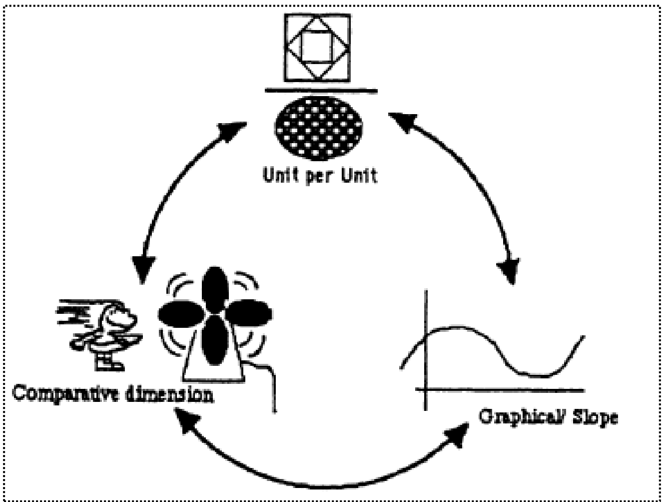


Figure 1: Rate as a coordination of multiple concepts (Confrey & Smith, 1994, p.54)

Other than the formal definitions, Stroup (2002) gave importance to the qualitative understanding of rate of change (how fast) which means intuitively experiencing and perceiving it in real situations. He indicated this with the notion of "qualitative calculus". Qualitative understanding of calculus involves ability to discriminate, and coordinate "how much" and "how fast" quantities represented on graphs. A student could make inferences about the speed of a moving car by looking at the distance-time graph and vice versa without needing a ratio-based conception of it. Although, it resonates with the *gross quantification* indicated by Thompson (1994b) and the *comparative dimension of rate of change* indicated by Confrey and Smith (1994), according to Stroup (2002) the notion of *qualitative calculus* is more inclusive and foundational, because rate, ratio and other formal mathematical concepts are only the mathematical models of ideas appearing in real situations. More details about the perspectives of Stroup (2002) and Thompson (1994b) were provided under conceptual framework section.

Concluding from the literature, rate of change can be seen in the form of a directly experienced quantity as “fastness”, a quantitative operation producing a new quantity resulting from the multiplicative comparison of two quantities, or a formal mathematical interpretation (e.g., slope, quotient of difference). Three of these views are incorporated in the descriptions of rate of change and the ability to form coordination between them have been identified as being critical for the robust conception of it (Confrey & Smith, 1995; Stroup, 2002; Thompson, 1994a).

2.3.2 Common difficulties and misconceptions with regard to rate of change

Rate of change is the more inclusive conceptualization of derivative which is used in interpreting derivative in real situations. However, it is the least understood interpretation of derivative and it is less stressed in textbooks. Research studies determined various difficulties and misconceptions that students have in conceptualizing rate of change such as confusing it with the arithmetic mean (Bezuidenhout, 1998), difficulty in interpreting the meaning of it (Orton, 1983; White & Mitchelmore, 1996), knowledge about it limited with the motion context (Herbert & Pierce, 2008, 2012), and difficulty in interpreting it in non-linear situations (Teuscher & Reys, 2012).

As mentioned previously, Orton (1983) determined high school students’ and pre-service teachers’ weak understanding of rate of change. He evidenced students’ difficulties in relating rate of change with difference quotient and slope. In the study by White and Mitchelmore, (1996), students’ difficulties in interpreting the symbols used for variables and their weak understanding of the contextual meaning of derivative were reported. In a study specifically focusing on rate of change, Bezuidenhout (1998) investigated first year university students’ conceptions of rate of change. 523 students from three universities in South Africa attending to service calculus courses were the participants of the study.

Question 6

The following table shows certain x-values and the corresponding function values of two continuous functions S and g which is such that $g'(x) = S(x)$:

x	-1	-0.5	0	0.5	1	1.5	2	3
$S(x)$	4	0.75	-1	-1.25	0	2.75	7	20
$g(x)$	0	1.125	1	0.375	0	0.625	3	16

Find (show all calculations):

6.1. the average rate of change of $g(x)$ with respect to x over the interval $[0, 3]$;
6.2. $\int_0^3 S(x)\,dx$;
6.3. the average value of $S(x)$ for $0 \leqslant x \leqslant 3$.

Figure 2: One of the questions used in the study of Bezuidenhout (1998, p.398)

He used Tall and Vinner's (1981) concept image perspective as the conceptual framework. The data revealed that the majority of students demonstrated misunderstandings or confusions between the ideas of "average rate of change", "average value of a continuous function", and "arithmetic mean". As an example, in the first part of the question asking average rate of change of a function (see, Figure 2), the majority of students tried to find the average rate of change of $g(x)$ by using the values of $g'(x)$. They calculated the average rate of change $g(x)$ by adding the values of derivative function ($g'(x)$) at the points in the given interval and dividing the result by the number of values (i.e. $\frac{g'(0)+g'(0.5)+g'(1)+g'(1.5)+g'(2)+g'(3)}{6}$). Students used some other irrelevant procedures such as summing the derivatives at the two endpoints and dividing by two, computing the mean of the values of $g(x)$ function, and finding the difference between the derivative values at the end points. These results clearly demonstrated that the majority of students consider average rate of change as if it was the arithmetic mean of rate function which was a misconception. In addition, students could not give meaning to rate of change in non-motion contexts such as distance versus velocity. They demonstrated a dominant image that "*derivative of distance is velocity*", but they could not interpret the rate of change in other contexts such as distance as a function of velocity. Bezuidenhout (1998) interpreted these results as students' insufficient images of covariation and he has taken the attention of researchers for finding innovative pedagogical approaches.

Recently, Teuscher and Reys (2012) examined high school advanced placement calculus students' understandings and misconceptions of rate of change coming from colleges using different mathematics curriculum approaches (i.e. integrated or single-subject). One of the groups completed their four year college preparatory with Core-Plus (integrated) mathematics curriculum, and the other group completed with a single-subject (having a sequence of algebra I, geometry, algebra II, and pre-calculus) curriculum. Two standardized achievement tests and two open-ended tasks were used as research instruments. The results of standardized tests, and open-ended tasks showed that there was not a significant mean score differences between students from both groups, and they had common difficulties and misunderstandings. Students from both group had common difficulties in interpreting the contextual meaning of rate of change and calculating it. Additionally, students from both groups had difficulties in representing and interpreting rate of change in non-linear functional situations. According to Stroup (2002), this is an expected result, because rate of change is generally introduced with linear functions. Teuscher and Reys (2012) pointed out that students from both groups have incomplete understanding of slope, rate of change, steepness and the relationship among them. In the study by Coe (2007), similar difficulties and weaknesses related to the conceptual understanding of rate of change have been reported for in-service mathematics teachers.

Young students may coordinate the changes in two covarying quantities involving rate of change without using a ratio-based reasoning. Johnson (2012) investigated a tenth grade secondary student's way of quantitative reasoning (numerical and non-numerical) process related to rate of change by using tasks involving multiple representations of covarying quantities. The student had not taken a calculus course or she was not instructed on calculus concepts. Based on the student's work across different tasks, the researcher characterized her way of reasoning about rate of change as follows: "*systematically varying one quantity and simultaneously attending to variation in the intensity of change in a quantity indicating a relationship between covarying quantities*" (Johnson, 2012, p.327). This study showed that a student can explain the nature of covariational relationship and rate of change by systematically varying one quantity (independent variable) and the simultaneous variation in the intensity of change in other quantity (dependent variable) without needing ratio-based reasoning, limit, and function. The way of student’s reasoning on rate of change characterized by Johnson (2012) is consistent with the ideas indicated by Stroup (2002) and Zandieh and Knapp (2006). That is, to be able determine the variation in rate of change of covarying situations; students do not always need ratio-based understanding which involves ratio, limit, and function.

In addition, the contextual meaning of derivative and rate of change is generally confused with the amount of change in the output variable (Rowland & Javanoski, 2004; Thompson, 1994a, 1994b; Thompson & Thompson, 1992; Zandieh & Knapp, 2006). In a study conducted by Rowland and Javanoski (2004), 59 first year undergraduate calculus students’ interpretation of the derivative in ordinary differential equations in modeling contexts were investigated. In their interpretations of the terms in differential equations, most of the students thought with “amount” in place of “rate of change of amount”. For instance, students interpreted the algebraic expression $\frac{dD}{dt}$ with the verbal expressions as such *“…is the amount of drug*” or *“…represents how much the amount of drug changes due time*” (Rowland & Javanoski, 2004, p.510). In a question, it was asked students to determine the algebraic expression of the growth in fish population that grows at a rate proportional to the population as a function of time, and where 5000 fish per year was removed from the pond for sale. Most of the students selected the algebraic expression $\frac{dP}{dt} = 500t$ instead of $\frac{dP}{dt} = 500P$. Students thought with the amount of change in the population, not with the rate of change, because $500t$ was giving the amount of fish removed after t years. This was also evidenced in the study by Zandieh and Knapp (2006). For the question asking the meaning of derivative, some of calculus students provided the explanation that “*derivative is a change*” (p.12). Similarly, in the studies by Thompson (1994b) and Thompson and Thompson (1992), young students considered speed not as a rate of change of distance with respect to time; rather they saw it as a distance to measure other distances. The study by Thompson (1994a) also revealed calculus students’ confusion between “amount of change” and “rate of change”. And lastly, in the study of Bezuidenhout (1998), calculus students were asked to interpret the algebraic expression $S'(80) = 1.15$ where S shows the stopping distance of a car after applying the brakes as a function of velocity, and some of the students interpreted 1.15 as the velocity, change in the velocity, deceleration, or amount of time to cover a distance of 80 km. All these studies show that students from various grades levels may have the misconception that "*rate of change (derivative) is the amount of change in the dependent variable*".

Another issue that have been reported by many research studies is related to frequent usage of motion context and the Physics concepts (i.e., speed, acceleration) for explaining the contextual meaning of derivative. Wilhelm and Confrey (2003) pointed out the little published research investigating rate of change outside the motion context. Students' conceptions of rate of change have been generally rooted in motion context (distance-time and speed-time) and they have difficulties in transferring these concepts into non-motion contexts (Gravemeijer & Doorman, 1999; Herbert & Pierce, 2008; Wilhelm & Confrey, 2004; Yoon, Dreyfus & Thomas., 2010). Motion context has been frequently used in mathematics instructions. Students are familiar with the concepts of kinematics from earlier grades. Students who have taken a calculus course know that the derivative of distance-time function is speed-time function. This familiarity may prevent students to think about in-depth meanings of speed and acceleration concepts. In their study with 10^{th} grade students, Herbert and Pierce (2012) observed that students may see speed "*as a single entity with little emphasis on the covariance of the variables of time and distance*" (p.476). Although motion context is the best context for the development of calculus concepts throughout the history, it is not cognitively demanding for them to think about the rate of change as the simultaneous covariation between distance and time.

Students from different grade levels have many difficulties and misconceptions with the concept of rate of change. The common difficulties and misconceptions are summarized below.

- Difficulty in giving meaning to the rate of change term (e.g., conceiving the term "average" as if it is the arithmetic mean) (Bezuidenhout, 1998; Orton, 2013;White & Mitchelmore, 1996)
- Knowledge of rate of change is limited with the motion context and students have difficulties in interpreting it in non-motion contexts (Herbert & Pierce, 2008, 2012; Wilhelm & Confrey, 2003; Gravemeijer & Doorman, 1999; Zandieh & Knapp, 2006)

- Lack of qualitative understanding of rate of change in real life situations (Stroup, 2002)
- Explaining rate of change as the amount of change in the dependent variable (Rowland & Javanoski, 2004; Thompson, 1994a, 1994b; Zandieh & Knapp, 2006)
- Difficulty in interpreting rate of change in non-linear situations (Teuscher & Reys, 2012)
- Difficulty in forming connections between rate of change, slope and difference quotient (Herbert & Pierce, 2008; Zandieh, 2000)

By the review of literature about the teaching of derivative and rate of change, we deduced the following argument as being the second principle that should be considered in the teaching of derivative.

Principle 2: Rate of change is more inclusive interpretation of derivative. However, it is generally underestimated or introduced only limited with the motion context which results in many difficulties. The concept of "rate of change" should be specifically focused on within non-motion real life contexts for conceptual understanding of derivative.

2.3.3 **Design-based or intervention studies on the teaching of rate of change**

Recently, some intervention studies using design research perspectives on the teaching of rate of change have reported various effective experiences (Arleback, Doerr & O'Neil, 2013; Doorman & Gravemeijer, 2008; Doerr & O'Neil, 2012; Herbert & Pierce, 2008; Hoffkamp, 2011). Doorman and Gravemeijer (2008) demonstrated young students' developing models of calculus ideas as they worked on the discrete total distance and displacement graphs. This was also observed in the study by Hoffkamp (2011) while students were studying on an interactive activity. In both studies, students developed mathematical sense for qualitative aspect of rate of change and they formed connections between amount and rate graphs. In the study by Herbert and Pierce (2012), secondary grade students clarified the meaning of speed as a rate of change in distance with respect to time, but their difficulty in transferring the model of rate of change that they developed in the motion context to the non-motion contexts continued. The results of these studies are concurrent with the qualitative calculus notion introduced by Stroup (2002) in terms of evidencing the possibility of developing students' informal and intuitive understanding of the relationship between rate and amount ideas. The intervention studies mentioned here generally used contextual or interactive activities for developing students' calculus related ideas of change, accumulation of change, and rate of change.

In their study, Gravemeijer and Doorman (1999) had introduced the discrete graphs of total distance and displacement as being the foundational idea in the development of standard calculus concepts of amount of change, rate of change, and accumulation of change. Depending on the same argument, Doorman and Gravemeijer (2008) examined 10th grade students' developing ideas of change, velocity and their graphical representations in a teaching experiment designed according to emergent modeling perspective of Realistic Mathematics Education (RME). They designed an instructional sequence including real life situations (i.e., tropical storm approaching to the coast). During the process of teaching experiment, by the use of guided reinvention, students have been directed to construct mathematical models for describing changing phenomena while studying on the discrete graphs of changes by using the Flash program. While working on the discrete total distance graph and displacement graph, students formed connections between two graphs. They also developed the sense that displacement graphs stands for velocity. After the teaching experiment, most of the students demonstrated the ability of using the difference quotient idea of rate of change, and they were able to interpret rate of change on graph by considering the dimensions of variables. Also, a change in students' use of language was observed from tentative to more formal.

In a recent study, Hoffkamp (2011) investigated 10th grade students' reasoning on calculus concepts as they worked on interactive learning activities in pairs by using dynamic geometry software. The students were familiar with the families of functions like linear, quadratic, exponential and trigonometric, but they had not been introduced the derivative, integration, and curve sketching routines yet. The researcher designed interactive activities as one is seen in the Figure 3 below by which change in the shaded area of a triangle is analyzed as a function of the distance of point D from point A. By this activity, researcher aimed at not only developing students' understanding of calculus ideas, but also developing a dynamic view of functional dependency which involves the comprehension of the covariation between simultaneously changing quantities. While the graph was showing the stock (or amount) of area as a function of the distance from point A, the dark shaded triangle was showing the rate of change in area.

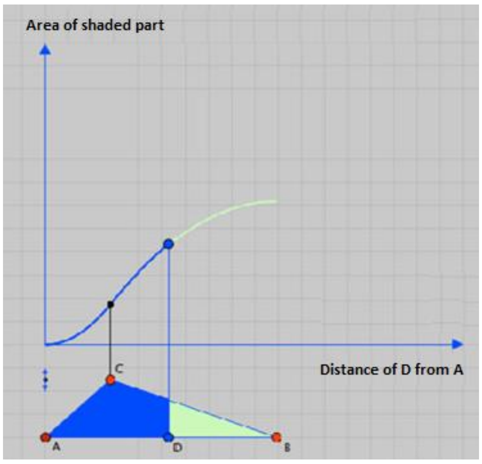


Figure 3: Screenshot of the interactive learning activity analyzing area of triangle as a function of distance of point D from point A (Hoffkamp, 2011, p.364)

While students were working on the task, they realized the increase in the shaded area and also the change in that increase, but they had difficulties in explaining their ideas. On the graph, they tried to explain the variation in the curvature with the terms as "slope decreases" or "slope increases". Some of the students indicated the difficulty of speaking about the slope on such a curved graph. Hoffkamp (2011) indicated that "*verbalization forced the students to negotiate the mathematical meaning*" (p.370). During this study, students' understanding of the functional dependency considerably changed towards a dynamic view. Hoffkamp (2011) argued that the activity seen in the figure above can serve students as a basis for understanding the fundamental theorem of calculus.

In the same way, Doerr and O'Neil (2012) and Arleback, Doerr and O'Neil (2013a) reported similar findings as they investigated a group of high school students' understanding of rate of change by a design-based research methodology. In both studies, a model development sequence has been designed aiming at developing students' understanding of average rate of change. Doerr and O'Neil (2012) reported that after the implementation of the model development sequence, there was a significant improvement in students' understanding of average rate of change. Doerr and O'Neil (2012) also identified some language difficulties of students while interpreting the average rates of change over three successive subintervals of an exponentially decaying function. Students obtained three negative signed values of average rate of change which were successively increasing as they become less negative; however they had difficulty in describing the function as "*decreasing at an increasing rate*". In exponentially decaying functional situations, the amount of decrease in the value of the function gradually decreases, but the signed rate of change increases as it becomes less negative. The difficulty was coordination of the change in the function and the rate of that change. When the rates are negative but increasing, the situation became further complex for students. The study by Arleback et al. (2013a) also revealed students' difficulties while describing the changes with negative rates. While students were working on a model-application activity involving the light intensity as function of the distance from the light source, some of them confused the change in the light intensity and the change in the average rate of change of it. These studies, in common, revealed that pre-calculus students' understanding of average rate of change developed in symbolic, graphical, and algebraic forms, but systematic difficulties observed while they were interpreting the variation in average rates of change in exponentially decaying functional situations.

2.4 Graphically Understanding the Connections Between a Function and Its Derivative

Graphical connections between a function and its derivative are covered by the applications of reversing between derivative and antiderivative graphs. There are some standard procedures of graph sketching such as determining the monotonicity (increasing or decreasing) of intervals, max-min points, inflection and cusp points, and the sign of the second derivative for deciding the concavity. Students in calculus courses have been generally thought these standard procedures. However, applications of those procedures did not develop students' understanding of the underlying ideas behind them such as what concavity means (Berry & Neyman, 2003; Stroup, 2002; Tall, 1992). Several studies investigated students' understanding of the graphical connections between a function and its derivative (Asiala, et al, 1997; Aspinwall et al., 1997; Baker et al., 2000; Haciomeroglu, 2007; Haciomeroglu et al., 2010; Ubuz, 2007). These studies commonly indicate students' difficulties in interpreting and reversing between derivative and antiderivative graphs. It has been understood from these studies that making inferences about the function or drawing the graph of it by looking its derivative function or making inferences about the derivative function by looking the original function is not trivial.

Asialaet al. (1997) investigated calculus students' understanding of a function and its derivative by using the APOS (action-process-object-scheme) framework. The results revealed students' difficulties and misconceptions in relating the slope of the tangent line with the derivative in the absence of an algebraic expression. Furthermore, Baker et al. (2000) analyzed students' construction of a function's graph when its analytical properties (intervals of monotonicity by first and second derivatives, limits and continuity) were given with the same group of participants in Asiala et al. (1997)'s study. They evidenced students' difficulties in handling the cusp point, vertical tangent at a point, removal of discontinuity, and giving meaning to second derivative when drawing the graph. In a similar vein, Ubuz (2007) investigated first year engineering students' conceptions and misconceptions in the process of their interpretations of the graph of a function and constructing its derivative graph. Seeing the slope of a tangent line as if it was the derivative function, relating concavity with the power of function, and distinguishing the meaning of inverse in "inverse function" were several of the misconceptions and difficulties. Ubuz (2007) identified prototypes students have, weak understanding of limit, process-product obstacle, and reversing between graphical and symbolic representations as being the sources and origins of students' difficulties and misconceptions.

In the study by Aspinwall et al. (1997), a student's difficulty in reversing between derivative and antiderivative graphs was observed in the absence of the algebraic formula. An engineering student who completed elementary calculus drew a curved graph, looking alike the shape of a cubic function for the derivative of the second-degree parabola graph. The researchers pointed out the need for using such graphical tasks involving the reversing between a function and its derivative graphs as the starting point. The study by Haciomeroglu et al. (2010) revealed that students having different thinking preferences demonstrated different difficulties. Whereas visual thinker tried to determine the antiderivative graph by looking the changing of slopes on the derivative graph, the analytic thinker thought with the algebraic expressions. The visual thinker had difficulties in interpreting how the vertical stretching of the antiderivative graphs changes the derivative graph. The analytic thinker, on the other hand, had difficulties related to discontinuity and differentiability while drawing the antiderivative graphs. The researchers indicated that it is essential for students to synthesize the analytic and visual thinking for having a complete understanding of differentiation and integration.

The results of the aforementioned studies indicated that reversing between derivative and antiderivative graphs is not trivial and it is an important aspect of conceptual understanding of the derivative. Stroup (2002) argued that students' qualitative and intuitive understanding of the graphical connections between a function and its derivative can be fostered by using learning tools from authentic situations. Stroup identified qualitative understanding of calculus as seeing the same system when looking at "how much" and "how fast" graphs. Likewise, various studies evidenced the effectiveness of using authentic tasks in developing students' understanding of the derivative and antiderivative graphs (Berry & Nyman, 2003; Doorman & Gravemeijer, 2008; Gravemeijer & Doorman, 1999; Hoffkamp, 2011; Stroup, 2002; Yoon et. al., 2010).

For example, Berry and Nyman (2003) investigated eight university students' way of thinking when reversing between derivative and antiderivative graphs while working on a contextual task. The students were provided with four graphs of derived functions (speed-time graphs) and they were asked to produce the displacement-time graphs by using the Calculator Based Ranger and a Texas Instruments graphic calculator. Students rarely used the formal mathematical language and they did not determine the max-min and inflection points procedurally. They generally used the terminologies as "slow", "fast", "start faster", "slower here", "walk back", and "parabola opening out". Students' intuitive understanding of the connection between distance and speed graphs developed. The researchers recommended the usage of such graph drawing tasks in pre-calculus courses for developing students' understanding of properties of graphs and helping them to develop a "physical (or calculus) feel". In a similar vein, Yoon et al. (2010) investigated calculus students' developing ideas of graphical understanding of derivative and integration as they engaged in a model-eliciting activity. Students were given a model-eliciting activity after they received the traditional instructions on derivative and integration. In the activity, students were provided a gradient-distance graph, and they were wanted to develop a method for drawing the distance-height graph of the track. The results showed that students could not use their formal knowledge of derivative and integration while drawing the graph of height-distance. As an example, initially they intuitively decided the levels of summits and valleys without using the area under gradient-distance curve. Although participants developed primitive verbal explanations for the use of area under the curve for deciding the levels of summits and valleys, they were unaware that their ideas were related to integration. Both studies (Berry & Nyman, 2003; Yoon et al. 2010) reported parallel findings. They indicated that even formally instructed students could not identify the formal calculus ideas in real situations, but both studies indicated a development in students' qualitative understanding of these ideas.

2.5 Summary of the literature

Research indicated that derivative is not an easy concept to understand, because it involves the concepts of function, slope, rate and ratio, and limit (Zandieh, 2000). Although students can perform variety of algebraic operations, students and teachers have many difficulties such as in giving meaning to symbolic expressions used for derivative (Santos & Thomas, 2001; White & Mitchelmore, 1996), in conceptualizing derivative with all its process-object pairs (Zandieh, 2000; Habre & Abboud, 2006), in forming connections between different representations of derivative (Herbert & Pierce, 2012b; Zandieh, 2000; Zandieh & Knapp, 2006), weak understanding of rate of change (Bezuidenhout, 1998; Orton, 1983; White & Mitchelmore, 1996), and weak understanding of the graphical connection between a function and its derivative (Asiala et al., 1997; Aspinwall et al., 1997; Baker et al., 2000; Haciomeroglu, 2007; Haciomeroglu et al., 2010; Ubuz, 2007).

Student difficulties in understanding the derivative are rooted in their weak understanding of function and rate of change concepts (Bezuidenhout, 1998; Tall, 1992; Zandieh, 2000). Covariational reasoning ability has been mentioned as being critical and prerequisite for developing students' understanding of dynamic view of functional dependency, rate of change, and derivative (Carlson et al., 2002; Confrey & Smith, 1994; Hoffkamp, 2011; Monk, 1992; Oehrtman, Carlson & Thompson, 2008; Saldanha & Thompson, 1998; Thompson, 1994a). Secondly, rate of change is the more inclusive conceptualization of derivative, but the literature pointed out that it is the least understood interpretation of derivative. Various difficulties and misconceptions related to rate of change have been reported such as confusing it with the arithmetic mean of the functions' values (Bezuidenhout, 1998), difficulty in interpreting the meaning of it (Orton, 1983;White & Mitchelmore, 1996), difficulty in interpreting it in non-motion contexts (Herbert & Pierce, 2008, 2012; Wilhelm & Confrey, 2003; Gravemeijer & Doorman, 1999; Zandieh & Knapp, 2006) and difficulty in interpreting rate of change in non-linear situations (Teuscher & Reys, 2012). Interestingly, it is less stressed in curricular documents (Bezuidenhout, 1998; Bingolbali, 2008).

In addition, student difficulties in graphical understanding of derivative and reversing between derivative and antiderivative graphs have been evidenced by many studies (Asiala et al., 1997; Aspinwall et al., 1997; Baker et al., 2000; Haciomeroglu, 2007; Haciomeroglu et al., 2010; Ubuz, 2007). On the other hand, by the use of interactive computer-based or real life learning environments, students' developing ideas related with the qualitative aspects derivative and antiderivative graphs have been reported (Berry & Nyman, 2003; Doorman & Gravemeijer, 2008; Gravemeijer & Doorman, 1999; Hoffkamp, 2011; Stroup, 2002; Yoon et al., 2010).

2.6 Conceptual Framework of the Study

This study investigated pre-service mathematics teachers' existing and developing conceptions of the big ideas involved in derivative as they engaged in the classroom experimentation of the model development unit. Four different perspectives were used in conjunction while determining the design principles and while analyzing the data. These are; (i) the concept of covariation and covariational reasoning (Carlson et. al., 2002; Confrey & Smith, 1994; Thompson, 1994a, 1994b; 2011), (ii) quantitative reasoning (Thompson, 1993, 1994b, 2011), (iii) qualitative calculus (Stroup, 2002), and (iv) mathematical modeling (Lesh & Doerr, 2003).

2.6.1 Covariational Reasoning

The concept of covariation and covariational reasoning was explained previously (see, Section 2.1). To remind again, Confrey and Smith (1994) defined the concept of covariation as "*moving operationally from* y_m *to* y_{m+1} *coordinating with the movement from* x_m *to* x_{m+1}" by which a function is understood as the juxtaposition of two data sequences (p.33). This definition indicates the discrete coordination of the successive changes in two variables. Saldanha and Thompson (1998), on the other hand, emphasized the developmental nature of covariational thinking evolving from coordination of two quantities for some discrete values to images of continuous coordination of two quantities. In a recent paper, Castillo-Garsow (2010) extended the understanding of the continuous covariational reasoning from the perspective of quantitative reasoning. According to Castillo-Garsow (2010), the ways of reasoning while coordinating the covarying quantities may appear in “chunky” or “smooth” forms. While “chunky” way of reasoning is inherently discrete (may be static and dynamic) in nature, “smooth” way of reasoning involves the continuous and dynamic coordination of changes in both variables (Johnson, 2012). When a person focuses on some particular amounts of covarying quantities, it involves a static way of reasoning on covariation; but when he focuses on the amounts of changes in covarying quantities; it involves a discrete dynamic coordination. The framework developed by Carlson et al. (2002) was the other attempt for characterizing covariational reasoning. In the framework, five levels of covariational reasoning was identified by the mental actions of (i) coordinating the quantities as dependent and independent variables, (ii) coordinating the direction of change, (iii) coordinating the amount of change, (iv) coordinating the average rate of change, and (v) coordinating the instantaneous rate of change. Covariational reasoning has been accepted as the foundational idea for the robust understanding of pre-calculus and calculus concepts one of which is derivative (Bezuidenhout, 1998; Carlson et. al., 2002; Confrey &Smith, 1994, 1995; Monk, 1992; Saldanha & Thompson, 1998; Thompson, 1994a, 1994b; Zandieh, 2000). Therefore, I benefited from the literature on the concept of covariation while deciding the first design principle used in designing the model development unit (see, Section 2.1). Additionally, I also benefited from the literature on covariational reasoning while analyzing the data in relation to the pre-service teachers’ ways of covariational reasoning.

2.6.2 Quantitative Reasoning

The second theoretical perspective that I benefitted from in the current study was Quantitative Reasoning conceptualized by Thompson (1993, 1994a, 1994b, 2011). Quantitative reasoning was also used in generating the design principles and in analyzing the data. Because the quantitative reasoning perspective has extensively explained the change, ratio, rate, rate of change concepts, I followed this perspective while analyzing and interpreting the data related to pre-service teachers' ways of covariational reasoning and their conceptions of rate of change.

Thompson (1994b) grounded his arguments on Piaget's (1980) notions of action, images, internalization and interiorization, mental operation, scheme, and reflective abstraction. According to Thompson (1994b), for comprehending a situation mathematically, identification of the quantities and the quantitative operations involved in the situation is required. Quantitative reasoning involves comprehending the situations in terms of the quantities and their qualities, constructing networks of relationships among quantities and so obtaining new quantities, and making inferences with them (Thompson, 1994b; 2011). Comparing, combining and coordinating the qualities of quantities are the primary quantitative operations. The basic constructs explained by the quantitative reasoning perspective are quantity, quantification, quantitative operation, and numerical operation and interpretations of them in different situations.

2.6.2.1 *Quantity*

According to Thompson (1994b), quantities are conceptual entities existing in people's conceptions of situations. Thompson (ibid.) characterized a quantity as being schematic that involves (i) an object, (ii) a quality of the object, (iii) an appropriate unit or dimension, and (iv) a process of assigning a numerical value to the quality. Quantities are attributes of objects that can be measured directly or indirectly and also they can be conceptualized without measuring them. Let's consider a moving train as an example. Here, the train is an object and the motion of it is the quality. Expressing the motion of the train with numerical values and with some units as 90 kilometer per hour involves the quantification of its quality (motion).

Schwartz (1988) distinguishes two types of quantities; quantities that can be measured or counted are *extensive* quantities, and those that cannot be measured or counted directly are *intensive* quantities. Whereas the amount of fuel a person buys for his car is an extensive quantity that is measured in liters, the fuel efficiency (fuel consumption per unit distance) of the car is an intensive quantity. While rate-related ideas involve intensive quantities, amount-related ideas involve extensive quantities (Stroup, 2002).

2.6.2.2 *Quantification*

Thompson (1994b) indicated quantification as "*a process by which one assigns numerical values to qualities*" (p.190). However, in a recent paper, Thompson (2011) pointed out the problematic nature of this definition, and mentioned about its inadequacy for conceptualizing the quantification of various science concepts. By giving the example of "torque" concept from Physics, he indicated the complexity of assigning numerical values to the qualities of some objects. It may take years, or generations for quantifying some intensive quantities. He modified the definition of the quantification as follows:

> Quantification is the process of conceptualizing an object and an attribute of it so that the attribute has a unit of measure, and the attribute's measure entails a proportional relationship (linear, bi-linear, or multi-linear) with its unit (Thompson, 2011, p. 8).

According to Thompson (2011), this definition forms a link between science and mathematics education in conceptualizing the quantification. Quantification involves conceptualizing an object and direct or indirect measurement of its quality. Two types of quantifications that are gross quantification and extensive quantification have been introduced by Thompson as follows.

> Gross quantification refers to a conception of a quality in ways that objects having it can be ordered by some experiential criteria (e.g., "appears bigger than"). Extensive quantification refers to a conception of a quality as being composed of numerical elements which arise by operations of unitizing or segmenting (Thompson, 1994b, p.190)

Gross quantification involves some form of perceptual and experiential criteria. On the other hand, extensive quantification is more mathematical. Let's consider a moving car. Whereas, the expression as "*the car moves very fast*" indicates a kind of gross quantification, the expression "*the car is moving at least 100 kilometer per hour*" includes features of extensive quantification of car's speed. According to Thompson (1994b), both kinds of quantification are important for comprehending situations quantitatively, but only using the gross quantification is inadequate for correctly interpreting the situations.

2.6.2.3 *Quantitative operations versus numerical operations*

Thompson (1994b) defined quantitative operation as "*a mental operation by which one conceives a new quantity in relation to one or more already-conceived quantities*" (p.190). Quantitative operations emerge with one's comprehension of a situation and they are non-numerical. Thompson (1994b) discriminated two types of operations that are quantitative operations and numerical operations. That is; quantitative operations are used for creating quantities, and numerical (arithmetical) operations, on the other hand, are used to evaluate quantities.

Table 2: Arithmetic and quantitative operations (Thompson, 2011, p.17)

Quantitative Operation by which a new quantity is obtained	Arithmetic Operation
A quantity is the result of an additive combination of two quantities	Addition
A quantity is the result of an additive comparison of two quantities	Subtraction
A quantity is the result of a multiplicative combination of two quantities	Multiplication
A quantity is the result of a multiplicative comparison of two quantities	Division
A quantity is the result of an instantiation of a rate	Multiplication
A quantity is the result of a composition of ratios	Multiplication
A quantity is the result of a composition of rates	Multiplication

Quantitative operations involve one's comprehension of a situation mathematically. Generally, a new quantity is obtained as a result of a quantitative operation. The obtained quantity and the original quantities operated upon form the quantitative structure that all related to each other. Various forms of quantitative operations and the corresponding numerical operations used to evaluate them are provided on the table above. For instance, ratio can be seen as a new quantity obtained as the result of the multiplicative comparison of two quantities. These three quantities form a quantitative structure (Quantity A, Quantity B, Result of comparing) (Thompson, 2011). For example, let's consider a particular amount lemon-water solution consisting of A unit of water and B unit of lemon. Comparing the amount of lemon with the amount of water multiplicatively, we obtain the ratio of A/B. Therefore, the quantities "amount of water", "amount lemon", and "ratio of amount of lemon to amount of water" (A, B, A/B) form a structure that all three quantities are in relation with each other. When any of these two quantities are known; the other one can be find. Difference is another quantity obtained as a result of the additive comparison of two quantities.

Furthermore, the terms “quantity” and “quantitative reasoning” are not synonymous with the terms “number” and “numerical reasoning” (Smith & Thompson, 2007). However, there is not a notational distinction between two. The conventional notation systems already being used in mathematics stand for both quantitative operations and numerical operations, and this is a source of confusion for many teachers and students (Thompson, 1994b). Performing arithmetical operations successfully in some situations do not guarantee students’ understanding of the quantity that they evaluated. Therefore, the quantitative reasoning perspective emphasizes the importance of giving meaning to the symbolic numerical operations in real situations, and developing students’ and teachers’ awareness in distinguishing between quantitative relationships and numerical operations.

2.6.2.4 *Rate of Change from the Quantitative Reasoning Perspective*

Because I dominantly used the Quantiative Reasoning Perspective while analyzing the data about pre-service teachers’ conceptions of rate of change, the concept of rate of change from this perspective should be mentioned here. For explaining rate of change, this perspective starts with the ratio and rate concepts.

In the literature, ratio is frequently explained as the comparison between quantities of like nature, while rate is explained as the multiplicative comparison of quantities of unlike nature (Thompson, 1994b). Quantitative reasoning perspective does not agree with such a distinction, and believe in that rates and ratios are the products of mental operations. Therefore, the rate and ratio distinction should be based on the mental operations by which people comprehend situations. In other words, the mental operations of a person who try to quantify distance traveled within the duration of trip (time) can be characterized as ratio or rate independent of the quantities having like or unlike nature. Thompson (1994b) characterized ratio as the result of multiplicative comparison of two non-varying quantities. Non-varying and static nature of the multiplicatively compared quantities is the central idea in the conception of ratio. However, rate involves the multiplicative comparison of two dynamically varying quantities. Thompson defined rate as “*reflectively abstracted* *constant ratio*” (p.192) and explained the difference between rate and ratio with the following words:

> When one conceives of two quantities in multiplicative comparison, and conceives of the compared quantities as being compared in their, *independent*, *static* states, one has made a ratio. As soon as one re-conceives the situation as being that the ratio applies generally outside of the phenomenal bounds in which it was originally conceived, then one has generalized that ratio to a rate (i.e., reflected it to the level of mental operations) (p.193).

Consideration of two simultaneous and dynamic covarying quantities so that their measures remain in constant ratio is identified as the mature image of rate (Thompson, 1994a). Mature image of rate also constitutes the concept of rate of change. Thinking within linear situations, rate of change of a function at some instances or over subintervals is constant. However, in non-linear situations, rate and rate of change is not constant. Rate as “*reflectively abstracted constant ratio*” view is also valid in such situations. Changing rate of change means that the instantaneous rates of change take different values for different input values within the interval (Coe, 2007). But, the idea of instantaneous rate of change also involves the ratio between accruals of changes in two functionally related quantities within an infinitesimal interval. So, rate can be conceived as a reflectively abstracted constant ratio within this infinitesimal interval.

The concept of rate of change in real situations can be quantified in two different ways that are gross quantification and extensive quantification (Thompson, 1994b). Confrey and Smith (1994) used *perceptual comparative* term in place of gross quantification, and *unit per unit comparison* term for extensive quantification of rate of change. For example, while perceiving speed directly as “seems faster” is an example of gross quantification, perceiving it as a new quantity resulted from the ratio between distance and time and stating it with the measured values as 50 kilometers per hour is an extensive quantification.

2.6.3 **Qualitative Reasoning on Calculus**

Gross quantification of rate of change emphasized by Thompson (1994b) has been more comprehensively explained by Stroup (2002) with the notion of “Qualitative Calculus”. Qualitative calculus concerns calculus related ideas of young learners emerging in real situations, but not linked to ratio-related ideas (e.g., slope, ratio, proportion) of standard calculus curriculum.

Stroup introduced the notion of qualitative calculus as a synthesis of learning research by which he voices the necessity of having young students to work in real situations for developing their understanding qualitative aspects of calculus concepts. The intensification of rate and forming reversibility between "how much" (amount) and "how fast" (rate) in different situations may be the examples of qualitative understanding. In standard calculus "how much" stands for a function $f(x)$ and "how fast" stands for the derivative of it $f'(x)$. As an example for intensive understanding of rate in motion context is "fastness". According to Stroup, rate as "fastness" is "*not yet organized as a ratio of changes*" (p.170), but still it is a powerful idea. The followings are general indicators of students' qualitative understanding of rate in their interpretations of how-much graphical contexts (Stroup, 2002, p.182):

- In a how much graphical context, lines getting "more and more up" mean motion gets faster.
- Faster in a positive direction is associated with "steeper" up.
- Losing the sense that the graph is a picture of something.
- Distinguishing between the kinds of curvature in describing increases or decreases (As the distance increases increasingly, it is represented by a concave-up increasing graph. As the distance increases decreasingly, it is represented by a concave-down increasing graph).
- Steepness is understood to be independent of how far something has traveled at this speed.
- A flat section of graph is understood as no change, rather than a constant rate of change.

Some other ways of reasoning may appear during interpreting the how-fast graphical contexts. Qualitative ways of reasoning about rate those may appear in how-fast graphical contexts are the followings (Stroup, 2002, p.189):

- The ordinate values (y-values) of the points standing on the graph of $f'(x)$ signifiy the fastness of how much quantity. The positive ordinate values imply the change in how much quantity at positive direction (increase) and the negative values imply decrease.

- The area under the curve of how fast graph signifies the amount of change in how much quantity.
- Cancelling the areas under the curve at negative and positive sides, and so reaching the total amount of change in how much quantity.

When we look at the qualitative ways of reasoning, all these are also expected with the standard calculus instruction. Because these ways of reasoning have been demonstrated by young students as they worked in real-life situations, qualitative calculus can be seen as a stepping stone for the standard calculus. However, Stroup sees qualitative calculus as being "*cognitively significant and structural in its own right*" (p.170) rather than being transitional to more formal ratio-based ways of thinking. This was stated by Stroup as follows:

> Qualitative calculus is not limited to being simply transitional. In a significant sense, it can be considered "foundational". That said, qualitative calculus is certainly not exhaustive or exclusive. It participates fully with other powerful forms of reasoning (including, of course, the construction of quantitative metrics and multiplicative structures) in supporting our sense-making related to the sum of our lived experiences. Rather than "go away" as part of more advanced stages of development, qualitative calculus remains a major "player" at the table of embodied, intra-related, sense-making. Through intra-connection, mediated in relation to experience, slope, ratio, and proportion serve to powerfully model the foundational ideas of qualitative calculus. (p.204)

Qualitative calculus is foundational in learning the contextual meaning of calculus concepts. Stroup (2002) repeatedly indicated that understanding qualitative calculus does not bring the development of ratio-based conceptions of calculus ideas together to front. The reverse is also valid. Slope, ratio and proportion have been used as the formal models of qualitative calculus ideas, and therefore qualitative understanding of these ideas is more critical for meaningful learning. From the perspective of qualitative calculus, Stroup also criticize starting with the linear situations while introducing the rate of change and slope in standard calculus courses. According to him, linear functions are very special case of the general idea of function and using linear situations for introducing rate of change is an obstructive factor in students' further learning. Instead, using non-linear situations involving varying rate have been offered for developing qualitative calculus reasoning and this is seen more powerful for young students.

According to Stroup (2002), qualitative aspects of calculus ideas (such as intensification of rate in real situations) are underestimated or assumed to be developed by the formal instruction. However, this is not the case in practice as evidenced by many research studies. So, the theoretical conjecture derived from the notion of qualitative calculus was the following.

Principle 3: Qualitative aspects of calculus concepts are critical as well as their formal mathematical representations. Therefore, students should be provided opportunities for covering qualitative aspects of calculus concepts. Directing students to work on the graphs of "how much" and "how fast" quantities in real situations have potentials to develop the concepts of slope, ratio, rate, rate of change, and also the basics of reversing between derivative and antiderivative graphs (Stroup, 2002).

The arguments indicated by Qualitative Calculus were also used in analyzing the data about pre-service mathematics teachers' understanding of the graphical connections between a function and derivative.

2.6.4 Mathematical Modeling

The concept of covariation and covariational reasoning emphasizes the coordination of simultaneous changes in two covarying quantities and so the dynamic nature of functions (Carlson et. al., 2002). The quantitative reasoning perspective explains the contextual meanings of mathematical concepts and related quantitative operations (Thompson, 1994b). Qualitative calculus points out the appearance and properties of calculus concepts in different real situations (2002). All these perspectives emphasize the role of context in conceptualizing mathematical concepts. For me, usage of mathematical modeling activities can be a common base for matching up the arguments and expectations of these perspectives. At that point, what is mathematical modeling, and what are the basic arguments behind the usage of modeling activities and their properties are some of the questions that come into mind.

In general, mathematical modeling is the process of mathematizing, interpreting, verifying, revising, and generalizing real life situations or complex systems (Lingefjard, 2002). There is not a homogeneous understanding of mathematical modeling in the literature and nuances in pedagogical, psychological, subject-related, and science-related goals create the different perspectives (Kaiser & Sriraman, 2006). Models and Modeling Perspective (MMP) proposed by Lesh and Doerr (2003) have been adopted in this study. This theory based perspective comes with comprehensive and novel ideas for all aspects of mathematics education. According to MMP, modeling activities provide students significant local conceptual developments and meaningful learning of basic mathematical ideas in real situations (Lesh & Doerr, 2003). Modeling activities (*model-eliciting* in their terms) involve sharable, modifiable, and reusable conceptual tools (e.g., models) for constructing, describing, predicting, or controlling mathematically significant systems (Lesh & Harel, 2003).

By the use of modeling activities, development of mathematical ideas from informal to formal within meaningful and realistic situations have been emphasized (Gravemeijer, 2002 ; Lesh & Doerr, 2003). However, not every problem asked within a real situation can be accepted as a modeling activity. Modeling activities should carry out some important properties. Lesh, Hoover, Hole, Kelly, and Post (2000) determined six principles that a good modeling activity should have, which are reality principle, model construction principle, self-evaluation principle, model-documentation principle, simple prototype principle, and model generalization principle. Reality principle indicates the meaningfulness of the real life situation for the students. Model construction principle emphasize that if the task awakes students' feeling of a need for constructing a mathematical model that can be modified, extended or refined. Assessment and judgment of the usefulness of the model in terms of describing the real situation involve the self-evaluation principle. Model-documentation principle of a modeling activity is related to if the task requires students to report their thinking and solution in detail or not. The simple prototype principle refers to the simplicity of the real situation, so the solution can be used a prototype for interpreting other structurally similar situations. And lastly, model generalization principle involves the generalizability, reusability, and sharable property of the conceptual tools produced for a particular situation to broader range of situations.

Instead of using modeling activities as standing alone problem solving applications, MMP perspective offers the usage of *model development sequence* aiming at teaching of a particular mathematical concept (Lesh et al., 2003). Model development sequences include structurally related modeling and follow-up activities, group discussions, student presentations, and classroom discussions about the structural similarities of mathematical ideas across the activities. Designing a model development sequence requires the development of structurally related well-designed modeling activities and their follow ups. The designed model development sequence can be used in research, as well as in assessment or instruction (Lesh et. al., 2003). A sequence of model development sequences aiming at teaching of a relatively extensive topic was called as *model development unit* (Lesh, 2010). In this study, a model development unit was designed and it was experimented for investigating pre-service teachers' developing conceptions of the ideas involved in derivative.

Putting together, Thompson's (1994a; 1994b) theory of Quantitative Reasoning remarked that mathematical concepts gain their meanings in real life situations. Stroup (2002) also emphasized the emergence of mathematical ideas while working on real situations. These ideas resonate with the general arguments with regard to teaching of mathematics indicated by MMP (Lesh & Doerr, 2003). Therefore, the last conjecture guiding our design can be stated as follows:

Principle 4: Students contextual understanding of the ideas involved in derivative seems problematic. Mathematical modeling problems from authentic real situations have potentials in eliciting students' meaningful conceptions of mathematical ideas (Lesh & Doerr, 2003).

CHAPTER 3

METHODOLOGY

This study investigated pre-service mathematics teachers' understanding of the big ideas involved in derivative as they engaged in a model development unit (Lesh et al., 2003; Lesh, 2010). A model development unit on the concept of derivative was designed, experimented, and evaluated from a design-based research perspective. Therefore, in this section the methodological issues and procedures of designing, implementing, and evaluating the model development unit were reported. First of all, the theoretical underpinnings of design-based research methodologies were discussed. Secondly, the conceptual framework of the study used in determining the design principles and also in analyzing the data was explained. It was followed by explaining the design process of the model development unit. Later, the experimentation of the model development unit in a classroom setting was described by also mentioning the data collection procedures, classroom setting, and participants. And finally, the data analysis procedures were articulated.

3.1 Design-based Research

Design (or artificial) science and analytic (or natural) science are accepted as different scientific endeavors (Collins, 1992; Colins, Joseph & Bielaczye, 2004). While design science deals with the products of human creativity, analytic sciences deal with the already existing phenomena (Lesh & Sriraman, 2005). Physics, biology, and chemistry can be seen as the majors of analytic sciences; engineering, artificial intelligence, computer science, and architecture are the examples of design sciences. In recent years, a growing body of researchers started to look educational studies as a form of design science which also resulted in paradigm shift in educational research methodologies. A research methodology perspective called with the notion of "design-based research" has been appearing in addition to conventional quantitative and qualitative research methodologies (Brown, 1992; Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003; Collins, 1992; Collins et al., 2004; Design-based Research Collective, 2003; Hjalmarson & Lesh, 2008; Lesh & Sriraman, 2005).In the literature, design-based research studies called with different terms such as design experiments, design-based research, design research, developmental research, engineering research, formative research and teaching experiments (van den Akker, 1999). However, I preferred “design-based research” as umbrella term hereafter.

Developing curriculum, preparing textbooks, creating innovative learning environments for teaching a specific topic, developing an assessment tool, or improving professional development courses for teachers are some of the professions carried out in educational studies each involve a kind of designing and engineering processes. Therefore, engineering metaphor has been used to describe the nature of design-based research in education (Cobb et al., 2003; Hjalmarson & Lesh, 2008; Lesh & Sriraman, 2005; van den Akker, 1999). Design-based studies generally involves designing innovative learning artifacts, researching by implementing the designed artifact, modifying and revising the artifact iteratively, and concluding domain-specific theoretical arguments related to more effective ways of teaching and learning. Design-based research may be seen as practical endeavor, and it aims at making both practical and theoretical contributions and shortening the gap between theory and practice in educational studies (Cobb et al., 2003; Cobb & Gravemeijer, 2008).

Design-based research studies not only focus on engineering artifacts, but also developing theoretical arguments and domain-specific learning theories by experimenting the designed artifact in a real setting (Cobb *et al.*, 2003; Collins et al., 2004; Hjalmarson & Lesh, 2008). During the experimentation, instead of using fixed standard procedures, it is started with planned procedures and with the materials which are not completely defined. The design of the artifact continues by the revisions depending on the successes or failures in practice. The revisions and modifications in the artifact indicate iterations in design which is also a common characteristic of design-based studies (Cobb, et al., 2003; Hjalmarson & Lesh, 2008). As the revisions and modifications occur in the designed artifact, new iterations of intervention are required. Therefore, design studies involve multiple iterations (Cobb & Gravemeijer, 2008). Lastly, design-based research studies bring together theoretical and practical knowledge bases, and shorten the gap between theory and practice (Cobb, et al., 2003; Hjalmarson & Lesh, 2008). Furthermore, design-based research studies try to look events from a holistic point of view instead of controlling particular variables or focusing on only a single variable (Collins et al., 2004). Qualitative and quantitative forms of data collection, analysis, and interpretation methods can be utilized (Brown, 1992). Generally, preparing for the experiment, experimenting in classroom, and conducting retrospective analysis of the data generated during the experimentation are three phases in conducting a design-based research (Cobb et al., 2003; Cobb & Gravemeijer, 2008; Hjalmarson & Lesh, 2008; van den Akker, 1999).

The current study carries out the features of design-based research in the form of a small-scale classroom experiment. A model development unit on the concept of derivative was designed as an artifact by the research team and it was tested in a real classroom setting by which students' conceptual understanding of the derivative and the workings and failures of the artifact were investigated. And thirdly, the data collected through experimentation of the model development unit was analyzed and reported. In the following sections, the conceptual framework of the study was introduced first. And then, it was followed by the explanation of the design, experimentation, and evaluation phases of the model development unit.

3.2 Designing the Model Development Unit

The first phase of the design-based research is engineering or preparation of an artifact (e.g., learning sequence). According to Hjalmarson and Lesh (2008), the designed artifact is the externalization of the interpretations and assumptions of the designers. Therefore, they can continually change as being tested in real settings. The detailed articulation of the design process of an artifact is one of the distinguishing aspects of the design-based research studies. In the current study, a model development unit on the concept of derivative was designed as an artifact. In this section, the design process of the model development unit was explained in detail.

This initial phase encompasses a range of issues that include determining the problematic situation, documenting the starting points, clarifying theoretical conjectures and interpretive frameworks that can guide engineering of the learning tools, and deciding the hypothetical learning trajectory (Cobb & Gravemeijer, 2008; Hjalmarson & Lesh, 2008; Simon, 1995). The problematic situation related to teaching or learning can create a need for innovative approach which can be a starting point for the design. After determining the problematic situation, the theoretical conjectures and interpretive frameworks should be clarified that will be used as a guide during the design process. The theoretical conjectures related to teaching of a concept include the issues such as determination of related sub concepts, the organization and sequence of them, the determination of appropriate contexts in which the concept can be introduced, and the mathematical skills required. In clarifying the theoretical conjectures, existing body of literature can be used in addition to other knowledge bases such as first-hand teaching experiences (Cobb et al., 2003; Hjalmarson and Lesh, 2008). The conjectures may be at a reasonable level of confidence drawn from well-studied domains, but they can be speculative as well. The determined conjectures involving starting points, elements of trajectory and prospective endpoints guide the design of the artifact.

3.2.1 Determination of the problematic situation & Design Principles

As extensively indicated in literature review part, the problematic situation is the weaknesses in students' conceptual understanding of derivative. Contextual understanding of derivative (i.e., Rate of change) and understanding the graphical connections between rate and amount functions seems the major problems. The current study deals with designing a learning tool (the model development unit) aiming at fostering students' conceptual understanding of derivative.

The design principles that shed light us in designing the model development unit was explained in due course in the literature review and in the conceptual framework parts. The four design principles are shown on the table below. The Quantitative Reasoning perspective (Thompson, 1994b), the notion of Qualitative Calculus mentioned by Stroup (2002), and the construct of Covariational Reasoning (Carlson et al., 2002) were the theoretical approaches that guided us in deciding the design principles. And lastly, MMP introduced by Lesh and Doerr (2003) was determined as an umbrella encapsulating the different ideas mentioned by three of the theoretical approaches. Therefore, the learning tool was designed under the guidance of MMP.

Table 3: Design Principles

Design Principles	Source
1) Covariational reasoning ability is a critical prerequisite in conceptualizing functional dependency, rate, rate of change, and so derivative. For developing covariational reasoning ability, dynamically changing real situations involving two simultaneously changing quantities can be utilized. When students worked on such situations, they can develop important ideas such as dynamic image of functional dependency and intuitive understanding of rate of change (Carlson et al., 2002; Confrey & Smith, 1994; Monk, 1992).	The literature on covariational reasoning (see, Section 2.1)
2) Rate of change is more inclusive interpretation of derivative. However, it is generally underestimated or introduced only limited with the motion context which results in many difficulties. The concept of "rate of change" should be specifically focused on within non-motion real life contexts.	Literature on derivative & Quantitative Reasoning
3) Qualitative aspects of calculus concepts are critical as well as their formal mathematical representations. Therefore, students should be provided opportunities for covering qualitative aspects of calculus concepts. Directing students to work on the graphs of "how much" and "how fast" quantities in real situations have potentials to develop the concepts of slope, ratio, rate, rate of change, and also the basics of reversing between derivative and antiderivative graphs (Stroup, 2002).	Qualitative Calculus
4) Students' contextual understanding of the ideas involved in derivative seems problematic. Mathematical modeling problems from authentic real situations have potentials in eliciting students' meaningful conceptions of mathematical ideas (Lesh & Doerr, 2003).	Mathematical Modeling

3.2.2 Product design: The model development unit

The product design phase involves the development the model development unit. The model development unit encapsulates development of the modeling and follow-up activities, sequencing of them, and organization of group presentations and classroom discussions. For developing a hypothetical learning sequence; we started to design appropriate modeling and follow up activities. In this section, the development of modeling activities and structurally related follow up activities was explained.

The modeling activities were developed by the research team consisting of ten mathematics education researchers within the scope of the project supported by TUBITAK (Grant no 110K250). The modeling activities explained here are four of about 60 activities either newly created or adapted by the research team. While designing the modeling activities, the project team worked collaboratively. The project team followed six principles characterizing the features of a good modeling activity determined by Lesh et al. (2000) as a guide during the development and evaluation of the activities. Each activity was developed by focusing on one or a few mathematical ideas. The development of all modeling activities involved three phases that are; (i) writing the first draft by two or three researchers, (ii) evaluation and revision of the activities by the project team during the weekly conducted meetings each lasted about 2 hours, (iii) and finally testing some of the activities in the field. After writing the first draft of an activity, we discussed about the activity as the project team during the weekly conducted meetings. At these discussions, the activity was evaluated in terms of appropriateness to the curriculum objectives and grade levels, consistency with the principles, and language and grammatical aspects. After the activities were revised according to the suggestions of the project team, the final versions of the activities were field-tested in a high school. During the field-testing process, a group of teachers checked over all of the activities and they provided feedbacks for revisions. Additionally, some of the activities have been implemented in a real classroom setting.

Three big ideas were determined to be covered in the model development unit as an integrated content of the course designed within the scope of the project supported by TUBITAK (Grant no 110K250). These were; (i) covariational reasoning, (ii) rate of change, and (iii) the graphical connections between a function and its derivative. Four modeling activities and structurally related follow up activities were developed for each of these ideas. In the following parts, I explained the model development unit and the design process of four modeling and follow up activities.

3.2.2.1 *The Model Development Unit*

The model development unit consisted of the series of four model development sequences each explained in following parts. The model development unit was designed and implemented as an artifact aiming at developing pre-service teachers' conceptual understanding of derivative. The general process of the *model development unit* is shown on the figure below.

The model development unit was started with the pre-test measuring conceptual understanding of derivative. The first model development sequence was related to the idea of covariational reasoning which has been determined as being the central idea in conceptual understanding of derivative. The second model development sequence was about the concept of rate of change. In the following sequences, developing pre-service teachers' understanding of the graphical connections between a function and its derivative was the main objective. The process ended with the administration of the post-test.

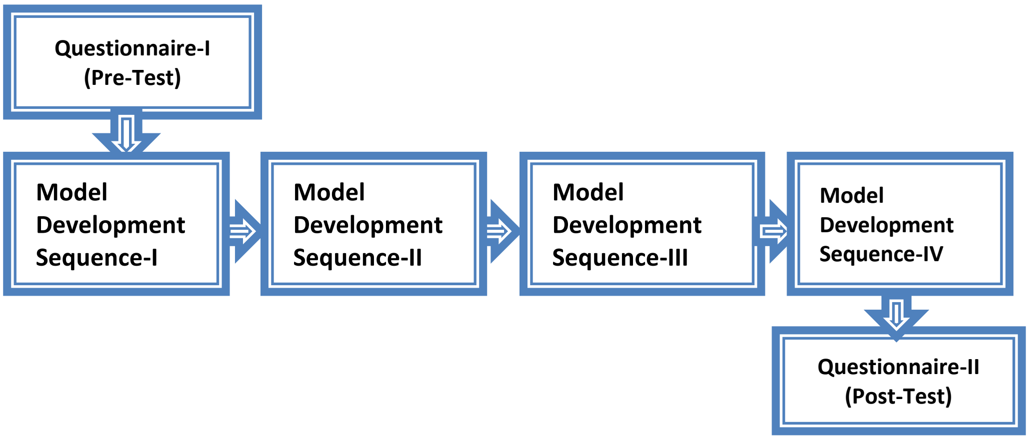


Figure 4: The process of Model Development Unit

In the following parts, the design process of each model development sequences was explained.

