PRE-SERVICE MIDDLE SCHOOL MATHEMATICS TEACHERS' UNDERSTANDING OF QUADRILATERALS THROUGH THE DEFINITIONS AND THEIR RELATIONSHIPS

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I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

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ABSTRACT

PRE-SERVICE MIDDLE SCHOOL MATHEMATICS TEACHERS' UNDERSTANDING OF QUADRILATERALS THROUGH THE DEFINITIONS AND THEIR RELATIONSHIPS

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The purpose of this study was to examine pre-service mathematics teachers' cognitive progress in constructing and evaluating quadrilateral definitions and the corresponding quadrilateral hierarchies under the support of the Geometer's Sketchpad learning activities. The study employed qualitative case study design of which data was collected from 5 pre-service middle school mathematics teachers during the spring semester of 2010-2011. The main data came from one-to-one clinical interview sessions assisted with Geometer's Sketchpad learning environment; and all sessions were hold in the Human-Computer Interaction Laboratory.

The findings revealed that the Geometer's Sketchpad supported learning tool developed in the light of related literature was effective to large extent in improving the pre-service teachers' thinking process of identifying the critical defining properties of quadrilaterals, evaluating the mathematical value of a definition, comprehending the relations between quadrilaterals, and constructing hierarchical classification of quadrilaterals. These findings implied that Geometer's Sketchpad assisted tasks which combined definitions, examples and figure constructions were effective enabling pre-service teachers to the definition construction process after several experiences with the concept, through which learners were able to construct meaningful geometric concept perceptions. Learning the availability of many different ways of defining a geometrical concept and discovering many different ways of classification through these definitions increased pre-service teachers' awareness of geometrical relations between concepts. So, the developed learning material is expected to fill the gap in teacher education programs in order to handle teachers' difficulties with concept definitions.

Key words: Pre-Service Middle School Teachers, Quadrilateral Definitions, Definition Construction, Quadrilateral Hierarchies, Dynamic Geometry

ÖZ

İLKÖĞRETİM MATEMATİK ÖĞRETMEN ADAYLARININ TANIMLARI VE ARALARINDAKİ İLİŞKİLER ARACILIĞIYLA DÖRTGENLERİ KAVRAYIŞLARI

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Bu çalışmanın amacı matematik öğretmen adaylarının dörtgen tanımlarını ve bu tanımlara bağlı dörtgen hiyerarşilerini değerlendirme ve oluşturmadaki düşünce süreçlerinin gelişimini, dinamik geometri programı destekli etkinlikler aracılığıyla incelemektir. Çalışmada araştırma deseni olarak, verileri 2010-2011 ilkbahar döneminde 5 ilköğretim matematik öğretmen adayından toplanan, niteliksel durum incelemesi kullanılmıştır. Asıl veriler, Geometer's Sketchpad öğrenme ortamının desteğinde, katılımcılarla birebir gerçekleştirilen klinik mülâkatlardan elde edilmiştir ve mülâkatların hepsi İnsan-Bilgisayar Etkileşimi Laboratuarı'nda yapılmıştır.

Bulgular, literatürdeki veriler doğrultusunda geliştirilen Geometer's Sketchpad destekli öğretim materyalinin, öğretmen adaylarının dörtgenlerin kritik tanımsal özelliklerini belirleme, dörtgen tanımlarının matematiksel doğruluğunu değerlendirme, dörtgenler arasındaki ilişkileri kavrama, ve dörtgen hiyerarşilerini oluşturmadaki düşünme süreçlerini geliştirmede büyük ölçüde etkili olduğunu göstermiştir. Bu bulgulara göre kavramların tanımlarını, örneklerini ve çizimlerini bir arada sunan Geometer's Sketchpad destekli etkinlikler, kavramla ilgili pek çok pratiğin ardından, öğretmen adaylarının tanım oluşturma sürecine girmesine olanak sağlamış ve böylece öğretmen adayları anlamlı geometrik kavram algısı oluşturabilmişlerdir. Bir geometrik kavramı tanımlamanın pek çok farklı yolu olduğunu öğrenmek ve bu tanımların işaret ettiği pek çok sınıflandırmanın yapılabileceğini keşfetmek geometrik kavramlar arasındaki ilişkiler konusunda öğretmen adaylarının farkındalıklarını arttırmıştır. Bu nedenle geliştirilen öğretim materyalinin, öğretmenlerin kavram tanımlarıyla ilgili zorluklarının üstesinden gelme konusunda öğretmen eğitim programlarındaki eksikliği gidermesi umut edilmektedir.

Anahtar Kelimeler: İlköğretim Matematik Öğretmen Adayları, Dörtgen Tanımları, Tanım Oluşturma, Dörtgen Hiyerarşileri, Dinamik Geometri

To My Parents

Hasan ÖZTOPRAKÇI and Hacer ÖZTOPRAKÇI

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LIST OF ABBREVIATIONS

CAI	Computer Assisted Instruction
CAS	Computer Algebra Systems
CTACI	Concurrent Thinking Aloud Clinical Interview
GSP	Geometer's Sketchpad
ISTE	International Society for Technology in Education
NCATE	The National Council for Accreditation of Teacher Education
NCTM	The National Council of Teachers of Mathematics
RTACI	Retrospective Thinking Aloud Clinical Interview
TTKB	Talim ve Terbiye Kurulu Başkanlığı

CHAPTER I

INTRODUCTION

Over the years many researchers in the field of cognitive psychology (e.g., Tall & Vinner, 1981; Fischbein, 1993) dealt with the individuals' mental process in concept formation and they developed cognitive models in which definitions took a crucial role. Tall and Vinner who introduced the terms "concept image" and "concept definition" to the literature in 1981 became one of the protagonists and paved the way for several researchers in the study of definitions (e.g., Burger & Shaughnessy, 1986; De Villiers & Govender, 2002; Fischbein, 1993; Fujita & Jones, 2007). The researchers defined the concept image as everything nonverbal associated in mind with the related concept, such as "visual representations, mental pictures, impressions and experiences" (Vinner, 1991, p. 68) and "set of properties associated with the concept" (Vong, 1989, p. 20). They also defined the *concept definition*, the other construct of their model, as "a form of words used to specify that concept" (Tall & Vinner, 1981, p. 152).

According to Vinner (1983), learning of a concept requires a two-way interaction between concept image and concept definition, but this interaction might not occur as intended every time, especially if learners are directly presented with concept definitions. He illustrates the probable interactions with an example. It might be the case that a learner may have developed a concept image for coordinate system as the two perpendicular axes; but when the teacher presents the formal definition of the coordinate system as two straight lines intersecting each other, not necessarily perpendicularly, three possible interactions between the concept image and concept definition in learner's mind are probable: (1) the learner will reshape his concept image so that it will also include non-perpendicular axes as the coordinate system; (2) the learner will not change the existing concept image, but he will temporarily learn teacher definition which will be forgotten after a while and the learner will only retrieve the concept image of coordinate system with perpendicular axes; (3) the learner will never change his concept image or will never internalize the definition provided by the teacher; so anytime he is asked he will retrieve perpendicular axes (Vinner, 1983). That is, if a learner is presented with a formal concept definition without considering his concept image, he may learn to respond with this formal definition in a restricted context, even though he has an inappropriate concept image (Tall & Vinner, 1981). However, when the learner meets the same concept in a broader context in future, he may not be able to cope with the conflict in his mind at that time (Tall & Vinner, 1981). Therefore, what we want to occur during the concept formation process is the first case in the Vinner's example; and for this interaction to occur there is need to engage learners in tasks that require consulting to not only the concept image but both concept image and concept definition so that learners can see the deficiency of their concept images and reshape them. So, it is clear that presentation of the definition of a concept at the beginning of a unit in a textbook or at the beginning of a lecture by the teacher would not help learners construct a meaningful representation of the concept; but engaging them in the definition construction process after several experiences with the concept would be an important part of mathematics learning to help students interpret the formal definitions given in the textbooks and build understanding of that concept (De Villiers, 1998; Shield, 2004).

In addition to their importance in the concept formation process, definitions have a crucial role in the teaching and learning process of mathematics since they are the central components of the mathematical skeleton. First of all, definitions remove the impurities so that individuals could see the mathematical situations and could distinguish examples and non-examples of a concept, which increases learners' awareness and understanding of the concept, and enhances the use of correct mathematical language (Morgan, 2005; Pimm, 1993). Moreover, definitions are the fundamentals of several mathematical activities such as problem solving, proof making; identifying mathematical objects and stating logical arguments (De Villiers, 1998; Silfverberg, 2003). Definitions also form the entry door to the construction of a theory as a mean to discover the properties of and relationships between the components of a theory (Mariotti & Fischbein, 1997). From the linguistic perspective, on the other hand, definitions are the most important tools to deliver the meaning of mathematical concepts so that written or oral communication can be ensured in the teaching and learning process (Thompson & Rubenstein, 2000). From the pedagogical side, knowing and using definitions effectively in the classroom teaching is accepted as an important component of teachers' knowledge for teaching mathematics (Ball, Bass, & Hill, 2004). That is to say, defining is a many-sided process including mathematical, pedagogical and linguistic aspects and therefore the significant role of definitions in teaching and learning of mathematics can not be denied.

1.1 Problem Statement and Purpose of the Study

According to French mathematician Poincaré (1952) definition and classification are intertwined issues in the sense that the properties used to define a concept allow us to include the concept into a class of objects; and the reason of defining a concept is to determine its place among the other concepts. This statement of Poincaré, emphasizing the close relationship between definition and classification, paved the way for several researchers to study definitions and classification as intertwined issues in the field of geometry (e.g., De Villiers & Govender, 2002; Erez & Yerushalmy, 2006; Fujita & Jones, 2007; Furinghetti & Paola, 2002). Besides, when the related literature examined it is seen that this close relationship between definitions and classification has been mostly investigated in the subject of quadrilaterals. There are several reasons for quadrilaterals to be studied. First of all, quadrilaterals provide a rich world of shapes to investigate the notion of equivalent definitions and the hierarchical or partition classifications through both verbalization or visualization processes (Furinghetti & Paola, 2002). Moreover, for the studies that investigate the definition and classification notions in the dynamic geometry environment, studying with the family of quadrilaterals is the best appeal to obtain rich data in order to underline the cognitive character of the dynamic geometry tools (Jones, 2000). However, the most two important reasons are the learner difficulties with quadrilaterals which remained unsolved till now due to the complex nature of quadrilaterals, and the disagreements in the literature on some quadrilateral related issues, such as classification of quadrilaterals (Jones, 2000; Wu & Ma, 2005). Therefore, there is still need today for conducting further detailed studies in the quadrilaterals topic, even though it has been studied since the time of Euclid.

Several studies that examined teachers' and students' definition construction and classification processes were conducted so far; however, these studies indicated that not only students but also mathematics teachers, who were assumed to have substantial understanding of their subject, had difficulties with constructing and evaluating the mathematical concept definitions and with using these definitions to explain relations between mathematical concepts (eg.,De Villiers & Govender, 2002; Erez &Yerushalmy,

2006; Fujita & Jones, 2007; Furinghetti & Paola, 2002; Jones, 2000; Shir & Zaslavsky, 2001). Literature revealed that constructing definitions and classifying mathematical objects were very difficult processes for learners to achieve. One of the reasons of the difficulty with definitions and classifications was the need for a strong deductive reasoning skill and a suitable concept image-concept definition interaction during identifying the critical and noncritical properties of different quadrilaterals, which was a high level cognitive skill that most of the individuals would not reach (De Villiers & Govender, 2002; Favailli & Romanelli, 2006; Fujita & Jones, 2007). In addition to this complex nature of the definition construction process, learners' creating prototypical shapes, inclusivity and exclusivity of definitions and the use of several attributes to define shapes were the other reasons that explained the difficulties of learners in this process (Hansen & Pratt, 2005). Moreover, difficulty in understanding geometry occurred when a geometric figure, which was determined by its formal mathematical definition, was confronted with a drawing which was the basis for the personal definition (Kuzniak & Rauscher, 2007). That is to say, the discrepancy between the learners' concept images and concept definitions was another reason of the experienced difficulties (Vinner & Dreyfus, 1989). However, even if there were such limitations in using definition construction process, being aware of these problems and difficulties which interfere with learners' concept learning could help educators to look for the ways to develop new methods and to improve their instruction.

Although literature indicated teachers' difficulty with concept definitions, many researchers argued from the pedagogical point that understanding the concept definitions was a fundamental component of teachers' knowledge for teaching mathematics; and their deficiency in knowing the well accepted definitions in the discipline and how to use their functions in teaching will unfavorably influence the quality of the instruction and thereby their students' learning (Ball, Bass, & Hill, 2004; Chinnappan & Lawson, 2005; Winicki-Landman, 2001; Zazkis & Leikin, 2008). Therefore, these unsatisfactory findings raised the concern about the sufficiency of teacher education programs. Then, it was necessary to take precautions at the university level before teachers started to practice in the real life school context, and this would only be possible through the qualified teacher education programs. So, pre-service teachers should be supported with efficient learning materials that will equip them with the necessary knowledge, skills and qualifications. However, the reviewed literature indicated that the studies conducted so far were generally limited in making a detailed analysis of the underlying reasons of learner difficulties and their reasoning skills during the definition construction process; and also they were generally limited in developing ways to improve lacking skills of teachers. Although some researchers consulted to the help of dynamic geometry, they could not find robust evidences to support the effectiveness of dynamic geometry in improving individuals' understanding and constructing definitions. That is to say, while some studies found the dynamic geometry as an effective tool to some extent (e.g., Jones, 2000; Furinghetti & Paola, 2002) some others could not get a clear argument about the effectiveness of it due to some contextual factors (e.g., De Villiers & Govender, 2002; Erez & Yerushalmy, 2007), which revealed a further need for an in-depth analysis of the mental process of individuals during definition construction and classification processes in a dynamic learning environment. Moreover, it was identified that there are not many studies in the Turkish context that specifically focused on the Turkish teachers' understanding of mathematical definitions.

Expecting to meet the need in this issue, the purpose of this study was to examine pre-service middle school mathematics teachers' cognitive progress in constructing and evaluating quadrilateral definitions and the corresponding quadrilateral hierarchies under the support of the Geometer's Sketchpad learning activities.

More specifically, answers to the following research questions were investigated:

1. What are the perceptions of pre-service middle school mathematics teachers regarding the definitions and the role of definitions' in the teaching and learning process, before engaging with dynamic geometry supported clinical interview sessions?
2. What are the understandings of pre-service middle school mathematics teachers regarding the *minimality, equivalence, inclusivity and exclusivity* nature of definitions, before and after engaging with dynamic geometry supported clinical interview sessions?
3. How do the pre-service middle school mathematics teachers improve their understanding of the quadrilateral concepts through definition construction and classification processes in the presence of a set of activities assisted by Geometer's Sketchpad learning environment?

4. How do the dynamic geometry supported learning activities contribute to the improvement of pre-service middle school mathematics teachers' ability to define, evaluate and classify quadrilaterals?
5. What are the impressions of pre-service middle school mathematics teachers about the definition construction in the Geometer's Sketchpad learning environment after having them engaged with clinical interview sessions?

1.2 Significance of the Study

Jean Pedersen described geometry as "a skill of the eyes and the hands as well as of the mind" (as cited in Mackrell & Johnston-Wilder, 2004, p.81). Similar to Pedersen, Duval also stated that "a geometric activity involves three kinds of cognitive processes which are visualization, construction and reasoning" (as cited in Laborde, Kynigos, Hollebrands & Strasser, 2006, p.276). That is to say, these requirements to learn geometry concepts make geometry a challenging issue which necessitates the learner to be able to use more than one skill in the process of meaningful learning. Therefore, it is important to provide learners with appropriate resources that will develop their hands-on, minds-on and eyes-on skills. Fortunately, today it is possible to combine these multiple skills in one learning environment with new technological developments; which makes learning and teaching geometry easier. This study aimed to solve the problems with the meaningful learning of quadrilateral concepts by the help of the learning activities developed in the ligh of the related literature; and in this activities it was considered to combine hands-on, minds-on and eyes-on skills of the learners in the dynamic learning environment to engage them into a very active learning process.

In the first hand, this study is expected to provide useful information about the pre-service teachers' current difficulties in the Turkish context; and the exploration of underlying reasons of these difficulties may give important clues about in what way to improve teacher education programs. It is expected that the dynamic geometry supported learning material developed for the purposes of this study will contribute to a large extent to the technology integration processes in our education system as well. If the prepared learning tasks are found to be effective to overcome teachers' difficulties with geometric concept definitions, the study may open a path through improving the quality of teacher education programs and accordingly student learning. If the teachers develop a sound understanding of the role of definitions through the developed material, this will probably affect their instructional decisions in a positive way and accordingly the learning of their students, as well. If both teachers and students would have a sound understanding of mathematical definitions, then their problems in other mathematical tasks which are underlined by concept definitions such as proof making, or differentiating examples and non examples of a concept will be able to overcome to some extent. In addition, the prepared learning tasks assisted with dynamic geometry can also be an effective tool for those teachers to teach the geometrical concepts through definition construction process to their students in classroom settings if it is adapted to the learners' level. Moreover, it can be a guide to develop better tasks in accordance with the curricular needs.

1.3 Definition of Important Terms

Pre-service middle school mathematics teachers

In this study, pre-service middle school mathematics teachers refer to the senior class teacher candidates attending to the 4-year undergraduate teacher education program at Middle East Technical University, Faculty of Education. During the 4-year undergraduate teacher education program, teacher candidates take several mathematics and education courses that equip them with the necessary skills to teach in the elementary schools. Seniors have already taken most of the courses offered by the program.

Dynamic geometry environment

In this study, the dynamic geometry environment refers to the use of Geometer's Sketchpad tool to support learners both visually and physically in the process of geometric thinking so that they can discover the object behaviors in relation to each other.

Mathematically workable definitions

Mathematically workable definitions refer to the formal definitions that are approved by the large body of mathematics community (Tall & Vinner, 1981). According to Van Dormolen and Zaslavsky (2003) the two most important criteria to accept a definition as a mathematically workable definition are hierarchy and minimality conditions. The hierarchy criterion requires defining a concept "as a special case of more general concept" (p. 94) and the minimality of definitions requires to not to define a concept with more than the necessary properties (Van Dormolen & Zaslavsky, 2003). So, this study will accept the definitions as mathematically workable when they have these two properties.

Correct economical definitions

De Villiers and Govender (2002) explain correct economical definition as "definition [that] has only necessary and sufficient properties… [and]… contains no superfluous information." (p. 5).

Minimality of definitions

In this study the minimality of definitions refers to not to define a concept with more than the necessary properties (Van Dormolen & Zaslavsky, 2003). Minimal definitions actually have the same meaning as correct economical definitions.

Equivalence of definitions

Equivalence of definitions refers to the arbitrariness of definitions which "refers to the existence (or choice) of different, alternative but correct definitions for the same concept" (De Villiers & Govender, 2002, p. 4).

Inclusivity and exclusivity of definitions

Usiskin and Griffin (2008) explain that when "one definition purposely excludes what the other definition includes, we call the one definition an *exclusive definition* and the other definition an *inclusive definition*" (p. 4). For example, if a trapezoid is defined exclusively as "a quadrilateral with exactly one pair of parallel sides" (p. 27), then parallelograms and trapezoids would be identified as disjoint subgroups of quadrilaterals. However, if the trapezoid is defined inclusively as "a quadrilateral with at least one pair of parallel sides" (p. 27), then all parallelograms would be a subgroup of trapezoids and trapezoids would include the parallelograms.

Classification

De Villiers (1994) defined *hierarchical classification* as "the classification of a set of concepts in such a manner that the more particular concepts form subsets of the more general concepts" (p. 11). On the other hand he defined *partition classification* as the classification where "the various subsets of concepts are considered to be *disjoint* from one another" (p. 11).

Defining properties

Defining properties are the necessary and sufficient attributes that characterize a particular concept and instances of that concept (Favilli & Romanelli, 2006). For instance, *two pairs of parallel sides* is a necessary and sufficient condition to identify a quadrilateral as a parallelogram and all instances of parallelogram (rectangle, rhombus, square) have this critical attribute. Another example can be given from the kites. *Perpendicular diagonals* is a necessary property to define kites, but it is not sufficient to characterize kites since there are other quadrilaterals with this property. However, when we add the property that *one diagonal is bisected by the other diagonal*, we can identify a quadrilateral as kite; and all instances of kite (rhombus, square) have the defining property of *one diagonal is a perpendicular bisector of the other.*

Cyclic and circum properties

A cyclic quadrilateral is "a quadrilateral whose four vertices lie on a circle" (Usiskin & Griffin, 2008, p. 63). For a quadrilateral to be cyclic, opposite angles must be supplementary; so rectangles, squares and isosceles trapezoids and kites (if congruent angles are 90°) must be cyclic quadrilaterals. Then, we can define these quadrilaterals using their cyclic property such as *an isosceles trapezoid is a cyclic quadrilateral (opposite angles supplementary) with at least one pair of opposite sides parallel* or *a square is a cyclic quadrilateral with congruent sides.*

A circum quadrilateral is "a quadrilateral circumscribed around a circle" (De Villiers, 2009, p. 42). The angle bisectors of circum quadrilaterals intersect at a point which is the incircle center; so, kite and all its special cases (rhombus, square, right kite) become circum quadrilaterals. For instance a kite can be defined as *a circum quadrilateral with two pairs of congruent adjacent sides.*

Unfamiliar quadrilateral shapes

Unfamiliar quadrilateral shapes refers to the quadrilaterals of which definitions are constructed by generalizing or specializing the properties of seven familiar quadrilaterals, namely, parallelogram, rectangle, square, rhombus, kite, trapezoid and isosceles trapezoid. The unfamiliar quadrilaterals that will be used in this study are *skew kites, skew trapezoid and trilateral trapezoid* which are defined by De Villiers (2009). However, in this study they will be named as quad 1, quad 2 and quad 3 respectively. These types of quadrilaterals are not very familiar to students and teachers since they do not exist in teaching programs in schools. However, constructing definitions of these unfamiliar quadrilaterals requires high-level of geometric thinking ability and for the purposes of this study this kind of activity can be a good practice to reflect pre-service middle school mathematics teachers' understanding of how to define a geometry concept.

Generalization and specialization

In this study generalization refers to the defining of a more general concept by leaving out some properties of another concept (De Villiers, 2009). For example a kite is defined as “a quadrilateral with two pairs of adjacent sides equal” (De Villiers, 2009, p.8). However, by leaving out some properties we can generate the definition of kite, such as “a quadrilateral with at least one pair of adjacent sides equal” (De Villiers, 2009, p.8). By this way, we generalize the concept of kite to a more general concept which can be named as skew kite; and kites become subsets of skew kites.

On the other hand, specialization refers to the defining of a more specific concept by adding some constraints to the definition of a concept (De Villiers, 2009). For instance, if we constrain the definition of the kite by adding the property of *being cyclic quadrilateral* we specialize the kite into a right kite of which congruent angles are right angles making it a cyclic kite (De Villiers, 2009).

CHAPTER II

LITERATURE REVIEW

The purpose of this study was to examine pre-service middle school mathematics teachers' cognitive progress in constructing and evaluating quadrilateral definitions and the corresponding quadrilateral hierarchies under the support of the Geometer's Sketchpad learning activities. Although this phenomenon was actually a complex one in its nature due to incorporating many cognitive processes within its scope, the underlying theory was the theory of concept formation. The meaningful formation of the quadrilateral concepts was scrutinized under the support of the dynamic geometry through definition construction, definition evaluation and classification cognitive processes each one of which required many cognitive skills.

To develop a better understanding of the scope of the study, theoretical background related to the concept formation process, important role of concept definitions in this process, the close relationship between the definition and classification, the role of dynamic geometry in learning environments and research studies related to the issue were presented throughout the chapter.

2.1 Concept Formation in the Cognitive Psychology

Over the years several researchers in the field of cognitive psychology dealt with the individuals' mental process in concept formation and they developed cognitive models which served as frameworks for several research studies in the literature. These cognitive theories will be examined in the following sections.

2.1.1 The Vinner Model

In 1980, Shlomo Vinner developed a cognitive model of *concept image* and *concept definition* in order to examine learners' mental process of learning a mathematical concept (Vong, 1989). However, the terms "concept image" and "concept definition" were first published next year in the journal article by Tall and Vinner (1981) and after introducing these terms to the literature the researchers became one of the protagonists paving the way for several researchers in the study of concept formation (e.g., Burger & Shaughnessy, 1986; De Villiers & Govender, 2002; Fischbein, 1993; Fujita & Jones, 2007).

Believing the importance of distinguishing between formal mathematical concept definitions and individuals' corresponding mental processes Tall and Vinner (1981) made a distinction between concept image and concept definition. They defined *concept image* as:

> […] the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures. (p. 152)

Namely, everything nonverbal associated in mind with the related concept, such as "visual representations, mental pictures, impressions and experiences" (Vinner, 1991, p. 68) and "set of properties associated with the concept" (Vong, 1989, p. 20) constitute an individual's concept image. Since concept images are acquired through different experiences by different individuals, the image associated with the same concept might differ from individual to individual; even the same individual might associate different images, not necessarily the correct ones, with a concept in different situations (Vinner, 1991). Due to the fact that development of concept image is a dynamic process and it is open to change when the individual meets new stimuli in different situations, different parts of the concept image may not be consistent at all times causing conflicting images to evoke at different times (Tall & Vinner, 1981). Referring to the reactions of brain to different stimuli, Tall and Vinner (1981) defined *evoked concept image* as "the portion of the concept image which is activated at a particular time" (p. 152).

The researchers defined the *concept definition*, the other construct of the model, as "a form of words used to specify that concept" (Tall & Vinner, 1981, p. 152), and they distinguished between two types of concept definitions. If the definition is a form of words used by students to explain their own evoked concept image, it is called *personal concept definition*; if it is a concept definition approved by the large body of mathematics community, it is called *formal concept definition* (Tall & Vinner, 1981).

According to Vinner (1983, 1991), learning of a concept requires a two-way interaction between concept image and concept definition (Figure 2.1); but this interaction might not occur as intended every time, especially when students are directly presented concept definitions. However, most of the teachers expect a one way relationship (Figure 2.2) expecting to evoke student's concept image by just giving the concept definition, even though it is not the case in practice.

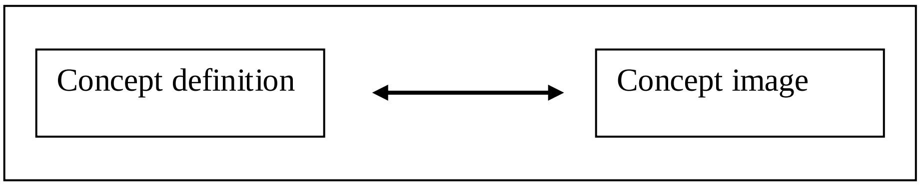


Figure 2.1 Interplay between concept image and concept. (Vinner, 1991, p. 70)

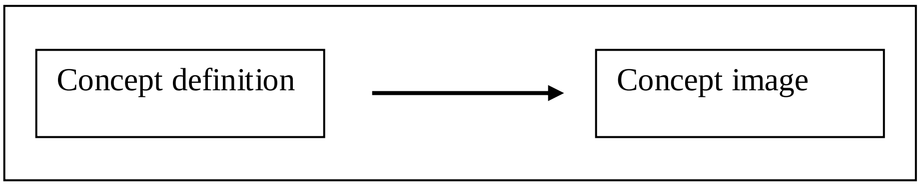


Figure 2.2 The cognitive growth of a formal concept. (Vinner, 1991, p. 71)

Vinner (1983, 1991) illustrates the probable interactions between concept image and concept definition with an example. It might be the case that a child may have developed a concept image for coordinate system as the two perpendicular axes; but when the teacher presents the formal definition of the coordinate system as two straight lines are intersecting each other, not necessarily perpendicularly, three possible interactions between the concept image and concept definition in child's mind are probable: (1) the child will reshape his concept image so that it will also include non-perpendicular axes as the coordinate system; (2) the child will not change the existing concept image, but he will temporarily learn teacher definition which will be forgotten after a while and the student will only retrieve the concept image of coordinate system with perpendicular axes; (3) the child will never change his concept image or will never internalize the definition provided by the teacher; so anytime s/he is asked s/he will retrieve perpendicular axes (Vinner, 1983, 1991). Similarly, the child may have only the concept definition but not any concept image of the coordinate system at the beginning, and when teacher presents concept image to child all three probabilities would be valid also in this case (Vinner, 1983, 1991).

In addition to the concept formation process, Vinner also explained the relationship between concept definition and concept image in the problem solving and task performance processes. According to Vinner (1991) many teachers believe that tasks and problems given to the students activate both concept image and concept definition, hereby students certainly consult to the concept definition before solving the problem (Figure 2.3, Figure 2.4, & Figure 2.5).

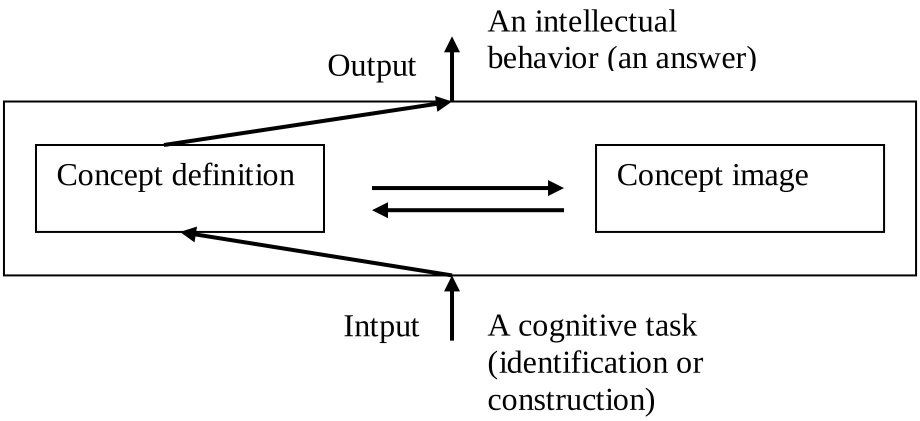


Figure 2.3 Interplay between definition and image. (Vinner, 1991, p. 71)

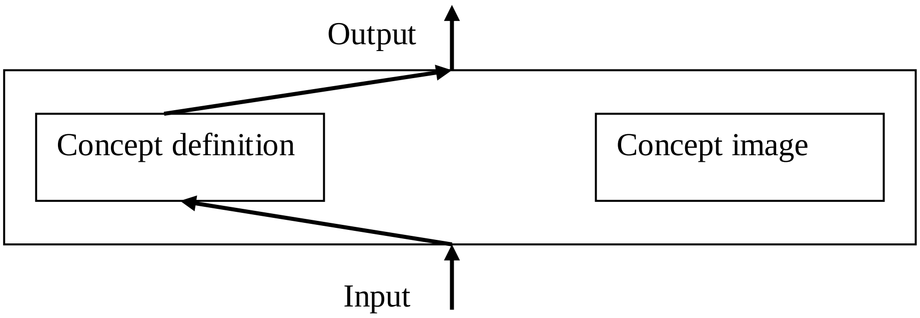


Figure 2.4 Purely formal deduction. (Vinner, 1991, p. 72)

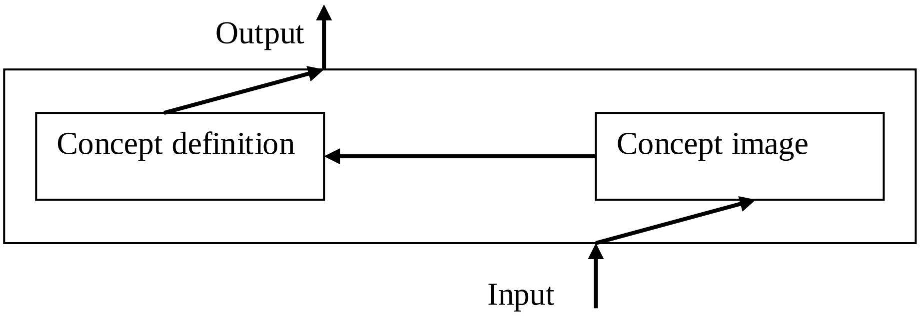


Figure 2.5 Deduction following intuitive thought. (Vinner, 1991, p. 72)

Of course, in formal learning context consulting to its definition before generating reasoning about a concept is a desirable process to prevent some pitfalls caused by the inappropriate concept images; however, in contrast to what many teachers expect, this is not always the case in practice (Figure 2.6) since students are unconscious about the need for consulting to the formal definition due to the sovereignty of the daily life thought habits (Vinner, 1991). That is to say, it is probable that everyday though habits will dominate the technical thought habits even in the technical context and when we ask students to explain concepts like rectangle or square, they generally will have a tendency to make judgments based on their pre-existing concept images instead of concept definitions (Alcock & Simpson, 2009; Tall, 1988). However, the pre-constructed image can give rise to development of inappropriate generalizations for a concept and can prevent students from using formal definitions; similarly, the use of everyday language to define a concept may cause them to develop an inappropriate naïve concept image (Alcock & Simpson, 2009). Therefore, definitions should be considered as important means to evoke correct concept images and to change daily life thought into more formal technical thought which is the aim of teaching mathematics (Vinner, 1991).

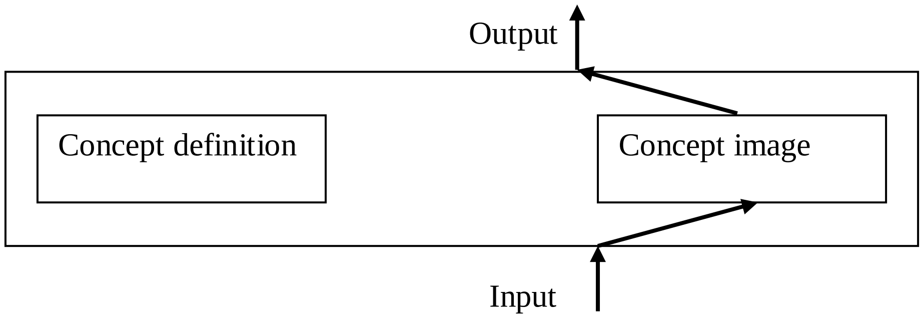


Figure 2.6 Intuitive response. (Vinner, 1991, p. 73)

As in the case of Vinner's (1983) example when students encountered with an old concept in a new context, they first remember the associated concept image which they developed through prior experiences (Tall, 1988). In some routine situations consulting only to concept image may not lead to incorrect reasoning automatically; however, if the concept image is not the correct one it causes learner to reason incorrectly about the concept of interest (Alcock & Simpson, 2009). In other words, if the examples associated with the concept in student's mind are not same as the examples determined by the formal definition, then students' reaction to the concept may differ from what the teacher expects (Vinner & Dreyfus, 1989) and their responses contradict with the formal theory (Tall, 1988). According to Tall and Vinner (1981) a concept image that is inconsistent with the other parts of the concept image or with the personal concept definitions or formal concept definitions is the source of the potential conflict factors in learner's mind. More specifically, the researchers named "a part of the concept image or concept definition which may conflict with another part of the concept image or concept definition, a *potential conflict factor*" (Tall & Vinner, 1981, p. 153).

Actually, the contradiction in student's mind need to be paid attention; because if a student is presented with a formal concept definition without considering his concept image, he may learn to respond with this formal definition in a restricted context, even though he has inappropriate concept image (Tall & Vinner, 1981). However, when the student meets the same concept in a broader context in future, he may not be able to cope with the conflict in his mind at that time (Tall & Vinner, 1981). On the other hand, if these conflict factors can be handled appropriately by the teachers, it is possible to reshape students' concept image in a correct way (Tall, 1988). Namely, when students come to school with their own concept images and when the correct formal definition and the corresponding correct images of the concept is presented to them, potential conflict factors will be activated (Tall, 1988). At this point, students will need to reshape their concept image to eliminate the confusion in mind (Tall, 1988). That is to say, through activating the conflict factors, it is expected to create concept images that are consistent with the formal definitions, in educational settings (Vong, 1989). At this point, what we want to occur during the concept formation process is the first case in the Vinner's (1983) example; and for this interaction to occur we should provide students with richer experiences and engage them in tasks that require consulting to not only the concept image but both concept image and concept definition so that students can see the deficiency of their concept images and reshape them (Vinner, 1991); but this is not so easy task at all (Tall, 1988).

Maybe it is not necessary to consult a definition in everyday context; because the experiences with everyday concepts such as car, chair, orange, etc., allow recognizing and using them without consulting a definition (Alcock & Simpson, 2009; Vinner, 1983, 1991). However, differently from the everyday context it is necessary to consult definitions in the technical context in order to prevent probable mistakes. Vinner (1991) explains this idea with the following example:

> Of course, there is no need to consult definitions (which do not exist) when trying to understand the sentence "among all the cars at the parking lot my green car is the nicest". However, it is necessary to consult definitions when trying to understand the sentence: "among all rectangles with the same perimeter the square is the one which has the maximal area." (p. 67)

Therefore, concept definitions have a crucial role in teaching and learning mathematics, because they may prevent some pitfalls caused by the concept images (Vinner, 1991). Moreover, without consulting to the formal definitions it is not possible to perform other mathematical tasks, such as differentiating between examples and non-examples of a concept, solving problems and conducting proofs. (Vinner, 1991). It is probable that students' daily life thought habits will be more dominant over the formal learning thought habits at the beginning of the learning process; however, the aim of mathematics teaching is to change daily life language to more formal mathematical language (Vinner, 1991).

2.1.2 Fischbein's Theory of Figural Concept

Similar to the Vinner model, another cognitive psychologist Fischbein (1993) also distinguished the two mental structures as *concepts* and *images* and explained their relation in the cognitive process through developing the theory of *figural concept*. However, Fishbein's theory is specific to the geometry concepts. (Walcott, Mohr, & Kastberg, 2009).

According to Fischbein (1993) "a concept…expresses an idea, a general, ideal representation of a class of objects based on their common features" (p. 139); on the other hand an image "…is a sensorial representation of an object or phenomenon" (p. 139). So, the *figural concept* is "the highest level in geometrical reasoning … in which the figural and conceptual constraints are perfectly harmonized" (Fischbein & Nachlieli, 1998, p. 1195). That is to say, figural concepts in their nature have properties of both concepts and images.

The *geometrical figures* can be thought as a highest level of figural concept owing to their possessing both conceptual and figural properties (Fischbein, 1993; Fischbein & Nachlieli, 1998). All the geometrical figures such as point, line, plane, circle, square, cube, etc., have the characteristics of generality, abstractness and absolute perfection which are the characteristics of the concepts; additionally, all geometrical figures are also images since they possess shape, location and magnitude which are the sensorial representations (Fishbein & Nachlieli, 1998). A robust geometrical reasoning requires the interaction between these conceptual and sensorial components even though this is the ideal situation which not always occurs due to psychological constrains (Fishbein, 1993; Fischbein & Nachlieli, 1998). During the geometrical reasoning process the figural aspects help to invent practical meaning in the mental process while the conceptual aspects help to formally ensure logical meaning and consistency (Fishbein & Nachlieli, 1998). Moreover, the most important characteristic of geometrical figures related to their conceptual aspect is that their images are controlled and imposed by their definitions (Fishbein, 1993). For instance the final decision that a square is actually a parallelogram is derived from the definition of a parallelogram, even though the figural appearances of a square and parallelogram seem different (Fischbein & Nachlieli, 1998). That is to say, while making a geometrical reasoning about the relationship between a concept of which meaning was constrained by a definition (parallelogram) and its possible specific cases (square), conceptual aspects, especially the definition, dominate the figural aspects; but the function of figural aspects still remains important (Fischbein & Nachlieli, 1998).

According to Fischbein (1993), figural concepts can be a useful tool to determine the student mistakes in geometrical reasoning process. Although images and concepts can sometimes cooperate or sometimes conflict in students' cognitive process, the aim of the given mathematical tasks should be to create cooperation and harmony between these constructs (Fischbein, 1993). Therefore, in mathematical education individuals need help to learn how to investigate the situations causing conflicts in their mind and to deal with them by using conceptual control (definitions) over the figural invention process (Fishbein & Nachlieli, 1998).

2.1.3 Other Theories of Concept Formation

In his book "Thought and Language" Vygotsky (1986) considered the concept formation as an aspect of cognitive development of children and he differentiated between two types of process, namely *scientific* and *spontaneous* that lead to formation of a concept. Scientific process originates from the educational instruction and involves imposing scientifically constructed concept definitions upon a child; on the other hand, spontaneous process involves concepts reflected by child's daily life experiences (Vygotsky, 1986, as cited in Daniels, 1996).

In educational settings children are expected to develop scientific concepts for their substantial development; however, this development requires a prior ability of children to understand concepts, which is connected with the development of their spontaneous concepts (Vygotsky, 1986). That is to say, child's daily life concept opens a door for the construction of scientific concept by making the elementary aspects of the concept clear to child; and then scientific concept gives the way to increase conscious and functional use of the concept (Vygotsky, 1986, as cited in Daniels, 1996). Moreover, Vygotsky (1986) argued that "It is a functional use of the word, or any other sign, as a means of focusing one's attention, selecting distinctive features and analyzing and synthesizing them, that plays a central role in concept formation" (as cited in Berger, 2004, p. 84).

Berger is one of the researchers who adopted Vygotsky's theory of concept formation in her study. For the purpose of explaining learners' understanding of a mathematical object through the various signs of its definition, Berger (2004) defined the terms *cultural meaning* and *personal meaning.* The cultural meaning of a definition is determined by the extent to which the signs (words or symbols) of the definition are congruent with mathematical community usage (Berger, 2004). Namely, culturally meaningful definition refers to the formal concept definition of Vinner's theory while personal meaning is the learner's own feeling or belief in grasping the cultural meaning of the definition even it is not the real case (Berger, 2004). For example, although misconceptions do not have any cultural meaning, they have a personal meaning for learners; moreover, a culturally meaningful definition may not be personally meaningful for the learners (Berger, 2004). Thus, personal meaning of a definition refers to the personal concept definition, generated by individuals' correct or incorrect concept images, in Vinner's theory.

According to Berger (2004) a good and satisfying mathematical practice occurs when the learner's personal meaning of a definition is coherent with its cultural meaning; because, in this case learner is able to use this definition functionally in different contexts. This is similar to the coherence between the concept image and concept definition for meaningful concept formation as being explained in Vinner's theory.

Embarking on the Vinner's definitions of concept image and concept definition, Fujita and Jones (2007) also developed two other terms. They called learners' own concept definitions and concept images which were developed through their prior experiences as *personal figural concepts*; and they called formal concept image and concept definitions which were validated by the mathematics community as *formal figural concepts*. The researchers stated that inappropriate personal figural concepts can result in incorrect judgments by the students, which prevents their understanding of formal figural concepts. Therefore, Fujita and Jones (2007) asserted that the gap between personal figural concepts and formal figural concepts should be focused on in educational settings.

De Villiers who dealt with the definition construction process in his studies (De Villiers, 1994, 1998, 2004, 2009; De Villiers & Govender, 2002) distinguished two different types of defining process. De Villiers (2004) explained the meaning of descriptive (posteriori) defining as "the concept and its properties have already been known for some time and is defined only afterwards" (p. 709). That is to say, learner already knows the properties of a concept and systematizes them in order to construct a definition from which all other properties of the concept that are not used in the definition can be deduced (Figure 2.7) (De Villiers, 2004, 2009).

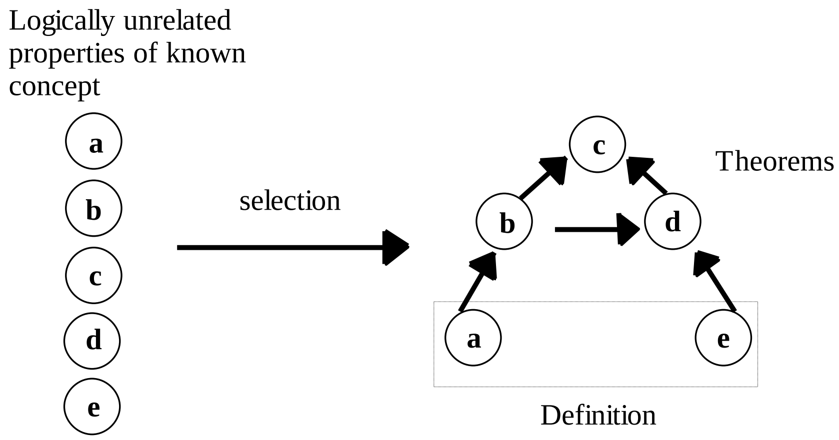


Figure 2.7 (De Villiers, 2009, p. 14)

He also related his definition to the Vinner's concept image and concept definition (Figure 2.8) and explained that in descriptive defining a concept definition is generated from an appropriately developed concept image of which properties have already been discovered (De Villiers, 2004).

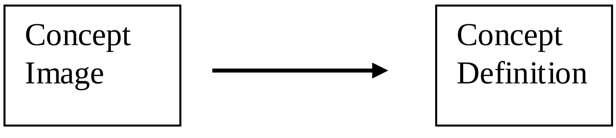


Figure 2.8 (De Villiers, 2004, p. 709)

On the other hand, De Villiers (2004) explained constructive defining as the following:

> Constructive (a priori) defining takes place when a given definition of a concept is changed through the exclusion, generalization, specialization, replacement or addition of properties to the definition, so that a new concept is constructed in the process. (p. 709)

That is to say, in constructive defining (Figure 2.9, Figure 2.10) learners are first given the definition of a known concept and then they are expected to change it by using appropriate variation (generalizing, specializing, etc.) to develop a new concept definition and corresponding new concept image (De Villiers, 2009).

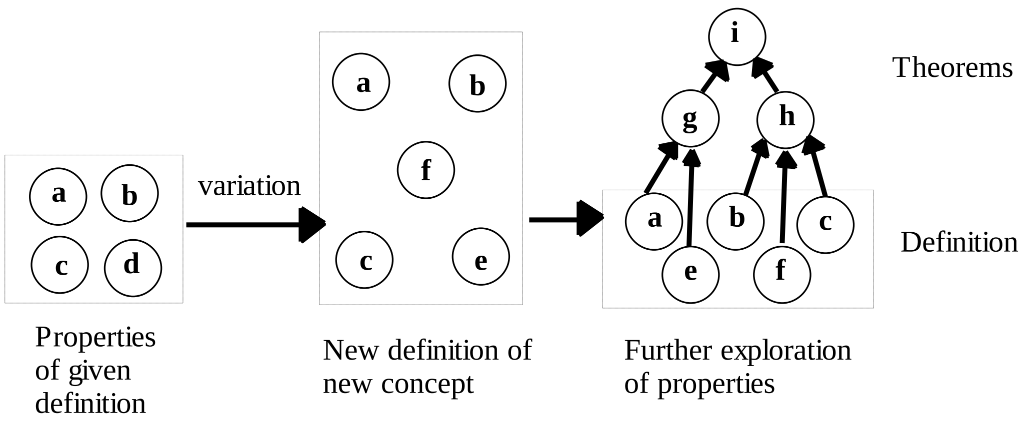


Figure 2.9 (De Villiers, 2009, p. 18)

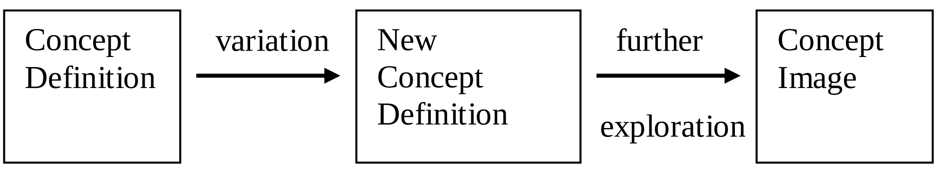


Figure 2.10 (De Villiers, 2004, p. 709)

Based on the historical perspective and on his personal experience, De Villiers (2004) stated that descriptive defining is the most common type in the process of concept formation. Further, these two types of definitions are possible to concur during the process of constructing a particular definition; so they are conjoint processes, in fact (De Villiers, 2004). From the constructivist point of view, if learners are expected to improve their understanding of geometrical concept definitions, they should be actively engaged in the definition construction process rather than being provided with pre-constructed formal definitions (De Villiers, 2004).

The other and very recent categorization of definitions was also made by Raychaudhuri (2008) who claimed the important aspects of a mathematical concept definition to be the context, form and property. According to the researcher two categories of the definitions are *static* and *dynamic* definitions. In static definitions objects can be determined by the properties which are used to define it; on the other hand, in dynamic definitions the definition of object itself is not enough to generalize that object. According to Raychaudhuri (2008) the dynamic definitions are dynamic, because they include interaction between the two mathematical processes: one is maintaining the essential property that stated clearly in the definition; and the other one is deducing the properties which are just implied in the definition without being clearly stated but needed to generate the object.

Walcott, Mohr, and Kastberg (2009) developed a conceptual framework combining Vinner's and Fishbein's theories; but differently from Raychaudhuri's (2008) dynamic figural concept they introduce dynamic figural concept as a non-static structure consisting of "the visual, verbal, written, symbolic, and/or formal properties of shape that are valued by an individual child" (p. 35) and that can change according to the experiences gained by the student. The dynamic figural concept is constructivist version of Fishbein's figural concept in the sense that it is constrained by individually constructed definition through students' sense making process (Walcott, Mohr, & Kastberg, 2009).

To sum up, throughout the years researchers examined the cognitive side of the concept formation process and they developed many theories to explain the issue. Moreover, these theorists also explained their recommendations to be followed in the educational settings. As it is obvious, Vinner's cognitive model was the most appreciated one by other researchers and it seems this model inspired other researchers to develop their own theories with little changes from the Vinner's concept image and concept definition explanations.

For the purpose of this study, learners' reasoning abilities in concept formation process which requires an interaction between concept image and concept definition (Vinner, 1983, 1991) will be investigated considering De Villier's (2004, 2009) descriptive and constructive defining processes.

2.2 The Importance of Definitions in Teaching and Learning Mathematics

The role of definitions in the teaching and learning of mathematics was accepted as crucial by several researchers. Definition construction is a mathematical activity as important as problem solving, conjecture making and proof making (De Villiers, 1998); because definition construction activity allows students to develop abilities of generalization, synthesizing and abstraction as well as to discover the relationship between the components of a concept (Ko, Lee, & Lew, 2007; Mariotti & Fischbein, 1997). Moreover, constructing a formal definition through using technical terms not only provides understanding a pre-existing concept, but also provides a mean to shape the concept so that it can be applicable for other particular purposes and for new contexts (Morgan, 2005; Wilson, 1990).

In addition to their role in constructing a basis for many other mathematical tasks, formal definitions are crucial in the process of reshaping naïve concept images so that these naïve concept images will agree with their formal definitions, (Vinner, 1991); that is, definition, together with intuition in students' concept images, plays a vital role in students' learning (Rolka & Rösken, 2007). Therefore, learners need to develop a better understanding of the definitions of mathematical concepts in order to be able to develop appropriate concept images and to further their understanding and improvement in other mathematical tasks (Vinner, 1991).

Furthermore, defining activities give rise to students' thinking the definitions as building parts of the mathematical theory (Furinghetti & Paola, 2002). Although their crucial role of definition is neglected in teaching, "it [definition] is the first gate to enter a theory" (Furinghetti & Paola, 2002, p. 392) due to its being the basic component of deduction process to discover the existing and new properties of and relationships between the objects of a theory (Mariotti & Fischbein, 1997; Morgan, 2005). So it is important to assist learners in developing an appropriate path to the construction of a mathematical theory by providing them with the processes of defining which is consistent with their mind (Furinghetti & Paola, 2002). Moreover, definitions are one of the basic components of the skeleton of the mathematical theory and they form the prerequisites for the theory and proof construction that are needed to develop mathematics as a deductive theory (Heinze, 2002, Vinner, 1991). This is because definitions form the basis of global proof structure by leading to logical arguments (Knapp, 2006); in other words, to be able to decide whether an argument leads to a proof there are four major elements to constitute: foundation, formulation, representation and social dimension; and definitions constitute the foundation part of these arguments while making a decision about whether the argument leads to a proof (Stylianides, 2007). Moreover, this relationship between proving and definitions was also supported by Moreno (2003) who found "a positive correlation between students' written definitions and their ability to prove mathematical statements" (p. 108). Therefore, it is important to engage learners in activities where they will interact with formal definitions and proofs and understand the important role of definitions to make logical arguments (Knapp, 2006).

A concept definition, which gained universal acceptance over the years by the struggle of mathematicians, has strength that "it gives form and meaning to a mere word, often allowing the same word (such as solution) to transcend the delineations of the contexts (such as from algebra to linear algebra to differential equations" (Raychaudhuri, 2008, p. 161). Moreover, "definition is one of a handful of meta-mathematical marker terms (others include axiom, theorem, proof, lemma, proposition, corollary), terms which serve to indicate the purported status and function of various elements of written mathematics" (Pimm, 1993, p. 261). That is to say, from the linguistic perspective, definitions are the most important tools to deliver the meaning of mathematical concepts, which makes them fundamental tools of mathematical language used for providing written or oral communication in the teaching and learning process (Shir & Zaslavsky, 2001; Thompson & Rubenstein, 2000). Moreover, definitions are the central components of the mathematical study in that they lead a meaningful communication whereby establishing the uniformity in the meaning of the concepts (Shir & Zaslavsky, 2001); because definitions remove the impurities so that individuals could see the mathematical situations and could distinguish examples and non-examples of a concept, which increases learners' awareness and understanding of the concept, and enhances the use of correct mathematical language (Morgan, 2005; Pimm, 1993).

In addition to its declaration by many researchers, the National Council of Teachers of Mathematics [NCTM] standards also pointed out the importance of moving students from everyday informal language to a more formal mathematical language and the role of definitions in this process as the following:

> As students articulate their mathematical understanding in the lower grades, they begin by using everyday, familiar language. This provides a base on which to build a connection to formal mathematical language. Teachers can help students see that some words that are used in everyday language, such as *similar, factor, area,* or *function,* are used in mathematics with different or more-precise meanings. This observation is the foundation for understanding the concept of mathematical definitions. It is important to give students experiences that help them appreciate the power and precision of mathematical language. Beginning in the middle grades, students should understand the role of mathematical definitions and should use them in mathematical work. (NCTM, 2000a)

That is to say, since mathematical language has a crucial role in teaching and learning of mathematics, it is important to make learners aware of the difference between meaning of a word in daily life use and in the mathematical use. That is, as Vinner (1991) stated the aim of mathematics teaching should be to change daily life language to more formal mathematical language (Vinner, 1991). Therefore, thinking that definitions are the most important tools to deliver the meaning of mathematical concepts, they can be considered fundamental bases of mathematical language; and students' interaction with definitions need to be given specific attention in order to underline the importance of mathematical language and to improve their understanding and using mathematical language.

To sum up, literature indicated that the crucial role that definitions play in the teaching and learning mathematics was approved by the researchers, theorists, educators and mathematicians. The researchers believed that definitions are the central components of the mathematical skeleton and without definitions it is not possible to communicate and to conduct activities such as constructing theorems, making proofs, solving problems, differentiating between examples and non-examples of a concept, generalizing, specifying, etc. That is to say, defining is a many-sided process including mathematical, psychological, pedagogical and linguistic aspects (Silfverberg & Matsuo, 2008; Van Dormolen & Zaslavsky, 2003) of which significant role in teaching and learning of mathematics can not be denied. So, it is crucial to attach importance on definitions and to prevent their negligence in the teaching and learning process by making educators and learners aware of their functions in this process.

2.3 Definitions and Classifications of Quadrilaterals

Poincaré (1952), a well known French mathematician, explained the close relationship between definitions and classification with the following words:

> The aim of each part of the statement of a definition is to distinguish the object to be defined from a class of other neighbouring objects. The definition will not be understood until you have shown not only the object defined, but the neighbouring objects from which it has to be distinguished, until you have made it possible to grasp the difference, and have added explicitly your reason for saying this or that in stating the definition. (p. 133)

It can be inferred from this statement that the definition and classification are intertwined issues in the sense that the properties used to define a concept allow us to include the concept into a class of objects which have these properties and the reason of defining a concept is to determine its place among the other concepts. Therefore, this statement of Poincaré paved the way for several researchers to study the definitions and the classification as intertwined issues in the field of geometry.

Furthermore, literature also revealed that several researchers found the concept of quadrilaterals to be the best subject to study the intertwined concept of definitions and classification. First of all quadrilaterals are popular because of their having been experienced both theoretically and educationally since the time of Euclid and they still keep this popularity due to the problems that could not be overcome so far (Furinghetti & Paola, 2002). Moreover, the topic *quadrilaterals* provides a rich world of shapes to investigate the notion of equivalent definitions and the hierarchical or partition classifications through both verbalization or visualization processes (Furinghetti & Paola , 2002). For the studies that investigate the definition and classification notions in the dynamic geometry environment to underline the cognitive character of the dynamic geometry tools and for the studies that investigate learners' geometrical reasoning ability in the concept of definitions on the bases of van Hiele reasoning levels, studying with the family of quadrilaterals is found to be the best appeal to obtain rich data (Furinghetti & Paola , 2002; Jones, 2000). From other point of view, there is a need for conducting further studies in the quadrilaterals topic even though it has been studied since the time of

Euclid; because due to the complex nature of quadrilaterals there are still unsolved learner difficulties and there are some disagreements on some quadrilateral related issues such as classification of quadrilaterals in the literature (Jones, 2000; Wu & Ma, 2005). Therefore, several researchers studied the close relationship between definition and classification issues in the concept of quadrilaterals (e.g., Athanasopoulou, 2008; Cannizzaro & Menghini, 2006; De Villiers, 1994; Fujita & Jones, 2007; Schwarz & Hershkowitz, 1999; Usiskin & Griffin, 2008; Welter, 2001).

Having analyzed the different types of classifications throughout the history, Athanasopoulou (2008) came up with that different definitions of quadrilaterals and corresponding different lines of reasoning led to different types of classification throughout the history. De Villiers (1994) also articulated that the process of defining and classification are depended to each other, and they are not isolated processes and differentiated between different types of classification. He defined *hierarchical classification* as "the classification of a set of concepts in such a manner that the more particular concepts form subsets of the more general concepts" (p. 11). On the other hand he defined *partition classification* as the classification where "the various subsets of concepts are considered to be *disjoint* from one another" (p. 11). He stated that both of these classifications and their corresponding definitions are accepted and employed equally in different fields of mathematics; in other words, none of the classifications are incorrect. Emphasizing the arbitrariness of definitions and so the corresponding classifications, De Villiers (1994) stated that using whether partition or hierarchical definitions and classifications depends on the personal purposes and preferences; however, he explained that he was in favor of hierarchical classification owing to the important functions of this type of classification. According to him, considering the hierarchical relationships provides a more general conceptual schema which makes it easier to deduce the properties of the more special concepts through the more general concepts and to construct different alternative definitions including the minimal properties generalizing a class of concepts.

Fujita and Jones (2007) also agreed with De Villiers (1994) on the economic function of hierarchical classification stating that a true statement for a concept in this type of classification will also be true for all specific subsets of the concept. Another reason of hierarchical classification's being more functional is that understanding a hierarchical relation improves the ability to realize the different classification ways for the same concept and ability to understand the transitivity, asymmetry, and opposite asymmetry of relations among the shapes (Fujita & Jones, 2007). For instance, understanding class inclusions between concepts requires the ability to define the concept differently in terms of other more general concepts; to make transitive reasoning such as if a square is a rectangle and a rectangle is an isosceles trapezoid then a square is an isosceles trapezoid; to understand lack of symmetry within the relations like a square is a rectangle but a rectangle is not a square; to understand the opposite inclusive relationship (Schwarz & Hershkowitz, 1999) between the concepts and their properties such as a square is a rectangle a rectangle is not a square; but while all properties of a rectangle are valid for a square, all properties of a square are not valid for a rectangle. Therefore, classifying objects is an important mathematical ability which is the result of students' better understanding the similarities and differences and inclusive relations between concepts and it helps learners to have better control over the concepts that are classified (Welter, 2001). However, complex nature of the relationships between the concepts makes it difficult for learners to understand the inclusive definitions and corresponding hierarchical classifications (Fujita & Jones, 2007; Schwarz & Hershkowitz, 1999).

In a very recent monograph devoted to the definition and classification of quadrilaterals, Usiskin and Griffin (2008) analyzed several textbooks from the year 1838 up to the present in order to identify the change in definitions through the years and they found several equivalent definitions for each quadrilateral, except for trapezoid. As a result of their analysis they concluded that the disagreement in the literature about the definition of trapezoid gave rise to the disagreement in the ways in which quadrilaterals are classified and related to each other.

Usiskin and Griffin (2008) explained that when "one definition purposely excludes what the other definition includes, we call the one definition an *exclusive definition* and the other definition an *inclusive definition*" (p. 4). For example, if a trapezoid is defined exclusively as "a quadrilateral with exactly one pair of parallel sides" (p. 27), then parallelograms and trapezoids would be identified as disjoint subgroups of quadrilaterals. However, if the trapezoid is defined inclusively as "a quadrilateral with at least one pair of parallel sides" (p. 27), then all parallelograms would be a subgroup of trapezoids and trapezoids would include the parallelograms. Therefore, as explained by Usiskin and Griffin (2008), while an inclusive definition leads to a hierarchical chain, an exclusive definition leads to a partition chain. Similar to the De Villiers (1994), Usiskin and Griffin (2008) also distinguished between the two types of classifications on the basis of the choice of inclusive or exclusive definitions. They explained *hierarchical classification* as the classification based on the inclusive definition of quadrilaterals and explained *partition classification* as the classification based on the exclusive definition of quadrilaterals. Examining the several geometry textbooks, researchers came up with that those books published before the 1930's were bounded by the exclusive definitions being influenced by the Euclidean definitions (Usiskin & Griffin, 2008). The Euclid's classification of quadrilaterals based on the definitions given in Book I of "Elements" was presented as the following:

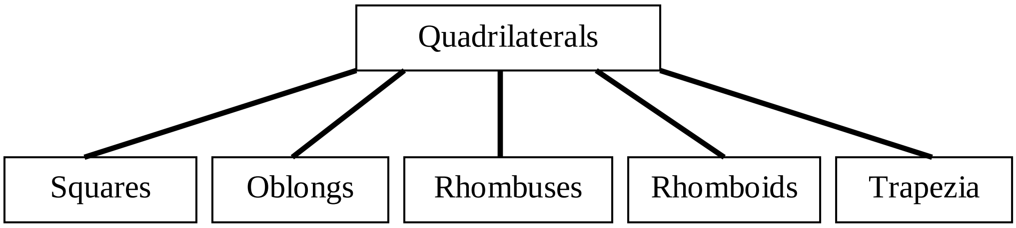


Figure 2.11 Euclid's hierarchy of quadrilaterals. (Usiskin & Griffin, 2008, p. 19)

As it is clear from the Figure 2.11, Euclid's classification of quadrilaterals, which influenced several textbook authors before 1930, was a partition classification based on the exclusive definitions as pointed out by Usiskin and Griffin (2008). Athanasopoulou (2008), who examined Euclidean definitions and classification, explained that Euclid did not include parallelograms in this classification, because he defined parallel lines after the definition of squares, oblongs, rhombuses, rhomboids and trapezia. Thus, he considered each shape as disjoint quadrilaterals. Knowing that Euclidean definitions and classification was used for decades, Athanasopoulou (2008), and Usiskin and Griffin (2008) appreciated the dominance of inclusive definitions and corresponding hierarchical classifications of quadrilaterals in the modern textbooks. However, they also stated that definitions used in the modern classification are still based on the Euclidean definitions; they are just modified version of Euclidean definitions. Canninzzaro and Menghini (2006) also mentioned the problem that because students have been generally taught the exclusive classification of quadrilaterals in the middle school, it becomes difficult for them to understand the inclusive relationships that form the basis of several activities in the high school.

Usiskin and Griffin (2008) structured the two type of general classification of eight special quadrilaterals (Figure 2.12 and Figure 2.13). In Figure 2.12, it is clear that the core center of the inclusive hierarchy is trapezoids since all other classifications are based on the inclusive definition of trapezoids. That is, five quadrilaterals, namely, isosceles trapezoids, parallelograms, rectangles, rhombuses and squares are the special cases of trapezoids; so they have all the properties of a trapezoid. Similarly, squares are both rectangles and rhombuses; rectangles are both isosceles trapezoids and parallelograms; rhombuses are both parallelograms and kites. On the other hand, in Figure 2.13 the core center of the exclusive hierarchy is parallelograms since the classifications are based on the exclusive definition of trapezoid. In this hierarchy, trapezoids, cyclic quadrilaterals, parallelograms and kites are the disjoint subgroups of quadrilaterals. As in the case of Figure 2.12, isosceles trapezoids are both trapezoids and cyclic quadrilaterals and rhombuses are both parallelograms and kites. Moreover, rectangles are both cyclic quadrilaterals and parallelograms.

Usiskin and Griffin (2008) implied the importance of definition of trapezoids saying that "[…] the choice of definition for trapezoid influences the amount of attention one should give to the various types of quadrilaterals" (p. 71). In addition to these two general types of classification, Usiskin and Griffin (2008) further examined the inclusive classifications by defining each quadrilateral in terms of their properties of angles, sides, diagonals and symmetry.

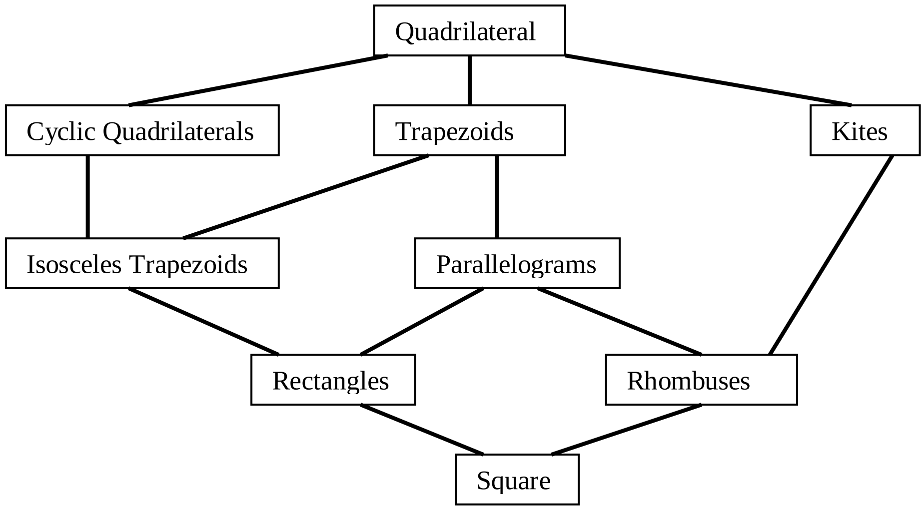


Figure 2.12 An inclusive quadrilateral hierarchy with eight special types of quadrilaterals. (Usiskin & Griffin, 2008, p. 69)

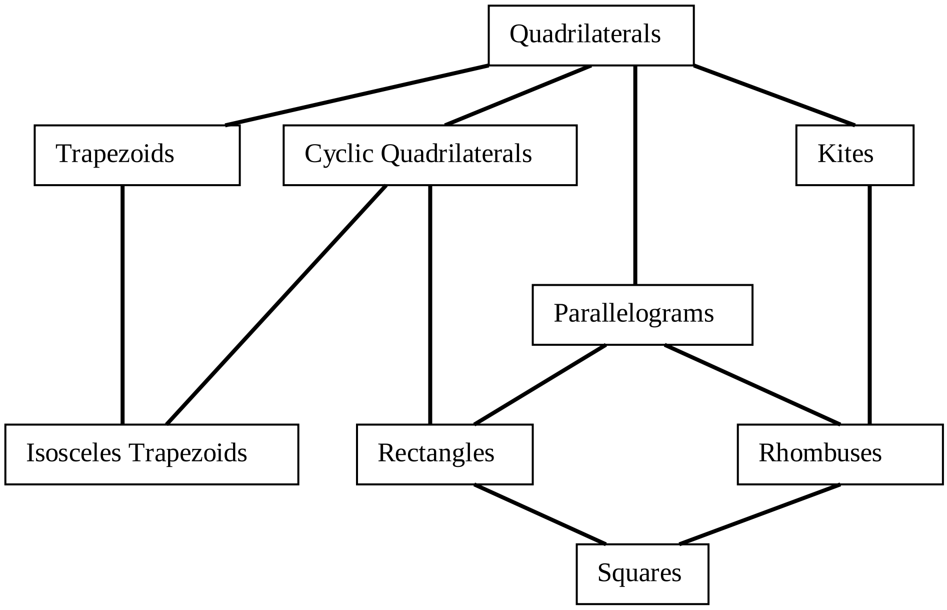


Figure 2.13 An exclusive quadrilateral hierarchy with eight special types of quadrilaterals. (Usiskin & Griffin, 2008, p. 69)

Another researcher who examined the different classifications of quadrilaterals was Graumann (2005). Knowing that there are several possible classifications of quadrilaterals, as he called "house of quadrilaterals" (p. 191), Graumann (2005) complained that students in schools are given an opportunity to examine just one stereotype of classification. He believed that engaging students in the process of developing different classifications of quadrilaterals in terms of their different aspects such as angles, sides, diagonals and symmetries would be a good way to develop their mathematical thinking abilities. Therefore, Graumann (2005) developed some different types of classifications of quadrilaterals which differed from the Usiskin and Griffin's (2008) classification with some additional attributes. In the Figure 2.14, below, Graumann (2005) ordered all types of quadrilaterals:

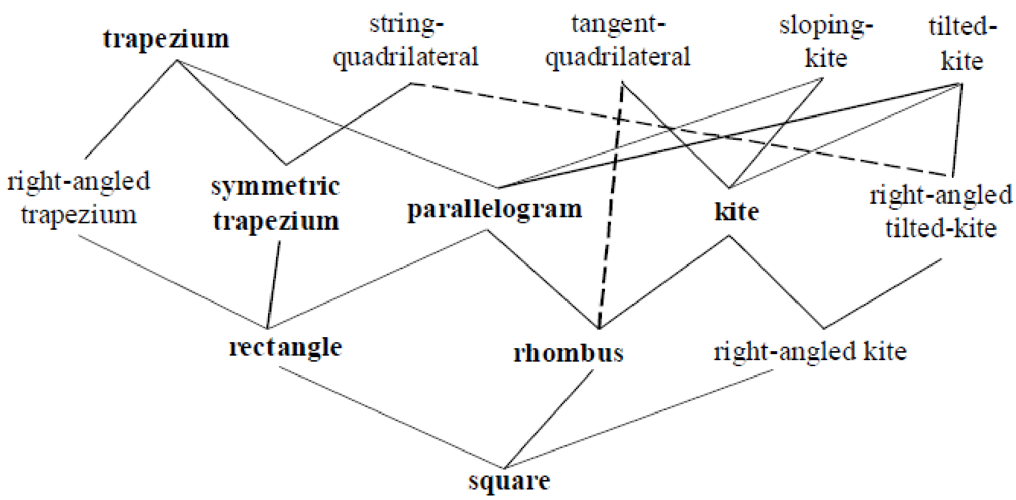


Figure 2.14 The "house of quadrilaterals" – An ordering tree concerning all important convex Quadrilaterals. (Graumann, 2005, p. 193)

It seems a little bit complex classification when compared to Usiskin and Griffin's (2008); because Graumann (2005) divided quadrilaterals as string- and tangent-quadrilateral; kites as sloping, right angled and tilted kite; and trapezium as right-angled and symmetric trapezium. He defined string-quadrilateral as "all four vertices are situated on one circle so that all sides are strings of this circle" (p. 192), and tangent-quadrilateral as “all four sides are tangent to one circle” (p. 192). He defined the tilted- kite as the kite which “we can tilt one of the two angles (together with its sides) without changing the characteristic” (p. 192); and called sloping-kite as “A quadrilateral where at least one diagonal bisects the other one” (p. 192).

Another classification of Graumann (2005) based on the sloping symmetries of the quadrilaterals (Figure 2.15). Graumann (2005) explained the sloping symmetry with the following words:

> With a sloping-symmetry there exists a reflection – not absolutely necessary orthogonal to the axis - which maps the quadrilateral onto itself. For such a sloping reflection the connection of one point and its picture is bisected by the axis and all connections lines point-picture are parallel to each other. (p. 191)

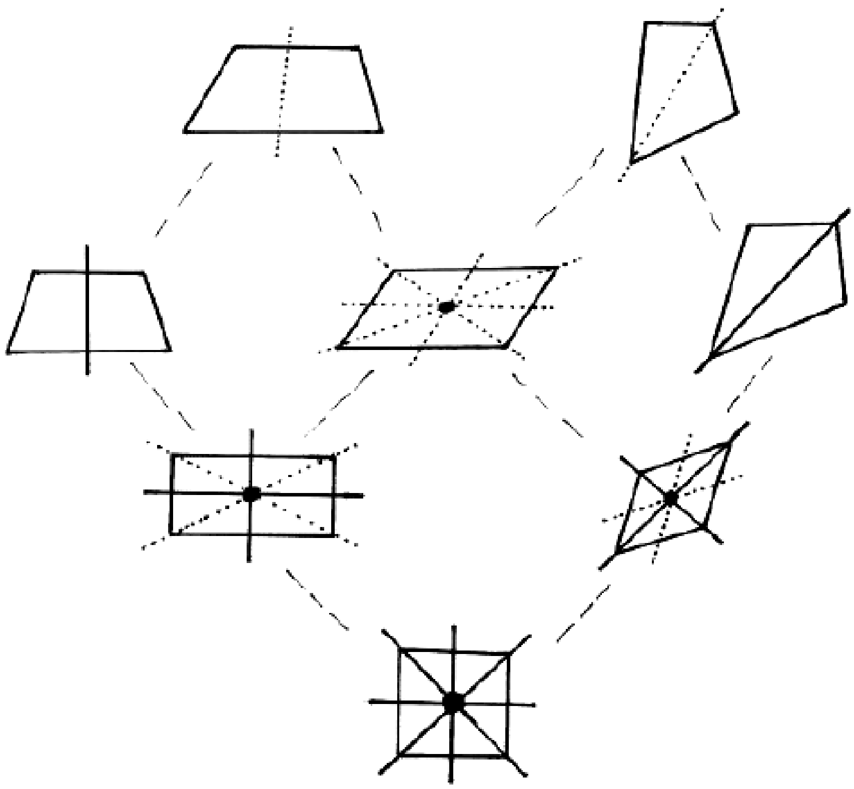


Figure 2.15 The “house of symmetric and convex quadrilaterals” completed with quadrilaterals with sloping symmetry. [The normal lines indicate normal (orthogonal) symmetry while the dotted lines indicate sloping symmetry] (Graumann, 2005, p. 193)

In addition to these two classifications, Graumann (2005) constructed one more classification focusing on the diagonals of the quadrilaterals as can bee seen in Figure 2.16.

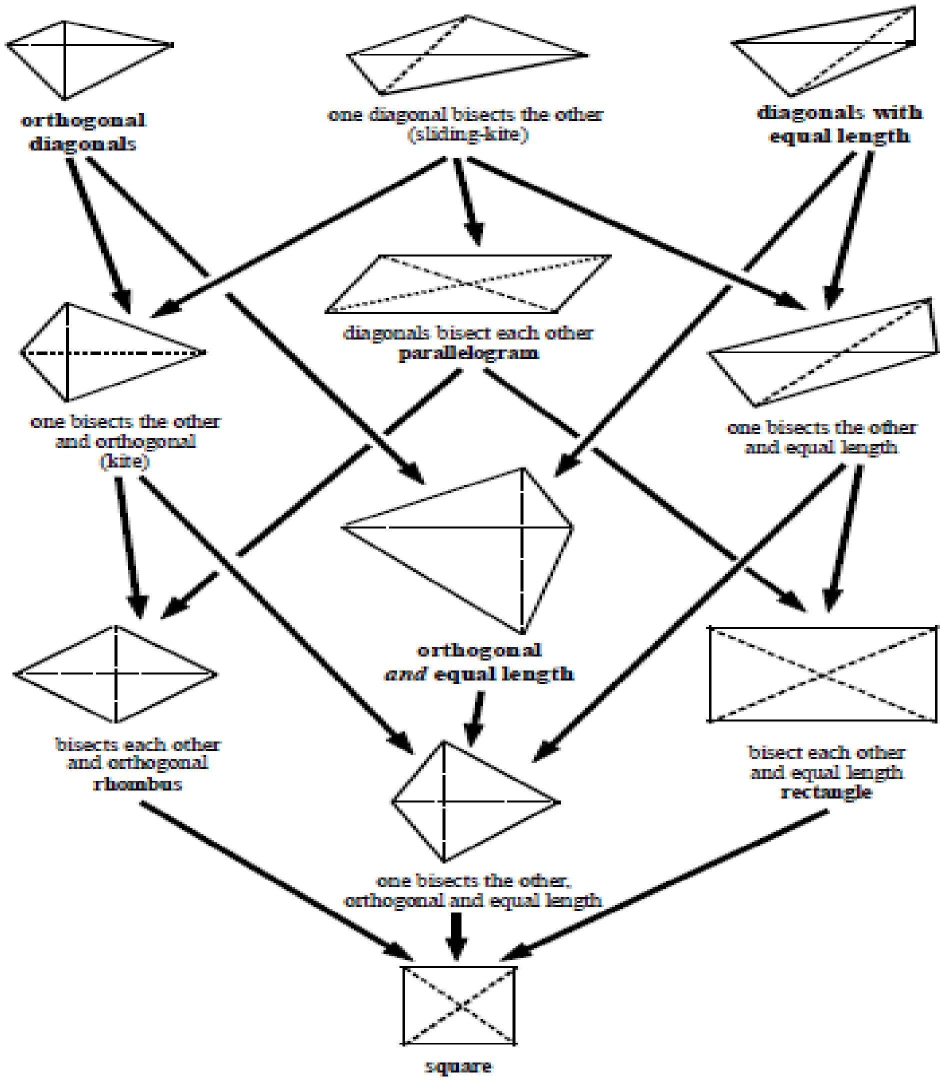


Figure 2.16 A "house of quadrilaterals" concerning diagonals. (Graumann, 2005, p. 194)

To sum up researchers in literature accepted that definitions and classifications are closely related to each other and they can not be treated as isolated process. Moreover, literature revealed that differences in the definitions of concepts led to different classifications to evoke throughout the history. So, many researchers found the quadrilaterals to be the best topic to reveal the different types of classifications based on the definitions of geometric concepts. For the purposes of this study the focus will be on the Usiskin and Griffin's (2008) inclusive and exclusive classifications to investigate learners' reasoning process through definition and classification relationship issue.

2.4 Research Studies

In the literature, several studies were conducted to investigate both pre-service teachers' and students' understanding of the definitions and the classifications of quadrilaterals. While many of the studies just examined the difficulties of learners, some others integrated the dynamic geometry into their study to see whether technology can provide a mediating learning environment to teach the definitions of and the relationships between the geometric shapes. These studies will be addressed in the following paragraphs.

De Villiers (1998), studies of whom were the source of the inspiration for this study, conducted several studies on definition construction in the concept related to quadrilaterals; and he believed that learners can develop their ability to define geometrical shapes if they are actively engaged in definition construction process. Namely, pre-constructed definitions should not be directly given to the students expecting that they will learn them immediately and will be able to use them functionally; instead learners should be allowed to discover the definitions on their own in a construction process. De Villiers (1998), by referring to Freudenthal (1973), defined two types of definitions: descriptive and constructive, which were explained before. In this study he focused on the descriptive defining process in which learners define a concept by organizing their existing knowledge about the concept and its properties, and investigated whether engaging students in a reconstruction process improved their ability to construct formal economical definitions when compared to direct instruction (De Villiers, 1998). For this purpose, De Villiers (1998) assigned 10^{th} grade students to the experimental group in which definitions were introduced with reconstruction, and to the control group in which definitions were introduced with direct instruction. The experimental study focused on the rhombus since students had already encountered with it, but they had not defined it before. When they were asked to define a rhombus at the beginning of the study, students defined it by listing all known properties which was identified as the correct uneconomical definitions and referred to the van Hiele level 2 by the researcher.

Then, to help students develop more economical definitions, the researcher engaged them into the deductive reasoning tasks in which they were given any parallelogram definition and were asked to derive logically all other properties that are not given in the definition. At the end of the study researcher gave some questions to evaluate students' development, and he detected that percentage of students who developed correct economical definitions was higher in the experiment group than the percentage of students in the control group.

In their survey study conducted with 158 trainee elementary school teachers in Scotland, Fujita and Jones (2007) investigated learners' understanding of definitions and their knowledge of inclusive relations between quadrilaterals. The survey questionnaire included two questions: one asked participants to explain their reasoning for the three statements as "Is a X a Y?" and the other one asked them to define some quadrilaterals considering the given definition of a kite and then to draw figures for these quadrilaterals. The researchers found a gap between personal figural concepts and formal figural concepts of the participants. Even though most of the participants were able to draw figures of the quadrilaterals, they had difficulty with defining and classifying quadrilaterals. Moreover, researchers offered for future studies to examine the usefulness of hierarchical classification of quadrilaterals to increase students' level of reasoning from level 2 to level 3.

In another study with 20 secondary school mathematics teachers Shir and Zaslavsky (2001) examined teachers' understanding of mathematical definitions through their judgment of the validity of given list of definitions for a square. Participants were given a questionnaire including eight equivalent statements and were asked to decide whether to accept or reject each statement as a definition of a square and to provide their reasoning for the decision. After working individually, participants worked in groups of 3-5 to discuss their answers and finally they were engaged in a whole class discussion. Despite all eight statements included necessary and sufficient conditions for a square, only five teachers judged all statements as valid definitions. Namely, the study showed that teachers were disagreeing on whether to accept a statement as a definition or not even when it is a simple concept like *square*.

In a recent study, very similar to Shir and Zaslavsky (2001), Zazkis and Leikin (2008) examined 40 prospective secondary mathematics teachers' understanding of the definition of a square and of the specific concepts involved in the definition. In the first task teachers were asked to write as many definitions as they can for a square. The findings for the first task showed that only 5 prospective teachers listed appropriate definitions which included necessary and sufficient conditions in addition to accurate mathematical terminology. However, 26 out of 40 teachers were able to write at least one appropriate definition. One week after the first task teachers were provided with a list of 24 definitions of square which consisted of expert definitions as well as definitions obtained from participants in the first task; however, half of the definitions were mathematically invalid. Teachers were first asked to judge the validity of each definition individually, and then in small groups, and finally within a whole class discussion. The analysis of the second task indicated a disagreement among the teachers to accept a definition as valid or not. That is to say, there was a disagreement among them to differentiate between the necessary and sufficient conditions as in the case of Shir and Zaslavsky's (2001) findings.

For the purpose of investigating the effect of mathematical skills, age and visual factors on the interaction between figural and conceptual properties of geometrical figures during the mathematical reasoning process, Fishbein and Nachlieli (1998) administered a questionnaire to two hundred and eighteen 9^{th}-10^{th} grade students and also interviewed with some of the participants. One of the questions in the questionnaire asked students first to define a specific parallelogram and then to identify the defined parallelogram among the presented images considering the relationships between them. The analysis showed that the percentage of students who defined the parallelograms correctly was higher than the percentage of students who identified the corresponding figures correctly. The researchers attributed this result to the students' dependence on the prototypes while identifying the figures. In another question students were provided with the definition of kite and were asked to identify the kites among the given images. Only 33% of the students were able to identify the kite according to the given definition. Moreover, the researcher found that students' interpretation of the figure on the basis of the definition was not affected by age (grade), but was highly affected by their mathematical capacity.

In another study, Moore (1994) examined university student's difficulties in the process of proving and detected out some difficulties. He found out that the students were not able to construct definitional statements including the correct mathematical language and notation; and they did not know how to initiate the proving process and how to use definitons to prove. Moreover, he detected out that the students did not have enough heuristic understanding of the concepts and sufficient concept images in order to actively take part in the proving process. As it is obvious from the Moore's (1994) findings, most of the problems have their source in students' difficulties with the definitions and this situation explains the crucial role that definitions play in achieving the other mathematical tasks.

In another study that focused on the issue in consideration, Wilson (1990) investigated 6th and 8th grade middle school students' use of mathematical definitions and examples. For that purpose students were administered a paper-pencil test in which students first asked to draw three different triangles, rectangles and squares and then they were asked to give written definitions for triangle, rectangle and square. In the next step, students were asked to select the rectangles, triangles and squares from the given collection of figures. Finally, students' agreement or disagreement for some statements was asked. To further explore students' mathematical thinking in the paper-pencil test, 21 students were interviewed. As a result of his analysis, Wilson (1990) found that across the different contexts, students exhibited ideas that are inconsistent with formal mathematical practice; because, there were inconsistencies between their definitions, identifying examples, and understanding of the squares, rectangles and the inclusive relationship between them. Moreover, most of the students were not aware of their inconsistent ideas. As an example of an inconsistent idea, Wilson (1990) found that "In a written context, 92 students agreed that all squares are rectangles and yet almost half of those students did not circle squares as examples of rectangles" (p. 41). Wilson (1990) attributed students' inconsistent ideas in different contexts to the complex nature of definitions and examples and to the students' insufficient knowledge about the definitions and examples. According to Wilson (1990) inconsistent ideas were originated from students' not understanding the necessary and sufficient conditions for a definition; and from their use of imprecise words and limited prototype figures.

Another researcher Heinze (2002) investigated 106 eight grade students' concept understanding, in particular, whether their understanding of the concepts of squares and rectangles were sufficient to recognize the equivalent definitions, to find counterexamples, and to identify the necessary and sufficient conditions for a definition. Data were collected through a paper-pencil test which included items measuring students' understanding the principles of "definitions, equivalent definitions, arguments and proof, logical implication and counterexamples" (p. 84). Heinze (2002) found out that students were bounded to their own concept image and ignored the concept definitions when they had to use the concepts. Moreover, students had a preference for the partitional classification for quadrilaterals which Heinze (2002) attributed to their misunderstanding the mathematical language and thinking.

In his study Pickreign (2007) conducted a survey with 40 pre-service elementary mathematics teachers to investigate their understanding the attributes of and the relationships between parallelograms. In the survey, teachers were only asked to give written definitions of rectangle and rhombus. The results of the survey showed that pre-service teachers lacked the ability to provide a complete description for these geometrical concepts.

That is, many studies revealed that both teachers and students have difficulty with understanding the nature of definitions of quadrilaterals and that their insufficiency with the definitions prevents them from understanding the relationships between the geometrical figures so that they can classify quadrilaterals.

Depending on the studies which indicated teachers' difficulty with the definitions, De Villiers and Govender (2002) conducted a study to investigate prospective teachers' understanding definitions of geometric shapes in a sketchpad context. They wanted to see whether evaluating the definitions in a dynamic geometry environment improved participants' understanding the nature of definitions. For that purpose the researchers obtained both qualitative and quantitative data from one-to-one task based interviews with 18 prospective teachers. Since previous study (De Villiers, 1998) had shown learners' difficulty with rhombus, it was again the focused concept in this study. For the purpose of investigating participants' prior understanding of the definitions, in the first interview session participants were first asked to define rhombus and then were asked to select possible definitions for a rhombus from a given list of statements. The analysis indicated that participants did not understand the possibility of alternative correct definitions for the same concept; their definitions were uneconomical; furthermore some participants' definitions were incomplete. Based on the analysis result, researchers concluded that participants were at van Hiele level 2 in terms of their geometric thinking.

In the second interview session participants were engaged in a process of evaluating definitions for a rhombus within a sketchpad context. By means of evaluating the definitions in the sketchpad, 17 out of 18 participants were able to identify the correct definitions of rhombus from the given list. The analysis of this session showed that participants developed their ability to differentiate between economical and uneconomical definitions and they developed a better understanding of the arbitrariness of the definitions. However, in the last interview session De Villiers and Govender (2002) investigated participants' ability to evaluate other definitions after the sketchpad session. They were given a correct but uneconomical definition of a rhombus and were asked to evaluate it. 14 out of 18 identified the definition as uneconomical, but they could not explain their reasoning. Moreover, when participants were asked to change the definition into a correct economical one half of these 14 participants were failed, which indicated that they were still at van Hiele level 2.

Although De Villiers and Govender (2002) concluded that sketchpad helped to evaluate the sufficiency of the definitions to some extent, they attributed participants' improvement in the sketchpad environment to their previous experience with the sketchpad, since they could not succeed in the last session. Moreover, having come across with some technical problems during the sketch construction, researchers recommended for the further studies to use sketches of which construction steps that can be seen by using the hide/show button rather than having participants construct the figures. Furthermore, they offered further studies to investigate the hierarchical and partition definitions since these concepts were come into question during their interviews.

Driskell (2004) believed that if students were actively engaged in drawing, constructing, and classifying activities in a dynamic geometry learning environment, they would be able to develop appropriate mental models of shapes and conceptual understanding of their properties. Therefore, in her dissertation Driskell (2004) studied with two pairs of 4^{th} grade students to investigate their ability to identify properties of shapes, ability to see the relationships between parallelograms, and their ability to define two dimensional shapes throughout the task-based intervention sessions with Shape Makers. As a result of her case study analysis of the four participants, Driskell (2004) concluded that as a result of working with Shape Makers students, to some extend, were able to move from informal everyday language to a more formal geometric language while defining the shapes. Moreover, this qualitative analysis indicated that Shape Makers helped students to develop better understanding of properties of quadrilaterals and to see the inclusive relationships between parallelograms.

Other researchers interested in the dynamic geometry, Erez and Yerushalmy (2007), believed that students' understanding the idea that dragging tool of the sketchpad preserves critical properties of a shape is important to construct appropriate concept image of shapes. Therefore, researchers conducted semi-structured interview with ten 5th grade students to investigate their understanding the idea of "properly constructed shape" under the manipulation of dragging tool and also to investigate their understanding the critical properties and hierarchical relations of quadrilaterals. Interviews consisted of two tasks. First, students were asked to identify the properly constructed parallelogram among the three pre-constructed parallelograms two of which were not proper construction of parallelogram. The researchers found that all students were able to identify the proper construction of parallelogram. In the second task, students were given pre-constructed quadrilateral shapes and were asked to answer the question "Do you agree that is an X? Why? Do you think you can turn it into Y by dragging it? Why?" (p. 279). As a result of their analysis researchers found that students were not able to understand the preserved attributes of the shapes when they were dragged. They come up with that it was difficult for students to change their concept image and to understand the idea behind the dragging tool simultaneously. The researchers attributed this difficulty to students' adherence to their previous knowledge and their tendency not to change it. Moreover, understanding the preserved attributes of the shape under dragging requires a formal thinking, but students at this age have a concrete thinking and they are mostly affected by the visual changes (Erez & Yerushalmy, 2007).

In his experimental study Han (2007) investigated the effectiveness of dynamic geometry over the traditional tools in improving students' understanding of quadrilaterals. The fifty seven 8th grade students in the experiment group were taught quadrilaterals through geometers sketchpad-based lessons; on the other hand forty 8th grade students in the control group were taught quadrilaterals through paper-pencil-based lessons in which ruler and protractor were used as teaching tools; however, both groups engaged in a discovery-based instruction. As a treatment session researcher developed two different versions of quadrilateral lessons for each group involving lessons about properties, definitions of quadrilaterals and relationships among them; diagonal properties and construction of quadrilaterals by the help of their properties. As a result of the analysis researchers concluded that Geometer's Sketchpad was more effective than the traditional tools in enhancing students' understanding of properties and definitions of quadrilaterals and the inclusive relationships among them.

In another study Furinghetti and Paola (2002) analyzed students' constructing and classifying quadrilaterals in a Cabri dynamic geometry context. After examining the data, researchers come up with that students have prototype static figures in their mind, but not the properties of the figures. Moreover, they did not tend to check correctness of their figures using the dragging tool; they just considered the particular position of their drawings. Yet, Furinghetti and Paola (2002) concluded that Cabri was an effective tool to identify students' strengths and weaknesses and to provide a learning environment which helps students to develop some kinds of thinking.

In a longitudinal study Jones (2000) examined 12 year old students' use of appropriate mathematical terminology for the geometric figures and their understanding of the relations among the figures in a dynamic geometry context. The researcher designed a teaching unit for the quadrilaterals in cooperation with the classroom teacher. Before and after the application of the teaching unit, students' geometrical reasoning ability levels were measured by means of the van Hiele test. The design of the study based on three phase. First phase of the study was aimed to provide students with a preliminary experience through the use of Cabri for 3-hours. In the second phase students were given visual prompts and they were asked to construct proper figures of rhombus, square and kite so that figures would not change under dragging; and to explain their reasoning of why the figure was a correct construction. In the last phase students were asked to explain the inclusive relationships between quadrilaterals and were asked to develop a hierarchical classification of quadrilaterals along with their explanations for the relationships. Jones (2000) came up with that the teaching unit together with the software helped students to move from everyday informal expressions to more formal mathematical explanations of the geometric concepts. Moreover, he concluded that working with dynamic geometry enhanced students' deductive reasoning skills so that they could better understand the hierarchical relationships between geometrical objects. However, Jones (2000) also warned about that achieving success with the use of software depend on very careful design of the tasks, a careful design of the encouraging classroom environment, and on a skillful teacher to manage the process; otherwise software could be a distracting tool.

