AN INVESTIGATION OF PRE-SERVICE SECONDARY MATHEMATICS TEACHERS' KNOWLEDGE OF STUDENTS' THINKING THROUGH ANALYZING STUDENT WORK

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I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

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ABSTRACT

AN INVESTIGATION OF PRE-SERVICE SECONDARY MATHEMATICS TEACHERS' KNOWLEDGE OF STUDENTS' THINKING THROUGH ANALYZING STUDENT WORK

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The purpose of this study was to investigate pre-service secondary mathematics teachers' knowledge of students' thinking within an undergraduate course context. The participants were twenty five pre-service mathematics teachers enrolled in this undergraduate mathematics education course which aims to develop pre-service mathematics teachers' knowledge in and about mathematical modeling in teaching and learning mathematics. The design of the study contained 4 two-week cycles. In the first week of each cycle, pre-service teachers worked on a non-routine mathematical task to develop their own solutions. The following week, they analyzed and discussed actual high school students' solutions produced on the same task through students' solution papers and video episodes, where students presented their solutions to their classmates. The data were collected through pre-service teachers' reflection papers, note-taking sheets, individual and focus group interviews, and observation notes. The findings showed many predictions of pre-service teachers were not consistent with students' actual ways of thinking, especially, at the beginning of this research. However, great portion of pre-service teachers' predictions have become consistent with them over time. In addition, analyzing students' works helped pre-service teachers value students' ways of thinking that was different from theirs, and develop process-oriented criteria to analyze students' works. Furthermore, the findings showed that pre-service teachers were able to interpret students' ways of thinking in three different ways "describing, questioning and explaining". This study revealed that analyzing students' works containing actual students' solutions for non-routine tasks provided pre-service mathematics teachers with rich opportunities to learn about students' thought processes in mathematics.

Keywords: Mathematics education, students' ways of thinking, pre-service mathematics teachers, mathematical modeling, practice-based instructional materials

ÖZ

ORTAÖĞRETİM MATEMATİK ÖĞRETMEN ADAYLARININ ÖĞRENCİ DÜŞÜNME ŞEKİLLERİ BİLGİLERİNİN ÖĞRENCİ ÇALIŞMALARINI İNCELEME YOLUYLA ARAŞTIRILMASI

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Bu çalışmanın amacı, lise matematik öğretmen adaylarının öğrenci düşünme şekilleri bilgilerini bir lisans dersi kapsamında incelemektir. Bu çalışmanın katılımcıları, öğretmen adaylarının matematik öğrenimi ve öğretiminde matematiksel modelleme bilgilerini geliştirmeyi amaçlayan matematik eğitimi dersine kayıtlı olan yirmi beş öğretmen adayıdır. Çalışmanın tasarımı dört tane iki haftalık döngüden oluşmaktadır. Her bir döngünün ilk haftasında matematik öğretmen adayları rutin olmayan matematik problemleri üzerinde çalışmış ve kendi çözümlerini geliştirmişlerdir. Sonraki hafta, matematik öğretmen adayları lise öğrencilerinin aynı matematiksel problemler üzerinde geliştirmiş oldukları çözümlerini, öğrencilerin çözüm kâğıtları ve öğrencilerin çözümlerini sınıf arkadaşlarına sundukları video görüntüleri aracılığıyla incelemiş ve tartışmışlardır. Çalışmanın verileri öğretmen adaylarının düşünme raporları, not alma kâğıtları, bireysel ve odak grup görüşmeleri ve gözlem notları ile toplanmıştır. Çalışmanın bulguları öğretmen adaylarının öğrenci düşünme şekillerine yönelik tahminlerinin öğrencilerin gerçek düşünme şekilleri ile özellikle çalışmanın başlangıcında tutarlı olmadığını, fakat zamanla bu tahminlerinin büyük bir kısmının tutarlı hale geldiğini göstermiştir. Aynı zamanda, öğrenci çalışmalarını

incelemeleri öğretmen adaylarına kendi düşünme şekillerinden farklı öğrenci düşünme şekillerine değer vermelerine ve öğrenci çalışmalarını analiz etmek için süreç odaklı kriterler geliştirmelerine yardımcı olmuştur. Ayrıca, bulgular öğretmen adaylarının öğrenci düşünme şekillerini "tanımlama, sorgulama ve açıklama" olarak üç farklı şekilde yorumlayabildiklerini göstermiştir. Bu çalışma, rutin olmayan problemler için gerçek öğrenci çözümlerini içeren öğrenci çalışmalarını analiz etmenin öğretmen adaylarının öğrencilerin düşünme süreçlerini anlamalarına zengin imkânlar sağladığını ortaya çıkarmıştır.

Anahtar kelimeler: Matematik eğitimi, öğrenci düşünme şekilleri, öğretmen adayları, matematiksel modelleme, uygulamaya dayalı öğretim materyalleri.

I dedicate this dissertation to my dear family

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TABLE OF CONTENTS

LIST OF TABLES

TABLES

LIST OF FIGURES

FIGURES

LIST OF ABBREVIATIONS

PK	Pedagogy Knowledge
SMK	Subject Matter Knowledge
PCK	Pedagogical Content Knowledge
CK	Curricular Knowledge
PST	Pre-service Teacher
PSTG	Pre-service Teacher Group
SG	Student Group
TBRP	Task Based Reflection Paper
STRP	Students' Ways of Thinking Reflection Paper

CHAPTER 1

INTRODUCTION

There are various views regarding how teaching is conceptualized: “Teaching is an interactive process” (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989), “teaching is a complex problem solving activity” (Carpenter, Fennema, & Franke, 1996), “teaching is an ill-structured domain” (Doerr & Lesh, 2003), “teaching is relational and multidimensional” (Franke, Kazemi, & Battey, 2007), or “teaching is seen an activity involving teachers and students working jointly” (Shulman, 1986). These views tell us teaching includes different kind of conceptions and defining it properly is not an easy task. However, what all these phrases have in common indicates that teaching is an activity for learning and has a complex nature. And, because the intent of mathematics teaching is to promote the learning of mathematics, the improvement of mathematics learning in classrooms is fundamentally related to development in teaching (Jaworski, 2006).

Learning to teach is a long learning process starting informally when teachers were students. Teachers spent many years in mathematics classes as students themselves (Ball, 1988). As the sociologist Dan Lortie (1975) points out, teachers go through a very long “apprenticeship of observation” since they observe how teachers teach during their all K-12 education (as cited in Kennedy, 1999). And, during their K-12 education teachers gain many ideas about subject matter, students and teaching. Therefore, before arriving at formal teacher education, they construct a teaching frame in their mind (e.g., Ball, 1988, 1990; Ball & Cohen, 1999; Kennedy, 1999). Then, when entering formal teacher education programs, teachers gain new ways of thinking about teaching and learning, and improve pedagogical ways of doing, acting and being as a teacher, and a method course is usually the starting point for their learning to teach mathematics (Ball, 1990). Next, they practice as well as improve their learning when they begin in their professions. Classrooms are the fundamental place where teachers have opportunities to engage with students and rigorous mathematics; that is why, learning to teach of teachers takes place in their classroom setting. In their classroom, they develop their knowledge, skills and dispositions they need to teach, which they learnt formally in teacher preparation courses.

In fact, teachers future practice is directly influenced from their frames shaped from their past experiences as students (Kennedy, 1999). The "image of teaching" developed by teachers in their primary and secondary school experiences is mostly quite limited and be problematic; especially, their knowledge about students is vague and mainly based on what they know about themselves (e.g., Ball, 1988). The classrooms observed by teachers during their primary and secondary education (K-12) are mostly conventional classrooms where the teachers' main work is to write problems on the board and to solve it, to make calculations or to prove theorem, and their instructions are mainly textbook-centered and writing on the board is their main works (Davis & Hersh, 1981). That means, they (prospective teachers as students) hardly observe a teacher teaching mathematics by focusing on students' thinking (Ball, 1988). As Ball (1988) emphasized, watching teachers in a conventional classroom does not provide an opportunity to learn how to teach mathematics or how students think. Therefore, their images should be altered during their pre-service teacher education (Kennedy, 1999). If their experiences are not altered during the pre-service teacher education, the likelihood of changes might be later very difficult. With this respect, pre-service teacher education programs has a crucial role in changing of teacher initial frames because it is located between teachers' past experiences as students in classrooms and their future experiences as teachers in classrooms (Kenndy, 1999).

However**,** several researchers (e.g., Ball & Cohen, 1999; Feiman-Nemser, 2001; Hiebert, Morris, & Glass, 2003; Masingila & Doerr, 2002) emphasize that sufficient number high quality reform-based classrooms are not available for pre-service teachers to be able to make these changes. According to Feiman–Nemser (2001) and Lampert and Ball (1998), in many countries, teacher preparation programs face several enduring problems. Many of these teacher preparation programs are conventional. The curriculum of these conventional programs is superficial as well as divides theory and practice. That means, the relationship between subject matter courses and field experiences is weak. Furthermore, in their field experiences, schools do not provide enough opportunities for teacher candidates to work on serious problems of practice and complex nature of teaching and learning. The impact of teacher preparation programs on the development of pedagogical content knowledge of teacher candidates is limited (Lee, Brown, Luft, & Roehrig, 2007; Nilsson, 2008).

According to Ball (1988; 1990) and Kennedy (1999), changing of pre-service teachers' teaching frames should be made by changing the ways their interpretations to particular situations and how to respond them. Therefore, as Crespo (1998) and Kennedy (1999) emphasized, in pre-service teacher education programs, pre-service teachers need a different design and pedagogy to see new ways, think and act. In this respect, Schorr and Lesh (2003) propose creating learning environments, in which pre-service teachers can analyze, interpret and discuss classroom situations, to make powerful interventions in teacher education. Similar to Schorr and Lesh (2003), Lampert and Ball (1999) recommend orienting teacher education programs around investigations of practices of teaching and learning instead of focusing on only providing knowledge and skills for teaching.

As it is previously stated, at universities, providing opportunities to pre-service teachers to work with K-12 students are difficult and demanding. Recently, several mathematics educators (e.g., Ball, 1997; Ball & Cohen, 1999; Son, 2013; Sowder, 2007) suggest the use of documentation of instructional practices such as copies of students' works, videotapes of classroom lessons, curriculum materials and teachers' notes as a way to address this issue. It is certainly possible that teachers can learn subject matter, knowledge of children, learning and pedagogy in a variety of courses and workshops; however, since such knowledge is situated in practice and cannot be learned entirely in advance or outside of practice, it should be learned in practice (Feiman-Nemser & Remillard, 1996). And, such kind of educational materials such as students' written work, videotapes of classroom lessons, curriculum materials and teachers' notes would provide pre-service teachers with opportunities for in depth exploration of teaching process and students' thinking in real classroom settings (Smith, 2001).

1.1 What do Teachers Need to Know for Teaching Mathematics (better)?

The knowledge base teachers need to posses deserves a special attention, since it influences both what the teachers learn and how they teach (Ponte & Chapman, 2002). Analyzing teacher knowledge has a critical value to understand teaching process, to evaluate teacher competence and to bring about fundamental change in how teachers teach (Carpenter, Fennema, & Franke, 1996). Various researchers indicate that there are several dimensions of teacher knowledge such as knowledge of and about content, knowledge of educational contexts, knowledge of educational ends and values, knowledge of students' understanding, errors and misconceptions and knowledge of curriculum (e.g., Ball, 1988; Grossman, 1990; Shulman, 1986; 1987). Among these, one of the fundamental knowledge base for teaching (mathematics) is the pedagogy as it deals with general principles of education such as learning theories, psychological, sociological, classroom management and assessment aspects (Liljedahl, 2009). On the other hand, knowledge of subject matter is a cornerstone of teaching and includes knowledge of facts, rules and concepts in a certain domain that are to be learned by students. In order to teach mathematics for understanding, teachers' subject matter knowledge (SMK) need to comprise the meanings and connections (Ball, 1997; Ball & Cohen, 1999). However, pedagogy and subject matter are not the only professional knowledge base that teachers need (Sowder, 2007). In order to teach the subject matter to the students, teachers also need to know about (their) students.

> A teacher who knows little of the content, or knows it in only narrow and rigid ways, may miss children's often wondrous insights. But paradoxically, a teacher with a considerable depth of knowledge may fail to hear the nonstandard perspective, the novel insight, listening only for 'the answer' (Ball, 1997, p.775).

Teachers need to know not only about their students' cultural and socio economical background, but also about their experiences, prior knowledge and understanding (e.g., what they are likely to find interesting and to have trouble with in particular domains) (Ball, 1997). As Ramsey (1991) stated, teachers should know students not only in general terms but also in moment (as cited in Ball, 1997). Knowledge about students is referred to by Shulman (1986, 1987) as one aspect of teacher pedagogical content knowledge (PCK). According to Shulman (1986), PCK includes "an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to learning of those most frequently taught topics and lessons" (p.9). As Ball and Cohen (1999) indicated, knowing students is not a simple matter; that is, knowing students requires knowing students' ideas about academic subjects and becoming insightful in listening to and interpreting them. So, teacher would need to see not only students as more capable of thinking and reasoning but also the subject matter through the eyes, hearts and minds of the students (Ball, 1997).

1.2 Knowledge of Students' Thinking

The efforts in mathematics education throughout the world stress the necessity and importance of "understanding of student thinking" (e.g., Ball & Cohen, 1999; Carpenter et al., 1988; Chamberlin, 2002; Even & Tirosh, 1995; Shulman, 1986). Teachers' understanding and interpretation of students' thinking provides many benefits to make appropriate instructional decisions before, during and after teaching. For instance, teachers may understand and interpret the mathematics from students' point of view; they may develop and select appropriate activities/worthwhile mathematical tasks for students; they may change their instruction from teacher centered didactical instruction to student centered problem solving instruction (Chamberlin, 2002; Even & Tirosh, 1995; Smith, 2001). On the other hand, Schorr and Lesh (2003) cited that

> The teacher who has insight into student's thinking can appreciate the sense in students' interpretations and representations of mathematical ideas, and can deal with them constructively. By contrast, the teacher who lacks understanding of student's thinking is left in a kind of pedagogical delusional state: the teacher understands a concept in a certain way, thinks that concept is being taught to the students, but the student is either not learning it at all, or in fact learning entirely different concept from one of the teacher has assumed (p.144).

Nevertheless, various studies (e.g., Bergqvist, 2005; Kılıç, 2011; Nathan & Koedinger, 2000; Tirosh, 2000) indicate inconsistency between how students think about particular topics in mathematics and what both teachers and pre-service teachers know about those. For example, Nathan and Koedinger (2000) stated that there were discrepancies with ways of students' algebraic reasoning and teachers' predictions about that. Similarly, examining teachers' expectation of students' reasoning and performance on conjecture tasks Bergqvist (2005) found that the teachers' tendency was to underestimate the students' reasoning levels. Furthermore, Kılıç (2011) investigated pre-service mathematics teachers' knowledge of students in terms of students' misconceptions and the sources of students' errors in a method course. The findings of her study revealed that pre-service teachers had difficulty in determining students' misconceptions and the source of the errors, and also producing effective ways to eliminate these misconceptions.

Because of the gaps between teachers/pre-service teachers' knowledge about students' thinking and students' actual thinking, in latest efforts of mathematics education, it is often emphasized that pre-service teachers and teachers need to increase their awareness across students' thinking. In fact, recently, with the increase in the importance of teachers' understanding of students' thinking, many professional development projects such as Cognitively Guided Instruction (CGI) (Carpenter, Fennema, & Franke, 1996), Multi-tiered Teaching Study (Schorr & Koellner-Clark, 2003), and Integrating Mathematical Assessment (IMA) (Gearhart & Saxe, 2004) has focused on students' thinking to foster teacher knowledge of student thinking . However, not only professional development programs for teachers but also teacher preparation programs for pre-service teachers should help them to increase their awareness across students' various thinking, and common correct and incorrect cognitive processes (e.g., Klein & Tirosh, 1997; Lampert & Ball, 1998; Sowder, 2007; Tirosh, 2000). It is suggested that the professional development projects designed for teachers can also be productive component of teacher preparation programs (Sowder, 2007). For example, Sowder's (2007) suggestion is that the CGI model, which is based on using the research-based knowledge of students' ways of thinking to develop teachers' instruction, can also be applied successfully with pre-service teachers. On the other hand, Ball and Cohen (1999) suggest that because knowledge of children, learning and pedagogy are situated in practice; therefore, pre-service teachers would investigate documentation of practice such as copies of students' works, videotapes classroom lesson, teacher's notes and curriculum materials. Similarly, in order to create opportunities for pre-service teachers to view different kind of instructions, Wilson and Ball (1996) recommend providing models of teaching, where video cases might discuss in teacher education class.

Vacc and Bright's (1999) research, based on CGI model, can be displayed as one of the examples in which pre-service teachers were provided an opportunity to analyze, interpret and discuss classroom situations. Another example is the study carried out by Masingila and Doerr (2002). In their study, they used the multimedia case materials (e.g., a video of class lesson, students' written work and video journal of the teacher's reflection) to understand how these materials can support pre-service teachers' development of pedagogical understanding, and they found that use of multimedia case materials not only enabled pre-service teachers to examine the teaching issues more deeply but also provided them a common experience to observe and interpret these teaching issues.

As it is seen in the research above, there are several kinds of ways for helping teachers/pre-service teachers to understand, to make sense of and to attend students' thinking. Case discussions, curriculum materials, students' oral or written explanations, students' drawings on the board, solutions and reflection produced by students during instruction or video-tapes of classroom lessons (Ball, 1997) can be used to create a context in which pre-service teachers examine information from a classroom. However, Ball (1997) points out that different from case and curriculum materials, investigating artifacts of teaching and learning from a classroom such as video tapes of classroom or students' written work may provide much more benefits to pre-service teachers because these materials offer direct information from classrooms, and they are "unnarrated as well as unstructured/uninterpreted" (p.811-812). In this way, teachers are placed in realistic positions for understanding and interpreting students' thinking. Teachers obtain an opportunity to watch students' body motion, hands or drawings, to listen their expressive talks, or to investigate students' puzzling comments or unclear writings; in this way, they make sense if what students are saying, thinking or writing (Ball, 1997).

1.3 Problem Statement and Research Questions

Knowledge of students' thinking is an important component of teachers' pedagogical content knowledge. However, it is emphasized that teacher education courses have a weak impact on the development of pre-service teachers' knowledge of students' thinking. On the other hand, the studies strongly suggest use of practice based sources (set of materials drawn directly practice) such as video cases from real classrooms, case discussions, curriculum materials, students' oral, written explanations or drawings on the board in the design of teacher education courses in order to support the development of pre-service teachers' knowledge of students' ways of thinking.

Therefore, in this study, I aimed to investigate pre-service secondary mathematics teachers' knowledge of students' thinking within an undergraduate course context where they first worked on non-routine mathematical tasks themselves and then examined actual solutions produced by high school students to these tasks through students' written work and video episodes of students' discussion.

The following research questions guided the study:

1. What do pre-service secondary mathematics teachers predict about students' ways of thinking before they engage in students' works about solutions for non-routine mathematical tasks?

2. What do pre-service secondary mathematics teachers identify about students' ways of thinking while they engage in students' works about solutions for non-routine mathematical tasks?

3. What do pre-service secondary mathematics teachers value in students' solutions produced for non-routine mathematical tasks?

4. How do pre-service secondary mathematics teachers interpret students' thinking as manifested in students' works about solutions for non-routine tasks?

5. What do pre-service secondary mathematics teachers focus in terms of students' ways of thinking in analyzing students' works about solutions for non-routine tasks?

1.4 Significance of the Study

The nature and quality of teachers' practices are closely influenced by the design of the teacher preparation programs (Hiebert et al., 2003). Research emphasizes that central task of pre-service teacher programs should build on current thinking what a teacher needs to know, care about and be able to do in order to promote substantial learning for all students. They have to develop pre-service teachers' not only subject matter knowledge for teaching but also pedagogical content knowledge, specifically, understanding of learners and learners' mathematical learning, to provide repertoire for reform minded teaching, and to form habit and skills necessary for ongoing study of teaching (Feiman-Nemser, 2001). Moreover, pre-service teachers not only are taught for the predictable parts of the practice but also have to be prepared for the unpredictable part of teaching, for example, what to say when a student gives a solution (Lampert & Ball, 1999). To do that, in these programs, pre-service teachers should be provided opportunities to reason in an about practice (Sowder, 2007); in addition, they should learn how to listen, hear and watch of the students and their ways of thinking.

Even though perceived wisdom that conventional teacher preparation programs are weak interventions, they can make a difference. They can prepare pre-service teachers to teach for understanding, and enhance pre-service teachers' knowledge of learner and learner's thinking if they designed the teacher education programs around an explicit and thoughtful mission (Feiman-Nemser, 2001). As suggested by the relevant literature, one possible way of improving teacher preparation is that "samples of authentic practice" taken from real classrooms would become a curriculum for teacher education programs, which provide opportunities pre-service teacher for observing and studying children's thinking about mathematics (Ball & Cohen, 1999; Smith, 2001; Sowder, 2007). In teacher education courses, pre-service teachers can begin developing observation, interpretation and analysis skills by analyzing documentation of practice such as copies of students' works, videotapes classroom lesson, teacher's notes and curriculum materials, comparing different curricular materials, and interviewing students to uncover their thinking (e.g., Ball & Cohen, 1999; Feiman-Nemser, 2001; Son, 2013; Sowder, 2007; Wilson & Ball, 1996).

In the light of these issues, in this research, pre-service teachers are provided a learning environment within an undergraduate course context in which they have an opportunity to work on samples of authentic practice materials. Therefore, this study suggests that with these initial experiences, pre-service teachers may understand how students think and reason mathematically. Furthermore, pre-service teachers are very likely to develop their pedagogical content knowledge by examining, conjecturing and exploring students' alternative mathematical thinking by working on students' works from real classroom settings. Moreover, they may establish their knowledge on students' thinking, and they may develop competencies as mathematics teachers before beginning their profession. They also may start feeling what it is like to be a teacher (Oliveira & Hannula, 2008). So, this study may show one of the basic steps to prepare teachers for teaching mathematics better, and the design of the course may be one of examples of reform based classrooms.

The results of this study has potential to be interest of (mathematics) teacher educators since they would understand how pre-service teachers learn about and from students' ways of thinking in the context of analyzing student work. Therefore, the design of this course may lead mathematics teacher educators in universities to design their courses. In addition, the knowledge generated may also help rearrangement existing pre-service teacher education curriculums in order to make teacher preparation programs more effective. Furthermore, this study also contributes to the growing body of research literature on use of students' actual work from real classroom setting in teacher education courses because few studies have explored the use of actual students' works to examine and develop pre-service teachers' knowledge for teaching in these courses.

1.5 Definitions of the Major Terms

Pre-service teachers: In this study, pre-service teachers are students who have not yet completed their training to be a teacher, but they are at their middle and last stages (3rd, 4th and 5th) enrolled in five year secondary mathematics education program.

Students' Ways of Thinking:

(i) Students' range of solution approaches/ways applied to solve given non-routine tasks (ii) Students' errors, difficulties or misunderstandings experienced in given non-routine tasks (e.g., Carpenter et al., 1988; Chamberlin, 2002; Fennema, Franke, Carpenter, & Carey, 1993; Kazemi & Franke, 2004; Koellner-Clark & Lesh, 2003).

Solution approach:

In this study, solution approach refers students' thought process that they would use to arrive at a solution instead of a specific method used to find a solution. That means, for example, probable mathematical concepts that students would use to arrive a solution, or the probable quantities, operations, and representations (a sequence of solution procedures) that students would select to reach a solution can be accepted as part of solution approach.

Pre-service Teachers' Knowledge of Students' Thinking: In this study, teacher knowledge of students' thinking refers to teachers' awareness of students' different solution approaches (or strategies), conceptions (representing and formulating concepts), difficulties and errors in given tasks. That is, what and to what extent they predict and identify students' solutions, conceptions, errors and difficulties. In addition, teacher knowledge of students' thinking includes teachers' understanding and interpreting students' ideas and solutions (e.g., Ball, 1988; Ball, Thames, & Phelps, 2008; Doerr & Lesh, 2003; Grossman, 1990; Shulman, 1986, 1987).

Undergraduate course context: In this study, undergraduate course refers to the specially designed mathematics education course where pre-service teachers worked on non-routine tasks (i.e., mathematical modeling tasks), and students' solution papers and video episodes taken from real classroom setting where these non-routine tasks were implemented.

Non-routine mathematical tasks: In this study, non-routine mathematical tasks refer to the mathematical modeling tasks pre-service teachers attempted to solve. The tasks differ from the traditional textbook word problems as the students were expected to interpret a complex real-world situation and formulate a mathematical description beyond producing short answers (Chamberlin, 2002; Lesh & Doerr, 2003).

Student work:

In this study students' works include:

(i) Students' written work was students' worksheets (solution papers) including their final solutions with all scratch papers that they generated during their solution process. These work display students' different ways of thinking emerged during their solution process.

(ii) Video episodes illustrated several issues such as students' explanations and discussions of their solutions, their confusions about the issues being discussed and their interaction with their teacher during their solution process. All generated video clips were 7-10 min long.

CHAPTER 2

LITERATURE REVIEW

In this chapter, I present the theoretical framework of the study and the relevant research literature. In this study, my theoretical framework and literature review have been shaped by research on (i) teacher knowledge for teaching mathematics (Ball, 1988; Ball et al., 2008; Grossman, 1990; Shulman, 1986, 1987), (ii) the development of teacher's pedagogical content knowledge (PCK), in particular, teacher knowledge of student's thinking (Ball, 1988; Ball et. al., 2008; Carpenter et al., 1988; Carpenter et al., 1989; Doerr & Lesh, 2003; Grossman, 1990; Koellner- Clark & Lesh, 2003; Shulman, 1986, 1987), and (iii) the use of practice-based materials in supporting the development of pre-service teachers' knowledge of students' thinking (Ball & Cohen, 1999; Smith, 2001).

2.1 Teacher Knowledge for Teaching Mathematics

Teacher knowledge is not monolithic; it includes many of the components such as subject matter knowledge, knowledge of learners, curriculum knowledge, pedagogical content knowledge and knowledge of general principles of instructions (Fennema & Franke, 1992). Several researchers have documented the components of teacher knowledge by categorizing them. Almost three decades ago, Shulman (1986) proposed a special dimension of teacher knowledge termed as pedagogical content knowledge, and he categorized the teacher content knowledge as subject matter content knowledge, pedagogical content knowledge and curricular knowledge. In addition to that, in his later discussion, Shulman (1987, p.8) pointed out teacher knowledge should include at least the following categories.

- content knowledge;
- general pedagogical knowledge, with special reference to those broad principles and strategies of classroom management and organization that appear to transcend subject matter;
- curriculum knowledge, with particular grasp of the materials and programs that serve as "tools of the trade" for teachers;
- pedagogical content knowledge, that special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding;
- knowledge of learners and their characteristics;
- knowledge of educational contexts, ranging from the workings of the group or classroom, the governance and financing of school districts, to the character of communities and cultures; and
- knowledge of educational ends, purposes, and values, and their philosophical and historical grounds.

Shulman's categories of teacher knowledge (Shulman, 1987, p.8)

On the other hand, as presented in Figure 2.1., Grossman (1990) stated four general areas of teacher knowledge, which are fundamental of the emerging work on the professional knowledge for teaching: general pedagogical knowledge, subject matter knowledge, pedagogical content knowledge and knowledge of context.

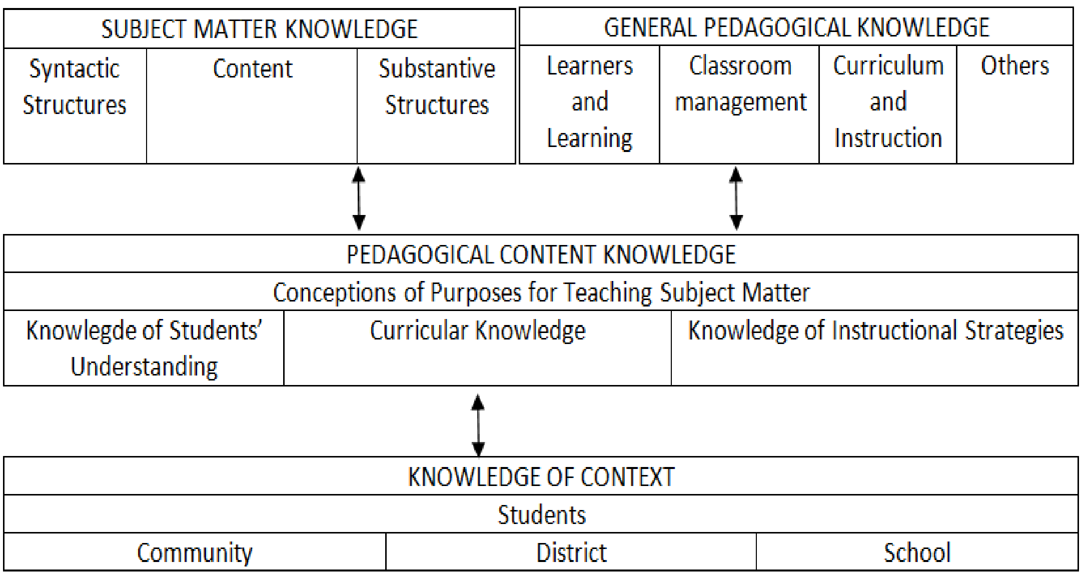

Figure 2.1 Grossman's model of teacher knowledge (Grossman, 1990, p.5)

Throughout the twenty years, pedagogical content knowledge has taken much more interest by many of the researchers, who have referred to pedagogical content knowledge in their research. However, according to Ball and her colleagues (2008), although PCK is widely used, it has lacked definition and empirical foundation that limits its usefulness. Therefore, they reconsidered Shulman's (1986) notion of pedagogical content knowledge and developed a practice-based theory of content knowledge for teaching. As a refinement of Shulman's (1986) categories, they proposed a new categorization of knowledge for teaching as presented in Figure 2.2.

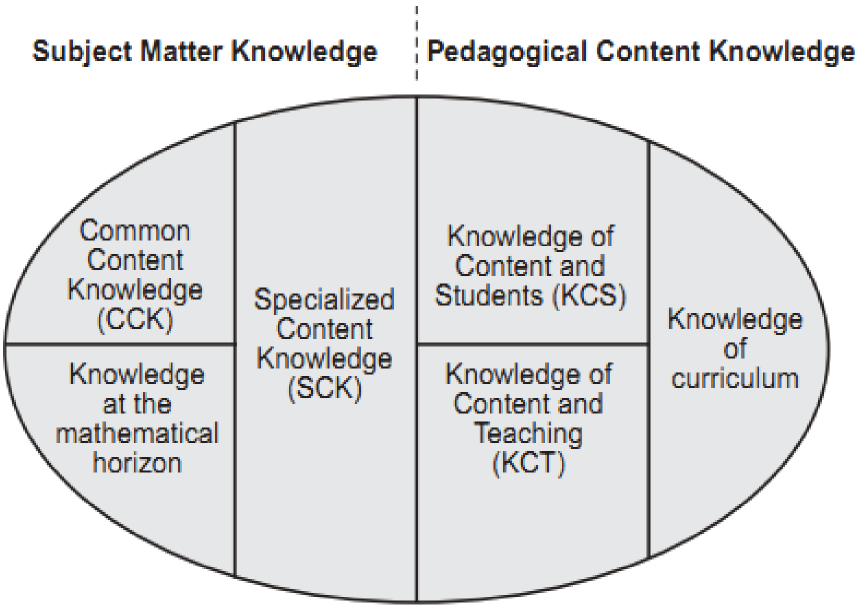

Figure 2.2 Domain map for mathematical knowledge for teaching (Ball, Thames, & Phelps, 2008, p. 403).

As shown in Figure 2.2, unlike Shulman's (1986) categories, Ball et al. (2008) divide pedagogical content knowledge into three categories, and address these three categories separately. In addition, they consider that knowledge of curriculum is a part of pedagogical content knowledge.

Briefly, all researchers' categorization of teacher knowledge commonly displays while pedagogical knowledge is content independent knowledge, other kinds of knowledge is content related knowledge.

2.1.1 Knowledge of pedagogy (PK)

Knowledge of pedagogy is independent of content and involves a body of general knowledge, beliefs and skills related to teaching (Grossman, 1990). Principles and strategies of classroom organization and management, general principles of instruction, knowledge of lesson structure, knowledge of learners' characteristics or knowledge of educational ends, purposes and values are the examples of pedagogy knowledge (Grossman, 1990, 1995; Shulman, 1987). Knowledge of classroom organization and management is related to teacher's effort to organize and maintain order in the classroom. On the other hand, knowledge of lesson structure involves teacher's making plan to teach lessons, making smooth transitions between different components of lesson or presenting clear explanation of content (Grossman, 1995).

2.1.2 Subject matter knowledge (SMK)

The subject matter knowledge (SMK) can be accepted as the most essential component of teacher knowledge because teachers have to know the subject they teach. If teachers do not know well the subject they teach, it is not likely to learn knowledge they need to help students (Ball et al., 2008). Shulman (1986) basically defined SMK as "the amount and organization of knowledge per se in the mind of the teacher" (p. 9), and he added "to think properly about the content knowledge requires going beyond knowledge of facts or concepts of a domain." (p. 9).

On the other hand, as shown in Figure 2.2, Ball and her colleagues (2008) consider the subject matter knowledge within three sub-domains. They call the first sub-domain "common content knowledge (CCK)" and define it "as the mathematical knowledge and skill used in settings other than teaching" (p.399). The basic property of domain is mathematical knowledge, which is not unique for teaching. It is related to the knowledge that teacher's recognition of the students' wrong answers, the inaccurate definitions presented in the textbooks, teacher's correct usage of term and notation and teachers' knowing of material they teach (Ball et al., 2008). They call the second domain "specialized content knowledge (SKC)" (p.400). Unlike first knowledge domain, it allows teachers "engagement in particular teaching tasks, including how to accurately represent mathematical ideas, providing mathematical explanations for common rules and procedures, and examining and understanding unusual solution methods to problems (Ball et al., 2005 as cited in Hill, Ball, & Schilling, 2008). Hill et al. (2008) stated that while CCK corresponds to subject matter knowledge in Shulman's categorization, SCK is conceptualized first time by Ball and her colleagues. However, they briefly indicated that both knowledge CCK and SCK are completely mathematical knowledge; therefore, they do not entail knowledge of students or teaching. And, the third sub -domain within subject matter knowledge they call "horizon knowledge". Horizon knowledge is "an awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (p. 403). Yet, although Ball and her colleagues (2008) define horizon knowledge within subject matter, they indicate that they are not sure if this category is part of subject matter knowledge. They suspect that whether horizon knowledge may be part of knowledge of content and teaching or it may run across the other categories.

In their definition, both Shulman (1986) and Ball (1988; 1990) emphasize the nature of the subject matter knowledge by focusing on the Schwab's approach. Namely, Shulman (1986) stressed that in order to teach subject, knowing its facts and concepts is not adequate. According to him, teachers must not only tend to transfer students the accepted truths in a domain but also they must be able to explain why they are worth knowing, how they can be related other propositions. Namely, teachers should have two kinds of understanding of subject matter "knowing that" and "knowing why" (Even & Tirsoh, 1995; Shulman, 1986). "Knowing that" is the most basic level of subject matter knowledge and it involves the rules, procedures or algorithms associated with mathematical topics in the school curriculum. On the other hand, "knowing why" is related to understanding the meaning of the concepts as well as understanding of why. Both types of knowledge are important for teachers to get pedagogical decisions (Even & Tirosh, 1995). For example, while a teacher decision about if a student's response is correct/wrong is related to "knowing that", understanding the reasoning behind students' conceptions and anticipating sources of correctness/wrongness is based on "knowing why" (Even & Tirosh, 1995).

In mathematics education, the importance of teachers' having a robust mathematical knowledge is mentioned by the many researchers (Fennema & Franke, 1992). Having a strong mathematical knowledge does not guarantee that one will be an effective teacher (Ponte & Chapman, 2002); however, teacher's mathematical knowledge affects the instructional practice closely. For example, Ball (1988) stated that "knowledge of mathematics is obviously fundamental to being able to help someone else learn it" (p.12). When teachers have an intensive knowledge of mathematics, they can know how to structure their own mathematics teaching. In addition, when they have more knowledge, they can ask more conceptual questions, in which students can draw relationship between concepts; when teachers have less knowledge, they can tend to write the exam questions focusing on memorizing (Fennema & Franke, 1992). Moreover, teachers' knowledge of mathematics is a quite crucial to understand and assess students' mathematical thinking, and to determine their conceptions, misconceptions, errors and difficulties. Grossman (1990) indicates that it also contributes to teachers to select of curriculum materials. Furthermore, Grossman (1990) adds if teachers have more confident in their subject matter knowledge, they are more likely to move away the strict organization of content found in textbooks. Namely, subject matter knowledge of teachers is very crucial to teach mathematics for understanding.

2.1.3 Pedagogical content knowledge (PCK)

As Ball (1997) stated, "responsibility to subject matter is only part of the equation" (p.773). A second kind of content related knowledge is pedagogical content knowledge (PCK) termed first time by Shulman (1986). Shulman (1986) stated that "PCK goes beyond knowledge of subject matter per se to the dimension of subject matter knowledge for teaching" (p.9). According to Shulman (1986, p.9), PCK includes,

> (i) for the most regularly taught topics in one's subject area, the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, and demonstration (ii) an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons.

Similarly, in the light of Shulman's (1986) definition, Carpenter et al. (1988) pointed out that PCK comprises "teachers' knowledge of techniques for assessing students' understanding and diagnosing their misconceptions" (p.386).

On the other hand, Grossman (1990) indicated four components of PCK (see Figure, 2.1). According to Grossman (1990), while the first component involves knowledge and beliefs about the purposes for teaching a subject at different grade levels, the second component includes knowledge of students' understanding, conceptions and misconceptions of particular topics in a subject matter. In addition, the third component is curricular knowledge involving knowledge of curriculum materials and knowledge of both horizontal and vertical curricula for a subject. And, lastly, the fourth component includes knowledge of instructional strategies and representations such as analogies or explanations for teaching.

However, according to Ball and colleagues (2008), although PCK has used by many researchers since 20 years, the nature of definition of pedagogical content knowledge was not obvious, it was usually not distinguished from the other forms of the teacher knowledge, and so it was lack of clarity. Therefore, they revised the concept of pedagogical content knowledge as well as other categories proposed by Shulman (1986), and they proposed a new categorization called as "mathematical knowledge for teaching" (see Figure 2.2). In their new domain, PCK includes "knowledge of content and students (KCS), knowledge of content and teaching (KCT) and knowledge of curriculum" (see Figure 2.2, right side of the oval). However, although knowledge of content and curriculum is placed in PCK, Ball et al. (2008) indicated that they are unsure whether or not this knowledge may be part of knowledge of content and teaching or it may be a category in its own right. Therefore, with respect to their domain PCK has two main categories as KCS and KCT.

The first category of PCK is KCS, where knowing about students intertwined with knowing about mathematics (Ball et al., 2008; Hill et al., 2008). Similar to Shulman (1986) and Grossman's (1990) specification of PCK, Ball et al. (2008) stress that knowledge of students' common conceptions and misconceptions about particular mathematical content is in centre of KCS. That is, teachers have to predict students' difficulties and confusions, and they also have to anticipate what students are interested in before choosing examples or whether students find the examples easy or hard. Moreover, teachers have to be able to listen, hear and interpret students' emerging and incomplete thinking when they are engaging in mathematical tasks (Ball et al., 2008; Hill et al., 2008). KCS does not involve the curriculum materials, and it is separable from the knowledge of teaching moves (Hill et al., 2008)

The second main category of PCK is KCT. Unlike KCS, KCT includes knowledge of teaching moves, and similar to KCS, it entails knowledge of content. Namely, this category combines knowledge of mathematics and knowledge of teaching. KCT involves teacher's choice of examples or mathematical tasks and use of analogies and representations that helps students to understand the content deeply (Ball et al., 2008).

In order to teach mathematics for understanding, having a strong mathematical knowledge is not sufficient. It also requires extensive knowledge of students, what they know in particular mathematical topics, how they learn them, and what kinds of informal and formal ways of mathematical thinking they have (Crespo, 1998). In order to make subject matter accessible for students, teachers also have to know their students (Ball, 1997). In order to understand how students' think about mathematics, teachers have to look mathematics from student's point of view (Smith, 2001). In sum, teachers' knowledge and beliefs about students' thinking and learning has a strong effect on teachers' instructional practices (e.g., Ball, 1988; Carpenter et al., 1988; Chamberlin, 2002; Even & Tirosh, 1995).

On the other hand, as an another component of teacher knowledge, whereas curricular knowledge (CK) is within pedagogical content knowledge in Grossman (1990) and Ball et al.'s (2008) model of teacher knowledge, it is taken place a separate category in Shulman's (1986) categorization of teacher knowledge. Curricular knowledge is related to knowledge of curriculum materials represented the full variety of programs designed for the teaching of particular subject and topics at a given level. In addition, the knowledge of instructional materials in relation to these programs is included by the curricular knowledge (Grossman, 1990; Shulman, 1986). Furthermore, curricular knowledge includes two additional aspects "knowledge of horizontal and vertical curricula for a subject". Shulman (1986) and Grossman (1990) state that while the vertical curricula knowledge is related to teacher's familiarity with the topics and issues which a teacher has taught and teacher will teach in later years in the school, knowledge of horizontal curricula is related to teachers' skills; namely, it means that how a teacher associates a content in a given course/lesson to topics being discussed simultaneously in other classes.

2.2 The Development of Pre-service Teachers' Pedagogical Content Knowledge in Teacher Education Programs

Pre-service teacher education programs are the place where the development of pre-service teachers' pedagogical content knowledge begins formally. According to Grossman (1990), during one's pre-service teacher education, different kinds of sources may contribute to the development of his/her pedagogical content knowledge. Grossman (1990) collects these sources under four categories which are apprenticeship of observation termed by (Lortie, 1975), disciplinary background, professional coursework and learning from teaching experience. The first source is "apprenticeship of observation" (Lortie, 1975). Teachers spend amount of time in their elementary and secondary classrooms during their apprenticeship of observation and they gain several ideas about what the school subject matter is like, how students learn and what they do or act in school (as cited in Grossman, 1990; Kennedy, 1999). In this respect, Grossman (1990) highlights the pre-service teachers' experience in undergraduate coursework and explains that apprenticeship of observation contributes to teachers' development of different components of the PCK. According to Grossman (1990), pre-service teachers' observations in their undergraduate coursework may be more powerful in their development of knowledge of instructional strategies since the memories of the strategies used in undergraduate classes may more clear as well as accessible than used in elementary and secondary school. Secondly, apprenticeship of observation contributes to prospective teachers' knowledge of students' understanding. While they are thinking about students' understanding, pre-service teachers generally rely on vague memories about what they had learn when they were students at elementary and secondary classrooms. On the contrary, Ball (1988) indicates that "what students know or how student think is unlikely to be gained through the apprenticeship of observation. That is, although the apprenticeship of observation influences pre-service teachers' understanding of students' thinking, pre-service teachers acquire only image of it, and they come to teacher preparation programs with this image. Apprenticeship of observation supplies very limited knowledge about students and students' thinking to pre-service teachers. And thirdly, Grossman (1990) and Kennedy (1999) point out that "apprenticeship of observation" effects prospective teachers' curricular knowledge since pre-service teachers rely on their experience as students to select curricular materials.

The second source of the development of PCK stated by Grossman (1990) is pre-service teachers' disciplinary background, namely, their subject matter knowledge acquired in the subject matter courses in teacher preparation programs. Teacher's decisions about particular content, selection of curricula, organization of curriculum materials and determination of students' difficulties and errors in particular content may have their roots in teachers' subject matter.

The third source is based on professional coursework. These courses of professional coursework are specially designed to help pre-service teachers to acquire knowledge of teaching. One of the examples of these courses is method course which covers the topics such as academic discipline, school curriculum, students' learning of particular topics and specific teaching techniques (as cited in Grossman, 1995). So these courses may be most basic place for pre-service teachers to acquire pedagogical content knowledge.

Lastly, the most fundamental source of development of PCK is actual classroom experience. While the other three sources are mainly related to acquire pedagogical content knowledge of pre-service teachers, teaching experience in actual classroom experience is more relevant to test as well as to improve the knowledge pre-service teachers acquired from other sources. That means, PCK develops through classroom practice (Grossman, 1990; Lee, Brown, Luft, & Roehrig, 2007; Nilsson, 2008). During their teaching experiences, teacher candidates may learn about students' conceptions, misconceptions and background of particular topics and curriculum. With this experience, pre-service teachers find an opportunity what instructional strategies for teaching particular topics work well; in addition, what metaphors and representations are useful as well.

Even though a pre-service teacher's own experiences both as a learner in an undergraduate coursework and as a teacher in their teaching practice may affect acquire and development of pedagogical content knowledge (Even & Tirosh, 1995; Grossman, 1990), researchers mainly agree that in teacher education courses, pedagogical content knowledge of teachers develops limited and at the minimum level (as cited in Lee et al., 2007).

Some research has focused on particularly the role of classroom teaching experience on the development of the prospective teachers and teachers' pedagogical content knowledge (e.g., Lee et al., 2007; Nilsson, 2008). These studies call attention to classroom teaching experience has a crucial role in the development of teachers' pedagogical content knowledge. In addition to teaching experience, the researcher indicates the development of PCK takes time.

In their research, for example, Lee et al. (2007) examined the PCK of beginning science teachers, who are in their first year of teaching, and the change in the level of their PCK over the first years. In order to beginning science teachers' PCK, they developed and employed a rubric, and they focused on "knowledge of student learning" and "knowledge of instructional strategies" as the dimensions of PCK. Teachers were observed throughout the year and they were also interviewed at the beginning and at the end of the school year. As a result, the collected data displayed that the PCK level of beginning teachers were limited level at the beginning of the school year and it changed the basic level during the year.

Unlike Lee et al. (2007), who worked with beginning teachers, Nilsson (2008) explored the development of teacher candidates' PCK during their pre-service education. The participants of the study were four teacher candidates in science and mathematics for primary and secondary schools. The participants were in their final year of 4.5 year of program, and had finished all their science courses and they had also been in schools for their practical training. For this study, these participants were participated a project to teach physics once a week over a year. Their lessons (one-third) were videotaped; the interviews were conducted with these participants using the videotape for getting pre-service teachers' reflection on their classroom practice with their conceptual understandings of physics. In this study, reflections of teacher candidates on own their classroom experiences were the context to explore the development of their PCK. The results of this study drew attention to the importance of teaching experiences which might contribute to the development of teacher candidates’ PCK. This study suggested to teacher educators that in order to support the development of PCK, they have to understand deeply the knowledge of teacher candidates needed for teaching and its relation to classroom practice.

On the other hand, several researchers (e.g., Feiman-Nemser, 2001; Lampert & Ball, 1998; 1999) touch on the ineffectiveness of formal pre-service teacher education for pre-service teachers’ learning to teach compared with teachers’ on-the-job experience. By referring what other researchers (e.g., Feiman-Nemser, 1983; Feiman-Nemser & Buchman, 1985) reported, Lampert and Ball (1998) and Feiman-Nemser (2001) stated that pre-service teacher education programs are weak interventions, and the programs are criticized because of the some conceptual and structural problems. For example, the researcher emphasize that the teacher education programs are mainly conventional, and their curriculums are quite superficial and lack of connection to practice. Traditional teacher education curriculum divides theory and practices both physically and conceptually; that is, theory is rarely examined in practice. In addition, the curriculum does not provide pre-service teachers common and shared experience (Lampert & Ball, 1998). Therefore, in teacher education courses, pre-service teachers mainly do not find opportunities to engage deeply and practically with alternative approaches to teaching and learning. That means, there is a weak relationship between academic courses and pre-service teachers’ teaching (field) experiences. The researchers also point out that although teaching experience is a crucial part of teacher training of teacher candidates, they usually can not find opportunities to observe, to analyze and interpret the classrooms, teachers as well as teaching regularly. Although faculties provide knowledge of content and pedagogy, knowledge of students or theories of learning, teaching practice mainly does not take place in their focus. It means there is a considerable disconnection between universities and schools. Teacher preparation programs usually do not balance between theory and in the realities of the practice.

On the other hand, the researchers state that the teacher education courses are too superficial to develop teacher candidates’ deep understanding of teaching (e.g., Feiman-Nemser, 2001; Lampert & Ball, 1998). In addition to that, they stress that wide variety of instructors, who educate teachers, do not think of this work as a part of their role. Although teacher educators teach the effective and appropriate learning and teaching approaches, they do not practice what they teach. So, pre-service teachers' image of teaching is influenced by the role of their instructors.

All these stated reasons create obstacles for teacher candidates to develop their PCK in advance level. Teacher candidates do not transfer what they learned in teacher preparation courses into the real classroom settings, and their knowledge about teaching acquired in teacher preparation classrooms does not help them to teach at all.

2.3 Knowledge of Students' Thinking

2.3.1 Research on pre-service teachers' knowledge of students' thinking

Ball (1988) indicates that pre-service teachers have not more ideas about knowledge of students such as what they know and what can they do. When they think about students, pre-service teachers prominently display an egocentric perspective; namely, they apply their own experience when they were students. On the other hand, pre-service teachers are not ready to interpret student work beyond identifying the correctness of the responses. Moreover, they are quite unprepared to see mathematics through the eyes of the students. Briefly, Ball (1988) emphasizes that what pre-service teachers know about students is vague as well as thin.

Some researchers investigated elementary and secondary pre-service teachers' knowledge and understanding of students' ways of thinking within a method course context in various kinds of mathematics subjects such as subtraction and slope (Ball, 1988), fraction (Tirosh, 2000), algebra (Kılıç, 2011), ratio and proportion (Son, 2013) and all these research commonly exposed pre-service teachers' lack of knowledge on students' ways of thinking.

In her research, for example, Ball (1988) presented pre-service teachers a student's paper, and asked them to look and to talk about their thoughts on student's understanding. The elementary pre-service teachers examined a second grade student work on subtraction with regrouping which included an important and common error (e.g., $52 - 29 = 37$, where student took the difference between the numbers) while secondary pre-service teachers examined a student paper concerning to slope and graphing. Ball's (1988) findings displayed that both elementary and secondary pre-service teachers' knowledge of students and of students' common confusions was very weak. Their interpretation of students' work was mainly based on restatement of it like "the student is switching the top and the bottom number" (p.282), and they mainly did not know and interpret the reason of the students' mistakes. In addition, they had struggled to make sense of students' mistakes, and they could not view the subject matter from the students' perspective.

Similarly, Kılıç (2011) investigated pre-service mathematics teachers' knowledge of students in terms of students' misconceptions, the sources of students' errors in a method course. And, like Ball's (1988) research, she did not apply any intervention and she presented the nature of the pre-service secondary mathematics teachers' knowledge of students appearing as a result of the examination of development their pedagogical content knowledge in a method course. The participants were six pre-service teachers, and the data collection sources were observations, interviews and written documents. The findings of the study revealed that pre-service teachers had difficulty in determining students' misconceptions and the source of the errors and also producing effective ways to eliminate these misconceptions.

Another study conducted by Akkoç, Yeşildere and Özmantar (2007) examined the prospective mathematics teachers' knowledge of student difficulties in limit process that was applied to define definite integral. For this aim, they observed four prospective teachers during their micro teaching activities. They collected the data with micro teaching videos, interviews, and written documents such as lesson plans and teaching notes. The analysis of data displayed that prospective teachers had lack of knowledge on students' misconceptions. All prospective teachers indicated the limit process was necessary for defining definite integral; therefore, they used limit process to teach definite integral in their lesson plans. However, when their lesson plans and micro-teaching process were examined, it was found that prospective teachers did not address any possible misconceptions of students in their microteaching activities. In addition, interviews conducted to get more insights on prospective teachers' knowledge displayed that prospective teachers could state students' difficulty only in a general manner.

As distinct from other research (Akkoç et al., 2007; Ball, 1988, Kılıç, 2011), Tirosh (2000) used the materials developed based on research findings to examine pre-service teachers' awareness the misconceptions held by students, and enhance their awareness. Actually, the initial aim of the Tirosh (2000) was to enhance prospective teachers' SMK in terms of division of fractions. Therefore, she designed a mathematics method course and 30 prospective teachers enrolled in the course. Initially, at the beginning of the academic year, she measured the prospective teachers' SMK and PCK of rational numbers with a questionnaire. In addition, she conducted interview with each participant to deepen her understanding. And, during the academic year, all prospective teachers participated in a method course. In order to improve prospective teachers' concepts, structures and relations about rational number, the materials developed based on research findings, were used. At the same time, these materials applied to improve teachers' knowledge about students' conceptions and misconceptions. The result of the study displayed that prospective teachers were not aware of the students' misconceptions in this domain before the instruction. Prospective teachers mostly mentioned only algorithmically based mistakes before they entered the course; however, most of them became aware of the intuitive based mistakes and were familiar with the sources of incorrect responses at the end of the course.

Kılıç (2011) and Tirosh (2000) commonly suggest that prospective teachers' knowledge on students' difficulties and misconceptions in different mathematical topics should be developed in pre-service teacher education programs. Prospective teachers should acquire knowledge of students' thinking in teacher preparation programs, and these programs should help pre-service teachers to increase their awareness across students' correct and incorrect common cognitive process and how they cause students to think in different ways.

Unlike other researchers (Akkoç et al., 2007; Kılıç, 2011; Tirosh, 2000), instead of examining pre-service teachers' knowledge of students' difficulties, errors or misconceptions, Son (2013) particularly investigated how pre-service teachers interpret and respond students' additive reasoning error on the topic ratio and proportion. Pre-service teachers in elementary, special and secondary mathematics education were the participants of this study. In this study, these pre-service teachers enrolled in one semester mathematics method courses that are designed to support their understanding of teaching and learning issues and approaches and aimed to learn to teach by focusing on students' thinking. In these courses, pre-service teachers were provided opportunities to discuss on fundamental teaching ideas and some students' work to analyze them. However, they did not discuss on the specific strategies in a particular mathematics topic. Two tasks were used in this study. The first one was to assess pre-service teachers' content knowledge, and they were asked to solve a question about "finding missing length of similar triangles" and explain their solution methods. And then, in order to assess their pedagogical content knowledge, they were given a student's incorrect responses on the same question and asked to interpret and respond to it. In this question, finding missing length in as similar rectangles includes both conceptual aspects and procedural aspects of similarity. In order to solve the question, students should have understood the concept of similarity, recognize the relationships, set up a proportion for representing similarity and make correct calculations. Both quantitative and qualitative analyses were conducted with respect to pre-service teachers' responses. The responses of pre-service teachers to student's work analyzed in terms of their ways of interpreting and responding to student errors. In addition, student's errors were addressed as conceptual-based and procedure-based errors in the data analysis process. Student's errors displayed that these errors mostly originated from student's limited understanding of similarity; that's why, they were mainly concept-based errors. However, the results displayed that more than half of the pre-service teachers (both elementary and secondary) identified procedure-based errors, and small number of pre-service teachers determined the student's errors stemming from the conceptual aspects of the similarity.

2.3.2 Research on teachers' knowledge of students' thinking

Teachers' understanding, hearing and attending to students' mathematical thinking is an important aspect of mathematics teachers' knowledge for teaching. Much research has focused on teachers' knowledge of students' thinking in terms of teachers' predictions of students' understanding/reasoning/performance (Bergqvist, 2005; Hadjidemetriou & Williams, 2002, Nathan & Koedinger, 2000; Şen-Zeytun, Çetinkaya, & Erbaş 2010), teachers' awareness of students' tendency (Even & Tirosh, 1995; Tirosh, Even, & Robinson, 1998) and teachers' listening and interpreting students' thinking (Even & Wallach, 2004; Wallach & Even, 2005) in different mathematical concepts and topics such as algebraic reasoning, covarational reasoning, graphs (e.g., slope-height confusion, linearity-smooth and y=x prototype, reversing and misreading coordinates), conjectures and simplifying expressions. However, the findings of these studies commonly showed that teachers had poor knowledge, and there were wide discrepancies (gaps) between their knowledge and students' mathematical thinking.

Nathan and Koedinger (2000), for instance, conducted a research on teachers' and researchers' beliefs about the development of algebraic reasoning. In this research, they also examined discrepancies between teachers and researchers' prediction and students' problem solving performance. Sixty-seven mathematics teachers who were responsible for teaching 7^{th} through 12^{th} grades and thirty-five mathematics education researchers participated in the study. Twelve mathematics problems were given to teachers in order to rank them individually from easiest to most difficult. On the other hand, researchers were also given six of the problems presented to the teachers and researchers requested to rank the problems starting with the most difficulties to easier for students. The problems included three formats as story, word equation and symbolic equations and also two unknown positions as start unknown and result unknown. The findings of the study displayed that both teachers and researchers tended to rank story and word equation problems as more difficult for students. Although teachers predicted that the story problems and word equation problems would be more difficult than symbol-equation problems, students found that symbol equation problems were more difficult. Therefore, this result showed that there were discrepancies with students' difficulties and teachers' predictions.

Another study carried out by Hadjidemetriou and Williams (2002) exhibited similar results. Their study addressed to assess teachers' pedagogical content knowledge in terms of teachers' perceptions of the difficulty of graphical items and students' real difficulty. Twelve teachers chosen as knowledgeable and leading teachers participated in this study. A diagnostic test was given to the teachers. They were asked to answer all the items and then predict "how difficult their children would find the items, suggest the possible errors and misconceptions with respect to students, and suggest several methods which they would apply to overcome students' these difficulties". In addition, semi-structured interviews were conducted with teachers in order to get information on the way how they introduce and teach graphs to their classrooms and students' difficulties in graphical conceptions. The results elicited that there was several mismatch of the teachers' perceptions of students' difficult and students' actual difficulties. While teachers underestimated technical difficulties of graphing, they overestimated the difficulty of interpretative.

Similarly, Bergqvist (2005) examined teachers' expectations of students' reasoning and performance on conjectures, and like others, he found the discrepancy between teachers' predictions and students' actual performance. In this study "the graphs of a linear function and a quadratic function always intersect at two points", "if the graph of a quadratic function cuts the x-axis at two points, there is a point between the intersections where the tangent to the graph is horizontal" and "the graph of a third degree polynomial always cut the x-axis" (p.174) were three conjectures presented to ten students. Students were asked to decide whether these conjectures were true or false. On the other hand, in this study, students' level of reasoning included four proof levels: Level 1 (naïve empiricism) and Level 2 (the crucial experiment) were called lower level reasoning; Level 3 (the generic example) and Level 4 (the thought experiment) were called higher level reasoning. In order to get teachers' expectations of students' performance and reasoning levels with the three conjectures, eight teachers, who had experience to teach the mathematical content related to conjectures, were interviewed. The interview comprised three parts. In the first part, teachers' thoughts on how students would handle three conjectures were asked. In addition, teachers were presented four invented examples of students' work to decide which would match their expectations best. In the second part, in order to comment on the students' performance and levels of reasoning, teachers were given the students actual work produced on three conjectures. Lastly, in the third part, a general discussion regarding activity types used in the study was made. As a result, Bergqvist (2005) found three fundamental results. First, the teachers' tendency was to underestimate the students' reasoning levels. That means, they expected that most of the students would use examples to convince themselves, and other would decide that the conjecture is true. Second, some of the teachers considered that if students approached algebraically and use advanced mathematics, they had high level performance. Third, all teachers' belief in the study was that only a few students were use their higher level of reasoning to decide if the conjectures are true or false.

Furthermore, Şen-Zeytun, Çetinkaya and Erbaş (2010) investigated covariational reasoning abilities of mathematics teachers and their predictions about students' covariational reasoning abilities in terms of students' possible solution strategies in solving problem, their possible mistakes and misconceptions. Their participants were five secondary mathematics teachers who had experience in teaching 12^{th} grade. The researchers collected the data through interviews. During the interviews, teachers were initially asked to work through a model eliciting activity to reveal teachers' own covariational reasoning abilities. The context of the task was related to the functions, particularly increasing and decreasing functions. In this task, teachers were expected to draw a volume-height graph corresponding to given bottle, and explain their reasoning while they were drawing the graph. Then, they were asked about "what might be students' possible solution strategies to solve this problem and their possible mistakes in solving problem?" and "which misconceptions might be hold by students?". In order to explore teachers' covariational reasoning abilities, the researchers used the covariational framework developed by Carlson (1998) and Carlson et al. (2002), and they coded the data with respect to behaviors described in this framework. Moreover, they also coded the transcripts of interview data to identify teachers' predictions of students' covariational reasoning abilities. As a result, they obtained that teachers had difficulty to solve the problem correctly and their covariational reasoning abilities were weak. In addition to that, teachers' predictions were quite limited and bounded by their own thoughts. All teachers predicted that students would have difficulty to construct the graph. Their predictions about students' possible solution strategies and mistakes were based on their own solution strategies or mistakes, and they could not go beyond their own thoughts.

On the other hand, Tirosh, Even and Robinson (1998) investigated 7^{th} grade teachers' awareness of students' tendency to "conjoin" or "finish" open expressions. They had four teachers, two of them were novice and two of them had more than 15 years experience, as participants. In their research, their attempts were to examine the issue teachers' knowledge of students' ways of thinking in the context of algebra, in particular, simplifying algebraic expressions. And all participant teachers were both teaching algebra in regular 7^{th} grade classes from the same textbook. The researchers collected data through three types of data sources lesson observation, lesson plan and post-lesson interviews. One of the researchers observed the three initial lessons on equivalent algebraic expressions of each of the four teachers, and the researcher took field notes during observation as well as audio-taped it. In addition, the teachers were requested to submit a lesson plan before their each lesson. After each lesson, the interviews were conducted with teachers and they were audio taped. In each interview, they were required to reflect the issues on their lesson plans, what they expect from their students and their feelings about lessons. And, teachers were asked two following specific questions "What are the main difficulties that students encounter when they learn to simplify algebraic expressions? and "What are the sources of these difficulties?". They analyzed and reported data for each teacher. Their findings showed that whereas two experience teachers were aware that students' tendency to conjoin or finish open expressions (knowing what) and their several sources (knowing why), the novices were unaware. However, none of these experienced teachers presented deeper level of explanation regarding sources of students' difficulties. They only attributed the sources to students' eagerness to finish expressions.

Similarly, Even and Tirosh (1995) concentrated on teachers' planned representations of the subject matter in their research. They were also interested knowledge about subject matter and knowledge about students as the main sources of pedagogical content knowledge. They emphasized those teachers both mainly do not understand the reasoning the students did in a given mathematical task and the sources of the students' responses. In the case, where they provided teachers with two common students' responses of 4/0, one of them is "4:0=4 and other is 4:0=0", Even and Tirosh (1995) found that most of teachers did not attempt to understand students' reasoning better. In addition, many of the teachers made judgment about students' answers just in terms of being right and wrong.