3.2.2.2 *Model development sequence-1*

As indicated by the first design principle, covariational reasoning has been determined as being a critical prerequisite for conceptual understanding of derivative. We determined the possible activities from the literature that can be used for developing students' covariational reasoning. In this sense, the "Water Tank" modeling activity was adapted from the study of Carlson et al. (2003) (see, Appendix-A1). The activity was asking about the height-volume relationship on a water tank. In the story of the activity, a firm that produces dynamic animations was commissioned to produce dynamic animations of a variety of water tanks and their graphs as they were filled with water. To be able to produce the scene as realistic as possible, the firm wanted mathematical explanations supported with the graphs from the mathematicians. Four sample tank figures were provided within the problem text. The activity has been controlled and evaluated by the research team by following the six-design principles proposed by Lesh et al. (2000).

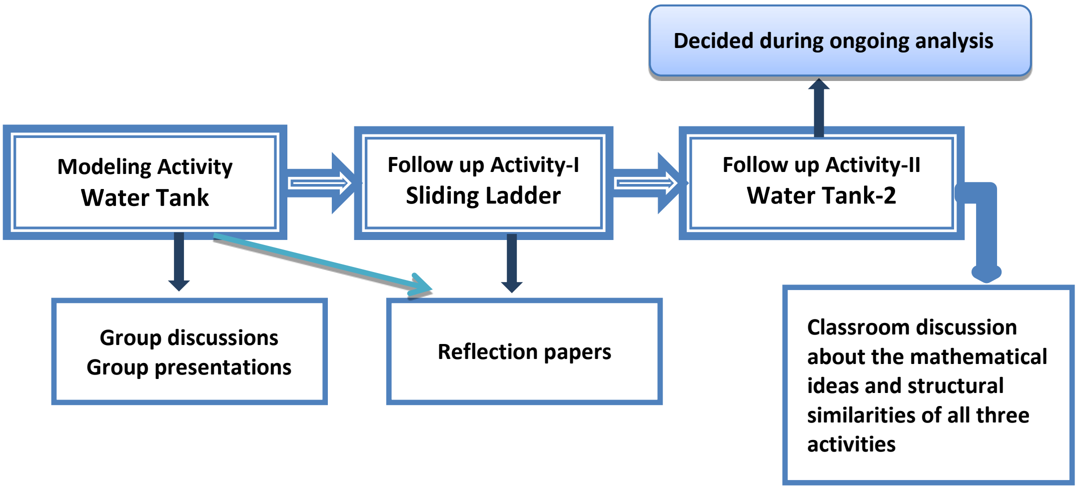


Figure 5: The process of Model Development Sequence-1

The “Sliding Ladder” problem was developed by the researcher inspiring from the study of Monk (1992) (see, Appendix-A2) as being the follow-up activity. The “Sliding Ladder” problem carries out structural similarities with the “Water Tank” modeling activity that both require covariational reasoning.

Additionally, during the ongoing analysis, pre-service teachers’ difficulties in interpreting a “concave-down decreasing graph” and representing it on the graph were observed. To evaluate the general state of all pre-service teachers in the class, we decided to implement the “Water Tank-2” problem as the second follow up activity (see, Appendix-A3). The problem was related to drawing and explaining the height-volume graph of an emptying water tank. The general process of the first model development sequence aiming at revealing and developing pre-service teachers’ covariational reasoning abilities is demonstrated on the figure above.

3.2.2.3 *Model development sequence-2*

By the second design principle, the concept of rate of change was focused on as being the second critical idea. A modeling activity was developed involving the analysis of the population change in Turkey (see, Appendix-B1).The first draft of the “Population of Turkey” activity was written by two researchers and the final version was decided by the project team going through a few iterations. The data were obtained from the official web site of Turkish Statistical Institute (TSI). Up to 2000, the population data was given according to the population censuses carried out once in five or ten year intervals. For later years, the yearly population data has been recorded by address-based population registration system. In addition to the concept of rate of change, the other mathematical ideas covered in this activity are the slope of a secant and tangent lines, and approximating the rate of change at a point from right or left sides and by mean value.

In developing this task, we initially reviewed the literature and read the research studies conducted on the teaching and learning of rate of change concept. The literature displayed us two critical points as being the possible sources of students' and teachers' difficulties in conceptual understanding of rate of change which are (i) the context used, and (ii) provision of data with one-unit increments in the independent variable. The literature indicated that kinematics (distance, velocity, acceleration) is frequently preferred in curricular documents for introducing rate of change concept. However, students' familiarity with this context from many earlier grades may prevent them to think in depth about the meaning of rate of change in distance with respect to time as also pointed out by many researchers (e.g., Bingolbali, 2008; Gravemeijer & Doorman, 1999). Therefore, we decided to develop the task by using a non-motion real life context. The second point was that the data provided in contextual rate of change tasks have been generally given with one-unit or equal increments in the independent variable. According to Cooney et al. (2010), this situation provides students great information about the variation in rate of change in such a way that he/she can easily see the pattern of change in the dependent variable without considering the change in independent variable. In other words, they do not need a multiplicative coordination between the changes of two variables. Therefore, we provided the population data for the years 1980, 1985, 1990, 2000, 2007, 2008, 2009 and 2010. The year intervals were varying between 1 to 10 years. When pre-service teachers tried to decide by using the "amount of change in population", they were expected to realize the inequality of the year intervals. At that point, to standardize the amount of change in different year intervals, they were expected to reach the idea of "(average) rate of change".

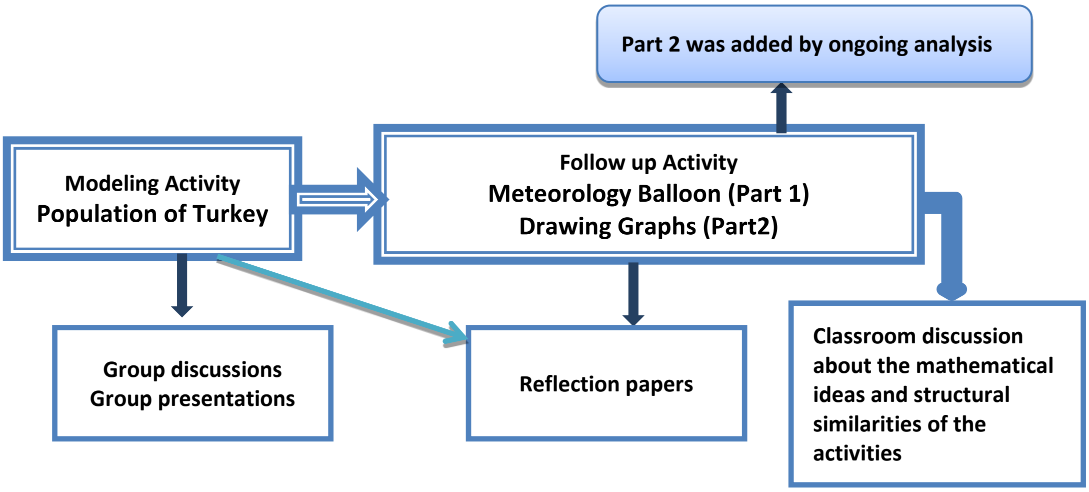


Figure 6: The process of Model Development Sequence-2

The follow up activity planned to be executing after the "Population of Turkey" modeling activity constituted two parts. In the first part, a tabular data was provided showing the height of a meteorology balloon with the corresponding time and pressure values of the balloon (see, Appendix-B2). The questions asked in this part were about finding "*rate of change in height with respect to time*" and "*rate of change in pressure with respect to height*" for various intervals. During the ongoing analysis, pre-service teachers' confusions and difficulties in discriminating additive and multiplicative rate of change were observed. To remove their confusion, the second part of the follow up activity was added. In the question, pre-service teachers were asked about two situations that are; (i) if the yearly population growth in percentage is constant what it means in terms of yearly population changes, and (ii) if the population grows yearly with equal increments, what it means in terms of yearly growth percentage. By these questions, pre-service teachers were expected to clarify and distinguish between the mathematical ideas of constancy in percentage and constancy in amounts of change.

3.2.2.4 *Model development sequence-3*

By the third design principle, graphical connections between a function and its derivative were the third idea covered in the model development unit. Two modeling activities were developed and planned to be integrated in the model development unit. The "Roller Coaster" modeling activity was the first one of them (see, Appendix-C1). In developing the "Roller Coaster" modeling activity, the mathematical ideas considered were “slope on curved-graphs”, “slope of tangent line & slope of secant line”, “derivative as function”, and “inflection point”.

In the story of the activity, the adventurous part of the railway path was planned to be constructed. It was asked help to construct the railway of roller coaster by taking into account the safety and minimum cost criteria. Safety criterion required that the absolute value of the slope of the track at any point could not be more than 5.67. The project team worked with the same procedures as explained for the previous tasks.

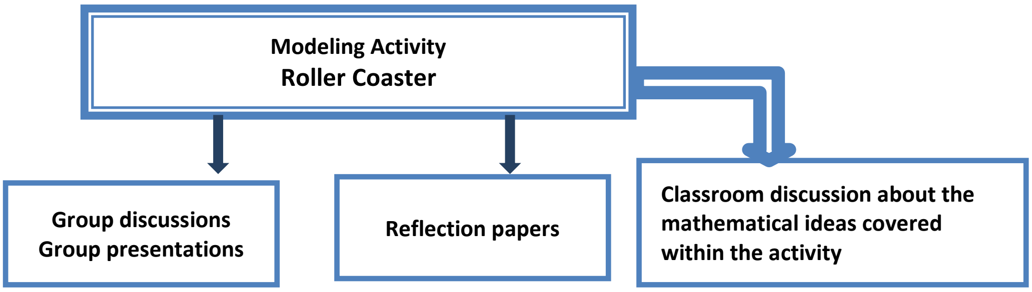


Figure 7: The process of Model Development Sequence-3

Because the “Roller Coaster” activity was expected to last more than two hours, no specific follow up activity was planned to be used after it. But, the follow up activity explained in the 4^{th} model development sequence served also for the “Roller Coaster” activity. The graphical interpretation of derivative was continued in the following model development sequence.

3.2.2.5 *Model development sequence-4*

One more model development sequence was planned for revealing and developing pre-service teachers’ understanding of the graphical connections between a function and its derivative. The “Tracking Track” activity was adapted from the study of Yoon et al. (2010). The activity was adapted by two researchers and then the final version was decided by the project team. In the activity, it was asked proposing a method for drawing the height-distance graph of a tramping track when the gradient-distance graph was already given (see, Appendix-D1). The distinguishing aspect of the task was that the distance-gradient graph was not stated explicitly as the derivative graph of the distance-height graph.

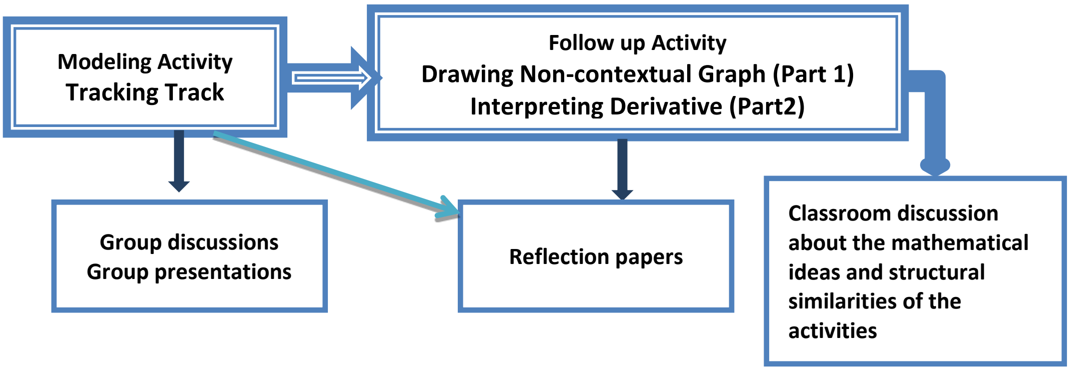


Figure 8: The process of Model Development Sequence-4

The follow up activity consisted of two parts (see, Appendix-D2). The first part was about drawing the graph of a function where its derivative graph was provided. This part was planned as a follow up for the mathematical ideas covered in "Roller Coaster" and "Tracking Track" activities. In the second part, volume-height and temperature-solubility graphs were provided with their derivative graphs. In the derivative graphs, the units (names) of the coordinates were not given and pre-service teachers were asked to assign the units. With this task, we aimed to assess pre-service teachers' interpretation of derivative in non-motion contexts. The second part of the follow up activity can also be thought as an evaluation tool for understanding development in pre-service teachers' contextual understanding of derivative.

3.3 System of use: Implementation of the Model Development Unit

The second phase of a design-based research is the experimentation of the designed artifact in a relevant system (Cobb et al., 2003; Hjalmarson & Lesh, 2008). This phase of design studies is critical in terms of gathering information that can be useful in developing domain-specific theoretical arguments as well as in developing the artifact. During the experimentation phase, data can be collected through the use of multiple methods such as interviews, classroom observations, video-recorded classroom and group discussions, pre-tests and post-tests, and written in-class materials (Cobb et al., 2003).

The model development unit was implemented in two iterations in a real classroom setting as part of a course offered for undergraduate pre-service mathematics teachers. The course was developed as a part of the project supported by TUBITAK (Grant no 110K250). The aim of the project was developing professional development programs about the pedagogical knowledge of mathematical modeling that can be used for in-service and pre-service teachers. Within the scope of pre-service component of the project, an undergraduate course that of “Mathematical Modeling for Teachers” was designed. The objectives of the course were improving pre-service teachers’ mathematical modeling abilities and providing them with pedagogical knowledge about the use modeling activities in teaching mathematics. The content of the course was started to be formed and first applied in 2010-2011 fall semester in a public university in Ankara. In 2011 fall semester, the final piloting of the course was executed with 20 pre-service primary school mathematics teachers. The main experimentation of the *model development unit* was executed during the final piloting of the course. The syllabus of the course during which the current study conducted was provided in Appendix-E.

3.3.1 **Pilot Study (Preliminary Iteration)**

The piloting of the first version of the *model development unit* was conducted as a part of the “Mathematical Modeling for Teachers” course which was administered in spring semester of 2010-2011 with 10 pre-service mathematics teachers. In this course, the first three weeks were devoted to the theoretical issues about mathematical modeling involving what mathematical modeling is, the difference between modeling and problem solving, and why mathematical modeling is important. In the following weeks, eight modeling activities were implemented during class periods. Four of the activities were “Water Tank”, “Population of Turkey”, “Roller Coaster”, and “Tracking Track” introduced in the artifact design section. These activities were implemented in that sequence between the sixth and tenth weeks. Pre-service teachers worked on the activities in three groups consisting of 3 to 4 person. Pre-service teachers’ individual and group reports were collected after the completing their studies. Also, they were asked to write a reflection paper explaining their thinking process in group and individual studies. This experimentation was accepted as the preliminarily iteration, because some of the tools such as Questionnaire-1&2 had not been prepared yet. In this preliminary iteration, the modeling activities and the appropriateness of the sequencing of them were field-tested. Concurrently, the design process of the assessment tools (i.e., Questionnaire 1&2) and the follow up activities continued.

This experience helped us in deciding the necessary modifications and revisions in the activities. Some minor or major revisions were decided in the “Water Tank”, “Population of Turkey”, and “Roller Coaster” activities. The first version of the “Water Tank” activity which was implemented during the pilot study resulted in some difficulties stemming from the complexity of the problem text. Two of the groups thought with more advanced concepts such as using the double integration for calculating the volumes of different tank figures at different heights. We understood from these results that the problem did not guide students to approach the task the way that we planned. Therefore, we revised it by simplifying the problem context (see, Appendix-A1).

In the “Population of Turkey” activity, we observed that rate of change can be interpreted in various ways such as multiplicative comparison of new to old, or quotient of differences in two quantities. We also added sub-questions asking the rate of change at a particular year (See, Appendix-B1). During the classroom implementation, it was emerged that students had difficulty in understanding what it means average rate of change in population. They confused the conventional understanding of average rate of change in population with respect to time, and the population growth in percent taking into account the preceding year. Therefore, we returned to the literature and realized different interpretations of rate of change as additive rate of change, multiplicative rate of change and the relation between two (Confrey & Smith, 1994). This pilot study showed us the contextual nature of rate of change interpretation as appeared in this task. In population contexts, rate of growth in population can be interpreted as yearly growth of population in terms of percent taking into account the preceding year, and also may be interpreted as the rate of change in population with respect to change in years. So, we decided that population context was an appropriate context for discussing the differences between the additive and multiplicative interpretations of rate of change as well. The pilot study also displayed us the necessity of technological support (spreadsheet, graphic calculator). In this way, the final version of the task was developed by considering the problems of students who encountered in class implementations and suggestions of teachers and project team.

In the solution of the "Roller Coaster" activity, pre-service teachers also lived difficulties in interpreting how to consider the boundary conditions. In general, this activity worked well in developing participants understanding of "derivative at any point", "inflection point", and how the slope of tangent line was changing according to the nature of curve. But some difficulties observed stemming from the unnecessary information provided within the problem context. Some modifications and revisions were made in the problem text (see, Appendix-C1).

Furthermore, the pilot study provided valuable information about the classroom environment during the implementation of modeling activities, and to anticipate possible shortcomings in a real classroom setting. For example, we observed a considerable amount of deviations in the time schedule that we presupposed for a particular modeling activity. More reasonable time schedule about the duration of modeling activities, group presentations, and classroom discussions were decided in the light of the experience from the pilot study. Also, observing students' ways of thinking during the pilot study contributed us in deciding the appropriate follow up activities for developing effective model development sequences. The pilot study also showed us the necessity of technological support (spreadsheet, or graphic calculator) during the implementation of modeling activities. For example, participants wanted us to provide a technological tool for conveying the tabular data provided in the "Population of Turkey" activity to the graphical form.

3.3.2 **The main experimentation of the model development unit (Iteration-1)**

The main iteration of the model development unit was implemented during the fall semester of 2011-2012 academic years as an integrated content of "Mathematical Modeling for Teachers" course. As indicated previously, the course was designed with the aim of developing pre-service mathematics teachers' pedagogical knowledge about the use of mathematical modeling in teaching mathematics. The implementation of the *model development unit* was started by the sixth week of the course. It is important to mention here that, before starting to implement the model development unit, pre-service teachers experienced with three modeling activities in the previous weeks. During the first 5 weeks, they got used to the group work, classroom environment, studying with the video cameras, the audio recorders, and how to work in groups while solving the modeling activities. Additionally, pre-service teachers experienced managing the time for group discussion, preparing the group report, and getting ready to group presentation. The experimentation progress of the *model development unit* is represented week by week on the table below. The experimentation was started with the Questionnaire-1 and ended with the Questionnaire-2 (see, Appendix-E).

Table 4: The experimentation progress of the model development unit

Week 6	**Week 7**	**Week 8**	**Week 9**
Questionnaire-I (Pre-test)	**MDS-1** • MA-1: Water Tank • FU: Sliding Ladder	**MDS-1 Continued** • FU: Water Tank-2 • Classroom discussion on MDS-1 **MDS-2** • MA-2:Population of Turkey	**MDS-2 Continued** • FU: Meteorology Balloon • Classroom discussion on MDS-2 • Student way of thinking application on MA-2
Week 10	**Week 11**	**Week 12**	**Week 13**
MDS-3 • MA-3: Roller Coaster	**MDS-3 Continued** • Classroom discussion on MDS-3 • Student way of thinking application on MA-3	**MDS-4** • MA-4: Tracking Track • FU: Drawing graph and interpreting derivative • Classroom discussion on MDS-4	Questionnaire-2 (Post-test)

FU: Follow-up Activity, MDS: Model Development Sequence, MA: Modeling Activity

In the following sections, participants of the study, course format, the role of the researchers and instructor, and data collection methods were explained in detail.

3.3.2.1 *Participants of the study*

Because this study is not aiming at reaching statistically valid generalizations, the sampling method can be described as purposeful sampling which has been accepted as a non-probabilistic way of sample choice (Cohen, Manion & Marrison, 2000). The participants of the study were 20 undergraduate or graduate pre-service mathematics teachers attending an elementary mathematics teacher education department in a public university in Ankara. In order to complete the department, students should take about 40 courses from various departments including mathematics, science, education, and history. There are 11 mathematics courses including Calculus-I, Calculus-II, Differential Equations, and Linear Algebra which should be taken from the department of mathematics. After graduating, they are entitled to a certificate for teaching mathematics for grades 5 to 8 in elementary schools. Three of the participants were graduate students one of which was an in-service teacher continuing their education for Master of Science degree, and 17 of them were senior students. 7 out of the 20 participants were male and 13 of them were female. All of the pre-service teachers successfully completed at least Calculus-I and Calculus-II courses. Additionally, the students already completed most of the courses offered by the teacher education program and they also had some school and teaching experiences.

In order to get deep insight about the pre-service teachers' understanding of the ideas involved in derivative as they engaged in the model development unit, 4 pre-service teachers were selected for in depth analysis. In selecting the participants for qualitative analysis, I used the purposeful sampling method by which the selection of information rich cases was aimed (Merriam, 1998). First of all, at the second week of the course, we selected one participant from each group for interviewing to understand the details of thinking process during the group discussion process. 6 interviewees, as being the representative of all groups, were selected on the basis of voluntary participation. They were interviewed just after the implementation of each modeling activity throughout the semester. The interviews were started by the 3rd week and all the interviews were conducted by the researcher. Because, more rich data was obtained during the interviews, I decided to select the participants for qualitative analysis among the 6 interviewees. During the first three interviews, I determined the persons according to their ability of expressing themselves with reasonable competence. The second criterion used was the quality of the group working process. Therefore, I decided to focus on the qualitative analysis of 4 participants who were having high communicative abilities. In addition to the descriptive results obtained from the whole classroom, I tried to provide a detailed and thick description of the results by reporting for 4 pre-service teachers.

3.3.2.2 *Course Format & Role of the Instructor*

The course was taught by an instructor who was also a member of the project team. The length of the course was planned to be weekly three hours and it was executed on Thursday afternoons. During this 3-hours period, one break was given generally lasting about 20 minutes. Each week, the course was executed in two classroom periods; each lasted about 75 minutes in average.

At the beginning of the semester, in addition to the syllabus, detailed weekly lesson plans of the course were prepared by three researchers including the instructor (see, Appendix-E, and Appendix-F). In these plans, all the details about the classroom works involving timing and duration of them and the materials needed were carefully considered. Weekly meetings were conducted at the day before the course about the details of the lesson and about the teacher guide for implementing a particular modeling activity. During these meetings, the planning and details about the classroom works were clarified, and the critical points about the roles of instructor and researchers were determined. In contrast to traditional methods, teacher's role during the application of modeling activities radically changes. Teacher's role can be summarized as assisting students when necessary, being a planner and organizer for efficient learning environments, being advisor, and the most important of all; learning together with the students. Instructor's role was particularly determined by the project team for demonstrating pre-service teachers a good role model. Primarily, before implementation of modeling activities, the research studies on teacher's role were elaborately examined and teacher guide for each modeling activity was prepared. The teacher guided students by open-ended questions such as "how did you do", "why did you think in that way", and "what was you assumption". The teacher selected the groups for presentations and he organized the classroom discussion.

Modeling activities were implemented in the form of group works consisting of 3-4 members, and pre-service teachers worked on with the same group throughout the semester. For each modeling activity, the time schedule was planned as follows: In the first 5 minutes, all pre-service teachers studied on the problem individually. After the individual work, the instructor asked to the classroom about the problematic situation in the activity. A few pre-service teachers summarized the problem situation under the directive of instructor and then the group works started.

The group works were planned to continue about 60-75 minutes. Pre-service teachers were wanted to prepare a group report for the solution on an A3-sized paper. Additionally, theywere allowed to use and provided with technological devices such as graphic calculator and other software programs. After the group works completed, a break time was given lasting about 20 minutes. During the second period of the course, groups presented their solutions. If there was no time restriction, all groups presented their solutions. If there was a time limitation, two or three of the groups selected by the instructor according to their solution approaches presented. The primary aim of individual working before group work, and group work before classroom discussion was increasing the quality and efficiency of the group discussion and classroom discussion. Group presentations were followed by the implementation of the follow up activity during which all pre-service teachers worked individually. After the group work reports and individual solutions to the follow up activities were collected, a whole class discussion carried out about the issues including the most reasonable solution method, the underlying mathematical ideas, and the structural similarities of modeling and follow up activities.

3.3.2.1 *Sitting Arrangement*

Because most of the in-class practices were executed in the form of group works, every week, about an hour before the course, three researchers arranged the desks in order to fit the classroom for six different groups' work. As seen in the figure below, the single desks and chairs were arranged in accordance with its appropriateness for three or four membered group works.

Figure 9: Sitting arrangement

The distance between groups was considered for minimizing the interaction. After the classroom became the most efficient for group work, three video cameras were placed in their pre-determined positions. One of the cameras was used as a portable camera. This camera was carried by one of the researchers while following the instructor and for recording the dialogues between instructor and groups. Two of the cameras were placed to fixed points by which two groups were observed.

3.3.2.2 *The Role of Researchers*

In addition to the instructor, the course was followed by three other researchers. These three researchers participated to the course as observers. They were also the members of the project team as scholars. The author of this study was one of those researchers.

The role of researchers can be characterized as assistance to the instructor before and during the course, collecting data about the course (e.g., taking observation notes), and helping on technical issues (e.g., video recording, the control of audio recording devices, cameras, projector, and other technical adjustments). Additionally, they observed the classroom and group discussions without participating in the conversations. Creswell (2003) identified this role "observer as participant". Each researcher was responsible for controlling a camera in order to reflect the group discussion in the best way. The duties and roles carried out by the researchers were the followings:

- Planning the course: Two of the researchers planned the course and the model development unit in detail week by week before the semester and shared with the instructor. The instructor and the researchers met before the course to overview the plan and procedures. Additionally, debriefing meetings were conducted just after the course during which the working and problematic parts of the lesson was evaluated and the prospective lessons were planned for. Teacher guides for each modeling activity was prepared by three researchers by analyzing students' ways of solutions obtained in the previous field-testing of modeling activities.
- Observing: During the lesson periods, unless it is necessary, the researchers did not communicate with the students. Students' questions were directed to

the instructor. The researchers' role was observer as participant in such a way that they only observed with participating at minimum level (Creswell, 2003).

3.3.3 Data Collection Procedures

This study investigated pre-service mathematics teachers' understanding of the ideas involved in derivative as they engaged in a model development unit. Because, we designed an artifact, implemented it in a real classroom setting, and revised it by a group of researchers, this study was qualified as a design-based research methodology. As indicated previously (see, section 3.1), design studies carry out the properties of mixed method approaches and benefit from multiple data collection methods (Collins et al., 2004; Design-based Research Collective, 2003; Hjalmarson & Lesh, 2008). Therefore, we used both qualitative and quantitative methods in data collection process.

Table 5: An overview of the research questions and data collection methods

Research Questions	Data Collection Tools
1) What is the nature of pre-service mathematics teachers' existing conceptions related to the big ideas involved in derivative prior to attending the classroom experimentation of the model development unit?	➢ Questionnaire-I ➢ Individual and group written solutions to modeling and follow up activities ➢ Classroom Observations
2) What conceptions with regard to the big ideas involved in derivative did pre-service mathematics teachers develop as they attended to the classroom experimentation of the model development unit?	➢ Questionnaire-II ➢ Individual and group written solutions to modeling and follow up activities ➢ Classroom Observations ➢ Reflection Papers ➢ Task-based clinical interviews

The data collection tools used in this study were: (i) Questionnaire-1 and Questionnaire-2 for evaluating pre-service teachers' conceptual understanding of derivative; (ii) Written group solution reports of modeling activities, and individual written solutions to follow-up activities; (iii) Reflection papers written by each participant after the solution of a modeling activity; (iv) Semi-structured interviews; and (v) Video-recorded classroom observations and field-notes of researchers. These multiple sets of data yielded a rich data, and the triangulation was satisfied by using various data collection methods (Cohen et al., 2000). The research questions and the data collection tools are summarized on the table above.

3.3.3.1 *Questionnaires for Evaluating the Conceptual Understanding of Derivative*

After deciding the modeling activities, model development sequences, and the whole model development unit by specifying the underlying mathematical ideas to be covered within the hypothetical learning trajectory, it became a necessity to develop an instrument for assessing pre-service teachers' understanding. In order to understand pre-service teachers' existing understanding of the concepts covered by the model development unit, and for evaluating the developments in their understanding, an assessment tool developed by the researcher in two parallel forms that are Quetionnaire-1 and Questionnaire-2 (see, Appendix-G1 & Appendix-G2). Questionnaire-I was administered in the 6^{th} week, and Questionnaire-II was administered in the 13^{th} week of the course during the regular course hours. All of the pre-service teachers who attended to the course completed both questionnaires. The items included in both questionnaires were decided according to the concepts covered within the model development unit. In order to increase the face validity of the questionnaires, a table of specifications including the mathematical ideas and the level of knowledge was provided in Appendix-G3. Rate of change and other different interpretations of derivative (e.g., slope of tangent line, speed, quotient of differences), covariational reasoning, and reversing between derivative and antiderivative graphs were considered to be the basic ideas related to conceptual understanding of derivative. Pre-service teachers' understanding of derivative was evaluated in terms of conceptual and procedural knowledge levels. In both questionnaires, the first four questions were aiming at procedural level while the remaining seven questions were at conceptual level (see, Appendix-G3). All of the questions were prepared in open-ended form. In the following parts, the details about some of the questions are explained.

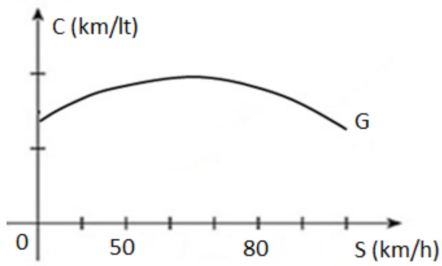
The first questions in both questionnaires were asking the symbolic expression of a tangent line passing through a point on the function. This question was prepared by the researcher in order to understand pre-service teachers' understanding of derivative as slope at the procedural level. The 2^{nd}question was also prepared by the researcher inspiring from the study of Orton (1983). Orton (1983) showed that university students could not interpret what the rate of change of a function means. By the second question in both questionnaires, pre-service teachers' interpretations of rate of change in a non-contextual function were planned to be observed. The second question was also accepted at procedural level because it only involves the application of the difference quotient and taking the derivative algorithms.

The 3rd and 4th questions were also prepared by the researcher and they involve the procedural level of knowledge. In the third question, the procedures of graph drawing by analyzing a symbolic function's analytical properties such as determining the increasing and decreasing intervals, maximum and minimum points, and the inflection points. In the fourth question, a symbolic function for the position of a ball with respect to time was provided, and the maximum height that a ball can reach was asked.

The 5th questions in both questionnaires were about determining the ability of covariational reasoning. As indicated by Monk (1992) and Carlson et al. (2002), having the covariational reasoning ability is a pre-requisite for understanding of functions and derivative. Therefore, by inspiring from a problem used by Lingefjard (2000), I developed the "Cassette Player" problem in which the radii of the reels were dynamically and simultaneously changing. For the parallel form of this question, I developed the "Space Shuttle" problem in which the change in the angle of camera with respect to change in the height of the space shuttle was asked. Because these questions involved a process of interpretation, they were accepted under the category of conceptual level of knowledge.

The graph on right side represents the function G. Let G(s) shows the fuel efficiency of a car as a function of its speed, s, in kilometers per hour.

Fuel efficiency (C) = Kilometers per liter of fuel that an automobile gets. Its unit is km/lit



Interpret the following statements in terms of the fuel efficiency of the car and its speed by also considering the units.

a) G(70)=24
b) What does the result obtained as a result of the operation $\frac{G(100)-G(70)}{30}$ means in the problem context? Explain.
c) What does the expression G'(80) = -0.3 means in the problem context? Explain.

Figure 10: The 6th question in Questionnaire-I (see, Appendix-G1)

The 6th questions in both questionnaires were involving the contextual interpretation of average and instantaneous rates of change. In the question asked in Questionnaire-I, the fuel efficiency of a car as a function of its speed was used as the problem context. The independent variable was speed of the car; time was not used as an independent variable. By using such a context, it was aimed to observe pre-service teachers' understanding of rate change and derivative in non-motion contexts. These questions were inspired from the similar structured problems used in the studies of Bezuidenhout (1998) and Goerdt (2007). In the 7th question, a tabular data of a differentiable function was provided and the derivative at a particular point was asked. By this question, pre-service teachers' understanding of the derivative as being the infinitesimal approximation by the difference quotient rule was aimed to be assessed. The question in Questionnaire-1 was inspired and the question in Questionnaire-2was adapted from the study of Hartter (1995).

In order assess pre-service teachers' contextual understanding of derivative, by the 8th question they were asked to interpret the daily verbal expressions by matching them with the appropriate symbolic expressions. The questions in Questionnaire-1 and Questionnaire-2 were adapted from the study of Goerdt (2007). The last three questions were related to drawing or interpreting the graph. The 9th question was about interpreting derivative ideas on a provided distance-time graph. This question was included in order to understand pre-service teachers' conceptions of derivative in graphically provided motion context and it was adapted from CPM Calculus (Dietiker, 2003, p.80). In the 10th question, two curved graphs were provided and pre-service teachers were asked to explain the graph verbally. By this question, pre-service teachers' graph interpretation abilities were considered. The last question (11th question) involved in drawing the graph of a function when its derivative graph was already provided. The Turkish version of all problems in Questionnaire-1 and Questionnaire-2 were provided in Appendix-G1 and Appendix-G2.

Expert opinions were used for ensuring the content validity of the questionnaire. Five mathematics education professionals who were continuing their doctorate education in that field, and one assistant professor reviewed the questionnaires. They controlled each of the items in terms of the consistency between the mathematical ideas they assessed for, and the appropriateness of the parallel tasks. There was about eighty percent agreement among the experts. After completing the first examination of the experts, the questionnaire was revised and two parallel forms were decided. The revised version of two parallel forms of questionnaires was field-tested with two different small groups of senior pre-service mathematics teachers. They were wanted to respond each question and they were asked to provide their comments about the problematic aspects of questions if they observed any. Some minor revisions also executed according to the feedbacks obtained during the field-testing. In the last phase, two parallel forms of the questionnaire were again reviewed by three mathematics education experts and a ninety-five percent agreement was reached. In short, for ensuring the validity, in addition to taking most of the questions from the related literature, I also benefited from expert opinions.

I used inter-rater reliability evidence for the reliability of the questionnaires. A detailed rubric for grading and evaluating the questionnaires were prepared by the researcher (see, Appendix-G4 and Appendix-G5). The details of the rubric preparation process are explained in data analysis part. The questionnaires were evaluated by two researchers according to the rubric and the correlation between two sets of scores was obtained as 0.92.

3.3.3.2 *Written Solutions to the Modeling and Follow up Activities*

Pre-service teachers' written solutions to the modeling and follow up activities were the two important data tools used in this study. For each modeling activity, all groups wrote down a report explaining their solution method. In the report, they were wanted to explain the way of solution by also mentioning about the assumptions and underlying ideas behind the solution. All groups were provided with A3 and A4-sized papers. The final group reports were written on A3-sized papers in order to demonstrate as a poster during the group presentations, and all other drafts of working sheets were also collected. During group presentations, the solutions of each group either demonstrated directly or reflected by an overhead projector. Individual solution papers to the follow up activities were also collected.

3.3.3.3 *Reflection Papers*

After each modeling activity, pre-service teachers were asked to write a reflection paper. A guide consisting of fifteen questions were prepared by the research team (see, Appendix-H). In the reflection paper guide, teacher candidates were requested to explain their solutions by also mentioning the stages they went over in detail. In some of the questions, they were asked to evaluate their solution comparing with other groups' way of solutions, and asked to evaluate the reusability of their solution method in other similar structured contexts. They were also asked to include their opinions about the issues such as what they have learnt during this process, their opinions about the group work and teacher's role, and what they may encounter if they applied in the classroom.

By the reflection papers, we tried to obtain more information about the group discussion process from each individual's point of view. In trying to answer all the questions in the reflection paper guide, pre-service teachers were expected to develop important ideas not only related to the way of solution for each modeling activity, but also related to pedagogical knowledge about the mathematical modeling aimed with the course. Reflection papers also provided us critical information for crosschecking the interpretations that we drew from the written group solutions.

3.3.3.4 *Video or Audio-Recorded Observations*

Observation is one of the invaluable ways of collecting data in educational research in terms of providing first-hand information to the researcher (Creswell, 2003; Cohen et al., 2000). According to Yin (2011), observation is a critical source of information for qualitative studies by which the researchers directly experience the process and become familiar with the context, and so they can draw their first and rough inferences. In this study, classroom observation was carried out by three researchers. During the course period, in addition to helping the instructor in technical issues, three researchers observed the group discussions and the classroom atmosphere in general. The focus of the observation was getting information about the model development unit and also about the mathematical reasoning of pre-service teachers. There was not a structured observation form used in the process. However, all the groups were audio-recorded. Previously determined two groups were observed by two researchers throughout the semester. These two groups were also video-recorded for the whole period of each class. The researchers took field-notes that they observed as interesting during the group discussions. The other researcher observed the classroom and he controlled the third video camera in recording the whole class discussions. He also followed the instructor for recording the dialogues between groups and instructor. The observations recorded through three cameras and six audio recordings went on in the way from beginning to the end of the semester.

During the observations, researchers can take various roles like complete participant, observer as participant, participant as observer, and complete observer (Creswell, 2003). In the current study, the role of researchers can be characterized as observer as participant which means the researcher's primary role was observing and he/she participated to the classroom works at minimum level. The researchers participated to the discussions at minimum level only when the instructor asked their ideas. During the data analysis, the classroom observation guided me in determining the critical points that should be focused on. In addition, video-records helped me in clarifying the conclusions drawn from other data tools. For example, I frequently returned to the video-records in order to remember or understand the details whenever a participant mentioned about a group discussion on a particular issue in the interview or in the reflection paper.

3.3.3.5 *Interviews*

In addition to the written reports, observations, and reflection papers; semi-structured task-based interviews were carried out by six pre-service teachers who were selected as being the representative from each group. Interviewing is an important source of data for qualitative and quantitative research methodologies. Structured and qualitative ways of interviewing have been mentioned as two types of interviews in general (Yin, 2011). Structured interviews involve asking a set of previously determined closed-ended questions generally utilized in survey studies. Qualitative interviews, on the other hand, do not foster asking a series of standardized questions. Although the researcher has a general mental framework of the questions to be asked, the nature of questions, the number of questions may change according to the context (Yin, 2011). The nature of questions to be asked is open-ended.

Clinical and task-based interviews are the qualitative types of interviews conducted for examining participants' mathematical understandings while they are working on particular tasks (Clement, 2000; Goldin, 2000). According to Clement (2000), clinical interview was the notion first introduced by Piaget including open-ended interviews and think-aloud protocols. During the critical interviews, an individual studies on a particular task designed for a certain purpose in front of an interviewer. As the individual studies on the task, the interviewer may probe about critical ways of thinking. The major aims with the clinical interviews are collecting detailed information about participants' ways of thinking, and eliciting their conceptual understanding in a natural way. Task-based interviews are another methodology used for examining participants' mathematical behaviors as they worked on a sequence of carefully designed tasks (Goldin, 2000).

> Task-based interviews typically do not focus on easily defined outcomes such as patterns of correct and incorrect answers by subjects. Rather, investigators try to observe, record, and interpret complex behaviors and patterns in behavior, including subjects' spoken words, interjections, movements, writings, drawings, actions on and with external materials, gestures, facial expressions, and so forth. (Goldin, 200, p.527)

According to Goldin (2000), task-based interviews serve for making systematic examination of participants' cognitive processes as they worked on a structured mathematical environment. They can also be used as an assessment tool.

In this study, I conducted weekly interviews with six pre-service mathematics teachers about the modeling and follow up activities. After the implementation of each modeling activity, the interviews were conducted within the following week period before the next class. The time schedule and place of each interview were planned at the second week of the course and executed according to that plan. Because I questioned about the ways of mathematical thinking that participants used in solving the modeling and follow up activities, the interviews carried out the properties of task-based and clinical type. A set of questions asked during the interview were the brief version of the reflection paper guide and previously determined by the research team (see, Appendix-I). The issues dealt with by these questions were the details of solution process, evaluation of their way of solutions, the mathematical ideas included in the problem, and their way of understanding them.

During the interviews, the list of interview questions were not asked in a strict manner; instead the researcher talked with the participant as friendly as possible and used the question outline as a general framework. The critical aspect of the interviews for this study was the specified questions for each participant about their particular ways of thinking, conceptions, misconceptions, and difficulties. To do that, the researcher carefully examined the written solutions, reflection papers, and video-records of the group works for each participant before conducting the interviews, and determined the particular set of questions to be asked. The questions to be asked differed according to the tasks and according to the person. There were very specific questions related to the tasks used in the model development unit in order to understand participants' mathematical reasoning processes. For example, in the "Water Tank" activity, most of the participants were asked about the graph at transition points because they draw it with sharp corners. Additionally, participants resolved the follow up problems by thinking aloud during the interview. Researcher asked questions related with the tasks and about the mathematical ideas included in the tasks both in the form of probing and prompting.

Selection of participants for interviewing was explained previously under the participants of the study part. The interviews lasted about 30 minutes in average. All the interviews were executed by the same researcher. By asking the consent of the participants, all the interviews were recorded by an audio-recorder. During and just after the interview, the researcher took notes about the critical points while the answers of the participant were still fresh in his mind. Before the analysis of data obtained by interviews, the same researcher transcribed the interviews of four pre-service teachers for reporting in this study according to the criteria of having the high communicative abilities, who were willing in their working, and whose group working processes were qualified as hard working. The data obtained by the interviews were used in conjunction with the written documents, video-recorded classroom observations, and reflection papers.

3.4 Data Analysis

The third phase of a design-based research involves evaluation of the experimentation by conducting retrospective data analysis. Because multiple data collection methods are used in design research studies, qualitative and quantitative methodologies can be used together (mixed-method approach) in analyzing the data (Brown, 1992). Two types of analysis that are ongoing analysis and retrospective analysis involve in a design-based research (Cobb & Gravemeijer, 2008). The *ongoing analyses are* conducted in the process of the experiment and it deals with the quick modifications of the artifact. In the current study, some modifications in the tools of the model development unit were decided during the experimentation process. The *retrospective analysis*, on the other hand, attempts to generate a coherent framework and tries to put into words more comprehensive theoretical arguments with regard to effective ways of teaching and learning of the domain aimed with the designed artifact. The data should be analyzed in a systematic way in the retrospective analysis. In this study, I systematically analyzed the data by using both descriptive and naturalistic data analysis methods. In this section, the quantitative analysis of the questionnaires was explained first, and it was followed by the explanation of qualitative data analysis.

3.4.1 Analysis of the Questionnaires

Before starting to the experimentation of the model development unit, Questionnaire-1 was administered to get information about pre-service teachers' pre-requisite level of conceptual understanding of derivative. All the questions included in both questionnaires were open-ended. For assessing and grading pre-service teachers' performances, a detailed rubric was prepared by the researcher and controlled by two experts (see, Appendix-G4). The same rubric was prepared for Questionnaire-2 (see, Appendix-G5). Pre-service teachers' scores from Questionnaire-I and Questionnaire-II were assessed according to the rubrics. Each question was evaluated out of 10 points and the total score was 110 for both questionnaires. The different scores for each question obtained according to the rubric then adapted in order to obtain 10 points for each question as the maximum score. Because the questionnaires were prepared appropriate for pre-test and post-test designs and administered on the same group at different times, the descriptive results related to the mean scores of the classroom were provided. The data obtained by the descriptive statistics was used for providing a general picture about the effectiveness of the model development unit on pre-service teachers' conceptual understanding of derivative. The details of the developments in pre-service teachers' conceptions were tried to be understood by the qualitative data analysis.

3.4.2 Qualitative data analysis

In qualitative studies, data analysis means making sense of the data for generating inferences and it is a complicated endeavor that even experienced researchers may have difficulties (Yıldırım & Şimşek, 2006). Although some differences may appear according to the design of the study, qualitative data analysis process generally involve the steps of (i) organizing the data for analysis, (ii) reading the data for obtaining a general sense, (iii) organizing the information by coding, (iv) collecting the codes under categories or themes, (v) deciding the way of representing the narrative, and (vi) interpreting the meaning of data (Creswell, 2003; Yin, 2011). According to Yin (2011), these steps are not occur in a linear sequence, rather they occurs in a recursive and iterative manner. In qualitative studies the data analysis should be done simultaneously with the data collection process (Merriam, 1998).

Furthermore, there exist some different methodological approaches in qualitative data analysis. Merriam (1998) mentioned six different data analysis strategies used in qualitative educational studies that are; (i) ethnographic analysis, (ii) narrative analysis, (iii) phenomenological analysis, (iv) constant comparative method, and (v) content analysis. The constant comparative method of data analysis strategy was developed by Glaser and Strauss (1967) which is generally used in the development of grounded theories. Constant comparative method of analysis indicates comparing the incidents obtained from different or same data sets. The main strategy used by constant comparative method is comparing and contrasting the incidents that form the preliminary categories obtained from a particular source of data with other incidents in the same or in another source of data to determine if a fit is possible (Merriam, 1998; Strauss & Corbin, 1998). In other words, constant comparative can be thought as the process of making initial conjectures during analyzing a data source and looking for further evidence from the other data source for confirming or disconfirming the conjecture (Cobb & Whitenack, 1996).

After organizing and preparing the data for analysis, the steps followed in constant comparative method are; (i) breaking down the data into discrete incidents or units and coding them, (ii) constructing categories by continuously comparing the incidents and codes that captures the recurring patterns, and (iii) developing grounded theories (Merriam, 1998; Glaser & Strauss, 1967; Strauss & Corbin, 1998). While coding the data as the first step, the researcher begins with carefully reading the data. As the researcher reads a particular set of data, for example the written documents of participants, he or she writes down notes and comments that are potentially relevant to the focus of the study. This also contributes to the researcher for getting a first impression about the possible codes and categories that may appear. Even, a tentative list of codes from the incidents, comments, and notes may be reached. When moving the second set of data (e.g., transcribed interviews), the researcher reads the data in the same manner by also keeping in mind the tentative codes that he obtained from the previous data source. During the reading of the second or the following data sets, the researcher may obtain new lists of tentative codes as well as observing recurring patterns of the incidents. After reading the complete data sets, by comparing the list of codes and incidents, the researcher should merge them into one list of codes obtained from all sets of data (Merriam, 1998). This list of the codes can be accepted as the primitive outline of the classification of recurring patterns in various data sets. Creswell (2003) defined coding process as "*the process of organizing the data into chunks before bringing meaning to those chunks*" (p.192). The second step involves the detailed examination of the codes and collecting them under categories by also revisiting the original data sources when necessary (Strauss & Corbin, 1998). In this step, the incidents and codes are examined in detail according to their relevance with the study and by considering the structural similarities. Merriam (1998) defined categories "*as the conceptual elements that cover or span many individual examples of the data*" (p.182). Categories are the abstractions concluded from the data that can be used for interpreting the general phenomena under investigation. This step also involves assigning names to the categories. The names of categories may come from the researchers, from the participants, and from the literature (Merriam, 1998). However, according to grounded theory perspective, using a classification scheme developed by others may be an obstacle in front of researchers for producing an extensive explanation of the situation under investigation (Glaser & Strauss, 1967). Although researchers can use already produced classification schemes, generating their own codes and categorization systems has been proposed as being the most appropriate way compatible with the nature of qualitative studies. The final step in constant comparative method is producing theoretical conclusions by interpreting the categories and their relevance with the study (Strauss & Corbin, 1998).

In the current study, pre-service teachers' written documents, reflection papers, observations, and interviews were the different data sets. Because multiple data collection methods and sources involved, the constant comparative method was followed in analyzing the data. The data analysis process started simultaneously with the experimentation process. The ongoing data analysis was in the form of getting the first impression about the data. Ongoing analysis involved careful reading of the interviewees' written solutions and reflection papers, and the field notes for determining the questions to be asked during the interview. After completing all the interviews, I started to transcribe the audio-recorded interviews. There were 16 interviews each lasting about 30 minutes. The transcription process lasted about one and half month. I also thought about the possible codes and categories by taking notes and comments during the transcription process. Afterwards, I prepared and organized the data sets for more systematic analysis. All of the written solutions for the questionnaires, modeling and follow up activities provided by the participants were scanned and saved in electronic format. The field notes arranged according to the issues mentioned. An electronic file for each pre-service teacher was prepared including all data sets belonging to them.

After organizing the data, the systematic data analysis was started. I should make clear that pre-service teachers' understanding of mathematical concepts, and their ways of reasoning were focused in this study. Therefore, the data analysis was executed at the cognitive level (Collins et al., 2004). The unit of analysis was the incidents such as a sentence, an episode from a dialogue, or a graph reflecting a mathematical idea clearly.

Covariational reasoning, rate of change, and graphical understanding of derivative was the sequence of big ideas covered with the model development unit. As a matter of course, there were three initially settled concepts that we tried to characterize pre-service teachers' ways of understanding. I started with the analysis of data for the first concept in our learning trajectory which was covariational reasoning. Because the research question was asking about the developmental understanding of pre-service teachers, I decided to analyze the data sets by considering the chronological order. The first data source was the 5th question in Questionnaire-I that was asking about the covariational relationship between radii of two reels. I analyzed solutions of the 20 pre-service teachers to the 5th question, and tried to make coding based on my interpretations and the related literature. During this process, I was aware of the covariational reasoning framework developed by Carlson et al. (2002), and the theoretical ideas indicated by Thompson (1994b), Comfrey and Smith (1994), and Monk (1992) with regard to the concept of covariation. Nevertheless, I frequently visited the theoretical explanations of these studies when I confronted to a new incident. By comparing and contrasting the incidents appeared in the Questionnaire-I, I obtained the first tentative list of codes.

Later, I started to analyze the written group solutions to the "Water Tank" modeling activity. I analyzed each groups' written solutions to the modeling activities and individual solutions to the follow up activities during which I continued coding by comparing the incidents obtained. The next data set was the reflection papers. It was continued with the analysis of transcribed interviews.

During the analysis of each data set, I obtained new codes, and I modified and revised the code list by comparing across different data sets. The process was not strictly linear; I frequently returned to the previous data sets until I recognize the recurring patterns. After reading all data sets step by step and reaching a saturated coding list by hand, I imported all of the data sources to the qualitative data analysis software Nvivo10. The program enabled me coding by using the stripes while reading the data on the screen. The program was appropriate for coding not only the word documents, but also for coding the images (scanned written solutions). Other facilities the program provided me were easily following which codes used for which incident at a first glance, monitoring the frequency of codes, the data sources they were used for, and flexibly organizing and revising the code list. I re-coded all of the data sources related to covariational reasoning by using the Nvivo10 software.

After the re-coding process completed by using Nvivo10, I started to group the codes according to their structurally similar characteristics. When I wanted to categorize the codes by using the covariational reasoning framework of Carlson et al. (2002), I realized many of the incidents in my data were not fitted with this way of categorization. For instance, in the "Water Tank" modeling activity, although it was asked about the height of water in the tank as a function of its volume, many of the pre-service teachers thought height as a function of time. It was impossible to label this way of reasoning under a category in Carlson et al.'s (2002) framework. Monk's (1992) study reminded me that this was related to the identification of the functional relationship between variables which was accepted as the one aspect of covariational reasoning in our categorization. The second aspect of covariational reasoning was determined to be the way of coordinating the variables. The third aspect of the covariational reasoning was deciding the variation in rate of change. The names of categories and sub-categories were assigned barrowing from the related literature of quantitative reasoning (e.g., Thompson, 1994b) and they were indicated in the form of propositions. For the details and explanations of each category see Appendix- J1.

The same procedures were followed for generating the categories of rate of change and graphical understanding of derivative. Before starting to coding, I read the data sources several times and thought about the possible codes and recurring patterns. All the data was coded by using the constant comparative method on the Nvivo10 software. All of the data sources including the covariational reasoning tasks were covered in order for looking pre-service teachers' understanding of rate of change and graphical derivative holistically. For assigning the names of categories, in addition to my own interpretations, I benefited from the related literature. For example, the names perceptual comparative, amount of change, and ratio-based reasoning used for characterizing pre-service teachers' conceptions of rate of change were barrowed from the studies of Confrey and Smith (1994) and Thompson (1994b). For the list of categories generated for rate of change and their explanations, see Appendix-J2. Additionally, see Appendix-J3 for the details about the categorization schemata used for interpreting and drawing graphs.

3.4.3 Validity and Reliability Issues (Trustworthiness)

Validity from quantitative research perspective means "*the appropriateness, meaningfulness, and usefulness of the inferences*", while reliability means "*the consistency of inferences over time, location, and circumstances*" (Fraenkel, Wallen & Hyun, 2012, p.458). There are standardized techniques for ensuring validity and reliability in quantitative studies. However, it is difficult to mention about the same techniques in naturalistic studies. The meanings of these terms change in qualitative studies, and *trustworthiness* is the key construct replacing the meaning of the conventionally used terms of validity and reliability (Lincoln & Guba, 1985; Yıldırım & Şimşek, 2006).

Trustworthiness, in general, means persuading the audiences in terms of the truth value, applicability, consistency, and neutrality of the study. "Credibility," "transferability," "dependability," and "confirmability" are the four naturalistic terms used in place of the conventional terms "internal validity", "external validity", "reliability", and "objectivity" respectively (Lincoln & Guba, 1985, p.300). In the same way, *trustworthiness* in design studies is related to the reasonableness, justifiability and credibility of data analysis (Cobb & Gravemeijer, 2008).

Table 6: Techniques for Establishing Trustworthiness (Lincoln & Guba, 1985, p.328)

Criterion Area	Techniques
Credibility (Internal Validity)	Prolonged engagement Persistent observation Triangulation (data sources, methods, investigators) Peer debriefing Member checking Reflexive journal
Transferability (External validity)	Thick descriptions Purposive sampling Reflexive journal
Dependability (Reliability)	Dependability audit Reflexive journal
Confirmability (Objectivity)	Confirmability audit Reflexive journal

The degree of being open to monitoring by other researchers and systematic nature of analysis of the extensive data contribute the trustworthiness of the study. Systematic analysis of data may involve issues such as working chronologically or episode-by episode. For credibility of data analysis, the coding criteria of the interpretive framework that is used for making claims should be explained clearly. The criteria and the common techniques for ensuring trustworthiness in naturalistic study are demonstrated on the table below. In this part, the meanings of these terms were explained first and then it is followed by articulating the techniques used in this study for ensuring each of them.

For ensuring the trustworthiness of a study, *credibility* is one of the most important concepts to be considered. Credibility indicates the congruence between reports provided by a researcher as the findings of an inquiry and the reality (Lincoln & Guba, 1985). In quantitative research, this is indicated by the term "interval validity". According to Lincoln and Guba (1985), there are several techniques for enhancing the credibility of a naturalistic study that are prolonged engagement, persistent (long-term) observation, triangulation, peer debriefing, and member checking. In this study, I used most of these techniques for ensuring the credibility. First of all, I participated to the course as an observer throughout the semester. This process provided me of developing the sense about the nature of data and the environment where the data was collected. By also looking the observation notes, I remembered easily the events and distortions occurred in the learning environment when I needed. Additionally, I found the opportunity of building trust with the participants during the period of prolonged engagement. Secondly, in addition to the prolonged engagement, I also conducted long-term observations. Each class period, I observed working process of a particular group, and classroom discussions by taking observation notes. These observations provided me in identifying the general characteristics of pre-service teachers' conceptions relevant to the focus of this study. Before starting to systematic data analysis, I had ideas about pre-service teachers' particular ways of thinking about the mathematical ideas covered with the model development unit.

The third technique that I used for ensuring the credibility was the triangulation. There are four different modes of triangulation that are using multiple sources of data, multiple methods of collecting data, or multiple investigators and theories (Cohen et al., 2000; Lincoln & Guba, 1985). In the current study, I used multiple sources of data, multiple methods and tools for collecting data, and investigator triangulation. There were 20 pre-service teachers as participants of the study as different data sources. In addition, 4 participants were selected for interviews as different cases. I also used different data collection tools including handwritten documents, questionnaires, reflection papers, observations, and task-based interviews. The inferences reported in results chapter were drawn from these multiple sets of data. The investigator triangulation was also used because the experimentation process was observed by two other researchers. These researchers observed the groups and the classroom by taking notes. These two researchers and the instructor also provided me valuable information by sharing their ideas about the methodological issues of experimentation and what they observed as interesting in pre-service teachers' ways of reasoning. These meetings were conducted just after the course period in the form of peer debriefing. During data analysis, I always considered the observation notes of these two researchers. I also used member checking technique for ensuring the credibility of my conclusions during or just after the task-based interviews. During or after the interviews, I gave the written solutions to questionnaire or follow-up activities to the interviewees and asked them if they were agreed what they had already written. I also asked about the ideas they mentioned in the reflection papers.

Transferability is the other concept that should be considered in order to increase the trustworthiness of a qualitative study. This is the term used in place of external validity. However, while external validity means making generalizations to the population, transferability means making analytic and argumentative generalizations "*depending on the degree of similarity between sending and receiving contexts*" (Linconln & Guba, 1985, p.297). From the design research perspective, the generalizability of design research studies is not related to the applicability of the artifact produced in diverse settings, rather it is closely related to reaching of sound domain-specific instructional theories and conjectures that can guide other researchers customizing the design or producing new ones (Cobb & Gravemeijer, 2008). The theory of *quantitative reasoning* is an example which was reached after a series of teaching experiments (Thompson, 1994b). In this study, I used thick description and purposive sapling techniques for ensuring the transferability of the study. The design principles and process of the model development unit, the experimentation, and the data analyses processes were explained in detail. Additionally, detailed results of four participants interpreted according to the data analysis frame were reported in long narratives supported with critically selected raw data reflecting particular ways of reasoning such as drawings, quotations from explanations, and episodes from the dialogues.

Dependability is the term used instead of reliability which also has meanings of consistency and repeatability. According to Lincoln and Guba (1985), the techniques used for credibility also serve for the dependability of the study. However, using an inquiry auditor or step-wise replication of the study are additional techniques that can be used. Because design studies accepted to be unique according to the setting they used or according to the group of participants, repeatability of design studies is also a controversial issue as it is in qualitative studies (Kelly, 2004). According to Cobb and Gravemeijer (2008), thick descriptions of settings, design principles, interpretive frameworks, and theoretical arguments serve the repeatability of the study. But, repeatability does not mean directly doing the same things; rather the detailed practical and theoretical information provided in the report can guide other researchers to customize and modify the design according to the needs of setting that they will use. In this study, in addition to the triangulation of data collection tools, two researchers continuing their PhD studies in mathematics education helped me during the development of research tools and experimentation process. I always consulted their ideas during the experimentation process, and they participated to the implementation process as observer. For example, the questionnaires were carefully examined by the two researchers and by the instructor. After obtaining the categories in the data analysis process, I prepared booklet explaining the data analysis frame in detail (see, Appendix-J1). Also, a small set of data was analyzed by a dependability audit who was an expert in mathematics education. After a brief instruction, the audit analyzed a small number of data sources from each data set according to the categories developed by me. The audit's selection of categories for particular incidents was nearly consistent with my categorization. But, when this was not case, the categories were revised.