Consequently, it can be inferred from these studies that while some researchers who studied for different purposes in different contexts, found the dynamic geometry environment to be effective to some extend in helping learners grasp the relationships between definitions and classifications of quadrilaterals (Driskell, 2004; Han, 2007; Jones, 2000; Furinghetti & Paola, 2002); some others did not found it effective for some reasons depending on the context of the study or on other related attributes (De Villiers & Govender, 2002; Erez & Yerushalmy, 2007). However, depending on the positive findings about the dynamic geometry and believing its ability to make some defining and classifying process easier for the learners, this study will use Geometer's Sketchpad as a dynamic geometry just as a helping tool, but not as the direct focus of the study.

2.5 Pedagogical Perspective on Teachers' Use of Mathematical Definitions

Not only from the mathematical point of view, but the notion of definition was also taken in hand from the pedagogical point of view in the literature and by the curriculum standards. The crucial role of teachers in teaching mathematical definitions through the mathematical language was articulated by The National Council of Teachers of Mathematics [NCTM] standards as the following:

> Teachers can help students see that some words that are used in everyday language, such as *similar, factor, area,* or *function,* are used in mathematics with different or more-precise meanings. This observation is the foundation for understanding the concept of mathematical definitions. It is important to give students experiences that help them appreciate the power and precision of mathematical language. (NCTM, 2000a)

Vinner (1976) stated that most of the time students come to classroom with their personal concept images which are built up through their prior experiences and which may or may not be consistent with the formal concept definitions. When it is the case that students have inappropriate concept images and the concept definitions, Poincaré (1952), emphasizing the teachers' role, offered the following:

> They [students] should be made to see they do not understand what they think they understand, and brought to realize the roughness of their primitive concept, and to be anxious themselves that it should be purified and refined. (p. 123)

That is, even though students come to school with their own comprehension of the images and definitions which are generally supposed not to be correct, it is the teachers' responsibility to change incorrect ideas and to help students develop a comprehension of what defines a shape by going beyond just identifying shapes (Welter, 2001). Therefore, Poincaré (1952) saw the teachers as the key actors to improve students' understanding and constructing meaningful concept definitions.

Thames (2006) pointed out that if teachers are lacking the skills such as examining the mathematical tasks, judging the validity of textbook definitions, providing students with appropriate examples and interpreting students' statements; they will probably mislead their students and also be misled by their students. Besides, he explained the crucial role of definitions in teachers' pedagogical proficiency as the following:

> In addition to knowing the mathematical definitions of terms, teachers must be able to use definitions effectively when teaching. In defining a term, they need to be able to find language that is meaningful to children, yet mathematically correct. (p. 1)

According to Thames (2006) teachers' awareness of the importance of definitions in teaching mathematics and in mathematical arguments; their knowledge of when to call for a definition; their ability to write accurate and usable definitions, and to build up equivalent definitions are the skills that compose the main part of mathematical knowledge for teaching. Since teachers need those skills for an effective teaching, the teacher education programs should be equipped with the appropriate courses that will contribute to the teachers' developing those skills (Thames, 2006).

Asserting that teachers' knowledge of their subjects and how they use this knowledge in the classroom have a significant effect on students' opportunities to learn, Ball and Bass (2003) also dwelled on the notion of definitions as being the central domain of mathematical knowledge for teaching. According to the researchers choice of definitions that are mathematically accurate and usable for students' ability level is an important aspect of teaching skills. In addition, Ball and Bass (2003) stated that when required teacher will resort to the textbook definitions; however, judging the validity of the definitions given by the textbook and assessing their appropriateness to the students' level is the responsibility of the teacher. If the given definitions are not in accordance with the students' level, or including terms that are beyond the students' knowledge, teacher should be able to construct a more suitable definition, which requires teachers' knowledge above a formal concept definition given in textbooks (Ball & Bass, 2003; Ball, Bass, & Hill, 2004). Moreover, Ball, Bass and Hill (2004) explained that mathematical definitions help individuals to make clear arguments so that they can communicate effectively, and that definitions are crucial for developing an accurate mathematical reasoning, too. For that reason, more important than learning formal definitions in mathematics courses, teachers should also understand definitions' role and function in the classroom and should know the ways how to integrate them into teaching process effectively so that these definitions would be more meaningful to students (Ball, Bass & Hill, 2004).

Having been investigated teachers' definition construction ability, Zazkis and Leikin (2008) also emphasized the definitions' being the building blocks of the subject matter knowledge of mathematics teachers. They believed in the impact of teachers' knowledge of mathematical concept definitions on their decisions of the ways of teaching and on their pedagogical sufficiency. Moreover, Chinnappan and Lawson (2005) argued that teachers' use of mathematical language associated with geometrical shapes plays a crucial role on students understanding these shapes and the relations between them. So, according to the researchers, teachers' use of words to define shapes reflects their geometric knowledge for teaching.

From the pedagogical side of view, De Villiers (1998) criticized the teachers' simply giving the mathematical definitions to their students without any emphasis on the definition construction process. He argued that students can understand the definitions and can use them functionally if only they are engaged actively in the definition construction process, but not by means of the direct teaching. Mariotti and Fischbein (1997) also saw the teachers as the key actors to guide definition construction and to create a mediating learning environment. Similarly, Winicki-Landman and Leikin (2001) stated that the choice and use of definitions in the classroom teaching is a fundamental part of a teachers' pedagogical content knowledge, and level of this knowledge plays a crucial role in their flexibility with teaching process and with different student reactions. Having been examined teachers' written definitions for some geometrical shapes; Pickreign (2007) came up with that teachers need help to further develop their understanding of the mathematical concepts and concept definitions. For that he criticized the insufficiency of teacher education programs and defended that teachers should be given opportunity to experience these geometric ideas through the effective teacher education courses. Gutierrez and Jaime (1999) also made some suggestions for the teacher education programs after they found out the deficiency in teachers' concept images and concept definitions of a specific geometrical concept. They recommended that teacher instruction programs should consider teachers' prior concept images and should provide them with required experience to eliminate the gap between their personal concept images and the formal mathematical definitions, if exists any. Moreover, teachers should be given opportunity to discuss their perceptions of mathematical concepts so that they can eliminate the conflicts in their minds. Teachers should also be provided with the real examples of students' concept images and should be asked to analyze them to reflect upon their own concept images; and teachers might be given an opportunity to analyze some research studies on the concept images and concept definitions to reflect on their own concepts (Gutierrez & Jaime, 1999).

To conclude, the importance of knowing and using definitions effectively in the classroom teaching is accepted as an important component of teachers' knowledge for teaching mathematics. However, the result of the several studies' revealing teachers' difficulty with understanding the definitions raise the concern about the insufficiency of teacher education programs. Being aware of that, researchers dealing with teacher education suggest improvements in teacher education programs (Gutierrez & Jaime, 1999; Pickreign, 2007; Thames, 2006), which is the expected contribution of this study.

2.6 How to Handle Definitions in Geometry Instruction

While the definitions' significant role in the learning geometry concepts has been accepted in the literature, how to handle definitions in instruction needs to be elaborated on. A concept image is the sensorial component of the geometrical concept which is formed through student intuition and it helps to invent the practical meaning in the mental activity of concept formation (Fishbein & Nachlieli, 1998); but as Kondratieva1 and Radu (2009) stated "… intuition cannot give us exactness, not even certainty…" (p. 216). So, to be able to ensure logical meaning and consistency in the arguments, there is need to introduce exactness in definitions which control the images (Fishbein, 1993; Kondratieva1 & Radu, 2009). Moreover, the concept definition alone does not ensure the conceptual understanding of the concept, that is to say, a student can state the correct definition of a parallelogram when asked but may not consider square, rectangle and rhombus as specific cases of parallelogram since the student has an incorrect concept image of a parallelogram which does not allow the all sides and/or angles to be equivalent (De Villiers, 2004). Therefore, a robust concept formation requires the interaction between these conceptual and sensorial components (De Villiers, 2004; Fishbein, 1993; Fischbein & Nachlieli, 1998). Although the images and concepts can sometimes cooperate or sometimes conflict in students' cognitive process, the aim of the given mathematical tasks should be to help learners how to investigate the situations causing conflicts in their mind and to deal with these conflicts by using conceptual control (definitions) over the figural invention process (Fishbein & Nachlieli, 1998) so that they can create cooperation and harmony between the images and definitions (Fischbein, 1993).

In addition to these mathematical considerations, daily life language of the learners should be changed into to more formal mathematical language by the help of definitions; because definitions are the fundamental bases of mathematical language due to their being most important tools to deliver the meaning of mathematical concepts (Vinner, 1991). Therefore, during mathematics teaching it is important to engage students in an interaction with definitions in order to underline the importance of mathematical language and to improve their understanding and using mathematical language (Vinner, 1991). So, "concept formation starts with the development of vocabulary and the recognition of shapes towards the identification and association of characteristics" (Kotzé, 2007, p. 23).

In the light of the literature at the outset of the instruction, the first thing to do should be to reveal whether students have an inappropriate concept image of a particular concept or not. This can be easily understood by asking students to judge the instances of a concept (Alcock & Simpson, 2009); for example, learners can be asked to identify parallelograms from a given set of shapes to see whether they identify the square and rectangle as instances of parallelogram. If they have appropriate concept image the next step would be moving them from using everyday language to more formal technical language and engaging them into the tasks where they will discover the further properties of the concept and use them to construct formal concept definitions themselves. However, if they are lacking the appropriate concept images the focus of instruction would be on helping students to reshape their naïve concepts through engaging into the activities which will require consulting to the concept definition as well to the concept image and will create a contradiction in their mind (Vinner, 1991).

Geometric concept formation is a multifaceted process which includes visual, spatial and measurement skills together; but visual ability is specific to the geometric concept formation which distinguishes it from the concept formation in other mathematics fields (Walcott, Mohr, & Kastberg, 2009). So, it is important to engage learners in activities in which they will both control the visual image and investigated properties of concepts to improve their geometrical thinking in the study of defining and classifying (Fujita, 2008). So, it is clear that presentation of the definition of a concept at the beginning of a unit in a textbook or at the beginning of a lecture by the teacher would not help students construct a meaningful representation of the concept; but engaging them in the definition construction process after several experiences with the concept of interest would be an important part of mathematics learning which would help students to interpret the formal definitions given in the textbooks and to build understanding of that concept (De Villiers, 1998; Shield, 2004).

Before engaging students in the process of defining geometrical concepts, there is a need for making a distinction between describing and defining for the purposes of instruction because they are two different processes requiring different abilities (Favilli & Romanelli, 2006; Monaghan, 2000; Raman, 2002) of which teachers and learners must be made aware of. When a concept is identified by listing all the properties of it, the concept is only described, which is a reasoning ability at van Hiele level 2 (De Villiers 1996, 1998). However, constructing a formal mathematical definition of the concept requires high level reasoning skills such as explaining the relations between the properties and making deductive reasoning among a set of properties in order to distinguish between necessary and sufficient properties for characterizing a concept (De Villiers & Govender, 2002; Favailli & Romanelli, 2006; Fujita & Jones, 2007) which is a reasoning ability at van Hiele level 3 (De Villiers, 1996, 1998). For example, if we define a parallelogram as *a quadrilateral with two pairs of parallel and congruent opposite sides, two pairs of congruent opposite angles and with diagonals bisecting each other*, we only describe the parallelogram. That is, just listing the describing properties of a concept is not defining process (Favilli & Romanelli, 2006); defining requires to understand the interrelations between the properties and make deductions such as *opposite angles equal* property comes from *opposite sides parallel* property and vice versa; or *congruent opposite sides* property implies *bisecting diagonals* and vice versa. After such kind of deductions one can come up with necessary and sufficient conditions to identify a concept and can construct formal mathematical definitions of a parallelogram such as *a quadrilateral with two pairs of opposite parallel sides* or *a quadrilateral with opposite congruent sides* from which all other properties can be deduced by logical reasoning.

According to Winicki-Landman and Leikin (2000) while defining a mathematical concept the following logical principles should be met:

1. Defining is giving a name. The name of the new concept is presented in the statement used as a definition and appears only once in this statement.
2. For defining the new concept, only previously defined concepts may be used.
3. A definition establishes necessary and sufficient conditions for the concept.
4. The set of conditions should be minimal.
5. A definition is arbitrary. (p. 17)

There are several necessary conditions that are the properties of a mathematical concept and among these necessary conditions there are several sufficient conditions that are sufficient to characterize the concept. By means of logical deductions between the properties of a concept, several equivalent classes of statements that include both necessary and sufficient conditions can be constructed. While any one of these equivalent statements can be used as definition of the concept the remaining statements in the equivalence class become theorems to prove. For example, if a rectangle is defined as *a quadrilateral with three right angles*, then a theorem like *if a quadrilateral has three right angles, the fourth angle is also a right angle* can be proved. Then we can say that an uneconomical definition including redundant information, let's say *a rectangle is a quadrilateral with four right angles,* includes a definition and at least one theorem (Van Dormolen & Zaslavsky, 2003). Namely, definition construction process is a very difficult process for learners since it requires high level reasoning as explained before and the most critical cognitive ability in defining is to distinguish between the properties that describe a concept and the properties that define it; and at this crucial point the use of dynamic geometry based tasks can help learners to overcome these difficulties to some extent (De Villiers & Govender, 2002; Favilli & Romanelli, 2006).

Believing the difficulty of learning geometry concepts Bender and Schreiber (1980) proposed the principle of operative concept formation (POCF) which combined three traditional ways of introducing concepts: "(a) by definition (language), (b) by giving examples (intuition), (c) by drawing (construction)" (p. 59). According to this theory a concept can not be isolated from the other neighboring concepts which influence the formation of one another; for instance to be able to define a straight line as the intersection of two planes, the concept of plane also needs to be defined (Bender & Schreiber, 1980). Therefore understanding the relationship between neighboring concepts requires "developing a terminology (definitions), acquiring factual knowledge (propositions, examples), and providing algorithms (for realizations, constructions, measurement)" (Bender & Schreiber, 1980, p. 79).

Moreover, Morgan (2005) claimed that using definitions for differentiating between examples and non-examples of the given concept increases the learners' awareness and understanding of the concept and enhances the use of correct mathematical language. Activities such as discussing the definitions and alternative definitions for a concept, making deductions from the definitions and proving arguments on the basis of the definitions are important to help students develop ways of mathematical thinking (Morgan, 2005). Furthermore, engaging students in tasks which ask them to organize several properties of a geometry concept to generate a definition; to deduce other properties from the definition of a concept and to classify shapes based on their properties would increase learners' deductive and inductive reasoning ability and would teach them how to define (Freudenthal, 1971).

According to Shield (2004) there are two effective way to consider in the definition construction process: to be able to define a concept with its unique critical properties and to be able to realize the grouping within definitions which will help students understand membership of their particular concept to a class of concepts. That is to say definition and classification issues are closely connected with each other in the sense that the properties given in the definition allow us to include the concept being defined to a class of objects which have these properties (Poincaré, 1952). The availability of many different ways of defining a geometrical concept and discovering many different ways of classifying in these definitions can increase the learners' awareness of important geometrical relationships and their understanding of the hierarchical relations between concepts (Shield, 2004).

Depending on the purpose of the instruction, definition can lead to hierarchical or partition classification based on its being an inclusive or an exclusive definition respectively, as explained before (De Villiers, 1994; Usiskin & Griffin, 2008). However, hierarchical classification is the mostly preferred one due to its several advantages such as it allows for more economical definitions and for several alternative definitions since the concept is defined in terms of more general concepts; moreover, a true statement for a concept in this type of classification becomes also true for all specific subsets of the concept. (De Villiers, 1994; Favilli & Romanelli, 2006; Fujita & Jones, 2007). On the other hand, while engaging students into the hierarchical classification activities their ability to realize the transitivity, asymmetry, and opposite asymmetry of relations among the shapes should be given specific attention (Fujita & Jones, 2007). For instance, understanding class inclusions between concepts requires the ability to define the concept differently in terms of other more general concepts; to make transitive reasoning such as if a square is a rectangle and a rectangle is an isosceles trapezoid then a square is an isosceles trapezoid; to understand lack of symmetry within the relations like a square is a rectangle but a rectangle is not a square; to understand the opposite inclusive relationship (Schwarz & Hershkowitz, 1999) between the concepts and their properties such as a square is a rectangle a rectangle is not a square; while all properties of a rectangle are valid for a square but all properties of a square are not valid for a rectangle. That is to say making students aware of this complex nature of definition and classification issues is a really difficult task and if these required abilities are not given specific attention, the teaching process can lead to misunderstanding of class inclusions.

Graumann (2005) complains that students in schools are given an opportunity to examine just one stereotype of classification. He believes that engaging students in the process of developing different classifications of geometric shapes in terms of their different aspects such as angles, sides, diagonals and symmetries would be a good way to develop their mathematical thinking abilities. Moreover, although inclusive definitions and corresponding hierarchical class inclusions are supported in the literature it would be a good practice to help learners realize the differences between the inclusivity and exclusivity of definitions and their role in the different classification ways of the concepts (Usiskin & Griffin, 2008). For instance, if a trapezoid is defined exclusively as "a quadrilateral with exactly one pair of parallel sides" (Usiskin & Griffin, 2008, p. 27), then parallelograms and trapezoids would be identified as disjoint subgroups of quadrilaterals. However, if the trapezoid is defined inclusively as "a quadrilateral with at least one pair of parallel sides" (Usiskin & Griffin, 2008, p. 27), then all parallelograms would be a subgroup of trapezoids; thus trapezoids would include the parallelograms. So, as explained by Usiskin and Griffin (2008), while an inclusive definition leads to a hierarchical chain, an exclusive definition leads to a partition chain. Therefore, allowing students to experience with exclusive and inclusive definitions would help them discover the hierarchical and partition relations between geometric objects.

To sum up, not in the everyday contexts but in the technical context it is expected from learners to consult formal definitions while reasoning about a concept so that they can change the everyday thought habits to the mathematical thought habits and reshape their naïve concept images to catch the consistency with the formal concept definitions. However, presenting students with the pre-constructed definitions does not help them to learn the related concept; but if they are allowed to explore the concepts and to organize their ideas about the concept through these explorations then they can construct their own definitions as a result of this process and reshape their concept images accordingly (De Villiers, 1998; Shield, 2004). Therefore, the most important use of definitions to help students meaningfully understand a concept and its relationships with other concepts would include asking students to define a concept with its unique critical properties and to organize several properties of a geometry concept to generate a definition; to distinguish between examples and non-examples of concepts and between attributes and non-attributes of figures; to deduce other properties from the definition of a concept and to classify shapes by identifying the grouping within definitions which will help students understand membership of their particular concept to a class of concepts (De Villiers, 1998; Freudenthal, 1971; Shield, 2004; Pimm, 1993; Shir & Zaslavsky, 2001;Vinner, 1983).

2.7 Possible Weaknesses and Limitations of Use of Definitions in Learning Concepts

Although the definition construction process is confirmed as being effective in concept formation process there is another side of the issue that needs to be considered: the use of definition may sometimes have some limitations or weaknesses due to some reasons. Hansen and Pratt (2005) claimed that “Understanding of geometric definitions is a complex area to study” (p. 408). According to the researchers learners’ creating prototypical shapes, inclusivity and exclusivity of definitions and the use of several attributes to define shapes are the three reasons of the complexity in the field of geometry (Hansen & Pratt, 2005). For instance, Leung (2005) claims that although the process of defining a concept with its necessary and sufficient conditions sounds effective in constructing a definition, the decision of which property is critical can be problematic to some extent. Vinner (1983) claims that if students are not given a chance to deal with several examples of a concept but only with specific set of examples they may develop wrong concept images. For instance, if we only present the isosceles triangles with a horizontal basis, when the learners encountered with isosceles triangles which do not have a horizontal base they probably will not see them as isosceles triangles (Vinner, 1983).

Another difficulty in the geometry concept learning might be the prototypes, a kind of a cognitive conflict, which may occur due to the opposite inclusive relationship between the geometrical concepts and between their attributes (Schwarz & Hershkowitz, 1999). For instance, quadrilaterals include parallelograms and parallelograms include squares; on the other hand, squares include all critical attributes of parallelograms and parallelograms include all critical attributes of the quadrilaterals (Schwarz & Hershkowitz, 1999). These structure of geometrical concepts leads to one or more prototypical instances of each concept and learners' concept image generally develop from these prototypical examples which are retrieved first when they are asked to reason about that concept (Schwarz & Hershkowitz, 1999). Learners' prototypical judgment might base on the visual image or on the self attributes of this prototype, which most probably causes them to come up with incorrect judgments (Hershkowitz, 1989 as cited in Schwarz & Hershkowitz, 1999). For example, the students may not identify an instance of a concept as an instance of that concept, because they may think that it does not fit to the visual appearance of their prototypes (Hoffer, 1983 as cited in Schwarz & Hershkowitz, 1999) or they may think that the instance of the concept does not have the self attributes of their prototypes (Hershkowitz, 1989 as cited in Schwarz & Hershkowitz, 1999). Students' judgment based on a prototypical image also prevents them from understanding the class inclusions since their prototypes will not carry all characteristic features of the class of concepts represented by this prototype (Kondratieval & Radu, 2009). Even the prototypical judgment can limit the dynamic geometry investigations if the technology is not supported with appropriate teaching strategies; so, even in the technological learning environment understanding the definition construction process can remain as a challenge (Connor & Moss, nd.).

Furthermore, although the definition construction process is important for improving learners' understanding of some concepts, some other concepts can be too complicated and they may not allow easy construction of a definition (Alcock & Simpson, 2009; Shield, 2004). Even if such kind of concepts are defined in some way in textbooks or somewhere else, these definitions probably will not be able to contribute learners' meaningful concept learning (Shield, 2004). For example, Shield and Dole (2002) found out that the definitions of the concepts of fractions, ratio, rate and proportion in the textbooks are not able to make students fully understand the nature of these concepts and relationships between them. That is to say, it is probable that the information given in a definition for multifaceted concepts would even make an impression on students that mathematics is a "meaningless, rule-dominated and highly specialized subject, accessible to few people" (Shield & Dole, 2002, p. 615). The use of formal definitions still important to make students understand such kind of concepts; however, they may not be sufficient every time due to the complex and multifaceted nature of the these concepts of interest (Shield, 2004).

To sum up, even though several type of definition activities are claimed to be effective in learning of geometry concepts, learning is a complex process. Therefore, it can not be claimed that the use of definitions in concept formation will lead to a full understanding of the concept; of course there will be some limitations or weaknesses of using definitions. To cope with the prototypes (Hansen & Pratt, 2005; Schwarz & Hershkowitz, 1999), with the complication of several attributes (Hansen & Pratt, 2005; Leung, 2005), and with the complex and multifaceted nature of some concepts (Shield, 2004; Shield & Dole, 2002) will probably interfere with the process. However, even if there are such limitations in using definition construction process, being aware of the problems which interfere with students' concept learning can help educators to better understand thinking and reasoning process of their students. By the way, along with these clues teachers would look for the ways to develop new methods and to improve their instruction.

2.8 Facilitator Role of Instructional Technology in Mathematics

It is clear that in the last decade incredible advances in technologies not only have caused a revolution in our life; but also technological developments provided education systems with tools that facilitate teaching and learning process. Along with these advances in educational technologies professional organizations such as The National Council of Teachers of Mathematics (NCTM), International Society for Technology in Education (ISTE) and the National Council for Accreditation of Teacher Education

(NCATE) started to release math standards that encourage the use of technology in mathematics classrooms. For example, NCTM encouraged the use of technology stating the principle: “Technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning” (NCTM, 2000). ISTE performance indicators offer teachers to become fluent users of technology and collaborate with others to support effective use of technology in their classrooms, to use digital media resources to communicate with students, parents and peers, to use digital tools to support research and learning (ISTE, 2008, p. 1)

Moreover, if the educational technology principles of the curriculum programs released by TTKB (2007) are examined, it is seen that innovative approaches in the sense of technology usage for educational purposes have taken their place in the program. In the new Turkish elementary mathematics education program it is stated that the aim of the program is not to make use of the information technologies, but to make an efficient use of them to reach the intended educational goals. Both the elementary and secondary grade programs aim to improve learners’ skills of using technology for the purpose of searching, finding, processing, presenting and evaluating the information. In the elementary and secondary curriculum program calculators, computer algebra systems (CAS), and computer assisted instruction (CAI) are encouraged; besides, computers are seen not as optional tools for CAI, but as complementary contributors of the whole system. Moreover, teachers are recommended to prepare effective tools consistent with the constructivist notions of CAI and to guide the activities to make learning and teaching process easier (TTKB, 2007).

Astonishing effect of new technological tools on mathematics teaching relies on their interactive, dynamic nature and capability to provide more than one form of information (multiple representations) at once on the computer screen (Suh & Moyer, 2007). That is, technology provides learners a different experience with mathematical concepts than the paper-based experiences due to its ability to present multiple representations of the problems and to allow discovering the patterns and conjectures by manipulating the data (Orrill, Ledford, Polly and Erbaş, 2004). Moreover, the capability of technology to provide multiple representations allows students to analyze the symbolic, graphic and numeric form of the data at once on a computer screen which helps them better see the relationships between mathematical concepts (Erbaş, 2005).

Today's technology with its dynamic, visual and manipulable nature also annihilates the abstractness of mathematical concepts through engaging students in the activities which facilitates mathematical connections by linking representations and connecting mathematics to the real life phenomena (Alagic, 2003). That is, technology's ability to attract students' attention by relating subject matter to their real life experiences makes technology very appealing to educators (Akkoyunlu, 2002). Moreover, technological tools are able to emancipate the repetitive and tedious work by acting out several time consuming procedures such as computing or graphic construction which allows students to focus on other vital cognitive thinking skills such as problem solving, pattern identification or conjecture making (Warwick, 1993; Orrill, Ledford, Polly, & Erbaş, 2004). The feature of technology to allow focusing on higher order thinking skills is also stated by NCTM as "As some skills that were once considered essential are rendered less necessary by technological tools, students can be asked to work at higher levels of generalization or abstraction" (NCTM 2000, p. 26).

In addition to the dynamic nature that allows users to flexibly construct and manipulate multiple representations, technological tools also provides *a student-friendly interface* since they provide easy use functions; *support for visualizations* since several visualization tools allow learners to give meaning to difficult concepts; and co*nstructive representations* since they provide constructive learning environments where students can construct their own understanding aligned with content and pedagogy (McCoy, 1996).

Jean Pedersen described geometry as "a skill of the eyes and the hands as well as of the mind" (as cited in Mackrell & Johnston-Wilder, 2004, p.81). Similar to Pedersen, Duval also stated that "a geometric activity involves three kinds of cognitive processes which are visualization, construction and reasoning" (as cited in Laborde, Kynigos, Hollebrands & Strasser, 2006, p.276). That is to say, geometry is a challenging issue and requires the learner to be able to use more than one skill in the process of meaningful learning; therefore, it is important to provide learners with appropriate resources that will develop their hands-on, minds-on and eyes-on skills. According to Ahmad & Zaman (n.d.) "teaching geometry with the assistance of computers would allow students to move from empirical to logical thinking; encourage students to make and test conjectures, facilitate precision and exactness in geometric thinking; encourage the development of autonomy, and act as a mirror, reflecting the geometric thinking of students for teachers and themselves" (Ahmad & Zaman, n.d., p. 2). So, benefiting from the facilities of technology in teaching geometry has a crucial importance on teaching and learning geometry concepts.

With the new developments, educational technologies went beyond the calculators and internet, and the new innovations such as spreadsheets, dynamic geometry and computer algebra software have become prevalent as effective mathematics teaching and learning tools all over the world (Haspekian, 2005). Among these tools dynamic geometry software such as Shape Makers, Geometer's Sketchpad and Cabri allow users to construct geometric objects of which geometric relations are preserved when dragged, which is a characteristic that helps user to make conjectures about geometrical relations and to discover them (Harper, 2003). Namely, dynamic geometry with its hands-on, minds-on and visual applications is the most appropriate resource to meet the needs of students in learning geometry and to overcome abstract nature of the geometry concepts.

For the purpose of this study, a dynamic geometry tool "Geometers' Sketchpad" will be used as a supportive instrument during the clinical interview sessions. This technology has been preferred due to its several functions with regard to the concept of quadrilaterals. Geometer's Sketchpad allows learner to construct geometric objects of which specific relationships are preserved under dragging (Erez & Yerushalmy, 2007; Warwick, 1993). Geometer's Sketchpad's function of preserving critical properties under moving objects is expected to help participants of this study to discover the critical attributes needed to define quadrilaterals and to grasp the idea of inclusive relations among the quadrilaterals

On the other hand, learning is a complex process in which the interaction of students with the technological tool is not sufficient to enhance learning. The appropriateness of the task for the use of technology, the social climate of the classroom, applied teaching methods; moreover, the skills and knowledge of the teacher for the effective use of the technology are all the complementary features for effective learning. Mackrell & Johnston-Wilder (2004) discussed exploratory and expressive approaches of using interactive geometry while other researcher Fischbein used different terms "figural and conceptual" to define the same approaches (as cited in Laborde, Kynigos, Hollebrands & Strasser, 2006, p.277). That is to say, in the expressive approach students are expected to construct their own figures using the software; while in the exploratory approach they are expected to explore the geometry concepts on pre-constructed figures (Mackrell & Johnston-Wilder, 2004).

Both approaches, expressive or exploratory, might have pros and cons in different situations. At first sight, the expressive approach seems more effective to develop students' creativity in the constructing process and it also seems effective to develop their better understanding of the properties of the figure which they created. On the other hand, application of expressive approach can be problematic in some way (Mackrell & Johnston-Wilder, 2004). For example, if students are not familiar with the properties of the software, they may have some troubles while constructing; and the process may become more complex for them. On the other hand, it can be more practical and less time consuming to provide students with pre-constructed files to work on them and to explore the geometric ideas; but of course, specific learning objectives should be considered while deciding on which approach to use. That is to say, technology usage in geometry teaching serves to different purposes; and this situation indicates that technology is just a tool to reach the intended goals, but the most important is to know when to use how to use and in what conditions to use the technology in order to reach the intended goals. I mean if the educational objectives are the destination, technology is the vehicle to take to the destination, but not the only one vehicle.

In this study, both expressive and exploratory approach will be used in order to obtain most useful information from the participants; but predominantly pre-constructed sketches will be used. The participants will construct their own sketches only when they are testing whether the properties given in the definition lead to the correct construction of the geometry concept. That is, participants will be asked to make constructions only when it is necessary; because it is not the focus of this study to examine the construction process of the figure in the dynamic geometry environment.

CHAPTER III

METHODS

The purpose of this study was to examine pre-service middle school mathematics teachers' cognitive progress in constructing and evaluating quadrilateral definitions and the corresponding quadrilateral hierarchies under the support of the Geometer's Sketchpad learning activities. Pre-service teachers' perceptions of the definitions and their views about the use of definitions in the teaching process, their understanding of the nature of definition construction, difficulties with the definition construction and underlying reasons of these difficulties, confidence about the definition construction skills and flexibility with the definitions, previous experiences with definitions, misconceptions related to the quadrilaterals and sources of these misconceptions, understanding of the inclusive and exclusive relationships between quadrilateral concepts, prototypical quadrilateral images as cognitive conflicts in the minds, and use of the mathematical language are all in the scope of this study.

Throughout this chapter, the method of the study and related issues such as the design of the study, the context in which study held, participants and their specific charactersitics, selection of the participants, the data collection process and instruments, analysis methods; and some quality issues such as trustworthiness, credibility, dependability, transferability and confirmability issues were addressed.

3.1 Design of the Study

This study aims to meet the need for an in-depth analysis to explore the pre-service middle school mathematics teachers' thinking process in defining and classifying geometric concepts by qualitatively analyzing the issue through case study design.

Case study is a qualitative design which aims an in-depth analysis of an issue through single or multiple cases and by means of detailed multiple data collection sources (Creswell, Hanson, Clark, & Morales, 2007). So, qualitative case study provides researchers with opportunities to understand many sides of a complex phenomenon within its context and to gain great deal of insight into a case (Baxter & Jack, 2008).

Yin (2003) differentiates between three types of case studies that can base on single or multiple cases: exploratory, explanatory, and descriptive. While exploratory case study aims to develop hypothesis for further investigations, explanatory case study aims to explain causal relations that are too complex for quantitative designs and the descriptive case study aims to describe a phenomenon within its own setting. Stake (1995) also differentiates between intrinsic, instrumental and collective studies. Intrinsic case study is used when the primary interest is to better understand a particular case. On the other hand, if the purpose of the case study is to intuitively grasp the inner nature of a particular phenomenon where the case is used to understand this phenomenon, Stake (1995) suggests using an instrumental case study. In instrumental case study, case is a "secondary interest; it plays a supportive role, facilitating our understanding of something else." (Stake, 1994, p. 445). That is to say, a case is studied in order to thoroughly understand and get insight into the phenomenon of interest. Moreover, a researcher can investigate the phenomenon by selecting more than one instrumental case which belongs to a particular group of cases and are similar in some ways, which is called collective case study or multiple case study (Stake, 2005). That is to say, multiple case study design is a collection of instrumental case studies. In multiple case study design, a researcher has two responsibilities: researcher as a director of the study should consider all cases collectively; however, researcher as data collector should focus all attention on each single case and try to understand single case at a time (Stake, 2005).

Stake (2005) invented an unfamiliar word "quintain" to define "an object, a phenomenon or condition to be studied" (p. 6). In multicase study, the researcher starts with the quintain; then s/he studies single instances of it and similarities and differences between these single instances to generate a better understanding of the quintain. So, multi-case study differs from the single case study in the sense that "the ultimate question shifts from 'What helps us understand the case?' toward 'What helps us understand the quintain?'"(Stake, 1995, p. 6). Rather than getting insight into the single case, the purpose of this study was to get insight into the quintain or phenomenon of *pre-service* middle school *mathematics teachers' defining and classifying processes of the geometric shapes.* So, in this study I used every cases as an instrument to better understand the functioning of this phenomenon through employing *descriptive instrumental multiple case study method.*

3.2 Participants and Setting

In multiple instrumental case study it is important to purposively select the unit of analysis which will provide the best information to get insight into the phenomenon of interest; because, while some cases do good job in providing required information, some others may restrict obtained information (Stake, 1995). In purposive sampling, researchers "use their judgment to select a sample that they believe, based on prior information, will provide the data they need" (Fraenkel & Wallen, 2006, p. 101). So, the participants of this study were purposively selected from the *pre-service middle school mathematics teachers* who had no experience with the Geometer's Sketchpad program before. The reason of setting this boundary, namely the Geometer's Sketchpad level, came from the literature. De Villiers and Govender (2002) found that the students seemed to conceptually understand the targeted mathematical ideas in the dynamic geometry context; but they could not succeed when they were asked to perform same mathematical ideas without using dynamic geometry. So the researchers came up with the idea that student's observed improvement of the mathematical ideas in dynamic geometry environment could be due to their high level proficiency in using the functions of the tool instead of being due to their developing conceptual understanding of the issue of interest. On the other hand, participants' lacking the basic skills of using the dynamic geometry tool also can be a potential restrictor of the collected data. For example, Erez and Yerushalmy (2007) found that not grasping the idea that the attributes of the shape were preserved under dragging in a dynamic geometry environment caused learners not to grasp the intended mathematical ideas. That is to say, while studying with the participants who have advanced ability in using the dynamic geometry tool might mislead the interpretations made about their real mental process in conceptualizing mathematical ideas; on the other hand, studying with the participants who are unable to understand basic properties of a dynamic geometry tool can also prevent the progress of the research study. Therefore, at the beginning of the study I conducted one-to-one Geometer's Sketchpad teaching sessions aiming to equip the participants with the same level of skills in using the dynamic geometry tool, not too professional or too amateur skills so as not to experience restricting problems during the study.

Yin (1994) analogically likens the multiple case study design in the qualitative research to the experimental study design in the quantitative research, in terms of their replication strategy. In multiple case study design, the cases are selected either to support preliminary theory if the similar patterns are found or to refute the preliminary theory if the different patterns are found between the replicative cases (Yin, 1994). When it comes to the selection of the cases, there are no strict rules about how many cases to be selected. According to Yin (1994), in some situations even 2 cases would be sufficient to satisfy a convincing support for the replication; but in general, he suggest to use from 6 to 10 cases. Although there is no ideal number, Eisenhardt (1989) suggests not using less than 4 cases claiming that less than 4 makes it difficult to generate a theory; and he suggests not exceeding 10 cases claiming that more than 10 makes it difficult do deal with the complex and huge data. Since there is no statistical reason for the sample selection, there is no concern for the representative sample; instead, the sample is selected considering the number of cases that are required to saturate the development of theories related to the phenomenon of interest, namely to saturate the replication strategy (Eisenhardt, 1989; Yin, 1994). That is to say, the data are continued to be collected till not obtaining any new finding from the cases. For the purposes of this study, I planned to start with six cases and then to reduce or increase the number depending on the power of the obtained data to saturate the findings.

As a first step of the sample selection, an announcement for the study was made through an e-mail sent to the all seniors in the Elementary Mathematics Education Department (EME), Faculty of Education, Middle East Technical University (METU). In the announcement there was no detatiled information related to the content of the study, but they knew that GSP would be taught and they would use this program funcionnally after their experience in this study. Then, those who answered to the announcement were asked why they volunteered to take part in this study; and the ones who wanted to learn the GSP were accepted.

As I planned I started with 6 cases; however, in the Geometer's Sketchpad Teaching Session conducted at the beginning of the study, I realized that one of the participants had an advanced experience with the use of GSP, in oppose to what she had said. According to her academic transcript, she had not also taken any elective course related to GSP; but I learned that she had dissembled from me participating in a paid one-week GSP seminer given by an expert. Since her situation was contrary to my theoretical boundary of GSP knowlegde level due to the possibililty of her GSP knowledge to interfere with my data, I eliminated this participant from the study and continued till the end of the study with other 5 participants. These 5 participants provided me enough data to saturate my phenomenon of interest such that there were no new significant finding after two cases, but I continued to collect data from all 5 cases to ensure the patterns.

As a result, participants of this study included 5 pre-service middle school mathematics teachers purposively selected among seniors in Elementary Mathematics Education (EME) undergraduate program at Middle East Technical University. It is a four-year teacher education program where the language of instruction is English. By the time of data collected, these pre-service teachers were in their their last academic term of the program and had completed almost all the courses that the EME program offers, except for the last semester courses. As a requirement of the program, participants took IS100 (Introduction to Information Technology and Applications), CEIT100 (Computer Applications in Education) and ELE329 (Instructional Development and Media in Mathematics Education) courses related to the technology, objectives of which are presented in the Table 3.1. The former course aimed to introduce the basic information related to the technology concepts and applications, and to make learners both computer and information literate. CEIT100 aimed to focus on the role of computers in society, organizations and education and to make learners aware of the computer literacy, word-processing, spreadsheet and presentation software. Moreover, the last course aimed to introduce instructional technologies like worksheets, transparencies, slides, and videotapes and encouraged learners to develop teaching materials through personal web pages, spreadsheets and posters. Geometer's Sketchpad Program (GSP) was only mentioned superficially in this course; but, participants did not have any experience with it. Moreover, none of the participants took any elective course related to GSP. Detailed information about all the courses offered by the EME program is also given in the Table 3.2.

Table 3.1 Explanations of the Technology Related Courses Taken by the Participants

IS100	INTRODUCTION TO INFORMATION TECHNOLOGIES AND APPLICATIONS Course Objective To introduce all METU students to the basic information technology concepts and applications in their preparatory school / freshman year, preparing them to use these skills during their undergraduate studies in their respective disciplines, as well as professional lives.
CEIT100	COMPUTER APPLICATIONS IN EDUCATION Course Objective The major goals of this course are to: 1. enable the students to understand the basic concepts of technology and concepts of computing in education, 2. examine the role of computers in society, organizations and education. 3. explain computer literacy, word-processing, spreadsheet and presentation software.
ELE329	INSTRUCTIONAL TECHNOLOGY AND MATERIAL DEVELOPMENT Course Objective The objectives of this course are to provide students with the basic knowledge and skills necessary to create and design effective intructional media materials, such as posters, www pages, PowerPoint presentations etc. The course also covers a range of instructional tecniques, models, and tools connecting them to the learning process.

Table 3.2. Courses Taken in the Elementary Mathematics Education Program

UNDERGRADUATE CURRICULUM			
FIRST YEAR			
First Semester		Second Semester	
MATH111	Fundamentals of Mathematics	MATH112	Discrete Mathematics
MATH115	Analytic Geometry	MATH116	Basic Algebraic Structures
MATH117	Calculus I	MATH118	Calculus II
EDS200	Introduction to Education	CEIT100	Computer Applications in Education
ENG101	English for Academic Purposes I	ENG102	English for Academic Purposes II
IS100	Introduction to Information Technologies and Applications		
SECOND YEAR			
Third Semester		Fourth Semester	
PHYS181	Basic Physics I	PHYS182	Basic Physics II
MATH219	Introduction to Differential Equations	MATH201	Elementary Geometry
STAT201	Introduction to Probability &Stat. I	STAT202	Introduction To Probability &Stat.II
ELE221	Instructional Principles and Methods	ELE225	Measurement and Assessment
EDS220	Educational Psychology	ENG211	Academic Oral Presentation Skills
Any 1 of the following set ..		Any 1 of the following set ..	
HIST2201	Principles of Kemal Atatürk I	HIST2202	Principles Of Kemal Atatürk II
HIST2205	History of the Turkish Revolution I	HIST2206	History of the Turkish Revolution II
THIRD YEAR			
Fifth Semester		Sixth Semester	
MATH260	Basic Linear Algebra	ELE310	Community Service
ELE341	Methods of Teaching Mathematics I	ELE329	Instructional Technology and Material Development
Any 1 of the following set ..		ELE342	Methods of Teaching Mathematics
TURK201	Elementary Turkish	EDS304	Classroom Management
TURK305	Oral Communication	Any 1 of the following set ..	
-	Elective	TURK202	Intermediate Turkish
-	Elective	TURK306	Written Expression
			Restricted Elective
FORTH YEAR			
Seventh Semester		Eighth Semester	
ELE301	Research Methods	ELE420	Practice Teaching in Elementary Education
ELE419	School Experience	EDS416	Turkish Educational System and School Management
ELE465	Nature of Mathematical Knowledge for Teaching	EDS424	Guidance
-	Restricted Elective	-	Elective
-	Elective		

Some information about the participants, such as age, gender, C.GPA are also given in the Table 3.3. Participants included 1 male and 4 females and only the participant 1 was at the age of 25 while the others were at the age of 22. This was because Participant 1 went to the university 2 years later and repeated at English preparatory year. While 3 of the participants graduated from Anatolian Teacher Education High School, 1 graduated from high school and 1 graduated from a private high school. Except for the school experience and practice courses offered by the program, only participant 5 had teaching experience in a private teaching institution for 1 year.

Table 3.3. Information about the Participants of the Study

	Participant 1	Participant 2	Participant 3	Participant 4	Participant 5
Age	25	22	22	22	22
Gender	Female	Female	Female	Female	Male
Undergraduate program	Elementary Mathematics Education	Elementary Mathematics Education	Elementary Mathematics Education	Elementary Mathematics Education	Elementary Mathematics Education
Year	4^{th} (senior)	4^{th} (senior)	4^{th} (senior)	4^{th} (senior)	4^{th} (senior)
C.GPA	1.82	3.11	2.99	3.34	3.03
Graduated high school	Anatolian Teacher Education High School	Anatolian Teacher Education High School	Anatolian Teacher Education High School	High School	Private High School
Teaching experience	No experience	No experience	No experience	No experience	1 year in private teaching institution
Technolgy related must courses taken	IS100 CEIT100 ELE 329	IS100 CEIT100 ELE 329	IS100 CEIT100 ELE 329	IS100 CEIT100 ELE 329	IS100 CEIT100 ELE 329
Technolgy related elective courses taken	No	No	No	No	No
GSP experience	Heard in ELE 329 course, but not practiced	Heard in ELE 329 course, but not practiced	Heard in ELE 329 course, but not practiced	Heard in ELE 329 course, but not practiced	Heard in ELE 329 course, but not practiced

3.3 Pilot Study

Before the main data collection, a pilot study was conducted for the purpose of developing researcher's interviewing, observing, data collecting, data analyzing and interpreting skills; for the purpose of establishing guiding hypotheses about pre-service mathematics teachers' types of thinking process in definition construction process; and for the purpose of assessing the strengths or weaknesses of the interview tasks and the difficulties with the dynamic geometry tool in order to make necessary revisions.

The participants of the pilot study included 2 pre-service middle school mathematics teachers who had recently started to study for the postgraduate degree in the Elementary Mathematics Education Program at Middle East Technical University. The two participants were selected as volunteers to take part in the study. Both participants had completed a 4-year undergraduate program at Elementary Mathematics Education Program and both participants had not had any experience with any of the dynamic geometry programs, including GSP.

The data for the pilot study was collected during the 2010-2011 fall semester and the data collection procedure included following steps:

1. ***GSP Teaching Session and Application of the"Questionnaire on Quadrilaterals I."*** At the beginning of the study, both particiants were taught the functions of the GSP program through one to one session. Each session was conducted in the seminar room of the Faculty of Education at the Middle East Technical University (METU) and the sessions approximately took 2,5 hours. Subsequent to the GSP teaching session, "Questionnaire on Quadrilaterals I" test was administered.
2. ***Initial Interview***. One week after the GSP teaching and the administration of the "Questionnaire on Quadrilaterals I," an interview was conducted at the beginning of the first clinical interview session. The interview was audio and video taped in the Human-Computer Interaction Laboratory in METU Computer Center .
3. ***Clinical Interviews (1, 2, 3, 4).*** The clinical inerview sessions were prepared in two diferent formats, namely the Concurrent Thinking Aloud Clinical Interview (CTACI) and Retrospective Thinking Aloud Clinical Interview (RTACI). One clinical interview format was used with one participant while the other format was used with the second participant in order to detect out which format would provide the most

useful information. In RTACI the participant was interviewed after completing the tasks unaided, while in CTACI the participant completed tasks at the time of being interviewed; but both interviews included same questions to reveal thinking process on the tasks. Moreover, the 4 separate sessions were conducted at consequent days.

4. ***Final Interview***. Subsequent to the completion of the last clinical interview session, a final interview was conducted.
5. ***Application of the "Questionnaire on Quadrilaterals II.."*** At the completion of the clinical interviews, the participants were administered "Questionnaire on Quadrilaterals II" test including different questions from the questions of "Questionnaire on Quadrilaterals I," but serving for the same purposes.

All data recorded during the pilot study were transcribed and organized as computer files and they were analysed in accordance with the original analysis process. Besides becoming very useful for determining and improving the deficient sides of the questions and activities, this pilot study also helped the researcher to impove interviewing, observing, data collecting, analyzing and interpreting skills.

As a result of analysis, it was also seen that both clinical interview formats had some advantages and disadvantages as well. RTACI was advantageous in that it provided the participant with time to think on her own and on her own speed. On the other hand, in CTACI the participant had to think and explain her thinking at the same time as soon as the question was asked. Therefore, while the RTACI would be more advantageous for the silent and introvert learners that have difficulty to explain their thinking, the CTACI would be advantageous for more extrovert learners who do not have difficulty with explaining thinking. Moreover, the RTACI also provided the researcher with enough time to decide on which probing and prompting questions to ask in order to guide learner's thinking; because while the participant was working on the test I was able to see all of her answers instantly from the camera. However, in CTACI I had to think very quickly and develop appropriate questions according to the answer of the participant, which required much more effort by my side.

The RTACI was disadvantageous in that the participant might have forgotten what she thought while working on the tasks and might not have explained her thinking process appropriately. On the other hand, in CTACI every idea or thinking was developing at the time of conversation, and it was possible to help with the small technical problems while it was not the case in the RTACI. In RTACI the participant had to handle every technical problem unaided. Besides, the RTACI sessions took almost twice much time than the CTACI session since the participant first worked on her self and then interviewed. Moreover, in CTACI it was easier to see the limit of the participant's thinking by asking additional questions at the time of thinking; but there was no such a chance in RTACI.

For the purposes of this main study, it was important to examine the thinking process through asking follow up questions in the heat of the moment; so the CTACI format was more advantageous for that. Moreover, I did not want the participant's thinking process to be interrupted because of an unimportant technical problem that could be handled simultaneously in the CTACI format. Although all of the geometer's sketcpad activities were prepared with a great care so as not to cause technical difficulties for the participants, in the case of the possibility of a technical problem or the problem with the constructions, or the participant's questions related to the function of the dynamic figures, I should have been ready to solve the problem at that moment, which was possible in the CTACI format. Moreover, I could easily handle the disadvantage of the CTACI by giving participants as much as time they want to think and answer at the moment of speaking. That is, I would let them have enough inner thinking to explain their thoughts. Besides, I could handle another disadvantage by estimating the possible answers of the participants and preparing possible questions to lead these answers; but for the unexpected answers, I need to be equipped with the enough skills and knowledge to scrutinize these answers instantly with the correct probing questions. I believe that as a researcher I have enough knowledge and got skilled enough during the pilot study to ask necessary leading questions in response to the participant answers; so asking instantly in response to the participant answers would not be a disadvantage in CTACI format.

As a result, considering the feasibility and all the advantages and disadvantages of these two methods that I experienced through this pilot study, I decided that I could eliminate the disadvantages of the CTACI to a larger extent and CTACI would provide the richer information to answer the research questions of the main study. Therefore, I preferred CTACI format in the main study.

3.4 Data Collection Process and Instruments

In order to get deeper insight into the phenomenon of interest, multiple sources were used to collect data as offered by Creswell, Hanson, Clark, and Morales (2007). In addition to the audio and video recordings of the clinical interviews, pre-post questionnaire and interview, field notes and participant's written work were used as data sources. The data for the study was collected during the 2010-2011 spring semester by following the same data collection procedure in the case of pilot study. More detailed information about these sources and the process were given in the following sections.

3.4.1 Geometer's Sketchpad Teaching Session

The process started with one-to-one GSP teaching session which aimed to teach some functions of the Geometer's Sketchpad (GSP) program in order to make the participants skilled enough in using the menu functionally and in making constructions. The most important point implied during the GSP teaching was the difference between drawing and constructing. This GSP sessions were conducted with each participant individually in a seminar room in the Faculty of Education, at Middle East Technical University and a teaching session took approximately 2 hours. Inside the room there were two laptops ready for use, one for me and one for the participant. The session started with a brief introduction to the menu tools and then some practices were made in order to imply the difference between drawing and construction. The main steps of whole GSP teaching session can be found in Appendix A. At the beginning of the session, I introduced the menu functions through demonstrations, but the participants also actively discovered the menus and the sub menus on their own screen. After examining all functions and practicing them, participants were given some construction tasks from the basic level to the high level. However, these construction tasks were chosen so as not to interfere with the research study; they included the essentials that they would use in the main constructions throughout the clinical interviews. At the end of the Geometer's Sketchpad instruction, all participants had been equipped with the essential knowledge and skills for the main constructions.

3.4.2 Application of "Questionnaire on Quadrilaterals I"

The "Questionnaire on Quadrilaterals I" was applied at the end of the GSP teaching session in the same seminar room and it included 6 open-ended tasks and subtasks developed by the researcher in the light of the literature (Appendix C). The purpose of this questionnaire was to determine the pre-service middle school mathematics teachers' prior understanding of constructing and evaluating the quadrilateral definitions and understanding of the relations between quadrilaterals before engaging them into the clinical interview sessions.

In the first task, participants were asked to define rhombus, rectangle, and square and to find their examples among the given quadrilateral shapes. This task was developed by the researcher in order to determine participants' ability to define quadrilaterals and to identify their examples from the given collections of shapes. In this question, their understanding the nature of definitions was also investigated. That is, whether they were defining a geometric shape by listing many redundant properties or by using necessary and sufficient properties; and whether they were aware of the inclusive relations between geometric shapes were examined.

The second task was prepared by the researcher in order to analyze the ability of organizing several properties to construct correct economical definitions of a shape by deducing some properties from another. The participants were given a list of properties of a quadrilateral and were asked to write two alternative definitions using the minimum number of properties.That is to say, this task measured participants' deduction ability of understanding how certain properties of a figure were interrelated. In addition, this task also measured participant's ability to construct alternative definitions for the same concept.

The third task was designed to determine participants' ability to evaluate mathematically workable definitions among a list of rhombus definitions. This task was taken and adapted from De Villiers and Govender's (2002) study in which the researchers evaluated the task in terms of whether participants were able to select full complement of correct definitions. The same evaluation made in this study; and additionally, participant's reasoning for unpicked definitions were asked.

The fourth task asked to create definitions including some quadrilaterals as examples while excluding some others. This task was designed by the researcher for the purpose of examining participants' ability to construct inclusive and exclusive definitions.

In the fifth task participants were asked to fill the gaps in the given statements in order to measure their ability to understand class inclusions and inclusive relations between quadrilaterals. This task was directly taken from the online Mathematics TEKS Toolkit.

Finally, the sixth task was prepared by the researcher in order to examine participants' ability to classify quadrilaterals based on the different properties of these quadrilaterals. For this purpose, participants were given 3 Venn diagrams representing the some properties and were asked to put the quadrilaterals into the correct regions on the diagram.

In order to construct credibility of this questionnaire, independent experts from the Faculty of Education were asked to match the questionnaire items with the related objectives. Moreover, experts checked the format of the instrument in terms of clarity of the language and directions, irrelevant information and physical appearance of the paper.

3.4.3 Initial Interview

One week after the GSP teaching and the administration of the "Questionnaire on Quadrilaterals I," the next step in the process was to interview with the participants to reveal their experiences with the concept definitions during their education years, their perceptions of the role of definitions in the teaching and learning process, their confidence for the definition construction proficiency, and their understanding of a good definition. The interview which included 7 questions took about 20 minutes immediately before the first clinical interview session.

Immediately after the audio and video taped initial interview, we moved on to the first clinical interview session.

3.4.4 Clinical Interviews

As stated before, the main data of the study came from GSP assisted clinical interview sessions. The use of clinical interview as a research tool in the field of education, originated from the studies of Jean Piaget who believed the inappropriateness of the naturalistic observation or standardized tests to investigate intellectual processes of a child (Ginsburg, 1997; Opper, 1977). Adapting the clinical interview method used in the field of psychiatry, he created his own research method that would allow researchers to understand underlying difficulties of learning (Opper, 1977).

Three goals of clinical interview are to describe learners' natural, spontaneous thoughts through observing their responses; to detect out learner's mental process while responding to special tasks and questions designed to test hypothesis; and to consider a whole mental context of learners while interpreting their responses, including their motivation, beliefs and other answers (Ginsburg, 1997). The most deterministic character of the clinical interview is that the researcher makes every effort to uncover reasoning process of the learner by introducing additional questions or new tasks and by presenting some counter arguments when it is necessary to lighten doubtful points (Opper, 1977). The researcher tries to understand the mental processes by focusing not only on the verbal explanations, but also on the observations of learner's actions while working on the given tasks (Opper, 1977). Namely, the process is being shaped according to the answers and reactions of the learners and every effort is put to encourage learners to provide maximum information about their mental process. This cycle continues until the researcher believes that s/he obtained sufficient data to reveal learner's thinking and reasoning process on the targeted issue. Of course, immediate interpretation of the learner during the clinical interview process requires a robust awareness of the related theory on the side of the researcher (Coben , 2000).

In order to be able to increase the comparability of the results, a partially standardized version of the clinical interview is preferred in most of the studies (Opper, 1977). In this version of clinical interview, the tasks and questions are standardized, but still permits the researcher freedom to reword or rephrase the questions to clarify the meaning; to ask additional questions to uncover the some interesting sides of the learner's mental process; and to introduce further probing questions, tasks, and extra items when the researcher feels some inconsistencies or doubts about whether the responses reflect real thinking of the learner (Opper, 1977). That is to say, partially standardized clinical interview is a combination of the structured observation and clinical interview due to its standardized and flexible nature.

As all data collection methods, clinical interview also has some advantages and disadvantages. The biggest advantage of clinical interview method is the direct interactive communication between the researcher and the learner (Hunting, 1997). As in all types of qualitative research, the measuring instrument of the clinical interview is the researcher, but with a particular type of flexibility (Ginsburg, 1997). Moreover, clinical interview provides an informal environment which reduces the anxiety of being tested and makes learners feel comfortable to verbalize their thinking; so this informal nature based on a mutual respect and rapport maximizes the information obtained from the learner through effective conversation (Opper, 1977). Furthermore, this method emphasizes the importance of language to clarify the meaning of what is being asked by the researcher and what is being verbalized by the learner (Hunting, 1977).

In addition to the several advantages clinical interview also has some disadvantages, too. For example, interpretation of the learner's verbalizations would be difficult when the learner rejects responding to the questions, even though all the effort are put by the researcher; or when the learner fabricates the answers that does not reflect the real mental process. Moreover, the clinical interviews are very time consuming and limited in terms of generalizability and comparability (Opper, 1977). After all, generalizing is not to do much with this type of research method since it is not a standardized process in its nature; however, comparability can be provided to some extent with the standardized version of the clinical interview (Opper, 1977). For the purposes of this study, standardized version of the clinical interview was used; namely, some pre-structured tasks were presented with high flexibility in order to investigate thinking processes of the learners. It was expected that this standardized version would increase the comparability in replication studies (Opper, 1977).

As for this study, each participant was individually taken into four clinical interview sessions in Human-Computer Interaction (HCI) Research and Application Laboratory in METU Computer Center. The sessions were conducted at consequent days. The overall purpose of the clinical interview sessions was to observe the participants' developmental thinking process in evaluating and constructing definitions of quadrilaterals and understanding the relationships between them.

Specifically, the first clinical interview session aimed to examine the participant's thinking process in comprehending the inclusive relations between kite, rhombus and square while discovering the properties of them in a sketchpad context. The session also aimed to examine the participants' ability to organize several properties to construct mathematically workable and correct economical definitions of these shapes (Appendix D). The questions used in this session were prepared by the researcher in the light of the related literature. However, some sub questions, which were used to guide students during their Geometer's Sketchpad work, were adapted from De Villiers and Govender (2002). In this interview session, participants were provided with a pre-constructed sketchpad file which included a dynamic figure of kite with all related measurements on the sketchpad screen and they worked on it to answer the questions throughout the interview. They were asked to investigate the properties of the kite, to drag the figure into other quadrilaterals so as to find out special kites satisfying all properties of a kite, to find out the inclusive relations between kite class, to construct inclusive definitions to define kite and all its descendants, to evaluate the validity of the definitions through making corresponding constructions on sketchpad, to find the sufficient and necessary defining properties of quadrilaterals through many trials on sketchpad and to draw the hierarchy diagram of the kite and its special instances.

The overall purpose of the second interview session was to examine the participant's thinking process in discovering the defining properties and inclusive relations between isosceles trapezoid, trapezoid, parallelogram, rectangle, square, and rhombus in a sketchpad context; and to examine the ability to organize several properties to construct mathematically workable and correct economical definitions of these shapes (Appendix E). They worked on the pre-constructed sketchpad file to answer the questions throughout the interview as in the case of first session.

The purpose of the third interview session was to examine the participant's thinking process in discovering the cyclic and circum quadrilaterals and constructing definitions in terms of these properties (Appendix F). All of the questions in this session were developed by the researcher for the purposes of this study. Participants were asked to investigate the special cyclic and circum quadrilaterals through dragging the pre-constructed ordinal cyclic and circum quadrilateral figures on the sketchpad to see which quadrilaterals were always cyclic and always circum. Then they were asked to construct definitions for these quadrilaterals in terms of cyclic quadrilateral and were asked to add the "cyclic" and "circum" quadrilateral categories into the hierarchy to indicate the corresponding relations.

The fourth session aimed to examine the participant's thinking process in discovering new shapes (skew kites, right kites, trilateral trapezoid) in the hierarchy through generalizing or specializing the known definitions (Appendix G). The use of skew kites, right kites and trilateral trapezoid was originated from De Villiers (2009). However, in this study this new figures were named as quad1, quad2 and quad 3 correspondingly.

All 4 clinical interview sessions were conducted at the Human-Computer Interaction (HCI) Research and Application Laboratory in METU Computer Center. The HCI lab consisted of a test room and a control room which were separated with a mirror. While the participant sat in the test room in front of a computer attached to an eye tracker (Tobii 1750) that would record everything done on the screen, the researcher sat in the control room in front of a computer screen that was connected to the participant's computer. In the test room, there were two camcorders which can rotate 360°: one to record face of the participant and another to record keyboard and mouse activity of the participant from the top view. These camcorders were controlled by the researcher in the control room through the control unit which was connected to a monitor and allowed managing the movement of the two camcorders. Moreover, the conversation between the researcher and the participant was done through the speakers and microphones controlled by a sound mixer. The lab provided the researcher with essential screen recordings and audio-video recordings.

3.4.5 Final Interview

At the completion of the fourth clinical interview session, participants were re-interviewed to reveal their general opinions about the quadrilateral learning experience in the GSP learning environment; their perceptions of the positive and negative sides of the GSP activities and the differences between learning in the classroom environment and in the GSP learning environment. Moreover, it was also aimed to reveal what they learned related to the definition construction process and whether they made progress in their definition construction skills when compared to the situation before the study. In addition, some questions were asked to learn their perceptions of the use of definition construction process as a mathematical activity to teach concepts. The interview which took approximately half an hour included 12 questions (Appendix H) and conducted at the Human-Computer Interaction Laboratory.

3.4.6 Application of "Questionnaire on Quadrilaterals II"

At the completion of the post interview, participants were administered "*Questionnaire on Quadrilaterals II"* including different questions from the questions of first questionnaire, but serving for the same purposes (Appendix I). Altough the partcipants'main progress was observed during the clinical interviews, this final questionnaire was administered as an addiditonal data source to determine the improvements through comparing it to the first questionnaire findings.

In the first question, participants were initially asked to define isosceles trapezoid, trapezoid and rectangle; and then they were asked to find the examples of the given list of definitions among the given quadrilateral shapes. This task was developed by the researcher in order to analyze participants' ability to define quadrilaterals and to identify examples of a definition from the given collections of shapes. In this question, their understanding the nature of definitions was also investigated. That is, whether they were defining a geometric shape by listing many redundant properties or by using necessary and sufficient properties; and whether they were aware of the inclusive relations between geometric shapes were examined.

The second task was prepared by the researcher in order to analyze the ability to organize several properties to construct correct economical definitions of a shape by deducing some properties from another. That is to say, this task measured participants' deduction ability to understand how certain properties of a figure were interrelated. Moreover, this task also measured participant's ability to construct alternative definitions for the same concept. For these purposes, the participants were asked to construct two economical definitions of the rhombus which included only the necessary and sufficient defining properties.

Then, the next question measured participants' ability to evaluate mathematically workable definitions through identifying the necessary and sufficient defining properties in the given list of statements. This task was taken and adapted from De Villiers' (2009) study.

The fourth task asked to create definitions which included some quadrilaterals as examples while excluded some others. This task was designed by the researcher with an inspiration of the fourth task of the pre-test on quadrilaterals. The purpose was to examine participants' ability to construct inclusive and exclusive definitions.

In the fifth task, participants were asked to fill the gaps in the given statements indicating the relations between quadrilaterals to measure participants' ability to understand class inclusions and inclusive relations between quadrilaterals. This task was developed by the researcher with an inspiration of the fifth task of the pre-test on quadrilaterals.

Finally, the sixth task was prepared by the researcher in order to examine participants' ability to classify quadrilaterals based on their different properties. For this purpose, participants were asked to construct a hierarchy diagram indicating the relationships between the parallelogram, rhombus, square, rectangle, trapezoid, isosceles trapezoid and kite based on the diagonal properties.

3.4.7 Field Notes

Field notes "are the researchers' written account of what they hear, see, experience, and think in the course of collecting and reflecting on their data." (Fraenkel & Wallen, 2006, p. 516). These notes can be descriptive or reflective in their nature. Descriptive field notes define everything in the setting such as activities, behaviors and facial expressions of the participants; any particular events during the study; physical appearance of the settings or manner of utilized materials, etc. (Bogdan & Biklen, as cited in Fraenkel & Wallen, 2006). Reflective field notes, on the other hand, reflect the researcher's own thinking and comments about what is being observed, such as the problems related to the analysis or design of the study; possible factors that might affect the study; or any kind of conflicts or concerns, etc. (Bogdan & Biklen, as cited in Fraenkel & Wallen, 2006).

For the purposes of this study, both descriptive and reflective field notes were collected to provide an ongoing evaluation and critique of the study progress. Generally, I took short notes during the interviews for the remarkable things and immediately after the interview I expatiated the notes and added personal comments as well.

3.4.8 Participants' Written Work

Any written work of the participants, obtained while they were working on the interview tasks and questionnaire tasks, were collected to utilize in analysis process. Moreover, every sketch constructed on the Geometer's Sketchpad by the participant was recorded in order to evaluate during the analysis process.

To sum up, in order to get deeper insight into the phenomenon of interest, multiple sources were used to collect data as offered by Creswell, Hanson, Clark, and Morales (2007). The main data for the study came from audio- and video-taped clinical interview sessions and the screen recordings obtained in these sessions. In addition to the audio and video recordings of the clinical interviews, questionnaires, initial and final interviews, researcher's field notes, and participant's written work were used as data sources.

3.5 Data Analysis

The analysis of the case study developed concurrently with the data collection process as in all qualitative studies (Baxtar & Jack, 2008). For multiple case studies, Creswell (2007) suggested describing each case and the themes in detail which he called "within-case analysis" (p. 75) and then making a thematic analysis which he called "cross-case analysis" (p. 75). For a more systematical path, Creswell's (2007) data analysis steps of case study which involves organizing data files, establishing initial codes, describing the cases in detail, establishing themes or patterns, making sense of the data through interpretation, and finally presenting a whole picture was followed for the data analysis of this case study.

According to Yin (1994) there is a need for following an analytical strategy to analyze qualitative data, and one of the offered strategies is relying on theoretical propositions. Propositions are very important to limit the edges of the analysis so as not to go out the scope of the study and so as to explore the rivalry explanations of the phenomenon, which increases the confidence of the findings (Yin, 2003). The propositions which form the foundation of the study "may come from the literature, personal/professional experience, theories, and/or generalizations based on empirical data" (p. 551), but each one of them should have a different focus (Baxter & Jack, 2008).

According to Yin (1994) pattern matching is one of the analytic techniques to be used in theoretical propositions strategy which includes comparing patterns or themes inferred from the study with the already discovered patterns in the literature. One way would be to examine whether the expected outcomes are confirmed by the study findings or some alternative patterns are found in the study; and another way would be to search for the rival explanations pattern in the data (Yin, 1994).

In the light of the Yin's (2004) theoretical propositions strategy, I compared the patterns and themes obtained from the data with the propositions inferred from the theory to see whether this study would find some consistencies or inconsistencies which explained the phenomenon of interest. As a first step of data analysis, all audio-taped interview sessions with each participant were transcribed and together with all other data sources they were organized and stored as computer files. Then, the next step was to reduce the data into more meaningful forms by forming initial codes. For that purpose all data set including transcripts, field notes, participants' written work, audio- and video-tapes were critically reviewed over and over to make sense of the data. During this review process, a short list of five or six categories, which Creswell (2007) called "lean coding" (p. 152), were formed as a starting point and then these categories were expanded up to 30 categories as the review of data was in progress. Then, these categories were classified into approximately 6-8 general themes. The code names were identified by the researcher through looking for code segments that best described the information, as offered by Creswell (2007).

The main theme of this study is the definition construction process. According to the related literature there are some criteria to consider while defining a concept so as to fit it into a mathematical deductive system and two of these several criteria included *minimality* and *hierarchy* (Van Dormolen & Zaslavsky, 2003). Conceptually orienting from the study of Aristotle, Van Dormolen and Zaslavsky (2003) state that *hierarchy* criterion requires defining a concept "as a special case of more general concept" (p. 94) and this criterion is very fundamental and a logical necessity of a deductive system. The logical importance come from the fact that one does not need to prove properties for the special case since they had already proved for the general concept (Craine & Rubenstain, 2003; De Villiers, 1994; Van Dormolen & Zaslavsky, 2003). According to Van Dormolen and Zaslavsky (2003), the *minimality* criterion requires a concept not to be defined with more than the necessary properties; however, this criterion is not a logical necessity, but the idea of a general mathematical culture. For example, when a concept is defined with minimal conditions, it leads to the discoveries of other properties and other instances of a concept through deductive reasoning. Namely, not using minimality condition will not create a contradiction in a deductive system, but it will lead to a higher level reasoning skills about the defined concept. However, the decision of accepting minimality as a defining criterion will depend on the instructional purposes (Van Dormolen & Zaslavsky, 2003).

Other researchers who defend the idea of considering minimal conditions to define a concept are De Villiers and Govender (2002). According to the researchers "A definition is incomplete if it contains necessary but insufficient properties. So an incomplete definition is also an incorrect one." (p. 5). Moreover, they state that

> A correct definition can be either economical or uneconomical. An economical definition has only necessary and sufficient properties. It contains no superfluous information. On the other hand, an uneconomical definition has sufficient, but some unnecessary properties. In other words, it contains more information than necessary (redundant) information. (p. 5)

According to De Villiers (1998), if the levels are arrayed starting from 1 to 5 instead of 0 to 4 students at the Van Hiele level 1 are able to construct visual definitions, namely, they define a shape referring only to the visual aspects of it. Students at level 2 are able to construct uneconomical definitions; they tend to list all properties of the shape without distinguishing between critical and noncritical attributes; and students at level 3 are able to construct correct, economical definitions; they include only necessary properties in the definition of the shape.

Although most of the codes emerged as the study progress, some pre-constructed categories were also used. In this study the definitions were analyzed considering the minimality and hierarchy criteria. Originating from the study of De Villiers and Govender (2002), the definitions produced by the participants were categorized as *incomplete* or *wrong definitions*, *uneconomical definitions* and *economical definitions* to evaluate them in terms of minimality criterion. Moreover, the definitions were also evaluated in terms of participants' having an *inclusive or exclusive* thinking, that is to say, in terms of *hierarchy* criterion. According to van Hiele theory, learners at van Hiele level 1 and level 2 of which thinking skills are expected at the grade band Pre-K-2 and 3-5 respectively (Genz, 2006) think exclusively since they evaluate geometrical concepts in terms of visual appearance and do not have mental ability to think of a concept to be two different things at the same time (Aichele & Wolfe, 2007). However, starting from van Hiele level 3, which is the expected level at grade 6-8 (Genz, 2006), learners can think inclusively and make class inclusions between geometric shapes (De Villiers, 1996). So in this study, a definition was evaluated as inclusive if it also defined special cases of the concept and as exclusive if it ignored the special cases of the concept and identified it as a distinct concept.

As mentioned in the literature review, different types of classifications emerged to show the relationships between quadrilateral shapes. In this study, the classifications were evaluated in terms of De Villiers'(1994) *hierarchical classification* which was the classification of the concepts so that the more general ones would include the more particular ones as their subsets and *partition classification* which was the classification of the concepts as disjoint objects. Namely, the relationship between the definitions and classification was evaluated in terms of whether the inclusive definitions lead to the hierarchical classification and exclusive definitions lead to the *partition classification* as explained by Usiskin and Griffin (2008).

After establishing the codes, categories, themes, and patterns for each session, more general meanings pertaining to the issue were formed by interpreting the data.

3.6 Trustworthiness

Lincoln and Guba (1985) assert that establishing the trustworthiness of a qualitative study is important to illustrate the worth of the study and the establishing the trustworthiness requires to meet credibility, dependability, transferability and confirmability criteria. Credibility is term used in preference to internal validity in the quantitative studies and this criterion requires to provide a true picture of what is bening investigated in the qualitative study. Dependability is used in preference to reliability and this criterion requires to show consistency and the repeatability of the findings. Transferability, on the other hand, is used in preference to generalizability and in qualitative study, transferability requires to provide sufficient detail of the context so that it could have applicability in other settings. Finally confirmability, in preference to objectivity, requires to provide a degree of neutrality.

Detailed information about how the criteria were met to establish the trustworthiness of the study is presented in the following sections.

3.6.1 Credibility and Dependability

One of the possible credibility threats in such kind of a qualitative study could be the insufficiency or inaccuracy of the collected data (Maxwell, 1992). I tried to handle this threat by collecting as much data as possible through different means and also by employing audio- and video-taping. According to Maxwell (1992) one another credibility threat in such kind of qualitative studies could be disregarding the alternative theories. I tried to eliminate this threat by seeking negative cases or theories, which may contradict with the study findings, through the present data or through the additional data collected from the literature.

Another possible credibility threat would be the incorrect interpretation of the data (Maxwell, 1992). To overcome this threat, the interview sessions were both audio-taped and video-taped and the dialogues were transcribed with a great attention. Moreover, the data were critically read over and over to be able to make an appropriate interpretation and the quotations were provided as evidence to the inferred interpretations. When it comes to the clinical interviews, Piaget offered several means to check the credibility of clinical interview data (Elkind, 1964). In order to understand whether the responses of the learner reflected his/her real thinking process, Piaget identified three response types that should not have been pursued by the researcher since they were misleading; and two valuable response types that were very important to reflect true mental ability (Elkind, 1964). One of the worthless answers was "answers at random" (p.42) that arouse if the learner said whatever came into mind randomly when s/he was disinterested or got tired and bored during the interview (Elkind, 1964). Another worthless response type was "romancing" (p. 42) which was inventing or making up an answer that did not reveal the real thinking. Finally, the last worthless answer type "suggested conviction" (p. 42) arouse when the learner gave the answers which s/he thought to please the researcher or the answers which were inferred from the clues (Elkind, 1964). On the other hand, there were two response types which were very important to reflect the real thinking of the learner. If the learner could develop a correct answer to a new unfamiliar question by using his/her existing thought, this type of reflection was called "liberated conviction" (p. 42) which was very valuable data for the researcher (Elkind, 1964,). The second significant response was called a "spontaneous conviction" (p. 42) which arouse when the learner gave an immediate answer without thinking about it since s/he had already grasped the idea to solve it (Elkind, 1964). Distinguishing between these mentioned two groups of answers was very important to obtain a valid data. So during the interview process, when I realized some doubtful or inconsistent reflections of the learner, I introduced some counter arguments or counter suggestions to check whether learner's thinking was consistent in this contradiction situation. So, this helped me detect out whether the answers were really based on liberated conviction or spontaneous conviction (Elkind, 1964; Ginsburg, 1997; Opper, 1977). Moreover, I also checked whether the response of the learner reflected true mental process or not. For this purpose I asked questions about related issues in order to see whether the response to these related issues fitted to the learner's existing mental schemas, which would indicate a true reflection of his/her thinking (Elkind, 1964).

Considering the issue of dependability of the clinical interview, Ginsburg (1997) argued that it was not possible to establish a general dependability, but some level of dependability under certain conditions might have been possible. One way to achieve dependability of the clinical interview was the inter-observer agreement to see whether independent observers did agree on the categorization of the data (Elkind, 1964; Ginsburg, 1997). Another index of dependability could be to check the consistency of the learner responses at different times (Elkind, 1964; Ginsburg, 1997). Moreover, checking whether the learner reflected the same thinking process in answering more than one interview questions measuring the same issue could be an evidence to provide internal consistency (Ginsburg, 1997). To some extend I tried to carry out these recommendations to provide some evidence for the dependability of the clinical interview, but it was not necessary to get evidence about it in clinical interview; because interpretations of the dependability could bias the interview findings on the other side (Ginsburg, 1997). For example, inconsistent responses to the questions of a clinical interview could be an indicator of the weak mental processing of the learner; that was, as Ginsburg (1997) used the term *reliability* instead of *dependability* "low internal consistency or test-retest reliability may have said more about the true state of the child than about the reliability of the measuring instrument" (pp. 171-172). Even Piaget, creator of the clinical interview, considered the credibility issue but neglected the dependability issue in his clinical interview studies (Elkind, 1964).

Dependability threats were also tried to be eliminated by means of the triangulation and member checking. Moreover, both for credibility and dependability, all the data and all the researcher activities were kept recorded which is known as audit trail (Lincoln & Guba, 1985).

3.6.2 Transferability

A thorough study of an individual in a case study design provides a great deal of insight that can not be obtained by other research means (Ginsburg, 1997); that is, particularization rather than the generalization is the main issue in case studies; because it is the power of case study to focus on the particular situations and to gather deeper understanding (Stake, 1995). Even though the aim of a case study is not to generalize findings from a case to a population, the transferability would be possible on case-to case basis when the scope of the study is thoroughly described so that the readers could replicate the study using the same context and the procedure. (Creswell, Hanson, Plano & Morales, 2007; Stake, 1978).

So, considering the transferability of the study, all research procedure and scope of the study were described in detail in order to provide readers a roadmap to replicate this study in their own broader or narrower contexts.

3.6.3 Confirmability

All data collection process with each participant was conducted by myself. Through a direct interaction with each participant, I tried to understand their thinking and reasoning process by actively involving into the process and directing the interview sessions.

As Ginsburg (1997) stated, the measuring instrument of the clinical interview is the researcher with a different flexibility. As a researcher, my role during the clinical interviews was to make every effort to elicit the real mental process of the participants. This effort included establishing rapport with each participant and providing them with a friendly athmosphere; introducing additional follow up questions or tasks in order to follow thoughts to the full extent; pushing them to think and explain what is going on in their mind, challenging them to clarify doubts and to evaluate their confidence in the answer; reading their mind through the body language, behaviours, actions, facial expressions and voice; encouraging them to speak by asking probe questions, but not to ask leading questions; keeping their motivation and interest alive; giving as much time as they need to think and answer; and allowing the participant to express thoughs without any judgement and implicit or explicit lead. That is to say, my role as an interviewer was as flexible as possible to get into the participant's mind, but it was definitely not to interfere with the participant's thoughts. My role was not like an instructor who wants to teach, my role was to reveal what they knew. That is, I was like a driving force to push them to show me the improvement in the mental process with regard to the tasks handled. The clinical interview is difficult to handle since it mostly depend on the skills of the researcher. In such kind of a flexible interview, it is very easy to make mistakes and to interfere in the participants' cognitive process. However, I was aware of my role and had improved my interview skills to a considerable extent during the pilot study. Moreover, I examined the videotaped clinical interview sessions over and over, and searched the data extensively for any interference by me. So, I believe that I did a honest effort not to bias study findings.

On the other side, sticking to the theories in the literature and trying to find the similar results would be another researcher bias. For instance, literature indicated learners' difficulty with understanding the definitions and the classifications of the quadrilaterals; and the facilitator role of the dynamic geometry environment in enhancing learners' ability to grasp the ideas in these concepts was stated. However, I was not under the influence of the literature and I did not try to find the same results in accordance with the literature, which could orient me to behave subjectively while conducting the study. Therefore, I searched for the negative cases which may disconfirm the literature findings in order to eliminate the researcher bias threat (Lincoln & Guba, 1985). Moreover, for the purpose of eliminating the researcher bias, the data collected from the different sources were analyzed to see whether they converge to the same findings, a technique known as triangulation (Lincoln & Guba, 1985). Furthermore, the participants of the study were asked to confirm interpretations made by myself, which is known as member checking method (Lincoln & Guba, 1985).

That is to say, as a researcher I tried to provide the comfirmability of the study becoming as objective as possible throughout the whole process.

CHAPTER IV

WITHIN-CASE ANALYSIS RESULTS

In this chapter, the study findings for each participant were explained in detail under 5 sections dealing with the research questions.

In the first section, findings of the initial interview which aimed to learn about participant's experiences with the concept definitions so far, their thoughts about their own proficiency level of defining, their perceptions of a good definition and beliefs about the importance of definitions; generally, their perceptions of the definitions and their role in the teaching and learning process were presented.

The second section was devoted to the findings of "Questionnaire on Quadrilaterals I" which was administered to determine the pre-service middle school mathematics teachers' initial understanding of the nature of the quadrilateral definitions and the corresponding hierarchies, before engaging them into the clinical interview sessions. Their definitions were analyzed considering the criteria that the knowledge of the quadrilateral properties, the use of the correct mathematical language to state properties, the use of the necessary and sufficient properties (minimal information) to define, the use of the different sufficient properties to construct more than one alternative equivalent definitions and inclusivity or exclusivity nature of the definitions. Moreover, their definition evaluation proficiency was analyzed considering the ability to make critical thinking on the properties so as to identify the necessary and sufficient defining properties.

The next section " Clinical Interviews" was divided into 4 subsections to reveal the analysis findings for each clinical interview session one by one. The subsection related to the first clinical interview revealed the findings about the participant's thinking process in comprehending the inclusive relations between kite, rhombus and square while discovering the properties of them in a sketchpad context; and about the participants' ability to organize several properties to construct mathematically workable and correct economical definitions of the shapes. In the subsection related to the second interview session, findings about the participant's thinking process in discovering the defining properties and inclusive relations between isosceles trapezoid, parallelogram, rectangle, square, and rhombus and about participants' ability to organize several properties to construct mathematically workable and correct economical definitions of these quadrilaterals were stated. The subsection of the third clinical interview session provided findings about the participant's thinking process in discovering the cyclic and circum quadrilaterals and defining quadrilaterals in terms of these properties. And in the fourth subsection, findings about the participant's thinking process in discovering new shapes in the hierarchy through generalizing or specializing the known definitions were presented with regard to the fourth clinical interview session.

Another section was devoted to the the participants' opinions about the positive and negative sides of the GSP activities, about the differences between learning quadrilaterals in the classroom environment and in the GSP learning environment and about their progress in definition construction are presented.

Finally, the last section reflected the findings of "Questionnaire on Quadrilaterals II" which was administered to determine the pre-service middle school mathematics teachers' understanding of constructing and evaluating the quadrilateral definitions and understanding of the relations between quadrilaterals, after engaging them into the clinical interview sessions. Their definitions were analyzed considering the same criteria used for analyzing "Questionnaire on Quadrilaterals I,"

4.1 Participant 1's Analysis Results

Findings related to the Participant 1's perceptions of the definitions and understanding of the quadrilateral definitions and the hierarchies before engaging them into the clinical interview sessions, her mental process and progress during the 4 clinical interview sessions, opinions about her experience in this study and the findings related to her understanding of the quadrilateral definitions and the hierarchies after the clinical interview sessions were stated in the following sections.

4.1.1 Participant 1's Initial Perceptions of the Definitions

According to the Participant 1, definitions were important to express the properties of the geometric shapes. However, she thought that a good definition must have included all known properties of the defined concept, which was a very common misled idea among learners (De Villiers, 1998).

Having been asked whether defining a concept indicated a meaningful understanding of that concept, she stated that not only defining a concept but also making inferences between the properties was necessary for a robust understanding. She also believed that concept definitions should have been accompanied with the concept images to make the concept more meaningful to the students; and at that point technological tools could be helpful.

Paticipant 1 told me that throughout her education life, definitions were just written on the board by the teacher at the beginning of the lesson and she just wrote them on the notebook without questioning. There was no discussion about the meaning of the definition. She also confessed that she was unaware of he importance of a definition till having been asked to construct definitions in an undergraduate course at the university. That is to say, she had been encouraged to think on the mathematical correctness of the concept definitions at the university level for the first time.

Moreover, the participant stated that she was not skilled enough to construct definitions at that moment of the interview, but she believed that she could improve her ability to define after being allowed to experience the definition construction process. When she was asked about her use of definitions in her own teaching, she stated that she would use the definitions that were already in the textbooks instead of constructing new ones.

4.1.2 Participants 1's Initial Understanding of the Nature of the Quadrilateral Definitions and the Hierarchies

It was seen that Participant 1 made a correct inclusive but uneconomical definition of rhombus saying that *"a rhombus is a quadrilateral of which opposite sides are parallel and all sides are equal."* Here, the condition of "all side lengths' being equal" would be enough to define a rhombus; because the most general quadrilateral that we can draw when we are told to draw a quadrilateral with all sides lengths equal would be the rhombus. The square also has this property, but we can specialize it if we know that at least one angle is 90 degree in addition to the all side lengths being equivalent. On the other hand, the first condition alone would not be sufficient to define a rhombus, but would lead to a prototypical parallelogram. Moreover, it was seen that the participant had some problems with the correct usage of the terms "equal" and "congruent" which indicated that she was not aware that side lengths could be equal, but sides could be congruent.

Although this definition was not economical, it was a correct inclusive definition since it included square as a special rhombus which indicated that the participant was aware of the inclusive relationship between rhombus and square. This finding was also supported in the next question, because she was able to identify squares as examples of rhombus along with the prototypical rhombus figures among the given group of quadrilaterals.

The participant 1 defined rectangle as *"a quadrilateral with parallel and equal opposite sides and with all 90° angles."* This was again a correct inclusive but uneconomical definition. "A quadrilateral with all 90° angles" would be the sufficient condition to characterize a rectangle. When a quadrilateral is drawn with this property, the most general figure would be the rectangle. Although the square also has this property, there is need to know an additional property of "all sides being congruent" to characterize it. Moreover, she again incorrectly used the word "equal" instead of "congruent" to point out the meaning of "congruent sides" or "equal side lengths."

However, though her definition included square as a special case of rectangle, participant1 did not identified the squares when she was asked to identify examples of rectangle among the given group of quadrilaterals (Figure 4.1); even, she could not identify the prototypical rectangle correctly. She selected the figure "s" and "f" as the examples of rectangle though the angle measures of the former figure were not 90°, but she did not select "c" and "t," namely the squares, as special rectangles. This indicated that the participant was not aware of the inclusive relationship between rectangle and square though her definition included squares as special rectangles. That is to say, her rectangle definition randomly became an inclusive definition since she had a prototypical concept image of the rectangle in her mind which did not let her accept squares as rectangles.

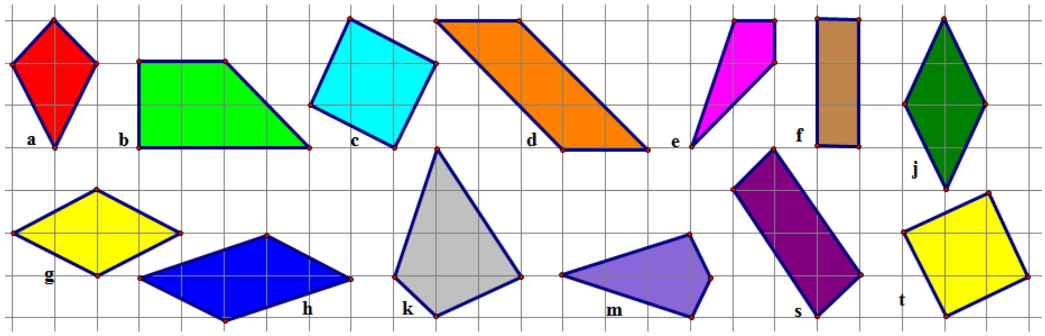


Figure 4.1 The given group of quadrilaterals in Questionnaire on Quadrilaterals I

Next, the participant 1 defined square as *"a quadrilateral with opposite sides are parallel, all sides are equal, and all angles are 90°."* She again listed all properties instead of the necessary and sufficient defining properties which lead to a "square description" rather than a "square definition." Moreover, she again used "equal" instead of "congruent."

The data obtained also indicated that the participant was able to identify necessary and sufficient properties to write alternative definitions for the concept of kite to some extent. Her first definition for the kite was *"a quadrilateral with two pairs of congruent adjacent sides and with diagonals perpendicular to each other."* This was a correct but uneconomical definition which included second property as the extra information. "Two pairs of congruent adjacent sides" is the sufficient defining property to characterize a kite, because it leads all other properties of the kite to be automatically correct. That is to say, if two pairs of congruent adjacent sides are constructed to form a quadrilateral, this quadrilateral will have perpendicular diagonals in any case.

Her second definition was *"a quadrilateral with one pair of congruent opposite angles and with the diagonals one is bisected by the other"* and this time she had come up with the correct economical definition which included only the necessary and sufficient properties without extra information. Neither of the two properties alone could be sufficient to characterize a kite. An ordinal quadrilateral can have only the "one pair of opposite congruent angles" property or only the "diagonals one is bisected by the other" property (Figure 4.2 and Figure 4.3).

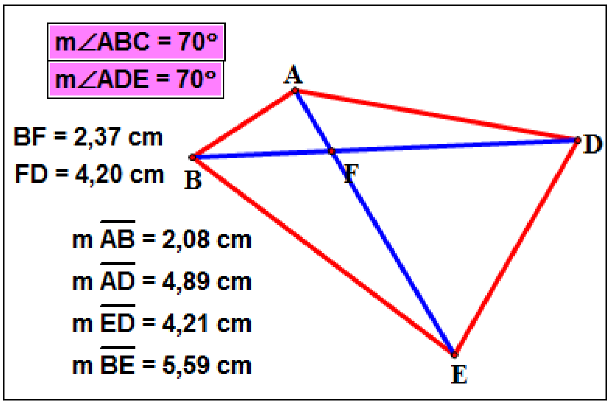


Figure 4.2 An ordinal quadrilateral with "one pair of opposite congruent angles"

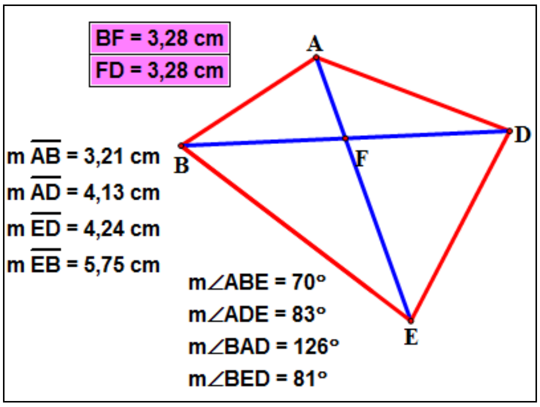


Figure 4.3 An ordinal quadrilateral with "diagonals one is bisected by the other"

However, a quadrilateral with both of these properties can only be a kite since these two properties lead to the other properties of kite to be true. If a quadrilateral is drawn using only these two properties, the figure also automatically satisfies "two pairs of congruent adjacent sides" property, for instance (Figure 4.4).

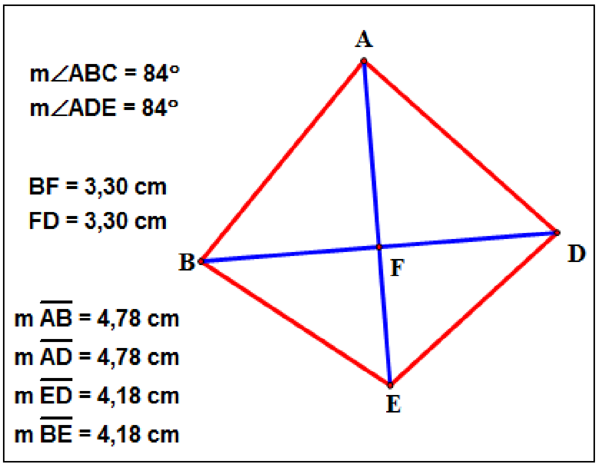


Figure 4.4 A quadrilateral with one pair of opposite congruent angles and with diagonals one is bisected by the other

When she was asked to evaluate the correctness of the given rhombus definitions in terms of critical defining properties, participant 1 was unsuccessful to identify the mathematically correct ones. For example, according to the participant "a rhombus is a quadrilateral with all sides congruent" does not define a rhombus even though it does. According to her, there is need to state also the opposite sides being parallel property. Indeed, she was not able to see that if all sides are congruent, opposite sides automatically become parallel. Moreover, she evaluated the definition that "a rhombus is a quadrilateral with to pairs of congruent adjacent sides" as correct, though it was not. This property is a defining property of a kite and to specialize a kite as a rhombus there is need to know an additional properties such as "all sides being congruent" or "diagonals bisecting each other."

On the other hand, she was very good at constructing inclusive and exclusive definitions which indicated that she can identify the defining properties that distinguish the group of figures from another group, but define the figures inside the same group.