As distinct from the other researchers (e.g., Bergqvist, 2005; Even & Tirosh, 1995; Hadjidemetriou & Williams, 2002) in their research, Even and Wallach (2004) and Wallach and Even (2005) aimed to examine teacher's hearing and interpreting students' ways of thinking. Even and Wallach (2004) analyzed several cases associated with assessing of teachers' learning and understanding of their students' mathematics by listening their talks and observing their actions. In the analysis of the cases, the researchers focused on problems and obstacles. They suggested two different kinds of problems in hearing students and they categorized them as problems in hearing students that might be overcome and that cannot be overcome. The reasons of one of the problems were “teachers were unable to make unplanned changes, teachers were lack of knowledge about common student conceptions and their possible sources and teachers did not attribute value to students’ ways of thinking”. The analysis of the two cases displays that Benny who is 7th grade novice teacher has evaluative listening in hearing his students because he are not able to deviate from his planning of the lesson. In addition, since Benny is not familiar with common student misconceptions, he is prevented from hearing his students. Moreover, the analysis of the third cases shows that Ahuva, who has 20 years teaching experiences in elementary schools, does not value to students’ ways of thinking because she does not expect her students to come up with original solution ways. All these problems have the potential to overcome but in order to overcome teachers need to professional support and teaching experience. On the other hand, the analysis of Ruth’s case, an elementary school teacher with 11 years teaching experience, exhibits teachers’ problems in hearing their students. Ruth has two different kinds of obstacles called “under hearing” and “over hearing” in hearing students. Namely, the researchers explored that teachers always hear trough various personal factors such as their mathematical knowledge and understanding or their beliefs related to teaching and learning mathematics. Therefore, teachers’ such kind of problems in hearing students cannot be overcome.

Likewise, Wallach and Even (2005) explored the potential meaning for teacher to hear students and to interpret their students’ talks and actions. That is, they aimed to understand the nature of teacher hearing and interpreting. In their study, 25 elementary teachers, who taught all subjects including mathematics, participated in the study. But, in this research they only reported the case of Ruth. Ruth was the teacher-college graduates and had eleven years experiences in upper elementary grades. The researcher selected Ruth as a case because she was openness as well as willing to participate in the various workshop activities; therefore, she provided rich and diverse data. The teachers enrolled in a weekly 4 months long in service mathematics workshop. In the workshop, the participants worked on several mathematical problems. They first solved these mathematical problems and then discussed their solutions in small groups. Later, they chose one of these problems and presented it to two students in their class as well as they observed and videotaped their students while they were working on problem. And, in this process, they did not give any comments, hints or advice to their students. Ruth who was the focus of this research selected to “Shirts and Numbers” problem and observed Sigal and Ore’s works. The written works of Ruth, the videotape of her students’ problem solving sessions, her individual interviews, the students’ written work and two focus group interviews in which she participated were the data sources of this study. As a result of the analysis of Ruth’s case, the researchers presented four types of teacher interpretation students’ talks and actions which were “describing, explaining, assessing and justifying” and five different characteristics of teacher hearing of the describing and explaining types of interpretation which were “over hearing, compatible hearing, under hearing, non hearing and biased hearing”. The researchers indicated that the nature of teacher’s interpreting and hearing of students’ talks and actions were problematic because Ruth’s concern for students’ success, her own conceptions of the problem and its solutions influence her hearing of students. Therefore, they suggested that teacher education programs and professional development programs should address that problem.

On the other hand, unlike all research presented above, Klein and Tirosh (1997) aimed to evaluate not only pre-service teachers but also in-service teachers’ knowledge of common difficulties of students (knowing that) and its possible sources on the multiplication and division word problem solving including national numbers (knowing why). 67 pre-service teachers and 46 inservice teachers were the participant of the study. All inservice teachers participated in this study were enrolled in two year “Expert Teaching Program (ETP)”. Their instruments were a diagnostic questionnaire (DQ) and semi-structured interview. They implemented DQ to the participants in two sessions of 90 minutes each during the method course. Although Klein and Tirosh’s (1997) main concern was to investigate pre-service and inservice teachers’ PCK of rational numbers, they analyzed participants’ subject matter knowledge as well. Their findings related to subject matter knowledge displayed that while most of inservice teachers (93%) expressions for the multiplication and division of the word problems were correct, the percentage of pre-service teachers responses were lower than teachers (average 69%). On the other hand, the responses of participants, whom were asked to describe possible sources of students’ mistakes, displayed that while only few of the pre-service teachers determined to possible source of the students' incorrect responses to the different kinds of problem, among teachers who are in their second year in ETP programs were able to list source of students' common incorrect responses. Therefore, they concluded that many of the prospective teachers had a dull knowledge of students' common incorrect responses as well as their possible sources. Furthermore, although most of the pre-service teachers were aware of students' common incorrect responses, they were not aware of their possible sources.

2.4 The Practice-based Materials Supporting the Development of Teacher Knowledge of Students' Thinking

In spite of the many benefits of attending to, understanding and interpreting students' thinking, Ball (1997) states that listening and attending to students' thinking is challenge for many teachers. One of the reasons of challenges indicated by Ball (1997) is that children are not same as adults; for example, the eight years old children's talks, gestures and moves are so special, and children think and see quite different than adults. In addition, children's thinking, representing their ideas and talking are shaped by their identities and experiences. So, listening to the children who have different gender, culture, religion or native language may create challenges for teachers. Other cited challenge is whereas students can think one thing under one set of conditions, they can think quite different under other conditions.

However, despite such kind of challenges, Ball (1997) points out that listening and attending to students' thinking is not an impossible task. Teacher can fairly improve their capabilities to figure out what their students know and learn. For this reason, Ball (1997) suggests three emerging approaches, all are intellectual sources for teachers to help them to understand and interpret the uncertainties of students' thinking. They are also common in learning of students' thinking in the real time practice. The approaches suggested by Ball (1997) are "discussing cases of student thinking, using redesign curriculum materials and investigating artifacts of teaching and learning" (pp. 808-811).

In the first approach "Discussing cases of student thinking" (p. 808), teachers work on written cases of episodes related to children thinking. Their focus is on developing a case knowledge of students' thinking. When teachers are working on case episodes collectively, they examine, conjecture and explore different kinds of interpretation of students' thinking. Group discussion and analysis of written episodes are the basic component of this approach. Moreover, in this process, teachers focus on not only students' thinking but also the mathematics, students' language and drawings as well as teacher questions. Therefore, discussing on case materials help teachers to both look and listen their students' thinking more closely and to examine and interpretation of it.

The second approach "using redesigned curriculum materials" (p. 809) helps teachers to prepare for hearing and making sense of students; however, unlike case materials, they usually do not provide detailed information about students. They are centrally situated in teacher practice and guide teachers to present the content. Moreover, they also help to decrease the inherent distance between teachers and students (Ball, 1997).

The third approach is "investigating artifacts of teaching and learning" (p. 811). Videotapes of classroom lessons, students' written work such as homework, quizzes and daily work, students' oral explanations during instruction, their drawing on the board and paper may be presented sample of artifacts of teaching and learning (Ball, 1997; Ball & Cohen, 1999; Smith, 2001). Different from the case materials, the information taken place in such work is uninterpreted and unstructured. Therefore, for instance, while teachers are watching students' body motions, hands or drawings in videotape of class discussion, they could make sense of what student are saying (Ball, 1997). On the other hand, Smith (2001) points out a videotape of the classroom could provide for teachers as a basis to work on several teaching issues. For examples, while watching the video, teacher can analyze the learning environment and try to respond the questions such as "What decisions did students make?" and "Who asked the questions?". In addition, teacher can analyze and try to understand what students seemed to be learning and how by responding the questions such as "what student's solution tell teacher about his/her understanding?" or "what are the student's misunderstanding or errors in solving the task?". Moreover, Smith (2001) stressed that when teachers examine students' work, they can realize that students' ways of interpreting and solving problem from their interpretations and solutions;

however, they are equally valid. Furthermore, teachers' ability to interpret students' solution strategies may be developed in examining of students' work (Smith, 2001).

Over the last decade, video has become quite popular artifact of practice in professional development programs since it has several potential roles to play in helping practicing teachers. For example, video has an ability to capture the richness and complexity of the classrooms. It allows recording small group interactions and teachers conversions with individual students. It provides teachers for shared and collaborative teaching learning events. Since everyone watches something in common, it provides a common basis for discussion. On the other hand, video recording allows teachers to closely observe students while they are working on task and to understand how they are thinking (e.g., Jaworski, 1990; Sherin, 2004; Sherin & Han, 2004; van Es & Sherin, 2008).

Several studies focused on how teachers' knowledge of students' mathematical thinking (e.g., An & Wu, 2011; Carpenter et al., 1988; Chamberlin, 2002; Koellner-Clark & Lesh, 2003; Sherin & Han, 2004; Steinberg, Empson, & Carpenter, 2004; van Es & Sherin, 2008) and pre-service teachers' knowledge of students' thinking (e.g., Crespo, 1998, 2000; Masingila & Doerr, 2002) develop when they engage in research based knowledge of students' thinking and different kinds of practice based materials such as students' written work, video cases from real classrooms, interviews with students and curriculum materials. These studies present evidence that practicing teachers and pre-service teachers' knowledge of students' ways of thinking remarkably enhances with their engagement of these materials.

2.4.1 Research focusing on the development of teacher knowledge of students' thinking

All research presented below revealed that teachers either acquired or enhanced their knowledge of student thinking when they closely engaged in students' thinking in different contexts such as analyzing student written work (e.g., Chamberlin, 2002; Hallagan, 2003; Kazemi & Franke, 2004; Koellner-Clark & Lesh, 2003), grading students' homework and analyzing misconceptions in them (An & Wu, 2011), implementing a student-centered curriculum and watching (Empson & Junk, 2004), analyzing as well as discussing videotapes of teachers' own classroom (Sherin & Han, 2004; van Es & Sherin, 2008).

The pioneering research of Carpenter et al. (1988) was situated in CGI (Cognitively Guided Instruction) project chain of inquiry. Their approaches were based on using the research-based knowledge to enhance teachers' instruction and students' achievement. The aim of the study was to investigate if teachers' instruction and students' achievement would improve when teachers were given knowledge derived from research on students' thinking about addition and subtraction. This study was an experimental, and 40 first grade teachers participated in the study in which twenty teachers were assigned randomly to experimental group (CGI teachers) and twenty of them were assigned to control group. The teachers in experimental group studied research findings concerning to the development of children's problem solving skills in addition and subtraction during 4-weeks summer workshop. On the other hand, control group teachers got two 2-hours workshops in which they focused on non-routine problem solving. In the following year, trained observers observed all teachers with their students during mathematics instruction. At the end of instructional year, teachers were asked their predictions on how their students would solve specific problems and then they matched their students' performance. In this way, teacher knowledge of their students was measured. Teachers' beliefs also measured with a test including 48 items. The result of the study indicated that CGI teachers' knowledge, beliefs and instructional practices changed in favor of building instruction on students' existing knowledge. During their instructions, in contrast to control teachers, CGI teachers presented problem solving based instruction in which they posed more questions, they listened their students and used different kinds of strategies to solve a certain problem. They also encouraged each student to propose a solution to the problems; in this way, they facilitated their learning in a meaningful ways for them. Consequently, the result suggested that research based knowledge on students' thinking and problem solving was an effective approach to improve classroom instruction and students' achievement.

In the design of the professional development program, Kazemi and Franke (2004) allowed for the principles of CGI (e.g., Carpenter et al., 1988), and they introduced CGI terminology as teachers made observations of their students' mathematical thinking. However, in this study, teachers are not provided knowledge of students' thinking derived from research. Rather, unlike them, the focus of Kazemi and Franke (2004) was to investigate what teachers learn through collective examination of student work. They engaged from ten teachers representing a range of grade level from one elementary school in a small urban district in ongoing professional development. The group of teacher met once a month throughout the school year. Before each meeting, teachers applied a common problem in the domain such as place value, addition, subtraction, multiplication and division in their classes. Then, for each meeting, they chose pieces of student work, and shared them with other teachers in the group. In addition, at the beginning of the each meeting, they made reflection about the issues such as why they posed this problem to their class or why they have chosen this problem to share with the group. In the meetings, the facilitator was a key role who invited teachers to share the different kinds of students' solution ways that they observed in their classrooms. Moreover, the facilitator consistently pushed teachers to describe the details of the students' solution ways and s/he wrote them down on chart paper. The data were collected in two settings "the work groups and classrooms" since researchers made informal visits in teachers' classrooms to provide them ongoing support, to build strong relationship and to learn more about student thinking. The data sources were transcripts from audio recording of seven workgroup meetings, copies of student work, teachers written reflections as well as the end of year interviews with teachers. In their data analysis, researchers' approach was case study and grounded theory. As a result of the data analysis, their findings displayed two major shifts in teachers' workgroup participations. The first shift was teacher started to attend to the details of students' thinking. That means, at the first workgroup meetings, they were quite unaware and uncertain of the diversity of their students' solution ways; however, they started to engage actively with students' solution ways, recognize the complexity of them as well as give the details of students' solution ways rather than describe it. The second shift was that teachers began to develop possible instructional trajectories in mathematics. For example, in their discussions, they began to reconsider how to relate students' mathematical understanding to classroom tasks.

Another teacher professional development approach, which emphasizes the importance of the teacher knowledge of students' thinking, was proposed by

Koellner-Clark and Lesh (2003). This approach is based on model and modeling perspective. Unlike Carpenter et al. (1988) research, Koellner-Clark and Lesh (2003) provide opportunities for teachers to work on their own students' work rather than provide research-based knowledge to teachers. The researchers of model and modeling approach called this type of teacher development as being on the "on-the-job" teacher development (Schorr & Lesh, 2003, p. 157). The activities used in professional development process are taken from the teaching. That is, teachers gain expertise in understanding their own student's thinking. In addition, models and modeling approach also stressed that when the researchers look at teachers' knowledge, they investigate the conceptual systems that teacher use to make of complex teaching and learning situation rather than simply looking at teacher's skills, attitudes and beliefs (Koellner-Clark & Lesh, 2003). They emphasize that new models for mathematics teaching develop over time and in stages instead they appear instantaneously. And, teachers' initial models can be both tentative and distorted. According to their approach, while teachers initially analyze and interpret their students' mathematical thinking, they cannot deeply interpret and they also oversimplify their students' ideas and conceptualizations. However, when they obtain experience, their understandings deepen; in this way, they can identify conceptions and misconceptions of students in detail. Their fundamental hypothesis is to provide opportunities teachers to improve their knowledge of how students interpret mathematics in order to design effective teaching. In this professional development approach, they used a multi-tiered teaching study in order to investigate teacher development. The research design of the project had three levels. While one level focused on teachers' interpretations on students' thinking, the second level focused on teachers evolving conceptions about teaching and learning process. On the other hand, third level's focus was researchers' evolving conceptions about the teachers' evolving knowledge and abilities. The project was 3-years project and it included upper-elementary teachers. During the intervention, teachers met regularly with researchers in a workshop, they produced solutions to the challenging activities, they applied these activities in their classrooms and they recognized and analyzed their students' evolving mathematical ideas during classroom implementations. They also worked and then assessed their students' work obtained classroom implementations. The data was collected through different kinds of sources such as teachers' responses to questions asked periodically, reflections of the researchers based on their observations and students' work from thought revealing activities. Results indicated that teachers were more aware of the different strategies applied by students in the problem solving situations. Teachers' interpretations of students' mathematical thinking became more explicit and deep over time. In addition, teachers' perceptions concerning to teaching experiences in terms of observing students when they engage in problem solving activities and helping them to reflect and assess their own work (Koellner–Clark & Lesh, 2003).

Similar to Koellner-Clark and Lesh's (2003) research, in her dissertation, Chamberlin's (2002) approach was based on model and modeling perspective. She examined the development collective interpretation of teachers' thinking of their students as a consequence of teachers' investigating their students' work as artifacts of teaching and learning. Seven middle grade teachers were participants of the study. During the study, teacher engaged in five investigations of their students' work where teachers interpreted their students' mathematical thinking as revealed in the students' work. Each investigation lasted approximately one month. For each investigation, teachers initially attended to the introductory workshop where they completed a thought revealing task and then made discussion on the issues such as mathematics inherent the activity, the expected students' responses as well as the implementation process. Following the introductory workshop, teachers implemented the tasks in their classrooms and observed their students' thinking. And then they examined students' written work from their implementations. In this process, the teachers were asked to create a student thinking sheet (STS) which would include both students' different mathematical strategies that used in solving tasks and excerpts from students' actual work. Following this process, teachers attended to follow up workshop where they shared students' different solution ways recorded in STS, discussed their interpretation of students' thinking and created "Consensus STS" by synthesizing all of students' different mathematical solution ways associated the tasks. She analyzed data from 1^{sth}, 3rd and 5th investigations to address research questions. Chamberlin (2002) used grounded theory approach for the data analysis, and she made open coding, axial coding as well as selective coding. Data analysis revealed that teachers' collective interpretations developed in three dimensions which were determining the students' thinking, determining the mathematics associated with thinking strategies and determining the effectiveness of each thinking strategies. As a result, the teachers' investigation of their students' work provided them insight into their students' perspective. Moreover, teachers mainly showed their appreciations their students' thinking even if the students' thinking was not completely correct or logical. So, the results showed that teachers are likely to improve their interpretations of students' thinking since they take on students' perspectives.

On the other hand, like Koellner-Clark and Lesh (2003) and Chamberlin (2002), Hallagan (2003) also used the model and modeling perspective as the framework of the study design. Yet, unlike them, Hallagan (2003) aimed to describe middle school mathematics teachers' interpretation of students' responses from selected algebraic tasks including the distributive property and equivalent expressions rather than to investigate the development of teachers' interpretation of students' thinking. In particular, she investigated what information middle school teachers gain about their students and how they interpret their students' algebraic thinking. In this study, the model eliciting activities were teachers' creation of "Ways of Thinking Sheet (WOT)" based upon students' answers to the given algebraic tasks and teachers' selection, analysis and interpretations of sample of students' work. The aim of the model eliciting activities was to reveal teachers' models of their students' algebraic thinking. Five teachers from two different middle schools (three male and two female) participated in this study. They were enrolled in two activities. In the first activity, they created WOT sheets by thinking of the subjects such as what mathematical concept might need to be retaught, the mistakes students made or students' unique solution ways. In the second activity, teachers created a library which includes samples of results of students' responses to the selected algebraic tasks. Since the nature of the study allowed qualitative techniques, the data was collected through semi-structured interviews, team discussions, WOT sheets, library of student work and observation/field notes. The grounded theory approach was used in the analysis of data. The findings of the study initially displayed that the models and modeling perspective was quite effective methodology to reveal teachers' models/interpretation of students' algebraic thinking. Moreover, the findings showed that there were major aspects in the focus of teachers' models of students' responses to tasks with equivalent expressions and the distributive property. As a result of investigations, teachers became aware of students' tendency to conjoin expressions, desired a numerical answer and their difficulties writing algebraic generalizations. Furthermore, teachers understood that visual representations were quite beneficial instructional tools. As a conclusion, Hallagan (2003) found that teachers participated in this study need more experience in analyzing and interpreting student work.

Similar to the aforementioned research, Krebs (2005) reported a professional development activity for teachers where they worked on mathematical tasks and made prediction on students' achievement on these tasks, listened and evaluated students' understanding. However, in her professional development activity, she used both students' written work and video clips as an artifact. About twenty middle grades teachers enrolled in the study, and they worked on 8th grade students' work. At the time the study was carried out, the 8th grade students had completed the linear relationship subject and they were working on exponential relationship subject. In this professional development activity, teachers first completed the mathematical task themselves, and then discussed on the possible gains of classroom teachers by using such kinds of tasks. Next, they worked on students' written work. They analyzed, explored and discussed on the students' way of thinking. After teachers completed students' written work, they wanted to more about students thinking, and they analyzed additional data which were video clips of students working on the tasks and video clips from interviews of students. As a result, Krebs (2005) reported that the engagement of teachers in students' work in a professional development activity was an opportunity to develop flexibility and proficiency in the assessment of their own students' work. Furthermore, she stated that teachers can recognize their students' weaknesses and strengths of the mathematical understanding with such type of experience.

Distinctively, An and Wu (2011) used "grading homework" approach and explored how teachers enhance their knowledge of students' thinking through grading students' homework as well as assessing and analyzing misconceptions in the homework . In this study, the researchers defined "homework as a task that provides students an opportunity to practice and reinforce their learned knowledge and skills in order to be ready for new lessons" (p. 722). On the other hand, according to them, "grading homework refers to a teacher evaluating students' understanding of their independent practice and analyzing their patterns of misconceptions" (p.722). Ten teachers at 5^{th} -8^{th} grade levels from four different schools participated to study. In this study, teachers were divided into two groups. These groups were called as experimental group and control group, and each of them included five teachers at different grade levels from 5^{th} to 8^{th}. The teachers in the experimental group were trained on how to group their students according to their achievement levels and to choose students from each group daily. Furthermore, they were trained on how they analyze students' errors. On the other hand, teachers from control group were trained to record their normal ways of grading students' homework with grading logs designed by authors. The data was collected through pre-post questionnaires for teachers and for students, classroom observations, interviews and teachers' daily grading logs. The teachers' pre-post questionnaires were similar to each other and involved four pedagogical content knowledge questions about fractions, decimals and percents. The focus of teachers' pre-post test questionnaires was teachers' knowledge of students' cognition and how to foster the growth of students' thinking. Moreover, while the aim of the pre-test for students was to identify their backgrounds in mathematics which provided information for experimental group teachers, the aim of the post-test was to determine the effects of grading homework on students' learning. In addition, the aim of daily grading log for the experimental group teachers was to help them to improve their knowledge of students' thinking. In order to analyze data both quantitative and qualitative method were used. The quantitative method was used to analyze students' pre- post test results, and the qualitative method was applied to analyze other data sources such as classroom observations, interviews or teachers' pre-post questionnaires. The data analysis displayed that grading homework through identifying and analysis of students' errors and misconceptions is one of the effective ways to enhance teachers' knowledge of students' thinking. Based on the findings, in this process, the researchers indicated that analyzing error plays a crucial role in teacher learning and understanding of students' thinking.

In their paper Steinberg et al. (2004) reported Ms. Statz's, who is fourth grade classroom teacher, change of engagement with children's thinking as a result of making mathematical discussions with her students in her instructions. In this research, the researchers focused on teacher-student talk to understand the nature of teachers' change. In this study, Ms. Statz taught mathematics using precepts of

Cognitively Guided Instruction (CGI) which is a professional development program designed by Carpenter et al. (1988) and where teachers are encouraged to apply research based knowledge about children's mathematical thinking to make instructional decisions. The data was collected at three points in time. While first author observed Ms. Statz for a five month period at the first point, at the second point, the following year, the second author observed Ms. Statz over a period of several months. And, several years later, at the third point, the researcher conducted interviews with Ms. Statz about her development as a teacher and think back her professional development from pre-service teachers up to now. The data was collected through classroom observations, audio-taped teachers' meeting with researcher and student assessments. In order to characterize teacher change, the researcher used "Levels of Engagement with Children's Mathematical Thinking" framework constructed by Franke, Carpenter, Levi and Fennema (2001). And as a result of analysis of data sources, the themes consistent with teacher change were determined with respect to this framework. The findings of the study revealed that Ms. Statz's knowledge about children's thinking dramatically changed in her third year of teaching. In the first phase, students talked about their solution strategies, and although Ms. Statz paid attention to these strategies, she scarcely challenged children to expand their thinking or she did not refer them in later discussions. In the second phase, after the participant researcher provided information about how individual children think, Ms. Statz recognized the differences between her knowledge of students' problem solving strategies and students' actual strategies. She noticed that while some of the students applied wrong strategies in solving problems, other used so basic strategies. In the third phase, while Ms. Statz started to spend much more time to talk to children about their thinking in her routine instruction, in the fourth phase she began to apply the knowledge of students' thinking obtained from one to one interaction in the group discussions in her instructions. Namely, the knowledge of students' thinking became guidance for her instructions. The researchers indicated that Ms. Statz's engagement with children's thinking to Level 3 according to Franke et al. (2001) scale at the beginning of the study, it went up to Level 4 at the end of her third year of teaching. And they emphasized that the change of Ms. Statz's engagement with children's thinking was closely related to her belief about the importance of students' thinking as well as her desire to know more about it.

Empson and Junk (2004) investigated the knowledge of teachers gains' after implementing a student-centered curriculum called as "Investigations in Number, Data and Space"; in particular, they examined what knowledge teachers applied to make sense of students non-standard strategies while they were implementing a student-centered curriculum and how the teachers' gains of this knowledge might be related to use of new curriculum materials. In this research, researchers indicated that the student centered curriculum means the curriculum designed to reveal students' ways of thinking. The subjects of the study were 13 elementary teachers at three, four and five grade levels at a single elementary school. All teachers had a teaching experience student-centered curriculum for one or two years at the time of the study. The researchers were interviewed all teachers once to learn about what teachers gained while implementing "Investigations in Number, Data and Space". The interviews included five open-ended questions. The first interview question was about what and how teachers learned after implementing student-centered curriculum, and the fifth question was about which unit of the student-centered curriculum helped the most teachers and students' learning. Second, third and fourth interview questions, which included three scenarios, were the focus of the inquiry. In the first scenarios (question 2), teachers were asked to produce at least three strategies for multi-digit multiplication students might apply. The question was aimed to understand the extent of teacher's knowledge of students' nonstandard strategies as well as their conceptual understanding of those strategies. On the other hand, in second and third scenarios teachers were presented nonstandard strategies of students to evaluate them. Specifically, whereas the aim of the second scenario (question 3) was to explore the depth of teachers' knowledge used to interpret student's novel strategy, the third scenario was to assess how teachers identified the validity of a novel nonstandard strategy. The analysis of data showed that teachers' knowledge of children's nonstandard strategies for multi-digit multiplication was broad and in some cases deep. Teachers' generation of different kinds of nonstandard strategies, their explanations of these strategies and their consistency with children's invented strategies reported in other documents were evidence for teachers' broad knowledge of children's thinking. Furthermore, the data showed that some of the teachers' knowledge of mathematics was not so connect to knowledge of children's thinking so that disconnection affected teachers' hypothesized responses of children's thinking. Moreover, the findings displayed that teachers' knowledge of children's thinking was related to implementing students centered curriculum.

On the other hand, different than engagement in students' written work and curriculum materials, the professional development programs where the video excerpts from teachers own classroom were used to investigate the changes of teacher learning of students' thinking. For instance, Sherin and Han (2004) examined teacher learning in the context of video club where a group of teachers watched and discussed videotapes of their classroom. This study was the part of professional development project "Fostering a community of teachers as learners" (Shulman & Shulman, 1994). Four mathematics teachers who had a range of teaching experience from 1 to 28 years were the participant of this study. The video clubs meeting were hold once a month from September through June and in total 10 video clubs meetings took place over the course of the year. Two researchers participated in the meetings as well where one's role was a facilitator and other's was participant observer. At each video clubs, the excerpts were selected from only two teachers' classroom. Prior to each meeting, one of the researchers videotaped one of the teachers' class and then following the videotaping, the researcher and teacher met together. While the camera focused on either student or teacher who are speaker during whole class discussions, during small group discussions or individual seatwork the camera focused on the teacher. At these meetings, they reviewed the videotapes and selected excerpts which were approximately six minutes long in order to show them at the video clubs. Each video club lasted approximately 40 minutes. The researcher who was the facilitator created a discussion environment by generally asking "any comments?" and "what did you notice?". The first seven video club meetings had same format and teachers watched and discussed only one video clip. On the other hand, the last three video club meetings was different format than first seven meetings. All video club meetings were videotaped. In the data analysis, the researchers focused on first seven video club meetings. They identified five topics which are pedagogy, student conceptions, classroom discourse, mathematics and other and analyzed the data with respect to these topics. The analysis of the data displayed the changes of what the teachers discussed in the video clubs and of how the teachers discussed the topics. Over time, in the discussions, teachers increased their attention to not only students' actions, ideas and thinking but also they moved the discussion of student thinking from statement of students' ideas basically to analysis of student thinking in depth.

Similar to Sherin and Han (2004), van Es and Sherin (2008) investigated the changes in teachers' thinking through video club designs. Seven fourth and fifth grade elementary teachers, who were in their third year of implementing reform based curriculum, were the participants of the study. Ten video clubs were carried out throughout the school year from October to May. All meetings had the same format. Prior to each meeting, two teachers' mathematics classes were videotaped by one of the researcher. While videotaping, the researcher focused on several central activities of the lessons. For example, during the whole class activities, the focus of the camera was interaction and discourse between teacher and students. In contrast, while during the small group work the camera zoomed out one or two groups students working together, during the individual seatwork teacher was the focus of the camera. After videotaping, the researcher who videotaped the classrooms looked at the tapes and determined the 5-7 min long excerpts. In the excerpts the following mathematical issues such as student's question about a particular concept and the teacher's corresponding explanation and students' confusions about mathematical issues were highlighted. At each meeting, the researcher's role was facilitator. The researcher not only provided the background about the teacher's lessons concerning to video excerpts but also helped teachers learn to notice and interpret students' thinking by prompting their thoughts. For example, the researcher asked several questions such as "What did you notice?" or "What do you think that says about student's understanding" (p.248). The data for this study involved transcripts of the video club meetings and two individual interviews with each teacher which are conducted before the first video club meeting and after the final video club meetings. In addition, the interviews with four elementary teachers from same school who were in the control group were the data source of this study. In order to examine how teachers learn to look at classroom interaction the researchers used to "Learning to Notice Framework" which includes three main aspects: (i) determining the important issues in teaching (ii) the use of one's knowledge about a context to reason about a situation (iii) making connections between particular events of classrooms and general principles of teaching and learning. The researchers analyzed firstly pre-and post-interviews in order to examine the changes over time. Examining video club data was the second stage of the data analysis process. The analysis of both pre- and post-interviews and video club meetings displayed the shifts in teachers' talk. Secondly, the interviews results revealed that teachers increased their comments on the students. In addition, while teachers commented on the issues of climate in the pre-interview, the focus of their comment were students' mathematical thinking in the post interview. The third shift was concerned to the dimension of stance. In the pre- interview, the nature of the teachers' remarks was descriptive; however, teachers were more interpretative in post-interview. Lastly, teachers' interpretation shifted from general to specific. Analysis of the video clubs also displayed shifts that were quite similar to shifts in the interview context. The percentage of comments about students and students' thinking increased over time. They began to make more specific and interpretative comments.

2.4.2 Research focusing on the development of pre-service teacher knowledge of students' thinking

Because of the ineffectiveness of the teacher education programs and the idea that teacher education could help pre-service teachers and teachers learn to construct knowledge in the context of practice, Lampert and Ball (1998) designed a multimedia environment which would provide an opportunity for pre-service mathematics teachers to investigate teaching in practice. The aim of Lampert and Ball (1998) was to provide pre-service teachers with more access to the complexities of the classrooms and to the various students' work each which has specific characteristics. In addition, they also wished that pre-service teachers were able to share common experience by working collaboratively. Therefore, in order to design an approach teaching called as "multimedia environment", they collected records of teaching for the whole year in two different classrooms. They chose their own classroom to videotape since they were willing to be recorded. The multimedia environment involved three sets of records which were named by researchers as "beginnings", "fractions" and "time/speed/distance". While beginning represents a set of teaching and learning in Ball's classroom in September, fraction represents other set from Ball's classroom in April through June of the same academic year. On the other hand, "time/speed/distance" was the third set of the records from Lampert's classroom in October and November. During the whole year, the researchers collected multiple records of teaching and each of set of records included them which were written notes (e.g., homework, quiz, daily notes) and interviews from third and fifth grade students, teachers' journals, video excerpts from lessons and interviews with teacher and students as well as transcripts of the video. The records of the practice in the multimedia environment were an opportunity for pre-service teachers to examine and understand teaching and learning in classrooms, to know more about children as well as mathematics so that they may never encounter such kinds of environment in their teacher education process.

Similar to Lampert and Ball's (1998) research aim, in their research, Masingila and Doerr (2002) explored how multimedia case studies can help pre-service teachers to make meaning of complex classroom experiences; in addition, to use student thinking for guiding instruction, how these case studies support them in developing strategies and rationales. For this aim, they developed case studies materials with their colleagues, which involved a four day lesson sequences in on 8th grade math class. The lesson sequences also included the video overview of the school setting, the teachers' lesson plan, video of class lesson, students' written work and a video journal of the teachers' reflection and anticipation on each lesson. On the other hand, the mathematical focus of the lesson sequences was ranking and weighting data. The aim of the use of these materials of the cases were not posed as specific dilemma for pre-service teachers to analyze and produce solutions; instead, they were created to engage the pre-service teachers in actively learning mathematics, discussing mathematical arguments and expressing their mathematical ideas about problematic situations. The participants of the study were nine intern pre-service mathematics teachers (grade 7-12) who were concurrently student teaching. For this study, pre-service math teachers enrolled in a seminar class during the five weeks and met once a week for two hours after their student teaching. The multimedia case study materials were used in seminar class. Pre-service teachers were requested to determine a specific issue related to both their own practice that they had been working on and teachers' practice that was addressed in the case studies. For each four weeks of the five weeks, pre-service teachers worked on the case studies materials and during the fifth week, they presented their papers that they created as a result of working on journal questions assigned pre-service teachers to answer after viewing the video cases. There were kinds of data sources such as transcript of all class discussions, a journal kept by instructor of the seminar and a questionnaire related to case study filled by pre-service teachers at the end of the semester. The researchers analyzed data in three phases. In the first phase, they developed a code scheme and coding the data independently by using inductive method. In the second phase, they made a detailed analysis of the issues of concern to pre-service teachers which they determined in the first phase. Lastly, in the third phase they compared the each pre-service teacher's issues and reasoning and created their categories. As a result of the data analysis they found that the data fit in three categories; namely, they made links between their own emerging practice and case study teacher's practice (i) pedagogical issues which include student participation and facilitation of classroom practice (ii) pedagogical issues from a mathematical perspective which involve controlling students' mathematical understanding, the role of questioning in improving student thinking and use of students' responses in furthering the teacher's mathematical agenda (iii) mathematical issues from a pedagogical perspective which include introduction and movement to mathematical ideas (p. 248). In addition, they found that pre-service teachers apply on their own perspectives to draw several dilemmas and tensions found in teaching. In conclusion, the researchers stated that the analysis of the case study materials offered pre-service teachers a shared experience to observe and interpret teachers. In this way, pre-service teachers went beyond to discuss the usual concerns with classroom management issues and they focused on more complex issues of classroom.

Similarly, Crespo (1998) explored the nature of pre-service elementary teachers' learning in inquiry based context. However, Crespo's (1998) context was based on "math penpal investigations". Therefore, she investigated pre-service teachers' learning through their math penpal investigations and the factors that influence their learning in this context. This study took place within the teacher education method course where pre-service teachers engaged in investigation project with 4th grade students. The focus of this investigation project was pre-service teachers' engagement in a math letter writing exchange with one or two students in fourth grade throughout the course. The participants of this research was one penpal teacher who is 20 years of teaching experience, eighteen (11 female and 7 male) penpal students who were nine and ten years of age in grade 4 as well as 13 pre-service teachers (12 female and 1 male). The main sources of the data were the math penpal letters written by both penpal students and pre-service teachers, the weekly math journals of pre-service teachers and final case study of pre-service teachers' project. An analytical framework was created based on pre-service teachers puzzles concerning to their problem posing, interpreting and responding practice and in order to find the patterns and changes in pre-service teachers' view and practice, this framework was used in the data analysis process. And, in the finding of the study, Crespo (1998) discussed the learning of pre-service teachers under three main issues as posing, interpreting and responding and three sub themes are associated with each theme. While the issues associated with posing were learning about challenging of responding to students' math work, to value problematic problems and to broaden goals and expectation of problems, the issues associated with interpreting were learning about the challenges of interpreting students' math work, to see and construct meaning from students' work and to question and revise claims about students' abilities and attitudes. Moreover, the issues pre-service teachers' learning associated with responding were learning about the challenges of responding to students' math work, to recognize and interrogate hidden message in their interactive discourse with students and to respond differently. Beside to themes associated with pre-service teachers, the founded factors which influence pre-service teachers learning involved (a) interactive experience with students, (b) engagement of collaborative exploration of problems and comparable students' work, (c) opportunity to revisit and interpret their experiences their experiences with students in multiple occasions.

Moreover, Crespo (2000) examined perspective teachers' interpretation of students' work and the change of their interpretation over time in the context of mathematics letter exchanges with fourth grade students. The aim of the letter exchange was to offer a context for prospective teachers to investigate and detect students' ways of thinking as well as communicating in mathematics. This study was conducted in mathematics teacher education course with 13 prospective teachers, took place each week and lasted 11 weeks. As a result, Crespo (2000) reported "the change in the focus of interpretation" and "the change in the interpretative approach" as two significant interpretative turns. That means, the findings of the Crespo's (2000) research, while prospective teachers' early interpretations were based on focusing on the correctness of students' answers, they started to focus on the meaning of the students' work in their later interpretations. Prospective teachers' interpretations became more detailed and exploratory. Moreover, whereas prospective teachers made conclusive judgments about their students' mathematical thinking, abilities and attitudes, their interpretation became more thoughtful.

On the other hand, Vacc and Bright's (1999) research can be displayed as another example where pre-service teachers were offered an opportunity to analyze, interpret and discuss classroom situations. However, unlike other research stated above, Vacc and Bright's (1999) focus was to investigate the change of pre-service teachers' beliefs about teaching and learning mathematics rather than their knowledge of student thinking and the change of their ability to design mathematics courses on the basis of students' mathematical thinking. Their research framework was based on CGI (e.g., Carpenter et al., 1988). Therefore, in their research, they provided an opportunity for pre-service teachers to work children's thinking in mathematics method course. Their research process was based on presented mathematics story problem to pre-service teachers and then pre-service teachers were asked to produce alternative solutions to the problem. Next, pre-service teachers were watched videotaped examples of solution strategies concerning to same mathematical problems and discussed on it. Their discussions were on both how children's solutions were similar with their solutions and on what following problems might be presented to children. The findings of the research revealed that pre-service teachers' belief was changed to constructivist orientation about learning mathematics. In addition, their beliefs about teaching and learning mathematics significantly changed.

D'Ambrosio and Campos (1992) examined to what extent the research experience could improve pre-service teachers' understanding of children's knowledge of mathematical concepts. Five senior pre-service teachers involved in the study. In this study, pre-service teachers had two kinds of role. First, they were the participants of macro study, namely, participants of this study and second they were a member of micro study; that is, they were enrolled in a small research project addressing children's understanding of mathematics concepts. It means as the participants of the study, the pre-service teachers formed a research group and conducted a study focusing on children's understanding of a certain mathematical topic. This micro study consisted of several consecutive steps such as select identification content of focus, familiar with existing research literature, determination of the subjects of the research, instruments to be used, administration of the instrument, data collection and data analysis. As focus of content of micro study, the pre-service teachers chose the fractions. Pre-service teachers read some researches involving both children's understanding of fractions and examples of various forms of data collection, and then discussed them. After reading and discussing the research, they started to develop their research. They began research to collect data and to examine children's work. They used a test on fractions which was developed by a researcher and was a multiple choice test. After they administered the test to large student sample at grade 5^{th}-8^{th}, they analyzed the data. In their analysis of data, pre-service teachers first were interested in looking at the percentages of students present specific answers, and then they decided to look the pattern in students' responses. In this process, pre-service teachers encountered several conflicts with the information they gathered. Therefore, they revised and developed the instrument to gather more robust information children's understanding of fraction concept. Next, they decided to make interview with students to gather information about their understanding. Because the information about children understanding of concept they obtained from interviews were quite different than gathered through the other instruments, several discussions were raised in the group. Namely, the micro study went on the sequence of such kinds of events. On the other hand, the macro study focused on evidence of situations in micro study where the conflicts appeared and how the pre-service teachers resolved the conflicts. The conflicts included the issues such as pre-service teachers' amazement of the results because of contradictions with their expectations, the differences their predictions and student responses, and the different ideas appeared in group discussions. The data for the macro study was collected by written documents and observation and field notes. Briefly, the results indicated that this research experience led pre-service teachers to question instructional practices and they became quite inquisitive of children's understanding of a topic. At the end of the study, they became familiar with children's understanding of fractions and they enhanced their knowledge about children's intuitive understanding of the topic. Briefly, research experienced provided them with rich learning environment to develop their effectiveness as teachers.

In their study, in order to enhance pre-service mathematics teachers' understanding of children's mathematical thinking, McDonough, Clarke and Clarke (2002) required pre-service mathematics teachers to conduct and analyze one to one mathematics assessment with primary children. The research conducted as a part of three year professional development project "The Early Numeracy Research Project (ENRP)" and the assessment tool was drawn from this project. The research carried out at two universities. At one of the university, pre-service teachers conducted interview individually a child in grade 1 or 2 as part of assessment for their first mathematics education unit which was the core unit. During the interview, pre-service teachers presented orally a range number of tasks. Next, in their final year elective unit, they again carried out interview with one child in grade 2. This interview was full interview and the tasks were about Number, Geometry and Measurement. At another university had interview individually with two children aged between 7 and 10 by using a series of addition and subtraction tasks. The aim of the interviews at both universities was to increase pre-service teachers' both attention and knowledge on what young children know, what they can do and what are their possible strategies used in the tasks as well as provide a model of tasks and questions that can be used to elicit children's thinking. In this study, the data was collected by two pages questionnaires which were given pre-service teachers who were in the first mathematics education unit and had conducted interviews. In addition, researchers had an interview with the five pre-service teachers in the first mathematic education unit and conducted focus group discussion with six pre-service teachers from the early numeracy elective unit. The analysis of the data suggested that use of one to one interviews provided important benefits with pre-service teachers. Pre-service teachers not only enhanced their appreciation of what young children know and can do but also increased their awareness of students' kinds of thinking and possible strategies that they used in the tasks. In addition, pre-service teachers gained insight into the power of giving children one to one attention and time.

2.5 Summary of the Literature

In this chapter, several fundamental bodies of theories and literature closely related to aim of this research have been addressed. Initially, the knowledge what teacher need to know for teaching mathematics was presented by focusing on relevant theories on teacher knowledge (e.g., Ball, 1988, 1990, 1997; Ball et al., 2008; Fennema & Franke, 1992; Grossman, 1990; Hill et al., 2008; Shulman, 1986, 1987).

In these theories, it was revealed that although teacher knowledge was differently categorized by the researchers, the researchers had common idea on the core knowledge base what teachers need to know for teaching mathematics. As one of the core knowledge of teacher in teaching and learning mathematics for understanding, these researchers stressed the necessity of pedagogical content knowledge. In addition, they emphasized teachers' knowing and understanding of students' conceptions, errors and difficulties as one of the important dimension of pedagogical content knowledge. Then, it was followed by the role of pre-service teacher education programs on the development of pre-service teachers' knowledge for teaching mathematics (Grossman, 1990; Kennedy, 1999). The relevant literature revealed that during their pre-service teacher education process, different sources such as apprenticeship of observation in their undergraduate courses, disciplinary background, professional coursework, and learning from teaching experience, contribute to the development of their knowledge for teaching. On the other hand, the literature also indicated that the deficiencies and ineffectiveness of teacher preparation programs in the development of pre-service teachers' pedagogical content knowledge. As the main source of the weak impact of pre-service teacher education programs, it was presented pre-service teachers' early experience, the structure of curriculum, institutional and programmatic contexts of teacher preparation programs (as cited in Lampert & Ball, 1998). Next, the research findings on both pre-service teachers and teachers' knowledge of students' thinking offered the supporting evidences on the ineffectiveness of the teacher education courses on the development of their knowledge for teaching mathematics because the findings indicated the gaps between how students think about particular topics in mathematics and what teachers/pre-service teachers know about those (e.g., Akkoç et al., 2007; Ball, 1988; Bergqvist, 2005; Even & Wallach, 2004; Hadjidemetriou & Willams, 2002; Kılıç, 2011; Klein & Tirosh, 1997; Tirosh, 2000; Tirosh et al., 1998). Then, the literature offered the possible ways of how teacher education programs help pre-service teachers to increase their awareness across students' various and sometimes incorrect common cognitive processes and thinking (Ball, 1997; Ball & Cohen, 1999; Jaworski, 1990; Smith, 2001; van Es & Sherin, 2008). And, lastly, the research findings which have focused on developing teachers' knowledge of students' mathematical thinking both teachers (e.g., Carpenter et al., 1988; Chamberlin, 2002; Hallagan, 2003; Kazemi & Franke, 2004; Koellner-Clark & Lesh, 2003; Sherin &

Han, 2004; Steinberg et al., 2004) and pre-service teachers (e.g, An & Wu, 2011; Crespo, 1998, 2000; Lampert & Ball, 1998; Masingila & Doerr, 2002; Vacc & Bright, 1999) displayed that teachers/pre-service teachers' knowledge, understanding and attending to students' thinking remarkably enhance when they engage in different kinds of practice-based materials such as students' written work, video cases from real classrooms, interviews with students etc.

All of the theoretical framework of the study and synthesizes the relevant research literature contribute to my understanding teacher education programs should help pre-service teachers to increase their knowledge of students' thinking, in particular, their awareness across students' various, common, correct and incorrect cognitive thinking processes.

CHAPTER 3

METHODOLOGY

In this chapter, I initially present the research context and the design of the research. Then, I discuss the data collection and data analyzing strategies and issues. Next, I state that how I provide trustworthiness of my research's findings, and how I address the ethical issues that can be appeared during my research.

This is a qualitative research study aimed to investigate pre-service teachers' knowledge of students' thinking within an undergraduate course context. So, the research questions are:

1. What do pre-service secondary mathematics teachers predict about students' ways of thinking before they engage in students' works about solutions for non-routine mathematical tasks?

2. What do pre-service secondary mathematics teachers identify about students' ways of thinking while they engage in students' works about solutions for non-routine mathematical tasks?

3. What do pre-service secondary mathematics teachers value in students' solutions produced for non-routine mathematical tasks?

4. How do pre-service secondary mathematics teachers interpret students' thinking as manifested in students' works about solutions for non-routine tasks?

5. What do pre-service secondary mathematics teachers focus in terms of students' ways of thinking in analyzing students' works about solutions for non-routine tasks?

3.1 Research Method

In this dissertation, the nature of the research questions lends itself to the use of qualitative research techniques. The research method of this study is case study.

According to researchers (e.g., Gall, Gall, & Borg, 2007; Yin, 1984), case studies rely on the contemporary events, and their main sources of evidence are “direct observation” and “systematic interviewing”. In case studies, the investigator does not manipulate behavior directly and systematically, and the investigator has little or no control on the events. Yin (1984) indicated that case studies mainly are preferred, when the researcher control is little over the events and when there is a contemporary phenomenon in the real life context as a research focus. Yin (1981a, 1981b) defines as “case study is an empirical inquiry that investigates a contemporary phenomenon within its real life context, when the boundaries between phenomenon and context are not clearly evident and in which multiple source of evidence are used” (as cited in Yin, 1984, p. 23). Similarly, Gall et al. (2007) define case study as “the in-depth study of one or more instances of a phenomenon in its real life context that reflects the perspective of the participants involved in the phenomenon” (p.447).

In the case studies, defining unit of analysis, that is defining what the case is, is one of the fundamental components. A case can be an individual, a group, an event, a person or a curriculum. For example, if an individual is primary unit of analysis of a case study, the information about the relevant individual would be collected. If the unit of analysis is a small group, the people involved by the group have to be distinguished from those who are not included.

In this study, the unit of analysis (case) is knowledge of student thinking of 25 pre-service secondary mathematics teachers enrolled in an undergraduate course. The contemporary phenomenon “pre-service teachers’ knowledge of students’ thinking” in the real life context as a research focus is investigated in this research in depth.

3.2 The Participants

This study took place at Secondary Science of Mathematics Education Department in a state university in Ankara, Turkey. The participants of this study were 25 pre-service secondary mathematics teachers enrolled in an elective mathematics education course called “Mathematical Modeling for Pre-service Teachers”. Data were collected during the second semester of 2011-2012 academic years.

The pre-service teachers enrolled in this research were in 3rd, 4th or 5th years of their mathematics education program. Of the 25 pre-service teachers, 16 were in 3^{rd} year, 7 were in 4^{th} year and 2 were in 5^{th} year. The 4^{th} and 5^{th} year pre-service teachers have completed most of their university level mathematic courses (e.g., Calculus I, II, Abstract Mathematics, Linear Algebra, Algebra, Analytic Geometry, Topology, and Complex Analysis). Then, they were in the progress taking the other mathematics courses (probability, abstract algebra and number theory); pedagogic courses (e.g., measurement and evaluation, educational statistic and quantitative research methods) and pedagogical content courses (e.g., methods of mathematics teaching I-II). On the other hand, the 3^{th} grade pre-service teachers were still taking the mathematics courses such as Number Theory and Topology, Introduction to Algebra, and Euclidean Geometry and they were also taking their initial pedagogic courses such as Theories and Approaches in Teaching and Learning. They took neither a method course in mathematics education nor a mathematics education course. Mathematical Modeling for Pre-service Teachers was the first mathematics education course taken by the 3^{rd} graders.

These participating pre-service teachers involved seven male and eighteen female. Moreover, the age range of the participants was 19-22. Average of their GPA was 2.66 on a 4-point scale with the standard deviation of 0.37.

3.3 Research Context

This research was conducted as a part of an undergraduate course designed to develop pre-service mathematics teachers’ knowledge in and about mathematical modeling in teaching and learning mathematics. The course was based upon a 3-year research project called “Mathematical Modeling in Secondary Mathematics Education: Pre-service and In-service Teacher Education.” The project aimed to fulfill three purposes:

1. To develop mathematical modeling tasks to use both in teacher education programs and mathematics classrooms at secondary level (grades 9-12) in accordance with objectives Turkish mathematics curriculum.

2. To develop and implement professional development program for inservice teachers about mathematical modeling and to examine the possible effects of the program on teacher’s knowledge, belief and practice.
3. To develop an academic course for pre-service secondary mathematics teachers and to examine the possible efffects of this course on pre-service teachers’ attitudes towards mathematical modeling and their knowledge and competencies in use of mathematical modeling tasks (activities) mathematics education.

The course mentioned in this study was designed and offered as part of the third aim.

3.3.1 The course setting

This course was a specially designed mathematics teacher education course called “Mathematical Modeling for Pre-service Teachers”. The university, where this research was conducted, was a state university in Ankara, Turkey and, in this university this course was offered first time at second semester of the 2011-2012 academic years. This was an elective course for pre-service secondary mathematics teachers (grade 9-12) enrolled in 5-year teacher education program at the Secondary Science and Mathematics Education Department of Faculty of Education. This course served to realize several purposes such as “pre-service teachers develop modeling competencies, understand the characteristics of modeling activities, learn how to use modeling activities in mathematics teaching and interpret students’ mathematical thinking in the context of modeling activities”. This course was scheduled once a week for 4 hours every Friday afternoon from 13.30 to 17.30. It ran for 15 weeks from February 10, 2012 to May 18, 2012. But, in the mid of the semester, one of the weeks was midterm week for the university and there were not any classes during this week. Therefore, the course duration extended one week more to May 25, 2012 and the last course ran at the same course hour of this day (see Appendix M). The classroom, where the course took place, was different from a regular lecture classroom. This classroom was called as “Mathematics Laboratory” classroom where was mainly taken for method courses due to its appropriateness for group working. It was large enough for 25 pre-service teachers. Figure 3.1 displays a scene from the classroom layout.

Figure 3.1 A scene of classroom layout where the course took place

Throughout the semester, pre-service teachers were expected to join the group work, and contribute to the group studies and class discussions by speaking, listening, observing, sharing ideas and reflecting on the assigned readings and related materials. Therefore, from the first week to last week of the course, pre-service teachers worked in groups of 3-4. In the first week of the course, which was the “Introduction” week, pre-service teachers were required to organize their groups. While the groups were forming, the researcher did not interfere at all, and the groups were formed with respect to pre-service teachers’ own preferences. As a result, pre-service teachers were organized into seven groups, while four of the seven groups included four pre-service teachers; three of the groups involved three pre-service teachers as presented in Table 3.1. The seating arrangement of pre-service teachers at each group is seen in Figure 3.2. And, throughout the semester pre-service teachers worked with the same group.

Table 3.1 The groups of pre-service teachers with their pseudonyms

Pre-service Teacher Group No	Pre-service Teachers (PSTs) in the Group	# of pre-service teachers
PSTG 1	PST1*, PST17, PST18	3
PSTG 2	PST13, PST14, PST15, PST16	4
PSTG 3	PST5, PST6, PST19	3
PSTG 4	PST10, PST11, PST12, PST20	4
PSTG 5	PST7, PST8, PST9, PST25	4
PSTG 6	PST21, PST22, PST23	3
PSTG 7	PST2, PST3, PST4, PST24	4

*PST1 means pre-service teacher numbered as 1.

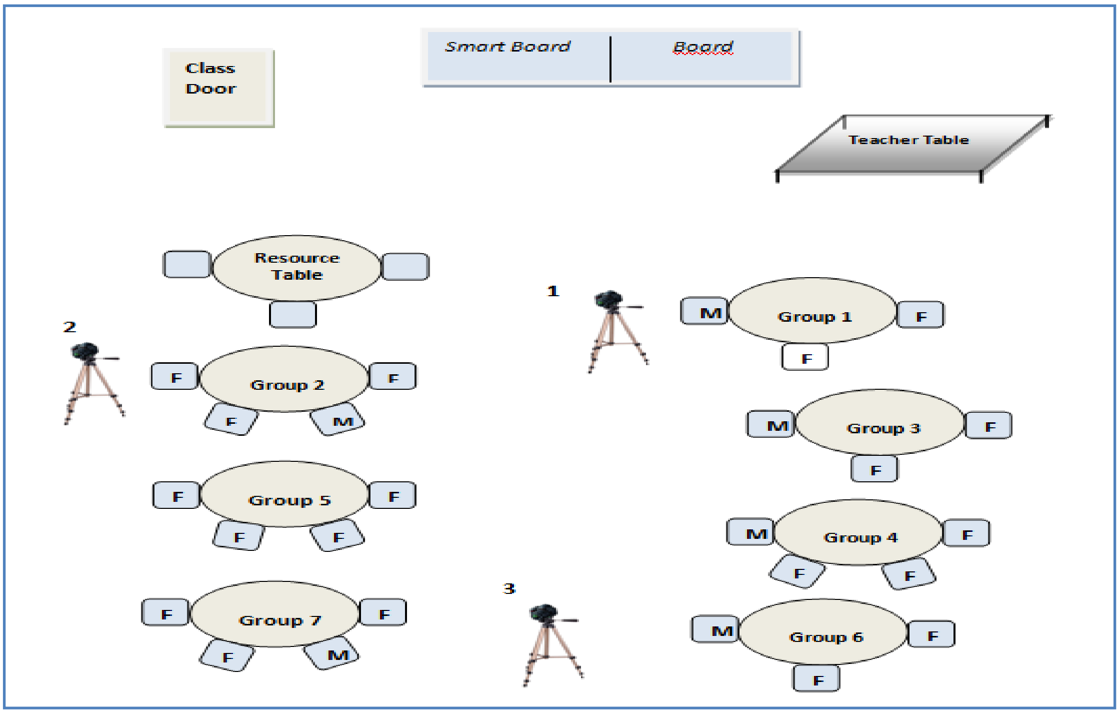

Figure 3.2 Seating arrangement of pre-service teachers

Pre-service teachers' attendance was mandatory because of the nature of this course. Class discussions of modeling tasks and students' ways of thinking for each week were significant parts of this course. Therefore, pre-service teachers were expected to attend all the class and arrive on time unless there was a valid reason that could be documented. Throughout the semester pre-service teachers were given several assignments such as writing a reflection paper on each modeling task and on students' ways of thinking, developing an authentic modeling task as well as a lesson plan to implement developed modeling task. All assignments had to be submitted in both paper (hard copy) and electronic format. Electronic versions of the assignments had to be sent via e-mail attachment to the instructor and the assigned course assistant(s).

Throughout the semester, all courses were videotaped and audio taped. The videotaping took place using three cameras and audio-taping took place using seven voice recorders. In general, one camera (#3) was used to record whole class, namely, it followed both the teacher and all groups of pre-service teachers during the course activities. The other two cameras (#1 and #2) followed two selected groups of pre-service teachers while they were working on the course activities (see Figure 3.2). In addition, these two cameras were also used to follow speakers (either teacher or students) during whole class discussions. On the other hand, seven audio recorders were used to record the interactions and discourse that took place among pre-service teachers in each of the seven groups.

3.3.1.1 Instructor (teacher) of the course

The instructor of the course was full time assistant professor of mathematics education at the Secondary Science and Mathematics Education Department in a state university in Ankara. He had 4.5 years of experience in teaching elementary schools and 8 years experience in teaching universities. He was student friendly instructor. He had given various mathematics education courses both graduate and undergraduate levels. The mathematics education courses offered by him were "Issues in Science and Mathematics Teacher Education" (graduate level), "Hands-on Activities in Mathematics Instruction" (undergraduate level), and "Mathematical Modeling for Teachers" (undergraduate level). In addition to them, the instructor had theoretical experience on teaching and learning mathematical modeling, as he had took a mathematical modeling course in their doctorate education. Furthermore, although the contents were different than the one offered for this study, the instructor had offered a mathematics education course twice on mathematical modeling at the university where he works.

3.3.1.2 The major components of the course

There were three major components of this course which were “solving mathematical modeling tasks (non-routine mathematical tasks), working on students’ ways of thinking on these tasks, and developing an authentic modeling task based on a real-world situation and implementing the developed modeling task in this course”. In addition to these three major components, use of technology in solving modeling task was a minor component of the course content. In the second week of the course some fundamental applications of Spreadsheet” (Microsoft Excel Program) and “Graphing calculator” were introduced to pre-service teachers since these programs would be used in the course activities. Furthermore, in order to support pre-service mathematics teachers’ knowledge about mathematical modeling in teaching and learning mathematics that they would acquire by working on modeling tasks, the theoretical knowledge regarding nature of modelling tasks, the role of teacher in teaching mathematical modeling, and the role of group working in modeling were presented by the instructor. That means, the instructor of the course made a presentation for each topic. In addition, class discussions were made based on these topics. Therefore, the presentations and classroom discussions based on the presentations were another minor component of this course (see Appendix M).

3.3.1.2.1 Solving non-routine tasks (mathematical modeling tasks)

In this study, the non-routine tasks refer to mathematical modeling tasks (or model eliciting tasks). They are realistic tasks and directly related to lives of students. These tasks have several significant characteristics that make difference them from traditional textbook problems (Chamberlin, 2002; Chamberlin & Moon, 2005; Lesh & Doerr, 2003; Lesh & Harel, 2003; Lesh, Kelly, Hoover, Post, & Hole, 2000). According to these researchers, for instance, in these tasks, students are asked to create a model and generalize their model to other situations. Students generally are required to work in a group and spend long period of time as well as much effort to complete their solutions. Moreover, each student group may produce their own unique solution by using their own method. Furthermore, students need to be in communication with their peers while they are working on the tasks and documenting their solutions. In such kind of tasks, because students interpret a complex real-world situation and formulate a mathematical description, they cannot solve the tasks by making simple calculations. They go beyond to produce short and brief answers. In these tasks, students also mathematize the realistic situations by developing an explicit mathematical interpretation to situations. And, the realistic nature of the tasks may increase students' creativity to solve the tasks. Furthermore, the final product of the students' solution is complex artifacts which are sharable and re-usable in other situations. The solution of the students produced to these tasks not only provides information about the final result but also whole thinking process of students that contributed to these final results. Therefore, unlike routine textbook problems, these tasks include rich information on students' thought processes. Briefly, these characteristics of modeling tasks allow them to be powerful and useful tools to understand ways of student thinking both for teachers and teacher candidates. Throughout the semester pre-service teachers completed six modeling tasks (non-routine mathematical tasks), which were developed for grades 9-12 mathematics curriculum and in accordance with grade level objectives and integrating multiple topics and concepts by the project team in varied subject matters such as trigonometry, trigonometric functions, derivative, average and exponential functions. Table 3.2 shows each non-routine mathematical task with their core mathematical domains engaged by pre-service teachers throughout the semester.

Table 3.2 The non-routine mathematical tasks and their core curriculum domains

Title of Non-Routine Mathematical Tasks	Related Curriculum Content
Summer Job	(Weighted) Average, rate, ratio
Ferris Wheel	Unit circle, trigonometric functions
Street Parking	Trigonometry, trigonometric relationships, geometry, the geometry of triangles,
Bouncing Ball	Exponential functions, exponential inequalities
Roller Coaster	Slope of tangents, derivative, graph of functions, curve analysis
Water Tank	Graph of functions, derivative

Pre-service teachers worked in groups of 3-4 on each activity and completed their activity sheet collaboratively. Before pre-service teachers started to work on the tasks, each task was introduced by the instructor of the course. He also asked students to read and discussed the task with their group members. In whole class introduction, the instructor strongly suggested pre-service teachers to read initially the task individually and to develop their own interpretations of the possible solution ways. For this reason, for each task, pre-service teachers were given 5 to 10 minutes. After pre-service teachers thought on their own solution individually, the instructor wanted them to work independently in their groups to complete the task. And, in order to work on the each task collaboratively, pre-service teachers were given approximately 120-150 minutes. When they were working together to complete task, pre-service teachers explained their points of view to each other, produced assumptions, expressed and revised their ways of thinking. Towards the end of given time, pre-service teachers were required to document their works on the given poster sheet; that is, they were required to write their all solution approaches with their final solutions and provide explanations for their answers. Then, the groups were required to present their solutions and one of the group members of the each group presented their group solutions to the class on the board. In this process, whereas the groups were explaining and justifying their solutions ways they developed, the remaining of the pre-service teachers and the instructor asked several questions and gave constructive feedback in order to clarify, discuss or extend the solution ways of the groups. Although in some activities, all groups could not present their group solutions because of the time deficiency, for each activity, at least five groups out of the seven could present their solutions. Pre-service teachers were given approximately 30-40 minutes in total to discuss their group solution ways so that each group had average 5 minutes.

As pre-service teachers were working on the each task, the instructor moved around the classroom to observe their interactions, listened to their group discussions and answered their questions when they needed. However, the instructor was careful not telling the pre-service teachers the correct solution way (s) and giving them hints at all if their either solution ways or conclusions were wrong. When they needed help, instructor asked pre-service teachers several probing or guiding questions to invite them to think in-depth or rethink about the issue and revise their thinking ways. That is, the instructor did not interfere with pre-service teachers' ways of thinking directly in this process. All whole class interactions and two of seven the group works were videotaped. In addition, group works were audio-recorded.

3.3.1.2.2 Working on students' ways of thinking

Pre-service teachers worked on students' ways of thinking through students' solution papers and video episodes obtained from high school students for four of the implemented six modeling tasks presented in Appendix A. This component of the research will be addressed in depth in the research process section (see Section 3.3.2.3).

3.3.1.2.3 Micro-teaching activity

As an assignment of the course, pre-service teachers were expected to develop an authentic modeling task based on a real-world situation. After constructing the modeling activity, pre-service teachers were expected to find a solution to this problem, prepare an implementation plan for the modeling activity, implement the modeling problem in class and discuss the implications of modeling in teaching and learning mathematics. Therefore, in the last two weeks of the course, pre-service teachers applied their modeling task that they developed in the course process. Each microteaching activity lasted approximately 45-60 minutes for each pre-service teacher group.

3.3.2 Research process

3.3.2.1 The setting where the students' works come from

In this study, for pre-service teachers, students' solution papers and video-episodes were used as two specific samples of students' works. Students' solution papers and video records belonging to these solutions were produced from classroom implementations with secondary students (9-12 grade level) where they had worked and discussed about non-routine thought revealing tasks as a part of ongoing three-year research project about mathematical modeling in secondary schools (see pp. 61-62). The classrooms, where the students' works were produced, were from three different secondary schools. While one school had been selected for the pilot study of the three-year project, other two schools had been selected for the main study of the project to fulfill project's first and second purposes. Whereas in the school selected for the pilot study of the project twelve modeling tasks were implemented, in the schools selected for main study of the project were seven (identical) modeling tasks were implemented.