The last construct of the trustworthiness is the *confirmability* which is used in place of the objectivity. In quantitative studies, researchers are expected to put inferences in an objective manner. However, studies involving naturalistic inquiry look events holistically considering not only the variables but also the participants, the context, and the researcher's role etc. Confirmability, at that sense, means if the researcher can make a logical explanation about the findings by supporting with the raw data (Yıldırım & Şimşek, 2006). Again, triangulation, and thick description of the data collection tools and the analysis of them contribute to the confirmability. Additionally, confirmability audit and reflexive journal of the researcher are mentioned as two basic techniques (Lincoln & Guba, 1985). In this study, I used a confirmatory audit who was a PhD student in mathematics education for controlling the inferences drawn from the data. He examined the categories and the related incidents from the raw data, and controlled for the consistency of the interpretations.

In summary, I used triangulation, prolonged engagement, in-depth methodological description, thick and rich narratives in reporting the results, peer debriefing, member checking, and dependability and confirmability audit techniques for ensuring the trustworthiness of the study. Namely, in this study, the argumentative grammar which is the term used for "*the scheme of argumentation that characterizes a particular methodology*" tried to be satisfied by showing how pre-service teachers developed sophisticated forms of reasoning and clearly documenting the major shifts in their understanding by also coupling them with the context (Cobb & Gravemeijer, 2008, p.85). The findings was reported as detailed as possible supporting with the narratives, episodes from dialogues, and handwritten drawings of participants.

3.4.4 **Ethical Issues**

Ethical issues were considered during the data collection and data analysis process. In qualitative studies, ethical problems may appear during data collection and analysis process (Merriam, 1998). During data collection, participants may be disturbed physically or psychologically. To ensure these problems, the participants should be informed at the beginning of the study about the data collection methods. In this study, at the first meeting of the course, pre-service teachers were informed about the study conducted as a part of the course. The usage of three vide-cameras and six audio-tapes for each group, reflection papers, questionnaires, and weekly interviews were explained in detail. After providing this information, 20 pre-service teachers decided to register to the course. At the second week, we delivered informed consent forms to the pre-service teachers explaining the aim of study and the data collection methods in detail (see, Appendix-K). They reminded to be free to withdraw being a participant of the study at any time in the process. All participants voluntarily signed the forms. Additionally, the researchers participated to the study as an observer without actively involving the process, and they tried to behave non-obstructively during the classroom periods.

I also considered the ethics in data analysis and reporting of findings. First of all, the names of pre-service teachers were not used explicitly in anywhere. In reporting the data analysis, pseudonyms were used for the four pre-service teachers whose processes were reported in detail. Only the researcher knew for which participant the pseudonyms were used for. The results of questionnaires and follow up activities were not used for grading pre-service teachers, and the results were not announced to them in public. However, the results were specially shared with the pre-service teachers who wanted to know about. At the end of the process, all participants were informed about the general findings of the study.

CHAPTER 4

RESULTS

This section was organized around four issues. The results of pre-service teachers' conceptual understanding of derivative were reported first by using the descriptive statistics from the questionnaires. The descriptive results of Questionnaire-1 and Questionnaire-2 were reported in order to provide a general idea about the developments in pre-service teachers' conceptual understanding of derivative. Later, qualitative analyses of data for four pre-service teachers were reported. Because developments in pre-service teachers' conceptions were focused, the chronological order of the activities was considered while reporting the results. To remind again, the following research questions guided this study:

1) What is the nature of pre-service mathematics teachers' existing conceptions related to the big ideas involved in derivative prior to attending the classroom experimentation of the model development unit?
    - What is the nature of covariational reasoning that pre-service teachers demonstrated prior to or at initial phases of the model development unit?
    - What is the nature of pre-service teachers' conceptions of rate of change prior to or at initial phases of the model development unit?
    - How did pre-service teachers interpret the graphical connections between a function and its derivative prior to or at initial phases of the model development unit?
2) What conceptions with regard to the big ideas involved in derivative did pre-service mathematics teachers develop as they attended to the model development unit?
    - How did covariational reasoning demonstrated by pre-service teachers change during the process?

- What is the nature of developments in pre-service mathematics teachers' conceptions of rate of change during the process?
- How did pre-service teachers' interpretations of the graphical connections between a function and its derivative change during the process?

The first research question finds its answer with the results of Questionnaire-1 and the early results from the group works on the modeling activities. The second research question finds its answer with the presentation of the results taking into account the developments in pre-service teachers' conceptions.

4.1 Quantitative results for pre-service teachers' understanding of derivative

Before starting to the implementation of unit, Questionnaire-1 was administered to get ideas about pre-service teachers' pre-requisite knowledge about the ideas involved in derivative covered in model development unit. The descriptive results are demonstrated on the table below.

Table 7: Descriptive Results of Questionnaire-1&2

		Quetionnaire-1		Questionnaire-2	
Questions	Out of	Mean	S.D	Mean	S.D
Q1	10	5.5	4.8	5.2	4.9
Q2	10	4.5	3.2	6.2	3.1
Q3	10	7.4	3.1	7.7	2.6
Q4	10	6.7	4.7	8.5	2.8
Q5	10	5.1	2.2	7.0	3.0
Q6	10	5.2	2.5	7.1	3.3
Q7	10	3.7	3.2	6.8	3.6
Q8	10	3.7	1.8	6.3	3.1
Q9	10	5.7	1.5	6.1	1.7
Q10	10	5.4	2.7	6.8	2.5
Q11	10	3.8	3.9	7.8	2.9
Total	**110**	**56.8**	**19.3**	**75.6**	**19.2**

The total mean scores and the mean scores for each question is presented on the table above. Looking the answers provided for some of the questions qualitatively, interesting results observed. In the 2nd questions (Q2), average rate of change of a context-free function was asked. In Questionnaire-1, only 4 participants answered the question by using the formula $\frac{f(5)-f(3)}{5-3}$. The other 16 participants could not understand the question or answered by using irrelevant procedures. In Questionnaire-2, 10 participants answered this question correctly. The other participants continued thinking with irrelevant procedures. The outstanding mean score increase was observed in the 7^{th} and 11^{th} questions. In the 7^{th} question, some values of a function were provided with a tabular form, and students were asked to approximate the derivative at a particular point. In Questionnaire-1, only two pre-service teachers could answer this question by using the difference quotient rule from left and right side. On the other hand, more than half of the pre-service teachers answered this question correctly in Questionnaire-2. Similarly, the 11^{th} question was answered by only 3 pre-service teachers in Questionnaire-1, but about two third of them could answer the question correctly in Questionnaire-2. The descriptive results of Questionnaire-1 and Questionnaire-2 indicate some developments in pre-service teachers' conceptual understandings and these results were provided as being supportive for the qualitative analysis. In the following sections, the qualitative analysis related to the details and nature of developments in pre-service teachers' conceptual understanding of derivative in the process of the model development unit.

4.2 Characterizing pre-service teachers' covariational reasoning

During the data analysis, a framework was obtained for characterizing pre-service teachers' covariational reasoning. In the framework obtained in this study, there are three dimensions of covariational reasoning; (i) identifying the variables, (ii) way of coordinating the variables, and (iii) quantifying the variation in rate of change. For the details of the categories and sub-categories, see Appendix-J1.

Table 8: Coding schema used for analyzing covariational reasoning

Categories	Sub-categories and Abbreviations
Identifying the variables (IV)	Thinking by primary variables (IV-PV) Thinking by secondary variables (IV-SV) Thinking input and output variables in reverse order (IV-RO)
Way of coordinating the variables (WOC)	Uncoordinated way of thinking (WOC-UC) Indirect Coordination (WOC-IC) Direct Coordination (WOC-DC) Direct and Systematic Coordination (WOC-DSC)
Quantifying the variation in rate of change (VRC)	Gross Quantification (VRC-GQ) Extensive Quantification with *Additive comparison (VRC-EQAC)* *Unit per unit comparison (VRC-EQUC)* *Multiplicative comparison (VRC-EQMC)*

In this section, pre-service teachers' ways of covariational reasoning were reported by considering the developmental process for each of the three dimensions. Therefore, the chronological order of the tasks was also considered in reporting.

4.2.1 Pre-service teachers' ways of reasoning while identifying the variables in the tasks involving simultaneously changing quantities

Pre-service teachers demonstrated three different ways of reasoning while identifying the variables in the contextual tasks involving simultaneously changing quantities. These are, (i) thinking by primary variables, (ii) thinking by secondary variables, and (iii) thinking input and output variables in reverse order. Thinking by primary variables involves consideration of the covarying quantities as dependent and independent variables. Thinking by secondary variables involves consideration of an unnecessary variable as the independent variable. For instance, in a question asking the surface area as a function of volume, considering "time" as the independent variable can be given an example for this. Thinking input and output variables in reverse order means changing the roles of dependent and independent variables. In the following parts, pre-service teachers' ways of reasoning while identifying the variables were reported by taking into account the changes in their reasoning in the process.

At the beginning of the model development unit, pre-service teachers were administered the Questionnaire-1.The “Cassette Player” problem was related with the covariational reasoning (see, Appendix-G1). When the answers for “Cassette Player” problem analyzed, although change in the radius of first reel with respect to the radius of second reel was asked in the question and these variables were expected to be considered as dependent and independent variables, pre-service teachers generally considered an unnecessary variable (i.e., time) as the independent variable.

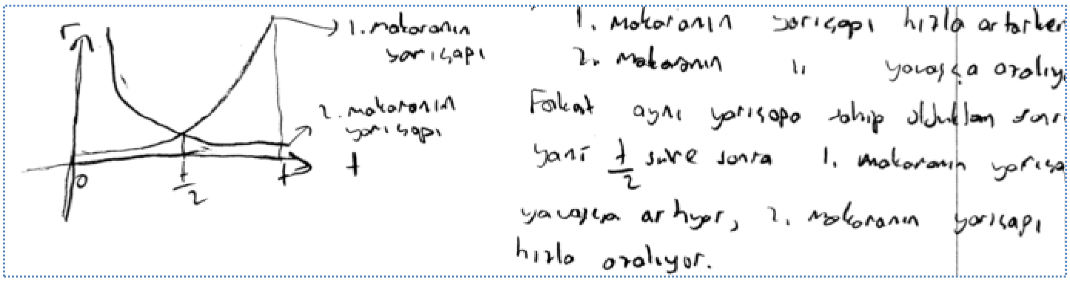


Figure 11: Halit’s answer for Q5 in Questionnaire-1

For instance, Halit considered “time” as the independent variable in the verbal explanation and on the graph as can be seen on the figure above. He did not form a functional relationship between the covarying variables, which were the radius of first reel (R1) and the radius of second reel (R2). Similarly, Beyza and Nilgün also considered “time” as the independent variable in their graphs and in their verbal explanations. As can be seen in the figure below, Beyza also used “time” as the independent variable. She provided time-based verbal explanation and she drew two graphs on which time was plotted as independent variable on the horizontal axes.

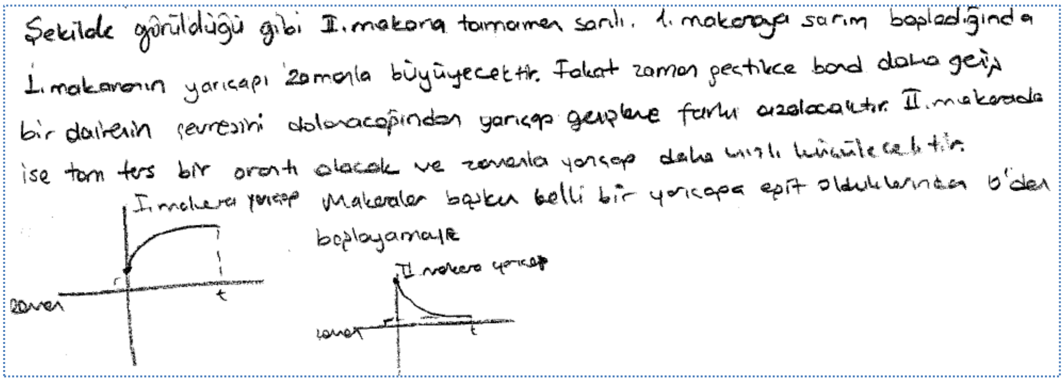


Figure 12: Beyza's answer for the 5th question in Questionnaire-1

Rana's way of reasoning was a little different. She plotted the covarying variables as independent and dependent variables on the graph. But, by the expression "*the rate of increase in the radius of the first reel is greater than the rate of decrease in the radius of second reel up to a point"* she indicated a secondary independent variable implicitly. Although Rana thought by considering the primary variables on the graph, she implicitly considered "time" in her verbal explanation.

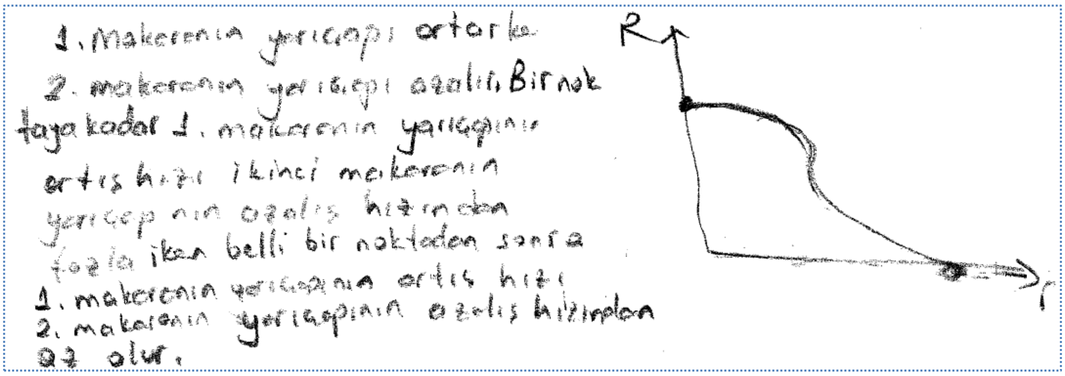


Figure 13: Rana's answer for the 5th question in Questionnaire-1

In the "Water Tank" modeling activity, pre-service teachers demonstrated three of the different ways of thinking. To begin with, Halit and his group's solution to this activity were depending on the radii of the cross-sectional circles inscribed in the shapes of tanks. They did not directly use volume and height as independent and dependent variables, instead they considered the radii of cross-sections. They explained their solution as follows:

> …If the radius (of cross-sectional area) is constant, the graph will be linear, if the radius is gradually increasing, the graph will be parabolic with decreasing slope, and if the radius is decreasing the graph will be parabolic with increasing slope. That is to say, while r is increasing, since increase in the volume take more time, the increase in the height is getting slower. (Halit's group report for the "Water Tank")

As can be seen in the excerpt from the group report, while deciding the height as a function of amount of water, Halit and his group considered radii of cross-sections in place of volume. They were depended their reasoning on the radius of shapes. In group discussion, they discussed about which was the independent variable (i.e., time, volume, or radius). Halit indicated this process as follows:

> R: Halit, you mentioned about a hesitation with your group on the issue that if the graph is height-volume or height-time. How did you conclude this discussion?
>
> Halit: We tried to introduce a general structure for height-volume graph (we thought for the first figure). At the beginning, we draw the graphs as height-volume, but we realized that although it can be interpreted from the figure, the volume values are not certain. Since the volume values are not certain some people offered drawing a height-time graph. Then I compared and thought about the difference between height-volume and height-time, and there is no difference…There may be some changes in the numerical values on the graph, but the character of the graph does not change.
>
> R: What stays unchanged on the graph?
>
> Halit: The character of the graph stays the same… because the values of volume are not certain and cannot be specified we focused on height-time graph

During the group discussion, they decided that it does not make any difference considering time or considering amount of water as independent variables. Halit also indicated that they wanted to think by considering "time" as the independent variable because of the inappropriateness of volume for scaling on the tank figure. He stated that "*we can scale height by measuring with a ruler on the paper, but it is difficult to scale, for example, equal amounts of water on a spherical tank figure*". Nevertheless, they decided to use volume as the independent variable when plotting on graph. But, in their verbal explanations and discussions, they used the radius of cross-sectional circle in the form of independent variable and they decided variation in volume by depending on the change in radius which is coded as *thinking with secondary variables*.

When we look at Beyza and her group's solution to the "Water Tank" modeling activity, they plotted volume and height as independent and dependent variables while drawing the graph, but they used time as independent variable in their verbal explanations. In the reflection paper, Beyza explained their reasoning on the variables as follows.

> …We confused regarding to the drawing of the graphs with respect to what. We confused with the expression that *height with respect to volume*. But later, by considering the flow rate of water constant, we discussed and realized the volume and time variables may be used interchangeably and so the height-volume and height-time graphs will be the same. (Beyza, Reflection Paper on Water Tank)

In the excerpt above from the reflection paper, Beyza explained their discussion about the independent variable. They confused considering time or volume and decided that, if constant flow-rate assumed, time-based or volume based thinking does not make any difference. This shows that they could not distinguish the difference between considering time and considering volume as independent variable. In the interview after the classroom, I questioned Beyza about the variables. Because, they considered time or volume as similar variables when the constant flow rate was assumed, I asked her to think about the volume-height relationship with varying flow rate. The episode is below.

R: Okay Beyza, let's imagine the following. Let's say the water poured drop by drop in the first region, it poured a little fast in the second region, and it poured rapidly in the third region (Speaking about the 3rd tank figure). How the height-volume graph changes in this condition?

Beyza: This is the situation when flow rate is different. In my opinion, the intervals change on height-volume graph. In other words, the type of curve does not change, since the tank enlarges the graph will be decreasingly increasing, but the intervals on the volume axes may change. For instance, when we accepted a constant flow rate, we accepted and plotted a short interval for 0-V1.

R: What is the corresponding height for V1?

Beyza: Let's say it is h. Then we can accept this one as 3h (the corresponding height forV2). But I wonder if these are proportional or not? For instance, if the flow rate for the first interval is 0,5V and the flow rate for the second interval is V, then they would be equal…

R: What is equal?

Beyza: The volume… No, no, the time for reaching the same height is equal…

R: Is that affecting the height-volume relationship?

Beyza: No, when drawing the height graph with respect to volume, hmmm, it is different of course. We considered time because of thinking with the constant flow rate…

R: What would be different if the flow rate was not considered as constant?

Beyza: The times would be different.

R: Can we expect any difference in other parts?

Beyza: In my opinion, hmm (thinking a few seconds), there would be no difference in volume, because the tank is constant and it has a particular shape.

R: What difference can be observed?

Beyza: the only difference would be the time. In other words, even if the flow rate is not constant, the height-volume graph does not change, yes…

At the beginning, Beyza stated that "*The graph does not totally change; only the lengths of intervals of volumes plotted at horizontal axis changes*". Thinking volume as a function of time is observable by this explanation. By emphasizing the increase in the length of intervals on the horizontal axis which represents the volume, she indicated variation in the period of time for filling particular amounts of volume. Although she plotted volume as an independent variable on the graph, she was still thinking by considering time as the independent variable implicitly. By assigning some particular and proportional symbolic values to the points plotted (h and 3h), I wanted her to compare what stays constant or changes if the flow rate is 0,5V/sec in the first interval and 1V/sec in the second interval. She realized that the time required for filling the first interval (0-h) and the second interval (h-3h) was equal, but the corresponding volumes in those intervals were not equal. I also asked if the volumes in those intervals changes or not with these different flow rates. She realized that the volume corresponding to a particular interval (represented on graph or figure) does not change with the time or flow rate and justified her claim by stating that the figure of the tank does not change. I then asked her again about the "*time*" variable that they used in their group report.

R: You used time variable in your group report, what do you think about its usage now?

Beyza: Actually, we had used time as an extra variable, it is an extra variable.

R: Why did you need the "time" variable?

Beyza: The reason for using "time" as variable is our previous experiences. We got used to draw time-dependent graphs. The variable on the horizontal axes generally occurs "time", and the variable on the vertical axes may be "height", "volume", "area" and something like these. The same situation appears in distance-time and velocity-time graphs. Plotting "time" on the horizontal axis is a habitual for us…

Beyza realized that considering "time" as the independent variable was not necessary for analyzing volume-height relationship. For the question asking why they needed time as independent variable, Beyza mentioned about their previous experiences with functions and graphs. She indicated that their previous experiences were limited with the use of time-based contexts as the contextual examples of functions and graphs.

Thinking differently on graphs and in verbal explanations was also observed in Rana's group. Rana and her group used volume and height as independent and dependent variables respectively when plotting on the graph, but they used time-based reasoning in their verbal explanations. For explaining the graphs, they used the expression that “the rate of increase in height is greater than the rate of increase in volume (both with respect to time)”. In this verbal explanation, they considered volume and height as two separate functions of time and compared the rates of change of these two functions. Rana explained their reasoning about the variables in the interview as follows:

> R: You said, in your reflection paper, that we thought about which variables should be labeled for horizontal and vertical axes. Could you explain the discussion process in detail?
>
> Rana: Firstly, we drew the coordinate axes. As I also indicated in the reflection paper, one of the variables generally exists as “time”. “We generally use “time” as independent variable. In that situation, should we consider time, because, in fact we are gradually filling the tanks by water. They are filling from bottom to top. In fact, there is time here.
>
> R: Yes
>
> Rana: But “time” was not the variable that we want to focus on. We plotted “time” to the x-axes at first. Then we thought on if we were looking at height as a function of time. If so, the variations in the tank figures were becoming meaningless. Then we decided once more to determine the difference when thinking with time or volume. For instance, if we think with the time, then the flow rate and other variables need to be considered. Because we could not know at what minute the water level passes to the second region, we left thinking with time. We considered the volume of water. When we consider time as independent variable, the tank can be filled not continuously or the flow rate may vary, and since there is no information provided related to these issues…

In the episode above, Rana stated that they thought time as the independent variable at first and plotted it on the horizontal axis. She explained the reason of their time-based reasoning with their previous experiences by stating that “... *we generally use time as independent variable”.* She mentioned about their confusion for considering time or volume as the independent variable. When we look at the expressions that “*when we consider time as independent variable, the tank can be filled not continuously…, the flow rate can vary*”, Rana seemed to realize the needlessness of considering time as independent variable. But still, Rana’s time-based reasoning continued from time to time.

Nilgün and her group also demonstrated three different ways of thinking about the variables. Nilgün and her group mentioned about time, radius of cross-sectional areas inscribed in tank figures, and volume as possible independent variables. Although they plotted volume and height as independent and dependent variables on the graph, they used radius and time as other independent variables. I asked Nilgün to explain their reasoning process while drawing the graphs.

> R: Can you explain how you draw the graphs?

Nilgün: We thought with the radius. Because the shape of the tank is not regular, it enlarges or narrows, the tank cannot be filled with constant volumes. As the shape of water tank enlarges through up, the speed of filling gets slower.

Nilgün indicated that they constructed the graphs by considering the change in the radius of cross-sectional circles. Their reasoning was as "*if the radius of cross-sectional circle does not change, the height-volume graph will linearly increase*". Taking the change in radius of cross-sectional circle into consideration, Nilgün and her group drew the graphs by plotting volume and height as independent and dependent variables respectively which was accepted as thinking with primary variables. However, they used radius and time as other independent variables. As clearly indicated by Nigün, they considered radius as an independent variable while drawing the graphs.

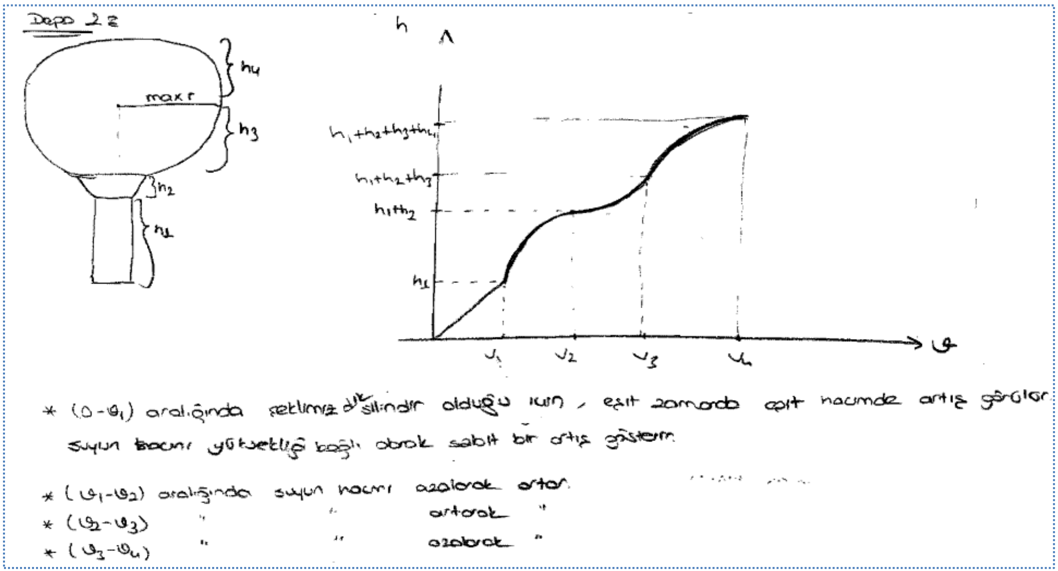


Figure 14: An excerpt from Nilgün's group report for the "Water Tank" activity

One more thing observed in Nilgün's group report was related to verbal expressions of the graphs. Although plotting on the graph correctly, Nilgün and her group indicated the functional dependency between volume and height reversely. The expression that they stated for the first interval "*Volume increases constantly with respect to height…*"indicates volume as dependent variable and height as the independent variable. In the verbal expressions for other graphs, they continued using dependent and independent variables in reverse order.

In the “Sliding Ladder” problem, pre-service teachers continued thinking by secondary variables such as “angle” or “time”. Halit, Beyza, and Rana considered “angle” as the independent variable. Moreover, although they plotted independent and dependent variables on the correct axes on the graph, they tried to use “*angle*” as the independent variable. As appeared in the episode below, Halit tried to determine and coordinate the relationship between the distance of point-A and the height of point-B by taking into account the changes in “*angle*” formed between ladder and ground.

> Halit: At the beginning, the angle will decrease as the ladder slides down...
>
> R: How long the angle is important here?
>
> Halit: The angle? The angle is important here. I considered the situation at 45^0. Let’s consider the following situation. For the 3-4-5 right triangle, what is the value of that angle? I am doing for showing the importance of 45 degree. The angle here is 53 degree. Later, I changed the position of ladder so formed a 4-3-5 triangle. The angle at that position is 37 degree. Later it will be 0 as the ladder lies on ground….

Similarly, although Beyza plotted independent and dependent variables on the correct axes, she used “*angle*” as the independent variable (a secondary variable in this context) while deciding the nature of curve.

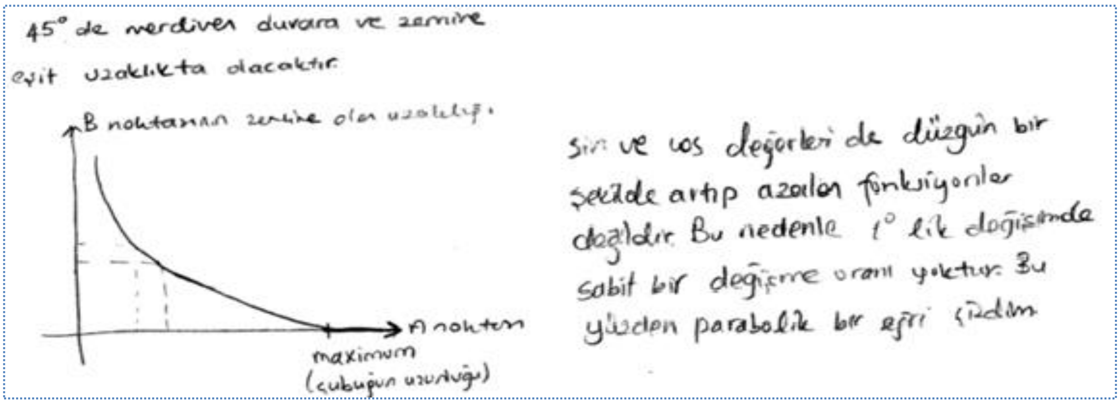


Figure 15: Beyza’s answer for the “Sliding Ladder” problem

Nilgün, on the other hand, continued thinking with “time” as the independent variable. As can be seen on the figure below, instead of thinking the height of point A from ground level with respect to distance of point B from the wall, she considered both variables as a function of time and drew two distinct linear graphs.

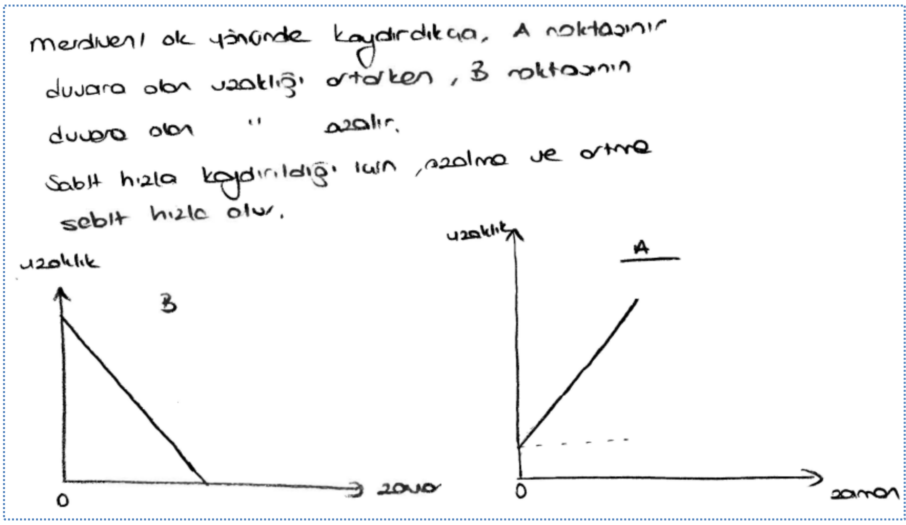


Figure 16: Nilgün's answer for the Sliding Ladder problem

Nilgün realized the possibility of forming direct mathematical relationship between covarying quantities by means of Pythagorean Theorem during the interview. By assigning some particular values to both variables, she also observed the functional relationship in detail.

Table 9: Pre-service teachers' ways of reasoning while identifying the variables across different tasks

	Task-1 Cassette Player	Task-2 Water Tank	Task-3 Sliding Ladder	Task-4 Water Tank-2	Task-5 Space Shuttle
Halit	IV-SV	IV-PV IV-SV	IV-PV (on Graph) IV-SV (Verbal)	IV-PV	IV-PV
Beyza	IV-SV	IV-PV (on Graph) IV-SV (Verbal)	IV-PV (on Graph) IV-SV (Verbal)	IV-PV	IV-PV IV-RO
Rana	IV-PV IV-SV	IV-PV (on Graph) IV-SV (Verbal)	IV-PV (on Graph) IV-SV(Verbal)	IV-PV	IV-PV
Nilgün	IV-SV	IV-PV (on Graph) IV-RO IV-SV (Verbal)	IV-SV	IV-PV	IV-PV

Pre-service teachers' ways of reasoning while identifying the variables appeared across the sequence of five tasks is seen on Table 9. Almost all of the pre-service teachers started to think by primary variables with the "Water Tank-2" problem and continued thinking in the same way while answering the "Space Shuttle" problem (the 5th question) asked in Questionnaire-2. Pre-service teachers' ways of reasoning while identifying the variables shifted from thinking by secondary variables to thinking by primary variables in the process.

4.2.2 Pre-service teachers' ways of coordinating the variables in the tasks involving simultaneously changing quantities

Pre-service teachers demonstrated four different ways of reasoning while coordinating the variables. These are, (i) uncoordinated way of thinking, (ii) indirect coordination, (iii) direct coordination, and (iv) direct and systematic coordination. Uncoordinated way of thinking involves consideration of covarying quantities as two separate functions with respect to another variable. Indirect coordination means deciding the nature of functional relationship between covarying quantities by using an extraneous variable as an implicit independent variable. Considering the radii of cross-sections of a bottle as the independent variable in place of volume while deciding the volume-height relationship in a filling bottle context can be an example for this. Direct coordination involves directly coordinating the primary variables and expressing a linear relationship. Direct and systematic coordination means systematically changing the input variable and observing the simultaneous variation in the output variable.

To begin with the "Cassette Player" problem in Questionnaire-1, Halit, Beyza, and Nilgün used an *uncoordinated way of thinking* while coordinating the covarying variables. As seen in Figure 11, Halit's time-based reasoning was observable. His verbal explanation "*the radius of first reel rapidly increases and the radius of second reel slowly decrease by the time*" clearly shows the uncoordinated way of thinking. By using "*time*" as the independent variable, the radius of first reel and the radius of second reel became two separate quantities changing as a function of time. This was also observed in the answer of Beyza seen in Figure 12more explicitly. Beyza considered the radius of full side (R1) and the radius of empty side (R2) as two separate functions changing with respect to time. Therefore, she explained the change in the radii of both reels with respect to time independently. Additionally, she constructed two separate radius-time graphs without relating the radius of full side and the radius of empty side directly. Uncoordinated way of thinking possibly stems from thinking by secondary variables.

Rana, on the other hand, *directly coordinated* the covarying quantities when plotting on the graph, but an *uncoordinated way of thinking* was observed in her verbal explanations. As seen in Figure 13, Rana determined radii of both reels as dependent and independent variables and plotted on the graph. But, in the verbal explanation "*the rate of increase in the radius of the first reel is greater than the rate of decrease in the radius of second reel up to a point…*" she indicated two separate functions. According to this expression, the radius of first reel (R1) is a function of a variable not indicated explicitly and the radius of second reel (R2) is a function of the same variable. She compared the rate of change in both reels with respect to the implicit variable.

When we consider pre-service teachers' ways of coordinating the variables in "Water Tank" modeling activity, the common way of thinking observed at initial phases was *indirectly coordinating* the height and volume variables. Halit and his group reasoned by considering the radii of cross-sections. As can be seen in Figure 17, thinking with "*radius*" as an independent variable resulted in *indirect coordination* between covarying quantities. Although they plotted amount of water as independent variable and height as dependent variable on the graph, they used the radius of cross-sectional circle while deciding the graph of height as a function of volume. That means, they indirectly related volume and height. In addition, from time to time an *uncoordinated way of thinking* was observed. In the reflection paper for the "Water Tank" activity, Halit provided the explanation that "*In the third region of the tank figure, while the volume increases more, the heights begin to increase less, and so we draw the graph with decreasing slope*". The expression "*while the volume increase more, the height begin to increase less*" shows uncoordinated way of thinking. He mentions about the change in the increase of volume and change in the increase of height independently, not interdependently.

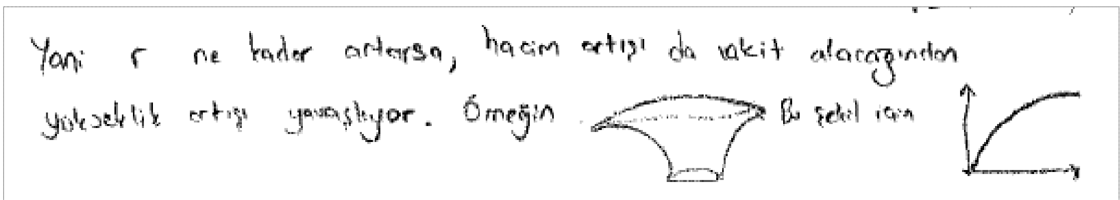

Yani r ne kadar artarsa, hacim artışı da vakit alacağından
yükseklik artışı yavaşlıyor. Örneğin Bu şekil için

Figure 17: An excerpt from Halit's group report for the "Water Tank" activity

In the same way, Rana with her group also used *indirect way of coordination* in “Water Tank” activity. They plotted the height-volume on the correct axes but, when we look at the verbal explanation, their time-based reasoning was observable. Rana explained their reasoning in the reflection paper as follows:

> Because the first region of the first tank figure is narrowing through upwards, in the process of filling the tank with water, change in height with respect to volume cannot be constant; contrarily change in the height gradually increases. Namely, the character of the curve for that interval must seem like . This curve shows that the amount of increase in height will become more than the amount of increase in volume as time passes. Because the third region of the first tank is enlarging through upwards, it can be said that the increase in height will gradually decrease when compared with the increase in volume. This shows the character of the graph should be like . (Rana, Reflection Paper, Water Tank)

In the excerpt above, Rana explained their reasoning in detail for drawing the graphs. She used enlargement or narrowing property of the tank shapes for explaining the curves. Also, she used time as the independent variable. For explaining the concave-up increasing graph, she stated that “*the amount of increase in height will become more than the amount of increase in volume as time passes*”. She explained various curves in the same way during the interview. For coordinating the covariation between height and volume, Rana indirectly coordinated the height and volume variables by using enlarging-narrowing property of tank shapes and time as independent variables. Nilgün with her group also used *indirect coordination* (radius-based reasoning). This is also pointed out by Nilgün in the reflection paper.

> While preparing the manuals and preparing the solution, we benefited from graph drawing and graph interpretation. By determining the maximum and minimum radii, we analyzed the nature of relationship between volume and height by also examining the height-radius and volume-radius relationships… In addition to these, we also benefited from the slope concept while preparing the manuals. When the figure of the tank changed, although the graphs have similar character, we explained the variations in graphs by the slope concept (Nilgün, Reflection Paper, Water Tank)

In the reflection paper, Nilgün restated their radius-based reasoning with the expression that “*we examined the relationships between height-radius and volume-radius and by depending on them we determined the relationship between volume and height*”. This expression shows Nilgün’s indirect way of coordination between height-volume by using the radius of cross-sectional circle as the other variable.

In the solutions of the “Sliding Ladder” problem, pre-service teachers demonstrated uncoordinated way of thinking, indirect coordination, and direct coordination thinking styles. Beyza, Halit, and Rana used *indirect coordination*. For example, as clearly seen in Figure 15, Beyza plotted point-A and point-B as independent and dependent variables respectively on the graph, but in the explanation she used “*angle”* as the independent variable. Her explanation that “*for equal amount of change in angle, the values of trigonometric functions do not change linearly*” shows using angle as an implicit independent variable for coordinating the direction of change between covarying quantities. Ran also used *indirect coordination* by taking into account “angle” as an independent variable. As seen in Figure 18, Rana assigned some symbols for the angles of the triangle, and also focused on the changes of these angles as the ladder slides down. Although, Rana plotted the vertical distance and horizontal distance as dependent and independent variables respectively while plotting on graph, she tried to decide the nature of graph by focusing on the change in the angles.

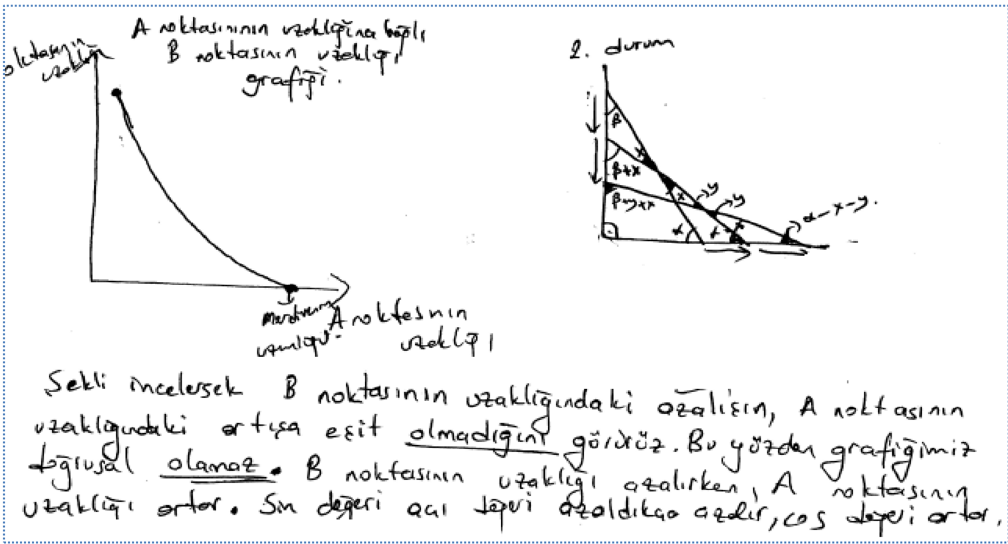


Figure 18: Rana’s solution for the Sliding Ladder problem

In the interview, I asked Rana about her reasoning on the “Sliding Ladder” problem and wanted her to rethink on it.

R: What are our (dependent and independent) variables? Think according to them.

Rana: Not angle. Is it constant? But, I tested it with 30 degree. It is not constant, but again is this because of degree?

R: It is asking about how the vertical distance changes with respect to the horizontal distance, but you are mentioning about angle as if it was a variable.

Rana: But there is angle when sliding the ladder, in fact when sliding the ladder per unit degree… Is it linear? But it was not linear when I tested at 30 degree. The increase here is more, or the increase at horizontal distance and the decrease in vertical distances are not equal.

R: Then, how does it occur?

Rana: My reasoning was as follows: for instance, let me consider the angle here, now it is 15 degree. Then the acute angles are 15, 75. Let me slide the ladder by 15 degree. Then the new angle here became 30 degree, and the other one became 60 degree. Here (the angle formed by the intersection of hypotenuses) is 15 degree. I thought such a way that, because these two triangles are not congruent, the changes in two distances are not equal.

R: Then, how?

Rana: I do not know, according to me, the change is not constant, I am still thinking like that. We can also see the inequality when thinking with Pythagorean Theorem, since there is a quadratic relationship.

R: How did you draw this graph?

Rana: I decided the change cannot be constant, because the length of hypotenuse is constant and while the x2 changes, y2 will also change accordingly. But, because the sum of squares will be equal to a constant; changes in two variables will not be equal. Namely, the change will not be constant; there will be a relation with squares. I randomly draw this graph.

R: Can't you transfer the idea in previous task to this context?

Rana: I cannot transfer. Change in vertical distance per unit time, no I cannot interpret.

R: Is it per unit time?

Rana: No, per unit distance. Is it unit centimeter? The unit centimeter change in point "a", with respect to point "b". When the point "a" increase per unit centimeter, the simultaneous change in the position of point "b"… Yes…

As seen in the episode above and in Figure 18, Rana reasoned by considering the changes in the angles as the ladder slides. She first assigned some particular angle values as 17-75-90 on the triangle, and then changed the base angle from 15^0 to 30^0 by sliding the ladder. The angles of the new triangle were 30-60-90. She tried to observe the variation in the rate of change by changing the values of angles. By this way, she observed the non-linear relationship, but constructed a concave up decreasing graphintuitively. She used indirect coordination by using angle between the covarying quantities. I prompted her to transfer the mathematical idea of unit per unit thinking for coordinating the covarying quantities. Although she mentioned changing the variable on the horizontal axes uniformly and trying to observe simultaneous variation in the variable on vertical axes, she could not apply. My question that "*Can you assign some particular values for the distances?"* behaved as a hint and she started to think by using *direct and systematic coordination*.

R: Can you assign some particular values?

Rana: hmmm, values for distances? Maybe, not as angles… Now, let me say this is 5-12-13. I think there will be no difference…

R: Okay, let's slide one unit.

Rana: This length is 13, and that one became 11. I should look the value at here.

R: You already slide down the point here.

Rana: hmm, I will look the value when this length is 6. Clearly, it cannot be 11. (Computing by Pythagorean Theorem), square root of 136, and then I slide one more unit, yeah its true, why I did not try in that way... Now it is 7, 7-13-what? Let me form a table, and the length of hypotenuse is always 13. I have to find the values for A and B, when A=5, then B=12, when A=6, then B=…, when A=8, then B= square root of 105.

R: Yes.

Rana: Now, I will look the rate of increase in horizontal distance, and rate of decrease in vertical distance. How can I interpret it? Let me say this is 144,

R: Why do you need looking the rate for each distance separately?

Rana: I can also look at the amounts, yes because the horizontal distance changes one unit increment. 9, 13, 15 (looking at the differences between radical expressions), it is increasingly decreasing, yes increasingly decreases. Okay, I made mistake since I thought with angles. The curve should be the inverse of what I already drew: What about this, this is decreasingly decreasing. Okay, I understood very well.

As seen in the episode above, Rana assigned particular values on the edges of the triangle formed by the position of the ladder. She slide down the ladder so that increasing the horizontal distance with one unit, and observed the necessity of finding simultaneous change in the vertical distance.

In the solution of the “Sliding Ladder” problem, Nilgün used *uncoordinated way of thinking* by taking into account “time” as the independent variable. As seen in Figure 16, Nilgün drew two separate graphs on which “time” was plotted as independent variable. She did not form a direct or indirect coordination between the height of point A and the distance of point B. In the interview, I asked Nilgün about her solution for the “Sliding Ladder” problem, and wanted her to rethink about it.

R: Let’s return the Sliding Ladder problem. You drew such a graph. If you solve again, how do you solve?

Nilgün: Now, it is indicated that by holding from the point A, the ladder is sliding on the ground with a constant speed… Hmmm, one minute, why did I accept the vertical and horizontal distances equal, it seems I have accepted them (interpreting her previous solution)…

R: You drew the graphs by taking into account the “time”. Should you consider the “time”?

Nilgün: If I reason with per unit time… (Thinking about 5 seconds)

R: You can use the basic idea covered in Water Tank…

Nilgün: Then I reason with per unit time, it is sliding with constant speed… Now let’s say this covered x amount of distance (horizontal axis). I thought in that way because it is sliding with constant speed. When this point came to this point per unit time, that point will slide through downward like that. Now, because this was pulled away with constant speed (the point on the horizontal axis), I wonder if the speed at that point (the point on the vertical axis) is different or not… (Thinking)…

R: I think the speed was not asked to you. It is asking the change in the distance of point B with respect to the distance of point A.

Nilgün: Here I try to interpret the relationship between these two distances by using the speeds… But I am not sure; I cannot see how the correct graph is… Are these wrong? (Asking about the linear drawings)

In the episode above, Nilgün continued reasoning with *time* as the independent variable. When I asked her the reason of considering "time" as independent variable, she indicated the constant speed given in the question as the reason. She also tried to compare the speeds at the two ends of ladder. When I reminded her that the question did not ask about the speeds, she stated that "*I want to interpret by using the speeds*". Because she did not reach an idea by comparing speeds, I prompted her to think without using "speed" or "time" as independent variables. I also recommended Nilgün using the height of point B and the distance of point A as dependent and independent variables.

R: You may think without considering the time… Think these two variables as the independent and dependent variables. …

Nilgün: Hmmm, if we slide that point with x amount, the other point will go here. When we slide one more x, the other point will come here. When sliding thirdly, the point will be here. The length of hypotenuse does not change… I said this length was h when that was x, and hypotenuse is the same. Now, this length will be 2x, when this part increased with equal increments, it seems as if the length of height will decrease with same amounts… At the beginning, the hypotenuse was $a^2=x^2+h^2$, later $(2x)^2$ plus something will be equal to the same hypotenuse. Namely, the height will decrease, at first it was $h^2=a^2-x^2$, subsequently it will be $h^2=a^2-4x^2$, and later it will be $h^2=a^2-9x^2$. That is to say, one minute… hmmmm… the height will decrease increasingly…

R: Okay, can you draw its graph?

Nilgün: Now, the height decreases starting from a particular point… I divided the horizontal axes unit by unit by equal x distances. Now, the height increasingly decreases here… (By plotting the points on the coordinate axes), let it was here first when this is x, later when the horizontal distance became 2x, the height decreased increasingly, let's plot here. Later it will decrease more and continue in that way. Then, the graph should be like that (drawing a concave-up decreasing graph) in my opinion, but I am not sure.

R: Okay.

Nilgün: Can you say if it is correct or not?

R: In fact you can decide by yourself, because you solved the critical part of the problem, and you used the unit per unit comparison idea too.

Nilgün assigned x and h symbols for the edges and a for the hypotenuse of the triangle formed with the leaning against position of the ladder. She drew two other static positions formed by sliding the ladder x unit horizontally. For each x unit increment on the horizontal axes, she could not assign the simultaneous change in vertical distance at first. But, she observed the necessity of finding simultaneous change in the vertical distance. Because the length of the ladder (the hypotenuse) was constant, she realized the Pythagorean Theorem as the appropriate mathematical operation for finding the change in vertical distance. By changing the horizontal distance with equal increments, she observed the simultaneous change in vertical distance using the Pythagorean Theorem. By obtaining the new h values as a result of change in horizontal distance as $h_0^{\,2} = a^2 - x^2$, $h_1^{\,2} = a^2 - 4x^2$ and $h_2^{\,2} = a^2 - 9x^2$, she observed the increasingly decreasing nature of the height with respect to horizontal distance. This was the first time that Nilgün used direct and systematic coordination.

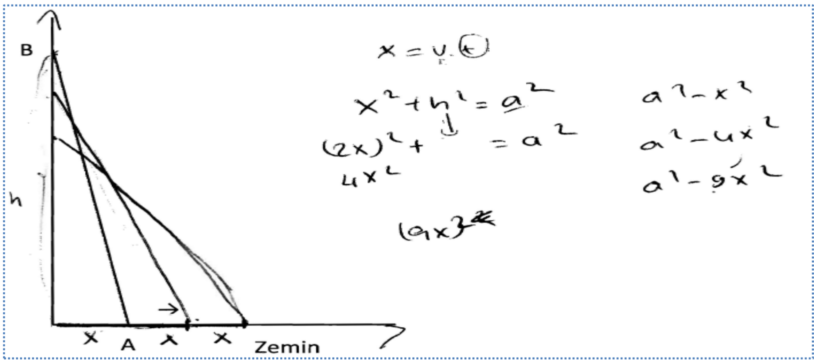


Figure 19: An excerpt from Nilgün's solution for the "Sliding Ladder" problem during interview

Halit also got the idea of *direct and systematic coordination* during the interview. While speaking on the solution of the "Sliding Ladder", I probed Halit to restate the main mathematical idea in the "Water Tank" modeling activity and wanted him to transfer this idea to the Sliding Ladder context.

> R: Can you transfer the main idea in that problem to the Sliding Ladder? What was the mathematical idea in Water Tank?
>
> Halit: The main idea was the change in height with respect to unit volume…
>
> R: The unit volume, at which axes is the volume?
>
> Halit: x-axes… the height with respect to unit distance hmmm (thinking a few seconds)… change in height with respect to distance. Then, I have to change the distance with equal amounts and tried to observe the change in height…

In the above episode, Halit stated "*change in height per volume*" as the main idea in "Water Tank" activity, and he tried to transfer this idea to a new context. Thinking for a while, he explicitly realized the procedure that systematically and uniformly changing (1 unit increments) the quantity of independent variable and observing simultaneous variation in the output variable.

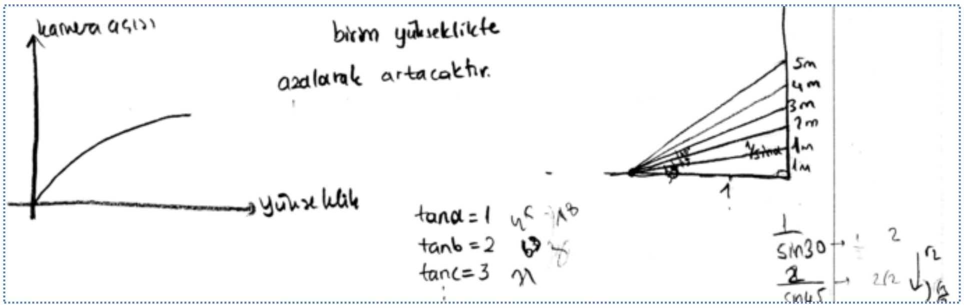


Figure 20: Beyza's answer for the Space Shuttle problem in Questionnaire-2

Before the administration of the "Water Tank-2" and the "Space Shuttle" problems, a comprehensive classroom discussion was conducted on the basic ideas involved in the "Water Tank" and "Sliding Ladder" problems. After that, most of the pre-service teachers started to use *direct and systematic way of coordination*. For example, in solving the "Space Shuttle" problem, Beyza systematically changed the height which is the independent variable and observed the simultaneous variation in the angle of camera (see, Figure 20).

Table 10: Pre-service teachers' ways of coordinating the variables across different tasks

	Task-1 Cassette Player	Task-2 Water Tank	Task-3 Sliding Ladder	Task-4 Water Tank-2	Task-5 Space Shuttle
Halit	WOC-UC	WOC-UC WOC-IC	WOC-IC WOC-DC WOC-DSC (Int)	WOC-DSC	WOC-DSC
Beyza	WOC-UC	WOC-IC	WOC-IC WOC-DSC (Int)	WOC-DSC	WOC-DSC
Rana	WOC-DC WOC-UC	WOC-IC	WOC-IC WOC-DSC (Int)	WOC-DSC	WOC-DSC
Nilgün	WOC-UC	WOC-IC	WOC-UC	WOC-DSC	WOC-DC

When we look at the process in general, it is clear that pre-service teachers learned the idea of systematically changing the independent variable and observing the simultaneous variation in the dependent variable. As can be seen on the table above, while they were thinking in an uncoordinated way or in the form of indirect coordination at initial phases, they started to apply direct and systematic way of coordination successfully.

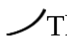
4.2.3 Pre-service teachers' ways of reasoning while quantifying the variation in rate of change

Quantifying the variation in rate of change in the contextual tasks involving simultaneously changing quantities was determined as the third critical dimension of covariational reasoning. Pre-service teachers demonstrated two different ways of quantification. These are; (i) gross quantification, and (ii) extensive quantification. Gross quantification means deciding the variation in rate of change of covarying quantities perceptually without providing a mathematical justification. Extensive quantification of the variation in rate of change involves the usage of some quantitative operations. In this study, pre-service teachers used three quantitative operations which are (a) additive comparison, (b) unit per unit comparison, and (c) multiplicative comparison. Extensive quantification with additive comparison means changing the input variable with equal increments and additively comparing the simultaneous change in the output variable. Extensive quantification with unit per unit comparison involves uniformly changing input variable, and observing the simultaneous change in output variable. Extensive quantification with multiplicative comparison appears when changing the input variable with equal increments and multiplicatively comparing the simultaneous change in the output variable. In the following parts, pre-service teachers' ways of reasoning while quantifying the variation in rate of change have been reported by taking into account the changes in their reasoning in the process.

To begin with the pre-service teachers' ways of reasoning while deciding the variation in rate of change for the first task that of "Cassette Player" question asked in Questionnaire-1, most of them used *gross quantification*. Halit, Beyza, Rana, and Nilgün all used gross quantification because their explanations were perceptual rather than depending on clear mathematical justifications. When we look at the answer provided by Beyza seen in Figure 12, it is observable that she used perceptual explanations related to the variation in rate of change. Rana's way of reasoning was also perceptual. As seen in Figure 13, Rana drew a decreasing graph starting with a concave down and continuing with a concave up curve. While explaining the graph, she stated that "*the rate of change in the radius of first reel is greater than the rate of change in the radius of second reel up to a point and…*" without offering an explicit mathematical operation to observe this relationship. According to Rana's perceptual explanation, the increase in the decrease of R1 gradually gets bigger temporarily up to the equilibrium point and then it gets smaller again. She perceptually decided the variation in rate of change and that's why she constructed an unreasonable graph.

In the process of the "Water Tank" modeling activity, gross quantification continued as being the most frequent way that pre-service teachers used for quantifying the variation in rate of change. Because most of the pre-service teachers thought by secondary variables, they could not apply direct and systematic way of coordination which is the critical part of extensive quantification. For example, Halit and his group thought by radius-based of cross-sections. As seen in Figure 17, although they constructed correct smooth graph, the explanation that "*as the radius increases, since volume will increase slowly, the increase in height is getting slower*" seemed to be perceptual. Furthermore, the explanation does not suggest an explicit mathematical method or quantitative operation to be able to justify the claim numerically. In reflection paper, Halit continued using these perceptual explanations by the expressions as "*when the shape is becoming wide, the height begins to increase less while the volume increases more*".

In the same way, Rana and her group used gross quantification in the "Water Tank" activity. In the group report, their explanation was as "*the rate of increase in height with respect to time is greater than the rate of increase in volume with respect to time*". Rana and her group tried to explain by comparing the rates of change in volume and height with respect to time separately. This explanation can be represented symbolically as $\frac{dH}{dt} > \frac{dV}{dt}$. However, the explanation does not involve a mathematical justification for this claim, rather it seems perceptually provided. Rana's blurred explanations continued in the reflection paper.

> … This curve indicates that the amount of increase in height gets gradually bigger than the amount of increase in volume. (Rana, Reflection Paper, Water Tank)

As seen above, Rana indicated differently the meaning conveyed by a concave up increasing graph. She indicated that "*…the amount of increase in height gets gradually greater than the amount of increase in volume with time*" for explaining various graphs. With this explanation, she compared additively the amount of change in height and the amount of change in volume with respect to time that can be represented symbolically as $H(t_2)-H(t_1)>V(t_2)-V(t_1)$. Surely, this is not always mathematically true also for concave up graphs. In the interview, Raba continued with some other elusive explanations.

> Rana: We looked at the general features of tank figures. We categorized the figures under three groups. If the tank figures are narrowing through up, we thought such a way that, it is really difficult to express, the ratio of the increase in height to the increase in volume gradually gets bigger. In other words, the ratio between the increase in height and the increase in volume will gradually grow. Height will increase faster when compared its previous levels.

In the episode above, Rana used a different kind of explanation for a concave up increasing graph. She indicated a multiplicative relationship between volume and height using time variable implicitly. Her expression that "*the ratio of the increase in height to the increase in volume gradually gets bigger*" shows ratio based comparison in the form of $\frac{\Delta H}{\Delta V}$. The inconsistencies between the explanations provided by Rana related to the variation in rate of change showed her perceptual way of reasoning. At the end of the "Water Tank" activity, the critical ideas, which were direct and systematic way of coordination and extensive quantification of the variation in rate of change, were appeared in one group's solution and most of the pre-service teachers realized these ideas during the group presentations.