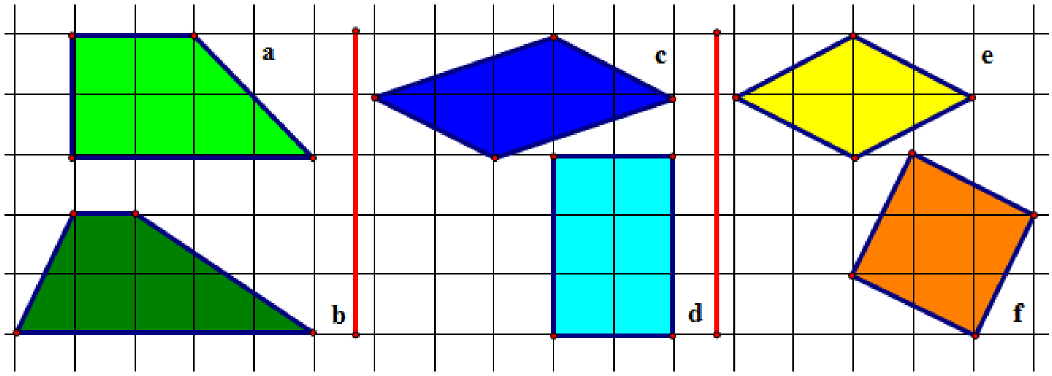
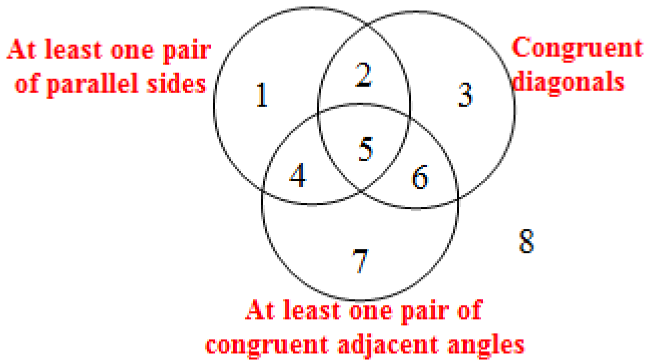


Figure 4.5 The given group of quadrilaterals for inclusive and exclusive defining

Having severed from the figures c, d, e and f, the participant was correctly able to construct a definition including only the figures a and b as *"quadrilaterals with at most one pair of parallel sides."* Next, she correctly constructed a definition including the figures a, b, c and d together but excluding the figures e and f as *"quadrilaterals with at least one pair of parallel sides but not all sides are equal"* The only thing she missed in this definition was using "equal" instead of "congruent" as she did before. Finally, she constructed a definition including all the figures as *"quadrilaterals with at least one pair of parallel sides."*

The analysis also indicated that the participant was very good at understanding the inclusive relations between the quadrilaterals through considering their properties. For example, she was able to state that a parallelogram had perpendicular and bisecting diagonals when it was a square. Moreover, she was able to explain that the diagonals of a rhombus were congruent when the rhombus was a square and that the rectangle had congruent adjacent sides when it was a square. This indicates that the participant accepts square as a special case of both rhombus and a rectangle.

When the participant was asked to think of the prototypes of each quadrilateral and to classify them based on their properties, she almost failed. For example, in the below diagram she put the parallelogram into the region 2 whereas it must be in the region 1 (Figure 4.6). This was because she had a misinformation that a parallelogram has congruent diagonals. Moreover, she incorrectly put the kite into region 7 although it does not have at least one pair of congruent adjacent angles and incorrectly put the rhombus into the region 2, although it does not have congruent diagonals. Furthermore, she matched the trapezoid with the region 8 although it has at least one pair of parallel sides and so must be in the region 1; and she matched the rectangle with the region 2, although it also has at least one pair of congruent adjacent angles and so must be in the region 5. That is to say, in this diagram she only put the square into the correct region, but failed for the others.



Participant's Answers

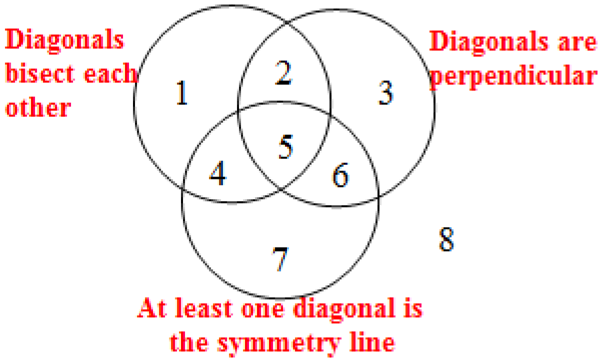
Parallelogram	**2**	Trapezoid	**8**
Kite	**7**	Isosceles Trapezoid	**6**
Square	**5**	Rectangle	**2**
Rhombus	**2**		

Correct Answers

Parallelogram	**1**	Trapezoid	**1**
Kite	**8**	Isosceles Trapezoid	**5**
Square	**5**	Rectangle	**5**
Rhombus	**1**		

Figure 4.6 Participant 1's first diagram of the classification of the quadrilaterals

In the second diagram she was better when compared to the first one, but she again failed to put the parallelogram, isosceles trapezoid and rectangle into the correct regions due to her misleading knowledge of the properties of these quadrilaterals (Figure 4.7).



Participant's Answers

Parallelogram	**7**	Trapezoid	**8**
Kite	**6**	Isosceles Trapezoid	**1**
Square	**5**	Rectangle	**4**
Rhombus	**5**		

Correct Answers

Parallelogram	**1**	Trapezoid	**8**
Kite	**6**	Isosceles Trapezoid	**8**
Square	**5**	Rectangle	**1**
Rhombus	**5**		

Figure 4.7 Participant 1's second diagram of the classification of the quadrilaterals

4.1.3 Clinical Interviews

Participant 1's cognitive progress during the clinical interview sessions and the effect of the GSP activities on the participant 1's cognitive improvement in understanding the quadrilaterals through definition construction and classification processes were described in detail in the following subsections.

4.1.3.1 Session 1 with Participant 1: Kite, Rhombus and Square

At the very beginning of the session, the participant was asked to remember the properties of a kite verbally and it was seen that she could correctly remember almost all the side, angle, diagonal properties. At this step, making measurements on the GSP figure just helped to ensure her knowledge related to the properties. When the participant was asked how she could define a kite to her students, she made a definition as

> *"A kite is a shape constructed by sticking the two isosceles triangles from their bases."*

The two combined isosceles triangles is actually a very common concept image of kite in the learners' mind and actually this one is one of the correct economical definitions of the kite.

After the initial definition of kite, the participant was asked to drag the dynamic kite figure to observe the changes in the measures so that she could detect out the preserved critical properties of kite and was asked to generalize these critical defining properties to all kites. Considering the side property, she came up with the statement that "two pairs of adjacent sides are equal." From this statement, it is seen that the participant was not aware that two line segments were congruent when their lengths were equal; that is, she used the terms "congruent" and "equal" interchangeably. When she was asked the difference between "two pairs of adjacent sides are congruent" and "two pairs of adjacent side lengths are equal" she first said that there was no difference; but then realized that sides could be congruent, but the side lengths could be equal. She also was able to correctly generalize other properties such as "one pair of opposite angle measures are equal, diagonals are perpendicular to each other and one diagonal is bisected by the other diagonal, one diagonal is angle bisector." Moreover, when the symmetry property was asked, she easily detected out the angle bisector diagonal as the symmetry line of a kite.

After this step, the participant was asked to think of which other quadrilaterals the kite figure could be dragged into; in other words, which other quadrilaterals had the kite properties or which other quadrilaterals were the special instances of kite. Before dragging the figure, she judged that the kite figure could be dragged into the square because a square also made up of two isosceles triangles. When she was asked whether the side property alone is enough to decide a square was a special kite; she thought and came up with that square carried all other properties of kite as well. Considering the rectangle-kite relationship, she correctly stated that a rectangle was not a special kite since the congruent adjacent sides were not congruent. Moreover, she thought that a parallelogram could not be a special kite because of not having perpendicular diagonals property. When she compared the properties of rhombus with the kite, she thought that rhombus also had all critical kite properties and so it was also a special kite. For the trapezoids, she thought except for the isosceles trapezoid the trapezoids could be special kites; because an isosceles trapezoid had one pair of opposite congruent sides instead of two pairs of congruent adjacent sides. However, she was not sure whether some trapezoids would be special kites. It was seen that the participant was good at comparing the properties to find the special instances of a quadrilateral. Before working on the dynamic figure, she correctly identified the square and the rhombus as examples of kite; but she was only doubtful about the trapezoids.

After mentation, it was time to confirm her thoughts by working on the dynamic figure. She easily dragged the figure into the square and rhombus, but not into the parallelogram and rectangle, which confirmed her previous thinking. Although she struggled to drag the figure into the trapezoid for a while, she could not do it. When I asked the reason, she stated that there were parallel sides in a trapezoid although a kite did not have. Upon this, I remind her that she detected square and rhombus as a special kite though they had parallel sides, too. Then, she examined the figure once more and realized that a trapezoid and isosceles trapezoid did not have to have any congruent adjacent sides, congruent opposite angles or perpendicular diagonals, so they did not have the critical properties which made them special kite. Our dialogue about whether trapezoids would be special kites proceeded as the following:

Researcher: Let's move on to trap ezoid, you've said that trapezoid could be a special kite…

Participant 1: No it did not. I can not drag the kite figure into a trapezoid.

Researcher: It did not… But you've said it could… why do you think so? How can you explain this?

Participant 1: Perhaps, it is because that there is no parallel sides of a deltoid.

Researcher: You say there is not any parallel sides of a kite… Are you sure?

Participant 1: Aaa… no, a rhombus is a kite but it has parallel sides… a square as well…

Participant 1: Can you compare the properties of kite and trapezoid once more, please?And then let's try to explain why a trapezoid is not a special kite.

Participant 1: Heaa…….I think it is because of the fact that a trapezoid does not have two pairs of congruent adjacent sides.

Researcher: What about the other properties?

Participant 1: Moreover, a trapezoid does not have any opposite congruent angles…

Researcher: Is'nt it? I need at least one pair of congruent opposite angles, but a trapezoid does not necessarily always have this property.

Participant 1: Moreover, the perpendicular diagonals property of a kite is not satisfied by the trapezoid.

After identifying the inclusive relationships, the participant was asked to show the hierarchy diagram between kite, rhombus and square; and she correctly came up with that the rhombus was the special case of a kite and a square was the special case of both (Figure 4.8)

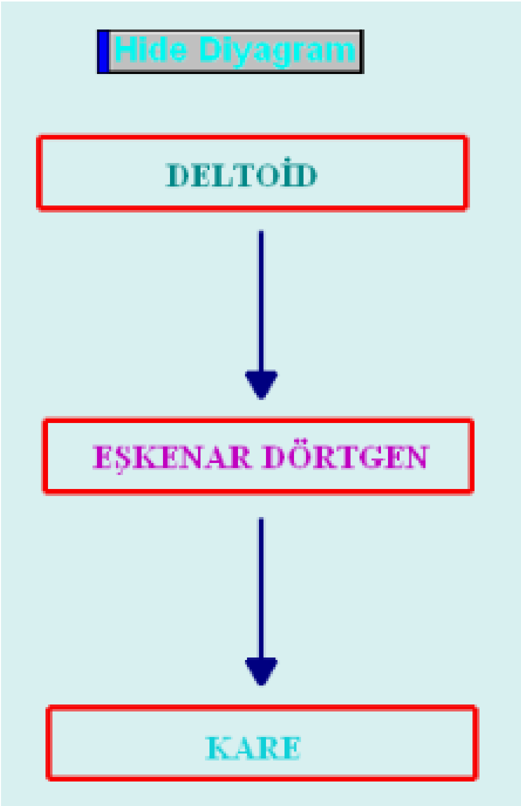


Figure 4.8 Inclusive relations between kite, rhombus and square

Then, the participant was asked to construct an inclusive definition of kite which included its descendants as well. She defined as :

> *"A kite is a quadrilateral with at least one diagonal is an angle bisector."*

Actually, she was able to construct a correct economical definition at her first attempt, but she was not sure whether it was a correct definition. For instance, when I asked her to find a counter example which satisfied this definition but not a kite, she claimed that a parallelogram also satisfied this definition since its one diagonal was a symmetry line. She was wrong due to her misinformation related to the diagonal property of parallelogram and in order to eliminate this misinformation I asked her to work on a pre-made parallelogram figure and to examine its diagonal symmetry property (Figure 4.9). As soon as she constructed the symmetrical part with respect to the diagonal, she realized that the parallelogram did not satisfy her definition and so it was not a counter example.

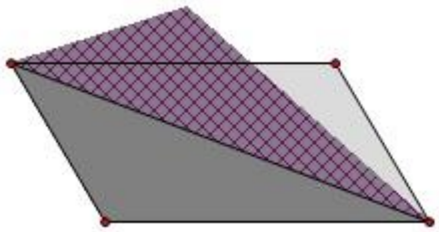


Figure 4.9 Participant 1's diagonal symmetry construction on the pre-made parallelogram figure

Then, I encouraged her to think of other possible counter examples; she could not find any other, but still was not sure. Therefore, I asked her to construct the quadrilateral only by using the property in her definition to see whether it was the sufficient defining property to generate a kite. That is, she tried to construct a quadrilateral with at least one diagonal was an angle bisector. She started the construction with the diagonal and a side, and then reflected the side with respect to the diagonal in order to form two congruent angles equally divided by the diagonal. After this step, she completed the figure into a quadrilateral and made the required measurements in order to be sure that the quadrilateral was a kite (Figure 4.10). Consequently, she came up with that having at least one angle bisector diagonal was enough defining property to characterize a kite; because this property alone satisfied all other kite properties. So, she stated that the definition was a correct economical definition of the kite.

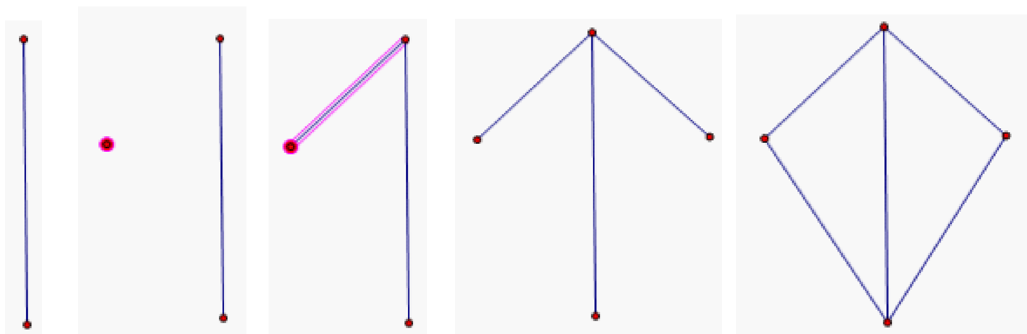


Figure 4.10 Participant 1's construction of a quadrilateral with at least one diagonal as an angle bisector.

After this step, the participant was presented with 4 pre-made kite definitions and was asked to evaluate whether these definitions were the correct ones including just necessary and sufficient properties.

Definition 1: A kite is a quadrilateral with perpendicular diagonals.

Although it was insufficient, the participant claimed that having perpendicular diagonals was the sufficient defining property to define a kite. Upon this incorrect answer, I asked her to construct a figure just by using this property in order to see whether having only this property would be enough to construct a kite. She started to construct two perpendicular lines; but she tried to make these perpendicular lines (diagonals) congruent, though it was not stated in her definition. At this stage, she realized that her definition was lacking some information. Then, she constructed noncongruent perpendicular lines since unless stated otherwise in the definition. Anyhow, she completed her sketch into a quadrilateral to see the result and she said that it visually looked like a kite (Figure 4.11). However, after making the related measurements and moving the figure, she identified that the figure did not preserved critical kite properties.

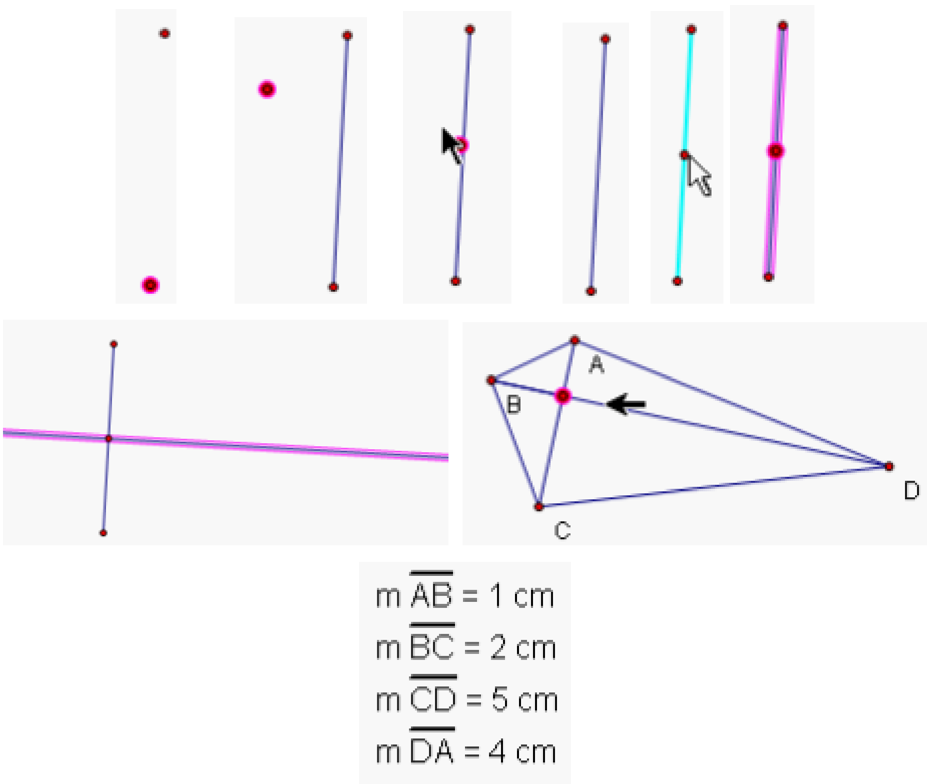


Figure 4.11 Participant 1's construction of a quadrilateral with perpendicular diagonals.

So, I asked her what other property needed; she correctly answered that "to be able to construct a kite, at least one diagonal must also be bisected by the other diagonal in addition to being perpendicular." As a result of this construction and dragging test, she came up with the second pre-made correct definition. Upon her answer, I asked her to construct a quadrilateral to confirm this second definition given as:

> *Definition 2: A kite is a quadrilateral with at least one diagonal is a perpendicular bisector.*

At that time, she tried to construct the additional property that *"at least one diagonal is bisected"* and after the related measurements and dragging, she identified the figure as kite. So, after adding the necessary property the definition included necessary and sufficient information to generalize a kite (Figure 4.12).

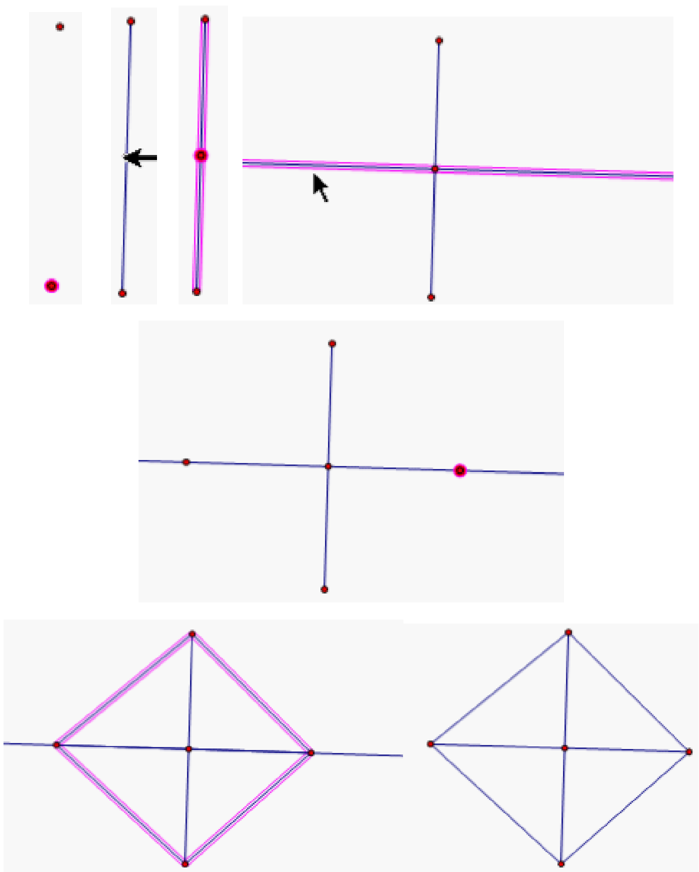


Figure 4.12 Participant 1's construction of a quadrilateral with at least one diagonal is a perpendicular bisector.

Definition 3: A kite is a quadrilateral with two pairs of congruent adjacent sides and one pair of congruent opposite angles.

When the participant was asked to think on the third definition, she firstly thought that it included the necessary and sufficient defining properties. When I encouraged her to think on whether each given property alone would be enough to define a kite, she answered that "one pair of opposite congruent angle" property might have been the result of "two pairs of congruent adjacent sides" property. However, she was doubtful about her idea, so I asked her to confirm her idea with the construction of the quadrilateral having only the first property. Although constructing two pairs of congruent adjacent sides was a challenging process for the participant, she achieved it with my probing questions.

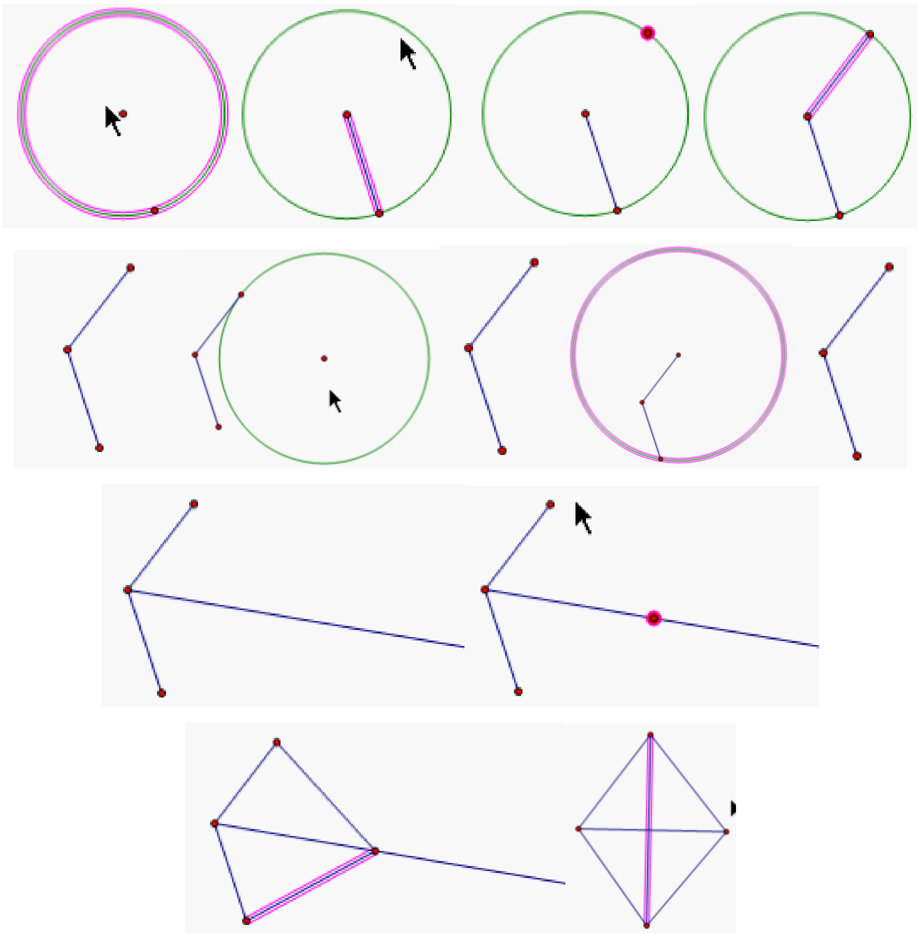


Figure 4.13 Participant 1's construction of a quadrilateral with two pairs of congruent adjacent sides and one pair of congruent opposite angles

After making the necessary measurements and dragging test to see the preserved properties, she detected out that "two pairs of congruent adjacent sides" property satisfied all other properties of a kite. So, she concluded that the given definition included more than necessary information; the first property was a sufficient defining property alone.

Finally, the participant was asked to evaluate the fourth definition based on the symmetry property and she directly stated that one was a correct economical definition and this property alone

Definition 4: A kite is a quadrilateral with at least one diagonal is the symmetry axis.

She stated that because of the symmetry, two pairs of adjacent sides would be congruent, one pair of opposite angle would be congruent and one diagonal would be bisected by the other. In order to confirm this, she also did the related construction as can be seen in the Figure 4.14. As a result of her measurements and dragging, she saw that the definition was the correct economical definition of the kite which had basis on the symmetry property.

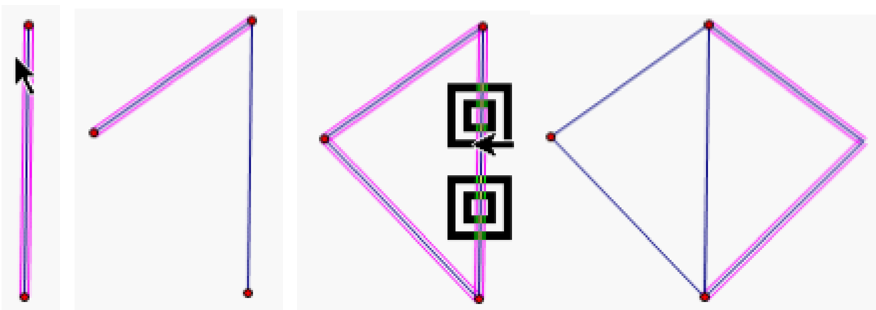


Figure 4.14 Participant 1's construction of a quadrilateral with at least one diagonal is the symmetry axis.

As a final step of this session, I asked the participant to construct a correct economical definition for the rhombus and she successfully defined it as *"A rhombus is a quadrilateral constructed by sticking the two congruent isosceles triangles on their bases."* This definition differed with a simple word "congruent" from the kite definition she said at the very beginning of the session. This indicated that she understood the importance of a word on the meaning of the definition; that is, very small changes make big differences in the meaning.

4.1.3.2 Session 2 with Participant 1: Parallelograms and Trapezoids

At the beginning of the session, there was an isosceles trapezoid on the screen and the participant was asked to remember the side, angle, diagonal and symmetry properties of it. She stated that an isosceles trapezoid had at most one pair of parallel sides, opposite equal sides which were not parallel to each other, and adjacent equal sides. She also stated that the diagonals were not symmetry axes and they did not bisect each other. When I asked whether there were any other symmetry lines she detected that the perpendicular bisector of the parallel sides was the symmetry axes. Moreover, she stated that the diagonals were looked like perpendicular, but she was not sure since she only visually conceived this idea. In her statements it was seen that she still used the terms "equal" and "congruent" incorrectly. Moreover, her using the term "at most" in the first property indicated that she did not think inclusively; that was a clue to estimate that she probably would not consider rectangle and square as special isosceles trapezoids. It was also observed that she almost remembered all the properties correctly, but had problems to express them using the correct mathematical language.

After brainstorming on the properties, she observed which properties remained unchanged under dragging and detected out the ones that she could not remember. She stated the preserved properties as "unparallel opposite sides are congruent, adjacent angles are congruent, diagonal lengths are equal, but they may or may not be perpendicular. Moreover, she found that "the diagonals intersected each other in the same ratio" and that "the perpendicular bisector of the parallel sides was the symmetry axis."

Next, the participant was asked to think about the descendants of the isosceles trapezoid. Through thinking each property one by one, she concluded that square and rectangle had all characteristic properties of an isosceles trapezoid. When it came to reason for a parallelogram, she successfully stated that it was not symmetrical with respect to the line passing through the midpoints of the parallel sides. For the similar reason, she also eliminated the rhombus as well. Moreover, she eliminated the prototypical kite since it did not always have to have any parallel sides. However, she incorrectly concluded that a prototypical trapezoid was a special isosceles trapezoid. She did not think that a prototypical trapezoid did not always have one pair of congruent sides, so it could not be a special isosceles trapezoid. As a result of this thinking process, she decided that square, rectangle and trapezoid were all special instances of an isosceles trapezoid; she was only mistaken for the prototypical trapezoid.

After, she worked on the dynamic figure to test which quadrilaterals preserved the isosceles trapezoid properties. She saw that she was right except for trapezoid and realized that a prototypical trapezoid could not be symmetrical if it was not an isosceles trapezoid, since its unparallel sides were not congruent. Moreover, she realized that adjacent angles of the trapezoid were not congruent as in the case of isosceles trapezoid. Finally, she correctly placed the isosceles trapezoid, square and rectangle on the hierarchy diagram (Figure 4.15).

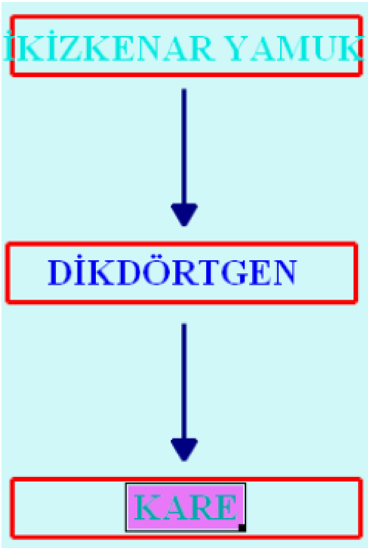


Figure 4.15 The diagram of the relations between isosceles trapezoid, rectangle and square

In the next step, the participant was asked to define an isosceles trapezoid inclusively so that the definition would include rectangle and square as special isosceles trapezoids. She constructed a definition like

> *"An isosceles trapezoid is a quadrilateral with at least two parallel sides and with at least two opposite congruent sides."*

When I asked her whether any other quadrilateral that was not in the isosceles trapezoid class would satisfy this definition she immediately found that a prototypical parallelogram was also included in this definition, though it was not an isosceles trapezoid. Upon this, she thought on how to use symmetry property as a defining property of an isosceles trapezoid. Then, she redefined as

> *"An isosceles trapezoid is a quadrilateral which is symmetrical with respect to the line passing through the midpoints of the parallel sides."*

She tried to find any counter examples that satisfied the definition out of the isosceles trapezoid class; but she could not find and decided that this definition was the correct economical definition.

In the next part of our presentation, I showed some pre-constructed isosceles trapezoid definitions and asked her to evaluate whether they included the necessary and sufficient defining properties to characterize isosceles trapezoid class. The first definition was

> *Definition 1: An isosceles trapezoid is a quadrilateral with at least one pair of parallel sides and with at least one pair of opposite congruent sides.*

However, we had already discussed this definition and she had justified that it was incorrect due to including parallelogram as counter example.

> *Definition 2: An isosceles trapezoid is a quadrilateral with one pair of parallel sides and with one pair of congruent but unparallel sides.*

As for the second definition, the participant stated that it did not include rectangle and square even though they were the specific isosceles trapezoids; but the definition only included the prototypical isosceles trapezoid. So, she did not accept this definition as a correct definition. Actually, it was not a correct inclusive definition but a correct exclusive definition which excluded the special cases of the isosceles trapezoid and just defined the prototypical isosceles trapezoid as a single object.

> *Definition 3: "An isosceles trapezoid is a quadrilateral with at least one pair of parallel sides and with opposite supplementary angles."*

For the third definition, she correctly concluded that the definition was a correct inclusive definition including the square and rectangle as special cases.

Definition 4: An isosceles trapezoid is a quadrilateral with two pairs of congruent adjacent angles.

She thought that the fourth definition was also a correct economical one; she tried to find any counter example, but could not. She claimed that if the adjacent angles were congruent the figure had to be symmetrical, but she was not sure. So, I encouraged her to make the corresponding construction for this definition. However, instead of using the congruent adjacent sides property, she constructed the figure using the symmetry property (Figure 4.16); so I asked her to write an economical definition using the symmetry property. She correctly defined as

"An isosceles trapezoid is a quadrilateral which is symmetrical with respect to the perpendicular line passing through the midpoints of the parallel lines."

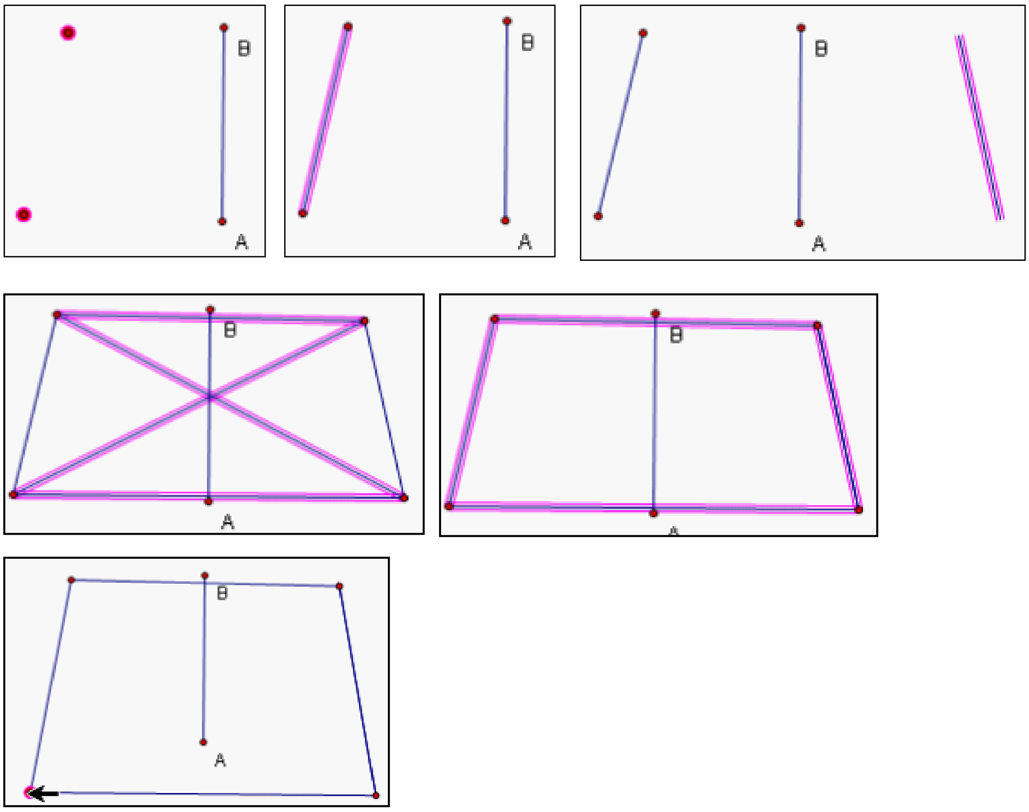


Figure 4.16 Participant 1's construction of a quadrilateral with two pairs of congruent adjacent angles.

She had already tested this definition and concluded that it was correct, but she still did not test the fourth definition. Therefore, I asked her to construct two pairs of congruent adjacent angles to test whether this information was sufficient to characterize an isosceles trapezoid. For this purpose, she first constructed an angle that could sweep 360 degree when dragged, and then she carried the same angle to the other end point of this segment to create the first pair of congruent angles. To be able to construct the second pair, she constructed a line which was parallel to the base segment and identified the intersection points of the parallel line and the side segments (Figure 4.17).

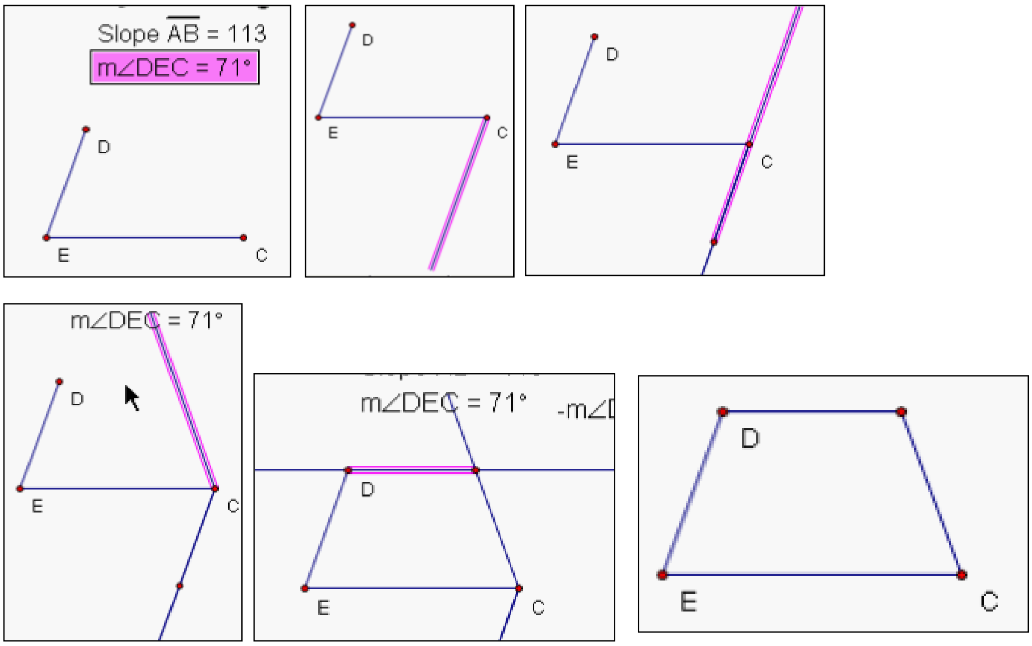


Figure 4.17 Participant 1's construction of a quadrilateral with two pairs of congruent adjacent angles

When she dragged the final quadrilateral, she observed that the figure always remained an isosceles trapezoid under dragging and all other properties were satisfied, so having two pairs of congruent adjacent angles property was enough defining property to define an isosceles trapezoid. When she was asked to evaluate the last pre-constructed definition which was

Definition 5: "An isosceles trapezoid is a quadrilateral with congruent diagonals,"

she could not make any reasoning. So I asked her to take help of a construction. She needed to construct two congruent segments intersecting at any point. In her first two attempts she could not construct what was wanted. At first attempt, she constructed a segment and constructed any point on it; then rotated the segment 90° around this point (Figure 4.18). However, this construction restricted the angle between diagonals to be 90° although it was not specified in the definition.

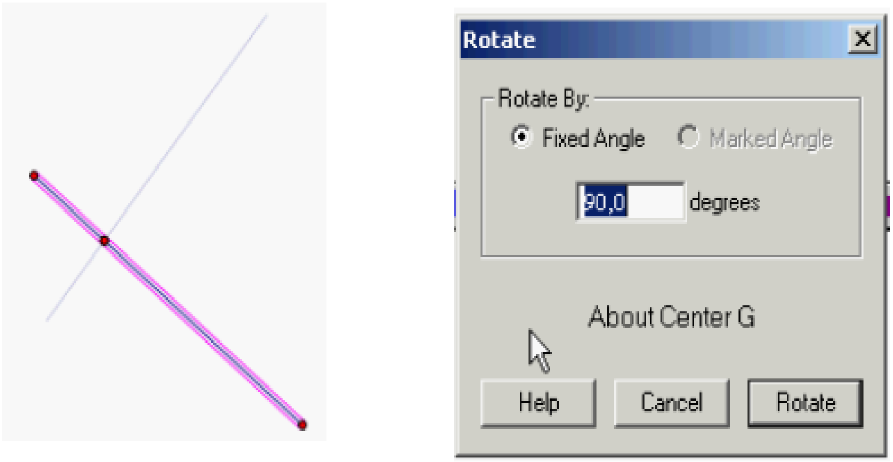


Figure 4.18 Participant 1's first attempt to construct of a quadrilateral with congruent diagonals

In her second attempt, she constructed a segment and measured its length as 5 cm (Figure 4.19). Then, she tried to construct another segment with a length of 5 cm by translating a point with a 5 cm fixed distance and -90 degree fixed angle. However, when she dragged the segments, just the second segment preserved its 5 cm length while the first segment's length changed. So, this was not the appropriate construction to meet the definition. Then, I encouraged her to think whether these two intersecting diagonals could be the diameters of a circle. Upon this, she constructed two diameters on a circle; but realized that the segments bisected each other although it was not specified in the definition. However, she realized that to construct an isosceles trapezoid, the diagonals had to intersect each other in the same ratio in addition to their being congruent. Since she found out the correct economical definition as *"An isosceles trapezoid is a quadrilateral with congruent diagonals that intersect each other in the same ratio,"* we did not continue constructing the figure.

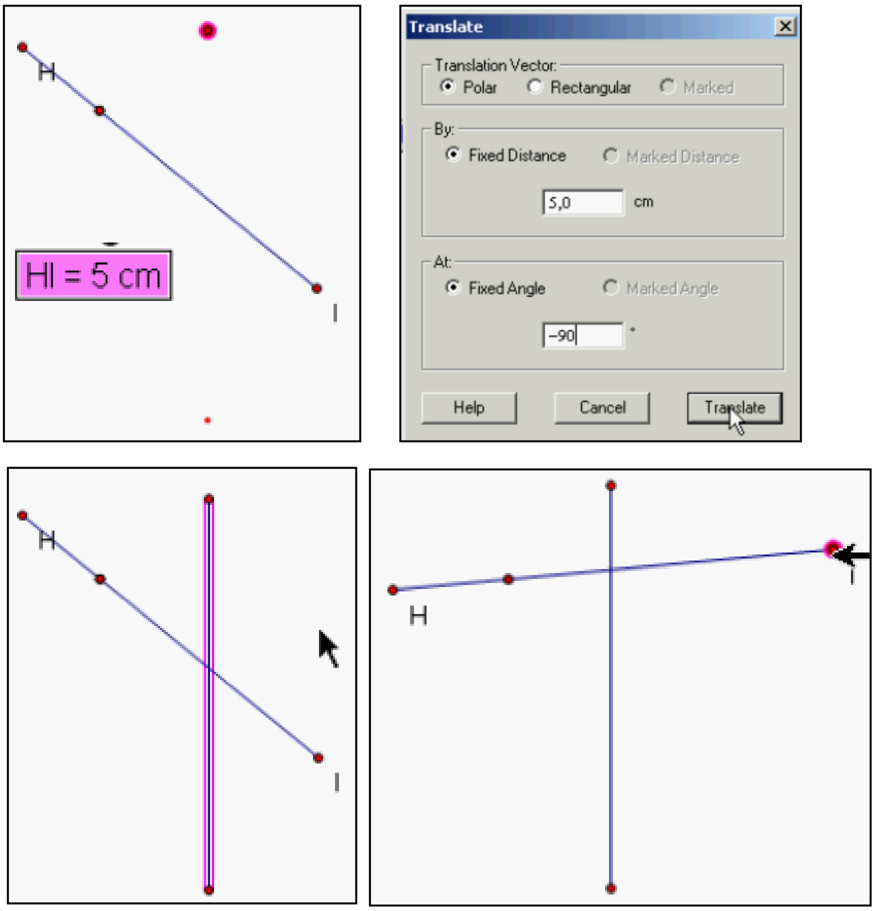


Figure 4.19 Participant 1's second attempt to construct of a quadrilateral with congruent diagonals

Having found the special isosceles trapezoids as rectangle and square and having defined the isosceles trapezoid inclusively, now the participant was asked to figure out special instances of a parallelogram. Surprisingly, she initially claimed that square was not a special parallelogram; but after thinking on whether the properties of a parallelogram were preserved, she changed her idea and stated that square was a special parallelogram. She also stated that a rectangle and a rhombus were also special parallelograms since they had two pairs of parallel and congruent sides and congruent opposite angles. Since deltoid did not have any parallel sides, she eliminated it. Moreover, she eliminated the trapezoid and isosceles trapezoid since they did not have to have two pairs of parallel sides and bisecting diagonals. As a result, she concluded that rectangle, square and rhombus were the special parallelograms and confirmed this by dragging the dynamic parallelogram figure into each quadrilaterals one by one.

The next step was to indicate these relationships with a hierarchy diagram. Parallelogram was the most general concept among these quadrilaterals, so she placed it on top of the hierarchy. Then, I asked what she would think about the relationship between the rhombus and rectangle; she incorrectly thought that rhombus was a special rectangle. Upon her answer, I provided her with dynamic rhombus and rectangle figures on the screen and asked her to investigate the relationship between them. However, when she tried to drag one figure over the other she saw that none of them could be dragged into the other. Then I asked her to compare a rectangle and rhombus in terms of their properties. She said that a rhombus has all congruent sides, so a rectangle could not be a special case of it. On the other side, a rhombus could not be the special case of a rectangle since it did not have right angles. As a result of such kind of reasoning, she concluded that there was not any hierarchical relation between rectangle and rhombus. So, she placed rectangle and rhombus separately under the parallelogram in the hierarchy. Finally, she stated that the square was the most specific quadrilateral included by all other quadrilaterals; so she placed the square at the bottom of the hierarchy (Figure 4.20).

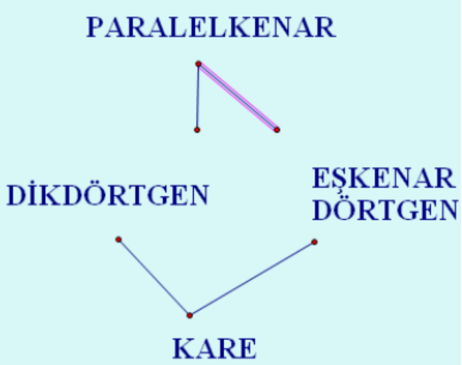


Figure 4.20 Participant 1's hierarchy diagram of the parallelograms

Then, the next step was to investigate the relationship between the parallelograms and trapezoid. She thought that a trapezoid needed to have at least one pair of parallel sides and parallelograms satisfied this property; so the parallelogram could be the special trapezoids. Then, she tested her idea by dragging the trapezoid into a parallelogram and saw that she could do it. I asked her whether there was any other special case of trapezoids other than the parallelograms. She stated that an isosceles trapezoid was also a specified trapezoid and tested this through dragging as well. Then I asked her how she could define trapezoid so that the definition would include parallelograms and isosceles trapezoid as special cases. She easily found the correct economical definition as *"a trapezoid is a quadrilateral with at least one pair of parallel sides."*

The next step was to show all relationships discovered so far on the hierarchy diagram. At first, she was mixed up and showed the rhombus as a special isosceles trapezoid on the diagram, but did not show it as a special kite (Figure 4.21). Upon this, I asked her to think of the relationship between isosceles trapezoid and rhombus; and she suddenly realized that rhombus was not symmetrical with respect to the line passing through the midpoints of the parallel sides, so it was not a special isosceles trapezoid. Finally, she changed the places of rhombus and rectangle and constructed the correct hierarchy (Figure 4.22).

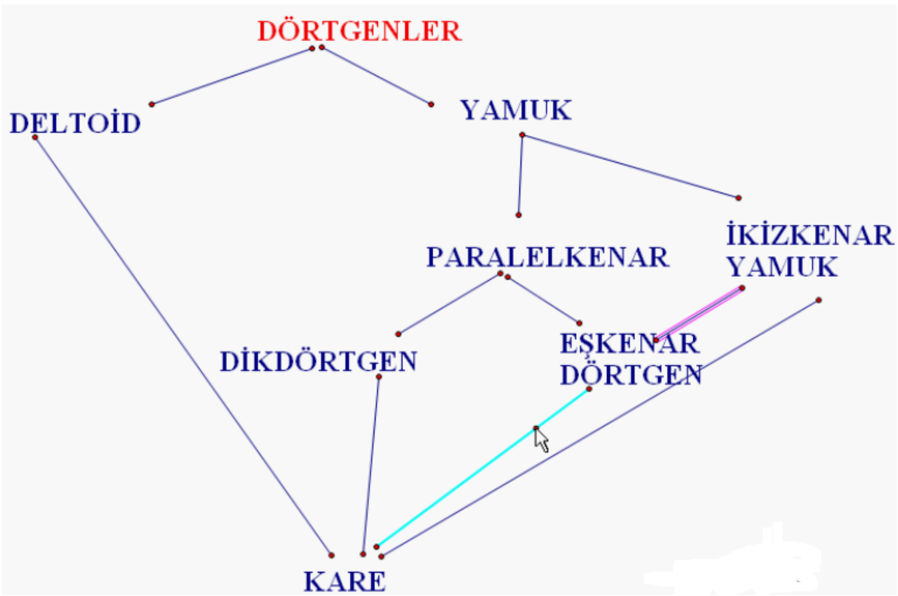


Figure 4.21 Participant 1's first attempt to construct quadrilateral hierarchy

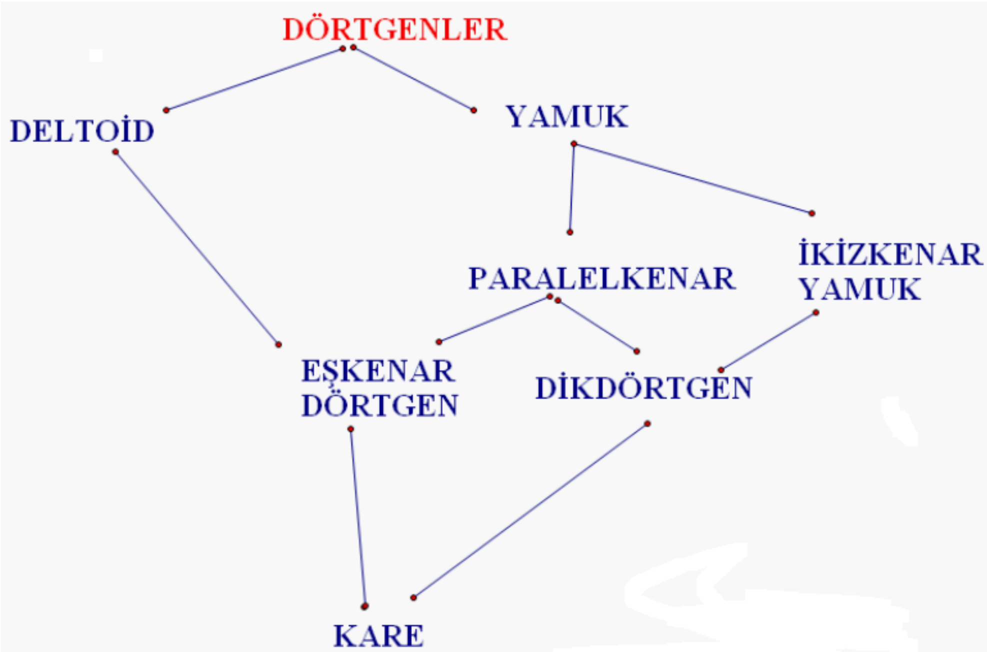


Figure 4.22 Participant 1's second attempt to construct quadrilateral hierarchy

So far, the participant was always asked to construct inclusive definitions which included the special instances of the defined concept. Now, it was time to construct *exclusive definitions* which excluded the special instances of the concept and defined it as a single object. Firstly, I asked her to define a parallelogram exclusively so that the definition would only define the prototypical parallelogram figure, but not its descendants. Moreover, I restricted her to use only the *diagonal property* of the parallelogram. She first remembered that the diagonals of a parallelogram did not have to be congruent, but they have to bisect each other. When asked to think on the diagonal properties of rhombus, square and rectangle, she stated that to be able to eliminate rhombus and square the diagonals must not have been perpendicular to each other. Moreover, she added that the diagonals must not have been congruent to eliminate the rectangle from the exclusive definition. So she defined parallelogram exclusively as *"a parallelogram is a quadrilateral of which diagonals bisect each other, but are not perpendicular and congruent."*

Next, she was asked to define rhombus exclusively so that the square would not be included in the definition, but with a restriction of using the *symmetry* property. She correctly defined as *"a rhombus is a quadrilateral of which both diagonals are symmetry* *axes but are not equal in length."* Finally, I asked her to define kite exclusively using *any* property; this time there were not any restriction. She defined as *"a kite is a quadrilateral which is symmetrical with respect to only one of its diagonals."*

When I asked her whether constructing inclusive or exclusive definitions would encourage more effective learning, she supported the inclusive definitions claiming that they increased analysis skills much more than the exclusive ones.

4.1.3.3 Session 3 with Participant 1: Cyclic and Circum Quadrilaterals

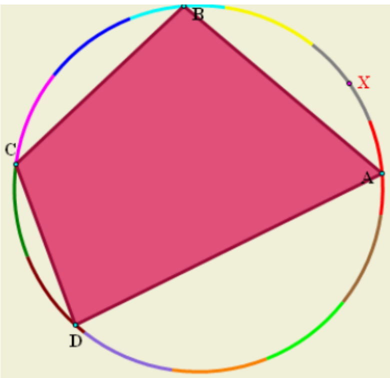


Figure 4.23 Dynamic cyclic quadrilateral image

When I asked what kind of a quadrilateral seen on the screen was (Figure 4.23), the participant called it as "irregular quadrilateral," but she did not know that it was called "cyclic quadrilateral." Then, I asked her how to define a cyclic quadrilateral, she defined as "a cyclic quadrilateral is a quadrilateral of which vertices are on the circle." This was a correct definition. Next, I asked her what the conditions were for a quadrilateral to be a cyclic quadrilateral; she thought a while, but said that she did not know the answer. Then, I asked her to think about the vertices of which special quadrilaterals could be placed on a circle. She quickly said a square could be a cyclic quadrilateral, but she could not explain the reason; she stated that she only imagined that all sides of the dynamic figure on the screen could be made congruent. That is, it seemed like she just decided based on the visualization without any mathematical reasoning. Moreover, she stated that the cyclic quadrilateral on the screen could be dragged into a rectangle, but not into a parallelogram. She explained that at least two vertices of a parallelogram could not be placed on the circle because of the parallelism. Upon this, I asked her why a rectangle could be placed but a parallelogram could not, though they both had parallel sides. Then, she stated that it was because of the right angles of the rectangle that it could be placed on the circle; she was getting close to the crutial point step by step. When I encouraged her to think on the angles, she easily reached to the conclusion that for a quadrilateral to be a cyclic quadrilateral it had to have opposite supplementary angles. Considering this criterion, she correctly stated why rhombus could not also be a cyclic quadrilateral. For the kite, she stated that it was not a cyclic quadrilateral since it did not have to have supplementary angles; however, she did not considered that there would be a special kite having congruent angles of 90 degrees each. She also correctly detected that a trapezoid was not a cyclic quadrilateral, but an isosceles trapezoid was cyclic due to having opposite supplementary angles. That is to say, she correctly detected out rectangle, square and isosceles trapezoid as cyclic quadrilaterals but not considered right kite.

In order to confirm arguments, she dragged the cyclic quadrilateral figure into the other quadrilaterals one by one. She saw that the figure could be dragged into a square and rectangle; but when she tried to drag the figure into a trapezoid, it became an isosceles trapezoid. Moreover, she could not drag the cyclic quadrilateral figure into a parallelogram and into a rhombus; when she tried to make a parallelogram it became a rectangle and when she tried to make a rhombus it became a square. When she tried to make a kite she realized that only a special kite of which one pair of congruent angles were 90 degrees each could be a cyclic quadrilateral. Then, we decided to call this special kite as "right kite" from then on.

Since a new kite was specialized, now it was time to define it with the correct mathematical language. So far the participants were asked to define each quadrilateral in terms of "quadrilateral," but from now on they were asked to construct definitions in terms of other quadrilaterals that are more general ones than the defined ones. Here, the participant was asked to define a right kite in terms of "kite," "quadrilateral" and "cyclic quadrilateral" separately. She started with defining it in terms of a quadrilateral; and defined as

> *"A right kite is a quadrilateral with one pair of congruent opposite angles measures of which are 90 degrees each."*

However, this definition included rectangle which was not a kite; when I asked her to think on the counter examples, she detected this fact and offered to add information about the congruent adjacent sides. Her next definition was

> *"A right kite is a quadrilateral with two pairs of congruent adjacent sides and with at least one pair of congruent opposite angles measures of which are 90 degrees each."*

When she was asked how she decided to use these defining properties, she stated that having two pairs of congruent adjacent sides property was the property that made it a kite and having congruent opposite right angles was the property that made it a special kite which was a cyclic quadrilateral. Then she defined the right kite in terms of "kite" as

> *"A right kite is a special kite."*

However, this was not a definition which could help someone, who did not know what a right kite was, to characterize the figure of right kite. So she redefined it as

> *"A right kite is a kite of which congruent opposite angles were 90 degrees each."*

This time it was a correct definition including the sufficient information to characterize a right kite for someone who knew what a kite was. In order to increase the participant's awareness about the difference between defining in terms of different quadrilaterals, I encouraged her to explain the difference between these two definitions. She was aware that she had to add kite properties when she defined it in terms of quadrilateral, but not when she defined it in terms of kite; because this type of a definition addressed to the ones who had already known what a kite was.

Next step was to write a definition explaining what kind of a cyclic quadrilateral the right kite was. In her fist trial she defined as

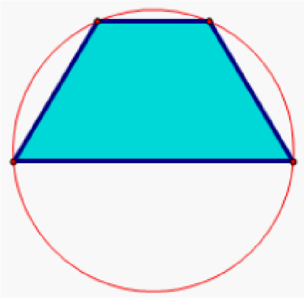
> *"A right kite is a cyclic quadrilateral with at least one pair of congruent opposite angles."*

Then I made her remember that the rectangle was a cyclic quadrilateral having one pair of congruent opposite angles but it was not a right kite. Then I encouraged her to think what property was automatically added to the definition when she defined it in terms of cyclic quadrilateral. She correctly stated that having opposite supplementary angles property was known if it was a cyclic quadrilateral. So, she realized that she needed to add only the property that made it a kite and redefined correctly as

"A right kite is a cyclic quadrilateral with two pairs of congruent adjacent sides."

Next, the participant was asked to define other cyclic quadrilaterals, namely, isosceles trapezoid, square and rectangle in terms of cyclic quadrilateral. She defined isosceles trapezoid as

"An isosceles trapezoid is a cyclic quadrilateral with two pairs of congruent adjacent angles."

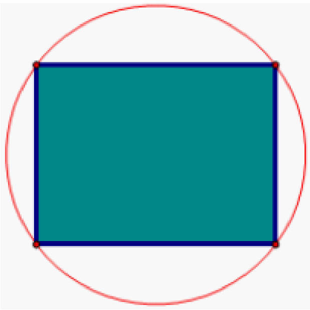


She explained that "if we know that the quadrilateral is a cyclic quadrilateral then we already know that it has supplementray opposite angles; and saying that it had congruent adjacent angles would be enough defining property." I asked her what if we only knew that one pair of the adjacent angles were congruent; after thinking a while, she realized that saying one pair of congruent adjacent angles would be enough property since the congruency of the other pair could be inferred from the given information. Therefore, she redefined as

"An isosceles trapezoid is a cyclic quadrilateral with at least one pair of congruent adjacent angles."

Then, she defined a rectangle as

"A rectangle is a cyclic quadrilateral with all congruent angles and with opposite congruent sides."



However, this definition included more than the necessary information in terms of angle property. Our conversation for this definition continued as the following:

Researcher: What property was automatically added to the definition when you said that a rectangle is a cyclic quadrilateral?

Participant 1: If it is a cyclic quadrilateral, I know that the opposite angles are supplementary… emm..

Researcher: And you added another angle property that all angles are congruent… That is, if we rewrite your definition in terms of a quadrilateral, we can say that "A rectangle is a quadrilateral with congruent opposite supplementary angles and with opposite congruent sides."

Participant 1: Ohh yes…Then… what if we say "a rectangle is a cyclic quadrilateral with one pair of congruent adjacent angles."

Researcher: Then, what is the difference between the isosceles trapezoid definition that you constructed a few minutes ago and this rectangle definition?

Participant 1: Hmmm…yes, this definition included isosceles trapezoid

Researcher: So, how you could change the definition?

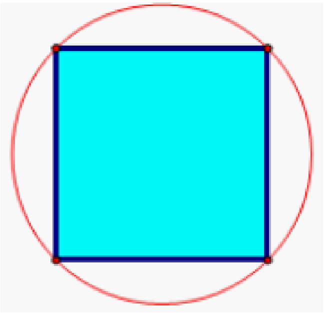
Participant 1: Mmm….. "a rectangle is a cyclic quadrilateral with two pairs of opposite congruent sides" would be enough, I think..

Researcher: Could you please explain how you decided?

Participant 1: The most general quadrilateral with two pairs of opposite congruent sides is parallelogram; however, we know that a cyclic quadrilateral has to have opposite supplementary angles… so, it can not be a parallelogram for this reason… Then, this information would be enough to infer all other information for the rectangle.

That is, she was able to detect out correct and minimal defining properties of a rectangle to define it as a special cyclic quadrilateral. Next, the participant was asked to define square, and she correctly defined at once as

"A square is a cyclic quadrilateral with all congruent sides."



Then, she was given the whole hierarchy diagram on the screen and was asked to explain the relations once more. She was able to explain the inclusive relations correctly. when she was asked to place cyclic quadrilateral and right kite categories into the correct places in the hierarchy, she did it correctly (Figure 4.24).

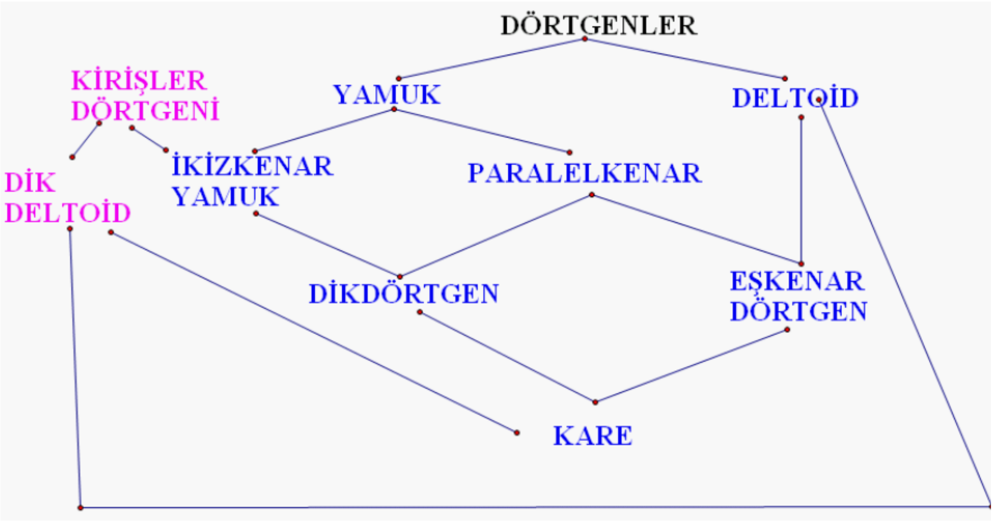


Figure 4.24 Participant 1's hierarchy diagram of quadrilaterals including cyclic quadrilaterals and right kite

In the next sketchpad screen an ordinal quadrilateral of which 3 sides were tangent to a circle was given and the participant was asked to investigate in which quadrilaterals the fourth side was always tangent to the circle (Figure 4.25). That is, she was asked to detect out the quadrilaterals that were always circum quadrilaterals with all 4 sides were always tangent to a circle. First, she was asked just to think before dragging the figure into other quadrilaterals. She stated that the diameter of the circle could be made as long as the one side length of a square, so a square was a circum quadrilateral. Then she said that a trapezoid could also be a circum quadrilateral, but she suddenly changed her mind and stated that not a prototypical trapezoid, but an isosceles trapezoid was always a circum quadrilateral. Moreover, she stated that she only made visual predictions without any mathematical explanation; that is, she did not know the conditions which made a quadrilateral a circum quadrilateral. Finally, she only thought that square and isosceles trapezoids could be the circum quadrilaterals; but she could not make any prediction about the other quadrilaterals since she did not grasp the mathematical condition.

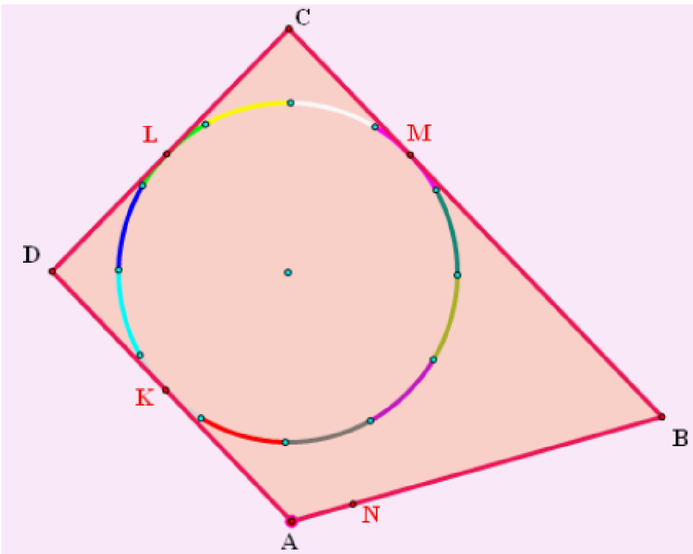


Figure 4.25 Dynamic circum quadrilateral image

She was very eager to drag the dynamic figure to see what happened. At first, she was able to drag the figure into a square, so she was right that a square was a circum quadrilateral (Figure 4.26). Then I asked her to drag the bottom side of the square straight down without changing the angles. She realized that it became a rectangle, but did not remained as a circum quadrilateral; so she concluded that a prototypical rectangle was not a circum quadrilateral, but its special case square was. As for the rhombus, she had difficulty to drag the figure and so she thought that it was not a circum quadrilateral; that is, a technical difficulty was about to cause her to conclude incorrectly. However, this was highly due to her not understanding the function of the dynamic figure exactly; because after working on it for a while, she was able to construct a rhombus easily.

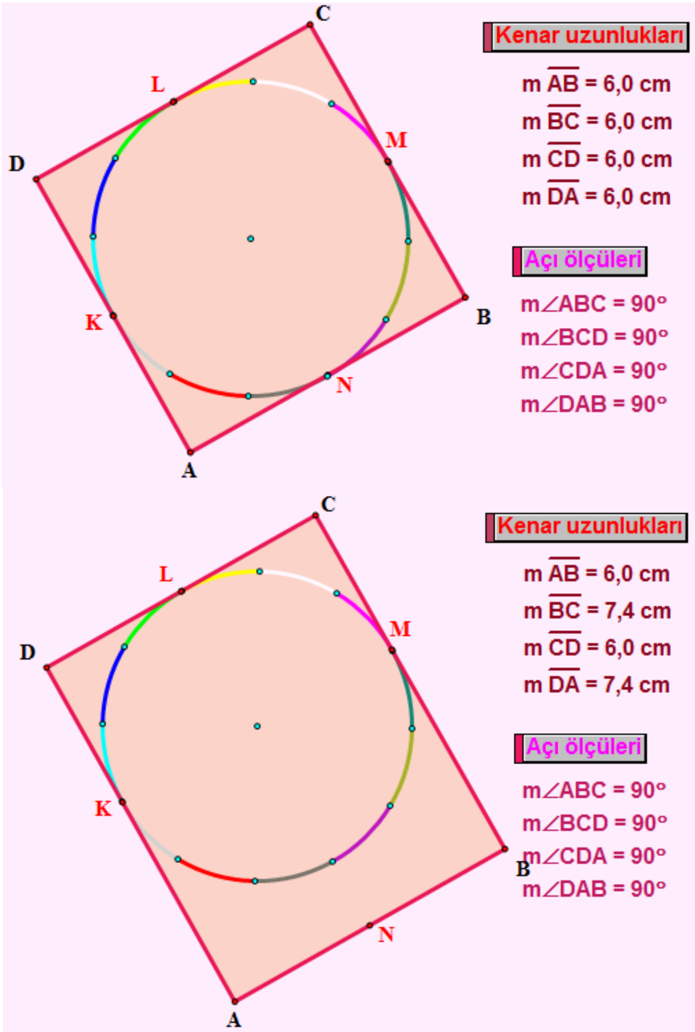


Figure 4.26 Participant 1's dragging the dynamic circum quadrilateral figure into square and rectangle

As in the case of square and rectangle; she also observed that when the side of the rhombus was dragged so that it would turn into a parallelogram, the figure lost the property of being a circum quadrilateral (Figure 4.27). Next, she was able to drag the figure into a kite and stated that then all special cases of kite had to be a circum quadrilateral since the property was preserved in all of its descendants. This conclusion was what I expected from the participants and indicating their understanding of the inclusive relations.

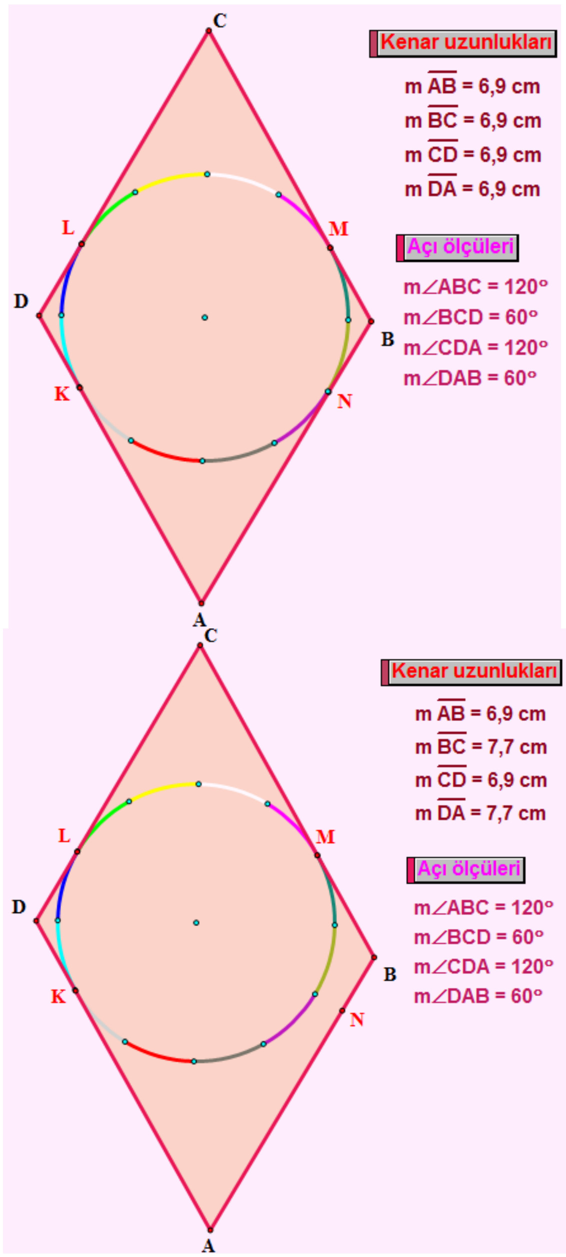


Figure 4.27 Participant 1's dragging the dynamic circum quadrilateral figure into rhombus and parallelogram

Next, she tried to drag the figure into trapezoid (Figure 4.28) and isosceles trapezoid and she was able to construct them; however, when I encouraged her to think of whether they could always remain as a circum quadrilateral she remembered the cases of rectangle-square and rhombus-parallelogram. Then she dragged the bases of the trapezoids and observed that they did not always remain as a circum quadrilateral. As a result of this process, she correctly detected out the circum quadrilaterals as kite class including the kite, rhombus, square and right kite.

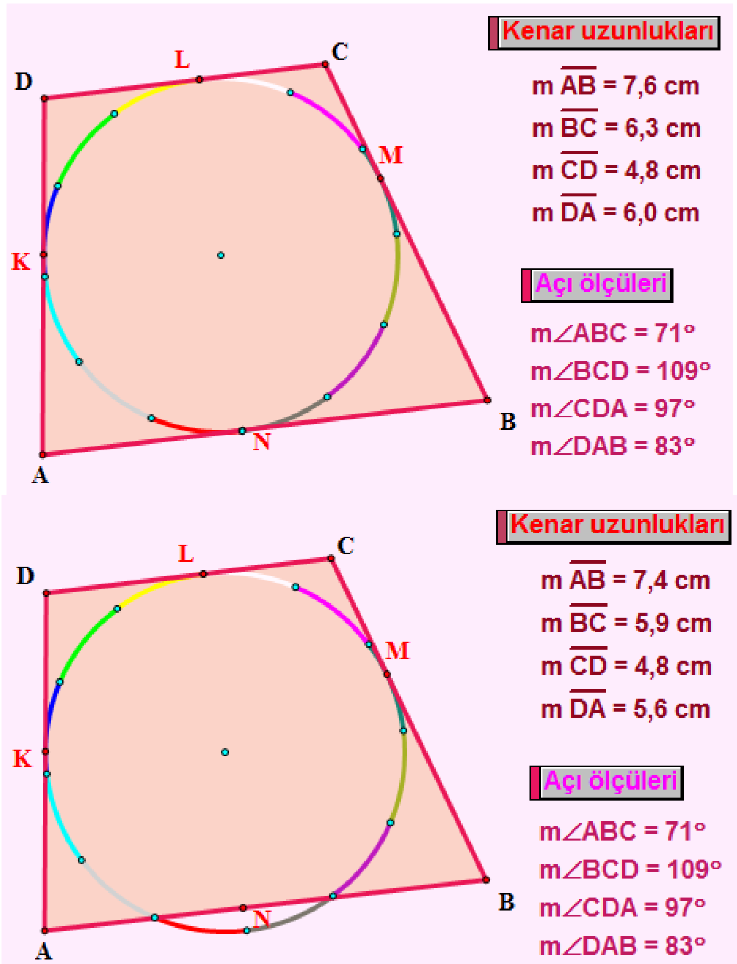


Figure 4.28 Participant 1's dragging the dynamic circum quadrilateral figure into trapezoid

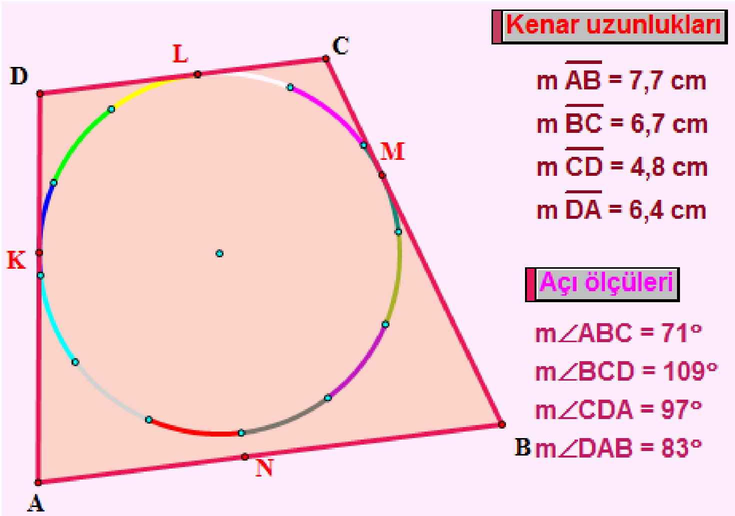


Figure 4.28 (continued)

Up to now, the participant detected out that the cyclic quadrilaterals were isosceles trapezoid, rectangle, square and right kite, while the circum quadrilaterals were the kite, rhombus, square and right kite. In the next screen, participant was given a quadrilateral of which intersection of the angle bisectors were drawn (Figure 4.29). The participant was asked to drag the quadrilateral into kite, right kite, rhombus, square, parallelogram, rectangle, trapezoid and isosceles trapezoid and to observe the intersection of angle bisectors.

Figure 4.29 Dynamic figure of a quadrilateral with intersection of the angle bisectors

As a result of her investigation, she found that the intersection was a point while the quadrilateral was kite, right kite, rhombus and square. Moreover, the intersection was a rectangle for parallelogram, an ordinary quadrilateral for trapezoid, right kite for isosceles trapezoid and square for rectangle.

Then we moved on to the next screen. In the next screen, she was given a quadrilateral of which intersection of the perpendicular bisectors of the sides were drawn (Figure 4.30) and she was asked to drag the quadrilateral into kite, right kite, rhombus, square, parallelogram, rectangle, trapezoid and isosceles trapezoid and to observe the intersection.

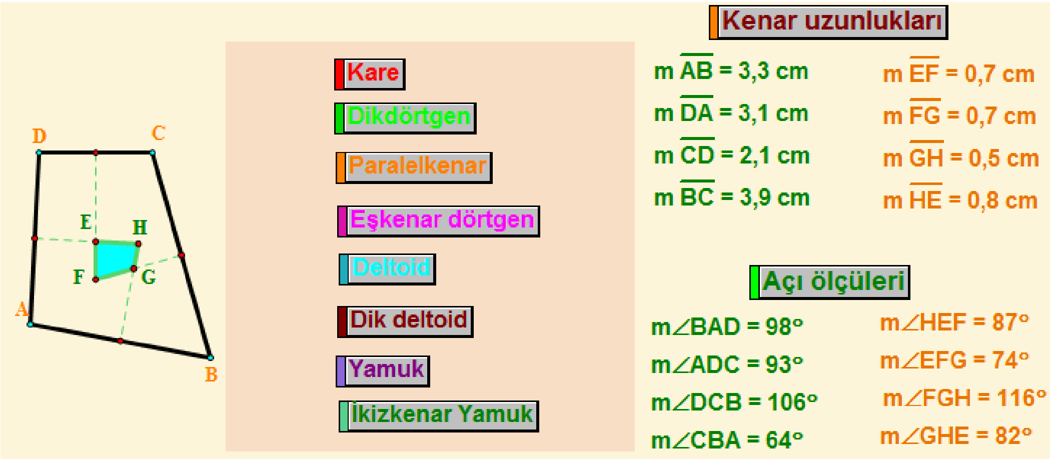


Figure 4.30 Dynamic figure of a quadrilateral with intersection of the perpendicular bisectors of the sides

As a result of her investigation, she found that the intersection was a point while the quadrilateral was isosceles trapezoid, rectangle, square and right kite. Moreover, the intersection was the same quadrilateral as the dragged quadrilateral for parallelogram, trapezoid, kite and rhombus.

In the next step the participant was asked to detect out the quadrilaterals of which intersection of both angle bisectors and perpendicular bisectors of the sides were points. She correctly identified that quadrilaterals of which intersection of the angle bisectors were point were all circum quadrilaterals. So, when asked, she was able to pass on a judgment that the condition for a quadrilateral to be a circum quadrilateral was that the intersection of the angle bisectors had to be the center of the circum circle. On the other side, she detected that for all of the cyclic quadrilaterals, the intersection of the perpendicular bisectors of the sides was a point; and she concluded that this point was the center of the cyclic circle. Moreover, she also realized that the square and right kite were included in both group which made them both cyclic and circum quadrilaterals.

Finally, she was able to add the circum quadrilaterals category in to the hierarchy diagram and was able to indicate all relations (Figure 4.31).

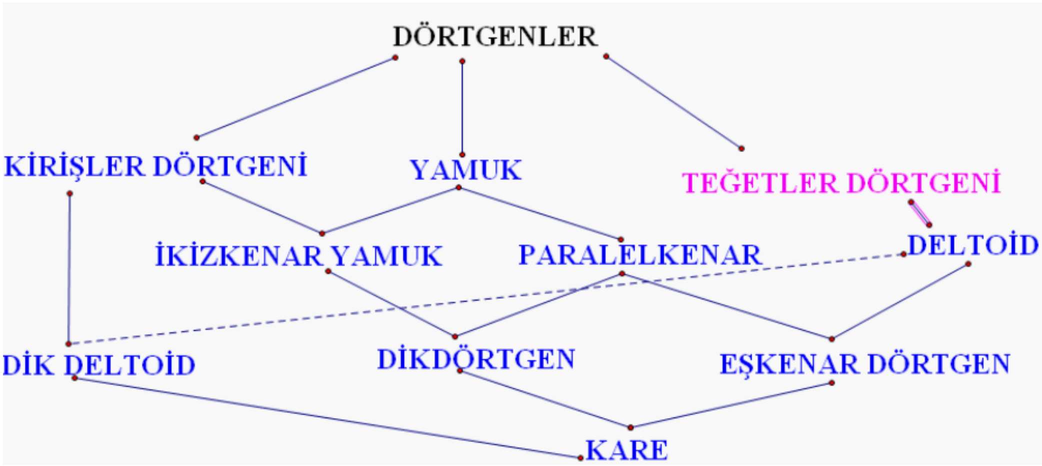


Figure 4.31 Participant 1's hierarchy diagram of quadrilaterals including circum quadrilaterals

4.1.3.4 Session 4 with Participant 1: New Quadrilaterals in the Hierarchy

At the beginning of the session, the participant was given the hierarchy constructed so far and was asked to explain the relationships (Figure 4.32). She correctly explained the inclusive relations in the hierarchy. She also correctly explained that the number of properties increased from top to the bottom of the hierarchy since the quadrilaterals specified more from top to the bottom. Then, I asked her whether the inclusive relationships between the quadrilaterals and the inclusive relationships between their properties were the same. At first, she could not grasp the idea, so I encouraged her to think of the relationship between rectangle and square specifically. She correctly stated that square was more specific than rectangle, so a square had much more properties which included the properties of rectangle. She was able to state that there was a converse inclusive relation between the quadrilaterals and between their properties such that while rectangle included square; the properties of square included the properties of rectangle.

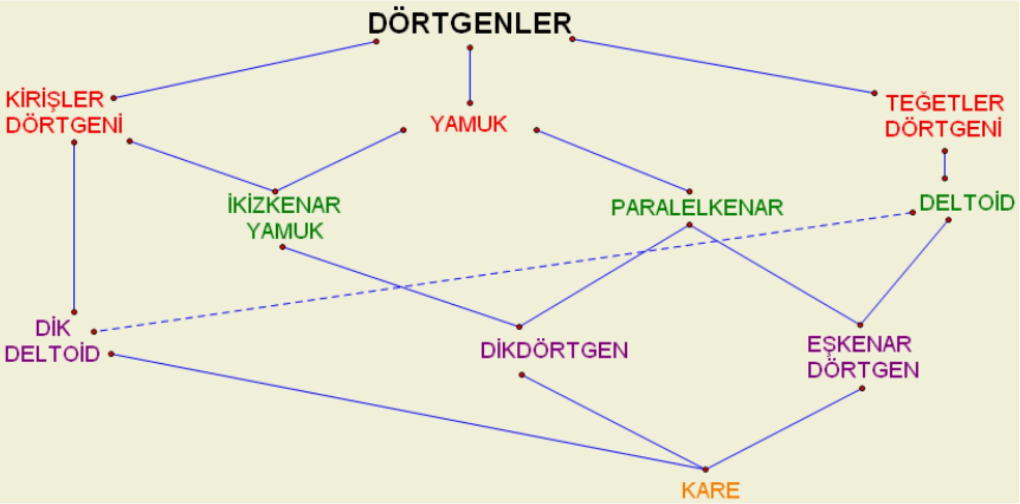


Figure 4.32 Hierarchy diagram of quadrilaterals

At the beginning of the session, the participant was given the hierarchy constructed so far and was asked to explain the relationships. She correctly explained the inclusive relations in the hierarchy. She also correctly explained that the number of properties increased from top to the bottom of the hierarchy since the quadrilaterals specified more from top to the bottom. Then, I asked her whether the inclusive relationships between the quadrilaterals and the inclusive relationships between their properties were the same. At first, she could not grasp the idea, so I encouraged her to think of the relationship between rectangle and square specifically. She correctly stated that square was more specific than rectangle, so a square had much more properties which included the properties of rectangle. She was able to state that there was a converse inclusive relation between the quadrilaterals and between their properties such that while rectangle included square; the properties of square included the properties of rectangle.

In the next step, it was time to define new shapes in the hierarchy. First, we started with generalizing the existing quadrilaterals and for this purpose the participant was asked to define a new shape (quad 1) that would include all kite class (Figure 4.33).

For this purpose she first remembered the definition of kite as *"a quadrilateral with two pairs of congruent adjacent sides."* She correctly reasoned that there was need to restrict the property used in the kite definition in order to define a more general quadrilateral.

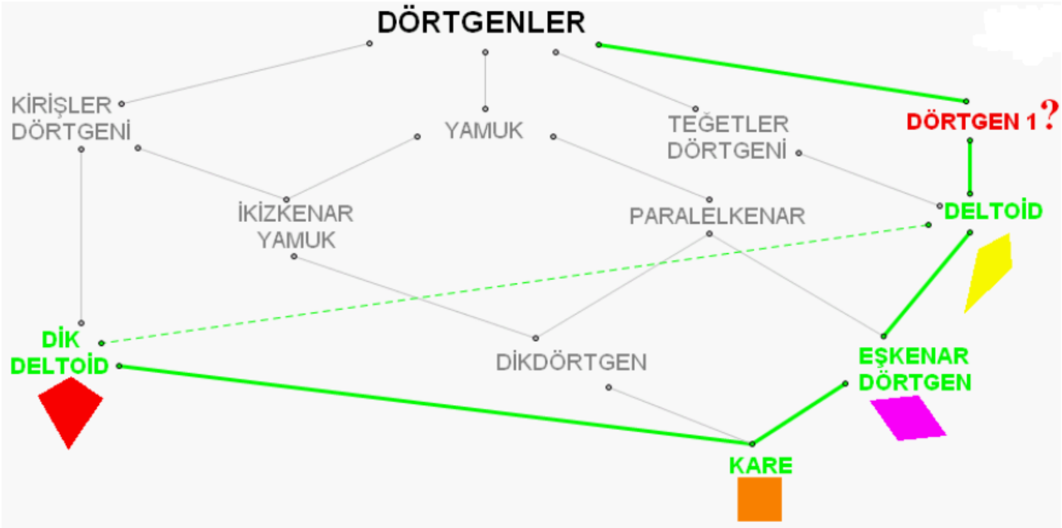


Figure 4.33 Hierarchy diagram of quadrilaterals including "quad 1"

She explained that since "quad 1" would be on the same level with cyclic quadrilaterals, circum quadrilaterals and trapezoids categories in the hierarchy; it would not have their properties. So, none of its sides would be parallel. Then she reduced the definition of kite to define quad1 as the following:

> *"A quad 1 is a quadrilateral with at least one pair of congruent adjacent sides."*

Next, he was asked to make only the drawings of some possible quadrilaterals other than the kite class, which were the examples of this definition (Figure 4.34).

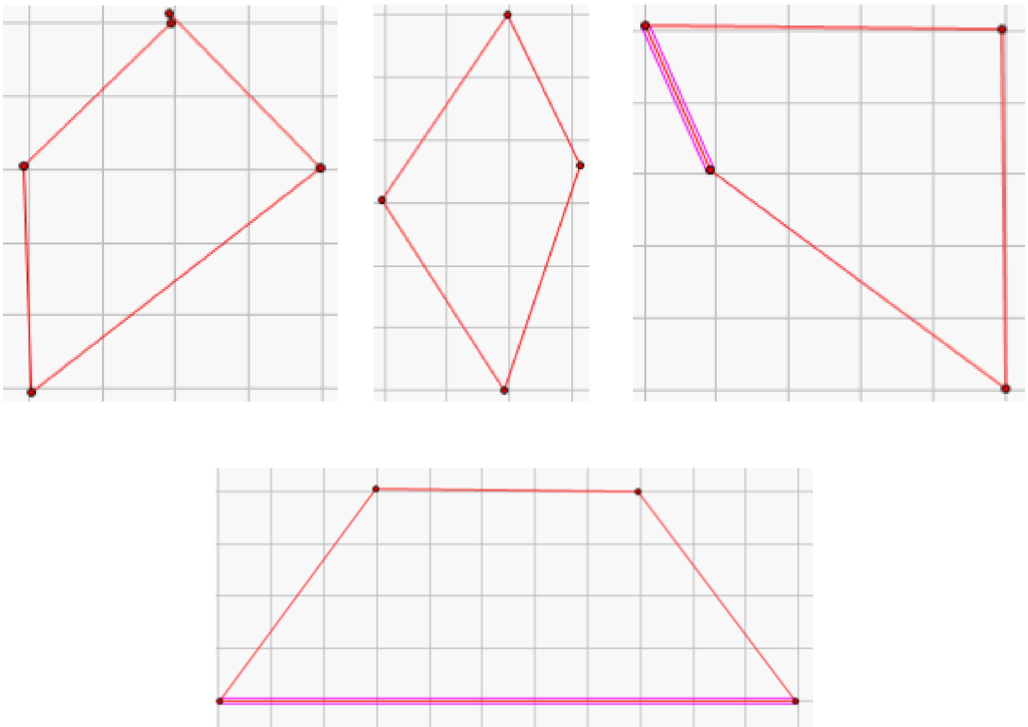


Figure 4.34 Participant 1's drawings of "quad 1"

Next, she was asked to define a "quad 2" which had the properties of both "quad1" and "trapezoid;" that is she was asked to specialize the definitions of "quad1" and "trapezoid." Moreover, she was asked to define it in terms of "quadrilateral," "quad1" and "trapezoid." Her definitions were the following:

> *"A quad2 is a quadrilateral with at least one pair of parallel sides and with at least one pair of congruent adjacent sides."*
>
> *"A quad2 is a trapezoid with at least one pair of congruent adjacent sides."*
>
> *"A quad2 is a quad1 with at least one pair of parallel sides."*

Surprisingly she was very good at defining the new quadrilateral in terms of other quadrilaterals; she did know how to command the properties. Then she also drew the possible quad2s (Figure 4.35). While drawing, she observed that rhombus was a special quad2 though it was not a special trapezoid.

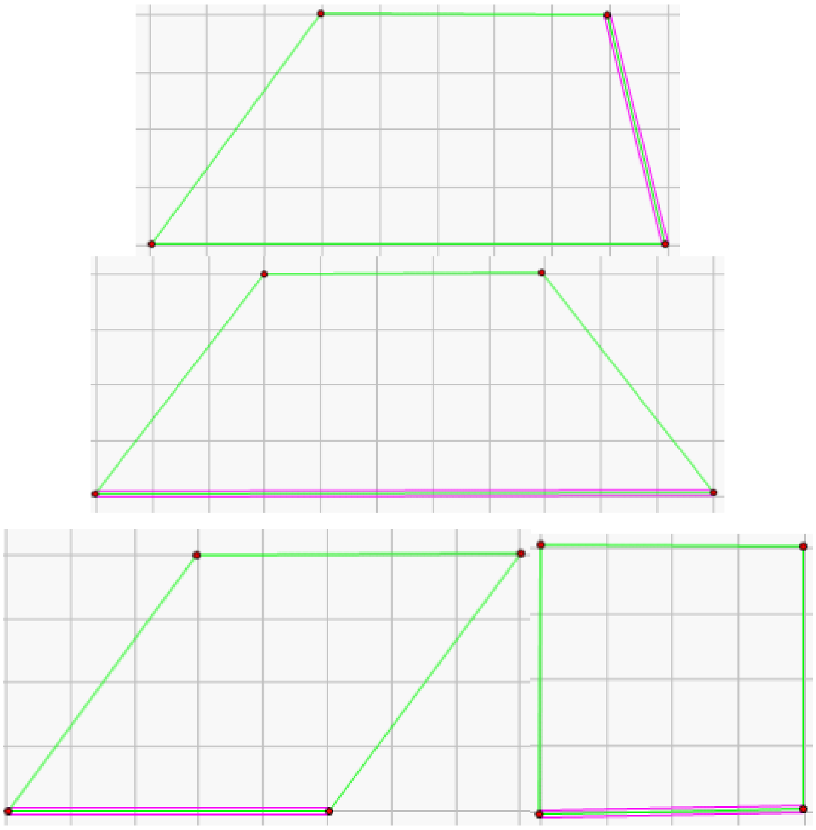


Figure 4.35 Participant 1's drawings of "quad 2"

Now, she was asked to define a "quad3" which was the special case of both isosceles trapezoid and quad2. So, she was going to define it in terms of isosceles trapezoid, quad2 and cyclic quadrilateral. Her definitions were the following:

> *"A quad 3 is an isosceles trapezoid with at least one pair of congruent adjacent sides."*

While defining in terms of isosceles trapezoid she stated that she needed to only add a defining property of quad 2 since properties of isosceles trapezoid were already included into the definition.

> *"A quad 3 is a quad2 with two pairs of congruent adjacent angles."*

This time she only added defining property of an isosceles trapezoid since the quad2 properties were already included in the definition since it was constructed on the base of quad 2.

She stated that to be able to define in terms of a cyclic quadrilateral, there was need to add properties of both isosceles trapezoid and quad2. She explained that she needed to add the properties that at least one pair of parallel sides, two pairs of congruent adjacent angles and at least one pair of congruent adjacent sides. Then she realized that these properties all together made 3 adjacent congruent sides and she defined it as

> *"A quad 3 is a cyclic quadrilateral with 3 congruent adjacent sides and with at least one pair of parallel sides."*

However, the definition was including extra information that was not necessary; she did not realize at this point that having 3 adjacent congruent sides and being a cyclic quadrilateral would already lead "at least one pair of parallel sides" property. When she was asked to think on reducing the definition, she wanted to think by drawing on the paper. Upon this, I asked her first to make the related drawings and then to decide whether the definition was including more than the necessary information. Her sketches for the quad3 were the following (Figure 4.36).