Likewise pre-service teachers enrolled in this study, high school students joined the group works and contributed to the group studies while they were producing their solutions in classroom implementation. Students worked in their groups to develop their solutions. Towards the end of the course, the groups presented their solutions and explained their solutions in depth. In order to clarify group's solution, the remaining student groups asked questions about the aspect of the solutions and made comment on the solutions. Although the number of the students in groups changed with respect to number of students in the implementation classes and the preference of the implementation teacher, student worked mainly in small groups of four to six students in each classroom of each secondary school approximately 90 minutes (two course hours) duration. Each mathematical modeling task implemented in the two different classrooms of each school. At the end of the each implementation, the students' solution papers with scratch papers from each group were collected.

During the implementations, three video cameras were set up in the classrooms. One of the video camera mainly focused on the class as whole as well as followed the teacher throughout the lesson, and the other two cameras focused on two student groups as they worked during small group activities. On the other hand, all these cameras focused on the groups who presented their solutions on the board as a whole at the end of the lesson. That is why, for each task, video records including three different contents were obtained: (i) whole class records (ii) records of presentations of solution developed and presented as a group (iii) records of collaborative activity of two groups.

3.3.2.2 Preparation of the students' works for the research

Prior to selecting of the students' solution papers and creating video episodes, first of all, all solution papers produced from these tasks and recording of presentations of solution as a group were matched each other by the researcher. Next, they were simultaneously examined. That means, while the researcher was examining the students' solution papers, she was watching these video records of the students at the same time. And then, the selection of the students' solution papers and producing video episodes were done consecutively. Namely, after the solution papers of students were decided to use in the classroom, the video records belonging to these solution ways were purposefully edited. In order to use in this research, four to six solution papers among all group solution papers were selected. Then, the video episodes belonging to these solution papers were created from the records of presentation of solutions and records of collaborative activity of two groups. Throughout the semester, pre-service mathematics teachers worked on students' works belonging the tasks "Street Parking", "Bouncing Ball", Roller Coaster" and "Water Tank" as presented in Table 3.3.

Table 3.3 The number of students' solution papers and video episodes belonging each task with students' grade

Modeling Tasks	# of students' selected solution papers	# of students' used video episodes	Grade level of students	The number of selected schools
Street Parking	5	4	11	1
Bouncing Ball	4	4	9 & 11	1
Roller Coaster	5	5	12	1
Water Tank	6	6	10 & 11	2

3.3.2.2.1 The selection of students' solution papers

Students' written work was students' solution papers including their final solutions and their all scratch sheets (worksheets) which they are generated by groups of students during their solution process (see Appendix B). These works display students' different ways of thinking emerged during their solution process.

Prior to selecting the students' solution papers for four non routine mathematical tasks used in the course, all students' solution papers for 10-12 groups which were produced from each classroom implementations of three different high schools were initially examined by the researcher in depth. Then, for each task, the solution papers belonging to 4-6 student groups, which had totally different from each other in terms of solution ways (either correct or incorrect) were purposefully selected. Furthermore, in the selection of the students' solution papers, it was carefully paid attention to the issues which represented students' different ways of thinking, such as the understanding and interpretation of the problem in the different ways, use of various kinds of mathematical subjects, concepts and representations in solving problem, taking place students' algebraic, logic error or intuition error, having difficulties and challenges on several subject and concepts.

3.3.2.2.2 The production of the video episodes

Analyzing students' thinking was not limited to students' solution papers; therefore, the video episodes were used as the complementary of these students' solution papers. The goal was to use video episodes as a vehicle through which pre-service teachers explore students' mathematical thinking in depth. In order to be an effective tool of the video episodes for this research, the video episodes were selected and produced purposefully. In general, the video episodes illustrated several issues such as students' explanations and discussions of their solution ways, their confusions about the issues being discussed and their interaction with their teacher during their solution process.

Prior to producing the short video episodes for using this research, all the records, which belonged to each solution papers of students and produced from classroom implementations with secondary students, were both reviewed two times by the researcher and brief excerpts highlighting mathematical issues that were raised in the implementations were identified. For example, the researcher selected a time slot in which students in a group discussed their ideas to produce solution(s) or in which there appeared to be some confusion on the part of the students about the mathematical issues being discussed. Other time slot illustrated students' questions about a particular solution ways and the teachers' corresponding explanations. The start and stop points of the all time slots were write down on the notebook by the researcher. Then, the all the videos, which were planned to edit, were loaded by the researcher to the video-editing program in sequence. After the start and stop points of the imported videotapes were adjusted, the videorecords were splitted into the wanted parts. In this way, the smaller video excerpts as needed were created. Then this process was followed in the same way for other selected videorecords and formed small video excerpts each has 3-5 minutes were created. Then, all the formed small excerpts were combined into a single excerpt; in this way, a new contiguous excerpt (episodes) was created in order to use in research. Because the duration of the each excerpts had 3-5 minutes, the duration of the combined excerpt was approximately 7-10 minutes. In order to edit videos, both "Movie Maker" and "Wondershare Video Editor" programs were used.

For this study, two different kinds of video episodes were produced. The first type of the video episodes was produced as a result of editing of the video records of presentation of students' solutions. Therefore, the theme of these video episodes was presentation of the students' solution ways in front of the board, and the content of these presentations was the briefly explanation and justification of solution ways of students they developed as a group. The second type of the video episodes produced from the records of two student groups as they were working during group activities. That is why; the theme of these video episodes was small group activity and they included the snippets from the whole solution process such as "the comprehensibility of the problem statement by the students, proposed first solution ways or mathematical ideas by students, the discussion of students on the task, their difficulties, errors and challenges".

The produced two different types of video episodes had different depth of the students' thinking and clarity. Although both created types of video episodes provided evidence of students' thinking, the first type provided less detail than the second type of video episodes. But, in both, students thinking were transparent and students' responses were far from focusing on correctness and use of algorithms by rote. Namely, in these video episodes, students were engaging in mathematical reasoning and problem solving and explaining and justifying their reasoning/thinking process. For all produced video episodes, a corresponding transcript was prepared. Furthermore, other issues such as the technical quality of the sound and the optimum length of time for a video clip were taken into consideration by the researcher while the videos were edited.

3.3.2.3 Working on students' ways of thinking

The aforementioned component of the course "Working on Students' Ways of Thinking" was the main site for this research. However, two major components "working on modeling task" and "working on students' ways of thinking" were intertwined for this study. Pre-service teachers were not familiar with the modeling tasks at the beginning of the course since they had not had any experience with modeling tasks prior to this course. So, it was aimed pre-service teachers to understand the characteristics and nature of the modeling tasks during the first three-four weeks of the course. The main data collection process started in the fifth week of the course and ended in the thirteenth week of the course; that is, it lasted eight weeks (because of no class at 10^{th} week) as presented in Appendix M. However, one week before and one week after of main data collection process, two self-report questionnaires were administered to the pre-service teachers. On the other hand, from the first week of the course to the last, the data were collected via video-cameras and audio recorders so that they supported the main data as well as presented information about pre-service teachers' mathematical background, level of research participation and attitudes towards research. The main data collection of this study consisted of 4 two-week long cycles. As an example, the Figure 3.3 shows one 2-week cycle of the data collection process.

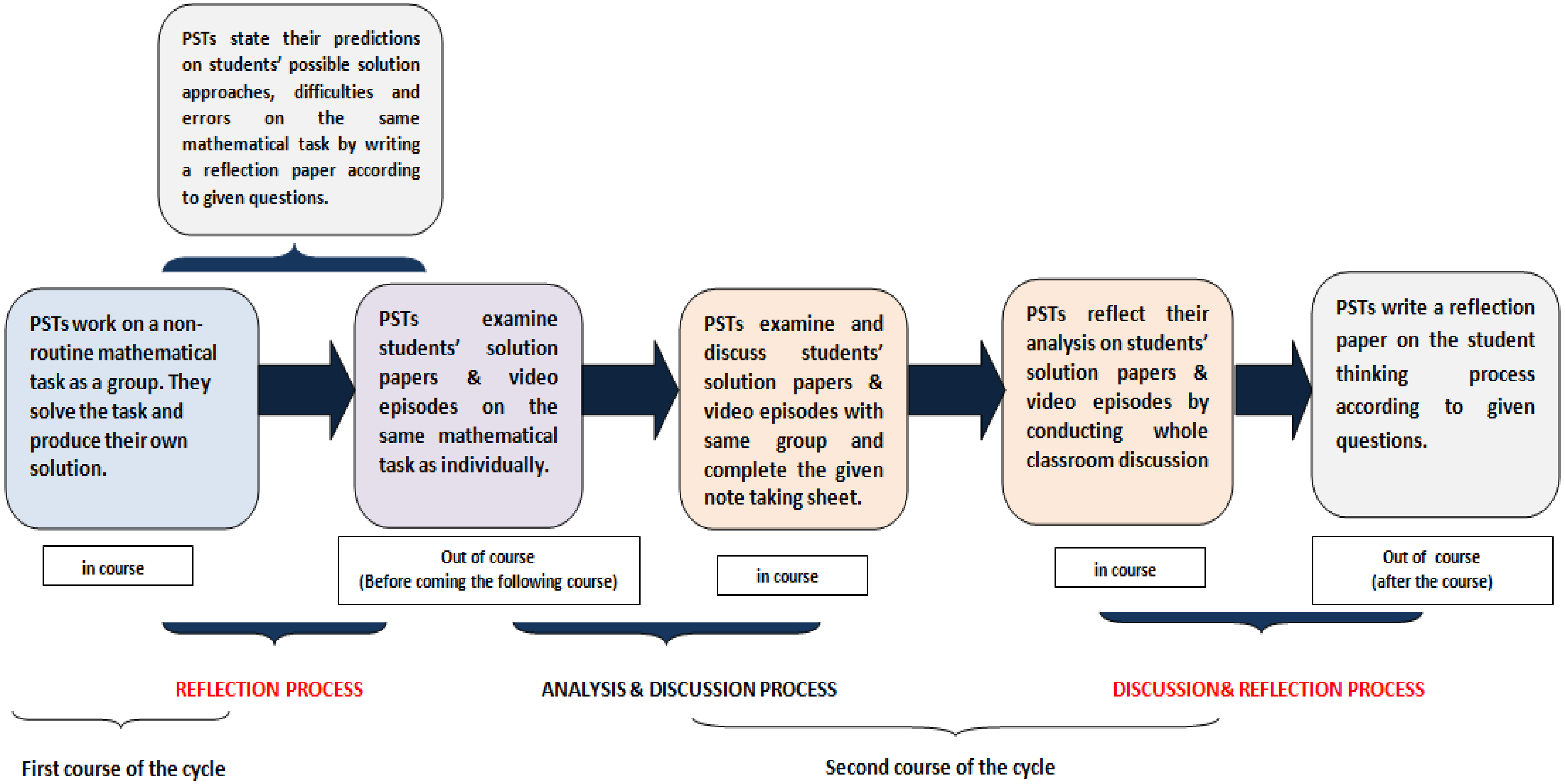

Figure 3.3 The example of 2-week cycle process

In each cycle, pre-service teachers first worked on a non-routine task in groups of 3-4 to produce their own solutions to the task as well as they presented their solutions to the other students and responded to the comments and questions related to their solutions. Pre-service teachers were given approximately 180 minutes in total, 150 minutes for working on each task as a group and approximately 30 minutes for presenting their group solution way, discussing on and getting questions related to it (see section 3.3.1.2.1, pp.67-68). For each week, after pre-service teachers produced their own solutions to the tasks, they were asked to write a reflection paper on their process as an assignment. As a guide 14 questions were provided to them in order to develop their reflection paper. Among these set of questions, the 10th question was used one of the main data source of this research where pre-service teachers were asked to reflect their expectation/predictions on students' possible difficulties, errors or solution ways they might produce in solving this task. In this question, the pre-service teachers were asked the following.

Think as a teacher,

- When you implement this task in the classroom setting, what might be your expectations on what knowledge the students acquire?
- What kind of solution ways might students produce to this task?
- How can you implement this task in the classroom setting?
- In such kind of classroom implementation,
  - What kinds of difficulties might students have?
  - What kinds of errors might students make?
- What will you do to solve your difficulties the students had and errors they encounter them in solving this task?

On one hand, the aim of these questions was to have pre-service teachers to think about students' ways of thinking before attending the next course on the "working on students' ways of thinking on the same task". On the other hand, the aim was to understand and evaluate pre-service teachers' knowledge of students' errors, difficulties and solution strategies as well as track the development of their knowledge over time.

Pre-service teachers were given 3 days to develop and submit their reflection papers. Immediately after all pre-service teachers submitted their reflection paper assignments, to have PST to prepare to the following week, the scanned copies of high school students' written work on the same task and link of produced video episodes to connect them were sent to pre-service teachers via e-mail by the researcher. That means pre-service teachers could access the students' written work and video episodes prior to attending to the each course on "working on students' ways of thinking" from any computer connected to internet. The link of the videos allowed pre-service teachers to watch videos at any time and from anywhere connected to the internet, however, due to the ethical consideration of the research, the format of the videos were adjusted to be not to allow downloading them. The students' works was sent prior to course due to the several reasons. One of the reasons was that pre-service teachers were wanted to look at students' solution papers and to watch video episodes related to these solutions individually and to have individual ideas on them before working and analyzing them collaboratively. Other reason was to inadequacy of course time to watch video episodes several times in depth and to understand the details of students' thinking ways in the videos. When pre-service teachers were working themselves, they had much more opportunity to pause, re-play, analyze and re-analyze the same instance of students' thinking.

Then, in the following course, pre-service teachers were provided with high school students' written work and classroom videos of students' discussions on the same task while they were solving it in groups. The pre-service teachers were asked to collaboratively analyze students' thinking manifested in their written work and the videos and to write about students' solution approaches/strategies, strengths and challenges of their solutions, mathematical concepts used in their works and other things they noticed. The note taking sheets to document students' ways of thinking presented in Appendix J were given pre-service teachers. The common process of use of students' solution papers and video episodes was shown in Table 3.4 for each task.

Table 3.4 The process of use of students' solution papers and video episodes

Steps	Artifacts		Time
Step 1	Analysis of students' solution papers	Each pre-service teacher was given photocopied students' solution papers and note taking sheets, and they were required to collaboratively analyze students' solutions and take notes on the note taking sheets. For pre-service teachers, the note taking sheet served as the focus of the class discussion in Step 4 for as well.	Approximately 25-30 min.
Step 2	Watching the video episodes	After pre-service teachers worked on students' worksheets 25-30 minutes, a break was made, and they were watched to all video episodes. While pre-service teachers were watching the videos, they were asked to take notes on ways of student thinking what they observed/noticed.	Approximately 30 min.
Step 3	Analysis and discussion on both students' solution papers and the videos	After pre-service teachers watched the video episodes, they continued to analyze students' work and complete the given note taking sheet.	Approximately 60 min.
Step 4		After they collaboratively analyzed students' work, they made class discussion on various aspects of these works such as "what the students think and why" or "what the students do/do not understand" and interpreted these ideas.	Approximately 45 min.

While pre-service teachers were working on students’ work collaboratively in groups, the instructor (teacher) of the course spent most of his time closely watching groups by walking among them without any interference. On the other hand, at the beginning of the each new step, instructor reminded pre-service teachers the purpose of each step as well as how it would be enacted. The instructor had also a facilitator role during the whole classroom discussion. The instructor also encouraged participation of all pre-service teachers in many ways, for example, he used a tone voice that was welcoming or he explicitly invited pre-service teachers to share their opinions who have not spoken to contribute to the conversation. The instructor redirected questions (see Appendix E) to pre-service teachers about solution strategies of each group in sequence and wanted to share their thoughts about different solution strategies, the mathematical ideas, the difficulties and errors of students’ solutions etc. that they explored as a result of analysis of students’ works. The instructor encouraged pre-service teachers to describe the details of the students’ solutions and to present the evidence to support their interpretations. Moreover, after a pre-service teacher provided an interpretation, the instructor asked if anyone else had a different explanation. While the pre-service teachers were sharing their interpretations, the instructor pushed the pre-service teachers to extend their thinking.

Similar to instructor (teacher of the course), the researcher as one of the teaching assistants of the course moved around the groups occasionally, observed groups’ interactions and discussions, and took field notes without interference. The other role of the researcher was to distribute photocopied students’ solution papers to all pre-service teachers at the beginning of the course and organized video-episodes to be presented to class during the course and control the video-records and audio-records in case of the possible technical problems.

As Kazemi and Franke (2004) indicated, in order to open up opportunities for learning, teachers should not just be brought together. The issues such as use of student work, the norm and habits of professional discourse have a potential impact on teacher knowledge and learning and they should be carefully paid attention. Therefore, in this research, for managing the discussion process effectively and maintaining its focus, a semi-structured protocol (guideline) was used that was prepared by the help of the existing protocols for looking at students’ works (Allen

& Blythe, 2004). It was an outline and the function of it was to guide for looking at collaboratively and carefully at student work (see Appendix E).

3.4 Data Sources

The data of this study were collected by using variety of data collection sources such as pre-service teachers' solution papers on modeling tasks, reflection papers, note taking sheets of pre-service teachers to document students' ways of thinking while working on students' works, video-taped focus group discussions, and video-taped individual interviews, video and audio-taped observations, questionnaires (pre-post self report) and observation notes kept by researcher.

3.4.1 Questionnaire (pre-post self report)

In this research, two questionnaires which were called as "Self Report Questionnaire" were used to gather data. The pre-self report questionnaire was implemented in the 4th week of the course before pre-service teachers started to work on students' works. The aim of this pre-questionnaire was to obtain pre-service teachers' own perception about their knowledge of students' ways of thinking, their knowledge of analyzing students' thinking as well as to learn their level of experience about students' way thinking before beginning to work on students' works (see Appendix G1). The post-self report questionnaire was implemented in the 14th week of the course after pre-service teachers completed all their works regarding students' ways of thinking. The aim of the post-questionnaire was to obtain pre-service teachers' self perceptions about the contributions of the students' works to their learning concerning students' thinking, (if there is any) the change of their predictions, understanding and interpreting of students' ways of thinking (see Appendix G2). These questionnaires were not implemented in the course. Instead, the researcher sent them to all pre-service teachers by e-mail, and collected them back by e-mail as well.

3.4.2 Pre-service teachers' solution papers on modeling tasks

Pre-service teachers' solution papers produced by pre-service teachers working on non-routine tasks in groups of 3-4 were the data source of the study. These solution papers had either one or two pages long and involved their either correct or incorrect solution ways with the understanding and interpretation of the problem in the different ways and usage of different kinds of mathematical idea, concepts and representations. For each activity, because there were seven groups, there were seven solution papers.

3.4.3 Note taking sheets of pre-service teachers to document students' ways of thinking

With respect to aim of the study, a note-taking sheet designed to help pre-service teachers to reveal their identification of students' ways of thinking was used as a data source. This note-taking sheet designed by researcher through research literature on "looking at students' work/investigation of students' work" (e.g., Allen & Blythe, 2004; Chamberlin, 2002; Hallagan, 2003). It included a number of cells where it was requested pre-service teachers to take notes on students' mathematical thinking such as students' solution strategies, strengths and challenges of their solutions, mathematical ideas/concepts used in their works (see Appendix J). The note taking sheets had four to six pages depending on the number of the students' solution papers. Pre-service teachers were given about 90-105 minutes in total to complete the note-taking sheet.

3.4.4 Reflection paper

Students were required to write two different types of reflection papers as an assignment throughout all semester. One type of them was task-based reflection paper (TBRP). After each modeling activity was completed, pre-service teachers were asked to write a reflection paper as an assignment. This assignment was given on Fridays which was the course day and were collected on the following Mondays via e-mail. Pre-service teachers used the set of questions that were provided as a guide when developing their reflection paper. There were fourteen main questions in total. In these questions, for example, pre-service teachers were asked to explain their solution process with their solution strategies and difficulties. Moreover, pre-service teachers were wanted to think like a teacher and they were asked to predict students' possible ways of thinking such as "how students might solve this activity, which solution strategies might they apply and what kind of errors might they make and in which part they might have difficulties". For this research, pre-service teachers' answers for question ten just were used as a data source (see Appendix F1).

Second type of the pre-service teachers' reflection paper was written on students' ways of thinking. After working on students' written work and video episodes, students were asked to write a reflection paper as an assignment. Similarly, pre-service teachers used the set of questions that were provided as a guide when developing their reflection paper. There were seven main questions in total. In these questions, for example, pre-service teachers were asked to explain "What students were thinking and why?" "What did you see in these students' works that was interesting or surprising?" and "what did you learn about how these students think?" (see Appendix F2). Like task-based reflection paper, this assignment also given on Fridays after pre-service teachers worked on students' works and the reflection papers were collected on Mondays. Pre-service teacher sent their assignments to researcher by e-mail.

3.4.5 Individual interview

Qualitative interviewing refers to conversation with the participants, and its format ranges from no structured to highly structured. But, regardless of the structure, the purpose of the interview is to collect information from the participants about the topic. There are several types of interview that are structured interview, semi-structured interview, the in-depth interview and unplanned (causal) interview. In structured interview, the questions and their format are same for all participants while in semi-structured interview the questions can change depending on the situations although there are general set of questions (Lichtman, 2006).

For this research, the interviews were conducted with the volunteered seven pre-service teachers. Each of the pre-service teachers was selected from each group. At the beginning of the semester, the time and date of the interviews were adjusted with respect to appropriate time of participant pre-service teachers. That means the pre-service teachers determined the appropriate time period for themselves to interview. The aim of the interviews was to understand the issues how pre-service teachers analyzed and interpreted students' works, what they learned as a result of their analysis in depth**.** Therefore, the focus of the interview was students' ways of thinking. It was semi-structured interviews, and the set of questions that were provided as a guide to interview (see Appendix H). On the other hand, pre-service teachers' reflection papers on students' ways of thinking guided to the interviews. Therefore, the date of the interviews was adjusted after pre-service teachers sent their reflection papers on students' ways of thinking. The date of interviews was mainly on "Wednesday and Friday". The interviews lasted approximately 30-40 minutes for each interviewee. All interviews were both audio and video recorded.

3.4.6 Focus group discussion

After pre-service teachers worked with students' solution papers and video episodes, the whole class discussions were made to elicit pre-service teachers' identifications and interpretations on students' ways of thinking. The class discussions were implemented immediately after pre-service teachers worked on students' solution worksheets and video episodes and completed note taking sheet working on students' thinking collaboratively. The instructor as a facilitator leaded the class discussion and used a set of questions. During the whole class discussion, the pre-service teachers discussed on and shared with whole class their collaboratively produced notes on their note taking sheet about students' thinking ways such as the aspects of these works such as "what the students do think and why" or "what the students do/do not understand", "what are the strength and weaknesses of their solution ways" etc. (see Appendix E). Each class discussion lasted approximately 50 minutes and was video-recorded by three cameras.

3.4.7 Observation note

In this study, the observations were done to gather information about the issues such as pre-service teachers' performance, attendance to class, level of participation to group studies and class discussions, attitudes towards the course. While participants were working and discussing on both non-routine mathematical tasks and students' solution papers and video episodes, the researcher observed all of them. The observation notes on the classroom context, physical conditions and pre-service teachers' behaviors were kept (see Appendix I).

In brief, table 3.5 shows which data sources were used to get data for each research question.

Table 3.5 Research questions and data collection sources

Research Questions	**Data Collection Sources**
1. What do pre-service secondary mathematics teachers predict about students' ways of thinking before they engage in students' works about solutions for non-routine mathematical tasks? **2.** What do pre-service secondary mathematics teachers identify about students' ways of thinking while they engage in students' works about solutions for non-routine mathematical tasks?	▪ Pre-service teachers' own solution papers on working modeling tasks ▪ Reflection papers ▪ Note taking sheets of pre-service teachers to document students' ways of thinking ▪ Video-taped focus group discussions ▪ Video-taped interviews ▪ Questionnaire (pre-post self report) ▪ Observation notes
3. What do pre-service secondary mathematics teachers value in students' solutions produced for non-routine mathematical tasks?	▪ Reflection papers ▪ Video-taped interviews
4. How do pre-service secondary mathematics teachers interpret students' thinking as manifested in students' works about solutions for non-routine tasks?	▪ Reflection papers ▪ Note taking sheets of pre-service teachers to document students' ways of thinking ▪ Video-taped focus group discussions ▪ Video-taped interviews ▪ Observation Notes
5. What do pre-service secondary mathematics teachers focus in terms of students' ways of thinking in analyzing students' works about solutions for non-routine tasks?	▪ Reflection papers ▪ Questionnaire (pre-post self reports) ▪ Observation notes ▪ Video-taped interviews

3.5 Data Analysis

3.5.1 Steps of data analysis

The analysis of data was conducted in four steps “data management, coding, re-reading and revising codes, and constructing categories/themes and drawing conclusions” as presented below.

3.5.1.1 Data management

Data analysis was started by managing the data what have been collected. The process of the data management is the following.

Transcription: Before transcription of the audio and video recorded data, it was initially decided what would be transcribed and what would be left out of the collected data with respect to aim of the research. And, then videotaped focus group interview (class discussion) and audio-video taped individual interview data were transcribed into written text by the researcher.

Organizing Data: Some of the other artifacts (pre-service teachers solution papers on modeling tasks, note sheets of pre-service teachers while working on students' work and observation notes) were scanned and created as PDF file and transferred into computer medium. The organization of the data was done with respect to each task used in this research. For example, for each tasks a new folder was created and named as “TASK 1_STREET PARKING” and then all related data sources with this activity were collected under this folder as seen in Table 3.6.

Table 3.6 An example of the data organization

Task	Data sources	# of data sources
TASK 1_STREET PARKING	• Task based reflection papers	25
	• Reflection papers on students' ways of thinking	25
	• Note Taking Sheets of pre-service teachers to document students' ways of thinking	7
	• Individual interviews	7
	• Focus group interview	1
	• Pre-service teachers' group solution ways of tasks	7

Reading Data. In spite of being familiar with the all type of data in the data collection process and in the transcription period, after the data were organized, the initial reading of the data was done. It was the skim reading. That means, quickly all the data was reviewed to gain insight to data before starting the analysis.

3.5.1.2 Coding

"Codes are tags or labels for assigning units of meaning to the descriptive or inferential information" compiled during the study (Miles & Huberman, 1994, p. 56). Codes, which are generally attached to words, phrases, sentences, whole paragraphs etc., are applied to organize data. There different types of codes which are descriptive, interpretative and pattern codes. In addition, different kind of methods to create codes such as "creating provisional start list of codes prior to field work", "inductive coding" or the method between them (Miles & Huberman, 1994).

In this research, in order to create codes, two different methods were applied. Initially, a provisional start list of codes was created prior to analyzing data. The code in this list came from the conceptual framework of the research, research questions and prior research related to problem statement of this research (Miles & Huberman, 1994). Although the pre-code list created, this list was not used to code the data at first. Rather, the analysis of the data began with open coding. Namely, the grounded theory approach was drawn in this process (Corbin & Strauss, 2008).

According to this method, during data analysis process, codes evolved from the data itself. First of all, several data randomly selected among from each different data sources. Then, the selected data were examined line by line, and the code names to the concepts what was describing in the data were given. The researcher was open-minded and context sensitive as doing open coding. In this process, the data was coded both by hand. Next, the codes created from the data and came from the theoretical part of the research were combined, and a new code list were created. At this time, the operational definition of codes was made that could be applied consistently by the researcher and other researchers, who would be thinking about the same phenomena while coding the same data (Miles & Huberman, 1994).

While coding the data, in order to find answer the research questions, I looked for evidences in the data. In coding process, in the light of research questions, the evidence of pre-service secondary mathematics teachers' knowledge about students' thinking was sought. The evidence was looked for what pre-service mathematics teachers know about students' ways of thinking, and if there was any change through examining of students' works. Therefore, the data were coded with respect to pre-service teachers' predictions and identifications about students' ways of thinking. In this process, what pre-service teachers predicted about students' possible solution approaches/strategies, errors and difficulties before they engage in students' work about solutions, and what they identified about students' possible solution approaches/strategies, errors and difficulties while they engage in students' work about solutions, were looked for. Moreover, while pre-service teachers were analyzing students' works and reflecting their findings on students' ways of thinking, what they appreciated and when they reflected their appreciation, were examined. In addition, the different characteristics of the interpretations relating to the problem statement were looked at. The nature of the pre-service teachers' comments about students' thinking reflected in their reflection papers, interviews or focus group discussions was examined. For example, "Did they just describe or restate of ways of students' thinking?" "Did they pay attention to the mathematical details in students' thinking?" or "Did they evaluate ways of students thinking or interpret?" etc. Moreover, the evidence for the aspects of pre-service teachers in their focus was searched in data from reflection papers and pre-post self-report questionnaires. What pre-service teachers had written about their primary focus of attention before, during and after the four two-week research cycles was looked at carefully.

The coding of the data was proceeded by the help of the qualitative data analysis software (NVIVO 10). All data sources “word documents and PDF files” were imported into NVIVO. The data sources were imported into NVIVO as shown in Table 3.6 for each task. Then, the created code list also entered into NVIVO before starting to code. In this way, all data were coded by the created code list via NVIVO. The data were coded task by task through created code list. First of all, all data sources relating the Task 1 were coded, and then the others were done in turn. For each task, the reflection papers were initially coded, and then it was followed by coding of notes sheets of pre-service teachers while working on students’ work, focus group interviews and individual interviews, and pre-service teachers’ group solution ways. In this process, because all types of data produced on students’ written, verbal and visual solution ways, students’ solution ways were carefully took into consideration as each data sources were being coded. Furthermore, the groups of sentences that maintain meaning were defined as the unit of analysis. As the sources were explored, if new codes were emerged, new nodes were created in NVIVO as well. On the other hand, the several reasons of the use of NVIVO in coding process were:

- It assisted better management of large amount of the data.
- It saved time and offered flexibility.
- It enabled to manipulate large amount of data in terms of changes of codes and categories and editing text.

3.5.1.3 Re-reading and revising codes

A large number of codes (more than 60) were emerged during the initial coding. After the initial coding the entire data from all data sources, initially the texts under each code were reread to check consistency of the codes with the excerpts. In addition, the code names and the definition of the codes were discussed with a mathematic educator (PhD) and the problems concerning to name of the codes were resolved. It was decided that some of the codes conveys similar meanings; therefore, these codes were combined and renamed. For example, the codes called “conclusive statement, judgmental statement, evaluative statement and general statement” were combined into one code called “evaluative/make judgment”. On the other hand, the code called “not to understand ideas/whole solution strategy” was renamed as “difficulty in understanding ideas/whole solution strategy”. As a result, the coding list was developed as presented in Appendix K.

3.5.1.4 Constructing categories/ themes and drawing conclusions

After the codes were modified, they were organized into categories by the grouping certain codes which can construct a major topic. In this study, the categories derived directly from the data instead of from the external framework or theories developed by researchers. The categories were also checked and discussed with the same mathematics educator (PhD) who got involved in revising coding process. Based on discussion at the meetings, some of the categories were rearranged and renamed because their meanings were not so clear. To illustrate, the constructed category “pre-service teachers’ reactions to students’ solution strategies” was decided to exclude. On the other hand, the category named as “Describe general feature of ways of students’ thinking” was decided to rename as “Describing and assessing students’ ways of thinking”. As a result of rearranging and renaming process, the following themes and categories were emerged from the codes.

1. Pre-service teachers’ awareness of students’ ways of thinking
   1.1. Pre-service teachers’ predictions and identifications of students’ solution approaches
   1.2. Pre-service teachers’ predictions and identifications of students’ difficulties and errors
2. Pre-service teachers’ valuing students’ ideas and solution approaches
3. Pre-service teachers’ ways of interpretation of students’ thinking
   3.1 Describing and assessing students’ ways of thinking
   3.2 Questioning students’ ways of thinking
   3.3 Explaining mathematical details of students’ ways of thinking
4. Pre-service teachers’ criteria for examining students’ works.

Lastly, the identified categories above were interpreted carefully; they were made sense and made inference from the data.

3.6 Researcher Role

In qualitative research, the background of the researcher is an important aspect. As the researcher of the study, I am PhD student and research assistant at Department of Secondary Science and Mathematics Education (SSME) in Faculty of Education in a public university. I had 4-year experience as research assistance and I enrolled in different kinds of the undergraduate (e.g., Mathematical Modeling for Teachers) and graduate courses (e.g., Critical & Analysis of Research in Science and Mathematics Education) as an assistant. I took a qualitative research course before collecting the data and learnt the qualitative research paradigms and qualitative research methodologies. I also learnt the use of data analysis software (NVIVO 10) during my PhD education. I also enrolled in three year research project “Mathematical Modeling in Secondary Mathematics Education: Pre-service and In-service Teacher Education”.

The role of the researcher in qualitative study is complex, and researcher is the key instrument in the data collection process. In addition, researcher is also responsible for identifying appropriate sites and obtains necessary information to be able to collect data (Fraenkel & Wallen, 2006; Gall et al., 2007). For this study, as the researcher, my role consisted of several main aspects. First, I had crucial role to identify appropriate research sites and participants. According to Marshall and Rossman (1999), observation refers to “the systematic noting and recording of events, behaviors and artifacts in the social setting chosen for the study” (p. 107). And, observing people in their natural settings helps researcher to understand the complex human behavior (Litchman, 2006). In qualitative research, the researcher, who is also observer, takes different types of role like complete participants, participants as observer, observer as participant and complete observer (Fraenkel & Wallen, 2006). In this study, I conveyed the role of participant-observer. Namely, I participated fully and actively in the activities and was in interaction with the participants; therefore, the participants knew that they were being observed. I observed pre-service teachers from the first week of the course until the last week of the course. I video- and audio-recorded the all courses and took observation notes. Moreover, I served as one of the research assistants of the course where the data were collected. I both distributed and collected all the research materials such as questionnaires or note taking sheets. I also sent their electronic versions of documents via e-mail when it was necessary. Furthermore, I conducted weekly interviews with seven pre-service teachers.

In addition, examination students' solution papers and watching students' video episodes and then making in depth analysis of students' solutions before the research process was my other crucial role. In addition, before each course regarding pre-service teachers' working on student work, I presented detailed information to instructor about students' solution approaches, difficulties and errors which I identified in students' works.

3.7 Reliability and Validity Issues

In qualitative research, the researchers prefer to use different terminology for the reliability and validity concepts (Shenton, 2004). For example, as distinct followed the conventional paradigm, naturalistic paradigm (e.g., Guba, 1981; Guba & Lincoln, 1985 as cited in Shenton, 2004) called these constructs are as in the following (1) credibility (internal validity) (2) transferability (external validity/generalisability) (3) dependability (reliability) (4) confirmability (objectivity). In this research, reliability, validity (internal and external) and objectivity concepts are used to show how to ensure quality of the research as seen in the following Table 3.7.

Table 3.7 Ensuring trustworthiness for qualitative research

Concepts	Techniques
Internal Validity (Credibility)	Prolonged Engagement
	Peer Debriefing
	Triangulation
	Ensure Honesty of Informants
External Validity (Transferability)	Thick Description
Reliability (Dependability)	Inter-coding (Dependability audit)
Objectivity (Confirmability/External Reliability)	Overcoming Researcher Bias

3.7.1 Reliability (Dependability)

In the qualitative research, the reliability is related to the consistency of the researchers' approach with different researchers and projects (Gall et al., 2007; Creswell, 2009). That is, as Merriam (1995) stated, "it is concerned with the question of the extent to which one's findings will be found again (p.55)". In order to provide reliability of the qualitative research, different kinds of strategies are suggested. For example, checking the transcripts, comparing the data with the codes to be sure that there is not a shift in the meaning of the codes and inter-coder agreement is the several of the fundamental strategies (Creswell, 2009). Therefore, in this study, I started to check all transcripts to be sure that they do not include any obvious mistake. Secondly, I defined all codes and then compared the data with the codes two times to see if there was a meaning shift of the codes that would become during the coding process. Lastly, I conducted the cross-checking provide inter-rater reliability of the coding process (see section "Re-reading and revising codes" and "Constructing Categories/themes and drawing conclusions").

3.7.1.1 Inter-coding

Inter-coding agreement is crucial process to ensure reliability of the research. "The inter-coder reliability is an agreement between multiple coders about how they apply codes to the data" (Kurasaki, 2000, p.179). According to Miles and Huberman (1984, p.63), (at least) two coders should code separately 5-10 pages of the transcribed data and look at the consistency. They also indicated that the initial expectation should not be more than 70% at first. On the other hand, according to them, although the agreement is better than 70% is acceptable, the inter-coding agreement should be at least 90% percent range in the end.

In this study, in order to ensure inter-coding agreement, I and the external coder, who is a teacher educator (PhD), held several meetings. In our meetings, I and external coder examined two reflection papers (among 96 reflection papers), one individual interview data (among 28), one focus group interview data (among 4) and one data obtained from note taking sheet (among 24). I randomly selected each data. The reflection papers were 2-3 pages long, the transcribed individual and focus group interview were 4-5 pages long, and the note taking sheet comprised five pages. The samples of data are presented in Appendix L. The documents were given to external coder with the coding booklet seen in Appendix K. Then, the sample of data were coded separately and compared. Initially we started to code a sample of reflection paper data, which were the most fundamental data source of this study. Then, we coded a sample of interview data, focus group discussion data and the data on note taking sheet. We mainly agreed with on the coding, but some disagreements were also appeared in, mainly in focus group discussion data. Initial agreement on the coding of reflection paper, interview, focus group discussion and note taking sheet data were about 80%. The emerging disagreements appeared in all kind of data sources were solved by discussing them until reaching consensus, and we reached about 95% agreement.

3.7.2 Validity

On the other hand, validity means that "the researcher checks for the accuracy of the findings by employing certain procedures" (Creswell, 2009, p.190). In this research, various strategies were used to ensure the validity.

3.7.2.1 *Internal validity (Credibility)*

Internal validity is one of the key criteria in the validity issues to ensure trustworthiness of the research. In internal validity, the researchers seek to ensure whether their research measured what is actually intended (Shenton, 2004). For example, Merriam (1995) addresses the internal validity by asking the question "how congruent are one's finding with reality?" (p.53). In this study, in order to strengthen the internal validity, the following strategies were employed.

3.7.2.1.1 Prolonged engagement

In this study, in order to ensure validity of the findings, as the researcher, I spent prolonged time in the field and the data were gathered over two months. In this way, I developed an in-depth understanding of the phenomenon. In addition, I did regular and repeated observations throughout the whole semester and spent more time with the participants in their actual settings.

3.7.2.1.2 Peer-debriefing

In order to enhance accuracy of findings, first of all, when this research was designing, two mathematics educators (teacher of the course and a research assistant) provided crucial feedbacks at each issue such as the duration of the research (time), the selection of appropriate tasks and materials, the sequence of use of materials (implementation). Especially, their feedbacks in the issues "how to use students' written work and video episodes, which one should be first used and how much time should be spent for each activity" were great feedbacks for the research design.

Secondly, during the data collection process, as researcher, I and these two mathematics educators talked and discussed about the content of the each course. Their experiences brought to these talks widened my vision and helped me to think alternative approaches and ideas.

Thirdly, I and one mathematics educator (PhD) discussed the clarity of code names and comprehensibility of the code definitions, and how they are connected with the research questions of the study. Before doing it, initially, as the researcher I organized my coding booklet, which includes all codes used to code different types of data and their definitions, to present the external coder (see Appendix K). After that, the mathematics educator as an expert read all the codes and their definitions to understand what they mean, and indicated his comments as well as his confusions and disagreements especially about the codes names. For example, we spent some time to discuss about the code called "pre-service teachers' appreciation of students' thinking". Although I defined this code to convey cognitive meaning, the external coder stated his disagreements because he thought that this code entailed the affective meaning. So, there was inconsistency between the code name and its definition. Therefore, initially, we reached agreements with all the codes. Lastly, the data analysis process and emerging findings are shared with the mathematics educator, who are expert in the field, and asked their comments and feedbacks on plausibility of the findings (Creswell, 2009; Gall et al., 2007).

3.7.2.1.3 Triangulation

"Triangulation" is another important method of confirming findings. There are different kinds of triangulations such as triangulation "by data source", "by method", "by researcher", "by theory" and "by data type" (Miles & Huberman, 1994, p.267). As I explained in method chapter, I obtained the information from multiple data sources including individual interviews, focus group interviews (classroom discussions), reflection papers, note taking sheets and observation notes. In this research, I collected data through different data sources and I examined evidences from these sources. To illustrate, what I observed and heart about the phenomenon in the focus group discussions (classroom discussions), I read them in pre-service teachers' reflection papers and heart them in interviews.

3.7.2.1.4 Ensure honesty of informants (participants)

In this research, as Shenton (2004) stressed, a rapport established between the researcher and participants (pre-service teachers) at the beginning of the research and the pre-service teachers encouraged being honest. They were told there was no right answer to the questions to reduce pre-service teachers' fears and concerns. In this way, they found an opportunity to be able to express their comments, ideas and experiences without fear.

3.7.2.2 External validity (Transferability)

The external validity is concerned with the generalizability of the findings of the study. That is, as Merriam (1995) expressed, "the extent to which the findings of a study can be applied to other situations (p.57)". Although external validity is relating to generalizability of the research findings, in qualitative research the discussion on that still continues because of random sampling rarely are preferred by qualitative researchers and the sample size is not adequate to generalize (Merriam, 1995; Shenton, 2004). Although the discussions on the limitations of qualitative research, there are several strategies such as thick description, multi-site design or modal comparison (Merriam, 1995) that are suggested to employ in order to strengthen the external validity issue. In this study, the following strategy employed.

3.7.2.2.1 Thick description

Thick description of the research context and research results is an important phenomenon for the external validity. Therefore, in this study, the research context, the physical settings, environment, and the characteristics (demographic), and experience of the participants the data collection instruments, schedules and temporal order of events, under which the results were interpreted, were described in depth. In addition, in order to provide thick description of the research results, in adequate long quotes from the pre-service teachers (participants), the dialogues excerpts of researcher-pre-service teachers and pre-service teacher-pre-service teacher were presented. At the same time, pre-service teachers' emotions, actions and relationship to each other were tried to reflect in these quotes and excerpts (Ponterotto, 2006).

3.7.3 Objectivity (Confirmability)

Objectivity is concerned with the researcher bias. It is another key issue that should be paid attention carefully because researcher bias is inevitable in qualitative research. In this study, for example, one bias of the researcher would be that she might have focused on what she wanted to see rather than what pre-service teachers actually did/told while she was observing. Another bias would be researcher's experience in interpreting qualitative data and data analysis. Moreover, reseracher's theoretical knowledge and belief about use of instructional documents in teaching and learning pre-service teachers in teacher preparation courses would be other bias. In this study, the researcher initially was self aware about all the possible biases. In order to provide objectivity, the researcher background (experience) and role in the study defined in depth (see section 3.6). Moreover, data were triangulated with several data collection methods rather than depending on only her observations. The method and procedures of research design, how the data analyzed and how categories were derived explicitly were described and the results and conclusions explicitly were linked with displayed data (e.g., Merriam, 1995; Miles & Huberman, 1994).

3.8 Ethical Issues

Different kinds of ethical problems can appear in the data collection process (Gall et al., 2007). In this study, two different types of ethical issues were considered as following:

The first ethical issue was related to participants of the study. In this respect, three aspects “avoidance of harm”, “confidentiality” and “informed consent” were paid attention carefully to prevent the possible of ethical problems and protect the participants (Miles & Huberman, 1994). Initially, because this course served as a part of ongoing three-year research project about mathematical modeling in secondary schools, the necessary permissions for implementing research had been gotten from the Ethical Committee. Then, in order to be understood clearly by the participants, the research objectives were articulated verbally and in detail. In the first week of the course, pre-service teachers were given the syllabus of the course and made in depth explanation about the content of the course. It was also informed that this course served as a part ongoing three-year research project about mathematical modeling in secondary schools. Therefore, pre-service teachers knew that they did participate in not only the course but also research aspect of the course. The written permission was received from all the participants of the study through the consent form presented in Appendix N. The permission of 25 pre-service teachers was gotten by signing the consent form after pre-service teachers have read it carefully. All participated in the research voluntarily. Moreover, the participants were informed of the all data collection devices such as audio recorders and video recorders. In that way, the possible psychological harm were tried to minimize. In addition to that, without participants’ permission, any conversations have not been recorded by using a hidden video and audio recorder or any mechanic devices. All participants have been treated with respect and never been lied them. Furthermore, all collected data were assured to no one else to reach them. The names of the participants have been used anonymously during reporting the results. And, also all participants were informed about their rights “to withdraw from the study or not to be used their data from the research” (Fraenkel & Wallen, 2006).

Second ethical issue was related to use of students’ written work and video episodes. Video data may contain images of the classrooms, for example, actual face and expressions of the students and teachers and several instructional materials. Because of this, one of the risks is to manage privacy (anonymity) and confidentiality of the students and teachers in relation to visual material. It is not so possible to give pseudonyms to teachers and students in the video materials. Therefore, in order to address this issue, the detailed and explicit permission and informed consent forms were given to teachers and students, and their permissions were obtained. Consent form includes the consent of teachers to use students' written work and several parts of the video data obtained from classrooms subsequently. The second one is possibility dissemination of data is very quickly. Due to concerns of security and confidentiality, several precautions were taken. Initially, the web links were generated for the video episodes, and instead of sending generated video episodes to the pre-service teachers, only the link of the produced video episodes were sent to pre-service teachers. Furthermore, the formats of the videos were adjusted to be not to allow downloading them. In addition to that, pre-service teachers were verbally informed by the instructor (teacher) of the course about the privacy of the videos, and in the given consent form, they were also informed that the use/dissemination of the videos were certainly forbidden for any reasons.

CHAPTER 4

RESULTS

In this chapter, I present the results of my research by considering my research questions. I initially present the results concerning pre-service teachers' awareness of students' ways of thinking in terms of their predictions and identifications of students' approaches, errors and difficulties. Then, I present the results regarding what kinds of thinking ways that appeared in students' works were valuable for pre-service mathematics teachers. Next, I present my results regarding pre-service mathematics teachers' interpretation of students' ways of thinking. And lastly, I display the results concerning the criteria applied by pre-service teachers for examining students' work.

4.1 Pre-service Teachers' Awareness of Students' Ways of Thinking

4.1.1 Pre-service teachers' predictions and identifications of students' solution approaches

In this part, the results regarding what pre-service mathematics teachers predict about students' solution approaches in solving non-routine mathematical tasks, and what they identify about students' solution approaches students produced for the same mathematical tasks are presented.

For the first task "Street Parking", the data analysis revealed that in their reflection papers, 14 of the 25 pre-service teachers explicitly predicted/expected about possible solutions students would have. However, 11 of them did not explicitly state their predictions/expectations about students' solutions in their reflection papers. Table 4.1 shows pre-service teachers' common predictions/expectations about possible solutions that students would have for "Street Parking" task.

Table 4.1 Pre-service teachers' predictions/expectations about students' possible solutions for "Street Parking" task.

Pre-service teachers' predictions/expectations about possible solutions*	Pre-service Teacher (PST)	# of the PST
Students would use "similarity of triangles"	PST10, PST15, PST17, PST20, PST25, PST2, PST9, PST7, PST8, PST13	10
Students would use "area of geometric figures"	PST2, PST3, PST4, PST8, PST15,PST10	6
Students would use "trigonometry"	PST13, PST5, PST7, PST8,PST9,PST12	6
Students would use "derivative"	PST25,PST8	2

* Students' approaches to making a solution for mathematical task: mathematical concepts, representations or set of procedures that would be used in solving non-routine task.

The data initially revealed that although pre-service teachers were asked to predict students' possible solution approaches, they mainly stated only the mathematical concepts that would be associated with solution rather than predicting a specific solution approach students would produce. As shown in Table 4.1, pre-service teachers predicted/expected that students would solve the non-routine mathematical task by using similarity of triangle, trigonometry, area or derivative concepts. Of fourteen, ten pre-service teachers predicted that students would likely solve this task by using similarity of triangle. This predicted approach was produced by four groups of the pre-service teachers while they were working on this task (see Appendix D1). That is, the data presented evidence that while pre-service teachers were predicting students' possible solution approach; they mainly preferred to predict their own solution approach as for students. The following excerpts illustrate pre-service teachers' predictions/expectations about students' solution.

> **PST10:** For this problem, students would construct similarity as we did. Next, they would try to find a solution approach by determining the area given in the figures [above] and to form a right triangle which has one of length of the edge is *x* meter and other is 3 meters as a dead region [unused region] for each car and by finding the areas of those dead regions **[TBRP_1].**

> **PST 17:** I expected that many of the students would use the similarity because students were given both angle and length **[TBRP_1].**

As seen in the excerpts above, while predicting, pre-service teachers used general expressions like "they might establish similar triangles like us", "they might use similarity of triangle and trigonometry like we did".

On the other hand, six pre-service teachers predicted that students would apply "area" to solve the mathematical task so that several of these pre-service teachers used the area of geometric figures in their solutions. Nevertheless, as seen in the excerpt of PST8, although these pre-service teachers predicted that students would use the "area" in their solutions, they did not make any explanation about how students would use the "area" in their solutions and what kind of solution process students would follow by using area. Therefore, their predictions were so superficial.

> **PST8:** Student would track different ways while producing a solution to the problem. For example, as I did, they would formulate equality and write an equation. Or, they would think to approach to the solution by using the values of angle, that is, by applying the trigonometry without formulating an equation. Because the area covered by cars is known, students would use the area to find a solution. Students would formulate a quadratic equation and then take derivative of this equation; they would try to solve it as max-min problems **[TBRP_1].**

In addition, three pre-service teachers (PST7, PST13 and PST14) indicated that "they have never supposed that students would use the area to solve the task".

> **PST13**: [...] I don't expect students would use the area while solving this problem. They would use similarity of triangle and trigonometry. On the other hand, I expected that few students would reach a solution. This is not a question asking the unknown directly; therefore, I expected that students would choose the wrong ways which I did **[TBRP_1]**

After examining and analyzing students' worksheets and video episodes containing actual student solutions as a group, the pre-service teachers identified students' actual solutions on "Street Parking" task. Table 4.2 summarizes the pre-service teachers' descriptions of students' actual solution approaches (either correct or incorrect) what they identified as a group.

Table 4.2 Summary of students' actual solution approaches pre-service teachers identified as a group.

Student group (SG)	Pre-service teachers' identification of solution approaches	Pre-service Teachers Group (PSTG)
SG1	*For the parallel parking method;* ▪ Students calculated 150/c. *For the angle parking method;* ▪ Students had used angles and similarity in their solution, and they found the length of "c" ▪ Students produced a solution by using sine function.	PSTG1, PSTG2, PSTG3,PSTG4, PSTG5, PSTG6, PSTG7
SG2	▪ Students used trial-error method. ▪ Students found the length of "c" and the number of cars for parking by using trigonometric values of special angle such as "30°,45° and 60°".	PSTG1, PSTG2, PSTG3, PSTG5, PSTG6, PSTG7
SG3	▪ Students used the area of rectangle and triangle.	PSTG1, PSTG2, PSTG3, PSTG4, PSTG5, PSTG6, PSTG7
SG4	▪ Students used the area of parallelogram.	PSTG2, PSTG3, PSTG4, PSTG5, PSTG6, PSTG7
SG5	▪ Students used the trial and error method, and they tried several angle values. ▪ Students used "Pythagoras" and "sine" to find the angle θ. ▪ Students found how many cars can park by finding the value of θ and calculating area.	PSTG1, PSTG2, PSTG4, PSTG5, PSTG6, PSTG7

When pre-service teachers' predictions in Table 4.1 were compared to students' actual solution identified by pre-service teachers in Table 4.2, many discrepancies were observed between them. That is, students' actual solution determined by pre-service teachers, and possible solutions predicted by pre-service teachers were not so consistent with each other. By contrast with the pre-service teachers' predictions, which were mainly about students' use of similarity of triangle concept, as pre-service teachers identified, students tended to use either area of triangle or area of parallelogram to solve the task. The fundamental mathematical idea in solution of three student groups (among five) was based on either use of the area of triangle and parallelogram (see Appendix C1).

Furthermore, in their individual reflection papers, pre-service teachers reflected the similarities and differences between their predictions and students' actual solution approaches they identified. Pre-service teachers' these reflective accounts also confirmed this inconsistency. Only three of the pre-service teachers stated that their predictions/expectations were similar to students' actual solutions. In fact, these similarities stated by pre-service teachers were not associated with students' possible solution approaches. Rather, they were only on whether or not students would solve the task (students' performance). On the other hand, fourteen of the pre-service teachers indicated that their predictions/expectations were quite different from what students actually did. The predicted/expected and unpredicted/unexpected students' solution as are presented in the following Table 4.3.

Table 4.3 The predicted/expected and unpredicted/unexpected solutions

Expected/predicted solution	Unexpected/unpredicted solutions
Students' use of similarity and trigonometry in their solution correctly	Students' use of area of geometric figures (e.g., parallelogram, triangle and rectangle)
	Students' use of trial-error method by trying specific angle values such as 30°, 45°, 60° and 75°.

The Table 4.3 shows that there were two types of differences. The means, the first type of differences is that pre-service teachers expected a solution approach that would (correctly) be produced by students; however, many of the students in the groups did not produce such kind of solution (approach). The following excerpt is an example for the first type of differences. PST11 explains the difference between what she predicted and what she observed in the students' solutions.

> **PST11**: Students' solution approaches, difficulties and errors they have experienced in solving problem, which I have identified after examining students' worksheets and video episodes, were rather different. I had thought that all students would have correct solution; they would use similarity of triangle at least and find the length of "c", and then they would decide that the parallel parking method as the most correct method like we did and other pre-service teacher groups did. Of course, I did not expect that all students would arrive at this result but I predicted that most of them would track this way. However, we did not observe this solution in the students' worksheets and video episodes which we watched **[STRP_1]**

The second type was that pre-service teachers have never predicted/expected several solution approaches, but majority of the students in the groups applied these solutions. In particular, as it is stated by PST12 and PST7 in the following excerpts, several pre-service teachers (PST2, PST7, PST12, and PST19) stated student used a solution approach that they have never expected such a way.

> **PST12**: As I stated in my task based reflection paper, I expected that student would solve this problem because it was not complicated and difficult. I predicted that students would find a solution way easily by using trigonometry. However, most of the students, who especially used trigonometry, made several errors. After I have seen, I understood that students would not arrive at correct solution by using trigonometry as I predicted. Some of the student groups arrived at solution by using area. That was an unexpected solution approach for me. In fact, we and our other friends did not produce a solution approach by using area, and we have generally used trigonometry. Therefore, arriving at a solution by using area it was quite good **[STRP_1]**

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> **PST7**: The students in Group-4 used a solution approach which I have not predicted at all. They equated the area of a parallelogram by using two different bases and their corresponding heights **[STRP_1].**

On the other hand, pre-service teachers' identification of students' various approaches allowed them to express their feelings, in particular, their amazement. For several solution approaches, pre-service teachers expressed their amazement as "it is very creative", "original", 'interesting solution" and "we have never thought such a way" both in their note taking sheets completed as a group and in their individual reflection papers. That is, these solution approaches of students were largely unpredicted. For example, two solution approaches produced by students in group-3 (SG3) and group-4 (SG4) surprised pre-service teachers the most. While solving the task, students in both groups had used the area of geometric figures (see Appendix C1), but their thinking processes and solution approaches were totally different than each other. In the following solution approach produced by students in group-4 (SG4), the fundamental mathematical idea was the use of area of parallelogram.

As shown in Figure 4.1, students wrote parallelogram's area formula by using two different bases of the parallelogram and the heights corresponding to the bases. Then, they equated one formula to another to find the value of x which let them to construct and solve a quadratic equation to be able to find the value of both x and c. As it is seen in students' solution approach, the rationale behind their solution approach is to find θ by the help of the x, and then to decide method of parking "parallel or angle" which would allow the most room for the parking.

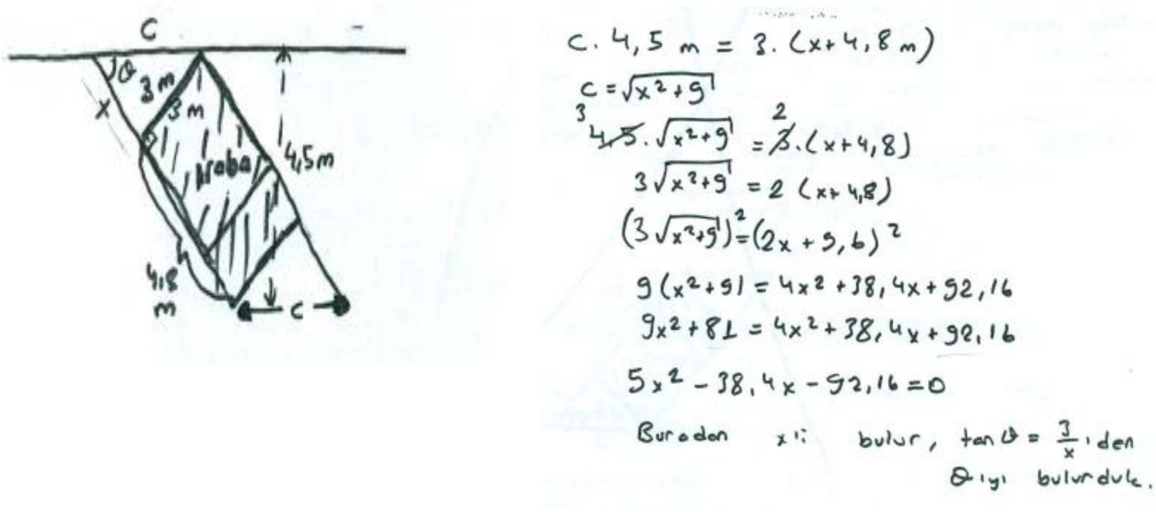


Figure 4.1 The example of solution produced by students in Group-4.

Although students' solution has not been produced fully correct, fourteen pre-service teachers expressed their amazement for this solution approach. They expressed that they were surprised because students calculated the area of parallelogram by using two different bases to find the length of c as seen in Figure 4.1. According to pre-service teachers, this was quite different and an original way of thinking, and this solution approach was not consistently emerged from both students in other groups and pre-service teachers in each group.

Here is an example of what PST8 reported about her/his amazement in the reflection paper on students' ways of thinking.

> **PST 8**: Generally, the groups, who solved the problem by using area, impressed me. The solution attempt of students in group-3, where they subtracted the area of triangular regions from whole area and then they divided the rest of the area to area of parallelogram and found the number of cars, was quite good. Moreover, the solution method of students in group-4, where they equated two areas of parallelogram calculated by using two different bases and their corresponding heights, was a way so that I can say that I never thought it. It was really a well thought solution approach. I was quite surprised since the students at high school level thought this solution approach **[STRP_1].**

For the second task "Bouncing Ball", eight of the pre-service teachers had predicted/stated their expectations individually about students' possible solutions, and fifteen of them neither made predictions nor stated their expectations about students' possible solutions at all. Similar to pre-service teachers' predictions in the first task, in this task, pre-service teachers stated the mathematical concepts that would be associated with a solution, or a thought process of students that would use to arrive at a solution. As shown in Table 4.4, eight pre-service teachers predicted/expected that students would solve the task by using exponential inequalities, five pre-service teachers predicted that students would solve this task by using sequences/series formulas, and two pre-service teachers predicted that students would produce a solution to the task by considering "the decrease of the height of the bouncing ball as constant amount after each bounce". On the other hand, even if several pre-service teachers predicted that students would use inequalities approach to find bounce rate, they expected that students would find the bounce rate as being equal a constant rate rather than finding it in an interval.

Table 4.4 Pre-service teachers' predictions/expectations about students' possible solutions for "Bouncing Ball" task.

Pre-service teachers' predictions/expectations about possible solutions*	Pre-service Teachers (PSTs)	# of the PST
Students would apply "Inequalities approach" (exponential inequalities)	PST11, PST13, PST16, PST17, PST25, PST2, PST8, PST9	8
Students would apply "Sequences/Series approach"	PST25, PST8, PST2, PST16, PST9	5
Students would approach the idea "the decrease of the height of the bouncing ball as constant amount after each bounce instead of the constant bouncing rate"	PST13, PST16	2

* Students' approaches to making a solution for non routine task: mathematical concepts, representations or set of procedures that would be used in solving non-routine task

The following excerpt illustrates what PST2 predicted/expected about students' possible solutions for this task.

> **PST2**: I predict that students would understand easily and get a certain solution at that moment since the problem is not very complex. For example, the students would draw the height, which is 52 meters above the ground, and determine where the observer stands. Then, they would count how many times the ball passes, and then they may find an interval with respect to bouncing number. Either, they would not think that the ratio would be an interval, and they would find just one (constant) ratio because they would think that both 8^{th} and 9^{th} bounces are equal to 15 meters. Either, if they do not draw a figure that represents the bounces, by the help of sequences and series, they would think "total total distance the ball has travelled". However, if they think logically, they recognize that they cannot go further. I mean if they think logically and carefully, they can interpret up to 8^{th} bounces. At the end, students would find a constant bounce rate rather than finding the bounce rate in an interval **[TBRP_2].**

On the other hand, though two of the pre-service teachers (PST4 and PST5) stated their predictions about students' possible solutions, their predictions were related to students' performance instead of students' possible solution approaches. Pre-service teachers expressed that students would not produce different kinds of solution approaches in solving this task. Below is an example from PST5.

> **PST 5**: I expected that most of the students in groups would have same and correct solution. Moreover, several students would make a mistake whether or not it is equal to 15 meter in its 8^{th} bounce **[TBRP_2].**

After analyzing students' worksheets and video episodes containing actual student solutions, the pre-service teachers identified students' solutions on "Bouncing Ball" task. Table 4.5 summarizes the pre-service teachers' descriptions of students' actual solution (either correct or incorrect) what they identified as a group.

Table 4.5 Summary of students' actual solution approaches pre-service teachers identified as a group

Student group (SG)	Pre-service teachers' identification of students' solution approaches	Pre-service Teachers Group (PSTG)
SG1	▪ Students used the trial-error method.	PSTG1, PSTG2, PSTG3, PSTG4, PSTG5, PSTG6, PSTG7
SG2	▪ Students used the trial-error method and inequalities approach.	PSTG1, PSTG2, PSTG3, PSTG4,PSTG5, PSTG6, PSTG7
SG3	▪ Students used inequality (exponential inequality) and sequencing.	PSTG1, PSTG2, PSTG3,PSTG4,PSTG5, PSTG7
SG4	▪ Students used the idea that "the decrease of bouncing ball's height was constant each time"	PSTG1, PSTG2, PSTG3,PSTG4, PSTG5, PSTG7

As shown in Table 4.5, pre-service teachers determined that students in two groups (SG1 and SG2) solved the task by using trial-error method. However, as seen in Table 4.4, none of the pre-service teachers have predicted/expected that students would use a trial-error method to solve the problem. Furthermore, whereas pre-service teachers identified that students in group-3 solved the task correctly by using exponential inequalities as they solved, students in group-4 solved the task incorrectly by thinking the decrease of bouncing ball's height is constant instead of by taking it constant rate (see Appendix C). That is, when pre-service teachers' predictions in Table 4.4 were compared to students' actual solutions identified by pre-service teachers in Table 4.5, some consistent predictions were observed. For example, as some of pre-service teachers predicted, students in group-3 (SG3) had solved the problem by using exponential inequalities. However, there were also inconsistencies between pre-service teachers' predictions and students' actual solutions identified by pre-service teachers.

On the other hand, in their reflection papers, pre-service teachers reflected on what they have predicted/expected before analyzing students' works and what they observed related to students' solution approaches when they analyzed students' works. The data showed that whereas five pre-service teachers indicated the similarities, eleven pre-service teachers stated the differences between their predictions and identifications. According to pre-service teachers' reflections, the differences between their predictions and identifications were as seen in the following table 4.6.

Table 4.6 The expected/predicted and unexpected/unpredicted solution approaches.