Coming to the "Sliding Ladder" problem, although many of the pre-service teachers continued thinking by *gross quantification*, they could not be sure with their explanations involving perceptual reasoning. The "Sliding Ladder" problem was challenging for pre-service teachers, because they needed to reason with explicit quantitative operations. This was not the case in "Water Tank" activity. For example, Halit determined and thought by the primary variables as dependent and independent variables as seen in Figure 21 below. The difficulty of Halit was deciding the variation in rate of change that is if the graph was concave-up or concave-down decreasing. Because he could not figure out perceptually, he assigned some particular numerical values and tried to decide the variation by changing these numerical values. He assigned numerical values 3-4-5 to the triangle and changed it to 4-3-5 triangle and later to an isosceles right triangle with 45 degree. He then tried to determine the changes in height and distance as a result of transforming 3-4-5 triangle to $\frac{5\sqrt{2}}{2}$ - $\frac{5\sqrt{2}}{2}$-5 and 4-3-5by using Pythagorean Theorem. In this process, because Halit did not keep the change in one of the variable constant, he tried to observe the variation by looking particular values in an unsystematic way. Although he predicted a varying nature of rate of change between covarying quantities as can be seen on the graph, he was not able to decide the true curve (see, Figure 21).

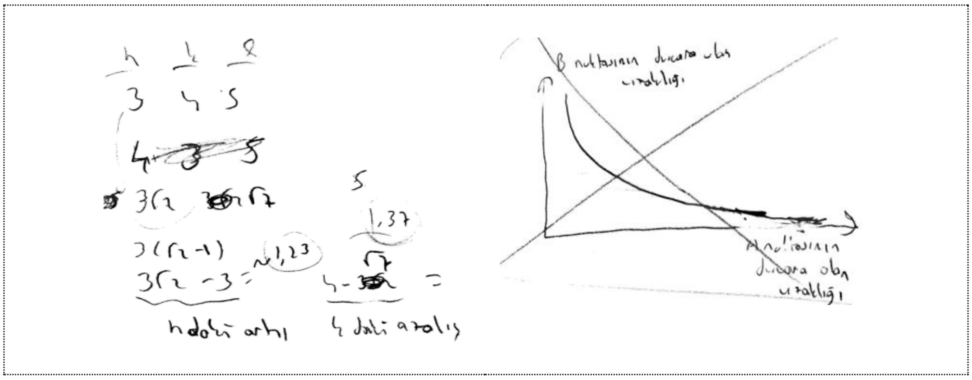


Figure 21: An excerpt from Halit's solution to Sling Ladder problem

In the interview, I questioned Halit and I wanted him to use more sound justifications. In the following episode from interview, Halit's emerging thinking styles while trying to find robust mathematical justifications were observed.

R: Can you transfer the main idea in that problem to the Sliding Ladder? What was the mathematical idea in Water Tank?

Halit: The main idea was the change in height with respect to unit volume…

R: The unit volume… At which axes is the volume?

Halit: x-axis… the height with respect to unit distance hmmm (thinking a few seconds)… change in height with respect to distance. Then, I have to change the distance with equal amounts and tried to observe the change in height…Okay, the idea here is the change in height with respect to distance, but the length of the ladder is constant. Therefore, a quadratic function appears here…

R: How do you know?

Halit: Because, 3-4-5 triangle comes into my mind. On this triangle, I called the x=3 as distance and h=4 as height. The distance of the ladder is 5. Later, I changed the x value to 4 by increasing 1 unit. Then h becomes 3. The length of the ladder stays constant. What did we realize in here? We did not realize anything, because as we increased x with 1 unit, h decreased with 1 unit. Now, I have to compare these values with something….

Halit first restated the idea that systematically changing the input variable and observing the simultaneous variation in the output variable as the main idea in the "Water Tank" activity. He emphasized that the first thing that comes into his mind is 3-4-5 triangle for this situation. He then transformed this triangle to 4-3-5. He realized that as he changed the triangle from 3-4-5 to 4-3-5, the vertical height decreased 1 unit and the horizontal distance increased 1 unit (see, Figure 22). He then realized that this operation was showing the numerical values of two particular amounts and did not provide him sufficient data to make a decision about the variation in rate of change.

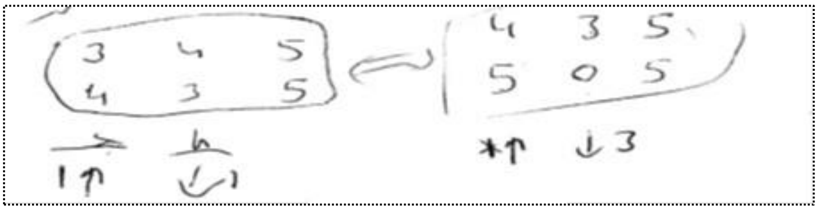


Figure 22: Halit's reasoning with particular numerical values on Sliding Ladder problem

As also represented in Figure 22, Halit realized that he needs to know at least two consecutive values of change in the dependent variable obtained as a result of changing independent variable with equal amounts. He continued his reasoning as follows:

> Halit: With what can I compare? Later, I considered the 4-3-5 triangle on which we obtained x=4. Then, by lying the ladder on ground, I obtained the x=5 and h=0. The length of the ladder is 5 again. Of course, the last condition is not a triangle, but it shows the real situation. Now we can compare these values (indicating the changes in height). As the x value increased from 4 to 5 with 1 unit, the height changed from 3 to 0, decreased with 3 units. Now we can make a comparison. As the value of x increased by 1 unit in both step, okay we came the Water Tank example again, that is how the height changes with respect to unit distance? Now, when the x value increased from 3 to 4 with one unit, the height also decreased with 1 unit, here as the x increased 1 more unit, the height decreased 3 units.
>
> R: What does that mean? At which step the change in height is more?
>
> Halit: The change in height gets more as the x (distance) approaches to 5.
>
> R: How do you convey this idea on graph?
>
> Halit: if we make a comparison here what does it mean? I drew this graph (concave up decreasing) similar with the drawing of SB (Halit's group member). If I draw like that (drawing a new concave up decreasing graph)….

In the interview, Halit used direct and systematic coordination. He equally and uniformly increased the horizontal distance and calculated the simultaneous values of vertical height. By this way, he realized additively comparing the successive changes in the output variable (vertical distance) while keeping the changes in the input variable constant. At the beginning, he was considering all the particular values of horizontal distance, vertical distance, and the length of ladder at the same time as can be seen in Figure 21 and Figure 22. Taking into account all the values resulted in a difficulty for organization of the thinking. Nevertheless, he could reach the idea of *extensive quantification* with the quantitative operation of *additive comparison* of changes in the dependent variable while keeping the change in the input variable constant.

Beyza also used gross quantification while answering the "Sliding Ladder" problem. As seen in Figure 15, she drew a concave-up decreasing graph supported with the explanation that is "*Since Sine and Cosine functions do not change linearly, for every one incremental change in the base angle, the change is not constant. That's why I drew such a curve*". This explanation clearly shows perceptual nature of Beyza's reasoning on the variation in rate of change. During the interview, I asked Beyza about her solution to this problem.

R: Do you remember the Sliding Ladder problem applied just after the Water Tank activity? Let me Show your solution…

Beyza: My solution may be incorrect.

R: Is there any relationship between this problem and the Water Tank?

Beyza: I could not realize any relationship. Of course they both are related to drawing graphs, but… there is not a linear increasing here. For instance, while going down from the point b on the wall to point a, it seems the decrease in point b will be more, it decreases more rapid, but from here, hmmmm….

R: What was asked in the question?

Beyza: It is asking the height of point B from the ground with respect to the distance of point A from the wall… Point A is on this axes and Point B is on that axes. Because it is indicated with respect to Point A, I labeled point A on the horizontal axes, and so I labeled the Point B on the vertical axes. Now I thought like that; point A and Point B, they will be equal when the angle of 45 degrees.

R: Yes, now, the point B is sliding on the ground. That is to say, the distance of point A from wall is increasing while the height of point B from ground is decreasing. We can easily see this relationship. The same situation was appearing here (indicating Water Tank). As the tank filled by water, the height was increasing with the increase in volume.

Beyza: It was easy to imagine in Water Tank. But, it is difficult to imagine and see how much the height of point B will decrease for a particular amount of change in point A, if the changes in both are proportional or not…

R: How can you quantify?

Beyza: I don't know how I can quantify this relation.

R: What was the mathematical idea in Water Tank activity?

Beyza: The main idea here was, hmmmm, we only benefited from the figures here. I did not use a specific mathematical knowledge.

R: Let me say the idea here, change in height per unit volume, unit volume, at what axes was the volume? It was on the horizontal axes.

Beyza: Yes, unit volume, putting here, how much the height of point B will change as the point A changed 1 meter or 1 centimeter. How much will it change? Could we know this? It seems to me as if the height of point B will change lesser with such an imagination. We also tried to imagine the real situation here.

R: Okay, how can you do?

Beyza: I thought like that: when the ladder is at that position, since the center of gravity changes, the ladder will slide fast, and it will collapse speedily on the ground after a while.

R: But the constant speed is emphasized in the question. Imagine as if the ladder is hold by someone so its sliding speed is constant.

Beyza: Hmmmm, it does not fall freely. As I said, I had difficulty in solving this problem. As the point A changed with little amounts, I thought the point B changes with large amounts. But I don't know why I thought like that, I did not use angle or any other thing.

In the episode above, I asked Beyza about the relationship between the "Water Tank" activity and this problem. She could not form any relationship between these two problems except drawing the graphs. But, she emphasized that the nature of the relationship between volume-height in the "Water Tank" activity was easy to visualize, but it was difficult to visualize in the "Sliding Ladder". She stated that "*when the point B slides, it is difficult to see how much the height of point-A changes".* I then asked her the main mathematical idea in the "Water Tank" activity. As an answer to this question, she mentioned about the tank figures and her previous experiences. I prompted the idea of change in height per volume and wanted her to reconsider this idea in sliding ladder context. She demonstrated the systematic coordination features in her reasoning by the expression "*if we change the point A with one unit increments*…" But, thinking by systematic coordination did not provide her enough information to reach a conclusion. She continued thinking with perceptual expressions as "*how can we know about the change of point-B, can we know this? When trying to visualize, it seems to me that the point-B will change less".* Although she realized the idea of systematically changing one of the variables, she could not find an appropriate mathematical operation for coordinating point-A and point-B directly. As seen in the following episode, I wanted Beyza to rethink about the solution.

R: How can you do now?

Beyza: If we say 60 degree here, this angle will be 30 degree. Then, if the length of the ladder is 2a, the height will be 1a, and the horizontal distance will be $a\sqrt{3}$.

R: You can assign some particular and big numbers.

Beyza: Let me say 10, 20 and $20\sqrt{3}$. But I cannot find the value of the base angle formed by the position of ladder when I slide it with vertically 1 unit.

After thinking for a while, with my suggestion, Beyza substituted some particular angles and assigned particular values for the edges of the triangle. Because she considered base angle of the triangle as a variable, she tried to find the corresponding angle change as a result of uniform changes in the height of point B. She stated the difficulty of computing the values of corresponding angle. I then asked her the reason for finding these angles. And by reminding the variable in the "Water Tank" activity that she systematically changed, I suggested her to change the variable on horizontal axis uniformly. The episode is below.

R: Which variable were you changing equally and uniformly in Water Tank activity?

Beyza: hmmm. Okay. I am seeking, how much point B will change as the point A slides with 1 unit? When I slide A with 1 unit, this point will become 11. Hmmmm, it is too easy, so point B will decrease. The length of ladder is 20 and it is constant. The square of 20 will be equal to the square of 11 and the square of unknown. What was the previous value of point A, it was $10\sqrt{3}$. Its value now is$\sqrt{289}$. It decreased.

R: Can you decide now?

Beyza: No, I cannot. I cannot see the general character of change; I have to perform one more step to be able to see the increase and decrease. If I slide the ladder with 1 more unit, the new values are 12, 20 and$\sqrt{246}$. Hmmm, this is 30, the difference here is 21. It means that, it decreases gradually more. Yes, increasingly decreases.

R: How can you show this on the graph?

Beyza: Increasingly decreasing, it is like that, no, no is it like that? That one… (*Drawing a concave-up decreasing graph*)

R: How?

Beyza: Isn't it like that? Increasingly decrease…

R: Which one is changing (decreasing) more?

Beyza: point B is decreasing more.

R: Is the point B decreasing more on your graph? Can you compare?

Beyza: No, it should be like that (drawing a concave-down decreasing graph), because it suddenly decreases. It decreases slowly on this graph (indicating concave-up decreasing graph). Okay…

In the above episode, with my question about the variable that she systematically changed in the "Water Tank" activity, she started to think by systematically changing the variable on the horizontal axes. She assigned particular values for the edges of the triangle (12-16-20), and changed the base edge on the horizontal axes with one unit increments. As she realized the applicability of the Pythagorean Theorem as mathematical operation, she directly computed the edges of new triangles. After realizing Pythagorean Theorem as the true mathematical tool for computing the values of edges, the solution of the problem become trivial for Beyza. By additively comparing the successive changes of the dependent variable with respect to uniform changes in independent variable, she decided the increasingly decreasing nature of the independent variable. After that point, Beyza's struggle for converting this mathematical idea to the graphical form was observed. Interview with Beyza revealed two critical issues related to covariational reasoning which are the possible effect of visual character of the situation and the role of selecting the true mathematical operation for extensive quantification.

The mathematical ideas involved in the “Water Tank” and “Sliding Ladder” was discussed in the next class. These two tasks provided pre-service teachers with the basics of covariational reasoning. In the following tasks, most of them reasoned by extensive ways of quantification. In the “Water-Tank-2” and “Space Shuttle” problems, Halit, Beyza, Rana, and Nilgün used extensive quantification with additive comparison or extensive quantification with unit per unit comparison while deciding the variation in rate of change. For example, Beyza decided the variation in rate of change by using the extensive quantification with *unit per unit* comparison in “Space Shuttle” problem asked in Questionnaire-2 (see, Figure 20).

Table 11: Pre-service teachers’ ways of reasoning while deciding the variation in rate of change across the set of tasks

	Task-1 Cassette Player	Task-2 Water Tank	Task-3 Sliding Ladder	Task-4 Water Tank-2	Task-5 Space Shuttle
Halit	VRC-GQ	VRC-GQ	VRC-GQ VRC-EQAC (Int)	VRC-EQAC	VRC-EQUC
Beyza	VRC-GQ	VRC-GQ VRC-EQAC VRC-EQUC (Int)	VRC-GQ VRC-EQAC (Int)	VRC-EQAC VRC-EQUC	VRC-EQAC VRC-EQUC
Rana	VRC-GQ	VRC-GQ VRC-EQMC	VRC-GQ VRC-EQAC	VRC-EQAC VRC-EQUC	VRC-GQ
Nilgün	VRC-GQ	VRC-GQ	VRC-GQ	VRC-EQAC VRC-EQUC	-

Four pre-service teachers’ ways of reasoning while quantifying the variation in rate of change across the set of tasks were summarized on the table above. It is observable that while pre-service teachers were commonly using gross quantification at the beginning, they started to use extensive quantifications in the process. The classroom discussion conducted after the “Water Tank” and “Sliding Ladder” tasks helped participants in developing the idea of extensive quantification.

4.3 Characterizing pre-service teachers' understanding of rate of change

The concept of rate of change was implicitly or explicitly involved in all of the modeling activities. As represented briefly on Table 12, the analysis of data from all data sets yielded various categories related to participants' conception of rate of change. For the details of about the categorization schemata, see Appendix-J2.

Table 12: Coding schema used for analyzing the conceptions of rate of change

Categories	Sub-categories and Abbreviations

Conceptions of Rate of Change	Difficulty in giving meaning to the term of 'rate of change' (TERC-D)
	Perceptual comparative (PR)
	Amount of change (AC)
	Ratio-based reasoning (RBR)
	Average Rate of Change
	    True Conception (ARC-T)
	    Misconception (ARC-M)
	Instantaneous Rate of Change
	    True Conception (IRC-T)
	    Misconception (IRC-M)

Forming Connections Between Different Representations	Geometric Slope (GS)
	    Using slope procedurally (GS-P)
	    Being aware of different interpretations of slope (GS-A)
	Difference Quotient Rule
	    Using difference quotient procedurally (DQR-P)
	    Being aware of different interpretations of difference quotient (DQR-A)
	Reasoning with Physics concepts (PC)

In this section, pre-service teachers' ways of thinking related to rate of change were reported by considering the developmental process for each of the two dimensions. Therefore, the chronological order of the tasks was also considered in reporting.

4.3.1 **Pre-service teachers' conceptions of rate of change**

The analyses of data revealed six different dimensions by which pre-service teachers' conceptions of rate of change can be described. These are; (i) difficulty in giving meaning to Turkish term of rate of change, (ii) perceptual comparative (e.g., seems faster) conception of rate of change, (iii) amount of change conception of rate of change, (iv) ratio between changes, (v) average rate of change, and (vi) instantaneous rate of change.

4.3.1.1 *Pre-service teachers' difficulties with giving meaning to rate of change term*

The difficulty of pre-service teachers in giving meaning of Turkish term of rate of change (i.e., değişim oranı) was observed in answering the Questionnaire-1 and while solving the "Population of Turkey" modeling activity. Most of the pre-service teachers perceived rate of change in population with respect to time as the percentage. Halit and Rana with their groups interpreted the rate of change in population context as the percentage of change in population with respect to previous year. Halit explained the difficulty they lived as follows:

> After answering the first question by using MS Excel, we continued with the second question. We had difficulty in this question, because what was the meaning of the expression that "the average rate of change in population with respect to time"? We could not understand the question yet… (Halit, Reflection Paper, Population of Turkey)

Halit clearly emphasized that he did not understand the meaning of the expression "*average rate of change in population with respect to time*". The same difficulty was observed in other groups. When the solution of Halit with his group was examined, they consistently used the ratio $\frac{P2-P1}{P1}x100$ for calculating the rate of change. In the interview, when I asked Halit the reason for thinking in that way, his explanation was "*when I read this expression, I need to compare one year with the other year*" shows his reasoning with multiplicative rate of change.

Rana with her group also interpreted the Turkish expression of rate of change as the percentage of population change. In the interview, I asked Rana about the meaning of rate of change expressed in the question text.

> R: How did you interpret the "rate (speed) of increase in population with respect to time" expression?
>
> Rana: We interpreted this as percentage. I mean, we did not use the yearly population changes. When we say amount of increase, we can find directly by subtraction such as the population increased by 5000 people. But when we say rate of increase, it is different. Because let me explain by a simple example; the change in population from 1000 to 1500 and the change in population from 500 to 1000 are not the same. Although, the amounts of change are equal for both, the population increased by hundred-percent for the one while it increased by fifty-percent for the other. I mean, when we hear rate of increase, we considered the amount of change in population with respect to the previous value.
>
> R: You preferred percentage for this reason. What was the alternative interpretation?

Rana: We preferred percentage instead of looking at yearly population changes. We did not say that, it increased by 5000 here, and it increased by 5000 here, so rate of increases are the same. We took into account the previous value.

In the episode above, Rana indicated their preference for percentage interpretation. In her argumentation, she emphasized the difference between amount of change and percentage of it, and indicated that they considered percentage of change in population with respect to previous year's population as the rate of change. When I asked about the other alternative, she indicated the yearly population change, but they considered it as amount of change. In the following parts of the interview, I wanted Rana to transfer the given tabular data on a graph and reinterpret the expressions on the graph.

R: All right, you thought with percentage interpretation… Now, if we draw such a graph (drawing a graph), this axis show years, this one shows population, and the graph is like that. This point is 1990, and this is 2000, what does the slope of line between these points give us?

Rana: It gives the rate of increase in population with respect to years.

R: All right, what had we asked in the question?

Rana: with respect to time, the same thing…

R: So, what are we computing here? (Emphasizing the slope on graph)

Rana: Slope, because we divide by 10, we find the yearly population increase. You mean the slope on graph gives us all the information…

…

Rana: hmmm, okay, this is actually what we already did… I got it; it is related with the idea of derivative with limiting process.

R: How?

Rana: We can use the definition of derivative. f(a) minus f(b) divided by a minus b. 2007 and 1990 will be assigned in places of a and b respectively… I mean it will be 2007 minus 1990 divided by 17.

R: Very well, what is the unit of this?

Rana: divided by year… (Thinking)… It means increase in population per year, people over year.

…

R: All right, what is the difference between these two reasoning?

Rana: I understood the difference between two. I mean, the increase in percentage seems to be increased instantly, but rate of increase; here I am looking unit per unit.

R: Unit, with respect to what?

Rana: The rate of increase is continuous I think because the derivative is at every point. But when we reason with percentage, there is a beginning point and there is an end point, we do not know about the points in the interval. It seems to me like that…

As can be seen in the episode, by translating the tabular data on a graph, I asked Rana about the meaning of the slope of secant line on the population-year graph. She answered as "*the rate of change in population with respect to years*". I then wanted her to return to the expression used in the problem text, and she realized the similarity. She indicated that the slope of secant line was giving the yearly population change between the intervals. Up to here, Rana was considering yearly population change as amount of change, not as a rate of change. After interpreting the slope of secant line on year-population graph, she realized the ratio nature of yearly population change. When I asked about the difference between these two interpretations, she explained by stating the discrete nature in the percentage interpretation and the continuity in slope interpretation.

On the other hand, Beyza and her group interpreted the Turkish expression of rate of change in population with respect to time as change in population per year. In the interview, I asked Beyza about their discussion process on the Turkish expression of rate of change. The episode is below.

R: Did you discuss in the group about the expression that “Average rate of change in population with respect to time”? What did you understand from this expression?

Beyza: From this expression, we computed with respect to per unit time, we tried to find the change per unit time. What I mean by per unit time is computing the yearly population change. We computed the yearly population change. But, when we mention rate, actually it is different…

R: How is it different?

Beyza: When we mention rate, it can be interpreted as follows: when we say rate of change in population, it may involve dividing the change in population with the population. Then it can be a rate of increase. That is to say, rate can be obtained as a result of dividing something by another thing.

R: Did you discuss about these issues in your group?

Beyza: …We thought like that … (thinking). Rate of increase in population will be different, because the population values are different for every year. Change in population, and the population values are different. We did not calculate it for each year separately. We thought such a way that we computed the yearly population values for 30 years,

R: As yearly population change?

Beyza: Yes, we computed as yearly population change, not as rate of change.

Beyza indicated that “*we tried to find change per year*” which clearly shows that they interpreted this expression as additive rate of change. But, later she confused a little and she needed to revise her current way of understanding to the percentage interpretation. According to her, rate should be obtained as a result of dividing one quantity by another quantity. Therefore, she did not accept yearly population change as a rate; instead she considered this value as an amount. I continued questioning Beyza about her understanding of rate of change expressed in Turkish.

R: Do you see any difference between the expressions “rate of increase in population” and “speed of change in population”? Speed was already provided with the parenthesis.

Beyza: I think there is no difference, because we mention about per unit time. It depends on time; therefore the speed is also a rate. Here, I did not consider the time since I thought with yearly populations.

R: So, what is that in this condition?

Beyza: I mean rate of increase is also the speed of population change.

R: Okay in this context, what have you found the highest rate of change in population and for what interval?

Beyza: Here, we did not find the rate (speed) of change in population, 771 people for the 2007 and 2008 interval which is the lowest. In my opinion, to find the rate … (*Thinking*)

R: What is this? What it means that the population increased by 771 in one year?

Beyza: It is speed. Speed… (*Thinking*)

R: How?

Beyza: the population increased by 771 in one year. For example, in motion context, we say that the car traveled 100 km in one hour. This means the speed of the car is 100 km per hour. If we think in that way, then 771 people can be interpreted as a speed.

R: Okay, what did the “rate” term used in the question indicate for you?

Beyza: Rate reminds me of following: hmmm, in fact we probably did the same thing, because for 771, if there were some years, I will divide it by the years. As I said before, if the population increases by 771 in one year, this means 771 over 1. This is also a kind of rate. For instance, I did not divide here because I already take it for one year. We can take 771 directly as speed; the same is true for rate. Yes, then speed also means rate …

Interestingly, Beyza did not accept yearly population change as a rate, but she accepted it as the *speed* of population change. Namely, as also observed in the episode, she did not see yearly population change as rate (“oran” in Turkish), but she accepted it as *speed* (“hız” in Turkish). Beyza used Turkish expression of *speed* in place of rate. In the last part of the episode, she realized the ratio within the expression of yearly population change. She stated that “*if the population increases by 771 in one year, this means 771 over 1. Actually this is also a kind of rate*”. She also realized *speed* and *rate* terms were expressing the same phenomena.

Although the same terminology was used in the “Meteorology Balloon” follow up activity, Halit, Nilgün and also anybody in the classroom did not use percentage interpretation. Generally, they applied the difference quotient rule for calculating the average rate of change in air pressure with respect to height. I asked Halit about this in the interview.

R: you worked on Meteorology Balloon problem last week… Although the same terminology was used in this, you and anybody in the classroom did not think in that way in terms of percentage. Why?

Halit: Because there was a graph in this problem.

R: There was no graph.

Halit: Sorry, there was a table. Here…hmm, how can I say, there was nothing to do (laughing).

R: Why did not you compare in terms of percentage by looking the previous value here?

Halit: (thinking about 3 seconds) Now, I think the population context confused us. Otherwise, as you said we could thought with the same idea, but when speaking about the increase in population, we always have read or heard from the news expressions like "the population increased by one percentage, 2 percentage"…

Halit pointed out that rate of change was interpreted as percentage in population context. The same expression was used in another context, but nobody used percentage interpretation. Halit indicated their previous experience withthe percentage interpretation ofpopulation change as the possible reason. Although most of pre-service teachers thought rate of change in population with percentage interpretation, during the group presentations, classroom discussions, and interviews they came up with the idea that yearly population change was a kind of rate.

4.3.1.2 *Pre-service teachers' perceptual comparative reasoning on rate of change*

Across the first three tasks that are "Cassette Player", "Water Tank", and "Sliding Ladder" related to covariational reasoning, most of the pre-service teachers decided the variation in rate of change perceptually without an explicit quantitative operation. As previously articulated in previous part and summarized on Table 11, Halit, Beyza, Rana, and Nilgün frequently used gross quantification of rate of change.

As an example, Halit tried to explain the variation in rate of change of two covarying quantities perceptually in the "Water Tank" activity by stating that "*when the shape is becoming wide, the height begins to increase less while the volume increases more".* In the same way, Beyza tried to explain the variation in rate of change of two covarying quantities perceptually in the "Sliding Ladder" problem by stating that "*since the values of Sine and Cosine functions do not change constantly as the angle changed with equal increments, I drew the graph as a curve*" (see, Figure 15). Rana also used perceptual comparative way of reasoning for deciding the variation in rate of change across the first three activities related to covariational reasoning. As an example, Rana tried to explain the variation in rate of change of two covarying quantities perceptually in the "Sliding Ladder" problem by stating that

"*When we examine the figure, it is seen that the amount of decrease in the height of point B is not equal to the amount of increase in the distance of point A. That's why the graph cannot be linear*" (see, Figure 18).

After classroom discussions and interview, pre-service teachers started to use extensive quantifications. As can be seen on the Table 11, the appearance of perceptually deciding the rate of change gradually decreased in the progress. Pre-service teachers realized and started to use additive comparison of change in the dependent variable and unit per unit comparison for deciding the variation in rate of change.

4.3.1.3 *Pre-service teachers' amount of change conception of rate of change*

Amount of change in the dependent variable was appeared as being the dominant conception that pre-service teachers demonstrated related to rate of change. This way of reasoning first appeared in Questionnaire-1 while interpreting the symbolic expressions of average and instantaneous rates of change. For example, Beyza interpreted the symbolic expression $G'(80) = -0,3$ in the 6thquestion asked in Questionnaire-1 as decrease in fuel efficiency at the given speed. As can be seen in Figure 23 below, she considered only the change in the dependent variable in her explanation.

G'(80)= -0,3 sonucu problem bağlamında ne anlama gelmektedir? Açıklayınız.

80 km/saat hızla gittiğimizde yakıt verimi azalıyor. (0,3 oranında)

Figure 23: Beyza's answer for a part of the 6th question in Questionnaire-1

Amount of change conception of rate of change also appeared in subsequent phases of covariational reasoning tasks while deciding the variation in rate of change in the form of additively comparing the changes in the dependent variable. Pre-service teachers were perceptually deciding the variation in rate of change at initial phases of covariational reasoning tasks. After classroom discussion they developed a systematic way of thinking that is varying equal amounts in independent variable and observing simultaneous variation in the dependent variable. From the perspective of rate of change, this way of reasoning was accepted as focusing *amount of change,* because while deciding the variation in rate of change, the additive comparison of the successive values of the output variable was considered.

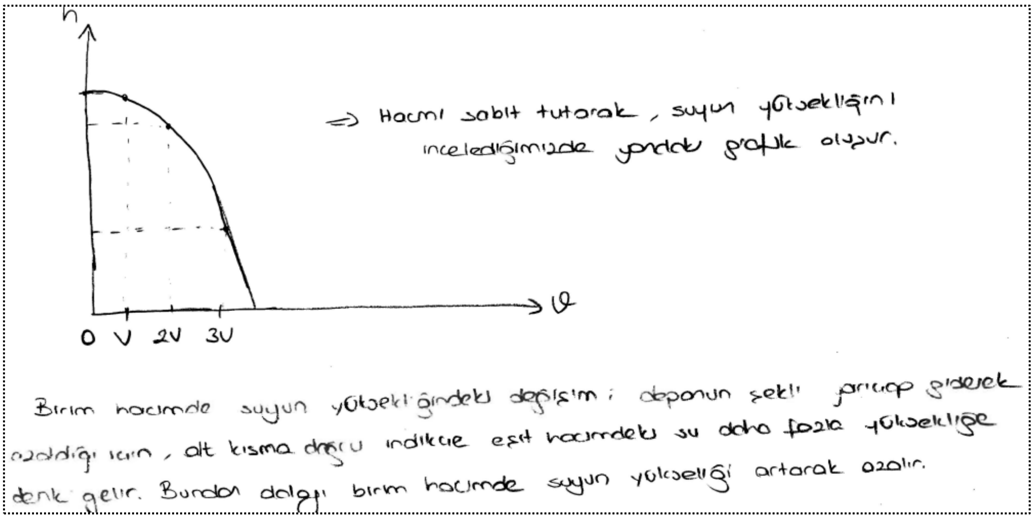


Figure 24: Nigün's answer for the "Water Tank-2" problem

As an example, Nilgün demonstrated this way thinking while solving the "Water Tank-2" problem. As can be seen in Figure 24, Nilgün explained that "*when the volume changed with equal increments, the decrease in height will gradually increase*". Nilgün did not form a ratio-based relationship between the change in volume and change in height. This way of reasoning was also observed in other pre-service teachers.

Pre-service teachers' amount of change conception of rate of change was also observed in "Population of Turkey" activity. Most of the participants considered yearly population change as an amount, not as a rate. For instance, Beyza with her group provided an answer by which they used a ratio-based reasoning for the "Population of Turkey" activity, but in the interview, she confused about if the yearly population change is a rate or an amount of change. She did not consider yearly population change as a ratio at first, but she realized in process by using the definition of speed as reference. Halit, Rana their groups reasoned in the same way.

In the follow up activity conducted after the "Tracking Track" activity, the derivative of height-volume graph was provided without plotting the names of variables on the axes. Pre-service teachers were asked to name and explain the variables to be plotted on the axes. As can be seen in Figure 25, on the derivative graph of the height-volume graph, Halit explained the variable on vertical axes of derivative graph of height-volume as "*increase in height*". He used amount of change in height (dependent variable) in place of rate of change in height with respect to volume. While Nilgün's answer was the same with Halit's, Rana and Beyza answered this question by demonstrating ratio-based conception of rate of change.

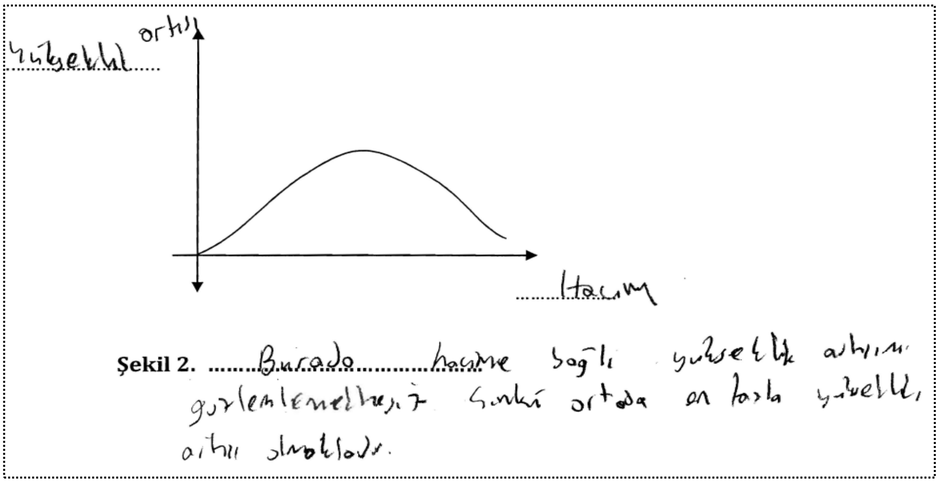
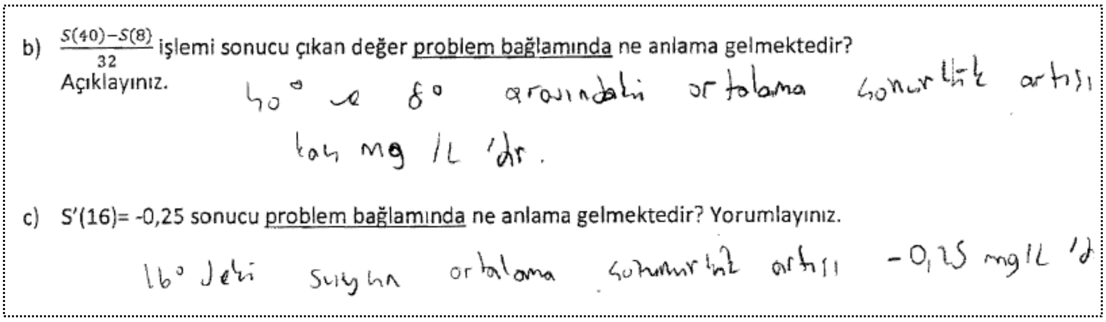


Figure 25: Halit's answer for the follow up of "Tracking Track" activity

When we look at the answers provided for the sixth question in Questionnaire-2, most of the pre-service teachers continued considering rate of change as amount of change in the dependent variable. Although Beyza's ratio-based conception of rate of change was evidenced in her written solutions and during interviews which will be explained in the following part, her interpretation of symbolic expressions of rate of change asked in the 6th question in Questionnaire-2 was still in the form of amount of change. She interpreted the symbolic expression of average rate of change as "*Change in solubility in the temperature interval 8-40 centigrade degree*". Likewise, she interpreted the symbolic expression $S'(16) = -0.25$ as "*Change of solubility decreases at the temperature of* $16\ \mathrm{C}^0$ *and its value is 0.25*". In both explanations, she only focused on the change in the dependent variable, and she did not indicate a multiplicative relationship in the form of a ratio between dependent and independent variable.


b) $\frac{S(40)-S(8)}{32}$ işlemi sonucu çıkan değer <u>problem bağlamında</u> ne anlama gelmektedir? Açıklayınız.

40° ve 8° arasındaki ortalama çözünürlük artışı kaç mg/L'dir.

c) S'(16)= -0,25 sonucu <u>problem bağlamında</u> ne anlama gelmektedir? Yorumlayınız.

16° deki suyun ortalama çözünürlük artışı -0,25 mg/L 'd

Figure 26: Halit's answer for the 6^{th} question in Questionnaire-2

Halit also used "*amount of change*" in the explanation for the symbolic expression of average rate of change in solubility-temperature context while answering the 6^{th}question asked in Questionnaire-2. As can be seen on the Figure 26, for explaining the symbolic expression of average rate of change, he used the expression "*average increase in solubility*". Nilgün's explanation for the same symbolic expression was also as amount of change. She interpreted the symbolic expression of average rate of change as "*Change in the solubility of water in the given temperature interval (8-40)*". All in all, some of pre-service teachers' conceptions of rate of change continued to appear in the form of amount of change in the dependent variable.

4.3.1.4 *Pre-service teachers' ratio-based reasoning on rate of change*

While answering the 6^{th} question in Questionnaire-1, only a few pre-service teachers demonstrated a ratio-based reasoning on rate of change one of who was Rana. Rana explained the meaning of symbolic expressions of difference quotient rule and derivative by using *"(average) rate of change or speed*" term. Most of the pre-service teachers explained the symbolic expressions involving rate of change as amount of change in the dependent variable or they used other irrelevant explanations. As indicated in previous part, in covariational reasoning tasks, almost all of the pre-service teachers perceptually decided the variation in rate of change at initial phases. Later, they started to compare amount of successive changes in the dependent variable. Pre-service teachers started to demonstrate ratio-based reasoning styles while solving the following covariational reasoning tasks.

Pre-service teachers' ratio-based reasoning started to appear with the "Water Tank" activity. For example, Rana and Beyza demonstrated ratio-based reasoning (in the form of unit per unit comparison) for the first time in the interview conducted after “Water Tank” activity. In the interview, I asked Beyza to explain the meanings of “*increasingly increasing”* or “*decreasingly increasing”* expressions that she used. Possibly, after seeing other group’s solutions and reflecting about the solution process during writing the reflection paper, she started to use *unit per unit thinking* consistently by connecting it with rate and slope. The episode is below.

> R: After drawing the graphs, you used expressions such as “decreasingly decrease”, “increasingly increase” in verbally explaining the graphs. Can you restate these expressions by also taking into account the height and volume?
>
> Beyza: Let me say, I should state it with respect to per unit: If the change in height gradually decreases with respect to unit volume… In other words, if the tank is filled by 1 meter cube water in t period of time and it will rise with h amount; if the tank is filled by 2 meter cube water in 2t period of time, then it will not rise with h amount.
>
> R: You again considered “time”, but okay I understood.
>
> Beyza: Yes, I considered time…
>
> R: You mentioned about speed and used expressions as the speed of rising increases or decreases. What does speed mean in volume-height context?
>
> Beyza: I thought like that: For instance, when we speak about the speed per unit time, a vehicle goes a particular amount of distance in a particular period of time. This gives us a constant speed. When we thought in that context, as the water level rose, there will be a speed of rising. For a cylindrical container, the speed will be constant because it will rise with same amounts for every 1 meter cube water addition.
>
> R: All right, can you explain the speed?
>
> Beyza: Speed in that context is the increase in height per unit volume, height per unit volume…
>
> R. Okay. While preparing the manual, you also mentioned about the variation in the slope of tangent lines…
>
> ……..
>
> R: What does the slope of tangent line gives us here?
>
> Beyza: It is the speed here. The speed of raise at water level in the container, because for instance…
>
> R: Then can you restate the meaning of speed?
>
> Beyza: It is the change in the height of water level per unit volume.

At the first parts of the episode, Beyza used additive comparison of the changes in height in relation to 1 m^3 increments in volume. For explaining the expression “*decreasingly increasing”,* she stated *“for every 1 m^3 increment in volume, the height increases with 2h and later 1h…*”Because Beyza’s time-based reasoning continued, I asked her the meaning of speed that they used in their group report. She explained the speed as “*change in height per unit volume*” by using again the speed concept as reference and transferring the definition of speed to volume-height context. In this explanation, Beyza’s way of reasoning for explaining the variation in rate of change was involving unit per unit comparison. She also related the unit per unit explanation with the slope of tangent line at a point, and also with the rate (speed) of change in height with respect to volume. The number of pre-service teachers who demonstrated a ratio-based reasoning increased in the “Water Tank-2” problem.

In the “Population of Turkey” modeling activity, as reported in previous part, most of the pre-service teachers interpreted rate of change as percentage of change in population. But, this activity also contributed pre-service teachers’ ratio-based reasoning on rate of change. In other words, they realized the ratio involved in the unit per unit thinking. They started to see the expression “*change in population per year*” as a rate. To exemplify, Beyza and her group calculated yearly population change for every interval, but they used “*amount of yearly population change*” in their verbal explanations without seeing it as a rate. In the interview, I asked Beyza about her understanding of rate of change in population context. She expressed that “*we did not find the rate, only the amount*” at first. I asked again the meaning of the expression “*771 people per year*” that they used. After thinking for a while, she conceived this expression as speed by transferring from the description of speed that is “*distance covered per unit time*”. Beyza clarified her understanding of yearly population change as a rate by using the speed as reference concept.

In the same way, Halit also clarified his understanding of rate of change in the interview conducted related to the “Population of Turkey” activity. I asked Halit about the basic mathematical ideas covered within this question.

> R: What were the mathematical ideas covered in this activity?
> Halit: Percentage, slope
> R: What does slope mean in this context?
> Halit: Average rate of change in population with respect to time
> R: Can you restate by using the expressions in Water Tank activity?
> Halit: Increase in population with respect to unit second or unit year. There is instantaneous velocity, average velocity in this question that we remember from Physics, but when the population is asked, yearly percentage is coming into my mind…

According to Halit, increase in percentage and slope were the basic ideas in “Population of Turkey” activity. I asked him the meaning of slope in population context. He expressed that “*slope is the average rate of change in population with respect to time*”. I later asked him to relate this expression with the mathematical idea covered in “Water Tank” activity; and he responded in a way that “*amount of* *increase in population with respect to unit time*". He detailed what he meant by unit time by stating unit second or unit year. Also, he connected with the Physics concepts. In the solution of "Meteorology Balloon" activity applied after "Population of Turkey" activity, almost all of the pre-service teachers correctly interpreted rate of change in height-temperature context.

When we came to the follow up activity asking about the names of axes on derivative graph of volume-height which was conducted after "Tracking Track", about half of the pre-service teachers explained the name of vertical axis as rate of change in height. The other half of pre-service teachers either used amount of change in height or some other irrelevant explanations. As seen in Figure 27, Rana identified the vertical axis of the derivative graph of volume-height as the "*rate (speed) of increase in the height*". She explained the graph by using the unit per unit comparison idea of rate of change which is "*speed of change in height per volume*".

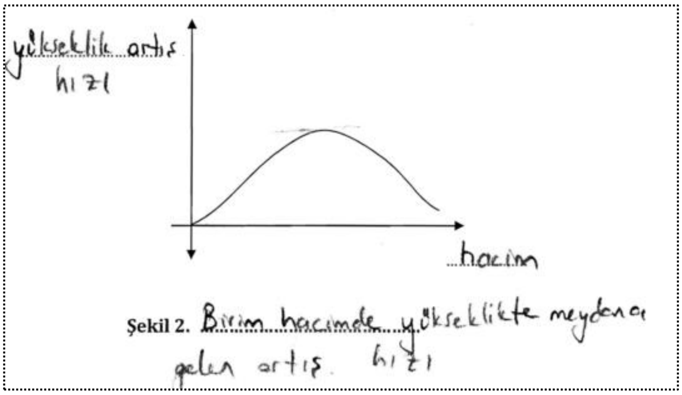


Figure 27: Rana's answer for the follow up problem of the "Tracking Track" activity

While explaining the symbolic expressions in the 6thquestion asked in Questionniare-2, about half of pre-service teachers demonstrated a ratio-based reasoning. The other half of pre-service teachers explained either by amount of change or by using some other irrelevant expressions. For instance, Rana answered this question by using the idea of rate of change properly. As seen on the Figure 28, Rana interpreted the symbolic expressions given in the form of difference quotient rule and instantaneous rate of change as rate of change.

b) $\frac{S(40)-S(8)}{32}$ işlemi sonucu çıkan değer problem bağlamında ne anlama gelmektedir? Açıklayınız.

18°C ve 40°C sıcaklıkları arasında oksijenin su içerisinde çözünürlüğünün ortalama değişme oranı.

c) S'(16)= -0,25 sonucu problem bağlamında ne anlama gelmektedir? Yorumlayınız.

16°C sıcaklığında oksijenin su içerisinde çözünürlüğünün değişme hızıdır.

Figure 28: Rana's answer for the 6th question in Questionnaire-2

In the progress of the model development unit, pre-service teachers demonstrated ratio-based conception of rate of change from time to time. But it is difficult to argue that they all developed a robust ratio-based conception for rate of change, because some of them continued to consider rate of change as amount of change in the dependent variable.

4.3.1.5 *Pre-service teachers' conceptions of average & instantaneous rates of change*

Pre-service teachers' lack of knowledge and some misconceptions about average and instantaneous rate of change observed. In subsequent phases of the model development unit, most of the pre-service teachers' conceptions developed.

To begin with the second question Questionnaire-1, pre-service teachers' misconceptions and unawareness with these concepts appeared. In this question, the average rate of change and instantaneous rate of change of a context-free function were asked. For the average rate of change part of the question, pre-service teachers used a variety of irrelevant procedures. Halit's answer to this part of the question was finding the average rate of change of the derivative function in the given domain.

3. $f(x) = 3x^2 - 4$ fonksiyonu reel sayılardan reel sayılara tanımlı bir fonksiyondur.

a) Bu fonksiyonun $x = 3$ ve $x = 5$ aralığındaki ortalama değişim oranı nedir?

$f'(x) = 6x \quad f'(3) = 18 \quad f'(5) = 30 \quad O.O = \frac{30-18}{5-3} = \frac{12}{2} = 6$

Figure 29: Halit's answer for a part of the 2nd question in Questionnaire-1

As can be seen in Figure 29, it was asked to find the average rate of change of the function between x=2, and x=5. Halit computed the values of derivative for these two end points and applied the difference quotient rule on the derived function in such a way that $\frac{f'(5)-f'(2)}{5-2}$. Beyza’s answer for the same question was a little different. She first computed the values of derivative function at the given points and took the difference between two as can be seen in Figure 30 below.

a) Bu fonksiyonun $x = 3$ ve $x = 5$ aralığındaki ortalama değişim oranı nedir?

$f'(x) = 6x \qquad f'(5) - f'(3) = 30 - 18 = 12$

Figure 30: Beyza’s answer for a part of the 2nd question in Questionnaire-1

Nilgün also used an irrelevant procedure that $\frac{f'(3)}{f'(5)}$ for computing the average rate of change. There were only a few pre-service teachers who properly applied the difference quotient rule for computing the average rate of change. Rana was one of them. She correctly interpreted the average rate of change of a function in the given interval by using the difference quotient rule in a way that $\frac{f(5)-f(3)}{5-3}$. Interestingly, pre-service teachers correctly answered the second part of this question asking about instantaneous rate of change. They directly found the answer by substituting the given point in the derivative formula.

Additionally, most of the pre-service teachers could not interpret the symbolic expression of $\frac{G(100)-G(70)}{30}$ involving average rate of change. In the same way, they had difficulty in interpreting the meaning of $G'(80) = -0.3$ in fuel efficiency-speed context. As can be seen in Figure 31, Halit defined the meaning of $G'(80) = -0.3$ as a decrease in fuel efficiency. Beyza’s way of reasoning was similar.

c) G′(80)= -0,3 sonucu problem bağlamında ne anlama gelmektedir? Açıklayınız.

80 km/saat'teki hıza ulaştığı andaki yakıt verimliliğinin gittikçe azalıyor olması.

Figure 31: Halit's answer for a part of the 6th question in Questionnaire-1

In the "Population of Turkey" activity, average rate of change and instantaneous rate of change concepts were involved. Although most of the pre-service teachers thought with percentage interpretation at the beginning, they realized the meaning of rate of change with this activity. They interpreted (average and instantaneous) rate of change as yearly population growth in percentage with respect to previous year. Namely, they did not consider yearly population change as a rate of change. But, after the group presentations and classroom discussion, pre-service teachers realized the average rate of change and instantaneous rate of change. For instance, Halit demonstrated his understanding of average rate of change in the interview. In the interview, I drew a concave down increasing time-population graph and assigned some of the population data on graph. I asked him the meaning of average rate of change between the years 1985-1990 on the graph. His explanation was as follows:

> R: Can you interpret the meaning of average rate of change in population in the interval of 5th and 10thyears, namely in the 85-90 year interval. Let's the graph is like that (Drawing a concave-down increasing graph)
>
> Halit: I understood better after Meteorology Balloon problem, when we say average; this is related to derivative and also slope. We are taking this point, the population at year 90, let's say f(90), and then we are taking f(85). f(90) minus f(85) over 90 minus 85 ($\frac{f(90)-f(85)}{90-85}$). That is to say, we are forming a triangle in that interval…

Halit clearly related average rate of change with the slope in this interval. Halit's reasoning of slope was emerged after the population data was converted to the graphical form. When the data was in tabular form, Halit insisted on thinking in percentage.

In the "Population of Turkey" activity, Halit with his group did not think with the instantaneous rate of change. In the interview, after converting the tabular data to the graphical form, I questioned again the meaning of rate of change at years 2000 and 2004 on the graph. Halit could not interpret the meaning of the rate of change at a year. While he was questioning about it, I asked him to interpret the speed at a point.

He directly mentioned about the distinction between the independent variables of these two contexts. His confusion was that one year is too large interval to mention about an instantaneous rate of change. He did not consider that instantaneous rate of change was a kind of average rate of change on infinite small intervals. I also asked the meaning of speed at a point and he said that "*it is just a procedure and we memorized it*", and he explained the instantaneous speed as the slope of tangent line.

Beyza also had the same difficulty in interpreting the instantaneous rate of change in population context. Beyza narrated their difficulty in interpreting rate of change of population at a year in the reflection paper as follows:

> …We had great difficulty in finding the rate of growth in population with respect to time at year 2000. We discussed in the group. According to me, we could find the rate of change at 2000 by comparing with the previous year. Other friend suggested finding the rate at 2000 by taking into account the previous and following years. Another friend suggested finding an average population increase at 2000 by comparing with all other years. As a result of the discussion, we decided to find rate of change in population at 2000 by using the data of previous and following years. We accepted this because the given population data was for the sixth month of each year. (Beyza, Reflection Paper, Population of Turkey)

In their group report, they calculated the rate of change at 2000 in percentage by taking the average of percentage of population change with respect to previous year and percentage of population change with respect to following year. I asked Beyza about the instantaneous rate of change in the interview.

> R: … Let's consider it is smoothly increasing on the graph. How do you interpret the increase in population at a particular point on the graph? Or, what does it mean saying that the rate of change in population is 771 people per year, or the speed is 50 m/s at a particular point on the graph?
>
> Beyza: At that point with respect to time… No, I am confused… Here, it means I move 50 meter away from the previous location.
>
> R: But, we mention about the instantaneous speed, if we thought 1 second later, okay it may go 50 meter.
>
> Beyza: (thinking)… Don't we compute by taking the averages here? I could not interpret what it means…
>
> R: In fact this is related to the question asking the approximate value of rate of change in population at 2000. How did you compute in population context?
>
> Beyza: We discussed for a long time on this question, and later we thought such a way that we should compare the population at 2000 with another year… We need another year in order to find respective population increase.
>
> R: Very well, according to which year did you compare?
>
> Beyza: We took according to previous and following year, and computed the averages of percentages.

In the episode above, Beyza had difficulty in explaining the rate of change at a point. She was frequently using the speed concept as reference while explaining the rate, but she could not interpret the instantaneous speed in motion context as well. When I asked about the meaning of the rate of change at a point on the graph, she could not provide a sophisticated answer. For explaining the speed at a point, she mentioned about the distance covered in 1 second. In population context, according to her, it was meaningless to mention about instantaneous rate of change without knowing another particular value to compare with.

In Questionnaire-2, some of the pre-service teachers' difficulties with average and instantaneous rates of change concepts continued. But, about half of the participants provided correct answers to the questions involving average and instantaneous rates of change.

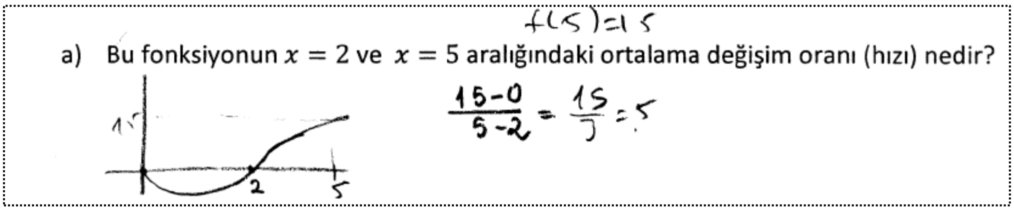


Figure 32: Beyza's answer for a part of the 2nd question in Questionnaire-2

When we look at the solution provided by Beyza to the 2nd question in Questionnaire-2asking the average rate of change of a context-free function, she used difference quotient rule. She first drew the graph of the function and plotted the given points and computed the slope of secant line. To remember again, while answering the equivalent question in Questionnaire-1, she computed the derivative at two end points and took the difference between two (see, Figure 30).

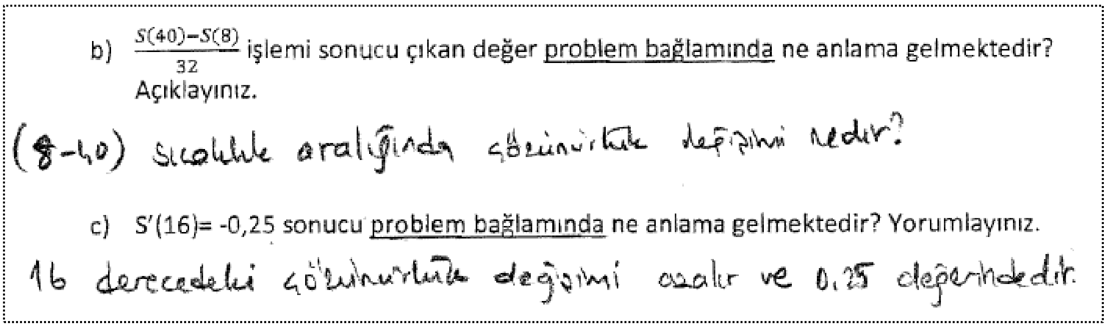


Figure 33: Beyza's answer for the 6th question in Questionnaire-2

On the other hand, while interpreting the symbolic expressions related to average and instantaneous rate of change in the 6th question of Questionnaire-2, for symbolic expressions involving average and instantaneous rates of change, Beyza used explanations indicating the change in the dependent variables (not rate of change).

Although Halit connected average rate of change with the slope of secant line and instantaneous rate of change with the slope of tangent line during the interview, while answering the 6th question in Questionnaire-2, he interpreted both symbolic expressions as *amount of change* in the dependent variable (see, Figure 26). Nilgün's explanations for this question were similar with Beyza's and Halit's answers. But, Rana explained these two expressions properly by using the terminology of average and instantaneous rate of change (see, Figure 28).

*Table 13**:** Pre-service teachers' conceptions of rate of change*

	Task-1 Pre-Test	Task-2 WT	Task-3 SL	Task-4 POT	Task-5 POT-F	Task-6 TT-F	Task-7 Post-Test
Halit	TERC-D PR AC ARC-M IRC-M	PR AC (int)	PR AC	TERC-D AC RBR (int) ARC-T (int)	PBR ARC-T	AC	AC RBR ARC-M ARC-T IRC-M
Beyza	TERC-D PR AC ARC-M IRC-M	PR AC (Int)	PR AC (Int)	TERC-D RBR AC (Int) ARC-M ARC-T (int) IRC-M	RBR ARC-T	RBR	AC RBR ARC-T ARC-M IRC-M
Rana	PR RBR ARC-T IRC-M	PR AC RBR (Int)	PR AC (Int)	AC RBR (int) ARC-M ARC-T (int) IRC-M IRC-T (Int)	RBR ARC-T	RBR	PR RBR ARC-T IRC-T
Nilgün	TERC-D PR AC ARC-M IRC-M	PR AC	PR AC (Int)	AC RBR (int) ARC-M ARC-T (int) IRC-M IRC-T (Int)	RBR ARC-T	PR	TERC-D AC PBR ARC-M IRC-M

Pre-service teachers' ways of reasoning related to rate of change appeared in the process of the model development unit is summarized on the table above. Following the pre-service teachers' ways of reasoning in progress, they had difficulty in giving meaning to rate of change (TERC-D). Most of them were unaware of what was expressed by rate of change. In subsequent phases, they learned about the meaning of rate of change. They also explained the situations involving rate of change by using the perceptual impressions (PR) or as the amount of change of change in the dependent variable (AC) at initial phases. Only a few pre-service teachers used a ratio-based conception of rate of change (RBR). The frequency of ratio-based reasoning demonstrated by pre-service teachers increased through the end. In addition, pre-service teachers' misconceptions with regard to average rate of change (ARC-M) and instantaneous rate of change (IRC-M) were frequently observed at initial phases. In subsequent phases, although some of pre-service teachers developed the true conception of average rate of change (ARC-T) and instantaneous rate of change (IRC-T), ways of reasoning involving misconceptions continued to appear. This data showed that although some developments observed in pre-service teachers' conceptions of rate of change, it is difficult to argue that they formed a robust conception of rate of change.

4.3.2 **Pre-service teachers' conceptions with regard to different interpretations of rate of change and the connections between them**

Although the rate of change concept was focused in this study, some developments also observed in pre-service teachers' understanding of other interpretations of derivative and in the ways of forming connections between them. The other formal mathematical representations and interpretations of rate of change that pre-service teachers developed important ideas during the process were; (i) geometric slope, (ii) difference quotient rule, (iii) Physics concepts (speed, acceleration), and the connections between them.

4.3.2.1 *Pre-service teachers' conceptions of slope*

Pre-service teachers' frequently used the slope interpretation when asked about the meaning of derivative. However, interesting results obtained related to pre-service teachers' conceptions of slope. They could not easily explain the meaning of slope in volume-height context, and they had difficulty in interpreting the slope on curved graphs. They realized the connections between slope of tangent line, rate of change, and difference quotient rule interpretations.

Pre-service teachers frequently used slope interpretation when interpreting the derivative throughout the sequence. For instance, while answering the second and seventh questions in Questionnaire-1, Beyza translated the symbolic function and tabular data in both questions to graph and then she tried to answer thinking with slope. In the seventh question, the approximate value of derivative at x=2 was asked provided with a tabular data. As seen on the figure below, Beyza first translated the tabular data on the graph and she mentioned about the tangent line at the point without providing a numerical answer.

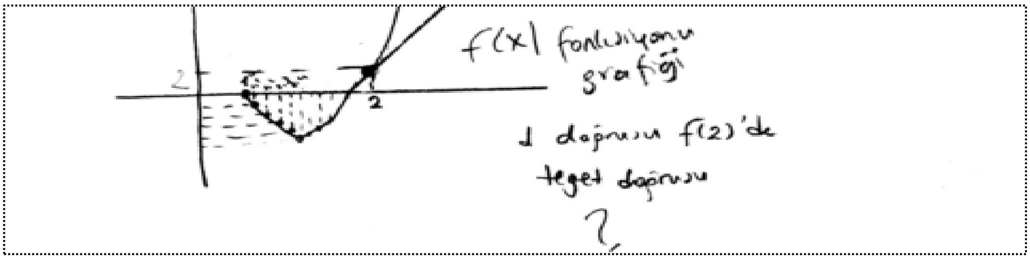


Figure 34: Beyza's answer for the 7th question in Questionnaire-1

Pre-service teachers' difficulties with the meaning of slope in different contexts first appeared with the "Water Tank" activity. For example, Halit could not easily explain the meaning of slope on volume-height graph. As described earlier, Halit and his group's solution to the "Water Tank" activity was involving indirect way coordination of variables. They explained the differences between graphs by the changes in slope. In the interview, I asked Halit about the meaning of slope in volume-height context.