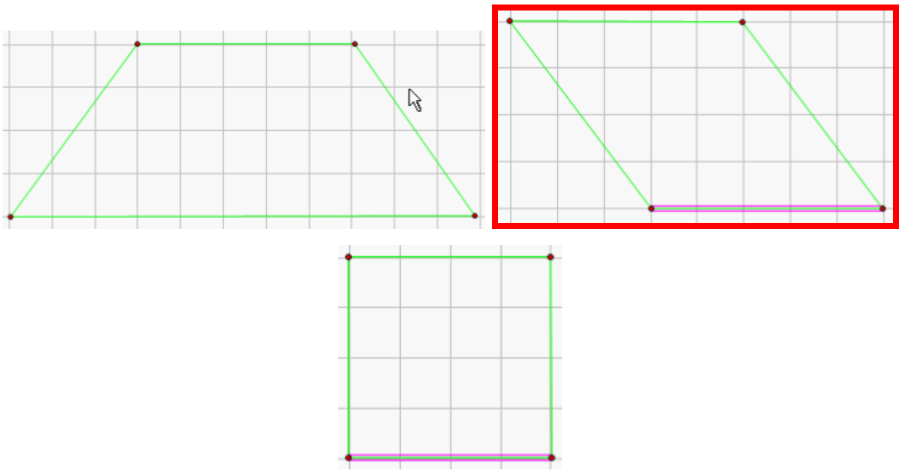


Figure 4.36 Participant 1's drawings of "quad 3"

In addition to an isosceles trapezoid with 3 congruent adjacent sides and a square, she also drew a rhombus although it was not a cyclic quadrilateral and so could not be considered as an instance of quad3. However, as soon as she realized that a rhombus was not an isosceles trapezoid and was not a cyclic quadrilateral since it did not have opposite supplementary angles. Then she decided that only special cases of the quad3 were a trapezoid with 3 congruent adjacent sides and a square.

After the related sketches, we went back on to defining quad3 in terms of cyclic quadrilateral. She then decided that following definition would be sufficient:

"A quad3 is a cyclic quadrilateral with three congruent adjacent sides."

She also thought of whether rhombus, kite and any other quadrilaterals would be counter example for the definition, but could not find. Then, she was asked to include quad3 into the hierarchy diagram and she correctly did and indicated its relations with the other quadrilaterals. The final hierarchy with the additional quadrilaterals was the following (Figure 4.37).

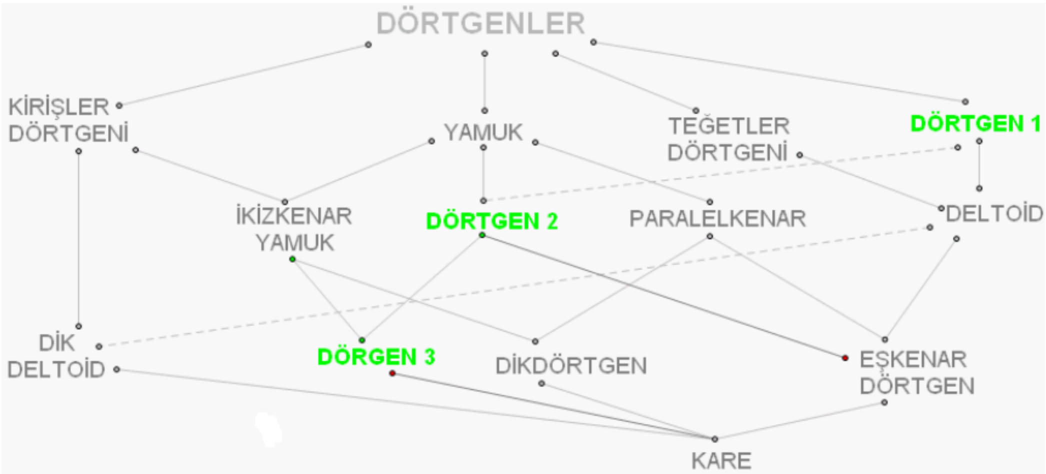


Figure 4.37 Participant 1's hierarchy diagram including "quad 1," "quad 2," and "quad 3"

4.1.4 Participant 1's Opinions about the Quadrilateral Learning Experience in the GSP Learning Environment

The participant 1 believed that learning with GSP was a great experience for her, since it added visual enrichness to the learning process. According to her, the most effective side of the GSP was constructing the figures based on the given definitions; because the construction process required to make inferences between the properties and also to think about the geometric relations used as building aids. Another effective side of the GSP was dragging the figure into other figures which revealed the special instances of quadrilaterals and inclusive relations between them.

When she was asked about the technical difficulties of using GSP in this study, she stated that not in other tasks but only in the circum quadrilateral task she had difficulty to drag the figure into its descendants. She thought that this difficulty was due to her not understanding the function of the figure. When asked what was new to her about the quadrilaterals, she explained that she was encouraged to think on many things that she had never thought on before. For example, she had not thought on inclusive relations between the quadrilaterals and had always evaluated them as single geometric objects; through this study she believed that she better understood the relations by thinking critically on the properties. Moreover, she had learned through this study to construct definitions of a quadrilateral in terms of other quadrilaterals, which she did not know before. She also stated that she had learned the nature of a definition that listing all of the properties was not a definition; but that constructing a definition required making inferences among the properties and identifying the defining properties. According to the participant, economic definition construction process improved the skills of critical analysis and reasoning to a large extent.

When the participant was asked to compare the traditional classroom teaching and the GSP assisted teaching, she told that in the traditional paper-pencil teaching a teacher could only give the properties and draw static figures on the board and ask some questions to encourage student to think. However, GSP assisted tasks provide a chance to move the figures dynamically in order to see the transformations between them and the changes in the measures to better discover the inclusive relations. That is, she thought that the dynamic nature of the GSP which provided seeing the effect of moving one object on the other related objects at the same time in one screen was the most effective side of the GSP assisted teaching.

She added also that she had never think so far that the GSP could be used so effectively in the definition construction process; and she would like to use it in her in-service teaching in a similar manner. She said that through this study she learned to construct mathematically correct economical definitions in many different ways that she did not think of before. Moreover, after this study she stated that she would encourage her students to think on the definitions and would encourage them to construct their own definitions instead of providing them with the pre-made definitions.

4.1.5 Participant 1's Understanding of the Nature of the Quadrilateral Definitions and the Hierarchies after the Clinical Interviews

She successfully constructed correct inclusive economical definition of trapezoid stating that *"a trapezoid is a quadrilateral with at least one pair of parallel sides."* Although her isosceles trapezoid and rectangle definitions included a little extra information they were very close to the most economical definition. She defined isosceles trapezoid as *"a trapezoid with two pairs of congruent adjacent angles."* If she had defined on the base of a quadrilateral this definition would be the most economical definition. Through defining on the base of a trapezoid, the defining property of trapezoid that "at least one pair of parallel sides" was automatically incorporated into the definition; so stating just "an isosceles trapezoid is a trapezoid with at least one pair congruent adjacent angles" would be enough to characterize an isosceles trapezoid. That is to say, "one pair of parallel sides" and "one pair of congruent adjacent angles" properties would automatically lead the other pair of adjacent angles to be congruent. The participant defined rectangle uneconomically as "*a quadrilateral with congruent and parallel opposite sides and with all congruent angles."* Actually, the most general quadrilateral with congruent angles was the rectangle, so it would be enough to say that "a rectangle is a quadrilateral with all congruent angles" Namely, there was no need to say that opposite sides are congruent and parallel. The first two properties together, by the way, most generally would define a parallelogram and so they would not be the sufficient defining properties to specialize a rectangle. However, we could say that "a rectangle is a quadrilateral with congruent opposite sides and with at least one perpendicular angle" which would be a definition including the necessary and sufficient defining properties to specialize a rectangle.

The participant was also good at identifying the examples of the given definitions among the given group of figures. She almost chose all correct figures, but she incorrectly chose the rectangle as an example of the definition that "a cyclic quadrilateral with at least one pair of congruent adjacent sides," though it did not have any congruent adjacent sides unless it was a square. Moreover, she also incorrectly did not choose the rhombus as an example satisfying the definition that "a trapezoid with at least 3 congruent sides," though it did. The participant might have misperceived the fact that although rhombus was not a special case of isosceles trapezoid, it was a special trapezoid.

When the participant was asked to construct two alternative economic definitions for a rhombus, she was able to construct one correct economical definition and one correct uneconomical definition. By means of using the diagonal property, she stated that "*a rhombus is a quadrilateral of which perpendicular diagonals are bisecting each other."* The properties she used were the sufficient diagonal properties to characterize a rhombus. "Perpendicular diagonals" property alone would not be enough; because kite is the most general special quadrilateral having this property and many ordinal quadrilaterals would also satisfy this property. Similarly, "bisecting diagonals" property alone would not be enough to define a rhombus; because when we ask someone to construct a quadrilateral with bisecting diagonals s/he can construct a parallelogram and rectangle even if we don't add that these diagonals are also perpendicular. So, the combination of these two properties makes the correct economical definition to characterize the rhombus as the most general quadrilateral satisfying this definition. If the definition is more specialized adding the property of the "diagonals' being congruent", someone could characterize the square as a special rhombus. She also constructed a correct but uneconomical definition that *"a rhombus is a quadrilateral of which opposite sides are parallel and all side lengths are equal."* All equal side lengths would be the sufficient property to characterize a rhombus; because when the all sides are congruent, opposite sides automatically become parallel for a quadrilateral. The most general quadrilateral in the hierarchy having all equal side lengths is the rhombus; and then the square as a special case of it.

Moreover, it was seen that the participant was very successfully identified the necessary and sufficient properties to characterize the quadrilaterals and made the correct explanations for her anaswers. For example, she stated that having congruent diagonals was necessary, but not a sufficient property to define a rectangle; because an isosceles trapezoid also had congruent diagonals. She also stated that the diagonals of a rectangle were bisecting each other in addition to being congruent, which specialized it as a rectangle. Moreover, she explained that having one pair of parallel sides and the other pair being congruent was not a sufficient condition to define a parallelogram since the isosceles trapezoid also had this property. She stated correctly that there was need to say two pairs of parallel sides to define a parallelogram.

She was also generally successful at constructing inclusive and exclusive definitions for the given group of quadrilaterals. For example, she defined the kite class by excluding the rhombus and square as *"quadrilaterals which are formed by combining the bases of the two different isosceles triangles"* which defined only the prototypical kite figure but not the rhombus and square. She also successfully defined all the figures involved in the kite class as "*quadrilaterals which are formed by combining the bases of the two isosceles triangles."*

However, she constructed a definition to include all kite class except for square as *"quadrilaterals which are formed by combining the bases of the two isosceles triangles and which has only one pair of opposite congruent angles."* However, this definition also excluded the rhombus from the kite class since it had two pairs of congruent opposite angles. To exclude only the square from the given group, she could state that "quadrilaterals which are formed by combining the bases of the two isosceles triangles but which do not have all congruent angles" or "quadrilaterals which are formed by combining the bases of the two isosceles triangles and which do not have congruent diagonals." She probably failed to notice rhombus among the given group of quadrilaterals.

On the other hand, the participant was good at understanding the hierarchical relationships between the quadrilaterals. For example, she was able to state that a square was always a kite and a cyclic quadrilateral. Moreover, she correctly stated that a kite was sometimes a cyclic quadrilateral but did not explained that it could be cyclic quadrilateral in the case that it was a right kite. Similarly, she said that a trapezoid was sometimes a cyclic quadrilateral but she did not state it could be a cyclic quadrilateral if and only if it was an isosceles trapezoid. I think she decided based on these explanations, but did not realize that she was asked to give examples in the question.

When it comes to classify special quadrilaterals with respect to their diagonal properties, the participant failed only to put parallelogram and isosceles trapezoid into the correct places in the diagram (see). By constructing the diagram like that she meant that a rhombus was an isosceles trapezoid whereas it was not due to not having congruent adjacent angles and congruent diagonals. None but interchanging the places of isosceles trapezoid and parallelogram would correct the diagram.

4.2 Participant 2's Analysis Results

In the following sections, findings related to the Participant 2's perceptions of the definitions and understanding of the quadrilateral definitions and the hierarchies before engaging them into the clinical interview sessions, her mental process and progress during the 4 clinical interview sessions, opinions about her experience in this study and the findings related to her understanding of the quadrilateral definitions and the hierarchies after the clinical interview sessions were stated.

4.2.1 Participant 2's Initial Perceptions of the Definitions

Having thought that definitions were not given enough attention, the participant 2 believed that they were the most important tools to identify a concept and to differentiate it from the other concepts. According to her, what a definition told us was very important to generate that concept.

The participant 2 thought that defining a concept did not mean it was meaningfully learned by the learner, because the learner might have just memorized it without understanding. According to her, the concept could be said to be meaningfully learned if a student could define the same concept in different situations and in different contexts.

As usual, participant 2 also stated that the definitions were not paid sufficient attention during her education life; they were just written on the board by the teacher at the beginning of the lesson, but never criticized. She was also doubtful about constructing correct definitions and had a low self-confidence at that moment of interview.

When asked how she would use definitions in her teaching, participant 2 stated that as done in her education life she would give the pre-constructed definitions at the beginning of the lesson, but would also encourage students to discuss on them. As traditional, instead of encouraging students to construct their own definitions the participant thought of providing them with the pre-constructed definitions. This was probably due to her not having experienced any other method of definition construction process throughout her education life. However, even encouraging learners to think on the definitions is a good point in the sense of understanding the meaning of the written words.

According to the participant 2, a good definition should have been precise that is, it should have given the exact meaning of the defined concept. Moreover, she stated that a good definition should have been clear and understandable so as not to enable misunderstandings and must include the appropriate mathematical language considering the target group level. However, the participant thought that the definition must have included all the properties of the defined concept, which was a common mislead idea among the participants.

4.2.2 Participants 2's Initial Understanding of the Nature of the Quadrilateral Definitions and the Hierarchies

Participant 2 defined rhombus as *"a quadrilateral with all side lengths are equal"* which was a correct economical definition including square as the special case. She also identified all squares among the given group of quadrilaterals as examples of rhombus in addition to the prototypical rhombus image. This indicated that she was aware of the inclusive relationship between the rhombus and square and did not have only the prototypical rhombus image in her mind. Moreover, she used the term "equal" correctly by stating the side lengths were equal but not the sides.

However, regarding the definition of rectangle, she came up with a correct inclusive but uneconomical definition by listing all known properties, which leaded to a description rather than a definition. She defined the rectangle as *"a quadrilateral with opposite side lengths are equal, opposite sides are parallel and all angles are 90°."* It was the same description as the Participant 1 and there was no need to use all these properties to characterize a rectangle. As explained in the case of Participant 1, "A quadrilateral with all angles 90°" would be the sufficient condition to characterize a rectangle and when a quadrilateral is drawn with this property, the most general figure would be the rectangle. Although the square also has this property, there is need to know an additional property of "all sides being congruent" to characterize it. It can be inferred again from this definition that the participant was aware of the fact that two line segments were congruent when their lengths were equal. Moreover, she correctly identified the squares as examples of rectangle among the given group of quadrilaterals which was an indication of her awareness of the inclusive relationship between rectangle and square.

That is, she knew that a square is a special rectangle due to having all the defining properties of the rectangle.

As in the case of the rectangle definition, the participant 2 also made a correct inclusive but uneconomical square definition, namely a description rather than a definition. She listed almost all properties as *"a quadrilateral with all side lengths are equal, opposite sides are parallel and all angles are right angles."* For example "a quadrilateral with all side lengths are equal and with at least one right angle" would be sufficient condition to generate a square. The first condition in this definition would generate a rhombus; so, to specialize it as a square the second condition needs to be added. It is possible to create many different economical definitions by using many different combinations of the necessary and sufficient properties.

The analysis also indicated that the participant 2 was not good at identifying the necessary and sufficient properties among the given list of properties to generate a correct economical definition for kite. The two alternative definitions of kite constructed by the participant were as the following:

> *"a quadrilateral with two pairs of congruent adjacent sides and with one pair of opposite congruent angles."*
>
> *"a quadrilateral with two pairs of congruent adjacent sides and with perpendicular diagonals."*

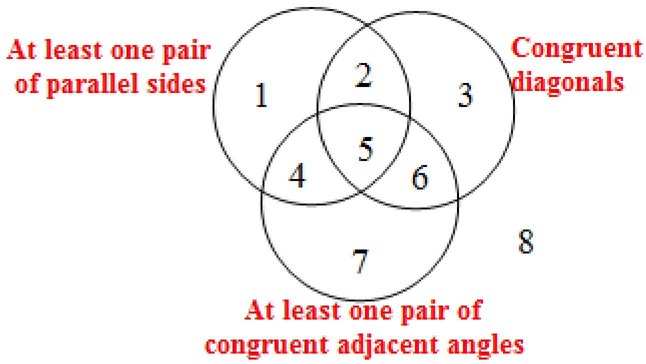
Both of her definitions were uneconomical and it seemed as if she randomly selected two properties and combined them to construct the definitions. In her alternative definitions the property of "two pairs of congruent adjacent sides" was common and it was actually the sufficient property alone to generate a kite; so there was no need to add the second properties in each definition. Because this property is enough to satisfy all other properties of the kite; if a quadrilateral has two pairs of congruent adjacent sides, then it automatically will have one pair of opposite congruent angles and perpendicular diagonals. But it seemed that the participant 2 could not make inferences between these properties and did not scrutinize whether one property automatically satisfied the other one.

On the other hand, the participant 2 was generally good at evaluating the correctness of the given rhombus definitions in terms of defining critical properties. However, she evaluated the definition that "a rhombus is a quadrilateral of which symmetry axes are the perpendicular lines passing through the opposite vertices" as incorrect whereas it was correct. According to her, not every quadrilateral with this property was a rhombus. However among the quadrilaterals with perpendicular diagonals, the rhombuses including the square as a special case are the ones where these perpendicular diagonals are also the symmetry lines. That is to say, this definition is a correct inclusive definition including the square as a special instance of rhombus. Besides, square has extra symmetry property of being symmetrical with respect to the lines passing through the midpoints of the opposite sides. Even there is no need to say that those diagonals are perpendicular in the definition; because if a quadrilateral is symmetrical with respect to both of its diagonals, then those diagonals must be automatically perpendicular and bisecting each other. That is to say, “ rhombus is a quadrilateral which is symmetrical with respect to the diagonals” would be the correct economical inclusive definition.

Moreover, she was able to construct all inclusive and exclusive definitions correctly which indicated that she can identify the defining properties that distinguish the group of figures from another group, but define the figures inside the same group. She constructed a definition which included figures a and b but not the others as *“quadrilaterals of which top and bottom sides are parallel.”* At first sight, this definition implied the same meaning as saying that “quadrilaterals with only one pair of parallel sides,” and so it was correct. On the other hand, this definition wouldn’t lead to a correct meaning when the position of the figures are changed so that the top and bottom sides would change. However, I evaluated this definition as correct believing that the participant 2 was aware of this nuance and meant the correct definition. In another definition, she successfully defined a, b, c and d as a group stating that “quadrilaterals with at least one pair of parallel sides and with at most two equal side lengths.” By saying “with at most two equal side lengths” she meant to say “not all four side lengths are equal.” Finally, she properly defined all the figures as a group in such a way that *“quadrilaterals with at least one pair of parallel sides.”*

Additionally, the participant was generally good at understanding the inclusive relations between the quadrilaterals through considering their properties. For example, she was able to state that a parallelogram had perpendicular and bisecting diagonals when it was a square and that a rhombus had congruent diagonals when it was a square. However, she stated that a rectangle could never have congruent adjacent sides which indicated that she disregarded square as a special rectangle.

On the other hand, it was seen that the participant 2 was very bad at classifying quadrilaterals with regard to their properties. In the first diagram she incorrectly placed parallelogram, kite and isosceles trapezoid, and correctly placed square, rhombus, trapezoid and rectangle (Figure 4.38). For example, she put the parallelogram into the region 5 with misleading information that a parallelogram has congruent diagonals and has at least one pair of congruent adjacent sides. Similarly, she thought that a kite has at least one pair of parallel sides, though it did not have any. She also did not know that an isosceles trapezoid has congruent diagonals.



Participant's Answers

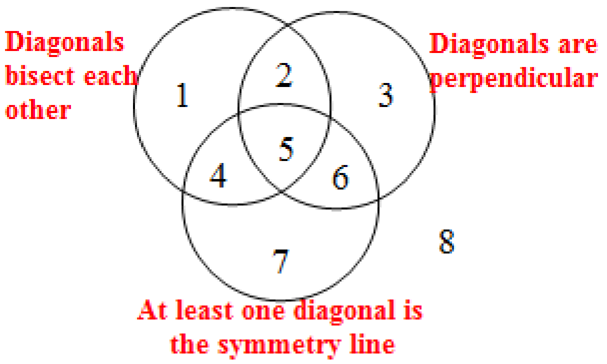
Parallelogram	**5**	Trapezoid	**1**
Kite	**1**	Isosceles Trapezoid	**4**
Square	**5**	Rectangle	**5**
Rhombus	**1**		

Correct Answers

Parallelogram	**1**	Trapezoid	**1**
Kite	**8**	Isosceles Trapezoid	**5**
Square	**5**	Rectangle	**5**
Rhombus	**1**		

Figure 4.38 Participant 2's first diagram of the classification of the quadrilaterals

In the second diagram she failed again and was able to place the parallelogram, square and rhombus into the correct regions, but she failed in the case of kite and rectangle due to her misinformation that a kite has bisecting diagonals and a rectangle has at least one diagonal as a symmetry line (Figure 4.39). Even worse, the participant could not place the trapezoid and isosceles trapezoid into any region.



Participant's Answers

Parallelogram	**1**	Trapezoid	**?**
Kite	**5**	Isosceles Trapezoid	**?**
Square	**5**	Rectangle	**4**
Rhombus	**5**		

Correct Answers

Parallelogram	**1**	Trapezoid	**8**
Kite	**6**	Isosceles Trapezoid	**8**
Square	**5**	Rectangle	**1**
Rhombus	**5**		

Figure 4.39 Participant 1's second diagram of the classification of the quadrilaterals

4.2.3 Clinical Interviews

Participant 2's cognitive progress during the clinical interview sessions and the effect of the GSP activities on the participant 2's cognitive improvement in understanding the quadrilaterals through definitions construction and classification processes were described in detail in the following subsections.

4.2.3.1 Session 1 with Participant 2: Kite, Rhombus and Square

When the participant 2 was asked which of the kite properties she remembered, she was just able to correctly say the diagonal properties; but she could not say anything related to the side, angle or symmetry properties at the beginning of the session. When she was asked to give a definition for kite, it was seen that the participant B also had a *"two isosceles triangles connected by their equal length bases"* concept image of the kite (Figure 4.40).

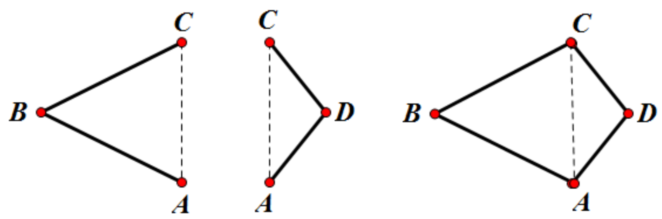


Figure 4.40 An image of two isosceles triangles connected by their equal length base

Next, the participant was asked to move the kite figure on the GSP screen and to determine the critical preserved properties of the kite under dragging. She was able to generalize the side property as "a kite has two pairs of congruent adjacent sides" after several trials on the figure. On the other hand, realizing the preserved angle properties of kite became more challenging for the participant. After many trials on the figure, she came up with that "a kite has one pair of opposite congruent angles." As oppose to the participant 1, the participant 2 used the terms congruent and equal appropriately which at first hand indicated that she was aware of the fact that two angles would be congruent when their angle measurements were equal. However, there was need to follow more statements of her for a good measure of the correct the usage of the terms “equal” and “congruent.”

In addition to the side and angle properties, the participant correctly determined the diagonal and symmetry properties of the kite by working on the dynamic figure. Then, she was asked to think of into which other quadrilaterals the dynamic kite figure could be dragged; namely whether there were any other quadrilaterals having the critical properties of the kite. She correctly thought that the kite figure could be dragged into a square, because all sides of the kite could be made congruent by dragging. When I asked her whether preserving only the side property was enough to decide a square was a kite, she started to think on the other properties. She stated that the diagonals of the square also perpendicular to each other and not only one diagonal, but additionally two of them were bisected. Moreover, she said that when the kite was dragged into a square not only one diagonal but two of them would be angle bisectors. When the participant thought of the rectangle, she stated that a kite could not be dragged into a rectangle, because a rectangle did not necessarily have perpendicular diagonals and congruent adjacent sides which were critical kite properties. She also correctly stated that a kite couldn’t be dragged into a parallelogram since its diagonals were not necessarily perpendicular. When it came to think of the rhombus, she incorrectly claimed that since “one pair of opposite congruent angles” property was not preserved in a rhombus, a kite could not be dragged into a rhombus. Moreover, she also incorrectly stated that a kite could be dragged into a trapezoid, but she could not explain the reason; she said she could imagine it mentally. However, she correctly stated that a kite could not be dragged into an isosceles trapezoid because the property of two congruent adjacent sides could not be preserved. That is to say, she correctly stated that a kite could be dragged into a square but not into a rectangle, parallelogram and an isosceles trapezoid. However, she incorrectly stated that a kite could be dragged into a trapezoid but not into a rhombus.

In the next step, she was asked to try dragging the dynamic kite figure into other quadrilaterals to determine which one of them could be the descendants of the kite. As she thought in the previous step, she was able to drag the figure into a square, but not into a rectangle and a parallelogram. However, when she was able to drag the kite into a rhombus she got surprised; but after thinking on the reason, she came up with the explanation that all the critical properties of a kite were preserved in the rhombus. She said that her incorrect idea stemmed from her misinformation about the diagonal property of the rhombus; because, she had thought that the diagonals of the rhombus were not perpendicular to each other. Similarly, when she tried to transform the kite into the trapezoid, she could not do it; upon this, she realized her mistake and explained the reason that a trapezoid did not have to preserve the "two pairs of congruent adjacent sides" and "perpendicular diagonals" properties. As she stated before, she also could not drag the figure into an isosceles trapezoid. Finally, by investigating on the dynamic figure, she detected out that the square and the rhombus had all critical properties of the kite, which made them special kites. Then, she correctly showed the hierarchical relations on the diagram explaining that the square was the most special quadrilateral and a special case of all other quadrilaterals.

Next, the participant was asked to construct a definition of kite that would also include its special cases. She uneconomically defined as *"a kite is a quadrilateral with at least two pairs of congruent adjacent sides and with perpendicular diagonals."* That is, her definition were including more than necessary information; more than the necessary and sufficient defining property. To make the participant realize the extra information in her definition, she was asked to think of whether these two properties together were necessary or one of them would be enough. She thought that having two pairs of congruent adjacent sides did not necessarily lead to perpendicular diagonals; or just having perpendicular diagonals did not necessarily lead to two pairs of congruent adjacent sides. Finally, she decided that the definition should include both properties to characterize a kite; but was not sure whether these two properties together were the correct defining properties. Upon her confusion, I asked her to make a construction just by using the each property separately as a defining property, and she started with the construction of the "two pairs of congruent adjacent sides" property (Figure 4.41).

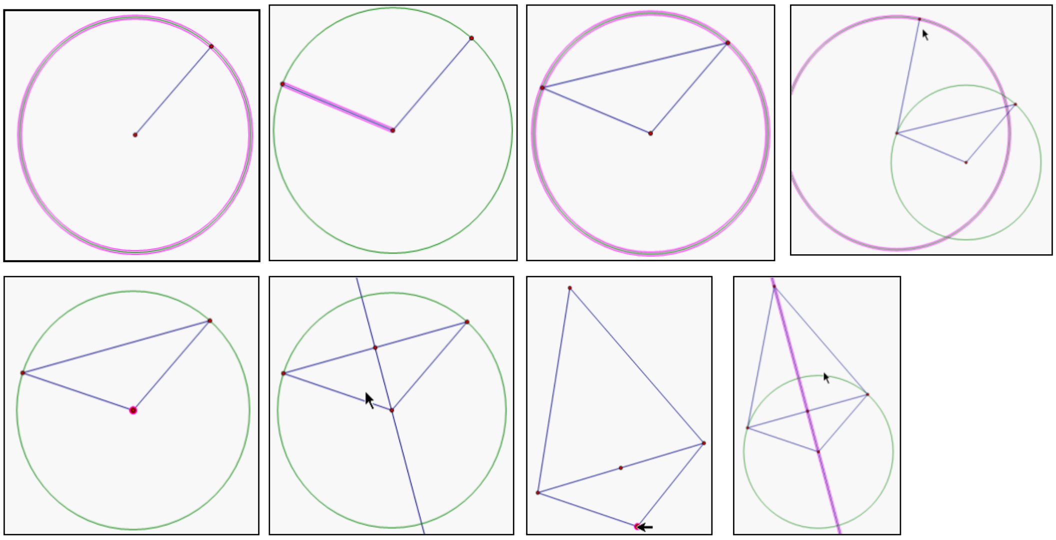


Figure 4.41 Participant 2's construction of "a quadrilateral with two pairs of congruent adjacent sides"

She easily constructed the first pair of congruent adjacent sides by constructing two radiuses on a circle. However, when it came to the construct the second pair of congruent adjacent sides, she actually got use of the perpendicular diagonals property although she should not used. Then, she dragged the figure to see whether it preserved kite properties and also made measurements on it to make sure. However, she still thought that at least two pairs of congruent adjacent sides did not ensure a kite, because there was a possibility that three adjacent sides could be congruent; but in this figure there was no possibility of three adjacent congruent sides. She made this decision without trying to drag the figure so that three adjacent sides would be congruent. When I asked her to do so, she saw that not only three adjacent sides but also all 4 of them could be congruent. After this construction and dragging test, she came up with the conclusion that this property alone was enough sufficient property to define a kite.

Next, I asked her to test by the appropriate construction whether the second property alone could be a defining property of kite. That is, she tested whether any quadrilateral with perpendicular diagonals would be kite. However, during the construction she drew the one diagonal perpendicular to the other diagonal through its midpoint, though the property that she was constructing did not include the information that one diagonal was bisected. Namely, she used extra information which was not given in the definition although she was asked to use only the given information. After my warning, she took any point on one segment and constructed the other perpendicular segment through this point and connected the end points with segments to complete the quadrilateral (Figure 4.42)

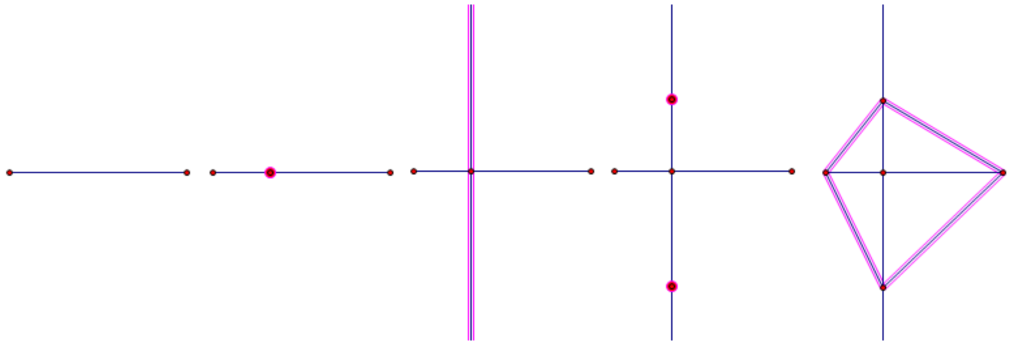


Figure 4.42 Participant 2's construction of a quadrilateral with perpendicular diagonals

Then, I asked her to think on whether "a kite is a quadrilateral with perpendicular diagonals" would be correct economical definition. After dragging the figure she easily realized that the figure did not remain a kite, but it was an ordinary quadrilateral. As a result, she decided that the second property in her definition was redundant information and the first property alone would be necessary and sufficient defining property for kite. She redefined kite inclusively as "a kite is a quadrilateral with two pairs of congruent adjacent sides."

After testing her own definition on sketchpad, I asked participant to evaluate the mathematical correctness of 4 pre-constructed definitions of kite and to consider if it included necessary and sufficient conditions or included redundant information.

Actually, she had already checked the first definition in her previous sketch, so she directly stated that having perpendicular diagonals was not sufficient property to be a kite.

Definition 1: A kite is a quadrilateral with perpendicular diagonals.

So, we moved on the second definition which additionally included the information that at least one diagonal was bisected. The participant thought that this definition included enough defining property of kite, but was not sure whether all other kite properties would be the result of this property and wanted to see through construction (Figure 4.43).

Definition 2: A kite is a quadrilateral with at least one diagonal is a perpendicular bisector.

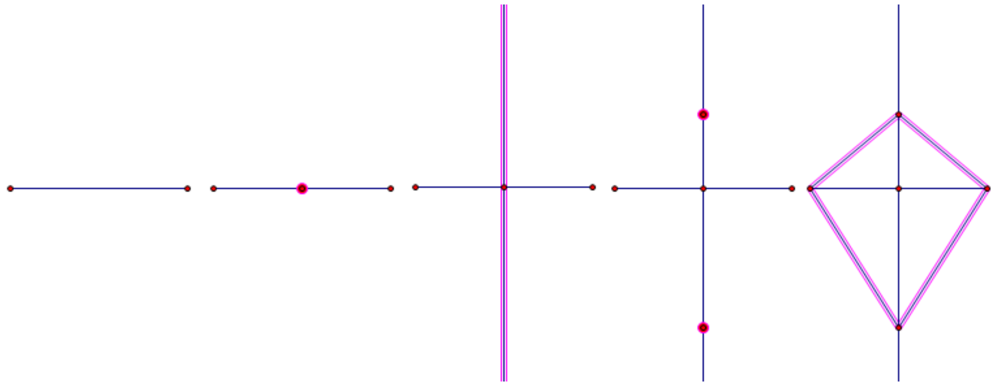


Figure 4.43 Participant 2's construction of a quadrilateral with at least one diagonal is a perpendicular bisector.

This time, she constructed the midpoint and drew the other perpendicular diagonal through this midpoint, and finally constructed segments to complete the figure into a quadrilateral. After observing the related measurements during dragging, she detected out that the figure preserved all kite properties, so the definition was correct economical definition.

The third given definition included two properties; namely, "two pairs of congruent adjacent sides" and "one pair of congruent opposite angles."

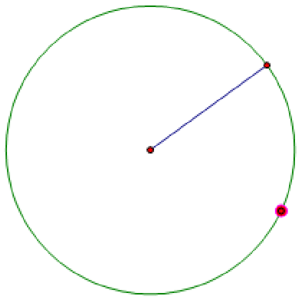
Definition 3: A kite is a quadrilateral with two pairs of congruent adjacent sides and one pair of congruent opposite angles.

Actually, the participant had already tested the first property while making the related construction for her own definition and had found that it was correct sufficient defining property which satisfied all other properties of kite. So, she stated that "a kite is a quadrilateral with two pairs of congruent adjacent sides" definition would be the economical definition that could be inferred from this uneconomical definition. Upon her answer, I asked her what about defining a kite as "A kite is a quadrilateral with one pair of congruent opposite angles." She was able to make the inference that constructing one pair of opposite congruent angles would not ensure that the two pairs of adjacent sides would be congruent. She stated that not necessarily a kite, but an ordinary quadrilateral could also have one pair of congruent opposite angles; so this property was not the sufficient defining property to define a kite.

For the last given definition, the participant correctly answered that when we drew a diagonal as a symmetry line we divided the kite into two congruent parts and all elements would be symmetrical according to this line.

> *Definition 4: A kite is a quadrilateral with at least one diagonal is the symmetry axis.*

However, when I asked her to confirm this with the appropriate sketch, she attempted to construct the symmetric sides as radius segments of a circle, although the definition did not included any information that there would be two pairs of congruent adjacent sides. So, she should not have used this information.



After my warning, she made the correct construction (Figure 4.44). However, before doing the related measurements on the constructed figure, she incorrectly thought that the figure would not remain as a kite. As soon as she dragged the figure and observed that kite properties were preserved, she stated that the definition was a correct economical one including just the necessary and sufficient defining properties of kite.

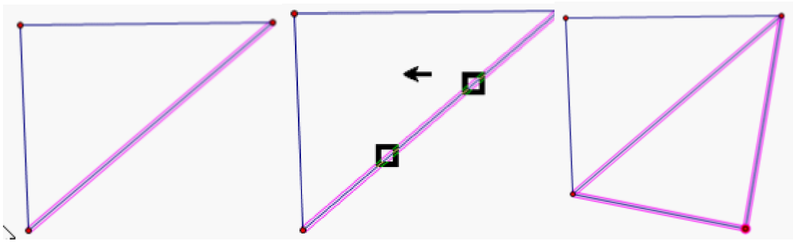


Figure 4.44 Participant 2's construction of a quadrilateral with at least one diagonal is the symmetry axis.

As a final step of this session, the participant was asked to construct an inclusive definition for rhombus so that this definition would include the special case square as well. She very easily constructed the definition that "a rhombus is a quadrilateral with all congruent sides," which was a correct economical rhombus definition based on the side property. Because the most general quadrilateral with all congruent sides would be the rhombus; and this definition would also include the square as a special case. Moreover, she constructed another definition based on the symmetry property as "a rhombus is a quadrilateral with both diagonals as the symmetry axes."

4.2.3.2 Session 2 with Participant 2: Parallelograms and Trapezoids

The participant was able to remember almost all isosceles trapezoid properties correctly. At first, she said that the side segments were congruent, but when I asked her to use more appropriate mathematical language she corrected this property saying that one pair of opposite sides were congruent. Then, she added that diagonals were congruent and the adjacent angles formed by the parallel bases and the common side segment were supplementary due to the parallelism. After learning her previous knowledge, I asked her to test the properties by working on the dynamic figure. In addition to what she had already known, she also investigated on the figure that the diagonal parts constructed by the intersection of diagonals were equal in length. She actually meant that diagonals bisect each other in the same ratio, but did not know how to express this with correct mathematical language. She also discovered that the isosceles trapezoid was not symmetrical with respect to the diagonals, but symmetrical with respect to the lines passing through the midpoints of the parallel sides.

In the next step, the participant was asked to think of the special instances of an isosceles trapezoid. She thought that square and rectangle were special isosceles trapezoids and she was able to explain the reasons over the preserved properties. When it came to think of the parallelogram, she reasoned as in the case of a rectangle and stated incorrectly that a parallelogram was also a special isosceles trapezoid, though it was not. She claimed that a parallelogram had parallel and congruent sides which satisfied the side properties of an isosceles trapezoid. She was only doubtful about the diagonal property, but when she worked on a parallelogram figure, she saw that the diagonals were bisecting each other and decided that parallelogram satisfied all definitional properties of isosceles trapezoid. However, she did not pay attention to the fact that a parallelogram did not have to have congruent diagonals, which eliminated its being an isosceles trapezoid. Moreover, she did not realized that only the opposite angles of a parallelogram were congruent not the adjacent angles. So, she incorrectly accepted that a parallelogram was a special isosceles trapezoid. At first, she also claimed that a rhombus was an isosceles trapezoid due to having two parallel and two congruent sides. Moreover, she stated that the diagonals of a rhombus also divided each other in equal ratio. Upon this, I asked her whether the diagonals of a rhombus were congruent as in the case of an isosceles trapezoid and she realized that a rhombus could not have been an isosceles trapezoid since its diagonals did not preserve the congruency. She eliminated trapezoid since it did not have to have any congruent sides and congruent diagonals. Similarly, she eliminated kite because of the non congruent diagonals property. As a result, she correctly detected out rectangle and square, but incorrectly detected out the parallelogram as special cases of an isosceles trapezoid.

During the dragging process, she saw that the dynamic isosceles trapezoid figure could not be dragged into a parallelogram and suddenly realized that it was because of the fact that the diagonals of a parallelogram had not have to be congruent. She also detected out that the symmetry and angle properties of an isosceles trapezoid were not met in a parallelogram. Therefore, she concluded that the only descendants of an isosceles trapezoid were rectangle and square. Then she correctly placed isosceles trapezoid on top of the hierarchy, then the rectangle and square at the bottom.

The next step was to define isosceles trapezoid inclusively so that definition would include its descendants as well. She defined as

"An isosceles trapezoid is a quadrilateral with at least one pair of parallel sides and with at least one pair of opposite congruent sides."

Then I offered her to search for the counter examples that satisfied the definition, but were not belonged to the isosceles trapezoid class. She at once found that parallelogram was a counter example for this definition which made it incorrect isosceles trapezoid definition. After thinking a while she redefined it as

"An isosceles trapezoid is a quadrilateral with at least one pair of parallel sides and with congruent diagonals."

Then she thought about whether there was any other quadrilateral with congruent diagonals and with one pair of parallel sides, but she could not find. After making some drawings on the draft paper, she concluded that if one pair of side was drawn parallel and diagonals were drawn congruent, these two properties would have automatically satisfied the other properties.

In the next step, I asked her to evaluate some pre-constructed isosceles trapezoid definitions. However, I skipped the first definition since we had already discussed it.

Definition 1: An isosceles trapezoid is a quadrilateral with at least one pair of parallel sides and with at least one pair of opposite congruent sides.

For the second definition she reasoned that the definition only included the isosceles trapezoid but not its descendants, namely rectangle and square; and accepted it as an incorrect definition.

Definition 2: An isosceles trapezoid is a quadrilateral with one pair of parallel sides and with one pair of congruent but unparallel sides.

She stated that the third definition was not a correct definition of an isosceles trapezoid. She claimed that a parallelogram also had opposite supplementary angles, but it was not an isosceles trapezoid. Upon her answer, I provided a parallelogram and asked her to measure its angles. As soon as she measured, she saw that the opposite angles were congruent but not supplementary in a prototypical parallelogram. Then, she thought about rhombus as a counter example, but eliminated it since its opposite angles were not supplementary but congruent. Not having found any counter example, she decided that the definition was a correct economical definition of isosceles trapezoid.

Definition 3: "An isosceles trapezoid is a quadrilateral with at least one pair of parallel sides and with opposite supplementary angles."

After thinking a while and making some drawings on the draft paper, the participant decided that the fourth definition included the necessary and sufficient information.

Definition 4: An isosceles trapezoid is a quadrilateral with two pairs of congruent adjacent angles.

She explained that the “two pairs of congruent adjacent angles” property was enough to prove all other properties; but she was not sure enough. Therefore, I asked her to test the sufficiency of the definition with appropriate construction. At first, she did not know how to construct the first congruent pair of angle; she just constructed two angles of which measures changed independently from one another (Figure 4.45). That is, she did not know how to carry the same angle to the other side of the base segment.

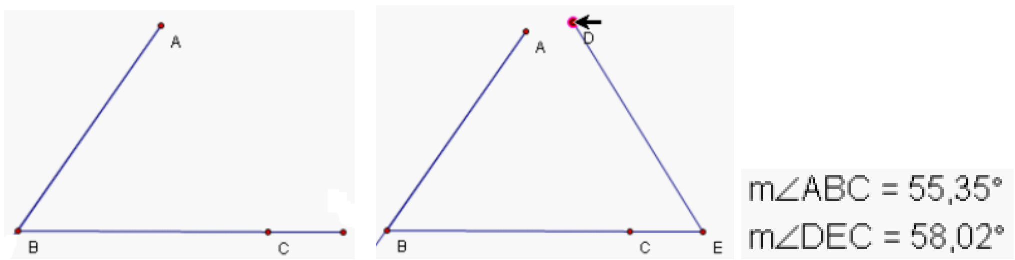


Figure 4.45 Participant 2’s first attempt to construct a quadrilateral with two pairs of congruent adjacent angles

I encouraged her to think of the rotation tool to carry the angle. This time the problem was to rotate which point around which point as much as the marked angle. In a few trials, she rotated in wrong direction (Figure 4.46). Then, she rotated with a fixed angle by fixing the angle to the measure of the first angle; but she did not think that when he changed the measure of the first angle the second angle would be fixed and so they would not remain congruent. However, she was able to construct the same angle finally.

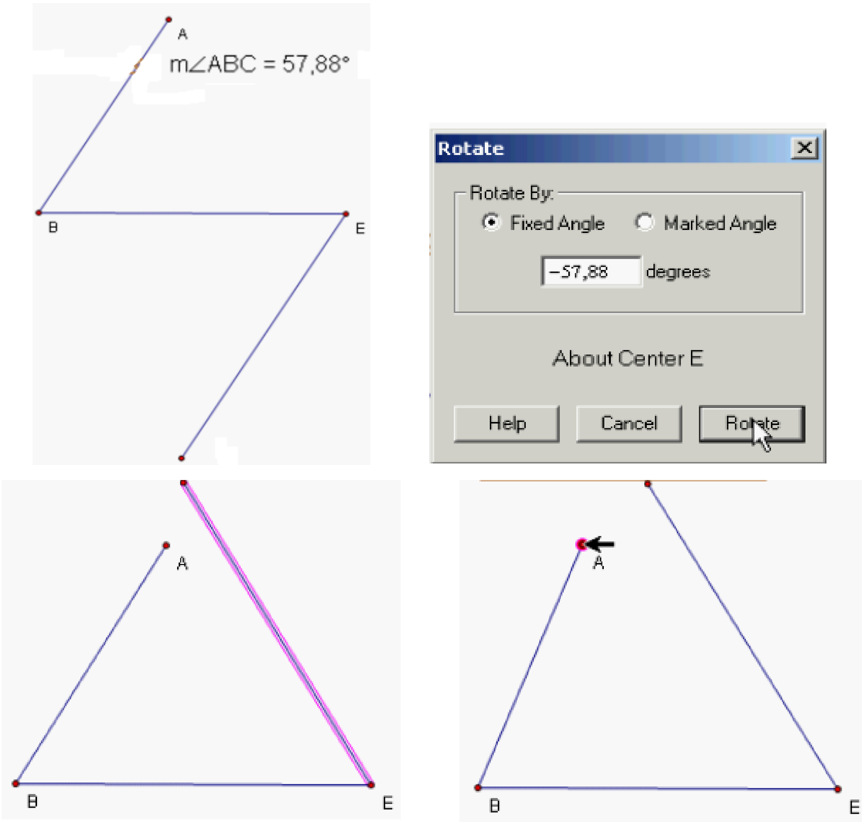


Figure 4.46 Participant 2's second attempt to construct a quadrilateral with two pairs of congruent adjacent angles.

Then, she understood that she needed to measure the negative of the angle and rotated with this angle. Finally to be able to construct the other pair of congruent angles, there was only one thing to do: to construct a parallel segment to the base segment (Figure 4.47). After dragging, she saw that the constructed figure protected isosceles trapezoid properties and so she concluded that the definition was the correct inclusive definition including the minimal defining property.

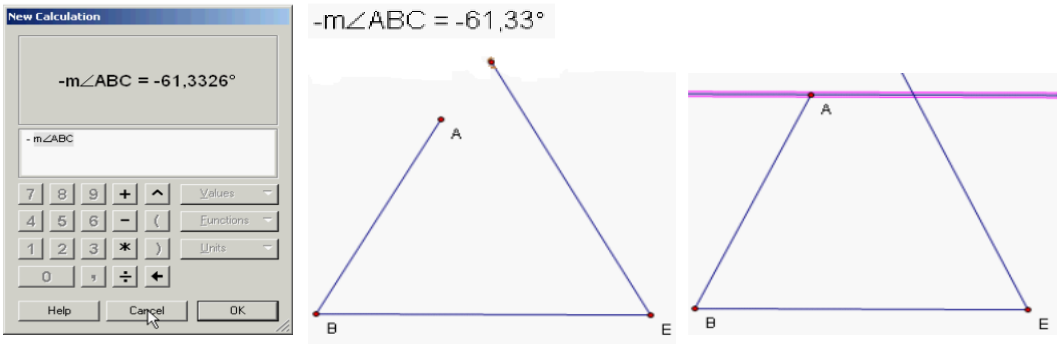


Figure 4.47 Participant 2's third attempt to construct a quadrilateral with two pairs of congruent adjacent angles.

She evaluated the last definition as an incorrect one, but she could not explain any reason for her answer. So, I asked her to construct the corresponding figure to see whether it was an isosceles trapezoid or not.

Definition 5: "An isosceles trapezoid is a quadrilateral with congruent diagonals."

She needed to construct two congruent intersecting segments, but did not know how to do. When I asked her where these two segments could be placed, she found out that they could be the radiuses or diameters of a circle. She first constructed the circle and a diameter on it. Then, there was need for a second segment intersecting the first one at any point but congruent to it. She reasoned that if the second diagonal was constructed on the same circle they would bisect each other, but this was not what she needed. So, she first constructed any point on the diameter as the intersection point of the diagonals and then constructed two circles centered at this point with diameters of the length between this point and the end points of the first diameter segment (Figure 4.48). Actually, in the definition there was not any information that the diagonals would intersect each other in the same ratio; but, during this construction process the participant realized that and added this property to her definition. Therefore, in this construction process, she had already concluded that the given definition was including the correct but not sufficient information. So, after the construction, she concluded that there was need to add the information that the diagonals had to intersect each other in the same ratio as well.

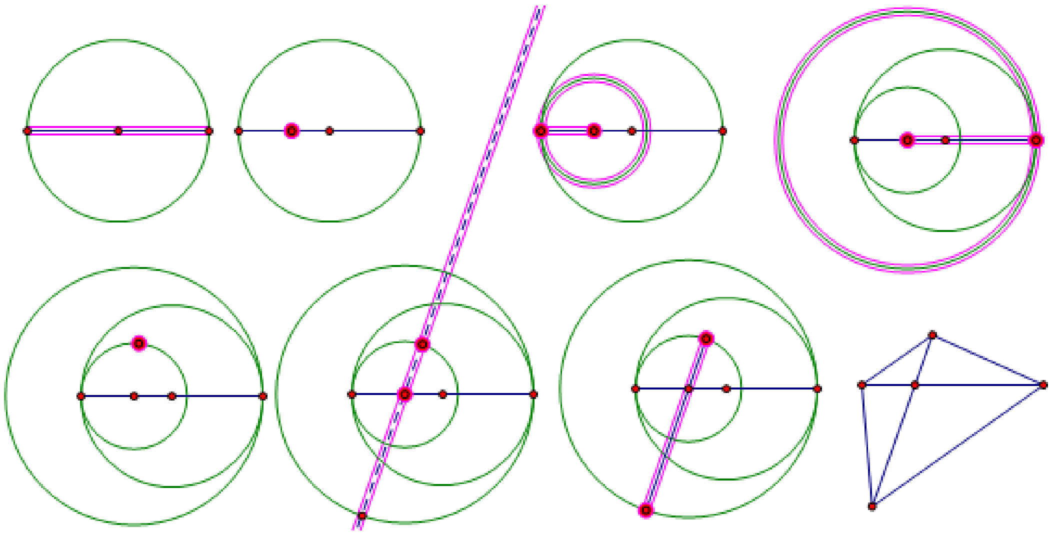


Figure 4.48 Participant 2's construction of a quadrilateral with congruent diagonals

She redefined the last definition as *"An isosceles trapezoid is a quadrilateral with congruent diagonals which intersect each other in the same ratio."*

In the next step, I asked the participant 2 identify special parallelograms and she had no difficulty to detect out them. She correctly stated that square, rectangle and rhombus were the special parallelograms and was able to explain the reasons considering the preserved parallelogram properties. When it came to show the relationship on the hierarchy diagram, she failed; because indicated the rhombus as a special rectangle, though it was not (Figure 4.49).

Figure 4.49 Participant 2's first hierarchy diagram of parallelograms

Upon her mistake, I asked her to explain which properties of a rectangle were preserved in a rhombus. She realized that to make a figure rectangle, it had to have right angles, but this property was not preserved in rhombus, so rhombus could not be a special rectangle. Moreover, she stated that a rectangle also could not be a special rhombus since it did not have congruent adjacent sides. Then she redesigned the hierarchy so that the rectangle and rhombus would be two different special classes of parallelogram (Figure 4.50).

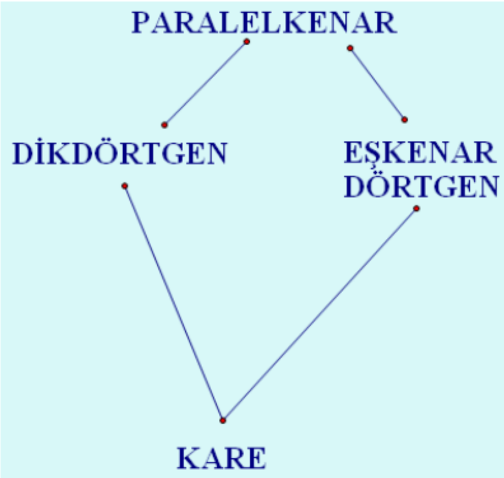


Figure 4.50 Participant 2's second hierarchy diagram of parallelograms

In the next step, we talked about the relationship between prototypical trapezoid and parallelograms; that is whether parallelograms could be the special trapezoids or vice versa. She correctly stated that a trapezoid could not be a special parallelogram since being a parallelogram required having two pairs of parallel sides. Then, she thought conversely and stated that only condition to be a trapezoid was having at least one pair of parallel sides and so parallelograms were the special trapezoids. When I asked her whether there were any other special trapezoids other than the parallelograms, she detected the isosceles trapezoid as a special trapezoid among the others. She also dragged the trapezoid figure on the other quadrilaterals to test her arguments.

In the next step, the participant was asked to define a trapezoid so that the definition would include parallelograms and isosceles trapezoid as special cases. She easily defined as *"A trapezoid is a quadrilateral with at least one pair of parallel sides."* She explained that while constructing this definition she considered all special cases and detected out one common property which was having at least one pair of parallel sides.

Then, it was time to show all relationships on a hierarchy diagram. It was a little difficult for her to place all the quadrilaterals but after a few trials she was able to construct the correct hierarchy (Figure 4.51).

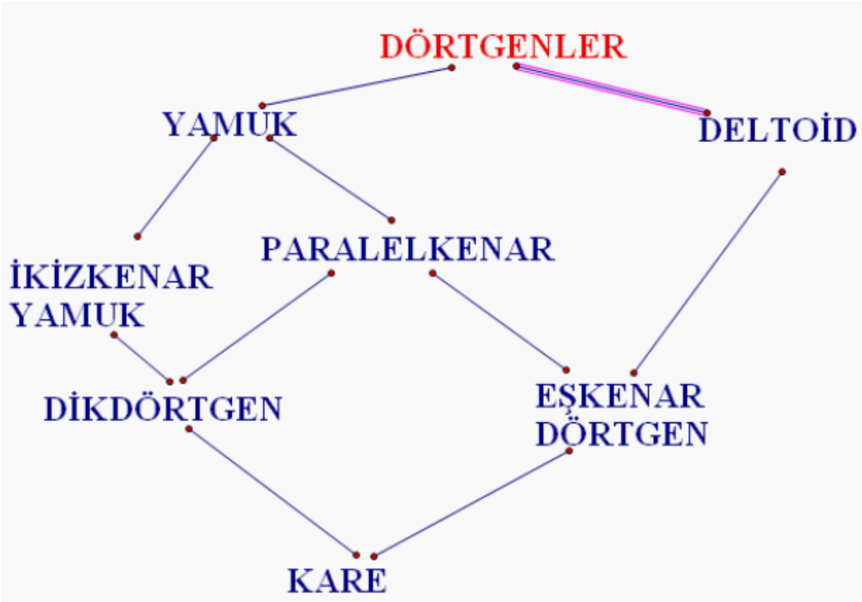


Figure 4.51 Participant 2's hierarchy diagram of quadrilaterals

Then I asked her how the hierarchy would change if the trapezoid was defined as "a quadrilateral with exactly one pair of parallel sides." She correctly reasoned that parallelograms would not be special tapezoids although isoscelels trapezoids would remain as special trapezoids

In the final step, the participant was asked to construct exclusive definitions with some property restrictions. She was first asked to define a *parallelogram* exclusively so that the definition would only include prototypical parallelogram, but not the rectangle, square and rhombus as special cases. However, the restriction was to use the *diagonal* property. She first remembered the diagonal property of the parallelogram saying that they were not congruent, but bisected each other. Then, I encouraged her to think how to eliminate special cases from the definition by using the diagonal property; upon this, she stated that the diagonals would not be congruent and perpendicular to each other. This reasoning of her had eliminated the rectangle, rhombus and square from the definition. Her definition was that *"a parallelogram is a quadrilateral with bisecting, but not congruent and perpendicular diagonals."*

Next, she was asked to define a *rhombus* exclusively using the *symetry* property. At first, she correctly stated that a rhombus was symmetrical with respect to both diagonals and correctly stated that a square had also the lines passing through the midpoints of the sides as symmetry axes. However, she then incorrectly reasoned that a rhombus also symmetrical with respect to the lines passing through the midpoints of the sides. Upon her misinformation, I provided her with a rhombus figure on the screen and asked to check for her argument (Figure 4.52). After her sketch, she understood that she was wrong in her argument and defined the rhombus exclusively as *"a quadrilateral which is symmetrical only with respect to the both diagonals."*

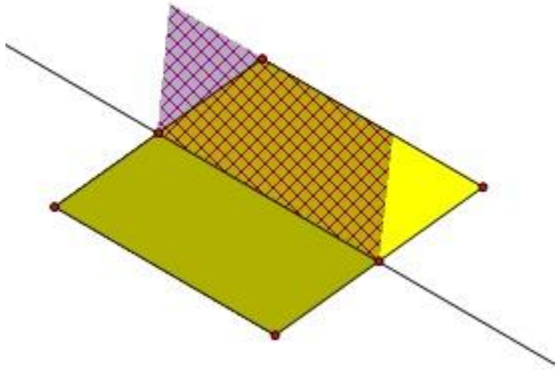


Figure 4.52 Participant 2's construction of the rhombus symmetry with respect to the lines passing through the midpoints of the sides

Finally, she was asked to define kite exclusively using any property she chose. She defined as *"a kite is a quadrilateral which has two pairs of congruent sides and which is symmetrical with respect to the only one diagonal."*Then she thought that "*a kite is a quadrilateral which is symmetrical with respect to the only one diagonal"* would be enough to define deltoid.

4.2.3.3 Session 3 with Participant 2: Cyclic and Circum Quadrilaterals

The participant called the figure she saw on the screen as irregular quadrilateral, but she could not remember that the quadrilateral placed on a circle was called cyclic quadrilateral. When I asked her what the sides of the quadrilateral are, she correctly stated that they were the chords of the circle and then she suddenly remembered that it was a cyclic quadrilateral. She initially defined a cyclic quadrilateral as

> *"A cyclic quadrilateral is a quadrilateral of which sides are the chords of a circle."*

Alternatively she constructed a second definition as

> "*A cyclic quadrilateral is a quadrilateral of which vertices are on a circle."*

When I asked her the condition of being a cyclic quadrilateral, she did not know. Then I asked her to think on which special quadrilaterals could be the cyclic quadrilaterals and why. She thought that a square and a rectangle were cyclic quadrilaterals due to having inscribed angles measured 90º. She explained that the vertices of the square and rectangle could be placed on a circle so that the diagonals would be the diameter of the circle. When asked, she stated that she was doing such judgments based on both thinking visually and considering the properties. Then, she continued saying that parallelogram was not a cyclic quadrilateral; because when two pairs of sides of the dynamic cyclic quadrilateral were dragged to be parallel, the figure would change into a rectangle or a square. As for the rhombus, our conversation continued as the following:

Researcher: What do you think about rhombus?

Participant 2: I am thinking its properties… I am thinking its angles… I guess it could be a cyclic quadrilateral.

Researcher: What about its angle property?

Participant 2: Opposite angles are congruent…

Researcher: Do you think you could make the opposite angles of the dynamic cyclic quadrilateral figure congruent?

Participant 2: Yes, I think I can… hmmmm… but it would be a square not a prototypical rhombus.

Researcher: *Why it would be a square but not a prototypical rhombus?*

Participant 2: The arc seen by the angle D… for instance the arc CBA is the one part of the circle… On the other hand, the arc CDA is seen by the angle B…When the measures of angle B and D are equal to each other, they both have to be 90º.

Researcher: What must be the sum of these two angle measures?

Participant 2: …in a circle…hmmm… the total arc measures seen by the angles B and D build up a circle… I mean the measures of the B and D are the half of the measures of the arcs they see…

Researcher: Then what is the sum of angle B and D?

Participant 2: Yes… It must be 180º… In a cyclic quadrilateral the opposite angle measures have to be supplementary and since a rhombus has congruent opposite angles, they can only be 90 º

Researcher: Then, what is the condition for a quadrilateral to be cyclic?

Participant 2: The opposite angles have to be supplementary angles.

That is to say, during this conversation she detected out the condition of being a cyclic quadrilateral. Next, she thought about kite and stated that a kite had at least one pair of opposite congruent angle and that for a kite to be a cyclic quadrilateral these congruent angles must be 90º each. That is to say, she correctly reasoned that only the special kites of which at least one pair of opposite congruent angles are 90 º could be cyclic quadrilaterals.

As for the trapezoid, she thought that it was easy to construct the one pair of parallel sides on a circle, so a prototypical trapezoid could be the cyclic quadrilateral. However, she did not consider the condition of being a cyclic quadrilateral; she just made a visual judgment. Then, she correctly made a judgment for the isosceles trapezoid stating that in an isosceles trapezoid opposite angles were supplementary so it was a cyclic quadrilateral.

As a result of this thinking process she correctly detected out all cyclic quadrilaterals except for the prototypical trapezoid. Next, she tested her arguments by dragging the dynamic figure into the other quadrilaterals one by one and confirmed that square, rectangle, kite with right congruent angles and isosceles trapezoid were the special cyclic quadrilaterals. Moreover, she realized that the figure could not be dragged into a prototypical trapezoid since its opposite angles did not need to sum up to 180º.

Next, she was asked to define the special kite, which was found to be cyclic quadrilateral and was called as *"right kite,"* in terms of *"quadrilateral," "cyclic quadrilateral"* and *"kite."* She defined it in terms of a quadrilateral as *"A right kite is a quadrilateral of which vertices are on a circle;"* however, as soon as having written the definition, she realized that it was a general definition of a cyclic quadrilateral and redefined right kite as

> *"A right kite is a quadrilateral of which vertices are on a circle and two pairs of adjacent sides are congruent."*

Actually, this was a correct definition but was not as clear as wanted; because it required knowing that if the vertices lied on a circle then opposite angles had to be supplementary. Therefore I encouraged the participant to clarify this point in the definition and she finally redefined it as

> *"A right kite is a quadrilateral of which two pairs of adjacent sides are congruent and opposite congruent angles are 90º each."*

Next, she defined a right kite in terms of a kite correctly in her initial attempt as

"A right kite is a kite of which opposite congruent angles are 90º each."

Finally, she defined in terms of a cyclic quadrilateral thinking that being a cyclic quadrilateral would have already made the opposite angles supplementary. So she thought there was need to add only the property which made it a kite and defined it correctly as

"A right kite is a cyclic quadrilateral which has two pairs of congruent adjacent sides.

After the definitions, I asked her to think of the special instances of a right kite which met the properties in this definition, and the participant easily detected that the square was the special right kite.

In the next step she was asked to define other cyclic quadrilaterals, namely isosceles trapezoid, rectangle and square, in terms of cyclic quadrilateral. Her definitions were the following

"An isosceles trapezoid is a cyclic quadrilateral with at least one pair of parallel sides."

"A rectangle is a cyclic quadrilateral with two pairs of parallel sides."

"A square is a cyclic quadrilateral with all congruent sides and with opposite parallel sides."

She correctly defined all three quadrilaterals in terms of cyclic quadrilateral, but only the square definition included more than the necessary information. However, when I asked her to think on whether all the properties were necessary, she said that *"A square is a cyclic quadrilateral with all congruent sides"* would be enough to define a square as a cyclic quadrilateral.

In the next step, she placed "right kite" and "cyclic quadrilaterals" category into the hierarchy diagram. It was seen that she was able to put them into the correct places easily (Figure 4.53).

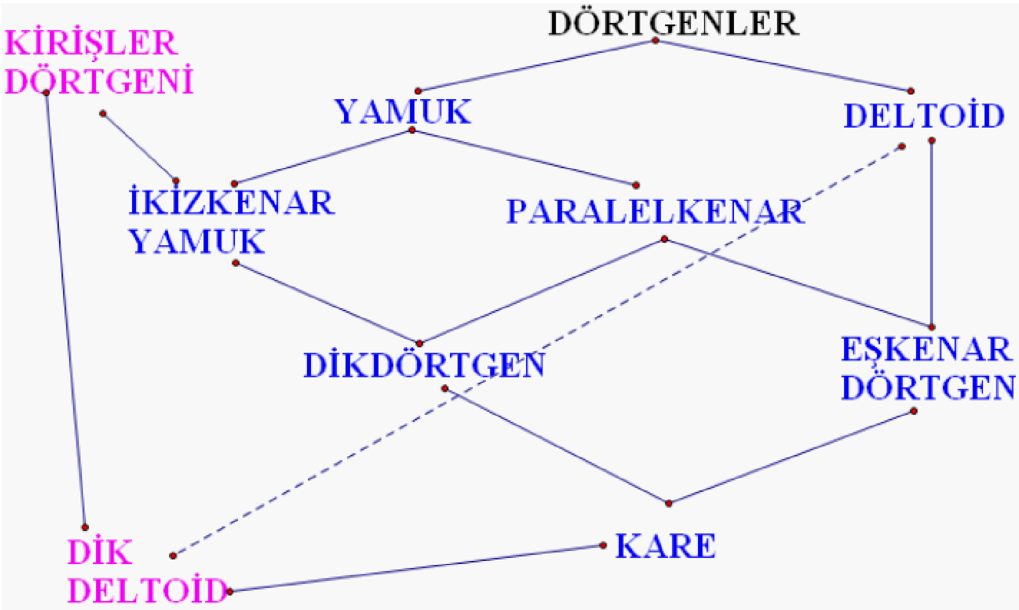


Figure 4.53 Participant 2's hierarchy diagram of quadrilaterals including cyclic quadrilaterals and right kite

In the second part of this session, same process was followed with the circum quadrilateral. The participant was asked to think of the quadrilaterals which were always circum quadrilaterals. She thought that a square was a circum quadrilateral since a circle could be drawn inside it so that the sides would be tangential to the circle; the side length of the square was the diameter of the circle. Moreover, she thought that a circle could not be placed inside a rectangle since the rectangle had two different pairs of opposite congruent sides; and so it was not possible to draw a circle with a fixed diameter. She also stated that a rhombus could be the circum quadrilateral but she could not explain why, since she did not know the mathematical condition to be a circum quadrilateral. She incorrectly thought that a parallelogram could be the circum quadrilateral. She stated that if the perpendicular segments from the center of the circle to the tangential points were drawn, they would be the radiuses of the circle; that is, she incorrectly concluded that the distance between the opposite sides would be the same. As for the prototypical kite, she again failed in her judgment saying that it would not be a circum quadrilateral, but she also added that she really have difficulty to make judgments about the circum quadrilaterals. Then I asked her to think whether the circle drawn in a kite would be tangential to all its sides. Thinking just visually, she stated that it would be possible; but she was not sure. She thought that a right kite was also a circum quadrilateral, but she could not explain any reason for this decision. She also thought that a prototypical trapezoid was also a circum quadrilateral. She explained that the length of the perpendicular segment between the parallel sides would be the diameter of the circle and the other two sides would be placed so that they would be tangential to the circle. On the contrary, she decided that an isosceles trapezoid was not a circum quadrilateral. She explained that the segments perpendicular to the tangential points would not intersect at the center of the circle. As a result of her judgments she concluded that square, rhombus, prototypical trapezoid, parallelogram, kite and right kite were always the circum quadrilaterals; but she could not make satisfactory explanations to confirm her judgments.

Next, it was time to test on the dynamic circum quadrilateral figure. She was able to easily make a square and when she extended the moving side of the square without changing the angles, she observed that the figure became a rectangle, but the sides were not tangential to the circle. She also easily constructed the rhombus by dragging the figure so that the congruent angle pairs were 120º and 60º. She also dragged the moving side without changing the angles and observed that the figure became a parallelogram, but it did not remain a circum quadrilateral. Then, she constructed a kite and right kite successively (Figure 4.54) and confirmed her judgment that they were the circum quadrilaterals.

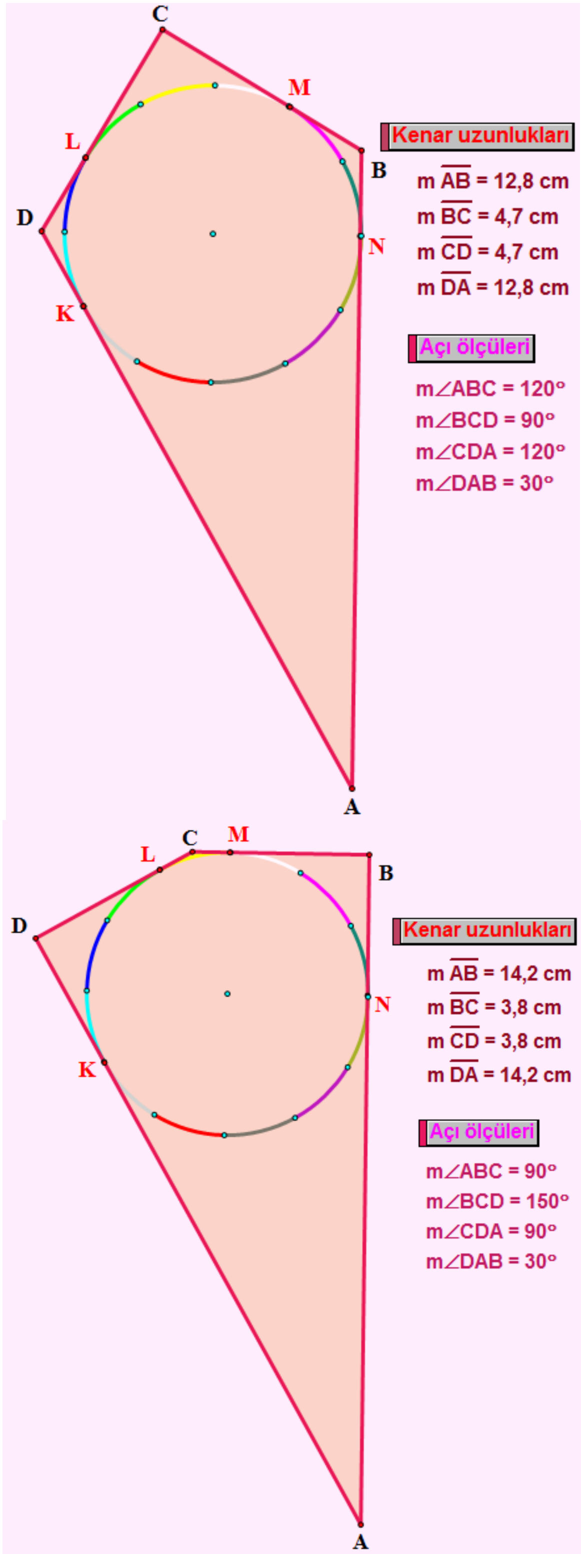


Figure 4.54 Participant 2's dragging the dynamic circum quadrilateral figure into kite and right kite

She had had an incorrect judgment about the trapezoid and isosceles trapezoid. When she was able to construct a trapezoid, she stated that she confirmed her judgment that trapezoid was a circum quadrilateral. However, when I asked her to drag the moving side upward or downward, she observed that the figure remained as a trapezoid but did not preserve being a circum quadrilateral property. So, she concluded that a trapezoid was not always a circum quadrilateral. Moreover, she had concluded that an isosceles trapezoid was not a circum quadrilateral; but she was able to construct an isosceles trapezoid. However, she understood that as in the case of trapezoid it did not remain a circum quadrilateral when the base segment was moved up and down. So she concluded that the circum quadrilaterals were square, rhombus, kite and right kite; that is, the whole kite class was the circum quadrilaterals.

In the next screen, the participant was given a quadrilateral of which intersection of the angle bisectors were drawn. As a result of her investigation, she found that the intersection was a point while the quadrilateral was kite, right kite, rhombus and square. Then she realized that all of them were the circum quadrilaterals and passed on a judgment that in all of the circum quadrilaterals the intersection of the angle bisectors was a point. When I asked her to combine these two situations, she easily generalized that if the angle bisectors of a quadrilateral intersect at a point, this quadrilateral would be a circum quadrilateral and the intersection point would be the center of the circle. Next, she was given a quadrilateral of which intersection of the perpendicular bisectors of the sides were drawn. After working on the figure, she detected out that the intersection was a point while the quadrilateral was isosceles trapezoid, rectangle, square and right kite all of which were the cyclic quadrilateral class. So, she was able to make the connection that if the intersection of the perpendicular bisectors of the sides of a quadrilateral was a point, then this point was the center of the circle and this quadrilateral would be the cyclic quadrilateral. Then she detected that for square and right kite both angle bisectors and the perpendicular bisectors of the sides intersected at the center of the inner circle and outer circle correspondingly, which would make them both cyclic and circum quadrilaterals. Finally, the participant correctly placed circum quadrilaterals category into the hierarchy (Figure 4.55).

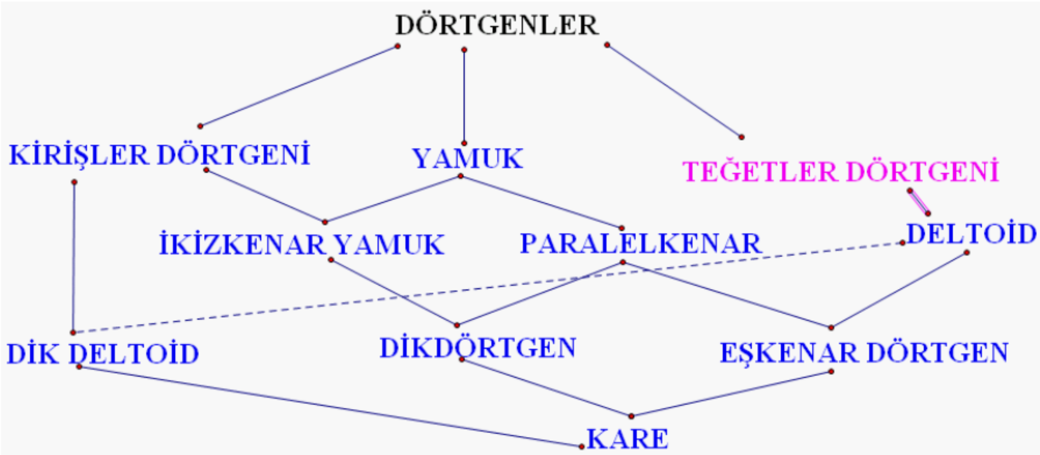


Figure 4.55 Participant 2's hierarchy diagram of quadrilaterals including circum quadrilaterals

4.2.3.4 Session 4 with Participant 2: New Quadrilaterals in the Hierarchy

The participant was able to explain the inclusive relations in the hierarchy diagram. However, she stated that the inclusive relationship between the quadrilaterals and between their properties was the same. When she was asked to think of the case of rectangle and square, she stated that a square is a special rectangle, so rectangle class included square and the properties of rectangle also included the properties of square. Upon her answer, I asked her to think of the properties of each and decide of which properties would be in the outer set and of which properties would be in the inner set. She first stated that the set of square properties would be the subset of the set of rectangle properties. Then I asked her to remember the properties of square and rectangle and as soon as she remembered the properties, she stated that square had many more extra properties which made it special case of a rectangle. So she understood that the inclusive relation between the properties of quadrilaterals was the inverse of the inclusive relation between their properties. She realized that the more the quadrilaterals were specified from top to the bottom of the hierarchy, the more properties they had.

Next, the participant was asked to define a more general quadrilateral called "quad 1" which would include all kite class. The participant first thought aloud the definition of kite and then she stated that a kite must have had at least two pairs of congruent adjacent sides. Then she correctly decided that this definition could be generalized for quad1 as

> *"A quad1 is a quadrilateral with at least one pair of congruent adjacent sides."*

After constructing the definition, the participant was asked to draw the special instances of the definition other than the kite class. Her drawings were the following (Figure 4.56).

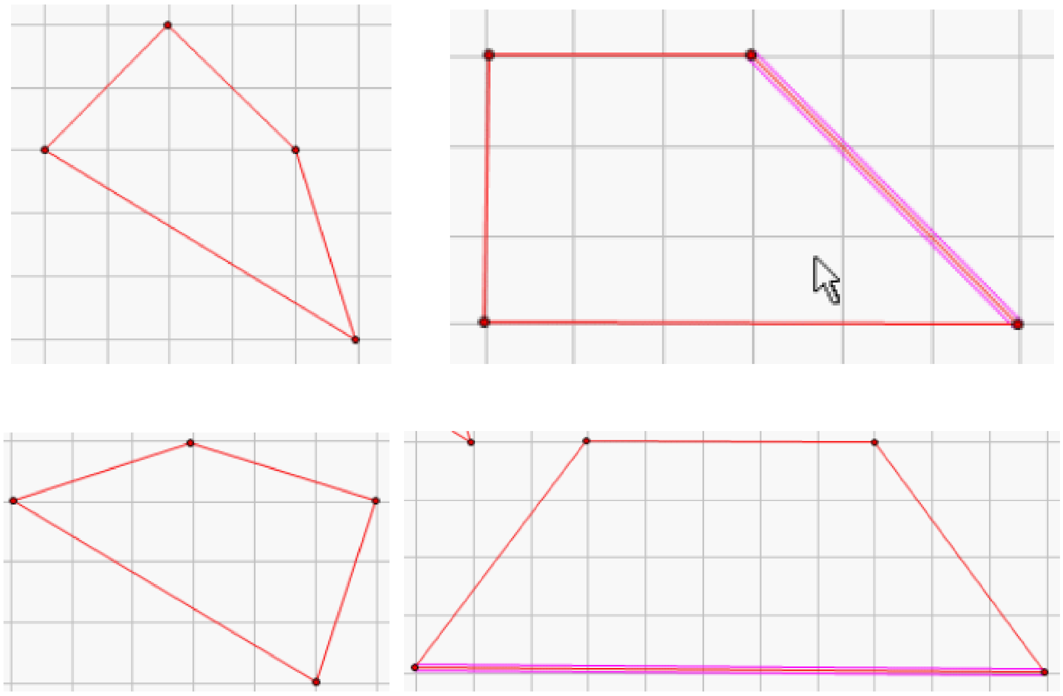


Figure 4.56 Participant 2's drawings of "quad 1"

In the next step, the participant was asked to define a "quad2" which was the specific case of both trapezoid and quad1. Her definition on the base of quadrilateral was

> *"A quad2 is a quadrilateral with at least one pair of congruent adjacent sides and with at least one pair of parallel sides."*

She explained that she added the defining properties of the both quadrilaterals since quad2 had to have both trapezoid and quad1properties. Moreover, the reason of saying "at least" was to include the special cases. Next, she correctly defined in terms of a trapezoid and quad1 as

"A quad2 is a trapezoid with at least one pair of congruent adjacent sides."

"A quad2 is a quad1 with at least one pair of parallel sides."

After the defining process, the participant put the quad2 category into the diagram and correctly indicated its special cases as square and rhombus. Then, she was asked to sketch the corresponding quadrilaterals other than the rhombus and square (Figure 4.57).

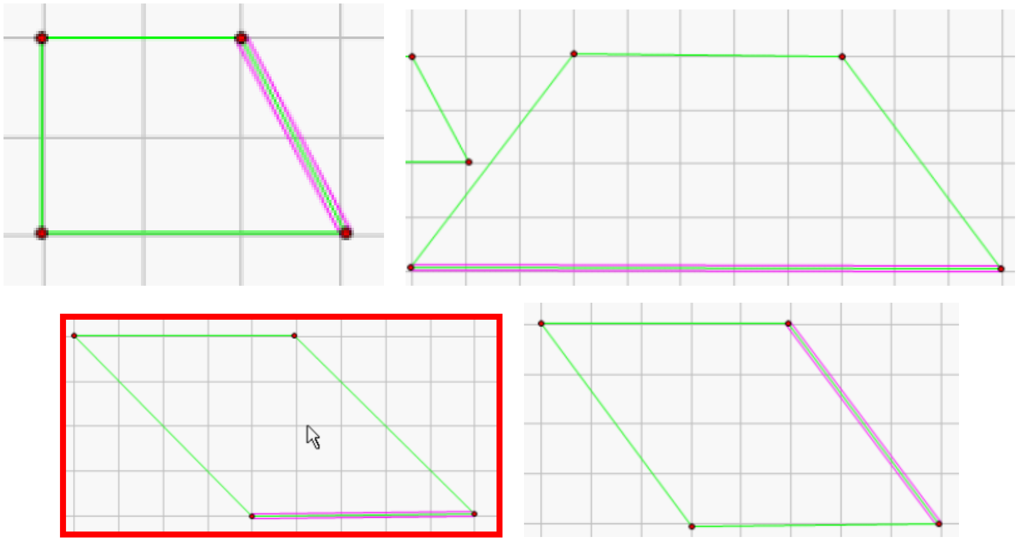


Figure 4.57 Participant 2's drawings of "quad 2"

However, she incorrectly sketched a parallelogram as an example of quad2 definition although it did not have one pair of congruent adjacent sides. She stated that she actually wanted to draw a rhombus not a parallelogram; so she drew a rhombus afterwards. Then, she incorrectly stated that she could also draw a kite, too; however, she immediately realized that it did not satisfy the definition due to not having any parallel sides.

In the next step, the participant was asked to define special case of both isosceles trapezoid and quad2 which was called "quad3." Her definitions in terms of isosceles trapezoid and quad2 were the following

"A quad3 is an isosceles trapezoid with at least one pair of congruent adjacent sides."

"A quad3 is a quad2 with two pairs of congruent adjacent angles."

She correctly constructed these two definitions thinking that while defining in terms of an isosceles trapezoid, only the defining properties of quad2 had to be added; and while defining in terms of quad2, only the defining properties of isosceles trapezoid had to be added. However, while defining quad3 as a cyclic quadrilateral she failed; because when I asked her to search for the counter examples, she detected that a right kite also satisfied this definition eventhough it did not satisfy the previous two definitions of quad3 she constructed.

> *"A quad3 is a cyclic quadrilateral with at least one pair of congruent adjacent sides."*

Then, she correctly redefined the definition as

> *"A quad3 is a cyclic quadrilateral with at least one pair of congruent adjacent sides and with at least one pair of parallel sides."*

However, according to the participant, rhombus and square were the examples of this definition; but she did not realize that a rhombus could not be a quad3 since it was not a special isosceles trapezoid and was not a cyclic quadrilateral as well. When I asked whether a rhombus could be a cyclic quadrilateral, she stated that it could not be a cyclic quadrilateral due to not having opposite supplementary angles. That is to say, the participant accepted this correct definition as incorrect since she detected rhombus as a counter example though it was not. However, she constructed another correct definition right after as

> *"A quad3 is a cyclic quadrilateral with at least three congruent adjacent sides.*

Next, she drew all possible quad3s which were a square and an isosceles trapezoid with 3 congruent adjacent sides (Figure 4.58).

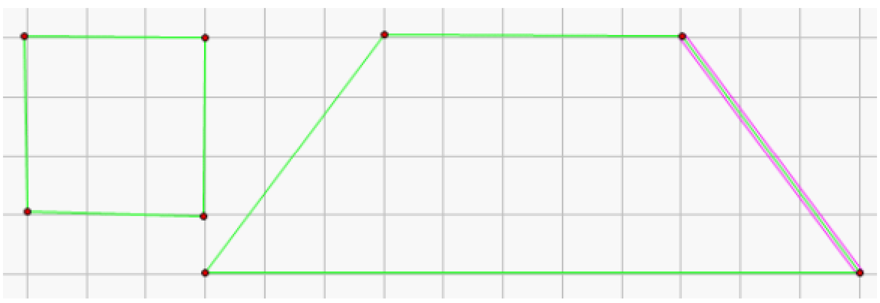


Figure 4.58 Participant 2's drawings of "quad 3"

Finally, the participant added quad3 into the hierarchy and indicated its special case as a square (4.59).

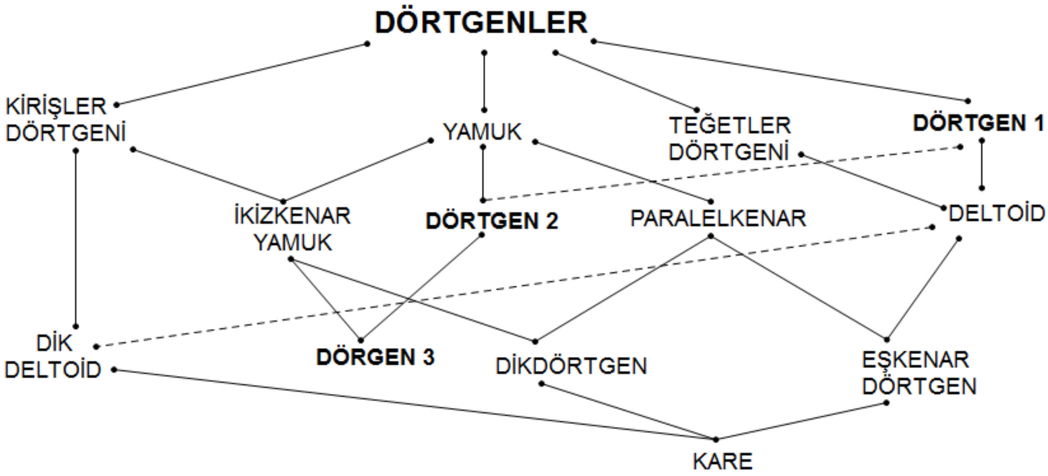


Figure 4.59 Participant 2's hierarchy diagram including "quad1," "quad2," and "quad3"

4.2.4 Participant 2's Opinions about the Quadrilateral Learning Experience in the GSP Learning Environment

According to the participant 2, experiencing tasks in the dynamic geometry learning environment was a great experience since she had not had such an experience before. She stated that GSP was most effective in investigating the special instances of quadrilaterals and in investigating the inclusive relations; because it allowed dragging the figures into the specific cases, which would not be achieved on paper. She said that the function of the GSP to preserve the defining properties of the figures under dragging enabled to discover the critical defining properties of the figure which were also carried by all its special cases.

The participant thought that due to not having a good command of the GSP menu, she had a little bit difficulty while constructing the figures based on the definition. Moreover, she found the circum quadrilateral figure difficult to drag into other figures since it required very sensitive arrangement of the measures.

When she was asked what was new to her, the participant stated that she had never thought about which quadrilaterals could be cyclic and which ones could be circum. She also added that before the study, a definition meant to her just listing the properties of the defined concept; however, throughout the study she learned how to define just using the defining properties. When she compared the traditional teaching with the GSP assisted teaching, she stated that GSP made it easier to understand the hierarchical relations since the GSP figures preserved critical defining properties of the figure when the figure was dynamically dragged into other figures. She believed that the dynamic nature of the GSP provided multiple representations which enhanced the limit of learning when compared to the paper-pencil learning.

She said that she had never imagined that GSP assisted tasks would be useful tools to teach definitions of the geometric concepts. So, after this study, she was more enthusiastic to use GSP in her in-service teaching. Finally, she added that the GSP assisted tasks that she was involved during this study were very effective to improve her skills to construct definitions and to understand the many different sides of the definition construction process.

4.2.5 Participant 2's Understanding of the Nature of the Quadrilateral Definitions and the Hierarchies after the Clinical Interviews

Although Participant 2 did well during the clinical interviews, she was not so good at constructing correct economical definitions. She only correctly and economically defined the trapezoid as *"a quadrilateral with at least one pair of parallel sides;"* however, she incorrectly defined isosceles trapezoid and rectangle. She incorrectly stated that *"an isosceles trapezoid is a trapezoid with at least one pair of opposite congruent sides,"* but this definition also included rhombus. Rhombus is a trapezoid with at least one pair of opposite congruent sides, but it is not an isosceles trapezoid due to not having congruent diagonals and two pairs of congruent adjacent angles. She also incorrectly defined rectangle as *"a trapezoid with congruent and parallel opposite sides."* First of all defining on the base of a trapezoid directly added to the definition that at least one pair of sides would be parallel as a defining property of the trapezoid; so there was no need to say it again. Moreover, this definition also included parallelogram and rhombus which are also trapezoids with congruent and parallel opposite sides. If we had defined rectangle as "a trapezoid with at least one pair of opposite congruent sides" or as "a trapezoid with two pairs of opposite congruent sides," parallelogram and rhombus would also satisfy this definition. Similarly, if we had defined rectangle as "a trapezoid with at least one pair of 90° *adjacent* angles" then a right trapezoid would also satisfy this definition. So we might define a rectangle on the base of a trapezoid as "a rectangle is a trapezoid with at least one pair of *opposite* right angles" or as "a trapezoid with at least 3 adjacent right angles."

The participant was generally good at identifying the examples of the given definitions among the given group of quadrilaterals. However, she incorrectly chose the rhombus as an example of the definition that "a cyclic quadrilateral with at least one pair of congruent adjacent sides." Rhombus figure does not satisfy this definition since it is not a cyclic quadrilateral due to not having supplementary opposite angles. Moreover, the participant did not choose the rhombus as an example of the definition that "a trapezoid with at least three congruent adjacent sides," although it satisfied this definition. That is to say she did not grasped the inclusive relation that rhombus is a special trapezoid, but not a special isosceles trapezoid.

The participant also very successfully constructed two alternative correct economical definitions for rhombus. She defined as *"a quadrilateral with all congruent sides,"* and as *"a quadrilateral with perpendicular and bisecting diagonals."*

When she was asked to decide whether the given condition was the necessary and the sufficient one to define the related specific quadrilateral, she was able to correctly identify all of them. For example, she stated a quadrilateral that has congruent diagonals did not always define a rectangle since it was not the only quadrilateral having this property. However, she did not mention about what information was missing, although she was asked to do so. She should have also said that in addition to being congruent, the diagonals also needed to bisect each other in order to define this quadrilateral as a rectangle. Similarly, she knew that a quadrilateral that has one pair of parallel sides and the other pair of sides congruent did not always define a parallelogram; because an isosceles trapezoid also satisfied this definition. However, she again did not explain that a parallelogram could be defined as "a quadrilateral that has one pair of opposite congruent and parallel sides" or "a quadrilateral with two pairs of parallel sides."

Moreover, Participant 2 successfully wrote a definition which included kites and rhombuses, but excluded square as *"quadrilaterals with at least two pairs of congruent adjacent sides and with at most one pair of right angles."* But, there was no need to add "at least," since a quadrilateral could not have more than two pairs of sides. However, she incorrectly wrote a definition including only the prototypical kite figure but excluding rhombus and square from the kite class as *"quadrilaterals with two pairs of congruent adjacent sides."* This is a definition for the kite class including kite, rhombus and square; in order to exclude rhombus and square from this definition, the participant should have said that "quadrilaterals with two distinct pairs of congruent adjacent sides." Finally, she correctly defined whole kite class including kite, rhombus and square as *"quadrilaterals with at least two pairs of congruent adjacent sides."*

She was also very successful at understanding the inclusive relationships between the quadrilaterals; she answered all of them correctly and gave reasoning for her answers. For example, she stated that a kite could be cyclic quadrilateral when it was a right kite having one pair of opposite right angles. She also knew that trapezoid could be a cyclic quadrilateral when it was an isosceles trapezoid having supplementary opposite angles and that the diagonals of a kite could bisect each other when it was a rhombus or a square.

As in the case of participant 1, participant 2 also put the parallelogram and the isosceles trapezoid in the place of one another which lead to an incorrect relationship that a rhombus was a special isosceles trapezoid. She should have changed the places of parallelogram and isosceles trapezoid so that the rhombus would be the special case of parallelogram, but not the special case of isosceles trapezoid.

4.3 Participant 3's Analysis Results

Findings related to the Participant 3's perceptions of the definitions and understanding of the quadrilateral definitions and the hierarchies before engaging them into the clinical interview sessions, her mental process and progress during the 4 clinical interview sessions, opinions about her experience in this study and the findings related to her understanding of the quadrilateral definitions and the hierarchies after the clinical interview sessions were stated in the following sections.

4.3.1 Participant 3's Initial Perceptions of the Definitions

Participant 3 stated that she realized the importance of the definitions at the university level when she was asked to critically think on them. She believed that definitions were important to learn the properties of the concepts. On the other hand, she stated that defining a concept did not necessarily mean that concept was learned, but constructing different definitions for the same concept indicated that the concept was learned.

As in the case of other participants, she stated that the definitions were just said verbally by the teachers without any sufficient attention during her elementary and secondary school years; but she came to realize the importance of them in the undergraduate years.

When she was asked whether she could construct more than one definition for the same concept, she was doubtful about whether she could; she had a very low self-confidence about her definition construction ability. She said that she would like to use the definitions more effectively in her in-service teaching, but she did not know how to do it. She thought that she would probably take the pre-constructed definitions from the text book and would encourage students to think on them.

According to the participant, a good definition should have included all the properties of the defined concept and should have separated the concept from the other concepts. Moreover, she believed that the mathematical language used in the definition should have been appropriate for the level of students.