Expected/predicted solution , but it was not produced by any students	**Unexpected/unpredicted solution**
▪ Students' use of geometric sequences/series	▪ Students' use of trial-error method

Some of the pre-service teachers indicated that students approached to the solution of the task totally different from what they expected. For example, any of the students in groups did use the geometric sequences/series in their solutions. Rather, the students in group-1 and group-2 tried to produce a solution to the task by using trial-error method. Here, PST14 reflects how students solved the task different from her expectations.

> **PST14:** Before I have seen students' worksheets and video episodes, I had thought that students would solve the problem in a different way. For example, I expected that there would be a solution approach through geometric series, but it was not a solution approach emerged from the students in groups. On the other hand, the students in group-2 found a solution by using trial and error method. I have never expected that [method] because there were many real numbers in the interval, and in my opinion trying these real numbers was quite difficult. But, students had used this approach **[STRP_2].**

On the other hand, pre-service teachers were surprised at the trial-error method used by students in group-1 (SG1) and students in group-2 (SG2). In the solution approach produced by students in group one; pre-service teachers were surprised at not only solution approach based on trial-error but also students' reasoning used in this solution approach. According to pre-service teachers, this solution approach was not the ordinary trial-error method. Rather, it was a systematical trial-error method. As seen in the solution presented in Appendix C2, students initially had began to solve by squeezing of bouncing rate between 15/52 and 1, and then calculated the average of these rates to find the wanted bouncing rate. The idea of "squeezing and calculate average" was quite intriguing for pre-service teachers.

Furthermore, while examining students' solutions, on the back pages of students in group-2 (SG2) and group-3's (SG3) worksheets, pre-service teachers observed several graphs sketched like a parabola graph and exponential function graph, and equations belonging these graphs as illustrated in Figure 4.2. Students' use of parabola graph formulating equation of the parabola and exponential function as a thinking way was both interesting and unexpected.

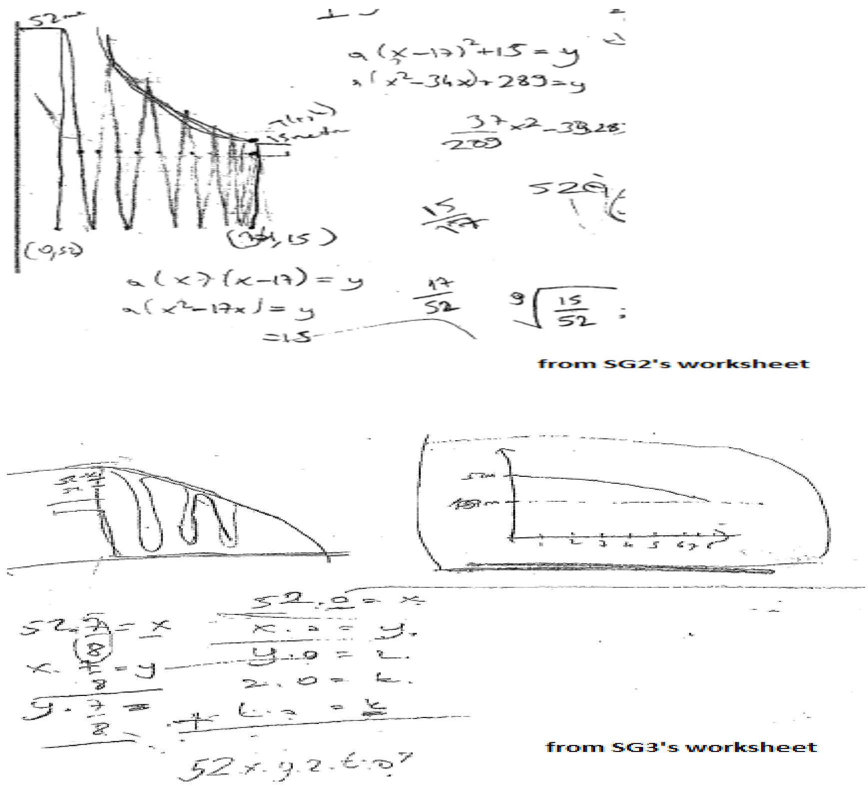


Figure 4.2 An example of graphs observed in worksheets of students in Group-2 and Group-3

Pre-service teachers stated that even if these ideas did not help students to produce a correct solution, they still found these ideas very logical. Pre-service teachers stated that none of them has ever thought about such kind of ideas while they were producing a solution to the task. PST2 reported that in the following.

> **PST 2:** In the solutions we analyzed, the thing really surprised me was students' ways of thinking in their solution processes rather than produced solutions. The idea of students in group two to use the parabola graph and formulate its equation was quite good. In addition, the idea of student group three was also really good. Although this group [group three] could not transfer their thoughts to their drawings by using this idea, it was a case which attracted my attention because it gave us an idea to produce a solution. I did not think that students in group one would develop a solution way by using trial and error method. Their mathematical operation skills were quite good **[STRP_2].**

For the third task “Roller Coaster”, eleven of the pre-service teachers had predicted/stated their expectations individually about students’ possible solutions as illustrated in Table 4.7.

Table 4.7 Pre-service teachers’ predictions/expectations about students’ possible solutions for “Roller Coaster” task.

Pre-service teachers’ predictions/expectations about possible solutions*	**Pre-service Teachers (PST)**	**# of the PST**
Student would focus on only design of roller coaster.	PST10, PST14, PST15, PST2, PST4, PST3	6
Students would design/create path of roller coaster using only straight lines.	PST18, PST14, PST8, PST9	4
Students would focus on slope and inflection point concepts.	PST8, PST7	2
Students would set up an equation of the sketched path (graph) of the roller coaster	PST8	1
Students would design the roller coaster by using both straight lines and curves.	PST16	1

* Students’ approaches to making a solution for non routine task: mathematical concepts, representations or set of procedures that would be used in solving non-routine task

As shown in Table 4.7, the main focus of majority of pre-service teachers’ predictions/expectations was that students would approach to the solution of the task either by considering only design of the roller coaster instead of thinking the slope and inflection point of the graph or by using straight lines to design wanted path of the roller coaster. Two pre-service teachers expected that students would produce a solution as they solved. That means, according to these pre-service teachers’ expectations, students would solve the task by considering slope of the curve and inflection point of the curve. On the other hand, while one pre-service teachers predicted/expected that students would solve the task by using both straight lines and curves together in their design of the roller coaster, one pre-service teachers predicted that students would try to formulate the equation of the graph, which are drawn to design the path of roller coaster, and would try to solve this graph. In the following, while the excerpt of PST18 illustrates his prediction about use of linear curves by designing roller coaster.

> **PST18**: In my opinion, students' solution approach would be based on linear path, and they would stay away from using parabolic paths which may be more complex for them. Even if they had an idea for parabolic paths, I predicted that it would not be a consistent representation **[TBRP_3].**

Similarly, the excerpt of PST4 exemplifies his prediction for students' possible solution approach.

> **PST4**: In this problem, students mainly would try to design a path because it would be enjoyable for them **[TBRP_3].**

After analyzing students' worksheets and video episodes containing actual student solutions as a group, the pre-service teachers identified students' actual solution approaches on "Roller Coaster" task. Table 4.8 summarizes the pre-service teachers' group descriptions of students' solution approaches.

Table 4.8 Summary of students' actual solution approaches pre-service teachers identified as a group.

Student group (SG)	Pre-service teachers' identification of students' solutions	Pre-service Teachers Group (PSTG)
SG1	▪ Students designed a roller coaster; they drew straight lines in their design and drew circles (quadrant) at the transition points of the segments. Students focused on thrill (factor).	PSTG1, PSTG2, PSTG3, PSTG4,PSTG5
SG2	▪ Students drew the path of roller coaster (graph) by using the given criteria in the task. ▪ Students determined the maximum heights of path of roller coaster (the graph) and finding the maximum slope. ▪ Students sketched the path of roller coaster by using scaling idea.	PSTG1, PSTG2, PSTG3, PSTG4,PSTG5
SG3	▪ Students analyzed the curve (the path of roller coaster they drew) and used the idea of inflection point.	PSTG1, PSTG2, PSTG3, PSTG4,PSTG5
SG4	▪ Student considered the thrill (factor) to design roller coaster. ▪ They analyzed the curve piece-wise and drew linear curves (lines).	PSTG1, PSTG2, PSTG3, PSTG4,PSTG5
SG5	▪ Students associated horizontal length of the coaster (displacement) with the height of roller coaster and formulated equations.	PSTG1, PSTG2, PSTG3, PSTG4,PSTG5

When pre-service teachers' predictions in Table 4.7 were compared to students' actual solutions identified by pre-service teachers in Table 4.8, consistent predictions were observed. Pre-service teachers observed some of what they predicted about students' possible solution approaches. As pre-service teachers predicted/excepted, except students in group-3 (SG3), none of students in other groups have paid attention to the fundamental mathematical ideas "inflection point and slope of curve" for solving task at all. Rather, while they were designing the roller coaster, they mainly focused on the thrill of the roller coaster. In addition, as several pre-service teachers expected, while students were creating their roller coaster, several students in groups used only straight lines (linear curves) and a few students in groups used both straight lines (linear curves) and curves together in their design without considering the real life situation.

In addition to that, in their reflection papers, pre-service teachers reflected on what they have expected related to students' solution approaches before analyzing students' works, and what they have observed related to them while they were analyzing. Six pre-service teachers indicated the similarities between their predictions and students' actual solution approaches they identified. The following excerpt illustrates the similarities between pre-service teachers' expectations/predictions and students' actual responses.

> **PST 13:** In my reflection paper, I have written in the following: Students would think that the dropping segments of the path would be linear rather than curvilinear; the bottom and top of the path would be flat, and then the ascending segments of the path would be linear with the slope 5.67. And, students would not think that the train would not move safely through such kind of way. When I watched the video episodes and examined students' worksheets, I have observed that I was correct. Except the students in group three, all student groups assumed that several segments of the path were linear. In addition, the slope of this straight line was 5.67 which was the criteria for safety **[STRP_3]**

On the other hand, although pre-service teachers made consistent predictions for this task, there were still several solutions which have not been predicted by pre-service teachers. Eight pre-service teachers stated the differences between their predictions about students' possible solutions and students' actual solutions as shown in Table 4.9.

Table 4.9 The expected/predicted and unexpected/unpredicted solution approaches

Expected/predicted solutions, but they were not produced by students	Unexpected/unpredicted solutions
-----	Students' use of circles (quadrant) at the transition points of the segments" in the design of the roller coaster. Students' use of "scaling" while drawing the path of roller coaster and calculating slope of the curves. Students' association of the horizontal length of the coaster (displacement) with the height of roller coaster, and in this process students' formulation of several equations and, students' use of the arithmetic/algebraic operations mostly to solve them.

"The use of (quarter) circles" of SG1 and "the use of scaling" of SG2 in their design of roller coaster path were unexpected/unpredicted approaches for pre-service teachers as exemplified by PST9.

> **PST9**: I could not predict that a segment of the roller coaster path designed by students in group one was drawn by using quadrant. It seems logical. The simplest curve we know is circle, and students had tried to draw it. However, the slope is more than 5.67 at least one point of the circle and safety laws are not provided **[STRP_3].**

In this analysis of students' works, roller coaster design of students in group-4, use of scaling idea to calculate slope in their solutions of students in group-2 and group 3, and use of quadrant in the design of the roller coaster of students in group-1 were other example of mathematical ideas/solution approaches that surprised pre-service teachers. All these solutions were mostly unexpected, creative and not common among students. On the other hand, pre-service teachers were also quite surprised a solution that was not compatible with the real life situations.

For the fourth task "Water Tank", 8 of the pre-service teachers stated their predictions/expectations about students' possible solution approaches. Table 4.10

illustrates pre-service teachers' individual predictions/expectations about students' possible solution approaches for the fourth task.

Table 4.10 Pre-service teachers' predictions/expectations about students' possible solutions for "Water Tank" task

Pre-service Teachers' predictions/expectations about possible solutions*	**Pre-service Teachers (PST)**	**# of the PST**
Students would produce a solution approach similar to our approach.	PST10, PST24, PST5 PST25, PST6, PST7	6
Students would sketch the graph by using properties of geometric figures (parts) of the water tanks.	PST10	1
Students would use intuitive approach.	PST24	1
Students would sketch the linear graphs.	PST7, PST13	2
Students would draw the mini models to sketch the graph.	PST15	1
Students would use of cross section and they would reduce dimension of water tanks from 3D to 2D.	PST10	1

* Students' approaches to making a solution for non routine task: mathematical concepts, representations or set of procedures that would be used in solving non-routine task

Six of the pre-service teachers expected/predicted that students would approach to the solution of task as they solved by stating "students would solve the task as we solved". In addition, for this task, while pre-service teachers stated their predictions, they still used general statements rather than provided detailed predictions about students' solution processes. However, their predictions were more diverse and specific to the task. On the other hand, a couple of pre-service teachers predicted that students' solutions would be similar, and so many different solution approaches would not be emerged among students. PST13 indicated her thought in the following.

> **PST13:** Because drawing a graph is wanted in the problem, students would not produce different solution approaches for this problem. However, the

differences would be the nature of the graphs sketched by student groups. For instance, students would not decide whether the graph linear or curvilinear is. Even if they decided, they would not decide to what type of curve (e.g., concave up, concave down) they would draw **[TBRP_4].**

On the other hand, PST24 indicated that it was rather difficult to predict students' solutions that they would produce for this task, and he predicted that students would sketch the graphs intuitively.

PST24: Because we could not produce various solutions and we did not make lots of errors for this problem, it is difficult how students would think or what kind of difficulties they would have. But, most likely, students can produce mathematical solution what we produced. Either, even if their solutions were not being so mathematical, they would produce more or less a solution based on this reasoning. They can draw the graphs of given tanks intuitively or logically; however, they cannot explain the reason of it [...] **[TBRP_4].**

After analyzing students' worksheets and video episodes containing actual student solutions as a group, pre-service teachers identified students' actual solution approaches on "Water Tank" task. Table 4.11 summarizes what the pre-service teachers identified as a group.

Table 4.11 Summary of students' actual solution approaches pre-service teachers identified as a group

Student group (SG)	**Pre-service teachers' identification of students' solution approaches**	**Pre-service Teachers Group (PSTG)**
SG1	▪ Students divided the water tanks into parts and compared these parts rather than examining each part in itself, and they sketched the less step graph correspond to wider parts and sketched a steeper graph corresponds to narrower parts.	PSTG2, PSTG3, PSTG4, PSTG5, PSTG7
SG2	▪ Students divided the height of the water tank in the equal parts and determined the change with respect to it and sketched line graphs.	PSTG2, PSTG3, PSTG4, PSTG5, PSTG7

Table 4.11 (continued)

SG3	▪ Students approach was "If the volume increases, the rate of increase in height decreases; if the volume decreases, the rate of increase in height increases".	PSTG2, PSTG3, PSTG4, PSTG5, PSTG7
SG4	▪ Students sketched the height versus volume graphs with respect to changes of water tank's shapes; getting narrower, getting wider or remaining constant.	PSTG2, PSTG3, PSTG4, PSTG5, PSTG7
SG5	▪ Students sketched the graphs with respect to changes of tank's shapes; getting narrower, getting wider or remaining constant.	PSTG2, PSTG3, PSTG4, PSTG5, PSTG7
SG6	▪ Students sketched the graphs with respect to changes of tank's shapes; getting narrower, getting wider or remaining constant.	PSTG2, PSTG3, PSTG4, PSTG5, PSTG7

When pre-service teachers' predictions in Table 4.10 were compared to students' actual solutions identified by pre-service teachers in Table 4.11, it was observed that pre-service teachers made consistent predictions. For instance, as six pre-service teachers predicted, students in groups three, four, five and six (SG3, SG4, SG5 and SG6) produced quite similar graphs to what pre-service teachers produced, and their graphs sketched for each water tank were almost correct. However, distinctively, pre-service teachers recognized that the students in these four groups intuitively drew the wanted graph rather than use of the mathematical notions "rate of change in height with respect to a fixed amount of water or width (cross-sectional area) of bottle at a given height". Furthermore, pre-service teachers identified students in group-1 (SG1) and group-2 (SG2) produced linear graphs in their solutions as they predicted.

On the other hand, the analysis of pre-service teachers' self evaluation of the similarities and differences between their predictions/expectations and students' actual solution approaches displayed that nine of the pre-service teachers stated that students mainly solved the task as they expected. The following excerpt of PST7 provides evidence to similarities between pre-service teacher's predictions/expectations about students' possible solution approaches and students' actual solution approach.

> **PST7**: I could say that students' solution approaches, which I predicted in my task based reflection paper, were almost similar to students' actual solution they produced. In my reflection paper, I had stated that students would present a solution approach similar to our solution approach, this is, students' solution approach in group three. Moreover, students in group four and five also produced a solution similar to our solution approach. These groups (four and five) were not able to construct a smooth curve; instead, they constructed contiguous curves on the graph. They produced the graphs exactly similar to our drawings; however, they could not amend their graphs. Moreover, in my reflection paper, I also indicated that students would draw straight lines rather than curves. We can observe that in the solution of students in group one and two. Student determined the transition points of the water tanks and then they drew the straight lines among these points. The students in group two tried to use the idea "unit volume and unit height" to draw graphs of water tanks; however, they could not understand that the graphs should be parabolic **[STRP_4].**

On the other hand, four of the pre-service teachers pointed out the differences between their predictions about students' possible solution approaches and students' actual solution approaches they identified as in the following Table 4.12.

Table 4.12 The predicted/expected and unpredicted/unexpected solution approaches

Expected/predicted solutions: They were not produced by students in any group	**Unexpected/unpredicted approaches**
▪ Student would not interpret the shapes of the water tank and sketch the height versus volume graph at all.	▪ The idea "sketch straight line graph by dividing the height of the water tank in the equal parts"

As seen in Table 4.12, the approach of students in group-2 *(it is not totally correct)* to sketch graphs was not predicted /expected by any of the pre-service teachers (see Appendix C4, STG4) where they sketched straight line graph by dividing the height of the water tank in the equal parts and determine the change with respect to it. PST1 reflects it in the following:

> **PST1:** […] Maybe, I could predict that students would not use only straight line, but I have never thought that students drew the graphs with respect to changes of volume by dividing the height of the water thank in the equal part **[STRP_4].**

On the other hand, for this task, producing correct graphs for the given water tanks were quite amazing for pre-service teachers because pre-service teachers in several groups could not construct a smooth curve. Therefore, students' correct interpretations and representing of graphs were unexpected for them.

Overall, the data analysis revealed that pre-service teachers started to become more familiar with students' possible solution approaches over time and over experience, and they started to make more consistent predictions with students' actual solutions. In addition, as a result of their identification of students' solution approaches, pre-service teachers recognized that students could produce more different and valid solution approaches, different ways of interpreting and use more powerful mathematical ideas than they had done. Moreover, pre-service teachers increased their awareness that students' tendency was approaching the solution either intuitively or use of an informal methods like trial-error rather than use of formal algebraic operations (e.g., construct and solve equations, functions etc.).

4.1.2 Pre-service teachers' predictions and identifications of students' difficulties and errors

Similar to presentation of findings in previous section, in this section, the results concerning what pre-service teachers predicted/expected about students' possible difficulties and errors in solving each mathematical task, and what pre-service teachers identified about difficulties and errors in solutions students produced for each mathematical task are presented.

The data analysis revealed that for the first task "Street Parking", before working on students' works, only two pre-service teachers explicitly made predictions in their reflection papers about students' possible difficulties and errors they would experience. While both pre-service teachers (PST5, PST6) were in the same group (PSTG3), their predictions were *"students would make a mistake while they are choosing the correct root of quadratic equation from two roots of it".* The examination of those pre-service teachers' own solutions on the same task showed that this predicted error was pre-service teachers' own error made while solving the task, and they expected the same error from students. That is, they predicted their own error as students' possible error.

After examining students' worksheets and video episodes containing student solutions in detail as a group, the pre-service teachers identified students' actual difficulties and errors on "Street Parking" task as presented in Table 4.13.

Table 4.13 Summary of the pre-service teachers' identification of students' difficulties and errors.

Student group (SG)	Pre-service teachers' identification of students' actual difficulties and errors	Pre-service Teachers Group
SG1	▪ Students could not construct triangle similarity correctly and they equaled the different values of sine to each other as "$\sin\alpha=\sin\beta$" where $\alpha\neq\beta$. ▪ Students ignored the length of the "c" given in the problem statement and calculated another length so that they called it as c. ▪ After students sketched the cars as an angle parking, they accepted two triangular areas, one of which was at the beginning and another was at the end of the parking area, as equal.	PSTG1, PSTG2, PSTG3, PSTG4,PSTG5, PSTG6, PSTG7
SG2	▪ Students miscomprehended of the length of secure park area which is 4.8 meter. ▪ Students used only the trigonometric values of special angles such as 30^0, 45^0 or $60^{\ 0}$, and they ignored the angles which are decimal numbers.	PSTG1, PSTG2, PSTG3, PSTG4,PSTG5, PSTG6, PSTG7
SG3	▪ Students could not use the Euclidean geometry relation correctly. ▪ Students solved the task in two different ways and obtained two different solutions, but they did not check why their solutions were different than each other. ▪ After students sketched the cars as an angle parking, they accepted the triangular areas at the beginning and at the end of the parking area as equal.	PSTG1, PSTG2, PSTG3, PSTG4,PSTG5, PSTG6, PSTG7

Table 4.13 (continued)

SG4	▪ Students incorrectly calculated the number of cars in parallel parking method to allow the most room for the parking of the cars. ▪ Students could not solve the quadratics equation which they set up.	PSTG1, PSTG2, PSTG3, PSTG4,PSTG5, PSTG6, PSTG7
SG5	▪ Students could not comprehend the task exactly; therefore, they thought the length of the secure parking space which is given 4.8 meter as a wrong length	PSTG1, PSTG2, PSTG3, PSTG4,PSTG5, PSTG6, PSTG7

As seen in the table 4.13, the data revealed that pre-service teachers could identify many errors and difficulties of students which students experienced in solving the Street Parking task.

On the other hand, after pre-service teachers identified students' difficulties and errors, in their reflection papers on students' ways of thinking, they reflected the differences and similarities between what their expectancies on students' possible errors and difficulties were (even if many of them did not state their predictions explicitly in their task based reflection papers before working on students' works) and what students had done, as seen in Table 4.14.

Table 4.14 The predicted/expected and unpredicted/unexpected students' errors and difficulties

Expected/Predicted	**Unexpected/Unpredicted**
Calculation Errors / Procedural Errors	Students' misperceptions of the length of secure park area
Difficulty in solving quadratic equation	Students' equalization of the different values of sine to each other as "$\sin\alpha=\sin\beta$, where $\alpha\neq\beta$"
Difficulty in use of trigonometry	After they sketched the cars as an angle parking, they accepted the triangular areas at the beginning and at the end of the parking area as equal.

The analysis of data revealed that five of the pre-service teachers indicated students made similar errors what they expected/predicted. As seen in Table 4.14, these predictions were students' calculation errors, students' difficulty in solving quadratic equations and students' difficulty in the use of trigonometry. However, tree of the five pre-service teachers have not explicitly stated their predictions in their task based reflection papers before working on students' works. As it was explained above, only two pre-service teachers made predictions, and their predictions were "students would have difficulty in solving quadratic equation".

As five pre-service teachers predicted, students experienced these errors or difficulties in solving the task. However, pre-service teachers' these predictions were very general in their nature. Their expressions were only like "students might make calculation errors". For example, as pre-service teachers predicted, students in almost all groups made calculation errors in their solution or students in several groups made error while trying to use trigonometry in their solution. However, pre-service teachers have never provided specific examples to explain "where students' calculations/procedural errors would be" or "which concept of trigonometry would be challenging for students". The following excerpt illustrates PST12's reflection whose prediction is very general

> **PST12:** I had thought that students would make calculation errors. I have examined sequentially if the groups [students] made calculation error. However, it was possible that there were groups who made calculation error. As I reported in my reflection paper, I predicted that students would have difficulty in solving equation. That was similar which I predicted. I have seen that students had difficulty in solving quadratic equation. For instance, the solution approach of students in group four is correct, and they formulated a quadratic equation in their solution approach. If they solved this equation correctly, they would finish their solution. However, because they had a difficulty in that point, they could not complete their solution way **[STRP_1].**

Furthermore, fourteen of the pre-service teachers reported that students made more errors and had more difficulties from their expectations. Some of the pre-service teachers indicated that they have never expected difficulties, errors or misunderstandings of students seen in Table 4.14. The following excerpt is an example of what PST14 indicated about her unexpected predictions/expectations.

PST 14: For this task, in their solution approaches, the errors students would make and the difficulties students would have that I have never predicted were the following: Students in group one equated the different angles of sine to each other as $\sin\alpha=\sin\beta$, where $\alpha\neq\beta$; students in group two did not check whether or not their results they calculated for different θ values corresponds to the lengths of the triangle sides; students in group three accepted that the triangular areas at the beginning and end of the parking area were equal after they sketched the cars as an angle parking; students in group four could not solve a quadratic equation; students in group five tried to make calculation for varied θ values although two edges and one angle of the triangle were known and constant **[STRP_1].**

Moreover, as shown in the excerpt, the nature of these unpredictable errors and difficulties displayed that they were mainly concept-based and reasoning-based errors or difficulties of students. For example, "students' acceptance of the triangular areas at the beginning and end of the parking area were equal after they sketched the cars as an angle parking" was an error relating to students' reasoning/assumptions while producing a solution to the problem. Moreover, students' misunderstanding/miscomprehension of the problem was also unexpected for some of the pre-service teachers as it is illustrated in the following excerpt.

PST1: As many [students] groups did, I have not expected that they made an error while they were labeling where 4.8 meter length was **[STRP_1].**

On the other hand, pre-service teachers reflected that they were surprised at or found interesting students' several conceptual difficulties in particular mathematical concepts, calculation errors, or misinterpretation of problem statement. When pre-service teachers reflected that they were surprised at students' difficulties, errors or misunderstandings, pre-service teachers generally did not explicitly explain the reasons why they were surprised at students' these errors, difficulties or misunderstanding. However, in their reflective accounts, pre-service teachers generally expressed that "I have never expected that students would make such an error" or "students had to know that, how they made such an error". Therefore, pre-service teachers' those expressions provided a clue that these students' errors or difficulties were quite unexpected for them. For example, the all unexpected errors seen in Table 4.16 were surprising for pre-service teachers. In addition to them, they were also surprised at the following error of students in group-1 so that "*students found 106 cars in total with angled parking*".

The following expert shows which error of students in group-1 was found interesting by PST2.

> **PST 2:** I have never predicted that the groups [student group] found such a high number of cars. It was quite interesting that student group one found 106 cars in total [53+53=106]. Even if the cars were parked at a right angle and it was paid attention to the criteria width of the parking space as 3 m, there would make 100 cars [50+50=100]; however, students presented a wrong solution way including a calculation error without thinking their mistake **[STRP_1].**

As seen in the excerpt, according to PST2, students produced a solution without thinking logically. In addition to that, PST20 stated that he was surprised at students' confusion regarding several mathematical concepts. According to him, the trigonometry should be known very well by students; however, he has explored that students were confused the trigonometry with the similarity.

> **PST20:** The most surprised thing for me was that some of the concepts were fragile for the students. In my opinion, a high school student at 11^{th} grade level has to know trigonometry very well. However, to be honest, I was quite surprised that the student group one who called it as "sine theorem" while they were constructing a similarity and the student group three who called it as "Euclidean relation" while they were calculating area **[STRP_2].**

Furthermore, several pre-service teachers indicated that they expected students would understand and solve the task correctly, but they have never expected students would make lots of errors.

For the second task "Bouncing Ball", the data analysis revealed that in their reflection papers, 18 pre-service teachers explicitly predicted/expected students' possible difficulties and errors that they would experience. The common, representative and most predicted/expected students' errors and difficulties were as in the following Table 4.15.

Table 4.15 The most predicted students' errors and difficulties

Prediction/Expectation about students' difficulties and errors	Pre-service Teachers (PST)	# of the PST
Students would use "equality" (=) rather than "inequality" by thinking that the last bounce is equal to 15 meters.	PST13, PST3, PST5,PST7, PST8	5
Students would think that the ball pass one time rather than two times while the ball is dropping and rebounding from the level of 15 meter above the ground at each time the ball bounces and then students may perceive that the ball bounces 17 times rather than passes 17 times.	PST14,PST20,PST24,PST3, PST4,PST9	6
Students would think a linear relationship and reasoning as "the decrease of the height of the bouncing ball as constant amount after each bounce instead of the constant bouncing rate"	PST13,PST14,PST16	3
Students would find a constant bouncing rate of the ball rather than finding the bouncing rate in an interval.	PST17,PST1,PST2	3
Students would make an error while formulating the algebraic inequality or while solving this inequality (e.g., $52.x^{n}>15$)	PST14,PST15, PST16, PST20, PST25, PST2,PST7, PST11	8
Students would make a procedural error while using the series/sequences formula.	PST14,PST23,PST7	3

As seen in table 4.1, the pre-service teachers could predict/expect various kinds of students' errors or difficulties. Moreover, the data displayed that pre-service teachers' some predictions regarding students' possible errors and difficulties were task-specific. That means, their predictions were not based on such an expression "students would make calculation errors".

After pre-service teachers examined students' worksheets and video episodes containing student solutions in depth, they determined students' actual difficulties and errors on "Bouncing Ball". The data showed that pre-service teachers could identify many of the students' errors and difficulties that students experienced in solving the Bouncing Ball task. Table 4.16 summarizes what pre-service teachers identified as a group.

Table 4.16 Summary of the pre-service teachers' identification of students' difficulties and errors

Student group (SG)	Pre-service teachers' identification of students' actual difficulties, errors or misunderstandings	Pre-service Teachers Group (PSTG)
SG1	▪ Students found only one/constant bouncing rate and not to think the solution (bouncing rate) should be in an interval.	PSTG1,PSTG2, PSTG3,PSTG4, PSTG5, PSTG6, PSTG7
SG2	▪ Students found only one/constant bouncing rate. ▪ Students thought that the last bounce is equal to 15.	PSTG1,PSTG2, PSTG3,PSTG4, PSTG5,PSTG6, PSTG7
SG3	▪ Students had difficulty in finding the 9^{th} root; therefore, they wrote the bounce rate in the interval as "1>x>0. 865"	PSTG2,PSTG3, PSTG4,PSTG5, PSTG7
SG4	▪ Students had difficulty in comprehending task/problem statement and think the decrease of bouncing ball's height was constant each time rather than thinking bouncing rate was constant	PSTG1,PSTG2, PSTG3,PSTG4, PSTG5,PSTG6, PSTG7

When the pre-service teachers' predictions in Table 4.15 was compared to students' common actual errors and difficulties described by pre-service teachers in Table 4.16, it was seen that some of actual errors and difficulties of students were consistent with the errors and difficulties predicted by pre-service teachers. Pre-service teachers observed many of the predicted errors and difficulties in students' solutions. For example, as some of the pre-service teachers predicted, students in two groups found only a constant bouncing rate rather than found the bouncing rate in an interval.

On the other hand, in their reflection papers, pre-service teachers reflected the differences and similarities between what their predictions/expectancies on students' possible errors and difficulties were (even if they did not state them explicitly in their reflection papers) and what students done as provided in Table 4.17.

Table 4.17 The predicted/expected and unpredicted/unexpected students' errors and difficulties

Expected/Predicted	Unexpected/Unpredicted
▪ Student accepted that the last bounce of the ball, namely 8th bounce was equal to 15 meter.	▪ The decrease of bouncing ball's height was constant each time.
▪ The decrease of bouncing ball's height was constant each time.	…..
▪ Students found only one/constant bouncing rate.	…..

The data obtained from those reflection papers also support the consistency shown in table 4.15 and 4.16. The data also revealed that while the error "the decrease of bouncing ball's height is constant each time" shown in Table 4.17 was the expected/predicted error for three of the pre-service teachers, five of pre-service teachers indicated that they have never predicted/expected such kind of error. Below an excerpt illustrates the reflection of PST9. She stated that she had never expected students would make such kind of error.

> **PST 9:** I have never predicted that the error made by students in group-4. At each bounce, students thought that the decrease of bouncing ball's height was constant. After they found that [constant] amount, they looked at the ratio of the height of the ball at each bounce to amount in the descent, and then they obtained different rates each time. They also calculated the approximate value of bounce rate which they found **[STRP_2].**

Moreover, this error was also surprising some of the pre-service teachers who did not predict/expect such kind of error. On the other hand, several pre-service teachers explained that while they expected students would make an error by thinking that the ball would rise 17 times above 15 meters rather than 8 times, pre-service teachers reflected that students did not make any such kind of error or have difficulty at all.

For the third task "Roller Coaster", the data analysis revealed that in their reflection papers, 16 pre-service teachers predicted about students' possible difficulties and errors they would experience. For this task, the predicted errors and difficulties of students are provided in Table 4.18.

Table 4.18 The most predicted/expected students' errors and difficulties

Prediction/Expectation about students' difficulties, errors or misunderstandings	Pre-service Teachers (PST)	# of the PST
Students would not either understand or interpret the givens in the tasks. For instance, what means the slope of path can be no more than 5.67?	PST14, PST15,PST24,PST3 PST4	5
Students would not understand (or have difficulty in understanding) where 100 meter distance is.	PST14,PST15,PST7,PST8, PST25	5
Students would have difficulty in determining the inflection point.	PST14, PST20,PST24,PST25 PST6,PST7	6
Students would interpret the slope of the curves as the slope of the straight lines without thinking the slope of the curve is different at different points.	PST13,PST15,PST24	3
Students would have difficulty in interpreting derivative concept and interpreting the slope concept.	PST10,PST21,PST8,PST2,	4
Students would not pay attention (or ignore) the safety criteria.	PST9	1

After pre-service teachers analyzed students' worksheets and video episodes containing student solutions as a group, the pre-service teachers identified students' actual errors and difficulties on "Roller Coaster" task. As table 4.19 summarized, the data revealed that pre-service teachers could identify many of the students' fundamental errors, difficulties or misunderstandings which students experienced in solving the Roller Coaster task.

Table 4.19 Summary of the pre-service teachers' identification of students' actual difficulties and errors

Student group (SG)	Pre-service teachers' identification of students' difficulties and errors	Pre-service Teachers Group (PSTG)
SG1	▪ Students misunderstood that 100 meter length was the length of path of designed roller coaster rather than total horizontal length of the roller coaster. ▪ Students designed the path of roller coaster using straight lines [*which does not satisfy the given restriction*], and they ignored safety criteria.	PSTG1, PSTG2, PSTG3, PSTG4,PSTG5
SG2	▪ While using scaling method, students did not paid attention to the restriction so that the slope of path could not be more than 5.67 ▪ Students could not determine where slope of path maximum was.	PSTG1, PSTG2, PSTG3, PSTG4
SG3	▪ Students could not interpret the concept of inflection point and interpreted the inflection point as the middle point of the curve (not to as a point at which concavity changes)	PSTG1, PSTG2, PSTG3
SG4	▪ Students designed a roller coaster path that was not safety and valid for a real life. ▪ Students sketched the all segments path of roller coaster as parabolic; however, they made several calculations as if they were straight lines.	PSTG1, PSTG2, PSTG3, PSTG4,PSTG5
SG5	▪ Students assumed that the slope of the curve was constant (5.67) at each point of the curve and they ignored the safety criteria. ▪ Students accepted that the slope was 80° at the top (peak point) of path of roller coaster.	PSTG1, PSTG2, PSTG3, PSTG4,PSTG5

When pre-service teachers' predictions in Table 4.18 were compared to students' common actual errors and difficulties described by pre-service teachers in Table 4.19, it was observed that some of the difficulties/errors of students could be predicted by pre-service teachers in advance. For instance, the misunderstanding of students as 100 meter length is the length of path of designed roller coaster rather than horizontal length of the roller coaster; the erroneous thought that the slope of the curve is same at each point of the curve and it is 5.67 as well as students' inattention to the safety criteria were the predicted error, difficulty or misunderstanding. On the other hand, after pre-service teachers examined students' works, in their reflection papers, they reflected the differences and similarities between what their predictions on students' possible errors and difficulties were (even if they did not state them explicitly in their reflection papers) and what students had done as presented in Table 4.20. The data obtained those reflection papers also support the consistency drawn from table 4.18 and 4.19.

Table 4.20 The predicted/expected and unpredicted/unexpected students' errors and difficulties

Expected/Predicted	Unexpected/Unpredicted
▪ Students' misunderstanding so that they accepted the 100 meter is the length of path of designed roller coaster rather than horizontal length of the roller coaster.	▪ Students' misunderstanding so that they accepted the 100 meter is the length of path of designed roller coaster rather than horizontal length of the roller coaster.
▪ Students' difficulty in interpreting the given restriction in the task so that slope of path which cannot be more than 5.67.	▪ Students interpreted the slope of the curves as the slope of the straight lines without thinking the slope of the curve is different at different points.
▪ Students' difficulty in comprehending the task completely and making computations in solving the task.	. .....
▪ Students' inattention to safety criteria.	......
▪ Students' thinking "the slope of the curve was 5.67 at each point of the curve" was incorrect.	…...

As shown in Table 4.20, whereas some of errors and difficulties of students were expected by some of the pre-service teachers, two of them had never been expected by other pre-service teachers. For example, students' misunderstanding "100 meter length was the length of path of designed roller coaster rather than length of overall horizontal displacement" was unexpected/unpredictable misunderstanding for PST13. Similarly, students' miscomprehension relating to slope of a curve and slope of a line was an unpredictable difficulty for PST14. These unexpected students' difficulties were also surprising for these pre-service teachers.

The following excerpt illustrates PST14's reflection of her/his expected and unexpected students' difficulty and error.

> **PST14**: The task we engaged in was to understand the students' image of slope. For instance, I have predicted that students would understand the distance as the length of path of roller coaster and would have difficulty in understanding the value of 5.67. However, in particular, the error, which students interpreted the slope of curve is the same at each point of the curve instead of interpreting the slope of a curve at a line, did not took place among errors which I thought **[STRP_3].**

Lastly, the data analysis revealed that for the fourth and last task "Water Tank", in their reflection papers, seventeen of pre-service teachers predicted students' possible difficulties and errors they would experience. Many pre-service teachers predicted that students would have difficulty in sketching the graphs of water tank. Particularly, pre-service teachers' prediction was that students would have difficulty in interpreting the graphs in terms of increasing or decreasing. Furthermore, three of the pre-service teachers also expected that students would have difficulty in producing wanted instruction manual to generalize their reasoning for any water tank. In addition to that, several pre-service teachers expected that students would sketch the graph of water tank intuitively without thinking the mathematical ideas such as "cross-sectional area of the geometric shapes (of water tanks) or unit volume". The common and most predicted/expected students' errors and difficulties are presented in the following Table 4.21.

Table 4.21 The most predicted students' errors and difficulties

Prediction/Expectation about students' difficulties and errors	Pre-service Teachers (PST)	# of the PST
Students would have difficulty in sketching increasing or decreasing graphs; students would sketch incorrect graphs; students would sketch the correct graphs but interpret the graphs incorrectly.	PST10, PST11,PST17,PST18, PST23,PST25, PST20, PST4,PST6,PST7, PST9	11
Students would not decide if they would sketch the linear or curvilinear graphs.	PST13, PST7	2
Students would sketch the graphs intuitively, and they would not make reasoning about the mathematical idea behind it.	PST24	1
Students would divide the water tanks into different geometric parts; however, they would have difficulty in sketching the graphs with respect to changes of water tanks' shapes.	PST2,PST3	2
Students would have confusion while interpreting dependent (output) and independent (input) variable.	PST14,PST17,	2
Students would have difficulty in producing wanted instruction manual to generalize their reasoning for any water tank.	PST12, PST25,PST4	3

Then, after pre-service teachers examined students' worksheets and video episodes containing actual student solutions in depth as a group, they identified students' actual difficulties and errors on "Water Tank" task. Similarly, as it was explored in other tasks, for this task, pre-service teachers could identify students' various errors and difficulties correctly. Table 4.22 summarizes what pre-service identified as a group.

Table 4.22 Summary of the pre-service teachers' identification of students' actual difficulties and errors

Student group (SG)	Pre-service teachers' identification of students' actual difficulties and errors	Pre-service Teachers Group (PSTG)
SG1	▪ Students sketched the graphs by using straight lines for all water tanks.	PSTG2, PSTG3, PSTG4,PSTG5, PSTG7
SG2	▪ Students sketched the graphs by using straight lines for all water tanks.	PSTG2, PSTG3, PSTG4,PSTG5, PSTG7
SG3	……..	……
SG4	▪ Students were not to be able to construct a smooth curve and interpret instantaneous rate of change.	PSTG2, PSTG3, PSTG5, PSTG7
SG5	▪ Students were not to be able to construct a smooth curve and interpret instantaneous rate of change.	PSTG2, PSTG3, PSTG4, PSTG7
SG6	▪ Students were not to be able to construct a smooth curve and interpret instantaneous rate of change.	PSTG2, PSTG3, PSTG4,PSTG5, PSTG7

When the pre-service teachers' predictions in Table 4.21 were compared to students' common actual errors and difficulties identified by pre-service teachers in Table 4.22, it was observed that pre-service teachers could predict some of the main difficulties and errors of students. For example, as several pre-service teachers expected/predicted as an error, students in two groups (SG1 and SG2) sketched all graphs of water tanks by using straight lines rather than curves. On the other hand, after pre-service teachers identified students' actual errors and difficulties they indicated that the errors and difficulties were the similar errors and difficulties what they expected from students as presented in Table 4.23.

Table 4.23 The predicted/expected and unpredicted/unexpected students' errors and difficulties

Expected/Predicted	Unexpected/Unpredicted
Students' difficulty in sketching the increasing and decreasing graphs.	………..
Students' difficulty in deciding to sketch linear or curvilinear graphs.	………..
Although students divided the water tanks into different geometric parts to sketch graphs, they could not transfer their reasoning correctly on the graph.	………..
Students divided the water tanks into different parts and drew the graphs of water tanks with respect to these parts; they could not interpret the instantaneous rate, and they could not construct smooth graphs; instead, they constructed contiguous curves on the graph with respect to parts of the water tanks.	………..
Students' difficulty in producing wanted instructional manual to generalize their reasoning.	………..

The data obtained those reflections support the consistency drawn from the tables 4.21 and 4.22. Below, an excerpt is an example of what PST12 was able to predict about students' possible difficulties or errors.

> **PST12:** Before examining the students' worksheets and video episodes, I expected that students would solve this problem because it was not very difficult and complicated. In addition, it was not required a lot of mathematical process to solve. Therefore, I predicted that student would this problem but they would make an error while sketching the graph(s). I predicted that students would not have difficulty in solving the problem; however, they would have difficulty in producing the instructional manual. Once I examined the students' worksheets and video episodes, I have seen that my thought was correct. On the contrary, each student group reached the correct mathematical idea for the solution of problem; however, they made an error while they sketching the graph(s) **[STRP_4].**

On the other hand, although some of the students' main errors were expected by the pre-service teachers, they were still surprised when they observed them. For example, pre-service teachers commonly reported that the error of students in group one “dividing the fourth water tank incorrectly and sketching line graphs for all type water tanks” were quite surprising for them.

In addition to increasing pre-service teachers’ awareness of students’ common errors, difficulties or misunderstandings on particular mathematical topics, during the investigation of students’ works, pre-service teachers also indicated that they gained several ideas about students’ knowledge of certain mathematical concepts. For example, while analyzing students’ works in “Roller Coaster” task, pre-service teachers recognized that students’ lack of (conceptual) knowledge of “inflection point”. In addition, while they were analyzing “Roller Coaster and Water Tank” tasks, pre-service teachers had learnt that students had difficulties with the concept of slope, particularly, they did not know the difference between the slope of a straight line and the slope of a curve; in addition, pre-service teachers became aware that students’ tendency was to accept the finding of the slope of a curve as same as the slope of line. The excerpts illustrating this case are presented as follows.

> **PST14:** [Water Tank] For example, students know, they know more or less what slope is. They are aware of using the slope. But they are not aware of how they convert that slope to graph. For instance, the reason would be that they were students at 10th grade level or they would not learn types of function graphs and their properties. Maybe, even if they learnt, they would not solve. That is actually “image of slope”. The reason may be “what was taught is very superficial and it was not connected to geometry”. However, the thought I have experienced is that; for instance, the student says *“it is getting wider”*. There is a mathematical language of it, but s/he does not express in that way. You know, h/she says “while *my water tank is getting wider, the water level (height) increases less. That means, the slope will be less steep. I will draw a less steep graph to a wider water tank*”. Hahh..S/he thinks and interprets the graph very well. However, at this time, s/he says *“I will sketch less steep but where the less steep should be?”* S/he says “where is the slope? The slope of a line...So, I can sketch the straight lines which have different slopes”. The student thought exactly like that. The logic of student is good. The basis of logic is robust. But, the image of slope should be comprehended better **[STRP_4].**

***

Researcher: What can you say about the image/perception of slope?

PST17: [Roller Coaster] For instance, here, the [students] group took those, I mean, when it was given 5.67, they drew a [right] triangle, and calculated a tangent [value] and thought that the slope is directly that tangent [value]. You know, when slope is said, both student groups thought about the tangent. You know, it is always said us, "Slope is tangent" **[Individual Interview_3]**

As seen in both of the excerpts, while analyzing students' works, pre-service teachers observed how students interpret a mathematical concept in real life context. In this way, the pre-service teachers recognized that students were confused by some mathematical concepts.

4.2 Pre-service Teachers' Valuing Students' Ways of Thinking

The data analysis displayed that pre-service teachers frequently expressed their appreciations for students' solution approaches they identified in students' solutions. This occasion was found to be linked with pre-service teachers' amazement of students' solution approaches and mathematical ideas in these solutions because some of the stated solution approaches and ideas were common. That means, pre-service teachers appreciated several solution approaches and mathematical ideas which they were surprised.

The data revealed that pre-service teachers valued students' solution approaches when

- pre-service teachers had never thought these solution approaches (e.g., use of area; trial-error method) neither as an individual nor as a group while producing a solution to the task.
- the solution approaches of students included original and creative mathematical ideas, for instance, the several graphs generated as one of the thinking way in students solutions, even if students' solutions were not entirely logical or correct.
- students produced similar or entirely same with their solutions and correct solution approaches.

The following table 4.24 summarizes the solution approaches and mathematical ideas of students so that pre-service teachers expressed their appreciation for them.

Table 4.24 The solutions and mathematical ideas valued by pre-service teachers

Tasks	Student Group (SG)	Most valued solutions	# of pre-service teachers
Street Parking	SG3, SG4	▪ Use of area of triangles to be able to produce a solution (SG3) ▪ Finding the area of parallelogram by using two different bases and their corresponding heights, and then equating them each other (SG4)	17
Bouncing Ball	SG2, SG3	▪ The parabola graph and equation (SG2) ▪ Use of second solution strategy to prove their first solution which are solved by trial-error method (SG2) ▪ Use of graphical representations to find the solution (SG2 & SG3)	10
Roller Coaster	SG3	▪ Use of the idea inflection point and use of the concept local maximum, local minimum and slope effectively correctly.	6
Water Tank	SG3	▪ Producing smooth graphs correctly for the given water tanks.	12

For example, when pre-service teachers analyzed students' works for "Water Tank" task, twelve pre-service teachers expressed their appreciations on the solution of students in Group-3 seen in Figure 4.3. As it is seen in the graph, students initially placed marks on the side of the water tank, then they divided water tank into four parts and labeled them as I, II, III and IV. On the other hand, they labeled the axes as amount of the water (independent) and height (dependent). That is, they sketched the graph the change of height with respect to amount of water (volume), and they constructed a smooth curve that is linear, concave down (increasing), concave down (increasing) and concave up (increasing).

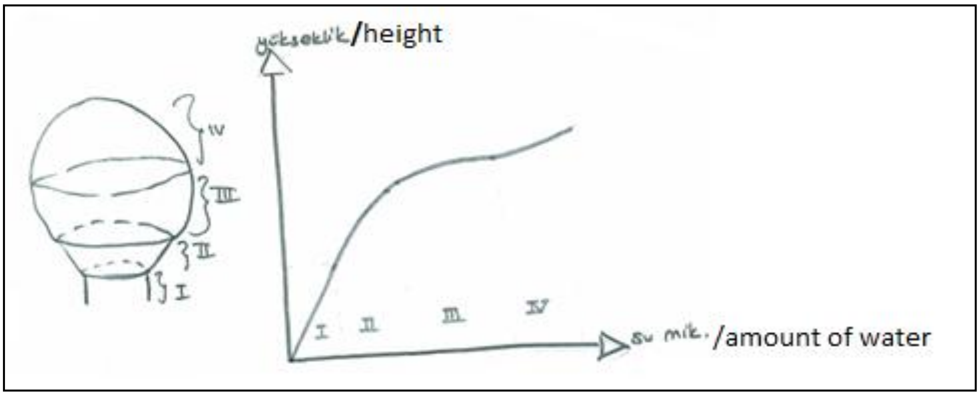


Figure 4.3 An example of sketched graph by students in Group-3

The stated reason why pre-service teachers valued these constructed graphs of students in group-3 (SG3) was that students considered the changing nature of the rate and constructed a smooth curve. On the other hand, although students have not learned the average rate of change/instantaneous rate of chance concepts yet because they were 10th grade students, their intuitive rationale for construction of graph were quite valuable for pre-service teachers. Some pre-service teachers, like PST14, expressed that students might have either interpreted the graphs as intuitively or relied on their previous knowledge obtained from in their physic courses.

> **PST14**: The students in Group-3 paid attention to the water poured at the constant rate. In addition, they identified not to change the amount of water per unit volume per unit of time. Because of these reasons, students interpreted logically that they could ignore the "time concept". Next, for all types of water tanks (bottles), they considered that the increasing rate of height decreases where the water tanks changes from getting to narrower. Then, they thought the corresponding graphs for these cases with their knowledge obtained in the physics courses. They constructed a smooth curve and they have not constructed discontinuous transitions between segments of the graphs. Furthermore, students interpreted the graph intuitively. That means, they did not use the unit volume or unit height concepts while they were interpreting their sketched graphs. Instead, they could interpret these graphs intuitively. Moreover, their explanations, which they wrote under each graph, were logical and valid. I mean, their

> mathematical expressions and language were more understandable than the students in group two. For example, their explanations under the graph of water tank four “the rate of amount of water to height increase over the radius because the volume become smaller and the graph of amount of water with respect to height increases by increasing were clear and understandable. In addition, the interpretation of mathematical concepts and connection between the concepts were quite good **[STRP_4].**

To summarize, the data analysis revealed that for each task, in depth examination of students’ works individually and collaboratively allowed pre-service teachers to value students’ solutions, which comprised rather different, creative, not conventional and logical mathematical ideas even if these solutions were not completely correct. Moreover, they also appreciated students’ solution when they produced completely correct.

4.3 Pre-Service Teachers’ Ways of Interpretation of Students’ Thinking

This part highlights the ways of pre-service teachers’ interpretation of students’ thinking over four two week cycle. The data analysis provided evidence that pre-service mathematics teachers displayed different characteristics of interpretations in nature that fits into the following categories.

4.3.1 Describing and assessing students’ ways of thinking

The reflections and comments of pre-service teachers provided evidence that especially pre-service teachers’ initial interpretations were based on the general features of students’ thinking reflected in their solution approaches. In their interpretations of students’ works, pre-service teachers sometimes described surface features of students’ ways of thinking and they did not pay attention to the mathematical details of students’ thinking at all such as “how students draw the figures/graphics to represent quantities” or “how students formulate the problem, mathematical statements” or “which assumptions students make and why”. They mainly described students’ ways of thinking in broad terms, and they did not make specific connections to students’ solution approaches while they were talking about them.

Moreover, several pre-service teachers described students’ solution process almost direct. That is, most of their interpretations were based on restatement of students’ solution approaches, and they described sequentially what the students had done in their solutions. As an example, the following excerpt presents an evidence for this case.

> **PST11**: If we look at the item b of students' solution in group one *(Figure 4.4),* students correctly put the angle "α" and angle "β"on the figure [triangles], and then they equated "sinα" to "sinβ". Next, they found x as 1.7 as a result of their calculations; they subtracted 3.89 from the total length which is 150 meter and then divided it to "length of c". Then, they obtained 53. By thinking the cars should be parked on the both sides of the road, they multiplied it by two **[STRP_1].**

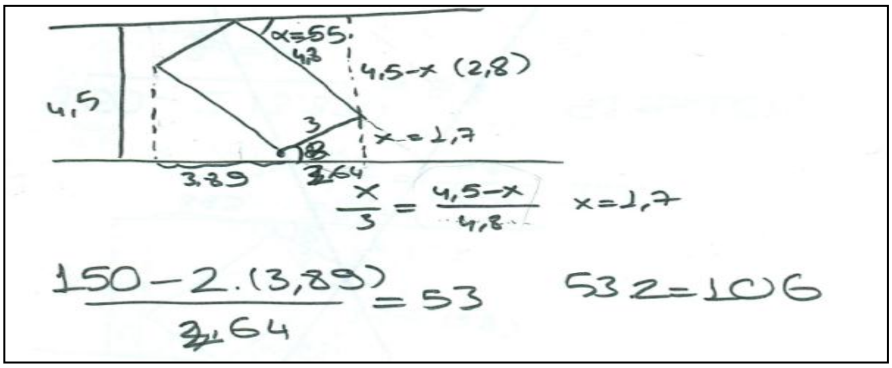


Figure 4.4 A section from students' solution in Group-1

As shown in Figure 4.4, which is the main part of students' solution in group-1, PST11 just explained the sequence of arithmetic and algebraic operations/procedures carried out by students instead of providing evidence about students' reasoning behind those mathematical operations.

In addition to that, some of the pre-service teachers focused on only final products of students' solution. They dealt with the correctness and wrongness of students' solutions; therefore, they described students' solutions by using judgmental statements as "incorrect", "terrible", or "good". For example, the following excerpt is a sample that provides an evidence of PST11's focusing on general features of students' solution and use of judgmental statement.

PST11: I think that students in Group-2 did not understand the problem at all. They approached the solution by trying the angles, which they were familiar with, in sequence. They did not paid attention to the other angles between those angles. In the end, they found 75. Because they did not understand the problem, they did not either worked on a mathematical idea or represent their ideas on the figure. Their solution was rather inadequate **[STRP_1].**

As shown in excerpt, PST11 focused on missing in the students' solution and just evaluated the errors in the solution approach. She has not attempted to understand students' reasoning behind this solution approach, and she directly claimed that students did not understand the problem and also indicated that students' solution was weak.

On the other hand, the following dialog obtained from focus group discussion on "Street Parking" task also illustrates that pre-service teachers evaluated students' errors instead of discussing students' ways of thinking while they were reflecting and sharing their comments as a whole class.

R: Well, What can you say about students' solution in Group-1?

PST18: If the values that they calculated were correct, they could produce quite different solution way that they would not notice it. You know, there is another rectangular area which they can fit in this rectangle. Students would say that how many rectangular areas should be there to be able to park cars. However, students should know that value correctly. However, *they incorrectly used that similarity.* They constructed a similarity to help their usage. If they constructed correctly, they would produce a better solution by using the areas.

PST20: Students told it as "sine rule" although they constructed similarity.

PST18: *The student divided everything to each other*

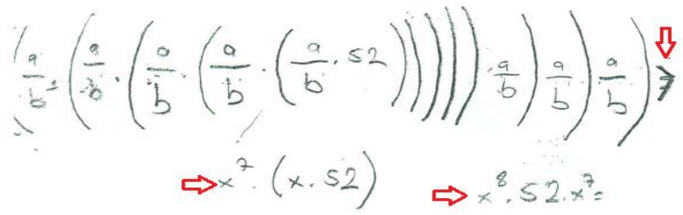
PST24: To be honest, *that student does not know what s/he does*. In my opinion, they constructed the similarity by determining the alpha (α) incorrectly. And then, s/he could not find any time to correct it; otherwise [...] **[Focus group discussion_1]**

Similarly, the excerpt below from the pre-service teachers' whole class discussion belonging to Bouncing Ball task illustrates the judgmental interpretation of preservice teachers regarding students' ways of thinking.

R: Yes, students in group two!

PST4: For example, they [*students in group two*] made lots of mistakes along the way. Students say that at first bounce, it goes down 52 meters and goes up 48 meters, at second bounce, it goes down 48 meters and it goes up 44.30 meters and at third bounce it goes down 44.30 meters. *There is a systematic error here*. It goes up to 17th bounce as they did there.

On the other hand, at the bottom (*of the sheet*) [*she shows while talking*], for example, they correctly wrote that expression; [*she refers the expression in the following and talking on it*].



But then, students suddenly used "x" rather than "a/b" in the second line, and wrote "x.52". And, when they used distribute property, they found x^8.52.x^7 at the end. *That is an error as well.* They don't know that what is greater than what. *I mean, there is nothing*. There is same thing here as well.

PST4: *I think they don't know the [mathematical] topic (silence).* There is also that one in previous sheet. No, they do not know the "associative property". There is a logical error here. They say "x time 52 times x" (x.52.x.). That's correct. They say x as a rate and when they multiply, it was x^2. But, please paid attention to 4th line, students write x^3.52 times x^2. However, they should be x instead of x^2. The next one..They say.. They should be x instead of x^3. That is also incorrect. I mean, they don't know not only associative property but also topic.

R: Well, the report on the front page, how did students arrive there?

PST4: As we told, they did something from somewhere but…

PST24: *They could hear somewhere;* they have heard and tried it.

R: The student says that because the ball goes up (bounces up) eight times, we multiplied it eight times.

PST9: They actually thought the ninth bounces up. It was written $(a/b)^9$ =15/52. You know, it is smaller than 15. They have noticed that $(a/b)^9 \leq 15/52$ but they did not either paid attention to it in their final report or keep their bounce rate, which they calculated, in that interval. However, they actually thought it.

PST24: They paid it attention. They found something but they did not use it. It means, *this solution does not belong to them.* **[Focus group discussion_2]**

As seen in the dialogue, in their discussions, pre-service teachers (PST4, PST9 and PST24) expressed that "*there is nothing; this solution does not belong to them; they could hear somewhere; I think they don't' know the [mathematical] topic"*. Their all expressions were judgmental, and their claims were mainly conclusive. They focused on the students' errors, which they identified, with students' lack of knowledge about "distributive property and associative property".

Above, all excerpts from pre-service teachers' individual reflections and whole class discussions about students' works were at surface level and evaluative in broad sense. Moreover, in their initial interpretations, some of the pre-service teachers generally tended to make no speculations about the reasons behind the students' solutions and understanding of specific mathematical concepts and procedures. On the other hand, the data provided evidence that pre-service teachers occasionally interpreted students' ways of solutions in broad terms and they oversimplified and judged students' ways of solutions in their later investigations of students' works.

4.3.2 Questioning students' ways of thinking

The findings indicated that although pre-service teachers' initial interpretations were sometimes descriptive, the data showed that in their later interpretations, especially in their second and third investigation of students' works, pre-service teachers raised questions and made broad discussions to understand the mathematically significant details of students' solutions. For example, the following excerpt belonging to whole class discussion illustrates how pre-service teachers questioned the details of solution produced by students in group-2 [SG2] to understand students' thinking process.

PST7: 45/52 was calculated and then students multiplied the numbers and they did…[*PST7 looks at students' solution paper*]

R : Students told they used trial and error method at the beginning.

PST24: If students used trial-error method, there would be several mathematical operations. Here, they say that we have found 45/46. We have searched…Students here…

R: Well, let's look at the last page of the worksheet.

PST24: Students tried something there; they told that we took 12/13 and then they did something by using it…it has not been…Then, they took another ratio…You know..umm..They heard something from somewhere.

R: There are three columns there. What did students try there? Did you understand these three columns? Which groups (pre-service teachers) did exactly analyze and understand it?

PST24: Here, students found the heights. Here, the first one is 52 meter. The coefficient is 12/13. They multiply 12/13 and 52 and then subtract it from 52 and s/he finds 45.94 as result. Then, they multiply this result and 12/13 again and obtain a result. ..Then they subtract that result and obtain a new result. They found such a length and equaled it to 15. You know, they try one by one. I mean, they look at the seventeenth value. Here, they try it.

PST3: But why 46? For example, the group one had calculated the mean and obtained 15/52 somehow. However, you know, they might have found it from somewhere; but, did they think the below or above anymore?

R: However, they had tried 12/13.

PST3: Yeah…

PST24: But, where did they find 12/13?

PST24: While they were telling that we tried and found it, they actually did not find anything.

R: They have tried both 46 and 44.

PST24: Then, they have tried both and thought that we could calculate the average; but, I think it is not like that.

PST2: Students took 45, but why 45? Here, there is not trial and error for that. They directly accepted 45.

PST3: For example, why did they think 45/52?

R: Do you think it is?

PST24: We [the pre-service teachers in that group] found like that but our other friends [other pre-service teacher groups] may have found something else.

PST3: Why is it not 35/52? Why is it not 45/52? [**Focus group discussion_2**]

As this excerpt shows, students' inexplicit responses served to raise pre-service teachers' awareness to understand and question the students' solution approach and the mathematical ideas behind this solution approach. Pre-service teachers became curious about students' ways of thinking. By the help of the instructor's several probing and prompting questions, pre-service teachers made in depth discussion to understand “why students follow this solution way” and “how did they calculated 12/13”. That is, this excerpt displays that pre-service teachers have changed the focus of their discussions from and they started to make deeper and more critical discussions rather than judgmental discussions.

On the other hand, the following excerpt from pre-service teacher’s reflection paper in their second investigation exemplified how the pre-service teacher interprets the details of students’ thinking by questioning.

> **PST14:** Students in group-2 initially arrived at a solution by trial-error method, and then they formulated an equation to check their solution. Next, when I examined students’ solution paper, I have observed that students tried to solve the task by drawing a parabola graph. I have thought that what students aimed to achieve to formulate equation of parabola. I mean, let’s think about that students formulated such an equation, how would they calculate the wanted bouncing rate?, or How would they relate the bouncing rate with equation of parabola? [...]**[STRP_2].**

As seen in the excerpt, pre-service teacher was curious about students’ way of thinking which she identified it in students’ solution. While pre-service teacher was talking about students’ solution, she asked question herself and tried to understand “why students used this idea” instead of directly assessing students’ ideas as correct or incorrect.

4.3.3 Explaining mathematical details of students’ ways of thinking

The data analysis displayed that in their first investigation of students’ works, most of the pre-service teachers’ tended simply to focus on general features of ways of students’ solution and they mainly did not paid attention to the details in their students’ ways of solutions. However, as pre-service teachers continued to work on various kinds of students’ works from different mathematical tasks, it was observed that pre-service teachers became more curious and sensitive to complexity of students’ ways of thinking, and they generally started to explore what students’ solutions may reveal about their thinking. That is, in their later investigations, they started to look the clues into understanding students’ ways of thinking as well as they increased their attention to the details in ways of students’ solutions. In addition to questioning students’ ways of thinking, they also speculated about students’ alternative ways of thinking produced in solutions. For example, in their individual reflections and during whole class discussions, they tried to provide alternative explanations for why students thought like that?"

An excerpt of PST14 relating to second investigation of students' thinking exemplified how pre-service teacher interpreted student way of students' solution in group-1 by tracing the mathematically important details of students' solution and predicting their possible thinking ways.