R: What does slope mean in volume-height context? For instance, you drew a graph for the first tank. What does the slope of tangent line at any point give us?

Halit: What does slope mean? On the graph of volume-height … (Pause, hmmm, 10 seconds). It gives us if the graph is increasingly increasing or decreasingly increasing. In fact, the slope here… hmmm… What does slope give us? Hmmm…

Halit could not find an answer to this question. He thought about 10 seconds without providing an answer. Later, he said that "*the slope gives idea about if the function is increasingly increasing, or decreasingly increasing*". But he did not satisfy with his explanation and continued thinking for a moment. After waiting for a considerable amount of time without an answer, I asked him what if this graph was a distance-time graph.

R: Think in such a way that this is a distance-time graph. Let's the variable on the x-axis is time, and H stands for the distance. What is slope in this situation?

Halit: It gives "speed".

R: How do you define "speed" in the motion context? And how do you explain it verbally in height-volume context?

Halit: hmmm… It is about at what amount of time a particular distance is covered with a value of speed...

R: Do not use the speed term, because you are explaining the speed…

Halit: hmmm….it is the distance per unit time.

R: How do you explain in volume-height context?

Halit: hmmm (thinking about 4 seconds), here the volume will be plotted on the x-axis, okay, and then we can say it is the height per unit volume.

In the above episode, Halit explained slope as speed in motion context. I then asked him the definition of speed. At first he tried to explain considering time and distance with the expression "*at what amount of time a particular distance is covered at a particular value of speed*". Later he explained by using unit per unit thinking and explained it as "*distance covered per unit time"*. He translated this way of reasoning to the volume-height context by associating the independent and dependent variables on graph. He reasoned in a way that because time and volume are both on the horizontal axis, volume should take the place of time. And so, slope in volume-height context means change in the height with respect to unit volume. At the beginning Halit could not explain the meaning of slope in volume-height graph, but he could interpret slope as speed in distance-time context. Also, he used unit per unit thinking while defining the meaning of speed. After all, he could reach the meaning of slope in volume-height by visiting the concept of speed and transferring the idea of unit per unit thinking. While translating the meaning of slope from motion context, he was depended his reasoning on the correspondence of variables on the horizontal axis. Nilgün also used slope interpretation of derivative, but she was not successful in relating it with other interpretations.

Beyza and Rana also used slope interpretation frequently and they were successful in relating slope with other interpretations of derivative. For instance, in the "Water Tank" modeling activity, Rana with her group mentioned about the variation in slopes for explaining the difference between graphs having the same character. In the interview, I asked Rana about the meaning of slope in the volume-height context.

R: You said that we decided by looking at the slopes, slopes are different. What do you mean by slope here? I mean, when you are explaining the difference between these two figures...

Rana: It is the increase in height per unit volume, I mean it is faster.

In the episode above, Rana could explain the slope on height-volume graph by stating "*the increase in height per unit volume…*" which can be accepted as an important indicator of ratio-based conception of rate of change. She related slope with the rate of change. Beyza explained the slope in volume-height context in the same way. During the interview related to the "Water Tank" activity, when she was mentioning about the changing slope of various curves, I asked her the meaning of slope in volume-height context.

> Beyza: hmmm, they did not find a general formula. In my opinion, it is enough because we also drew by considering the slope. That is to say, the slopes of these graphs are different.
>
> R: What do you mean by slope?
>
> Beyza: Slope here is the speed (rate) of increase in the height of water level. The slope on the volume-height graph gives us the speed (rate) of increase in the height water level…

In the episode above, Beyza explained the slope as the rate of change in height with respect to volume. She related the slope with the speed (rate) of height increase with respect to volume. She generally used "speed" and "rate of change" terms interchangeably.

In the "Population of Turkey" activity, pre-service teachers interpreted rate of change in population as the slope after translating tabular data to the graphical form. For example, Halit and his group interpreted rate of change as the percentage of change in population in their solution. They did not use the given tabular data as a different representation of a functional relationship between time and population. In the interview, I asked Halit that "*if the given data was plotted on a graph in the form that year at the horizontal axis and population at the vertical axis, how do you interpret the "average rate of change in population with respect to time" in this graphical context?"* Halit directly interpreted this expression as slope and explained the difference quotient rule for calculating it. The same situation was observed with Rana and Beyza. As indicated earlier, in the "Population of Turkey" modeling activity, Rana with her group calculated yearly population change on tabular data, but they did not consider it as a rate. Rana accepted yearly population change as a rate after translating the given tabular data to a graphical form and after relating it with geometric slope.

Most interesting results related to pre-service teachers' conception of slope were obtained during the interpretation of the maximum slope constraint in "Roller

Coaster" activity. Most of pre-service teachers thought as if the slope was 5.67 on every point of a curved graph as figured out in Figure 35. Halit, Beyza, Rana, and Nilgün reasoned in the same way.

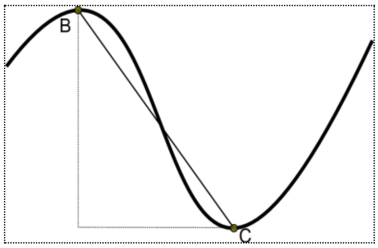


Figure 35: Conception of slope as a line on a curved graph

For instance, Halit with his group created a curved railway for roller coaster, but they reasoned as if the slope was 5.67 at every point (linear). As seen in the figure above, they formed a triangle between top and valley points of the curve and tried to calculate the highest point of railway. Halit explained their reasoning process in the reflection paper as follows:

> …Because we thought the slope was constant at everywhere, the greater the height of hill, the greater will be the horizontal distance. We still did not realize the slope cannot be same at every point. Later, as the instructor came and asked some questions, we realized something going wrong. The first of them was that at where the slope was getting 5.67, and the instructor stated that the height is not true if the slope is not constant at everywhere. Later, the instructor wanted us to draw two graphs, one is linear and the other one is parabolic, both starting from the same point. According to these graphs, the slope of linear graph was 5.67 at every point, and the other's slope was getting this value at only one point. Accordingly, we decided that the heights should be smaller. The instructor also asked that at what point the slope of tangent line gets its maximum value. I drew a graph starting with a concave-up and continuing with a concave-down curve by considering the safety criteria. We realized that the slope was getting its maximum value at the transition point between these two curves. Namely, it was the inflection point where the second derivative gets zero. Finally, we understood that we should draw a graph starting with a concave up curve and continuing with a concave down, and the heights of the hills should be less. (Halit, Reflection Paper, Roller Coaster Activity)

Halit emphasized that they had difficulty in interpreting that maximum slope constraint. During the group discussion process, although they designed a curved railway for roller coaster, they considered as if the slope was 5.67 at everywhere. They continued this way of thinking until the question *"which is the point at where the slope is 5.67 and how do you know it is maximum?"* of the instructor. After a group discussion started with this question, they realized the changing nature of slope of tangent at every point of the curve. After realizing the changing slope on the curve, Halit emphasized that they developed the idea of maximum slope as the inflection point, and he expressed that "*maximum slope is at the point where the curve changes from concave up to concave down*". This can also be accepted as an important idea for the development of derivative as a function in a way that for any point on the curve, the slopes of tangent line (derivative) take different values.

The ratio between opposite edge and adjacent edge of a right triangle was observed as being the dominant image of slope that some of pre-service teachers have. In the "Roller Coaster" activity, Beyza considered the maximum slope as the slope of triangle formed by tying the top and valley points with a line. This way of reasoning was exactly the same with Halit's way of reasoning depicted with a graph in Figure 35. During the modeling process, with the questions of the instructor and by group discussion, Beyza got the idea that slope of tangent line was changing at every point of the curve, and she reasoned on how to approximate its value. In the interview, this was expressed by Beyza as follows:

> R: Okay… you stated that you learned the meaning of slope on curved graphs. Can you detail this?
>
> Beyza: We generally compute the slope as we do previously. We draw a right triangle, difference here, difference here, and we can find the slope directly by dividing two… I do not know the slope on curved graphs; especially I cannot compute the slope at a particular point as observed in previous weeks. Here, we should approximate by forming smaller and smaller triangles, I could not think in that way at the beginning.
>
> R: But you did here.
>
> Beyza: I applied in this problem in an improvisation way. By these problems, I had the opportunity to see a real situation, and also applied in that situation. I did not experience previously such applications of approximating to the slope of tangent line at a point, although I know formally.

In the above episode, Beyza explained that her understanding of slope was the ratio between two sides of a right triangle as they applied for calculating the height. She emphasized her inexperience with the analysis of changing slopes on curved graphs. Her understanding of slope was dominantly shaped within linear contexts.

During the group or classroom discussions, pre-service teachers' awareness of the changing nature of slope of tangent line on curved graphs increased. "Roller Coaster" and "Population of Turkey" activities helped pre-service teachers in realizing the changing nature of slope on a curve, and in understanding the possible ways for approximating the slope at a point.

4.3.2.2 *Pre-service teachers' understanding of difference quotient rule*

Difference quotient rule known as the function difference divided by the point difference is the essential idea for understanding of the derivative with its formal definition. Difference quotient is the symbolic or algebraic interpretation of rate of change.

In Questionnaire-1, the 7th question was directly related to application of difference quotient. In this question, pre-service teachers were provided a tabular data with 0.1 unit increments in the x-variable for a continuous and differentiable function, and they were asked to find the approximate value of derivative at point 2. Most of the pre-service teachers could not use the difference quotient rule for approximating the derivative at the given point. For instance, as seen in Figure, Beyza translated the tabular data to graph but she could not use difference quotient rule from left or right side for approximating the value of derivative at the given point. Ran wrote out the symbolic following formulas; $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ and $f^{'}(2)=\frac{f(x)-f(2)}{x-2}$, but she could not use these formulas for approximating the value of derivative at point (2, 2) on the given tabular data.

In the "Population of Turkey" activity, the slopes of secant and tangent lines, approximating them from right or left side and by Mean Value theorem were the basic ideas discussed. Unfortunately, most of the pre-service teachers' ways reasoning was in percentage and during group solutions they did not discuss these ideas. Pre-service teachers realized these ideas after transferring tabular data to graphical form during classroom discussions. For instance, Rana realized that the rate of change can be interpreted as slope on the graph. She explicitly mentioned about the definition of derivative and used the difference quotient rule for calculating the average rate of change in population between the years 1990 and 2007. By the discussions in the interview and classroom discussion many other pre-service teachers realized the relation between difference quotient rule and slope, and used this idea when explaining the way of approximation to the derivative at a point on a curve.

Pre-service teachers started to use approximating the slope of tangent line from left and right sides. During the interview on the “Roller Coaster” activity, I asked pre-service teachers about the way of approximating the value of slope at the inflection point by using only a meter stick. Halit’s reasoning for this question was as follows:

> R: Let’s say, you have meter stick. You can measure the vertical and horizontal distances by the meter stick. How can you approximately calculate the slope of tangent line at the inflection point?
>
> Halit: We can calculate by two ways; we can approximate by taking another point from right side very near to the inflection point. That is to say, we take another point 2 cm further from the inflection point, and we can say h/x by using these two points. Or, we can take the 2 cm down of the inflection point, and we again obtain an h/x. Moreover, we can take two points having equal height distances from inflection point, one is 5 cm higher and the other one is 5 cm lower, we can find a slope by drawing a triangle between these two points. If you ask which of them will give a better approximation, it changes according to me.
>
> R: How it changes?
>
> Halit: I am thinking such a way that if the other point taken from the above of inflection point is far away, then we can obtain a very bad result. For instance, we can obtain 3 not 5.67. We should take the point very near to the inflection point, which is the reason why we say as h goes zero, and this is also the requirement for continuity.

For the question asking calculating the slope of tangent line approximately at the inflection point by using only a meter stick, Halit provided a detailed explanation. He mentioned approximating from left or right side by also emphasizing the smallness of the interval. He also mentioned about approximating by calculating the slope of secant line of two points very near to the inflection point one is from left and side one is from right side.

I also asked Beyza about the way of approximate the slope of tangent line at a point using some particular values. Beyza’s answer for this question was as follows:

> Beyza: Okay, let’s the unit of lengths is in centimeter, for instance 9 and 11. After finding the heights corresponding to these two points, actually I can find the slope at these points.
>
> R: How?
>
> Beyza: I mean I find the slope here and here by dividing the height with the distance.
>
> R: How can you find the slope at that point?
>
> Beyza: (thinking)…hmm I cannot find. I mean, when I say finding the slope, if I divide… (Thinking)
>
> R: Okay, let’s say we accepted these two distances are 9 and 11 cm. The height here is 15.2 meter, and the height here is 14.9 meter.
>
> Beyza: Ahaa, last week we mentioned about the Mean Value theorem, this is very similar to the last weeks, and don’t we go by using the limit? To be able to use limit, we should know the formula of the function.

R: But there is only a meter stick here, and we can measure some particular values.

Beyza: I can do like that; if I find the slope between these two points, actually I will find the approximate value of the slope at that point.

R: Okay, can you do by assigning the particular values?

Beyza: 15.2 minus 14.9, I can do like that, when the distance here divided by the distance here, this gives me the slope. For instance, I say 11 minus 9, and I find the slope between these two points, and this will be the best approximation for the slope at the point between two. I mean, the reason of forming a triangle was also doing this computation.

In the episode above, she first mentioned about dividing the value of dependent variable to the value of independent variable at that point, but she realized her fault. After assigning some particular values, she directly used the difference quotient rule $(\frac{15,2-14,9}{11-9})$ for calculating the slope between two points. Beyza interpreted this slope as an approximate value of the slope at point 10 which is between 9 and 11. She applied the central idea indicated by Mean Value theorem.

Developments in pre-service teachers' understanding of difference quotient rule and its relationship with derivative was observed in Questionnaire-2. While answering the 7th question involving an approximation for the derivative at a particular point, most of pre-service teachers successfully applied the difference quotient from left side, from right side, and by considering the idea of Mean Value. This was clearly observed in answers provided by Halit, Beyza, Rana, and Nilgün.

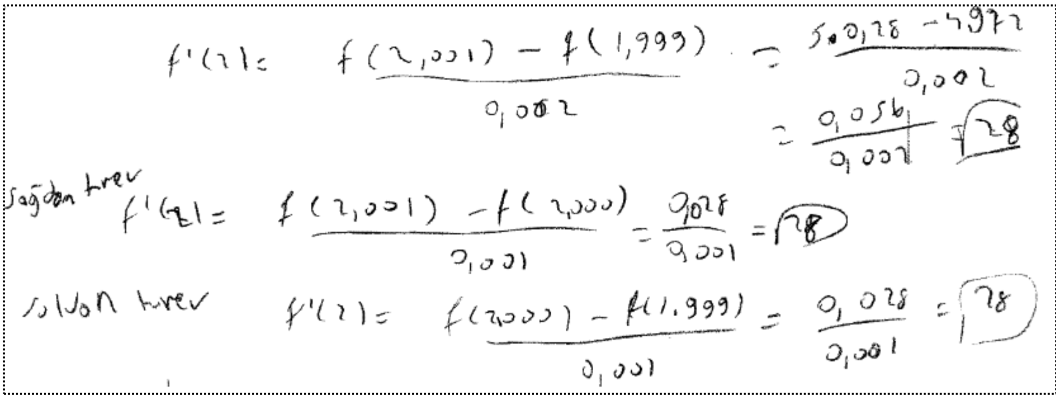


Figure 36: Halit's answer for the 7th question in Questionnaire-2

As seen in the figure, Halit successfully used difference quotient rule in Questionnaire-2 for approximating the derivative value at a point from right and left side and in the form of mean value. In the interview, Halit emphasized important points about what he learned by these activities. From Halit's point of view, his understandings related to the meaning of tangent line, what inflection point is, and the way of approximating the slope of tangent line by using difference quotient rule developed. He also emphasized that he already knows almost all the ideas covered during this sequence, but he did not have any idea about how to approximate to the slope of tangent line at a point. He added that "*Although I know the expression* $\frac{f(x+h)-f(x)}{h}$ *and I can apply it for solving various problems, but I could not use it for approximation*".

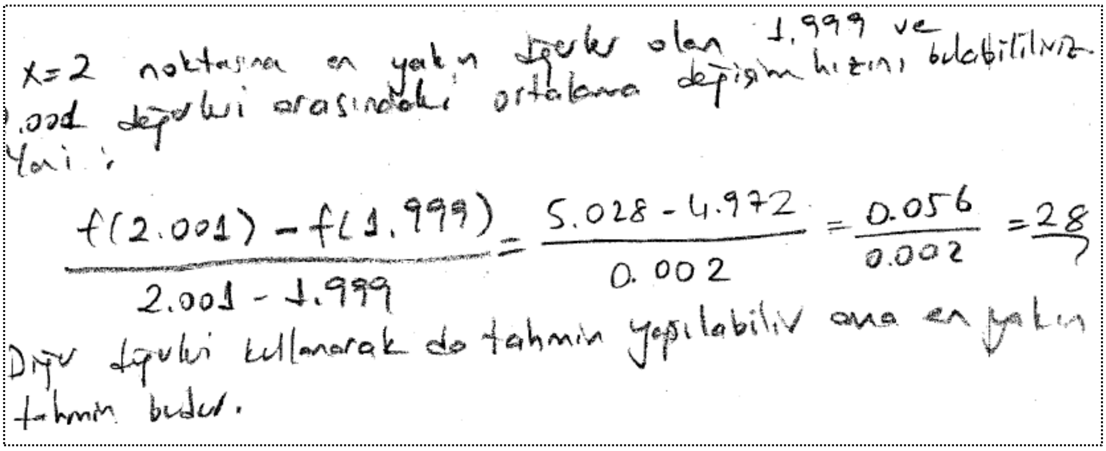


Figure 37: Rana's answer for the 7th question in Questionnaire-2

Rana also answered the 7th question in Questionnaire-2 by successfully applying the difference quotient. As seen in Figure 37 above, she successfully used difference quotient rule in the form of Mean Value Theorem for approximating the derivative value at a point. Also, she verbally stated that "*for getting the better derivative value at x=2, we can use two adjacent points from left and right side and calculate the average rate of change between these two points*".

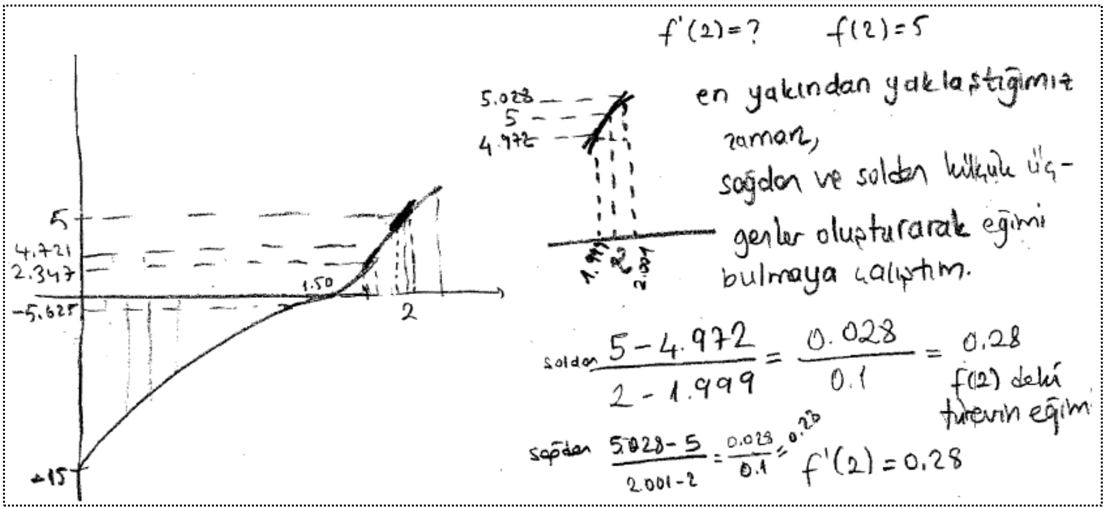


Figure 38: Beyza's answer for the 7th question in Questionnaire-2

Beyza also used difference quotient rule for approximating the derivative value at a point from right and left sides as seen in the figure above. She first translated the tabular data to the graphical form, plotted the given points on the graph, and applied the difference quotient rule from left-right sides. In the interview, I asked Beyza about the concept of derivative and wanted her to evaluate what she learned by these activities.

R: Okay Beyza, by the way, if I ask you what the derivative is what would be your answer?

Beyza: Derivative let me show concretely. For instance, I can ask the derivative of some particular points on a curved or linear graph. But, how can I define the derivative? We have learned that the limit from right and left sides should be equal for the differentiability. I mean, the function should be continuous, I can say the slopes of tangent lines…

R: Okay, you said the slopes of tangent lines. What else you can say when considering the "Water Tank", "Population of Turkey" and this height-distance context…

Beyza: In the "Population of Turkey", it was asked at what year the population change will become stable. We used derivative there, for instance we had obtained a graph, and we interpreted the stability of population as the point where the derivative is getting zero. Because, I thought by using the derivative such a way that when the derivative becomes zero, there will be a top point.

R: Yes, for instance, what does derivative mean in volume-height context? What is derivative at a point?

Beyza: Volume-height, for instance this is height and this is volume. Is it with respect to per unit time…? I have already indicated derivative as the slope, or rate.

R: What is rate?

Beyza: For instance, it is the increase in volume per unit height

R: increase in volume per unit height, so how do you express the "rate" in height-distance context?

Beyza: I can say it is the increase in height when I horizontally go 1 unit further…

In the episode above, Beyza explicitly used different interpretations of derivative in the same context by forming meaningful connections between them. She started with the slope of tangent line interpretation of derivative and mentioned about the rules of differentiability (e.g., continuity and equality of derivatives from left and right sides). When I asked her to explain derivative contextually, she mentioned about rate. She explained rate as *change in volume per unit height* by also interrelating this interpretation with slope and derivative. Beyza seemed to realize the interrelationship between *unit per unit thinking* that she used in covariational reasoning tasks, rate of change, slope, and difference quotient rule as being the different interpretations of derivative. But interestingly, when interpreting the symbolic expression of average and instantaneous rate of change in Questionnaire-2, she did not use ratio-based explanations.

In short, as studying on the model development unit, pre-service teachers developed important ideas related algebraic interpretation of derivative during group and classroom discussions. Most importantly, they formed the connection between symbolic expression of difference quotient rule and the slope of secant (or tangent) line, and they re-conceptualized the idea of $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ as secant line approximates to the tangent. However, it still difficult to argue that pre-service teachers' developed a robust conception about the relationships between different interpretations of rate of change. Because, some of pre-service teachers' difficulties continued while interpreting the given symbolic expression of difference quotient in a context as reported under amount of change conception of rate of change.

4.3.2.3 *Pre-service teachers' ways of reasoning with the Physics concepts*

Pre-service teachers frequently used the speed concept while explaining the contextual meaning of derivative. They used the description of speed concept as reference for explaining the meaning of derivative or slope in other contexts. They generally visited the speed concept when they had difficulty in explaining the rate of change in non-motion contexts. For example; Rana consistently used the "speed" concept in place of "rate" as can be seen in explanations in Figure 27. During the interview conducted after the "Water Tank" activity, I asked Rana about the meaning of "speed" that she used for indicating the rate of change.

R: What do you mean by "speed"? How is it related to speed-time in motion context?

Rana: It is the speed (rate) of increase in height; it is not related with the speed in motion context. It other words, it is the speed (rate) of increase in height, for instance, the height doubly or trebly increases. The increase in height is getting faster.

R: Could you express the speed (rate) here with its units?

Rana: How?

R: For instance, what is "speed" in motion context and what is its' unit?

Rana: Meter over seconds. Hmmmm speed at this place…. (Thinking)… may be meter cube over minutes. Or is it centimeter cube over seconds?

R: What was the variable on the x-axis?

Rana: Volume… what can be its' unit? Ahaaa, centimeter over centimeter cube…. I am a little confused.

R: Let's use liter as the unit of volume

Rana: Then, it is centimeter over liter.

R: Now, let's write centimeter over liter. How do you explain this?

Rana: The rate of increase in height as a function of volume with respect to unit time.

R: But you put time in your expression.

Rana: Hmmm, I should not use "per unit time". I should say "with respect to per unit liter", this is a past habit, because we generally used "per unit time"… The true expression is change in the height per unit volume.

As can be seen in the episode above, Rana used the Turkish version of "speed" term in place of rate for explaining the change in height with respect to volume. I asked what speed means in volume-height context. She first stated that its meaning is different from the meaning in motion context and described it as the ratio between successive changes in dependent variable. When I asked about the meaning of speed in motion context with its units and wanted her to explain the rate of change in height-volume, she explained as "*change in height with respect to unit volume*". Rana generally used the terminology of "speed" in place of rate of change.

Beyza also used the description of speed as reference while explaining the rate of change, slope, or derivative in different contexts. For instance, she used the "speed" term in place of rate for explaining the change in height with respect to volume. When I asked about the meaning of speed in volume-height context, she first described the meaning of speed as "*distance covered with respect to unit time*" and applied this way of thinking for explaining the rate of change in height with respect to volume as "*change in height with respect to unit volume*".

*Table 14**:** Pre-service teachers' preferences of different representations for rate of change*

	Task-1 Pre-Test	Task-2 WT	Task-3 POT	Task-4 POT-F	Task-5 RC	Task-6 TT	Task-7 TT-F	Task-8 Post-Test
Halit	GS-P		GS-P DQR-P DQR-A	DQR-A	GS-P GS-A DQR-A	GS-P GS-A		DQR-A
Beyza	GS-P	GS-P PC	GS-P DQR-P DQR-A PC	DQR-A	GS-P GS-A DQR-A	GS-P GS-A	PC	GS-A DQR-A
Rana	DQR-P	GS-P PC	GS-P DQR-P DQR-A	DQR-A	GS-P GS-A DQR-A	GS-P GS-A	PC	DQR-A PC
Nilgün			GS-P DQR-P	DQR-A	GS-P GS-A DQR-A	GS-P GS-A		DQR-A PC

In summary, pre-service teachers' conception of derivative was slope of tangent line at the beginning. Their contextual understanding of derivative was limited with the Physics concepts. In the progress, pre-service teachers first realized the difference quotient interpretation of slope and derivative. They also linked connection between difference quotient and the concept of rate of change.

4.4 Pre-service teachers' ways of interpreting graphs and their conceptions of the graphical connections between a function and its derivative

All of the modeling activities implicitly or explicitly involved reading, interpreting, and drawing graphs. Although "Water Tank" activity was directly related to covariational reasoning, graph drawing and graph interpretation was also involved in this activity. The "Population of Turkey" activity also involved graph interpretation when the tabular data is converted to the graphical form. The last two modeling activities (i.e., "Roller Coaster" and "Tracking Track") were planned directly aiming to cover and develop pre-service teachers' graph interpretation skills and their conceptions of graphical connections between a function and its derivative. But, because important datum obtained also in other activities with regard to pre-service teachers' conceptions of graphs, the results have been introduced considering the chronological order of the other activities as well.

*Table 15**:** Coding schema used for analyzing the graphical understanding of derivative*

Categories	Sub-categories and Abbreviations
Interpreting graphs	Verbal explanation of a graph *Non-mathematical verbal explanation (VA-NM)* *Inconsistent verbal explanation (VA-I)* *Consistent verbal explanation (VA-C)*
Drawing graphs	Transition between different curves *Transition with sharp corners* (TR-SH) *Transition with (nearly) smooth corners* (TR- SM) Critical Points *Determining all max-min points or reflecting on graph successfully* (CP-MS) *Inability to determine max-min points or reflecting them on graphs* (CP-MI) *Determining inflection point successfully and reflecting it on the graphs* (CP-IPS) *Difficulty in determining the inflection point or reflecting it on the graphs* (CP-IPI) Increasing and Decreasing Intervals *Deciding by using the sign of derivative function* (INT-SDF) *Deciding intuitively without systematically using sign of derivative* (INT-IN) The area under the derivative curve *Deciding the top and valleys by using area (integration)* (AR-I) *Not considering the area under curve for deciding top and valleys* (N-AR) Reversing between rate and amount functions (in context) *Slope-based reasoning* (RV-SBR) *Realizing derivative-antiderivative relationship* (RV-RD)

The categories and sub-categories obtained during the data analysis are shown on the table above with the related abbreviations. For the details about the coding schemata of curve analysis, see Appendix-J3. In order to be reader friendly, I did not use abbreviations of codes in the presentation of results. I benefited from abbreviations only for summarizing the results on a table obtained in the process for all participants.

4.4.1 Pre-service teachers' ways of reading and interpreting the contextual graphs

When we looked at pre-service teachers' ways of reading and interpreting the graphs across the tasks, three different categories were emerged. These were; (i) non-mathematical verbal explanation involving improper usage of language such as depending on the physical attributes of the graph, (ii) inconsistent verbal explanations when compared with the original graph, and (iii) consistent verbal explanations involving the proper usage of mathematical language. It has been observed in this study that while pre-service teachers were using non-mathematical verbal explanations at initial phases, they started to use consistent verbal explanations in subsequent phases of the model development unit.

To begin with the tenth question in Questionnaire-1, most of pre-service teachers tried to explain the graphs by using non-mathematical verbal explanations. For instance, Beyza did not explain the curve by taking into account the area and volume as independent and dependent variables. As can be seen in Figure 39, Beyza explained the volume-area graph by the physical attributes of container.

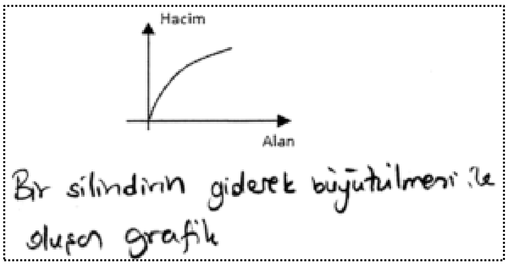


Figure 39: Beyza's answer for the 10th question in Questionnaire-1

Nilgün's explanation of the same graph was involving some improper usage of language. She explained the first graph with the expression that *"As the area increases, an increase towards negative direction occurs in volume"*. In this explanation, she did not functionally coordinate area and volume variables as independent and dependent variables. Rana explained the same volume-area graph by using the expression that "*The volume increases as the area increases. But, the speed of increase is high at the beginning and this speed gradually decreases*". She provided an explanation taking into account the area and volume variables as independent and dependent variables respectively with the first sentence, but, when trying to explain the curve, she mentioned about the speed of increase in volume.

With the development in covariational reasoning, pre-service teachers started to use more sophisticated and consistent verbal explanations. They began to use explanations as "*the dependent variable increasingly increases with respect to unit change in independent variable*" and they constructed graphs consistent with the verbal explanations. While at the beginning, they were explaining the graphs by mentioning the dependent and independent variables uncoordinatedly or by depending on the physical attributes of graphs, later they developed and used the idea of unit per unit thinking. For instance, Nilgün's explanation of the graphs in Questionnaire-2 was as follows:

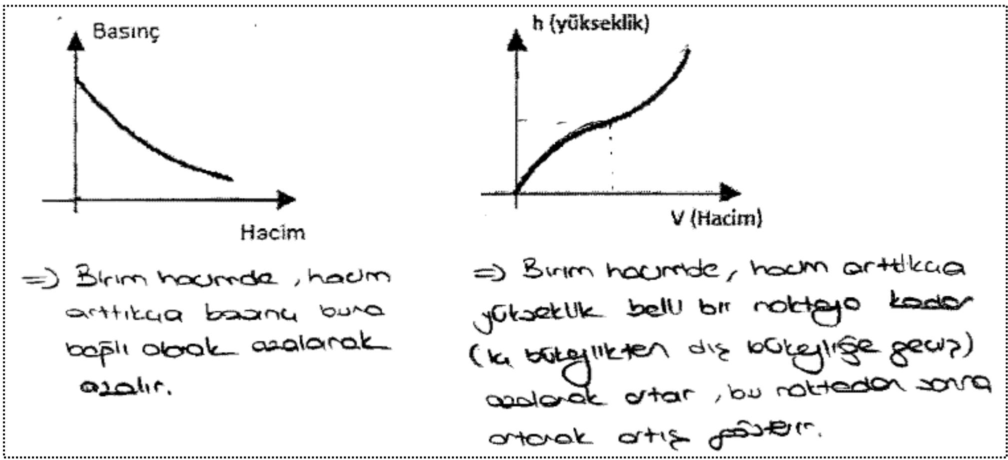


Figure 40: Nilgün's answer for the 10th question in Questionnaire-2

When we look at the verbal explanation provided by Nilgün for the graphs in the 10th question of Questionnaire-2, as can be seen in Figure 40, she used unit per unit thinking in the explanation that "*Pressure decreasingly decreases with respect to unit volume*". When compared with the explanation used in Questionniare-1, the mathematical sophistication of the verbal explanation increased considerably. Most of the pre-service teachers' demonstrated this way of explanation in Questionnaire-2.

However, pre-service teachers' difficulties in explaining the decreasing graphs were appeared. Although they hold the idea of unit per unit thinking and used this idea in their explanations, they inconsistently explained the decreasing graphs. For instance, Beyza's verbal explanations for the decreasing curves were inconsistent with the concave-up or down character of the curve. Her difficulty with verbally explaining the decreasing graphs continued throughout the sequence. She could not match the "*decreasingly decreasing*" and "*increasingly decreasing*" verbal explanations with the *concave-up decreasing* and *concave-down decreasing* graphs respectively.

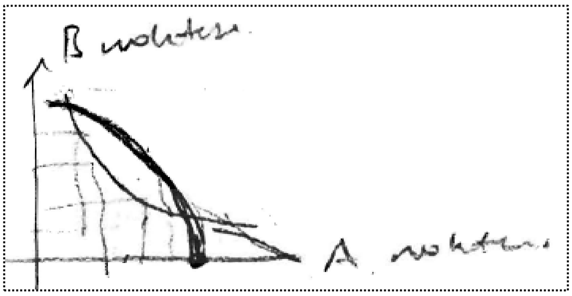


Figure 41: Beyza's drawing for the "Sliding Ladder" problem

Beyza: No, I cannot. I cannot see the general character of change; I have to perform one more step to be able to see the increase and decrease. If I slide the ladder with 1 more unit, the new values are 12, 20 and$\sqrt{246}$. Hmmm, this is 30, the difference here is 21. It means that, it gradually decreases more. Yes, increasingly decreases.

R: How can you show this on the graph?

Beyza: Increasingly decreasing, it is like that, no, no is it like that? That one… (Drawing a concave-up decreasing graph)

R: How?

Beyza: Isn't it like that? Increasingly decrease…

R: Which one is changing (decreasing) more?

Beyza: point B is decreasing more.

R: Is the point B decreasing more on your graph? Can you compare?

Beyza: No, it should be like that (drawing a concave-down decreasing graph), because it suddenly decreases. It decreases slowly on this graph (indicating concave-up decreasing graph). Okay, it is like that, this is zero (drawing the correct graph)…

As seen in Figure 41 and in the episode, Beyza confused about the nature of curve. Although she determined the increasingly decreasing character of the point-B with respect to unit changes of point-A in "Sliding Ladder" problem, she constructed a concave-up graph at first. I prompted her with the question that "*which variable is decreasing more?*" She quickly answered this question as the height of point-B. I wanted her to compare the changes in two distances on the graph as well. After thinking for a while, she corrected the curve and sketched the bold curve as seen in Figure 41. Beyza's confusion continued with the "Water Tank-2" follow up activity She explained the situation in a way that "*the height increasingly decreases with respect to unit decrease in amount of water*", but she constructed a concave-up decreasing graph. Beyza again used the expression "*the pressure increasingly decreases with respect to unit volume*" for the concave-up decreasing graph as seen in Figure 42.

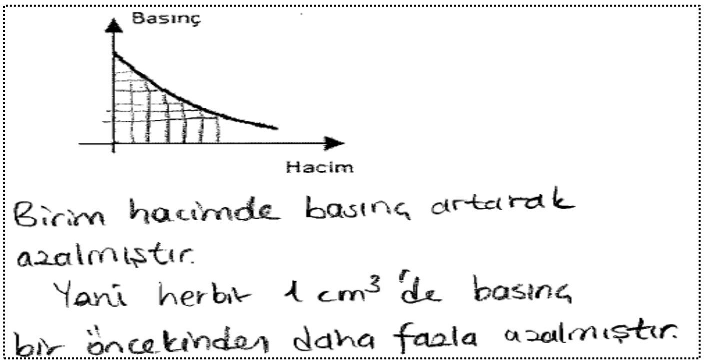


Figure 42: Beyza's answer for the 10th question in Questionnaire-2

The same difficulties while explaining the decreasing graphs were observed with Halit, Rana, and Nilgün. For instance, although Ran started to use unit per unit thinking consistently in the explanations for increasing graphs successfully, she lived some difficulties in explaining the decreasing graphs.

R: Can you draw a graph indicating that the height increasingly increases with respect to unit volume?

Rana: (drawing correctly)

R: decreasingly increase?

Rana: (drawing correctly)

R: increasingly decrease?

Rana: increasingly decrease, there is not such an expression. Just a minute, increasingly decreasing, the height will decrease, it will decrease increasingly. The height decreases, let's think as if we are pumping water...

R: Well, when the water decreased with unit amounts…

Rana: Let's think as if we are bleeding water from the filled containers.

R: Very well.

Rana: What was it, increasingly decrease. Can I draw a figure of container first?

R: Of course…

Rana: It will be a narrowing down shape like that. Is it like that (drawing a concave-up decreasing graph), or is it like that (drawing a concave-up decreasing graph), I should decide this at first. It will decrease increasingly, (she became uncertain between two graphs)… I will interpret like that, as the volume increases here…

R: Decrease the volume if you want, because the volume is decreasing

Rana: Here, the height nearly does not change, not this graph, that one (pointing out the correct graph), is it correct? It increases more here as the volume decreases. Okay, this one.

The difficulty of Rana while sketching and interpreting an increasingly decreasing (concave-down decreasing) graph has been observed in the episode. She confused it with a decreasingly decreasing (concave-up decreasing) graph. In the interview conducted after the “Water Tank” modeling activity, she emphasized her understanding and awareness of the interpretation of graphs having various curves developed. Then, I asked Rana about the graphs of increasingly increasing, decreasingly increasing and increasingly decreasing. She successfully drew the increasing graphs, but she could not give meaning to an increasingly decreasing graph. She first constructed a concave up graph, but she confused if the graph was as concave-up or concave-down. After thinking for a while by drawing a container figure and also by assigning some values on graph, she realized the difference between in the verbal expressions of a concave-up and a concave-down decreasing graphs.

Table 16: Pre-service teachers' ways of interpreting graphs

	Task-1 Pre-Test	Task-2 WT	Task-3 POT	Task-4 POT-F	Task-6 TT	Task-7 Post-Test
Halit	VA-NM (Q10)	VA-NM VA-C (Int*)	VA-I VA-C (Int)	VA-C	VA-C	VA-C
Beyza	VA-NM	VA-NM VA-C (Int*)	VA-I VA-C (Int)	VA-I	VA-C	VA-I VA-C
Rana	VA-NM	VA-NM VA-C (Int*)	VA-I VA-C (Int)	VA-C	VA-C	VA-I VA-C
Nilgün	VA-NM	VA-NM VA-I VA-C (Int*)	VA-I VA-C (Int)	VA-C	VA-C	VA-C

Pre-service teachers' ways of reading and interpreting graphs in the process of the model development unit is summarized on Table 16 above. At the beginning, most of the pre-service teachers used non-mathematical explanations while interpreting the graphs (VA-NM). In subsequent phases, they started to use the language correctly and properly by demonstrating the unit per unit thinking and by considering the functional relationship between variables while explaining the graphs (VA-C). Verbal explanations for the graphs shifted from explaining with physical attributes to considering the variables as dependent and independent variables within the explanation. However, some inconsistencies between the graph and the verbal explanation have been observed in pre-service teachers' interpretations (VA-I). Some of pre-service teachers had difficulty in explaining the decreasing graphs. Although they continued using the more sophisticated language, they confused the verbal expressions for concave-up decreasing and concave-down decreasing graphs.

4.4.2 **Pre-service teachers' understanding of graphs and their conceptions of the graphical connections between a function and its derivative**

The analysis of data revealed various dimensions by which pre-service teachers' understanding of graphs and their conceptions of the graphical connections between a function and its derivative can be evaluated. These are, (i) the way of considering the fluency at the transition points, (ii) ways of handling the critical points, (iii) the ways of deciding the increasing and decreasing intervals, (iv) the way of considering the area under the derivative graph, and (v) the way of reversing between derivative and antiderivative graphs. Interesting results obtained for each of these dimensions through the model development unit.

4.4.2.1 *Transition between different curves*

The first results related to graph drawing were obtained in the "Water Tank" modeling activity. In this activity, pre-service teachers were asked to draw the height-volume graphs of filling water tanks. This activity was involving the construction of a continuous piecewise function not necessarily differentiable at every point. There were some problems in pre-service teachers' drawings at the transition points. Most of the pre-service teachers did not pay attention to the transition points and they fell into error in their drawing.

In the group solution for the "Water Tank" activity, Halit and his group drew a continuous graph formed by linking the different shaped curves. As can be seen in Figure, they drew the graph with sharp corners at the transition points. The problem with this drawing was not the sharp corners, but the appearance of the corner point indicating that the rate of change just before the point and just after that point was not consistent with the original situation.

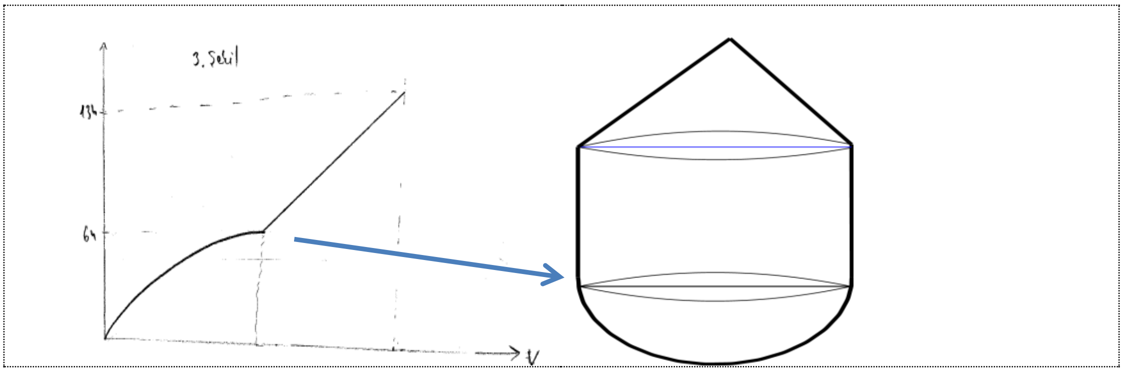


Figure 43: An excerpt from Halit's group solution: An example of sharp transition

During the group discussion, Halit and his group could not get the idea of (nearly) smooth transition. Because of time limitation during the classroom implementation, all the group solutions could not presented their solutions and also the classroom discussion was postponed to the next class. The remaining group solutions were presented at the beginning of the next class. So, we conducted the interview with Halit before the classroom discussion. In the interview, I asked Halit about these transition points.

> R: For example, let's consider the following situation. Look at the regions just before and just after the transition point at where the figure of water tank changes from spherical to the cylindrical. What is the difference between the increase in height per unit volume, just before the transition point and just after that point?
>
> Halit: (Thinking, about 6 seconds)…Now we are taking this point, just before and just after, I mean there seems no difference actually, is there? Because, there will be a slight difference, they are almost the same.
>
> R: All right, isn't there any slight difference? When we compare this point and this point, at what point it should be faster according to the water tank?
>
> Halit: Is it faster here? (Pointing out the spherical part)
>
> R: Very well, let's turn your graph again. According to your graph, at which point the increase in height seems faster, here or here (asking on the graph)?
>
> Halit: Hmmm, this part should go from downward. At the transition point, we say it will decrease, but we are drawing from upward. This is a fault.
>
> R: So, how should it be?
>
> Halit: We should draw from downward, hmm I got the idea. When we compared the very small amounts of change in height with respect to unit volume, they are almost equal, and this already gives the slopes at these points. This appears when we compared these two. I think, this is an expected fault for us, because we did not experience questions showing these kinds of faults…

In the episode above, I tried to attract Halit's attention to the sharp corners at the transition points. I asked him that "*What is the difference between the increase in height per unit volume, just before the transition point and just after the transition* *point?"* After silently thinking for a while, he answered "they are nearly the same", and he realized the small difference between two. I then asked him to reconsider their graphs as if it shows the small difference. Halit realized their fault and he stated that the second (linear) part of the curve must be drawn from downward. He also explained the situation by using unit per unit thinking and connecting it with the slope by stating that "W*hen we compared the very small amounts of change in height with respect to unit volume, they are almost equal, and this already gives the slopes at these points*". In the interview, Halit used the idea of unit per unit thinking with infinite-small thinking for deciding the smoothness of the graph. In short, Halit realized the idea that the curve must be (nearly) smooth at these transition points[1].

As also observed in other groups, in group solution of the "Water Tank" modeling activity, Nilgün with her group drew continuous graphs, but the graphs were having sharp corners at the transition points where the tank figures change forms. For the third tank, Nilgün with her group drew the graph as can be seen in Figure 44.

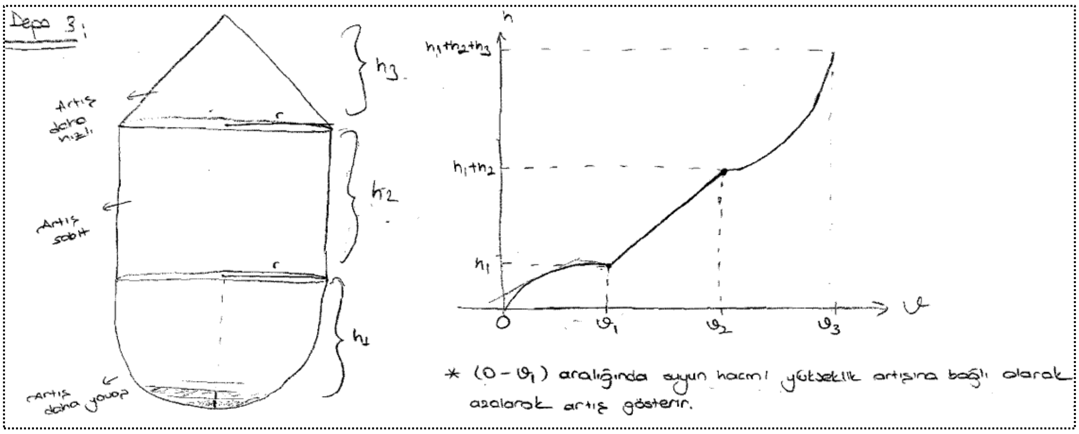


Figure 44: Nilgün and her group's drawing for the "Water Tank" activity

The graph constructed by Nilgün with her group shows the height-volume relationship correctly in general, but there seems problem at the transition points between the intervals. When we look at the point where the transition from first region to second or from second region to third one occurs, a dramatic change in

[1]The term "smooth" used here does not mean differentiability at that point.

height with respect to volume is observable according to the graph. In the interview, I took her attention to their drawing at these transition points. Nilgün realized their mistake by comparing the rate of change in height with respect to unit volume just before and just after the transition point and tried to remove the cambered view of the graph by drawing it again.

The same mistakes were observed in Beyza's and Rana's group solutions. Pre-service teachers realized their mistake and corrected their graphs by comparing the rate of change in height with respect to volume just before the transition point and just after that point. Some of pre-service teachers realized during the interviews, but most of them realized during the group presentations and classroom discussion.

4.4.2.2 *Pre-service teachers' ways of reasoning when reversing between a function and its derivative*

Almost all of the pre-service teachers lived difficulty in interpreting the derivative-antiderivative relationship in the "Tracking Track" modeling activity. In individual and group studies, they tried to construct the height-distance graph by slope-based reasoning without realizing the derivative-antiderivative relationship. During the individual and group studies, all participants spent a considerable amount of time on discussing about the way of constructing a distance-height graph from the given distance-gradient graph. They could not easily realize the relationship that the distance-gradient graph is the derivative graph of the distance-height. The dominant usage of slope-based reasoning resulted in various difficulties. Firstly, pre-service teachers had difficulty while transferring the concave-up increasing part of the gradient-distance graph to the distance-height graph. In other words, they could not realize that an increasing distance-gradient graph on positive side (even if it is concave-up increasing, concave-down increasing, or linear increasing) will yield a concave-up increasing distance-height graph. Pre-service teachers generally considered the variation in the slopes, although it was not necessary. The second difficulty was interpreting the negative parts of gradient-distance graph. The details from group solutions, group discussions, and interviews were provided below.

Most of the pre-service teachers did not realize a derivative-antiderivative relationship between the two graphs. For instance, Halit and his group, at the beginning, did not consider derivative, anti-derivative relationship. They constructed the distance-height graph intuitively by focusing only on the slope (gradient)[2]. In the process of constructing distance-height graph with slope-based reasoning, they lived the difficulties stated before.

> When we look at the region B, the slopes were decreasingly decreasing. In spite of a decrease in the slopes, the height was still increasing; therefore we drew a decreasingly increasing graph for this region. Then we came to the issue that we spend the longest time during the group discussion. That was about the region C. At the beginning, we decided that the height decreases in this region. But, me and a friend of mine thought that the distance-height graph should decrease decreasingly, because we were also looking at the slopes of distance-gradient graph. However, the slope was decreasing in negative direction. Later, we realized our mistake by comparing the slopes of graph drawn by SB (a group member) with the other graph. The mistake was that we were looking at the slopes of gradient graph. But, we had already known the intervals where the slopes were increasing, decreasing, negative, or positive. Thanks to that we divided the distance-gradient graph into 8 regions and we tried to draw the distance-height graph. (Halit, Reflection Paper, Tracking Track Activity).

Halit indicated in the reflection paper that they tried to construct the distance-height graph by only focusing on the slope on distance-gradient graph. They did not directly determine, for example, the points intersecting x-axis on distance-gradient as the relative maximum or minimum points on height graph. Similarly, they did not consider the maximum or minimum points on the gradient-distance graph correspond to the inflection points on the height-distance graph which were the procedures used in reversing between derivative and antiderivative graphs. Their reasoning was depended on the changing nature of slope appeared in the verbal expressions as "*the slope increases increasingly*". But they lived difficulty for transferring and representing the increasingly increasing (or decreasingly decreasing) part of distance-gradient graph to the distance-height graph. For the positive parts, they intuitively sketched the graph with slope-based reasoning, but for the negative parts it did not work.

[2]Slope and gradient was used interchangebly, but it does not mean these two terms are mathematically equivalent.

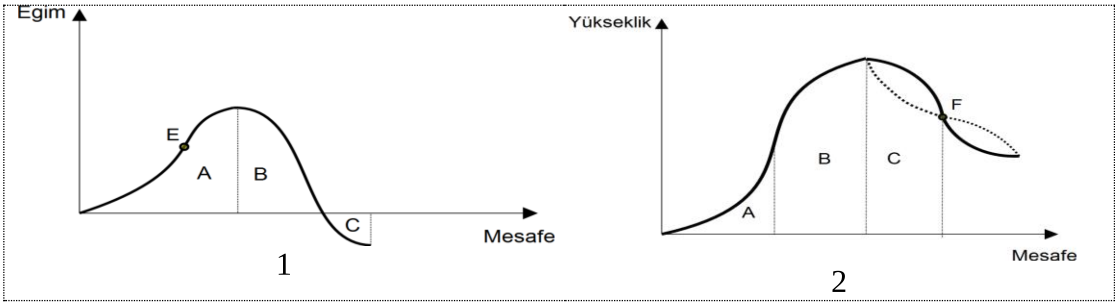


Figure 45 : The difficulty in reversing between two graphs for the region C

As can be seen in Figure 45, they discussed about the way of drawing the distance-height graph for the region C. Their reasoning was that since the slope decreasingly decreases in this region, also the height should decrease in the same manner, and constructed the graph as the dashed curve on the second curve in Figure 45. The group discussion process was as follows:

SG: Now, the graph is decreasingly decreasing here, isn't it? Decreasingly decreasing (Region-C), I mean it is like that…

SB: The absolute value of slope increases.

SG: the slope is decreasingly increasing, isn't it? I mean, the graph may be like that ( )

SB: If we reverse this curve

SG: If we reverse this graph, then it will be like that ( ). Hmmm. The slope increased decreasingly… (Silence, 5 seconds). Now, is the slope increasing here, what is happening?

Halit: Hey Friends, the inflection point on gradient curve is deceptive, do not consider it. The inflection points on the first graph are deceptive.

SG: Now, the slope is decreasingly increasing here, isn't it? (Indicating the region C)

Halit: We should speak in terms of absolute values… Is it increasingly increasing or decreasingly increasing? Let's see, it is decreasingly increasing.

SG: Decreasingly increasing… all right, is that graph decreasingly increasing of increasingly increasing? (Asking the thick lined curve in region-C on the second graph seen in Figure 47)

SB: It is decreasing, look at here; the slopes are increasing (showing by drawing tangent lines). Did you see, the slope increased. The slope increased, but decreasingly increased.

SG: Did it decreasingly increase? (Silence, 5 seconds)

SB: (By erasing his drawing beginning from the region-C), I am confused…

SG: The slope will decrease… The slope will increase by a decreasing pattern.

Halit: Look at me, this is deceptive. It is not useful unless confusing you. We should interpret like that, SB drew the graph here. At what point the slope is greater, here or here? (*While pointing out a point very near to the top, and a point very near to the finishing point in region-C*)

SG: But, isn't it important how I drew the graph in that interval?

Halit: Is the slope greater here or here?

SG: It is greater at the second point… (Silence)

SB: (Drawing, by drawing the lined curve on the second graph in Figure 47)

Halit: Its true, continue in that way.

SG: Is it like that (asking the region-C). I am really confused…

Halit: What is happening here, if the slope increasing or decreasing?

SG: Increasing.

Halit: Increasingly increase or decreasingly increase, these are not important. If it is decreasing or increasing, it is increasing. What happens to the height when the slope increases?

SG: It increases

Halit: It increasingly increases. (The instructor came). Instructor, you put an inflection point on the gradient graph, it results in confusions. You did not provide any other information. I mean, it may be a cubic function, or a fourth or fifth degree function, we cannot be sure…

SG expressed the decreasingly decreasing nature of slope in region-C. She was focusing on the variation in the slopes on the gradient-distance graph which is related to second derivative. The thinking style that "*If the slope increases increasingly, the height also increases increasingly*" worked in region-A, but it resulted in difficulties in region-B, and especially in region-C. Later, Halit noticed her friend about the unimportance of changing nature of gradient for constructing the height graph. He first emphasized the confusion resulted from the given inflection point on the gradient-distance graph by stating "*the inflection point on gradient curve is deceptive, do not consider it*". Later he explained that "*the increasing or decreasing nature of change in slope is unimportant, what is important here is if the slope is increasing or decreasing and positive or negative*". Halit tried to explain his friends that if the slope increases and it is positive (even if it is concave-up increasing, concave-down increasing, or linear increasing), then the height will increase increasingly. They at first constructed the curve represented with the dashed line by thinking in a way that "*if the slope decreases decreasingly, the height also changes in the same manner*". With the explanation of Halit, they only focused on the increasing or decreasing nature of slope without considering the variation in that change.

Similar processes have been lived in other groups. For example, Rana her group lived difficulty in "Tracking Track" modeling activity. As also observed in other groups, they tried to construct the height-distance graph by slope-based reasoning without realizing the derivative-antiderivative relationship. They divided the distance-gradient graph to nine regions by considering the top and valley points, inflection point, and the x-intercepts. Although they determined these critical points on gradient graph, they did not proceed with the procedures of constructing anti-derivative graphs for drawing the height-distance graph. Their reasoning was purely slope-based and this resulted in many difficulties and confusions. In the group report, their slope-based reasoning for each interval was observable. Their reasoning for the first interval was as "*because the slope increases increasingly, the height also increases increasingly*". They also continued by this way of thinking for all intervals.

The first difficulty of Rana's group with slope-based reasoning was confusion in transferring the concave-down increasing part of the gradient-distance graph to the height-distance graph. For the concave-down increasing part, their explanation was as "*because the gradient decreasingly increases, the height also decreasingly increases*". They inconveniently generalized the idea that they used for the first region to the other regions. They could not interpret the simple idea that for all positive increasing gradient-distance graphs having various curves (i.e., linear, concave-up, or concave-down); the character of height-distance graph should be increasingly increasing (concave-up) curve. Rana realized this idea during the interview with my prompt as seen in the following episode.

> R: Rana let me ask you that: let's f'(x) is a linear graph, a concave-down increasing, and a concave up increasing. Now, can you think about the graph of f(x) for each of the three situations? I mean think about the graphs roughly…
>
> Rana: The derivative here is continually increasing. I mean, the slope is constantly increasing. If the slope constantly increases … increasing continuously… Is it like that? (Drawing a concave-up increasing graph)
>
> R: Very well, can you draw for that one?
>
> Rana: Slope 1, 5, 8, 10… Is it the same with the previous one? Let me look at the slopes, it is greater here, and it decreases here. Okay, the graph will be similar.
>
> R: Okay, let's draw for this…
>
> Rana: The slope is increasingly increasing here. Is this also the same? Actually, the slope is increasing for three of them… Hmmm, the graphs of antiderivative function for these three graphs have the same character… I mean, the antiderivative graph will be concave-up increasing even if the derivative function is linear, concave-down increasing, or concave-up increasing…

Another difficulty with slope-based reasoning that Rana lived with her group was in transferring the negative parts of gradient-distance graph to the height-distance graph. For the negative parts of the gradient-distance graph, they first drew a graph with a corner point. Although they corrected at the end, graphs with corner points and sharp transitions as seen in Figure 46 was apparent in their earlier drafts of drawings.