4.3.2 Participants 3's Initial Understanding of the Nature of the Quadrilateral Definitions and the Hierarchies

In all her answers throughout this test, the participant 3 actually completed her definitions with "geometric shape" instead of "quadrilateral", but I think she was actually thinking of a quadrilateral. Yet, if we were not agree on what a quadrilateral was, everything mentioned would not make sense. So, I changed her definitions from "a geometric shape…." to the "a quadrilateral…"

It was seen that definitions that the participant 3 constructed were including some redundant information. For example, she did not actually constructed a definition, but described rhombus by listing all the properties in such a way that *"a rhombus is a quadrilateral with congruent opposite angles, with the adjacent angle measures summing up to 180°, and with all equal side lengths."* However, the condition of "all side lengths' being equal" would be enough to define a rhombus; from this property, all other rhombus properties could be extracted.

Participant 3 also constructed an uneconomical definition of rectangle as "a quadrilateral with opposite side lengths are equal and with all angles are 90°." It is seen that the first condition was redundant since the property that "A quadrilateral with all angles 90°" would be sufficient to characterize a rectangle. Because, opposite side lengths automatically become equal if all angles are 90°. Although square also has this property, there is need to know an additional property of "all sides being congruent" to characterize it.

In contrast to the first two definitions, the participant 3 was able to construct a correct economical definition of square as "a quadrilateral with all side lengths are equal and with all angle measures are 90°." If she had used only the first condition it would define a rhombus; and if she had used only the second condition it would define a rectangle. However, the combination of these two properties is enough to generalize a square. Indeed, "at least one angle measure is 90°" would be enough instead of "all angle measures are 90°."

It is also seen here that the participant appropriately used the term "equal" in all three definitions which indicated that she was aware that sides could be congruent but side lengths could be equal. Moreover, she also identified all squares among the given group of quadrilaterals as examples of rhombus and a rectangle.

The two alternative definitions of kite constructed by the participant 3 were the following:

> *"a quadrilateral with two pairs of congruent adjacent sides and with one pair of opposite congruent angles."*
>
> *"a quadrilateral with two pairs of congruent adjacent sides and with one of the diagonals is a symmetry line.*

The analysis of these two definitions indicated that the participant was not good at identifying the necessary and sufficient conditions to characterize a kite. This was because she did not make inferences between the properties and did not examine whether one property automatically resulted in the other one. As in the case of Participant 2, this participant also used the property of "two pairs of congruent adjacent sides" in both definitions, but combined this property with different properties to create alternative definitions. However, "two pairs of congruent adjacent sides" property was enough to satisfy all the other properties of the kite and so, there was no need to add another property to define kite. So, both definitions included redundant information which was like saying some property twice in a definition.

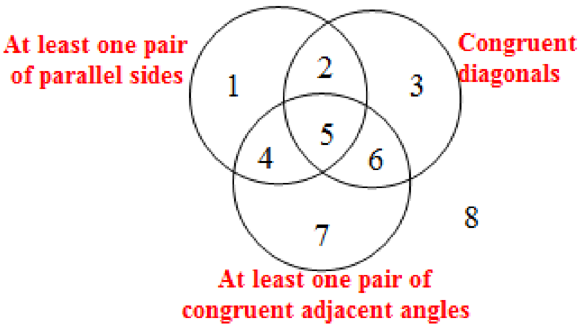
When the participant was asked to evaluate the mathematical value of the given rhombus definitions, she was generally good at it. For example when she was asked whether a rhombus is a quadrilateral with two pairs of congruent adjacent sides, she was able to explain that the "two pairs of congruent adjacent sides" was not a sufficient property to define a rhombus and to be able to define a rhombus there was need to specialize it as "all sides are congruent." She also correctly evaluated that parallel opposite sides was not a defining property alone, but 4 congruent sides was. However, she did not evaluated the definition that "a rhombus is a quadrilateral of which symmetry axes are the perpendicular lines passing through the opposite vertices" as correct, whereas it was. She thought that it was not enough to define rhombus, but did not explain further.

It was also determined that the participant was not successful at constructing inclusive and exclusive definitions. She correctly constructed a definition which included figures a and b but not the other figures as *"quadrilaterals with only one pair of parallel sides"* However, as to the definition that included a, b, c, and d but not the e and f she failed; because she excluded figures e and f from all quadrilaterals instead of excluding a, b, c, and d from e and f with her definition that *"quadrilaterals of which all 4 sides are not equal to each other."* That is to say, while defining a, b, c, and d as a whole group she stated which property they should not have, but did not stated what property they should have separating them from all other quadrilaterals. So, if someone is told to construct quadrilaterals based on the participant's definition s/he can construct many quadrilaterals out of a, b, c, and d. Hence, she should also have added a common defining property, namely the "at least one pair of parallel sides" to define a, b, c, and d.

It is also seen that the participant was not able to construct a definition including all 6 figures shown. She defined this group of figures as *"quadrilaterals with 4 sides of which interior angles sum up to 360°"* which included very general properties common to all quadrilaterals, but not only to this group of figures; so, she failed again.

On the other hand, the participant 3 was very good at understanding the inclusive relations between the quadrilaterals through considering their properties. The analysis indicated that she was aware that square was a special rhombus and so a rhombus could have congruent diagonals if it was a square. Moreover, she knew the fact that a rectangle could have congruent adjacent sides if and only if it was a square, which indicated that she also accepted square as a special instance of a rectangle.

When the participant was asked to classify quadrilaterals based on their properties she badly failed. In the first diagram she placed only trapezoid and isosceles trapezoid into the correct regions (Figure 4.60). Her answers indicated that she had many misleading information related to the properties. As an example, she had misleading information that a parallelogram had congruent diagonals and a kite had at least one pair of congruent adjacent angles. They could have these properties in their special cases, but here participants were asked to think prototypical shapes of each quadrilateral. For example, a parallelogram could have congruent diagonals when it is a rectangle and a square; and a kite could have at least one pair of congruent adjacent angles when it was a square.



Participant's Answers

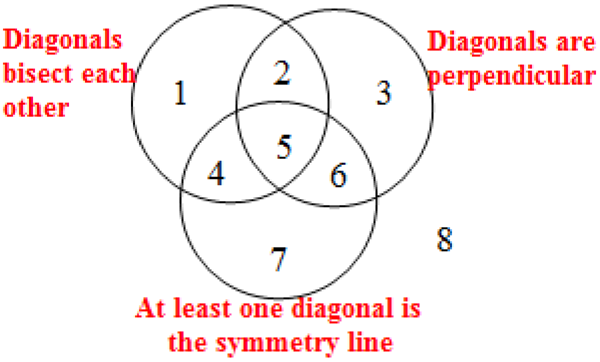
Parallelogram	**3**	Trapezoid	**1**
Kite	**7**	Isosceles Trapezoid	**5**
Square	**3**	Rectangle	**3**
Rhombus	**?**		

Correct Answers

Parallelogram	**1**	Trapezoid	**1**
Kite	**8**	Isosceles Trapezoid	**5**
Square	**5**	Rectangle	**5**
Rhombus	**1**		

Figure 4.60 Participant 3's first diagram of the classification of the quadrilaterals

The participant failed again in the second diagram (Figure 4.61). This time she could not place the trapezoid and isosceles trapezoid in any region and incorrectly placed kite, square and rectangle.



Participant's Answers

Parallelogram	**1**	Trapezoid	**?**
Kite	**5**	Isosceles Trapezoid	**?**
Square	**2**	Rectangle	**2**
Rhombus	**5**		

Correct Answers

Parallelogram	**1**	Trapezoid	**8**
Kite	**6**	Isosceles Trapezoid	**8**
Square	**5**	Rectangle	**1**
Rhombus	**5**		

Figure 4.61 Participant 3's second diagram of the classification of the quadrilaterals

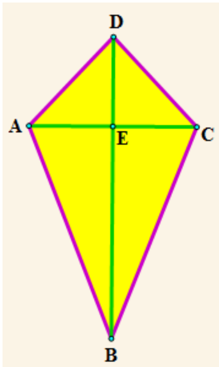
4.3.3 Clinical Interviews

Participant 3's cognitive progress during the clinical interview sessions and the effect of the GSP activities on the participant 3's cognitive improvement in understanding the quadrilaterals through definitions construction and classification processes were described in detail in the following subsections.

4.3.3.1 Session 1 with Participant 3: Kite, Rhombus and Square

At the beginning of the session, the participant was able to remember only that a kite had perpendicular diagonals and two pairs of congruent adjacent sides. Then, she was asked to work on the figure to confirm these properties and also to find out the ones that she could not remember. After observing the changes in the measurements under dragging, she correctly determined the preserved kite properties. She stated that "two pairs of adjacent side lengths are equal," "one pair of opposite angle measures are equal," "one diagonal is the perpendicular bisector of the other diagonal, and the perpendicular bisector diagonal is the angle bisector and the symmetry axes."

In the next step, the participant was asked to think of which other quadrilaterals could be the special kites that had all critical defining properties of the kite. At first, she stated that the dynamic kite figure could be dragged into a square thinking that a figure needed two pairs of congruent adjacent sides to be a kite. However, when she thought that square has right angles, she said the exact opposite that a square could not be a special kite since DCB and DAB angles could not be dragged into right angles. Upon her thought, I warned her about that the kite figure was constructed so that one pair of opposite angles would always remain congruent and it did not matter whether their measures would be 80º, 90 º, 120º, etc. That is to say, when the figure was dragged angle measures would change, but the angles always remained congruent. Moreover, the figure preserved all critical kite properties under dragging, not only the one pair of congruent angles property. Then, she said that the figure could be dragged into square if these opposite equal angle measures could be made 90 º both. The participant considered this property as a sufficient defining property of a kite which meant that a figure having only this property would be a kite, even though it was not like that. Then, I asked her to think about rectangle and she stated that "if I can drag a kite into a square, I can also drag it into a rectangle too, because a rectangle is a special square." When I asked her to explain why a rectangle was a square, she changed her mind instantly and said that a square was a special rectangle. When she was asked whether this inclusive relation between square and rectangle required a rectangle to be a kite, she answered "yes." She stated that to make a figure rectangle is easier than to make it a square, because square requires more properties to be satisfied. Then the conversation followed like that:



Researcher: Are all the critical defining properties of kite preserved in a rectangle? Angle, diagonal, side, symmetry properties for example…

Participant 3: Well, the diagonals will be perpendicular in a rectangle…other properties…yes… so, I think I can drag kite into the rectangle. Well, in my opinion I can drag the kite figure into a rectangle, if I can drag it into a square.

Researcher: Why do you relate the inclusive relationship between rectangle and kite with the square?

Participant 3: Well, lets'say I have dragged the figure into a square… That is, I have made all 4 sides congruent…by fine adjustment of the sides, I believe that I can make a rectangle.

Researcher: What about parallelogram, then? Will you say "if I can drag the dynamic kite figure into a rectangle, I can also drag it into a parallelogram?

Participant 3: Yes, the kite can also be dragged into the rectangle then.

Upon her incorrect answers, I encouraged the participant to think of the critical side property of kite which should have been preserved for all of its descendants. She correctly stated that a quadrilateral needed to have two pairs of congruent adjacent sides if it was a kite; but she still insisted on that a parallelogram was a special kite. This thinking process of the participant indicated that she did not grasped the fact that GSP figures preserved constructed properties under dragging, so two pairs of congruent sides of kite would always remain congruent which did not allow dragging it into a rectangle and into a parallelogram. She thought that the kite figure also could be dragged into a rhombus, trapezoid and isosceles trapezoid. As for the rhombus she was right, but it was a random estimation since her thinking process was wrong. In the case of trapezoid and isosceles trapezoid, she thought that she could make one pair of opposite sides of the kite parallel, but she did not think that having one pair of parallel side was not a critical defining property of a kite.

After learning her mental representations related to the special kites, the participant was asked to check her thoughts by dragging the kite figure into the other quadrilaterals and to detect out its descendants. She saw that the kite was dragged into a square and she was able to explain the reason that all the properties of kite preserved in square. When she dragged the dynamic kite figure for a rectangle, she could not make it a rectangle and mentally retrieved that it was because of the fact that a rectangle did not have two pairs of congruent adjacent sides though it was a preserved property for a kite. She also added that the diagonals of a rectangle were not perpendicular, even though they must have been for a kite. When she tried to drag the kite figure into a prototypical parallelogram, she could only drag it into a special parallelogram, namely rhombus. She correctly explained this situation again with the critical side property of the kite. While a rhombus have two pairs of congruent sides, a prototype parallelogram did not have. Similarly, she saw that the figure could not be dragged into a trapezoid and isosceles trapezoid due to the critical side property.

As a result of this dragging process, the participant saw that square and rhombus are the special kites having all critical kite properties. Moreover, she said that square also was a special rhombus since it had all properties of it and correctly showed the hierarchical relationship between kite, rhombus and square.

The next step was to construct an inclusive definition of kite which will include its special cases rhombus and square. The participant defined kite inclusively as "*kite is a quadrilateral with at least one pair of equal adjacent side lengths, with at least one pair of equal opposite angle measures and with perpendicular diagonals."*

As it was expected, the participant described kite rather than she defined it; she listed all the properties she knew without considering whether each property was necessary and sufficient defining property. In addition to including redundant information, her definition also included incorrect information. The participant was mistaken saying that "…at least one pair of equal adjacent side lengths…," because two pairs of the adjacent side lengths must be equal in a kite. However, at the time she reread her definition, she realized her mistake and corrected it as "*kite is a quadrilateral with two pairs of equal adjacent side lengths, with at least one pair of equal opposite angle measures and with perpendicular diagonals."*

At this point my aim was to encourage her to think of the properties used in the definition and to decide whether each one alone would be the enough defining property of the kite. After thinking a while on the properties used in her definition, she correctly made an inference that having two pairs of congruent adjacent sides property required one pair of congruent adjacent angles property, and so there was no need to use both of them in the same definition. Therefore, she shortened the definition as

"kite is a quadrilateral with two pairs of equal adjacent side lengths, and with perpendicular diagonals."

After that, she thought of whether "two pairs of congruent adjacent sides" property would lead to perpendicular diagonals, but she could not come to a conclusion. Therefore, I asked her to make a construction using each property alone and to see whether it would be enough to construct a kite figure. She first checked whether "two pairs of equal adjacent side lengths" property would be the sufficient defining property alone so that all other critical kite properties could be drawn from it. She began with constructing the first congruent adjacent pair as the radiuses of a circle; however, for the construction of second pair she did not know what to do (Figure 4.62). She drew another circle, but then realized that she could not make a real construction in this way. Then, she thought aloud that the fourth vertex, which was the intersection point of the second congruent pair, had to be located in equal distance to the end points of the first pair. She then stated that this point must be on the angle bisector of the angle between the first congruent pair, so she constructed the angle bisector and took any point on it as the fourth vertex. Finally, she completed the figure into a quadrilateral by connecting the vertices with segments. Then the participant made the related measurements and dragged the figure to detect out whether it always preserved kite properties under dragging. She observed that the figure was a real kite and came to the conclusion that having two pairs of congruent adjacent sides was sufficient defining property to characterize a kite; this property alone satisfied all other properties of kite. Therefore, she decided that "a kite is a quadrilateral with two pairs of congruent adjacent sides" would be a correct inclusive definition of a kite; there was no need for any other information to generalize a kite.

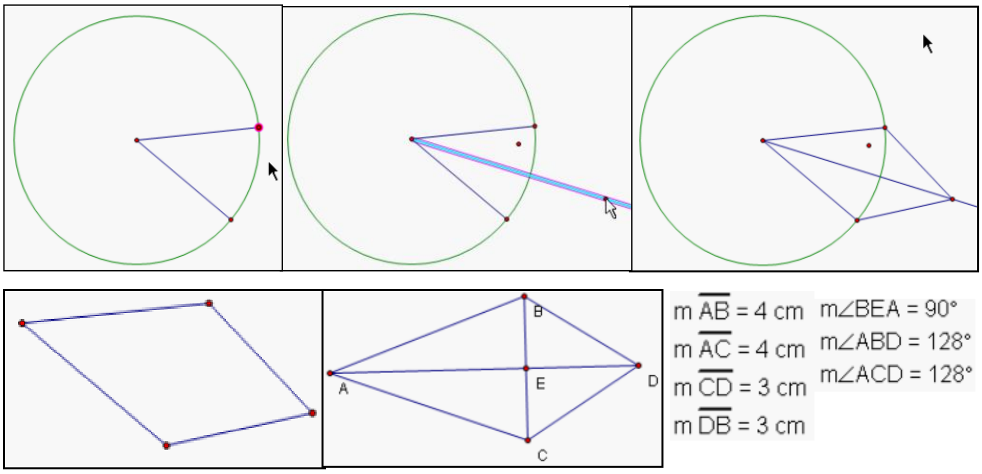


Figure 4.62 Participant 3's construction of a quadrilateral with two pairs of congruent adjacent sides

Next, she tested whether "a kite is a quadrilateral with perpendicular diagonals" would be correct definition; namely whether having perpendicular diagonals was specific defining property from which all other properties could be inferred. For this construction, she constructed two perpendicular line segments as diagonals which intersected each other at any point, because there was not any specific information about the intersection point (Figure 4.63). Then, she constructed two points on the perpendicular line and completed the figure into a quadrilateral.

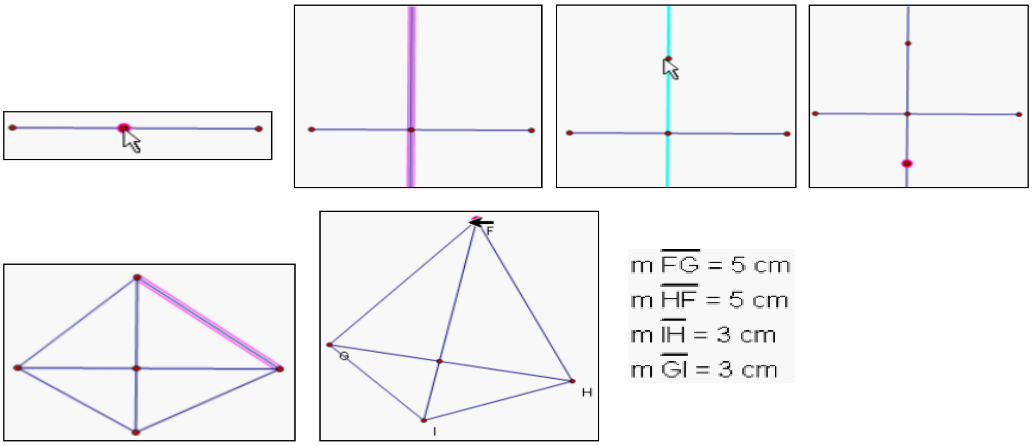


Figure 4.63 Participant 3's construction of a quadrilateral with perpendicular diagonals

When she observed the side lengths under dragging, she saw that the two pairs of congruent adjacent sides did not remain congruent to each other. Therefore, the figure was not a real kite since it did not preserve the critical kite properties and it was clear that having perpendicular diagonals was a necessary but not a sufficient property to define kite. As a result of this construction activity, the participant was able to construct a correct economical definition from her initial description by eliminating the redundant information.

In the next step, the participant was asked to evaluate the given definitions in terms of their including necessary and sufficient information to make it a correct inclusive definition. Since we had already discussed the first definition in the previous part, I skipped it and continued with the second definition.

Definition 1: A kite is a quadrilateral with perpendicular diagonals.

Definition 2: A kite is a quadrilateral with at least one diagonal is a perpendicular bisector.

The participant accepted the second definition as correct economical definition. She correctly thought that if one diagonal was a perpendicular bisector of the other diagonal, the triangles of which common base was the bisected diagonal would be the isosceles triangles and this would satisfy the two pairs of congruent adjacent sides property. She also tested her idea with the GSP construction and saw that the constructed figure remained a kite under dragging if one diagonal was constructed as the perpendicular bisector of the other diagonal.

Definition 3: A kite is a quadrilateral with two pairs of congruent adjacent sides and one pair of congruent opposite angles.

We had also discussed the third definition and she had already detected out that “two pairs of congruent adjacent sides” was the sufficient defining property of kite and there was no need for any extra information.

Definition 4: A kite is a quadrilateral with at least one diagonal is the symmetry axis.

For the fourth definition she said that it was a correct economical definition including the sufficient information to define kite. She thought that when one diagonal was the symmetry axes, the sides on both sides of this diagonal must be equal in length, which satisfied the side property of kite. To make sure, she also constructed the related figure using the information given in the definition (Figure 4.64). She first constructed the symmetry axes, then took a point on one side and reflected it to the other side with respect to the symmetry line. After completing the figure into a quadrilateral, she dragged it and observed that all critical properties of the kite were preserved.

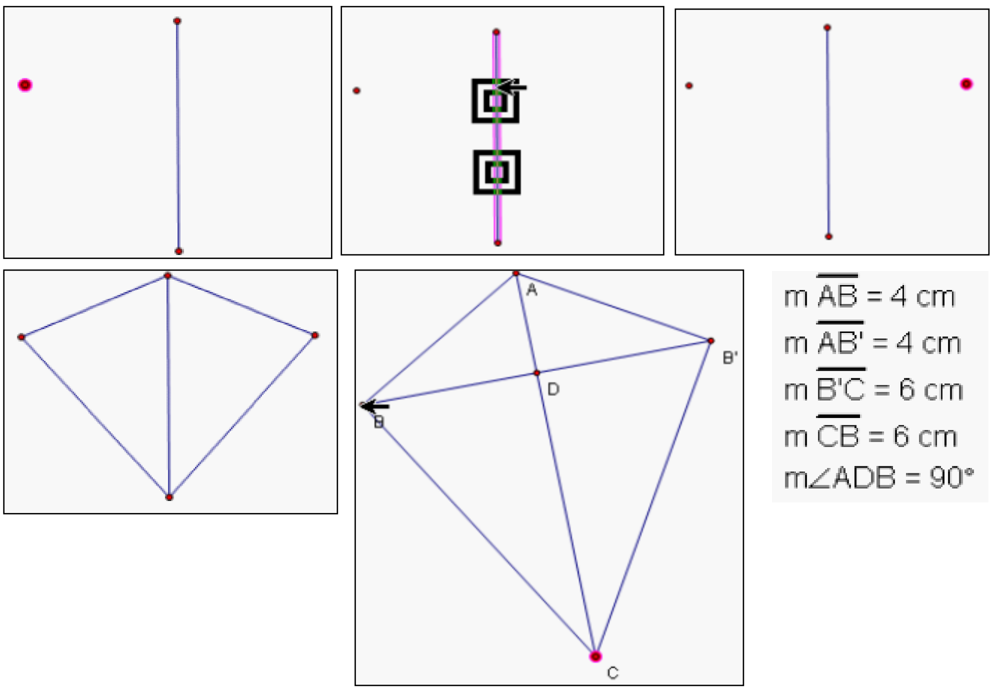


Figure 4.64 Participant 3’s construction of a quadrilateral with at least one diagonal is the symmetry axis

At the end of this session, I asked the participant to construct an inclusive definition for the rhombus and she correctly defined as *“a rhombus is a quadrilateral with all congruent sides”* which also included square as a special example.

4.3.3.2 Session 2 with Participant 3: Parallelograms and Trapezoids

When the properties of the isosceles trapezoid were asked, the participant stated that top and bottom bases were parallel to each other, one pair of opposite side lengths was equal and two pairs of adjacent angle measures summed up to 180 degrees. Moreover, she correctly remembered that the diagonal lengths were equal. She remembered almost all properties correctly, yet she was asked to observe the preserved side, angle, diagonal and symmetry properties. In addition to the ones she had remembered she also observed that the intersection point of the diagonals was the common vertex of two isosceles trapezoids, which was another way of saying that “diagonals intersected each other in the same ratio.” At first, she said there was not any symmetry property, but then realized that not the diagonals but the line passing through the midpoints of the parallel sides was the symmetry axes.

When she was asked how she would define an isosceles trapezoid to her students, she defined as *“a quadrilateral of which top and bottom sides are parallel and of which at least one pair of opposite side lengths are equal.”* However this initial definition of her was an incorrect one since it included prototype parallelogram and rhombus as counter examples. That is to say, the definition did not only define the isosceles trapezoid class. However, at this step I did not ask further about its correctness and moved on.

Next, she was asked to think of the special isosceles trapezoids. She said that square was an isosceles trapezoid since it had at least one pair of parallel sides and at least one pair of opposite congruent sides. Here, she only considered side property although she had to consider all critical preserved properties to make sure. For instance if we only considered the side property we should have say that a rhombus also was an isosceles trapezoid though it was not. That is to say, the participant was not aware of the fact that if a quadrilateral did not satisfy even one property, it could not be a special isosceles trapezoid. So, I asked the participant to consider all other properties and then she explained which other properties were preserved, too. She also compared all properties of an isosceles trapezoid and rectangle, then decided that rectangle preserved all critical defining properties of it. When it came to think on the parallelogram, it was a common tendency of all participants to consider a parallelogram as a rectangle and to directly state that it was also a special isosceles trapezoid, without considering the properties. This participant had the same tendency; that is, she incorrectly thought that if rectangle was an isosceles trapezoid, then a parallelogram was as well. Upon her answer, I encouraged her to think on the properties. While thinking aloud on the angle properties, the participant realized that a parallelogram did not have two pairs of congruent base angles as in the case of isosceles trapezoid and at this point she changed her mind that a parallelogram could not be a special isosceles trapezoid. Similarly, she was able to easily state that a rhombus was not also a special case due to not preserving the critical angle property of the isosceles trapezoid. She also eliminated the prototypical trapezoid, since it did not have to have congruent opposite sides; and the kite, since it did not have congruent diagonals.

Then, she tested all the arguments on the dynamic figure by dragging the figure into the other quadrilaterals and concluded that the only special isosceles trapezoids were rectangle and square. After that, she correctly indicated the inclusive relations on the hierarchy diagram (Figure 4.65).

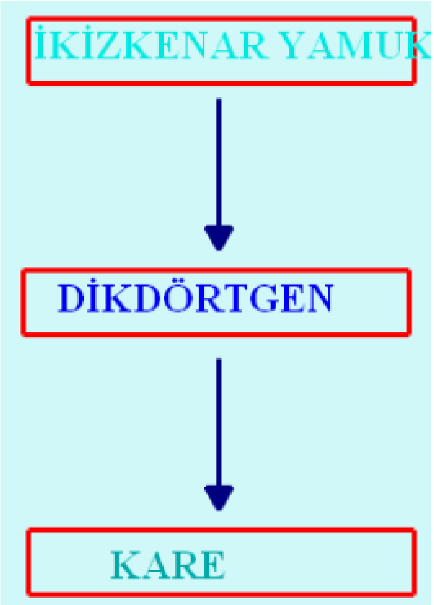


Figure 4.65 Participant 3's hierarchy diagram of isosceles trapezoid, rectangle and square

Next, she was asked to construct an inclusive definition of isosceles trapezoid right after investigating the inclusive relations. She defined as

> *"An isosceles trapezoid is a quadrilateral with at least one pair of parallel sides and two pairs of congruent adjacent angles."*

This definition was a correct definition but included more than the necessary property that could be inferred from the other property. In order to make the participant realize the redundant information I encouraged her to think on it. I first asked how she decided to use these properties as defining properties. She explained that having at least one pair of parallel side was not sufficient since a parallelogram also had this property; so she needed to add the angle property. Before I was about to ask her to consider each property in the definition separately, she said that "what if I remove the at least one pair of parallel side property?... Two pairs of congruent adjacent angles property may be sufficient…hmm…." Since she was not sure, I asked her to test whether only the angle property would be sufficient to construct an isosceles trapezoid.

After constructing the first angle, she did not know how to construct the congruent adjacent angle, as in the case of other participants. That was the trickiest part of this construction for all participants, so I needed to give some clues. After constructing the first angle, the participant wanted to reflect it with respect to the perpendicular line passing through the midpoint of the base segment; however, if she had used this method, she would have used the symmetry property although there was not any information about it in the definition. So, I asked her to think about doing a rotation with the same angle. By the help of this clue, she awakened and rotated the point B around point C with respect to the negative rake of the first angle (Figure 4.66). She also stated that the last pair of angles could be made congruent if the fourth side would be parallel to the base segment. Finally, she constructed the quadrilateral and observed some measurements under dragging to make sure whether it preserved all isosceles trapezoid properties. She concluded that her initial definition included redundant information and reduced the definition as

> *"An isosceles trapezoid is a quadrilateral with two pairs of congruent adjacent angles."*

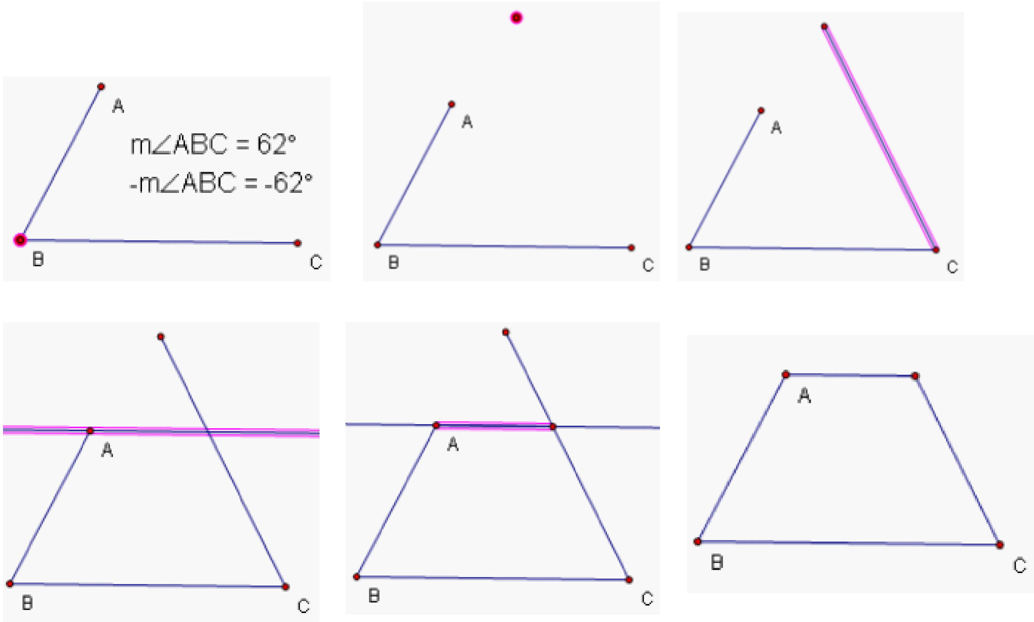


Figure 4.66 Participant 3's construction of a quadrilateral with two pairs of congruent adjacent angles

In the next step, she evaluated the given definitions. Since we had already discussed the first definition, she directly stated that parallelogram was a counter example which satisfied the definition but it was not an isosceles trapezoid. So, her final decision was that this definition was not a correct one.

> *Definition 1: An isosceles trapezoid is a quadrilateral with at least one pair of parallel sides and with at least one pair of opposite congruent sides.*

For the second definition, she was not sure whether the angle property would be satisfied if a quadrilateral would have these properties given in the definition. When I asked her whether the definition included all isosceles trapezoid class, she realized that it did not include rectangle and square, but only the prototypical isosceles trapezoid.

> *Definition 2: An isosceles trapezoid is a quadrilateral with one pair of parallel sides and with one pair of congruent but unparallel sides.*

For the third definition, she had a tendency to reduce the used properties and asked whether saying only "a quadrilateral with opposite supplementary angles" would be enough. Upon this, I asked her to think about any counter examples to refute the definition. She eliminated the parallelogram and rhombus, then thought about angles of kite and stated that one pair of congruent angle measures could be 90 degrees each and in this special case it would be a counter example for this definition. Having considered this reasoning, she decided that the given definition was a correct inclusive one including necessary and sufficient information; there was no need to reduce the properties.

Definition 3: "An isosceles trapezoid is a quadrilateral with at least one pair of parallel sides and with opposite supplementary angles."

For the fourth definition, she thought that definition also included rectangle and square as specific cases. When she thought of the other quadrilaterals as counter examples, she correctly eliminated parallelogram and rhombus since they had two paris of opposite congruent angles. She also eliminated kite due to having only one pair of opposite congruent angles. That is, she correctly evaluated this given definition through searching for the counter examples and so there was no need to construct the corresponding figure.

Definition 4: An isosceles trapezoid is a quadrilateral with two pairs of congruent adjacent angles.

For the last definition, she found parallelogram as an incorrect counter example claiming that the diagonals of a parallelogram were also congruent though they were not. So I provided her a parallelogram on the screen to make the related measurement and she understood that she was wrong. She was also doubtful about the diagonals of a rhombus; but after measuring, she made sure that its diagonals did not have to be congruent. She decided that the definition was correct economical one to define an isosceles trapezoid; but she was wrong since this property was not sufficient defining property.

Definition 5: "An isosceles trapezoid is a quadrilateral with congruent diagonals."

Having constructed two congruent segments intersecting each other at any point and having completed the figure to a quadrilateral, she saw that it did not remain as an isosceles trapezoid (Figure 4.67). So, as a result of the construction, she decided that any ordinary quadrilateral could have congruent diagonals, not necessarily the isosceles trapezoids. Then, I asked her what was needed to make it a correct definition; she answered that the intersecting diagonals had to form two isosceles triangles by which she meant to say the diagonals had to intersect each other in the same ratio. Then she also constructed the quadrilateral corresponding to redefined definition and saw that the figure was a real isosceles trapezoid (Figure 4.68). However, I needed to give some clues during the construction since it was difficult one.

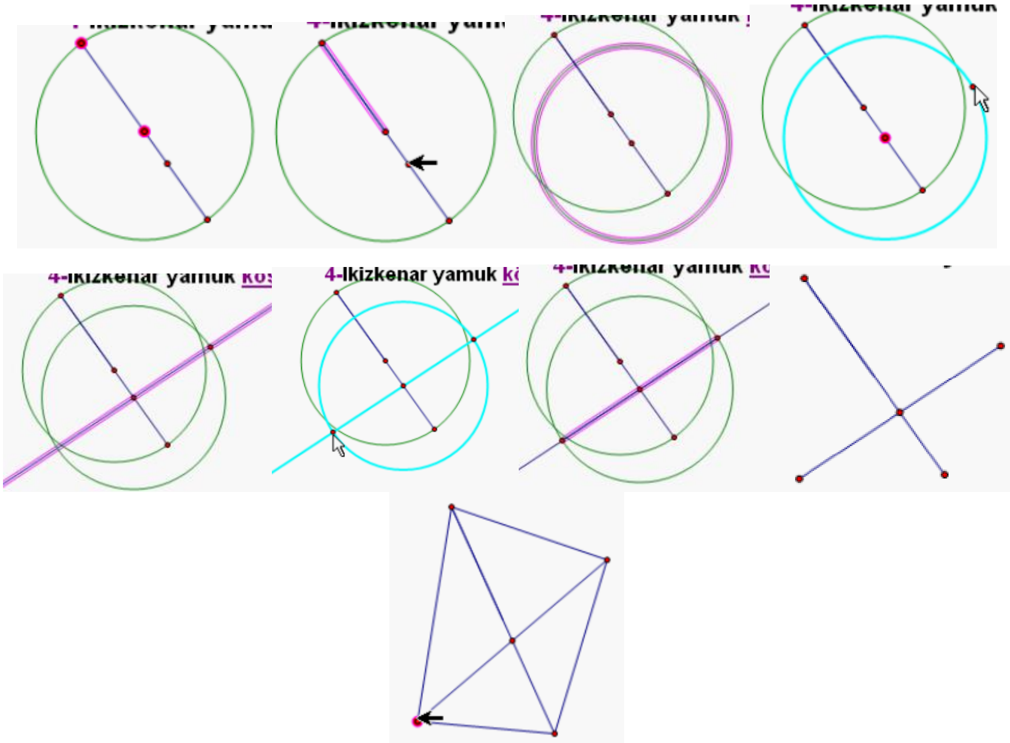


Figure 4.67 Participant 3's construction of a quadrilateral with congruent diagonals

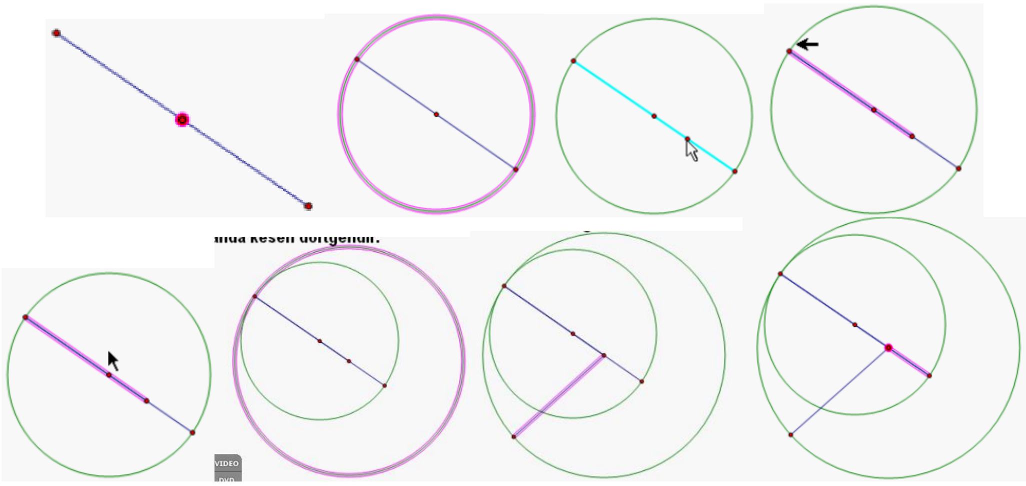


Figure 4.68 Participant 3's construction of a quadrilateral with congruent diagonals intersecting each other in the same ratio

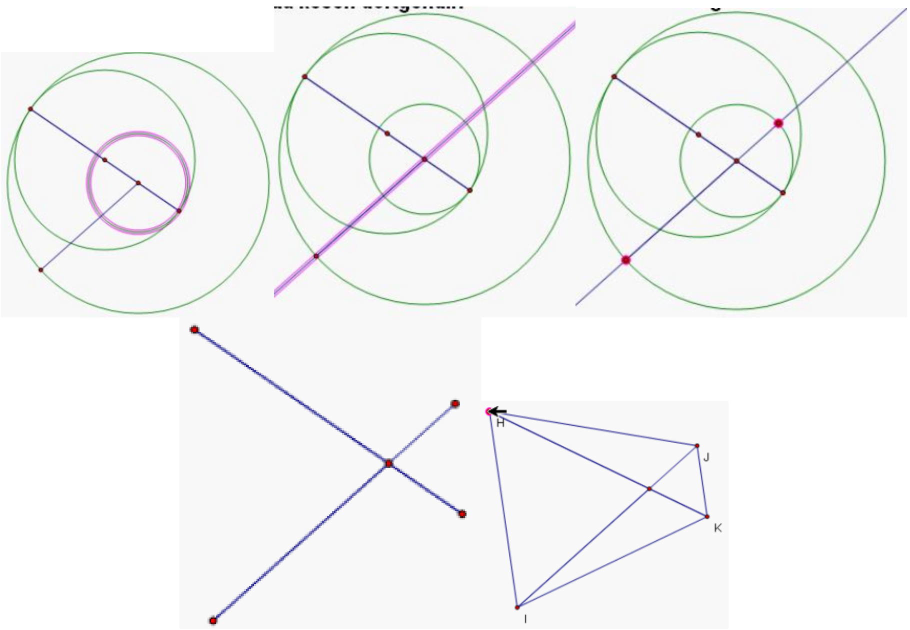


Figure 4.68 (continued)

Activity in the next step was to identify the special parallelograms. At first hand, participant 3 directly stated that rectangle and square were special parallelograms and explained the reasons correctly. After thinking about the properties of the rhombus she also correctly stated that it was also a special parallelogram. Moreover, she explained that a kite could not be a parallelogram since it did not have parallel sides; similarly trapezoid and isosceles trapezoid could not be parallelograms since they did not have to have two pairs of parallel sides. That is, she correctly detected out all special parallelograms and also confirmed her thoughts through dragging the parallelogram into the other quadrilaterals. As for showing the relationships on the hierarchy diagram, she also doubted about the relationship between the rectangle and rhombus as in the case of all other participants. She correctly reasoned that a rectangle could not be a rhombus since all sides were not congruent and a rhombus could not be a rectangle since it did not have right angles. After thinking a while, she correctly formed the hierarchy (Figure 4.69).

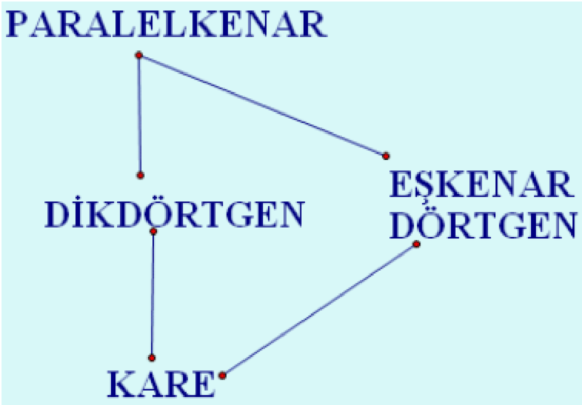


Figure 4.69 Participant 3's hierarchy diagram of parallelograms

Next, she was asked to think of the relationship between parallelograms and the trapezoids and easily answered that parallelograms were the special trapezoids since it was required only to have one pair of parallel side to be a trapezoid. Moreover, she stated that a trapezoid could not be a special parallelogram since it did not have to have two pairs of parallel sides which was the critical defining property of parallelogram. She said that isosceles trapezoids were already special trapezoids. Finally, she decided that parallelograms and isosceles trapezoids were the descendants of the trapezoid class (Figure 4.70).

Then, the participant constructed an inclusive definition of a trapezoid as

> *"A trapezoid is a quadrilateral with two pairs of parallel sides and with at least one pair of congruent opposite sides."*

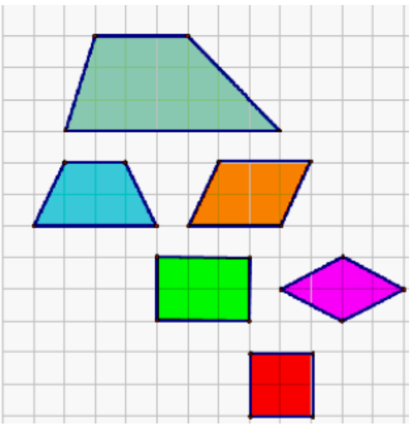


Figure 4.70 Trapezoid class

However, the definition she constructed was an incorrect one; it did not define a prototypical trapezoid. Then, she reduced the definition as *“A trapezoid is a quadrilateral with two pairs of parallel sides”* thinking that property would be sufficient; however, she said that she totally mixed her mind. She stuck to the idea that to be able to include the parallelogram into this definition she needed to add the “two pairs of parallel sides” property. She thought that when she said “at least one pair of parallel sides,” she could not define the parallelogram. I asked her to think on what if the definition would be *“A trapezoid is a quadrilateral with at least one pair of parallel sides,”* but she stated that this definition would not define the trapezoid class. Upon her claim, I asked which quadrilaterals in this class would not be included in this definition and she said “parallelogram.” Then I asked why we had accepted parallelograms as special trapezoid. She felt mentally alert and said that stating only “at least one pair of parallel side” property meant there could be more that one pair; so parallelograms were included. She accepted that she initially defined only the parallelograms.

Next, she worked on the hierarchy diagram to show all relationships discovered so far (Figure 4.71). At first hand, she constructed the hierarchy of parallelograms without any difficulty, but she did not know how to connect kite, parallelograms, trapezoid and isosceles trapezoid.

Figure 4.71 Participant 3’s first attempt of constructing the hierarchy of the quadrilaterals

Although she had found that only rectangle and square were special isosceles trapezoids, she incorrectly placed all parallelograms under isosceles trapezoid. Then, she realized that rhombus and parallelogram could not be isosceles trapezoid since they did not have congruent adjacent angles. Although it became a challenging process for the participant to remember all relationships, she constructed the correct hierarchy in the end (Figure 4.72). She sometimes needed to go back to remember the relationships.

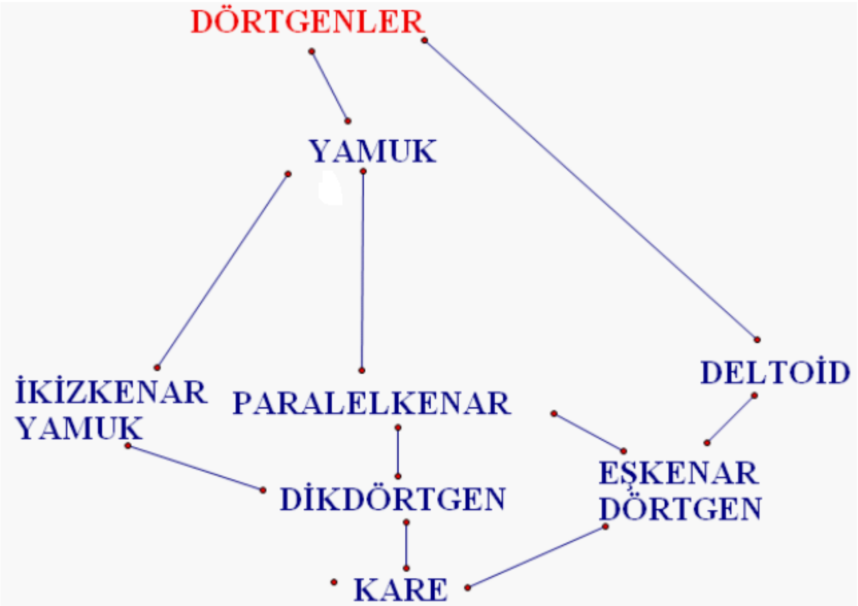


Figure 4.72 Participant 3's second attempt of constructing the hierarchy of the quadrilaterals

After completing the hierarchy, I asked her how the hierarchy would change if trapezoid was defined as *"A quadrilateral with exactly one pair of parallel sides."* She stated that parallelograms could not be special isosceles trapezoids then.

Next we moved on to the exclusive definitions. She exclusively defined *parallelogram* using the *diagonal* property as *"a parallelogram is a quadrilateral of which diagonal lengths are different from one another."* However, while she was writing the definition, thought aloud that a rhombus also had this property. Then she redefined as *"a parallelogram is a quadrilateral which has two pairs of parallel sides and of which diagonal lengths are different from one another."* When I asked her to think of the counter examples to refute this definition, she found out that rhombus also satisfied this definition though she wanted to eliminate it. She had eliminated rectangle and square since they had congruent diagonals; but she had to eliminate the rhombus as well. After thinking on it, she correctly redefined as "*a parallelogram is a quadrilateral of which diagonals bisect each other but not congruent to each other and not perpendicular to each other."* That is, by saying the diagonals were not perpendicular to each other, she eliminated rhombus as well.

In the next step, she defined *rhombus* exclusively by using the *symmetry* property as *"A rhombus is a quadrilateral of which both diagonals are the only symmetry axes."* This was a correct exclusive definition of rhombus eliminating the square.

Finally, she was asked to define *kite* exclusively using *any* property she liked. Her definition was that *"a kite is a quadrilateral which has two pairs of congruent adjacent sides and only one diagonal bisecting the other diagonal."* She also tried to find counter examples but could not. So this definition was correct definition defining only prototype kite, but not rhombus and square as its descendants. However, just the second property, namely "having only one diagonal bisecting the other diagonal" would be enough defining property to define prototype kite.

4.3.3.3 Session 3 with Participant 3: Cyclic and Circum Quadrilaterals

As soon as she saw the figure on the screen, the participant knew that it was called cyclic quadrilateral. However, when she was asked to define it, she said its angle measures summed up to 360º even though this was true for all quadrilaterals. Then she defined as

> *"A cyclic quadrilateral is a quadrilateral of which opposite angle measures sum up to 180º."*

In contrast to the other participants, this participant detected the criterion to be a cyclic quadrilateral at the very beginning of the session. Then she was asked to find out the special cyclic quadrilaterals. As for the square, she unexpectedly stated that it was not a cyclic quadrilateral because it was not possible to place its vertices on a circle. When I asked the reason, she suddenly changed her mind and stated that the vertices of a square could be placed on a circle when the diagonals were the diameter of the circle. However, she added that she was not much sure since she was just making a visual judgment. When it came to rectangle, she was sure that it was a cyclic quadrilateral. She explained that she was doubtful about whether the sides could be made congruent to place a square on a circle, but she thought that it was possible to make two pairs of congruent opposite sides in the case of rectangle. Next, she thought about the parallelogram and concluded that a parallelogram could not be a cyclic quadrilateral. When I asked the reason, she actually detected out the condition that “for a quadrilateral to be a cyclic quadrilateral the opposite angles had to sum up to 180º, and so a parallelogram was not a cyclic quadrilateral. Then she reconsidered square and rectangle and stated that since they had opposite supplementary angles they were the cyclic quadrilaterals. She evaluated that a rhombus also was not cyclic quadrilateral due to not meeting this condition. At this point, she could detect out that a kite could be cyclic quadrilateral if the congruent opposite angles were 90º each. Initially, she could not decide whether a trapezoid would be the cyclic quadrilateral. After thinking a while, she stated that a prototypical trapezoid would not be cyclic since the opposite angles would not necessarily sum up to 180º; however, she was sure that an isosceles trapezoid would be a cyclic quadrilateral. That is to say, through thinking, she was able to correctly detect all cyclic quadrilaterals; namely square, rectangle, and isosceles trapezoid and right kite. She also tested her arguments by working on the dynamic figure and confirmed that she was right .

In the next step the participant was asked to define *right kite* in terms of *“quadrilateral,” “kite”* and *“cyclic quadrilateral.”* She defined in terms of a quadrilateral as

> *“A right kite is a quadrilateral with one pair of opposite right angles.”*

Upon her definition, I asked her whether I could construct a rectangle based on this definition. Then, she realized that the definition needed to be specified a bit more and redefined it as

> *“A right kite is a quadrilateral with one pair of opposite right angles and with diagonals only one is bisected by the other diagonal.”*

I asked her to search for the counter examples which was not a right kite but included in this definition; but she could not find. However, this was an exclusive correct definition of kite which did not included square as a special right kite since she used “only”. When I asked her how to include square into the definition, she easily stated that when “only” was removed the definition would include square as a special case. Her final inclusive right kit definition was

"A right kite is a quadrilateral with one pair of opposite right angles and with diagonals one is bisected by the other diagonal."

Next, she defined a right kite in terms of a kite as

"A right kite is a kite with one pair of congruent right angles."

That was a correct inclusive definition; because she did not restricted one pair of congruent right angles by saying "only" or "exactly." So, the definition included square as special right kite. Since it was known to be a kite, she only added the property which made it a cyclic quadrilateral. When I asked her to explain the difference between defining in terms of a quadrilateral and a kite, she was able to explain that if it was known to be a kite, all kite properties were automatically added to the definition and there was need to add only the property which made it a cyclic quadrilateral. Next, she tried to define what kind of a cyclic quadrilateral the right kite was. She said that she should only have added the property that made it a kite and constructed the definition as

"A right kite is a cyclic quadrilateral with one pair of opposite congruent right angles and with diagonals one is bisected by the other."

As soon as she wrote the definition, she decided to change it as

"A right kite is a cyclic quadrilateral with diagonals one is bisected by the other."

However, the property that "one diagonal is bisected by the other" was not enough defining property to make a quadrilateral kite. To make her realize this, I asked her to drag the dynamic cyclic quadrilateral figure so that it would have one diagonal bisected by the other and to observe whether the constructed cyclic quadrilateral with this property was always a right kite. When she constructed a cyclic quadrilateral with one diagonal bisected by the other, she obtained an ordinary quadrilateral which was not a right kite (Figure 4.73).

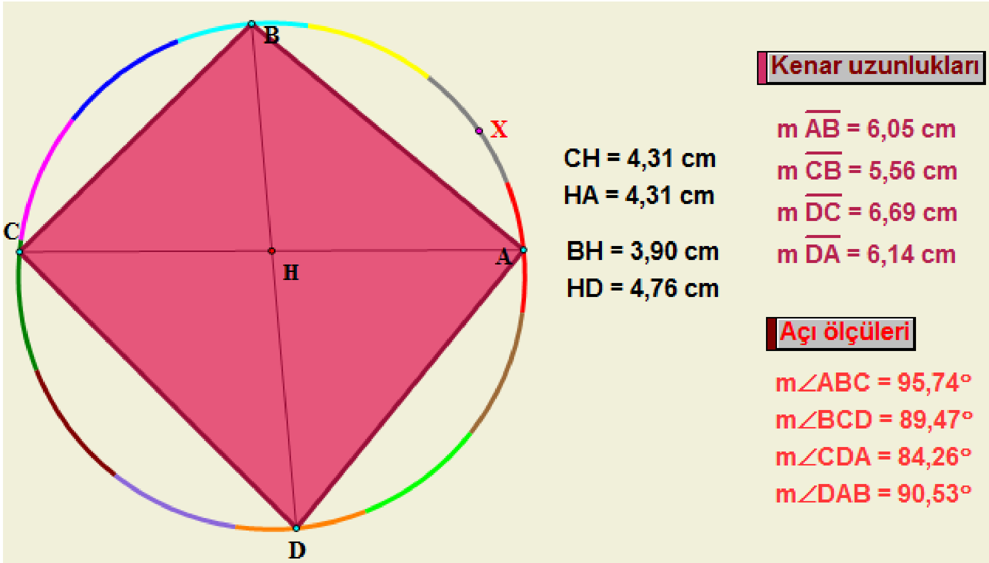


Figure 4.73 Participant 3's dragging the dynamic cyclic quadrilateral figure into a quadrilateral with one diagonal bisected by the other

During this dragging, she realized that a kite must additionally have perpendicular diagonals. So, she redefined as

> *"A right kite is a cyclic quadrilateral with perpendicular diagonals one is bisected by the other."*

This time she used the correct sufficient defining property of kite and since we know that the opposite angles of a cyclic quadrilateral were supplementary, then this required the opposite congruent angles to be 90º each.

Then the next step was to define isosceles trapezoid, rectangle and square in terms of cyclic quadrilateral. Her isosceles definition was

> *"An isosceles trapezoid is a cyclic quadrilateral with two pairs of congruent adjacent angles."*

I asked her to remember the definition of isosceles trapezoid in terms of a quadrilateral and she remembered that it would be defined as "a quadrilateral with two pairs of congruent adjacent angles" and understood that there was no difference between her definition on the base of quadrilateral and cyclic quadrilateral. She thought that being a cyclic quadrilateral required having opposite supplementary angles and decided that it would be sufficient to add the property that "one pair of congruent adjacent angles." Then she redefined as

"An isosceles trapezoid is a cyclic quadrilateral with at least one pair of congruent adjacent angles."

Then she correctly defined rectangle and square as the following:

"A rectangle is a cyclic quadrilateral with two pairs of congruent opposite sides."

"A square is a cyclic quadrilateral with all congruent sides."

When I asked her to think not "the cyclic quadrilateral with all congruent sides," but "the quadrilateral with all congruent sides", she correctly stated that this time it would be the definition of a rhombus. After the definition construction process, the participant was able to place "cyclic quadrilateral" and "right kite" categories into the hierarchy after few trials.

In the next screen there was a dynamic figure of a circum quadrilateral and the participant was first asked to think of the special circum quadrilaterals if there were any. She first stated that a square was a circum quadrilateral. However, when she was asked the criterion, she first asserted an incorrect condition that *"for a quadrilateral to be circum quadrilateral it needs to have at least one pair of supplementary adjacent angles… That is, it needs to have at least one pair of parallel sides…"* Then, she stated that she did not make a judgment based on a mathematical criterion, but based on a visual imagining. As for the rectangle she stated that a rectangle could not be a circum quadrilateral since its two pairs of sides which have different length. She stated that in this case, the circle drawn inside a rectangle would not be tangential to all sides. As for the parallelogram she judged in the same way as she did for the rectangle and stated that a parallelogram also was not a circum quadrilateral. At first, she could not decide whether a rhombus would be the circum quadrilateral. After toing and froing on it for a while, she decided that it was a circum quadrilateral due to having congruent sides. As for the kite our conversation followed as

Researcher: What would you say for kite?

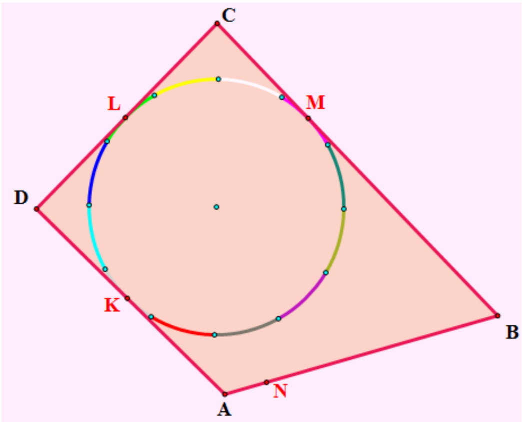
Participant 3: For kite…. It does not seem to be a circum quadrilateral…

Researcher: Do you make an evaluation based on the visual thinking again?

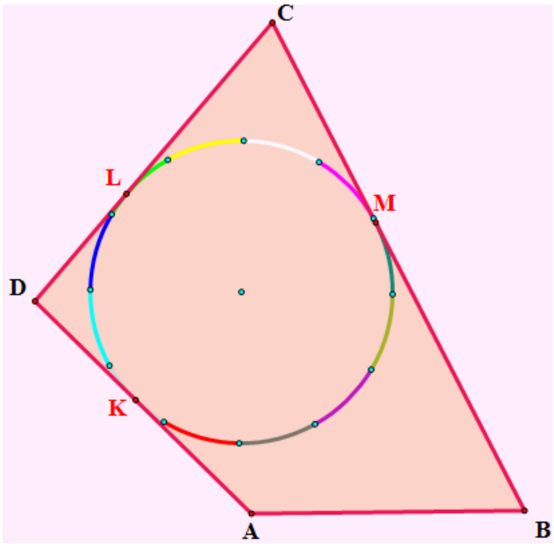
Participant 3: Yes… Well, can' t be a circum quadrilateral?….. if I think of the diameter…. No, no it is not… a circum quadrilateral needs to have at least one pair of parallel sides, but a kite does not have any parallel sides.

Researcher: How do you know that a circum quadrilateral have to have at least one pair of parallel sides? What is your anchor point?

Participant 3: I inferred it from the circum quadrilateral figure I see on the screen…



Researcher: But it is a dynamic figure and it changes.. Let me drag it a little bit so that no sides would remain parallel… Do not be mislead by the figure.



Participant 3: Ohh yes, I was wrong… I think a kite is a circum quadrilateral.

Researcher: Why do you think so, then?

Participant 3: I am just thinking that I could place the sides of this dynamic figure so that it would be a kite and all sides are tangential to the circle.

Researcher: You just imagined a kite around a circle and decided that a kite is a circum quadrilateral… Ok, what about right kite, a special quadrilateral?

Participant 3: It seems I can not place a circle inside it…

Researcher: You said a kite is a circum quadrilateral a few minutes ago… Why is a right kite not a circum quadrilateral? How did you decide?

Participant 3: I can not make any explanation right now… ☺, I really do not know the condition, I just predict based on visualizing in my mind. I had never thought on circum quadrilaterals before, so it is a different concept to me that I am not familiar with. I can not think any property, but I am just trying to imagine drawing a circle inside the quadrilaterals.

As it is obvious, the participant's predictions did not base on any mathematical idea, she just tried to make visual judgments in her mind. Finally, she considered the trapezoid and isosceles trapezoid and decided that a trapezoid would not be a circum quadrilateral. She stated that if she would draw a circle inside a trapezoid, the segments drawn between the parallel sides would be the diameter, but this circle would not be tangential to the other sides. Moreover, she stated that the same reason was true for the isosceles trapezoid, so it was not a circum quadrilateral.

In the next step she tried to drag the dynamic figure into the other quadrilaterals to detect out the circum quadrilaterals. She was able to drag it into square and rhombus, but when she extended the moving side so that the figure would be a rectangle and parallelogram correspondingly, she saw that the circle did not remain tangential to all sides. She was able to construct a kite and right kite, too and saw that the circle was always tangential to their sides. Next, she tried to contruct trapezoid and decided that it could be a circum quadrilateral; and she realized that when she dragged the moving side, the figure remained a trapezoid but not a circum quadrilateral. She saw that the similar situation was true for the isosceles trapezoid. As a result, she found out that the quadrilaterals which always remained as a circum quadrilateral were the kite, right kite, rhombus and square; namely the kite class.

In the next step, the participant examined the intersection of the angle bisectors on the dynamic figure and found out that for isosceles trapezoid, rectangle, square and right kite the angle bisectors intersect at a point. Then, she also examined the intersection of the perpendicular bisectors of the sides on the dynamic figure and found out that for kite, rhombus, square and right kite the perpendicular bisectors of the sides intersected at a point. Then, the participant was able to come to the conclusion that the angle bisectors of circum quadrilaterals intersected at a point and this point was the center of the circle; the perpendicular bisectors of the sides of the cyclic quadrilaterals intersected at a point and this point was the center of the circle. Then she also found that square and rectangle were both cyclic and circum quadrilaterals.

As a final step of this session, the participant placed the circum quadrilaterals category into the hierarchy correctly (Figure 74).

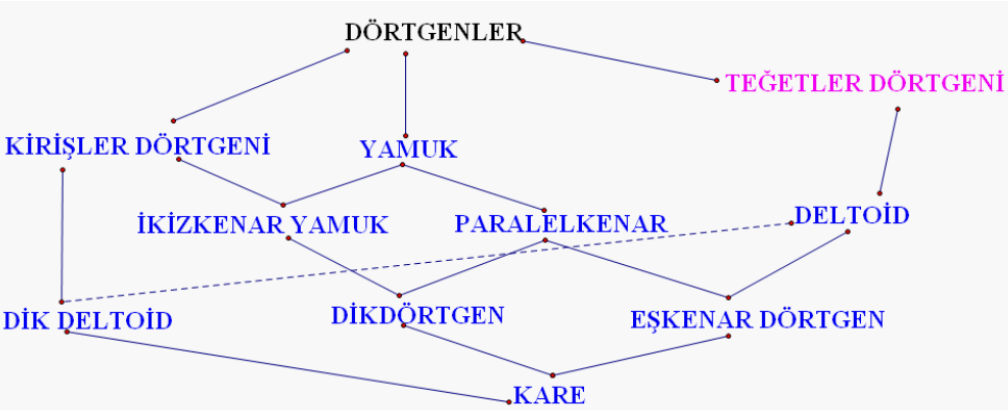


Figure 4.74 Participant 3's hierarchy diagram of quadrilaterals including circum quadrilaterals

4.3.3.4 Session 4 with Participant 3: New Quadrilaterals in the Hierarchy

At the beginning of the session, the participant correctly explained the inclusive relations in the hierarchy diagram which she constructed throughout the study. When she was asked about the inclusive relations between the quadrilaterals and between their properties, she confused. Although she correctly stated that square had many more properties due to being more special than the rectangle, she argued that the properties of rectangle included the properties of square. Here, she could not grasp the opposite inclusive relation between the quadrilaterals and between their properties; she thought that a rectangle included square and so the properties of rectangle included the properties of square. However, when I asked her to show this in terms of set-subset relationship, she stated that the set of properties of square would include the properties of rectangle as subset. She was able to explain that there was an inverse relationship between the quadrilaterals and between their properties; because the more general the quadrilateral was, the less properties it had.

In the next step, she easily was able to generalize the definition of kite in order to define a quad1 which was a more general concept than the kite including all kite class. She grasped the idea that she needed to reduce the properties used in the kite definition to define a more general concept.

> *"A quad1 is a quadrilateral with at least one pair of congruent adjacent sides."*

Having defined the quad1, the participant also drew the all corresponding quadrilaterals as well. Although it was known that all kite class was the special examples of this definition, she also constructed them in addition to the ones not included in the kite class (Figure 4.75).

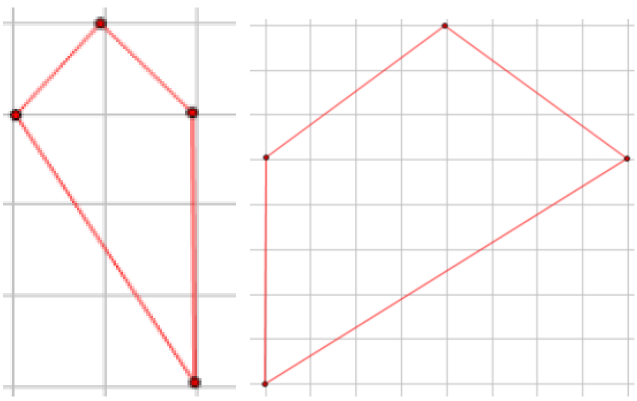


Figure 4.75 Participant 3's drawings of "quad 1"

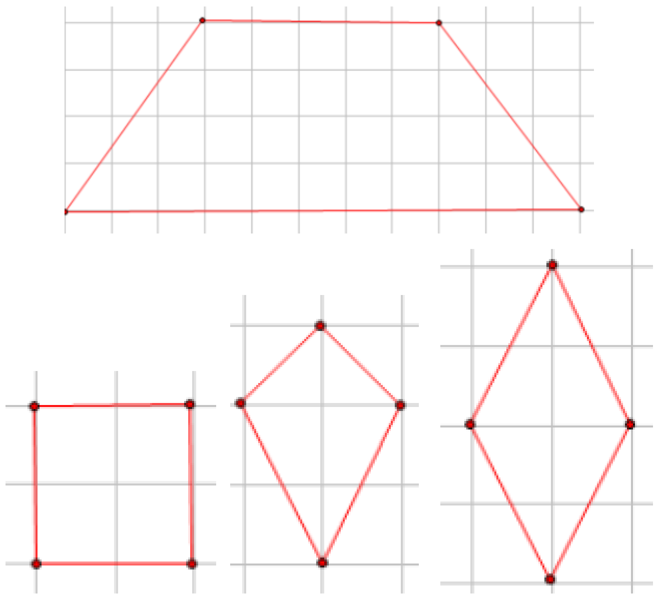


Figure 4.75 (continued)

In the next step the participant was asked to define a quad2 which was a special case of both quad1 and trapezoid. She correctly defined in terms of quadrilateral by using the defining properties of both trapezoid and quad1.

> *"A quad2 is a quadrilateral with at least one pair of congruent adjacent sides and with at least one pair of parallel sides."*

Then she defined quad2 in terms of trapezoid by using the defining property of quad1 thinking that the properties of trapezoid had already been added to the definition since it was defined as a trapezoid.

> *"A quad2 is a trapezoid with at least one pair of congruent adjacent sides."*

And finally the participant defined quad2 as a quad1 as

> *"A quad2 is a quad1 with at least one pair of parallel sides."*

After the definition process she also correctly drew the all possible quadrilaterals which were quad 2s according to this definition (Figure 4.76) and she indicated the special cases of quad 2 in the hierarchy diagram (Figure 4.77).

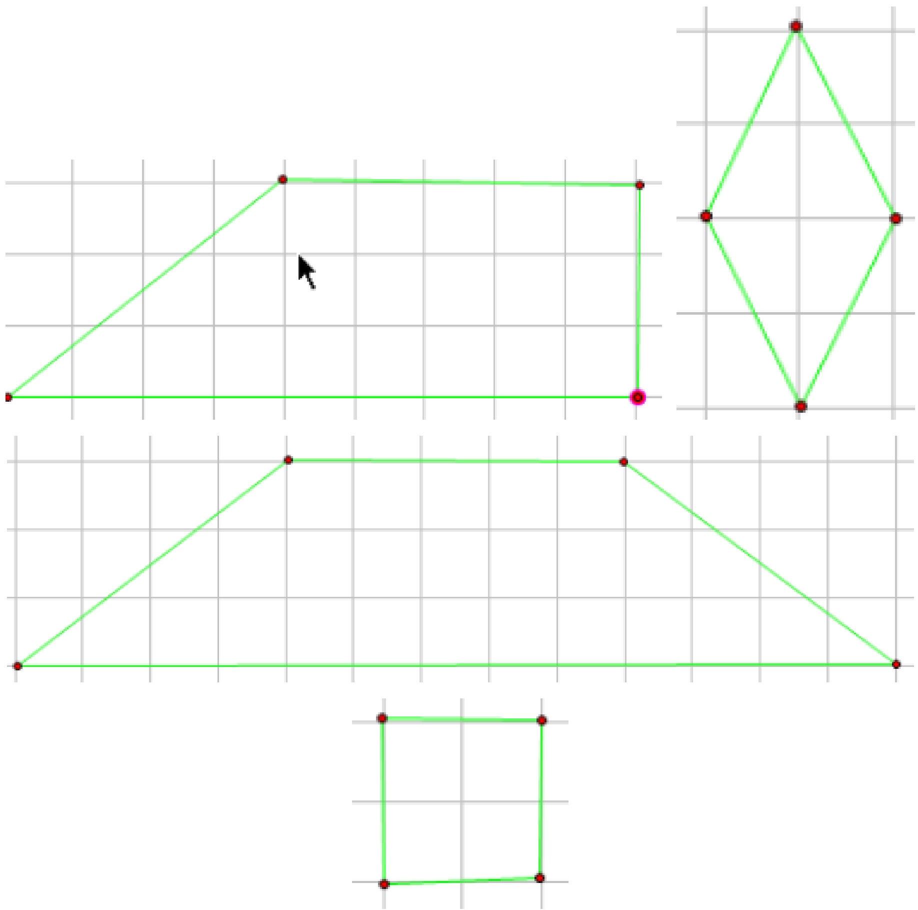


Figure 4.76 Participant 3's drawings of "quad 2"

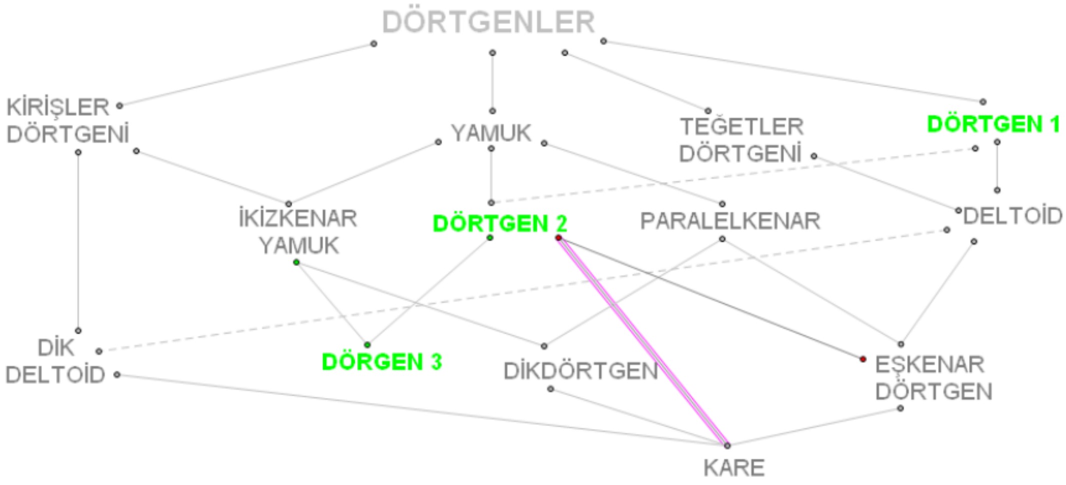


Figure 4.77 Special case of "quad2"

The last quadrilateral to be defined was quad3 which was a special case of both quad2 and isosceles trapezoid. The participant constructed all definitions quite easily considering that when it was defined in terms of a more general quadrilateral all properties of it was directly included in the definition, so there was no need to add the properties of it again. Her definitions followed as:

> *"A quad3 is an isosceles trapezoid with at least one pair of congruent adjacent sides."*
>
> *"A quad3 is a quad2 with opposite supplementary angles."*
>
> *"A quad3 is a cyclic quadrilateral with at least 3 congruent adjacent sides.*

Next, she sketched the quadrilaterals which satisfied the properties of a quad3. However, she also sketched a rhombus as an example of quad3 although it did not have opposite supplementary angles to be a cyclic quadrilateral. When asked, she stated that a rhombus was not also an isosceles trapezoid due to not having two pairs of congruent adjacent angles. So, she was able to only draw a square and an isosceles trapezoid with 3 adjacent congruent sides as examples of quad3 (Figure 4.78).

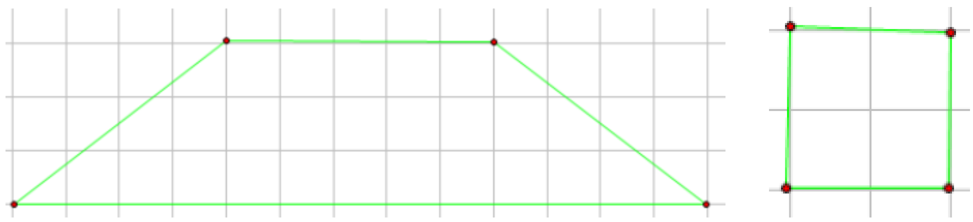


Figure 4.78 Participant 3's drawings of "quad 3"

When she was asked whether the definition *"a trapezoid with 3 congruent adjacent sides"* would be the definition of quad3, the participant correctly judged that this definition would include rhombus and a rhombus was not a quad3 due to not being a cyclic quadrilateral. Moreover, she stated that if the definition was changed as *"an isosceles trapezoid with 3 congruent adjacent sides,"* rhombus would be eliminated from the definition and so the definition would be the correct definition of quad3.

Finally, the participant indicated quad3 and its special cases on the hierarchy diagram (Figure 4.79).

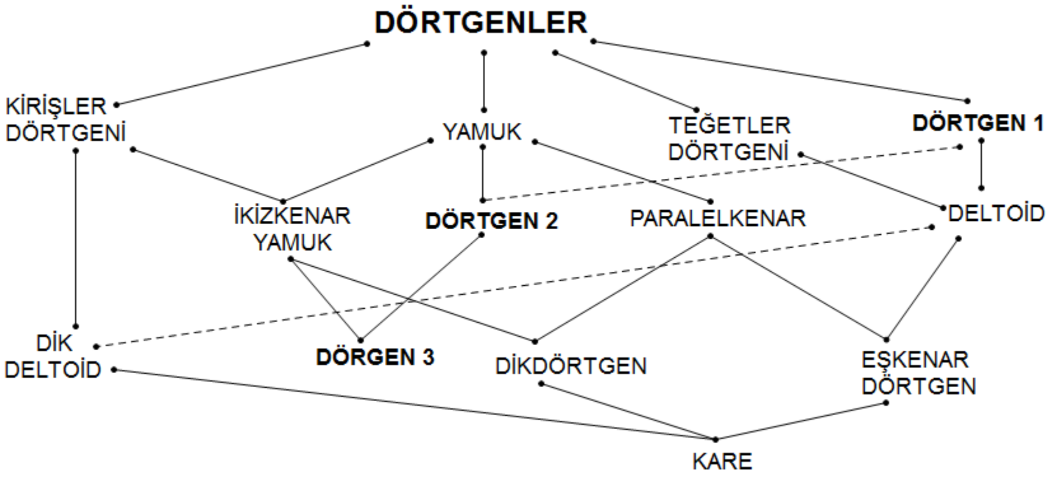


Figure 4.79 Participant 3's hierarchy diagram including "quad1," "quad2," and "quad3"

4.3.4 Participant 3's Opinions about the Quadrilateral Learning Experience in the GSP Learning Environment

The participant found the GSP assisted learning very effective and very enjoyable at the same time. She said that she learned many new things about definitions that she could not imagine to learn by the help of GSP. According to her, the most effective property of the GSP was that the defining properties of the figures preserved under dragging into the other figures, which enabled to identify the defining properties and the inclusive relations. She stated that she only experienced a little bit technical difficulty while arranging the measures of the circum quadrilateral figure.

She said that before this study, she had not thought on the cyclic and circum quadrilaterals so deeply and she had not also thought about the hierarchical relations between quadrilaterals. She also added that she had higher self-confidence than she had at the beginning of the study and believed that she could construct good definitions by identifying the defining properties.

After the study, she believed that the definitions could be effective teaching tools if the students were encouraged to actively take part in the definition construction and evaluation process. She added also that although she had disregarded the GSP before the study, she wanted to use it to teach definitions in her in-service teaching.

4.3.5 Participant 3's Understanding of the Nature of the Quadrilateral Definitions and the Hierarchies after the Clinical Interviews

The participant was able to write a correct economical definition of trapezoid as "*a quadrilateral with at least one pair of parallel sides."* She defined the isosceles trapezoid as *"a quadrilateral with at least one pair of parallel sides and with at least one pair of equal opposite sides."* However, this definition also included rhombus which was not an isosceles trapezoid, so it did not define only the isosceles trapezoids. Moreover, the participant also used the term "equal" instead of the correct term "congruent," which indicated that the participant still had problems with the correct terminology. The participant also wrote a correct but uneconomical definition of a rectangle as *"a quadrilateral with two pairs of equal side lengths and with all right angles."* At that time she used the correct term "equal" for the side lengths. However, the second condition was the sufficient one to characterize a rectangle; because the most general quadrilateral having all right angles would be the rectangle; and this property automatically satisfied the "two pairs of equal side lengths property. Therefore the definition could be economized as "a rectangle is a quadrilateral with all right angles." Moreover, the participant correctly identified all figures among the given group that were the examples of the given definitions.

She was able to construct two alternative correct but uneconomical definitions of rhombus as *"quadrilateral with all equal sides and with two pairs of equal opposite angles"* and as *"quadrilateral with all equal sides and bisecting diagonals."* First of all, in both of these definitions, participant used the term "equal" although the sides that had equal side lengths could be congruent. Moreover, the first common condition in both of these definitions would be sufficient to generalize a rhombus; so the remaining information was unnecessary.

On the other hand, the participant was very good at identifying the necessary and sufficient defining conditions of the quadrilaterals; however her answers were lacking the deficient property when it was the case that given property was not sufficient. For instance, she knew that having congruent diagonals was a necessary but not sufficient condition for a quadrilateral to be a rectangle; an isosceles trapezoid also had congruent diagonals, but it was not a rectangle. Although she knew the underlying reason she did not add the information that a quadrilateral with congruent diagonals could be a rectangle if its diagonals were also bisecting each other. Moreover, she identified that a quadrilateral with one pair of parallel sides and the other pair congruent could not always be a parallelogram; it could also define an isosceles trapezoid. However, she did not explained that to define a parallelogram there was need to say that it could have at least one pair of congruent and parallel sides or two pairs of parallel sides.

Unfortunately, she was very bad at constructing inclusive and exclusive definitions for the kite class, but I think she did not understand the question since she answered considering these group of given figures as a whole quadrilateral set. For instance, she defined the group including the kite and rhombus but excluding the square from them as "*quadrilaterals of which all angle measures are not 90°.*" This was an incorrect definition because there were many other quadrilaterals of which all angle measures were not 90°. That is, she did not include the properties of the quadrilaterals that were being defined. Similarly, she defined prototypical kite figure by excluding the rhombus and square as special cases as "*quadrilaterals of which all side lengths were not equal to each other;*" however, there were many other quadrilaterals which were not kites but did not have all congruent sides. That is, she did not include the defining property of kite into the definition.

When she was asked to construct a definition including all the given group of figures, the participant tried to make a very general definition, namely she tried to define all quadrilateral set as *"closed figures with 4 sides;"* Even, this was not a clear statement to define a quadrilateral; because it was lacking some information like whether the sides would be linear or whether the sides would be in the same plane etc.

Moreover, She perfectly identified the inclusive relations among the quadrilaterals considering their different properties. For example, she knew that a deltoid could be a cyclic quadrilateral if and only if it was a right kite or she knew that a square was always a cyclic quadrilateral and a kite as well. However, she totally failed to complete the hierarchy on the diagram

4.4 Participant 4's Analysis Results

In the following sections, findings related to the Participant 4's perceptions of the definitions and understanding of the quadrilateral definitions and the hierarchies before engaging them into the clinical interview sessions, her mental process and progress during the 4 clinical interview sessions, opinions about her experience in this study and the findings related to her understanding of the quadrilateral definitions and the hierarchies after the clinical interview sessions were stated.

4.4.1 Participant 4's Initial Perceptions of the Definitions

As usual, this participant also complained that the definitions were ignored during her elementary and secondary school years till she started to think on the definitions at the university level. She believed that the definition of a concept is the most important tool for conceptual understanding, since it was the basis of every activity related to this concept.

Having been asked whether being able to define a concept implied that the concept was learned, participant 4 asserted that it did not mean that concept was learned, because it might have been just memorized without understanding. According to her, we could understand whether the concept was learned through engaging students with different and contradictory examples of the definitions so that they could criticize their correctness.

The participant 4 also accepted that she was not good at definition construction since she was not encouraged to realize the importance of definitions for a long time, till the university years. That is, she had a low self-confidence for her defining skill at the beginning of the study. When she was asked how she could use definitions in her teaching with her current ability level of defining, she stated that she would try to find the best pre-constructed definitions from the textbooks and would list all the related properties. She also added that she would check the definition considering its appropriateness to the students' level and if it was not appropriate she would make some changes. She also believed that the definitions would be important teaching tools if they were used effectively through good examples.

According to the participant 4, a good definition should not be too long and it should be as basic as possible. She believed that the definition should not have included more than necessary information; the extra information could be inferred from the definition. Different from the other participants, this participant was the only one who thought that a definition should only include the necessary defining properties, instead of a list of all properties.

4.4.2 Participants 4's Initial Understanding of the Nature of the Quadrilateral Definitions and the Hierarchies

The participant 4 defined rhombus as *"a closed shape of which all 4 sides are equal"* First of all, the participant used the term "equal" incorrectly; because sides can be congruent when their lengths are equal. Secondly, and most importantly, she did not define rhombus in terms of quadrilateral, but defined in terms of a closed shape, which made her definition an incorrect one. On the other hand, if she had constructed the same definition saying that "rhombus is a quadrilateral of which all 4 side lengths are equal," this would be the correct economical definition of rhombus including minimum sufficient property to define a rhombus. However, her using the "closed shape" instead of "quadrilateral" was a failure in terms of a definition constructon process in which the use of mathematical terms had crucial importance to give the exacat meaning to characterize that concept. She also defined the following quadrilaterals in terms of a closed shape and her definitions in terms of closed shape were accepted as incorrect; however, I also analyzed them thinking that she defined them in terms of "quadrilateral"

The participant defined rectangle as *"a closed shape with equal opposite sides and with all angles are 90°."* The same critics are true also for his definition; she used the term "equal" incorrectly and defined in terms of "closed shape." Even if we assume that she meant that "a rhombus is a quadrilateral with congruent opposite sides and with all angles 90°," this would be an uneconomical definition since "a quadrilateral with all angles 90°" would be enough to define a rectangle.

Finally, she defined square in such a way that "a closed shape with all sides congruent and with all angles 90°" If we think that she defined in terms of a quadrilateral this would be accepted as a correct economical definition, even better, "at least one angle measure is 90°" would be enough instead of "all angles 90°."

Although her definitions were lacking some criteria, she was able to identify the square as a rhombus and a rectangle which indicated that she was aware of the inclusive relations among these quadrilaterals. However, she incorrectly identified the figure "s" as a rectangle though it was not, as in the case of participant 1.

It was also seen that the participant was not successful in identifying necessary and sufficient properties to construct more than one definition for the same concept. Her two alternative kite definitions were the following:

"a quadrilateral of which adjacent sides are equal and the angles formed by the merge of unequal sides are congruent."

"a quadrilateral with one pair of opposite equal angles, with the angle bisector diagonal passing through the unequal angles and with perpendicular diagonals."

First of all, in both of the definitions she used the term "equal" inappropriately. In the first definition it was not clear how many adjacent sides were congruent, so one could understand that all sides are congruent. However, in the second condition, she used the term "unequal sides" which meant that not all sides would be congruent. In this case, she should have clearly stated that "two pairs of congruent adjacent sides" for the first property. Moreover, her description of the one pair of opposite congruent angle was a little bit complicated for a definition. Even if we assume that she corrected her definition as "a quadrilateral with two pairs of congruent adjacent sides and with one pair of congruent opposite angles," this definition again would be uneconomical; because just the property of "two pairs of congruent adjacent sides" is a sufficient property to characterize a kite.

In the second definition, she used much more redundant information which made the definition a description. Just saying for example "a quadrilateral with one diagonal as an angle bisector of the one pair of opposite angles" would be enough to define a kite; because this property would make this diagonal also a symmetry line and thus satisfies the "two pairs of congruent sides" and "one pair of opposite congruent angles" properties.

It was determined that the participant 4 was generally good at evaluating the correctness of the given rhombus definitions in terms of critical defining properties. For example, she stated that having two pairs of congruent adjacent sides would not be a sufficient property to define a rhombus, because there would need to know also that all sides were congruent. Moreover, she stated also that having parallel opposite sides would not be enough to define a rhombus. However, she did not accepted the definition that "a rhombus is a quadrilateral of which symmetry axes are the perpendicular lines passing through the opposite vertices" as a correct one although it defined a rhombus. She reasoned incorrectly that there were rhombuses of which symmetry axes were not perpendicular; but all rhombuses including square have perpendicular diagonals as their symmetry lines.

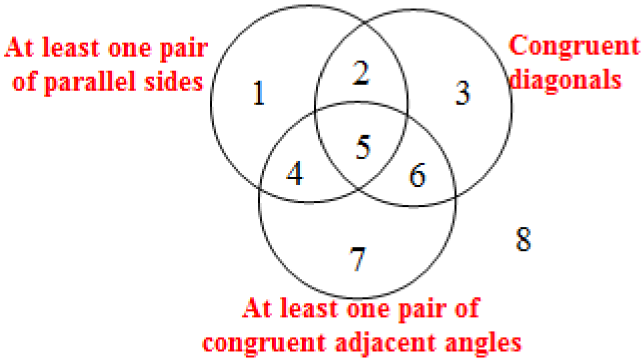
When compared to the others, this participant was the most unsuccessful in constructing inclusive and exclusive definitions. She defined a and b by excluding them from the others as *"quadrilaterals with one pair of parallel and equal opposite sides,"* but this was an incorrect definition since neither a nor b has congruent opposite sides. If she defined this group as "quadrilaterals with only one pair of parallel sides" this would be enough to define them.

She also constructed an incorrect definition to define quadrilaterals a, b, c, and d in such a way that *"quadrilaterals with at most one pair of equal opposite sides and with at least one pair of parallel sides."* First of all, she again made a mistake by using the term "equal" instead of "congruent" Moreover, it is seen from the figures that a and b even do not have any congruent opposite sides and c and d have two pairs of congruent opposite sides, which refutes her definition.

Finally, she defined all figures as *"quadrilaterals with at most 2 pairs of equal opposite sides and with at least one pair of parallel opposite sides."* It is seen that the first condition does not make any sense since a quadrilateral can not have more than two pairs of sides. However, when the first condition is ignored, the second condition would be enough to define a, b, c, d, e and f all together.

She was very good at understanding the inclusive relations between the quadrilaterals through considering their properties. She was able answer all questions correctly which indicated that she did not have a prototypical image of rhombus, rectangle and parallelogram and accepts square as a special case of all these quadrilaterals. However, she said that the diagonals of a parallelogram become perpendicular and bisect each other if and only if all the angles are 90°. Here there was a mistake in her reasoning such that the cases of a parallelogram having all 90° angles are rectangle and square, but the diagonals of rectangle are not perpendicular. So, the only case that satisfies this condition would be the square.

When compared to the others, the participant 4 did better in this question; but she had some mistakes especially in the first diagram (Figure 4.80). Her answers indicated that she had misleading information that a parallelogram had at least one pair of congruent adjacent angles. Similarly, she mislead that a rhombus had congruent diagonals and at least one pair of congruent adjacent angles. Moreover, her answers indicated that she did not know that an isosceles trapezoid had congruent diagonals.



Participant's Answers

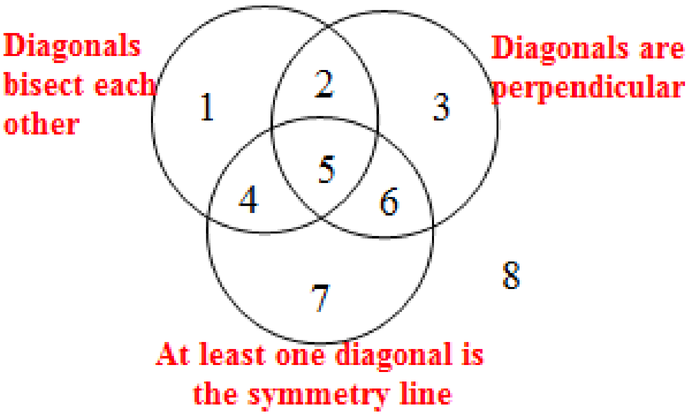
Parallelogram	**4**	Trapezoid	**1**
Kite	**8**	Isosceles Trapezoid	**4**
Square	**5**	Rectangle	**5**
Rhombus	**5**		

Correct Answers

Parallelogram	**1**	Trapezoid	**1**
Kite	**8**	Isosceles Trapezoid	**5**
Square	**5**	Rectangle	**5**
Rhombus	**1**		

Figure 4.80 Participant 4's first diagram of the classification of the quadrilaterals

She better performed in the second diagram and only placed the rhombus incorrectly into the region 1 instead of the correct region 5 (Figure 4.81). This indicated that she did not know that a rhombus had perpendicular diagonals both of which were symmetry axes.



Participant's Answers

Parallelogram	**1**	Trapezoid	**8**
Kite	**6**	Isosceles Trapezoid	**8**
Square	**5**	Rectangle	**1**
Rhombus	**1**		

Correct Answers

Parallelogram	**1**	Trapezoid	**8**
Kite	**6**	Isosceles Trapezoid	**8**
Square	**5**	Rectangle	**1**
Rhombus	**5**		

Figure 4.81 Participant 4's second diagram of the classification of the quadrilaterals

4.4.3 Clinical Interviews

Participant 4's cognitive progress during the clinical interview sessions and the effect of the GSP activities on the participant 4's cognitive improvement in understanding the quadrilaterals through definitions construction and classification processes were described in detail in the following subsections.

4.4.3.1 Session 1 with Participant 4: Kite, Rhombus and Square

When the participant was asked to remember the properties of a kite verbally at the beginning of the session, she was able to remember almost all of the properties correctly. She knew that at least one pair of opposite angle measures were equal, adjacent side lengths were equal, diagonals were perpendicular and one diagonal bisected the other diagonal; one diagonal was the angle bisector.

When she was asked how she would define kite to her students, like other participants she also said "A kite is a quadrilateral constructed by sticking the two isosceles triangles from their bases" which was actually the correct economical definition of kite. Using the properties of isosceles triangles, she also explained how all other properties of the kite could automatically be inferred from this definition.

Then, the participant was asked to detect the preserved side, angle, diagonal and symmetry measurements of kite under the dragging of dynamic kite figure. She stated the preserved properties as "a kite has two pairs of equal adjacent sides, perpendicular diagonals, one diagonal is bisected by the other, one pair of opposite angles are equal, one diagonal is the angle bisector and this diagonal is the symmetry line at the same time." From her statements it can be inferred that she did not know the difference between the terms "congruent" and "equal." Except for this incorrect term use, she easily identified all preserved critical defining properties of kite correctly.

After identifying the properties of kite, I asked the participant her opinions about which other quadrilaterals could be special kites having all critical properties of a kite. She correctly thought that square was a special kite since it is made up of two isosceles triangles, which satisfied the side property and hereby all other defining properties of kite. She directly eliminated rectangle and parallelogram since they were not formed of two isosceles triangles and so did not satisfy the side property. Moreover, she said they also do not have perpendicular diagonals, too. When it came to think of the rhombus, she stated that a rhombus was a special parallelogram, but incorrectly stated that a rhombus could not be a special kite because its diagonals were not perpendicular. As for trapezoid and isosceles trapezoid she correctly explained that they could not be special kites since they did not have congruent adjacent sides. That is to say, the participant was able to identify only square as a special kite but she could not identified rhombus due to her misinformation about the diagonal property of rhombus.

Now it was time to test her ideas on the dynamic figure. She dragged the kite figure into other special quadrilaterals one by one to detect out which of them preserved the critical defining properties of kite that would make them special kites. As a result of her dragging activity, she observed that the dynamic kite figure could be dragged into square and rhombus, but not into the other quadrilaterals. She had already been sure about square but she had been wrong about rhombus. Having made a rhombus under dragging the kite figure, she wanted to measure the angle between diagonals and understood that they were also perpendicular; as a result, she found out that square and rhombus were special kites. As a last step, she indicated these relationships between kite, rhombus and square on the hierarchy diagram correctly.