> **PST14:** Student group one did not look the problem just at mathematical point of view. They paid attention to the physical and environmental conditions and gravity factors where the experiment was conducted. I mean they exactly interpreted the problem in accordance with real life. Then, they defined the bounce rate of ball as the broadest. According to their interpretation, this bouncing rate should be bigger than $\frac{15}{52}$ so that the ball can be over 15 meter when it bounces; on the otherhand, this bouncing rate should be smaller than $\frac{52}{52}$ so that the ball should not bounce increasing. Therefore, they found an interval as $\frac{15}{52} < x < \frac{52}{52}$. There were infinite numbers in this interval and when I examined the solution ways of students so far, I wondered how students would determine the x and which value of x they would take. In order to ease to examine the numbers in the interval, students had determined the x values and they tried to see whether or not all these numbers verify the case of 8th bounces. I mean, students tried to find that whether or not a person observing the bouncing ball at a height of 15 meters feet above the ground can see the ball at it 8^{th} bounces. At this point, what drew my attention was that students preferred to choice the denominator of the ratios, which they looked at, as 52. I suppose that students either ignored the infinite number in this in interval or they just wanted to be interested in the ratios of which denominator was 52. After they determined that interval, in order to ease their investigation, they divided the interval into half; then by averaging, they decided to determine the most reasonable/appropriate x value. Students were able to solve the problem successfully and logically and to apply the trial-error method successfully. Next, the other point surprised me and drew my attention was, although students found a bounce rate which was valid, they later had constructed an inequalities and they checked their bounce rate by the help of this inequalities to see whether or not their bounce rate was valid. When I observed this case first time, I initially could not explain it because if a student thought such an equation for this problem, they would think it at the beginning while solving the problem and they could easily find an "x bounce rate" **[STRP_2]**

As shown in the excerpt, PST14 not only described students' mathematical procedures but also interpreted why students would apply those procedures and what would be their reasoning behind this way of thinking. Namely, PST14 became more understanding of students' solution attempts rather than made description or judgment as correct, incorrect or good. Similarly, below, there are excerpts from reflection papers written by pre-service teachers after their third investigation to exemplify their interpretation of details of students' solution in group-1.

PST15: For students in group one; students initially decided to sketch the path of roller coaster. Because students thought the thrill (criteria), they consistently sketched the descents (drops). While doing it, students had a difficulty in designing the ascents. Then, however, because students thought that the calculation would be easier, they accepted that the part between ascents and descents as a quadrant. As shown in the figure 4.5, because students' drawings were different at first, they changed their drawings. Here, we can see that students lack knowledge of parabola or curve. I understood that from their worksheets where they reflected their abstract thinking **[STRP_3].**

Figure 4.5 An example of students' drawing appeared in their solution approach

PST16: For instance, students in group two drew the figure, which was in their mind on, the graph paper. Next, 38 square were formed at horizontal distance of designed figure, and students equated 38 square to 100 meter (38x=100). Then, they calculated the length of one side of square as 2.63 meter. Next, students wrote this value corresponding places (height and the horizontal distance) on their graphs and calculated the real values. In a sense, they displayed us the miniature of the path of real roller coaster with 100 meter length. Students applied the scaling idea. This idea is quite different and this solution way is hard to think. I was impressed the creativity of students.

Students told that the slope should be 80 degrees. However, they did not pay attention to given restriction so that the slope (ascent and descent) should not be more than 5.67 while they were drawing the linear parts. If you produced a solution based on scaling, the error in the drawing should not be accepted. Therefore, if students were careful, it would be better. In the part of students' worksheets, I have seen the formulas of kinetic energy and potential energy. I was surprised that even these were thought by students. Then, I thought a different solution way. I wonder that whether or not I could solve the problem through energy conservation. When the train was moving down, they would have both potential and kinetic energy. We could write the height as "h" in the system based on scaling; however, because we have not any information about speed, this idea could not develop [*see figure 4.6 sketched by students associated with the pre-service teacher's interpretation]* **[STRP_3].**

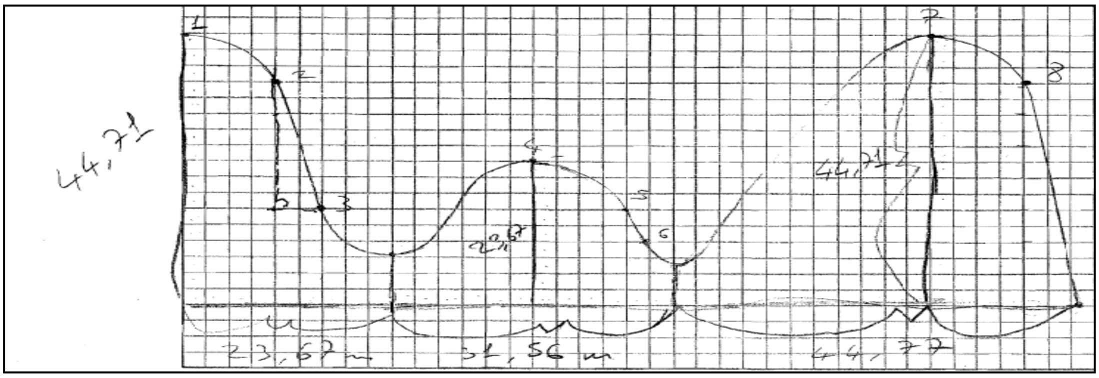


Figure 4.6 An example of students' sketch of path of roller coaster

Both excerpts showed that PST15 and PST16 tried to interpret in detail, what the students had done, by presenting several evidences/examples. Lastly, the excerpt of PST2's belonging to fourth and last investigation of students' works provided evidence that pre-service teachers increased attention to the details of students' ways of solutions.

PST2: For students in group one; they told that we sketched a steeper graph corresponds to narrower parts of water tanks [*students' refers the following figure 4.7*]. But, their error in their graphs was that they indicated the increase or decrease of the slope as steadily. Their difficulty with the concepts "increases by less and less and increases by more and more" was observed while they were presenting their

> solutions. It is quite obvious that they did not understand these concepts. I mean, according to students, the slope was constant in each interval that students identify for breakpoints. One segment of the graph is steeper than other segment, but here students logic might be there that the slope was constant on each point of one segment […][STRP_4].

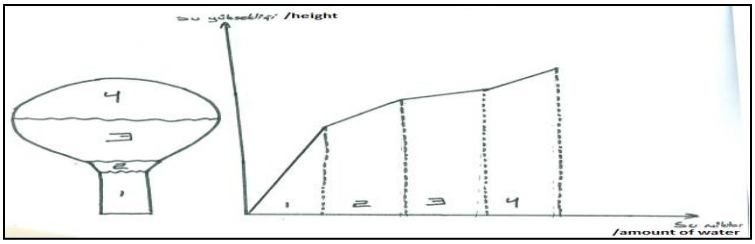


Figure 4.7 An example of the graph of water tank-2 drawn by students

As shown in the expert of PST2, she initially explained which concepts (increased by increased) were not understood by students, and then she provided several alternative explanations why students had difficulty and did the error while students were drawing the graph.

Briefly, in the interpretation of students' works for each task, pre-service teachers displayed all three characteristics of interpretation. However, the data showed that pre-service teachers' later interpretations of students' ways of thinking were less descriptive and judgmental; rather they were more interpretative and based on alternative interpretations for both students' mathematical weakness and strengths in their solution attempts. They also traced entire process of the solutions with the mathematical details rather than looked at only the final product of students' solutions. Although some of the pre-service teachers sometimes either briefly described or restated ways of students' thinking, they also noticed the details of students' ways of thinking, and tried to understand and interpret how students would solve the problem, what would be their reasoning behind their mathematical procedures.

Lastly, in addition to all the results regarding pre-service teachers' ways of interpretation of students' thinking, although pre-service teachers interpreted students' ways of thinking in three different ways, the data showed that during the investigations of students' works, a number of the pre-service teachers occasionally not only had a difficulty in understanding and interpreting students' ways of thinking but also misinterpreted students' ways of thinking. For example, below is an excerpt to exemplify PST17's difficulty and misinterpretation of students' solution approach while she tried to understand students' solution.

PST 17: Let's look at group 4. For instance, this group thought "c" (*trying to understand*).

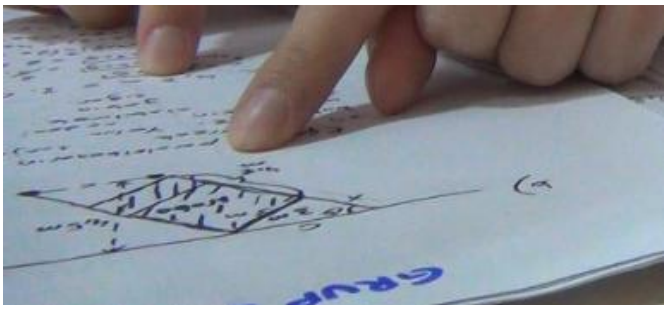


Figure 4.8 An example from students' solution in Group-4

R: How do students approach?

PST 17: If student took that this was c value, I could evaluate that the student understood and learnt the parallelogram. Maybe, it is insignificant; however, it is good to see it without just looking at the solution way. I suppose that it was done… [*Pre-service teacher examines again and tries to understand of the solution way of student group four*]…c times 4.5...Student constructed the similarity, I suppose that c over…It is quite difficult to understand [*pre-service teacher looks at but she does not understand*]. While I am thinking that, I think that this method [*pre-service teacher refers to examine students' thinking*] is useful in the following aspect. That is to say, for example, I might have solved the problem correctly in my exam paper, but, our teacher did not understand. For example, the similarity constructed by students might be correct, but I don't understand it at the moment **[STRP_1].**

As it is seen in the excerpt, PST17 struggled to make sense of students' solution approach shown in the figure 4.8. She tried to make several interpretations. Here, pre-service teachers interpreted that students constructed a similarity. However, the solution of students was produced by using the area of parallelogram. Therefore, this interpretation of pre-service teachers was not consistent with the solutions presented by students at all.

Similarly, in their fourth and last investigation of students' works, same pre-service teachers (PST17) had difficulty in understanding and interpreting the idea relating the line graphs drawn by students in group-2 where they divided the height of the water tank in the equal parts and determined the change with respect to it. In fact, because she could not understand students' reasoning behind the sketched graph, her interpretations were not consistent with the details of mathematical ideas presented by students. The following excerpt obtained from individual interviews illustrated a dialog between researcher and PST17. Here, this pre-service teacher had quite difficulty in understanding and interpreting the mathematical idea behind the graphs drawn by students in group-1 and group-2 as well as she misinterpreted it.

PST17: For instance, I am starting to student group one. Students initially divided the water tanks into parts [geometric]. For instance, this group [group one] sketched all the graphs as linear. Here, students thought that the steepness of the line (slope) should decrease and they associated this idea with the slope. It was that group who tells the volume of the figure (*I think it was not that*). Anyway, there is nothing much in this group [*PST looks at the other pages of the worksheet of group one and see the graph of fourth water tank*] For instance, that group divides the water tank into three [*PST talks about the following graph produced by student group one for the fourth water tank given in the task*].

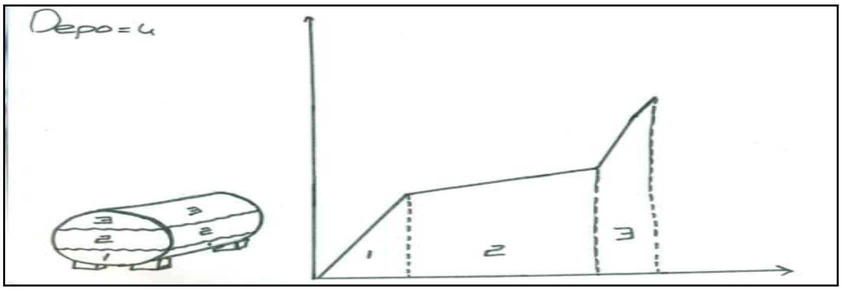


Figure 4.9 An example from students' solution interpreted by pre-service teacher

R: How did you interpret it?

PST 17: I guess that students thought "the thing"...It is more different in the second case because they approached intuitively, it is different than one and three. They understood that it increases by decrease in the first part [of the water tank]. I am stuck, how did they think? I was surprised how they thought.

R: What do you see, how do you interpret when you look at it?

PST17: As I said before, something is uncertain. For instance, while we were solving the problem, we solved it according to cross-sectional area. However, I could say that students thought all cross-sectional areas were the same and divided the water tank into parts according to this idea. But, then, students did not apply this idea. I suppose that they thought the flow of the water and thought that there would be a sudden change as seen in their graph (see figure 4.9), because the graph is getting flatter in second segment. You know, the shape of water thank changes in the second part, it becomes more arched. They thought that something is different there and they continued to draw. Then, they thought that the graph is increasing quickly from second segment to third, I mean; they drew this segment of the graph steeper, more inclined. [*After that, PST 17 changes her focus and starts to interpret the following graph in Figure 4.9 drawn by students in group two for the second water tank given in the task*]

In addition, for example, the students in this group also divided it into six parts. I read their all explanations below the graph but I thought they did not explain the reason why they divided it into six parts.

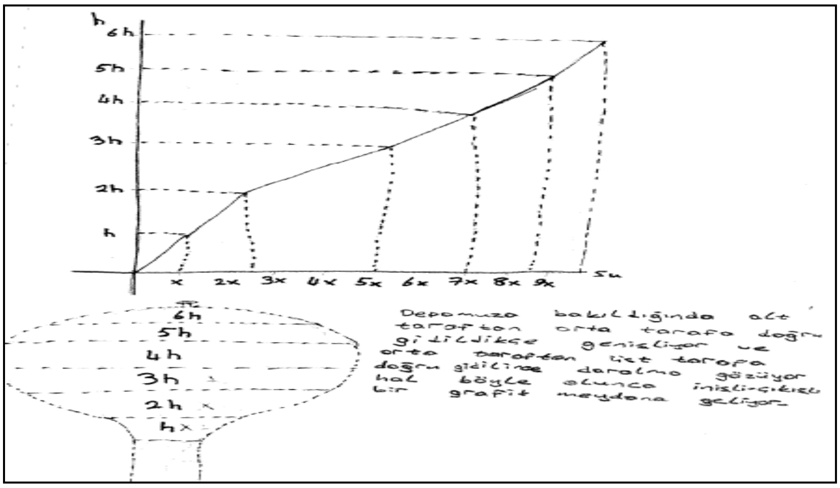


Figure 4.10 An example from students' solution in Group-2

R: Well, what did students divide into six parts?

PST17: I mean the water tank.

PST17: The height of water tank as h, h and h. You know, when I was initially examining… I looked at the graphs at first. I thought that the group one and two had same solution way. Next, when I looked at the reports and the axes [of graph], I actually noticed that the ideas were different than each other; their graphs just were same. For example, here, they took that h [height] was constant. For instance, here, students did not think that while x correspond to h, 2x may correspond to 3h. You know they always thought that it was linear as h, 2h, 3h

R: But, Does 2h (height) correspond to the amount of water "2x"?

PST17: I think that students did it considering the figure. Therefore, I wondered how connection students made so that they corresponded 2x to h. In my opinion, for this graph, h should be there and amount of water should be there. For example, this one, I mean I did not understand it. Why do students say that 2x corresponds to h? Let me look at, I may understand at the moment.

R: What are those x's? What did student think as x?

PST17: Did I understand it? Let me look at it…the amount of water…Umm... I think, for example, student told x, and then they told x for that as well. But, is it 2x if it is 2h in total? Did they think like that?

As it is seen in the dialog between pre-service teacher and researcher, PST17 struggled to understand the reasoning behind the students' solution. However, she was quite confused by the work of students in group-2 [SG2]. At first, she started to make general interpretation about the graphs of the student group one by looking at the front page of students' worksheet and she attempted to briefly describe "what students did". Then, she decided that there was nothing to say in the solution of this group; therefore, she started to look at the back pages of the worksheet and graph of fourth water tank drawn by students came to her attention. However, after she began to interpret the graph, she was stuck. Her interpretations were not so meaningful, and seemed not so consistent with the students' reasoning. Then, she jumped to the solution of students in group-2, and started to talk about the graphs in this solution. In this process, her initial focus was water tanks. She wondered "why students divided the water tank into the six parts"; however, she did not put forward an idea about that. Afterward, she tried to understand the associated graph with this water tank. Although she tried to provide possible reasons, she decided to "she did not understand what students did, how and why" after a certain time.

On the other hand, in their third investigation of students' works, many of the pre-service teachers had difficuly in understanding mathematical operations used students in group-1 (SG1) to solve the task (see Appendix C3, SG1), and they also had difficulty in interpreting these operations. In fact, they could not obtain a common idea when they discussed students' mathematical operations as a whole class. For example, the following excerpts exemplify PST25 and PST21's difficuly in understanding "how students of group one calculated the radius of circle as 0.34 cm".

> **PST25:** Students in group one is not clear either in their worksheets or presentation of their solution. For example, we could not understand where students found 0.34 or why they wrote it. It was not stated in their report [worksheets] and it was not explanatory to say "it should be 0.34" during their presentation [of solution way] **[STRP_2].**
>
> *******
>
> **PST21**: Students calculated the radius of circles as 0.34 meter. However, it did not make any sense while we were discussing either as a group or whole class. In addition to that, because nine quadrant and three paths, the length of the paths was calculated as 31, 73 meter [STRP_2].

Overall, the data showed that not all but several pre-service teachers had a difficulty in understanding the reasoning behind students' solutions over four study cycles; therefore, their interpretations sometimes were not consistent with students' ideas and reasoning reflected in their solutions.

4.4 Pre-service Teachers' Criteria of Examining Students' Works

Pre-post self report questionnaires and reflection papers on examining ways of students' thinking were the primary data sources to examine the criteria/aspects to what pre-service mathematics teachers attended. That is, these findings only obtained the analysis of pre-service teachers' self report data on the aspects that they focused on in analyzing of students' written work and video episodes. Pre-service teachers' criteria for examining students' works were analyzed separately before, during and after the four two-week research cycles; therefore, the findings are presented in this way.

Firstly, in the questionnaire called as pre-self report, pre-service teachers were asked to report about their previous experience in working on any students' works such as exam papers, homework or daily work including students' ways of thinking. In addition to that question, in order to learn pre-service teachers' primary criteria focused on, pre-service teachers were also requested to report the following question. "*As a teacher candidate, if you examined and analyzed any students' works including ways of their thinking, to what would you attend? and "Which criteria would be important for you?".* The analysis of pre-service teachers' self report data given for these questions showed that none of the pre-service teachers had an experience working on students' works to examine and assess and that was their first experience.

Moreover, the data revealed that there were no certain and robust criteria on pre-service teachers' focus. However, pre-service teachers commonly indicated that the students' solution ways, approaches or processes that they may follow and also students' errors that they may do in their solution would be main aspects in their focus if they examined any of the students' works. Furthermore, in addition to these aspects, several pre-service teachers also reported that the correctness of students' answers (PST4), students' understanding of problem statement and the mathematical topics (PST18, PST24) as well as the concepts used by students (PST14, PST17) would be in their focus.

Below an excerpt illustrates that PST17's primary aspect.

> **PST17:** First of all, I would determine where students made errors. Next, by paying attention to those errors, I would determine the topics which were challenges for the students. And then, I would start to solve students' those problem by considering them **[STRP_1].**

Similarly, PST1 indicated that looking at the misconception of students because it may cause students' errors would be important for her. She reported that:

> **PST1:** Initially, I would find the misconceptions of students which may cause the students' errors, and then I would try to determine either the mathematical topics or the lessons where students were taught that creates those misconceptions. Except the errors, I would also expect that student express their mathematical ideas logically. All would be my evaluation criteria [STRP_1]

On the other hand, similar to many pre-service teachers' focus, the focus of PST13 was the students' solution. The following excerpt illustrates it.

> **PST13:** I initially would pay attention to where the student starts to the solution because it shows what student think at first. Next, I would look at which solution strategy student follows and try to understand what student thinks. I also would pay attention whether the student has the missing in the solution and where the student makes an error. In this way, I would understand which part of the mathematical topic student has difficulty. Thus, I would see my missing and understand where should focus on while teaching the mathematical topics.

Lastly, the following excerpt exemplifies PST14's aspect that would be her/his focus if she examines and assesses the students' works in a particular mathematical topic.

> **PST14:** I would examine which mathematical concepts students might use and which concepts students might associate with each other in their solution processes. If I make an evaluation, the steps of students' solution would be more distinctive criteria for me. For example, how students solve the problem would be more important than the correct answer of any problem in the exam result.

As seen in the excerpt, actually, looking at student's solution process is important for him/her. Therefore, she gives priority the mathematical concepts that students might use in the solution of a given problem.

Secondly, throughout four two-week study cyles, for each modeling task, pre-service teachers examined and analyzed students' solution worksheets and video episodes in depth both individually and collaboratively. While pre-service teachers were examining and analyzing students' works individually before coming to class, they examined and analyzed them in a group for 3-4 when they were in the course. After pre-service teachers worked on students' works, they were asked to reflect in their reflection papers "What were in your focus while you were examining the students work?" Although this question was for their both individually and collaboratively examination process of students' works, the data displayed that pre-service teachers mainly answered it by considering their individual examination of students' works before attending to the courses. The analysis of pre-service teachers' reflective accounts showed that the main aspects the pre-service teachers focused in examining/analyzing students' works were shown in the following table 4.25 for each task.

Table 4.25 The examination criteria of students' works reported by pre-service teachers

		Examination of Students' Works			
	Pre-service Teachers' Examination Criteria of Students' Works	**Examination -1 # of PST**	**Examination - 2 # of PST**	**Examination-3 # of PST**	**Examination-4 # of PST**
Process-oriented	Checking if students' understand the problem	7	9	5	9
	Checking if students apply drawing/graphing to solve the problem	3	1	6	1
	Examining students' solution approaches.	12	12	9	8
	Examining the mathematical concepts and topics preferred by students while solving the task.	5	5	3	2
	Examining the initial mathematical ideas/operations produced by students while solving the task	5	4	1	5
	Examining the errors/difficulties of students in their solution	9	4	2	4
Product-oriented	Checking the correctness/wrongness of the students' answers/final products.	2	1	1	7

As seen in Table 4.25, the frequency of the criteria paid attention by pre-service teachers changes for each task. While pre-service teachers were examining the students' works, they focused on various kinds of criteria at the same time. For example, while analyzing students' works for the first activity, the main focus of PST13 were the aspects "checking if students understand the problem", "looking at the initial mathematical ideas/operations produced by students while solving the task" as well as "looking at the mathematical concepts and topics preferred by students while solving the task". However, as a result, as seen in Table 4.25, while they were examining the students' works, the seven aspects both process and product oriented emerged as the common focus of the pre-service teachers. In addition, the frequencies of criteria focused on by pre-service teachers, which were "if students understand the problem" and "the solution approaches preferred by students", were relatively high for each task. That is, for each activity, these two criteria were the major focus among the seven criteria for pre-service teachers. For example, two excerpts below are the examples belonging to PST11 from her first and third examination of students' works.

> **PST11:** While I was examining the students' worksheets and video episodes, first of all, I paid attention what students did for the solution. I tried to understand the solution of students. I looked at if students understood the problem and what they tried to do for that **[STRP_1].**

> **PST11:** While I was examining and assessing the students' worksheets and video episodes, I initially looked at the students' solution and the correctness and the validity of their solutions **[STRP_3].**

As seen in the excerpts above, in her first experience in working on students' works, PST11 stated that what students did for the solution of task was important for her/him. Although she gained experiences over time, she still kept her/his focus same while examining students' solution.

Like PST11, PST25 and PST15 reported that students' solution process was the crucial aspect for them to attend. The examples are below.

> **PST25**: While I was examining and evaluating students' works, I initially paid attention to students' solution. I focused on what students thought and what students did for the solution of task **[STRP_1].**

***

PST15: First of all, I started to examine students' worksheets. Here, I initially focused on the solution strategy of students. The reason of was to be able to understand that which concepts in mathematics the student associated with a certain [real-life] concepts. Next, I started to examine the video-episodes of students and my initial focus was to pay attention to how student understood and comprehend. The reason of that was to be able to understand students' missing and to show the students alternative solution as well as help him/her to ensure the relationship with other concepts **[STRP_1].**

Similar the solution of students, how students understand or interpret the problem statement was on the focus of many pre-service teachers during the examination of students' works for each task. For example, PST2 explained what she paid attention to in their second investigation of the students' works in the following.

PST2: While I was examining the students' solutions, the things that I paid attention to how students interpreted the problem, how they formed their thoughts and how they evaluated their mathematical ideas after they had read the problem. While I was watching the group discussion of students, my focus was what students paid attention or not the givens in the problem. I thought that students would interpret the expression "*ball had passed 17 times from where the employee stands"* as "*bounce*" so that I have seen that students understood this expression easily and followed more different ideas. For example, student group one [SG1] had associated the average with bounce and follow up a way with trial and error; student group two [SG2] paid attention to decreasing height and perceived it as an equation of parabola. Their solution approaches were quite interesting and different (*we have never thought those ideas at that moment*) […]**[STRP_2].**

Similar to PST2, PST15 also reflected that her/his primary focus to pay attention was if students understood the problem because she wanted to see how the students associated the real life with the mathematics.

PST15: […] I paid attention to whether or not students understood the problem. While I was paying attention to how students started to solve the problem as well as how students started to explain their solutions because I wanted to see how a relation students made between the real life and mathematical world. For example, when I started to examine the worksheets of students in group five, I paid attention to whether or not they understood the problem and I tried to understand that by watching the video episodes of them. After students have read the problem, they engaged in generally the numerical expressions rather than the verbal expressions. Therefore, they could not be aware of exactly what was asked in the problem. For example, students had the following dialogue (in the box) while working as a group **[STRP_3].**

Do you know what we are to do? You know that area is…It should be 5.67 to 1 so that if the opposite of tan80 is 5.67meter, the bottom should be 1. In this respect, the aim is to find how many decreases with 80 degrees have to we sketch at most.

On the other hand, the data provided evidence that not all, but some of the pre-service teachers, who stated the reasons why they noticed students' drawings, commonly reported that they examined carefully the students' drawings in their solutions because these drawings helped them to see if students understood the problem correctly and how students thought and how they interpreted to the problem. The examples of what pre-service teachers reflected are in the following:

> **PST23:** For this activity, while I was examining the students' solution papers, the first thing which I paid attention to was if students calculated correctly the operation how many times the ball was dropped and rebounded. When I examined the groups [the worksheets of students' groups], I have seen that many of the groups understood it. Next, it was important if students completely understood the problem. In this understanding process, I primarily paid attention to whether students drew the figures **[STRP_2]**

Here, as it is seen in the excerpt of PST23, the figures drawn by students represent students' understanding of the problem. In order to determine students' understanding of the problem, his prior aspect was students' drawings. Similarly, PST13 indicated that in order to see if students understood the problem, she examined initially the figures sketched by students while she was examining the third task.

> **PST13:** While I was examining the students' worksheets and video episodes, I examined them simultaneously. In this way, I found an opportunity to understand why students did what they did and in which order they did. The primary thing of mine while examining this works was whether or not students understood the problem. In order to see whether or not students understood the problem, I initially examined the figures drawn by students and I look at whether or not students correctly used the given information. In addition, in order to understand whether their design [of roller coaster] ensured the safety criteria, I focused on whether students used the curve paths in their design **[STRP_3].**

In addition to students' drawings, the initial mathematical ideas/operations produced by students while solving the task was an indicator to understand whether or not students were on the right track, what students understood when they read the problem or which way(s) they followed to solve it. Moreover, some of the pre-service teachers expressed their thoughts/beliefs about if there was an error in the solution process; this error came more likely from students' initial mathematical ideas/attempts produced to solve the problem. Others also indicated that students' initial mathematical ideas were most fundamental so that those ideas occupied students' minds all time while they were solving the problem.

The data also revealed that some pedagogical aspects "teacher's approach towards students and given time to solve the question" were occasionally in the focus of some of the pre-service teachers.

Lastly, in order to evaluate their own processes, after pre-service teachers have completed four two-week research cycles on students' ways of thinking, they were implemented a post questionnaire called 'Self-Report II'. In one of the questions of the questionnaire, pre-service teachers were wanted to reflect "*When you think all the work relating to students' works experienced in this course, which criteria/aspects became important for you to examine any of students' works? Please indicate them as order of importance and explain your reasons*". The data analysis showed that as a result of examination of students' works for four tasks, pre-service teachers developed several criteria which should be closely paid attention in the examination of students' works when they are teachers. The data displayed that of 25, 11 of the pre-service teachers (PST13, PST16, PST18, PST1, PST15, PST6, PST8, PST3, PST21, PST4 and PST23) reported that if I examined the students' solution produced for any mathematical task, my primary criteria would be "checking if students understand the problem". Moreover, the data showed that many of them they also reported as the second criteria paid attention to "students' thinking process in their solutions". Congruently, 11 of the pre-service teachers (PST11, PST12, PST14, PST15, PST19, PST25, PST7, PST2, PST9, PST2, PST17 and PST10) stated that my prior criteria would be "ways of student thinking and students' thinking process". That is, they indicated that not only "which way did students follow" but also "why students did follow this way and how did students think" became important aspects for them. For instance, PST7 reported that:

PST7: First of all, the most important point for me is which thought/idea students follow for the solution. As I said, the correct solution produced as a result of incorrect mathematical ideas does not make any sense. Next, the steps taken for this thought/idea became important for me. How students systematize their thinking and from where they start the solution became second important aspects for me. Because these provide a basis for the solution, each wrong step causes the error. Next, the calculations done for the solution were the aspect which I paid attention to. Especially, I have seen that students made lots of procedural errors in the questions which are complex and include numerical data **[SRQ_2].**

The following excerpt shows that although PST8 indicated students' understanding of the question was primary aspect to pay attention for her, she also equally emphasized the importance of students' solution process.

PST8: My experience was not very depth before. But, as an idea, I generally focused on either students solved the problem correct or incorrect. I have never paid attention to "why students made errors", what difficulties students have" or "where their misunderstandings are". Yet, while we were examining the students' worksheets, we tried to understand students all solution process and then to find their errors because there were not just one solution strategy of the students. In that respect, I noticed that there was a change. For instance, we complain much that if the result is incorrect, it is directly given 0 (zero) as a score. Actually, I could have obviously seen that it should be scored by evaluating "how students solved the problem", "why solved the problem in that way" and "in which step the student made an error. Therefore, first of all, my focus is whether or not the problem understands, and then if it is not understood, where is the problem? Is it due to the close structure of problem statement or due to the students' perceptions? Which thinking way students prefer to solve the question [...][**SRQ_2].**

On the other hand, several pre-service teachers reported that those criteria were not new criteria for them. Before working on students' works, they had an idea about the importance of focusing on students' thinking process in their work. However, they stressed that with this experience, they have strengthen those ideas.

PST14: As I mentioned above, the ideas came up with the students and how they associated with these ideas each other rather than the correctness of the solution became the most important aspect that I paid attention to. Anyway, before attending this course, my thoughts was that; however, supporting my thoughts with the examples and videos has been more productive for me **[SRQ_2].**

PST2: Before this experience, while I was approaching the solutions, my criterion was not to pat attention the correctness of the solution. However, this experience contributed me in terms of looking at the importance of students' solution once more again. In my opinion, in order to help students' recognizing where the students' missing is important. Is it in ways of students' thinking or in students' procedural abilities? I think it is also important to help students by knowing these missing of students. In this respect, with this experience [working on students' works] I have understood the importance of each step of students' solution process should be examined and evaluated. Seeing where students make errors helps us to think about asking the questions which help students to find the correct solution. I understood that we can gain students' great thinking by investigating the small details in their solutions. I mean, even if there is not correct solution strategy, I should examine to see what students think. The reason is that correct ideas may emerge from the students' errors **[SRQ_2].**

Furthermore, "initial mathematical ideas produced by students (PST5), the mathematical concepts and topics, if students understand the mathematical concept to be taught (PST24)" were the other criteria for the rest of pre-service teachers.

PST24: For me, the first one is whether or not the concept questioned in the problem is understood. Next one is the solution strategy and the last one is the correctness of the solution **[SRQ_2].**

Lastly, one of the pre-service teachers (PST22) did not reflect their primary criteria; instead, she reported that she learnt the importance of examining students' solution without judgment.

PST22: First of all, not to make judgment by just looking at the students' solution became an important aspect for me. I mean, I understood that I should pay attention to "why" and "how" questions. I mean, I understood that I should make interpretation **[SRQ_2].**

As a result, the following figure 4.11 summarizes the pre-service teachers' examination process of students' works before, during and after investigation of students 'work.

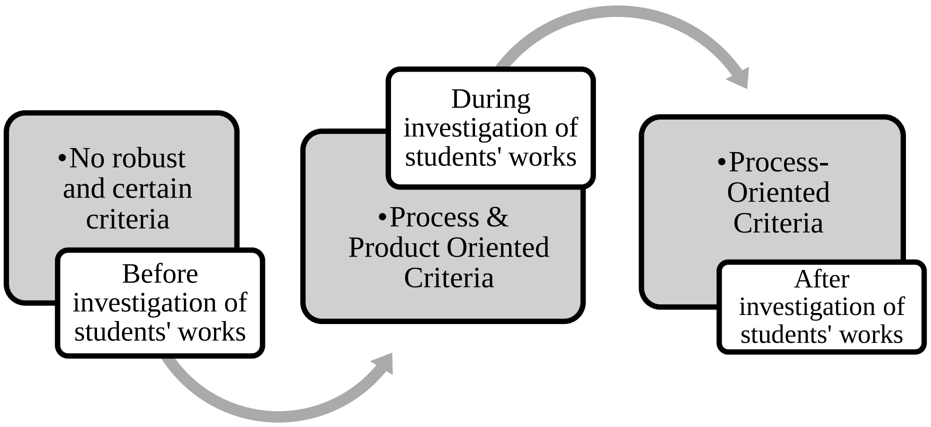


Figure 4.11 The nature of pre-service teachers’ criteria for examining students’ works

To sum up, although pre-service teachers had several criteria before working on students’ works, with this experience, they developed more robust and process oriented criteria, which were also common for many pre-service teachers, to examine students’ works.

CHAPTER 5

CONCLUSION, DISCUSSION AND IMPLICATIONS

In this final section, I initially present my conclusions drawn from the study and discuss them. I start with my discussion about pre-service teachers' awareness of students' ways of thinking. Then, I discuss the findings of pre-service teachers' valuing students' ways of thinking and interpreting students' thinking, the criteria for examining students' works. This is followed by a discussion of possible factors that might have helped to detect pre-service teachers' knowledge of students' thinking and to develop their knowledge. Next, I present the limitations and implications of the study, and make suggestions for the further research.

5.1 Conclusion and Discussion

As highlighted in several studies (e.g., Grossman, 1990, Lampert & Ball, 1998; Nilsson, 2008), the development of teachers' PCK mostly occurs in practice, and classroom teaching experience has an important role in the development of teachers' PCK.

Although the development of pre-service teachers' PCK formally begins during pre-service teacher education programs, it develops at limited there (as cited in Lee et al., 2007) because these programs are weak intervention to balance between theory and practice, and the structural features and contextual factors of them do not always provide adequate opportunities for pre-service teachers to support this development (Feiman-Nemser, 2001; Lampert & Ball, 1998). In addition, these programs are criticized for the fact that although they provide pre-service teachers with knowledge of content and pedagogy, students and theories of learning, pre-service teachers usually can not find regular opportunities transfer this knowledge into practice (Feiman-Nemser, 2001; Lampert & Ball, 1998). Yet, as Ball and Cohen

(1999) state, “being/learning in practice” does not necessarily mean learning in real classrooms. Therefore, in teacher education programs, some opportunities can be created for pre-service teacher to be in practice, for example, “being in practice” would also be provided by the help of documentation of practice such as students’ written work, video excerpt of classroom lessons, teacher’s notes or curriculum materials. “Using such things could locate the curriculum of teacher education in practice” (p.14) and examining students’ thinking can be considered as one of the core activities of practice.

In this study, while analyzing students’ ways of thinking, pre-service teachers had completed different phases: (i) working on the given non-routine task to produce their own solution, (ii) making predictions about students’ possible solution approaches, errors and difficulties before looking at students’ works on the same task, (iii) analyzing students’ thinking individually and collaboratively in terms of students’ solution approaches, weaknesses of their solutions, mathematical concepts used in their solutions and other things they noticed, (iv) discussing collaboratively and reflecting on their analysis results. Because pre-service teachers predicted, observed, analyzed, identified and interpreted complexities of students’ ways of thinking in these phases, this entire process provided a rich learning environment about students’ ways of thinking. Correspondingly, this study provided an opportunity to pre-service teachers for “being and learning in practice”.

The findings of the study allowed us to understand pre-service teachers’ existing PCK with respect to students’ ways of thinking, and the findings revealed that pre-service teachers’ knowledge of students’ thinking is not robust. Therefore, pre-service teachers need to develop robust knowledge in teacher preparation courses. On the other hand, the findings also allowed us to understand how use of documentation of instructional materials (students’ written work and video episodes) taken from real classrooms supports the development of pre-service teachers knowledge of students’ thinking during their pre-service teacher education. While pre-service teachers examined of students’ works in depth, not all pre-service teachers gained the same level of knowledge; however, analysis of students’ actual work contributed to each pre-service teacher’s pedagogical content knowledge with respect to knowledge of students’ ways of thinking.

5.1.1 Discussion of findings regarding pre-service teachers' awareness of students' ways of thinking

5.1.1.1 Pre-service teachers' predictions and identifications of students' ways of thinking

The findings of this study initially revealed that pre-service teachers were unprepared to predict students' possible solution approaches, and to see mathematics through the eyes of students, especially, at the beginning of this research. In their initial predictions, many of the pre-service teachers' level in predicting students' mathematical thinking were not high, and many of their predictions were not consistent with their actual solution approaches. This result was consistent with many research's findings conducted with teachers (e.g., Berqvist, 2005; Hadjimetriou & Williams, 2002; Nathan & Koedinger, 2000; Şen-Zeytun, Çetinkaya, & Erbaş, 2010). Moreover, the nature of pre-service teachers' predictions was mainly based on their own solutions. Similar to teachers in Şen-Zeytun, Çetinkaya and Erbaş's (2010) research, pre-service teachers in this research generally could predict students' possible solutions by focusing on their own solution approaches which left them unable to evaluate the solution of tasks from students' perspective. On the other hand, although pre-service teachers had difficulty in predicting students' possible solution approaches, especially, in their initial examination, the data displayed that pre-service teachers did not generally encounter much difficulty in identifying students' solution approaches. That means, pre-service teachers were able to identify most of the students' solution approaches correctly while examining students' works for each task.

Moreover, similar to previous studies conducted with both teachers and pre-service teachers in similar contexts (Chamberlin, 2002; Koellner-Clark & Lesh, 2003; Lampert & Ball, 1998; Steinberg et al., 2004), the findings suggested that over time, great portion of pre-service teachers' predictions have become more consistent with what students actually did; in addition, pre-service teachers started to predict from students' point of view. Furthermore, they increased their awareness concerning of how should approach the problem through students' point of view. Moreover, pre-service teachers in this study have deepened their understanding of students' different solution approaches. As Smith (2001) expressed, pre-service teachers recognized that students would solve the given problem in different ways and their understanding would be different, but these different solutions may be equally correct and valid. However, the findings also revealed that from first analysis to the last, the number of the pre-service teachers, who stated their predictions/expectations about students' approaches, did not increase. In addition, in their last analysis of students' works, pre-service teachers still made general predictions without explaining the details of the students' possible solution approaches.

In addition, analysis of data also displayed that after pre-service teachers examined and identified students' actual solution approaches; they expressed their emotions towards either several solution approaches or mathematical ideas in these solutions produced by students. Although some pre-service teachers sometimes expressed that they felt disappointed in students' solutions, many of the pre-service teachers' emotions towards students' solutions were mainly positive. Pre-service teachers were surprised at students' solution approaches when they identified different, creative and nonconventional solutions, and solutions including logical mathematical ideas. They were also sometimes surprised at a solution which they have never thought or a solution produced by completely correct.

Secondly, as for the results related with pre-service teachers' prediction and identification of the students' errors, difficulties or misunderstandings, the findings showed that consistent with many research's findings (e.g., Akkoç et al., 2007; Ball, 1988; Kılıç, 2011; Klein & Tirosh, 1997; Tirosh, 2000), the pre-service teachers in this study were not aware of some of students' possible errors and difficulties before analyzing students' works. In addition, the data also displayed that those pre-service teachers' initial predictions were not diverse, they were brief and superficial. When pre-service teachers were asked to predict students' possible errors, difficulties or misunderstanding in the given task, only two pre-service teachers explicitly stated their predictions in their reflection papers of their first task. Although several pre-service teachers told that they thought about certain predictions/expectations concerning students' difficulties and errors, they did not reflect on them in their reflection papers. One possible reason can be pre-service teachers' lack of self confidence and experience about how to reflect their predictions. Interestingly, the number of pre-service teachers, who stated their expectations about students' possible errors and difficulties, increased in the following tasks which are probably due to gaining some sort of experience. For the rest three tasks, more than 15 pre-service teachers stated their predictions/expectations about students' possible and common errors and difficulties.

Furthermore, it was found that, although they expected students' calculation errors or difficulties, especially, students' conceptual and logical errors, as well as difficulties in comprehending problems were quite unexpected for pre-service teachers. That means, similar to Son's (2013) results, pre-service teachers were able to predict students' possible procedural-based errors better than possible concept-based errors or logical errors. On the other hand, pre-service teachers emphasized that their expectation for students' level of understanding and knowledge were high; because in their first analysis of students' works, they stated that students made much more errors than their expectations. Moreover, pre-service teachers' identification of students' errors, difficulties and misunderstandings revealed that although pre-service teachers had difficulty in predicting students' possible errors and difficulties, they generally were able to identify many of the errors, difficulties or misunderstanding that students experienced.

In addition to that, the data displayed that over time, the diversity of pre-service teachers' predictions has increased and their predictions also became more consistent with students' actual errors and difficulties. Furthermore, their predictions became more detailed and specific.

Pre-service teachers' lack of familiarity with students' solution approaches, errors and difficulties was not surprising since they had no formal instruction and experience in learning them until enrolling in this course. However, the reasons why pre-service teachers' identification of students' ways of thinking is relatively better than their predictions can be explained by the following reasons: One of the reasons could be the adequacy of pre-service teachers' SMK to identify students' solution approaches, errors and difficulties. As researchers (e.g., Ball et al., 2008; Even & Tirosh, 2005; Hill et al., 2008; Son, 2013) indicated, teachers' recognition of students' wrong answers, understanding of students' unusual solutions are related to teachers' SMK. Therefore, it can be considered that pre-service teachers may have sufficient SMK to be able to identify students' solution approaches, difficulties, errors especially procedure-based errors rather than concept-based errors. Another fundamental reason for this finding may be pre-service teachers examined students' works not only individually but also collaboratively. In addition, they spent sufficient time to do this. Before coming to class, pre-service teachers worked on students' solution papers and video episodes individually. Then, during the course, they discussed students' solutions with their groups of 3-4, and they tried to both understand and identify students' solution approaches, errors and difficulties collaboratively by spending more than one hour. In the collaboratively working process, each pre-service teacher was able to observe, identify and discuss the different details about students' solution approaches, errors and difficulties reflected in those solutions, they were able to ask each other for help, and they shared their thinking with each other. That is, as Kazemi and Franke's (2004) pointed out in their research, collective examination of students' works had a significant role to support the deeper understanding and the better identification of students' thinking.

Furthermore, pre-service teachers recognized students' different solution approaches, errors and difficulties because they had opportunity to observe students' variety of answers, many diverse reasoning skills, errors, difficulties and their different level of understanding while examining students' actual solution approaches in depth in students' solution papers and video episodes. In addition, in this process, while pre-service teachers were identifying students' solution approaches, they also might have found an opportunity to recognize that students deal with multiple ways of interpreting the problem situations (Doerr & Lesh, 2003). These findings can also be closely associated with the fact that the consistency between pre-service teachers' predictions of students' solution approaches, errors or difficulties with students' actual solution approaches, errors or difficulties increased. That means, before attending to this course, pre-service teachers had no engagement with students' ways of thinking. With the attending of the course, pre-service teachers' noticing of students' multiple ways of interpreting the problem situations and students' conceptions, difficulties and errors may have helped them make better predictions about both students' possible solution approaches as well as their errors and difficulties. That is, pre-service teachers gained experience in understanding students' ways of thinking. However, here, it is not claimed that pre-service teachers' prediction level highly developed, it was slight at first, but this shift was evidence that analyzing students' works on non-routine tasks helps pre-service teachers to improve their predictions about students' ways of thinking in this context.

The results of the study also suggested that while pre-service teachers were able to identify various students' errors and difficulties, and to improve their awareness of students' common possible errors and difficulties related to diverse mathematical topics, they could not sufficiently foster their knowledge about the sources of the students' errors and difficulties during the course. This can be explained by Even and Tirosh's (1995) interpretation of the terms "knowing that" and "knowing why" in the context of teachers' knowledge about students. Even and Tirosh (1995, p.17) pointed out that *knowing that* refers to "research based and experienced based knowledge about students' common conceptions and ways of thinking in the subject matter" and *knowing why* refers to "possible sources of a certain students' response". Therefore, in this study, the pre-service teachers' knowledge enhanced relatively in terms of "knowing that" rather than "knowing why" (Even & Tirosh, 1995). The reason can be explained that pre-service teachers could not find enough time to attempt to understand the sources of students' errors and difficulties in depth during their investigation of students' works. Even though pre-service teachers predicted and identified "which errors were done by students" and "which difficulties were encountered by students", they could not discuss efficiently on "why students made these errors or had these difficulties" and "what the sources of their errors and difficulties would be".

In this respect, though, the findings of this study do not claim that pre-service teachers learnt the specific students' errors, difficulties or misconceptions related to several specific mathematical topics. The findings claim that pre-service teachers learnt what kinds of errors students would make in particular mathematics topics used in this study context, and pre-service teachers also increased their awareness that students may make some errors different from their expectations. For instance, specifically in this study, although pre-service teachers thought that students would easily use the similarity of triangle without making any mistakes, they noticed that students had difficulty in the use of similarity of triangle. Moreover, pre-service teachers have recognized the students' difficulties and challenges in several topics such as use of trigonometric concepts, the slope concepts or drawing the function graphs. For example, pre-service teachers also recognized students' confusion and difficulty in the concept of slope. They observed that many of the students had difficulty in distinguishing between slope of line and slope of a curve.

On the other hand, the data provided evidence that some of the students' errors and difficulties were expected for some of the pre-service teachers, the same errors and difficulties of the students were unexpected for the other pre-service teachers. This case may be mainly related that if pre-service teachers experienced various errors and difficulties while they were solving the given tasks, they could predict that students would experience the same errors and difficulties.

5.1.2 Discussion of findings regarding pre-service teachers' valuing students' ways of thinking

In this study, pre-service teachers' analysis of students' works led them to appreciate students' thinking. This result obtained from pre-service teachers confirms Chamberlin's (2002) result obtained from teachers. Similar to Chamberlin's (2002) finding, in the investigation of students' works for each activity, pre-service teachers made comments about valuing students' some of the mathematical ideas and solutions even if they were not completely true or logical. Pre-service teachers also appreciated the ideas and solutions of students which they have never thought while they were producing their own solution to the given non-routine mathematical task. Pre-service teachers also valued the ideas and solutions of students so that they considered original and creative. That is, the solutions and mathematical ideas valued by pre-service teachers were generally the solutions and mathematical ideas that they were found surprising and interesting when pre-service teachers identified them.

The appreciation of pre-service teachers regarding students' ways of thinking can be considered as an important finding as it shows teacher is actually paying attention to what students are saying during the instruction. As Wallach and Even (2005) indicated that the purpose of the assessment has changed in recent years, and teachers' assessment of student learning as an integral part of instruction has become crucial. Therefore, teachers should understand that students' understanding and learning can not be assessed only by administrating paper-pencil test at specific time. Teachers are expected to listen to their students while students are solving mathematics problems and discussing them in the lessons, and explaining their solutions to their teachers and peers. However, as Even and Wallach's (2004) indicated in their study, teachers' not attributing value to students' ways of thinking is one of the obstacles to hearing students. Similarly, Wallach and Even (2005) stressed that several possible sources of hearing her/his student arise from teacher's own conception of the problem, teacher's concern for her/his students' success and his/her familiarity with students. In classrooms, teachers usually do not expect students to come up with original solutions. Rather, they typically expect that students would solve a problem as they are taught; therefore, they usually do not value students' different solution approaches. And, if teachers do not value to students' solutions different from their expectations, they usually do not listen and try to understand students' different solutions. In fact, they assess these more likely as incorrect answer since they consider them complicated and puzzling as it is in Even and Wallach's (2004) research.

5.1.3 Discussion of findings regarding pre-service teachers' interpretation of students' ways of thinking

Although some of the pre-service teachers in the study showed that some of the pre-service teachers could interpret ways of students' thinking, some of them did not. The analysis revealed that over study cyle pre-service teachers had interpreted students' thinking in three different ways: (1) describing and assessing ways of students' thinking (2) questioning ways of students' thinking (3) explaining ways of students' thinking.

The findings about pre-service teachers' interpretation of students' ways of thinking in this study displayed similar characteristics to interpretation of teachers and pre-service teachers in other research (Chamberlin, 2002; Crespo, 1998, 2000; Kazemi & Franke, 2004; Koellner-Clark & Lesh, 2003; Wallach & Even, 2005). However, as distinct from Crespo (1998, 2000) and Kazemi and Franke's (2004) research, where they documented the change in the pre-service teachers' interpretation ways over time and over experience, this research did not display the change in pre-service teachers' interpretation of students' thinking. Rather, this research reported pre-service teachers' ways of interpretation while they were analyzing students' works. Similar to what all this research indicated, in this study while pre-service teachers were reporting on the students' responses, pre-service teachers occasionally tended to describe the steps of a procedure in the solutions of students rather than interpreting students' ways of thinking. As Koellner-Clark and Lesh (2003) pointed out, especially in their initial analysis and interpretation students' thinking, pre-service teachers could not deeply interpret, and sometimes oversimplified, students' mathematical ideas and conceptualizations. Pre-service teachers also focused on the correctness and wrongness of the students' responses and just assessed them as "*correct, incorrect, good, terrible*" etc. as in Crespo's (1998; 2000) research.

Additionally, the results of the study revealed that pre-service teachers tended to interpret students' ways of thinking beyond focusing on its correctness. The data revealed that pre-service teachers raised questions to understand the meaning of the students' ways of thinking in their reflection papers and whole class discussions. Similar to pre-service teachers in Crespo's study (1998; 2000), students' inexplicit responses were one of the fundamental reasons for raising questions. Pre-service teachers raised questions when they had difficulty in understanding what/how students did in solving task and when they became curious about students' unclear responses. In addition to that, the instructor's role and questions can be considered another fundemantal reason for pre-service teachers' raising questions. During whole class discussion, the instructor asked several prompting and probing questions to pre-service teachers, and he also asked pre-service teachers to point out their evidence while they were talking about students' ways of thinking. Therefore, the instructor's questions helped pre-service teachers move away from their general descriptions to raising questions about students' ways of thinking to understand better. Moreover, pre-service teachers in this study were able to make explicit and in-depth interpretations, by explaining the mathematically significant details of students' solutions and students' reasoning behind these solutions. This way of interpretation was consistent with the interpretations of pre-service teachers in Crespo's (1998; 2000) research, and the interpretations of teachers in Kazemi and Franke (2004) and Koellner-Clark and Lesh's (2003) research. Both teachers and pre-service teachers in above studies focused on the meaning of students' responses and interpreted the details of students' responses. This study suggested that pre-service teachers' engagement in students' ways of thinking in depth was an important factor to help pre-service teachers focus on the details of students' thinking instead of focusing on only its correctness.

On the other hand, the data also revealed that several pre-service teachers had difficulty in understanding the reasoning behind students' solutions over four study cycles. Therefore, they sometimes interpreted ways of students' thinking inaccurately. Pre-service teachers' difficulty in interpreting students' thinking may be due to several reasons. Pre-service teachers said to students' inexplicit, unclear and sometimes incomplete responses on their written work made interpreting students' ways of thinking different for them. In addition, it was found that students' messy solution papers were challenging for some of the pre-service teachers to interpret students' thinking because these papers did not offer enough information for revealing students' ways of thinking. Pre-service teachers also indicated that they could not hear any explanation about these inexplicit mathematical operations and reasoning while they were watching the students' videos. Other explanation of this result may be pre-service teachers' insufficient subject matter knowledge to recognize and interpret ways of students' thinking (e.g., Bartell, Webel, Bowen, & Dyson, 2013; Jacobs, Lamb, & Philipp, 2010). Bartell et al. (2013) examined the role of pre-service teachers' content knowledge to recognize students' conceptual understanding, and they found that content knowledge of pre-service teachers played an important role to recognize students' mathematical understanding. Their research displayed that content knowledge supported especially pre-service teachers' analysis of students' understanding when students' responses involved students' conceptual understanding or misconceptions. Therefore, in this study, especially in the investigation of students' works relating to curve analysis, covarational reasoning and interpreting the graph of function (in the third and fourth tasks), the subject matter knowledge of several pre-service teachers may be insufficient to be able to recognize and interpret students' solution approaches and mathematical ideas associated with these solutions. However, as Bartell et al. (2013) also indicated, subject matter knowledge is necessary but not a sufficient factor to analyze and interpret students' understanding. Thus, in this study, other explanations of this result may be pre-service teachers' attention to students' solutions, and how much time was spent by pre-service teachers to investigate students' works. These factors may not be considered as the sole contributing factors, but they may have a prominent role in pre-service teachers' difficulty in interpreting ways of students' thinking and making inaccurate interpretations. Furthermore, the observation notes data suggest that pre-service teachers' attendance to the course was other explanation because several pre-service teachers who had difficulty in understanding and interpreting students' solutions did not attend to the course regularly, and they could not follow each activity.

5.1.4 Discussion of findings regarding pre-service teachers' criteria for examining students' works

The data provided evidence for analyzing students' works that pre-service teachers' criteria changed over time. At the beginning of the four two week study cycle, that is, before pre-service teachers analyzed any students' works, each pre-service teacher expressed several criteria to examine students' works, but their criteria were very general, fragile and uncertain. Moreover, every pre-service teacher had different set of criteria. In other words, the criteria of pre-service teachers were no commonly stated criteria used by all pre-service teachers. As the main reason, it might be considered that because many of the pre-service teachers enrolled in this study were in their third year, they had not taken any course related to "measurement and evaluation in mathematics education". Therefore, they did not have any theoretical knowledge on how to examine and assess students' performance on open-ended tasks. That's why, their stated criteria were based on mainly their own previous experiences as learners, namely, what they observed when they were students at high school or elementary school.

Then, during four two week cycle, pre-service teachers commonly expressed several criteria in their focus. Although pre-service teachers focused on diverse criteria at the same time during investigation of students' works for each task, the nature of their criteria appeared to be either process or product oriented.

Yet, at the end of four two week cycle, the data showed that pre-service teachers commonly improved their attention to two process oriented criteria which were "checking if students understand the problem" and "looking at students' solution approaches". This study suggested that pre-service teachers' engagement in students' ways of thinking in depth could be a fundamental reason why pre-service teachers focused on these criteria. In the process of analyzing students' works on non-routine tasks, pre-service teachers observed various students' original and correct thinking ways in students' solutions, and they recognized that students would solve the given problem in different ways. In addition, they started to value students' ways of thinking. In this way, pre-service teachers increased their attention to students' solution processes while they were analyzing student work, and they developed examination criteria based on students' thinking process rather than their final products.

Creating criteria to examine students' works should be considered another important finding of this study supporting the development of PCK during pre-service teacher training in terms of pre-service teachers' knowledge of assessment. In schools, many teachers' tendencies were mainly to assess only students' final answer in terms of correctness or wrongness to mark them. As the findings of this study and several studies (e.g, Ball, 1988; Crespo, 2000) indicated, pre-service teachers also had similar tendency while looking at students' works. That means, teachers and pre-service teachers' tendency is to evaluate students' final products rather than processes. In this study, the nature of the criteria applied and developed by pre-service teachers to examine students' works showed that these criteria were mainly process oriented. Therefore, this finding can be interpreted such that examining students' works in depth may have helped to understand the value of examining students' whole process rather than correctness of final products.

Furthermore, creating examination criteria may also have contributed to support of pre-service teachers' knowledge to develop a rubric as an assessment tool to assess students' works because each criterion which was a focus of pre-service teachers' attention may be an item in a rubric.

However, at this point, another issue should be appeared to be paid attention. In this study, during the investigation of students' works, pre-service teachers created their own criteria to examine students' works. Yet, probably, the note taking sheet designed by the researcher on "looking at students' works/investigation of students' works" (e.g., Allen & Blythe, 2004; Chamberlin, 2002; Hallagan, 2003) was the main source for helping pre-service teachers to create their own criteria. Moreover, while examining students' works in-depth, noticing the important mathematical ideas in students' solutions as well as original and creative solution approaches most likely helped pre-service teachers to create mainly process oriented criteria to examine students' works. That is, in this study, pre-service teachers were not provided any theoretical knowledge existing in literature about examination and assessment of students' works produced from open-ended tasks, and they have gained only experience based knowledge. However, experience based knowledge is not solely enough to support pre-service teachers PCK in terms of knowledge of assessment. Therefore, their knowledge also should be support with the theoretical knowledge to provide robust understanding about the assessment of students' performance.

5.1.5 Discussion of the factors contributing to pre-service teachers knowledge of students' thinking

In this study, pre-service teachers' knowledge of students' thinking was investigated within an undergraduate course context where pre-service teachers first worked on non-routine mathematical tasks, and then examined students' written work and video episodes from real classroom setting.

The previous part, pre-service teachers' awareness, interpretation and examination of students thinking have been highlighted and discussed. In the research questions of this study was not particularly intended to investigate the factors (components of the course) which contributed to pre-service teachers' knowledge in this context. However, various factors might contribute to improve pre-service teachers' awareness of students' ways of thinking, to elicit their ways of interpretations and to develop process oriented criteria to examine students' works. Therefore, in this part, the factors, which might have contributed to improvement of pre-service teachers' knowledge of students' thinking, will be highlighted and discussed.

5.1.5.1 Use of mathematical modeling tasks as non-routine tasks

In this study, pre-service teachers produced solutions to four non-routine mathematical tasks, and then examined students' works on the same tasks. This study suggested that the use of these non-routine tasks have played a fundamental role pre-service teachers' recognition of students' various solutions and multiple thinking paths, the recognition of the students' conceptions, errors and difficulties, and the gaining an ability to interpret students' ways of thinking.

Doerr and English (2006) and Crespo's (2000) research supported this claim. In their study, Doerr and English investigated how the task features supported the improvement of teachers' knowledge while their students were working on the task. In their study, they used modeling tasks. As a result, their data revealed that modeling task features allowed teachers to gain a new understanding of the mathematical content. In addition, the modeling task features also enabled teachers to understand the ways where students' ideas developed, and to change the teachers' listening to their students from evaluative to interpretative. Similar to findings of Doerr and English (2006), in her study, Crespo (2000) examined which factors might have helped pre-service teachers to move away them from their evaluative interpretations. According to her, the use of unfamiliar thought inviting tasks was one of the influential factors that contributed to pre-service teachers' learning about students' thinking. She stated that because unfamiliar tasks were likely to produce unfamiliar responses from students, the likelihood of pre-service teachers investigating students' mathematical thinking increased.

As it is in Doerr and English's (2006) research, the used tasks in this study were mathematical modeling tasks. The features of the modeling tasks used in this study were different from the traditional and routine textbook problems. Therefore, these tasks were unfamiliar for students. Additionally, unlike traditional textbook problems, modeling tasks comprise rich information on students' thought processes and they have potential to reveal students' multiple ways of thinking (Chamberlin, 2002; Chamberlin & Moon, 2005; Lesh & Doerr, 2003; Lesh & Harel, 2003; Lesh et al., 2000). Therefore, in this study, the thought revealing nature of modeling tasks allowed students to approach by using many different ideas to the solutions. In addition, students documented their whole solution processes as well as final products of their solutions. In this way, in both students' written works and video episodes, pre-service teachers were able to observe and examine the multiple ways of students' thinking. Moreover, pre-service teachers could find an opportunity to examine the novel solution approaches of students in their work. That is, all these features of the tasks might have supported pre-service teachers' learning about students' ways of thinking; therefore, they became more aware that students think about the tasks in various ways that might be totally different from their own solution ways/approaches.

5.1.5.2 Solving the non-routine mathematical tasks collaboratively before working on students' works

Pre-service teachers' solving the modeling task collaboratively before investigating students' works on the same task may be considered as another prominent contributing factor for developing knowledge of students' thinking. Because the nature of the modeling task allows students to work in a group, and to spend long period of time to produce their solution, in this study pre-service teachers worked in a group for 3-4 and spent approximately two hours for solving each task. In this process, similar to students' processes, pre-service teachers described, explained, and justified their own thinking ways in their solutions collaboratively. And, because pre-service teachers discussed the problem statement in the task as a group, they might have gained a depth understanding to the problem. Moreover, because pre-service teachers shared their own point of view with each other while they were mathematizing given problem situation, they might have found opportunity to deal with multiple thinking ways and offer alternative perspectives on students' solution approaches, errors and difficulties. In addition, while pre-service teachers were engaging in the task, they might have clarified various possible mathematics and mathematical ideas, concepts and topics to use in solving task, and they might have discussed the concepts and different thinking ways elicited from their collaborative solutions to the problem (Koellner-Clark & Lesh, 2003). On the other hand, some of the pre-service teachers might have produced the same types of error and encountered difficulty that the students would have while they were engaging in the task.

5.1.5.3 Use of students' actual solution papers and video episodes of students' discussion about their solutions to analyze students' thinking

Investigating artifacts of teaching and learning such as videotape of classroom lessons, students' daily work, home work, exam papers or students' oral explanations is one of the suggested way for teachers and pre-service teachers both to learn about students and students' ways of thinking, to improve their ability to listen and interpret students thinking ways and to develop strategies for using students' ways of thinking in their instructions (e.g., Ball, 1997; Ball & Cohen, 1999; Lampert & Ball, 1998; Masingila & Doerr, 2002; Smith, 2001). In this study, as the artifacts of teaching and learning, students' solution papers and video-episodes belonging to these solution papers taken from real classrooms were used concurrently. That is, the video episodes were used as the complementary of students' solution papers. In their research, Masingila and Doerr (2002) used multimedia case study materials including teacher's lesson plans, student written work, video of students' working and teachers' reflection on lessons etc., to investigate how these materials help pre-service teachers to make meaning of complex classroom experiences. Their research showed that the analysis of case study materials support pre-service teachers to examine complexities of teaching more deeply. As in their research, in this research, analysis students' solution papers and video episodes might be considered as another prominent factor for revealing preservice teachers' existing knowledge of students' thinking and contributing the development to their knowledge of students' ways of thinking.

In this study, each solution papers produced by students was at least 3-4 pages, and the content of students' written solutions was rich and detailed because the sample of students' written work purposefully selected on the basis of diversity of solutions. Selected students' solution papers comprised the students' different understanding and interpretation of the problem in the different ways, unusual solution of students, students' use of various kinds of mathematical subjects and representations in solving problem and students experienced several difficulties and challenges. Therefore, these selected students' solution papers to analyze might be a powerful influencing factor to recognize correct and incorrect students' various solution approaches, students' common errors and difficulties. According to many of pre-service teachers in this study, the students' solution papers had useful artifact to learn about ways of students' thinking; however, they included several limitations to understand and interpret students' ways of thinking and to learn about students more. As teachers indicated in Krebs' (2005) professional development activitiy, almost all participant pre-service teachers agreed that additional information provided by video episodes was useful in evaluating and interpreting the students' understanding.