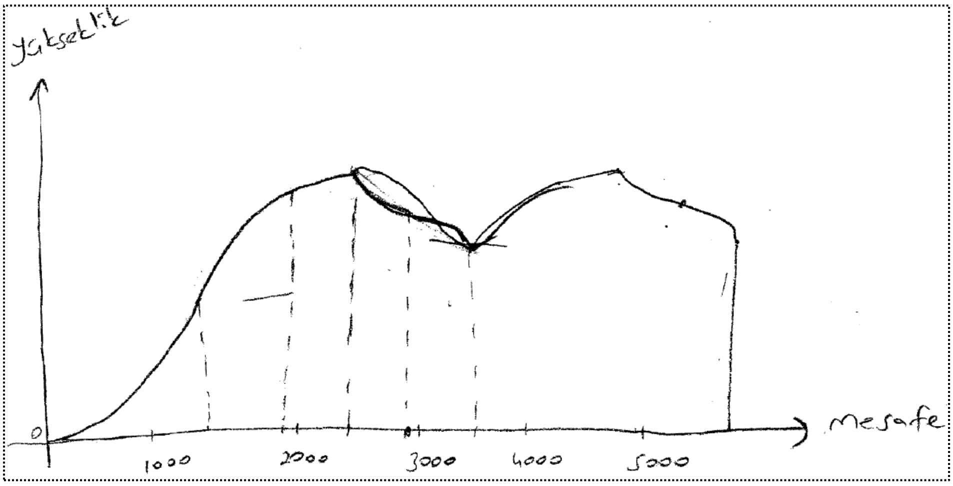


Figure 46: Earlier version of Rana and her group's drawing in the "Tracking Track"

Interpreting the negative parts with slope-based reasoning was difficult. Rana with her group used this way of thinking and they constructed incorrect graphs in group discussion process. In the interview I asked Rana about these drawings.

R: Can you mention about your wrong drawings?

Rana: That interval, as I said before, there is a decreasingly decrease here, now we say decreasingly decrease if it was positive, we thought as if the height-distance will also decrease decreasingly…yes, but there is an increase here, I mean the slope is increasing…

R: Yes, it is increasing at negative direction.

Rana: Yes, it is increasing at negative direction, like -1, -2, and -3. This was the reason for our mistake; we had difficulty in interpreting this negative part. We drew like that (Pointing out Figure 46) at first. But the graph seemed us a little strange. There is no such a track, how it will be at this point? Yes, the shape of graph seemed us a little strange. I mean, that point is like a scarp. The same is true for the other point; there is a point here that one should jump…

R: Yes, you turned again, and you interpreted the slopes again….

Rana: Yes, we decided about the mistakes, and then we draw as a symmetric…

As seen in the episode, Rana indicated that with slope-based reasoning they drew the height-distance graph as the graphs seen in Figure 46. They expressed the concave-up decreasing part of gradient-distance graph at negative side (see, region 4 in Figure 46) as "*decreasingly decreases*" as if it is in positive side and they drew a concave-up decreasing curve on the height-distance graph. Rana indicated that they transferred the decreasingly decreasing gradient-distance curve to the height-distance by keeping the same curve shape. She also indicated that they intuitively realized their mistake by looking the strangeness of graph shape having sharp transitions and corner points. They decided to draw the symmetries of curves at these negative regions for getting rid of this strangeness. Interestingly, during the process of obtaining the final graph, they did not form any relationship with the derivative and antiderivative.

The discussion in Halit's group, Rana's way of reasoning shows how slope-based thinking (without realizing derivative-antiderivative relationship) resulted in difficulties when reversing derivative graph to the anti-derivative in a context. Although Halit, Rana, Beyza and their group friends drew successful anti-derivative graph of a context-free derivative graph asked in Questionnaire-1, they had difficulties in constructing the distance-height graph in this activity. The thinking style that "*If the slope increasingly increases, the height also increasingly increases*" worked in the first region, but it resulted in difficulties in other regions. The problem with this way of thinking was that pre-service teachers are unnecessarily focusing on the variation in changing nature of the gradient graph which is related to second derivative. However, pre-service teachers thought in depth for reversing from gradient-distance to height-distance for each region which could not be observable in procedural derivative and anti-derivative graph sketching applications. Most importantly, they realized the derivative-antiderivative relationship and they covered the mathematical ideas behind the procedures of graph drawing by using the derivative. Most of the pre-service teachers correctly drew the graph of antiderivative function asked in the 11th question in Questionniare-2.

4.4.2.3 *Pre-service teachers' realization of the idea of integration and the derivative-antiderivative relationship*

While solving the "Tracking Track" activity, most of the pre-service teachers did not realize the derivative-antiderivative relationship at first. They tried to draw the graph by only focusing on the changing nature of slope. Similarly, most of the pre-service did not realize the idea of integration while deciding the levels of top and valleys on height-distance graph despite some of them used the area under graph intuitively.

A few group used the idea of area under gradient-distance graph for deciding the levels of top and valleys, but they had difficulty in relating it with the idea of integration. Halit's group was one of the two groups who discussed about the area under gradient-distance graph. The discussion process in Halit's group lasted for a considerable amount of time on this issue. Halit and his group discussed about the way of deciding the levels of top and valleys on the height-distance graph constructed from gradient-distance graph. While discussing on if the area under gradient curve is related to height or not, they also discussed the derivative, anti-derivative relationship.

> While drawing the distance-height graphs for each region, it was also important to see how the height will be drawn according to the distances. I mean, for example for the region-C, how much the graph will be dropped, and how much it will be dropped for the region-D. At that point, the idea of integration came into my mind. If the area under speed-time graph or integration gives us the distance, then the area under distance-gradient graph should give the height as well. I immediately shared this idea with my friends, but they did not convict. The graph drawn by SB was related with the area under distance-gradient graph… (Halit, Reflection Paper, Tracking Track)

Halit indicated in the reflection paper that he intuitively realized the relationship between the area under the distance-gradient graph and the distance-height graph. He used the speed concept for supporting his idea. But, when he explained this to his friends, they were not convicted with the explanation and a group discussion began and lasted about fifteen minutes. An excerpt from group discussion is provided below.

Halit: if the area here is greater than the area here, then the height level is higher. Haven't you still convicted about the idea of area?

SG: No, because you could not explain. You are saying that the area here gives the height...

Halit: I argue this.....

SG: No, I am asking how you say this.

SB: Think about an explanation, you have not explained yet.

Halit: Look at here, take a point; I looked at the area until that point. Let's say this area is A. Let's look here now, to the point 2000. Between 1000 and 2000, as I see here, the area is 2A or may be 3A. I mean, the height increased by 3A here. While it was increasing by A at the beginning, it increased by 3A later. And the horizontal distances are equal.

SG: Why, I am asking why it is?

Halit: When we look at the 0-1000 and 1000-2000 intervals, it can be observed explicitly.

SG: Now, are you thinking in such a way that the height increased because of the shaded area increased?

Halit: Okay, let me explain by another way. Think about a speed-time graph. The area under this graph gives us the distance. For instance, at the fifth second...

SG: All right, why does it give the distance? Because of the equation X=V.t (writing down). So, is there a relationship that Height=Slope*horizontal distance here? Is this the height?

Halit: ... (Thinking)... it is not height exactly, but approximately. Because, let's think as a triangle, you are climbing on a hill. What are we saying, slope*horizontal distance... (*Thinking about 7 seconds*)

SG: Okay, let's consider a triangle (drawing a triangle). You tell me that the area here gives us the height. This is slope (point out the hypotenuse), 3-4-5 triangle, and h=3. Let's look h*5… (*Thinking the length of hypotenuse as the slope*)

Halit: You are writing X=V*t here, but of course we cannot write such a formula here.

SG: No, I am telling you that the equality of X=V*t is the reason for why we interpret the area here as distance. If you explain by depending on this, then I ask you about that. Then, the result of multiplying the slope by the horizontal distance should give the height.

Halit: I mean, when the graph of height have the formula of x over 4, then the slope graph will have the formula of 16x over 3. I am saying this, it is not necessary to write X=V*t.

SG: What did you do here, did you take derivative?

Halit: Yes

SG: Okay, you are taking derivative, but I am telling you that I do not understand why the area under this graph gives the height. I am not arguing you are telling wrong, but explain it.

In the episode above, Halit claimed that the area under gradient graph gives the height of track. Other group members questioned Halit to explain this claim. As the first attempt, Halit tried to explain his reasoning by comparing the height differences between two intervals (0-1000 and 1000-2000). He could not convict his friends, because he was using the area under curve as an argument in his explanation. He was using the area under graph as a justification for his claim which was already been questioned by others. Halit then decided to use the relationship between velocity and distance from the motion context. SG emphasized that the area under speed-time graph gives the distance because of the $x = v \cdot t$ relationship and she wanted Halit to provide a formula-based mathematical explanation in the form of "*Height=Slope*Horizontal Distance*". While Halit was thinking about this, SG constructed a 3-4-5 triangle and determined 3 as height, slope as 5 (the length of hypotenuse), and asked again Halit if there is such a relationship. Halit stated the impossibility of writing such a formula (height=slope*horizontal distance) and continued with his claim by stating that if the distance-height has the formula of $4x^4$, then the distance-gradient will be $16x^3$. Halit was continuing to argue the derivative-antiderivative relationship without providing a sophisticated explanation. This discussion continued for a long time in the form Halit claimed insistently, and other group members asked for a mathematical explanation until asking for help from the instructor.

Halit: I will ask to the instructor. Instructor, SG want me to convict her, we could not solve the situation. I really believe in that this is related to derivative and integration, but I could not prove it.

Inst: SG, why don't you be convicted?

SG: Because, he could not provide a sophisticated explanation. He intuitively realizes this. He is telling that if this area is A, then this is 3A, but why?

Halit: In my opinion, we can say, it is explicit.

Inst: SG says that why we obtain this height when we compute the area here by multiplying slope and distance (Drawing a triangle). So there is a slope here, m, and this distance is x. She is asking why $\frac{m \cdot x}{2}$ gives the height.

SG: Okay, this is a right triangle. Here is height, and slope…

Inst: Yes, isn't this a slope on this graph? Therefore, if we consider this as linear, then does the area obtained by $\frac{m \cdot x}{2}$ give us the height?

Halit: Instructor, it is true when this is linear. It is not true for parabolic or curved graphs. We will directly accept it if this was linear.

Inst: Okay, SG asks you how this relation gives the height, think about it… (The instructor left)

SG: Just a minute, what did the instructor say? He said that the equality $\frac{m \cdot x}{2}$ is equal to the height. How?

Halit: This is m, look at here not that one, this is the slope. $\frac{m \cdot x}{2}$,

SG: If we accept this equality, we will accept this anyhow.

Halit: … For example, when I take this point (pointing a point on the linear graph), when I multiply it with this value, $\frac{slope \cdot x}{2}$ gives the height. The area of that triangle…

SG: How is it giving height? If I understand on the linear graph, I will already understand that (indicating the curved graph). I did not understand.

Halit: (Thinking)… (By drawing a line and plotting a point on it) how do you find the height here?

SG: (Drawing a triangle, labeling the h value, labeling the hypotenuse as the slope, m)

Halit: Where is slope? It is not slope…

SG: (correcting and showing the angle), (after writing $h = \frac{m \cdot x}{2}$, she is silently thinking).

Inst: (The instructor came back) what is the slope here? $\frac{h}{x}$ or $\frac{\Delta h}{\Delta x}$. Put the slope in the expression, what is happening?

SG: (thinking)… ooo, okay. We considered as if the hypotenuse is slope. Okay, I got it. It is satisfied when replacing it.

Halit wanted help from the instructor as seen in the episode. SG was asking Halit to provide a mathematical explanation for the argument behind "*height is related to slope multiplied by horizontal distance*". SG insisted for this justification and she wanted to see this relationship in at least one situation (on linear or curved graph). The instructor drew a linear graph and constructed a triangle to determine height, slope and horizontal distance. And then, he asked Halit about the reason of obtaining height with the relation $\frac{m \times x}{2}$ in this linear context. By this question, they continued to discuss about this relationship, but since they did not consider slope as the ratio between height and horizontal distance, they could not reach a conclusion. The instructor asked again the meaning of slope in this context, and explained by himself as $m = \frac{h}{x}$. With this explanation, they reached the idea that height can be obtained as the result of multiplying slope (gradient) with the horizontal distance. During this process, group members themselves constructed linear gradient-distance graphs and tried to see $h = m \cdot x$ relation. But, since they looked slope as an isolated object or assigned the length of hypotenuse as the slope, they could not reach the idea. The instructor directed them to consider slope as the ratio between height and horizontal distance. After realizing this relationship, all group members convicted about derivative-antiderivative relationship and started to draw the distance-height graph by taking into account the procedures of derivative graphs.

They constructed the distance-height graph as can be seen in Figure 47. They transferred all the critical points on distance-gradient graph to the distance-height graph successfully. They also decided the levels of top and valleys according to the area under gradient-distance graph.

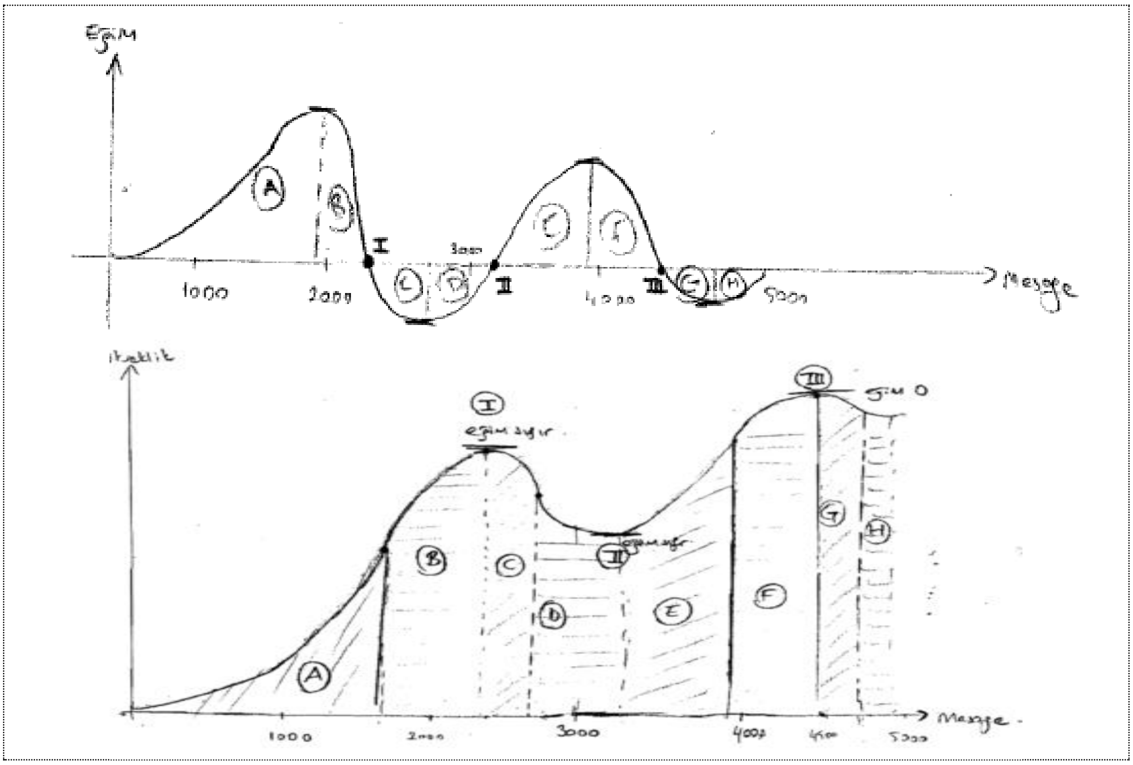


Figure 47: Height-distance graph in Halit's group report

Beyza and her group used the idea of area under the derivative curve for deciding the summits and valleys in the "Tracking Track" activity, but they did not mention about integration in their written report and in the reflection papers. They decided that the height level of second top on the height-distance by comparing the areas under positive and negative sides of the gradient-distance graph. In the interview, I asked Beyza about the reason for and the way of using the area under graph.

R: Yes, okay, you drew the graph. You said in the reflection paper that the second summit is lower than the first summit. You decided this by using the area. Could you please explain how did you reason?

Beyza: We first looked at region where the slopes were positive. We do not know the area exactly, but roughly we can see the area at positive side is greater than the area at negative side. That is to say, we are at a higher point when compared with the starting point. Later, the graph continued with the positive side and it passed to the negative side. This means, we will reach a higher summit when compared with the starting point.

R: How did you use the area, could you explain again Beyza?

Beyza: positive negative positive negative, when I compared the areas at positive side and the areas at negative side, I am still at the positive side. I mean, I will be at a higher point.

R: Is it because the areas at plus sides are greater?

Beyza: I am not at the height level when I started. If the areas at positive and negative sides were equal, then I will be at the same height level.

R: What is this about? When you mention about the area under curve, which mathematical idea is this related with?

Beyza: Is it integration?

R: Did you consider the idea of integration in any way?

Beyza: No, we did not consider the integration. We directly thought with the area. Of course the area under curve is related with the integration, but we directly compared the area at positive side and the area at negative side.

R: But the integration did not come into your mind, interesting. How did you reason without integration, I wonder?

Beyza: We did not think the concept of integration. Yes, but in my opinion, these are related to our problem solving habits in calculus. We did not aim revealing a mathematical concept and using it; instead we only applied them while solving the questions. I mean our knowledge seems as memorization…

I asked Beyza to explain how and why they used the area under the gradient-distance graph while drawing the distance-height. She expressed that since the area of first positive region is greater than the area of the first negative region, the height level is higher than the starting point. She also compared the levels of top and valley points by comparing the areas of positive and negative regions. I then asked the mathematical concept behind their area-based reasoning. She reluctantly asked me if it was integration. When I asked if they used and mentioned about integration in any way during group discussion, she stated that we did not use integration. Beyza seemed to realize the idea of integration explicitly after I asked about it. Up to my question, she used the area under distance-gradient curve for deciding the levels of top and valley points without explicitly relating it with integration.

In a similar vein, Rana and her group also used the area under gradient-distance graph while deciding the levels of tops and valleys on height-distance graph without mentioning about integration in their group report. In the interview, I asked Rana the same question about the way of using the area under gradient-distance graph.

R: All right, you determined the summits by using the area as you said in the reflection paper. How did you think with the area?

Rana: We intuitively decided the heights of those points. Now we are at the zero points and this part indicates a climb. The area here indicates climbing towards up. This part, on the other hand, indicates a continual descent. This means, the climbing part is more than the ascending part. Therefore, we will be higher level than the starting level at that point… I mean, we compared here and here, we compared the levels of these two summits…

R: I see, what is the mathematical idea that you used here?

Rana: hmmm, is it integration? We did not realize the idea of integration here as we did not realize the derivative. Now, I got it.

As seen in the episode, Rana indicated that they intuitively used the area under gradient-distance graph without a robust mathematical argumentation behind it.

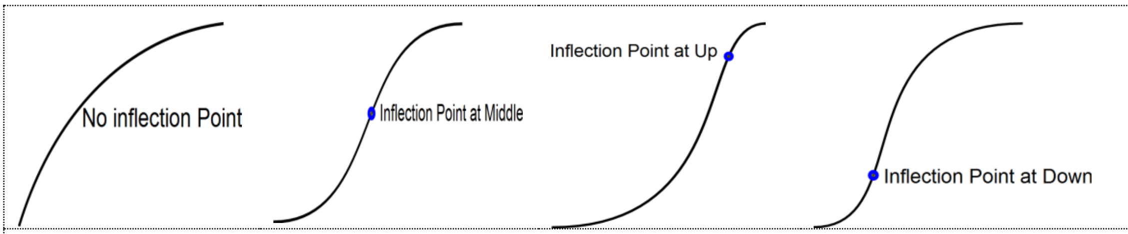
While explaining again, she compared the heights of top and valley points on height-distance graph by comparing the areas of positive and negative regions on gradient-distance graph. I then asked her the mathematical concept behind their area-based reasoning. She mentioned about integration and but added that they did not mention about integration during group discussion process. Rana indicated that they intuitively thought with integration without realizing it explicitly.

In the "Tracking Track" activity, most of pre-service teachers intuitively used the area under graph for deciding the levels of top and valleys without relating it with the idea of integration. Some of them realized the explicit concept of integration in later phases of group discussion and the others realized during the group presentations or during the classroom discussion. This also resulted in realizing the derivative-antiderivative relationship between gradient-distance and height-distance graphs as seen in the group discussion process of Halit. Beyza, Rana, and Nilgün's ways of reasoning with their groups were similar. It was interesting to observe that pre-service teachers could not easily see the multiplicative relationship between height, horizontal distance, and gradient. However, pre-service teachers reached the big ideas involved in the "Tracking Track" activity during the classroom discussion.

4.4.2.4 *Pre-service teachers' ways of reasoning on critical points and increasing-decreasing intervals*

In Questionnaire-1, there were two questions directly related with the drawing the graph of a function when its derivative function was provided or vice-versa. While the 3rd question was asking the analytic properties of a symbolic function, the 11th question was related to drawing an antiderivative graph. Most of the pre-service teachers successfully answered the 3rd question, but they had difficulty in drawing the antiderivative graph by using the derivative graph. While Halit, Beyza, Rana, and Nilgün could answer the 3rd question involving the analysis of a function in terms of its increasing-decreasing intervals, its critical points, and its inflection points, only Halit could draw the antiderivative graph by using the derivative graph in the 11th question.

In the "Roller Coaster" activity, pre-service teachers clarified their understanding of inflection point. At the beginning, their knowledge of inflection point was limited to "the point where the second derivative gets zero". Halit, Beyza, Rana, and Nilgün clarified their understanding of the inflection point. They all had similar discussion processes in the group works. For instance, at the beginning, Halit and his group did not consider the meaning of maximum slope given in the problem context. They constructed curve as seen in Figure, but they accepted the maximum slope as the slope of line between top and valley points (see, Figure 35). With the prompts of the instructor, they began to think on the maximum slope. They first realized that the slopes were different at every point on a curve. They then started to discuss about the location of maximum slope, and the way of determining it. During group discussion, they constructed various curves and tried to observe the location of maximum slope.



Curve 1 Curve 2 Curve 3 Curve 4

Figure 48:The discussion of Halit's group on the location of maximum slope for various curves

As a result of these discussions on the curves seen in Figure 48, Halit and his group came up with the idea of inflection point where the value of slope gets maximum value. In the following episode, Halit explained their discussion process.

R: On this curve, you said that we decided the location of the slope (i.e., 5.67) was at the inflection point, how did you decide it?

Halit: The discussion started with the question of the instructor which was "Where is the point having the slope of 5.67?" During this process, we decided that the slope 5.67 will be satisfied at only one point, and the slopes will get fewer values at all other points. Later we discussed on the location of maximum slope. For example, we draw a graph like that (as Curve 1 in Figure 48), it was only increasing…

R: Yes, I remember.

Halit: At what point the slope will be 5.67? I mean, it is continuously increasing, will it be at the top point, but we know the slope is 0 there.

R: Yes it will be 0

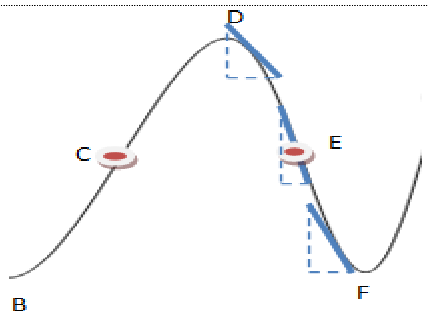
Halit: So, if the slope starts with zero, and it gets zero at the top point again, then the slope must be 5.67 at a point which means there should be an inflection point at somewhere. Later, we discussed where the maximum slope may be on different graphs, and we observed it can be at the middle, at the lower, and at the upper points. We felt this idea from the given picture, because while going down through downwards, we observe an increasingly decreasing graph at first and later a decreasingly decreasing graph. I mean, there is an inflection point here.

R: How do you say?

Halit: We drew the inflection point like that (by drawing Curve 2); we determined that point as the inflection point. I mean, we determined the slope 5.67 at the inflection point, which is the point where the graph changes from increasingly increasing to decreasingly decreasing. The maximum slope will be obtained at that point. We can also observe by drawing tangent lines.

In the episode above, Halit emphasized that they focused on the location of maximum slope with the question of instructor. They discussed on the maximum slope on various curves. The maximum slope is the inflection point where the curve changes form from concave up to concave down was the idea that they reached as final agreement. Halit also emphasized this way of reasoning with verbal expression as "*inflection point is the point where the graph change from increasingly increasing to decreasingly decreasing*".

A similar discussion process has been lived in the group of Beyza in the "Roller Coaster" modeling activity. They considered the railway as merging of lines each have a constant slope at first, but later they realized the railway of a roller coaster should be as a curve and so the slope of tangent lines should be different at every point. In the group report, they did not mention about inflection point in any way. Although they did not mention about inflection point explicitly, they approximately determined its location calling it as the middle point.



…Bölgeler üzerinde aldığımız teğetlerin en yüksek olanına yani orta noktalarına 5,67 eğimli olsun dedik. Dolayısıyla diğer teğetler daha az eğime sahip olacaklardır.

When the slopes taken from each interval was considered, we decided the location of the maximum slope 5.67 at the middle point. So, the slopes at other points will get fewer values.

Figure 49: Beyza's explanation of the maximum slope in Roller Coaster activity

As seen in Figure 49, while deciding the location of maximum slope, they determined it as the middle of the curves at ascending and descending parts of the railway.

R: All right, in this activity, you said that the slope will get its maximum value at the middle points. How did you decide?

Beyza: Actually, we directly benefited from the figure. We drew small triangles, and we observed the increase in the steepness towards the middle of the track… That's why we reasoned that the maximum slope was at the middle point on the curve.

Beyza described the maximum slope of a curve as the middle point of the curve starting with concave-up and continuing with concave-down or vice versa. They determined roughly the maximum slope as the middle point of a curve by comparing the slopes of tangent lines that they draw at various points. Beyza and other pre-service teachers realized the inflection point during the classroom discussion.

While solving the "Tracking Track" activity, most of the pre-service teachers' ways of reasoning was slope-based as indicated previously. Pre-service teachers tried to draw the graph of height-distance from the gradient-distance without realizing the derivative-antiderivative relationship. They intuitively determined the critical points such as maximum and minimum points on gradient-distance graph correspond to the inflection points on height graph. Although they divided the distance-gradient graph to the intervals and determined some points, they did not call them as increasing-decreasing intervals, max-min or inflection points. Their way of reasoning for constructing the height graph was slope-based, such as "*if slope increases, the height increases; if slope is zero, the level of height remain steady*". Also they made some mistakes appeared in their verbal expressions as "*if gradient decreasingly increases with respect to distance, the height also decreasingly increases*" by which they also considered the changing nature of gradient-distance graph which was related to second derivative. They sometimes fell in doubts by the slope-based reasoning as explicitly seen on the graph in Figure 46 drawn by Rana's group. Some of the groups constructed the height-distance graph correctly after realizing the derivative-antiderivative relationship.

The "Roller Coaster" activity helped pre-service teachers in clarifying the contextual meaning of inflection point. They also observed the varying nature of slope on curved graphs. The "Tracking Track" modeling activity provided pre-service teachers an experience of curve sketching without using the procedures. By this experience, pre-service teachers explored the procedures of reversing between derivative and anti-derivative graphs in detail. They remembered the meanings of increasing-decreasing intervals, critical points, and area under the graph. Most of the pre-service teachers have successfully drawn the antiderivative graph of a derivative graph in the follow up activity conducted after the "Tracking Track" activity. In the same way, most of the pre-service teachers successfully answered the 11th question in Questionnaire-2, which involved drawing an antiderivative graph of a derivative function.

Table 17: Pre-service teachers' conceptions of the graphical connections between a function and its derivative

	Task-1 Pre-Test	Task-2 WT	Task-4 RC	Task-5 TT	Task-6 TT-F	Task-7 Post-Test
Halit	CP-MS CP-IPI INT-SDF N-AR	TR-SH TR-SM(Int)	CP-IPI CP-IPS	CP-MI CP-MS INT-IN INT-SDF AR-I RV-SBR RV-RD	CP-MS CP-IPS INT-SDF AR-I	CP-MS CP-IPS INT-SDF AR-I (Q11)
Beyza	CP-MI CP-IPI INT-IN N-AR	TR-SH TR-SM (Int)	CP-IPI CP-IPS	CP-MI CP-MS INT-IN INT-SDF AR-I RV-SBR RV-RD	CP-MS CP-IPS INT-SDF AR-I	CP-MS CP-IPS INT-SDF AR-I
Rana	CP-MS CP-IPS INT-SDF INT-IN N-AR	TR-SH TR-SM (int)	CP-IPI CP-IPS	CP-MI CP-MS (int) CP-IPI INT-IN INT-SDF N-AR AR-I (Int) RV-SBR RV-RD (Int)	CP-MS CP-IPS INT-SDF AR-I	CP-MS CP-IPS INT-SDF AR-I
Nilgün	CP-MI CP-IPI INT-SDF N-AR	TR-SH TR-SM (int)	CP-IPI CP-IPS (Int)	CP-MI INT-IN INT-SDF AR-I RV-SBR RV-RD	CP-MI INT-SDF AR-I	CP-MS CP-MI INT-SDF AR-I

Pre-service teachers' graph drawing skills are summarized on the table above. To summarize in general, the model development unit supported pre-service teachers' understanding of graphs from various aspects. First of all, in Questionnaire-1, pre-service teachers' performances in reversing between derivative and antiderivative graphs were weak. They could not easily remember the procedures of determining maximum-minimum points, inflection points, and increasing and decreasing intervals. In addition, the "Water Tank" modeling activity helped pre-service teachers to realize the necessity of smooth drawing at the transition points. This idea also contributed their understanding of instantaneous rates of change. The “Roller Coaster” activity helped them to develop a robust idea about the meaning of slope of tangent lines on a curve and the idea of inflection point. To remember again, most of pre-service teachers had considered the given maximum slope as the slope of line between top and valley points. The “Tracking Track” modeling activity helped pre-service teachers to reconsider the ideas of positive slope, negative slope, determining increasing and decreasing intervals, inflection point, and the area under curve for deciding the levels of max and min points. By this activity, pre-service teachers covered and revised their understanding of the reasoning behind the procedures and most of them could perform well in the Questionnaire-2.

4.5 Summary of Results

To begin with, important developments in pre-service teachers’ covariational reasoning abilities were observed (see, Table 9, Table 10, and Table 11). Initially, most of the pre-service teachers had difficulties in identifying the variables. They were considering some unnecessary (secondary) variables (e.g., time in volume-height context) as independent variables. They were using either uncoordinated way of thinking or indirect coordination while deciding the functional relationship between the variables. In the process, pre-service teachers started to reason with the primary dependent and independent variables. Also, they learned and started to coordinate the covarying variables directly and systematically by which they focused on the changes in the dependent variable in relation to unit or equal amounts of change in the independent variable. Also, at the beginning, most of pre-service teachers were deciding the variation in rate of change perceptually (gross quantification) without providing an explicit mathematical explanation. By coordinating the variables directly and systematically, they started to decide the variation in rate of change by using extensive ways of quantifications.

Additionally, important developments were observed in pre-service teachers’ conceptions of rate of change (see, Table 13 and Table 14). First of all, while they were unaware of the concept of rate of change and its connection with derivative at initial phases, they realized its connections with the slope, derivative, Physics concepts, and difference quotient. While they were reasoning on rate of change perceptually or with amount of change in the dependent variable, during the study, they started to demonstrate ratio-based reasoning in the form of unit per unit comparison while interpreting the graphs and explaining the meaning of rate of change in different contexts. However, some of the pre-service teachers continued to confuse rate of change with the amount of change in the dependent variable and they could not form robust connections between different interpretations of rate of change although they demonstrated them in fragmentary nature. Nevertheless, it was observable that most of the pre-service teachers developed important conceptions about different interpretations of derivative. For instance, they started to use the difference quotient rule for approximating the derivative at a point and connected it with the slope and average rate of change.

The data of the current study showed important developments in pre-service teachers' graph interpretation abilities (see, Table 16). At the beginning, most of the pre-service teachers were explaining the curved graphs either with their physical attributes or by taking into account the unnecessary variables. In the process, they started to explain the graphs in a sophisticated way by taking into account the dependent and independent variables. Pre-service teachers' understanding of the graphical connections between a function and its derivative also developed in the progress (see, Table 17). Their awareness of the contextual meanings of the standard procedures used when reversing between the graphs of rate and amount functions involving the inflection points, the monotonicity intervals, and the maximum and minimum points developed.

CHAPTER 5

DISCUSSION, CONCLUSION, AND IMPLICATIONS

The purpose of the current study was to investigate pre-service teachers' understanding of the big ideas involved in derivative while they engaged in a model development unit. The basic ideas covered in the model development unit were the concept of covariation, rate of change, and the graphical connection between a function and its derivative. Characterizing pre-service teachers' developing knowledge related to these concepts from a design-based research perspective was the overall goal of this study. In this chapter, firstly, the findings related to pre-service teachers' developmental understanding of covariation, rate of change, and graphs were discussed by comparing and contrasting with the existing body of literature. Some critical theoretical arguments related to teaching and learning of derivative has been provided. Eventually, the general evaluation of the model development unit involving possible revisions in the learning tools and in the design principles for further studies were discussed. It is followed by the issues of major conclusions drawn, implications, limitations, and suggestions for further studies.

5.1 The Nature of Pre-service Teachers' Covariational Reasoning

The concept of covariation and covariational reasoning ability has been mentioned as being foundational for understanding of functions, derivative and other calculus concepts. In this section, findings related to initial states and developments in pre-service teachers' ways of covariational reasoning were discussed. Furthermore, a revision for the framework developed by Carlson et al. (2002) was provided and discussed in detail.

5.1.1 Pre-service teachers' initial states of covariational reasoning

The data analysis revealed that pre-service teachers in the current study demonstrated weak covariational reasoning abilities in the beginning, and important developments were observed in the progress of the model development unit. Initially, most of the pre-service teachers had difficulties in conceptualizing the covariation of variables as a functional relationship and identifying the dependent and independent variables. They frequently thought by considering some unnecessary variables as being the independent variable (e.g., time in volume-height context). This resulted in thinking about simultaneously changing variables as being two separate functions with respect to the unnecessary variable. Pre-service teachers' awareness of the functional relationship between covarying variables noticeably shifted in the progress. The second weakness observed in their reasoning was the way of coordinating the variables. At initial phases, changing the input variable with equal amounts and observing the simultaneous variation in the output variable did not seem trivial for pre-service teachers. They started to coordinate variables by focusing on the changes in the dependent variable in relation to unit or equal amounts of change in the independent variable in subsequent phases. Another difficulty observed in the beginning was deciding about the variation in rate of change or concavity on the graphical representation. In later phases, pre-service teachers demonstrated more explicit ways of thinking in deciding the variation in rate of change and representing it on the graph.

According to Carlson et al.'s (2003) framework, the mental action for the first level of covariational reasoning was identified as "*an image of two variables changing simultaneously*" (p.467). The behaviors supporting this mental action were determined to be labeling the variables on the axes or using verbal expressions of coordinating two variables. In other words, this level of covariational reasoning can be accepted as deciding the dependent and independent variables. Although this level can be seen as the trivial one, the data of the current study suggested that identification of the variables was not trivial and it was the beginning of many difficulties in pre-service teachers' ways of covariational reasoning. In the study of Carlson et al. (2002), many of calculus students considered "time" as the input variable when working on the relationship between height and volume in a filling-bottle problem. Similarly, in a following study, the study by Carlson et al. (2003) revealed that pre-service teachers thought by the cross-sectional area as being the independent variable when working on a model-eliciting version of the filling bottle problem. The general results obtained which cannot be explained by their framework were; (i) treating height as the input variable, and (ii) considering time as input variable. These findings also appeared in a recent study conducted by Zeytun et al. (2010) with in-service secondary mathematics teachers by using the same modeling activity. All of these results were related to the way of identifying the variables and the way of forming the functional relationships. The findings obtained at the initial phases of the current study resonate with the findings of aforementioned studies. Considering "time" or sometimes any other irrelevant variable as being the independent variable was dominantly observed in the analysis of the first two problems. It was clearly observed that pre-service teachers have a strong tendency using "time" as the independent variable although it was not mentioned in the problem context. Pre-service teachers' previous experiences limited with the contextual examples of time-dependent functions may be an important factor in their time-based reasoning (Herbert & Pierce, 2008, 2012; Wilhelm & Confrey, 2003; Gravemeijer & Doorman, 1999; Zandieh & Knapp, 2006). This was indicated by Beyza as "*Plotting time on the horizontal axes is a habit for us*". Halit, Nilgün, and Beyza plotted "time" at horizontal axis in the "Cassette Player" problem as also did by majority of pre-service teachers. In the "Water Tank" activity, replacing "time" as the independent variable in place of "volume", considering "radius of cross-sectional area" as the independent variable, and considering the "height" as the independent variable were the frequently observed instances. Sometimes, more complicated ways of thinking were observed such as using "time" as independent variable in verbal explanations, but plotting "height" and "volume" as dependent and independent variables respectively on the graph. These instances were relatively decreased in the following activities.

The second mental image in Carlson et al.'s (2002) framework was loosely coordinating the changing nature of variables with respect to each other. The following mental actions require systematically coordinating the amounts of changes, the rate of change for contiguous intervals, and the continuously changing rate over the entire domain in both variables. Carlson et al. (2002) showed that calculus students were able to coordinate the changes in output variable in tandem with the equal amounts of change in input variable. However, they had difficulties in coordinating the instantaneous rate of change of output variable with continuous change in the input variable. Students also had difficulties in explaining the smoothness of the curve and the meaning of inflection point in terms of variables. Zeytun et al. (2010) also determined teachers' difficulties in interpreting the relationship between varying rate of change and the concavity. Additionally, the study by Carlson et al. (2003) revealed that despite producing correct smooth graphs, students could not explicitly demonstrate how they constructed a smooth curve. In the current study, pre-service mathematics teachers had similar difficulties in deciding the variation in rate of change and representing this variation on the graph. In addition, at the initial phases, pre-service teachers could not easily demonstrate the mental action of coordinating the amount of change in the output variable with respect to equal or uniform incremental changes in the input variable which was accepted as the central idea in covariational reasoning. For example, there was only one correct answer involving the coordination of amounts of change in both variables supported with a correct smooth curve for the "Cassette Player" problem. In the "Water Tank" activity, pre-service teachers drew correct smooth graphs, but the variables assigned to the axes were different than the simultaneously changing ones. Moreover, there were sharp corners on the graph at transition points showing the graphs were drawn without taking into account the variation in the instantaneous rates of change.

In the following problems, deciding the variation in rate of change and representing it on the graph continued as being the major difficulty. Roughly and perceptually decided graphs for covarying quantities were the general finding in the current study that most of the pre-service teachers demonstrated. For explaining the graphs in the "Cassette Player" and "Sliding Ladder" problems, many of the pre-service teachers indicated intuitive expressions such as "*I feel like that*", "*I roughly draw it*" without providing a robust mathematical justification. This was also the major finding in the study conducted by Monk (1992) by a group of calculus students. According to Monk (1992), students' difficulties in drawing and interpreting the curved graphs of dynamically changing situations may be because of their dominant image of pointwise (or correspondence) view of function. This was characterized as having a static view of functional relationship. However, covariational reasoning entails a dynamic view of functional dependency (Castillo-Garsov, 2010; Johnson, 2012; Saldanha & Thompson, 1998). Restricted images or weak understanding of functions, as also observed in the current study, have been determined as one possible source of students' difficulties in conceptual understanding of derivative and other calculus concepts (Confrey & Smith, 1994; Monk, 1992; Tall, 1992, 1996). The data, at the initial phases of the current study, supported the literature by evidencing that because of their dominant image of static view of functions, pre-service teachers had difficulties in interpreting and graphing the functional relationship between simultaneously changing quantities.

5.1.2 The covariational reasoning framework proposed by the current study

As explained in the literature review and conceptual framework parts, various researchers tried to explain the nature of covariational reasoning (Carlson et al., 2002; Castillo-Garsow, 2010; Confrey & Smith, 1994; Saldanha & Thompson, 1998). Carlson et al. (2002) proposed a more comprehensive framework for characterizing students' covariational reasoning abilities. However, different ways of thinking on covarying situations could not be explained by using this framework such as considering the implicit "time" as input variable and thinking input and output variables in reverse order (Carlson et al., 2002; Carlson et al., 2003; Strom, 2006; Zeytun et al., 2010). Furthermore, the average rate and instantaneous rate levels seem to be vague for identifying one's way of covariational reasoning. For instance, the mental action of "*constructing contiguous secant lines for the domain*" which was supposed to indicate the level of "*coordinating average rate*" do not appear explicitly or implicitly in students' ways of thinking (Carlson et al., 2002, p.357; Carlson et al., 2003). Therefore, in the current study a new characterization of covariational reasoning was obtained by taking into account the descriptions offered in the literature and also it was used in analyzing the data (see, Appendix-J1). When compared with the arguments stated in the literature (e.g., Carlson et al., 2002; Carlson et al., 2003; Castillo-Garsow, 2010; Confrey & Smith, 1994; Saldanha & Thompson, 1998), the covariational reasoning framework proposed and used in the current study worth mentioning in detail. The framework proposed in this study try to characterize a person's covariational reasoning from three aspects, which are; (i) identification of the variables, (ii) ways of coordinating the quantities, and (iii) deciding (or quantifying) the variation in rate of change.

Identification of the variables and understanding the functional relationship between covarying variables was determined as being the first dimension of covariational reasoning. Identification of the variables involves determining the dependent and independent variables, labeling them on the true axes, and taking care of the dependency-independency relationship in the verbal expressions. The data of this study revealed three different ways of reasoning while identifying the variables, which were (i) thinking with primary variables, (ii) thinking with secondary variables, and (iii) thinking input and output variables in reverse order. *Thinking with primary variables* indicates determining the covarying quantities as dependent and independent variables (see, Appendix-J1). For example, in "Cassette Player" problem, considering the radius of second reel (R2) as a function of the radius of the first reel (R1), and labeling them on the true axes is an instance for thinking with primary variables. *Thinking with secondary variables* occurs when considering an unnecessary (external) variable as being the independent variable. The typical example of thinking with secondary variable is considering "time" as an independent variable although it is unnecessary. And lastly, *thinking input and output variables in reverse order* means changing the places of dependent and independent variables. The data of the current study showed that determining the dependent and independent variables and thinking with them was not trivial in covariational reasoning of pre-service teachers.

The second dimension of covariational reasoning appeared to be the way of coordinating the variables. Four different ways of reasoning, which are; (i) uncoordinated way of thinking, (ii) indirect coordination, (iii) direct coordination, and (iv) direct and systematic coordination, were specified in the current study. *Uncoordinated way of thinking* stems from one's consideration of an unnecessary (secondary) variable as the independent variable. In the "Cassette Player" problem, although it was asked about the radius of the second reel (R2) in relation to radius of reel one (R1), most of the pre-service teachers used "*time*" as the independent variable. Therefore, they thought R1 and R2 as being two separate variables changing as a function of time. *Indirect coordination* also means thinking by using an external variable apart from the covarying ones, but differently from the uncoordinated way of thinking, there is a direct coordination between the covarying variables on the graph. For instance, in the "Water Tank" activity, some of the pre-service teachers considered radius of the cross-sectional area inscribed in the water tank as an independent variable. In deciding the character of the curvature of height-volume graph, they reasoned with the radius of cross-sectional area in place of the volume. *Direct coordination* was the third way of coordinating the variables. Labeling the covarying variables on true axes, roughly drawing a linear or curved graph, and looking at the values of both variables at some particular points without coordinating the changes are some of the instances for direct coordination. Because it does not involve the coordination between the changes of the covarying variables, direct way of coordinating carry out the features of static view on functional dependency (Castillo-Garsov, 2010; Johnson, 2012; Monk, 1992). *Direct and systematic coordination* involves systematically changing the input variable and observing the simultaneous variation in the output variable. This way of coordination clearly appears in the verbal expressions like "*change in height per volume*". The independent variable may be changed uniformly, or non-uniformly, but the simultaneous variation in the dependent variable is carefully considered. Surely, directly and systematically coordinating the variables is the most critical idea in covariational reasoning. In contrast to the direct coordination, because students focus on the changes in two variables, the direct and systematic way of coordination includes the dynamic view of functional dependency (Castillo-Garsov, 2010; Monk, 1992). This does not mean the weakness of the other ways of coordinating the variables. For instance, in the "Water Tank" activity, if pre-service teachers' consideration of the radius of cross-sectional area as being the independent variable can be transformed to the idea that cross-sectional area is the infinitesimal change in volume, then indirect way of coordinating may also result in mathematically powerful ideas. Studies on covariational reasoning evidenced that students and teachers did not perform well in covariational reasoning tasks, because of their lack of dynamic view of functional dependency (Carlson et. al., 2002; Carlson et al., 2003; Hoffkamp, 2011; Zeytun et al., 2010). Dynamic view of function involves the concept of covariation that entails the coordination of simultaneous small (and infinitesimal) changes in the input and output variables. The correspondence approach and formula-based applications dominantly used in teaching does not support the idea of dynamic simultaneous variation of quantities (Confrey & Smith, 1994; Monk, 1992; Tall, 1996; Thompson, 1994b). Our framework suggests that, when a student thought with the primary variables and when he/she coordinated the variables directly and systematically, this is a strong indicator for him having the dynamic view of functional dependency.

Deciding the variation in rate of change and connecting it with the character of curved graphs was specified as being the third and most critical dimension of covariational reasoning. Looking from the quantitative reasoning perspective, rate of change is a quantity obtained as a result of multiplicative comparison of changes in two quantities (Thompson, 1994b; Thompson, 2011). Therefore, in deciding the variation in rate of change, a kind of quantification process is required where quantification was defined as "*conceptualizing an object and an attribute of it*" (Thompson, 2011, p.8). According to Thompson (1994b), there are two types of quantifications that are gross quantification and extensive quantification. While *gross quantification* means experientially or perceptually conceptualizing the quality of an object, *extensive quantification* refers to conceptualizing the quality of the object by using some quantitative operations such as comparing, composing, unitizing or segmenting. In the current study, both of gross and extensive ways of quantifications were observed in pre-service teachers' ways of reasoning while deciding the variation in rate of change of covarying quantities. At the initial phases, most of the pre-service teachers used gross quantification in deciding the variation in rate of change. In the following tasks, as a result of guided group discussions, pre-service teachers started to use extensive quantifications. Three different extensive quantifications were observed with different quantitative operations that are (i) additive comparison of changes, (ii) unit per unit comparison of changes, and (iii) multiplicative comparison of changes. Systematically changing the independent variable with equal increments and additively comparing the simultaneous change in the dependent variable was called as *additive comparison*. Here, the successive changes in the dependent variable were additively compared while keeping constant the changes in the independent variable. As also evidenced by Johnson (2012), although this way of reasoning does not involve ratio-based conception of rate of change, it is robust enough for determining the variation in the intensity of change in a variable involving rate of change that even middle school students can demonstrate. Additive comparison of successive changes in the output variable can be accepted as thinking within one measure space as also evidenced in the study by Thompson (1994b). Pre-service teachers who used additive comparison only focused on the changes in output variable as appeared in the expressions like "*the height increasingly increases*". They were implicitly considering the independent variable. Extensive quantification by additive comparison can be associated with the "chunky" way of thinking in Castillo-Garsow's (2010) terms, and because changes in the variables are coordinated in discrete intervals, it is discrete dynamic in nature (Johnson, 2012). The second way of extensive quantification was the *unit per unit comparison* which means uniformly (with small units) changing the independent variable and observing the simultaneous change in the dependent variable. Here, the smallness of the units is critical while the amounts of changes are relatively larger in additive comparison. Extensive quantification by unit per unit comparison requires being aware of that the units can be taken as smaller as possible according to the context. It appeared in the verbal expressions as "*the height increasingly increases per unit volume*". *Multiplicative comparison* of the simultaneous changes in two variables was determined as the other quantitative operation that was used for quantifying rate of change. It involves changing the independent variable with constant and equal increments and multiplicatively comparing the simultaneous change in the dependent variable. There is an explicit ratio-based reasoning here appeared in the verbal expressions as "*the height increases at an increasing rate with respect to volume*". The quantification of rate of change by using the quantitative operations of *unit per unit comparison* and *multiplicative comparison* can be seen analogous to "smooth" way of reasoning on covariation (Castillo-Garsow, 2010), and both involves continuous dynamic coordination of variables (Johnson, 2012; Saldanha & Thompson, 1998).

Determining the appropriate quantitative operation is critical in correctly deciding the variation in rate of change. But, it does not guarantee coming up with a decision about the nature of variation in rate of change. The quantitative operation should be finalized with a numerical (mathematical) operation. For instance, in "Sliding Ladder" problem, some of the pre-service teachers thought with primary variables, followed with direct and systematic coordination, and decided to use unit per unit comparison, but, they could not easily obtain the appropriate mathematical operation by which they can reach some numerical values. Until they realized the usability of Pythagorean Theorem, they could not decide the nature of variation in rate of change. At that point, we impressed that this may be related to the visual character of the situation. The context in "Water Tank" activity seemed to be easy to imagine for pre-service teachers, so most of them drew correct smooth graphs without using extensive quantifications, while the situations in "Cassette Player" and "Sliding Ladder" were not easy to visualize. In "Cassette Player" and "Sliding Ladder" problems, all of the pre-service teachers who tried to draw intuitively or perceptually created incorrect graphs, while almost all of them created the correct graphs in "Water Tank". The interpretations related to visual character of the situation were the impressions that the data of the current study suggested, but still the role of visual character of the situation on covariational reasoning requires further investigation.

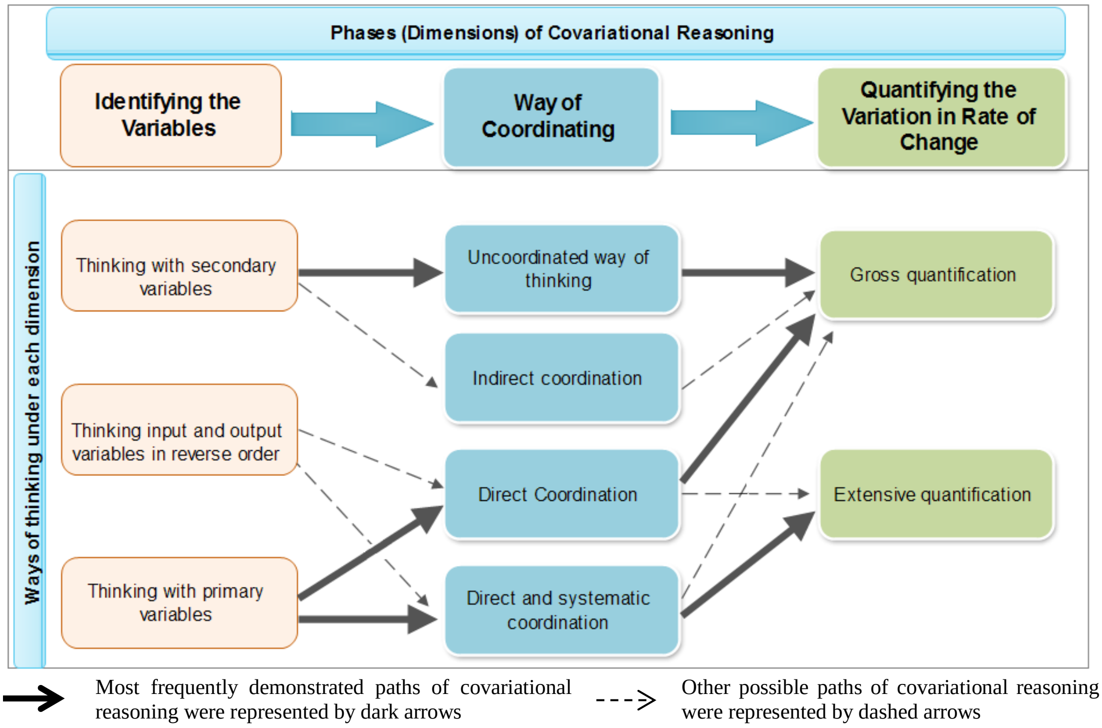


Figure 50: Covariational reasoning framework proposed in this study

The framework mentioned in the current study not only explains the robust level of covariational reasoning, but also the possible source of difficulties (see, Figure 50). For example, when a student thought with a secondary variable as the independent variable, most probably, he or she will follow with an uncoordinated way of thinking or indirect coordination. As explained earlier, uncoordinated way of thinking involves considering the covarying variables as two separate functions with respect to the secondary variable and results in inability to consider the simultaneous changes in both variables. Thinking with primary variables followed with a direct coordination results in gross quantification of rate of change. The robust level of covariational reasoning can be characterized as thinking with primary variables, using direct and systematic coordination, and quantifying the rate of change by quantitative operations of unit per unit comparison or multiplicative comparison of simultaneous changes in both variables.

5.1.3 Developments in pre-service teachers' covariational reasoning ability

Looking from the perspective of the framework used in this study, there were some important developments in pre-service teachers' covariational reasoning. Initially, most of them were considering unnecessary (secondary) variables (e.g., "time") as independent variables, and so they were using uncoordinated way of thinking or indirect coordination while coordinating the variables. In the progress, almost all of the pre-service teachers started to reason by considering the primary dependent and independent variables. Also, they learned and started to coordinate the covarying variables directly and systematically by which they focused on the changes in the dependent variable in relation to unit or equal amounts of change in the independent variable. For deciding the variation in rate of change, at the beginning, most of the pre-service teachers were deciding perceptually (gross quantification) without providing an explicit mathematical justification. Pre-service teachers started to reason with extensive quantifications for deciding the variation in rate of change after they realized the importance of thinking by primary variables. Extensive ways of quantifications appeared concurrently with the thinking by primary variables and using the direct and systematic way of coordination. Additive comparison of change in the output variable and unit per unit comparison of changes were the most frequently observed quantitative operations. However, it should be noted that, even in solving the last covariational reasoning problem that of "Space Shuttle", some of the pre-service teachers (e.g., Rana) continued using the gross quantification despite they demonstrated the reasoning styles involving extensive quantifications in previous tasks. This can be interpreted in a way that perceptions or experiences of participants may be strong enough so they do not need an extensive quantification to decide about the mathematical relationships. In such situations, using gross quantification does not always show one's weakness.

The concept of covariation and covariational reasoning has been proposed as the fundamental idea that students should have for understanding functions, rate of change, and other calculus concepts (Thompson & Thompson, 1992; Thompson, 1994b; Confrey & Smith, 1994; Monk, 1992; Carlson et al., 2002; Cooney et al., 2010; Saldanha & Thomson, 1998; Zandieh, 2000). On the other hand, students' weak understanding of dynamic view of functional dependency and process conceptions of functions was attributed to their lack of covariational reasoning ability (Carlson et al., 2002; Monk, 1992; Hoffkamp, 2011; Zandieh, 2000). It should be discussed here that whether and if the covariational reasoning ability develops with the conceptual understanding of function and rate of change concepts, or is it the prerequisite for conceptual understanding of them? The framework provided in the current study indicates that the covariational reasoning ability also encapsulates conceptual understanding of the function and rate of change concepts. Additionally, ability of covariational reasoning, and conceptual understanding of function and rate of change develops concurrently. In other words, employing students on dynamically and simultaneously changing situations not only develops their covariational reasoning ability, but also supports their understanding of dynamic view of functional dependency and rate of change concept. The data of this study revealed that considerable developments in pre-service teachers' understanding of functional dependency. However, it is difficult to argue the same development in conceptual understanding of rate of change. Although pre-service teachers made sense the qualitative aspects of rate of change, most of them could not reach the ratio-based conception. They decided the variation in rate of change by additively comparing the changes in the output variable with respect to the equal increments in the input variable. This way of coordination between the changes in both variables can be robust enough for deciding the nature of concavity of graph, but not yet involve the ratio-based conception of rate of change (Johnson, 2012; Stroup, 2002). Nevertheless, focusing on the simultaneous changes in two covarying variables can easily be directed to a ratio-based reasoning.

Consequently, covariational reasoning is important in conceptual understanding of rate of change. In order to develop covariational reasoning ability, the usage of learning environments involving physical models, conetextual tasks, or computer-based simulations of dynamically changing situations have been suggested (Carlson et al., 2002; Monk, 1992; Hoffkamp, 2011). The data of the current study supports this idea. For example, instead of asking students to calculate areas of individual rectangles many times, teachers can make students to study on the situation in which the length of one rectangle is stretched smoothly (Hoffkamp, 2011). Working on the tasks involving dynamically and simultaneously changing quantities also contributed to knowledge of pre-service teachers who already completed many advanced mathematics courses.

5.2 The nature of change in pre-service teachers' conceptions of rate of change

Developing pre-service teachers' conceptions of rate of change was the other central goal of the model development unit implemented in the current study. Almost all of the activities within the model development unit explicitly or implicitly covered the idea of rate of change. Although some important developments observed in pre-service teachers' conceptual understandings in subsequent phases, the data of the current study revealed particular difficulties and weaknesses. At the beginning, most of the pre-service teachers could not interpret the meaning of the Turkish expression (i.e., degişim oranı) of rate of change. They were unaware of the concept and term of rate of change and its relationship with the derivative. In covariational reasoning tasks, most of the pre-service teachers decided the variation in rate of change either perceptually or additively comparing the successive changes in the output variable. Additionally, most of the pre-service teachers confused rate of change with the amount of change in the dependent variable. They explained symbolic expressions of rate of change provided by difference quotient form as *amount of change* in the dependent variable.

Another point was that pre-service teachers' dominant image of derivative was geometric slope of tangent line at initial phases. They had difficulties in making connections between different representations of derivative. They frequently benefited from the definitions of speed (the distance per time) for giving meaning to derivative in different contexts (e.g., fuel efficiency as a function of speed). In the subsequent phases of the model development unit, most of them realized the meaning of rate of change and they got the idea of rate of change as an interpretation of derivative. While they were reasoning on rate of change perceptually or with amount of change in the dependent variable, they started to demonstrate ratio-based reasoning in the form of unit per unit comparison while interpreting the graphs and in explaining the geometric slope and the meaning of rate of change in different contexts. Additionally, pre-service started to use the difference quotient rule for approximating the derivative at a point and connected it with the slope and average rate of change. However, it was still difficult to say that they systematically demonstrated a ratio-based conception of rate of change across different tasks involving different contexts and representations. Some of the pre-service teachers continued to explain derivative as being the amount of change in the dependent variable. This result showed that some of the pre-service teachers could not form robust connections between different interpretations of rate of change although they demonstrated them in fragmentary.

5.2.1 **Pre-service teachers' initial interpretations of rate of change**

The first interesting result obtained at the initial phases of the current study was pre-service teachers' difficulty in giving meaning to the Turkish version of "rate of change" term. In answering the questions related to rate of change in Questionnaire-1, most of the pre-service teachers used irrelevant procedures such as taking the average of the function, summing the derivatives at the two endpoints and dividing by two, and finding the difference between the derivative values at the end points. These findings are concurrent with the findings obtained in the study by Bezuidenhout (1998). In a study specifically focusing on rate of change, Bezuidenhout (1998) determined that a majority of university students demonstrated misunderstandings between the concepts of "average rate of change", "average value of o continuous function", and "arithmetic mean".