The next step was to construct an inclusive definition for the kite so that the definition would include the square and rhombus as special kites. Her definition was

> *"A kite is a quadrilateral with congruent adjacent sides and with at least one pair of opposite congruent angles."*

She used the term "congruent" appropriately, but in the first property it was not clear how many adjacent sides had to be congruent, all of them, just one pair or two pairs. When I asked her clarify this information, she corrected the definition as

> *"A kite is a quadrilateral with two pairs of congruent adjacent sides and with at least one pair of opposite congruent angles."*

This definition was including more than the necessary information. So I asked the participant why she had used both properties in the same definition. She thought aloud like that:

> *"Hmmm… why I used both properties…first of all I have to say the two pairs of congruent adjacent sides property since the kite is made up of two isosceles triangles. I need this information… hmmmm…… Actually I think, I don't have to say the angle property, because it can be inferred from the first property…Yes, I can remove the second property. The first property is enough…"*

Then, I encouraged her to think of whether any other quadrilateral except for the kites would satisfy the reduced definition which was *"a kite is a quadrilateral with two pairs of congruent adjacent sides."* She tried to find any counter example which was not a kite but satisfied this definition, but she could not find and decided that this would make a correct economical inclusive definition of kite. To make sure that the information given in the definition was sufficient, I asked her to make the corresponding construction, too.

At first, she did not know how to construct the first adjacent congruent pair. Instead of a construction, she tried to make a drawing by connecting two segments; but the segments were even not congruent pairs. When I encouraged her to think about the GSP tools, she found out that she could construct the two pairs as the radiuses of a circle. Then, I asked her where the fourth vertex should be placed so that its distance to the end points of the first pair would be equal. She easily thought that the fourth vertex must lie on the angle bisector of the angle formed by the first pair. She constructed the angle bisector, took any point on it as the fourth vertex and completed the figure into a quadrilateral (Figure 4.82).

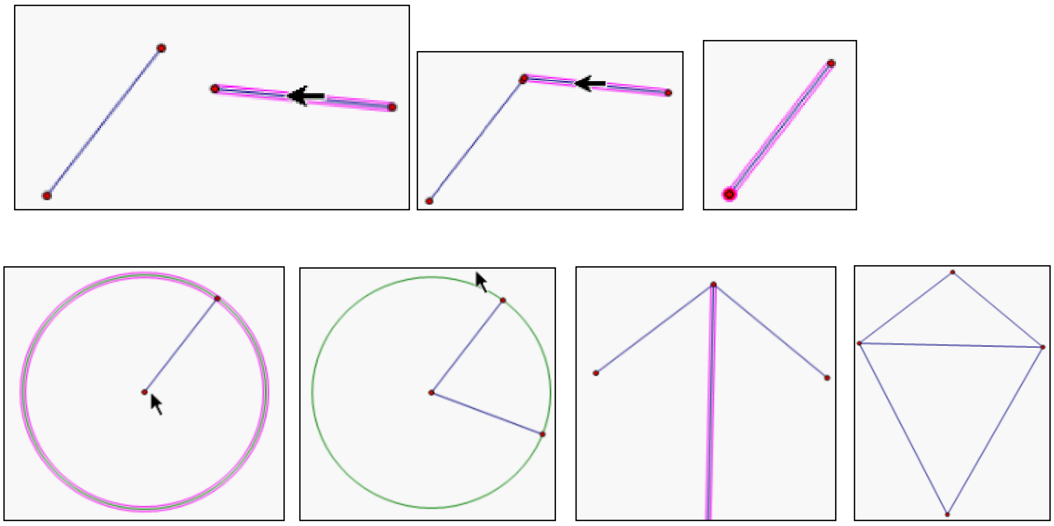


Figure 4.82 Participant 4's construction of a quadrilateral with two pairs of congruent adjacent sides.

After the construction, she made the related measurements and observed whether the figure preserved kite properties under dragging. She decided that the definition was a correct economical definition since it included the sufficient information to characterize a real kite figure.

In the next step, I asked her to evaluate the four pre-constructed kite definitions.

Definition 1: A kite is a quadrilateral with perpendicular diagonals.

She wanted to make a construction to see whether two perpendicular diagonals would be enough defining property to define a kite. She constructed any point on a segment and then constructed the perpendicular line through this point (Figure 4.83). Then she constructed any two points on the perpendicular line and combined the vertices with the segments. However, when she dragged the figure, she observed that it did not remained a kite and said that any ordinary quadrilateral could also have perpendicular diagonals. So, she decided that the definition did not include the sufficient information to define a kite.

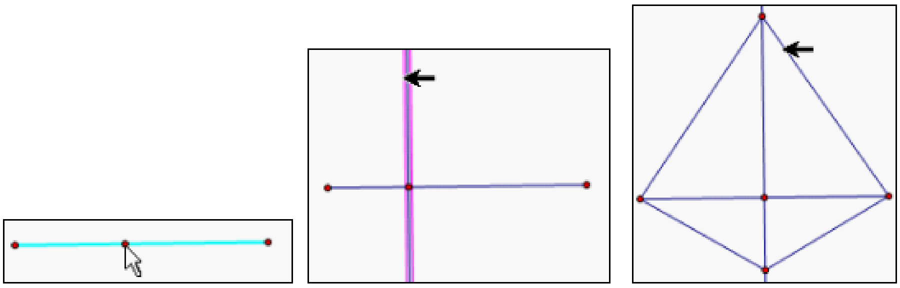


Figure 4.83 Participant 4's construction of a quadrilateral with perpendicular diagonals

Upon this, I encouraged her to think about which other property or properties could be added to make the information in the definition sufficient to define a kite. She easily stated that in addition to being perpendicular, one diagonal had to be bisected by the other diagonal. That is, she changed the definition as

"a kite is a quadrilateral with perpendicular diagonals where one diagonal is bisected by the other diagonal."

This was actually similar to the second definition I was about to ask her, but she founded it herself, so there was no need to discuss on the second definition. For good measure, she also constructed the diagonals corresponding to this information and saw that this information would make a real kite.

Definition 2: A kite is a quadrilateral with at least one diagonal is a perpendicular bisector.

We had already discussed the third definition as well and she had already known that the first property in this definition was enough defining property.

Definition 3: A kite is a quadrilateral with two pairs of congruent adjacent sides and one pair of congruent opposite angles.

The last definition was the following:

Definition 4: A kite is a quadrilateral with at least one diagonal is the symmetry axis.

For this definition she thought that when one diagonal was the symmetry line, there had to be two isosceles triangles combined through a common base which would satisfy the critical side property of kite. Moreover, she checked for the counter examples which might have satisfied this definition. However, she could not find any other special quadrilateral out of the kite class. I asked her to think of whether any ordinary quadrilateral could have this property other than the kites. She correctly explained that it was not possible to construct any other quadrilateral rather than the kites if one diagonal was a symmetry line; because when one side of the quadrilateral was infolded onto the other side throughout the symmetry diagonal, they had to cover each other. This was only possible when the parts of the quadrilateral on each side of the diagonal were congruent to each other. So, the definition was a correct inclusive definition including the minimal information from which all other necessary information could be inferred. She also confirmed her thinking through the related construction, as well (Figure 4.84).

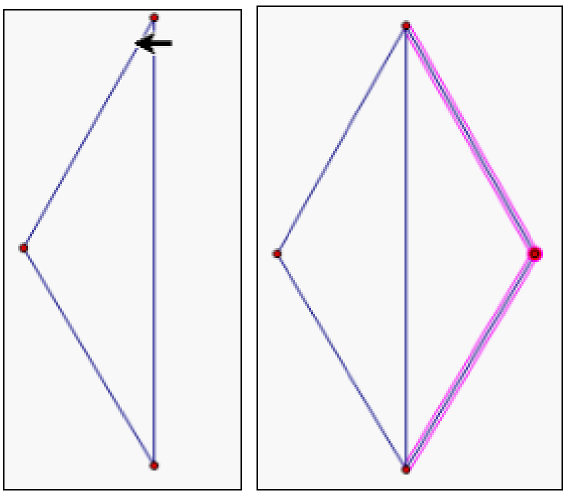


Figure 4.84 Participant 4's construction of a quadrilateral with at least one diagonal is the symmetry axis.

As a final step of this session, she easily was able to construct a correct economical definition as *"a rhombus is a quadrilateral with four congruent sides."*

4.4.3.2 Session 2 with Participant 4: Parallelograms and Trapezoids

When she was asked to remember the properties of isosceles trapezoid, the participant stated almost all properties correctly. She explained that an isosceles trapezoid had one pair of parallel sides and one pair of opposite congruent sides; moreover it had congruent base angles. In contrast to the other participants, she easily detected out that the diagonals intersected each other in the same ratio. All other participants had difficulty to express this property using the correct mathematical language. The participant also added that an isosceles trapezoid was formed up from two triangles or from one triangle and one parallelogram. After listing the properties she dragged the dynamic figure to detect out preserved critical defining properties of an isosceles trapezoid and detected all correctly without any problem.

Then, pushed her for thinking on which quadrilaterals had all critical properties of an isosceles trapezoid. She said that a square and a rectangle satisfied the properties of an isosceles trapezoid and explained correctly why. As for the parallelogram, she quickly realized that the angle property was not preserved, so it could not be a special isosceles trapezoid. This participant was better than all other participants to detect out inclusive relations. She just thought a little bit for the rhombus; and asked for a rhombus figure to remember its properties. After thinking a while, she said that similar to the parallelogram, rhombus did not preserve angle property which eliminated it from being an isosceles trapezoid. She also eliminated regular trapezoid since it did not have congruent sides; and kite since it did not have parallel sides. After thinking on the inclusive relationships, the participant also tested her ideas through dragging the dynamic isosceles trapezoid figure on other quadrilaterals and finally decided that rectangle and square were the two special isosceles trapezoids. She also constructed the hierarchy diagram correctly.

After discovering the inclusive relations between isosceles trapezoid and other quadrilaterals, I asked the participant to define isosceles trapezoid so that the definition would include special instances of it. She defined it as

"An isosceles trapezoid is a quadrilateral with at least one pair of parallel sides and with at least one pair of congruent sides."

However, this was an incorrect definition since it included quadrilaterals other than the isosceles trapezoid, such as rhombus. Moreover, it was not clear whether the congruent sides were the opposite or adjacent sides. On the other hand, her using the term "at least" was an indication of thinking rectangle and square as special cases. When asked why she had used this term, she explained that rectangle and square had two pairs of parallel and opposite sides, so there was need to add "at least." Then, I asked participant to think of counter examples to refute the definition and she quickly found out parallelogram. Upon this refutation she redefined as

"An isosceles trapezoid is a quadrilateral with at least one pair of parallel sides, with at least one pair of congruent sides and with two pairs of congruent adjacent angles."

However, the definition turned into a description where several properties were listed. So I encouraged her to think on whether each property was necessary to define an isosceles trapezoid. She focused on the last property and thought that having two pairs of congruent adjacent angles would be sufficient defining property. She explained her thinking process as the following:

"First, I constructed a base segment in my mind…then I thought two segments at the end points of the base segment so that they would make the same angle in different directions…then I constructed another base segment on the top so that it would make the same angle with the two side segments and so that the side segments would be congruent…However, the sum of the internal angles had to be 360 degrees for this figure to be a closed quadrilateral... I would sum up the measures of the bottom base angles and subtract this sum from 360 and then would divide the result into two… then, I would construct the top segment so that the congruent angle pair at the top would have the measure I calculated. Let's say the top pair of congruent angles are 60 degrees each…60 plus 60 sum up to 120 degrees. So the congruent pair at the bottom would be 120 degrees each… In this situation the angle pairs measured 60 and 120 complemented each other to 180 degrees and this indicated that top and bottom bases are parallel to each other…

That is to say, she tried to explain that if a quadrilateral would have two pairs of congruent adjacent angles, all other properties of an isosceles trapezoid would be met; that is angle property would be the sufficient defining property from which all other properties could be infered. Although she could find out the defining property among the others with visualization in her mind, I wanted her to speak on something more concrete and asked her to make a real sketch on GSP. After a few technical clues, she constructed two pairs of congruent adjacent angles and made related measurements to see whether isosceles trapezoid properties were preserved under dragging (Figure 4.85). She confirmed that she was right in her think aloud process in that this angle property was sufficient to infer all other properties. So she reduced and redefined an isosceles trapezoid as

> *"An isosceles trapezoid is a quadrilateral with two pairs of congruent adjacent angles."*

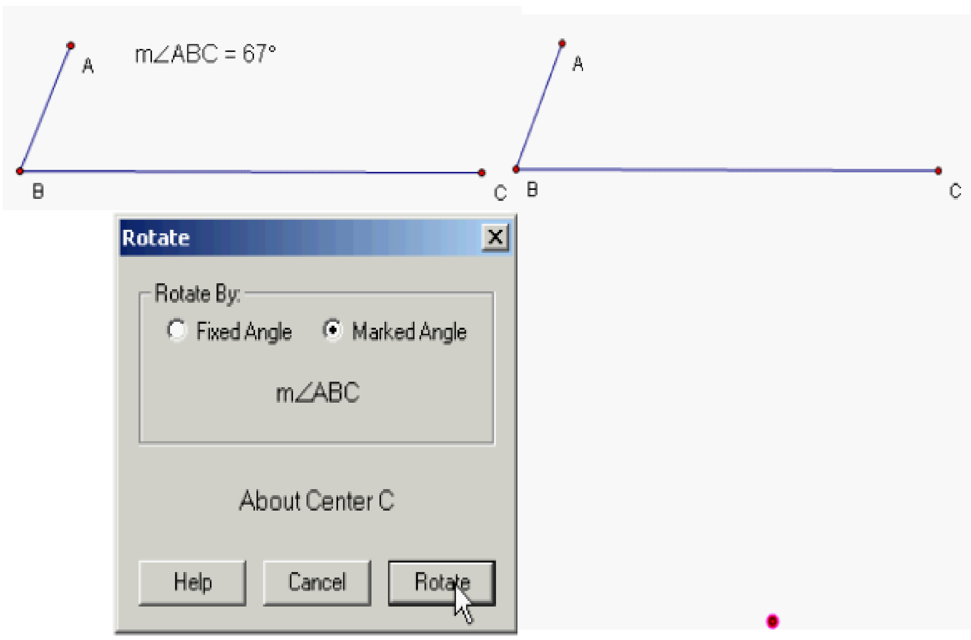


Figure 4.85 Participant 3's construction of a quadrilateral with two pairs of congruent adjacent angles

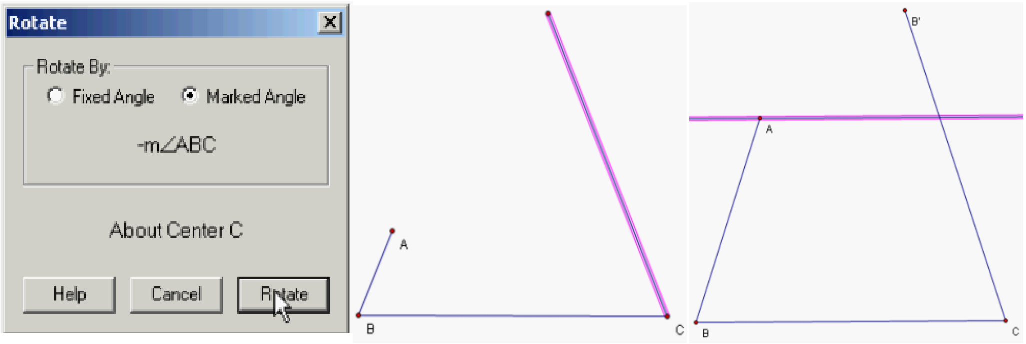


Figure 4.85 (continued)

In the next step I asked her to evaluate some definitions. Since we had already discussed the first definition I skipped it.

> *Definition 1: An isosceles trapezoid is a quadrilateral with at least one pair of parallel sides and with at least one pair of opposite congruent sides.*

For the second definition, she realized that the definition only included prototypical isosceles trapezoid but not its special instances, namely rectangle and square. She accepted this definition as correct definition if the purpose was to define only isosceles trapezoid without including its special instances into the definition.

> *Definition 2: An isosceles trapezoid is a quadrilateral with one pair of parallel sides and with one pair of congruent but unparallel sides.*

When she was asked to evaluate the third definition, she first thought aloud that it sounded like the definition of a parallelogram. When I asked her to show the opposite supplementary angles, she showed adjacent angles on the parallelogram figure. That is to say, she knew correctly which angles were supplementary in a parallelogram but she had a misunderstanding of the mathematical terms "opposite" and "adjacent." After making the differentiation between opposite and adjacent angles, she stated that the parallelogram did not satisfy this definition since it had adjacent but not supplementary opposite angles. After considering the properties of parallelogram, rhombus, trapezoid and kite, she decided that the only quadrilaterals that satisfied this definition were isosceles trapezoid, rectangle and square, so she considered this definition as a correct economical inclusive definition of isosceles trapezoid.

Definition 3: "An isosceles trapezoid is a quadrilateral with at least one pair of parallel sides and with opposite suppplementary angles."

For the fourth definition, she made the correct evaluation that it was correct economical definition. She stated that the definition included the rectangle and square as descendants and did not include other special quadrilaterals as a counter example.

Definition 4: An isosceles trapezoid is a quadrilateral with two pairs of congruent adjacent angles.

Finally, she evaluated the following definition.

Definition 5: "An isosceles trapezoid is a quadrilateral with congruent diagonals."

She tried to refute this definition with a counter example, but when she considered the properties of special quadrilaterals she could not find any example and accepted this definition as a correct definition. Upon her incorrect decision, I asked her to make a construction corresponding to the definition. At first, she tried to construct congruent diagonals as radiuses of a circle, but then realized that it would be easier to consider them as diameters (Figure 4.86).

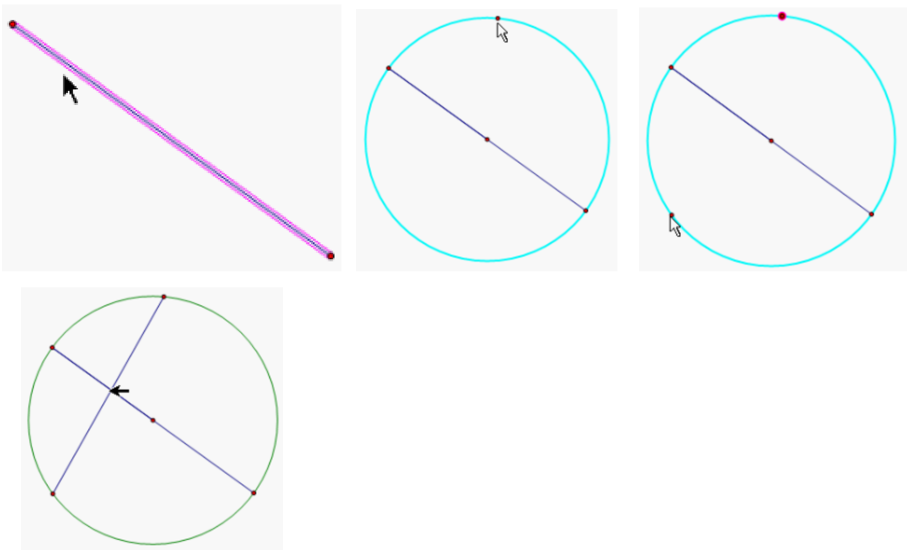


Figure 4.86 Participant 4's first attempt to construct a quadrilateral with congruent diagonals

In her first trial, she constructed the second diameter as a chord of which length was not always the same as the first diameter unless it was a diameter. After observing her mistake, the participant constructed any point on the diameter (the first diagonal) and then constructed another circle centered at this point with the same diameter of the first circle and constructed its diameter as well as the second diagonal. That is to say, she constructed two congruent line segments as the diagonals intersecting each other at any point, and completed the figure into a quadrilateral (Figure 4.87). However, when she dragged the figure and observed the properties, she saw that it did not remain an isosceles trapezoid and turned into any ordinary quadrilateral. While dragging the figure, she realized that it did not remain an isosceles trapezoid, because the diagonals also needed to intersect each other in the same ratio to make a figure an isosceles trapezoid.

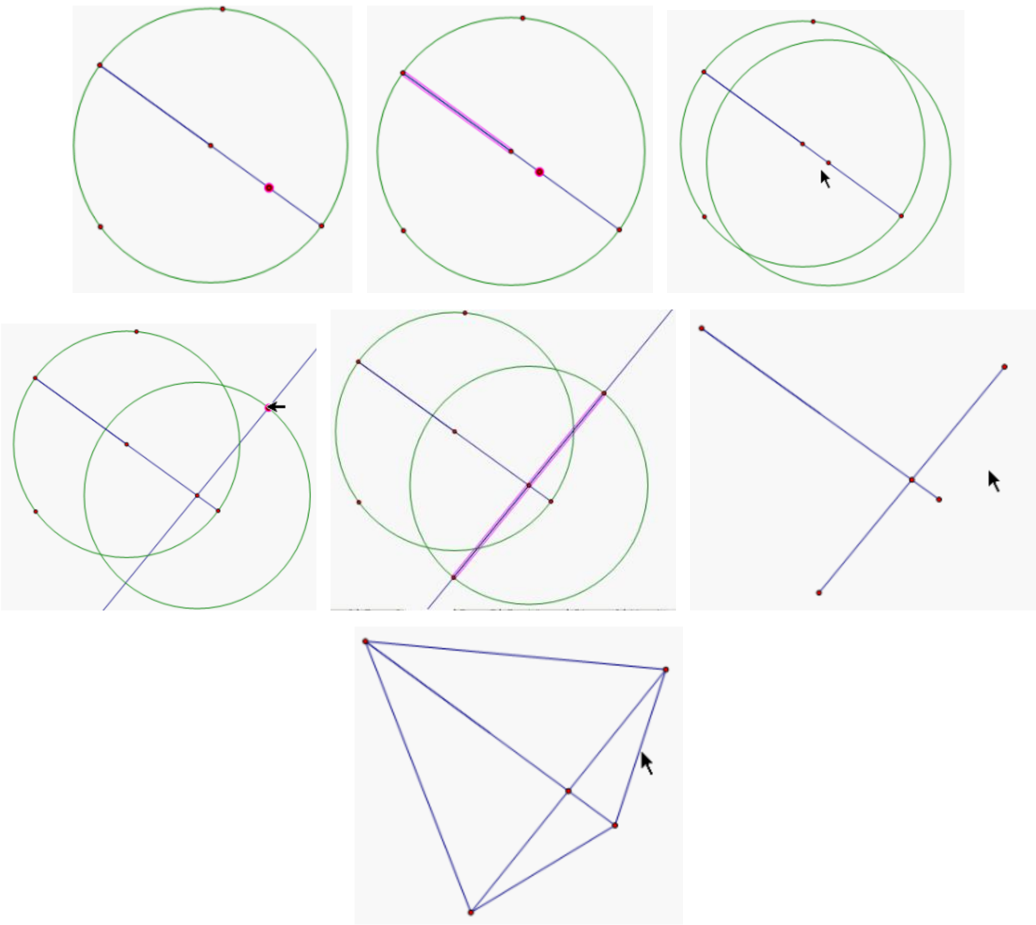


Figure 4.87 Participant 4's second attempt to construct a quadrilateral with congruent diagonals

So she corrected the insufficient definition as *"An isosceles trapezoid is a quadrilateral with congruent diagonals intersecting each other in the same ratio."* Moreover, she also constructed the quadrilateral corresponding to this reconstructed definition and confirmed the sufficiency of it.

After evaluating the given definitions, the participant was asked to detect out special parallelograms. She correctly stated that square, rhombus and rectangle were the special parallelograms and explained the reasons why they were, but not the others. However, when it came to show the relations on the hierarchy diagram, she failed (Figure 4.88). She indicated rectangle as a special rhombus.

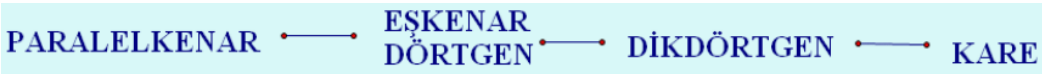


Figure 4.88 Participant 4's first attempt to construct hierarchy diagram of parallelograms

When I asked her whether the rectangle was a special rhombus, she correctly stated that it could not be a rhombus due to not having congruent adjacent sides. Upon this, she indicated a rhombus as a special rectangle (Figure 4.89). However, after thinking on the properties she realized that a rhombus did not have to have right angles as in the case of rectangle; so it could not be a special rectangle, too.

Figure 4.89 Participant 4's second attempt to construct hierarchy diagram of parallelograms

She mixed her mind and thought aloud the properties once more and realized that they were two different special cases of parallelogram. Finally, she constructed the correct hierarchy (Figure 4.90).

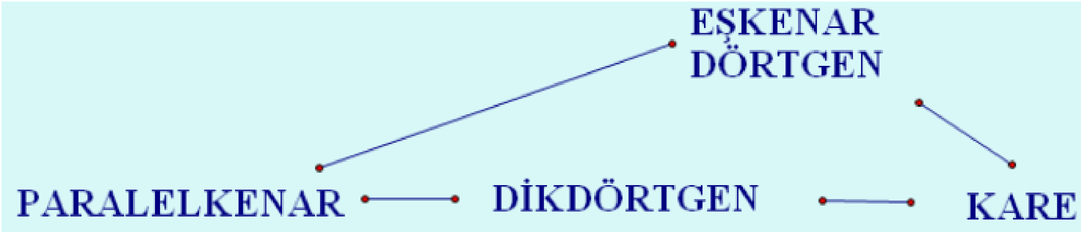


Figure 4.90 Participant 4's third attempt to construct hierarchy diagram of parallelograms

In the next step, the participant was asked to talk about the relationship between parallelograms and trapezoid. She said that both parallelograms and trapezoid had at least one pair of parallel sides. Then she reasoned that a parallelogram had one pair of parallel side which was the only property required to be a trapezoid; but a trapezoid did not have complementary adjacent angles like a parallelogram had. That is, she decided that parallelograms were the special trapezoids. When asked, she also stated that isosceles trapezoids were the special trapezoids in addition to the parallelograms. Then, she correctly defined a trapezoid inclusively as *"a quadrilateral with at least one pair of parallel sides."*

When it came to construct the hierarchy indicating all relationships discovered so far, she confused as expected, but after a few trial she was able to form the correct diagram (Figure 4.91).

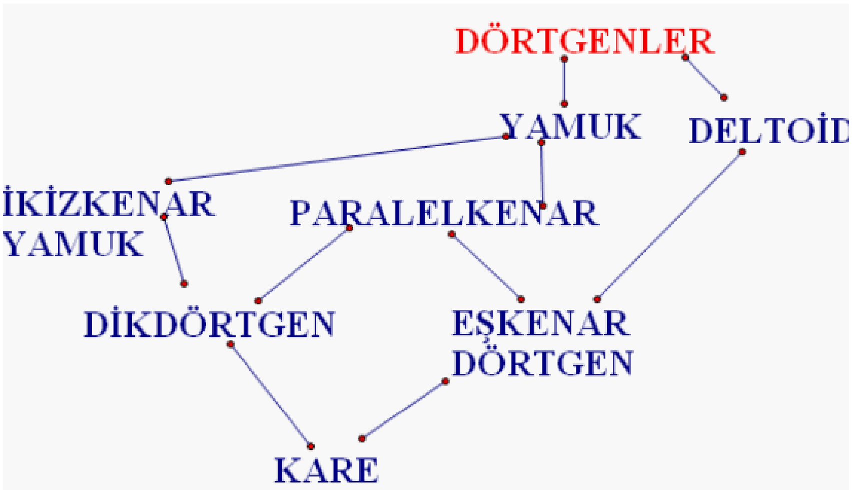


Figure 4.91 Participant 4's hierarchy diagram of quadrilaterals

When she was asked how the hierarchy would change if the trapezoid was defined as *"a quadrilateral with exactly one pair of parallel sides,"* she correctly thought that parallelograms would not be the special trapezoids, but isosceles trapezoids would remain as special trapezoids.

In the next step, the participant was asked to define *parallelogram* exclusively by using its *diagonal* property. She had no difficulty to construct the definition as *"A parallelogram is a quadrilateral with bisecting but not congruent and perpendicular diagonals."*

Then, she initially defined *rhombus* exclusively using the *symmetry* property as *"A rhombus is a quadrilateral which has only the diagonal symmetry."* However, she realized that she needed to specify the definitional property more, because a kite also had only the diagonal symmetry. So she redefined it as *"A rhombus is a quadrilateral which is symmetric with respect to the both diagonals."* However, this time she detected that square was also symmetric with respect to the both diagonals and said that she could eliminate this misunderstanding by adding the word "only" to the definition. Then, she constructed the final exclusive rhombus definition as *"A rhombus is a quadrilateral which is symmetric only with respect to the both diagonals."* She stated that she understood the importance of even one word on the meaning of a definition.

Finally, the participant was asked to define a *kite* exclusively using *any* property she liked. She first defined it as *"a kite is a quadrilateral with only one pair of congruent opposite angles"* thinking that square and rhombus were excluded. She also thought about the properties of other quadrilaterals, but did not think that any ordinary quadrilateral could also have exactly one pair of congruent opposite angles. So I asked her to think on whether any ordinary quadrilateral could have only one pair of congruent opposite angles. She thought this situation by working on a kite figure on the screen (Figure 4.92). She stated that any quadrilateral is formed from two triangles and so did kite. Then, she tried to construct opposite congruent angles on it by carrying the sides, and realized that any ordinary quadrilateral could have this property. So she decided that the definition did not only included prototype kite, but also any ordinary quadrilateral. Therefore, she added that the definition lacked the information that one diagonal had to be the angle bisector which would differ it from an ordinary quadrilateral.

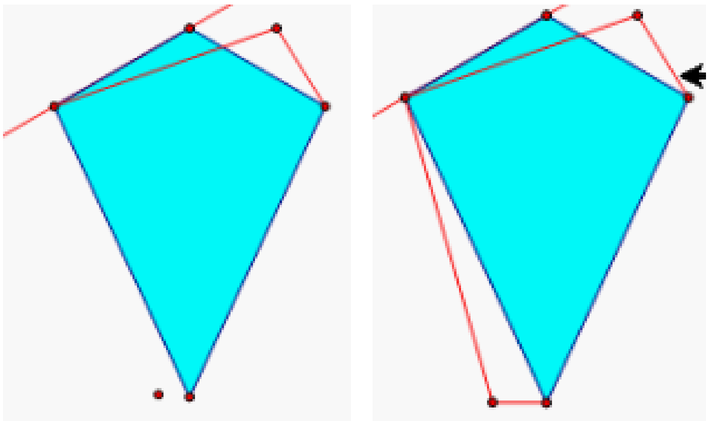


Figure 4.92 Participant 4's sketch on the kite figure to construct an ordinary quadrilateral with one pair of congruent opposite angles

Her final definition was "*a kite is a quadrilateral with only one pair of congruent opposite angles and with an angle bisector diagonal bisecting the non congruent angles."* She then used side property and defined a kite exclusively as *"a kite is a quadrilateral with congruent adjacent sides all of which are not have the equal length."* After thinking, she wanted to shorten this definition as "*a kite is a quadrilateral of which only the adjacent sides are congruent."* However, this definition included square and rhombus as well, since all sides were adjacent to each other. Then, the participant defined as "*a kite is a quadrilateral of which two different pairs of adjacent sides are congruent."*

4.4.3.3 Session 3 with Participant 4: Cyclic and circum Quadrilaterals

Initially, the participant could not remember the type of the quadrilateral she saw on the screen. However, when I asked her what the sides of the quadrilateral were called in a circle, she suddenly realized that it was a cyclic quadrilateral of which sides were the chords of the circle. She defined a cyclic quadrilateral as *"a quadrilateral of which sides are the chords of a circle."* Moreover, when she was asked the condition of being a cyclic quadrilateral she easily answered that the opposite angles had to be supplementary. When compared to the other participants, this participant had a better understanding of the subject. Based on the criterion she stated, she also correctly detected out all the cyclic quadrilaterals as square, rectangle, and isosceles trapezoid except for the right kite. At this point she could not notice that a kite could also be a cyclic quadrilateral when its congruent angles were 90º each.

Then she tested her arguments by dragging the dynamic cyclic quadrilateral figure into the special quadrilaterals. She was able construct a square, rectangle and isosceles trapezoid as she expected. However, before dragging the figure into a kite, she again stated that it was not a cyclic quadrilateral due to not having opposite supplementary angles. Yet, she got surprised when she could do it. Having observed the figure, she realized that if one pair of opposite congruent angles was right angles, then this special kite would be a cyclic quadrilateral. From then on, we decided to call this special kite as right kite.

In the next step, the participant was asked to define a right kite in terms of quadrilateral, kite and cyclic quadrilateral. She initially defined based on a quadrilateral as

> *"A right kite is a quadrilateral with one pair of opposite congruent right angles."*

However, as soon as she wrote the definition, she realized that it was lacking some information. Then I asked her to think of any counter example to refute this definition and she stated that a rectangle for example would be included in this definition. So she redefined as

> *"A right kite is a quadrilateral with one pair of opposite congruent right angles and with a diagonal as the angle bisector of the non congruent angle pair."*

She thought that having one pair of opposite right angles could not be enough defining property to generalize a right kite; and adding that one diagonal would be the angle bisector of the non congruent angles would make the definition sufficient one. She also searched for the counter example and stated that only the square fitted to this definition, but it was already a special kite. Next, she defined right kite in terms of kite as

> *"A right kite is a kite of which opposite congruent angles are 90º."*

She explained that when right kite was defined in terms of a kite, we knew that at least one pair of opposite angles were congruent, so there was need for adding only the missing information which made it a special kite; namely, the measure of these congruent angles. However, it became challenging for the participant to define right kite in terms of cyclic quadrilateral. Our conversation was the following:

Researcher: How would you define a right kite in terms of cyclic quadrilateral?

Participant 4: There is no need to say something… it is a cyclic quadrilateral.

Researcher: Then a rectangle is also a cyclic quadrilateral, a square and an isosceles trapezoid as well. How can I differentiate right kite among them as a cyclic quadrilateral?

Participant 4: Yes… hmmm.

Researcher: Assume that I do not know what a cyclic quadrilateral is and you define it to me as "a right kite is a cyclic quadrilateral." If I try to draw it I can draw any ordinary cyclic quadrilateral… I mean you must be able to define it so that the most general quadrilateral I can draw will be the right kite. What kind of a cyclic quadrilateral is a right kite? You should specify it.

Participant 4: Hmmm... I am going to say the same thing… It is a cyclic quadrilateral with at least one pair of opposite congruent angle. That is, when I say it is a cyclic quadrilateral…mmm... opposite angles are congruent… at least… hmm its diagonals… I don't know..

Researcher: We know that right kite is both a cyclic quadrilateral and a kite; it is a special case of the both groups like square, being the special case of both rectangle and rhombus… You defined right kite in terms of a quadrilateral and in terms of a kite. What is the difference between these two definitions?

Participant 4: When I defined in terms of quadrilateral I used the properties that one pair of opposite right angles and one angle bisector diagonal.

Researcher: Think of why you used these properties…

Participant 4: When defining in terms of a quadrilateral I added the defining properties which make it both cyclic quadrilateral and a kite…When defining in terms of kite, the kite properties had already included in the definition; so I only added the property which made it a cyclic quadrilateral… hmmm I think I got it… I should define as "A right kite is a cyclic quadrilateral with one pair of congruent opposite angles and with a diagonal which is the angle bisector of the non congruent angles."

Researcher: What did you think while writing this definition?

Participant 4: Here, we know that it is a cyclic quadrilateral, so we can infer that the congruent opposite angles need to be 90º each. So I only added the properties which were inherited from the kite.

In the next step, the participant was asked to define isosceles trapezoid, square and rectangle in terms of a cyclic quadrilateral. The isosceles trapezoid definition was the following:

"An isosceles trapezoid is a cyclic quadrilateral with one pair of parallel sides and with congruent base angles."

The participant stated that she had to say that the base angles of an isosceles trapezoid were congruent from which the congruency of the non parallel sides could be inferred; moreover, there was need to add the defining property of isosceles trapezoid which was the one pair of parallel sides. Upon this, I asked her to think whether all the properties were necessary. After thinking a while she stated that one pair of parallel sides would be enough to define; because being a cyclic quadrilateral it would have opposite supplementary angles and so the base angles would automatically be congruent. Moreover, the congruency of one pair of opposite sides could be inferred from this information. Therefore, her final definition was that

"An isosceles trapezoid is a cyclic quadrilateral with one pair of parallel sides"

She defined rectangle as

"*A rectangle is a cyclic quadrilateral with congruent and parallel opposite sides."*

Then I asked her again to think of whether all the properties used were necessary or they could be reduced. She thought on whether saying only "*A rectangle is a cyclic quadrilateral with parallel opposite sides"* would be enough to define a rectangle. At first, she thought that it would not be sufficient defining property. Then, she thought that it could be a parallelogram just by drawing a parallelogram in a circle on the paper. Here, she did not consider supplementary angle property. So I asked her to make a construction on the sketchpad. She constructed a circle and then constructed one pair of parallel segments inside it (Figure 4.93). She stated that the arcs between the parallel segments were congruent, so the side lengths would be congruent. Next, she constructed one of these segments (BC) and then a parallel line to this segment from the end point (A) of the top parallel segment. When she made the related measurements and dragged the point D to complete the figure into a quadrilateral, she saw that the figure only became a rectangle. That is, she observed that having two pairs of parallel sides would require these sides to be congruent in a cyclic quadrilateral and the most general "cyclic quadrilateral with two pairs of parallel sides" could be the rectangle, not a parallelogram.

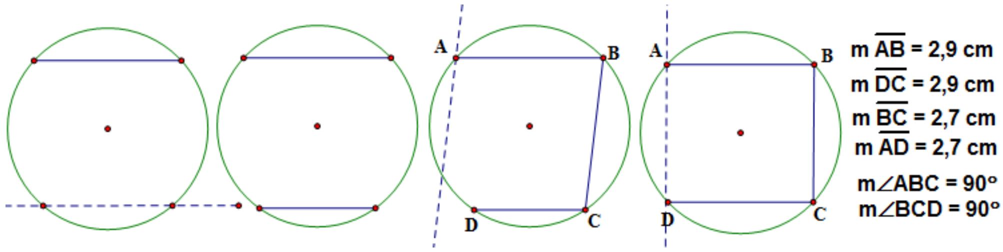


Figure 4.93 Participant 4's construction of a cyclic quadrilateral with two pairs of parallel sides

Finally, she correctly defined square as

"A square is a cyclic quadrilateral with all congruent sides."

When she was asked what would be "a quadrilateral with all congruent sides," she correctly stated that it would be a rhombus. Then, she correctly added the cyclic quadrilateral and right kite categories into the hierarchy (Figure 4.94).

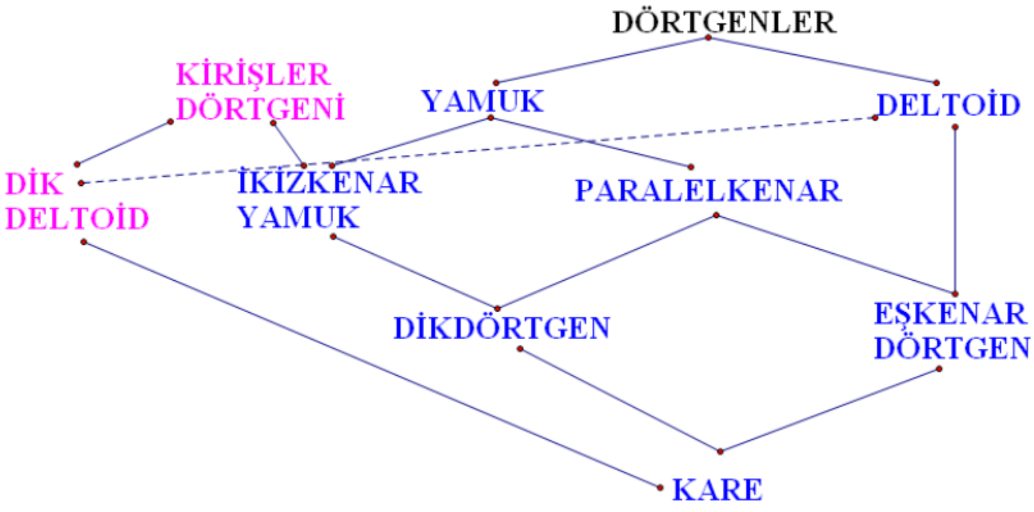


Figure 4.94 Participant 4's hierarchy diagram of quadrilaterals including cyclic quadrilateral and right kite

In the second part of the session, she was asked to think of the circum quadrilaterals. She correctly stated that a square was a circum quadrilateral, because a circle with the diameter of square side length could be placed into it. Moreover, she detected out that a rectangle could not be circum quadrilateral since it had two different pairs of opposite sides which prevented the circle not to be tangential to the one side. However, as for the rhombus she mixed her mind and stated that a rhombus could not be a circum quadrilateral since its diagonals did not pass through the center of the circle. Then, she drew a rhombus on a paper and drew a circle in it then she drew he perpendicular segments between the center and the tangential points. She thought that a rhombus was a circum quadrilateral but stated that she was not sure.

She was sure that a circle could not be placed inside a parallelogram so that it would be tangential to all sides. When it came to think of a kite, she thought that it could be a circum quadrilateral; but also added that she did not have an anchor point to give a reason. She stated that she was just doing visual evaluation. As for the trapezoid and isosceles trapezoid she was sure that they would not be circum quadrilaterals. She thought that the circle did not always have to touch to one of the parallel sides even if it touched to one of them.

As a result of her thinking process, she correctly detected only square as a circum quadrilateral but was not sure for the rhombus and kite. On the other hand, she was sure that rectangle, parallelogram, trapezoid and isosceles trapezoid were not circum quadrilaterals. Next, she tested her arguments by dragging the dynamic figure into the special quadrilaterals to observe whether they would always remain a circum quadrilateral or not. She confirmed her correct arguments, but also observed that a rhombus, kite and right kite were the circum quadrilaterals. That is to say, she detected out the kite class as the circum quadrilaterals. At this point, she still did not know the criterion that made a quadrilateral a circum quadrilateral which was discovered in the next step.

In the next step, she was asked to detect out the quadrilaterals of which angle bisectors intersected at a point. For this purpose, she dragged the dynamic quadrilateral into other quadrilaterals and observed the intersection. She saw that angle bisectors of the kite class intersected at a point. Then, she was asked to observe the intersection of the perpendicular bisectors of the sides for each quadrilateral and found that it was a point in the isosceles trapezoid, rectangle, square and right kite. When asked, she easily was able to make the connection that angle bisectors of all circum quadrilaterals intersected at a point which was the center of the inner circle; on the other hand, perpendicular bisectors of the sides of cyclic quadrilaterals were intersected at a point which was the center of the outer circle. Moreover, she observed that square and right kite were both cyclic and circum quadrilaterals since both the angle bisectors and perpendicular bisectors of the sides intersected at the points which were the centers of inner and outer circles correspondingly. Finally, she placed the circum quadrilaterals category into the hierarchy correctly (Figure 4.95) .

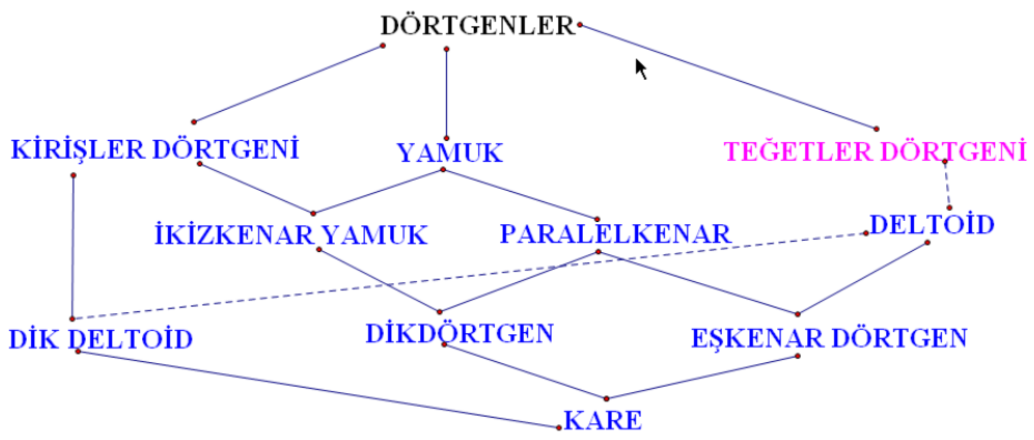


Figure 4.95 Participant 4’s hierarchy diagram of quadrilaterals including circum quadrilateral

4.4.3.4 Session 4 with Participant 4: New Quadrilaterals in the Hierarchy

The participant correctly explained the relationships on the hierarchy diagram. Moreover, she was able to explain the opposite inclusive relations between the quadrilaterals and between their properties.

Next, she defined a quad1 including all kite class as

"A quad1 is a quadrilateral with two pairs of congruent adjacent sides."

She thought that if "at least" was removed from the definition of kite, the definition would be generalized. However, when I asked her the quadrilaterals included by this definition, she was able to only list kite class but not any other quadrilateral; and so she decided that the definition was not general enough to include more general quadrilateral concepts than the kite class. After thinking a while, she correctly redefined as the following and then sketched the corresponding quadrilaterals other than the kite class (Figure 96).

"A quad1 is a quadrilateral with at least one pair of congruent adjacent side."

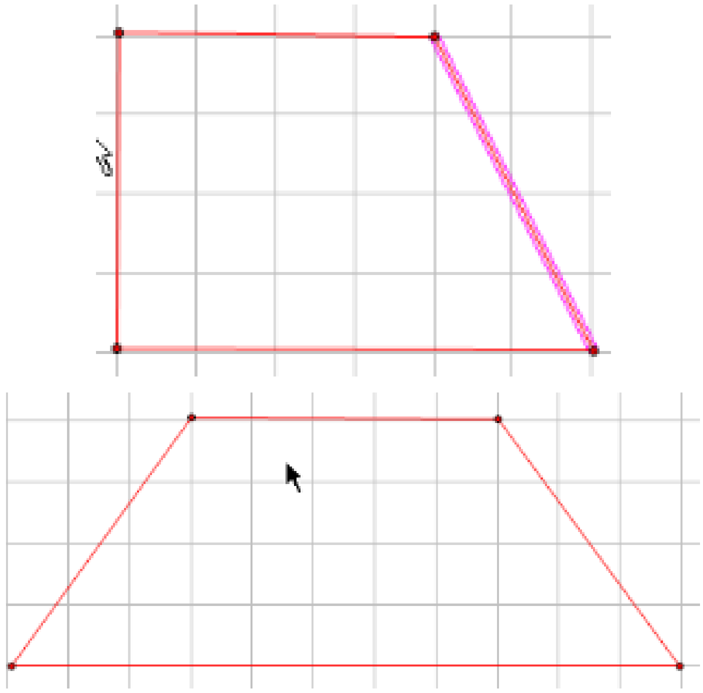


Figure 4.96 Participant 4's drawings of "quad 1"

In the next step she was asked to define a quad2 which was the special case of quad1 and trapezoid. She correctly constructed the definitions and then drew the related quadrilaterals as in the Figure 4.97 and also she indicated the special cases of quad2 as rhombus and square on the diagram.

"A quad2 is a quadrilateral with at least one pair of parallel sides and with at least one pair of congruent adjacent sides."

"A quad2 is a trapezoid with at least one pair of congruent adjacent sides."

"A quad2 is a quad1 with at least one pair of parallel sides."

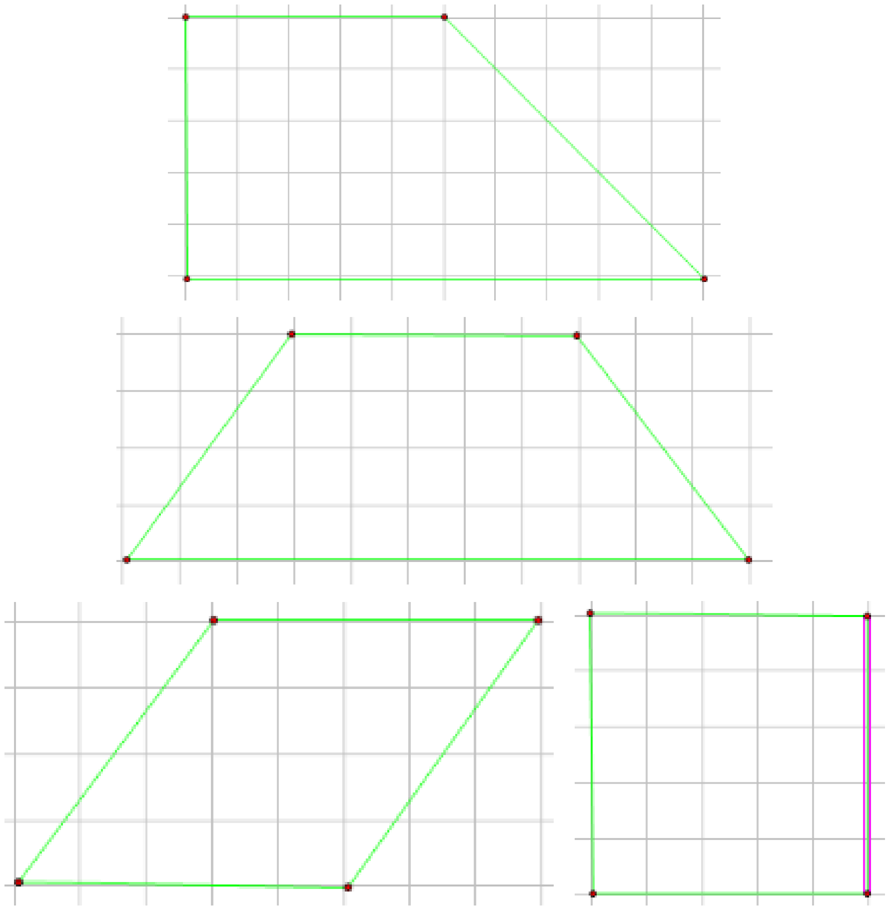


Figure 4.97 Participant 4's drawings of "quad 2"

Finally, she was asked to define a quad3 as a special case of quad2 and isosceles trapezoid. Her first definition was

> *"A quad3 is an isosceles trapezoid with at least one pair of congruent adjacent sides."*

She thought that since quad3 was defined in terms of an isosceles trapezoid it had been already known that one pair of sides had to be parallel, so there was no need to add this property. So, she had only added "one pair of congruent adjacent sides" property which would be enough to make it a quad2 besides isosceles trapezoid.

Next, she constructed two alternative quad3 definitions on the base of quad2.

> *"A quad3 is a quad2 with opposite supplementary angles."*

For the first definition above, she thought that defining in terms of quad2 would add on to the definition the properties of having one pair of parallel sides and one pair of congruent adjacent sides; so adding the angle property of isosceles trapezoid would make it an isosceles trapezoid as well.

Then, it was time to define a quad3 in terms of a cyclic quadrilateral. She correctly defined as

> *"A quad3 is a cyclic quadrilateral of which at least one pair of sides are parallel and the remaining opposite sides are congruent."*

However this definition included rectangle as acounter example; it was a cyclic quadrilateral but did not have congruent adjacent sides. After thinking on this counter example the participant redefined it as

> *"A quad3 is a cyclic quadrilateral with at least one pair of congruent adjacent sides."*

When I asked him to check for the counter examples he found right kite as a counter example, it was a cyclic quadrilateral but it did not have parallel sides. Upon this, he correctly constructed the definition as

> *"A quad3 is a cyclic quadrilateral with at least one pair of congruent adjacent sides and with at least one pair of parallel sides.*

Next, he drew the related quadrilaterals included by the definition of quad3. As in the case of other participants this participant also drew a rhombus but as soon as possible he realized that due to the angle property it could not be an isosceles trapezoid and a cyclic quadrilateral. So there were only 2 quadrilaterals as examples of quad3 (Figure 4.98).

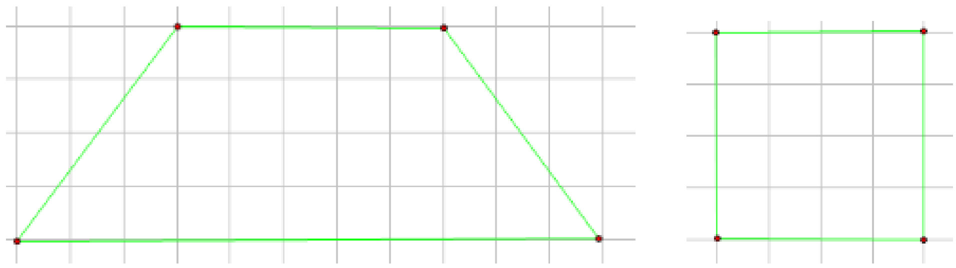


Figure 4.98 Participant 4's drawings of "quad 3"

When I asked her the common property of these two quad3s, she stated that they have at least 3 congruent adjacent sides and redefined quad 3 as

"A cyclic quadrilateral with at least 3 congruent adjacent sides."

When I asked her whether the definition would be correct if it was defined as "A quad3 is a trapezoid with at least 3 congruent adjacent sides," she immediately reasoned that this definition would include rhombus as counter example. Moreover, she stated that the definition would be correct quad3 definition if it was defined as

"A quad3 is an isosceles trapezoid with at least 2 congruent adjacent sides."

Finally the participant completed the hierarchy diagram by indicating the inclusive relation between quad3 and square with a connection segment (Figure 4.99).

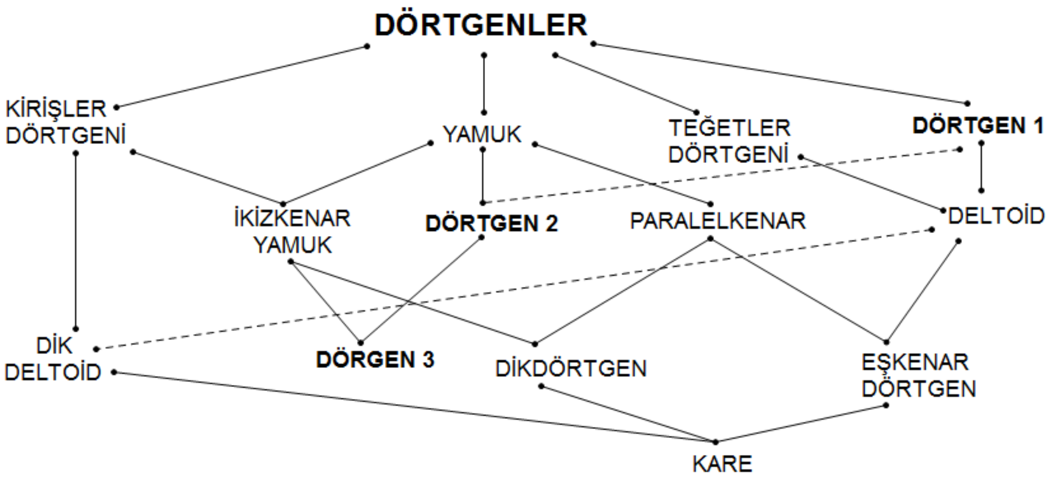


Figure 4.99 Participant 4's hierarchy diagram including "quad1," "quad2," and "quad3"

4.4.4 Participant 4's Opinions about the Quadrilateral Learning Experience in the GSP Learning Environment

The participant found her GSP assisted experience very effective and believed that the most effective side of the GSP was the dragging the figure into other figures without changing the constructed critical properties. Moreover, she stated that GSP allowed for many trials on a dynamic environment and provided visual and mathematical representations on one screen so that the relations could be seen better, which was not possible with paper-pencil tasks. Moreover, she stated that constructing the figures with the GSP encouraged thinking on the construction properties.

In contrast to the other participants, this participant said that she did not experience any technical difficulty with the GSP figures. However, the process of constructing economical definitions through eliminating the unnecessary properties was the most difficult one according to the participant.

She stated that before the study, she would define by listing all the properties and she did know nothing about the economic nature of the definitions; but through this experience, she had a good command of economizing them by critically thinking on the defining properties. Moreover, she added that she would use the definition construction process in her own teaching by enriching the learning environment with critical examples and non examples of the concept. She believed that engaging the students with the definition construction process would make the figures and the relations between them more meaningful, and would enable more meaningful conceptual learning.

Having considered her experience in this study, the participant believed that the GSP learning was superior to the paper-pencil learning in the sense that it provided figures and their related measures on the same screen and allowed to determine the properties through dynamic changes. That is to say, she found the dynamic nature of the GSP, which allowed observing the relationship between the objects, effective when compared to the paper-pencil learning. So, the participant stated that she was very enthusiastic to use GSP in her in-service teaching by adapting the tasks to the student level.

According to the participant, both pre-service and in-service teachers had much prejudice about the use of technology assisted learning since they thought that it would mix the students' mind up and would cause a waste of time; so, they did not want to give up traditional methods easily. She also added that throughout this study, she saw that instead of making things difficult, the sketchpad made things easier and that experiencing such kind of GSP assisted tasks would eliminate the prejudice of the teachers to a large extend, as in the case of herself.

4.4.5 Participant 4's Understanding of the Nature of the Quadrilateral Definitions and the Hierarchies after the Clinical Interviews

The participant very successfully constructed correct economical definitions for trapezoid and isosceles trapezoid. She defined trapezoid as "*a quadrilateral with at least one pair of parallel sides"* and defined isosceles trapezoid as "*a trapezoid of which base angles are congruent."* However, she defined rectangle as *"a quadrilateral which has opposite parallel sides and all interior angle measures 90°"* which was a correct but uneconomical definition. Only stating that "a quadrilateral with all interior angle measures 90°" would be enough condition to generalize a rectangle; because if a quadrilateral is constructed using only the information that all angles are 90°, the opposite sides would be automatically parallel to each other. Square also satisfies this angle condition, but to be able characterize a square there is need to add the extra condition that all sides are also congruent.

The participant was also good at identifying the example quadrilaterals among the given group that satisfied the given definitions. However, she incorrectly identified the rectangle as an example of the definition that "a cyclic quadrilateral with at least one pair of congruent adjacent sides," though a rectangle did not have any congruent adjacent sides. Moreover, she did not identify right kite among the given group of quadrilaterals as an example of this definition, though it was.

Moreover, when she was asked to construct two alternative definitions for a rhombus, she constructed one correct economical and one incorrect definition; respectively as *"a quadrilateral with all sides congruent"* and *"a quadrilateral with all angle measures 90° and with perpendicular diagonals."* A prototypical rhombus figure does not have any right angle, so this definition is incorrect; but a special case of rhombus, namely square, satisfies this definition. If the participant had defined as "a quadrilateral with perpendicular and bisecting diagonals", this would be sufficient to define a rhombus based on its diagonal property.

On the other hand the participant generally failed to evaluate given definitions. For example, according to her "having congruent diagonals" was a necessary and sufficient condition for a quadrilateral to be a rectangle. However, an isosceles trapezoid which is not a rectangle could be a counter example that satisfied this condition. To be able to define a rectangle, there was need to add the condition of bisecting diagonals, as well. Moreover, she stated that having one pair of parallel sides and one pair of congruent opposite angles were not sufficient conditions to characterize a parallelogram. According to her, unless we did not say "two pairs of parallel sides" an isosceles trapezoid would also satisfy this condition. However, she missed that while one pair of sides was parallel and one pair of opposite angles was congruent, the other pair of sides would automatically be parallel. Moreover, in an isosceles trapezoid not the opposite angles, but the adjacent base angles were congruent which was missed by the participant.

It was also seen that the participant was very successful at constructing inclusive and exclusive definitions; all her definitions were the correct economical ones. For instance, she defined the kite class by excluding the square from this class as *"quadrilaterals with two pairs of congruent adjacent sides and with at most one pair of congruent opposite angles."* She also successfully defined the prototypical kite by excluding its special cases rhombus and square as *"kites of which opposite sides are not congruent."* Moreover, she constructed a correct economical definition to include all quadrilaterals in the kite class (including kite, rhombus and square) as *"quadrilaterals with two pairs of congruent adjacent sides."*

Further, the participant perfectly achieved identifying hierarchical relationships regarding the properties. She knew that a kite could be a cyclic quadrilateral when it was a right kite, that a trapezoid could be a cyclic quadrilateral when it was an isosceles trapezoid and that the diagonals of a kite bisect each other when it was a rhombus. Moreover, she also stated that a square was always a special kite and a special cyclic quadrilateral. However, she also failed to show the hierarchical relations on the given diagram. Although the places were incorrect in the hierarchy, the relations between the parallelogram, rectangle and square was correctly shown.

4.5 Participant 5's Analysis Results

Findings related to the Participant 2's perceptions of the definitions and understanding of the quadrilateral definitions and the hierarchies before engaging them into the clinical interview sessions, her mental process and progress during the 4 clinical interview sessions, opinions about her experience in this study and the findings related to her understanding of the quadrilateral definitions and the hierarchies after the clinical interview sessions were stated in the following sections.

4.5.1 Participant 5's Initial Perceptions of the Definitions

According to the participant 5, definitions were very important in teaching and learning mathematics, because they made us understand what that concept really was. He thought that it was not possible to achieve any other mathematical activities related to that concept without understanding its definition. He stated that defining a concept did not mean that concept was learned; in addition, there was a need for executing its similarities and differences from the other concepts depending on its definition.

Participant 5 stated that until he came to the university, the definitions were just written on the board by the teacher at the beginning of the lesson as usual, but they did not discuss anything focusing on the definitions. He also said that he was not sure whether he could construct correct definitions; namely, he had a low self-efficacy belief of his definition construction skill. Moreover, participant 5 stated that he would use pre-constructed definitions in the textbooks, but would make students to think on the meaning of the definitions. However, he did not believe that the definitions could be used as effective teaching tools.

According to the participant 5, a good definition should have been precise; it should not lead up to the confusions related to the concept.

4.5.2 Participants 5's Initial Understanding of the Nature of the Quadrilateral Definitions and the Hierarchies

This participant was the most successful one among the others to construct the correct shortest definitions of rhombus, rectangle and square. He defined them as the following:

"A rhombus is a quadrilateral with all equal sides."

"A rectangle is a quadrilateral with all 90° angles."

"A square is a regular quadrilateral."

He only made a mistake by using the term "equal" improperly; but, all of his definitions were nearly the perfect economical definitions, especially the definition of square. He constructed a different, but correct square definition which would be understood if the meaning of the regular polygon was known. He also identified the squares as examples of both rectangle and rhombus which indicated that he was aware of the inclusive relationships among these quadrilaterals. However, he identified the figure "s" as a prototypical rectangle although its angles were not 90°. I think this was just a visual carelessness.

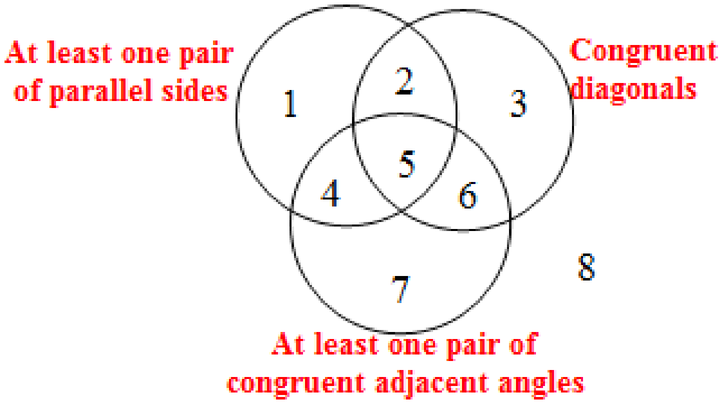
The analysis indicated also that he totally failed to identify the necessary and sufficient defining properties to write alternative definitions for the same concept. From the given properties, he inferred that this quadrilateral could be defined as "a regular quadrilateral;" however, the only regular quadrilateral is square and the given properties were not belong to square, but belong to kite. So, a kite can not be defined like this. Alternatively, he defined kite as "a quadrilateral with equal and perpendicular diagonals" which was again an incorrect definition because of the fact that a prototypical kite does not have diagonals of equal length. Besides, there was not any property such that "diagonals are congruent" among the given properties, but he wrote this incorrect property into the definition. Further, he used "equal" improperly, the correct use of which would be "a quadrilateral with equal diagonal lengths" or " a quadrilateral with congruent diagonals."

Moreover, when he was asked to evaluate the correctness of the given rhombus definitions in terms of critical defining properties, he only correctly identified the definition that "a rhombus is a quadrilateral with all sides congruent" and was able to explain that this was correct inclusive definition since it included square as special instance of rhombus. However, he did not account for accepting the other definitions as incorrect. For example, although the definition that “a rhombus is a quadrilateral of which symmetry axes are the perpendicular lines passing through the opposite vertices” was the correct one, he did not identified it as correct and could not give a reason for that.

On the other hand, the participant totally failed to construct inclusive and exclusive definitions. He could not construct any definition to define a and b by excluding them from the other given group of figures. He defined a, b, c, and d as “rhombuses” whereas the rhombuses were the figures e and f. He defined all figures as “polygons with 4 vertices,” which was a general definition to include not only the given figures but whole quadrilateral family.

It was seen that the participant 5 was moderately good at understanding the inclusive relations between the quadrilaterals through considering their properties. However, he had some misleading information related to the properties of rectangle and rhombus which prevented him to answer incorrectly to some questions. For instance, he stated that a rectangle always had congruent adjacent sides although the prototypical rectangle did not have congruent adjacent sides. Moreover he had a misinformation that the diagonals of a prototypical rhombus do not bisect each other except for the special case of square. On the other hand, he knew that square was a special rhombus with congruent diagonals and special parallelogram with perpendicular bisecting diagonals.

When the participant was asked to classify quadrilaterals based on their properties, he almost failed. In the first diagram he only was able to put the square into the correct region in the diagram (Figure 4.100). Although he was told to think each quadrilateral separately it is seen that the participant considered the special cases while deciding the regions. For instance, he thought that a parallelogram, when it was a rectangle or square, could have at least one pair of congruent adjacent angles and so he put the parallelogram into region 4.



Participant's Answers

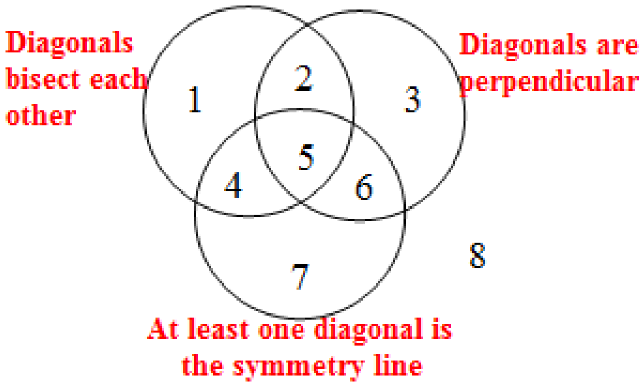
Parallelogram	**4**	Trapezoid	**4**
Kite	**1**	Isosceles Trapezoid	**4**
Square	**5**	Rectangle	**4**
Rhombus	**4**		

Correct Answers

Parallelogram	**1**	Trapezoid	**1**
Kite	**8**	Isosceles Trapezoid	**5**
Square	**5**	Rectangle	**5**
Rhombus	**1**		

Figure 4.100 Participant 5's first diagram of the classification of the quadrilaterals

In the second diagram he was able to put square, trapezoid and isosceles trapezoid into the correct regions, but not the others (Figure 101). Different from the first diagram, his mistakes were not due to his inclusive thinking, but due to his incorrect knowledge of the properties. For instance, because of having misinformation that the diagonals of a kite are not perpendicular and that one diagonal is not the symmetry line, he put the kite into the region 8. Moreover, he put the rhombus into 4 since he did not know that a rhombus has perpendicular diagonals.



Participant's Answers

Parallelogram	7	Trapezoid	8
Kite	8	Isosceles Trapezoid	8
Square	5	Rectangle	7
Rhombus	4		

Correct Answers

Parallelogram	1	Trapezoid	8
Kite	6	Isosceles Trapezoid	8
Square	5	Rectangle	1
Rhombus	5		

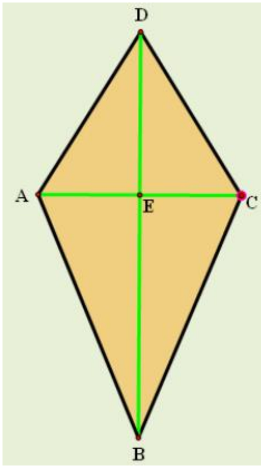
Figure 4.101 Participant 5's second diagram of the classification of the quadrilaterals

4.5.3 Clinical Interviews

Participant 5's cognitive progress during the clinical interview sessions and the effect of the GSP activities on the participant 5's cognitive improvement in understanding the quadrilaterals through definitions construction and classification processes were described in detail in the following subsections.

4.5.3.1 Session 1 with Participant 5: Kite, Rhombus and Square

The participant was able to correctly remember that one diagonal of a kite was bisected by the other diagonal. However, he had some misinformation that the area measures of the triangles above and below the bisected di agonal were equal to each other. This was only true when it was a special kite, namely when the kite was square or rhombus. However, it was not a defining property of a kite.



Then, he dragged the dynamic figure to detect out the preserved properties of kite by observing the side, angle, diagonal properties. As in the case of other participants he had a tendency to explain the properties using the statements specific to the figure on the screen, so I encouraged her to make general statements. That is, instead of saying "AB is congruent to BC and AD is congruent to DC," I expected him to generalize this side property as a general statement like "a kite has two pairs of congruent adjacent sides." Then he correctly stated that there were two pairs of congruent adjacent sides, one pair of opposite congruent angles; one diagonal bisected the other diagonal, diagonals were perpendicular, bisecting diagonal was also the angle bisector and the symmetry line.

After determining the properties of kite, I asked him how he would define a kite to his students. He wrote the definition that

> *"a kite is a special quadrilateral with two pairs of congruent adjacent sides and with perpendicular diagonals."*

This was a correct but not a minimal definition including more than the necessary information which could be inferred from the other information. After his definition we moved on to discovering the inclusive relations. I asked his opinions about whether there were any special kites among the quadrilaterals. At first he correctly said that a square was a special kite. However, he incorrectly added that if a square was a kite than a rectangle had to be a kite as well, since a square was a special rectangle. He correctly stated that a square had all properties of a rectangle but a rectangle did not have the properties of square; that is, square was like a subset of rectangles. Upon this, I asked him whether the critical defining properties of kite were preserved in a rectangle. He answered that not all of the properties of the kite were preserved in rectangle, but he thought that the kite figure could be dragged into the rectangle. After thinking a while, he came to the conclusion that rectangle did not satisfy the perpendicular diagonal property of a kite unless it was a special rectangle, namely square. That is, he correctly came to the conclusion that a kite could only be dragged into rectangle's special case square, but not into the prototypical rectangle. With a similar thinking, he reasoned that a kite could not be dragged into a prototypical parallelogram, too. He also correctly said that a rhombus was a special kite since it preserved all characteristic properties of kite. He said that he did not have any idea about whether a trapezoid was a special kite; but when I asked him to think about the preserved properties, he realized that not a prototypical trapezoid but a special case of it could be a deltoid and this special quadrilateral would only be the square. Similarly, he reasoned that an isosceles trapezoid was not a special kite because it did not have congruent adjacent sides. When he tested his arguments by dragging the dynamic kite figure into the other quadrilaterals, he observed that he was right saying that the only special kites were rhombus and square. After discussing the inclusive relationships, he also correctly placed kite, rhombus and square on the hierarchy diagram which indicated that a square was a special rhombus, a rhombus was a special kite and so a square was a special kite as well.

Next, it was time to construct an inclusive definition of the kite so that the definition would include rhombus and square as special cases. In his first attempt he was able to write a correct inclusive definition including the minimal information to characterize a kite. He also tested the sufficiency of his property with the corresponding construction.

"A kite is a quadrilateral with two pairs of congruent adjacent sides."

Then, I asked him to evaluate the following definition

Definition 1: A kite is a quadrilateral with perpendicular diagonals.

At first he accepted this definition as the correct definition since he could not find any other example out of the kite class that fitted to this definition. Upon this, I asked him to construct the corresponding figure just by using the information given in the definition. He constructed a segment, took any point on it and constructed the perpendicular line. Then he constructed any two points on the perpendicular line and finally combined the vertices with segments to construct the quadrilateral. He said that the figure looked like a kite, but when he dragged it and observed the related measurements, the figure did not remained a kite and turned into any ordinary quadrilateral (Figure 4.102).

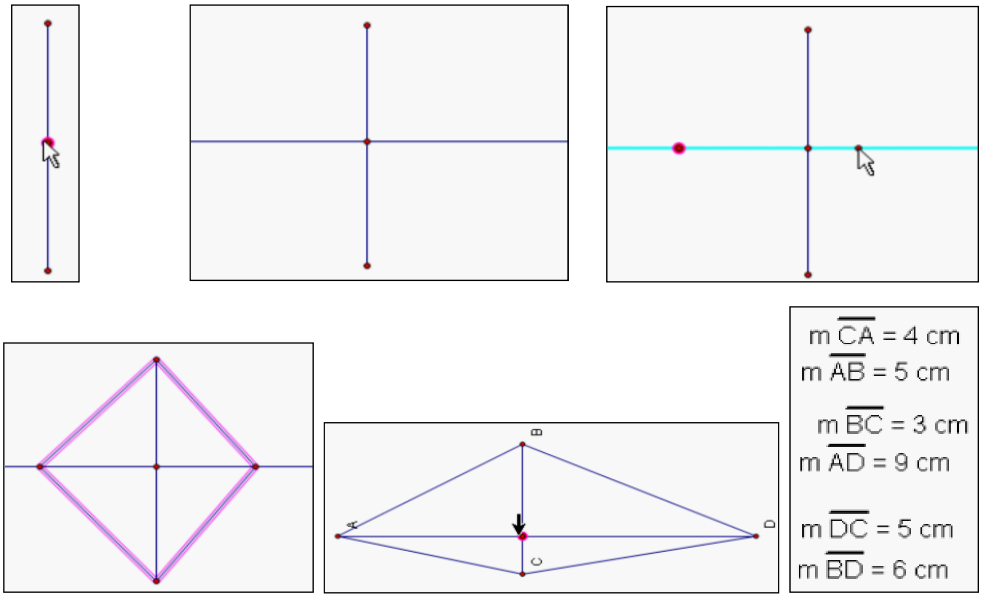


Figure 4.102 Participant 5's construction of a quadrilateral with perpendicular diagonals

Then, he realized that having perpendicular diagonals was not a sufficient property; in addition, one diagonal had to be bisected by the other. As a result, he corrected the definition as

> *"a kite is a quadrilateral with perpendicular diagonals where one diagonal is bisected by the other diagonal"*

which was the second definition I was about to ask him to evaluate. Then, he also constructed the related figure and confirmed that the information given in the definition was enough to characterize a kite (Figure 4.103).

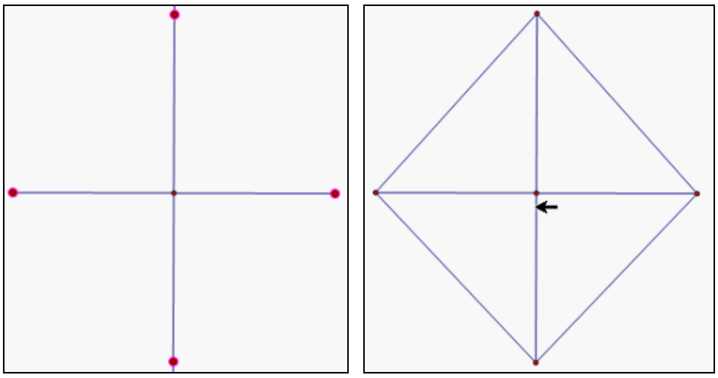


Figure 4.103 Participant 5's construction of a quadrilateral with perpendicular diagonals where one diagonal is bisected by the other diagonal

Next, he thought about the third definition and said that it was not a minimal definition, but was not sure whether it was correct.

Definition 3: A kite is a quadrilateral with two pairs of congruent adjacent sides and one pair of congruent opposite angles.

Then he remembered his initial definition he did at the beginning of this session and reasoned that if a quadrilateral had two pairs of congruent adjacent sides, one pair of opposite angles had already been congruent. So there was no need for the second property which could be inferred from the first property.

He claimed that the last definition did not include sufficient property and there was need to add that the diagonals had to be perpendicular to each other. That is, he could not infer that symmetry property would make the diagonals perpendicular. Upon this, I encouraged him to test this definition with a construction. When he constructed the figure, he realized that the diagonals automatically become perpendicular and all other properties were also satisfied. Therefore he decided that the definition was correct inclusive definition including the minimal information.

Definition 4: A kite is a quadrilateral with at least one diagonal is the symmetry axis.

As a final step of this session, he defined a rhombus as *"a rhombus is a quadrilateral which is symmetric with respect to the both diagonals,"* which was a correct economical definition. He tried to check the definition by searching for the counter examples, but he concluded that this property was only a defining property of rhombus.

4.5.3.2 Session 2 with Participant 5: Parallelograms and Trapezoids

The participant 5 did not remember most of the properties of the isosceles trapezoid. He only stated that it had two pairs of parallel and two pairs of congruent sides. He also indicated on the figure which angles were congruent, but he could not express this angle property with the appropriate mathematical language. Moreover, he was not sure about the correctness of the properties he said. So, I asked her to work on the dynamic figure and detect out the preserved properties. During the process, he intended to explain the properties based on the specific figure on the screen rather than making general statements using the mathematical language. I encouraged him to use mathematical terms as much as possible; for instance when he said the sides CB and DA had equal length, I asked what kind of sides they were, namely, opposite or adjacent sides. At the end of this process, he listed the properties as "an isosceles trapezoid has one pair of opposite congruent sides and one pair of parallel side, two pairs of congruent adjacent angles, complementary opposite angles, congruent diagonals." He also grasped that the diagonals intersected each other in the same ratio; but it was seen that he had difficulty to express this with the correct mathematical language. Moreover, he also found that the perpendicular line passing through the parallel sides and through the intersection of the diagonals was the symmetry line.

After detecting out the critical defining properties of the isosceles trapezoid, the participant was asked how he could define it. His initial definition was "An isosceles trapezoid is a quadrilateral which has one pair of parallel sides and one pair of congruent sides." However, this definition was not a correct definition since it included quadrilaterals out of the quadrilateral class; in the following steps he realized his mistake after discovering the inclusive relations. When he was asked which quadrilaterals could be the special isosceles trapezoids, he correctly stated that square and rectangle were special instances since they had all properties of an isosceles trapezoid. Then he claimed that because of its diagonal property, a parallelogram could not be a special isosceles trapezoid. He explained that the diagonals of the parallelogram intersected each other with a one to one ratio and they were not congruent; but the diagonals of an isosceles trapezoid did not have to intersect each other with a one to one ratio and they were congruent. He also eliminated rhombus and kite due to their non congruent diagonals. Moreover, he correctly detected out that an isosceles trapezoid was the special instance of a trapezoid but not the vice versa. Then he tested his arguments by dragging the dynamic isosceles trapezoid figure into the other quadrilaterals and confirmed his thoughts. He also indicated these relations on the hierarchy diagram correctly.

Next, he constructed an inclusive definition of an isosceles trapezoid which included rectangle and square as

> *"An isosceles trapezoid is a quadrilateral with one pair of congruent opposite sides, one pair of parallel sides and equal diagonals.*

That is to say, he added the congruency of the diagonals as an extra defining property to his initial definition. First of all, he used the term "equal" instead of congruent; second, the definition included more than the necessary information. When he was asked to think on the properties he used, he thought that one of the side properties could be removed; but he could not decide which one together with the diagonal property would make a defining property. Then he decided that saying *"An isosceles trapezoid is a quadrilateral with one pair of parallel sides and congruent diagonals"* would be enough to define isosceles trapezoid. He tried to refute the definition with counter examples, but he could not. Then I asked him what he would think if the parallel sides property would be removed rather than the congruent sides property. That is, he also tried to refute the definition that *"An isosceles trapezoid is a quadrilateral with one pair of opposite congruent sides and congruent diagonals."* Since he could not find any counter example, he accepted this definition as correct definition including the necessary and sufficient information. Since the participant reasoned correctly using the counter example search in this process, I did not need to ask him to make related constructions.

In the next step, the participant 5 was asked to evaluate the given definitions. Since the first pre-constructed definition was his initial definition and was discussed before, I provided him with the second definition.

Definition 1: An isosceles trapezoid is a quadrilateral with at least one pair of parallel sides and with at least one pair of opposite congruent sides.

For the second definition, he said that there was a problem with this definition, because it did not included the rectangle and square, but only defined prototype isosceles trapezoid. He accepted this definition as an insufficient definition.

Definition 2: An isosceles trapezoid is a quadrilateral with one pair of parallel sides and with one pair of congruent but unparallel sides.

For the third definition he tried to search for the counter examples to refute the definition, and stated that the parallelogram, rhombus and deltoid did not satisfied this definition but rectangle and square did. For the trapezoid, he wanted to draw a figure on the paper and think on it; and finally evaluated this definition as correct inclusive definition including the minimal information.

Definition 3: "An isosceles trapezoid is a quadrilateral with at least one pair of parallel sides and with opposite supplementary angles."

For the fourth definition, he considered each quadrilateral one by one and evaluated which ones satisfied the information given in the definition. He concluded that only isosceles trapezoid class satisfied the definition. I asked him to think on whether any ordinary quadrilateral would have the properties given in the definition. Upon my question, he stated that he would not be able to answer this without making a construction; so he started to make the corresponding construction (Figure 4.104). He needed to construct two pairs of congruent adjacent angles; however, it became a difficult process for him to construct which required my technical help. He first constructed and measured an angle and then calculated the inverse of this angle. Then, he rotated B around C with this marked angle in order to construct the congruent adjacent angle. After constructing the first pair, he easily realized that he could make the other pair congruent to each other only when he drew a parallel line to the base segment. As a result of his construction, he concluded that the definition included sufficient information to characterize an isosceles trapezoid.

Definition 4: An isosceles trapezoid is a quadrilateral with two pairs of congruent adjacent angles.

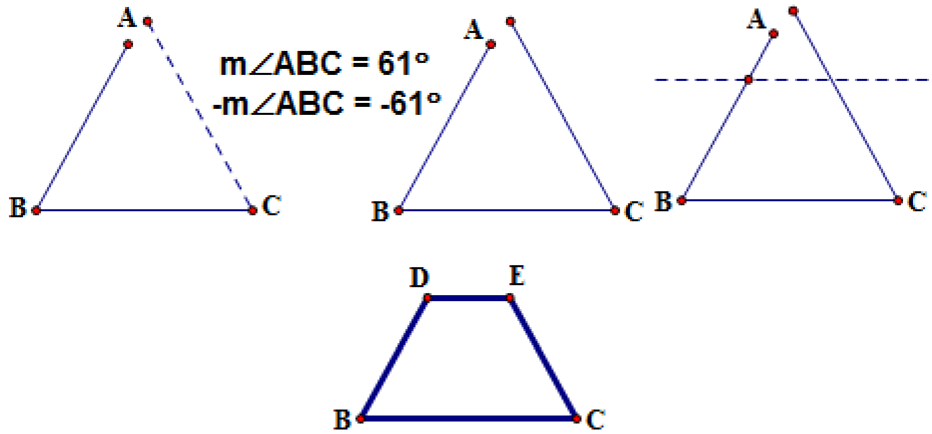


Figure 4.104 Participant 5's construction of a quadrilateral with two pairs of congruent adjacent angles

To evaluate the last definition he made some drawings on the paper and stated that this definition was incorrect since he could construct any ordinary quadrilateral with congruent diagonals. When I asked him to correct the definition, he easily detected out that there was a need to add that the diagonals were also intersected each other in the same ratio. When he intended to make a drawing on the paper, I encouraged her to make a real construction on the sketchpad.

Definition 5: "An isosceles trapezoid is a quadrilateral with congruent diagonals."

He first constructed a segment as the first diagonal and constructed any point on it. Then he measured any angle and rotated the segment around the point with this marked angle. After the construction, he observed the properties under dragging and confirmed that it remained as an isosceles trapezoid (Figure 4.105).

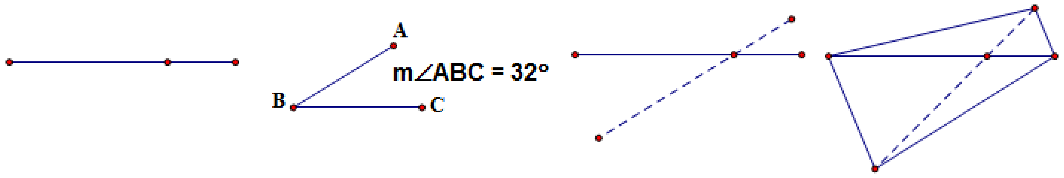


Figure 4.105 Participant 5's construction of a a quadrilateral with congruent diagonals

In the next step, the participant was asked to find out the special parallelograms. He directly stated that square, rhombus and rectangle were the special instances of a parallelogram; while deciding this, he considered side properties. Moreover, he correctly reasoned that a kite and trapezoid could not be parallelograms since they did not have two pairs of parallel sides.

Having tested on the sketchpad, he verified that he detected all special parallelograms correctly. Next, it was time to show the relationships with a hierarchy diagram and he perfectly constructed it without any doubt about the rectangle-rhombus relationship which had become a problem for other participants. That is, he indicated rectangle and rhombus as distinct parallelograms of which common special case was the square (Figure 4.106).

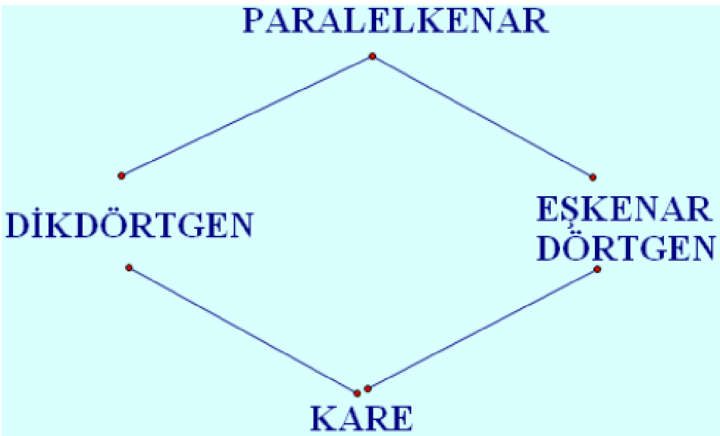


Figure 4.106 Participant 5's hierarchy diagram of parallelograms

Then, he was asked about the relationship between prototypical trapezoid and parallelograms. He explained that for a quadrilateral to be a trapezoid, there was need for one pair of parallel sides; so he correctly stated that each parallelogram was a special trapezoid, but not vice versa. Moreover, he detected that isosceles trapezoids were also the special trapezoids other than the parallelograms. Then, he correctly defined a trapezoid as "a quadrilateral with at least one pair of parallel sides."

In the next step, he was asked to construct the whole hierarchy indicating the relationships discovered so far. It became a little bit difficult for him to organize the places of the quadrilaterals on the hierarchy. He mixed his mind for a while, but he was able to place all of the quadrilaterals into the correct places. Moreover, when I asked her how the hierarchy would change if a trapezoid was defined as "a quadrilateral with only one pair of parallel sides," He correctly stated that the parallelograms would not be placed under the trapezoids, since they would not be special trapezoids any more.

In the final step of this interview, the participant was asked to define quadrilaterals exclusively so that the definitions would not include special instances; but they were restricted to use specific properties for each quadrilateral. He first defined *parallelogram* using the *diagonal* property as "a parallelogram is a quadrilateral with diagonals intersecting each other in the same ratio," but then he realized this definition was an inclusive one including the special cases of it. He thought for a while, but did not know how to eliminate square, rhombus and rectangle from the definition. Then, he correctly remembered the diagonal properties of rhombus, square and rectangle and then decided that he could define a parallelogram exclusively using the diagonal property as "a parallelogram is a quadrilateral with diagonals intersecting each other in the same ratio but not perpendicular to each other." However, this definition only eliminated square and rhombus, but not the rectangle. When I asked how to eliminate rectangle with the diagonal property he said that the diagonals must not be congruent. So he corrected the definition as *"a parallelogram is a quadrilateral with diagonals intersecting each other in the same ratio but not perpendicular to each other and not congruent to each other."*

Next, he was asked to define *rhombus* using the *symmetry* property, but he was not sure whether the lines passing through the midpoints of the sides were the symmetry lines; so he worked on the figure to detect it out. Then he constructed the definition as *"a rhombus is a quadrilateral which is symmetrical with respect to the both diagonals but not to the lines passing through the midpoints of the opposite sides."* Then I encouraged him to think on how to construct a shorter definition and after thinking a while he correctly defined as *"a rhombus is a quadrilateral which is symmetrical only with respect to the both diagonals."*

Finally, he defined *kite* using any property he liked as *"a kite is a quadrilateral with two pairs of congruent adjacent sides and with only one diagonal as a symmetry axes;"* however, this definition included redundant information. Then, he thought whether the special cases were eliminated or not, and confirmed that they were eliminated from the definition. When I asked whether both properties were required, he stated that saying *"a kite is a quadrilateral with one diagonal as the only symmetry axes,"* would be sufficient to define a kite exclusively.

4.5.3.3 Session 3 with Participant 5: Cyclic and Circum Quadrilaterals

The participant correctly stated that the quadrilateral he saw on the screen was a cyclic quadrilateral and defined it as *"A cyclic quadrilateral is a quadrilateral of which all vertices lie on a circle."* Then I asked him to explain his ideas about which quadrilaterals would be cyclic. He stated that a square was a cyclic quadrilateral. According to him, the measure of an inscribed angle was the half of the measure of the arc it saw and when he considered a square, the angles were 90º each and they lead to a total measure of 360º arc which formed a circle. When I asked him to generalize this criterion for all cyclic quadrilaterals, he correctly stated that the sum of the measures of the opposite angles had to be 180º.

Next, he stated that a rectangle was a cyclic quadrilateral since it had all right angles, but a parallelogram and rhombus were not; because the sum of the opposite angles was not 180º. Moreover, he correctly evaluated that a kite could be a cyclic quadrilateral if the congruent angles were 90º. As for the trapezoid, he thought that when two parallel sides were drawn in a circle a trapezoid could be constructed, but he was not sure whether all trapezoids would be cyclic. When he thought on the isosceles trapezoid, he changed his idea in the way that not all trapezoids, but only isosceles trapezoids would be cyclic. He explained that when one pair of parallel sides was drawn into a circle then the arcs between them had to be congruent which meant that the side lengths had to be congruent. That is to say, he correctly detected all cyclic quadrilaterals through correct reasoning. Moreover, he also confirmed his arguments by dragging the dynamic cyclic quadrilateral into the other quadrilaterals one by one.

In the next step he tried to define *right kite* on the bases of *"quadrilateral," "kite"* and *"cyclic quadrilateral."* He defined as

"A right kite is a quadrilateral with one pair of opposite right angles."

Then I asked him to search for counter examples that could refute this definition. He stated that only right kite would fit to this definition. Then I asked him to construct a quadrilateral with one pair of opposite right angles. However, when he constructed the quadrilateral he observed that it did not remained a right kite; it could turn into an ordinary quadrilateral as well (Figure 4.107).

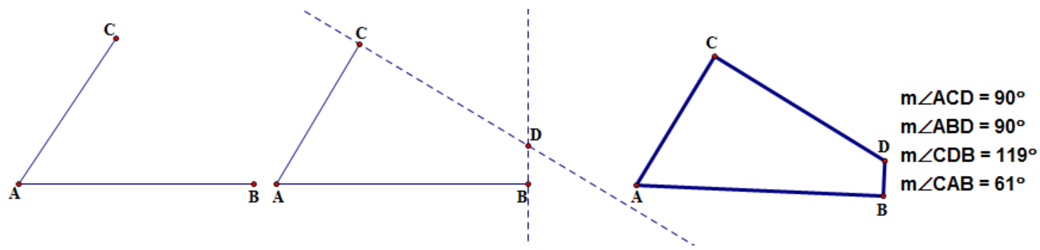


Figure 4.107 Participant 5's construction of a quadrilateral with one pair of opposite right angles

Upon this construction, he realized that the definition was not specific enough to generalize a right kite. I encouraged him to think on which properties made a right kite both a cyclic quadrilateral and a kite. He stated that having two pairs of congruent adjacent sides made it a kite while having one pair of opposite right angles made it a cyclic quadrilateral. Next, he defined as

> *"A right kite is a quadrilateral with one pair of opposite right angles and with a bisected diagonal passing through these right angles."*

This was a correct definition. If the additional property was added to the last construction it could be seen that the figure turn out to be a right kite (Figure 4.108).