As many of the studies (Ball, 1997; Masingila & Doerr, 2002; Sherin, 2004; Smith, 2001; van Es & Sherin, 2006) suggest, use of video as an artifact of practice in professional development programs provides a productive environment for helping teachers' learning. Video records of practice allow teachers multiple viewings to make productive discussions with their colleagues. Moreover, because the video records can be stopped, replayed and manipulated, they also allow teachers to examine and analyze issues relating to students or classrooms in depth.

Similar to these aforementioned affordances of video, the video episodes used in this study might have supported the pre-service teachers' understanding about ways of students' thinking. Video episodes enabled pre-service teachers to examine students' ways of thinking closely. The reflection papers written by pre-service teachers for each task revealed that video episodes were rather useful for pre-service teachers. Pre-service teachers highlighted several important benefits of watching videos including students' ways of thinking. They indicated that their awareness about students' ways of thinking increased more. For example, pre-service teachers mentioned that while they were watching the video episodes, they found an opportunity to hear students' talks, to observe their actions and to understand students' thoughts better. They also stated that video episodes afforded them to observe the interaction of small groups of students while students were producing their solution, and to hear their emerging mathematical ideas, questions, conflicts and resolutions while they were reaching consensus to produce a common solution. They also indicated that they found an opportunity to hear some of key mathematical ideas, which were not written in their solution papers, to interpret students' thought process accurately. This claim is also confirmed by Masingila and Doerr's (2002) findings. Similarly, Masingila and Doerr found that video journals of teachers and video of students' working on small groups enabled pre-service teachers to hear, observe and understand how students think and how teacher use students' ideas in making instructional decisions.

Therefore, by complementing each other, video episodes and students' solution papers may be considered an effective tool to increase pre-service teachers' awareness of students' ways of thinking. Both artifacts comprised uninterpreted and unstructured information about students' thinking because they were directly taken from real classrooms as Ball (1997) and Smith (2001) stressed. In addition, "samples of these students' works were concrete demonstration of what is known and what is not known" (Evan, 1993, p.72).

However, how to choose and use of these artifacts for teacher preparation programs and professional development programs should be paid attention. In order to use them effectively, these artifacts should be carefully selected to meet specific goals for both teachers and pre-service teachers' learning.

5.2 Limitations

Students' ways of thinking is not directly observable construct. Therefore, this research findings are limited what students documented about their thinking ways in their work in terms of interpretations of the given problem situation, selection of quantities, operations and representations, students' errors, difficulties or misunderstandings in given problem; how and to what extent pre-service teachers analyzed them, and how the researcher interpreted them.

In addition, the curriculum content of the each non-routine mathematical task (modeling task) used in this study did not follow a particular mathematics topic. Therefore, pre-service teachers' knowledge of students' ways of thinking and its development was investigated and interpreted in different mathematics subjects rather than in a particular mathematics subject.

5.3 Implications and Suggestions for Further Research

This study has several implications for mathematics teacher educators, curriculum developers, and suggestions for further research.

5.3.1 Implications for mathematics teacher educators

The teacher education literature suggests that in conventional teacher preparation programs there are some obstacles which decrease the effectiveness the preparation of teacher candidates for teaching (Feiman-Nemser, 2001). Although the student teaching (field experience) is one of the most valuable parts of their preparation, pre-service teachers mostly do not find adequate opportunity to work regularly with students in schools, and to learn about students.

This research was conducted as a part of an undergraduate mathematics education course. In this research, pre-service teachers worked on non-routine mathematical tasks, reviewed and analyzed students' written responses, watched video episodes taken from real classrooms while students were working on the same tasks, and then pre-service teachers discussed on students' ways of thinking. That means, pre-service teachers were being in practice through the instructional documentation of practice taken from real classrooms (Ball & Cohen, 1999; Smith, 2001). Therefore, this design may be considered as one of the examples of a reform-based teacher preparation course design, which is different from the traditional course design, for pre-service teacher education programs. And, in this context, the results of this study showed both pre-service teachers' lack of PCK with respect to students' ways of thinking, and how pre-service teachers' knowledge of students' ways of thinking improves through watching videos and examining worksheets containing actual student solutions for modeling tasks. Therefore, this study implies that the design of this research may be interest of mathematics teacher educators to design mathematics education courses for pre-service teachers and professional development programs for teachers. In order to help pre-service teachers learn students' common conceptions, errors and difficulties, and nontraditional solution approaches by engaging in students' ways of thinking**,** mathematics teacher education instructors may consider planning to use a similar design.

On the other hand, this study implies that knowledge of students' thinking should also be addressed in in-service teacher education programs. Therefore, the design and findings of this study may inform mathematics teacher educator to develop professional development activities focusing on eliciting and interpreting students' ways of thinking.

However, as Ball and Cohen (1999) stated, "Simply looking at students' works would not ensure that improved ways of looking at and interpreting such work will ensue" (p.16). In the light of the strength and limitations of this study design, this study implies that some issues should be considered carefully while constructing learning experience for teachers and pre-service teachers based on these materials as presented following.

Firstly, the nature and quality of mathematical tasks are crucial to learn about students' ways of thinking and interpreting students' understandings. For instance, in this study, mathematical modeling tasks used as non-routine problems had features to provide rich information on students' thought processes and creative solutions of students (e.g., Chamberlin & Moon, 2005). Based on the findings of this study, modeling tasks would serve as valuable tools for understanding the nature of pre-service teachers and teachers' knowledge of students' ways of thinking. In addition, these tasks would offer pre-service teachers with opportunities to gain about students' thought processes in mathematics. This study implies that in teacher preparation course, mathematical modeling tasks should be considered using in order to foster both teachers and pre-service teachers' knowledge of students' thinking.

Secondly, solving the non-routine mathematical tasks collaboratively before working on students' works has an impact on pre-service teachers' understanding of students' thinking better. Therefore, this study implies that before pre-service teachers and teachers work on students' works on non-routine mathematical task, they should try to solve the task, particularly in collaboration with their peers.

Thirdly, this study implies that mathematics teacher educators may also design a pre-service teacher education course or a professional development activity based on video case context, where pre-service teachers and teachers may watch and discuss on the video cases taken from classrooms. As in this study, the video cases allow teachers and pre-service teachers to hear students' talks, to observe their actions, and to understand students' thoughts better. It is also suggested that these video cases should include the snippets regarding the comprehensibility of the problem statement by the students, proposed solution ways or mathematical ideas by students, the discussions of students on the task, students' difficulties, errors and challenges.

Fourthly, the study also implies that facilitator is an important role to understand better from students' works. In a professional development program, a participating teacher or mathematics teacher educator may be facilitator. In this study, the instructor as facilitator knew his/her role well. For example, the role of instructor was guide to the process without imposing his/her own perspective. The instructor carefully monitored and actively listened to conversation among pre-service teachers while they were talking about students' works. Instructor also encouraged the pre-service teacher to share their ideas to elicit their understanding about students' ways of thinking. In addition, the instructor of course used "student ways of thinking protocol" as presented in Appendix E. Use of protocols described "the set of steps that prescribe how a group will interact" (Allen & Blythe, 2004, p.11) to look carefully at students' works should be considered while teachers and pre-service teachers work on students' works. As Allen and Blythe (2004) stated, the protocol used in this study was a guide for instructor to specify the roles of the pre-service teachers participating in this course, and also provided the instructor a structure to encourage for effective discourse. Moreover, this protocol helped the instructor accomplish the purpose of the study. Therefore, this study implied that use of protocols may enable teachers and pre-service teachers to understand deeply about students' thinking ways and teaching issues.

Lastly, duration (span of time) of the course or professional development programs would be other important factor to enhance teachers and pre-service teachers' understanding of students' thinking because learning takes time. In addition, fostering teachers as well as pre-service teachers' learning is demanding task. As Smith (2001) pointed out, a sustained effort needs to be able to gain insight into students' ways of thinking over time. Therefore, at least one semester courses (14-15 week) or long term professional development activities should be considered by mathematics educators. Furthermore, the mathematics teacher education courses based on pre-service teachers' understanding of students' thinking may be designed in pre-service teachers' each level of education "sophomore, junior and senior level" to observe and support pre-service teachers' development of knowledge of students' ways of thinking.

5.3.2 Implications for curriculum developers

This study highlighted the need for mathematics education courses or method courses in teacher preparation programs to provide learning opportunities pre-service teachers to know about students more and to recognize students' solution approaches, errors and difficulties. In addition, the findings of the study showed that examining students' ways of thinking through students' written work and video episodes might be one of the useful ways to do it. That's why; this study may also help curriculum developers to rearrangement of existing pre-service teacher education programs, and use of teaching and learning artifacts taken from real classrooms could be located in teacher preparation curriculums.

5.3.3 Suggestions for further research

Research regarding understanding of pre-service teachers' knowledge of students' ways of thinking and its development in the context of use of teaching and learning artifacts from real classrooms is limited because few studies have explored the use of this pedagogy to understand and develop pre-service teachers' knowledge of students' thinking in teacher education courses. Firstly, this study suggests to the researchers to continue investigating pre-service teachers' knowledge of students' thinking in similar contexts. In addition, in the light of this research context and limitation of this research, the following issues for the further research are suggested.

Another suggestion would be to investigate particularly the impact of the pre-service teachers' content knowledge in their understanding, evaluation and interpreting students' ways of thinking. In addition, conversely, examining and analyzing students' works would also contribute to pre-service teachers' content knowledge in this context. Therefore, in the further research, it is suggested to explore "how does pre-service teachers' content knowledge develop through examination of students' works?"

Third suggestion is based on the limitation of this study. Since the curriculum content of each modeling task used in this study was different than each other, this study did not allow tracking pre-service teachers' knowledge of students' common conceptions, misconceptions or concept development in a particular mathematical topic. Therefore, focusing on students' works in a specific content area would be useful to investigate the development of pre-service teachers pedagogical content knowledge in terms of their knowledge of students' common conceptions, misconceptions and understanding of particular topics in a subject matter (Grossman, 1990; Shulman, 1986, 1987).

Next, it would be useful to examine how pre-service teachers' beliefs and attitudes to mathematics, and teaching and learning mathematics influence their understanding about students' ways of thinking over the duration of their teacher training.

Lastly, the duration of the research was important to present robust evidence to understand the pre-service teachers' knowledge of students' thinking the and its development, and the number of the pre-service teachers enrolled in the course was also a factor to make in-depth investigation about pre-service teachers' developing knowledge of students' thinking. Therefore, this study should be replicated with same research tools and same research context, but with less pre-service teachers and extended time (e.g., at least over one academic semester). By doing so, pre-service teachers' knowledge of students' thinking may also be examined individually.

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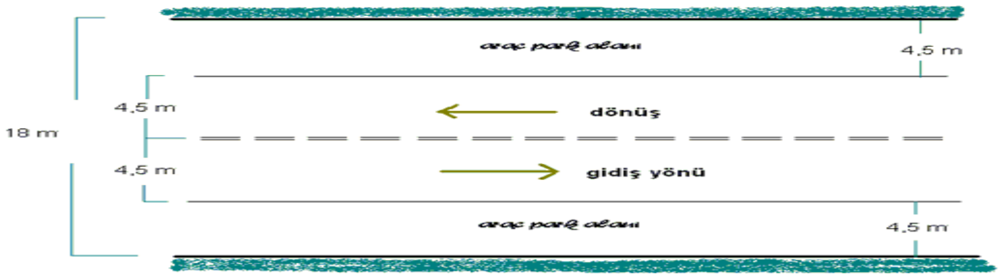
APPENDIX A

MODELING TASKS

A1. Modeling Task “Street Parking”

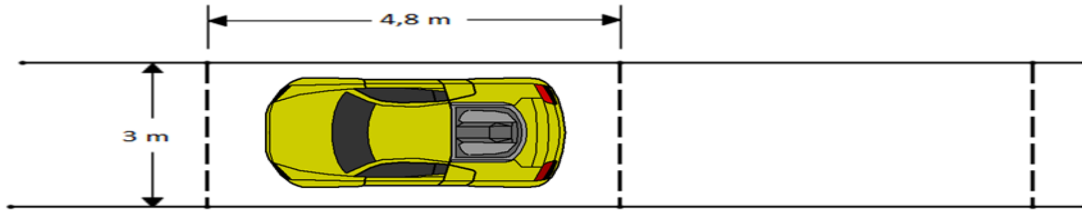
Caddede Park Yeri

Bir şehir planlamacısı iki yönlü bir yolun kenarında, evlerin önünde araba park yeri tasarlamak için sizden yardım istiyor. Şehir plancısının amacı caddede park edilebilecek araç sayısının en fazla olacağı düzeni sağlamaktır. Park edilecek yer yolun 150 metrelik kısmını oluşturuyor. Yolun toplam genişliği aşağıdaki çizimde görüldüğü gibi 18 metredir. Bu yolda hem iki yönlü trafik işlemeli, hem de iki tarafında arabalar park edebilmelidir. Şekil 1’de görüldüğü gibi yolun bir şeridi şerit çizgisi dâhil 4,5 metre ve yolun kenarındaki bir araç park alanının genişliği de 4,5 metredir. Bir arabanın güvenli bir şekilde park edilebilmesi için şerit çizgileri dâhil 3 m genişliğinde 4,8 m uzunluğunda bir alan ayrılmalıdır. Bu alan, yola paralel olabileceği gibi (bkz. Şekil 2a) açılı olarak da tasarlanabilir (bkz. Şekil 2b) ancak bu durumda araçlar yola taşmamalıdır.

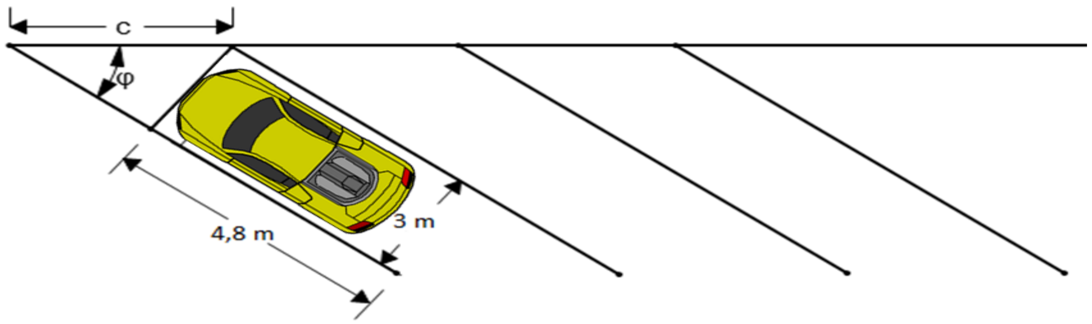


Şekil 1. Araba park alanı ve yol planı

Sizden istenen yolun bu 150 m'lik kısmına en fazla sayıda araç park edilebilecek şekilde yola <u>*paralel veya açılı park*</u> *yerleri tasarlamanızdır. Araba park yeri tasarımınızda aşağıdaki çizimlerden yararlanabilirsiniz.*



Şekil 2a. Paralel araba park yeri tasarımı



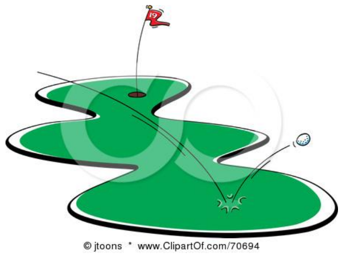
Şekil 2b. Açılı araba park yeri tasarımı

Eğer araç park alanının genişliği için verilen 4,5 metre sınırlaması olmasaydı şehir planlamacısına en fazla sayıda araç park edilebilmesi için nasıl bir park tasarımı önerisinde bulunurdunuz? Nedenleriyle açıklayınız.

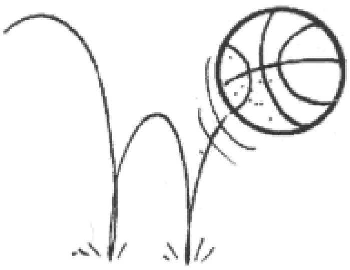
A2. Modeling Task “Bouncing Ball”

Zıplayan Top

Birçok popüler spor dalı bir çeşit top kullanımı gerektirir. Spor dallarında kullanılan topları tasarlarken göz önünde bulundurulması gereken en önemli etkenlerden birisi de topun iyi zıplayabilmesi, yani esnekliğidir. Örneğin, bir golf topu sert bir yüzeye çarptığında düştüğü yüksekliğin yaklaşık $\frac{2}{3}$'ü kadar sıçramalıdır.



Çeşitli spor dallarında kullanılmak üzere toplar üreten bir firmanın ARGE birimi çalışanları, esnekliğini test etmek için yeni geliştirdikleri bir topu, 52 metre yüksekliğindeki bir binanın çatısından aşağı doğru bırakıyor. Binanın bir katında gözlem yapan bir görevli de topun, yerden 15 metre yüksek olarak belirlenen gözlem seviyesinden 17 kez geçtiğini rapor ediyor. ARGE bölümünün matematikçisi olarak sizden, bu verileri kullanarak test edilen topun zıplama oranının ne olabileceğini bulmanız istenmektedir. Bunu yaparken, topun düz bir zemine çarparak her zıplayışta bir önceki yüksekliğinin belli ve sabit bir oranına ulaştığını varsayın.



A3. Modeling Task “Roller Coaster”

Ücretsiz Lunapark Treni

Ankara’da yeni kurulacak olan bir eğlence parkında yer alması düşünülen lunapark tren yolunun mesafeye göre yüksekliğini içeren tasarımı için yarışma açılacağı ve kazanana ömür boyu ücretsiz biniş hakkı verileceği tüm basın-yayın organlarında duyurulmuştur. Yarışmayı kazanma kriteri, tasarımın trene binen yolcuları ölesiye korkutarak heyecanlandıracak kadar eğimli, fakat onları sağ salim geri getirecek kadar da güvenli olmasına bağlı. Yolcuların heyecanlanması bu yolun yukarı ve aşağı doğru ani ve keskin değişimlerle harekete imkan vermesine bağlıyken, güvenlik kurallarına göre, yolun eğiminin mutlak değeri 5,67 den fazla olmamalı.

Siz de bu yarışmaya, bir grup mühendisle birlikte kendi tasarımınızla katılmak istiyorsunuz. Zamandan tasarruf etmek amacıyla, üçerli gruplar halinde çalışmanız gerekmekte. Her grup, bu yolun bir parçasını tasarlayacak, daha sonra bu parçalar birleştirilerek uzun bir yol elde edilecek. Sizin de içinde bulunduğunuz grup, bu eğimli demiryolunun sadece inişleri ve çıkışları olan, virajı olmayan, başlangıç noktasının yüksekliği 6 metre bitiş yüksekliği 9 metre olan 100 metre mesafelik bir bölümünü tasarlayacak. En az üç yerde ani aşağı doğru iniş içerecek olan bu yolun, hangi bölümlerinde heyecanın arttığını, hangi bölümlerinde azaldığını içeren bir rapor da hazırlamanız beklenmekte.

A4. Modeling Task “Water Tank”

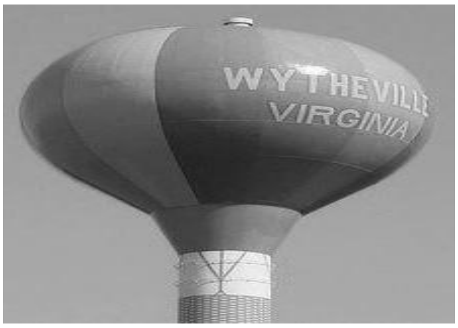
Su Deposu

Bir şirket bilgisayar destekli eğitim amaçlı yazılımlar hazırlamaktadır. Şirketteki bir ekibe öğrencilerin grafik çizme ve yorumlama becerilerini geliştirmeye yardımcı olacak bir su deposu doldurma animasyonu üzerinde çalışma işi verilmiştir. Ekibin bu animasyonu oluşturabilmesi için depo suyla dolarken depoda biriken su miktarına bağlı olarak suyun yüksekliğini gösteren bir grafiğe ihtiyacı bulunmaktadır.

Ekibin matematikçi üyesi olarak sizden istenen ekte verilen örnek depolar için istenen türden bu grafikleri yaklaşık olarak çizmeniz ve sonrasında herhangi bir şekle sahip bir su deposu için su miktarına bağlı olarak suyun yüksekliğini gösteren grafiğin nasıl çizileceğini açıklayan bir yönerge hazırlamanızdır.

Depo Şekilleri

Depo-1



Depo-2

Depo-3

Depo-4

Acknowledgments about Modeling Tasks:

In order to fulfill one of the purposes of the research project supported supported by the Turkish Scientific and Technical Research Institution (TUBITAK), grant number 110K250

- **"Street Parking"** task was adapted from **"**Swetz, F. & Hartzer, J.S., (1991). *Mathematical modeling in the secondary school curriculum: A resource guide of classroom exercises*. Reston, VA: NCTM."

- **"Bouncing Ball"** task was adapted from " Bouncing Ball": http://intermath.coe.uga.edu/topics/nmcncept/ratios/r16.htm"

- **" Roller Coaster"** task was adapted from "Cabana, C., et al. (2000). *College prepatory mathematics: Calculus*. CPM Educ. Program."

- **"The Water Tank"** task was adapted from "Carlson, M., Larsen, S., & Lesh, R. (2003). Integrating models and modeling perspective with existing research and practice. In R. Lesh & H. Doerr (Eds.), *Beyond constructivism: A models and modeling perspective* (pp. 465-478). Mahwah, NJ: Lawrence Erlbaum Associates."

APPENDIX B

AN EXAMPLE OF STUDENTS' SOLUTION PAPER

Bu bilgilerden yararlanarak, ilk önce deneme yanılma yöntemi ile $\frac{45}{52}$'ye
...nın bir oran bulduk.

Daha sonra bir formül çıkartmak için oranımıza "$\frac{a}{b}$" dedik. Oranımızı, 8 defa
...kış hareketi olduğu için, sayımızı, oranımızla 8 defa çarptık ve bu işlem son
...kan sayının 15'den büyük veya eşit olması gerekiyordu.

Buna göre çıkan formül;

$$\left(\frac{a}{b}\right)^8 \cdot 52 = x \geqslant 15$$

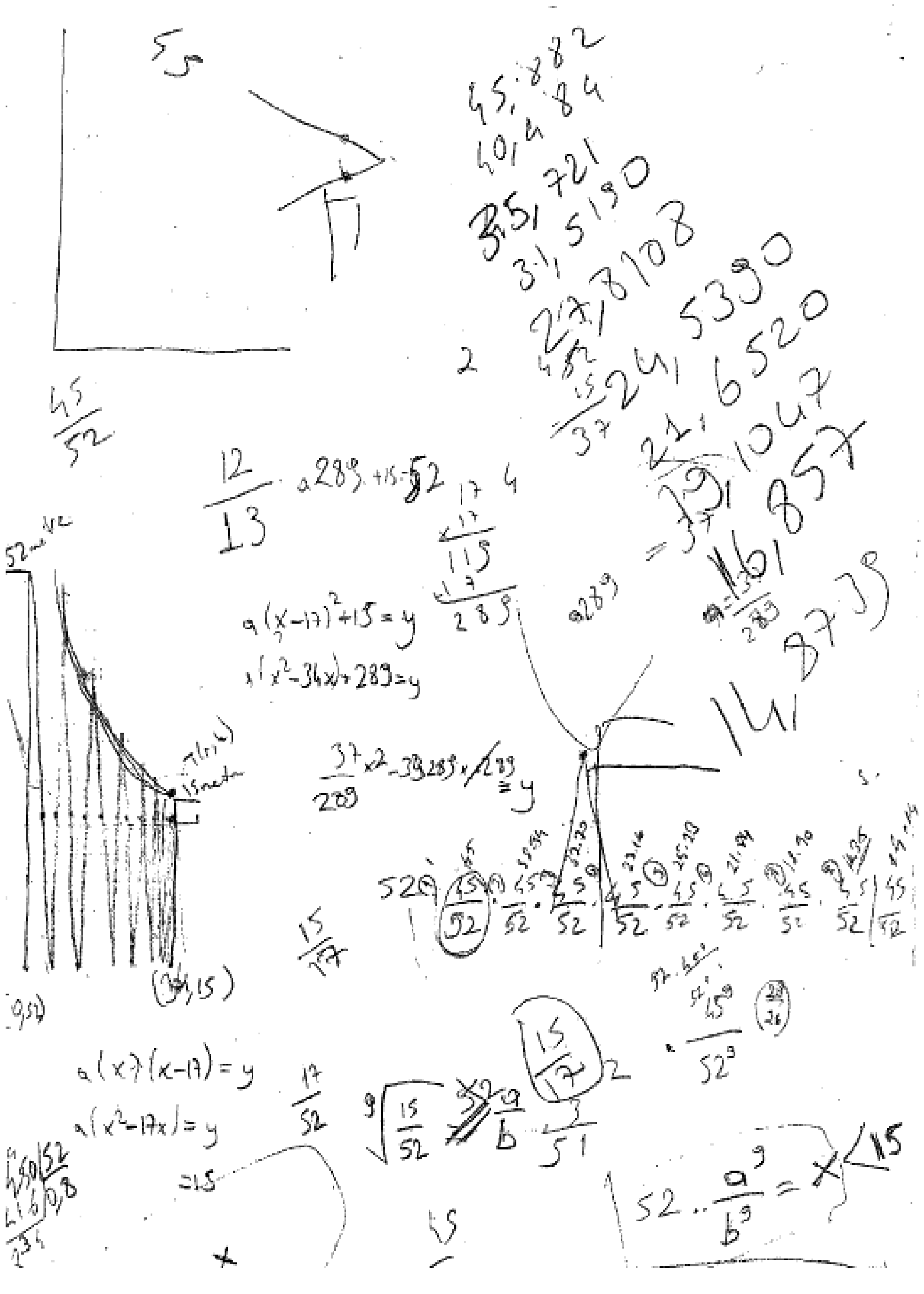
$$\left(\frac{a}{b}\right)^8 \cdot 52 \geqslant 15$$

$$\left(\frac{a}{b}\right)^8 \geqslant \frac{15}{52}$$

$$\frac{a}{b} \geqslant \sqrt[8]{\frac{15}{52}}$$

$$\frac{45}{52} = 0,86538$$

$\sqrt[8]{\frac{15}{52}} \rightarrow$ bu sonucumuz yaklaşık 0,85607... bulduk



1nci zıplama 48 iniyor 44,30 çıkıyor
2nci zıplama 44,30 iniyor

48	52 46	3 1
45,94	40,6	2 6
38,91	35,9	2 2
33,7	31,8	1 9
29,16	28,1	1 6
25,23	24,9	1 3
21,84	22,0	
18,90	19,5	
$\boxed{16,35}$	17,9	
14,8	15,2	

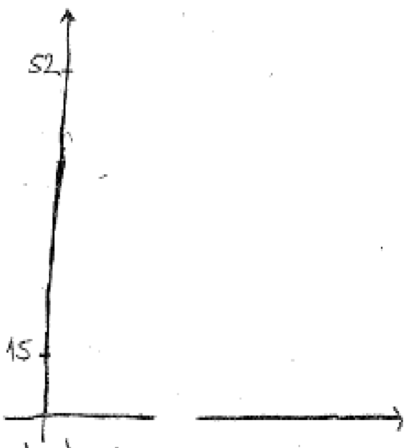
53

$\frac{45}{52}$ 0,2721967

(x . 52)

8

$\frac{a}{b} = x$



$$\left(\frac{a}{b}\cdot\left(\frac{a}{b}\cdot\left(\frac{a}{b}\cdot\left(\frac{a}{b}\cdot 52\right)\right)\right)\frac{a}{b}\right)\frac{a}{b}\Big)\frac{a}{b}\Big)$$

$x^{15} \cdot 52 =$

$\frac{1}{2} \cdot 20 = 10$

$\frac{1}{2} \cdot 10 = 5$

İlker Abla

15 metre

$x^7 \cdot 52^6 = y$

$x \cdot 52 = 155$

$x \cdot b = 155$

52

$\frac{23}{26}$

46

5 kez girdi

$\frac{1}{2} \cdot 5 = 2,5$

10m

3,1

4 metre

45 15 0

2, 4, 13, 26, 52

$x^8 \cdot (x) \cdot 52 = y$

4 334

$(x \cdot 20) \cdot x = y$ $\frac{1}{2}$

5 kez girdi

$\frac{1}{2}$ 52 = 25

$\frac{x}{4} \cdot 52 =$

$\frac{1}{2} \cdot 20 = y$

1,08

$\frac{1}{100}$

$x^5(x \cdot 20) = y$

45

$10 \cdot \frac{1}{2} = y$

x → sabit

$x \cdot 52 = y$ → 15

$x \cdot 20$

$\frac{1}{2} \cdot 20 = 10$

y = 15'den büyük sayılar

$\frac{1}{2}$ = oran = x

x → sabit bir sayı olmak üzere;

20 = atılan yükseklik

52 atılan yükseklik

y = 15 ve yukarı

bir örnek;

dışsız sayılar

$\frac{1}{2} \cdot 20 = y$

y dışında

$10 = y$

$x \cdot 52 = y$

$10 \cdot \frac{1}{2} = y$

$10 = (x \cdot 20)$

$10 \cdot \frac{1}{2} = 5$

$(x \cdot 20) \cdot x = y$

$(x \cdot 52) \cdot x = y$

$x^7 \cdot 52x^6 = 15$

$(x \cdot 20) \cdot x = y$

$x^2 \cdot 52x = y$

$x^3 \cdot 52x^2 = y$

$x^4 \cdot 52x^3 = y$

$x^5 \cdot 52x^4 = y$

$x^? (x \cdot 52) = y$

$x(x \cdot 52) =$

$x^3(x \cdot 52)$

$x^2 \cdot 20x = y$

$x^3 \cdot 20x = y$

4,20

45

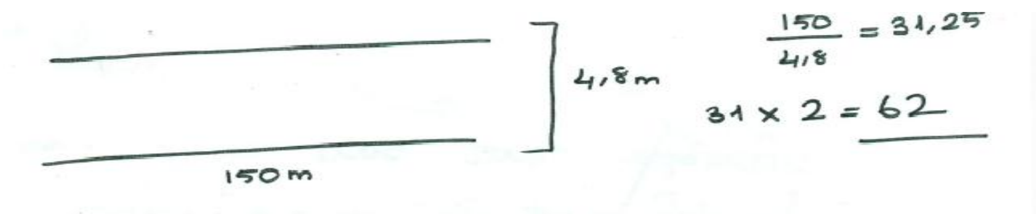
APPENDIX C

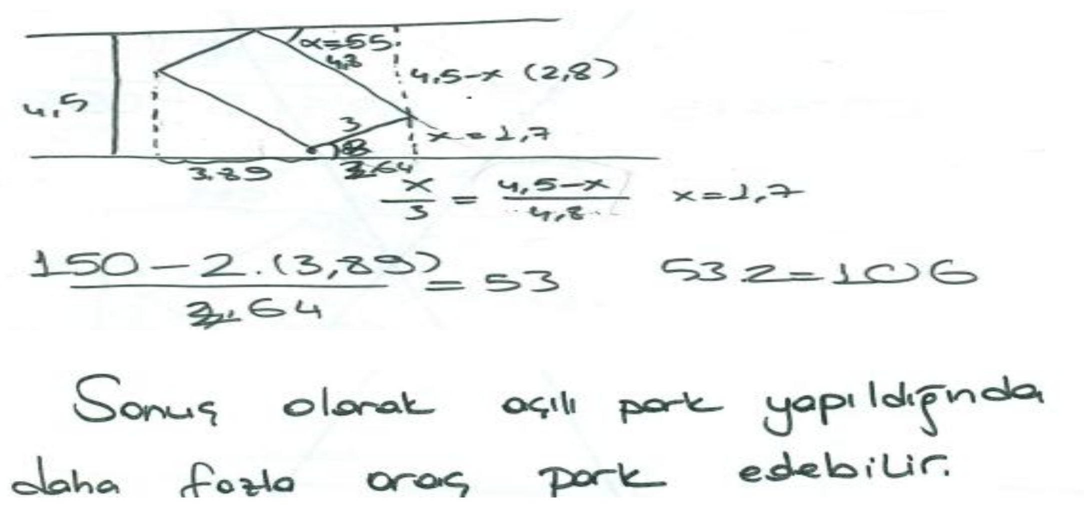
STUDENTS’ SOLUTION APPROACHES

C1. Students’ solutions for “Street Parking”
[Caddede Park Yeri Öğrenci Çözümleri]

GRUP 1 [SG1]: Kullanılan farklı çözüm yaklaşımı / Öğrencilerin neyi anlayıp neyi anlamadıkları /kullanılan matematiksel konu ve gösterimler/ öğrenci çözümlerinin güçlü ve zayıf yönleri;

Örnek çözüm:





Çözüm süreci:

a) Arabanın boyu 4,8 olduğu için $\frac{150}{4,8}$den hesaplama yapılıyor**.**

b) Arabanın kesinlikle dikdörtgen olduğu için alan formulü kullanmayacaklarını söylüyorlar ve dikdörtgensel bölge çiziyorlar park yerine. Verilen uzunlukları, 4,5'i ve 3'ü yerleştiriyorlar. Sonra 4,8'i dikdörtgenin uzun kenarı olacak şekilde yerleştiliyorlar. Daha sonra 4,5 m uzunluğu, x ve $4,5-x$ olarak bölüyorlar. Açıları yerleştiriyorlar ve benzerlik yapıyorlar oluşan iki üçgen arasında ve x dedikleri uzunluğu buluyorlar.

Çözüm yaklaşımı:

Benzerlik (geometri) kullanarak araç sayısını bulmak için gerekli uzunlukları bulma ve araç sayısını hesaplama.

Öğrenci çözümlerinin güçlü yönleri:

- Paralel park hesabının doğru yapılması
- Arabanın yerleşeceği güvenli park alanının uzunluğu yani 4,8 m'nin yerini doğru algılamışlar.
- Ölü alanları çıkarmaları gerektiğini hesaba katma

Öğrenci çözümünün zayıf yönleri:

Hatalı çözüm süreci ve yanlış sonuç bulma

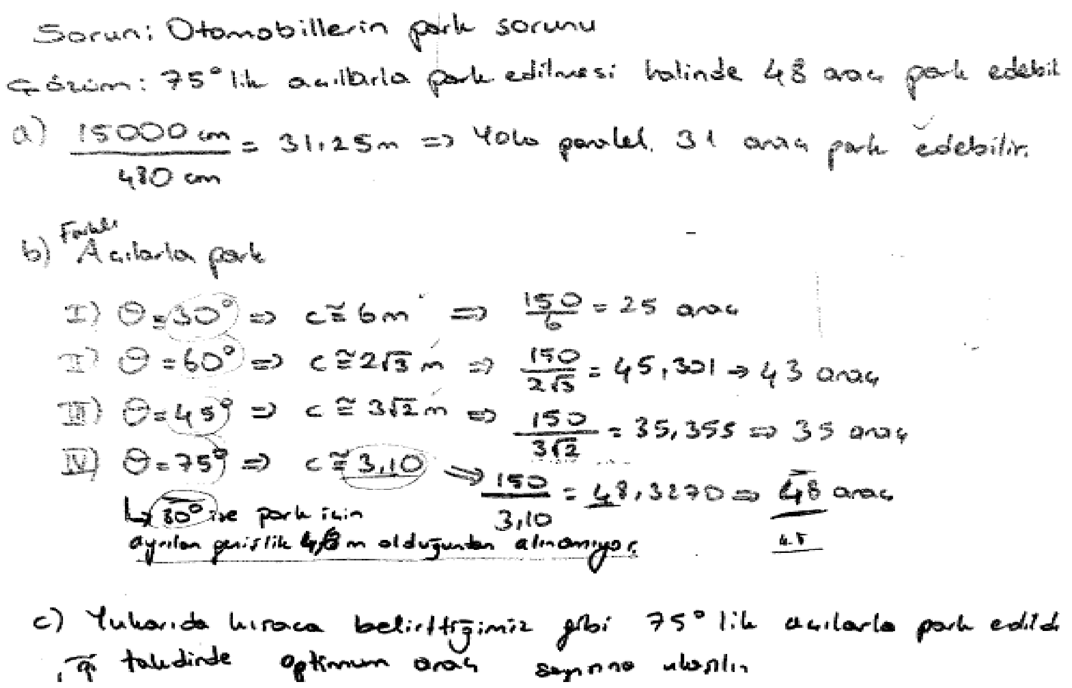
- Üçgenler arasında benzerliği yanlış yapma ve bu sebeple uzunlukları yanlış bulma
- Araçların duruşunu iyi modellemedikleri için buldukları, sığması gereken gerekli aracı bulmak için bölmeleri gereken "c" uzunluğunun doğru olmaması.
- Soruda istenen farklı açı değerleri için araç sayısını hesaplamaları gerektiğini göz ardı etmeleri açıları hiç kullanmamaları
- Şıkları ayrı ayrı değerlendirmeleri; soruya ortak gelmediler;hepsini genelleyip bir sonuca varmadılar..

Kullandıkları matematiksel konu/kavramlar:

- Geometri-benzerlik

GRUP 2 [SG2]: Kullanılan farklı çözüm yaklaşımı / Öğrencilerin neyi anlayıp neyi anlamadıkları /kullanılan matematiksel konu ve gösterimler/ öğrenci çözümlerinin güçlü ve zayıf yönleri;

Örnek çözüm:



Çözüm süreci: En çok bilinen açılardan yani 30°, 45°, 60° derece gibi açılardan, yola çıkarak çıkarak c değerini hesaplama.

$\text{Sin}\,\theta = \frac{3}{c}$ ve $\frac{150}{c}$ ' den hesaplama. Örneğin;

$\theta = 30$, $c = 6$ ve 25 araç

$\theta = 45$, $c = 2\sqrt{3}$ ve 43 araç şeklinde hesaplayıp, 75° açı için en çok araba park ettiklerini buluyorlar. 80° için ayrılan genişliğin uygun olmadığını belirtiyorlar.

Çözüm yaklaşımı:

- Bilinen trigonometrik değerlerden, açıların trigonometrik değerlerinden yararlanılarak çözme. Trigonometrik oranlar kullanılarak c bulunmuş (sin teta) sonrada $\frac{150}{c}$ den araç sayısı bulunmuş.

Öğrenci çözümlerinin güçlü yönü:

▪ Paralel park hesabının doğru yapılması

Öğrenci çözümünün zayıf yönleri:

▪ Sadece bilinen tirgonemetrik açı değerlerinden yararlanılmış

▪ Park alanı için ayrılan genişlik kriteri (4,5 olması) sadece 80° için dikkate alınmış, diğer açı değerleri göz ardı edilmiş; fakat 80° içinde nasıl o kriteri sağlayıp sağlamadığını anlamaları da yine çözüm kâğıtlarında görünmüyor.

▪ Şıkları ayrı ayrı değerlendirmeleri; soruya ortak gelmediler; hepsini genelleyip bir sonuca varmadılar.

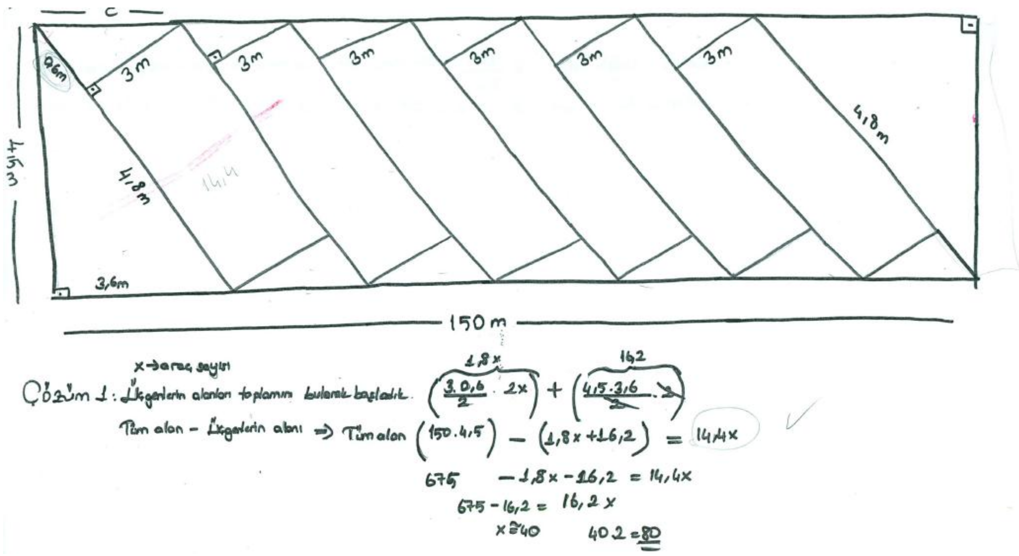
NOT: Görselleştirmek için çizim yaptıkları çizim yok. O sebeple park tasarımını nasıl yaptıkları ve uzunluk değerlerini nasıl aldıkdıkları açık değil.

Kullandıkları matematiksel konu/kavramlar:

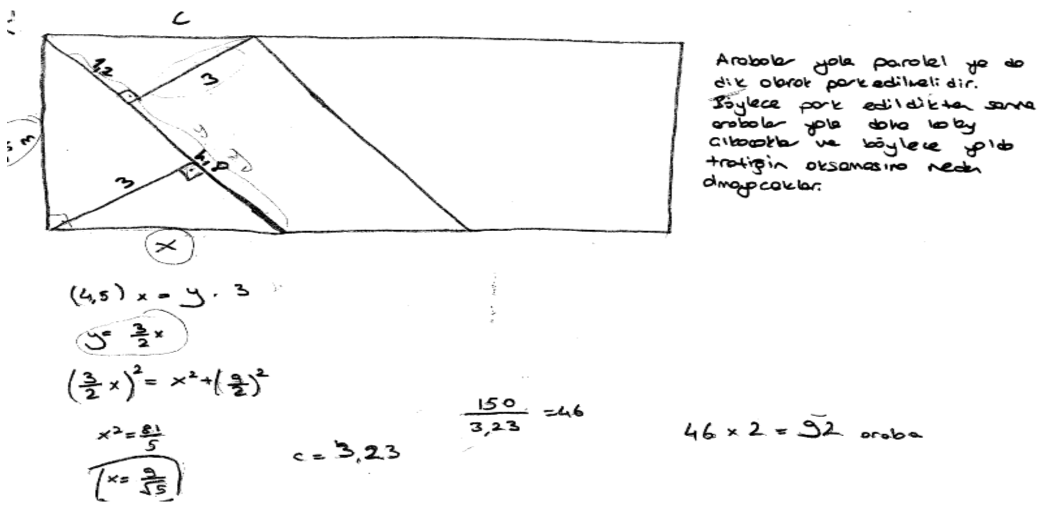
• Trigonometri

GRUP3 [STG3]: Kullanılan farklı çözüm yaklaşımı / öğrencilerin neyi anlayıp neyi anlamadıkları /kullanılan matematiksel konu ve gösterimler/ öğrenci çözümlerinin güçlü ve zayıf yönleri;

Örnek Çözüm 1:



Örnek Çözüm 2:



Çözüm süreci: a) pararel park yeri hesaplamasını yapıyorlar. $\frac{150}{4,8}$

b) Araba paralel olarak bir açıyla gireceği için iki kısımda boşluk olacak; aynı zamanda yolun bir başında bir de sonunda iki tane üçgensel büyük boşluk olacak. Yolun tüm alanını buluyorlar $(150 \times 4,5)$ ve buradan tüm üçgenlerin hem küçük üçgenler hemde 2 tane büyük üçgenin alanını buluyorlar (araç sayısı x)

Çözüm yaklaşımı:

- Toplam park alanından açılı yerleştirmede ölü alanların toplamı atılıp (iki büyük üçgen ve küçük üçgenler) geriye kalan park eden arabaların toplam park alanı bulunmuş
- Alandan yola çıkarak denklem kurma

Öğrenci çözümlerinin güçlü yönleri:

- Paralel park hesabının doğru yapılması
- Arabanın yerleşeceği güvenli park alanının uzunluğu yani 4,8 m'nin yerini doğru algılamışlar
- Ölü alanları çıkarmayı göz önünde bulundurma; yanlış kullanımda olsa alan fikrinin kullanımı

Öğrenci çözümünün zayıf yönleri:

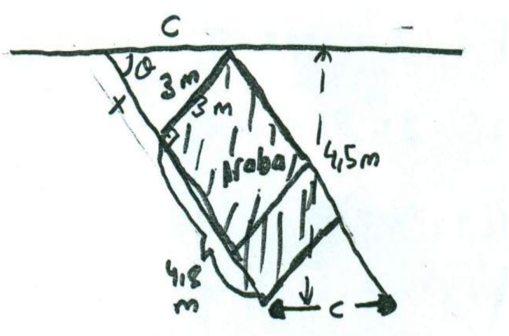
- 0,6 metreyi nasıl buldukları; uzunluk değerleri sağlamıyor $(4,5)^2+(3,6)^2 \neq (5,4)^2$
- Öklit teoreminden bulduklarını söylüyor 3,6, nasıl? Öklit kullanabilecekleri bir geometrik durum yok (geometri bilgilerini kullanmadaki hataları)
- 0,6 m olarak belirttikleri uzunluğu sabitlemişler, açı değerine bağlı o uzunluğun değişken olması durumu var
- Farklı açı değerlerinden yararlanmamışlar
- Şıkları ayrı ayrı değerlendirmeleri; soruya ortak gelmediler; hepsini genelleyip bir sonuca varmadılar.

Kullandıkları matematiksel konu/kavramlar

- Üçgenin Alanı, dikdörtgenin alanı (Geometri)

GRUP 4 [SG4]: Kullanılan farklı çözüm yaklaşımı / Öğrencilerin neyi anlayıp neyi anlamadıkları /kullanılan matematiksel konu ve gösterimler/ öğrenci çözümlerinin güçlü ve zayıf yönleri

Örnek çözüm:



Çözüm süreci: dikdörtgensel bölge olarak aldıkları arabanın güvenli park yerinde x olarak belirledikleri uzunluğu pararel kenarın alanından yola çıkarak hesaplamaya çalışıyorlar. Öğrenciler buradan bir denklem yazıyorlar. *x*'in değerini bulduktan sonra, tanjant kullanarak θ açısını bulabileceklerini düşünüyorlar, fakat kurdukları denklemi çözmedikleri (çözemedikleri) için sonuca ulaşamıyorlar.

Çözüm yaklaşımı: pararlel kenarın alanından yola çıkarak denklem yardımıyla sığabilecek maksimum araç sayısını bulmaya çalışma

Öğrenci çözümlerinin güçlü yönleri:

- Doğru, farklı ve mantıklı bir çözüm yaklaşımı
- Arabanın yerleşeceği güvenli park alanının uzunluğu yani 4,8 m'nin yerini doğru algılamışlar.
- Paralel kenarın alanını kullanarak, denklem kurma ve x değerini bulmak isteme (ilginç ve farklı)
- Çizim yardımıyla görselleştirme
- Geometri ve trigonometri ve cebirsel işlemler; konular arası ilişki kurma

Öğrenci çözümlerinin zayıf yönleri:

- Denklemi çözememe ve bu sebeple sonuca ulaşama
- "θ" açısını ne kadar 90°ye yakın alırsak o kadar fazla araba sığar düşüncesi (yanlış değil ama bu verilen şartlarda bu düşünce geçerli değil).

Kullandıkları matematiksel konu/kavramlar

- Geometri, Alan (Paralel kenarın alanı)
- Trigonometri
- Cebir, ikinci dereceden denklemler

GRUP 5 [SG5]: Kullanılan farklı çözüm yaklaşımı / Öğrencilerin neyi anlayıp neyi anlamadıkları /kullanılan matematiksel konu ve gösterimler/ öğrenci çözümlerinin güçlü ve zayıf yönleri:

Örnek çözüm:

a) Yola paralel parkedilme durumunda;

$\frac{150}{4,8} = 31,25 \rightarrow$ 31 araba park edilebilir. (Tek taraf)

Toplam = 62 araç parkedilebilir.

b) $(4,8)^2 = (4,5)^2 + x^2$

$x = 1,67$

$\tan \theta = \frac{4,5}{1,67} = 2,69$

$\theta \cong 70°$

$\theta = 30°$ için;

$* \ \frac{1}{2} = \frac{4,5}{x} \quad x = 9m$

Araba için gereken uzunluğun çok fazla olması gerekir. Yani az araç sığar.

$\theta = 90°$ için;

$* \ \sin 90° = 1$

Araba boyu = 4,5 m

Ama 4,8 m alınması gerektiği için güvenli park yapılamaz.

$\theta = 70°$ için;

$* \ \frac{4,5 \cdot 1,67}{2} \cdot 2 = 7,515 \, m^2$

* Bir araba için paralelkenarın alanı

$4,8 \cdot 3 = 14,4 \, m^2$

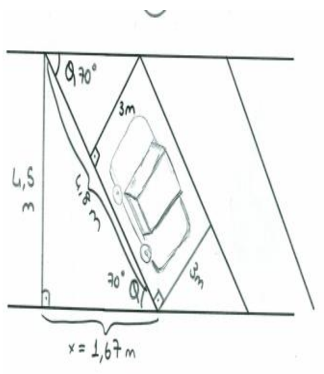
$* \ 14,4x + 7,515 = 150 \cdot 4,5$

$x = 46,353125$

Tek taraf 46 araç

2 taraf 92 araç sığar.

c) $\theta = 70°$ olmalı ve araçlar açılı olarak yerleştirilir. Böylelikle en fazla sayı araç sığdırılarak en güvenli park yapılmış olur.



Çözüm süreci:

a) Yola paralel park edilme durumu $\frac{150}{4,8}$ (yolun boyu/güvenli park yeri uzunluğu)

b) Ölü bölgenin alanı ve paralel kenarın alanını toplayıp (x tane araç olduğunu düşünerek), araçların park ettiği tüm dikdörtgensel bölgenin alanına eşitleyip kaç tane araç bulabileceklerini hesaplıyorlar ve burada 46 araç çıkıyor (tek taraf için). Aynı zamanda θ açısınıda hesaplıyorlar ve $70°$ olarak buluyorlar. (Önce küçültüp sonra büyülterek) Açıyı küçültüp 30° ve büyültüp 90° yapıp değişik açı değerleri için deniyorlar (bu süreçte bilinen açı değerleri veriliyor). Ve aracın boyunun yola taştığını görüyorlar ve en uygun açının 70° olduğunu görüyorlar.

Çözüm yaklaşımı:

- Trigonometrik değerler kullanılmı, açıların trigonometrik değerlerinden yararlanmış varsayımlar üzerinden hareket edilmiş (bazı açı değerleri)

Öğrenci çözümlerinin güçlü yönleri:

- Paralel park durumunu doğru çözme
- Değişik açı değerleri için denemeye ve yorumlamaya çalışma

Öğrenci çözümlerinin zayıf yönleri:

- 4,8 metrenin yanlış yerde almaları ve yanlış sonuca ulaşma ve bu durumda en idealini açılı park olarak bulma
- Trigonometri cetvelini kullanmada zorluk (videodan)
- Benzerlik kullanımında zorluk (videodan)

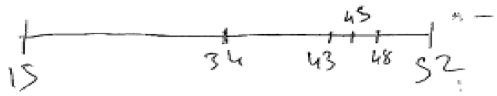
Kullandıkları matematiksel konu/kavramlar

- Alan
- pisagor teoremi
- Trigonometri

C2. Students' solutions for "Bouncing Ball"

[Zıplayan Top Öğrenci Çözüm Yaklaşımları]

GRUP 1 [SG1]: Kullanılan farklı çözüm yaklaşımı / Öğrencilerin neyi anlayıp neyi anlamadıkları /kullanılan matematiksel konu ve gösterimler/ Öğrenci çözümlerinin güçlü ve zayıf yönleri;



oran = $\frac{45}{52}$ Neden?

* İlk olarak topun zıplama oranını en geniş biçimde tanımladık. $\frac{15}{52} < x < \frac{52}{52}$ oran $\frac{15}{52}$'den büyük olmalı ki zıplayışı 15 metrenin üstüne çıksın. $\frac{52}{52}$'den küçük olmalı ki artarak gitmesin. Daha sonra ortalama bularak $\frac{34}{52}$'den büyük olması gerekiyordu. Çünkü zıplama sayısı 8 olmalı ve bu yeterli değildi. Bu yüzden tekrar bir ortalama aldık ve $\frac{43}{52}$'den hesapladık. Bu da yeterli olmayınca $\frac{48}{52}$'den gittik. Bu da fazla gelince $\frac{45}{52}$'yi denedik ve topun tam 8 kere zıpladığını gördük. Böylece top 15 metrelik yükseklikteki gözlemcinin önünden 17 kere geçmiş oluyor. Ayrıca $52 \cdot \frac{x}{52} = x \Rightarrow x \cdot \frac{x}{52} = \frac{x^2}{52}$ denklemlerinden ilerlersek $\frac{x^8}{52^7} \geq 15$? denklemine ulaştık. Bu denklemde $\frac{45}{52}$ oranını doğruluyor. (1)

Çözüm süreci: İlk olarak zıplama oranı aralığını en geniş olacak şekilde tanımlıyorlar ($\frac{15}{52} < x < \frac{52}{52}$). Daha sonra bu aralığı ortalama alarak daraltıyorlar ve top 8. kez yukarı çıktığında 15 metre ve yukarısı olabilsin ki bakan kişi topu görsün. Buldukları oranlar için deneme yapıyorlar (sayısal işlemler) ve şartları sağlayıp sağlamadığına bakıyorlar. Ve en son $\frac{45}{52}$ oranınını deneyerek topun 8 kez yukarı zıpladığında 15. metrede olduğunu ve 9.kez yukarı zıplayışta 15 metrenin altında kaldığını görüyorlar ve oranı $\frac{45}{52}$ olarak buluyorlar.

Çözüm yaklaşımı:

- Deneme yanılma yöntemi oranları rastgele, sistematik atama ve deneme. İlk başta $\frac{15}{52}$ alınıp, daha sonra sürekli ortalama alarak gerçek orana ulaşmaya çalışma (sistematik deneme yanılma)

Öğrenci çözümlerinin güçlü yönü:

- Zıplama oranı aralığı tanımlama
- Sistematik deneme yanılma
- Çizerek görselleştirme

Öğrenci çözümünün zayıf yönleri:

- Eşitsizlikle gösterimleri hatalı ($52.\frac{x}{52} = x$; x. $\frac{x}{52} = \frac{x}{52}^{2}$ .....$\frac{x}{52^7}^{8} \geq 15$ denkleminde $\frac{45}{52}$ oranı'na ulaşma)

Kullandıkları matematiksel konu/kavramlar:

- Oran
- Eşitsizlik
- Üslü sayılar
- Ortalama

GRUP2 [STG2]: Kullanılan farklı çözüm yaklaşımı / Öğrencilerin neyi anlayıp neyi anlamadıkları/kullanılan matematiksel konu ve gösterimler/ öğrenci çözümlerinin güçlü ve zayıf yönleri

Örnek Çözüm:

metredeki gözlemci atılan topu 17 kez gördüğüne göre, atılan top
kez iniş 8 kez çıkış hareketi yapmıştır.
Bu bilgilerden yararlanarak, ilk önce deneme yanılma yöntemi ile $\frac{45}{52}$'ye
kın bir oran bulduk.
Daha sonra bir formül çıkartmak için oranımıza "$\frac{a}{b}$" dedik. Oranımızı, 8 defa
kış hareketi olduğu için, sayımızı, oranımızla 8 defa çarptık ve bu işlem son
kan sayının 15'den büyük veya eşit olması gerekiyordu.

Buna göre çıkan formül;

$$\left(\frac{a}{b}\right)^8 \cdot 52 = x \geqslant 15$$

$$\left(\frac{a}{b}\right)^8 \cdot 52 \geqslant 15$$

$$\left(\frac{a}{b}\right)^8 \geqslant \frac{15}{52}$$

$$\frac{a}{b} \geqslant \sqrt[8]{\frac{15}{52}}$$

$$\frac{45}{52} = 0{,}86538$$

Çözüm süreci: Öğrenciler soruyu çizerek zihinlerinde canlandılar-düşüşleri göstermede soru netleşti. Eşitsizliği yazmadan bir oran oluşturdu. Eşitsizliği yazmadan önce kendince bir oran oluşturdu. Orana bir değişken atadı. Bir denklem buldu ve buradan sonra eşitsizliğe gitti. Deneme yanılma yöntemi ile $\frac{45}{52}$'ye ulaşıp, bunun formülünü çıkarıyorlar.

Çözüm yaklaşımı:

- Deneme yanılma ile başlamış sonrasında formülize etme (Oranları rastgele atıyorlar, kendileri deniyorlar; $\frac{12}{13}$ gibi)

Öğrencilerin Çözümün güçlü yönleri:

- Çizerek soruyu anlama
- Deneme yanılma yönteminden elde ettikleri sonucu, matematiksel olarak (formülle) kanıtlama/destekleme

Öğrencilerin Çözümün zayıf yönleri:

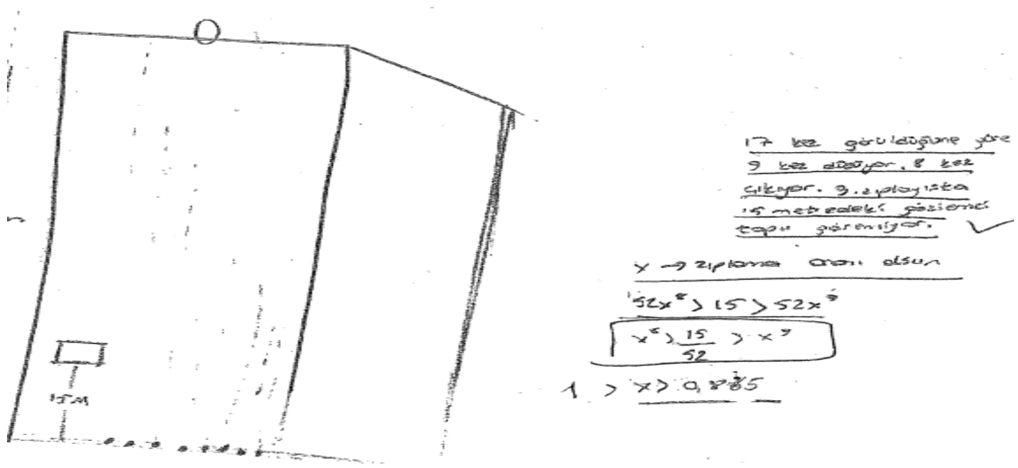
- Eşitsizliği tek yönlü alıyorlar.

Kullandıkları matematiksel konu/kavramlar:

- Oran
- Eşitsizlik
- Üslü sayılar

GRUP3 [STG3]: Kullanılan farklı çözüm yaklaşımı / Öğrencilerin neyi anlayıp neyi anlamadıkları /kullanılan matematiksel konu ve gösterimler/ öğrenci çözümlerinin güçlü ve zayıf yönleri;

Örnek Çözüm:



Çözüm süreci:

- x'i zıplama oranı olarak alıyorlar. $52.x$ birinci zıplamadaki son seviye, $52.x^2$ $52.x^2$ ikinci zıplamadaki son seviye, böyle devam ederek $52.x^8$ son zıplamadaki son seviye ve bakan kişinin görebilmesi için $52.x^8 > 15$. Diğer taraftan $52.x^9 < 15$ olmalı.

Çözüm yaklaşımı:

- Eşitsizlik kullanarak cebirsel olarak modelleme

Öğrenci çözümlerinin güçlü yönleri:

- Soruyu doğru anlama
- Çizim yaparak görselleştirme
- Çift taraflı eşitsizliği kullanma
- Doğru formülüze etme
- Doğru şekilde akıl yürütme

Öğrenci çözümünün zayıf yönleri:

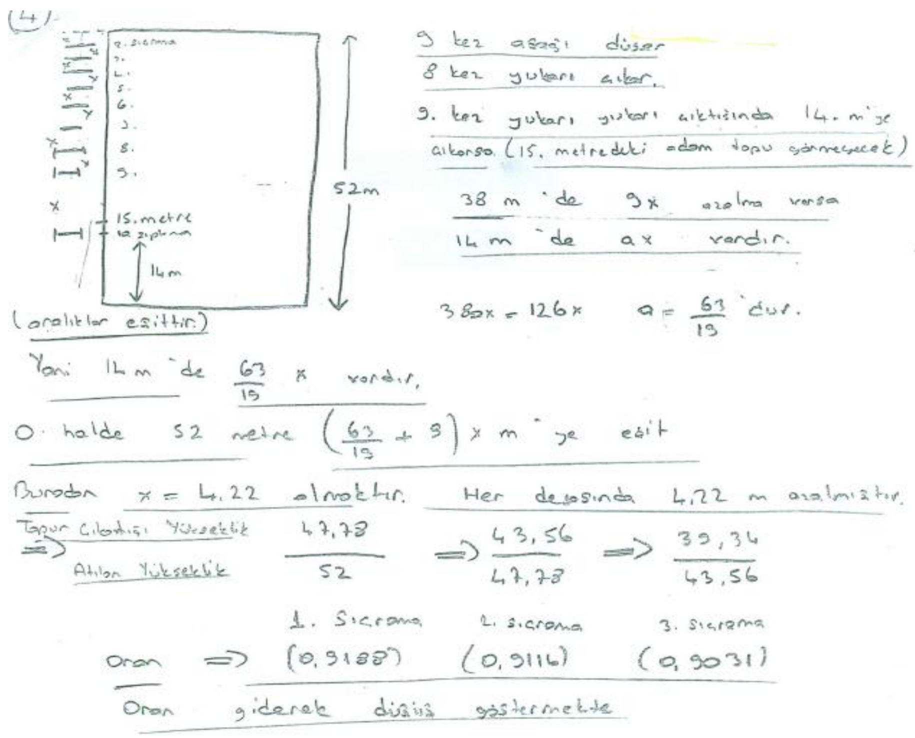
- Büyük ve küçük yerine büyük eşit; küçük eşit alınabilir

Kullandıkları matematiksel konu/kavramlar:

- Eşitsizlik
- Üslü sayılar
- Oran

GRUP 4 [STG4]: Kullanılan farklı çözüm yaklaşımı / Öğrencilerin neyi anlayıp neyi anlamadıkları /kullanılan matematiksel konu ve gösterimler/ öğrenci çözümlerinin güçlü ve zayıf yönleri;

Örnek Çözüm:



Çözüm süreci: ilk olarak çizerek 17 zıplama için 9 kez aşağı ve 8 kez yukarı çıkacağını görüyorlar. 15 m olduğu için 18.zıplayışta 15 m'nin altında olacağı düşüncesiyle, 9. Kez yukarı çıkışta 14 metreye kadar çıkacağının düşünüyorlar. Ve her zıplayışta, zıplayıştaki azalış miktarını 9 metre sabit olarak kabul ederek, oran orantı ile 38 metrede $(52-14=38)$ 9x azalma varsa, 14 metrenin içinde kaç x azalma vardır şeklinde bir hesaplama yapıyorlar ve $\frac{63}{19}$ şeklinde sonuç buluyorlar. $9x+\frac{63x}{9}$ metre yani toplamda 52 metrede 4,22 metre sabit miktarda azalma kaydettiğini buluyorlar. 52 metreden 4,22 yi çıkarark bir sonraki yüksekliği buluyorlar ve tekrar 4,22 metreyi çıkarma işlemini takip ederek işlemleri yapıyorlar.

İkinci yüksekliği birinci yüksekliğe, üçüncüyü ikinciye şeklinde devam ederek oranlama yaparak oranı buluyorlar.