Difficulty in giving meaning to the rate of change term was also appeared in many research studies (Bezuidenhout, 1998; Orton, 1983; White & Mitchelmore, 1996). In the study by White and Mitchelmore, (1996), students' difficulties in interpreting the symbols used for variables, and their weak understanding of the contextual meaning of derivative as rate of change were reported. Orton (1983) determined that high school students' and pre-service teachers' could not relate rate of change with difference quotient and slope. The same situations were observed in the current study. Most of the pre-service teachers did not use the difference quotient rule properly in answering the rate of change questions. Furthermore, in the "Population of Turkey" activity, most of the pre-service teachers interpreted the Turkish expression of rate of change as the percentage of change in the dependent variable or they could not give any meaning.

Considering rate of change as the percentage of change involving the ratio between successive values of dependent variable can be explained by the notion of multiplicative rate of change introduced by Confrey and Smith (1994). As indicated by Confrey and Smith (1994), the conventional understanding of rate of change is additive rate of change, but it is difficult to interpret it in some contexts involving exponential growth. Multiplicative rate of change is easier to interpret in such contexts. In the current study, most of the pre-service teachers' preference for the use of percentage of yearly population change supports the idea indicated by Confrey and Smith (1994). However, pre-service teachers' preference for the percentage interpretation may also be due to the fact that they did not accept "change in population per year" as a rate; rather they considered it as an amount. Pre-service teachers' different conceptions of rate of change were discussed in the following section.

5.2.2 **Perceptual, Amount of change, & Ratio-based conceptions of rate of change**

In the current study, three different conceptions of rate of change, which were (i) perceptual comparative, (ii) amount of change in the dependent variable, and (iii) ratio-based reasoning, were observed. As explained in literature review part, rate of change was defined as an intensive quantity obtained as a result of multiplicative comparison of changes in two quantities (Thompson, 1994b; Thompson, 2011). For conceptualizing rate of change as a quantity, quantification of the quality as "fastness" is necessary. According to Thompson (1994b), gross quantification and extensive quantification are the two types of quantifications. Throughout the first three covariational reasoning activities, perceptual comparative way of reasoning for deciding the varying rate of change was observed in most of the pre-service teachers' ways of reasoning. Perceptual way of thinking on rate of change was only observed in covariational reasoning tasks involving dynamically changing situations. This way of reasoning was not observed in the answers for the questions directly asking the interpretation of difference quotient or symbolic derivative. This result can be interpreted in a way that pre-service teachers' perceptual conceptions of the rate of change such as "fastness" involve the qualitative aspect of its formal interpretation, and they could not connect it with the formal mathematical representations (Stroup, 2002). As indicated by Stroup (2002), considering qualitative aspects of formal calculus concepts is "*cognitively significant and structural in its own right*" (p.170) rather than being transitional to more formal ratio-based ways of thinking. Pre-service teachers' usage of perceptual ways of reasoning on deciding the variation in rate of change shows that the dynamic nature of covariational reasoning tasks may serve students to clarify the qualitative aspect of rate of change. To be able analyze covarying situations; students do not always need ratio-based understanding of derivative involving ratio, limit, function (Stroup, 2002; Zandieh & Knapp, 2006). Some other ways of reasoning focusing on qualitative aspects of simultaneously changing quantities may appear and they are as powerful as the ratio-based conception of derivative.

Pre-service teachers frequently confused rate of change with the amount of change in the dependent variable. This has been also observed in many research studies (Bezuidenhout, 1998; Herbert & Pierce, 2012; Rowland & Javanoski, 2004; Thompson, 1994a, 1994b; Zandieh & Knapp, 2006). For instance, in the follow up activity conducted after the "Tracking Track" activity, the derivative of height-volume graph was provided without plotting the names of variables on the axes. Pre-service teachers were asked to name and explain the variables to be plotted on the axes. Some of them explained the variable on the vertical axes of the derivative graph of height-volume as "*increase in height*" (e.g., Halit, Nilgün). They used amount of change in height (dependent variable) in place of rate of change in height with respect to volume. This way of reasoning continued in answering the 6th question in Questionnaire-2 for the symbolic expression of average rate of change in solubility-temperature context. In the same way, the study by Rowland and Javanoski (2004) evidenced that most of the calculus students interpreted differentiation as being the amount of change in the function. For instance, students interpreted the algebraic expression $\frac{dD}{dt}$ with the verbal expressions as such "…*is the amount of* *drug*" or "…*represents how much the amount of drug changes due time*" (Rowland & Javanoski, 2004, p.510). In the study by Zandieh and Knapp (2006) some of calculus students answered the question of *"what is derivative?*" in such a way that "*derivative is a change*" (p.12). In the studies by Thompson (1994a, 1994b) and Thompson and Thompson (1992), young students considered speed not as a rate of change of distance with respect to time; rather they saw it as a distance to measure other distances. Thompson (1994a) indicated that calculus students confused "change" with the "rate of change". The findings of the current study and the previous studies show that students from various grades levels may have the misconception that "*rate of change is amount of change in the dependent variable*". On the other hand, it should be indicated that reasoning with amount of change in the dependent variable sometimes helped pre-service teachers to reach correct interpretations about the nature of curve and variations in rate of change. In covariational reasoning tasks, most of them decided the variation in rate of change by additively comparing the successive changes in the dependent variable with respect to implicit systematic variation in the dependent variable. The findings of this study supported the argument indicated by Johnson (2012) that even young students who has not taken a calculus course can explain the nature of covariational relationship and rate of change by systematically varying one quantity (independent variable) and the simultaneous variation in the intensity of change in other quantity (dependent variable) without needing ratio-based reasoning, limit, and function. The data of the current study additively showed that students who already taken calculus courses also decided the variation in rate of change by additively comparing the changes in the dependent variable with respect to equal increments in the independent variable without needing ratio-based reasoning.

Another conception of rate of change observed in the later phases of the model development unit was ratio-based reasoning. Ratio-based conception of rate of change involves the multiplicative comparison of changes in two quantities and being aware of that the quantity obtained is the ratio between two changes (Bezuidenhout, 1998; Thompson, 1994b). Pre-service teachers started to demonstrate unit per unit comparison and multiplicative comparison of changes in two quantities for quantifying the variation in rate of change first in covariational reasoning tasks. They also used ratio-based reasoning while explaining the symbolic expressions of average and instantaneous rate of change. For instance, while most of the pre-service teachers were considering "*change in population per year*" as an amount (Herbert & Pierce, 2012; Zandieh & Knapp, 2006), in the progress, they realized this was a ratio between change in population and change in year. Additionally, the data of the current study showed the critical role of verbal expressions used for indicating the formal mathematical concepts such as rate of change or derivative (Aerlback, Doerr & O'Neil, 2013b). Most of the pre-service teachers were unaware of the similarities and differences between various forms of verbal expressions. In other words, they did not use unit per unit comparison while explaining the symbolic expressions provided in the form of difference quotient and symbolic derivative. For instance, in explaining 6th question in Questionnaire-2, pre-service teachers who demonstrated ratio-based conception of rate of change interpreted the symbolic expression given in the form of difference quotient rule as "*average rate of change in the solubility with respect to temperature*". They did not use such an expression "*change in the solubility per unit change in the temperature*" despite they demonstrated this way of thinking in many other tasks. Some of the pre-service teachers realized the similar nature of these two expressions during the interviews.

5.2.3 Pre-service teachers' ways of reasoning on the connections between different representations of rate of change

At the beginning, most of the pre-service teachers' were unaware of the concept and term of rate of change and its relationship with the slope and derivative. As the participants' awareness of rate of change increased, they also realized the connections among different representations. Pre-service teachers dominantly used the slope in their explanations. After realizing the slope interpretation of rate of change, they also started to use the difference quotient rule. By the "Population of Turkey" activity, pre-service teachers developed the ideas of approximating the instantaneous rate of change from left side, right side, and by Mean Value theorem by gradually narrowing the interval. Thereby, they also realized the relationship between derivative, instantaneous rate of change, slope of tangent line, and difference quotient with the limiting process. The "speed" term was frequently used by the pre-service teachers in place of "rate of change" and they benefited from the definition of speed for explaining rate of change in different contexts.

In the "Population of Turkey" activity, pre-service teachers did not interpret rate of change in population in an interval as being the slope of secant line; rather they conceived it as the percentage of change in population with respect to previous year. Confrey and Smith (1994) explained this way of reasoning as multiplicative rate of change. Furthermore, most of the pre-service teachers did not consider "*yearly population change*" as a rate, ratio, or slope. They realized the relationship between "*yearly population change*", the slope of secant line, and the average rate of change after converting the tabular data to the graphical form. This result is also obtained in the studies by Zandieh (2000) and Zandieh and Knapp (2006). In the study by Zandieh and Knapp (2006), for answering the question of approximating to the derivative at a particular point by using the given values on a table, students used slope of tangent line interpretation after transferring the given tabular data to the graphical representation. The data of this study also revealed pre-service teachers' inexperience for interpreting rate of change, slope, or derivative when provided with tabular representations. This finding shows the need for benefiting from multiple representations in the teaching of derivative (Santos & Thomas, 2001; White & Mitchelmore, 1996).

Furthermore, at initial phases, when asked the meaning of the derivative, most of the pre-service teachers described it by the slope of tangent line and almost none of them described it as rate of change. Slope of tangent line was observed as being the pre-service teachers' dominant image of derivative in this study. This result is in line with the findings revealed by many earlier studies in the literature (Berry & Nyman, 2003; Habre & Abboud, 2006; Orton, 1983; Tall, 1992; White & Mitchelmore, 1996). This may be explained by the institutional orientation which was indicated in the study by Bingolbali et al. (2007) in such a way that students' understandings of derivative from different departments have different orientations. However, the role of geometric slope oriented methods for explaining derivative and ignorance of the other interpretations such as rate of change in curricular documents should also be considered as a possible reason (Berry & Neyman, 2003; Bingolbali, 2008; Teuscher & Reys, 2012).

Although pre-service teachers frequently used slope of tangent line while explaining the derivative, some weaknesses were also observed in their understanding of slope. For example, pre-service teachers could not explain the meaning of slope in height-volume context. Additionally, in the "Roller Coaster" activity, most of the pre-service teachers interpreted a particular numerical value of slope by forming a triangle between summits and valleys of a curved graph. They thought as if the slope value was yielding a linear graph. Teuscher and Reys (2012) and Coe (2007) determined that calculus students and mathematics teachers had difficulties in interpreting rate of change in non-linear situations. The data of the current study confirmed this finding, but additionally indicated that calculus students not only have difficulties in interpreting rate of change, but also they have difficulties in giving meaning the slope in non-linear situations. According to Stroup (2002), this is an expected result, because rate of change and slope was generally introduced with the linear functions in the curricular documents.

Pre-service teachers generally used and visited the definition of "speed" when they had difficulty in interpreting the meaning of derivative or rate of change in non-motion contexts. In explaining the meaning of slope or rate of change in height-volume context, they borrowed from the "*distance covered per unit time*" definition of speed and transferred it to the height-volume context as "*change in height per unit volume*". This can be the result of the frequent usage of motion context for explaining contextual interpretation of derivative (Bingolbali, 2008; Gravemeijer & Doorman, 1999; Herbert & Pierce, 2008; Wilhelm & Confrey, 2004; Yoon et al., 2010). Students' conceptions of rate of change are generally rooted in motion context (distance-time and velocity-time) and they have difficulties in projecting these concepts into non-motion contexts (Bezuidenhout, 1998; Gravemeijer & Doorman, 1999; Herbert & Pierce, 2008; Wilhelm & Confrey, 2004; Yoon et al., 2010; Zandieh & Knapp, 2006). Students are quite familiar with this context from early grades. This familiarity may prevent students to think about in-depth meanings of speed and acceleration concepts. Herbert and Pierce (2012) observed that students may see speed "*as a single entity with little emphasis on the covariance of the variables of time and distance*" (p.476).The frequent usage of motion context may also foster the idea that "time" is always an independent variable which results in confusions in non-temporal situations as observed in covariational reasoning tasks. Thompson (1995) mentioned the difficulty and the need for further abstraction for the image of rate in non-temporal situations which was also supported by the data of the current study.

Forming connections between rate of change, slope, difference quotient rule, and symbolic derivative is important for conceptual understanding of derivative (Herbert & Pierce, 2008; Zandieh, 2000; Zandieh & Knapp, 2006). However, the data of the current study evidenced that pre-service teachers do not have a connected knowledge about different interpretations of derivative provided with different representations. Pre-service teachers could not easily realize the connections between slope, difference quotient rule, derivative, rate of change, and the daily or verbal expressions used for them. Knowledge of derivative as slope of tangent line, being able to apply the rules of derivative in symbolic or idealized real contexts, and being able to solve symbolic procedural derivative problems does not bring together the learning of the different representations of derivative. I do not mean pre-service teachers did not know about derivative, slope, rate of change or difference quotient rule; rather, although they fragmentarily knew about different interpretations, they were not fluent enough in realizing and forming connections among them. Forming connections among different interpretations of rate of change (or derivative) should be an explicitly stated learning objective in curricular documents and the textbooks should contain many examples directly aiming at those connections.

5.3 Pre-service teachers' understanding of graphs and their ways of reasoning on the graphical connections between a function and its derivative

Pre-service teachers' graphical understanding of derivative involves two dimensions, which are (i) reading and interpreting the graphs, and (ii) understanding the graphical connections between a function and its derivative while drawing the graphs by reversing between derivative and anti-derivative functions.

5.3.1 Interpreting and reading the graphs

The data of the current study showed important developments in pre-service teachers' ways of reading and interpreting graphs. At the beginning, most of the pre-service teachers' were explaining the curved graphs either with their physical attributes or by taking into account some unnecessary variables. For instance, in explaining a concave-down increasing graph of volume-area, some of them indicated "*this graph can be formed by gradual enlargement of a cylinder*". They did not explain the curve by taking into account the area and volume as independent and dependent variables. In the progress, pre-service teachers' verbal explanations for the graphs shifted from explaining with general and daily expressions to more sophisticated expressions considering the variables as independent and independent variables and involving unit per unit thinking.

However, some additional difficulties appeared stemming from the complexity of the mathematical language. For instance, pre-service teachers generally preferred the expression "*increasingly increasing function*" for concave-up increasing graphs and "*decreasingly increasing function*" for concave-down increasing graphs. These explanations were not the already introduced terms; rather emerged and used naturally by the pre-service teachers. As repeatedly indicated in the covariational reasoning part, this way of reasoning involves focusing on the successive changes in the dependent variable while implicitly considering the equal increments in the independent variable. Looking from the concept of rate of change, this way of explanations does not involve a ratio-based coordination of changes, and it can be labeled as thinking within one-measure space (Thompson, 1994b). The true mathematical explanation for a concave-up increasing function should be "*increasing at an increasing rate function*" which includes the ratio-based conception of rate of change. The expressions "*increasingly increasing function*" and "*increasing at an increasing rate function*" both indicate a concave-up increasing graph, but there is a slight difference in their mathematical meaning. While the former expression focus on the successive changes in the function (dependent variable), the later focus on the rate of change of the function.

Furthermore, although pre-service teachers started to demonstrate more sophisticated verbal expressions involving the coordination of the variables, their confusions and difficulties in expressing the decreasing graphs continued. Although they could determine the increasing pattern in the decrease of the dependent variable in relation to the uniform increments in the independent variable, which was expressed as "*increasingly decreasing function*", they had difficulties in drawing the graph of this relationship. The same finding was appeared in the study by Monk (1992). This can be interpreted with the complexity of daily expressions and the true mathematical language while explaining a mathematical relationship. For a decreasing function having a concave-down curve, the expression "*increasingly* *decreasing function*” produces a complexity because it involves an “*increasing*” term for describing a decreasing function. This situation becomes more complex when expressed with the true formal mathematical expression. Again for the concave-down decreasing function, the true mathematical expression should be “*decreasing at a decreasing rate function*” which is the equivalent of “*increasingly decreasing function*”. As also observed in the studies by Doerr and O’Neil (2012) and Arleback et al. (2013a-2013b), it is more difficult for students to comprehend the true mathematical explanation, because the “*decreasing rate*” is related to the negative sign of rate of change. Namely, although the absolute value of rate of change increases, the signed rate of change decreases as it becomes more negative. Interpreting the variation in rate of change for decreasing functions produces complexities in either forms of reasoning. However, it can be argued that the true mathematical expressions for decreasing functions such as “*decreasing at an increasing rate function*” equivalent to “*decreasingly decreasing function*” involve extra difficulties, because it necessitates consideration of the sign of rates of change. As the data of the current study yielded and as reported by many other studies (Arleback et al., 2013a; Doerr & O’Neil, 2012), the true usage of language and being aware of the nuances between different expressions is important in order to convey the mathematical ideas accurately. In addition to teaching formal mathematical ideas, the true usage of mathematical language should be the aim of teaching mathematics. But, of course, teachers’ knowledge and awareness of the true mathematical language should be guaranteed first.

5.3.2 **Pre-service teachers’ ways of reasoning while reversing between the graphs of a function and its derivative**

Drawing inferences about the function or constructing the graph of it by looking its derivative graph or making inferences about the derivative function by looking the original function is not trivial. Several difficulties reported in the literature with regard to students’ understanding of graphical connections between a function and its derivative (Asiala et al., 1997; Aspinwall et al., 1997; Baker et al., 2000; Haciomeroglu et al., 2010; Ubuz, 2007). Reversing between graph of a function and its derivative graph is an important aspect of the conceptual understanding of derivative and relating it with the idea of integration. In the teaching of the graphical connections between a function and its derivative, there are standard procedures thought in calculus courses such as determining the increasing and decreasing intervals, max-min points, inflection and cusp points, and deciding the concavity according to the sign of second derivative. However, applications of those procedures do not develop students' understanding of the meaning of mathematical ideas such as what concavity or inflection point means (Berry & Neyman, 2003; Stroup, 2002; Tall, 1992). The data of the current study also evidenced that although pre-service teachers had knowledge about the standard procedures, they were unable to interpret and use these ideas in real situations. For instance, in solving the "Tracking Track" activity, most of the pre-service teachers could not follow the standard procedures. Additionally, in the Questionnaire-I, although they were successful in determining the analytical properties of a function provided with algebraic form, most of them were unable to draw the graph of a function by using its derivative graph.

More interestingly, in the "Tracking Track" activity, most of the pre-service teachers could not realize the derivative-antiderivative relationship. Therefore, they could not use the standard graph sketching procedures. Their reasoning was slope-based such as "*If the slope increasingly increases, the height also increasingly increase*". Slope-based reasoning worked in the positively increasing parts of the derivative graph, but resulted in difficulties when interpreting the way of reversing from rate (derivative) graph at the decreasing and negative parts to the amount (original function) graph. Some misinterpretations appeared such as "*if the slope values decreasingly decreases, the height also changes in the same way*". In deciding the top and valleys of the height-distance graph, some of the pre-service teachers used the area under gradient-distance graph without relating it with the idea of integration. Most of the pre-service teachers realized the derivative-antiderivative relationship during the group discussions or classroom discussion. The "Tracking Track" modeling activity helped pre-service teachers to revisit the mathematical meanings behind the sign of slope for determining increasing and decreasing intervals, inflection point where the second derivative gets zero and the area under curve for deciding the levels of top and valleys. Some parts of these findings resonate with the study of Yoon et al. (2010). But, the slight difference was that participants of the current study were not aware of the derivative-antiderivative relationship at the beginning.

Pre-service teachers' emerging ideas with regard to reversing between the graph of a function and its derivative can be explained by the Stroup's (2002) qualitative calculus, or by the "*physical or calculus feel*" indicated by Berry and Nyman (2003). Pre-service teachers thought intensively in such a way that as the slope gets greater, height increases more. This was a qualitative way of reasoning for reversing between rate and amount situations. Findings of this study supported Stroup's (2002) characterization of qualitative calculus as being "*cognitively significant and structural in its own right*". The data of the current study also supported the idea indicated by Berry and Nyman (2003) that drawing a function graph from its slope graph can be used for removing students from only reasoning with the algebraic schemes to developing a graphical sense in them supported by their intuitions and informal ways of thinking. Qualitative understandings of calculus ideas can be supported by working students in real situations, and this may be also a possible way of preventing them from many of difficulties.

5.4 Conclusions and Implications

This study investigated pre-service teachers' understanding of the ideas involved in derivative as they engaged in a model development sequence by adopting a design-based research perspective. Designing an artifact, implementing and evaluating it constitute the processes of design research. In the light of the major results obtained in this study, the major conclusions and implications can be expressed from various aspects.

First of all, pre-service teachers' weaknesses in covariational reasoning, their weak understanding of rate of change, and their difficulties while reversing between a function and its derivative graphs were observed in this study. They had difficulties in covariational reasoning tasks. They were unaware of the rate of change concept or their conceptions were not ratio-based. Additionally, they were unable to connect "rate of change" with other formal representations as slope, difference quotient, and derivative. They also had difficulties in interpreting the graphs and in reversing between the graphs of a function and its derivative. These findings force me to make the conclusion that even university students who already completed calculus courses may have not learned the essential ideas involved in derivative.

In the experimentation progress of the model development unit, there were remarkable developments in pre-service teachers' covariational reasoning abilities. While they had difficulties in deciding the dependent and independent variables at initial phases, they started to assign the functional relationship between covarying variables properly. The model development unit also supported them in attending the essential idea of covariational reasoning. Additionally, while pre-service teachers were using gross quantification at initial phases, they started to demonstrate extensive ways of quantification for deciding the variation in rate of change. Based on the findings in this study, it can be concluded that covariational reasoning is the big idea for comprehending the concepts of rate of change, derivative, and functional dependency. Therefore, one's covariational reasoning ability can be developed by considering his/her understanding of functions and rate of change at the same time. They all develop concurrently.

The model development unit also contributed to pre-service teachers in developing their conceptual understanding of rate of change. While they were unaware of the concept of rate of change, in subsequent phases, they demonstrated various conceptions including the ratio-based one. Pre-service teachers frequently used the definition of speed while giving meaning to the derivative in other contexts. Concurrently with the literature, the data of this study showed that introducing rate of change limited with the motion context does not support students understanding of rate of change in non-motion contexts (Bingolbali, 2008; Gravemeijer & Doorman, 1999; Herbert & Pierce, 2008; Wilhelm & Confrey, 2004; Yoon et al., 2010). Although perceptual and amount of change conceptions continued to appear in pre-service teachers' ways of reasoning, they developed the ratio-based conception of rate of change. More importantly, they realized the connections between rate of change, slope, difference quotient, speed, derivative, and the daily and mathematical language for expressing all. Some of the pre-service teachers started to be able to explain derivative in different contexts by being aware of its other interpretations.

Pre-service teachers' graph interpretation skills and their understanding of the graphical connections between a function and its derivative considerably changed in the process. While pre-service teachers were reading and interpreting the graphs by focusing on the physical attributes of them, they started to interpret graphs by taking into account the input and output variables and by indicating the variation in rate of change. The model development unit also helped pre-service teachers to cover the mathematical ideas behind the procedures of graph sketching while reversing between derivative and antiderivative functions.

Consequently, in the current study, the positive role of modeling activities and model development sequences in eliciting pre-service teachers' own conceptions of mathematical concepts was observed (Lesh & Doerr, 2003; Lesh et al., 2003; Gravemeijer, 2002). Contextual nature of the modeling problems not only supported pre-service teachers in developing conceptual understandings, but also helped them to realize the qualitative aspects of the calculus concepts (Berry & Nyman, 2003; Stroup, 2002).

Based on the findings obtained in this study, it can be mentioned about several implications with regard to teaching of derivative. Firstly, traditional methods of teaching functions, derivative, other calculus concepts do not supports students' understanding of these concepts in real life contexts. The formal mathematical concepts are the mathematical models used for describing some real life situations. However, introducing the formal concepts first and then expecting students to transfer them in real situations is an injustice. As also voiced by many researchers (e.g., Doorman & Gravemeijer, 2008; Gravemeijer & Doorman, 1999; Hoffkamp, 2011; Lesh & Doerr, 2003; Lesh et al., 2003; Stroup, 2002), students should be provided with the opportunity of developing their own mathematical concepts by directly experiencing them in authentic situations. To do this, mathematical modeling activities or interactive learning environments may be useful. The symbolic and rule-oriented applications in the teaching of derivative should be reconsidered. Although pre-service teachers in this study can differentiate diversity of symbolic functions, they had difficulties in interpreting what the derivative means in different contexts. The data of the current study indicated that the mathematical ideas behind the derivative could not be fostered by the rule-oriented way of teaching. Therefore, instead of teaching comprehensively with a collection of rules and procedures, derivative should be taught deeply and limited with the basic ideas involved in it.

Secondly, the role of curricular materials could not be underestimated in terms of giving directions to the practices of teachers. When the introduction of derivative examined, most of the textbooks ignored the rate of change interpretation or they only used the motion context (Bingolbali, 2008; Herbert & Pierce, 2012b; Teuscher & Reys, 2012). The concept of rate of change should be included more comprehensively in the curricular materials. And, it should be introduced by using non-temporal contexts. Additionally, textbooks and other curricular materials should include the tasks involving the dynamically and simultaneously changing quantities. These kinds of tasks are essential for developing students' conceptual understanding of the ideas such as dynamic view of functional dependency and covariational reasoning involved in derivative. Qualitative aspect of derivative concept can be fostered by employing students to work on authentic situations involving simultaneously changing quantities (Stroup, 2002).

Thirdly, some specific implications were drawn regarding to the teaching of some specific mathematical concepts. Slope is one of the critical concepts for conceptual understanding of derivative and rate of change (Stump, 1999). However, in this study, some weaknesses were observed in pre-service teachers' conceptions of slope. For instance, they could not interpret the meaning of slope in height-volume context and they had difficulty in interpreting the slope on curved-graphs. These results showed that pre-service teachers' dominant image of slope was rooted in linear contexts, and also they did not relate it with rate of change. While teaching the slope, in addition to the linear situations and graphs, non-linear situations involving curved-graphs should be used as well. As also indicated by Stroup (2002), linear functions are very special cases of different real life situations, and learning the slope limited with the linear situations can result in further difficulties as also observed in the current study. Therefore, we strongly suggest revisiting the slope when introducing the polynomial functions or starting to introduce it with the non-linear functions. In addition, the slope concept should be introduced not only geometrically, but also its rate of change interpretation should be emphasized in different real life situations. By this way, students can start forming connections between slope, rate of change, and derivative from many earlier grades.

5.4.1 Evaluating the model development unit and the design principles

The model development unit covered the ideas of covariational reasoning, rate of change, and graphical understanding of derivative. The "Water Tank" modeling activity was related to covariational reasoning. This activity worked well in revealing pre-service teachers' covariational reasoning styles. But, because it was easy to visualize, most of the pre-service teachers could draw correct graphs without demonstrating higher order covariational reasoning abilities. In further iterations of the experimentation of the model development unit, a different modeling activity can be used such as "Cassette Player" or "Sliding Ladder" problems for which it is relatively difficult drawing the correct graphs intuitively (by imagining). The data of the current study also provided information us for revising the first design principle, which is about covariational reasoning (see, Table 3). Namely, covariational reasoning encapsulates understanding of functional dependency and rate of change. Ability of covariational reasoning, conceptual understanding of function, and rate of change develops concurrently

The "Population of Turkey" activity, as a part of the second model development sequence, was aiming at revealing and developing pre-service teachers' conceptions of rate of change including the average and instantaneous rate of change. In general, this activity worked well for developing pre-service teachers' in clarifying the difference between the meanings of "amount of change" and "rate of change". Also, this activity and its follow up contributed pre-service teachers in realizing the connections between different representations of derivative. On the other hand, it also revealed pre-service teachers' difficulties in interpreting the difference between multiplicative and additive rates of change, and it resulted in some confusions. Therefore, for further iterations, I recommend using a different modeling activity if the main aim is developing the conventional idea of rate of change. But, if the aim is developing the different interpretations of rate of change in different contexts, this is an appropriate activity.

In the last two model development sequences, the "Roller Coaster" and the "Tracking Track" activities were related to interpreting graphs and the graphical connections between a function and its derivative. The "Roller Coaster" activity worked well in revealing and developing pre-service teachers' conceptions of slope on curved-graphs, "inflection point", "the changing nature of slope of tangent line", and so the "derivative at any point". The "Tracking Track" activity also worked well for revealing and developing pre-service teachers' understanding of the graphical connections between a function and its derivative. By this activity and the discussions conducted after it, pre-service teachers' awareness of the ideas behind the standard procedures used for reversing between the graphs of a function and its derivative was considerably increased.

The data of the current study also indicated the need for consideration of the way of teaching slope as a new design principle. Weakness in pre-service teachers' understanding of slope and their difficulties in connecting slope with its different representations indicated that the concept of slope should also be focused in the teaching of derivative. The desing principle that we can suggest for customizing the model development unit for further iterations is the following:

Suggested Principle: The concept of 'slope' is a foundational idea for conceptual understanding of derivative. Therefore, stundents' understanding of slope, connections between its different mathematical representations, and the meaning of slope in non-linear situations should be taken into consideration in the teaching of derivative.

5.4.2 Suggestions for future studies

In the current study, pre-service elementary mathematics teachers' understanding of the ideas involved in derivative was investigated as they engaged in a model development unit by adopting a design research perspective. A model development unit on the concept of derivative was designed in light of the arguments drawn from the literature. By the experience of the current study, some possible directions for further studies were identified.

First of all, this study was conducted with 20 senior or graduate pre-service elementary mathematics teachers as a part of the "*Mathematical Modeling for Teachers*" course. The main objective of the course was teaching pre-service teachers about the pedagogical issues of mathematical modeling. Teaching mathematical topics was not the primary focus of the course. Because the model development unit was not experimented as a part of a regular mathematics course; this can be accepted as being the main limitation of this study. Therefore, further studies are needed in clarifying the effectiveness of the designed model development unit in other contexts, especially, when experimented as an integrated part of regular mathematics courses. Although the data of the current study evidenced positive developments in pre-service teachers' conceptual understanding of derivative, further experimentation of the model development unit is needed as a part of mathematics courses for determining how it works when supported with formal mathematical instructions. The further studies can be conducted by direct implementation of the model development unit proposed in the current study with a different group of participants in a regular mathematics course, or it can be implemented by customizing the modeling and follow-up activities according to the design principles. Additionally, these kinds of learning tools, environments, and artifacts can be designed for other mathematical topics.

In the current study, a revision for the covariational reasoning framework has been offered. Also, the data of this study was analyzed by using the framework. However, the framework should be tested with different sets of covariational reasoning tasks and with different groups of participants. In addition, the data of the study implied that the visual character of the tasks involving covarying situations may affect students' ways of reasoning for deciding the variation in rate of change. The context in the "Water Tank" activity seemed to be easy to imagine for pre-service teachers, so most of them drew correct smooth graphs without using extensive ways of quantification. But, the situations in the "Cassette Player" and "Sliding Ladder" tasks were not easy to visualize. In these problems, almost all of the pre-service teachers perceptually sketched the graphs and they were generally incorrect. These interpretations related to visual character of the situation were the impressions that the data of the current study suggested, but still empirical studies are required investigating the role of visual character of the situation on persons' ways of covariational reasoning.

Pre-service teachers' unawareness of rate of change and their difficulties in giving meaning to this term were observed. The possible role of using a single term (i.e., Oran) in Turkish for both "ratio" and "rate" terms on this difficulty worth further investigation. In other words, pre-service teachers' difficulty in giving meaning to rate of change term can be related to terminology used in Turkish to represent rate and ratio. In addition, in the "Population of Turkey" activity, most of the pre-service teachers interpreted rate of change as being the percentage of change in population in successive years, and they did not find it reasonable thinking by the conventional interpretation of rate of change. This was indicated by Confrey and Smith (1994) in such a way that multiplicative interpretation of rate of change is easier for students in exponential situations. This study showed that pre-service teachers had strong tendency using multiplicative rate of change in population context, and they had difficulty in interpreting the difference between multiplicative and additive rate of change. Therefore, further studies are needed for investigating students' difficulties in clarifying the difference between additive and multiplicative rates of change. Studies should also focus on more effective pedagogical approaches for teaching students the difference and the relationship between multiplicative and additive rate of change.

Findings of the study also revealed the necessity of further studies in investigating the use of mathematical language in mathematics classrooms and its possible effects on students' learning. It was observed that pre-service teachers had difficulties in explaining the contextual graphs by using an appropriate mathematical language. Proper usage of language is important in teaching mathematics for conveying the mathematical ideas to students in an appropriate way. However, the data of the current study and other studies in the literature revealed pre-service teachers' and calculus students' difficulties with using the true language for explaining graphs, derivative, slope, and rate of change (Arleback et al., 2013a-2013b; Doerr & O'Neil, 2012). A comprehensive study is needed in analyzing teachers' ways of using the language in the teaching of various mathematical concepts, and its possible effects on students' difficulties and misconceptions. Additionally, proper usage of the language for each mathematical concept should be taken into consideration in curricular materials and in the teacher education programs.

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APPENDIX A

MODEL DEVELOPMENT SEQUENCE-I

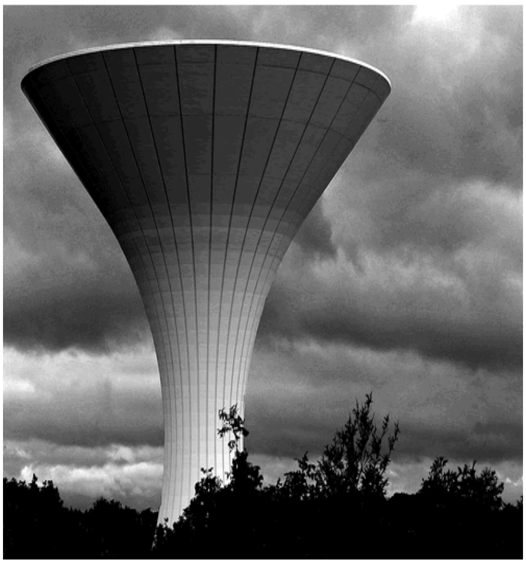
A.1: Water Tank

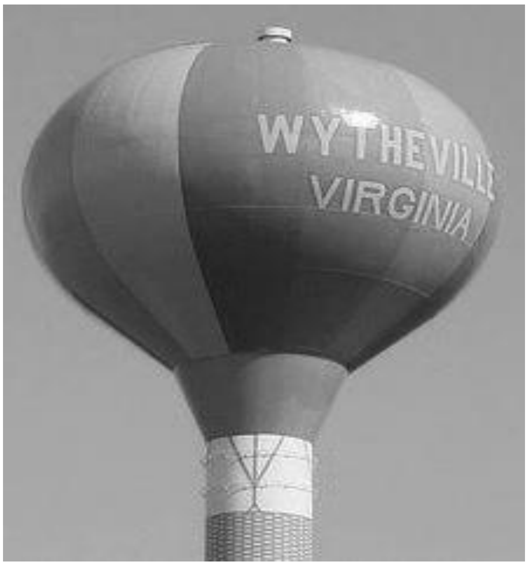
Su Deposu

Bir bilgisayar şirketi eğitim kurumlarına bilgisayar destekli eğitim amaçlı programlar hazırlamaktadır.Şirkete bağlı bir ekip öğrencilerin grafik çizme ve yorumlama becerilerini geliştirmeye yardımcı olacak bir su deposu doldurma animasyonu üzerinde çalışmaktadırlar.Ekibin bu animasyonu oluşturabilmesi için su deposu doldurulurken depoda biriken suyun hacmine bağlı olarak su yüksekliğini gösteren bir grafiğe ihtiyacı bulunmaktadır.

Ekibin matematikçi üyesi olarak sizden istenen ekte verilen depolardan birkaçı için bu grafikleri yaklaşık olarak çizmeniz ve herhangi bir şekle sahip bir su deposu için su miktarına bağlı olarak suyun yüksekliğini gösteren grafiğin nasıl çizileceğini açıklayan bir yönerge hazırlamanızdır.

Depo Şekilleri

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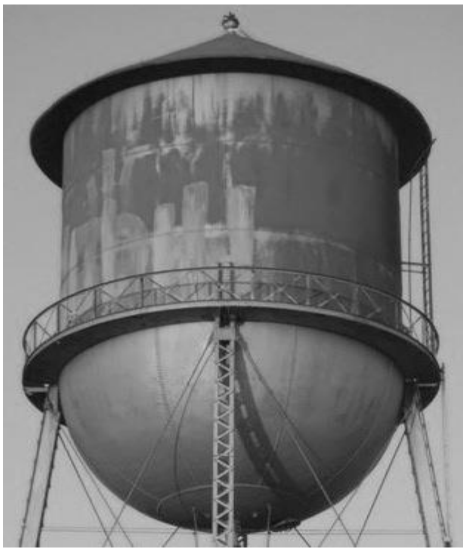
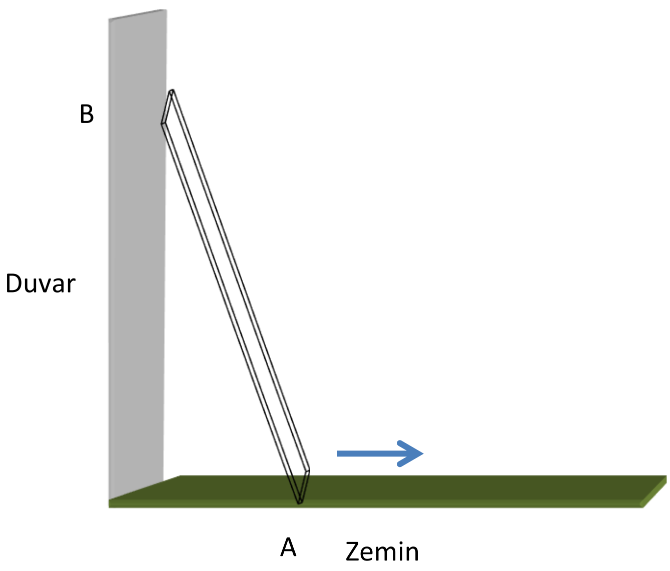


A.2: Sliding Ladder Problem

Merdiven Problemi

Duvara karşı dik pozisyonda duran bir merdivenin uç noktaları şekilde görüldüğü üzere A ve B olsun. Merdiven A noktasından tutularak zemin üzerinde ok yönünde sabit hızla çekilmektedir. Merdiven zemin üzerinde kaydıkça;

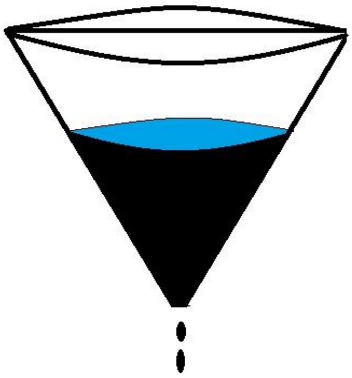
Merdivenin B noktasının zemine olan uzaklığı A noktasının duvara olan uzaklığına bağlı olarak nasıl değişmektedir? Matematiksel olarak açıklayınız ve grafiğini çiziniz.



A.3: Water Tank-2

Su Deposu-2

Yanda verilen ters koni şeklindeki bir su deposu alt kısımdaki sabit debili (hızlı) bir musluk ile boşaltılmaktadır.Başlangıçta tam dolu olan bu depo için boşalan su miktarına bağlı olarak depodaki suyun yüksekliği nasıl değişmektedir? Matematiksel olarak açıklayınız, grafik üzerinde gösteriniz.



APPENDIX B

MODEL DEVELOPMENT SEQUENCE-II

B.1: Population of Turkey

Gelecek Yüzyılda Türkiye

Devlet Planlama Teşkilatı (DPT), gelecek yüzyıl için stratejik planlar yapmaktadır. Bu çerçevede, Türkiye nüfusunun yıllara göre nasıl değiştiği ve önümüzdeki 100 yıl içerisinde demografik yapının nasıl farklılaşacağı gelecekte yapılacak yatırım planlarını yönlendirmede belirleyici bir etkendir.Çünkü mevcut durum gözetilerek yapılan yatırımlar yıllar sonra nüfus yapısındaki değişmeden dolayı kullanışsız kalabilmektedir.DPT, sizden, aşağıdaki sorular doğrultusunda bir rapor hazırlamanızı istemektedir.Raporda geçmiş yıllardaki nüfus değişimini inceleyip gelecekteki nüfus yapısının nasıl oluşabileceği konusu irdelenmelidir.

- Türkiye'nin nüfus artışı yıllara göre nasıl değişmektedir? Nüfustaki artışta nasıl bir değişim söz konusudur?
  - ➢ 1980-1985 yılları arasında ve diğer yıl aralıklarında nüfustaki artış miktarı nedir?
  - ➢ Zamana bağlı ortalama nüfus artış oranının (hızının) en yüksek ve en düşük olduğu yıl aralıkları hangileridir?
  - ➢ 2000 yılındaki zamana bağlı nüfus artış oranı (hızı) yaklaşık olarak nedir?
  - ➢ 2004 yılındaki zamana bağlı nüfus artış oranı (hızı) yaklaşık olarak nedir?
- Genç ve yaşlı nüfus yıllara göre nasıl bir değişim göstermektedir?
- Bazı uzmanlar gelecekte Türkiye nüfusunun durağanlaşacağını iddia ediyorlar. Sizce böyle bir durum söz konusu mudur? Durağanlaşma olacaksa bunun ne zaman gerçekleşmesi beklenebilir?

Tablo 1: Yaş grubuna göre nüfus

		Yaş grupları			
Yıllar	Nüfus*(×1000)	0-19	20-39	40-59	60 ve üzeri
1980	45586	22556	13745	6756	2529
1985	50660	24467	15015	7922	3256
1990	55971	25981	17500	8456	4034
2000	67800	27438	22458	12215	5689
2007	70786	24938	23604	15176	7068
2008	71557	24996	23927	15551	7083
2009	72641	25133	24225	15832	7451
2010	73724	25155	24482	16265	7822

*Yıl ortası nüfus verileri (örn. 1990 yılına ait veriler o yılın 6. ayına aittir.

B.2: Meteorology Balloon

1. Meteoroloji Balonu

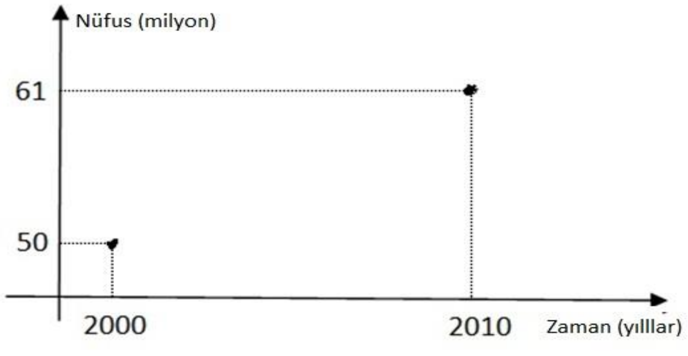
Meteoroloji balonları basınç, nem ve sıcaklık değerleri gibi birçok bilgiyi almak için günlük olarak havaya bırakılmaktadır. Balon yükseldikçe belirli zaman ve yüksekliklerde basınç, nem ve sıcaklık değerleri balonda bulunan elektronik verici aracılığı ile merkeze gönderilmektedir. Aşağıdaki tablo bir balonun yükselme hareketi boyunca kaydettiği altı ölçüm değerlerini göstermektedir.

	Zaman (dakika)	Yükseklik (metre)	Basınç (milibar)
1. ölçüm	0	Yer seviyesi	1000
2. ölçüm	360	260	925
3. ölçüm	650	440	850
4. ölçüm	1400	890	700
5. ölçüm	2750	1790	500
6. ölçüm	3600	2240	400

- Buna göre;[2. ölçüm-3. ölçüm] arasında balonun zamana bağlı yüksekliğindeki artış oranı (hızı) nedir? Birimini koymayı unutmayınız!
- Basıncın yüksekliğe bağlı değişim oranı (hızı) nedir? Birimini koymayı unutmayınız!
  - [4 ölçüm-5.ölçüm] arasında
  - [1.ölçüm-6.ölçüm] arasında

2. Nüfus Artış Oranı

Bir ülkenin 2000 yılındaki nüfusunun 50 milyon, 2010 yılındaki nüfusunun ise 61 milyon olduğu bilinmektedir. Bu durum aşağıdaki grafik üzerinde de gösterilmiştir.



- Bu ülkenin nüfusu <u>yüzdelik olarak</u> her yıl eşit oranda artıyorsa 2000-2010 yılları arasında bu durumu gösteren zaman-nüfus grafiğini çiziniz ve yorumlayınız?
- Bu ülkenin nüfusu <u>her yıl eşit miktarlarda</u> artıyorsa, 2000-2010 yılları arasında bu durumu gösteren zaman-nüfus grafiğini çiziniz ve yorumlayınız. Bu durumda nüfustaki <u>yıllık yüzdelik artış oranı</u> nasıl değişir?

APPENDIX C

MODEL DEVELOPMENT SEQUENCE-III

C.1: Roller Coaster Activity

Lunapark Treni

Ankara'da yeni kurulacak olan bir eğlence parkına bir lunapark tren yolu yapılması planlanmaktadır. Tasarım aşamasında olan parkurun maceralı kısmı (sadece iniş ve çıkışların bulunduğu, sağa yada sola viraj olmayan kısım) yatayda 180 metrelik düz bir bölüme inşa edilecektir. Tren yolunun bu maceralı bölümü için bir tasarım yarışması düzenlenmiştir. Tren yolu tasarımlarında yolun 3 tepeden oluşması, başlangıç noktasının yüksekliğinin 9 metre ve bitiş noktasının yüksekliğinin 6 metre olması istenmektedir.

Tasarımın yolcuları heyecanlandıracak kadar eğimli, fakat aynı zamanda güvenli olması gerekmektedir. Yolcuların heyecanlanması bu yolun yukarı ve aşağı doğru ani değişimlerle harekete imkân vermesine bağlıyken, güvenlik kurallarına göre, yolun eğiminin mutlak değeri 5,67 den fazla olmamalı.

Siz de bu yarışmaya, bir grup mühendisle birlikte kendi tasarımınızla katılmak istiyorsunuz. Tasarım için belirtilen şartları ve güvenlik koşulunu sağlamak şartıyla yolun 180 metrelik bu bölümünü tasarlayınız. Tasarımınızda güvenlik kriterini nasıl sağladığınızı açıklayınız. Tasarladığınız yolun farklı bölümlerinde yükseklik ve eğimin nasıl değiştiğini ve bu değişimin trenin hareketine nasıl yansıyacağını açıklayınız.

APPENDIX D

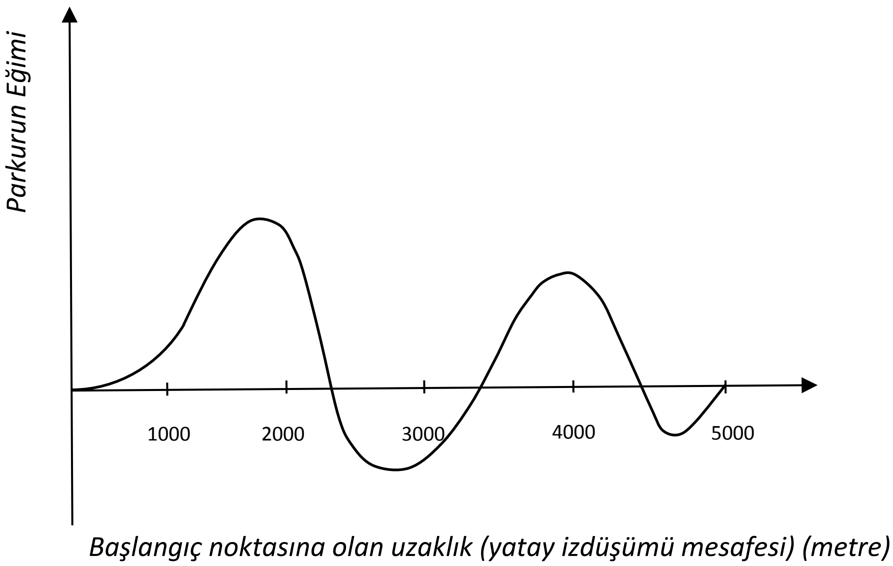
MODEL DEVELOPMENT SEQUENCE-IV

D.1: Tracking Track

Doğa Yürüyüşü Parkuru

Son yıllarda, Trekking'e (doğa yürüyüşü) olan ilginin artmasıyla bu yürüyüşleri organize eden acentelerin sayısı da artmaktadır. Bu acenteler her daim yeni bir parkur daha keşfedip, kişiye özel farklı zorluklarda parkur önerileri sunmak için çalışıyorlar.Bu amaçla yeni kurulan bir acente şirketi de bilinenlerden farklı parkurlar belirleyerek müşterilerin hizmetine sunmayı planlıyor.Bu acente için çalışan sporcular farklı parkurlarda yürüyüşler yapmaktadır.Sporcuların yürüyüşleri uydu yoluyla anlık takip edilebilmektedir.Yürüyüş başlangıcından itibaren sporcuların gittikleri mesafe ve bulundukları konumun eğimi, mesafe-eğim grafiği formatında kaydedilmektedir.Bu yürüyüşlerden bir tanesi için uydu kayıt verilerinin oluşturduğu taslak grafik aşağıda verilmiştir.

<u>Mesafe-Eğim Grafiği</u>

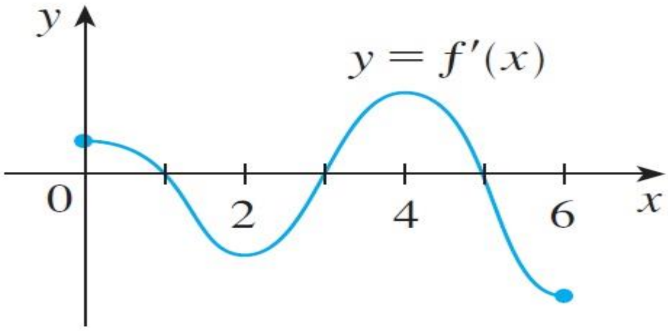


Yukarıdaki grafik verisi kullanılarak parkurun taslağı (kroki), mesafe-yükseklik grafiği şeklinde çizilebilmektedir.Bu konuda acente bir matematikçi olarak sizden yardım istemektedir.Yukarıdaki grafik verisini kullanarak parkurun taslağını (mesafe-yükseklik grafiğini) oluşturmalarına yardımcı olacak bir yöntem geliştiriniz ve bu yöntemi acente çalışanlarına açıklayınız.

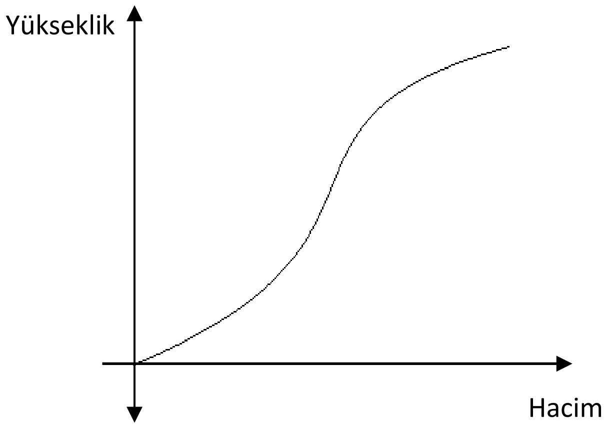
D.2: Graph Drawing

Grafik Çizme ve Yorumlama Devam Etkinlikleri

1. Aşağıda f (x) fonksiyonunun türev fonksiyonunun grafiği verilmiştir. Bu grafiği kullanarak ve aşağıdaki soruları da cevaplayarak f (x) fonksiyonun grafiğini yaklaşık olarak çiziniz. Not: f(0)=3
   a) f(x) fonksiyonunun artan ve azalan olduğu aralıklar bulunuz.
   b) f(x) fonksiyonunun yerel maksimum ve minimum noktaları hangileridir?
   c) f(x) fonksiyonun büküm noktaları hangileridir?

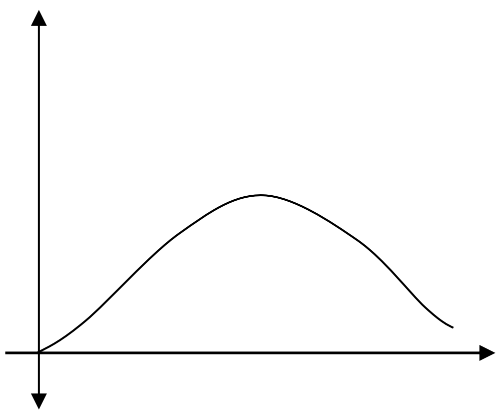


2. Bir depo akış hızı sabit bir musluk ile dolduruluyor. Bu depo dolarken, deponun içerisindeki suyun hacmine bağlı depodaki suyun yükseklik grafiği Şekil 1'deki gibi verilmiştir. Şekil 2'de ise bu grafiğinin türev grafiği yaklaşık olarak oluşturulmuştur. Şekil 2'deki grafiğin eksenlerini isimlendiriniz ve nasıl isimlendirdiğinizi açıklayınız.



Şekil 1. Hacim-Yükseklik grafiği

……………………………



……………….

Şekil 2…………………………………………

APPENDIX E

The Syllabus of the SSME 455 Course

Mathematical Modeling for Teachers (3-0)3

Fall 2011

Catalog Description: Models and modeling approach on mathematics problem solving, learning and teaching.Use of technological tools through modeling activities.Applications of mathematics to model real-world problem situations of different branches of mathematics. Training teachers in modeling process: understanding the real-world problem situations, making assumptions, formulating mathematical problem, solving the mathematical problem, interpreting the solution, verifying the model, reporting, explaining and making predictions based on the model. Implications of modeling in the classrooms, training prospective teachers for teaching modeling in the classrooms.

Course Objectives: Upon successful completion of the course, students should be able to;

- Develop their modeling competencies such as, understanding the real-world problem, setting up a model based on the reality, solving mathematical questions within the mathematical model, interpreting mathematical results in a real situation and making decisions about the results whether they need revisions or extensions, and whether they satisfy the conditions and assumptions given in the problem.
- Learn to apply their mathematical knowledge and skills to solve real-world problems.
- Develop their reasoning and communication skills using mathematical language, notation, diagrams, and graphs.
- Improve their knowledge about the use of technology in teaching and learning mathematics.
- Understand the characteristics of modeling activities.
- Learn how to use modeling activities in mathematics teaching

COURSE SCHEDULE

Wee k	Date	Topic	Assignment
1	Sep. 29, 2011	Overview and organization of the course Concept Map-I	-
2	Oct. 6, 2011	“Summer Job” Activity Technology and mathematical modeling: An overview of MS Excel Discussion of first impressions	-
3	Oct. 13, 2011	Technology and mathematical modeling: An overview of ClassPad. Nature of Mathematical Modeling (3 modeling activities will be studied by groups)	Reflection papers on Summer Job
4	Oct. 20, 2011	“Ferris Wheel” Activity Nature of mathematical modeling	-
5	Oct. 27, 2011	“Street Parking” Activity	Reflection Papers on Ferris Wheel
6	Nov. 3, 2011	Students Thinking Styles related to “Street Parking” Activity Questionnaire-I	Reflection Papers on Street Parking
7	Nov. 17, 2011	“Water Tank” Activity	-
8	Nov. 24, 2011	“Population of Turkey in Next Century” Activity	Reflection Papers on Water Tank
9	Dec. 1, 2011	Discussion on the nature of mathematical modeling, the characteristics of mathematical modeling activity and principles that have to be considered in developing a modeling activity.	Reflection Papers on Population of Turkey
10	Dec. 8, 2011	“Roller Coaster” Activity	-
11	Dec. 15, 2011	Classroom Discussion on Roller Coaster activity Students’ ways of thinking	Reflection Papers on Roller Coaster Project drafts
12	Dec. 22, 2011	“Tracking Track” Activity Discussion on nature of mathematical modeling and role of group work	-
13	Dec. 29, 2011	Questionaire-II Discussion on the roles of teachers in modeling process Developing plan for classroom application of a modeling activity	Reflection papers on Tracking Track Report on Models and Modeling
14	Jan. 06, 2012	Presentation of projects.	-
15		Presentation of projects. The general evaluation of the semester.	Projects

APPENDIX F

An Example of Lesson Plan for Week-7

7. Hafta Ders Planı

Tarih	
Dersin Amacı	Etkinlik Uygulaması-IV
Süre	Derste yapılacakların açıklanması: 5 dk Yeni etkinlik çözümü: 90 dk Sunumlar: 50 dk Dersin toparlanması: 15 dk
Araç ve Gereçler	Kameralar, ses kayıt cihazları ve doküman kamerası Su Deposu etkinlik kâğıdı (20 adet)

Ders Öncesi Hazırlık

Ses kayıt cihazları ve kameralar kontrol edilecek ve hazırlanacak (bir gün önceden ya da sabah)
Su Deposu etkinlik kâğıtları çoğaltılacak
A3-A4 kâğıtlar hazırlanacak
Zımba, uzatma kablosu, kamera ve tripotlar
Doküman kamera kurulumu
Sınıf düzeninin oluşturulması (sıra düzeni, kameraların kurulumu ve ses kayıt cihazları)
Dersin işleyişi ile ilgili araştırmacıların görüşmesi (1 gün önce)

- Araştırmacıların rolünün belirlenmesi
- Uygulanacak etkinlik için öğrenci düşünme şekilleri formunun hazırlanması (araştırmacılar tarafından) ve hocayla paylaşılması

Uygulama (Dersin İşleyişi)

1. Derste yapılacakların açıklanması (5 dk)
2. Su Deposu etkinlik uygulaması
   - Bireysel çalışma (5 dk)
   - Sorunun anlaşılıp anlaşılmadığı üzerine sınıf tartışması (15 dk)
   - Grup çalışması (90 dk)
3. Ara (20 dk)
4. Sunumlar (50 dk)
5. Dersin toparlanması (15 dk)

Grupça Hazırlanacak Proje Önerileri Teslimi (1.Versiyon)

Ödevler

Etkinlik sonrası düşünce raporu-6

APPENDIX G

QUESTIONNAIRES FOR EVALUATING CONCEPTUAL UNDERSTANDING OF DERIVATIVE

G.1: Questionnaire-I

Questionnaire-I

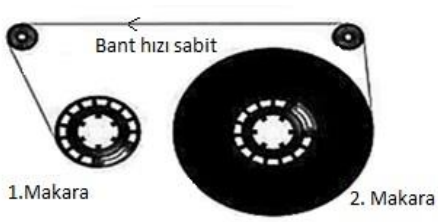
1. $f(x) = x^2 - 2x - 3$ fonksiyonunun grafiğine $(2, -3)$ noktasından çizilen teğet doğrusunun denklemini bulunuz.