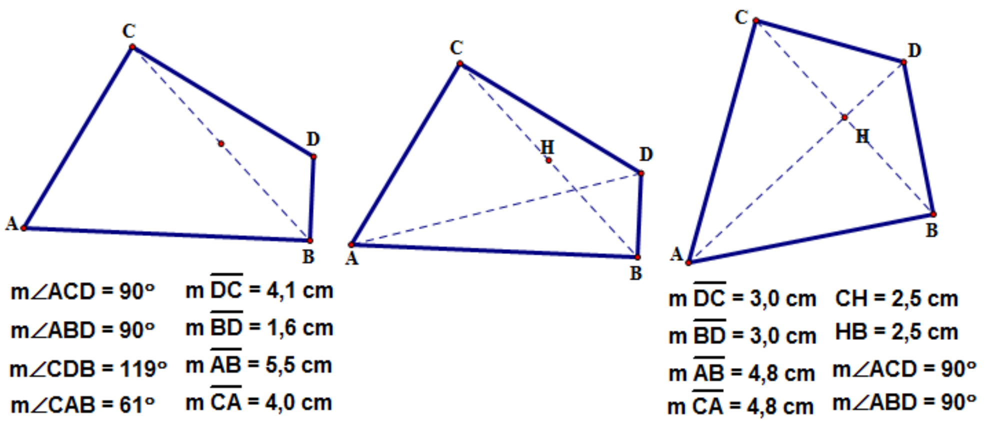


Figure 4.108 Participant 5's construction of a quadrilateral with one pair of opposite right angles and with a bisected diagonal passing through these right angles

He also made another alternative definition which was

"A right kite is a quadrilateral with one pair of opposite right angles and with two pairs of congruent adjacent sides."

Next, he correctly defined right kite in terms of kite as

"A right kite is a kite with one pair of opposite right angles."

Finally, he defined in terms of a cyclic quadrilateral as

"A right kite is a cyclic quadrilateral with at least one pair of opposite right angles."

Upon his definition, I asked him to think of any counter example and he stated that rectangle and square also would fit to this definition. He found that rectangle refuted this definition and also remembered his previous construction of any ordinary quadrilateral with one pair of opposite right angles. So he redefined as

"A right kite is a cyclic quadrilateral with at least one pair of opposite right angles and with at least one pair of congruent adjacent sides."

When I asked him to think on whether it was necessary to use both angle and side properties together, he thought that saying

"A right kite is a cyclic quadrilateral with at least two pairs of congruent adjacent sides,"

would be enough. He explained that when it was defined by the defining side property, we would know that it was a kite; and also its being a cyclic quadrilateral would require this kite to have right congruent angle pair.

In the next step, it was time to define other special cyclic quadrilaterals in terms of a cyclic quadrilateral. He correctly defined isosceles trapezoid and rectangle in his first trial as

"An isosceles trapezoid is a cyclic quadrilateral with at least one pair of parallel sides."

"A rectangle is a cyclic quadrilateral with two pairs of opposite congruent sides."

He defined a square as

"A square is a cyclic quadrilateral of which diagonal was the diameter of the outer circle."

He thought that this definition was correct since there was no any other example fitting it. Upon his answer, I asked him to drag the dynamic cyclic quadrilateral figure into a rectangle so that he could observe that the diagonal of a rectangle was also the diameter of the circle (Figure 4.109). Moreover, he also dragged the figure into an ordinary quadrilateral of which diagonal was the diameter of the circle.

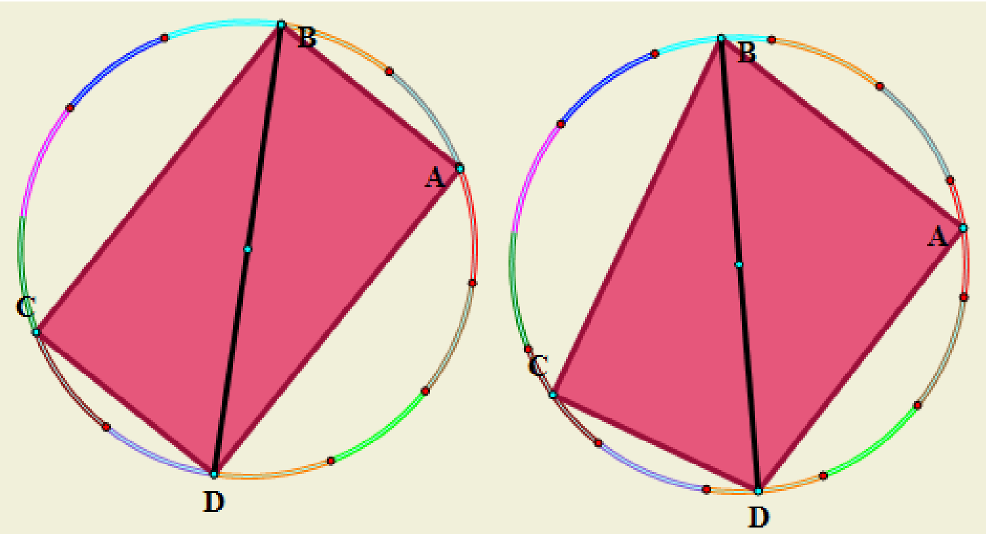


Figure 4.109 Participant 5's dragging the dynamic cyclic quadrilateral figure into a rectangle and an ordinary quadrilateral of which diagonal was the diameter of the circle.

Then, he defined as

> *"A square is a cyclic quadrilateral of which both diagonals are the diameters of the outer circle."*

However, as soon as he wrote this definition, he realized that a rectangle also fitted to the definition again. Then, he thought that square was different from the other cyclic quadrilaterals in terms of having all congruent sides and redefined as

> *"A square is an equilateral cyclic quadrilateral."*

After the definitions, he also correctly placed cyclic quadrilateral and right kite categories in to the hierarchy diagram (Figure 4.110).

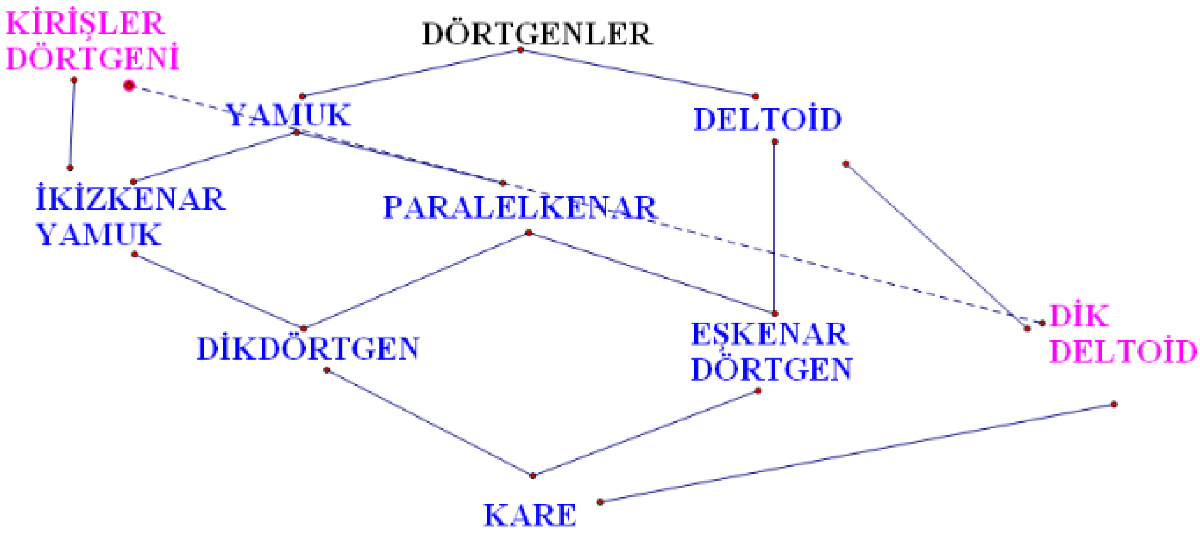


Figure 4.110 Participant 5's hierarchy diagram of quadrilaterals including cyclic quadrilateral and right kite

In the next step, there was a circum quadrilateral on the screen and the participant did know that it was called a circum quadrilateral. Then he was asked to think on which quadrilaterals would be the circum quadrilaterals. At first he stated that a square would be a circum quadrilateral but a rectangle would not because of the relation between the side lengths and the diameter of the circle. As for the parallelogram he stated the same reason he did for the rectangle. Then, he incorrectly concluded that a rhombus would not be a circum quadrilateral thinking that the circle placed inside it would not be tangential to all sides. Moreover, he had no comment for the kite and right kite due to not having any idea about the place of the center of the inner circle. For the trapezoid and isosceles trapezoid he was not sure whether the circle with a diameter of the distance between the parallel sides would touch to the other sides.

Next, he discovered the special circum quadrilaterals through dragging the dynamic figure. He was able to drag it into a square and rhombus, but not into the rectangle and parallelogram and he realized that his initial judgment for rhombus was wrong. Then he was able to construct a prototypical kite and a right kite as well, and detected that they were circum quadrilaterals. As in the case of other participants, he also was able to construct a trapezoid of which inner circle was tangential to all its sides and decided that it was a circum quadrilateral without considering the improbable cases.

However, when I asked him to drag the base segment without distorting the shape and he observed that the figure remained a trapezoid, but it did remain a circum quadrilateral any more. He also observed the same situation for the isosceles trapezoid. As a result, he found out kite class as circum quadrilaterals.

In the next step he worked on a dynamic quadrilateral of which angle bisectors were constructed in order to observe the intersection of the angle bisectors in each quadrilateral and he found that the intersection was a point in all circum quadrilaterals. Then, he worked on another dynamic figure of which perpendicular bisectors of the sides were constructed in order to observe the intersection of them in all special quadrilaterals and found out that the intersection of the perpendicular bisectors was a point in all cyclic quadrilaterals. Although he could make the connection that the intersection of angle bisectors would be the center of the inner circle for the circum quadrilaterals, he was still doubtful about the kites. Then I asked him to construct angle bisectors on a kite figure (Figure 4.111) and after the construction he was sure that the intersection point was the center of the inner circle.

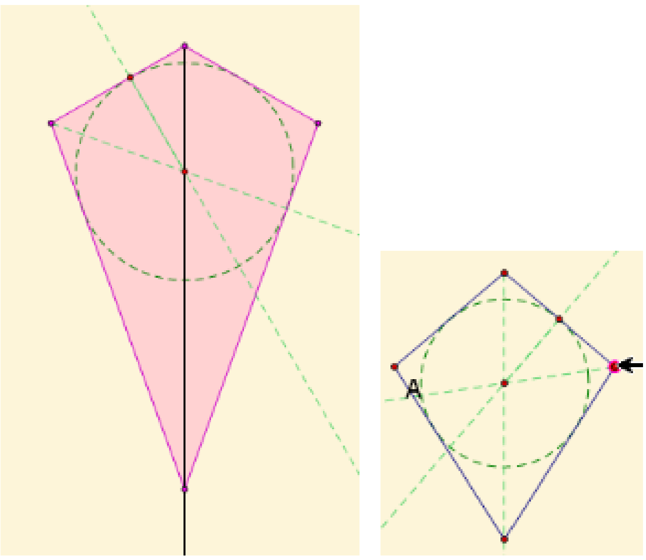


Figure 4.111 Participant 5's constructing construct angle bisectors on a kite figure

Then he was easily able to make a judgment that the intersection point of the perpendicular bisectors of the sides was the center of the outer circle for the cyclic quadrilaterals. Before I asked the participant to think of the quadrilaterals which were both cyclic and circum quadrilaterals, he made a judgment himself. He stated that when the diameter of the circle was scaled up and scaled down some quadrilaterals would be both cyclic and circum quadrilaterals. When I asked these quadrilaterals, he correctly said square and right kite.

Finally, he placed circum quadrilaterals category into the hierarchy diagram and indicated the inclusive relationships correctly (Figure 4.112).

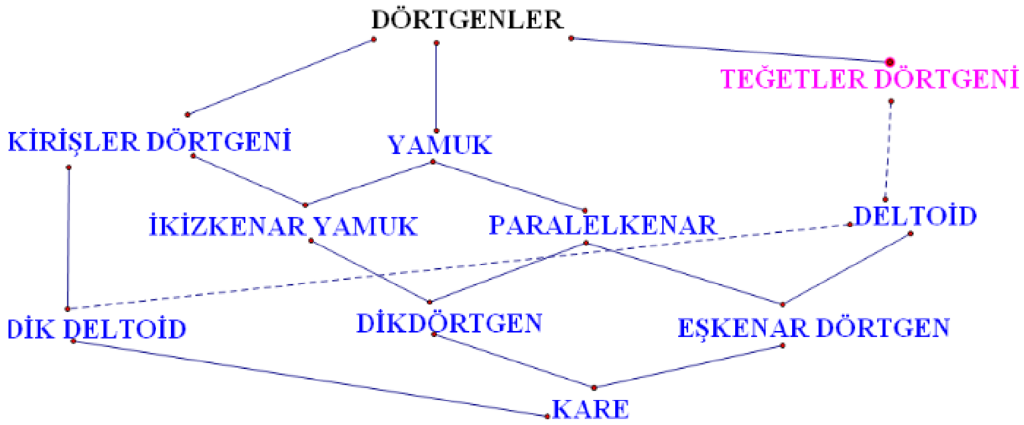


Figure 4.112 Participant 5's hierarchy diagram of quadrilaterals including circum quadrilateral

4.5.3.4 Session 4 with Participant 5: New Quadrilaterals in the Hierarchy

The participant correctly explained the inclusive relationships in the hierarchy diagram and stated that the quadrilaterals were specialized more as they got more property from top to the bottom of the hierarchy. He also was aware of the opposite inclusive relations between the quadrilaterals and their properties in terms of inclusive hierarchies. He stated that the properties of a special case included the properties of a more general quadrilateral.

In the next step the participant was asked to define quad1 which was a more general concept than the kites and including all kite class. He defined easily as

"A quad1 is a quadrilateral with at least one pair of congruent adjacent sides."

Then he drew the quad1s except for the kite class (Figure 4.113).

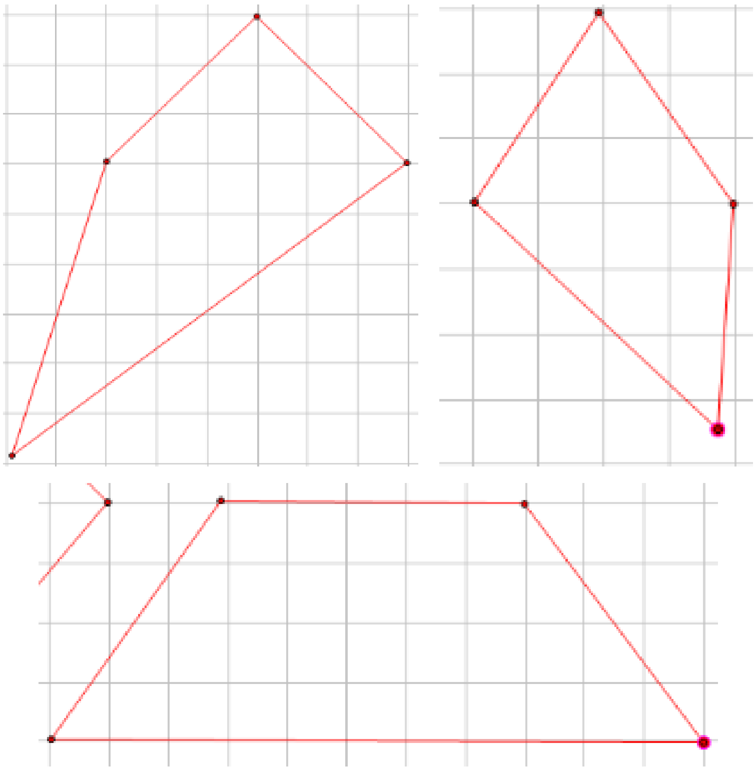


Figure 4.113 Participant 5's drawings of "quad 1"

In the next step, the participant defined quad2 as the special case of both quad 1 and trapezoid. He was asked to define quad2 as a trapezoid, quad1 and quadrilateral and he correctly constructed the definitions as

"A quad2 is a quadrilateral with at least one pair of congruent adjacent sides and with at least one pair of parallel sides."

"A quad2 is a trapezoid with one pair of congruent adjacent sides."

"A quad2 is a quad1 with at least one pair of parallel sides."

Then, he drew all possible quad2s as in the Figure 4.114.

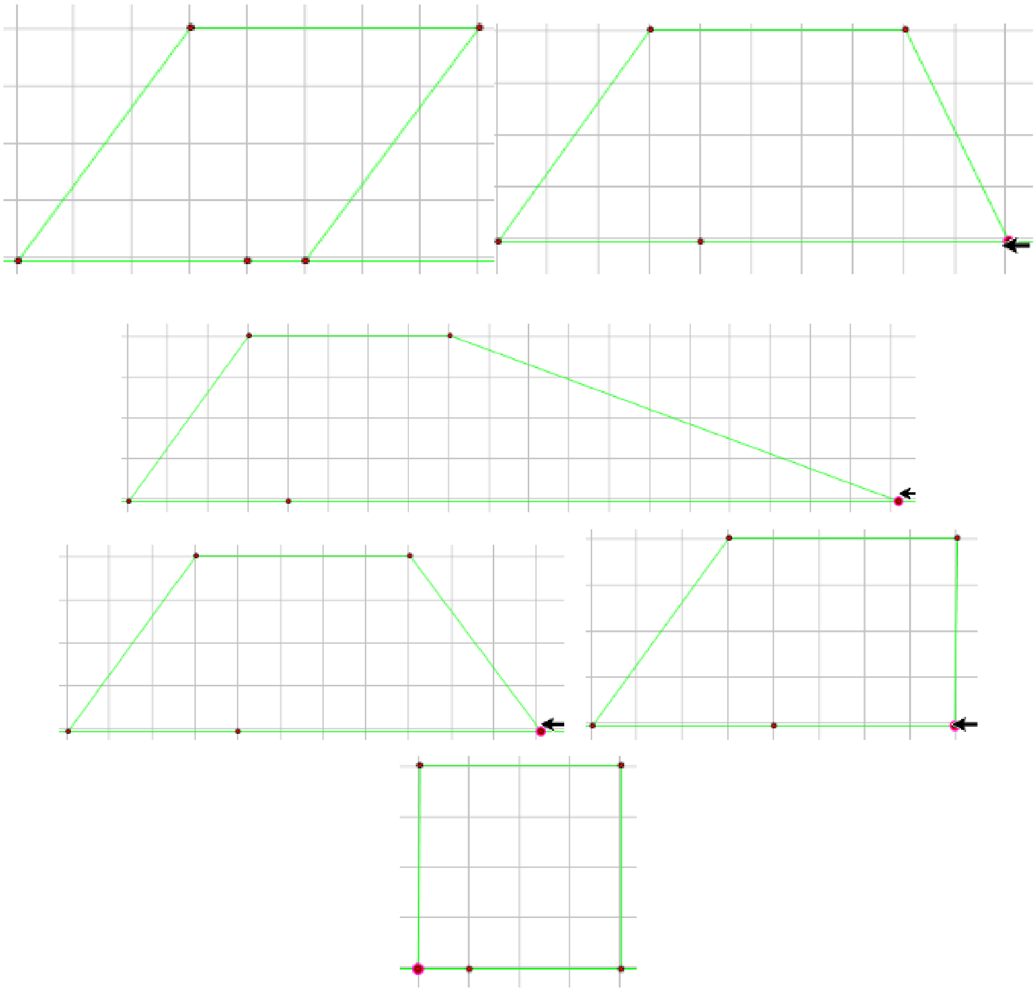


Figure 4.114 Participant 5's drawings of "quad 2"

In the next step, he was asked to define a quad3 as a special case of both quad2 and isosceles trapezoid. As an isosceles trapezoid and quad2 she correctly defined as

> *"A quad3 is an isosceles trapezoid with at least one pair of congruent adjacent sides."*
>
> *"A quad3 is a quad2 with opposite supplementary angles."*

In terms of a cyclic quadrilateral, the participant initially defined as

> *"A quad3 is a cyclic quadrilateral which is a trapezoid."*

However, when I encouraged him to think of the quadrilaterals satisfying this definition, he realized that rectangle also included by this definition though it did not have any congruent adjacent sides and so was not a quad3. After thinking a while he could not construct a definition and decided to make the sketch first and then think on the definition. He was able to draw only two quadrilaterals including an isosceles trapezoid and a square (Figure 4.115).

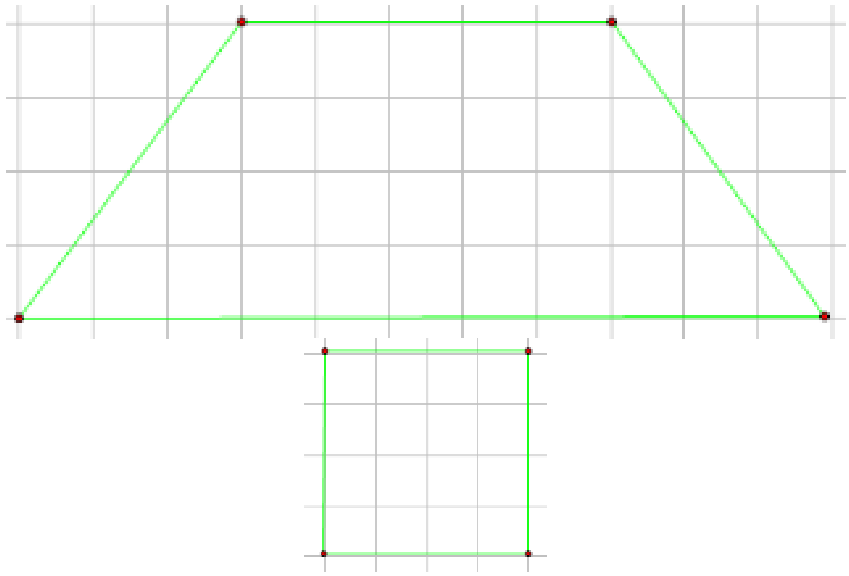


Figure 4.115 Participant 5's drawings of "quad 3"

After drawing the figures he redefined quad3 correctly as a cyclic quadrilateral as

> *"A quad3 is a cyclic quadrilateral with at least 3 congruent adjacent sides."*

Then, he wrote another definition that *"A quad3 is a cyclic quadrilateral with at least one pair of congruent adjacent sides."* However, when he was asked to think on any counter example which was not among the drawn quad3s, he realized that a right kite was also included in this definition. That is to say, a cyclic quadrilateral with at least one pair of congruent adjacent sides did not necessarily have to have one pair of parallel sides. Upon this judgment he redefined as *"A quad3 is a cyclic quadrilateral with more than two congruent adjacent sides;"* however, this definition was just another way of saying *"A quad3 is a cyclic quadrilateral with at least 3 congruent adjacent sides."*

When I asked him whether quad3 would be defined as *"a trapezoid with 3 congruent adjacent sides,"* he first responded yes; however, after looking at the quad3 sketches, he realized that rhombus also was included in this definition though it was not a quad3. Moreover, he stated that the rhombus could only be eliminated if the definition was changed as *"an isosceles trapezoid with 3 congruent adjacent sides."* Finally, he indicated square as a special case of quad3 on the hierarchy diagram.

4.5.4 Participant 5's Opinions about the Quadrilateral Learning Experience in the GSP Learning Environment

The participant found the GSP assisted learning experience amazing, because it provided an environment where one could see the things that could only be imagined in the paper pencil learning. He believed that the GSP made the abstract things more concrete due to its visual richness and the dynamic properties. The most effective side of the GSP according to the participant was making the relationships between the quadrilaterals more understandable due to its ability to preserve the properties while dragging the figures into its descendants. Moreover, he said that he did not encountered any technical or other problem while using the GSP, but that the construction process required having a good command of the construction tools.

According to the participant, the GSP learning differed from the traditional learning in the sense that it provided rich visual representations and allowed concurrently to observe the change in one of them when the others were changed. He stated that before the study he was positive towards the use of technology in mathematics lessons; however, after the study he became more positive and believed that technology assisted learning would make things easier if used properly. He said that the final hierarchical relationships that he came up with as a result of his GSP assisted experience impressed him to a large extent.

When the participant was asked about the new things that he learned in this study, he stated that in every task he learned new things that he did not know before. For example, in addition to the relationships between the quadrilaterals, he learned how to define concepts using their critical defining properties.

The participant stated that he would like to apply in his own teaching what he learned in this study related to the definitions. He would like to make students to comprehend the relationships considering the definitions.

4.5.5 Participant 5's Understanding of the Nature of the Quadrilateral Definitions and the Hierarchies after the Clinical Interviews

While the participant was able to construct correct economical definitions of trapezoid and rectangle, he wrote an incorrect definition of isosceles trapezoid. He defined trapezoid as *"a quadrilateral with at least one pair of parallel sides"* and defined rectangle as *"a quadrilateral with all right angles"* which were the correct economical definitions to characterize the related quadrilaterals. When it comes to the definition of isosceles trapezoid he incorrectly stated that an isosceles trapezoid was *"a trapezoid with one pair of opposite congruent sides,"* but this definition also included rhombus which was a trapezoid, but not an isosceles trapezoid due to not having congruent diagonals and congruent adjacent base angles.

When the participant was asked to identify examples of the given definitions among the given set of quadrilaterals, he did not identify right kite as an example of the definition that "a cyclic quadrilateral with at least one pair of congruent adjacent sides" and did not identify rhombus as an example of the definition that "a trapezoid with at least 3 congruent adjacent sides," although these quadrilaterals satisfied the related definitions.

Moreover, the participant 5 very successfully constructed two alternative correct economical definitions of a rhombus as "a quadrilateral with all congruent sides" and as "a quadrilateral perpendicular and bisecting diagonals." That is to say, all his definitions included the sufficient conditions to define rhombus in a different way.

It was also seen that the participant perfectly identified all the necessary and sufficient defining properties of quadrilaterals and explained the reasons very well. For instance, he stated that having congruent diagonals was not a sufficient condition to generalize a rectangle since an isosceles trapezoid could be generalized as well. He said that the diagonals also needed to bisect each other besides being congruent in order to generalize the rectangle. Furthermore, the participant stated that a quadrilateral which had one pair of parallel sides and the other pair congruent was not always a parallelogram, because an isosceles trapezoid also satisfied this condition. So he explained that the other pair of sides also must have been parallel to generalize a parallelogram.

As in the case of participant 3, Participant 4 also considered the given group of figures as a whole quadrilateral set, so he excluded the quadrilaterals which he was asked to exclude, from the whole quadrilateral set rather than the given group of quadrilaterals. For instance, when he was asked to construct a definition including kite and rhombus, but excluding the square, he wrote that *"quadrilaterals with at most 2 right angles."* However, this definition was not sufficient since there were many other quadrilaterals other than kite and rhombus having at most two adjacent or opposite right angles; a right trapezoid, for instance. In order to be able to define just kite and rhombus, there was need to add their defining property such that "two pairs of congruent adjacent sides." Moreover, he defined just the prototypical kite by separating it from the special cases of rhombus and square as "*a quadrilateral of which diagonals do not bisect each other,*" however, he missed that there were many quadrilaterals other than the prototypical kite satisfying this definition. This definition could be corrected as "a quadrilateral of which diagonals were perpendicular and only one diagonal is bisected by the other diagonal." He also wrote an alternative definition that "*a quadrilateral which is not equilateral,*" but this definition included all quadrilaterals except for rhombus and square. So it could not be the definition only for prototypical kite. Finally, he incorrectly defined whole kite class including the prototypical kite, rhombus and square as "*shapes with 4 sides,*" which indicated again that he considered whole quadrilateral set incorrectly.

He also perfectly identified hierarchical relations considering the properties of the quadrilaterals. He knew that a kite could be cyclic when it had one pair of opposite right angles and a trapezoid could be cyclic when it was an isosceles trapezoid. Similarly, he knew that a kite could have bisecting diagonals when this is a special kite, namely rhombus. He also stated that a square was always a kite and cyclic quadrilateral. Moreover, Participant 5 was the only participant who correctly placed all quadrilaterals on the hierarchy diagram.

CHAPTER V

CROSS–CASE ANALYSIS RESULTS

In the previous sections, each case was examined in detail and a thick description was presented in order to develop an in-depth analysis of the phenomenon under the study. In this section, data was analyzed across the cases in order to find out the differences and similarities between them, which helped to generate a wholistic interpretation and to further understand the phenomenon of interest.

The cross-case analysis results were stated under 5 headings which addressed the comparison of the participants' initial perceptions of the definitions, initial understanding of the nature of the definitions and the hierarchies; their cognitive progress during the clinical interviews, opinions about the experience in this study and final understanding of the nature of the definitions and the hierarchies.

5.1 Initial Perceptions of the Definitions

For a better comparison of the similarities and differences between the cases, a comparison matrix table was presented (Table 5.1). The cross-case analysis of the results indicated that all 5 pre-service teachers had very common perceptions about the definitions; regarding their perceptions of a good definition and beliefs about the importance of the definitions, their thoughts about their own proficiency level of defining, their experiences with the concept definitions so far, and their future plan for in what way to use the definitions in their own teaching.

Table 5.1 Matrix Comparing the Initial Perceptions of the Definitions between 5 Cases

	Participant 1	Participant 2	Participant 3	Participant 4	Participant 5
Importance of definitions	important	important	important	important	important
	defining a concept did not necessarily mean that concept was learned. making inferences between the properties and examining both concept definitions and concept images was also necessary.	defining a concept did not necessarily mean that concept was learned. memorization without understanding would be the case; defining the same concept in different situations and in different contexts was also necessary.	defining a concept did not necessarily mean that concept was learned. constructing different definitions for the same concept would indicate that the concept was learned.	defining a concept did not necessarily mean that concept was learned. memorization without understanding would be the case; engaging students with different and contradictory examples of the definitions was necessary.	defining a concept did not necessarily mean that concept was learned. there was a need for executing its similarities and differences from the other concepts depending on its definition.
Good definition	all known properties must be included in the definition.	all known properties must be included in the definition, but it must be as clear as possible.	all known properties must be included in the definition but the language used in the definition should be appropriate to the student level.	definition must not include more than necessary information; the extra information could be inferred from the definition.	all known properties must be included in the definition but it should not lead up to the confusions .
Believes in definition construction skills	disbelief in good definition construction skills.	disbelief in good definition construction skills.	disbelief in good definition construction skills.	disbelief in good definition construction skills.	disbelief in good definition construction skills.

Table 5.1 (continued)

Possible causes of disbelief in defining skills	definitions were not paid sufficient attention throughout elementary and secondary education.	definitions were not paid sufficient attention throughout elementary and secondary education.	definitions were not paid sufficient attention throughout elementary and secondary education.	definitions were not paid sufficient attention throughout elementary and secondary education.	definitions were not paid sufficient attention throughout elementary and secondary education.
	realized the importance of a definition in the undergraduate courses.	realized the importance of a definition in the undergraduate courses.	realized the importance of a definition in the undergraduate courses.	realized the importance of a definition in the undergraduate courses.	realized the importance of a definition in the undergraduate courses.
Possible use of the definitions in in-service teaching	would give the pre-constructed definitions in her own teaching.	would give the pre-constructed definitions in her own teaching but would encourage students to discuss on them.	would take the pre-constructed definitions from the text book and would encourage students to think on them.	would take the pre-constructed definitions from the text book but would check its appropriateness to the students' level; also would make some changes if necessary.	would take the pre-constructed definitions from the text book but would make students to think on the meaning of the definitions.

Findings indicated that all 5 participants had very common perceptions related to the definitions. At the beginning of the study, all 5 participants believed the importance of definitions in teaching and they explained why the definitions were important as the following:

Participant 1: "definitions are important to express the properties of the geometric shapes"

Participant 2: "definitions are important to identify a concept and to differentiate it from the other concepts; and what a definition tells us is very important to generate that concept.

Participant 3: "definitions are important to learn the properties of the concepts.

Participant 4: "definitions are important to identify concepts."

Participant 5: "definitions are important because they make us understand what that concept really is."

Although all of the 5 pre-service teachers stated that definitions were important in the teaching and learning process and they explained in what way they were important, they also agreed on that only knowing the definitions of a concept did not mean that the concept was meaningfully learned. The pre-service teachers explained what else was needed to learn a concept in addition to knowing its definition as the following:

Participant 1: "there is also need to make inferences between the properties and to accompany definitions with the concept images."

Participant 2: "there is also need to define the same concept in different situations and different contexts."

Participant 3: "there is also need to construct alternative definitions for the same concept."

Participant 4: "there is also need to criticize the correctness of different and contradictory examples of definitions."

Participant 5: "there is also need to identify the similarities and differences of the defined concept."

Actually, these abilities that participants considered as important to meaningfully learn a concept are all included in a definition construction process; but none of them were aware of what mental processes a definition construction included at that moment of the research study.

When it came to their perceptions of a good definition, except for the Participant 4, all other participants thought that a good definition should include all the known properties of the defined concept. Only participant 4 stated that a good definition shouldn't include more than the necessary information and all the other extra information should be inferred from the definition. That is to say, participant 4 did not consider a definition as a description; thereby she got a head start on the definition construction process. It is also seen that she tried to construct definitions using the minimal properties rather than listing all the properties in the later steps of the study.

Another stricking finding was that although all pre-service teachers had an idea about what a good definition was, without any exception all of them stated their disbelief in their definition construction skills. That is, except for the participant 4, all others perceived a long list of the properties as a good definition; but when they were asked about their belief in their ability to construct mathematically workable definitions, they stated a disbelief. It can be inferred from their answers that participant 4 had a disbelief in her ability to construct definitions using the minimum defining properties, while all others had a disbelief in their ability to list the properties of the concepts, which was an indication of their difficulty with the properties of the concepts.

In common, all participants associated their disbelief in defining to not being encouraged to critically think on the definitions and to construct their own definitions so far. As a common application in the Turkish education system, all pre-service teachers had been provided with the textbook definitions by their teachers and wrote those definitions into their notebooks without questioning them. Participants stated also that they questioned the definitions neither in terms of their subject knowledge nor in terms of their pedagogical skills as being teacher candidates; but they had understood the importance of definitions at the university level, when they were challenged to make proofs in the mathematics courses.

As for their own plans for the use of concept definitions in their own teaching, all 5 pre-service teachers stated that they would use the textbook definitions instead of constructing their own definitions. For example, Participant 1 stated that she would unquestioningly trust in the textbook definitions and would not need to check their correctness or appropriateness for the students. She thought that there was no need for further revisions on the textboook definitions since they were written considering the level of the students. On the other hand, Participant 2, Participant 3, and Participant 5 stated the least that they would encourage their students to think on the textbook definitions, while the participant 4 was the only one who would check the appropriateness of the textbook definitions to the students' level and would make revisions if necessary. Participant 4 was also the only one who knew what a good definition should entail and she stated that she would evaluate the textbook definitions before presenting them to her students. This finding revealed that she knew what a good definition was and could evaluate a textbook definition in terms of being a good definition and in terms of appropriateness of the language to the students; but she did not believe her ability to contruct definitions. That is, she could evaluate a given definition, but could not construct definition.

5.2 Initial Understanding of the Nature of the Quadrilateral Definitions and the Hierarchies

In this section, the findings of the "Questionnaire on Quadrilaterals I" were compared across the cases in terms of the cognitive abilities of identifying sufficient defining properties, using the minimality criterion in the definitions, using the correct language, constructing more than one alternative definitions for the same concept, understanding the inclusive relations, constructing inclusive and exclusive definitions and in terms of classifing quadrilaterals based on the different properties. To get a holistic picture easily, findings were displayed with a matrix comparison table for each question of the "Questionnaire on Quadrilaterals I" (Table 5.2, Table 5.3, Table 5.4, Table 5.5, Table 5.6, & Table 5.7).

The cross-case analysis result of the question 1 (see Table 5.2) indicated that when the pre-service teachers were asked to define rhombus, rectangle and square, all the definitions of the Participant 1 included more than the necessary information while all the definitions of the participant 5 included only the sufficient defining properties correctly. Participants 2 and 3 also generally had a tendency to list the properties of the defined concept instead of identifying the necessary and sufficient defining properties. That is to say, participant 5 was the most successful one to construct the correct economical definitions. As for the participant 4, the situation was different. She defined all 3 quadrilaterals in terms of "a closed shape" instead of " a quadrilateral" which totally changed the meaning of the definition and prevented it from characterizing the quadrilateral which it was intended to define. Her definitions of the rhombus, rectangle and the square were as the following:

> *Rhombus is "a closed shape of which all 4 sides are equal.*
>
> *Rectangle is "a closed shape with equal opposite sides and with all angles are 90°."*
>
> *Square is "a closed shape with all sides congruent and with all angles 90°."*

Table 5.2 Matrix Comparison of the Question 1 Findings of the Initial Questionnaire

	Participant 1	**Particaipant 2**	**Participant 3**	**Participant 4**	**Participant 5**
• **Mathematical correctness and the minimality nature of the rhombus, rectangle and square definitions** • **Inclusive relations** • **Use of the terms**	***Rombus:*** correct but uneconomical definition (description).	***Rombus:*** correct economical definition .	***Rombus:*** correct but uneconomical definition (description).	***Rombus:*** incorrect if defined as “a closed shape..” correct economical if defined as “a quadrilateral”	***Rombus:*** correct economical definition .
	Rectangle: correct but uneconomical definition (description).	***Rectangle:*** correct but uneconomical definition (description). (Same definition with the Participant 1).	***Rectangle:*** correct but uneconomical definition (description).	***Rectangle:*** incorrect if defined as “a closed shape..” correct uneconomica l if defined as “a quadrilateral”	***Rectangle:*** correct economical definition .
	Square: correct but uneconomical definition (description).	***Square:*** correct but uneconomical definition (description).	***Square:*** Correct economical definition.	***Square:*** incorrect if defined as “a closed shape..” correct economical if defined as “a quadrilateral”	***Square:*** correct economical definition .

Table 5.2 (continued)

	Inclusive Relations: Correct identification of square as rhombus but not as rectangle.	**Inclusive Relations:** Correct identification of square as rhombus and rectangle.	**Inclusive Relations:** Correct identification of square as rhombus and rectangle.	**Inclusive Relations:** Correct identification of square as rhombus and rectangle.	**Inclusive Relations:** Correct identification of square as rhombus and rectangle.
	Incorrect use of the terms "equal" and "congruent."	Correct use of the terms "equal" and "congruent."	Correct use of the terms "equal" and "congruent."	Incorrect use of the terms "equal" and "congruent."	Correct use of the terms "equal" and "congruent."
	NOTICE: She was the **least successful** participant.		**NOTICE:** She completed her definitions with "geometric shape" instead of"quadrilateral" I changed her definitions from "a geometric shape…." to the "a quadrilateral …"	**NOTICE:** She completed her definitions with "closed shape" instead of"quadrilateral" I analized the definitions in both ways, but I accepted the "closed shape" while evaluating.	**NOTICE:** He was the **most successful** participant among the others who constructed correct inclusive economical definitions for all three quadrilaterals

If her definitions were considered as if they were defined in terms of a quadrilateral, all 3 definitions would be the correct economical definitions; however, saying "a closed shape" made things different. For example, from her rhombus definition one can infer that it will be a closed four-sided figure and all 4 sides will be congruent; however based on this definition, as Pereira-Mendoza (1993) stated in a different context, it is possible to construct many different closed shapes which are not quadrilaterals (Figure 5.1).

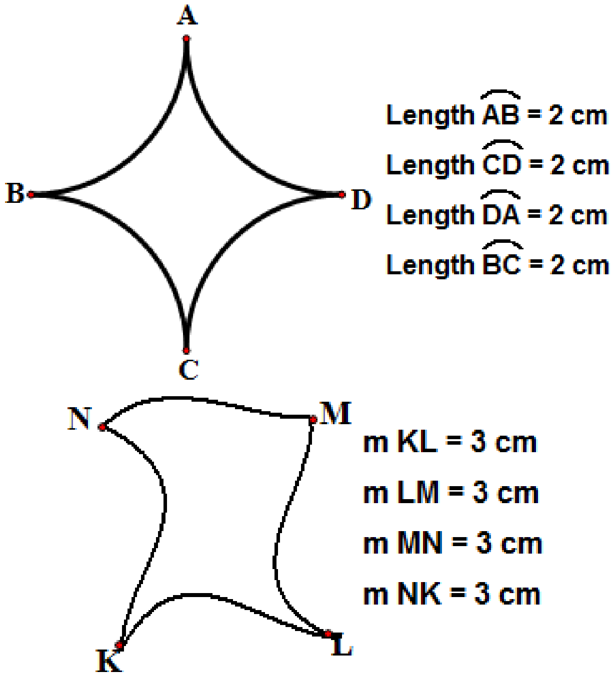


Figure 5.1 Closed shapes which are not quadrilaterals

That is to say, participant 4's definition in this form does not necessarily refer to a quadrilateral. Then, what is needed to be made clear is to state that the sides are linear line segments in such a way that *"a rhombus is a closed shape with congruent linear line segments for sides."* However, even this definition would not be enough to generate a quadrilateral; because one can still construct a figure of which all vertices are not in the same plane as the following (Figure 5.2):

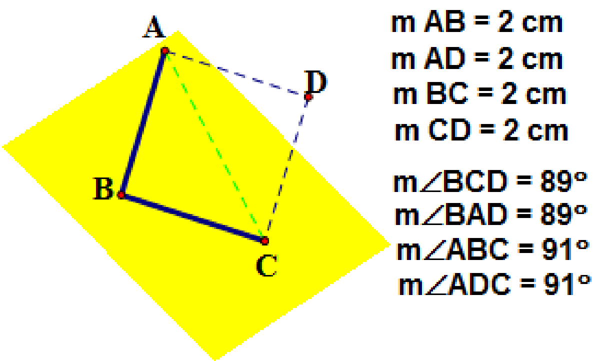


Figure 5.2 A closed shape with linear sides of which all vertices are not in the same plane

So, to be able to generate a quadrilateral there is also need to know that all vertices lie on the same plane. Finally, the definition in terms of a closed shape would be like *"rhombus is a closed shape with 4 congruent line segments for sides all of which are lying on the same plane."* Other than that, it would be defined in terms of quadrilateral as *"a rhombus is a quadrilateral with all congruent sides."* Because it is already known that a quadrilateral is a four sided closed shape of which all sides are linear line segments and all the vertices lying on the same plane. It is seen from this finding that even an incorrect use of a word makes crucial differences on the meaning of a definition and this finding highly recommends to teacher educators to make the pre-service teachers, who are expected to have a substantive knowledge of the terms in their field, aware of this nuances.

On the other hand, when the pre-service teachers were asked to identify the rhombuses, rectangles and squares among the given group of quadrilaterals, except for the Participant 1, other participants were able to identify square as special rhombus and rectangle which was an indication of that they did not consider only the prototypical shapes, but also considered the inclusive relations between them. Only, Participant 1 did not identified square as a special rectangle. Due to making judgments based on the visual appearance of the prototypes, she had had an inconsistency between her concept image and concept definition. For example, although her correct, but uneconomical definition of rectangle as "a quadrilateral with parallel and equal opposite sides and with all 90° angles" had included the square as an example, she only identified prototype rectangle figures but not the square figures as examples accompanying with the definition. This indicated an inconsistency between her concept definition and concept image, which was caused by her prototypical image of rectangle. That is to say, she did not identify square as an instance of rectangle due to the reason that square did not fit to the visual appearance of her prototype concept image of rectangle.

As for the cross-case analysis of organizing the several properties through identifying the sufficient defining properties and for constructing more than one equivalent definition using the different defining properties, a comparison matrix was presented in Table 5.3. It is seen that almost all participants failed; only Participant 1 was successful to some extent among the others. In their first attempt, Participant 5 constructed an incorrect kite definition due to his misinformation about the regular quadrilateral; although the only quadrilateral was the square, he defined kite as a regular quadrilateral. The first definitions of the other participants were the descriptions rather than the definitions; only the definition constructed by the Participant 1 was more closer to a correct economical definition. Moreover, participant 4's definition was also including deficient information in addition to including redundant information; because she did not make it clear in one of the defining properties how many adjacent sides were congruent. Moreover, in their second attempt to define a kite, Participant 1 was the only one to be able to write a correct economical definition; but all the remaining definitions were the descriptions including more than the necessary information. Same as their first definitions, participant 4's definition again included deficient property statement and participant 5's definition was incorrect. Moreover, it was also seen that the participant 4 and participant 5 incorrectly used the terms "congruent" and "equal" in the definitions.

Table 5.3 Matrix Comparison of the Question 2 Findings of the Initial Questionnaire

		Participant 1	**Particaipant 2**	**Participant 3**	**Participant 4**	**Participant 5**
• Sufficient defining properties • Equivalent alternative definitions and their minimality nature		Good at identifying necessary and sufficient defining properties to some extend.	Not good at identifying necessary and sufficient defining properties.	Not good at identifying necessary and sufficient defining properties.	Bad at identifying necessary and sufficient defining properties.	Very bad at identifying necessary and sufficient defining properties.
		Good at making inferences among the properties to some extend.	Not good at making inferences among the properties.	Not good at making inferences among the properties.	Bad at making inferences among the properties.	Very bad at making inferences among the properties.
		Best performer among the other participants.				Worst performer among the other participants.
	Definition 1	uneconomical (description)	uneconomical (description)	uneconomical (description)	uneconomical (description)	incorrect
		redundant information	redundant information	redundant information	redundant and deficient information	misinformation of regular quadrilateral

Table 5.3 (continued)

					incorrect use of the terms "equal" and "congruent"	
	Definition 2	economical	uneconomical (description)	uneconomical (description)	uneconomical (description)	incorrect
		no redundant information	redundant information	redundant information	redundant and deficient information	incorrect use of the terms "equal" and "congruent"
					incorrect use of the terms "equal" and "congruent"	

Cross-case analysis of the question 3 indicated that while participant 2, participant 3, and participant 4 were good at evaluating the given rhombus definitions, participant 1 and participant 5 failed (Table 5.4). Moreover, it was found that all 5 participants did not know the symmetry property of rhombus and this caused them to fail to evaluate the rhombus definition, constructed based on the symmetry property, as an incorrect one though it was. The definition that *"a rhombus is a quadrilateral of which symmetry axes are the perpendicular lines passing through the opposite vertices"* included the symmetry property as the sufficient defining property to generate a correct economical definition of the rhombus. A rhombus is the most general quadrilateral with symmetry axes as the both perpendicular diagonals. Any ordinary quadrilateral can have perpendicular diagonals, but these diagonals would not be the symmetry axes. A kite also has perpendicular diagonals, but only one diagonal is the symmetry axis, not both of them. On the other hand, a square also has perpendicular diagonals as the symmetry axis, but additionally these diagonals are congruent to each other, which specializes it as a special rhombus. That is to say, the given definition was a correct economical definition of the rhombus including the symmetry property as a defining property; however, it was seen that participants' misinformation about the properties of the concepts prevented them from constructing or evaluating the correct concept definitions.

Table 5.4 Matrix Comparison of the Question 3 Findings of the Initial Questionnaire

• Evaluating the given rhombus definitions	**Participant 1**	**Particaipant 2**	**Participant 3**	**Participant 4**	**Participant 5**
	Bad at evaluating the given rhombus definitions.	Good at evaluating the given rhombus definitions.	Good at evaluating the given rhombus definitions.	Good at evaluating the given rhombus definitions.	Bad at evaluating the given rhombus definitions.
	Failed at many of the definitions.	Just failed at the definition including the symmetry property as defining property.	Just failed at the definition including the symmetry property as defining property.	Just failed at the definition including the symmetry property as defining property.	Failed at many of the definitions.

For the analysis of the cognitive ability to construct inclusive and exclusive definitions, the findings were summarized in the Table 5.5. When they were asked to write some definitions that included some quadrilaterals, but excluded some others, it was found that participant 5 totally faile. That is to say, he could not write any definitions due to not being aware of the exclusive and exclusive nature of the definitions. Participant 3 and Participant 4 also failed, but they tried the least to construct definitions; even participant 3 was able to write one correct definitions though the others were incorrect. On the other hand, participant 1 and participant 2 were able to construct all correct definitions; however, participant 1 also used the terms “congruent” and “equal” interchangeably at this step of the defining in addition to participant 3 and participant 4.

Table 5.5 Matrix Comparison of the Question 4 Findings of the Initial Questionnaire

	Participant 1	**Particaipant 2**	**Participant 3**	**Participant 4**	**Participant 5**
• **Inclusive and exclusive definition construction**	Good at constructing inclusive and exclusive definitions.	Good at constructing inclusive and exclusive definitions.	Bad at constructing inclusive and exclusive definitions.	Failure Very bad at constructing inclusive and exclusive definitions.	Total failure.
	All 3 definitions are correct.	All 3 definitions are correct.	Just one correct among 3.	All 3 definitions are incorrect.	No definitions.
	Incorrect use of “equal” and “congruent.”	Best performer.	Incorrect use of “equal” and “congruent.”	Incorrect use of “equal” and “congruent.”	

In terms of evaluating the given statement considering the inclusive relations, the findings for each case were presented in the Table 5.6. When they were asked to complete the statements reflecting the inclusive relations between the quadrilaterals, it was seen that participant 1, participant 2, participant 3 and participant 4 were able to identify the inclusive relations. This was an indication of their not having any cognitive conflict related to the prototypes, which was the cause of incorrect judgements (Schwarz & Hershkowitz, 1999). Participant 2 only failed to consider a square as a special rectangle, but it was not due to the prototypes, but due to the carelessness. However, participant 5 was again the worst performer among the others due to his misinformation about the properties of the rhombus and rectangle.

Table 5.6 Matrix Comparison of the Question 5 Findings of the Initial Questionnaire

• Inclusive relations between the quadrilaterals • Prototypes	**Participant 1**	**Particaipant 2**	**Participant 3**	**Participant4**	**Participant 5**
	very good at understanding the inclusive relations between the quadrilaterals through considering their properties.	good at understanding the inclusive relations between the quadrilaterals through considering their properties.	very good at understanding the inclusive relations between the quadrilaterals through considering their properties.	very good at understanding the inclusive relations between the quadrilaterals through considering their properties.	not very good at understanding the inclusive relations between the quadrilaterals through considering their properties.
	no prototypical concept images.	disregarded square as a special rectangle.	no prototypical concept images.	no prototypical concept images.	misinformation about the properties of rhombus and rectangle.

As for the cross case analysis of the last question in which participants were asked to classify quadrilaterals based on the different properties of these quadrilaterals, all 5 participants failed (Table 5.7). While Participant 4, Participant 1 and Participant 5 were able to put some of the quadrilaterals into the correct places in the diagram, they generally failed, as well. This finding was an indication of the participants' difficulty with the quadrilateral properties. However, the possible cause of the failure in this question could be the participants' not reading the caution that they have to consider the prototypes of the each quadrilateral while classifying the quadrilaterals depending on the given properties. Not considering the prototypes, but also considering the quadrilateral classes could have caused them to reason incorrectly in this question.

Table 5.7 Matrix Comparison of the Question 6 Findings of the Initial Questionnaire

	Participant 1	Particaipant 2	Participant 3	Participant 4	Participant 5
• Classifying	Bad performance of classifying quadrilaterals based on their properties.	Very bad performance of classifying quadrilaterals based on their properties.	Very bad performance of classifying quadrilaterals based on their properties.	Better performance than the other participants in classifying quadrilaterals based on their properties.	Bad performance of classifying quadrilaterals based on their properties.
					Although he was told to think of prototypes for each quadrilateral, he considered the special cases while deciding.

5.3 Clinical Interviews

In this section, cross-case analysis of the findings obtained from the clinical interview sessions was holistically presented with reference to the matrix comparison tables prepared for each clinical interview sessions separately. These tables can be seen in the appendices sections J, K, L, and M.

The findings of the clinical interviews indicated that the pre-service teachers had difficulty with the properties of the quadrilaterals. For example, when they were asked to tell the properties of the kite at the beginning of the first session, it was seen that Pariticipant 2, Participant 3 and Participant 5 could not tell the side, angle, diagonal and symmetry properties; even Participant 5 had misleading information related to the properties. Participant 5 stated for example that “the area measures of the triangles above and below the bisected diagonal are equal to each other;” though it was not a defining property, but a special case of a kite when it was a rhombus or square. Moreover, while the Participant 1 knew all the properties correctly, Participant 4 could not state a few of them. Another finding revealed also that participants’ misconceptions regarding the properties sometimes caused them to make incorrect evaluations as well. For instance, when Participant 1 was asked to evaluate the given kite definition which was “A kite is a quadrilateral with at least one diagonal is an angle bisector,” she incorrectly thought that parallelogram had one diagonal as the symmetry axis; however, this misconception regarding the symmetry property of the parallelogram caused her to evaluate this correct definition as incorrect since she detected parallelogram as a counter example to that definition. By the same token, Participant 2 had difficulty to construct exclusive definition of kite using its symmetry property, because she had a misleading information that a rhombus was symmetrical with respect to the lines passing through the midpoints of the sides. However, it was found that the dragging the dynamic figure of the related quadrilateral and observing the changes in the measurements helped Participant 2, Participant 3 and Participant 5 to discover the critical side, angle, diagonal, and symmetry properties that they could not stated before this activity. While the Participant 1 and Participant 4 checked the correctness of the properties that they had stated by the dragging activity, Participant 4 also discovered the ones that she could not have stated. That is to say, observing the figures in the dynamic learning environment was found to be helpful to handle the pre-service teachers’ problems with the properites of the quadrilaterals.

On the other hand, just discovering the properties and knowing them was not enough to construct definitions, there was also need to use the correct mathematical language to state them. For example, while stating the side property of the kite, all participants had a tendency to make a statement which was specific to the figure that they saw on the GSP secreen. Rather than saying that “sides AB and AD are congruent to each other and sides BC and CD are also congruent to each other,” the participants were encouraged to make generel statements describing not only the side property of the figure on the secreen but a statement true for all kites; such as “a kite has at least two pairs of congruent adjacent sides.” However, the analysis of the data indicated that most of the participants had difficulty to express the properties with the correct mathematical language. The most difficult property to state was found to be the diagonal property of the isosceles trapezoid that “the diagonals bisected each other in the same ratio.” For example, it took much time for the Participant 2 to explain this information with the correct words and finally she was able to state that “the diagonal parts constructed by the intersection of diagonals were equal in length.”

On the other side, particiapant's incorrect or deficient use of the mathematical language sometimes caused them to incorrectly construct and evaluate the definitions, too. For instance, although she had known side property of the kite correctly, participant 3 could not express her knowledge with the correct terms and this caused her to give a different meaning in the definition. In one of her kite definitions she was mistaken saying that *" kite is a quadrilateral with at least one pair of equal adjacent side lengths...,"* though two pairs of the adjacent side lengths must be equal in a kite. Similarly, Participant 4 constructed an incorrect isosceles trapezoid definition due to the missing information in her property statement. She stated in the definition that *"An isosceles trapezoid is a quadrilateral…….with at least one pair of congruent sides;"* however, her property statement was lacking the information whether the opposite or adjacent sides were congruent. There was need to this little information in order to be able to infer the correct quadrilateral from the definition. In addiditon to the improper use of the terms like "opposite" and "adjacent," or "one pair" and "two pairs," there was another very common incorrect use of the terminology like "equal" and "congruent." Throughout the clinical interviews all 5 participants oftenly used these two terms interchangeably; however, their consciousness about the difference between these terms were increased step by step and they started to use them correctly to a large extent. Their improper usage of the terms "congruent" and "equal," probably based on their previous experiences where their attention was not drawn on the difference between these terms and where they did not heard the correct usage of these terms from their teachers. Moreover, their improper use of the terms "opposite" and "adjacent" could also be due to their carelessness while expressing their thinking or due to their unawareness of the effect of a single word on the meaning. The misuse of such kind of mathematical language by the teachers is a very important problem in the sense that they can deliver the incorrect meaning of mathematical concepts and can mislead learners' understanding of the concept defined. Unfortunately, there was nothing to do with the GSP to teach them the correct terms; however, drawing their attention on their teminology through redirecting them questions over and over was helpful to make them aware of the incorrect terminolgy they used. For example, when they left the statement just saying "a kite has equal angles," they were asked subsequently "which angle measures are equal in a kite?" and "which angles are congruent in a kite?" It was seen that when they made similar mistakes, this kind of probing questions helped them to realize the need for specifying whether they meant the opposite or adjacent angles. Besides, hearing the two different usage of the terms "congruent" and "equal" as in the case of above example, encouraged them to think on the difference between the meaning of these two terms. After many trials of such kind, almost all participants became more cautious about the use of words and started to use the correct language to deliver the correct meaning.

Another cognitive ability analyzed across the cases was understanding the inclusive relationships between the quadrilaterals. For example, after determining the critical properties of the kite in the first session, participants were encouraged to critically think on the properties and to decide which other quadrilaterals the kite figure could be dragged into; in other words, which other quadrilaterals had the kite properties or which other quadrilaterals were the special instances of kite, if there any. Through comparing the properties one by one mentally, they tried to find out in which other quadrilaterals kite properties were preserved. However, it was seen that only participant 5 was able to correctly identify all special kites; while the participant 3 incorrectly identified all of them. Other participants commonly identified square as a special kite, but some of them could not identify rhombus or incorrectly identified trapezoid as a special kite. For example, participant 3 decided just looking at the dynamic figure on the screen visually without considering the preserved properties and she decided that a square could not be a kite; she thought that the congruent angles of the dynamic kite figure on the screen could not be dragged into right angles. This thought of the participant was due to not grasping the idea that the dynamic kite figure preserved the critical defining property of one pair of opposite congruent angles whatever the angle measures were. That is to say, when the figure was dragged, angle measures would change; but the angles always remained congruent. This finding indicated that the participant did not understand the nature of a dynamic figure that it could be dragged into the other figures which had all of its critical properties. When she saw that those congruent angles could be dragged into any angle measures, she uncounsciously accepted the square as a special kite. Although a quadrilateral has to have all the properties of a general quadrilateral in order to be the special case of it, participant had only considered the angle property, but had not checked whether the other kite properties were preserved. So, her decision was correct, but it did not depend on a sound reason. Further, it was found that participant 3's misconception also caused her to make other incorrect decisions as well; for example, she identified the rectangle as a kite, too. She explained the reason as "if I can drag a kite into a square, I can also drag it into a rectangle too, because a rectangle is a special square." That is to say, her reason was also including the misunderstanding of the relation between rectangle and square. Misconceptions followed other misconceptions in her mental process and she also stated that *"if I can drag the dynamic kite figure into a rectangle, I can also drag it into a parallelogram as well."* That is to say, she was totally confused and her mental process was full of the misconceptions and conflicts. It was obvious that she just visually imagined dragging the figure into other quadrilaterals, but she did not consider the relationships in terms of the mathematical properties through critical thinking. She also herself confirmed that she only imagined as if she dragged the kite figure in order to make her decisions. That is, she was just chucking the answers around without a sound base for reasoning. In addition to her not grasping the role of dynamic figure, participant 3 had said that *"I always thought that a rhombus is a rhombus; but I had never thought that a rhombus is a kite, since they looked visually different."* That is to say, she had the prototypical images of the quadrilaterals which prevented her from seeing the inclusive relations between the quadrilaterals.

As for the participant 2, she also had some misconceptions related to the inclusive relations; but at least she had understood the criteria to be a special case of a quadrilateral to some extent. For example, she correctly detected square as a special kite and a rectangle as not a special kite; more importantly, she had correctly explained the reasons by thinking about all side, angle, diagonal properties. However, Participant 2 thought that rhombus did not have at least one pair of congruent opposite angles, though it was one of the requirements to be e special kite. Her incorrect mental process for the rhombus was due to her incorrect knowledge that rhombus did not have opposite congruent angles. Moreover, participant 2 incorrectly determined trapezoid as a special kite; but she could not explain the reason.The misdecision of the Participant 2 was again due to her deficient knowledge about the properties of the trapezoid. However, at that moment of the interview, she had not worked on the properties of the trapezoids yet; she had just used her initial knowledge. That is to say, Participant 2 knew what made a quadrilateral a special case of another quadrilateral, but she reasoned incorrectly due to her insufficient knowledge of the properties.

When they were asked to examine the descendants of the isosceles trapezoid in the second session, participants performed better, but there were still some problems. All of the participants were able to correctly identify square and rectangle as special isosceles trapezoids; and they were able to explain the reasons correctly. However, some of them incorrectly identified prototypical trapezoid and parallelogram as special isosceles trapezoids. For example, Participant 1 was able to correctly eliminate parallelogram and rhombus due to their not satisfying the symmetry property of the isosceles trapezoid. However, she thought that prototypical trapezoid was a special isosceles trapezoid, eventhough the case was the reverse; and this was probably due to her carelessness about the properties. On the other hand, Participant 2 identified parallelogram as isosceles trapezoids incorrectly. She related the parallelogram to the rectangle and claimed that parallelogram also satisfied the side and diagonal properties of the isosceles trapezoid. However, Participant 2 did not pay attention to the fact that only the opposite angles of a parallelogram were congruent, not the adjacent angles. Moreover, she did not realized that a parallelogram did not have to have congruent diagonals, which eliminated its being an isosceles trapezoid, although she had eliminated rhombus considering this property. That is to say, her misdecision for parallelogram was probably due to her insufficient knowledge about the properties of the parallelogram. Similarly, Participant 3 also chose the parallelogram as a special isosceles trapezoid for the same reasons as participant 2. So, it was seen that it was a common tendency of almost all participants to consider a parallelogram as a rectangle and to directly state that it was also a special isosceles trapezoid, without considering the properties. They had a tendency to ignore thinking on the angle property of the parallelogram and isosceles trapezoid, which mostly caused them to fail.

Another finding that was very surprizing was the participants' misconceptions about the relationships between the parallelograms, though parallelograms were the most familiar quadrilaterals. All of the participants were able to identify rectangle, rhombus and square as parallelograms, but when it came for the relationship between the rectangle and rhombus, they failed. For example, Participant 1 and Participant 2 both thought that rhombus was a special rectangle at first. However, when they were encouraged to compare their properties, they realized that neither a rhombus, nor a rectangle could be the special cases of each other. Participant 4, on the other hand, identified rectangle as a special rhombus at first; but after thinking on the properties she stated that a rhombus was a special rectangle. Similar to the other participants, she was able to realize that there was no relation between the rectangle and square except for their common descendant square. Participant 3 also was doubtful about the relationship, but she was able to correctly identify it by critically thinking on the properties. Dragging the dynamic figures onto each other to understand whether one of them would be the special case of the other helped all participants to identify those relationships easily and straightened their doubts out.

After explaining the relationships between the quadrilaterals, the next cognitive process analyzed across the cases was the ability to construct inclusive and exclusive quadrilateral definitions. However, as the inial findings indicated before, it was found that all participants were initially making descriptions rather than the definitions. Only quadrilateral that almost all participants were able to construct correct economical definition in their first attempt was the kite. This was because of their concept images that a kite was formed by connecting the bases of two isosceles triangles. They constructed a correct economical definition of the kite in their first attempt since the property of their concept image of the kite was sufficient defining property. That is to say, participants who defined kite correctly in their first attempt was not aware of they were constructing correct economical inclusive definition. When they were asked to change their descriptions into definitions, all 5 participants were able to identify the necessary and sufficient defining properties by the help of the GSP constructions made for the properties given in the definitions. However, after many trials of constructing inclusive economical definitions for different quadrilateral concepts, it was seen that all 5 participants learned to make reasoning among the properties to identify the sufficient defining properties whithout need for the GSP construction any more. That is to say, identifying the critical defining properties with the GSP contruction moved all participants from making descriptions towards making definitions by themselves.

As for the cognitive ability of exclusive defining, participants were first asked to evaluate a pre-constructed correct exclusive definition of isosceles trapezoid in order to see their reactions before moving onto the exclusive definition construction process. All of them were able to easily detect out that the definition was only including the prototypical isosceles trapezoid, but not its special cases rectangle and square. However, it was seen that all participants accepted the inclusive definition as the correct definition and did not know that both types of the definitions were accepted as correct depending on the educational purposes. When the participants were engaged into the exclusive definition construction process of some quadrilaterals like trapezoid, parallelogram, kite and rhombus so that the definition will not include their special instances, it was found that the most difficult process for all 5 participants became identifying the properties that would eliminate the special cases from the definition. For example, in her first attempt participant 3 had defined parallelogram exclusively as *"a parallelogram is a quadrilateral of which diagonal lengths are different from one another;"* but she could not eliminate the rhombus from the definition yet. She failed again in her second attempt, because her definition that *"a parallelogram is a quadrilateral which has two pairs of parallel sides and of which diagonal lengths are different from one another"* still included rhombus as a special instance. It was easy for her to eliminate rectangle and square since both of them have congruent diagonals, but eliminating the rhombus became a little challenging for her. However, critically thinking on the diagonal properties, she was able to come up with the exclusive definition that "*a parallelogram is a quadrilateral of which diagonals bisect each other but not congruent to each other and not perpendicular to each other."* Participant 4, for example, had no difficulty to define parallelogram; however, defining rhombus using the symmetry property became challenging for her. She first thought that she could eliminate the square by defining rhombus as *"A rhombus is a quadrilateral which has only the diagonal symmetry;"* however, while eliminating the rhombus, she realized that she added a more general concept "kite" into the definition.So, in order to eliminate kite from the initial definition, she redefined as *"A rhombus is a quadrilateral which is symmetric with respect to the both diagonals;"* but this time she again added the square into the definition. She thought really long time on how to define only the rhombus and in the end, she was able to correctly define it as *"A rhombus is a quadrilateral which is symmetric only with respect to the both diagonals."* During this process she tried to find the small but important word "only" to define what she intended to define, and she did after thinking critically.

Finally the cross-case analysis of the last clinical interview also indicated that all 5 participants improved their ability to define in different ways and ability to understand the relations between the quadrilaterals such that they all were capable of defining new quadrilaterals in the hierarchy. It was seen that all 5 participants were able to construct many alternative definitions for the new quadrilaterals in the hierachy through defining them in terms of other quadrilaterals.

5.4 Opinions about the Quadrilateral Learning Experience in the GSP Learning Environment

The comparison matrix for the participants' opinions regarding their quadrilateral learning experience were summarized in the Table 5.8

Table 5.8 Matrix Comparison of the Final Interview Findings

	Participant 1	**Particaipant 2**	**Participant 3**	**Participant 4**	**Participant 5**
Reflections	This study was a great experience for her. Increased confidence for definition construction.	This study was a great experience for her. Increased confidence for definition construction.	This study was a very effective and very enjoyable experience for her. Increased confidence for definition construction.	This study was a very effective experience for her. Increased confidence for definition construction.	This study was a amazing experience for him. Increased confidence for definition construction.
Advantages and disadvantages of GSP	GSP was most effective in evaluating the given definitions through construction and in evaluating the inclusive relations through dragging the figure into other figures. can be difficult to plan lessons and to evaluate student work.	GSP was most effective in investigating the the inclusive relations; discovering the critical defining properties and providing multiple representations. no disadvantage.	GSP was most effective in identifying the defining properties and inclusive relations due to preserving the properties under dragging. can be difficult to plan lessons.	GSP was most effective in dragging the figure into other figures without changing the critical properties; in providing many trials on a dynamic environment and in providing visual and mathematical representations on one screen. can be difficult to evaluate student work	GSP was most effective in making the abstract things more concrete; in providing rich visual representations which allow concurrently to observe the change in one of them when the others were changed. can be difficult to evaluate student work

Table 5.8 (continued)

Technical difficulties	circum quadrilateral figure was difficult to drag technically.	circum quadrilateral figure was difficult to drag technically.	a little bit technical difficulty while dragging the circum quadrilateral figure.	no technical difficulty with the GSP figures. constructing economical definitions through eliminating the unnecessary properties was the most difficult part.	no technical difficulty or other problem with the GSP figures.
What learned from this study	This study taught to construct mathematically correct economical definitions in many different ways that she did not think of before.	This study taught which quadrilaterals could be cyclic and circum;and how to define in many different ways just using the defining properties.	This study taught cyclic and circum quadrilaterals and hierarchical relations between quadrilaterals.	This study taught to economize definitions by critically thinking on the defining properties.	This study taught relationships between the quadrilaterals, and how to define concepts using their critical defining properties. everything he learned in this study was new to him.
Will for use of GSP in teaching	willing to use GSP in her teaching in a similar manner.	enthusiastic to use GSP in her in-service teaching.	although she had disregarded the GSP before the study, she was enthusiastic to use it to teach definitions in her in-service teaching.	very enthusiastic to use GSP in her in-service teaching by adapting the tasks to the student level.	very enthusiastic to use GSP in his in-service teaching. .

Table 5.8 (continued)

Use of definitions as mathematical activity	would encourage ther students to construct their own definitions instead of providing them with the pre-made definitions.	believed that GSP assisted tasks would be useful tools to teach definitions of the geometric concepts.	believed that the definitions could be effective teaching tools. Much more self-confidence to teach definitions.	would use the definition construction process in her own teaching by enriching the learning environment with critical examples and non examples of the concept.	would like to apply what he learned in this study related to the definitions in his own teaching.

The cross case analysis revealed that when their opinions about their quadrilateral learning experience in the GSP Learning Environment was asked, without any exceptions all 5 participants stated that it was an effective, great and amazing experience for them and that they learned many things that they had not thought of before. It was the common idea among all participants that they learned how to contstruct mathematically correct economical definitions in many different ways that they did not think of before. Participant 2 and Participant 3 also stated that she had not thought about the cyclic and circum quadrilaterals so deeply before this study and had never thought before that the quadrilaterals could be defined in terms of their cyclic or circum properties. Participant 3 and Participant 5 also said that through critically thinking on the relationships between the quadrilaterals in this study, they had realized the exclusive and inclusive nature of the definitions and corresponding classifications for the first time. Participant 5 stated that everything he learned through this experience was new to him, and that this experience broadened his perspective of teaching geometric concepts. Moreover, the findings also revealed that when participants were asked to compare their believes in their definition construction skills before and after the study, all 5 participants stated a very increased confidence in their ability to define.

On the other side, when the participants were asked about their opinions about the effectiveness of the GSP, Participant 1 stated that GSP was most effective in the process of evaluating the definitons and in the process of discovering inclusive relations between the quadrilaterals. Similarly, Participant 2 also stated that that the GSP was most effective in discovering the inclusive relations, and further stated that GSP was also effective in discovering the critical definiting properties of the quadrilaterals to construct economical definitions. Participant 2, Participant 4 and Participant 5 were also agreed on that GSP learning environment was effective since it provided many trials with visual representations and the related measurements on one screen and allowed to observe the effect of one change on all other representations. As for the disadvantages of the GSP learning environment, Participant 1 and Participant 3 stated that planning such kind of activities would be very difficult since they required very critical planning on each detail. Moreover, Participant 1, Participant 4 and Participant 5 stated their concerns about the difficulty of evaluating students'work while using the GSP activities in the classroom setting.

In terms of the technical difficulties they encountered during the study, while Participant, Participant 2 and Participant 3 found the dragging of the circum quadrilateral a little difficult, Participant and Participant 5 stated no encountered technical difficulty. After this GSP experience, all 5 Participants also stated that they were very enthusiastic to use the GSP learning environment in their own in-service teaching. Moreover, participants' thoughts about the use of the definitons in the teaching and learning process was also changed. They all stated similar reflection that they would use the definition construction as an effective teaching activity and instead of providing their students with the textbook definitions, they would encourage them to construct their own definitions and to think on the meaning of the definitions.

5.5 Understanding of the Nature of the Quadrilateral Definitions and the Hierarchies After the Clinical Interviews

After the final interview, the participants were administered a final questionnaire, "Questionnaire on Quadrilaterals" which included the questions which were different from the initial question, but measuring the same cognitive abilities. In this section, the cross case analysis of the findings related to the each question were discussed with reference to the comparison matrix tables prepared to see the similarities and differences between the cases more holistically.

In the first question of the final questionnaire participants' ability to construct correct inclusive economical definitions and ability to define quadrilaterals in terms of other more general quadrilaterals were analyzed. The matrix comparison for the findings were provided in the Table 5.9. When the participants were asked to define trapezoid in terms of a quadrilateral and to define isosceles trapezoid and rectangle in terms of a trapezoid, it was found that participant's definitions did not included a long list of the properties any more, and it was seen that they improved their ability from making descriptions towards making definitions by trying to use the minimum defining properties that were characterizing the quadrilaterals. Moreover, they were considering the inclusive relations in their definitions, as well. All 5 participants were aware of the difference between the definition and description and they were making effort to determine the critical defining properties that were sufficient to define the related concept and its descendants. For example, when they were asked to define trapezoid in terms of a quadrilateral, all 5 participants were able construct the correct inclusive economical definitions. However, defining the isosceles trapezoid in terms of a trapezoid was the most difficult one for the participants. Participant 1 and Participant 4 were able to construct the correct definitions, but others failed to notice that their definitions included the rhombus eventhough it was not a special isosceles trapezoid due to the angle property. It was also difficult for them to realize the rhombus as a counter example during the clinical interviews, but they had got the help of the GSP dragging activity in that case. As for the rectangle, although the definitions included very little extra information, all participants were able construct the correct definition very close to the most economical definition, except for the Participant 2. Only Participant 2 incorrectly defined rectangle in terms of a trapezoid. However, all 5 participants performed very good at finding the quadrilaterals among a group of quadrilaterals which were included by the related definition. That is, they were good at understanding the inclusive relations based on the inclusive definitions.

Table 5.9 Matrix Comparison of the Question 1 Findings of the Final Questionnaire

• Mathematical correctness and the minimality nature of the trapezoid, isosceles trapezoid and rectangle definitions • Defining in terms of other quadrilaterals • Inclusive relations	**Participant 1**	**Particaipant 2**	**Participant 3**	**Participant 4**	**Participant 5**
	Trapezoid: Correct inclusive economical definition.	**Trapezoid:** Correct inclusive economical definition.	**Trapezoid:** Correct inclusive economical definition.	**Trapezoid:** Correct inclusive economical definition.	**Trapezoid:** Correct inclusive economical definition.
	Isosceles Trapezoid: Very close to economical definition but just included a little bit extra information .	**Isosceles Trapezoid:** Incorrect definition .	**Isosceles Trapezoid:** Incorrect definition since included rhombus.	**Isosceles Trapezoid:** Correct inclusive economical definition.	**Isosceles Trapezoid:** Incorrect definition since included rhombus.
	Rectangle: Very close to economical definition but just included a little bit extra information .	**Rectangle:** Incorrect definition .	**Rectangle:** Correct uneconomical definition.	**Rectangle:** Correct uneconomical definition.	**Rectangle:** Correct economical definition .
	Identfying examples of the definition: Very good at identifying examples.	**Identfying examples of the definition:** Very good at identifying examples.	**Identfying examples of the definition:** Very good at identifying examples.	**Identfying examples of the definition:** Very good at identifying examples.	**Identfying examples of the definition:** Very good at identifying examples.

In terms of their constructing alternative equivalent definitions for the same concept, cross case anlaysis indicated that Participant1, Participant 2, Participant 3 and Participant 5 were able to construct both correct inclusive definitions of rhombus, while the Participant 4 could not construct the second one (see Table 5.10). Moreover, when the participants were asked to evaluate the given statements in terms of the sufficient defining properties of the quadrilaterals, except for the participant 4, all participants were able to evaluate the statements correctly and were able to provide the reasons of their answers (see Table 5.11).

Table 5.10 Matrix Comparison of the Question 2 Findings of the Final Questionnaire

		Participant 1	Particaipant 2	Participant 3	Participant 4	Participant 5
• Sufficient defining properties • Equivalent alternative definitions and their minimality nature	Definition 1	Correct inclusive economical definition.	Correct inclusive economical definition.	Correct Uneconomical definition.	Correct inclusive economical definition.	Correct inclusive economical definition.
	Definition 2	Uneconomical definition.	Correct inclusive economical definition.	Correct Uneconomical definition.	Incorrect definition.	Correct inclusive economical definition.

Table 5.11 Matrix Comparison of the Question 3 Findings of the Final Questionnaire

	Participant 1	Particaipant 2	Participant 3	Participant 4	Participant 5
Evaluating the given statements in therms of critical defining properties	Very good at identifying critical defining properties.	Very good at identifying critical defining properties.	Very good at identifying critical defining properties.	Generally failed at identifying critical defining properties.	Very good at identifying critical defining properties.
	Correct explanations for answers.	Insufficient explanations for the deficient properties.	Insufficient explanations for the deficient properties.	Insufficient explanations for the deficient properties.	Correct explanations for answers.

The fourth question in the questionnaire was about participants'ability to construct inclusive and exclusive definitions of the quadrilaterals which were compared in the Table 5.12. It was seen that while the Participant 1, Participant 2 and Participant 4 were good at inclusive and exclusive defining Participant 3 and Participant 5 failed. However, their failure was not due to their inability in inclusive and exclusive defining, but due to their incorrectly considering the given group of figures as a whole quadrilateral group. That is, anlaysis of their answers indicated that both participants were mistaken because of their misunderstanding of what was asked in this question.

Table 5.12 Matrix Comparison of the Question 4 Findings of the Final Questionnaire

	Participant 1	**Particaipant 2**	**Participant 3**	**Participant 4**	**Participant 5**
Inclusive and exclusive definition construction	Good at constructing inclusive and exclusive definitions.	Good at constructing inclusive and exclusive definitions.	Bad at constructing inclusive and exclusive definitions.	Very good at constructing inclusive and exclusive definitions.	Bad at constructing inclusive and exclusive definitions.
	Definition 1: Correct Definition 2: correct Definition 3: incorrect (included rhombus).	Definition 1: correct Definition 2: incorrect Definition 3: correct.	Definition 1: incorrect Definition 2: incorrect Definition 3: incorrect	Definition 1: correct Definition 2: correct Definition 3: correct	Definition 1: incorrect Definition 2: incorrect Definition 3: incorrect
			did not understand the question and considered the given group of figures as whole quadrilateral group.		did not understand the question and considered the given group of figures as whole quadrilateral group.

Another cognitive ability analyzed acrossly was the understanding the inclusive relations between the quadrilaterals through the properties in the given statements (Table 5.13). It was seen that all 5 participants were able to correctly identify the inclusive relations and were able to explain the reason of their answers.

Table 5.13 Matrix Comparison of the Question 5 Findings of the Final Questionnaire

Inclusive relations between the quadrilaterals	**Participant 1**	**Particaipant 2**	**Participant 3**	**Participant 4**	**Participant 5**
	Good at understanding the inclusive relations between the quadrilaterals through considering their properties.	Vey good at understanding the inclusive relations between the quadrilaterals through considering their properties.	Very good at understanding the inclusive relations between the quadrilaterals through considering their properties.	Very good at understanding the inclusive relations between the quadrilaterals through considering their properties.	Very good at understanding the inclusive relations between the quadrilaterals through considering their properties.
	Some missing explanations for the answers.	All correct explanations for the answers.	All correct explanations for the answers.	All correct explanations for the answers.	All correct explanations for the answers.

Only participant who classified the quadrilaterals based on the diagonal property was the Participant 5 (Table 5.14). Although he had totally failed in classifying the quadrilaterals in the first questioannaire due to not understanding the question, this time he performed very well. However, it was seen that other participants still had difficulty with the properties of the quadrilaterals in the absence of the GSP.

Table 5.14 Matrix Comparison of the Question 6 Findings of the Final Questionnaire

classification	**Participant 1**	**Particaipant 2**	**Participant 3**	**Participant 4**	**Participant 5**
	Incorrect classification of quadrilaterals based on diagonal property.	Incorrect classification of quadrilaterals based on diagonal property.	Incorrect classification of quadrilaterals based on diagonal property.	Incorrect classification of quadrilaterals based on diagonal property.	Very good performance of classifying quadrilaterals based on their properties.

CHAPTER VI

CONCLUSION, DISCUSSION, AND IMPLICATIONS

The purpose of this research study was to examine pre-service mathematics teachers' cognitive progress in constructing and evaluating quadrilateral definitions and the corresponding quadrilateral hierarchies under the support of the Geometer's Sketchpad learning activities. An in-depth analysis and description of the participants' constructing and evaluating the definitions of geometric concepts in the presence of dynamic geometry supported activities, their using the definitions to explain relations between the geometric concepts, their conceptions and misconceptions during the whole process, and the effect of dynamic learning environment on their progress were handled in the previous chapters.

In this chapter, important findings were reviewed and discussed with respect to the findings in the literature. Moreover, the implications of the findings, potential limitations of the study and recommendations for further research studies were addressed. The important research findings were reviewed and discussed under the 5 sections in relation to the following research questions. The questions 3 and 4 were discussed in the same section since they were intertwined issues.

1. What are the perceptions of pre-service middle school mathematics teachers regarding the definitions and the role of definitions' in the teaching and learning process, before engaging with dynamic geometry supported clinical interview sessions?
2. What are the understandings of pre-service middle school mathematics teachers regarding the *minimality, equivalence, inclusivity and exclusivity* nature of definitions, before and after engaging with dynamic geometry supported clinical interview sessions?
3. How do the pre-service middle school mathematics teachers improve their understanding of the quadrilateral concepts through definition construction and classification processes in the presence of a set of activities assisted by Geometer's Sketchpad learning environment?

4. How do the dynamic geometry supported learning activities contribute to the improvement of pre-service middle school mathematics teachers' ability to define, evaluate and classify quadrilaterals?
5. What are the impressions of pre-service middle school mathematics teachers about the definition construction in the Geometer's Sketchpad learning environment after having them engaged with clinical interview sessions?

6.1 Participants' Initial Perceptions of the Definitions

In this section, findings related to the participants' perception of the definitions, the definitions' role in the teaching and learning process, meaning of a good definition according to them and their previous experiences with the definitions were discussed depending on the first research question.

It is seen that the most common opinion was that the concept definitions were important to identify a concept among the other concepts; however, all of the pre-service teachers attended to the study agreed also on that definitions alone were not enough to learn a concept. Actually, these cognitive abilities that pre-service teachers considered as important for the meaningful learning of a concept were all the essential cognitive processes of the definition construction process; however, the pre-service teachers were not aware of what a definition construction process meaned at that moment of the research study. The analysis findings revelaed that although pre-service teachers believed the importance of the definitions in the teaching and learning mathematics; they had a disbelief in their ability to construct correct mathematical definitions delivering the exact meaning of what a concept really was. That is, pre-service teachers believed that they could not construct mathematically workable definitions, eventough they were expected to do so and to use them effectively in their teaching as part of their subject matter and pedagogical content knowledge (Ball, Bass & Hill, 2004, Thames, 2006). Besides knowing how to define, preservice teachers were also expected to evaluate textbook definitions in terms of the mathematical correctness and the appropriateness of the mathematical language considering their students'level; and if necessary, they should be able make necessary revisions or redefine the concepts (Ball & Bass, 2003; Ball, Bass, &

Hill, 2004). That is to say, in order to be able to use definitions flexibility in their teaching, preservice teachers are expected to have skills above knowing the definitions or even above knowing how to define. However, analysis results indicated that far from knowing how to effectively use definitions in their teaching, pre-service teachers even did not believe and trust in their ability to define. As a result of this, it was seen also that pre-service teachers were slanted towards unquestioningly using the pre-constructed text book definitions instead of their own definitions in their teaching process. Of course in such a case, it was not a surprising finding that pre-service teachers also did not know what a good definition really was and what it should have entailed. They had thought that a good definition must have included all the known properties of the defined concept; that is, pre-service teachers had a perception of good definition as a list of all the properties of the defined concept, which was a common misleading idea among the teachers (De Villiers & Govender, 2002).

The underlying reasons of pre-service teachers' disbelief in their ability to construct definitions, their unawareness of what a good definition was and their dependency on textbook definitions were associated to their school experiences by the participants themselves. The findings indicated the common complain of the pre-service teachers was that they were not asked to think and discuss on a definition throughout their elementary and secondary school years; and the definitions remained as sentences written on the board or in their notebooks which had to be memorized when required. Pre-service teachers' similar complains on this issue is an indication of the common problem about the ignorance of definitions in the teaching and learning process in the Turkish context, as well. That is, teachers' difficulty with the definitions mostly stemmed from the applications in our education system. Moreover, findings revealed also that pre-service teachers realized the importance of a definition only at the time they came to the university and needed to understand the definitions to make proofs or to understand mathematical theories in the pure mathematics courses. This finding supported the statement of De Villiers (1998) that defining was an important mathematical activity as important as other mathematical activities such as proof making and problem solving; without understanding the definitions, success could not be achieved in other mathematics activities. In other words, their lack of ability to critically think on the definitions and to meaningfully conceptualize them could cause the pre-service teachers to fail in other mathematical thinking processes, as well. Considering the importance of the problem, the findings of this study recommends teacher educators to show much more attempt to focus on this problem in the Turkish context. So, if their difficulties are not overcome and if they are not equipped with the necessary knowledge and skills through teacher education programs, pre-service teachers will probably follow the lead of their teachers; thus, they will probably ignore the definitions as effective teaching tools in their own teaching process and use textbook definitions as written sentences on the board. If some precautions are not taken in the teacher education programs, this vicious cycle seem to be continued from generation to generation.

6.2 Participants' Initial Understanding of the Nature of the Quadrilateral Definitions and the Hierarchies

In this section findings related to the pre-service teachers initial understanding of the minimality, inclusivity, exclusivity and hierarchy nature of the quadrilateral definitions, their initial understanding of the nature of the relationships between the quadrilaterals and their initial knowledge about the definition construction and the ways of contructing equivalent definitions for the same concept were discussed with respect to the second research question.

As presented in detail in the analysis chapters, the initial questionnaire findings also supported the literature finding that pre-service teachers did not know what a good definition was and what it should have entailed (De Villiers & Govender, 2002). The findings revealed that pre-service teachers' initial understanding of a good definition was not a definition, but it was a description including a long list of properties of the defined concept. This result supported the preliminary theory that pre-service teachers had a tendency to define the given quadrilateral concept by listing all the properties of it which was accepted as a description required a van Hiele level 2 skill, but not a definition which required thinking ability at van Hiele level 3 (De Villiers 1996, 1998). So, this finding revealed the need for increasing pre-service teachers' geometric thinking levels from level 2 to level 3 so that they could learn how to define. The underlying reason of pre-service teachers' initial tendency to construct descriptions rather than definitions was found to be their inability to make inferences between the properties of a concept in order to identify the sufficient defining properties. When they were given a list of properties of a quadrilateral and were asked to construct alternative definitions of that quadrilateral, pre-service teachers randomly wrote a few of the given properties to define the quadrilateral instead of determining the sufficient defining properties among the given list, and they generally could not wrote more than one correct definitions. They could not make inferences about whether one property would result in the other property and so there would not be need to use both of them in the same definition. That is to say, pre-service teachers did not know to differentiate between the properties that describe a concept and the properties that define it, which was the most critical cognitive ability to achieve definition construction process (De Villiers & Govender, 2002; Favailli & Romanelli, 2006; Fujita & Jones, 2007,Winicki-Landman &Leikin, 2000).

Another stricking finding was that pre-service teachers had problems with the properties of the quadrilaterals and the most difficult property to determine was the symmetry property. Pre-service teachers' inadequate information or misinformation about the properties of the quadrilaterals clearly inhibited them from constructing and evaluating the definitions and from classifying the concepts based on their properties. Properties are the fundementals to characterize a concept through the definitions, so problems with the properties are the reasons of the problems with the definitions. So, these findings recommended that firstly teachers' insufficiency about the properties of the concepts needed to be overcome in order to make them use of these properties effectively to define and classify concepts.

The analysis of the results also indicated that pre-service teachers initially seemed to have an inclusive thinking to some extent. That is, they seem to be aware of the inclusive relations between the quadrilaterals to some extent. When they were asked to identify the rhombuses, rectangles and the squares among the given group of quadrilaterals, pre-service teachers were able to identify square as special rhombus and rectangle which was an indication of that they did not consider only the prototypical images, but also considered the inclusive relations between them. That is, pre-service teachers' concept images allowed them to think of the rectangle and rhombus with all congruent sides. However, the analysis results of the last question in the questionnaire indicated that they were unable to show the inclusive relations between the quadrilaterals with the venn diagrams. The reason of the participants' failure in this question could be due to the use of the venn diagrams, because in the pilot study it was found that use of venn diagrams could mislead the learners' reasoning by causing them to think with the set logic. For example, one participant in the pilot study had confused with whether the rhombus was a special parallelogram or not. Thinking with the set logic, she had said that rhombus was not a parallelogram, because square set was the subset of the rhombus set and also was the subset of parallelogram set; so the difference of the rhombus set from the square set would include rhombuses that might not have been a parallelogram. She was right that a special case could have extra property which the quadrilateral that included this special case did not have to satisfy; but all the properties of the general quadrilateral must be satisfied by its special case. That is, she could say that if rhombus is a parallelogram then a square also must be a parallelogram since it satisfies all properties of the rhombus; but she could not say if a square is a parallelogram then a rhombus must/must not be a parallelogram. Actually she mentioned about a probability that rhombuses excluding the squares might not have been a parallelogram; but she did not consider the other probability that rhombuses excluding the squares might also have been a parallelogram. Her mistake here was to make modal comparisons between rhombus set and square set rather than making property comparisons between rhombus and parallelogram as she did for other quadrilaterals. So, pilot study had indicated that use of Venn diagrams to show relationships was a situation on thin ice. Considering the pilot study findings, participants of the current study were asked to use line diagrams rather than the Venn diagrams to show the relationships during the clinical interview sessions and it was seen that, line diagrams did not have any negative effect on their thinking process, as Venn diagrams did in the pilot study. The Venn diagrams were consciously used in the initial questionnaire to see whether they would affect the thinking of the participants as expected.

Although the participants were seemed to be aware of the inclusive relations to some extent, when they were asked to construct inclusive or exclusive definitions for some group of quadrilaterals they failed. This findings were an indication of the participants' insufficiency to use the correct mathematical discourse to give the correct meaning to what they knew. Even if they were aware of the correct relationships, pre-service teachers were insufficient to express these relationships with the correct mathematical language. That is, finding revealed that pre-service teachers had difficulty to use the correct mathematical language to express the ideas, even these ideas were correct ones.

To sum up, the analysis of the results obtained from the initial questionnaire indicated that at the beginning of the study, pre-service teachers did not know the nature of a good definition, they thought that a good definition was a list of all the properties of the defined concept; they had a tendency to make description instead of a definition and they did not know that defining required to use minimal properties which were the sufficient defining properties to characterize the concept. Moreover, they had problems with the properties and with the mathematical language which prevented them from constructing and evaluating the definitions and from classifying the quadrilaterals with respect to the properties. Moreover, they were aware of the inclusive relations between the quadrilaterals to some extent, but they did not know exclusively and inclusively defining.

The analysis findings discussed so far reflected pre-service teachers' initial knowledge about the quadrilateral concepts that were questioned throughout the study. That is, findings discussed so far informed about the current situation. In the following section, more detailed analysis of the results considering the pre sevice teachers'cognitive thinking process in comprehending the inclusive relations between kite, rhombus, square, isosceles trapezoid, parallelogram, and rectangle while discovering the properties of them in a sketchpad context, the pre-service teachers' ability to organize several properties to construct mathematically workable and correct economical definitions for these quadrilaterals, their thinking process in discovering the cyclic and circum quadrilaterals and defining quadrilaterals in terms of these concepts, pre-service teachers' thinking process in discovering new shapes in the hierarchy through generalizing or specializing the known definitions and finally their progress in the presence of dynamic learning environment were discussed with respect to the third and fourth research questions.

6.3 Participants' Cognitive Progress in Defining and Classifying Quadrilaterals in the Dynamic Geometry Learning Environment

Findings from the clinical interviews supported the initial findings that pre-service teachers had problems with the properties of the quadrilaterals. Throughout the sessions it was seen that they sometimes did not know the properties; sometimes had misinformation related to the properties; and sometimes had difficulty to explain the properties with the correct mathematical language, eventhough they knew the properties.

As stated before, misinformation about the properties sometimes caused participants to make incorrect evaluations as well. The findings revealed that pre-service teachers made incorrect judgements due to their incorrect knowledge about the properties while critically thinking on whether the properties used in the definition were characterizing the concept to be defined or not. That is, they sometimes incorrectly found some quadrilaterals as counter examples to refute the correctness of the definition claiming that the definition included the counter example which was not intended to be defined. However, the findings indicated that working on the dynamic sketchpad figure to investigate properties of quadrilaterals either helped pre-service teachers to correct their misleading knowledge of the properties or to detect out other properties that they could not remember or could not infer from other known properties. By dynamically moving the figure, teachers easily detected critical preserved properties of the quadrilaterals; that is to say, the function of the GSP that all constructed properties were preserved under dragging helped the teachers to detect out critical side, angle, diagonal, and symmetry properties of the quadrilaterals easily. It is needed to be explained here that the function of the GSP at this step was just to show the correct measurements and the changes in these measurements under dragging; but it was the responsibility of the learner to observe the preserved properties and to use the correct mathematical language to state these properties. There was nothing to do with the GSP with respect to the teachers' use of the correct mathematical language to state the discovered properties.