Çözüm yaklaşımı:

- Her zıplamada eşit miktarda azalıyor kabul etme ve doğrusal düşünüp oran-orantı kurarak çözme

Öğrenci çözümlerinin güçlü yönleri:

- Çizim yapma, görselleştirme
- Mantık yanlış olmasına rağmen, kurdukları mantığa göre çözüm ve işlem basamaklarının doğru olması

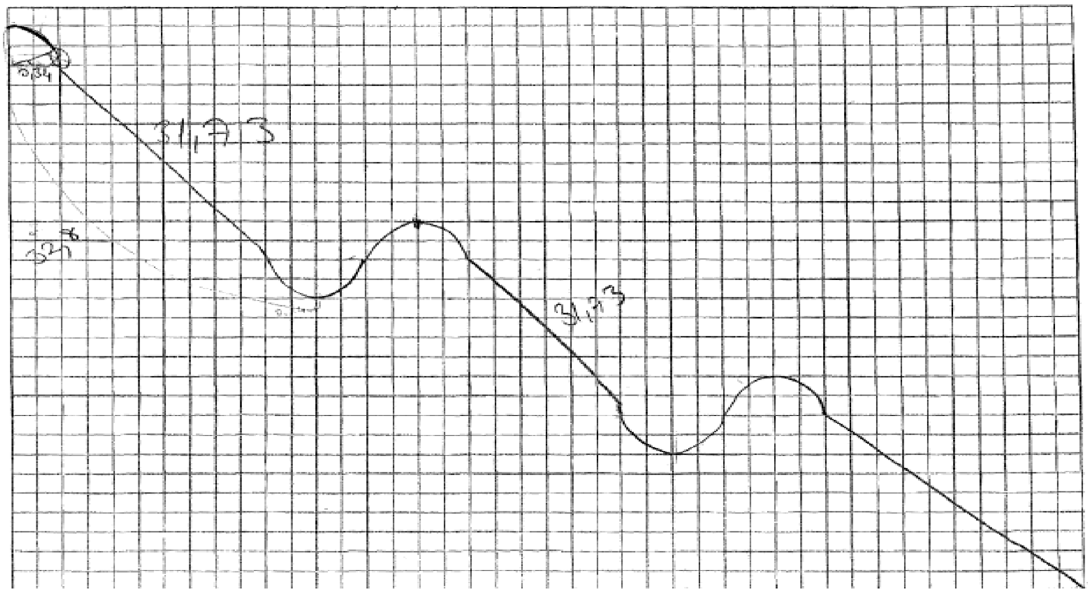
Öğrenci çözümünün zayıf yönleri:

- Soruyu yanlış yorumlama; azalma miktarı sabit değil, oran sabit
- Sabit azalma miktarı alıyorlar bu nedenle sabit oranla azalmıyor, fakat öğrenciler oranları ortalamasını alarak sabit bir oran kabul ediyor
- Oran orantı kullanma ve eşitsizliği görememe
- 15 metrenin altını ≤ yerine 14 olarak sabit değer alma

C3. Students' solutions for "Roller Coaster"
[Lunapark Treni Öğrenci Çözüm Yaklaşımları]

GRUP1 [SG1]: Kullanılan farklı çözüm yaklaşımı / Öğrencilerin neyi anlayıp neyi anlamadıkları /kullanılan matematiksel konu ve gösterimler/ öğrenci çözümlerinin güçlü ve zayıf yönleri;

Örnek Çözüm:



→ Öncelikle heyecan yaratmak için temel olarak inişleri kullanırken, çıkışları sadece inişin monotonluğundan kurtulmak için minimum miktarda kullandık. Başlangıçta trenimize irtifa kazandırmaya çalışıp mesafemizi harcamamak için trenimizi "O" noktasından direkt olarak aşağı indirdik. Kullandığımız çeyrek çemberlerin yarıçapları 0,34 iken, trenimizin ilk doğrultu değiştirdiği yerin başlangıç noktasına uzaklığı 32,8 ve ilk çeyrek çemberimizin (başlangıç) ikinci çeyrek çemberimize uzaklığı 31,73 m'dir.

Çözüm yaklaşımı:

- Tren yolunun uzunluğunu maksimum eğimi göz önünde bulundurarak sayısal hesaplamalarla hesaplamaya çalışma ve yolu tasarlama

Öğrenci çözümlerinin güçlü yönleri:

- Tasarım yapma, fikir üretme,

Öğrenci çözümünün zayıf yönleri:

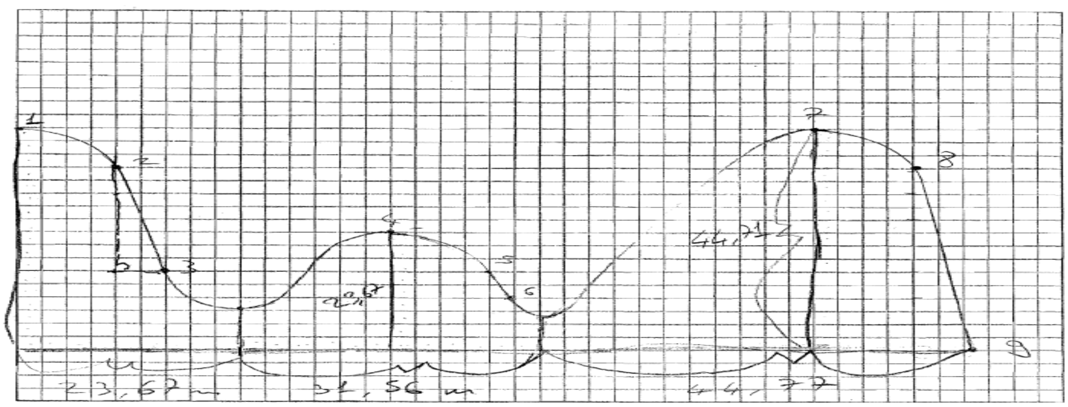
- Sorunun yanlış anlaşılması: 100 metrenin eğrinin uzunluğu olarak alınması (hatalı
- Eğrinin eğiminin doğrusal gibi değerlendirilmesi (eğride eğimin değişken olduğu göz ardı edilip, eğim sabit gibi düşünülmesi)
- Eğimin maksimum olduğu dönüm noktası fikri hiç yer almıyor

Kullandıkları matematiksel konu/kavramlar:

- Sayısal hesaplamalar
- Eğim (ortalama değim)

GRUP 2 Kullanılan farklı çözüm yaklaşımı / Öğrencilerin neyi anlayıp neyi anlamadıkları /kullanılan matematiksel konu ve gösterimler/ öğrenci çözümlerinin güçlü ve zayıf yönleri;

Örnek Çözüm:



Çözüm süreci:

- 5,67'nin tanjant 80^0 olmasından dolayı, 80-10-90 üçgeni oluşturarak yani eğriyi doğruya ingirgeyip inişlerde maksimum eğimi elde etmeye çalışıyorlar. 3 tane tepe noktası var ve 1. tepeden iniş başlatarak yani bu durumda üç inişi, iki çıkışı olan üç bölümden oluşan bir tren yolu tasarlıyorlar. Ölçeklendirme yapmaya çalışıyorlar, her bir kare 2,63 metre olacak şekilde alıp, üç bölüm için yükseklik ve mesafe uzunlukları belirliyorlar. Örneğin 1. Bölme dokuz kare olduğundan $9 \times 2,63 = 23,67$ metre, ve yükseklik=$17 \times 2,63 = 44,71$ metre.

Çözüm yaklaşımı:

- İnişlerdeki eğrileri doğruya indirgeyerek basitleştirme ve ölçeği (başarısız olarak) kullanarak eğimin 80° olmasını sağlama.

Öğrenci çözümlerinin güçlü yönü:

- Tasarım için fikir üretme

Öğrenci çözümünün zayıf yönleri:

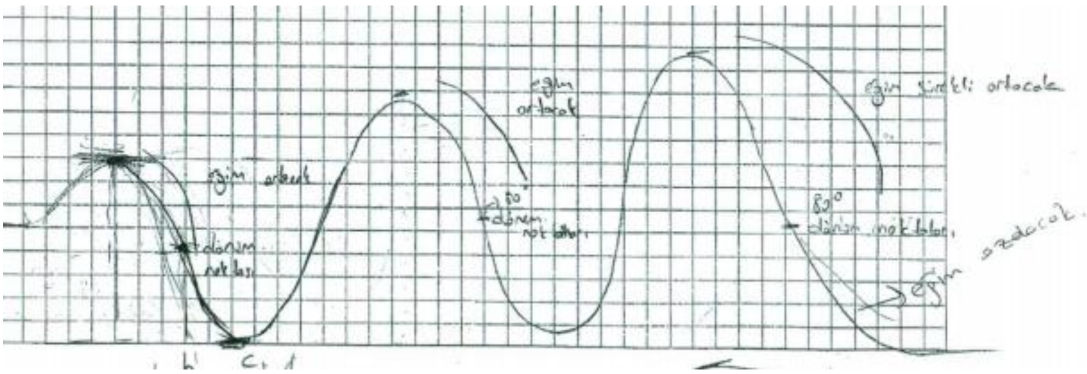
- Anlık eğim yok, ortalama eğim, doğruya indirgeme becerisi
- İnişlerdeki eğimi doğruya indirgeyerek basitleştirme; eğimin hesabını kolaylaştırıyorlar,
- Büküm noktası ile alakası yok
- Ölçeklendirme hatalı
- 2, 5 ve 8 noktalarının olduğu yerde nasıl 80° alıyorlar???

Kullandıkları matematiksel konu/kavramlar:

- Eğim (ortalama değim) (tan 80)
- Üçgen-hipotenüs uzunluğu

GRUP 3 [STG3]: Kullanılan farklı çözüm yaklaşımı / Öğrencilerin neyi anlayıp neyi anlamadıkları /kullanılan matematiksel konu ve gösterimler/ öğrenci çözümlerinin güçlü ve zayıf yönleri

Örnek Çözüm:



Çizdiğimiz eğrilerin maksimum eğim noktası 80° olmalıdır
Eğrilerin eğimlerini eğrilere çizdiğimiz teğetlerin normalle yaptığı açı olmalıdır.
Yapmak istediğimiz yüksekliğin maksimum olması için maksimum eğimin yani 80°
'nin orta noktada olması gerekir.

Eğim tepede minimum olacak daha sonra aşağıya inerken
eğim sürekli artacak, a-b aralığında eğim sürekli artarken, b-c aralığında
eğim azalmaktadır.

* Eğim orta noktada yani b noktasında maksimum eğim olacak. (Daha fazla heyecan)
dolayısıyla adrenalin en fazla orta noktamızda olacak.

* a -> yerel max c - yerel min b - ise dönüm noktası çünkü, a'dan b'ye
inerken eğim sürekli artar, b'den c'ye giderken eğim sürekli azalacaktır.
b'nin dönüm noktası olma nedeni orta noktadan sonra eğimin azalmaya başlaması.

Çözüm yaklaşımı:

- Eğrilerin büküm noktalarındaki teğetlerin eğimlerini 80 derece olarak ayarlayarak yol tasarlama

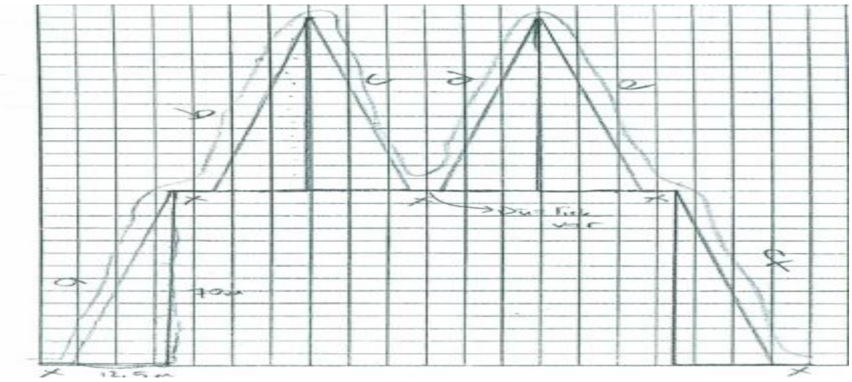
Öğrenci çözümlerinin güçlü yönleri:

- Eğriye teğetler çizmek, eğri analizi yapmak ve eğriyi doğrusallaştırmamak
- Eğimin tepede minumum ve büküm noktasında maksimum olma durumunu kullanmak

Kullandıkları matematiksel konu/kavramlar:

- Eğrinin eğimi (anlık eğim)

GRUP 4 (SG4): Kullanılan farklı çözüm yaklaşımı / Öğrencilerin neyi anlayıp neyi anlamadıkları /kullanılan matematiksel konu ve gösterimler/ öğrenci çözümlerinin güçlü ve zayıf yönleri;



x bölümü trenin boyunda düzlüklerdir. Her vagon yarım metre
up 10 vagon bulunmaktadır.

1) a ⇒ 5,67 eğimli (apsisde 12,5 metre ordinatta 70 metre) çıkış vardır.

2) x ⇒ Bu çıkışın sonunda 5 metre bir düzlük bulunmaktadır.

3) b ⇒ 5,67 eğimli bir çıkış düzlükten sonra başlamaktadır

) c ⇒ 5,67 eğimli bir iniş vardır.

5) x ⇒ 5 metrelik düzlük bulunmaktadır,

) d ⇒ 5,67 eğimli bir çıkış

) e ⇒ " " bir iniş

) x ⇒ 5 metrelik bir düzlük vardır.

) f ⇒ 5,67 eğimli bir iniş vardır. (Son 20 metre sürtünmeli)

) x ⇒ 5 metre son düzlük ⇒ sürtünmeli

Çözüm süreci:

3 tane iniş, 3 tane çıkış ve maksimum yükseklik 140 metre olarak almışlar. Trenin boyunu 5 metre kabul etmişler ve 5 metrelik bir mesafe (düzlükle) ile başlamışlar

tren yolu tasarımına (eğime geçmeden önce). 70 metre uzun kenar, 12.5 metre kısa kenar olacak şekilde; yani $\frac{70}{12,5} = 5,67$'lik maksimum eğimi sağlamak için; ayarlama yapmışlar ve çıkış yapıyorlar, sonra aynı özelliklerde tekrar bir çıkış ve bunun inişi, sonra tekrar çıkış ve başlangıç yüksekliğine getirmek için iki iniş olacak şekilde tasarlamışlar.

Çözüm yaklaşımı:

Ölçeklendirme yaparak eğimi 5, 67 (80°) olacak şekilde dik üçgenler oluşturma ve bu üçgenlerin hipotenüslerini birleştirerek yol inşa etme

Öğrenci çözümlerinin güçlü yönleri:

- Yorumları; kendi tasarımlarını yapmaları
- Ölçekli çizim yapmışlar, her kare 5 metre kabul etmişler

Öğrenci çözümünün zayıf yönleri:

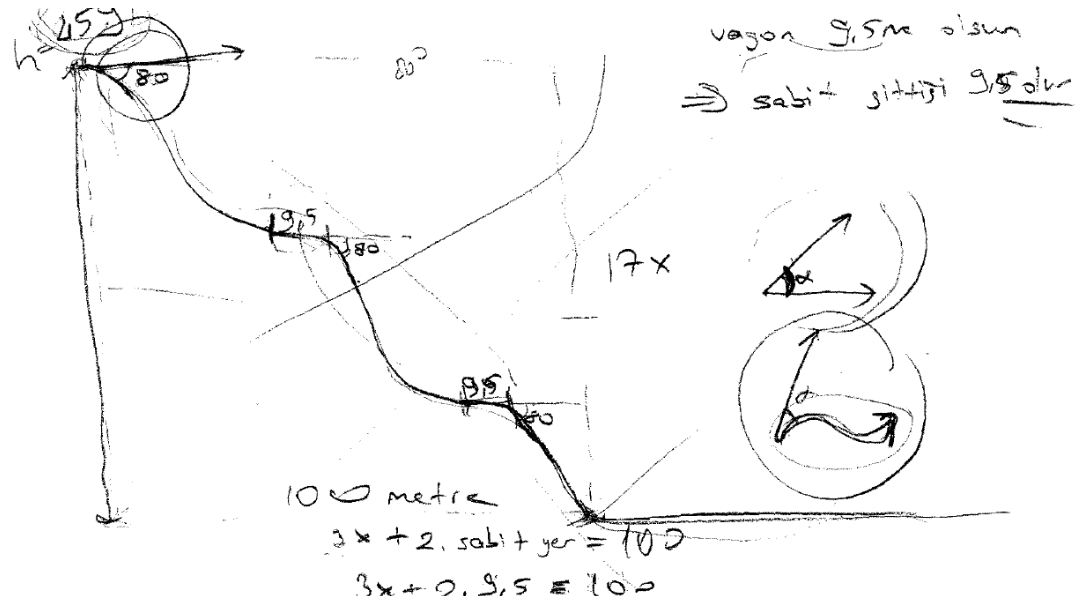
- Eğim eğrinin değil doğrunun eğimi algısı, sabit; basite indirgeme
- Eğim için mesafe yükseklik oranı kullanma, sayısal değerler verme
- Dönüm noktası fikrinin hiç yer almaması
- Gerçek hayat durumunu; özellikle tepe noktasında; göz ardı etme

Kullandıkları matematiksel konu/kavramlar:

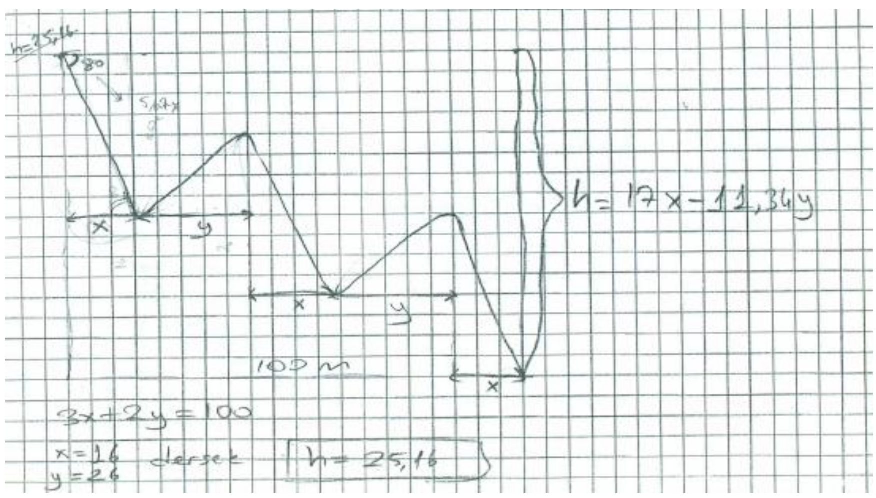
- Eğim (ortalama değim)

GRUP 5 (SG5-Odak grup video): Kullanılan farklı çözüm yaklaşımı / Öğrencilerin neyi anlayıp neyi anlamadıkları /kullanılan matematiksel konu ve gösterimler/ öğrenci çözümlerinin güçlü ve zayıf yönleri;

Örnek Çözüm 1:



Örnek Çözüm 2:



Çözüm süreci:

1. Çözüm: Önce 5, 67 değerinin tan 80 olduğunu bulma. Sonra tepelerden açı 80° derece olacak şekilde alarak sadece inişleri olan bir yol tasarlama. Vagon uzunluğunu 9,5 metre kabul etme, mesafenin 100 metre olmasından dolayı, 3 inişin yataydaki mesafesine 3x diyerek, 2 tane 9.5m trenin sabit gittiği yolu 100 metreye eşitleme ve x uzunluğunu bulma. x in değerinden ve 80° açı değerinden yararlanarak trenin başlangıç yüksekliğini 459 hesaplama.(eğim: 5,67x'e x oranından yüksekliği hesaplama). Gerçek hayat durumuna uygun olmadığından ve sadece inişlerin değil çıkışlarında bulunması gerektiğinden çözümğ değiştirme.

2. Çözüm: iniş ve çıkışlar yapma, 1.çözümün mantığı ile aynı fakat $3x+2y$=100 (yatay mesafe=100); x ve y'yi 100e eşit olacak şekilde değerler verme ve yatay mesafe uzunluklarını hesaplama ve buna bağlı olarak tren yolunun başlangıç yüksekliğini hesaplama. Dikey mesafe: $h=17x-11.34y$; yani x' mesafesine bağlı yükseklik $5,67x\times 3=17,01=17x$ ve y yatay mesafesine bağlı yükseklik $5.67y\times 2=11,34y$. Toplam yükseklik: $117x-11,34y$

Çözüm yaklaşımı:

- Dikeyde ve yatayda mesafeleri cebirsel olarak modelleme, eğrileri doğrulara indirgeyerek eğimi hesaplama; denklem kurma ve yüksekliği ve yatay mesafeyi işin içine katma

Öğrenci çözümlerinin güçlü yönleri:

- Tasarımları

Hata:

- Dikey mesafenin $117x-11,34y$ olarak alınması
- Eğimin doğrunun eğimi gibi alınması

Kullandıkları matematiksel konu/kavramlar

- Ortalama eğim

Öğrenci çözümünün zayıf yönleri:

- Eğrileri doğruya indirgeme (oldukça basite indirgeme)
- Gerçek hayat durumunu göz ardı etme
- Eğrinin eğimi, eğri analizi; dönüm noktası fikrinin hiç yer almaması

Cebirsel işlemler (birinci dereceden iki bilinmeyenli denklemler)

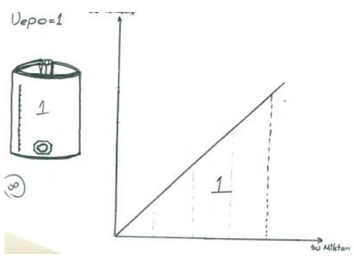
C4. Students' solutions for "Water Tank"

[Su Deposu Öğrenci Çözüm Yaklaşımları]

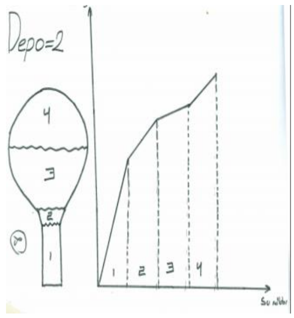
GRUP 1 [SG1] Kullanılan farklı çözüm yaklaşımı / Öğrencilerin neyi anlayıp neyi anlamadıkları /kullanılan matematiksel konu ve gösterimler/ öğrenci çözümlerinin güçlü ve zayıf yönleri.

Örnek Çözüm:

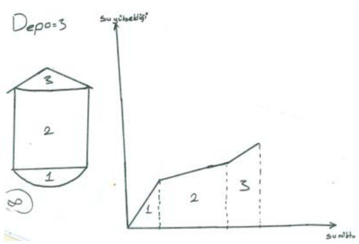
Örnek Depolar:

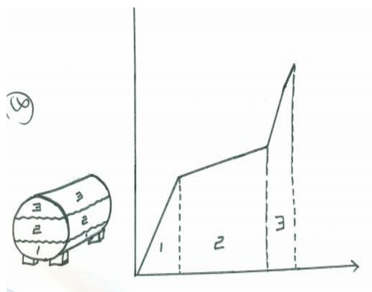


Depo-1



Depo-2





Depo-3

Depo-4

Öncelikle şekildeki depolara bakarak, kırılma noktalarını belirledik. Daha sonra hangi kırılma noktaları arasında suyun daha çok yada az olduğunu belirledik. Depolardaki dar yerleri grafikte daha eğimli şekilde gösterdik. Geniş yerlerde ise daha az eğimli gösterdik.

Grubumuzdaki herkesin fikir ve çizimlerini göz önünde bulundurarak ortak bir karar aldık.

Öğrenci çözümünün zayıf yönleri:

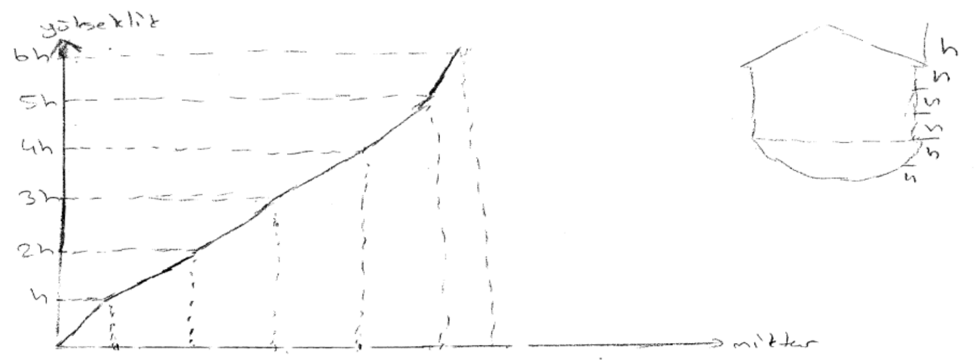
- Sadece deponun geometrik şekillerinin değiştiği geçiş noktalarına göre düşünmeleri ve tüm depo şekillerinde (doğrusal grafiği olan olmayan) grafiği doğrusal çizmeleri (ör: depo 2)
- Depo 4'ü yanlış bölmelendirmeleri ve grafiği çizimleri (ör: depo 4)

Kullandıkları matematiksel konu/kavramlar:

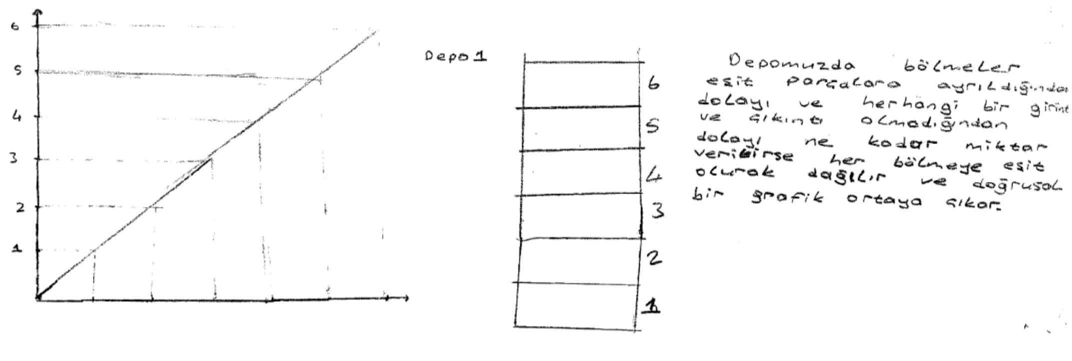
- Eğim
- Grafik yorumlama

GRUP 2 [STG2]: Kullanılan farklı çözüm yaklaşımı / Öğrencilerin neyi anlayıp neyi anlamadıkları /kullanılan matematiksel konu ve gösterimler/ öğrenci çözümlerinin güçlü ve zayıf yönleri;

Örnek Çözüm: Örnek Depolar: Depo 1 ve Depo 3



Depo-1



Depo-3

En alt küre gibi yani yarım küre olduğu için daha kısa sürede daha fazla yükseklik az miktar dolacak, sonra, sabit bir şekilde yükselecek, 6h da ise koni şeklinde olduğu için daha kısa sürede fazla h'la olacak ama miktar fazla olmayacak.

Çözüm yaklaşımı:

- Depoyu eşit yüksekliklere bölerek yani birim yükseklik fikrini kullanarak grafikleri çizme

Öğrenci çözümlerinin güçlü yönü:

- Birim yükseklilk fikri

Öğrenci çözümünün zayıf yönleri:

- Grafikleri doğrusal çizmeleri

Kullandıkları matematiksel konu/kavramlar:

- Eğim
- Birim yükseklik

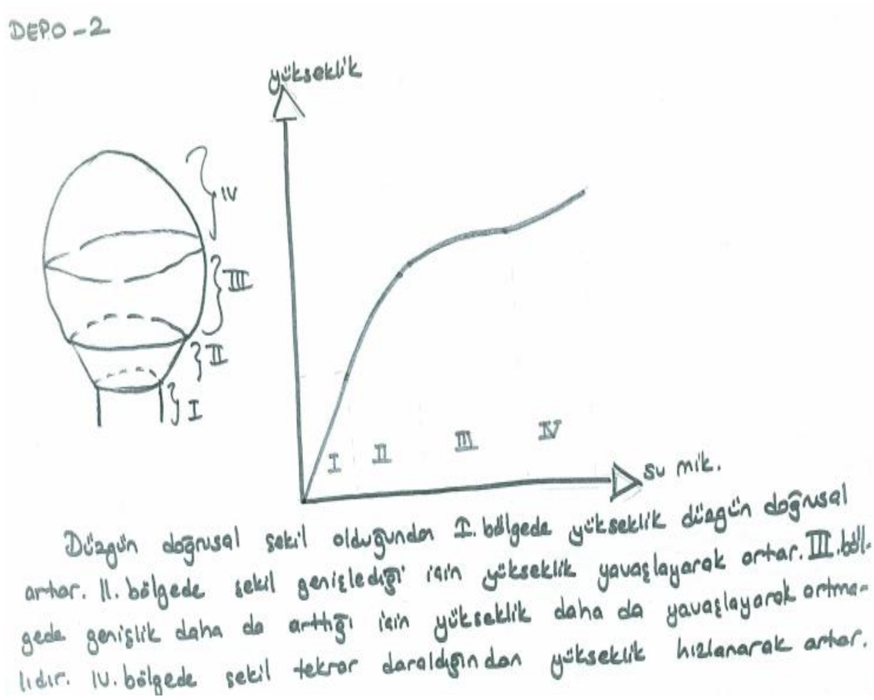
GRUP 3 [STG3]: Kullanılan farklı çözüm yaklaşımı / Öğrencilerin neyi anlayıp neyi anlamadıkları /kullanılan matematiksel konu ve gösterimler/ öğrenci çözümlerinin güçlü ve zayıf yönleri

Örnek Çözüm: Öğrencilerin grafiklerle ilgili yazdığı rapor

ilk olarak grafği çizerken şunları
dikkate aldık:
- Musluk sabit debili olduğu için birim
zamanda birim hacme düşen su miktarı
değişmediğinden grafiklerde su miktarı
ile zaman aynı birimin göstergesidir.

- Deponun şekli ne olursa olsun hacmin
tabandan itibaren arttığı bölgelerde su
yüksekliğinin artış hızı yavaşlar.
Hacmin tabandan itibaren azaldığı böl-
gelerde ise su yüksekliğinin artış hızı
artar.
Bu yönergeye dayanarak artan ve
azalan fonksiyonlarla suyun yüksekliğini
gösteren grafik çizebiliriz.

Örnek Depo



Depo-2

Öğrenci çözümlerinin güçlü yönleri:

- Artarak artan azalarak artan eğrilerini doğru şekilde çizmeleri
- Bölümler arasındaki eğim değişimini dikkate alıp grafiği kırıksız bir şekilde çizmeleri

Kullandıkları matematiksel konu/kavramlar:

- Eğim
- Grafik Yorumlama (artarak artan, azalarak artan, doğrusal artma)

GRUP 4 [ÖG4]: Kullanılan farklı çözüm yaklaşımı / Öğrencilerin neyi anlayıp neyi anlamadıkları /kullanılan matematiksel konu ve gösterimler/ öğrenci çözümlerinin güçlü ve zayıf yönleri.

Rapor:

- 1. grafik için;

Şekle baktığımızda deponun düzgün bir silindir olduğunu gördük. Bu da bize su miktarı ile yüksekliğin doğru orantılı olduğunu gösterdi. Ekle[nen] su miktarı kadar yüksekliğinin de arttığını göz önünde bulundurarak grafiği çizdik.

- 2. grafik için;

Deponun başı silindir olduğundan 1. grafikte olduğu gibi arttı. B bölümü kesik bir koni gibi olduğundan suyun yüksekliği azalarak arttı. C bölümünde ise eğim daha da arttığı için eklenen su yüksekliği daha da az arttırdı. D bölümünde şeklin daralması ile eklenen su yüksekliğin gittikçe daha da artmasını sağladı.

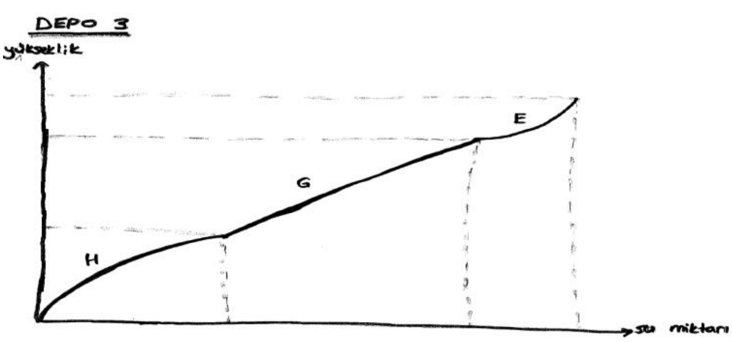
- 3. grafik için;

Şeklin tabanı yarım bir küreye benzediği için eklenen su gittikçe yüksekliğin daha da az artmasına sebep oldu. Depo 3'ün G kısmı düzgün bir şekil olduğundan grafiğin o kısmı düzgün artan oldu. E kısmı ise gittikçe daraldığı için grafikte eklenen su miktarına göre normalden daha da çok yükseklik arttı.

- 4. grafik için;

Şekle yandan baktığımızda gördüğümüz şeklin daire olduğunu anladık. Bu yüzden de önce yükseklik azalarak arttı. Dairenin ortasından itibaren yükseklik artarak arttı.

Örnek Depo: Depo-3



Öğrenci çözümlerinin güçlü yönleri:

- Değişimleri doğru bir şekilde yorumlayabilmeleri
- Ölçekli çizim yapmışlar, her kare 5 metre kabul etmişler

Öğrenci çözümünün zayıf yönleri:

- Eğimlerdeki değişimlerde keskin geçişler yapmışlar

Kullandıkları matematiksel konu/kavramlar:

- Eğim
- Grafik yorumlama
- Geometrik şekillerin özellikleri (kesik koni, küre, selindir)

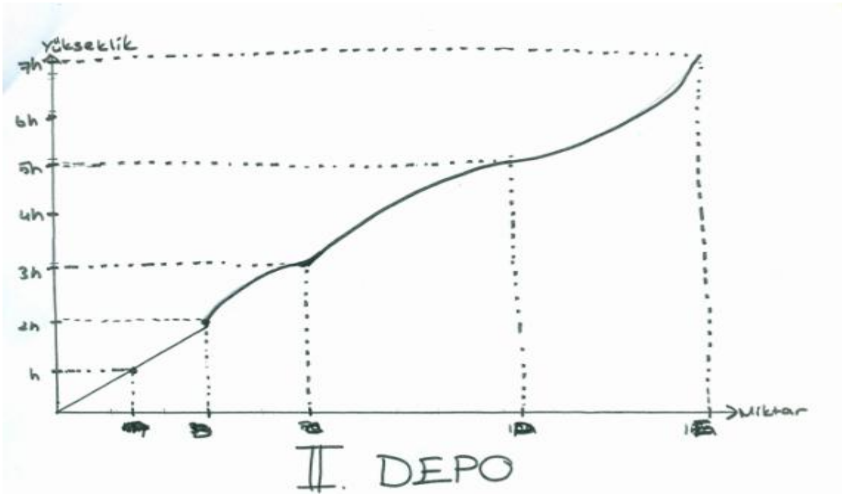
GRUP 5 [ÖG5]: Kullanılan farklı çözüm yaklaşımı / Öğrencilerin neyi anlayıp neyi anlamadıkları /kullanılan matematiksel konu ve gösterimler/ öğrenci çözümlerinin güçlü ve zayıf yönleri

Rapor:

Depo 1'de yaptığımız olan hesaplamaları sonucu düzgün silindirik yapıya sahip olan su deposunun sabit debili bir su akışı ile doldurulmasında suyun yüksekliği ile hacmi arasında bir doğru orantı olduğunu bulduk. Bunun sonucunda depo 1'e ait grafik düzgün doğrusal artan bir grafik oldu.

3 numaralı depo bir yarım küre ve bir silindir olmak üzere iki kısımdan oluşuyor. Yarım küre olan parça altta olduğu için sabit debili bir su akışı ile bu bölümün doluşunun grafiği azalarak artan grafik olacaktır. Silindir kısmın grafiği ise depo 1'inki gibi düzgün doğrusal artan bir grafik olacaktır.

Örnek Depo-2



Depo 2 klasik sıcak hava balonu şekline sahiptir Ayrıca deponun altı düzgün silindirik şekle sahiptir. Bu sebeple grafiğimizin ilk kısmı düzgün doğrusal artan bir artış gösterir. Deponun ikinci kısmı ise kesik koni şeklindedir. Bu nedenle grafiğin ikinci kısmı azalarak artmaktadır. Kesik koninin üstündeki düzgün kürenin eğiminin, koninin eğimine göre daha fazla olması sebebiyle ise azalarak artan grafikte keskin azalma görülür. Bu kısımdan sonra grafik hızlanarak artmaktadır.

Öğrenci çözümlerinin güçlü yönleri:

- Depoları doğru şekilde bölümlere ayırmaları
- Artarak artma, azalarak artma kavramlarını doğru kullanmaları ve grafikleri doğru çizmeleri

Öğrenci çözümünün zayıf yönleri:

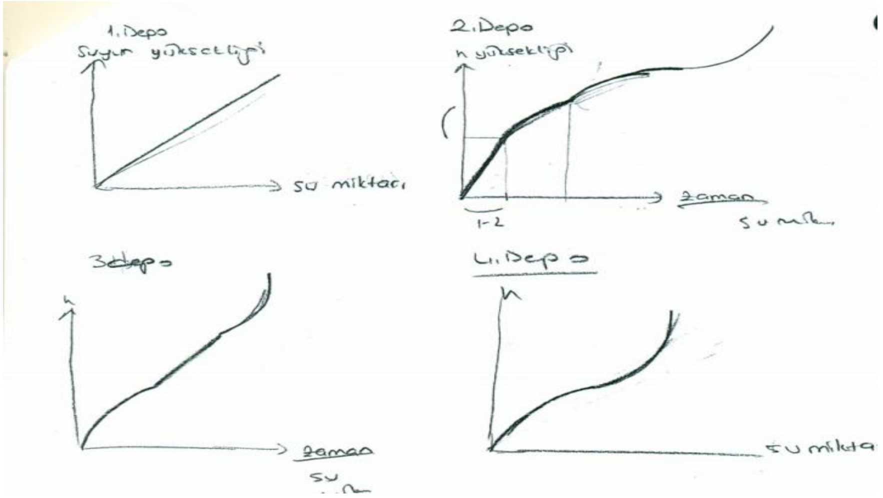
- Grafiğin karakterinin değiştiği noktalarda (kırılma noktalarını) keskin geçişler yapmışlar

Kullandıkları matematiksel konu/kavramlar

- Grafik yorumlama (artarak artma, azalarak artma); eğim

GRUP 6 [STG6]: Kullanılan farklı çözüm yaklaşımı / Öğrencilerin neyi anlayıp neyi anlamadıkları /kullanılan matematiksel konu ve gösterimler/ öğrenci çözümlerinin güçlü ve zayıf yönleri

Örnek Çözüm:



Depoları şekillerine göre nasıl grafikler çizilebileceğini düşündük

1. Depo için: Silindirik bir yapıya sahip olduğundan dolayı, su miktarına bağlı olarak su yüksekliği düzenli bir artış gösterecektir. Grafik doğrusaldır. Eğim sabittir (Grafik 1)

2. Depo): Şekli düzgün olmadığından dolayı 3 ayrı bölümde inceledik 1) Silindirik yapı 2) Kesik koni 3) Balonun alt kısmı 4) Balonun üst kısmı

1.) Silindiriktir yapılıdır. Bu yüzden su miktarına bağlı h yüksekliği sabit eğimle doğrusal bir biçimde artacaktır.

2) Şekil yukarı doğru genişlemektedir. Bu yüzden su miktarına bağlı h yüksekliği azalarak artacaktır. Bu yüzden eksene paralelleşecektir.

3) Şekil yukarı doğru genişlemektedir. 2. bölüme göre daha geniş olduğundan aynı zaman aralığında daha çok su dolar. Buradan dolayı su mik. bağlı yükseklik grafiğinin eğimi azalır.

4) Şeklin yukarı doğru genişliği azalır. Bu yüzden h yüksekliği artış hızı artar grafiğe dikkatle bir parabol çizilir.

Öğrenci çözümlerinin güçlü yönleri:

- Depoları doğru şekilde bölümlere ayırmaları
- Artarak artma, azalarak artma kavramlarını doğru kullanmaları ve grafikleri doğru çizmeleri
- Bağımlı ve bağımsız değişkeni doğru kullanmaları

Öğrenci çözümünün zayıf yönleri:

- Grafiğin karakterinin değiştiği noktalarda (kırılma noktalarını) keskin geçişler yapmışlar

Kullandıkları matematiksel konu/kavramlar:

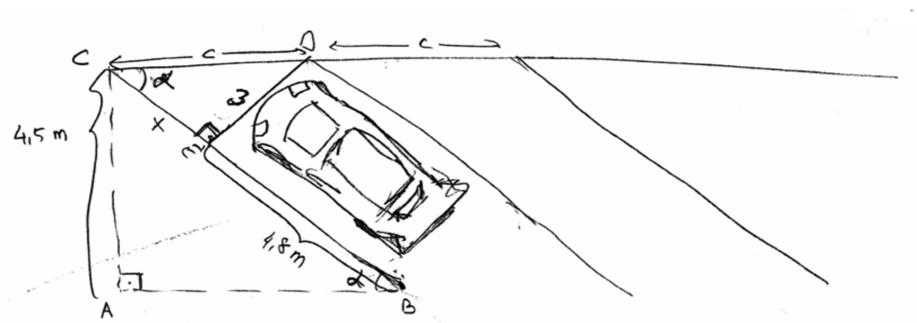
- Eğim
- Grafik yorumlama (artarak artma, azalarak artma)

APPENDIX D

PRE-SERVICE TEACHERS' SOLUTION APPROACHES

D1. Pre-service Teacher's solutions for "Street Parking"

Çözüm yaklaşımı 1: Trigonometrik Fonsiyonlar ve Denklemler



$x = c \cdot \cos\alpha$

$\overset{\triangle}{ABC}$'den $\sin\alpha = \dfrac{4,5}{c \cdot \cos\alpha + 4,8} \Rightarrow c \cdot \sin\alpha \cdot \cos\alpha + 4,8 \sin\alpha = 4,5 \quad \ldots(1)$

$\overset{\triangle}{CED}$ den $\sin\alpha = \dfrac{3}{c}$. $\cos\alpha = \dfrac{\sqrt{c^2-9}}{c}$

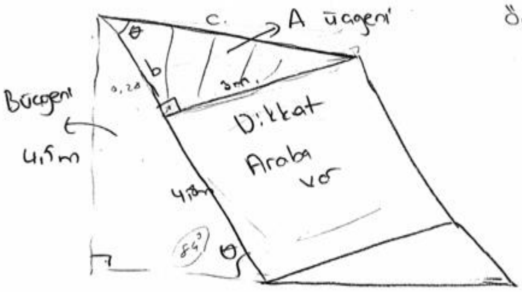
$c = \dfrac{3}{\sin\alpha} \quad \ldots(2)$

(1) denkleminde (2) denklemiyle c için bulduğumuz eşitliği yerine yazarsak,

$3 \cdot \cos\alpha + 4,8 \cdot \sin\alpha = 4,5$

Öğretmen adayları burada $\hat{ABC}$ üçgeninden ve $\hat{CED}$ yararlanarak yazdıkları $\sin\alpha$ fonksiyonlarını eşitleyerek $\sin\alpha$ ve cos α'ya bağlı elde ettikleri trigonometrik denklemi çözerek verilen kriterlere uygun maksimum park edebilmek için gerekli açı olan α açısının değerini hesaplayarak, park edilebilecek araba sayısını bulmuşlardır.

Çözüm yaklaşımı 2: Benzerlik ve İkinci Dereceden Denklemler



Öncelikle A üçgeninden pisagor yaparak $c^2 = b^2+9$ dan $b=\sqrt{c^2-9}$ buldu.

. Daha sonra A ve B üçgeninde benzerlik kullanarak (Açıları aynı)

$$\frac{c}{\frac{24}{5}+\sqrt{c^2-9}} = \frac{3}{\frac{9}{2}}$$

$$\Rightarrow \frac{c}{\frac{24}{5}+\sqrt{c^2-9}} = \frac{2}{3} \Rightarrow 3c = \frac{48}{5} + 2\sqrt{c^2-9}$$

$$\Rightarrow 3c - \frac{48}{5} = 2\sqrt{c^2-9}$$

$$\Rightarrow 15c - 48 = 10\sqrt{c^2-9}$$ (her iki tarafın karesi alınırsa)

$$\Rightarrow 225c^2 - 1440c + 2304 = 100c^2 - 900$$

$$\Rightarrow 125c^2 - 1440c + 3204 = 0$$ ikinci derece denklemidir

Δ bulunursa; $\Delta = \sqrt{b^2-4ac} = \sqrt{(1440)^2 - 4 \cdot 125 \cdot 3204}$

$$= \sqrt{2073600 - 1602000}$$

$$= \sqrt{471600} \Rightarrow \boxed{\Delta = 686,7}$$ bulunur.

$c_1, c_2 = \frac{-b \pm \sqrt{\Delta}}{2a}$ formulü kullanırsak;

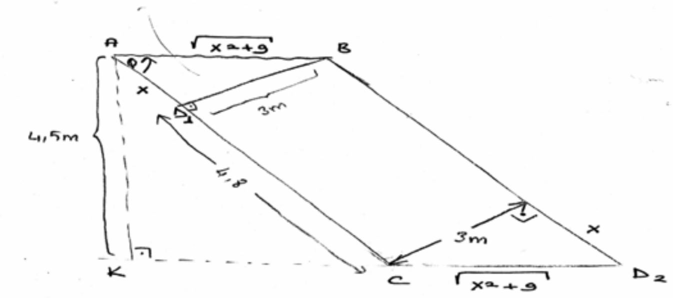
$$c_1 = \frac{1440 + 686,7}{2 \cdot 125} \quad \text{ve} \quad c_2 = \frac{1440 - 686,7}{2 \cdot 125}$$

$$c_1 = \frac{2126,7}{250} = 8,504 \quad \text{ve} \quad c_2 = \frac{753,3}{250} = \cancel{3,0132}$$ alınamaz ($c > 3,2$ den)

c_1 ve c_2 den C_1 yi almalıyız.

Öğretmen adayları bu çözüm yaklaşımında iki benzer üçgenden yararlanarak ikinci dereceden bir denklem oluşturmuşlar ve bu denklemin kökleri olan c uzunluğunu elde etmişlerdir. Bu denklemin iki kök değeri olan “c” uzunluğundan verilen kriterleri sağlayan doğru c uzunluğunu tespit ederek bu “c” uzunluğu yardımıyla parl edilebilecek maksimum araç sayısına ulaşmışlardır.

Çözüm yaklaşımı 3: Paralel Kenar Alanı



Park parkurunu paralelkenar olarak düşündük ve kendimize yüksekliği verilen değere göre 4,5 m olarak aldık. Bu durumda;

$4,5 \cdot \sqrt{x^2+9} = A(ABCD)$ ve

$\frac{3x}{2} + (4,8 \cdot 3) + \frac{3x}{2} = A(ABCD)$ olduğundan

$4,5 \cdot \sqrt{x^2+9} = 3x + (3 \cdot 4,8)$

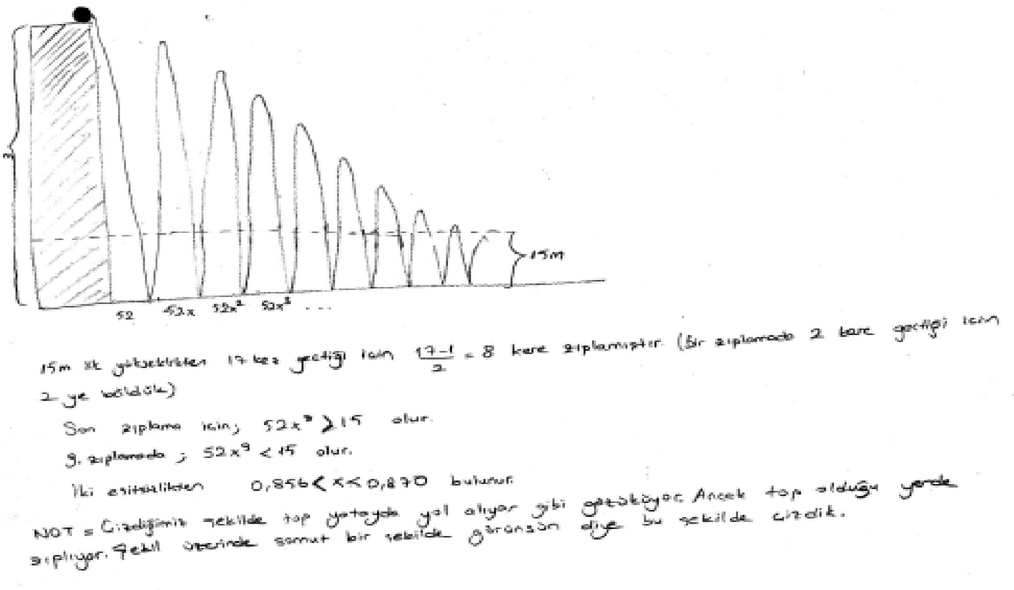
$\Rightarrow x_1 = +\frac{9}{25} \cdot \sqrt{131} - \frac{96}{25}$

$x_2 = \frac{9}{25} \cdot \sqrt{131} + \frac{96}{25} \cong 7,96$ m olarak bulunur.

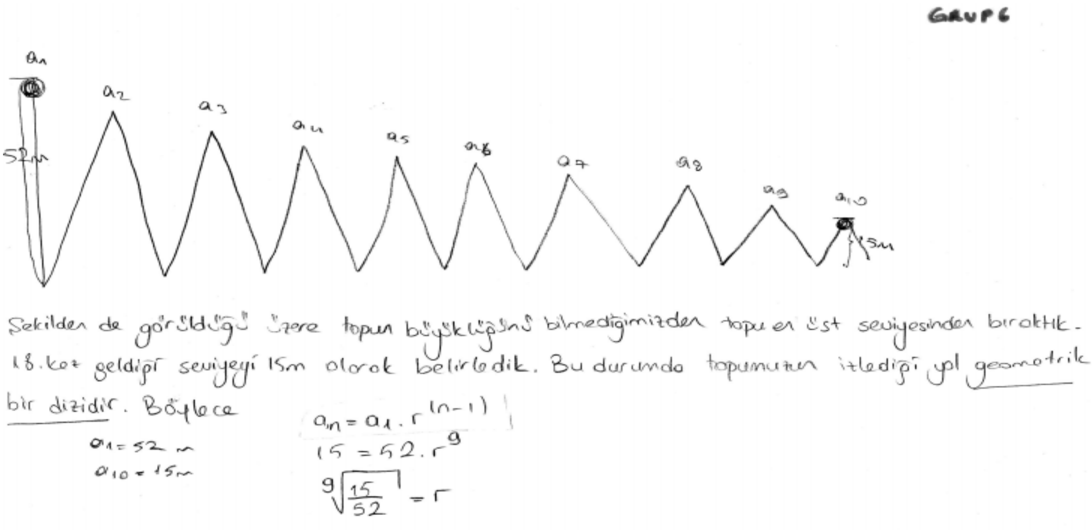
Öğretmen adayları bu çözüm yaklaşımından paralel kenarın alan formülünden yola çıkarak bir çözüm yaklaşımı sergilemişler ve verilen kriterlere uygun olacak şekilde maksimum park edilebilecek araba sayısına ulaşmışlardır.

D2. Pre-service Teacher’s solutions for “Bouncing Ball”

Çözüm Yaklaşımı 1: Eşitsizlik/Üslü Eşitsizlik

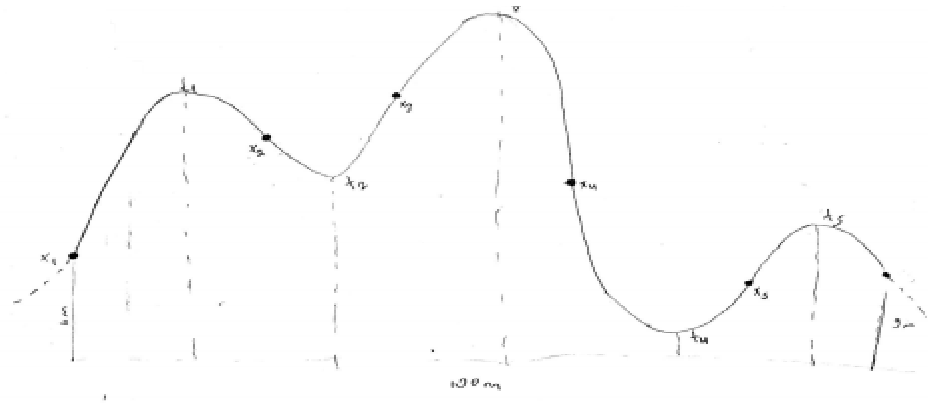


Çözüm Yaklaşımı 2: Geometrik Dizi



D3. Pre-service Teacher's solutions for "Roller Coaster"

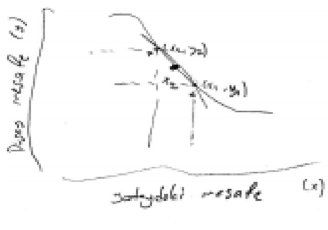
Çözüm Yaklaşımı 1:



Yukarıda ki düşündüğümüz model de x_1, x_2, x_3, x_4, x_5 noktaları bulundukları eğrinin max eğim noktalarıdır. Eğer ki bu noktaların sıfır dan küçük olduğunu garanti edebilirsek tren için istenilen güvenlik koşulunu sağlamış oluruz. Çünkü bu noktalar tren yollarının üzerine kurulacağı eğri parçalarının iç bükeylikten dış bükeyliğe geçtiği noktalardır.

Not: İç bükeylikten dış bükeyliğe geçiş noktaları eğrilerin eğimlerinin max-min ya da min-max geçişleri yaptığı noktalardır. Fakat bizim şeklimizde ki noktalar eğimin max-min geçişi yaptığı noktalardır.

Şimdi bu noktalardaki eğimin nasıl hesaplanacağını gösterelim



Şekilde ki x_2 noktasının eğimini bulmak için;

x_2'nin δ komşuluğundaki iki noktanın x ve y değerleri bulunur. Bu değerler bulunduktan sonra iki nokta arası eğim teoremi kullanılarak x_2 nin eğimi hesaplanır (ara değer teoremi)

$y_2 - y_1$ olarak bulunur

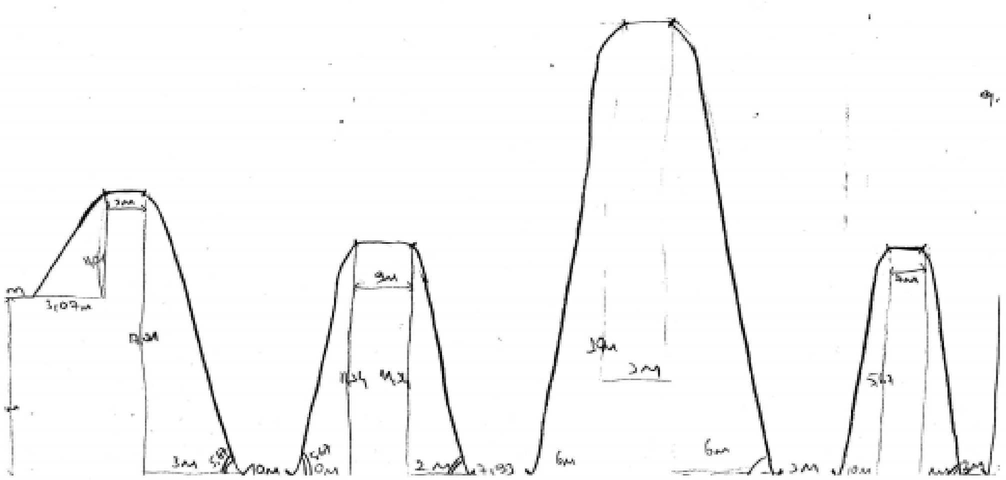
Çözüm Yaklaşımı 2:

Grup 6

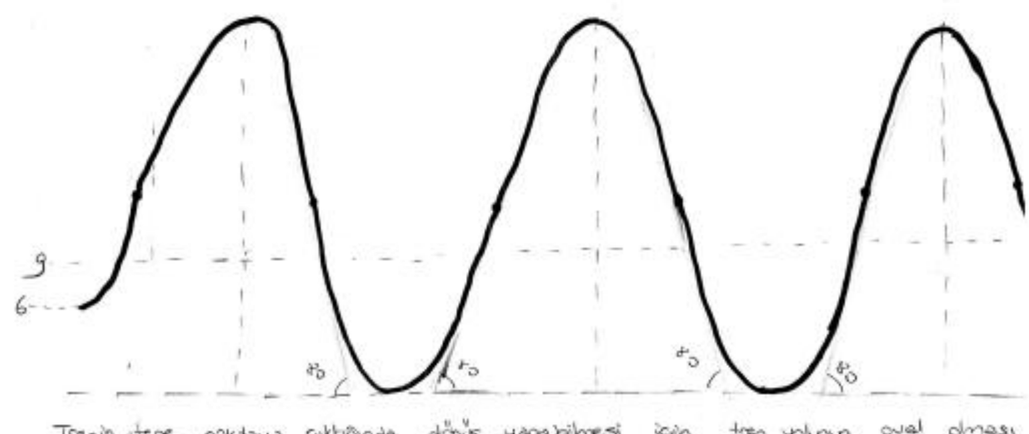
LUNAPARK TRENİ

(Sağ Çıkana Hediye)

Tanjant değerlerini dikkate alarak ve 100m'yi tamamlamaya çalışarak değerler verdik. Ani inişlerin olduğu kısımlarda eğimi maksimum değer olan 5,67'yi alarak belirledik ki heyecan faktörü en üst düzeyde olması için.



Çözüm Yaklaşımı 3:

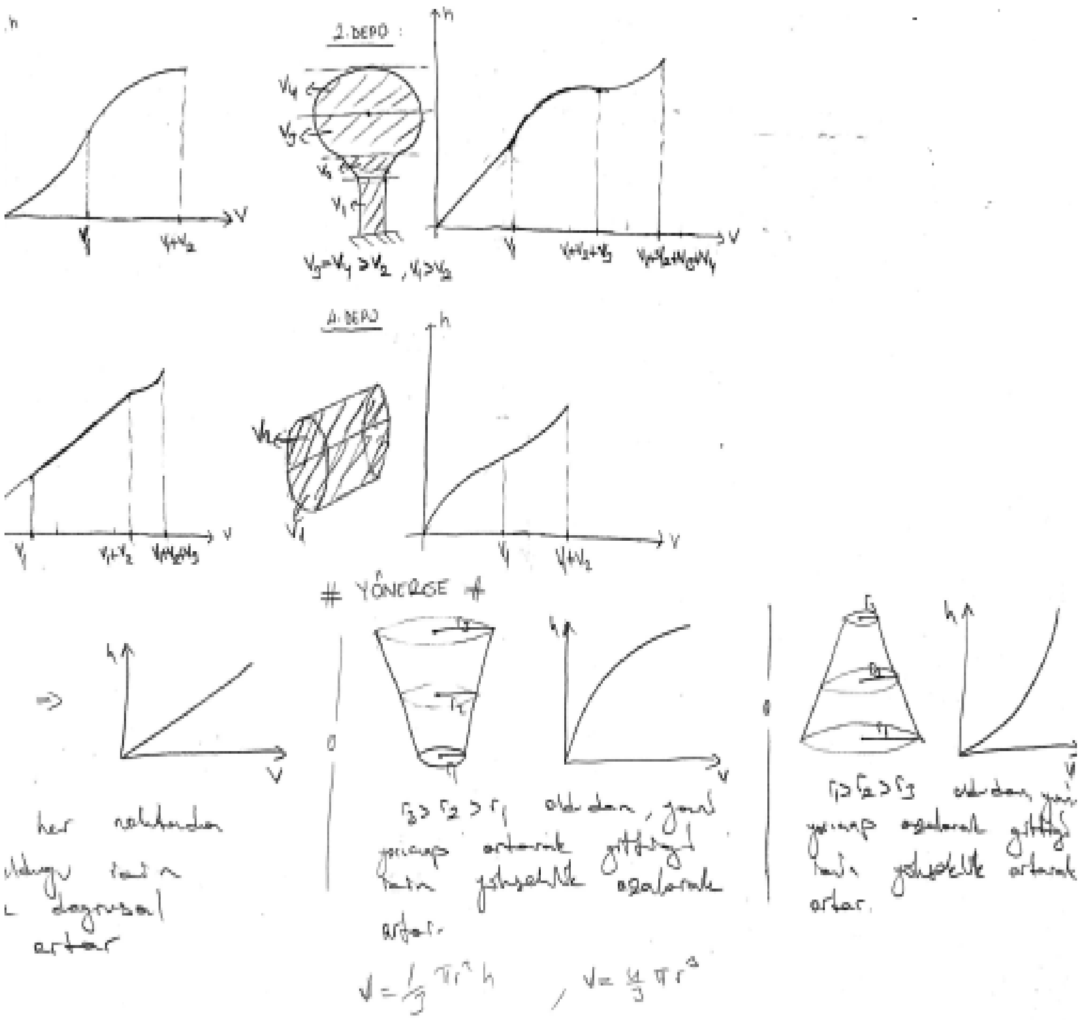


Trenin tepe noktaya çıktığında dönüş yapabilmesi için tren yolunun oval olması gerekir. Trenin inişi en uç noktadaki yolun da dönüş yapabilmek için oval olması gerekir. Yani en yüksek ve en alçak noktada eğim minimumdur hatta 0 dır.

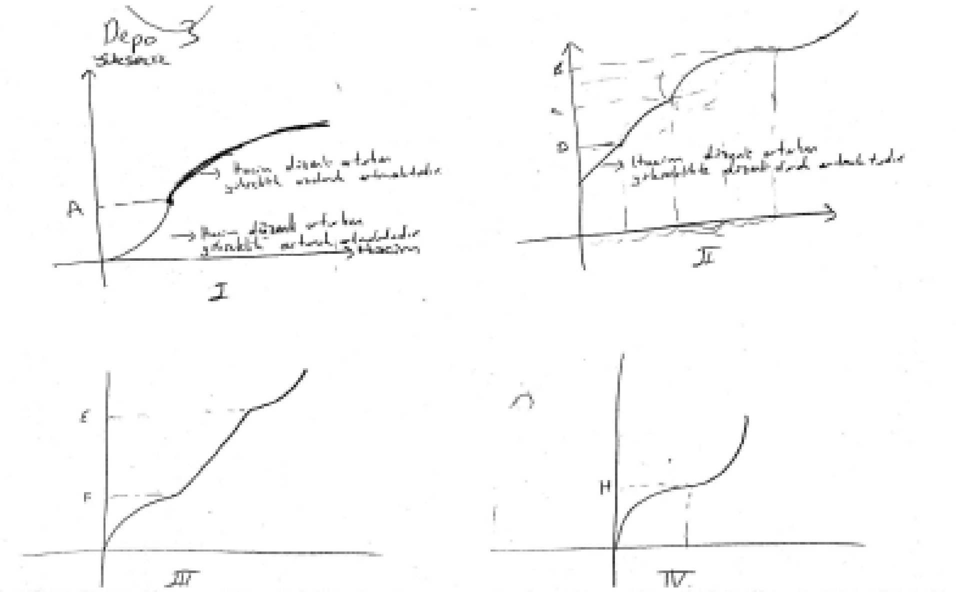
Tren 6m den yükselişe geçtiği zaman eğimi 5,67 olan noktaya kadar eğimi artarak devam eden bir yol izler. Tepe noktaya ulaşana kadar eğimi azalacak bir yol izler. Tekrar inişe geçtiğinde yine eğimi 5,67 olan noktaya kadar eğimi artan bir yol izler. 5,67'ye ulaştıktan sonra tekrar eğim azalacak şekilde bir yol izler. Bu 3. inişe kadar bu şekilde devam eder.