2. $(x) = 3x^2 - 4$ fonksiyonu reel sayılardan reel sayılara tanımlı bir fonksiyondur.
   a) Bu fonksiyonun $x = 3$ ve $x = 5$ aralığındaki ortalama değişim oranı nedir?
   b) Bu fonksiyonun $x = 6$ noktasındaki değişim oranı nedir?

3. $f(x) = \frac{x^3}{3} + 2x^2 - 5x + 6$ fonksiyonu için;
   a) Fonksiyonun artan ve azalan olduğu aralıkları bulunuz.
   b) Fonksiyonun yerel maksimum ve yerel minimum noktalarını bulunuz.
   c) Fonksiyonun büküm noktasını bulunuz.

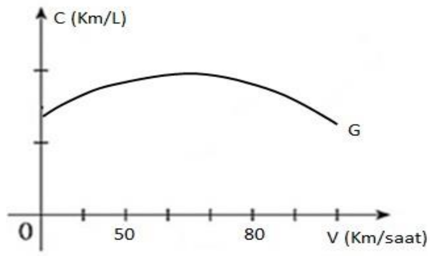
4. Yukarı doğru fırlatılan bir topun zamana bağlı yükseklik fonksiyonu $H(t) = -16t^2 + 128t + 320$ şeklindedir. Topun ulaştığı maksimum yükseklik nedir?

5. Bir teyp kaseti düşününüz. Teyp, kasetteki makaraları döndürerek bir makaradan diğerine <u>sabit hızla</u> bant sarımı yapmaktadır. Yanda şekilde de gösterilen bir kasetin başlangıçtan ilk yüzü bitinceye kadar olan süreçte; **İkinci makaranın yarıçapı birinci makaranın yarıçapına bağlı olarak nasıl değişmektedir?** Sözel ve grafiksel olarak açıklayınız. *Not: Boş durumda her iki makaranın yarıçapları eşittir.* (*Inspired from the study of Lingefjard (2000)*)



6. Yanda grafiği verilen G fonksiyonu bir otomobilin hızına bağlı olarak yakıt verimliliğini gösteren bir fonksiyon olsun.

   Yakıt Verimliliği (C)= Birim litre yakıt ile alınan yol, Km/L



   Aşağıdaki ifadeleri yakıt verimliliği ve otomobilin hızı bakımından birimlerini de koyarak yorumlayınız.

   a) G(70)=24
   b) $\frac{G(100)-G(70)}{30}$ işlemi sonucu çıkan değer problem bağlamında ne anlama gelmektedir? Açıklayınız.
   c) G'(80)= -0,3 sonucu problem bağlamında ne anlama gelmektedir? Açıklayınız.

   (*Inspired from the study of Goerdt (2007) and Bezuidenhout (1998)*)

7. F fonksiyonunun tanım kümesi reel sayılar olup sürekli ve türevlenebilir bir fonksiyondur. F(x) fonksiyonunun bazı değerleri aşağıda tabloda verilmiştir.

x	F(x)
1.0	0.00
1.1	-0.22
1.2	-0.46
1.3	-0.70
1.4	-0.90
1.5	-1.03
1.6	-1.02

x	F(x)
1.7	-0.81
1.8	-0.30
1.9	0.60
2.0	2
2.1	4.04
2.2	6.89
2.3	10.70

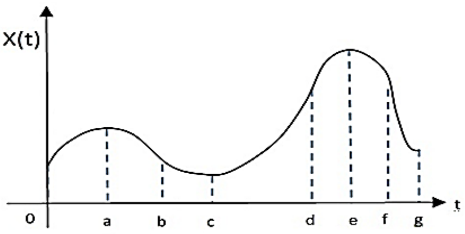
x	F(x)
2.4	15.67
2.5	22.03
2.6	30.02
2.7	39.90
2.8	51.97
2.9	66.56
3.0	84.00

F(x) fonksiyonunun $x = 2$ noktasında türev değeri için tablo değerlerine bakarak mümkün olan en iyi tahmini yapınız.Yaptığınız işlemleri ayrıntılı olarak gösteriniz. (*Inspired from the study of Hartter (1995)*)

8. Eğer H(t) sağlık harcamalarının zamana bağlı değişen bir fonksiyonu ise "*Sağlık harcamaları 2011 yılında da artmaya devam ediyor, fakat 3 yıl öncesine kıyasla daha yavaş oranda*" ifadesi aşağıdakilerden hangisi ya da hangilerini doğrular? Neden?
   1. $H'(2011) < 0$
   2. $H'(2011) < H'(2008)$
   3. $H''(2011) > 0$

   (*Adapted from the study of Goerdt (2007)*)

9. Yanda grafikte zamana bağlı yol grafiği verilen bir araç için aşağıda belitilen noktaları veya aralıkları belirleyiniz.

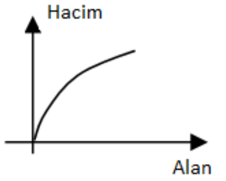


a) Hız pozitif
b) Hız en yüksek
c) İvme negatif

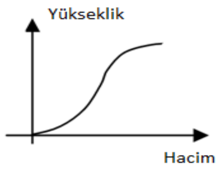
(*Adapted from CPM Calculus, Chapter 2, p.80*)

10. Aşağıda değişkenleri farklı iki fonksiyonun grafiği verilmiştir. Bu grafikleri sözel olarak açıklamak durumunda kalsanız matematiksel olarak nasıl ifade edersiniz?

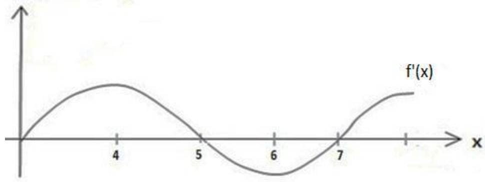
A



B



11. Aşağıda türev fonksiyonunun grafiği verilen ve f(0)=1 değeri bilinen fonksiyonun grafiğini yaklaşık olarak çiziniz.



G.2: Questionnaire-II

Questionnaire-II

1. $f(x) = x^2 - 4x + 3$ fonksiyonun grafiğine (4, 3) noktasında teğet olan doğru denklemini bulunuz.
2. $f(x) = x^2 - 2x$ fonksiyonu reel sayılardan reel sayılara tanımlı bir fonksiyondur.

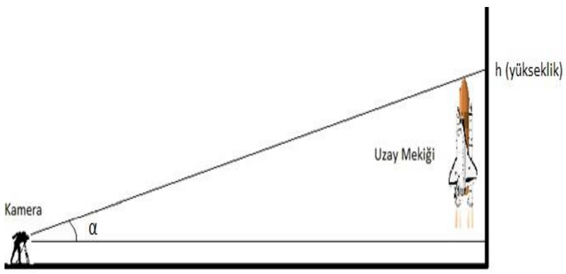
   a) Bu fonksiyonun $x = 2$ ve $x = 5$ aralığındaki ortalama değişim oranı (hızı) nedir?
   b) Bu fonksiyonun $x = 5$ noktasındaki değişim oranı (hızı) nedir?

3. $f(x) = \frac{x^3}{3} - 2x^2 + 3x - 10$ fonksiyonu için;

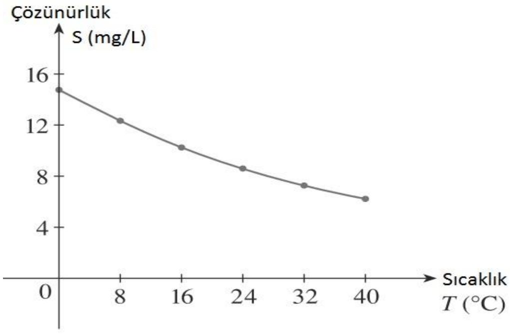
   a) Fonksiyonun artan ve azalan olduğu aralıkları bulunuz.
   b) Fonksiyonun yerel maksimum ve yerel minimum noktalarını bulunuz.
   c) Fonksiyonun büküm noktasını bulunuz.

4. Yukarı doğru fırlatılan bir füzenin zamana bağlı yükseklik fonksiyonu $H(t) = -4t^2 + 48t + 240$ şeklindedir. Füzenin ulaştığı maksimum yükseklik nedir?

5. Bir uzay mekiğinin uzaya fırlatılmasını görüntülemek isteyen bir kameraman kameranın açısını uzay mekiği görüntüden kaybolana kadar sürekli değiştirmek zorundadır. **Buna göre <u>kamera açısı (α) uzay mekiğinin yerden yüksekliğine bağlı olarak</u> nasıl değişmektedir?** Matematiksel olarak açıklayınız ve grafiğini çiziniz.



6. Oksijenin su içerisinde çözünürlüğü (mg/L) suyun sıcaklığına bağlı olarak değişmektedir. Yanda verilen grafik oksijenin suda çözünürlüğü olan S (mg/L) değerlerinin suyun sıcaklığı T (C^0)'ye bağlı olarak nasıl değiştiğini göstermektedir.



Aşağıdaki ifadeleri oksijen çözünürlüğü ve su sıcaklığı bakımından birimlerini de koyarak yorumlayınız.

a) S(32)=6

b) $\frac{S(40)-S(8)}{32}$ işlemi sonucu çıkan değer problem bağlamında ne anlama gelmektedir? Açıklayınız.

c) S'(16)= -0,25 sonucu problem bağlamında ne anlama gelmektedir? Yorumlayınız.

(*Inspired from the studies by Goerdt (2007) and Bezuidenhout (1998).*)

7. F fonksiyonunun tanım kümesi reel sayılar olup sürekli ve türevlenebilir birfonksiyondur. F fonksiyonunun bazı değerleri aşağıda tabloda verilmiştir.

x	F(x)
0.00	-15
0.90	-11.643
0.99	-11.069
1.00	-11
1.01	-10.929
1.10	-10.237
1.50	-5.625

x	F(x)
1.900	2.347
1.990	4.721
1.999	4.972
2.000	5
2.001	5.028
2.010	5.282
2.100	7.953

F(x) fonksiyonunun x=2 noktasında türev değerini için tablo değerlerine bakarak mümkün olan en iyi tahmini yapınız. Yaptığınız işlemleri ayrıntılı olarak gösteriniz. (*Adapted from the study by Hartter (1995)*)

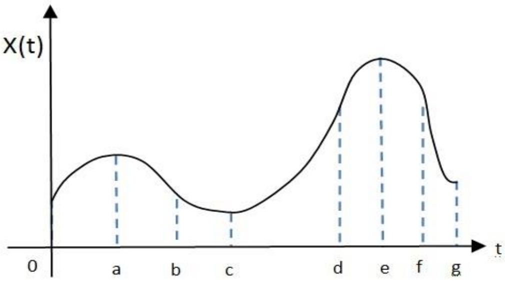
Zaman(t)	0	1	2	3	4	5
S(t)	28.5	25.5	24.0	23.2	22.9	22.8

8. Bir ürünün satışından elde edilen 5 yıllık gelir yukarıda tabloda gösterilmiştir (milyon TL). Tablo değerlerine göre aşağıdaki ifadelerden hangisi ya da hangileri doğrudur? Açıklayınız.
    1. S'(1) >0
    2. S''(1)>0
    3. S(1)>S'(1)

(*Adapted from the study of Goerdt (2007)*)

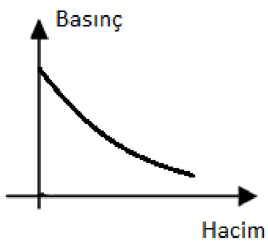
9. Yanda grafikte zamana bağlı yol grafiği verilen bir araç için aşağıda belitilen noktaları veya aralıkları belirleyiniz.
    a) Hız negatif
    b) Hız en yüksek
    c) İvme pozitif

(*Adapted from CPM Calculus, Chapter 2, p.80*)

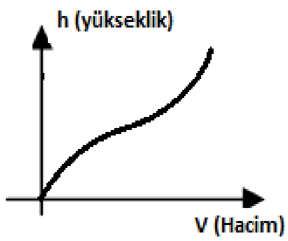


10. Aşağıda değişkenleri farklı iki fonksiyonun grafiği verilmiştir. Bu grafikleri sözel olarak anlatmak durumunda kalsanız matematiksel olarak nasıl ifade edersiniz?

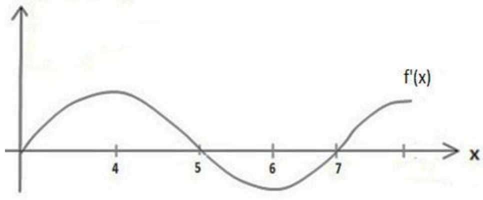
A



B



11. Aşağıda türev fonksiyonunun grafiği verilen ve f(0)=3 değeri bilinen fonksiyonun grafiğini yaklaşık olarak çiziniz.



G.3: Table of Specifications

Table of Specifications

Table of specifications for evaluating conceptual understanding of derivative is provided below. Conceptual understanding of derivative is evaluated from various aspects involving a robust level of covariational reasoning, knowledge about different interpretations of derivative (e.g., slope, rate of change, difference quotient rule), being able to transfer these knowledge in different contexts, knowledge about procedural operations on symbolic expressions, and graphical understanding of derivative. The item numbers and the mathematical ideas included in each items are the same for Questionnaire-I and Questionnaire-II.

Table 18: Table of Specifications

Mathematical ideas	Level of Knowledge and Item Number	
	Procedural	Conceptual
Covariational Reasoning		Q5 Q6-a
Derivative as slope	Q1	
Rate of Change	Q2-a Q2-b	Q6-b Q6-c
Difference Quotient Rule		Q7
Symbolic Derivative		Q8
Derivative in Motion Context		Q9
Application of derivative (Max-min Problems)	Q4	Q6-c Q8
Drawing Graph	Q3	Q11
Interpreting Graph		Q10

G.4: Assessment Rubric for Questionnaire-I

Scoring Rubric for Questionnaire 1

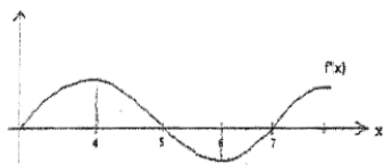
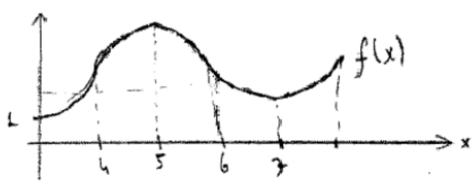
<table>
<tr><th colspan="4">Question 1 (*5)</th></tr>
<tr><th colspan="2">Correct answer</th><th colspan="2">Scoring</th></tr>
<tr><td colspan="2" rowspan="3">Step1: Calculating the value of f'(2) and determining as the slope of tangent line
$$f'(x) = 2x - 2$$
$$f'(2) = 2.2 - 2 = 2$$
Step 2: Writing the line equation
$$y + 3 = 2(x - 2)$$
$$y = 2x - 7$$</td><td>Correct solution with steps 1&2
Procedurally correct with some operational mistakes</td><td>2</td></tr>
<tr><td>Only step 1 without step 2</td><td>1</td></tr>
<tr><td>Blank, totaly wrong or irrelevant</td><td>0</td></tr>
<tr><th colspan="4">Question 2 ((a+b)*10/3)</th></tr>
<tr><th colspan="2">Correct answer</th><th colspan="2">Scoring</th></tr>
<tr><td rowspan="3">a</td><td rowspan="3">Average rate of change of $f(x) = 3x^2 - 4$ between x=3 and x=5 is;
$$\frac{f(5) - f(3)}{5 - 3} = \frac{71 - 23}{2} = 24$$</td><td>Correct solution</td><td>2</td></tr>
<tr><td>Taking the average $\frac{f'(5)+f'(3)}{2}$
Difference $f'(5) - f'(3)$</td><td>1</td></tr>
<tr><td>Blank or totaly irrelevant</td><td>0</td></tr>
<tr><td rowspan="2">b</td><td rowspan="2">Instantaneous rate of change of f at x=6 is;
$$f'(x) = 6x$$
$$f'(6) = 36$$</td><td>Correct solution</td><td>1</td></tr>
<tr><td>Blank or totally irrelevant</td><td>0</td></tr>
<tr><th colspan="4">Question 3 ((a+b+c)*2)</th></tr>
<tr><th colspan="2">Correct Answer</th><th colspan="2">Scoring</th></tr>
<tr><td rowspan="3">a</td><td rowspan="3">Determining the monotonocity intervals
Step 1: Finding the derivative function.
$$f'(x) = x^2 + 4x - 5$$
$$f'(x) = (x - 1)(x + 5)$$
Step 2: Deciding the positive and negative intervals and determining increasing and decreasing intervals

f function is increasing in the intervals $(-\infty, -5) \cup (1, \infty)$
f function is decreasing in the interval (-5, 1)</td><td>Correct solution with steps 1&2
Procedurally correct with some operational mistakes</td><td>2</td></tr>
<tr><td>Only step 1 is correct</td><td>1</td></tr>
<tr><td>Blank, totaly wrong or irrelevant</td><td>0</td></tr>
<tr><td rowspan="3">b</td><td rowspan="3">Finding the relative max and min points of the function
Step 1: Determining critical points.
$$f'(x) = (x - 1)(x + 5) = 0$$
$$x_1 = 1 \quad x_2 = -5$$
Step 2: Checking the function at critical points by using Step 2 in part a.
$\big(-5, f(-5)\big)$ maximum point and $\big(1, f(1)\big)$ minimum point</td><td>Correct solution with steps 1&2
Correct with some operational mistakes</td><td>2</td></tr>
<tr><td>Only step 1 is correct</td><td>1</td></tr>
<tr><td>Blank, totaly wrong or irrelevant</td><td>0</td></tr>
<tr><td rowspan="2">c</td><td rowspan="2">Finding the inflection point
Inflection point is where the second derivative is 0.
$$f''(x) = 2x + 4 = 0$$
$$x = -2$$</td><td>Correct solution</td><td>1</td></tr>
<tr><td>Blank, totaly wrong or irrelevant</td><td>0</td></tr>
</table>

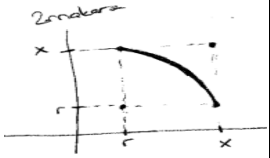
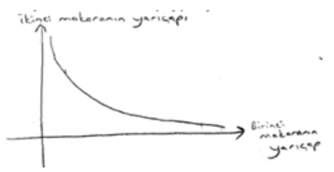
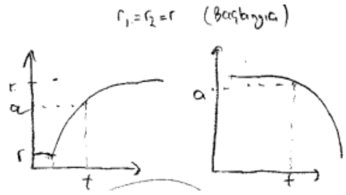
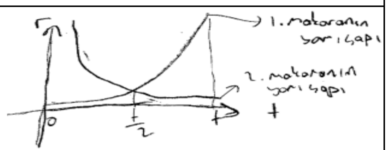
Question 4 (*5)		
Correct Answer	Scoring	
Step 1: Finding the heighest point by using derivative. The heighest point is where the first derivative gets 0. $H(t) = -16t^2 + 128t + 320$ $H'(t) = -32t + 128 = 0$ $t = 4$ **Step 2:** Substituting the value in the function. $H(4) = 576$	Correct solution with steps 1&2	
	Correct solution with steps 1&2 including some operational mistakes	2
	Only Step 1	1
	Blank, totaly wrong or irrelevant	0

Question 8 ((a+b+c)*10/6)			
	Correct Answer	Scoring	
1	$H'(2011) < 0$<u>is not true</u>, because the function is still increasing. $H'(2011) > 0$	Correct answer with correct explanation	2
		Correct answer without explanation	1
2	$H'(2011) < H'(2008)$<u>is true</u>, because the rate of change is gradually decreasing.	Correct answer with correct explanation	2
		Correct answer without explanation	1
3	$H''(2011) > 0$<u>is not true</u>, because the function is concave down and the rate of change is gradually decreasing which means a negative second derivative.	Correct answer with correct explanation	2
		Correct answer without explanation	1

Question 9 ((a+b+c)*10/8)			
	Correct Answer	Scoring	
a	Speed of the car is positive in the intervals; (0,a), (c,d) and (d,e)	For each of the intervals (1 point)	3
b	(Negative or positive) Speed of the car is possibly highest at the points; b, d, f	Expressing one or more of these points	1
c	Negative acceleration; (0,a), (a,b), (d,e) and (e,f)	For each intervals (1 point)	4

Question 11 (*10/3)		
Correct Answer	Scoring	
 	Correct graph with all critical points plotted	3
	Correct smooth graphing without ploting inflection or max-min points	2
	Determining all the points but graphing the curve with sharp transitions at critical points Incorrect graph having a few correct features	1
	Totally wrong or irrelevant graph	0

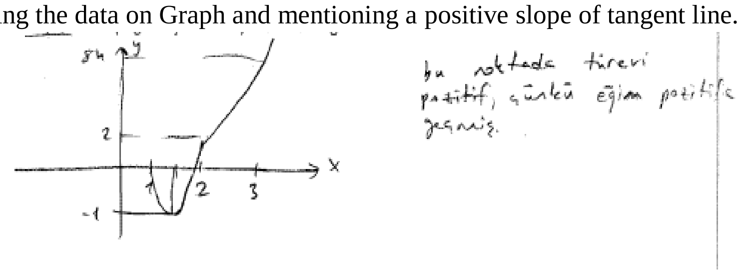
Rubric for Question 5 (*10/3)

Scr		Criteria and Cases
3	Correct	Correct graph with sophisticated verbal or other form of mathematical explanation. *Birinci makaranın yarıçapındaki birim artışa karşılık, 2. makaranın yarıçapı artarak azalmaktadır. Yani örneğin birinci makaranın yarıçapının her bir birim artışında 2 makaranın yarıçapı önce 1 birim, daha sonra 2 birim, 3 birim şeklinde azalmaktadır.*  Correct explanation without graph or correct graph without explanation *Birinci makaranın yarıçapı başlangıçta küçük olduğu için 2. Makara bir tur döndüğünde bıraktığı bant 1. Makara tarafından daha fazla dönerek sarılmalıdır. Dolayısı ile 1. Makaranın yarıçap artışı daha fazla döndüğü için ikinci makaranın azalmasına oranla daha fazla olacaktır.*
2	Partially Correct	Partially correct explanations and/or partially correct graphs emphasizing the changing rate of change. *Bir noktaya kadar birinci makaranın yarıçap artış hızı ikinci makaranın yarıçapının azalış hızından fazla iken belli bir noktadan sonra tam tersi olur.* Graphs (not "time" dependent) with explanations that emphasizes the changing nature of rate of change, but indicating an inverse concavity (Concavity confusion).  Correct time dependent graph with correct explanation *Birinci makaranın yarıçapı birim zamanda azalarak artarken, ikinci makaranın yarıçapı birim zamanda artarak azalmaktadır.* 
1	Including a few correct parts	"Time" dependent partially correct explanations or graphs emphasizing the changing nature of rate of change, but for example, indicating an inverse concavity (Concavity confusion).  Expressing a linear relationship (not "time" dependent)**with** graph or emphasizing only the direction of change without mentioning the intensity (changing nature) of change and with graph. *İkinci makaranın yarıçapı birinci makaranın yarıçapındaki artışa bağlı olarak doğru orantılı bir şekilde azalmaktadır.* Expressing a linear relationship (not "time" dependent)**without** graph. *İkinci makaranın yarıçapı birinci makaranın yarıçapındaki artışa bağlı olarak doğru orantılı bir şekilde azalmaktadır.* *Birinci makaranın yarıçapı artarken ikinci makaranın yarıçapı azalır.*
0		Blank or irrelevant answer.

Rubric for Question 6 (a+(b+c)*4/3)

Part a		
Score		Criteria and Cases
1	Correct	Understanding the function and interpreting the expression $G(70) = 24$ by its variables and units. *Otomobil 70 km/saat hızla giderken, yakıt verimliliği 24 km/litredir.* *Araç 70 km/saat hızla gittiğinde birim litrede 24 km yol almaktadır*
0		Blank or Irrelevant *Araç 70 km/saat hızla giderse yakıtı maksimum seviyede tüketir.*
Part b (*4/3)		
Score		Criteria and Cases
3	Correct	Average rate of change in Fuel efficiency in the interval of the given speeds. *70 km/s ve 100 km/s hızları arasında yakıt verimliliğindeki ortalama değişim oranı.*
2	Partially	Change in fuel efficiency or average change in fuel efficiency between 70 km/h and 100 km/h speeds without mentioning "rate". *70 km/s ve 100 km/s hızları arasında ortalama yakıt verimliliği değişimi.*
1	Slope	Slope without (average) rate of change and without units $m = \frac{G(100) - G(70)}{100 - 70}$
0		Blank or Irrelevant e.g., Fuel efficiency or average fuel efficiency in the given interval without mentioning "change" and "rate".
Part c (*4/3)		
Score		Criteria and Cases
3	Correct	(Instantaneous) rate of change in fuel efficiency at the given speed (80 km/h) *Hızın 80 km/saat olduğu anda yakıt verimliliğindeki hıza bağlı anlık değişim oranının -0,3 olması.* Average rate of change in fuel efficiency at the speed of 80 km/h *80 km/saat hız ile gidildiğinde yakıt verimliliğinde oluşan ortalama değişim oranı.*
2	P.C	Change in fuel efficiency at the speed 80 km/h without mentioning "rate". *80 km/saat hızla gittiğimizde yakıt verimliliği azalıyor.*
1	Slope	The value of slope of tangent line at that point. *80 km/saat hıza karşılık gelen grafik üzerindeki noktaya atılan teğetin eğiminin -0,3 olması.*
0		Blank or Irrelevant

Rubric for question 7

Scr		Criteria and cases
3	Correct	Applying the difference quotient rule for small intervals on the given data from both left and right sides. And then deciding the best approximation by using mean value theorem. Case: $m_1 = \frac{F(2,1)-F(2)}{2,1-2} = \frac{4,04-2}{0,1} = 20,4$ $m_2 = \frac{F(2)-F(1,9)}{2-1,9} = \frac{2-0,60}{0,1} = 14 F'(2) \cong \frac{F(2,1)-F(1,9)}{2,1-1,9} = 17,2$ is the best approximation Applying the difference quotient rule for small intervals on the given data from left or right sides, and reporting one of this values as the best approximation. Case: $\mathrm{F}'(2) \cong \frac{\mathrm{F}(2,1)-\mathrm{F}(2)}{2,1-2} = \frac{4,04-2}{0,1} = 20,4$ $\mathrm{F}'(2) \cong \frac{\mathrm{F}(2)-\mathrm{F}(1,9)}{2-1,9} = \frac{2-0,60}{0,1} = 14$ Because, $2 - 0,60 < 4,04 - 2$ *the second one is the best approximation.*
2	Partially Correct	Applying difference quotient rule on a large intervals. Case: $F'(2) \cong \frac{F(2)-F(1)}{2-1} = 2$ Taking the averages of slopes of secant lines from left and right sides Case: $m_1 = \frac{F(2,1)-F(2)}{2,1-2} = \frac{4,04-2}{0,1} = 20,4$ $m_2 = \frac{F(2)-F(1,9)}{2-1,9} = \frac{2-0,60}{0,1} = 14 F'(2) \cong \frac{m_1+m_2}{2} = 17,2$
1	A few correct parts	Tranlating the data on Graph and mentioning a positive slope of tangent line.  Reporting only the positive sign of derivative at that point by determening the increasing nature of the data. Case: *x=1,5 noktasından sonra fonksiyon artan bir özellik göstermektedir. Bu nedenle* $F'(2)$ *pozitif bir değer alır.* Trying to find a symbolic equation for the given data and calculating the derivative at the given point
0		Blank or irrelevant E.g, Case: $F(2) = 2 = F'(2)$

Rubric for Question 10

<table>
<tr><th>Score</th><th></th><th>Criteria and Cases</th></tr>
<tr><td>3</td><td>Correct</td><td>Explaining by rate of change
The volume is increasing with respect to area, but rate of change in volume with respect to area is getting smaller.

Explaining by unit per unit comparison. Comparing the increments in the output variable with respect to unit increments in the input variable ($\frac{V_2-V_1}{2-1} > \frac{V_3-V_2}{3-2}$)
The volume is decreasingly increasing with respect to unit changes in area.

Explaining by the multiplicative rate of change, e.g, Comparing the ratios of the simultaneous values of both variables ($\frac{\Delta V_2}{\Delta V_1} < \frac{\Delta A_2}{\Delta A_1}$)
The volume is increasing with respect to area. But changing rate in area is greater than changing rate in volume
Alan arttıkça hacim de artmış, fakat hacmin artış oranı alanınkinden az.</td></tr>
<tr><td>2</td><td>Partially Correct</td><td>Thinking within one measure space. Explaining the changing nature of dependent variable without mentioning about the independent variable.
The volume is decreasingly increasing (Hacim azalarak artmaktadır).

Explaining by using the motion terminologies (speed, acceleration).
Increasing speed of volume is getting smaller and smaller.

Concavity Difficulty. Explaining by rate of change or unit per unit thinking, but confusing the concave up or down property of the original curve (e.g, although the curve is concave up, the explanation is for a concave down graph) , Inconsistency between explanation and graph</td></tr>
<tr><td>1</td><td>A Few Correct Parts</td><td>Uncoordinated way of thinking, looking changes in both variables independently
Hacim yavaş yavaş artarken, alan hızlı artmaktadır
Time based explanations involving RC or unit per unit thinking (Emphasizing the changing rate of change, but time dependent)
Hacim birim zamanda azalarak artmaktadır.
Lineer thinking, expressing only the direction of change
Alan hacimle doğru orantılı olarak artmaktadır.
Hacim arttıkça alan da artmaktadır.
Partially correct time based explanations.
Hacim zamana bağlı olarak artmaktadır.
Describing the curve by using physical expressions.
Birinci grafiğe gore alan arttıkça hacimde negatif yönlü bir artış gerçekleşiyor.</td></tr>
<tr><td>0</td><td></td><td>Blank or irrelevant</td></tr>
</table>

G.5: Assessment Rubric for Questionnaire-II

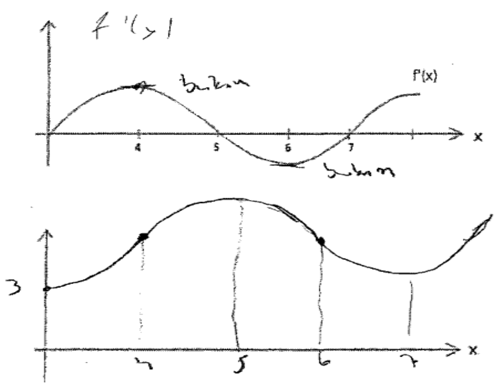
Scoring Rubric for Questionnaire-II

	Correct answer	Scoring	
	Question 1 (*5)		
	Correct answer	Scoring	
	Step1: Calculating the value of f'(4) and determining as the slope of tangent line $f'(x) = 2x - 4$ $f'(2) = 2.4 - 4 = 4$ Step 2: Writing the line equation $y - 3 = 4(x - 4)$ $y = 4x - 13$	Correct solution with steps 1&2 Procedurally correct with some operational mistakes	2
		Only step 1 without step 2	1
		Blank, totaly wrong or irrelevant	0
	Question 2 ((a+b)*10/3)		
	Correct answer	Scoring	
a	Average rate of change of $f(x) = x^2 - 2x$ between x=2 and x=5 is; $\frac{f(5) - f(2)}{5 - 2} = \frac{15 - 0}{3} = 5$	Correct solution	2
		Taking the average $\frac{f'(5)+f'(2)}{2}$ Difference $f'(5) - f'(2)$	1
		Blank, wrong or irrelevant	0
b	Instantaneous rate of change of f at x=6 is; $f'(x) = 2x - 2$ $f'(5) = 8$	Correct solution	1
		Blank, wrong or irrelevant	0
	Question 3 ((a+b+c)*2)		
	Correct Answer	Scoring	
a	Determining increasing and decreasing intervals of the function **Step 1:** Finding the derivative function $f'(x) = x^2 - 4x + 3$ $f'(x) = (x - 1)(x - 3)$ **Step 2:** Deciding the positive and negative intervals and determining increasing and decreasing intervals. f function is increasing in the intervals $(-\infty, 1) \cup (3, \infty)$ f function is decreasing in the intervals (1, 3)	Correct solution with steps 1&2 Procedurally correct with some operational mistakes	2
		Only step 1 is correct	1
		Blank, totaly wrong or irrelevant	0
b	Finding the relative max and min points of the function **Step 1:** Determining critical points. $f'(x) = (x - 1)(x - 3) = 0$ $x_1 = 1 \quad x_2 = 3$ **Step 2:** Checking the function at critical points by using Step 2 in part a. $(1, f(1))$relative maximum point $(3, f(3))$ relative minimum point	Correct solution with steps 1&2 Correct with some operational mistakes	2
		Only step 1 is correct	1
		Blank, totaly wrong or irrelevant	0
c	Finding the inflection point Inflection point is where the second derivative is 0. $f''(x) = 2x - 4 = 0$ $x = 2$	Correct solution	1
		Blank, totaly wrong or irrelevant	0

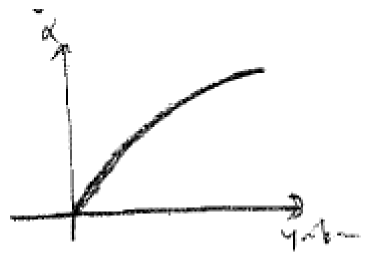
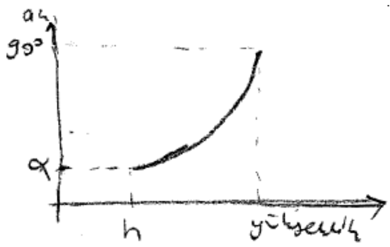
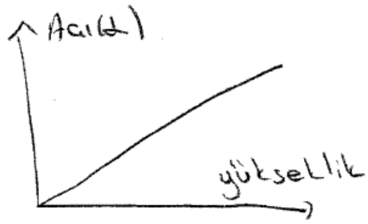
Question 4 (*5)		
Correct Answer	Scoring	
Step 1: Finding the heighest point by using derivative. The heighest point is where the first derivative gets 0. $H(t) = -4t^2 + 48t + 240$ $H'(t) = -8t + 48 = 0$ $t = 6$ **Step 2:** Substituting the value in the function. $H(6) = 384$	Correct solution with steps 1&2 Correct solution with steps 1&2 including some operational mistakes	2
	Only Step 1	1
	Other	0

Question 8 (((a+b+c)*10/6)			
	Correct Answer	Scoring	
a	$S'(1) > 0$ <u>is not true</u>, because the function is decreasing at that point. $S'(1) < 0$	Correct answer with correct explanation	2
		Correct answer without explanation	1
b	$S''(1) > 0$<u>is true</u>, because the graph of function is concave up and the rate of change is gradually increasing .	Correct answer with correct explanation	2
		Correct answer without explanation	1
c	$S(1) > S'(1)$ <u>is true</u>, because $S(1) > 0$ and $S'(1) < 0$	Correct answer with correct explanation	2
		Correct answer without explanation	1

Question 9 ((a+b+c)*10/8)			
	Correct Answer	Scoring	
a	Speed of the car is negative in the intervals; (a,b), (b,c), (e,f) and (f,g)	For each of the intervals (1 point)	4
b	(Negative or positive) Speed of the car is possibly highest at the points; b, d, f	Expressing one or more of these points	1
c	Positive acceleration; (b,c), (c,d), and (f,g)	For each intervals (1 point)	3

Question 11 ((*10/3)		
Correct Answer	Scoring	
	Correct graph with all critical points plotted	3
	Correct smooth graphing without ploting inflection or max-min points	2
	Determining all the points but graphing the curve with sharp transitions at critical points Incorrect graph having a few correct features	1
	Totally wrong or irrelevant graph	0

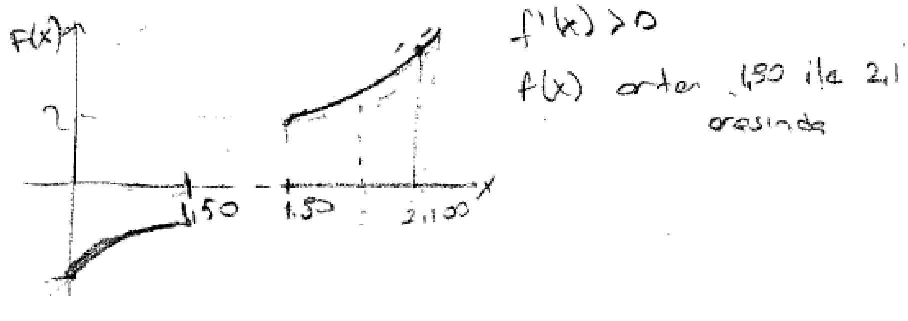
Rubric for Question 5 (*10/3)

Scr		Criteria and Cases
3	Correct	Correct graph with sophisticated verbal or other form of mathematical explanation. *Case: The angle of camera increases at a decreasing rate per unit change in the heigt.*  Correct explanation without graph or correct graph without explanation
2	Partially Correct	Partially correct explanations and/or partially correct graphs emphasizing the changing rate of change. *Case: Kamera açısı, uzay mekiği fırlatıldıktan sonra sürekli artmak zorunda. 90 derece olduğunda ise gözden kaybolana kadar uzay mekiğini gözlemleyebilir.*  Graphs (not "time" dependent) with explanations that emphasizes the changing nature of rate of change, but indicating an inverse concavity (Concavity confusion).
1	Including a few correct parts	Expressing a linear relationship (not "time" dependent) **with** graph or emphasizing only the direction of change without mentioning the intensity (changing nature) of change and with graph. *Case: Uzay mekiği yerden yükseldikçe, kamera açısı da artacaktır.* $\tan\alpha = \frac{h}{a}$ *olduğundan, yükseklik ve açı doğru orantılı olarak değişir.* 
0		Blank or irrelevant answer.

Rubric for Question 6 (a+(b+c)*4/3)

Part a		
Score		Criteria and Cases
1	Correct	Understanding the function and interpreting the expression $S(32) = 6$ by its variables and units. *Case:Oksijenin 32 santigrad derecedeki su içerisinde çözünürlüğü 6 mg/l dir.*
0		Blank or Irrelevant
Part b (*4/3)		
Score		Criteria and Cases
3	Correct	Average rate of change in the solubility of oxygen in the given tempretature intervals. *Case: 40 C^0 ile 8 C^0 sıcaklıkları arasında oksijenin su içerisinde çözünürlüğündeki değişim oranı*
2	Partially	Change in the solubility or average change in the solubility without mentioning "rate". *Case: 40 C^0 ile 8 C^0 sıcaklıkları arasında oksijenin su içerisinde çözünürlüğündeki değişim.*
1	Slope	Slope without (average) rate of change and without units $m = \frac{S(40) - S(8)}{40 - 8}$
0		Blank or Irrelevant
Part c (*4/3)		
Score		Criteria and Cases
3	Correct	(Instantaneous) rate of change in the solubility at the given temprature (10centigrate) *Case: 16 C^0 sıcaklıkta oksijenin sudaki çözünürlüğünün anlık değişim oranı* Average rate of change in the solubility at the temperature of 10 centigrate *Case: 16 C^0 sıcaklıkta oksijenin sudaki çözünürlüğünün ortalama değişim oranı*
2	P.C	Change in the solubility at the given temperature without mentioning "rate". *Case: 16 C^0 sıcaklıkta oksijenin sudaki çözünürlüğünün değişimi*
1	Slope	The value of slope of tangent line at that point. *Case: Çözünürlük-Sıcaklık grafiğinde 16 C^0 sıcaklık değerine karşılık gelen noktadan çizilen teğet doğrusunun eğimi.*
0		Blank or Irrelevant

Rubric for question 7

Scr		Criteria and cases
3	Correct	Applying the difference quotient rule for small intervals on the given data from both left and right sides. And then deciding the best approximation by using mean value theorem. Case: $m_1 = \frac{F(2{,}1)-F(2)}{2{,}1-2}$ $m_2 = \frac{F(2)-F(1{,}9)}{2-1{,}9}$ $F'(2) \cong \frac{F(2{,}1)-F(1{,}9)}{2{,}1-1{,}9}$ is the best approximation Applying the difference quotient rule for small intervals on the given data from left or right sides, and reporting one of these values as the best approximation. Case: $\mathrm{F}'(2) \cong \frac{\mathrm{F}(2{,}1)-\mathrm{F}(2)}{2{,}1-2}$ $\mathrm{F}'(2) \cong \frac{\mathrm{F}(2)-\mathrm{F}(1{,}9)}{2-1{,}9}$
2	Partially	Applying difference quotient rule on a large intervals. Case: $F'(2) \cong \frac{F(2)-F(1)}{2-1}$ Taking the averages of slopes of secant lines from left and right sides Case: $m_1 = \frac{F(2{,}1)-F(2)}{2{,}1-2}$ $m_2 = \frac{F(2)-F(1{,}9)}{2-1{,}9}$ $F'(2) \cong \frac{m_1+m_2}{2}$
1	A few correct parts	Transfering the data on graph and mentioning a positive slope of tangent line. Reporting only the positive sign of derivative at that point by determening the increasing nature of the data. 
0		Blank or irrelevant

Rubric for Question 10

Scr.		Criteria and Cases
3	Correct	Explaining both graphs by rate of change *Case: The pressure is decreasingat an increasing rate with respect to volume.* Explaining both graphs by unit per unit comparison. Comparing the increments in the output variable with respect to unit increments in the input variable ($\frac{V_2-V_1}{2-1} > \frac{V_3-V_2}{3-2}$) *Case: The pressure is decreasingly decreasingper unit volume.* Explaining by the multiplicative rate of change, e.g, Comparing the ratios of the simultaneous values of both variables ($\frac{\Delta P_2}{\Delta P_1} < \frac{\Delta V_2}{\Delta V_1}$)
2	Partially Correct	Thinking within one measure space. Explaining the changing nature of dependent variable without mentioning about the independent variable. *Case: The pressure is decreasingly decreasing (Basınç azalarak azalmaktadır).* Explaining by using the motion terminologies (speed, acceleration). *Case: The speed of decrease in pressure is getting smaller and smaller.* Concavity Difficulty. Explaining by rate of change or unit per unit thinking, but confusing the concave up or down property of the original curve (e.g, although the curve is concave up, the explanation is for a concave down graph) , Inconsistency between explanation and graph
1	A Few Correct Parts	Uncoordinated way of thinking, looking changes in both variables independently *Case: Hacim yavaş yavaş artarken, alan hızlı artmaktadır* Partially correct time based explanations. Time based explanations involving RC or unit per unit thinking (Emphasizing the changing rate of change, but time dependent) *Case: Basınc birim zamanda azalarak azalmaktadır.* Lineer thinking, expressing only the direction of change *Case: Basınç hacimle ters orantılı olarak azalmaktadır.*
0		Blank or irrelevant

APPENDIX H

REFLECTION PAPER GUIDE

You are expected to write a detailed reflection report in which the group study process is explained by proving examples. You can also add graphs, tables and equations to your reflection reports that you used in the group solution. You can use the following list of questions for preparing your report. You do not have to follow the given sequence of the questions, but try to answer all of the questions. You can also write your ideas and critics (if any) that is not considered by these questions.

1. The definition of the problem situation:

   What was the problem that you studied on? What was the aim?

2. Your personal ideas before starting to group study.

   *What did you think about the problem? Could you understand the problem? What was the first solution method that come into your mind (**indicate even if it is wrong**)? Did you think in a way that I can (or cannot) solve the problem? Why did you think in that way? etc.*

   In the 3rd, 4th, and 5th questions your thoughts are asked about the group solution process. Please try to answer these questions in detail so that reflects the group discussion process. You can emphasize steps that you (as group members) go over, different approaches appeared during group discussion and how the final solution is decided.

3. Explain problem solution process and your ideas about the process. Could you explain in detail your groups' solution process from beginning to the end and different way of thinking emerged during this process? (***indicate even if they were wrong***)

   a. How did you started to the solution?

   b. How did you analyze the problem situation? What were the difficulties (if any) that you faced during the solution process? Were there any points that you get stuck? What were they? What did you do to overcome these points?

   c. What were your assumptions that you considered? How did you determine these assumptions? What factors influenced your decisions (group discussions, previous knowledge etc.)?

   d. What were the mathematical concepts, ideas, and strategies that you used during the problem solving process?

   e. How did you utilize different mathematical representations (graphs, tables, pictures, equations, etc.) during the solution and verification phases?

4. How did you change your way of solution when you could not reach a solution? Did you check you solution steps in the process? Explain.

5. How do you evaluate your solution? How can you explain validity of your solution and its usability in other real situations?

6. What did you learned after solving this question? How do you evaluate your performance in this work? Explain.
7. When you consider other groups' way of solutions, if necessary how do you develop your own solution? Explain.
8. What were the mathematical ideas and concepts covered with this problem?
    a. Did you learn a new idea (concept), or all the ideas were known for you?
    b. If the ideas (concepts) were already known, is there any change with these ideas? What kinds of change occurred?
9. ***If you look from a teacher perspective;***
    a. If you apply this problem in a classroom context, what objectives do you expect students to acquire?
    b. What possible solution methods can students provide?
    c. How do you apply this problem in a classroom context?
    d. In such a classroom application of this problem;
        i. Where and what kind of difficulties students may live?
        ii. What kinds of mistakes do you expect from students?
    e. What could you do to overcome the difficulties and mistakes that students have?
10. Did you use any technology while solving the problem? What can you say about the advantages and (or) disadvantages of technology in such situations?
11. What do you think about the group work? Do you think that the group work was fruitfulfor this activity? How and in what ways? How your solution can be different if you solve the problem individually?
12. Can you compare and contrast this problem with the problems that you encounter up to now?

APPENDIX I

INTERVIEW QUESTIONS

Sorular

1. Bu haftaki soru ile ilgili genel olarak ne düşünüyorsunuz?
2. Problemin çözüm süreci ve bu süreç hakkındaki düşüncelerinizi almak istiyoruz. Çözümün başından sonuna kadar geçtiğiniz süreçleri anlatır mısınız
    a. *Problem durumunu tam anlayabildiniz mi? Eğer anlayamadıysanız, anlamak için neler yaptınız?*
    b. *Problemin çözümü için ilk aklınıza gelen yol neydi* (***yanlış da olsa belirtiniz***)*?*
    c. Problemi formüle ederken problem durumu ile ilgili dikkate aldığınız durumlar ve varsayımlar nelerdi? Bu varsayımları belirlemede ne etkili oldu?
    d. Matematiksel kavramlar, fikirler ve kullandığınız stratejiler nelerdi?
    e. (Raporlar ve çözüm kâğıtları incelendikten sonra) Şu yöntemi kullanmışsınız, neden bunu kullandınız? …
    f. Problem üzerinde uğraşırken karşılaştığınız zorluklar nelerdi? Bunları aşmak için ne yaptınız?
    g. Çözüm sürecinde grubunuzda ortaya çıkan farklı düşünceleri anlatır mısınız?
    h. Derste geliştirdiğiniz çözüm ile ilgili şuanda ne düşünüyorsunuz? Diğer grupların çözüm yaklaşımlarını nasıl değerlendirdiniz. Kısaca açıklar mısınız?
3. Size göre çözüm sürecinizi olumlu ya da olumsuz yönde etkileyen faktörler nelerdi?
4. Bu derste daha önce yapılan etkinlikler bu haftaki çözüm yaklaşımınıza nasıl katkı sağladı?
5. Bu probleme getirdiğiniz çözümü ve matematiksel fikri benzer başka durumlara genelleyebilir misiniz? Örnek verir misiniz?
6. Tüm grup çözümlerini de göz önüne aldığınızda ve bu problemde ön plana çıkan matematiksel kavram ve fikirleri düşündüğünüzde;
    a. Yeni bir kavram ya da bir fikir öğrendiniz mi?
    b. Bu kavramlarla ilgili sizin bilgilerinizde bir değişiklik oldu mu?
7. Bu etkinlikte grup çalışmasının sizin açınızdan verimli olduğunu düşünüyor musunuz?
    a. Soruyu bireysel çözmeye çalışsaydınız çözümünüzde nasıl bir farklılık olurdu?
8. Bu problemi bu güne kadar gördüğünüz problem türleri ile benzerlikleri farklılıkları açısından değerlendiriniz. *Etkinlik sonrası düşünce raporunda yetersiz veya eksik ifadelerin anlaşılması için sorulacak*).
9. *Bir öğretmen gözüyle bakmanız gerekirse;*
    a. Bu problemi sınıf ortamında uygularsanız öğrencilerin hangi kazanımlara ulaşmasını beklersiniz?
    b. Bu problemi sınıf ortamında nasıl uygularsınız?
    c. Bu soruya öğrencilerin getireceği çözüm yaklaşımları neler olabilir?
        i. Nerelerde ve ne tür zorluklar yaşayabilirler? Ne tür hatalar yapmasını beklersiniz?
    d. Öğrencilerin yaptıkları hataları ya da yaşadıkları zorlukları aşması için neler yaparsınız?
10. Eklemek istediğiniz başka bir şey var mı?

APPENDIX J

DATA ANALYSIS FRAMES

J.1: Covariational Reasoning Frame

Table 19: Coding schema used for analyzing covariational reasoning (Extended)

Categ.	Sub-Categories	Abbrev.
Identifying the variables (IV)	*Thinking with primary variables* Determining dependent and independent variables and labeling on graph	IV-PV
	Thinking with secondary variables Considering unnecessary variables as independent variable (time as independent variable)	IV-SV
	Thinking input and output variables in reverse order Using dependent and independent variables reversely	IV-RO
Way of coordinating the variables (WOC)	*Uncoordinated way of thinking* Using an unnecessary variable as an independent variable explicitly	WOC-UC
	Indirect Coordination Using an unnecessary variable as an independent variable implicitly ("Radius" of cross-section as an implicit independent variable)	WOC-IC
	Direct Coordination Directly coordinating the primary variables and expressing a linear relationship	WOC-DC
	Direct and Systematic Coordination Systematically changing input variable and observing simultaneous variation	WOC-DSC
Quantifiying the variation in rate of change (VRC)	*Gross Quantification* Deciding perceptually without providing a mathematical justification	VRC-GQ
	Extensive Quantification with	
	Additive comparison Changing the input variable with constant and equal increments and additively comparing the simultaneous change in the output variable	VRC-EQAC
	Unit per unit comparison Uniformly changing the input variable, and observing the simultaneous change in the output variable	VRC-EQUC
	Multiplicative comparison Changing the input variable with constant and equal increments and multiplicatively comparing the simultaneous change in the output variable	VRC-EQMC

J.2: Coding Schema for Analyzing Conceptions of Rate of Change

Table 20: Coding schema used for analyzing conceptions of rate of change (Extended)

Cat.	Sub-categories	Abbrev.
Conceptions of Rate of Change	*Difficulty in giving meaning to Turkish term of rate of change* Percentage; the ratio between the successive values of a variable. e.g., $\frac{x_2}{x_1}$ or $\frac{x_2 - x_1}{x_1}$	TERC-D
	Perceptual Using expressions as "seems faster than…" without forming a functional relationship between variables.	PR
	Amount of change in the output variable (there is no ratio) Focusing only change in the output variable without considering the input variable, additively comparing the changes in the output variable while keeping changes in the input variable constant, but not forming a ratio based relation. e.g., $x_2 - x_1 = \Delta x_i$ and $x_3 - x_2 = \Delta x_k \Rightarrow \Delta x_i \leq \Delta x_k$	AC
	Ratio-Based Reasoning Considering rate of change as a *ratio* either in the form of multiplicative rate of change or additive rate of change. Thinking with *unit per unit comparison* (e,g. the output variable increasingly increases per input variable) e.g., $\frac{\Delta y}{\Delta x}$	RBR
	Average Rate of Change (ARC) *True Conceptions of ARC;* The ratio between change in the output variable and change in the input variable in an interval (Difference Quotient Rule (DQR)). $\frac{f(x_2) - f(x_1)}{x_2 - x_1}$, Slope (of secant line).	ARC-T
	Misconceptions of ARC ; ARC of Derivative Function ($\frac{f'(x_2) - f'(x_1)}{x_2 - x_1}$) The difference between derivatives of the end points ($f'(x_2) - f'(x_1)$) The average of values of function at the interval $\frac{f(x_1) + f(x_2) + \ldots.. f(x_n)}{n}$	ARC-M
	Instantaneous Rate of Change (IRC) *True Conceptions of IRC* Slope of tangent line (at the given point), Relating it with difference quotient rule (DQR) with limiting notation	IRC-T
	Misconceptions of IRC Change in the dependent variable,	IRC-M

Cat.	Sub-categories	Abbrev.
	Stating IRC as if it is ARC	
Forming Connections Between Different Representations	*Geometric Slope*	
	Using slope procedurally (GS-P) Difficulty in relating it with rate of change and difference quotient	GS-P
	Using slope by being aware of its different interpretations (GS-A) Relating the slope with difference quotient, rate of change, and symbolic derivative	GS-A
	Difference Quotient Rule	
	Using difference quotient procedurally Difficulty in relating it with rate of change, slope, and derivative	DQR-P
	Using difference quotient by being aware of its different interpretations Relating the difference quotient with slope, rate of change, and symbolic derivative	DQR-A
	Reasoning with Physics concepts Using definition of speed or acceleration as reference	PC

J.3: Coding Schema for Analyzing the Ways of Reasoning on the Graphical Connection between a Function and Its Derivative

Table 21: Coding schema used for analyzing graphical understanding of derivative (Extended)

Cat.	Sub-Categories	Abbrev.
Interpreting Graphs	*Non-mathematical verbal explanation* Explaining by using the physical attributes of graph without considering the variables (concave-up, concave-down)	VA-NM
	Inconsistent verbal explanation Explaining the curve by taking into account the dependent and independent variables, but indicating the variation in rate of change incorrectly (drawing a concave-up decreasing graph, but saying "the dependent variable increasingly increases with respect to independent variable")	VA-I
	Consistent verbal explanation Explaining the curve by taking into account the dependent and independent variables and indicating the variation in rate of change correctly	VA-C
Drawing Graphs	*Transition between different curves*	
	Transition with sharp corners At the transition points where the tank figures change form, the transition between curves can be drawn so as producing sharp corners (not differentiable)	TR-SH
	Transition with (nearly) smooth corners At the transition points, the curves can be drawn nearly smooth. Smoothness here does not mean differentiability; it means that the rate of change just before the point and just after point is consistent with the original situation.	TR- SM
	Critical Points	
	Determining all *max-min* points and reflecting on both (derivative and anti-derivative) graphs *successfully*	CP-MS
	Inability to determine *max-min* points or reflecting them on both (derivative and anti-derivative) graphs	CP-MI
	Determining *inflection point successfully* and reflecting it on both (derivative and anti-derivative) graphs, and being able to define the inflection point contextually	CP-IPS
	Inability or difficulty to determine the *inflection point* or reflecting it on the graphs, defining the inflection point as the middle point on the curve	CP-IPI

Cat.	Sub-Categories	Abbrev.
	Increasing and Decreasing Intervals (INT)	
	Deciding by using the *sign of derivative function*, or by using the positivity or negativity of slope of tangent line.	INT-SDF
	Deciding *intuitively* without systematically using sign of derivative function, or sign of slope of tangent line	INT-IN
	The area under the derivative curve (AR)	
	Deciding the heights of summits and valleys by using the area under curve (*integration*)	AR-I
	Deciding intuitively, *without considering the area* under curve, for the heights of summits and valleys.	N-AR
	Reversing between rate and amount functions (in context) (RV)	
	Slope-based reasoning without realizing derivative-antiderivative relationship	RV-SBR
	Realizing derivative-antiderivative relationship and using the procedural knowledge	RV-RD

APPENDIX K

CONSENT FORM

GÖNÜLLÜ KATILIM FORMU

Bu ders, Doç. Dr. Ayhan Kürşat Erbaş tarafından yürütülen "**Ortaöğretim Matematik Eğitiminde Matematiksel Modelleme: Hizmet İçi ve Hizmet Öncesi Öğretmen Eğitimi**" projesi kapsamında içeriği oluşturulmuş matematiksel modelleme konusunda hizmet öncesi öğretmen eğitimini amaçlamaktadır. Matematik öğretmen adaylarının matematik öğretiminde matematiksel modelleme kullanımı ile ilgili bilgi, beceri ve tutumlarını ortaya çıkarma ve bunlardaki gelişimi ve değişimi tasarlanan hizmet öncesi eğitim programları aracılığıyla inceleme proje çalışmasının konularını oluşturmaktadır. Bu amaçlar için tasarlanan ders kapsamında 14 hafta sürmesi planlanan çalışma süresince (i) modelleme testi, (ii) türev testi (iii) anket, (iv) kavram haritası, (v) modelleme etkinlikleri için grup çalışma raporları, (vi) devam soruları için bireysel çözüm kâğıtları, (vii) ses kayıt ve video kayıt cihazlarıyla desteklenmiş gözlemler, (viii) görüşmeler, (ix) etkinlik sonrası düşünce raporları, (x) gruplarca hazırlanan modelleme soruları ve bu soruların uygulama planları (xi) öğretmen adaylarının sunumları (mikro-öğretim) temel veri kaynakları olacaktır. Bu kapsamda türev konusu ile ilgili toplanacak veriler Araş.Gör.Mahmut Kertil'in doktora tez çalışmasında kullanılacaktır.

Çalışma süresince toplanacak veriler tamamıyla gizli tutulacak ve sadece araştırmacılar tarafından değerlendirilecektir. Elde edilecek bulgular tez çalışmasında ve bilimsel yayımlarda kullanılacaktır. Çalışmaya katılım tamamıyla gönüllülük temelindedir. Çalışma süresince katılımcılar için potansiyel bir risk öngörülmemektedir. Ancak, katılım sırasında farklı amaçlarla toplanan veya alınan dersin gerekleri olarak toplanacak verilerin bilimsel çalışma ve tez çalışması amaçları çerçevesinde kullanılmamasını isteyebilirsiniz. Bu durum ders performansınızın değerlendirilmesinde kesinlikle negatif bir durum oluşturmayacaktır.

Çalışma hakkında daha fazla bilgi almak için ODTÜ Eğitim Fakültesi Ortaöğretim Fen ve Matematik Alanları Eğitimi Bölümü öğretim üyeleri Doç.Dr Ayhan Kürşat Erbaş (kursat@gmail.com), Y. Doç. Dr. Bülent ÇETİNKAYA (Tel: 210 3651; e-posta: bcetinka@metu.edu.tr) ve doktora öğrencisi Mahmut Kertil (e-posta: mahmutkertil@yahoo.com) ile iletişim kurabilirsiniz. Bu çalışmaya katıldığınız için şimdiden teşekkür ederiz.

Bu çalışmaya tamamen gönüllü olarak katılıyorum ve istediğim zaman yarıda kesip çıkabileceğimi biliyorum. Verdiğim bilgilerin bilimsel amaçlı yayımlarda kullanılmasını kabul ediyorum. (Formu doldurup imzaladıktan sonra uygulayıcıya geri veriniz).

İsim, Soyad	Tarih	İmza	Alınan Ders
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CURRICULUM VITAE

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PUBLICATIONS

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