D4. Pre-service Teacher's solutions for "Water Tank"

Çözüm Yaklaşımı 1: Depoların Geometrik Şekillerinin Özellikleri ve Yarıçap



Çözüm Yaklaşımı 2: Kesit Alanı



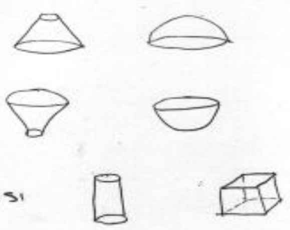
1) Depolar parçalara ayrılır. Bu ayırma işlemi aşağıdaki kriterlere göre belirlenir.

2) Eğer ele aldığımız parçadan aşağıdan yukarıya doğru kesitler aldığımızda bu kesitlerin tabana paralel bu düzlem kesiştirdiğimizde arakesitin alanı

a) Azalması

b) Artması

c) Sabit kalması



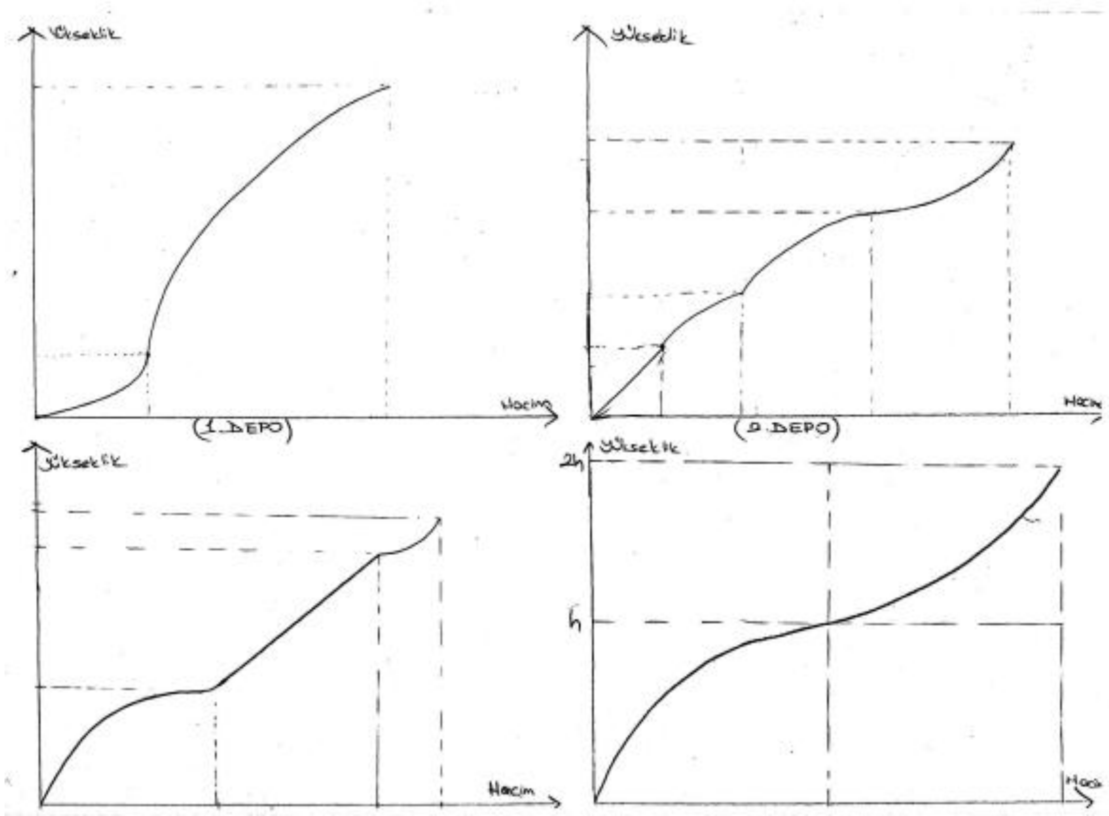
durumları göz önüne alınır.

3) Yukarıdaki durumlarda

Hacim = Taban alanı x Yükseklik

kullanılarak grafikler

Çözüm Yaklaşımı 3: Sezgisel ve Depoların bölmeleri arasında kırıklı geçiş



APPENDIX E

STUDENTS' WAYS OF THINKING PROTOCOL

ÖĞRENCİ ÇALIŞMALARINI İNCELEME PROTOKOLÜ

AMAÇ: Öğretmen adaylarının, öğrencilerin matematiksel düşünme süreçlerini ***fark etme***, anlama ve ***yorumlamalarını*** (becerilerini) arttırmak.

SÜRE: …

ROLLER:

A. Sunucu ve Yönetici [**Dersi Yürüten Öğretim Üyesi**]

- Kuralların hatırlatılması
  - Gruplar işbirliği ve fikir birliği içinde çalışması
  - Her bir bölüm için ayrılan sürenin hatırlatılması
- Sürecin yönetilmesi ve rehberlik
  - Soruların sorulması
  - Öğretmen adaylarının öğrenci kâğıtlarına verilen grup numaralarını belirterek açıklama yapmasını sağlama
  - Öğrenci kâğıtları ile videoların nasıl eşleşmiş olduğunu açıklanması.
- Öğrenci çalışmaları ile ilgili belli bilgilerin verilmesi
  - Kaçıncı sınıf, öğrenci çalışmalarının oluştuğu ortam, ortalama ne kadar zamanda çözdükleri bilgisinin verilmesi

B. Araştırmacı

- Öğrenci çözüm kâğıtlarının öğretmen adaylarına dağıtılması
- Öğrencilerin not almaları için amaç doğrultusunda hazırlanmış "öğrenci düşünmeleri şekilleri değerlendirme formu" öğretmen adaylarına dağıtılması

  (*öğrenciler bunu öğrenci kâğıtlarının analizi sürecinde doldurması)
- Video kesitlerinin hazırlanması ve sunulması

YÖNTEM:

Başlama/Giriş ve Öğrenci çalışmalarının sunulması

- Dersin öğretim üyesi kısaca kuralları açıklar ve zaman hakkında bilgi verir.
- Araştırmacı tarafından seçilmiş, 4-5 farklı gruba ait öğrenci kağıtları öğretmen adaylarına dağıtılır.

Uyarı: Öğrenci çalışmaları hakkında başlangıçta bazı bilgilerin verilmesi; sınıf seviyesi, grupça çalışmaları ve grupların kaç kişilik olduğu, öğrenci çalışmalarının oluşturulduğu ortam.

Öğrenci çözüm kâğıtları ve öğrenci sunumlarına ait videoların gösterilmesi

1. Adım:

Öğrenci çözüm kâğıtları ön analizi: Öğretmen adayları kendilerine verilen çözüm kâğıtlarını analiz etmeye başlarlar ve kendilerine dağıtılan "öğrenci düşünme şekilleri değerlendirme formunu" doldururlar. Öğretmen adaylarına öğrenci çözüm kâğıtlarının ön incelemeleri için 25-30 dk süre verilir. Öğretmen adayları ön incelemelerini tamamladıktan sonra, dersin öğretim üyesi aşağıdaki soruyu yönlendirerek 5-10 dakikalık kısa bir sınıf tartışması yapar.

- Öğrenci kâğıtlarını incelediğiniz; Neler görüyorsunuz, neleri fark ettiniz, neler söyleyebilirsiniz? Öğrenci çalışmalarından örnekler (yerler) göstererek bu söylediklerinizi destekleyebilir misiniz?

2. Adım:

Video Kesitleri: Öğrenci düşünme süreçlerini içeren çözüm kâğıtları ile ilgili video kesitleri öğretmen adaylarına izletilir. Video görüntülerinin uzunluğuna bağlı değişmekle birlikte, bu aşama için ortalama 30 dk ayrılır.

- Öğretmen adayları, "*odak gruba ait video görüntülerini*" izler ve kendilerine dağıtılan öğrenci düşünme şekilleri değerlendirme formuna (genel) notlar alır, bu süreçte notları bireysel alırlar.
- Öğretmen adayları "*sunumlara ait*" video görüntülerinini izlerler.
- Gruplar kâğıtlarla ilgili öğrenci sunumlarını dinler ve kendilerine dağıtılan öğrenci düşünme şekilleri değerlendirme formuna (genel) notlar alır.

3. Adım:

- **Öğrenci çözüm kağıtları ve video görüntüleri:** Öğretmen adayları video görüntülerinden elde ettikleri notlarla birlikte aralarında tartışarak (grupça) öğrenci kâğıtları üzerinde çalışmaya devam ederler ve verilen form üzerindeki istenenler doğrultusunda notlarını detaylandırırlar. Öğretmen adayları bu süreçte mümkün olduğu kadar öğrenci çalışmalarından bilgi toplamaya çalışırlar.Bu aşama için ortalama 60 dk süre ayrılır.

4. Adım:

Öğrenci düşünme şekilleri ile ilgili sınıf tartışması

Uyarı: Bu süreçte öğretmen adayları, aşağıdaki sorulara cevap verecek şekilde öğrenci çalışmalarını yorumlayarak öğrenci çalışmalarından bu yorumlarını destekleyecek kanıtlar sunacaklardır. Dersin öğretim üyesi öğretmen adaylarından öğrenci çalışmalarından örnekler göstererek bu söylediklerinizi desteklemerini istemelidir.

1. Öğrenciler soruyu ne kadar iyi anlamış?

2. Öğrencilerin kullandıkları farklı çözüm yolları nelerdir? (her bir öğrenci çözüm kâğıdı için ayrı ayrı cevaplanacaktır)

- Hangi matematiksel konu ve gösterimlerden yararlanmışlar?
- (Varsa) Sorunun çözümünde kullanabilecekleri hangi gerekli matematiksel bilgi/beceri/konuyu göz ardı etmişler?

3. Öğrencinin çözüm ve düşünme süreçlerinde güçlü gördüğünüz yerler nerelerdir? Öğrenciler sorunun hangi kısmında/kısımlarınsa en fazla çaba göstermişler?

4. Öğrenci çözüm ve düşünme süreçlerindeki zayıf gördüğünüz yönler/problemler nelerdir?

- Öğrenciler sorunun hangi kısmında/kısımlarınsa en az çaba göstermişler?
- Nerede zorlanmışlar/ ne tür hatalar yapmışlardır? Hangi kavramlar/matematiksel süreç onlar için zor gelmiştir?

5. Öğrencilerin çözümlerinden/düşünme süreçlerinde size ilginç gelen /şaşırtan bir yaklaşım var mı?

6. Sizin beklentilerinizden/tahminlerinizden farklı öğrenci düşünme şekilleri (çözüm yaklaşımları, hatalar, zorluklar) nelerdir?

APPENDIX F

REFLECTION PAPERS

F1. Etkinlik Sonrası Düşünce Raporu- Soru 10 [Question 10 from task-based reflection paper]

Bir öğretmen gözüyle bakmanız gerekirse; **[***Think like a teacher;]*

a. Bu problemi sınıf ortamında uygularsanız öğrencilerin hangi kazanımlara ulaşmasını beklersiniz? [*When you implement this task in the classroom setting, which learning objectives do you expect students to achieve?]*

b. Bu soruya öğrencilerin getireceği çözüm yaklaşımları neler olabilir? [*What kind of solution approach might students produce to this task?*]

c. Bu problemi sınıf ortamında nasıl uygularsınız? [*How do you implement this task in the classroom setting?*]

d. Böyle bir sınıf uygulamasında öğrenciler [In such a classroom implementation]

- Nerelerde ve ne tür zorluklar yaşayabilirler? [*What kinds of difficulties might students have?*]
- Ne tür hatalar yapmasını beklersiniz? [*What kinds of errors might students make?*]

e. Öğrencilerin yaptıkları hataları ya da yaşadıkları zorlukları aşması için neler yaparsınız? [*What do you do to solve your students' difficulties and errors they experienced in solving this task?*]

F2. Öğrenci Düşünme Şekillerini Değerlendirme Raporu

[Reflection Paper on Students' Ways of Thinking]

Öğrenci düşünme şekilleri sürecindeki tartışmalarınızı tekrar düşünerek, aşağıdaki sorulara cevap veren, mümkün olduğu kadar **detaylarıyla ve örneklendirerek** açıklayan bir rapor yazmanız beklenmektedir. Listedeki soruların hepsinin cevaplanmasına özen gösteriniz, ancak listedeki sıralamayı takip etmek zorunda değilsiniz. Bununla birlikte***, sorulara karşılık gelmeyen ekleyeceğiniz başka düşünceleriniz olursa*** kendinizi bu sorularla kısıtlamadan, etkinlikle ilgili her türlü düşünce ve eleştirilerinizi de yazabilirsiniz.

[*You are expected to write a report which should be as possible as detailed and include examples by thinking your discussions on students' ways of thinking. Please, pay attention to respond to all questions in the list; however, you must not follow the sequence in the list. In addition to that, if you have further ideas, you can add all your ideas/thoughts and comments without limiting yourself*]

1. Öğrencilerin çalışmalarını yani öğrenci çözüm kâğıtlarını ve bu çözüm kâğıtları ile ilgili video kesitlerini incelerken ve değerlendirirken çözüm kâğıtlarında ve videolarda öncelikli olarak nelere dikkat ettiniz (odaklandınız)? Açıklayınız.

[*While you were examining and assesing students' works, that is, students' solution sheets and video episodes, what did you initially notice? Please, explain.*]

2. "Etkinlik sonrası düşünce raporunuzda, öğrencilerin bu soruya getireceği farklı çözüm yaklaşımlarını, öğrencilerin ne tür zorluklar yaşayacağını ve yapabilecekleri olası hatalar ile ilgili beklenti ve tahminlerinizi" ifade etmiştiniz. Öğrenci çözüm kâğıtlarını ve video görüntülerini incelemeden önceki sizin beklentileriniz/tahminleriniz ile inceledikten sonraki gördüğünüz "öğrencilerin çözüm yaklaşımları", "sorunun çözümünde karşılaştıkları zorluklar" ve "yaptıkları hatalar" arasında farklılıklar var mıydı? Varsa, bu farklılıkları ***çözüm kâğıtlarından ve videolardan örneklerle destekleyerek açıklayınız.***

[*In your task based reflection paper, you have stated your expectations/predictions relating to students' possible solutions they would produce, difficulties and errors they would experience. Were there any differences between your predictions/expectations on students' possible solutions, errors, and difficulties before you examined the students' works and students' actual solutions, errors and difficulties you observed after you examined the students' works? If there is any,* *please explain the differences by supporting with the examples from students' worksheets and video episodes*].

3. Öğrencilerin ortaya koyduğu bu çözüm yollarından, "*bu şekilde düşüneceğini gerçekten de düşünemezdim; beni çok şaşırttı."* dediğiniz bir çözüm yaklaşımı (matematiksel düşünme süreci) var mıydı? Varsa, hangi çözüm yaklaşımı olduğunu ***nedeniyle birlikte açıklayınız*. [***Was there any solution approach (ways of thinking) among all solutions in students' works that was interesting or suprising? If there is, please explain which solution approach is and why.*]

4. İncelediğiniz tüm öğrenci çözüm kâğıtlarını ve video görüntülerini göz önüne aldığınızda, öğrencilerin matematiksel olarak nasıl düşündüğü, neler bildiği ve bilmediği hakkında neler öğrendiniz? Öğrenci çözümlerinden (kâğıtlardan ve videolardan) ***<u>örneklerle</u>*** açıklayınız? [*What did you learn about how students think, what students know and don't know when you consider all students' worksheets and video episodes? Please, explain with the examples from students' works.]*

5. Ders sürecinde incelediğiniz kâğıtları ve videoları değerlendirdiğinizde; [*When you consider the students' worksheets and video episodes which you examined in the course*;]

a. Öğrenci çözüm kâğıtları *hangi yönleri* ile sizin öğrencilerin düşünme süreçlerini anlamanıza ve yorumlamanıza yardımcı oldu? *[What aspects of students' solution papers did help you to understand and interpret students' thinking?]*

b. Öğrenci videoları hangi yönleri ile sizin öğrencilerin düşünme süreçlerini anlamanıza ve yorumlamanıza yardımcı oldu? *[What aspects of students' video episodes did help you to understand and interpret students' thinking?]*

6. Grup ortamında çalışmanızın (grup içi tartışmaların) öğrencilerin düşünme süreçleri ile öğrendiklerinize katkı sağladığını düşünüyor musunuz? ***<u>Nasıl ve Neden</u>***? [*Do you think that working in a group contributes to your understanding of students' thinking?* ***<u>How and why</u>?***]

7. Sınıf tartışmalarınızın öğrencilerin düşünme süreçleri ile öğrendiklerinize katkı sağladığını düşünüyor musunuz? ***<u>Nasıl ve Neden?</u>*** [*Do you think that whole class discussion on students' ways of thinking contributes to your understanding of student thinking?* ***<u>How and why?</u>***]

APPENDIX G

PRE-POST SELF REPORT QUESTIONNAIRE

G1. Ön-Öz Değerlendirme Anketi [Pre-Self Report Questionnaire]

Adı-soyadı: **Tarih:**

Grubu:

KİŞİSEL RAPOR- I

1) Daha önce aldığınız eğitim, matematik eğitimi veya özel öğretim yöntemi derslerinde öğrencilerin matematiksel düşünme süreçlerini içeren öğrenci çalışmalarını (örneğin; öğrenci çözümlerini içeren ödevler, yazılı kâğıtları ve ya öğrenci çözümlerini açıklayan video kesitleri) incelediniz ve tartıştınız mı? Cevabınız evet ise, süreci kısaca açıklayınız?

[*Have you ever examined and discussed on students' works (e.g., homeworks, exampapers or video episodes) including student mathematical thinking in your previous "mathematics education and method courses"? If your response is "yes", please briefly explain the process you experienced?]*

2) Bir öğretmen adayı olarak, verdiğiniz bir soruya/probleme ait öğrencileriniz matematiksel düşünme süreçlerini içeren öğrenci çalışmalarını (***örneğin***; öğrenci çözüm kâğıtlarını içeren ödevler, yazılı kâğıtları ve ya öğrenci çözümlerini açıklayan video kesitleri) inceleyecek/değerlendirecek olsanız nelere dikkat edersiniz? Hangi kriterler sizin için önemli olacak, öne çıkacaktır? ***Nedeniyle birlikte açıklayınız?*** **[***As a teacher candidate, if you examined and analyzed any students' works including students' ways of thinking, what would you attend to/focus? and Which criteria would be important for you?.Please explain it with its reasons*]

3) Bir öğretmen adayı olarak, [*as a teacher candidate;]*

a) Verdiğiniz (herhangi bir matematiksel konuya ait) bir soruda/problemde öğrencilerin ortaya koyabileceği farklı çözüm yaklaşımlarını ne derece tahmin edebileceğinizi düşünüyorsunuz? ***Açıklayınız.*** **[***What do you think about that to what extent you predict students' solution approaches they would make in a particular mathematical topic? Please, explain*]

b) Verdiğiniz herhangi bir matematiksel konuya ait bir soruda/problemde öğrencilerin ortaya koyduğu doğru veya yanlış) farklı çözüm yollarını ne derece anlayıp, yorumlayabileceğinizi düşünüyorsunuz? ***Açıklayınız.*** **[***What do you think about that to what extent would you understand and interpret students' different solutions produced in a particular mathematical topic? Please, explain*]

c) Verdiğiniz (herhangi bir matematiksel konuya ait) bir matematik problemini/sorusunu çözerken öğrencilerin karşılaşacağı matematiksel zorlukları ve yapabilecekleri hataları ne derece tahmin edebileceğinizi düşünüyorsunuz? ***Açıklayınız.*** **[***What do you think about that to what extent would you predict students' difficulties and errors they would experience in a particular mathematical topic? Please, explain*]

d) a, b ve c şıklarına verdiğiniz cevapları düşünerek, öğrencilerin matematiksel düşünme süreçlerini anlayabilmek açısından bilginizi değerlendirdiğinizde kendinizi hangi düzeyde (nasıl) görüyorsunuz? ***Açıklayınız.*** [*By considering your responses given the items a, b and c, what do you think about your own level of knowledge in terms of understanding students' thinking?]*

G2. Son- Öz Değerlendirme Anketi [Post-Self Report Questionnaire]

Adı-soyadı: **Tarih:**

Grubu:

KİŞİSEL RAPOR-II

1. Gerçek sınıf ortamından elde edilmiş öğrenci çözüm kâğıtları ve video kesitlerini incelediğiniz ve tartıştığınız bir öğrenme ortamında çalışmanın sizin öğrencilerin matematiksel düşünmelerini (düşünme şekillerini) anlamanıza,

[*Do you think that working in a learning environment, where you examined and discussed students' actual solution papers and video episodes, contributed you to understand students' ways of thinking?]*

- Katkı sağladığını düşünüyorsanız hangi açılardan katkı sağladı? Bu süreçte, neler öğrendiniz? Açıklayınız? *[If so, in what aspects did this learning environment contribute to you? Please, explain*]
- Katkı sağlamadıysa, neden katkı sağlamadığını açıklayınız? [*If not, please explain its reasons*]

2. a. Gerçek sınıf ortamından ve öğrenci çalışmalarından elde edilmiş, öğrenci çözüm kağıtları ve video kesitleri ile çalışmadan önce öğrencilerin çözüm yaklaşımlarını değerlendirirken başta dikkate aldığınız kriterler neydi? [*What was your criteria to assess students' solutions on students' solution papers and video episodes taken from real classrooms before working on them*]

b. Bu çalışma ile bu kriterlerde nasıl bir değişme oldu? Öğrencilerin çözüm yaklaşımlarını değerlendiriken hangi kriterler sizin için önemlidir? ***Nedeniyle açıklayınız?*** **[***How did your criteria change after attending to this activity? Which criteria are important to assess students' solutions?* ***Please explain the reasons*****]**

3. Gerçek sınıf ortamından ve öğrenci çalışmalarından elde edilmiş, öğrenci çözüm kâğıtları ve video kesitleri ile çalışmadan önceki öğrencilerin matematiksel düşünme süreçlerini (örneğin; kullanabilecekleri çözüm yaklaşımı, çözümlerinin zayıf yönlerini) tahminleriniz ile ders kapsamında yaptığımız bu tür etkinliklerden sonraki tahmin etme düzeyinizi karşılaştırarak açıklayınız. [*Please compare and explain your level of prediction (e.g., students' possible solution ways, errors and difficulties) before and after working on students' solution papers and video episodes taken from real classrooms*].

4. Gerçek sınıf ortamından ve öğrenci çalışmalarından elde edilmiş, öğrenci çözüm kâğıtları ve video kesitleri ile çalışmadan önceki öğrencilerin matematiksel düşünme süreçlerini anlama ve yorumlamalarınız ile ders kapsamında yaptığımız bu tür etkinliklerden sonraki anlama ve yorumlamalarınızı karşılaştırarak açıklayınız.
[*Please compare and explain your level of understanding and interpretation of students' ways of thinking before and after working on students' solution papers and video episodes taken from real classrooms*].

5. Öğrenci çözüm kâğıtları ve video kesitleri ile çalışma süreciniz boyunca öğrencilerin matematiksel düşünmelerine dair neler öğrendiğinizi düşünüyorsunuz? Öğrenci düşünme şekilleri ile ilgili yaptığınız tüm çalışmaları düşünerek örneklerle açıklayınız. [*What did you learn about students' ways of thinking while you worked on students' solution papers and video episodes?]*

6. Öğrencilerin matematiksel düşünme şekillerini tahmin etme, anlama ve yorumlama açısısıdan bireysel kazanımlarınızı etkileyen faktörler nelerdi? ***Nedenleriyle açıklayınız?*** [*What factors did impact your individual gains in terms of predicting, understanding and interpreting of students' ways of thinking? Please explain with its reasons*]

7. Verdiğimiz sorularda öğrencilerin ortaya koyduğu matematiksel düşünme süreçlerini tahmin edebilmenizde, anlayıp değerlendirebilmenizde soruyu önceden öğrenciler gibi grupça çözüp, diğer grupların çözüm yaklaşımlarını görmenizin nasıl bir etkisi oldu? [*You have first worked on non-routine tasks themselves as students and you have observed other pre-service teachers' group solutions. How did this process have an impact on your predictions, understanding and assessing students' ways of thinking?]*

8. Öğrenci kâğıtları ve video görüntüleri ile yaptığımız etkinlikleri düşündüğünüzde, öğrenci kâğıtlarının ve video kesitlerinin en önemli ve en önemsiz gördüğünüz yönleri nelerdir, değerlendiriniz? [*Please evaluate, what aspects of students' solution papers and video episodes were significant and insignificant for you?*]

APPENDIX H

INTERVIEW QUESTIONS

Birebir Görüşme Soruları

1. “Caddede Park Yeri/Zıplayan Top/ Lunapark Treni/Su Deposu” etkinliği ile ilgili öğrenci çözüm kağıtları ve bu kağıtlara ait video görüntüleri incelediniz ve tartıştınız. Yaptığınız bu çalışmayı değerlendirirseniz genel olarak ne söylersiniz? Ne düşünüyorsunuz? *[You have already examined students’ solution papers and watched video episodes belonging to those solution papers on Street Parking/ Bouncing Ball/ Roller Coaster/ Water Tank. Could you please evaluate students’ ways of thinking activity? What do you think*?]

2. Öğrenci düşünme şekillerini değerlendirme raporunuzda, öğrenci çözüm kağıtlarını/videoları izlerken **“...........”** gibi durumlara öncelikli olarak odaklandığınızı söylediniz. Peki, bu çalışmada sizin ilk olarak kâğıtlarda/videolarda bu durumların dikkatinizi çekmesi/odaklanmanızın sebebi nedir?

[*You reported that you have initially focused on the following aspects “….” in your reflection papers on students’ ways of thinking. Well, why did you focus on these aspects?]*

3. Bu etkinlikle ilgili 5-6 farklı gruba ait çözüm kâğıtları ve videoları izlediniz/incelediniz. [*You have examined the solution papers and watched video episodes belonging five/six different student groups.]*

a. Öğrencilerde bu soruya ne tür çözüm yaklaşımları getirmişler.

[*What kinds of solution approaches did students have?]*

b. Bu öğrencilerin düşünme süreçleri arasında herhangi bir ilişki görebiliyor musunuz? (ortak noktalar, benzerlikler ve farklılıklar)

[*Did you observe any relationship among students’ ways of thinking? If so, what kind of relationship did you observe, what were the similarities and differences?]*

c. Öğrencilerin ne bildikleri/ne bilmedikleri/ nasıl düşündükleri hakkında ne öğrendiniz? *[What did you learn about what students know and how students think?]*

4.

a) Senin grup arkadaşlarınla çalışmanın bu öğrendiklerine katkısı neler oldu? *[How did working in a group contribute to your learning about students' ways of thinking?]*

b) Sınıfça tartışmanızın bu öğrendiklerine katkısı neler oldu**?** *[How did class discussion contribute to your learning about students' ways of thinking?]*

c) Öğrencilerin düşünme süreçlerini anlamada kendinizin önce bu çözüm sürecinden geçmenizin nasıl bir katkısı olduğunu düşünüyorsunuz? *[How did solving the task collaboratively before working on students' works contribute to your understanding of students' ways of thinking?]*

5. Sınıf tartışmalarınız sırasında "öğretmen rolü" üzerinde fikirleriniz belirttiniz ve bu noktaya odaklandınız. [*In your class discussion, you focused on "teacher role"and offered your comments about it]*

- Bu etkinlik uygulamasında öğretmenin rolü ile ilgili neler söylersin? *[Particularly, for this activity, what can you say about teacher's role?]*

6. Böyle bir çalışma yapmak (öğrenci kâğıtları ve videoları ile ilgili) sana katkı sağladı mı? Sağladıysa hangi açılardan katkı sağladı? Örnek verebilir misin? *[Did attending such kind of activity contribute to you? If so, in what aspects? Please give examples]*.

7. Ders boyunca videolarla/öğrenci çözüm kâğıtları ile çalıştınız? Bu süreçte neler verimli/neler verimsizdi? Bu süreç nasıl daha etkili hale getirilir, geliştirilir, ne söylersiniz?

[*Throughout the course, you worked on students' solution papers and video episodes. In this process, what were productive and what were unproductive? How should this process improve?]*

APPENDIX I

THE SAMPLES FROM OBSERVATION NOTES
[Gözlem Notları Örnekleri]

ZIPLAYAN TOP [BOUNCING BALL]

30.08.2012

ZIPLAYAN TOP ALAN NOTLARI_ ÖĞRENCİ DÜŞÜNCELERİ

13.40 → ders başlangıcı

13.50 → öğrenci kağıtları dağıtıldı. grupça incelemeleri için öğrencilerden istendi. İncelemeleri için form verildi. gruplar aralarında tartışarak formu dağıttılar.

14.10 → grup çalışması videosu ilk olarak gösterildi grup 4'ün kağıdını ellerine alıp incelemeleri izlemeleri beklendi. (Bölüm 1 gösterildi)

* gruplar arası tartışmalar iyi. videoları izliyorlar

* (grup 1: [illegible] incelemeden izlemeden gelmiş.

daha sonra grup 1, grup 2 ve grup 3'ün video görüntüsünü gösterdik. öğrenciler /öğretmen adayları videoları izliyorlardı. ilgililerdi. genel olarak sorun yoktu.

APPENDIX J

NOTE TAKING SHEET OF PRE-SERVICE TEACHERS

ÖĞRENCİ ÇALIŞMALARI İNCELEME VE DEĞERLENDİRME FORMU			
Etkinlikle ilgili 5 farklı öğrenci grubuna ait çözüm kâğıtlarını incelediniz, bu çözüm kâğıtlarına ait öğrenci sunumlarını ve bir öğrenci grubuna ait video görüntülerini izlediniz. Her bir grup için aşağıdaki kriterler çerçevesinde öğrenci çözümlerini inceleyiniz ve form üzerinde ilgili kısımlara öğrenci çözümlerinden örneklerle destekleyen notlar alınız.			
GRUP NO:			
Sorunun öğrenciler tarafından anlaşılmış mı? Anlaşılmamışsa öğrenciler nereyi anlamamışlar?			
Öğrencilerin çözüm yaklaşımları		Bu çözüm yaklaşımında; öğrencilerin kullandıkları matematiksel konu ve gösterimler	
Öğrenci çözümlerinin güçlü gördüğünüz yönleri		Öğrencilerin sorunun çözümünde kullanabilecekleri fakat göz ardı ettikleri başka bilgi, beceri ve ya konular	
Öğrenci çözümlerinin zayıf gördüğünüz yönleri: öğrencilerin hataları ve zorlukları		Öğrenci düşünme süreçlerinden size ilginç gelen/ sizi şaşırtan durumlar.	

APPENDIX K

CODING BOOKLETS

No	CODE	ABBR.	DEFINITION OF THE CODE	
1	Predictions/expectations of possible solution approach	pst -PES	Pre-service teacher states his/her predictions/expectations of students' possible solution approaches before working on the students' works.	**RQ1**
2	Predictions/expectations of students' solution approach : Differences	pst -PESD	The differences between pre-service teacher's predictions/expectations on students' possible solution approaches and students' actual solution approaches [from pre-service teachers' point of view]	
3	Predictions/expectations of solution approach: Similarities	pst -PESS	The similarities between pre-service teachers' predictions/expectations on students' possible solution approaches and students' actual solution approaches [from pre-service teachers' point of view]	
4	Identification of students' solution approach	pst -IDS	Pre-service teacher's identification of students' solution approaches while examining students' works.	
5	Egocentric view on students' solution approach	pst -EVS	Pre-service teacher's predictions of students' solution approaches or mathematical ideas only from their own point of view; for example, s/he indicates that students would solve the problem like we did"etc.	
6	Finding interesting/surprising students' solutions	pst -FISS/SSS	Pre-service teacher's statement of his/her amazement at students' solution approaches.	
7	Finding interesting/surprising students' mathematical ideas	pst-FIMI/SMI	Pre-service teacher's statement of his/her amazement at students' mathematical ideas	

No	CODE	ABBR.	DEFINITION OF THE CODE	
8	To be disappointed of the solution	pst -DPS	Pre-service teacher's statement of his/her dissatisfaction with students' solution ways	
9	Predictions/expectations of students' possible errors and difficulties	pst-PESW	Pre-service teacher states his/her predictions/expectations about students' possible difficulties and errors before working on students' works.	
10	Predictions/expectations of students' possible errors and difficulties: Differences	pst-PESWD	From pre-service teachers' point of view, the differences between pre-service teacher's predictions/expectations on students' possible errors and difficulties, and students' actual errors and difficulties.	
11	Predictions/expectations of students' possible errors, difficulties or misunderstanding : Similarities	pst- PESWS	From pre-service teachers' point of view, the similarities between pre-service teacher's predictions/expectations on students' possible errors, and difficulties, and students' actual errors and difficulties.	
12	Identification of students' common errors and difficulties	pst -ISW	Pre-service teacher's identification of students' errors and difficulties while analyzing students' works.	**RQ2**
13	Egocentric view on students' solution errors and difficulties	pst-EV	Pre-service teacher's predictions of students' errors and difficulties only from their own point of view; for example, s/he indicates that "it was difficult question for us, therefore, students would have difficulty while solving it" etc.	
14	Finding interesting/surprising students' common errors or difficulties	pst-FISW/SSW	Pre-service teacher's statement their amazement at students' errors and difficulties while analyzing students' works.	
15	Appreciate students' mathematical ideas	pst-APMI	Pre-service teacher's appreciation for several mathematical ideas used by students in their solution approaches.	**RQ3**
16	Appreciate students' solution approach	pst -APS	Pre-service teacher's appreciation for students' (whole) solution approach	
17	Misinterpretation of students' mathematical ideas	pst-MSM	Pre-service teacher's misinterpretation of students' several mathematical ideas/thinking processes reflected in students' solutions; the inconsistency between what students did and what pre-service teacher interpreted.	**RQ4**
18	Misinterpretation of students' solution approach	pst-MSS	Pre-service teacher's misinterpretation of students' whole solution strategies	

No	CODE	ABBR.	DEFINITION OF THE CODE	
19	Difficulty in understanding mathematical ideas in the solution approach	pst-DUMI	While interpreting students' mathematical ideas reflected in solution approach, pre-service teacher's difficulty in understanding what students did and why.	**RQ4**
20	Difficulty in understanding whole solution approach	pst-DUSS	While interpreting students' solution approaches, pre-service teacher's difficulty in understanding what students did and why.	
21	Restatement of students' solution steps	pst-RSST	Pre-service teacher's restatement of students' thinking in a sequence (word by word).	
22	Evaluating/making judgment	pst-EJ	Pre-service teacher's evaluation of students' thinking processes by using the expressions as good, correct, incorrect etc without delving them into.	
23	Questioning students' solution(s)	pst-QSS	Pre-service teacher's curiosity about ways of students' thinking "how and students produce the mathematical ideas used in their solutions" and question them to understand and interpret better.	
24	Noticing the mathematical details of the solution strategy	pst-NMD	Pre-service teacher's recognition and interpretation of the mathematical details of students' thinking used in their solution processes.	
25	Speculating/providing alternative explanations	pst-PAE	While interpreting students' ways of thinking, pre-service teacher provides different explanation about students' ways of thinking.	
26	Pre-service teachers' focus of attention	pst-FA	Pre-service teacher's statement their aspects focused in examining students' thinking, before working on any students' works, during the investigation process and after investigation.	**RQ5**

APPENDIX L

INTER-CODING DOCUMENTS

L1. SAMPLE OF FOCUS GROUP REFLECTION PAPER DATA

EXPERT INFORMATION:

Name-Surname:

Title:

Area of Expertise:

Date:

DIRECTION:
Dear expert, you are requested to code the pre-service teacher's one of reflection papers written on ways of students' thinking through coding booklet developed for this study. If you need new codes, please add them to coding booklets.

YÖNERGE:
Sayın uzman, bu çalışma için geliştirilen kodlama kitapçığını kullanarak öğretmen adayının öğrenci düşünme şekilleri üzerine yazmış düşünce raporlarınında birini kodlamanız istenmektedir. Eğer yeni kodlara ihtiyaç duyarsanız lütfen onları kod listesine ekleyiniz.

Öğrencilerin video kesitlerini incelerken öncelikle kâğıtlara çizdikleri şekiller gözüme çarptı ki nasıl çizip şekli nasıl oluşturduklarını merak ettim. Videoda aralarında tartışırken ilk olarak nereden başladıkları (grup 5 in arabayı 4,8 e 3 m olarak yerleştirememeleri, açılardan mı gidelim, sinüs falan hesaplayacağız, alan sormuyor ki diye düşünüp tartışmaları gibi), neyi dikkate aldıklarını anlayabilmek için konuştuklarına, aralarındaki diyaloglara dikkat etmeye çalıştım. Zaten soruya ilk baktıklarında (izlediğimiz grup içi tartışmada) konuşup tartışma üzerinde yoğunlaştılar. Şekil çizmekle pek uğraşmadılar çünkü mantığını anlamamışlardı.
(4,8 neresi işte neyi kullansak diye düşünmekten mantıklı düşünmeye pek vakit ayıramadılar. Hemen sonuca şuradan gidilir buradan gidilir şunları belirleyelim, işte 4,8m, 4,5 m Pisagor kullanabiliriz diye düşünüyorlar. Aslında birinin alanlardan da gidilir demesine rağmen 4,8 m lik kısma yanlış karar vermeleri sonucu pisagordan açıya gidersek diye düşünmeleri onların düşünmekten kaçıp basit bir noktayı

gördüklerinde oradan hemen sonuca ulaşma istekleri sonucu yanlışa karar veriyorlar.) Genel olarak sunum kâğıtlarını açtığımda ilk dikkatimi çeken şekiller oldu. Çizdikleri ve yazdıkları denklemlerin mantığına odaklaşmaya çalıştım. Ki bazen zorlandım çünkü işlem hatalarını bu kadar beklemiyordum ve bazen mantıklarına anlam vermek zor oldu. Örneğin; 3.grubun 0,6 1,2 gibi değerleri üçgenin kenarlarına yazıp nereden geldiğini belirtmemeleri gibi. Videolarda grup içi tartışmalarda söylenenlere odaklaştığımda şuradan düşün diye kendimi kaptırdığım çok oldu. Burada çocukların niye böyle düşünüyorlar diye yanlışlarına dikkat etmeye çalıştım; ancak sürekli birinin atılması ve birbirlerinin de akıllarını karıştırmaları, konuşmaların, cümlelerin tam olmaması nedeniyle bir bütünlüğü onlar da tam sağlayamadı. Bende hata nedenlerini tam anlayamadım (sonradan konuştuklarını yazılı olarak inceleyince aslında kimilerinin doğru düşündüğünü ancak grup tartışmasının bazen yanlışa götürebileceğini görmek kolaydı). Düşünme aşamalarını göz ardı etsem bile sunum kâğıtlarındaki yazdıkları çözümlere daha çok odaklaştığımı ve hatalarını görebildiğimi söyleyebilirim.

Yazdığım raporda öğrencilerin ne tür zorluk yaşayacağı, yapabileceği hatalarla ilgili söylediklerim izlediğim video ve incelediğim sunum kâğıtlarıyla pek örtüşmedi. Bu sorunun lisede uygulanabilir olduğu hakkında şüpheye düştüm bu kadar farklı sonuç beklemediğimden belki de kendim gibi düşünüp onlarında görebileceğini düşünmem doğru değildi. Ama belirttiğim gibi düşünme yolları çok zor değildi ancak kavramlar tam oturmuş ve bilgiler yeterli olsa, biraz da dikkatle daha doğru çözümler çıkabilirdi diye düşünüyorum hala. Hata olarak onlarda işlem hatası ve geometrik bilgilerde böyle eksiklik beklemiyordum. Örneğin; grup 1 in sunum kâğıdındaki sin$\propto$ yı sinB ya eşitlemeleri, açıyı yanlış yere koymaları beni şaşırttı. Böyle bir hata beklemezdim. (Ki şekli oluşturduktan sonra bence çözüm bulamayınca şekilde bilinmeyen sayısı çoğalmasın diye o an öyle kabul edip sonuç için ilerlemiş de olabilirler.) Ben bilgileri bu konuda tamdır diye düşünüp denklemle uzunlukları bulurlar diye düşünüyordum ki denklemi çözemeyeceklerini tahmin ediyordum aman en azından o karmaşık denkleme kadar gelebilirler (grup 4 ün elde ettiği gibi) diye düşünüyordum. Orada hesap makinesini kullanırlar ya da hocaya sorarlar diye tahmin ettim(bizim yaptığımız gibi). İşlem hatası olarak ise grup 5 in tan$\propto$ değeriyle $\propto$ açısının uyuşmamasına şaşırdım. Raporumda tan$\propto$ yı belirlemek zor olabilir diyordum ama hesap makinesi kullanırlar diye düşünmüştüm. Sunum kağıtlarını incelediğimde çözüm yaklaşımlarımın öğrencilerinkinden çok farklı olmadığını gördüm. Dediğim gibi paralelkenarı kullanıp alanlar üzerinden yoruma giden grup 3 vardı ve bizim başta düştüğümüz hataya onlarda düşmüştü. Bu konuda karşılaşacağı zorluklar tahmin ettiğim gibiydi başta ve sonda eşit üçgenler kabul etmişlerdi. Ancak burada kenar uzunluklarını bulamama gibi bir sıkıntı beklememiştim. 0,6 m deyip nasıl buldukları belirsizdi. Karşılaşacağı zorluklar olarak 4,8 m yi anlamamalarını düşünmüyordum ki bize verilen şekille onlarınki farklıydı. Şekil onlara tam olarak verilmediğinden grup 5 açılı şekilde güvenli park kavramını değerler doğrultusunda şekle aktaramamıştı. Ben araç sayısı olarak grupların bu kadar yüksek sayı bulmalarını hiç tahmin etmezdim. Grup 1 in 53+53=106 bulması gerçekten ilginçti. Dik açıyla yerleştirildiğinde bile 3 m sınırını göz önüne aldığında 50+50=100 yapar ki onlar hiç düşünmeden işlem hatasıyla yanlış bir çözüm sunmuşlardı. Raporumda benzerliği z kuralı gibi kavramları anladılarsa diye belirtmiştim ki grup1 in bu konuda eksikliği belirgindi. Genel olarak herkesin yola paralel olarak park yapabileceklerini düşünmüştüm ki öyle oldu. Yalnızca grup 4 bunu yapamamıştı. Açılı yerleştirmede mantıklı düşünüp tek doğru denklemi ettikleri halde paralelde sonuca varamamaları ilginçti. Grupların açı değeri vererek ilerlemelerini tahmin etmiyordum ki grup 2 nin böyle ilerlemesi farklıydı ama çocuklar c değerini belirlemeyi hedef almışlardı (150m yi bölünce cevap hemen geliyor diye düşünmüş olabilirler.) Ve belirlerken diğer kenarları göz ardı etmişlerdi. Benim düşünmediğim yolu onlar düşünse bile yine dikkatsizlik ya da bilgi eksiklikleri ve şekli çizmeden orada üçgen oluşur mu düşünmeden hareket etmeleri sonucu yanlış bir sonuç bulmuşlardı. Bu yolu denerken daha dikkatli olsalar ya da başka bir yöntem bulsalar daha iyi olabilirdi.

Öğrencilerin çözüm yollarından beni en çok şaşırtan grup 1-2-3 de gördüğüm hataların üstüne (yukarıda bahsettiklerim) grup 4 ün gerçekten güzel bir şekilde paralelkenarın alanından c ile birlikte iki eşitlik kurup c yi yerine yazarak hareket etmesiydi. Denklem gayet güzel ve doğru elde edilmişti. Ayrıca hata olarak beni şaşırtan grup 1 in benzerlikteki hatası ve grup 5 in şekle güvenli kavramını oturtamamalarıydı. Ancak grup 5 in bu hatasını göz ardı edebilirim, bizde olan şeklin onlarda olmamasından kaynaklı olabilir. Ben öğrencilerin bu kadar farklı sonuçlar bulacağını hiç düşünmemiştim sonuçları incelerken nerdeyse kendimden şüphe edecektim. Bizim sonucun yanında grup 1 106, grup 2 96, grup 3 80, grup 5 82 gibi sayılar bulmuştu ki bu kadar çoğunluğun böyle farklı bulması ilginçti.

Öğrenci çözüm kağıtları, çizdikleri şekiller ve yazdıkları değerler düşünce süreçlerini yorumlamamızı sağladı. Bazı durumda şekli oluşturmuşlardı (grup3) ama değerlerin nereden geldiği belirsizdi ki değerler kuralların yanlış kullanılmasıyla belirlene de biliyordu (grup1); bazı durumlarda şekillerle denklemler doğruydu (grup4), bazı durumlarda şekli hiç kullanmadan ilerlemeyi denemişler (grup2), bazen de şekli yanlış oluşturmuşlardı (grup5). Burada videodan çok sunum kağıtları üzerinden onların düşünme yöntemlerini anlamaya çalıştım, tabi yanlışları bulmak ve fark etmek doğruyu bulmaktan daha zor olacağından ve her grubun videosu olmadığından sunum kağıtlarını incelemek dikkat istiyordu. Grup 5 in yanlış bir şekle karar verdikleri algılamak videosunu izledikten sonra kolay oldu ama zaten şekli görünce 4,8 m lik kısmı yanlış yorumladıkları fark ediliyordu. Videonun faydası onların düşünce tarzlarını fark ettirmekti çünkü aralarından bazıları doğruyu fark etseler bile bence burada grup içi tartışma onları yanlış yere yönlendirdi. Çünkü tartışırken kimi 4.8mnin öyle olmadığını söylüyor. Burada video sayesinde öğrencilerin doğruyu da ara ara yakaladıklarını fark ettim ama grup tartışması bir katkı sağlamamıştı. Bence sınıf tartışması öğrencilerin düşünme sürecine bir katkı sağlamamıştı (izlediğimiz sunum videoları için). Çünkü ortada doğru dürüst bir tartışma yoktu gruplar sunum kağıtlarını okuyup buldukları yöntemi anlattılar sadece hocalar bile fazla karışmadığından kimseye bir katkı sağladığını düşünmüyorum.

İncelediğim bu durumlardan sonra öğrencilerin gerçekten neyi bilip bilmediklerini, bilmediklerini öğrendim. Örneğin; grup 3 ün 0.6 değerini yazarken nerden geldiğini onlarda bilmiyor gibi geldi. Çünkü orada sağlama yapsalar (Pisagor teoreminden) değerler tutmuyordu. Geometri de basit diyebileceğim açıları yerleştirip benzerlik kurma kavramlarında sıkıntı yaşanabileceğini anladım.(grup1) öğrencilerin işlem hatasında bulunabileceklerini, soruya dikkat etmeden hareket edip üçgen kurallarını göz ardı edebileceklerini(grup2 açılara değer verirken), soruyu dikkatsizce okuyup şekil veremeyecekleri durumlar olabileceğini (grup5), bazen mantıklı düşünmeyi unutabileceklerini anladım. Bizde öğrenci olduk ama o anda böyle durumları yapmazdım diyemem. Çünkü hatalarla gelişir insan illa ki böyle hatalar olabilir, ben henüz öğrencilerin nasıl cevaplar vereceği hakkında birçok şeyi tahmin edemeyeceğimi düşünüyorum. Bun öğretmenliğe başladıktan sonra daha iyi görüp uygulamalı olarak öğreneceğimi ve bolca şaşıracağımı fark ettim. Bu etkinliklerin bizi gelecekteki şaşırmalarımız önlemek ve öğrencilere daha verimli olup onları anlayabilmek, büyük tepkiler vermemek açısından iyi olacaktır diye düşünüyorum

L2. SAMPLE OF FOCUS GROUP INTERVIEW DATA FOR INTERCODING

EXPERT INFORMATION:

Name-Surname:

Title:

Area of Expertise:

Date:

DIRECTION:
Dear expert, you are requested to code the document of focus group discussion data coding booklet developed for this study. If you need new codes, please add them to coding booklets.

YÖNERGE:
Sayın uzman, bu çalışma için geliştirilen kodlama kitapçığını kullanarak odak grup görüşme datası dökumanını kodlamanız istenmektedir. Eğer yeni kodlara ihtiyaç duyarsanız lütfen onları kod listesine ekleyiniz.

Instructor (I): Başka ekleyecek olan yoksa 2. grupa geçeceğiz? İkinci grupla ilgili neler söylersiniz? İkinci grubun sunumundaki çözüme baktığımız zaman farklı bir durum var çözümün detaylarına baktığımız zaman farklı bir durum var.

PST7: hocam arkada şu parabolü çizmiş olmaları yani şu şekilde çok değişik geldi bana gerçekten hani…ama bunu ne düşünerek nasıl yaptılarda buradan 45/52 oluşturdular onu bilemiyoruz tabi. Sadece onu o noktaları belirleyerek parabol oluştuğunu ve ne yapmaya çalıştıklarını görüyoruz..

I: Parabolün denklemini mi yazmaya çalışmışlar..

PST7: evet

PST24: iki farklı parabolün denklemi yazıp birbirine eşitlemeye yoluna gitmişler heralde. $a\times(x-17)$ filan.

I: oradan bir şey çıkarmışlar mı sonra ne yapmışlar..

PST7: yani hocam anlamadım ondan sonrasını ama bilmiyorum ama zannetmiyorum oradan çıkartacaklarını..

PST7: $\frac{45}{52}$ gelmiş bir yerde sonra onları çarpa çarpa şey yapmışlar (kağıda bakıyor inceliyor)

I: Deneme yanılma filan yapıldığını söylüyorlar en başında..

PST24: burada deneme yanılma yapılsa bir şekilde işlemler olur $\frac{45}{46}$ yı bulduk diyorlar..Biz aradık şurada adam orta ortaya $\frac{45}{52}$ mi neyse artık...

I: Peki en son sayfaya bakalım!!

PST24: Ya hocam orada bir şeyler denemişler ama $\frac{12}{13}$alalım demişler ona göre bir şeyler yapmışlar, olmamış..Sonra başka bir oran almışlar..Hani..ııı...bir yerde bir şey duymuşlar…

I: orada 3 tane kolon var ne yapmaya çalışmışlar orada.. siz anladınız mı o 3 kolon var ne yapmaya çalışmışlar. Hangi gruplar tam olarak çözdü, anladı?

PST24: Hocam burada aldıkları yükseklikleri bulmuşlar... burada işte 1.ci 52; katsayısı $\frac{12}{13}$ olsun.. $\frac{12}{13}$ ile 52 yi çarpıyor 52 den çıkartıyor sonucu işte 45,94 buluyor bunu tekrar $\frac{12}{13}$ile çarpıp bir sonuç buluyor..onu çıkartıp öyle bir uzunluk bulup 15 e kadar eşitleme yoluna gidiyor hani teker teker deniyorlar yani 17.değere bakıyorlar burada, burada onu deniyorlar..

PST3: ama neden 46…mesela ilk grup hani aritmetik ortalama alarak ne tarafın kaldığına filan bakmıştı ve $\frac{15}{52}$ye bir şekilde ulaşmıştı… ama.. Bunlar hani ya buldular bir yerden artık altını mı üstünü mü aldılar!

I: ama $\frac{12}{13}$’ü denemişler

PST3: evet

PST24: ama $\frac{12}{13}$’ü nereden buldular..

PST24: biz bunu denedik bulduk derken deneyerek bir şey buldukları yok.

I: 46 yı da denemişler, 44 ü de denemişler!

PST24: Sonrada ikisini denemişler ortasını alalım demişler yani ama bence öyle değil bence…

PST2: 45 i alıyor ama niye 45’i aldık onun deneme yanılması yok burada..Direkt 45’i kabul etmişler

PST3: Niye akıllarına $\frac{45}{52}$gelmiş meselea!!

I: Öyle mi gelmiş sizce?

PST24: Biz o şekilde bulduk hani başka arkadaşlar $\frac{45}{52}$bulduklarına dair bir şey buldularsa..

PST3: Niye $\frac{35}{52}$dememiş niye $\frac{45}{52}$değil..

I: Tamam soralım..Siz ne düşündünüz duygu?bunların çözümü hakkında

PST12: Yani hani en başta deneme yanılma yoluyla gitmişler ama..

I: Neyi denemişler?

PST12: İşte sayıları mesela filan...o oranı kafalarına göre gitmişler herkes 45 üzerinden gidiyor..Ama bunu nereden bulmuşlar bütün gruplar öyle gidiyor acaba hocaları mı bir fikir verdi? Direkt 45 den..

I: Efkan sen ne düşünüyorsun..

PST19: Bencede hocam direkt

I: Direkt 45 e mi bakmışlar

PST19: Yok..Bence denemişler..

I: Neyi denemişler hangi sayıları denemişler..

PST19: 8, 12, 13...Bir gruptan büyük bir ihtimalle duymuşlar hocam bencede..

I: 48 var, 46 var 44 var... Böyle ikişer ikişer inmiş olabilirler mi?

PST23: benim dediğim belkide buradaydı..44 ile 46 nın arasını almışlar..

I: evet..Peki neden 44 ile 46 nın arasını almış olabilirler?

PST24: 44 de 15 den büyük gelmiş; 46 da 15 den büyük gelmiştir... 44'de 15 den küçük gelmiştir. Aradaki mesafeden bu büyük geldi bu küçük geldiğine göre bu direkt 45 dir diye fikir yürütmüş olabilirler.

PST3: Yani 8. Zıplamada 13 oluyor 44 de

I: 44 de 8. Zıplamada 13 de

PST3: yani 15 in altında kalıyor normalde üstünde kalması gerekiyor

PST13: hocam 45,94 ile başlamışlar bir tanesinde 45,94 ü almışlar 16,35'e çıkmış orada 8.cisi 15 in üstünde çıkmış; 9.cu su 15 in altında çıkmış.. Aslında 45,94 doğru bir değer olmuş..Bunu yaptıklarına göre...yani o değer sadece $\frac{45}{52}$yi alıyor ya...Aslında 45,94 de sağlıyor...45 sağladığı gibi..Yani aralık almayı düşünebilirlerdi o arada..Yani o da doğru çıkmış o arada..45,94 ü hesaplamışlar oda doğru çıkmış..

I: 45.94ü hesaplamışlar oda doğru çıkmış, ama çözümde almamışlar yani..Evet

PST13: Çözümde aralık olarak almamışlar direkt 45 i almışlar..

PST20: hocam mesela *(2,4,6,8 sayıyor*) 8 tane rakam bulup 8. Rakamın 15 den büyük olup olmadığını, 13.çıkmış sonuncusu...

I: bir dakika..44 ve 37 yi nasıl bulmuşlar sence

PST20: Onu kafadan bulmuşlar...

I: Sizde öyle mi anladınız…orada 44 ve 37 yi nasıl bulmuşlar..yani rastgelemi yazmışlar...

PST20: her seferinde azalma miktarı daha yüksek düştüğü için miktarda düşecek ya o yüzden öyle bir şey yapmış olabilirler...

I: Diğerlerinde virgül var bunda yok o yüzden mi ratgele seçtiklerini düşünüyorsun..Mehmet ali sen ne dersin?

PST16: hocam bu oranın azalarak azalması gerekiyor ya..Acaba 15 i 8 kerede nasıl ulaşabiliriz diye 8 çıkarmış, 7 çıkarmış, 6, 5, 4, 3... Öyle 15 i acaba 8 kere kullanabilecek miyiz diye (anlaşılmıyor ses kaydına bakkk) 16'ı 15e yakın demekki bizim oranımız

I: onda öyle yaptı peki ötekinde nasıl yaptı 46 da yandaki kolonda hemen...

PST13: hocam...44e de yaklaşık oranlar aynı çıkıyor zaten $\frac{37}{44}$, 0, 84 çıkıyor...Yani Yaklaşık olarak oran aynı çıkıyor aslında..

I: oran aynı çıkıyor..o zaman öyle bir şey yapmış olabilir mi?

PST9: o ilk denemeleri olabilir... acaba çünkü 8, 7, 6, 5, 4 diye düşünmeye kalkmışlar

I: Selin başka bir noktaya dikkatimizi çekmeye çalışıyor; bir daha söyle

PST13: $\frac{37}{44}$; $\frac{31}{37}$; $\frac{26}{37}$ yani yaklaşık olarak 0,84 çıkıyor oran aynı çıkıyor hani sabit çıkması gerekiyor ya zaten.

I: neyi neye bölüyorsun?

PST13: ya hani şey ya...bir şeyin belli bir oranı kadar tekrar zıplıyorlar ya..Sonrakini baştakine bölerek...

I: Sonrakini baştakine...37 yi 44 e bölüyorsun… Ve 31'i 37 ye bölüyorsun.. ve bunlarda sabit oran çıkıyor..

PST13: neredeyse yani sabit aynı oran 0.84 gibi bir şey çıkıyor.

I: öyle bir şey yapmışlar olabilirler mi o zaman?

PST20: hocam baştan mesela oran $\frac{12}{13}$ olsaymış..Oran $\frac{12}{13}$ olsaymış

APPENDIX M

COURSE SCHEDULE WITH INTEGRATED RESEARCH PROCESS

Weeks-Date	Course Activities	Course Assignments	Research Process
1-10.02. 2012	No Course (due to inadequate participants)		
2-17.02.2012	Introduction Information about research project, objectives of course, context and processes. Implementation of mathematical modeling test-I		
3-24.02.2012	Introduction of MS Excel program and Graphing Calculator Implementation of Modeling Task "Summer Job"	Task-based Ref. Paper	
4-02.03.2012	Implementation of Modeling Task "Ferris Wheel"	Task-based Ref. Paper	Implementation of self report questionnaire-1
5-09.03.2012	Implementation of Modeling Task "Street Parking"	Task-based Ref. Paper	**Week1:**
6-16.03.2012	Students' Ways of Thinking Activity 1: Analyzing Students' Works for "Street Parking"	Student ways of thinking reflection paper	Week2: Completing the note taking sheets & making class discussions
7-23.03.2012	Implementation of Modeling Task "Bouncing Ball" Discussion on the nature of modeling tasks A presentation about (by instructor) the nature of modeling tasks.	Task-based Ref. Paper	Week3

8-30.03.2012	Students' Ways of Thinking Activity 2: Analyzing Students' Works for "Bouncing Ball"	Student ways of thinking reflection paper	Week4 Completing the note taking sheets &making class discussions
9-06.04.2012	Implementation of Modeling Task "Roller Coaster"	Task-based Ref. Paper	Week5
10-13.04.2012	No Course (Mid-Terms Week)	….	…
11-20.04.2012	Students' Ways of Thinking Activity 3: Analyzing Students' Works for "Roller Coaster"	Student ways of thinking reflection paper	Week6 Completing the note taking sheets & making class discussions
12-27.04.2012	Implementation of Modeling Task "Water Tank"	Task-based Ref. Paper	Week7
13-04.05.2012	Students' Ways of Thinking Activity 4: Analyzing Students' Works for "Water Tank". Developing an Implementation Plan for Modeling Tasks	Student ways of thinking reflection paper	**Week8** Completing the note taking sheets & making class discussions
14-11.05.2012	General evaluation of the course context Implementation of mathematical modeling test-I	Developing authentic modeling tasks	Implementa tion of self report questionnai re-2
15-18.05.2012	Micro Teaching Activity: Implementation of modeling task		
16-25.05.2012	Micro Teaching Activity: Implementation of modeling task.		

APPENDIX N

CONSENT FORM
[GÖNÜLLÜ KATILIM FORMU]

Bu ders, Doç. Dr. Ayhan Kürşat Erbaş tarafından yürütülen "**Ortaöğretim Matematik Eğitiminde Matematiksel Modelleme: Hizmet İçi ve Hizmet Öncesi Öğretmen Eğitimi**" projesi kapsamında içeriği oluşturulmuş matematiksel modelleme konusunda hizmet öncesi öğretmen eğitimini amaçlamaktadır. Matematik öğretmen adaylarının matematik öğretiminde matematiksel modelleme kullanımı ile ilgili bilgi, beceri ve tutumlarını ortaya çıkarma ve bunlardaki gelişimi ve değişimi tasarlanan hizmet öncesi eğitim programları aracılığıyla inceleme proje çalışmasının konularını oluşturmaktadır. Bu amaçlar için tasarlanan ders kapsamında 14 hafta sürmesi planlanan çalışma süresince (i) modelleme testi, (ii) anket, (iii) modelleme etkinlikleri için grup çalışma raporları, (vi) öğrenci düşünme şekilleri kişisel değerlendirme raporları, (v) öğrenci düşünme şekilleri değerlendirme raporları, (vi) ses kayıt ve video kayıt cihazlarıyla desteklenmiş gözlemler, (vii) görüşmeler, (viii) etkinlik sonrası düşünce raporları, (ix) gruplarca hazırlanan modelleme soruları ve bu soruların uygulama planları, (x) öğretmen adaylarının sunumları (mikro-öğretim) temel veri kaynakları olacaktır. Bu kapsamda öğretmen adaylarının öğrencilerin matematiksel düşünme şekillerini incelemeleri ile ilgili toplanacak veriler Araş. Gör. M. Gözde Didiş' in doktora tez çalışmasında kullanılacaktır.

Çalışma süresince toplanacak veriler tamamıyla gizli tutulacak ve sadece araştırmacılar tarafından değerlendirilecektir. Elde edilecek bulgular tez çalışmasında ve bilimsel yayımlarda kullanılacaktır. Çalışmaya katılım tamamıyla gönüllülük temelindedir. Çalışma süresince katılımcılar için potansiyel bir risk öngörülmemektedir. Ancak, katılım sırasında farklı amaçlarla toplanan veya alınan dersin gerekleri olarak toplanacak verilerin bilimsel çalışma ve tez çalışması amaçları çerçevesinde kullanılmamasını isteyebilirsiniz. Bu durum ders performansınızın değerlendirilmesinde kesinlikle negatif bir durum oluşturmayacaktır.

Çalışma hakkında daha fazla bilgi almak için ODTÜ Eğitim Fakültesi Ortaöğretim Fen ve Matematik Alanları Eğitimi Bölümü öğretim üyeleri Doç. Dr. Ayhan Kürşat Erbaş (kursat@gmail.com), Y. Doç. Dr. Bülent ÇETİNKAYA (Tel: 210 3651; e-posta: bcetinka@metu.edu.tr) ve doktora öğrencisi Makbule Gözde Didiş (e-posta: mgozde@metu.edu.tr) ile iletişim kurabilirsiniz. Bu çalışmaya katıldığınız için şimdiden teşekkür ederiz.

Bu derste kullanılacak olan görsel ve yazılı materyalleri ders dışında izinsiz olarak kullanmayacağım ve yaygınlaştırmayacağım. Bu çalışmaya tamamen gönüllü olarak katılıyorum ve istediğim zaman yarıda kesip çıkabileceğimi biliyorum. Verdiğim bilgilerin bilimsel amaçlı yayımlarda kullanılmasını kabul ediyorum. (Formu doldurup imzaladıktan sonra uygulayıcıya geri veriniz).

İsim, Soyad	Tarih	İmza	Alınan Ders
	----/----/-----		

CURRICULUM VITAE

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- B.S. Mathematics Education, Hacettepe University, Ankara, Turkey, June 2007
- M.S. (without thesis) Mathematics Education, Hacettepe University, Ankara, Turkey, June, 2007
- Ph.D Mathematics Education, Middle East Technical University, Ankara, Turkey, February, 2014

Experience:

- 2007-2014 *Research Assistant* at Secondary Science and Mathematics Education Department, Middle East Technical University, Ankara, Turkey.
- 2013 *Visiting Researcher* at Mathematics Education Centre (MEC), Loughborough University, Loughborough, UK.

Research Project

- TÜBİTAK, (2010-2013). Mathematical Modeling in Secondary Mathematics Education: Preservice and Inservice Teacher Education. Supported by the Turkish Scientific and Technical Research Institution, grant number 110K250 [as research student].

Research Interest:

- Teacher Education
- Pedagogical Content Knowledge
- Mathematical Modeling

Foreign Languages

English and German