Although pre-service teachers grasped the properties correctly through the GSP, analysis of the clinical interviews indicated that pre-service teachers had difficulty to express the properties with the correct mathematical language. Eventhough the definitions were accepted as the means of establishing the uniformity in the meaning of the concepts (Shir & Zaslavsky, 2001), pre-service teachers' inability to express the properties with the correct words also caused them to deliver the incorrect meaning through the definitions. The misuse of the mathematical language by the teachers is a very important problem in the sense that they can deliver the incorrect meaning of mathematical concepts and can mislead learners' understanding of the concept defined. Clinical interview analysis revealed that encouraging the pre-service teachers to think about the meaning of their statements and the use of the terms over and over was to some extent helpful to make them realize their deficiencies and correct them; but yet, there is need to handle these problems in the teacher education programs. I believe that if they had heard the correct terminology from their teachers and had been allowed to think on the meaning of their statements and had been made aware of the mathematical terminology they would not have difficulty in using the correct mathematical language now. So, considering that the purpose of teaching mathematics is to move learners from daily life language to more formal mathematical language and teachers are the key actors to deliver the correct language to their learners (Vinner, 1991), these findings increases the concern about the teacher education programs. Thinking that it is the teachers' responsibility to make learners aware of the difference between meaning of a word in daily life use and in the mathematical use (NCTM, 2000a) and the use of the correct mathematical language is the most important skill that a teacher should have (Ball, Bass, & Hill, 2004), this finding highly recommends to integrate mathematical terminology and discourse courses into the teacher education programs so as to teach the use of the correct terminology and language to the pre-service teachers.

Another cognitive ability observed during the clinical interview sessions was the participants' understanding the inclusive relationships between the quadrilaterals. Although the intial questionnaire findings had indicated that pre-service teachers had understood the inclusive relations to some extent, more detailed analysis of the issue through the clinical interview results revealed their real mental processes and problems with identifying the inclusive relations between the quadrilaterals. It was seen that pre-service teachers could not show even the inclusive relationships between the parallelograms and this was mostly caused by the prototypical concept images which caused to a conflict in their mind. That is to say, pre-service teachers had a tendency to make decisions based on a visual appearance of the figures instead of considering their properties, and their prototypical concept images did not allow them to accept a quadrilateral as special case of another quadrilateral due to the difference in their visual appearance. This finding supported the theory of Schawardz and Hershkowitz (1999) that such kind of prototypical judgments based on the visual image is the most important causes of learners' coming up with the incorrect perceptions of the concepts and of the relations between them. According to Schwarz and Hershkowitz (1999), understanding class inclusions between concepts required the ability to define the concept differently in terms of other more general concepts; the ability to make transitive reasoning such as if a square is a rectangle and a rectangle is an isosceles trapezoid then a square is an isosceles trapezoid; the ability to understand lack of symmetry within the relations like a square is a rectangle, but a rectangle is not a square; and the ability to understand the opposite inclusive relationship such as a square is a special rectangle, but the properties of the square includes the properties of a rectangle. Another reason of the pre-service teachers' not explaining the inclusive relations between the quadrilaterals was found to be their not grasping the idea of the opposite inclusive relations between the quadrilaterals and their properties. However, findings revealed also that the pre-service teachers were able to handle the problems with the this cognitive conflict and understand the logic behind the opposite inclusive relations between the quadrilaterals through the dragging activity on sketchpad. During this dragging activity on sketchpad, they investigated the special instances of each quadrilateral and the inclusive relations, as well. The tricky point of this activity was to understand the fact that when a correctly constructed figure was dragged, all characteristic properties of the figure remained unchanged. So, it was only possible to drag a figure into the figures which kept all of its properties, namely the special cases of that figure. However, more important than determining the special instances of a quadrilateral, when they were encouraged to think on the explanations for the inclusive relations, the pre-service teachers were able to give the correct reasoning behind those relationships through comparing the attributes of these quadrilaterals. Connor and Moss' (nd) stated their concern that prototypical judgement can limit the dynamic geometry investigations if they are not supported with the appropriate teaching strategies; but the findings of this study indicated that prototypes and misconceptions about the inclusive relations could be handled by the thinking path opened by the GSP activity.

When it comes to the definitions, analysis results of the clinical interviews also confirmed the pre findings that participants did not know what a good definition really was and what it should have entailed. Their initial definitions were descriptions including a long list of properties instead of the sufficient defining properties of the defined concept. The most critical cognitive ability in defining was to distinguish between the properties that describe a concept and the properties that define it; and at this crucial point the use of dynamic geometry helped pre-service teachers to overcome these difficulties and this result supported the findings in the literature (e.g., DeVilliers & Govender, 2002; Favilli & Romanelli, 2006). Through making a construction on the sketchpad just by using the information given in their definition, pre-service teachers were able to realize that there was no need to use all of the given information to characterize that quadrilateral. So, this process encouraged them to make inferences among the properties in order to decide the use of which properties together would satisfy all other characteristic properties of that quadrilateral. At this decision process, construction activity again helped them to verify whether they came up with the correct defining properties or not. That is, the findings indicated that through testing with the sketchpad figure construction, pre-service teachers learned to use minimum defining properties to generalize a concept and they moved from making descriptions towards making definitions.

Another important issue in the defining process was to be able define a concept in terms of other concepts; because hierarchy criterion of the defining process requires defining a concept "as a special case of more general concept" (Van Dormolen & Zaslavsky, 2003, p. 94). That is to say, while defining a geometrical shape X as "X is a quadrilateral…," we accept that what a quadrilateral is already known. Throughout the study, participants mostly defined concepts in terms of quadrilateral concept, but it was also important to be able to define in terms of other quadrilaterals and to see the fact that defining in terms of another quadrilateral already adds all its properties to that definition. Analysis of the results indicated that although it was hard to define in terms of any other special quadrilateral at the beginning of the related activity, pre-service teachers were able to easily handle it after grasping the tricky point that by defining in terms of a quadrilateral, all of its defining properties are added to the definition automatically. Paticipants were able to make sense of this fact when they were asked to think on what the difference was between saying "a right kite is a cyclic quadrilateral…" and "a right kite is a quadrilateral," and then they grasped the idea of defining in terms of other special quadrilaterals. By the way they learned another way of constructing alternative equivalent definitions for the same concept in addition to way of using different defining properties.

As for the evaluation of the correctness of the pre-constructed definitions, the findings revealed that pre-service teachers were evaluating the correctness of a definition just by checking the correctness of the properties used in the definition rather than checking whether they were the necessary and sufficient properties to define a concept; so this was misleading their judgments many times. However, when they evaluated the definitions in the dynamic geometry environment by constructing the figure accompanying with the definition, they were able make more reasonable judgment about the mathematical correctness of the definitions. By means of sketchpad construction, they were able to check whether the definitions were including only the necessary and sufficient defining properties or not. However, constructing the related sketch was not the only method they used to evaluate the given definition; in addition to the sketchpad construction, searching for the counter examples was also found to be an effective method to determine whether the definition was characterizing what it is intended to define. It was seen that use of both methods in combination was more effective to make them sure of their judgments and to come up with correct judgments.

Pre-service teachers were mostly encouraged to construct inclusive definitions to define a concept and all of its instances; because, inclusive definitions were favored in the literature by many researchers due to their advantages on thinking skills (e.g., De Villiers, 1994; Fujita & Jones, 2007; Schwarz & Hershkowitz, 1999). Although the inclusive definitions and inclusive hierarchies were supported in the literature, the pre-service teachers were also asked to define a quadrilateral as single objects by setting it apart from the others in order to make them aware of the difference between inclusive and exclusive defining. However, the findings indicated that the pre-service teachers were so accustomed to constructing inclusive definitions until this step and they did not accepted exclusive definition as a correct definition and they defended the necessity of including the special instances into the definition. That is, they were not aware of the correctness of both inclusive and exclusive nature of the definitions depending on the educational purposes. When they were engaged into the exclusive definition construction process, it was seen that the most difficult process was to consider the properties that would eliminate the special cases from the definition. It was found that throughout this process pre-service teachers used counter example search method to assure whether their definition really only defined that quadrilateral but not any other quadrilateral. They did not need to do the related sketchpad constructions. Participants also saw the effect of inclusive and exclusive definitions on the hierarchy diagrams by constructing the corresponding diagrams. As a result of experiencing this process, their awareness of the inclusive and exclusive nature of the definitions and their role in the hierarchy diagrams increased to a large extent. However, when they were asked whether they would be in favor of inclusive and exclusive definitions in their own teaching, almost all them favoured inclusive definitions with the same reasons as mentioned in the literature. That is pre-service teachers realized that the inclusive defining process was serving to increase thinking of the learners higher levels much more than the exclusive defining process.

In addition to learning many things related o the definitions and relations that they had not been aware of and had not thought before, the analysis of the final session revealed that pre-service teachers extended the limit of their thinking by defining new unfamiliar quadrilaterals in the hierarchy through generalizing or specializing the present definitions. Pre-service teachers were very successful at defining these new figures, which indicated their improvement in definition construction process through this study and the improvement of their thinking level to higher levels.

In this study, the dynamic geometry supported teaching activities were found to be effective in helping participants to overcome their difficulties and to improve their thinking to higher levels in defining and evaluating the concept definitions. The most effective contribution of GSP was found to be in finding the preserved characteristic properties of quadrilateral through the dragging activity; in discovering inclusive relations through the dragging activity and in testing with a construction whether the information given in the definition was necessary and sufficient. Moreover, GSP allowed for many trials on a dynamic environment and provided visual and mathematical representations on one screen so that the relations could be seen better.

In addition to the important role of GSP, other methods such as searching for counter examples, making inferences between the properties were also helpful as much as GSP. But, what made all these methods effective were the influential guidance, direction and endeavor put forward by the researcher. Without correct guidance or encouragement to think, neither GSP tasks nor other methods would make sense for the learners. Therefore, besides improving their technical skills to use such kind of activities in teaching, it is necessary for the teachers to improve their skills in directing learners through correct probing questions, as well. That is to say, learning is a complex process in which the interaction of students with the technological tool is not sufficient to enhance learning. The appropriateness of the task for the use of technology, the social climate, applied teaching methods, and the skills and knowledge of the teacher for the effective use of the technology are all the complementary features for effective learning.

Although definitions have been generally ignored in the teaching process so far, I expect this study will be an example of the effective usage of the definitions in teaching geometrical concepts. I hope such kind of constructive activities will be included in the teacher education programs in order to equip them with the related knowledge and skills, which will probably affect their instructional decisions and accordingly the learning of their students, as well.

6.4 Participants' Understanding of the Nature of the Quadrilateral Definitions and the Hierarchies after the Clinical Interviews

After the clinical interview sessions, it was seen that the participants improved their abilities to construct and evaluate definitions, and to classify concepts based on their definitions. Different from the pre-clinical interview situation, they were now much more conscious about the nature of the definition and about what a good definition was. Instead of listing all the properties of the concept as they did before, it was seen that now they were thinking on the ways of using minimal information to define a concept by critically thinking on the properties for identifying only the sufficient defining properties. That is, they were using high level thinking skills during the definition construction activity. It was seen that their definitions were generally correct economical definitions or very close to the most economical definition, but they were not descriptions any more. In contrast to the initial questionnaire findings, participants were able to handle the given list of properties to construct correct economical alternative definitions for the same concept using the different defining properties. Moreover, now, they were easily able to construct inclusive and exclusive definitions for the same quadrilateral and more easily able to explain the inclusive relations between the quadrilaterals through constructing hierarchical classification diagrams. However, there was still a problem with classifying the quadrilaterals based on their properties using the diagrams as in the initial questionnaire. Only Participant 1 correctly placed most of the quadrilaterals, but not all of them. This was probably due to their misunderstanding of the question; because, they all stated that they did not understand whether to think inclusively or exclusively while placing the quadrilaterals into the correct regions of the diagram. Although it was stated in the question as a notice that all the quadrilaterals would be evaluated as single geometric objects without considering their inclusive relations with the other quadrilaterals, participants did not take care of that notice and missed it.

6.5 Implications of the Study

Meaningful construction of a concept is a very complicated process which has been tried to be explained and improved by severeal researchers for several years, and the defitinions of the concepts took a crucial role in the concept formation process according to many theories (Fishbein 1993, Tall &Vinner, 1981, Vygotsky, 1986). Beyond any doubt, one of the most difficult field the concepts of which are rather challenging for learners is the field of geometry due to its requiring minds-on, eyes-on and hands-on skills together (Pedersen, 2004; Duval, 2006). So far, many studies examined the learners'difficulties with the geometric concepts, and the reserchers of these studies made recommendations in the light of their findings. This study aimed to carry the findings of these studies one step further by making an in depth analysis of the geometric concept formation process through the definitions and the relations of those concepts. The learner difficulties were identified with reference to the related literature and some dynamic geometry supported activities were prepared in the consideration of the recommendations in the literature expecting to get a deeper insight into the problem, to understand the underlying reasons and to see whether the developed activities were effective to reveal an improvement. The dynamic geometry supported activities prepared for the purposes of the current study were prepared binding the recommendations from the literature together and they aimed to engage pre-service teachers into the cognitive processes of defining a concept with its unique critical properties and organizing several properties of a geometric concept to generate a definition; distinguishing between examples and non-examples of the concepts and between the attributes and non-attributes of figures; deducing other properties from the definition of a concept and classifying shapes by identifying the grouping within definitions to understand the membership of the particular concept to a class of concepts (De Villiers, 1998; Freudenthal, 1971; Shield, 2004; Pimm, 1993; Shir & Zaslavsky, 2001;Vinner, 1983).

The questions of this chapter are what practical implications the findings of this study offer and what the scholarly contribution of this study is. The findings of the current study have important implications for the people who some way have a role in the preparation, development and application processes of the teacher education programs and the curriculum programs; namely, policy makers, program developers, teacher educators, teachers, etc.

Although knowing the definitions and using them flexibily and effectively in the teaching were accepted as an important component of the skills for teaching mathematics (Ball, Bass & Hill, 2004; Thames, 2006), the findings of this study revealed the pre-service teachers' current difficulties with the definitions in the Turkish context. The finding that definitions are ignored in the teaching and learning process and they just remain as written sentences on the board indicates the incompetencies of the teachers with the definitions and with understanding the role of the definitions in the teaching and learning process; and this increases the concerns about the teacher education programs and implies the need for taking precautions in the teacher education programs.

The finding that pre-service teachers feel incompetency for their skills to construct mathematically workable definitions due to their unawareness of what a good definition is and unawareness of the nature of the definitions implies the need for increasing their awareness through the teacher education programs. First of all, there is a need for revising the teacher education programs and integrating effective courses which will equip the pre-service teachers with the necessary knowledge and skills in the considered issue. Through these courses, teacher educators need to encourage pre-service teachers to critically think on the definitions of the concepts and to engage them into the active definition construction practices where they will practice how to deliver the correct meaning of the defined concepts through the correct mathematical language using many different alternative ways. Through these courses, pre-service teachers' attention need to be drawn on the importance of the correct use of the mathematical language and they have to be engaged into the activities which would increase their awareness of how absence or existence of even a little word could change the whole meaning they intended to deliver.

The prepared GSP supported activities have shown themselves to be quite effective in improving pre-service teachers' definition construction skills and in their meaningful formation of the quadrilateral concepts. This finding highly implies the need for supporting the process with the multiple visual, spatial and constructive representations which will help to remove the abstractness of the issue to some extent. The findings suggest to use GSP especially in the cognitive processes of finding the preserved characteristic properties of the geometric concepts; discovering inclusive relations between the geometric concepts and in identifying the necessary and sufficient defining properties. However, it needs to be taken into consideration that integrating the dynamic geometry supported activities into the teaching is not an easy process since it requires considering many things and requires very well planning. The findings imply that for the success of the GSP supported activities the activities need to be prepared with great care; even the arrangement of the decimal points in the measurements need to be considered for their functionality. The activities prepared for the purposes of this study were revised critically over and over considering the feedbacks of the pilot study participants and the observed technical deficiencies during the pilot applications, and the revisions took at least one year. For all that, dragging the circum quadrilateral figure was a little bit challenging for the pre-service teachers, though they were able to overcome difficulties through many trials on the figure. Fortunately, the participants of this study did not experienced any serious technical difficulty that would negatively affect their thinking process, but it is possible if such kind of activities are not prepared with great care. So, there is need to say here that such technical problems need to be considered with great care since they can mislead the thinking process of learner. That is, if the learner can not drag the figure due to technical problems, this can cause her to change the correct idea into an incorrect idea, or to support her incorrect idea. For example, a learner may think correctly that a square is a cyclic quadrilateral; however if she can not drag the cyclic quadrilateral into a square due to technical difficulties, this will mislead the learner. She may think that she was wrong saying that a square is a cyclic quadrilateral, if she considers the sketchpad dragging result but not the criterion to be a cyclic quadrilateral. For another example, a learner can incorrectly say that a square is not a cyclic quadrilateral when she does not consider the criterion of the supplementary opposite angles. Upon this incorrect though, if she can not drag the cyclic quadrilateral into a square, this will support her incorrect idea and she will be misled. So, for those who are about to use such kind of activities, this study implies the need to consider all pros and cons with regard to the teaching purposes.

Another implication this study reveals is that the use of line diagrams instead of venn diagrams to show the relations between the geometric concepts is more effective, and it is suggested for the teacher educators and teachers to avoid using the Venn diagrams in such kind of geometric activities. Educators should pay attention to the fact that use of Venn diagrams for showing the relationships among several geometric concepts can lead to a very complicated visual material that hindered relationships between quadrilaterals and moreover, the use of Venn diagrams can cause one participant to interpret inclusive relations incorrectly through the set logic.

The current study findings also implies the possibility of adaptation of the dynamic geometry supported activities into other concepts other than the quadrilateral concepts. It is thought that the activities used in this study can open a path for developing similar activities or for adapting them to different concepts and to different learners. If they can be adapted to the students level, they can also be used in the classroom settings by the teachers to engage the students into active thinking process on the quadrilateral concepts. Moreover, these activities can be improved or new ones can be developed for the meaningful formation of the other concepts considering the study findings which reveal the pros and cons of these activities. The definition construction process used in this study can also be adapted into the concepts of numbers, algebra and probability other than the geometry.

Moreover, the findings of this study imply that definition construction is a very important cognitive process including many mental skills together and it is possible to improve meaningful concept formation of the geometry concepts through definition construction acitivities. Therefore this study suggests not to ignore the definitions in the teaching and learning process, and to use them as effective teaching tools instead. That is, there is need to put emphasis on the concept definitions in the teaching contexts and to increase the awareness of the teachers about definitions' role of being effective teaching tools through the teacher education courses.

If the implications revealed by the findings of the study taken into consideration and put into practice by the targeted people, it is possible to go one step further in solving the problems and improving the quality of the education.

6.6 Suggestions for Further Research

This study investigated pre-service teachers' thinking processes in constructing quadrilateral definitions and quadrilateral hierarchies through the dynamic geometry assisted learning material in a case study design. To be able to make a detailed investigation, it was necessary to study on the case base. However, since the learning material developed through this study is expected to be used as an instructional material in teacher education programs, it would be contributory to investigate the effectiveness of the tasks in a classroom teaching environment.

Moreover, this study aimed to overcome teachers' difficulties with the definitions of the mathematical concepts by teaching them the nature of definitions and how to define. However, there is a further need to investigate how this study will affect their teaching skills and in what way they will use definitions in their classroom teaching; whether they will left the definitions as sentences written on the board to be memorized by their students or whether they will encourage their students to construct their own definitions and to think and discuss on these definitions in an active learning environment. That is to say, there is need to investigate teachers' use of definitions in the real classroom settings.

6.7 Limitations

The main limitations of this study are the number of participants and the restricted content. First of all, since this case study was limited by the data obtained from 5 prospective middle school mathematics teachers, the number of participants can restrict the study findings to the specific conditions used in this study. Although it is criticized for the weakness of generalizability, a thorough study of an individual in a case study design provides a great deal of insight that can not be obtained by other research means (Ginsburg, 1997). That is, particularization rather than the generalization is the main issue in case studies; because it is the power of case study to focus on the particular situations and to gather deeper understanding (Stake, 1995). Even though the aim of a case study is not to generalize findings from a case to a population, the transferability can be made on case-to case basis if the case is thoroughly described so that the readers will be able to see the similarities with their own cases and generalize the results to their cases (Creswell, Hanson, Plano & Morales, 2007; Stake, 1978). That is to say, it is the responsibility of readers to consider the conditions of this study and decide whether the findings can be generalized to their own contexts.

Secondly, this study's investigation of the learners' definition construction process in the presence of dynamic geometry learning environment was limited to the content of quadrilaterals; so the findings can not be generalized to other content areas.

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APPENDICES

Appendix A

Geometer’s Sketchpad Teaching Session

Instructions for the teacher:

- Introduce the work area and the function of the menu items like File, Edit, Display, Construct, Graph, Measure, Transform, Window, Help.
  - Show the commands under each menu items.
  - Underline the fact that some commands will be activated when the related objects are selected.
  - Explain the purpose of dialog boxes.
- Introduce the toolbox including Pointer, Dot, Circle, Segment, Text, Custom tools and let the learner to practice them.
  - Ask the learner to construct segments, rays and lines.
  - Show the ways of constructing a circle and let the learner practice.
- Ask the learner to construct the following:
  - Construct two intersecting lines and the intersection point.
  - Construct two intersecting circles and the intersection points.
  - Construct a circle using the circle tool and construct a point on it. Then explain the effect of dragging every three points on the circle.
  - Construct two intersecting segments which are perpendicular to each other.
  - Construct two perpendicular segments one of which is bisected by the other.
  - Construct two perpendicular and bisecting segments.
  - Construct two perpendicular, bisecting and congruent segments.
  - Construct a triangle. Then name the vertices, measure the side lengths, measure the angles and calculate the sum of the angle measures.
  - Construct a right angle arms of which are congruent segments.
  - Contruct a square.
- Underline the difference between a drawing and a construction by showing the effects of dragging on the drawn and constructed squares.

    - Encourage the learner to realize the fact that all defining properties of a figure are preserved under the construction, but they are not preserved under the drawing.
- Ask the learner to construct an isosceles and an equilateral triangle.
    - Ask the learner to explain which group could include the other group as a special case.
    - Ask whether all isosceles triangles could be the equilateral triangles or vice versa, and why.
- Introduce the animation and tracing function. For example, animate a circle on another circle and encourage the learner to construct different animations. For example, ask to animate a circle on a segment.
- Introduce transformations including reflection, rotation, dilation, translation, iteration with examples.
    - Show examples of both polar and rectangular translations.
    - Emphasize the fact that a center of dilation and a scaling factor are needed to construct a dilation
    - Encourage the learner to construct some fractals using the iteration menu.

Appendix B

Turkish Version of Initial Interview Questions

1. Matematik ve geometri kavramlarını öğrenme ve öğretme sürecinde tanımların rolü nedir?

2. Bir öğrencinin bir kavramı öğrenip öğrenmediğini nasıl anlarız? Kavramı tanımlayabilmesi o kavramı öğrendiğini gösterir mi? Neden? Başka ne gereklidir?

3. Bugüne kadarki öğrenim hayatında kavram tanımları ne şekilde sunuldu?

  - **a.** Tanımlar dersin hangi aşamasında sunuldu?
  - **b.** Tanımlar üzerinde ne derece duruldu?
  - **c.** Bu sunum şekli senin kavram tanımlarını öğrenmende ne derece etkili oldu? Etkili oldu mu?

4. Sen bir öğretmen adayı olarak tanım yapma becerini nasıl değerlendirirsin?

  - **a.** Örneğin, bir kavramı birden fazla değişik şekilde tanımlayabilir misin?
  - **b.** Bu konuda kendini eksik hissediyor musun?

5. Sen bir öğretmen adayı olarak tanımları öğrencilerine nasıl sunmayı planlıyorsun?

  - **a.** Ders kitaplarında verilen tanımlara güvenip doğrudan kullanır mısın, tanımın doğruluğunu, öğrenci seviyesine uygunluğunu süzgeçten geçirir misin?
  - **b.** Örneğin ders kitabındaki bir tanımı inceledin ve içerisindeki bazı terimlerin öğrencilerinin seviyesine uygun olmadığını düşündün, ne yaparsın bu durumda?

6. İyi bir tanım nasıl olmalıdır? İyi bir tanımın özellikleri ne olmalıdır?

  - **a.** Bir tanımı kritik ederken nelere bakarsın?
  - **b.** Kavrama ait bütün özellikler tanımda sıralanmalı mıdır?

7. Tanımları etkili bir öğretim aracı olarak kullanabilir miyiz?

Appendix C

Turkish Version of "Questionnaire on Quadrilaterals I"

1. Aşağıdaki dörtgenleri nasıl tanımlarsınız?

a. **Eşkenar dörtgen:**

____

____

____

____

Dikdörtgen:

____

____

____

____

Kare:

____

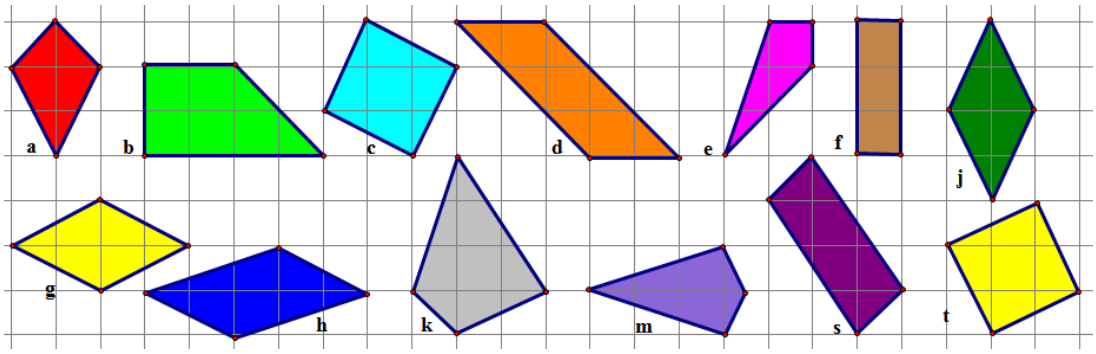
____

____

____

b. Yukarıda yapmış olduğunuz tanımlara göre

- Aşağıdaki dörtgenlerden hangisi ya da hangileri *eşkenar dörtgendir*? __________
- Aşağıdaki dörtgenlerden hangisi ya da hangileri *dikdörtgendir*? ____________
- Aşağıdaki dörtgenlerden hangisi ya da hangileri *karedir*? _______________



2. Özel bir dörtgenin özellikleri aşağıda verilmiştir

- İki çift komşu kenarı eş
- Bir çift karşılıklı açı eş
- Köşegenler birbirine dik
- Köşegenlerden biri diğerini ortalar
- Köşegenlerden biri geçtiği köşelerdeki açıların açıortayı
- Köşegenlerden biri simetri ekseni

a. Bu dörtgeni verilen özelliklerden mümkün olan en az sayıda özelliği kullanarak nasıl tanımlarsınız? 2 değişik (alternatif) tanım yazınız.

Tanım1:____

____

____

Tanım2:____

____

____

3. Hiç bilmeyen birisine eşkenar dörtgenin ne demek olduğunu ifade etmek için aşağıdaki tanımlardan hangisi ya da hangilerini seçerdiniz? Seçme veya seçmeme nedenlerinizi her bir ifade için açıklayınız.

a. Eşkenar dörtgen karşılıklı kenarları paralel olan dörtgendir.

____

____

____

b. Eşkenar dörtgen karşılıklı köşelerden geçen ve birbirine dik olan 2 simetri eksenine sahip dörtgendir.

____

____

____

c. Eşkenar dörtgen bütün kenarları eş olan dörtgendir.

____

____

____

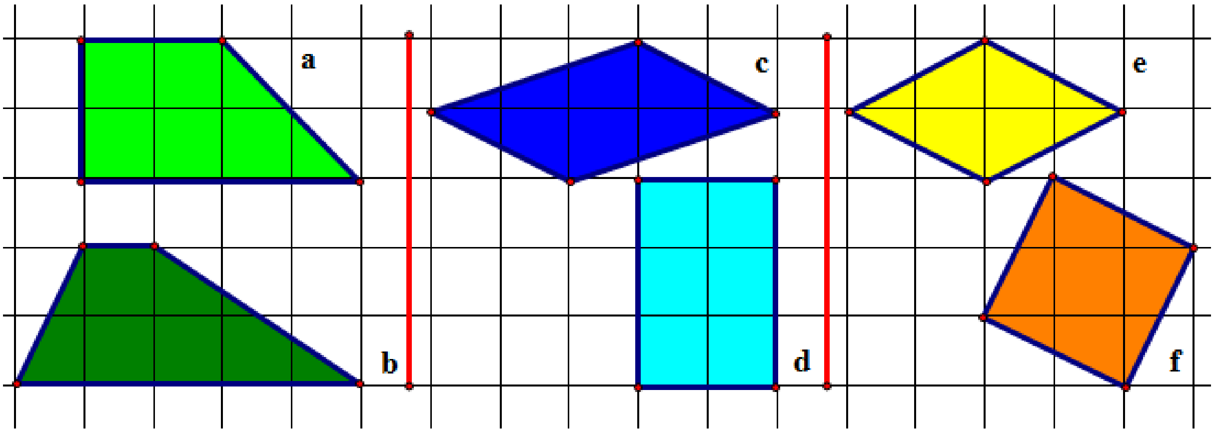
d. Eşkenar dörtgen iki çift komşu kenarı eş olan dörtgendir.

____

____

____

4. Aşağıdaki dörtgenlere bakınız.



a. a ve b dörtgenlerini tanımlayan fakat c, d, e ve f dörtgenlerini tanımlamayan bir tanım yazınız.

Tanım:____

____

____

b. a, b,c ve d dörtgenlerini tanımlayan fakat e ve f dörtgenlerini tanımlamayan bir tanım yazınız.

Tanım:____

____

____

c. a, b, c, d, e ve f dörtgenlerini tanımlayan bir tanım yazınız.

Tanım:____

____

____

5. Aşağıdaki ifadelerin doğruluğu için "her zaman," "bazen," ya da "hiçbir zaman" seçeneklerinden birini seçiniz. "Bazen" seçeneğini kullanırsanız olası durumlara örnek veriniz.

a. Kare bir eşkenar dörtgendir. *Her zaman* *Bazen* *Hiçbir zaman*

---

b. Eşkenar dörtgenin köşegenleri eştir. *Her zaman* *Bazen* *Hiçbir zaman*

---

c. Dikdörtgen eş komşu kenarlara sahiptir. *Her zaman* *Bazen* *Hiçbir zaman*

---

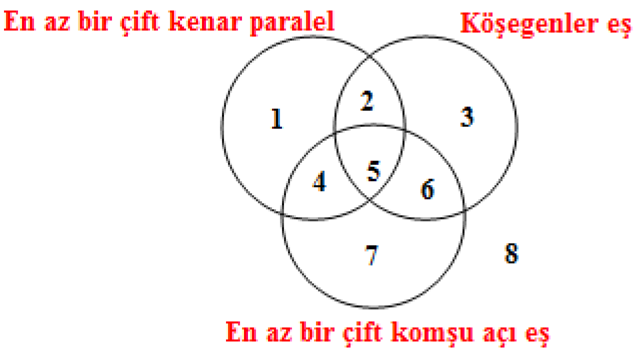
d. Eşkenar dörtgenin köşegenleri birbirini ortalar. *Her zaman* *Bazen* *Hiçbir zaman*

---

e. Paralelkenarın köşegenleri birbirini dik keser ve ortalar. *Her zaman* *Bazen* *Hiçbir zaman*

---

6. Özel dörtgenler olan paralelkenar, dikdörtgen, eşkenar dörtgen, kare, yamuk, ikizkenar yamuk ve deltoidi özelliklerine göre aşağıdaki Venn diyagramlarında numaralandırılmış bölgelerden hangilerine konması gerektiğini karşısına yazınız.



Paralelkenar	___	Yamuk	___
Deltoid	___	İkizkenar yamuk	___
Kare	___	Dikdörtgen	___
Eşkenar dörtgen	___		

Paralelkenar	___	Yamuk	___
Deltoid	___	İkizkenar yamuk	___
Kare	___	Dikdörtgen	___
Eşkenar dörtgen	___		

Appendix D

Turkish Version of Clinical Interview Session 1

1. Karşında bir deltoid çizimi görüyorsun şu an (Deltoid 1 sekmesi). Deltoidin bildiğin, hatırlayabildiğin kenar, açı, gibi özellikleri neler, söyleyebilir misin? Hangi özelliklerini biliyorsun?

2. Peki, farzet ki sınıftasın ve öğrencilerine deltoidi tanımlaman gerekiyor. Nasıl tanımlardın?

3. Şimdi sketchpad ortamında bu şekli incelemeye başlayalım ve hem senin söylemiş olduğun özellikleri teyit edelim, hem de eğer varsa diğer özelliklerini bulalım. Sketchpad ekranında önceden hazırlanmış deltoid çizimini görüyorsun. Bu çizim üzerinde yapılmış bazı ölçümler ekrandaki butonların altında gizli.

3.1. İlk olarak kenar ve açı ile ilgili butonlara tıklar mısın? Şekli köşelerinden çekip hareket ettirdiğinde kenar ve açı özelliklerinden değişmeyen hep korunan bir özellik var mı? Kenar uzunlukları ve açı özellikleriyle ilgili hangi yargıya varıyorsun?

3.2. Şimdi de köşegenlerle ilgili olan 2 butonu tıklayalım. Şekli hareket ettirdiğinde köşegen ile ilgili özelliklerin hangileri korunuyor? Köşegenlerin oluşturduğu doğru parçalarını ve açılar ile ilgili hangi yargılara varıyorsun?

3.3. Bu dörtgenin (deltoid) simetri özelliği ile ilgili ne söyleyebilirsin? (Gerekirse "transform" menüsünü kullanması teşvik edilir).

3.4. Bulduğun bu özellikleri daha sonra hatırlayabilmek için verilen kağıttaki deltoid üzerinde gösterebilirsin.

4. Şimdi Deltoid-2 isimli sekmeyi tıklar mısın? Ekranda gördüğün deltoidi ne şekilde hareket ettirirsen ettir o her zaman bir deltoid ve deltoide ait az önce saydığımız bütün özellikleri korunuyor.

4.1. Peki, sence bu deltoidi köşelerinden çekerek butonlar üzerinde ismini gördüğün dörtgenlere dönüştürmek mümkün olur mu? Hangi dörtgenlere dönüştürülebileceğini düşünüyorsun? Neden?

4.2. Şimdi bunu deneyerek test edelim. Butonlara bastığında ilgili dörtgeni ekranda göreceksin. Sırayla her bir dörtgenin üzerine deltoidi koyup köşelerinden çekerek o dörtgene dönüştürüp dönüştüremeyeceğini test eder misin?

4.2.1. Neden sence kareye/ eşkenar dörtgene dönüştürüldü? Nasıl açıklarsın? Bu kanıya varmanda sketchpadin hangi özelliği yardımcı oldu.

4.2.2. Deltoid özellikleri kare ve eşkenar dörtgen için nasıl değişti? Nasıl özelleştiler? (Tekrar dönüştürüp ölçümlere bakabilirsin).

4.2.2.1. Örneğin deltoidin kenar özelliği neydi ve bu özellik karede/eşkenar dörtgende ne oldu?

4.2.2.2. Kare deltoidin bütün özelliklerini taşıyor mu? Deltoid karenin özelliklerini taşıyor mu? Bu durumda hangisi diğerinin özel durumu olur?

5. Deltoid-3 isimli sekmeyi tıklar mısın? Eşkenar dörtgen ve kare arasındaki ilişki için ne söyleyebilirsin? Eşkenar dörtgen ve kareyi çekip hareket ettirerek çalışabilirsin istersen.

5.1. Birini diğerine çevirebilir miyiz? Hangisini?

5.2. Hangisinin bütün özellikleri diğeri için de geçerli?

5.3. Hangisi diğerinin özel durumudur?

6. “Show diyagram” butonunu tıklayalım. Deltoid, kare ve eşkenar dörtgen arasındaki hiyerarşik (sıradüzensel) ilişkiyi ekrandaki diyagramda nasıl gösterirsin? Genelden özele doğru nasıl yerleştirirsin bu dörtgenleri?

7. Deltoid 4 sekmesini tıklar mısın? Şimdiye kadar üzerinde konuştuğumuz bütün özellikleri düşünerek deltoidi özel durumları olan kare ve eşkenar dörtgeni de kapsayacak şekilde nasıl tanımlarsın? Yani az önce diyagramda gösterdiğin deltoid sınıfını nasıl tanımlarsın? Düşündükten sonra tanımı ekrana yazar mısın?

7.1. Tanımı yazma aşamasında hangi özellikleri kullanacağına nasıl karar verdin?

8. Deltoid-5 isimli sekmeye tıklar mısın? Sadece yazdığın deltoid tanımında yer alan özellikleri kullanarak sketchpad’de bir deltoid inşa eder misin?

8.1. Tanımındaki özellikleri kullanarak doğru inşa edilmiş bir deltoid elde ediyor musun?

8.2. Çizdiğin dörtgeni çekerek hareket ettirdiğinde daima bir deltoid olarak kalıyor mu?

8.3. Tanımında kullandığın özellikler gerçek bir deltoid oluşturabilmen için yeterli mi? Neden?

8.4. Deltoidi inşa ederken tanımında veriğin her bilgiyi kullandın mı? Eğer kullanmadıysan hangi bilgileri kullandın? Hangi bilgileri kullanmadın ve neden?

8.5. Yazdığın deltoid tanımı ekonomik (mümkün olan en az özelliği kullanarak oluşturulmuş) mi değil mi? Ekonomik değilse ekonomik olacak şekilde yeniden tanımlar mısın?

9. Şimdi sana bazı deltoid tanımları göstereceğim ve tanımların doğruluğunu, ekonomik mi ekonomik olmayan bir tanım mı olduğunu değerlendirmeni ve nedenini açıklamanı isteyeceğim. İstersen her bir tanımda verilen bilgiyi kullanarak ilgili constructionı yaparak karar verebilirsin.

9.1. Tanımda kullanılan özellikler deltoidi tanımlamak için gerekli ve yeterli özellikler mi? Tanım ekonomik mi ekonomik değilse nasıl ekonomik hale getirirsin?

9.2. Bu tanıma uyan fakat deltoid olmayan bir dörtgen olabilir mi? Karşı bir örnek bulabilir misin?

(1) Deltoid köşegenleri birbirine dik olan dörtgendir.

(2) Deltoid en az bir köşegeni diğerini dik ortalayan dörtgendir.

- *En az ile yalnız bir köşegeni dik ortalayan arasındaki fark nedir?*

(3) Deltoid iki çift komşu kenarı ve bir çift karşılıklı açısı eş olan dörtgendir.

- *Bu iki özellik de gerekli mi? Yalnız biri tanımlamaya yeter mi?*

(4) Deltoid en az bir köşegeni simetri ekseni olan dörtgendir.

10. Eşkenar dörtgen için mümkün olan en kısa tanımı nasıl yaparsın? Doğru ve ekonomik bir tanım olup olmadığını sketchpad’de test eder misin?

Appendix E

Turkish Version of Clinical Interview Session 2

1. Karşında bir ikizkenar yamuk çizimi görüyorsun şu an (İkizkanar yamuk 1 sekmesi) ikizkenar yamuğun bildiğin, hatırlayabildiğin kenar, açı, gibi özellikleri neler, söyleyebilir misin? Hangi özelliklerini biliyorsun?

2. Peki, farzet ki sınıftasın ve öğrencilerine ikizkenar yamuğu tanımlaman gerekiyor. Nasıl tanımlardın?

3. Şimdi sketchpad ortamında bu şekli incelemeye başlayalım ve hem senin söylemiş olduğun özellikleri teyit edelim, hem de eğer varsa diğer özelliklerini bulalım. Sketchpad ekranında önceden hazırlanmış deltoid çizimini görüyorsun. Bu çizim üzerinde yapılmış bazı ölçümler ekrandaki butonların altında gizli.

3.1. Butonlara sırayla bastıktan sonra ölçümleri değerlendirerek ikizkenar yamuğun kenar, açı, köşegen ve simetri özellikleri hakkında ne söyleyebilirsin? Şekli köşelerinden çekip hareket ettirdiğinde korunan özellikleri nelerdir?

4. Sence ikizkenar yamuğuğun özelliklerini taşıyan baska dörtgenler var mı? Varsa hangileri? Neden? (Sözel olarak yorum alınır).

5. "İkizkenar Yamuk-2" isimli sekmeyi tıklar mısın? Ekrandaki butonlara bastığında üzerinde yazılı olan dörtgeni ekranda göreceksin. İkizkenar yamuğu her bir şekil üzerinde çekip hareket ettirerek paralelkenar, dikdörtgen, deltoid, eşkenar dörtgen ya da kareye dönüştürüp dönüştüremeyeceğini, yani bu şekillerin özel birer ikizkenar yamuk olup olamayacağını inceler misin?

5.1. İkizkenar yamuğu neden kareye/dikdörtgene dönüştürebildin sence?

5.2. İkizkenar yamuğun dikdörtgen ve kare dışındaki dörtgenlere dönüştürülemeyişinin sebebi sence nedir?

6. Şimdi "Diyagram 1" sekmesine geçelim. İkizkenar yamuk, dikdörtgen ve deltoid arasındaki hiyerarşik ilişkiyi diyagram üzerinde gösterebilir misin?

7. "İkizkenar yamuk 3" sekmesine geçelim. İkizkenar yamuk ve özel durumları olan kare ve dikdörtgeni de kapsayacak en kısa tanımı nasıl yaparsın? Yani en az sayıda özelliği bir arada kullanarak; sadece gerekli ve yeterli özelliklerini kullanarak tanımlamanı istiyorum. Düşündükten sonra tanımı ekrana yazar mısın?

7.1. Tanımda kullandığın özellikler ikizkenar yamuğu tanımlamak için gerekli ve yeterli özellikler mi? Nasıl karar verdin?

7.2. Bu tanıma uyan fakat ikizkenar yamuk sınıfında bulunmayan başka bir dörtgen olabilir mi? Düşünür müsün?

7.3. Hala karar veremediysen "Tanım Çizim" sekmesinde çizim yapabilirsin

8. "Tanımlar" sekmesine geçelim. Şimdi sana 3 tane ikizkenar yamuk tanımı göstereceğim. Bu tanımların doğru ve yeterli bir ikizkenar yamuk tanımı olup olmadıkları konusunda değerlendirme yapmanı istiyorum. (Sözel olarak yorumlam alınır. Karşıt örnek bulmaları istenir, gerek duyulursa sketchpad kullanılır).

8.1. Bu tanımlara uyan fakat ikizkenar yamuk sınıfından olmayan dörtgen var mı, düşünür müsün?

8.2. İkinci tanım (exclusive) özel durumlarını kapsamıyor. Sence bu tanımı doğru mu yanlış mı kabul etmeliyiz?

- İkizkenar yamuk en az bir çift kenarı paralel ve en az bir çift karşılıklı kenarı eş olan dörtgendir.
- İkizkenar yamuk en az bir çift kenarı paralel ve diğer çift kenarı eş fakat paralel olmayan dörtgendir.
- İkizkenar yamuken az bir çift karşılıklı kenarı paralel ve karşılıklı açıları bütünler olan dörtgendir.

9. "Paralelkenar" isimli sekmeye geçelim. Hangi dörtgenlerin paralelkenar olduğunu düşünüyorsun? (Doğru cevap veremezse şekil üzerinde test etsin)

9.1. Bunu sketchpaddeki paralelkenar üzerinde test ederek gösterir misin?

9.2. "Diyagram 2" sekmesini tıklayalım. Paralelkenar ve özel durumları arasındaki ilişkiyi hiyerarşi diyagramıyla nasıl gösterirsin? Dörtgenleri genelden özele doğru (hiyerarşik) sıralarken aralarına doğru parçası çizerek birbirine bağlayabilirsin.

10. "Yamuk ve Paralelkenar" isimli sekmeyi açlım. Az önce ikizkenar yamuk ile paralelkenar arasındaki ilişkiye baktık. Peki ikizkenar olmayan bir yamuk ile paralelkenar arasındaki ilişki için ne söyleyebilirsin?

10.1. İstersen ekranda gördüğün paralelkenar ve yamuk çizimlerini bu ilişkiyi anlamada kullanabilirsin.

11. "Yamuk tanım" sekmesini açalım. Özel durumları olan ikizkenar yamuk ve paralelkenarları da kapsayacak şekilde yamuğu nasıl tanımlarsın? Düşündükten sonra ekrana yazar mısın?

12. "Diyagram 3" sekmesini açalım. Daha önce deltoid sınıfını, ikizkenar yamuk sınıfını, paralelkenar sınıfını ve şimdi de yamuk sınıfını bulduk. Bu sınıfları aynı diyagramda birleştirerek aralarındaki ilişkiyi nasıl gösterirsin? Dörtgenleri genelden özele doğru (hiyerarşik) sıralarken aralarına doğru parçası çizerek birbirine bağlayabilirsin.

13. Şimdi "show tanım" butonuna tıklar mısın? Eğer yamuk "**yalnız** bir çift kenarı paralel olan dörtgen" olarak tanımlansaydı bu hiyerarşi nasıl değişirdi? Diyagramı yeniden çizerek gösterir misin?

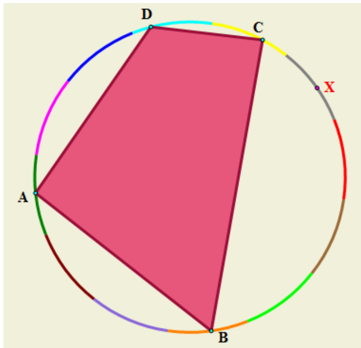
14. "Paralelkenar tanım" isimli sekmeyi açalım. Paralelkenarın *köşegen* özelliğini kullanarak dikdörtgen, eşkenar dörtgen ve kareyi özel paralelkenar olarak kapsamayan bir tanımı nasıl yaparsın?

15. "Eşkenar dörtgen tanım" isimli sekmeyi açalım. Eşkenar dörtgenin *simetri* özelliğini kullanarak kareyi özel bir eşkenar dörtgen olarak kapsamayan bir tanımı nasıl yaparsın?

16. "Deltid tanım" isimli sekmeyi açalım. Deltoidin *herhangi bir* özelliğini kullanarak eşkenar dörtgen ve kareyi hariç tutan bir tanımını nasıl yaparsın?

Appendix F

Turkish Version of Clinical Interview Session 3



1. Sketchpad ekranında ABCD kirişler dörgenini görüyorsun. Kirişler dörtgenini nasıl tanımlarsın?

2. Sence bir dörtgenin kirişler dörtgeni olabilmesi için hangi özelliklere sahip olması gerekir? Hangi dörtgenlerin kirişler dörtgeni olduğunu düşünüyorsun? Neden? (Sadece düşünce alınır).

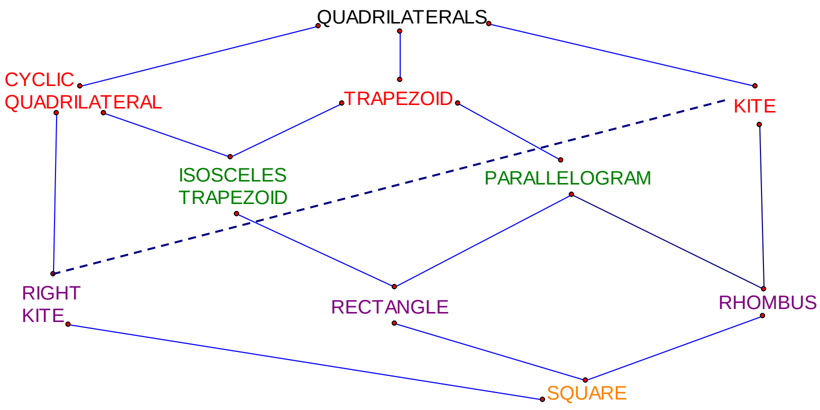
3. Ekranda verilen sıradan bir kirişler dörtgenini çekip hareket ettirerek ve açı, kenar ölçülerinden faydalanarak hangi özel dörtgenlere dönüştürebileceğini test eder misin? Çember şekli daha rahat dönüştürmene yardımcı olmak üzere 30°'lik yaylara ayrılmıştır. Bu açıları kullanabilirsin dönüşümleri yaparken.

   3.1. Dikdörtgen, kare, ikizkenar yamuk ve dik deltoide neden dönüştürebildin?
   3.2. Paralelkenar, eşkenar dörtgen ve yamuğa neden dönüştüremedin?
   3.3. Oluşturduğun deltoidin farkedebildiğin özel bir durumu var mı? Varsa nedir?
   3.4. Kirişler dörtgenini bu özelliği taşımayan diğer deltoidlere dönüştürmen mümkün mü? Neden?

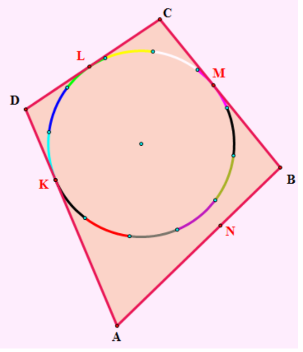
4. "Tanımlar 1" isimli sekmeye geçelim. **Dik deltoidi** dörtgen, kirişler dörtgeni ve deltoid bazında 3 farklı şekilde nasıl tanımlarsın?
   4.1. Bir tanımı "dörtgendir" ve "kirişler dörtgenidir" diye bitirmek arasındaki fark nedir?

5. "Tanımlar 2" isimli sekmeye geçelim. **İkizkenar yamuk**, **kare** ve **dikdörtgeni** kirişler dörtgeni bazında nasıl tanımlarsın?

6. "Diyagram 1" isimli sekmeyi tıklar mısın? Şu a daha önceki mülâkatta oluşturduğumuz hiyerarşiyi görüyorsun. Dik deltoid ve kirişler dörtgenini de yeni birer kategori olarak ekleyip yeni ilişkileri gösteren hiyerarşi diagramını nasıl çizersin?



7. "Teğetler Dörtgeni" isimli sekmeye geçelim. Ekranda gördüğün çember dörtgenin 3 kenarına teğet olacak şekilde çizildi, AB kenarı serbest bırakıldı. Senden istediğim listedeki her bir dörtgene sırayla çevirmeye çalışıp AB kenarının hangi dörtgenlerde **her zaman** çembere teğet olduğunu bulman. Yani bir dörgenin her zaman teğetler dörtgeni olup olmadığına karar verirken olamadığı durumlar var mı yok mu diye de incelemen gerekiyor. Ayrıca çember, şekli daha rahat dönüştürmene yardımcı olmak üzere 30°'lik yaylara ayrılmıştır. Bu açıları kullanabilirsin dönüşümleri yaparken.



7.1. Ama önce hangi dörtgenlerin teğetler dörtgeni olabileceğini düşündüğünü sorabilir miyim?

7.2. Şimdi ekranında verilen sıradan teğetler dörtgenini çekip hareket ettirerek ve açı, kenar özelliklerinden faydalanarak hangi özel dörtgenlere dönüştürebileceğini test eder misin?

8. "Açıortaylar" isimli sekmeye geçelim. Sketchpad ekranında iç açıortaylarının kesişim noktaları verilen bir dörtgen var. Bu dörgeni listedeki dörtgenlere üzerinde çekerek dönüştürüp EFGH bölgesini herbir dörtgen için gözlemlemeni istiyorum. Sana verilen kağıtta her bir dörtgenin karşısına EFGH bölgesiyle ilgili gözlemini yazmanı istiyorum.

8.1. Hangi dörtgenlerde açıortayların kesişimi bir nokta oluyor?

8.2. Açıortayların kesişimi bir nokta olmadığında ne gözlemliyorsun?

8.3. Açıortaylarının kesişimi nokta olan dörtgenler özel bir sınıf oluşturuyor mu?

8.4. Kirişler ve teğetler dörtgeni olma özelliği açısından bu sınıf hangi gruba giriyor?

8.5. O zaman teğetler dörtgeninin açıortaylarının kesişimiyle ilgili vardığımız yargı ne olur?

9. "Kenar orta dikmeleri" isimli sekmeye geçelim. Sketchpad ekranında kenarlarının orta dikmelerinin kesişim noktası verilen dörgeni çekip hareket ettirerek özel dörtgenlere dönüştürmeyi dener misin? Sana verilen kağıtta her bir dörtgenin karşısına EFGH bölgesiyle ilgili gözlemini yazmanı istiyorum.

9.1. Hangi dörtgenlerde kenar orta dikmelerinin kesişimi bir nokta oluyor?

9.2. Kenar orta dikmelerinin kesişimi bir nokta olmadığında ne gözlemliyorsun?

9.3. Kenar orta dikmelerinin kesişimi nokta olan dörtgenler özel bir sınıf oluşturuyor mu?

9.4. Kirişler ve teğetler dörtgeni olma özelliği açısından bu sınıf hangi gruba giriyor?

9.5. O zaman kirişler dörtgeninin kenar orta dikmelerinin kesişimiyle ilgili vardığımız yargı ne olur?

10. Hem açıortaylarının hem de kenarorta dikmelerinin kesişimi nokta olan dörtgenler var mı? Neden?

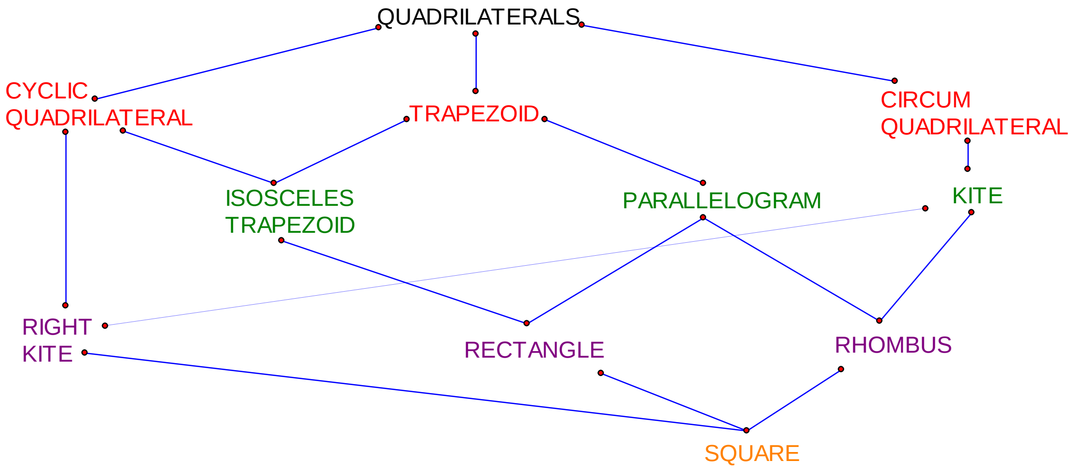
11. Kirişler ve teğetler dörtgenlerinin açıortay ve kenar orta dikmelerinin kesişimiyle ilgili nasıl bir genelleme yapabilirsin?

12. "Diyagram 2" sekmesini açalım. Teğetler dörtgenlerini de yeni bir kategori olarak hiyerarşiye ekleyip diagramı yeniden düzenler misin?

Appendix G

Turkish Version of Clinical Interview Session 4

1. Şu an ekranda gördüğün dörtgenleri incelemiş aralarındaki ilişkileri en son böyle bir diyagramla göstermiştik hatırlıyorsan. Bu diyagramı bana yorumlar mısın? Bu diyagram dörtgen ilişkileri ile ilgili sana ne anlatıyor?



1.1. Bu hiyerarşinin özellikleri nedir? Aşağı ve yukarı doğru gidildiğinde özellik sayısı nasıl değişir?

1.2. Dörtgenler arasındaki kapsama ilişkisi ile bu dörtgenlerin özellikleri arasındaki kapsama ilişkisi aynı mı? Düşünür müsün?

1.2.1. Mesela, kare ve dikdörtgen üzerinden örnek verebilirsin. Bütün kareler bir dikdörtgen midir? Karenin bütün özellikleri dikdörtgenin de özellikleri midir?

2. "Tanım 1" sekmesini açalım. Önceki mülâkatlarda deltoidi kenar özelliğine göre nasıl tanımlamıştın hatırlıyor musun? (bulamazsa sen söyle).

2.1. Bu tanımı genelleyerek hiyerarşide olmayan fakat bütün deltoidleri kapsayan yeni bir dörtgen sınıfını (Dörtgen 1) nasıl tanımlarsın?

2.1.1. Sence bir dörtgenin tanımını genelleştirebilmen için ne yapman gerekir? Yeni özellik mi eklemelisin yoksa mevcut özellikleri sınırlandırmalı mısın?

2.2. Şimdi “D1” isimli sayfa üzerinde çalışıp bu yeni tanıma uyan bütün dörtgenleri çizer misin? (Sırayla 2 kenarı, 3 kenarı ve 4 kenarı eş olan olası dörtgenleri çizmesi istenir).

3. Bu yeni dörtgen sınıfı Dörtgen 1’i hiyerarşiye ekleyebilmek için onun diğer dörtgenlerle ilişkilerini incelememiz gerekiyor.

3.1. “Tanım 2” sekmesini açalım. Yamuğu “en az bir çift kenarı paralel olan dörtgen” olarak tanımlarsak hem **yamuğun** hem de **Dörtgen 1’in** özelliklerini taşıyan yeni bir dörtgen tanımlanabilir mi? Nasıl?

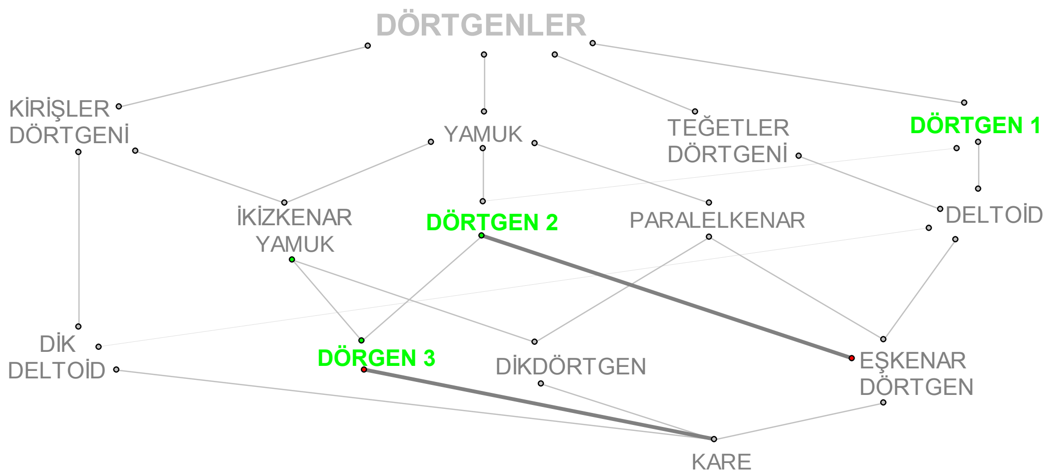
3.2. Şimdi D2 sayfası üzerinde çalışıp bu yeni tanıma (Dörtgen 2) uyan bütün dörtgenleri çizer misin?

3.3. Oluşturduğun bu Dörtgen 2’leri inceleyip **Dörtgen 2** ve **deltoidin** her ikisinin özelliklerine sahip (iki grubun kesişimi) dörtgenlerin olup olmadığı hakkında ne söyleyebilirsin?

4. Hem **Dörtgen 2** hem de **ikizkenar yamuk** özelliklerini taşıyan bir dörtgeni (Dörtgen 3) nasıl tanımlarsın? Tanıma uyan dörtgenleri çizer misin?

4.1. Bu dörtgen 3’ü “en az üç kenarı eş yamuk” olarak tanımlayabilir miyiz? Niçin?

5. Dörtgen1, dörtgen 2 ve dörtgen 3 dörtgenlerinin hiyerarşideki bağlantılarını da gösterir misin?



Appendix H

Turkish Version of Final Interview Questions

1. GSP ortamında dörtgenleri öğrenmek senin için nasıl bir tecrübeydi?

2. Dörtgenleri öğrenme sürecinde GSP'nin en etkili, en faydalı yönü neydi sence?

3. Dörtgenleri öğrenme sürecinde GSP'nin en zor yönü, en çok zorlandığın yönü neydi çalışırken?

4. GSP ortamında çalışma tecrübenin ardından dörtgenlerle ilgili daha önce bilmediğin şeyler öğrendiğini düşünüyor musun?

- **a.** Dörtgenler ve bunların tanımlarıyla ilgili yeni şeyler öğrendin mi?
- **b.** Tanımın kısa olması uzun olması, özelliklerini listelemek ya da kısaltmak gibi konularda yeni şeyler öğrenebildin mi?
- **c.** Ekonomik tanım yapma becerin daha önce de var mıydı bu çalışmada mı öğrendin?

5. GSP ortamında dörtgenleri öğrenme ile GSP olmadan sınıf ortamında doğrudan anlatarak dörtgenleri öğrenme arasında ne gibi farklar olduğunu düşünüyorsun?

6. Genel olarak baktığında GSP geometri öğrenme ve öğretmeye yönelik bakış açını nasıl etkiledi? Çalışmanın öncesini ve sonrasını düşündüğünde bakış açında bir değişme oldu mu?

7. Geometri öğretirken GSP öğretmenlik yaşantında kullanmayı düşünür müsün? Diğer öğretmen arkadaşlarına geometri öğretirken bu programı kullanmayı tavsiye eder misin?

8. Bu çalışmadan önce ve sonraki kavram tanımlarına ve bu tanımların kavramları öğrenme ve öğretmedeki rolüne bakış açının nasıl değiştiğini değerlendirebilir misin?

- **a.** Bu çalışmadan sonra tanımların dörtgen kavramlarını anlamada bunlar arasındaki ilişkileri keşfetmede etkili bir öğretim aracı olarak kullanılabileceğini düşünüyor musun?

9. Öğretmenlik yapsan bu tanımları öğrencilerine ne şekilde sunarsın?

- **a.** Tanımı öğretmen mi vermeli öğrenci kendisi mi oluşturmalı?

10. Bu çalışmadan sonra kavramları tanımlama becerinde bir gelişme olduğuna inanıyor musun? Kendi tanım yapma becerini arttırdın mı sence bu çalışma sonunda? Eksiklerini kapatabildiğine inanıyor musun?

11. Bir tanımı ekonomik olarak tanımlayabilmek sence hangi becerilerini geliştirir öğrencinin, hangi düşünme süreçlerini etkiler?

12. Bu çalışmayla ilgili söylemek istediğin ya da nasıl geliştirileceği yönünde tavsiyelerin var mı?

Appendix I

Turkish Version of “Questionnaire on Quadrilaterals II”

1. Aşağıdaki dörtgenleri nasıl tanımlarsınız?

a. Yamuk

____

____

____

İkizkenar yamuk

____

____

____

Dikdörtgen:

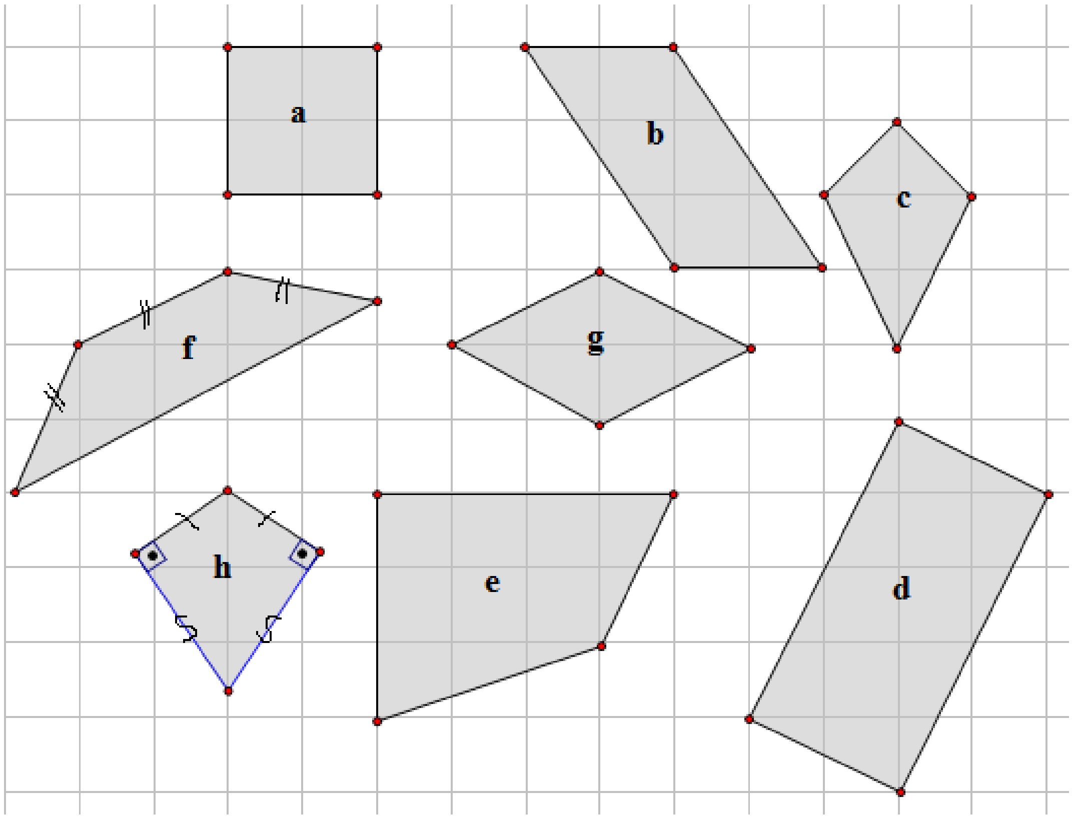
____

____

____

b. Aşağıda verilen bir grup dörtgen içerisinden verilen tanımların her birinin karakterize ettiği dörtgen(ler)i bulunuz.

NOT *: Birim karelerle ölçülemeyecek olan <u>eşit kenar uzunlukları</u> ve <u>dik açılar</u> şekil üzerinde gösterilmiştir. Gösterilmeyenleri birim kareleri kullanarak sizin bulmanız gerekir.*



(1) **Tanım**: En az bir çift komşu kenarı eş olan kirişler dörtgeni.

____

(2) **Tanım**: En az üç kenarı eş olan ikizkenar yamuk.

____

(3) **Tanım**: En az üç kenarı eş olan yamuk.

____

2. Eşkenar dörtgenin sadece *gerekli ve yeterli* olan tanımlayıcı özelliklerini içeren (ekonomik) 2 adet tanımını yapınız.

*Tanım1*____

____

____

____

*Tanım2*____

____

____

____

3. Aşağıdaki ifadeleri *(1) Gerekli ve yeterli (2) Gerekli fakat yeterli olmayan* seçeneklerinden biri ile tamamlayınız. *"Gerekli fakat yeterli olmayan"* seçeneğini kullandığınızda gerekçenizi açıklayınız ve eksik olan koşulu belirtiniz.

(a) Bir deltoidin kirişler dörtgeni olması için en az bir çift karşılıklı açısının dik açı olması ________________________________bir koşuldur.

____

____

____

(b) Eş köşegenlere sahip olma bir dörtgenin dikdörtgen olması için ____________________ bir koşuldur.

____

____

____

(c) Bir çift kenarın paralel ve bir çift karşılıklı açının eş olması bir dörtgenin paralelkenar olması için __________________ bir koşuldur.

____

____

____

(d) Bir çift kenarın paralel ve diğer çiftin eş olması bir dörtgenin paralelkenar olması için __________________ bir koşuldur.

____

____

____

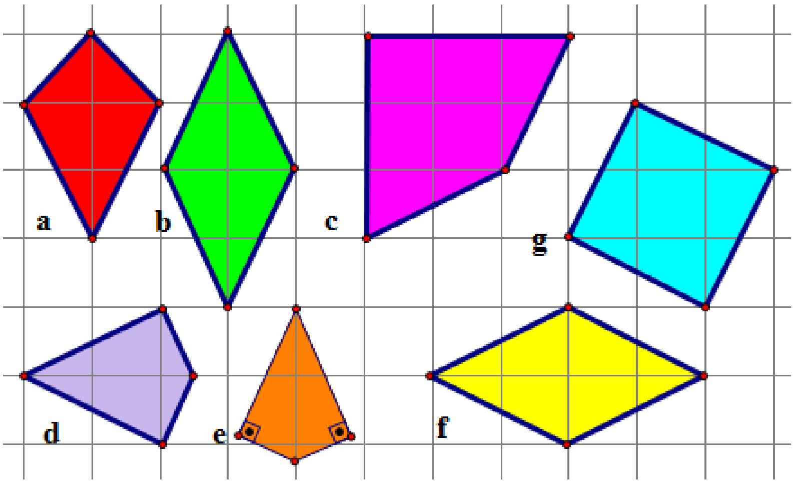
(e) Bir çift karşılıklı kenardan ve bir çift karşılıklı açıdan geçen 2 doğruya göre simetrik olma bir dörtgenin kare olması için __________________ bir koşuldur.

____

____

____

4. Aşağıdaki dörtgenlere bakınız.



(a) Yalnız a, b, c, d, e ve f dörtgenlerini kapsayan g dörtgenini dışlayan bir tanım yazınız.

____

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____

(b) Yalnız a, c, d, ve e dörtgenlerini kapsayan b, f, g dörtgenini dışlayan bir tanım yazınız.

____

____

____

____

(c) a, b, c, d, e, f, g dörtgenlerinin hepsini kapsayan bir tanım yazınız.

____

____

____

____

5. Aşağıdaki ifadelerin doğruluğu için "her zaman," "bazen," ya da "hiçbir zaman" seçeneklerinden birini seçiniz. "Bazen" seçeneğini kullanırsanız olası durumlara örnek veriniz.

a. Kare bir deltoittir. *Her zaman* *Bazen* *Hiçbir zaman*

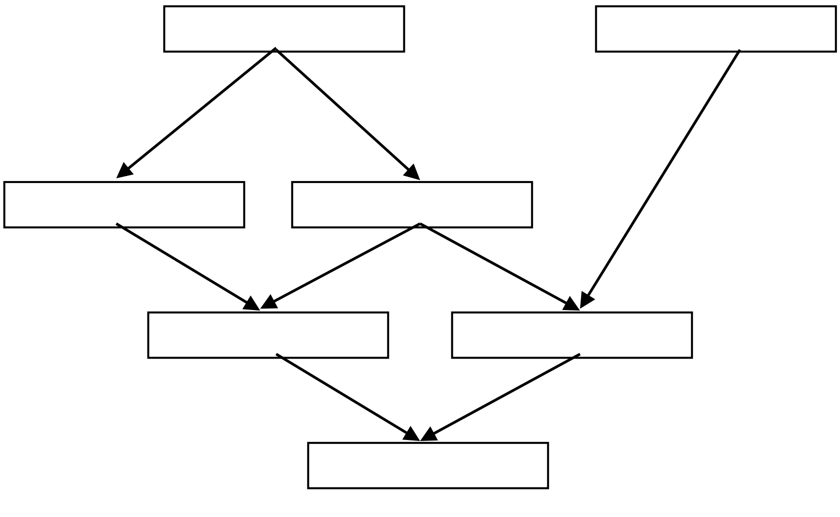
b. Deltoid bir kirişler dörtgenidir. *Her zaman* *Bazen* *Hiçbir zaman*

c. Yamuk bir kirişler dörtgenidir. *Her zaman* *Bazen* *Hiçbir zaman*

d. Deltoidin köşegenleri birbirini ortalar. *Her zaman* *Bazen* *Hiçbir zaman*

e. Kare bir kirişler dörtgenidir. *Her zaman* *Bazen* *Hiçbir zaman*

6. Özel dörtgenler olan paralelkenar, dikdörtgen, eşkenar dörtgen, kare, yamuk, ikizkenar yamuk ve deltoidi, köşegen özelliklerine göre aralarındaki hiyerarşik sınıflandırmayı gösteren aşağıdaki diyagramda doğru yerlere yerleştiriniz.



Appendix J

Matrix Comparison of the Findings of Clinical Interview 1

CLINICAL INTERVIEW 1					
	Participant 1	**Participant 2**	**Participant 3**	**Participant 4**	**Participant 5**
Initial kite definition	Correct inclusive economical (2 isosceles triangles).	Correct inclusive economical (2 isosceles triangles).	No initial definition.	Correct inclusive economical (2 isosceles triangles).	Correct but uneconomical.
Properties of kite	Correctly stated all properties but used the terms "congruent" and "equal" interchangeably while expressing the properties.	She did not remember the properties. She detected the preserved properties after observing the measures by dragging the figure. Used the terms "congruent" and "equal" correctly.	She did not remember the properties. She detected the preserved properties after observing the measures by dragging the figure. Used the correct terminology.	Correctly remembered almost all preserved properties and checked them through dragging the dynamic figure, but used the terms "congruent" and "equal" interchangeably while expressing the properties.	He had misinformation about properties. He detected the preserved properties after observing the measures by dragging the figure. Used the correct terminology
Special kites	Correctly identified square and rhombus, but incorrectly identified trapezoid.	Correctly identified square but could not identified rhombus; and incorrectly identified trapezoid.	Incorrectly identified all quadrilaterals as special kites. Could not provide reasons.	Correctly identified square but could not identified rhombus.	Correctly identified square and rhombus as special kites.
Dragging	Dragged the figure into rhombus and square but not into the trapezoid and other quadrilaterals. Explained the reasons.	Dragged the figure into rhombus and square but not into the trapezoid and other quadrilaterals. Explained the reasons.	Dragging helped to detect square and rhombus as special cases. Explained the reasons.	Dragging helped to detect square and rhombus as special cases. Explained the reasons.	Dragging helped to confirm his thoughts. Explained the reasons.

Hierarchy diagram		Correct diagram.	Correct diagram.	Correct diagram.	Correct diagram.	Correct diagram.
Inclusive definition of kite		Constructed correct inclusive economical definition but incorrectly identified parallelogram as counterexample since she has misconception that a parallelogram has one diagonal as symmetry axis.	Constructed correct inclusive but uneconomical definition.	Uneconomical definition (description) Included incorrect information as well. By thinking about the properties she correctly shortened it, but was not sure.	Constructed correct inclusive but uneconomical definition. By thinking about the properties she correctly shortened it, but was not sure.	Constructed correct inclusive economical definition.
Construction		Correct construction helped to confirm her misconception.	Correct construction helped to economize the definition.	Correct construction helped to economize the definition. (to eliminate the redundant information).	Correct construction helped to confirm her definition.	Correct construction helped to confirm her definition.
Pre-made definitions	**Definition 1**	Incorrectly evaluated the definition but correct construction helped to correct her evaluation.	Correct evaluation. No need for construction.	Correct evaluation and correct construction in the previous step.	Correct evaluation with the direct help of correct construction.	Incorrectly evaluated the definition but correct construction helped to correct her evaluation.
	Definition 2	Correct evaluation and correct construction.	Correct evaluation but not sure. Correct construction helped to sweep her doubts away.	Correct evaluation and correct construction.	Correct evaluation in the previous step and correct construction.	Correct evaluation and correct construction.

	Definition 3	Incorrectly evaluated the definition but correct construction helped to correct her evaluation.	Correct evaluation. No need for construction.	Correct evaluation and correct construction in the previous step.	This was the same as her definition. So no need for evaluation.	Correct evaluation. No need for construction.
	Definition 4	Correct evaluation and correct construction.	Correct evaluation and correct construction.	Correct evaluation and correct construction.	Correct evaluation and correct construction.	Incorrectly evaluated the definition but correct construction helped to correct her evaluation.
Inclusive definition of rhombus		Correct inclusive economical definition with correct terminology.	Correct inclusive economical definition with correct terminology.	Correct inclusive economical definition with correct terminology.	Correct inclusive economical definition with correct terminology.	Correct inclusive economical definition with correct terminology.

Appendix K

Matrix Comparison of the Findings of Clinical Interview 2

CLINICAL INTERVIEW 2						
		Participant 1	**Participant 2**	**Participant 3**	**Participant 4**	**Participant 5**
Properties of isosceles trapezoid		Almost remembered all properties.	Almost remembered all properties.	Almost remembered all properties.	Almost remembered all properties.	Did not remember almost any properties.
		Problems with mathematical language.	A little problem with mathematical language.	No problem with mathematical language.	No problem with mathematical language.	Problems with mathematical language.
		No inclusive thinking. (use of "at most")				
Preserved critical properties		Detected all correctly under dragging.	Detected all correctly under dragging.	Detected all correctly under dragging.	Detected all correctly under dragging.	Detected all correctly under dragging.
Descendants of isosceles trapezoid		Correctly identified square and rectangle but incorrectly identified prototypical trapezoid.	Correctly identified square and rectangle but incorrectly identified prototypical parallelogram. At first incorrectly identified rhombus but then corrected by critical thinking of the properties.	Correctly identified square and rectangle but incorrectly identified prototypical parallelogram.	Correctly identified square and rectangle.	Correctly identified square and rectangle.

Dragging dynamic figure		Helped to confirm ideas and realize that trapezoid is not an isosceles trapezoid. Explained the reasons.	Helped to confirm ideas and realize that parallelogra m is not an isosceles trapezoid. Explained the reasons.	Helped to confirm ideas and realize that parallelogra m is not an isosceles trapezoid. Explained the reasons.	Helped to confirm ideas.	Helped to confirm ideas.
Hierarchy diagram of isosceles trapezoids		Correctly showed relations between isosceles trapezoid, rectangle and square.	Correctly showed relations between isosceles trapezoid, rectangle and square.	Correctly showed relations between isosceles trapezoid, rectangle and square.	Correctly showed relations between isosceles trapezoid, rectangle and square.	Correctly showed relations between isosceles trapezoid, rectangle and square.
Inclusive isosceles trapezoid definition	**First attempt**	Incorrect (included parallelogra m as counter example).	Incorrect (included parallelogra m as counter example).	Correct but uneconomic al.	Incorrect (language is not clear, included rhombus and parallelogra m)	Correct but uneconomic al. Incorrect use of terms “equal” and “congruent. ”
	Second attempt	Correct inclusive economical.	Correct inclusive economical.	Eliminated redundant information through the construction .	Uneconomic al (description) .	Eliminated redundant information through the counter example search.
	Third attempt				After critically thinking on the properties she constructed correct inclusive economical definition.	
Pre-constructed isosceles trapezoid definitions	**Definition 1**	This was her own definition so no need for evaluation.	This was her own definition so no need for evaluation.	Correct evaluation, no need for construction .	This was her own definition so no need for evaluation.	This was his own definition so no need for evaluation.

	Definition 2	Correct evaluation, no need for construction. (inclusive thinking).	Correct evaluation, no need for construction.	Correct evaluation, no need for construction. (inclusive thinking).	Correct evaluation, no need for construction.	Correct evaluation, no need for construction. (inclusive thinking).
	Definition 3	Correct evaluation, no need for construction.	Incorrect evaluation. Thought parallelogram as counter example though it was not. After critical thinking she corrected the evaluation.	Correct evaluation, no need for construction.	Incorrect evaluation at first. Thought parallelogram as counter example due to misinformation about the parallelogram property. After critical thinking she corrected the evaluation.	Correct evaluation, no need for construction.
	Definition 4	Correct evaluation and correct construction.	Correct evaluation but doubts. Construction process was difficult to her, but helped to confirm her idea.	Correct evaluation, no need for construction.	Correct evaluation and correct construction.	Correct evaluation, but doubts. Construction process was difficult to her but helped to confirm her idea.

	Definition 5	No evaluation at first, but correct evaluation during the construction process.	Correct evaluation but no reasons. Construction process was difficult to her but helped to confirm her idea.	Incorrect evaluation. Construction process was difficult to her but helped to correct her evaluation.	Incorrect evaluation. She could not find counter example so accepted as correct definition. By contruction she corrected the definition.	Correct evaluation, but doubts. Construction process was difficult to him but helped to remove his doubts.
Descendants of parallelogram		Correctly identified and confirmed by dragging the dynamic figure.	Correctly identified and confirmed by dragging the dynamic figure.	Correctly identified and confirmed by dragging the dynamic figure.	Correctly identified and confirmed by dragging the dynamic figure.	Correctly identified and confirmed by dragging the dynamic figure.
Hierachy diagram of parallelograms	**First attempt**	Incorrectly thought that rhombus was a special rectangle.	Incorrectly thought that rhombus was a special rectangle.	Incorrectly thought that rhombus was a special rectangle.	Incorrectly thought that rectangle was a special rhombus.	Correct diagram.
	Second attempt	After critical thinking, she correctly showed the relationships.	After critical thinking, she correctly showed the relationships.	After critical thinking, she correctly showed the relationships.	Incorrectly thought that rhombus was a special rectangle.	
	Third attempt				After critical thinking, she correctly showed the relationships.	
Inclusive relation between parallelograms and trapezoids		Correctly identified.	Correctly identified.	Correctly identified.	Correctly identified.	Correctly identified.

Inclusive definition of trapezoid		Correct economical.	Correct economical.	Incorrect. She defined a parallelogram instead of trapezoid. After critical thinking, she corrected the definition.	Correct economical.	Correct economical.
Hierarchy diagram of all quadrilaterals		Put rhombus and rectangle into wrong places but then corrected.	At first attempt, incorrectly placed some quadrilaterals, but then constructed correct hierarchy.	At first attempt, incorrectly placed all parallelograms under isosceles trapezoid. Challanging process for her. After thinking, she constructed correct hierarchy.	At first attempt, incorrectly placed some quadrilaterals, but then constructed correct hierarchy.	At first attempt, incorrectly placed some quadrilaterals, but then constructed correct hierarchy.
Exclusive definitions	**Parallelogram**	Correct.	Correct.	First two definitions were incorrect. (both included rhombus). Correct definition at third attempt.	Correct.	First definition incorrectly included all descendants. Second definition still included rectangle. Correct definition at third attempt.

	Rhombus	Correct.	Misinformation about the symmetry property of rhombus. after working on the rhomus figure she corrected the definition.	Correct.	First definition included square, but then she correctly defined.	After working on the rhombus figure to remember the symmetry property, he constructed correct definition.
	Kite	Correct.	Correct.	Correct.	First definition was incorrect since it included any ordinary quadrilateral, but then she correctly defined.	First definition was exclusive, but uneconomical. Then he shortened the definition correctly.
Inclusive or exclusive defining		In support of inclusive defining thinking that it increases analysis skills.				

Appendix L

Matrix Comparison of the Findings of Clinical Interview 3

CLINICAL INTERVIEW 3						
		Participant 1	**Participant 2**	**Participant 3**	**Participant 4**	**Participant 5**
Cyclic quadrilateral figure		Did not know the name. Defined correctly. Did not know the condition to be a cyclic quadrilateral.	Did not know the name. Defined correctly. Did not know the condition to be a cyclic quadrilateral.	Did know the name. Defined correctly. Did know the condition to be a cyclic quadrilateral.	Did know the name. Defined correctly. Did know the condition to be a cyclic quadrilateral.	Did know the name. Defined correctly. Did know the condition to be a cyclic quadrilateral
Special cyclic quadrilaterals		Correctly identified square and rectangle; but decided based on the visualization.	Correctly identified square and rectangle; decided by thinking visually and considering the properties.	Correctly identified rectangle but not square by viasualization.	Correctly identified all descendants except for right kite.	Correctly identified all descendants.
		While comparing rectangle and paralleleogram she found out the condition.	While thinking about the rhomus, she found the condition.	While thinking about the parallelogram she found the condition.		
		Detected isosceles trapezoid based on the condition.	Detected isosceles trapezoid and right kite based on the condition	Detected square and isosceles trapezoid based on the condition.		

		Did not considered right kite.	Incorrectly accepted trapezoid as cyclic quadrilateral.	Did not considered right kite.	Did not considered right kite.	
Dragging cyclic quadrilateral figure		Dragging helped to confirm her findings and to discover that a right kite is a cyclic quadrilateral, too.	Dragging helped to confirm her findings and to discover that a prototypical trapezoid is not a cyclic quadrilateral.	Dragging helped to confirm her findings and to discover that a right kite is a cyclic quadrilateral, too.	Dragging helped to confirm her findings and to discover that a right kite is a cyclic quadrilateral, too.	Dragging helped to confirm his findings.
Definition of right kite	**in terms of quadrilateral**	First definition was incorrect (included rectangle).	First definition was incorrect (defined a cyclic quadrilateral in general).	First definition was incorrect (included rectangle).	First definition was incorrect (included rectangle).	First definition was incorrect (included ordinary quadrilateral).
		Second definition was correct economical.	Second definition was correct but not clear enough.	Second definition was correct economical but an exclusive definition (excluded square).	Second definition was correct economical.	After the construction he constructed correct definition.
			Final definition was correct economical.	Final definition was correct economical.		

	in terms of kite	First attempt was not a definition. Second definition was correct economical.	Correct economical.	Correct economical.	Correct economical.	Correct economical.
	in terms of cyclic quadr.	First definition was incorrect (included rectangle). Second definition was correct economical.	Correct economical.	First definition was incorrect (included ordinary quadrilateral). After dragging test on the dynamic figure, she constructed correct economical definition.	She could not define. It was challenging for her. After critical thinking she constructed the correct definition.	First definition was incorrect (included rectangle). Second definition was correct uneconomical Finally he constructd the correct economical definition.
Definitions in terms of cyclic quadrilateral	**Isosceles trapezoid**	First one was correct uneconomical Second one was correct economical.	Correct economical.	First one was correct uneconomical Second one was correct economical.	First one was correct uneconomical Second one was correct economical.	Correct economical.
	Rectangle	First one was correct uneconomical Second one was correct economical.	Correct economical.	Correct economical.	First one was correct uneconomical Second one was correct economical, but she was not sure. Construction helped to confirm the definition.	Correct economical.

	Square	Correct economical.	First one was correct uneconomic al Second one was correct economical.	Correct economical.	Correct economical.	First definition was incorrect (included rectangle). Second definition was incorrect (included rectangle). Finally he constructed the correct economical definition after construction on the dynamic figure.
Hierarchy diagram		Explained all relations correctly and added cyclic quadrilatera ls category correctly.	Explained all relations correctly and added cyclic quadrilatera ls category correctly.	Explained all relations correctly and added cyclic quadrilatera ls category correctly.	Explained all relations correctly and added cyclic quadrilatera ls category correctly.	Explained all relations correctly and added cyclic quadrilatera ls category correctly.
Circum quadrilateral figure		Did not know the condition to be a circum quadrilatera l.	Did not know the condition to be a circum quadrilatera l.	Did not know the condition to be a circum quadrilatera l.	Did not know the condition to be a circum quadrilatera l.	Did not know the condition to be a circum quadrilatera l.

Special circum quadrilaterals		By visual prediction she correctly detected only square but incorrectly detected isosceles trapezoid.	By visual prediction she correctly detected square, kite, right kite and rhombus, but incorrectly detected parallelogram and trapezoid.	By visual prediction she correctly detected only square and was not sure for rhombus. She did not accepted kite and right kite as circum quadrilateral.	By visual prediction she correctly detected only square and was not sure for rhombus and kite.	By visual prediction she correctly detected only square, but incorrectly he did not accept rhombus as circum quadrilateral. He did not know kite and right kite. He was not sure for trapezoid and isosceles trapezoid.
Dragging the dynamic figure of circum quadrilateral.		Dragging was technically difficult but she correctly determined all circum quadrilaterals.	Dragging was not difficult, she correctly determined all circum quadrilaterals.	Dragging was not difficult, she correctly determined all circum quadrilaterals.	Dragging was not difficult, she correctly determined all circum quadrilaterals.	Dragging was not difficult, she correctly determined all circum quadrilaterals.
Intersection of angle bisectors		Correctly identified for each quadrilateral.	Correctly identified for each quadrilateral.	Correctly identified for each quadrilateral.	Correctly identified for each quadrilateral.	Correctly identified for each quadrilateral.
Intersection of perpendicular bisectors of the sides		Correctly identified for each quadrilateral.	Correctly identified for each quadrilateral.	Correctly identified for each quadrilateral.	Correctly identified for each quadrilateral.	Correctly identified for each quadrilateral.
Relation between intersections and cyclic-circum quadrilaterals		Correctly determined the two conditions.	Correctly determined the two conditions.	Correctly determined the two conditions.	Correctly determined the two conditions.	Correctly determined the two conditions.

Hierarchy diagram		Correctly added circum quadrilateral category into the diagram.	Correctly added circum quadrilateral category into the diagram.	Correctly added circum quadrilateral category into the diagram.	Correctly added circum quadrilateral category into the diagram.	Correctly added circum quadrilateral category into the diagram.

Appendix M

Matrix Comparison of the Findings of Clinical Interview 4

CLINICAL INTERVIEW 4						
		Participant 1	**Participant 2**	**Participant 3**	**Participant 4**	**Participant 5**
Hierarchy		Correctly explained inclusive relations.	Correctly explained inclusive relations.	Correctly explained inclusive relations.	Correctly explained inclusive relations.	Correctly explained inclusive relations.
Opposite inclusive relations		Correctly explained opposite inclusive relations.	Confused with the opposite inlusive relations. After discussing the square and rectangle, she was able to explain.	Confused with the opposite inlusive relations. After discussing the square and rectangle, she was able to explain.	Correctly explained opposite inclusive relations.	Correctly explained opposite inclusive relations.
Quad1 definition		Correctly generalized kite definition.	Correctly generalized kite definition.	Correctly generalized kite definition.	Correctly generalized kite definition.	Correctly generalized kite definition.
Sketches of quad1		Drew all possible quad1s.	Drew all possible quad1s.	Drew all possible quad1s.	Drew all possible quad1s.	Drew all possible quad1s.
Quad2 definition	**Quadrilateral**	correct	correct	correct	correct	correct
	Quad1	correct	correct	correct	correct	correct
	Trapezoid	correct	correct	correct	correct	correct
Sketches of quad2		Drew possible quad2s. Discovered that rhombus was a quad2.	Incorrectly drew parallelogram but then corrected it as a rhombus.	Drew possible quad2s. Discovered that rhombus was a quad2	Drew possible quad2s. Discovered that rhombus was a quad2	Drew possible quad2s. Discovered that rhombus was a quad2

Quad3 definition	**Isosceles trapezoid**	correct	correct	correct	correct	correct
	Quad2	correct	correct	correct	correct	correct
	Cyclic quadrilateral	First definition was uneconomical but after the drawing process she economized the definition.	First definition was incorrect (included right kite). Second definition was correct, but he accepted rhombus as counter example though it was not. Redefined a third definition as the correct one.	correct	First definition was incorrect (included rectangle). Second definition was incorrect (included right kite). Redefined a third definition as the correct one.	First definition was incorrect (included rectangle). After the drawing process she corrected the definition. Then, he wrote an alternative definition but it was incorrect, (included right kite as counter example) Redefined the alternative definition correctly.
Sketches of quad3		Drew possible quad3s but incorrectly drew rhombus . Then realized that rhombus is not an isosceles trapezoid so it can not be a quad3.	Drew possible quad3s.	Drew possible quad3s but incorrectly drew rhombus . Then realized that rhombus is not an isosceles trapezoid so it can not be a quad3.	Drew possible quad3s but incorrectly drew rhombus . Then realized that rhombus is not an isosceles trapezoid so it can not be a quad3.	Drew possible quad3s.

Final hierarchy		Correctly included quad1, quad2 and quad 3 and indicated relationships.	Correctly included quad1, quad2 and quad 3 and indicated relationships.	Correctly included quad1, quad2 and quad 3 and indicated relationships.	Correctly included quad1, quad2 and quad 3 and indicated relationships.	Correctly included quad1, quad2 and quad 3 and indicated relationships.

Appendix N

Curriculum Vitae

PERSONAL INFORMATION

Surname, Name: Öztoprakçı, Seçil
Nationality: Turkish (TC)
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EDUCATION

Degree	Institution	Year of Graduation
BS	METU Elementary Mathematics Education	2007
High School	Lüleburgaz High School, Ankara	2001

WORK EXPERIENCE

Year	Place	Enrollment
2011- Present	Gazi University	Research Assistant

FOREIGN LANGUAGES

Advanced English, Elementary Spanish

PUBLICATIONS

1. Öztoprakçı, S. & Çakıroğlu, E. (2013). Dörtgenler. In İ. Ö. Zembat, F. M. Özmantar, E. Bingölbali, H. Şandır, & A. Delice (Eds), *Tanımları ve Tarihsel Gelişimleriyle Matematiksel Kavramlar* (pp. 251-272). Ankara: Pegem Akademi.

2. Öztoprakçı, S. (2012). Investigation of prospective elementary mathematics teachers' definition construction process in the presence of dynamic geometry supported tasks. *The International Journal of Science, Mathematics and Technology Learning,* 19*(4),* 53-66.

HOBBIES

Tango, Salsa, Swimming, Origami

Appendix O

Turkish Summary

GİRİŞ

Geometri hem öğrenilmesi hem öğretilmesi zor olan bir alandır (Hansen ve Pratt (2005) Geometrik kavramların zihinde oluşturulması görsel, uzamsal, ölçümsel becerileri bir arada gerektiren çok yönlü bir süreçtir. Bunlardan görsel beceri ise geometri alanına özgü olup onu diğer alanlardaki kavram oluşturma sürecinden ayırır (Walcott, Mohr ve Kastberg, 2009). Bireyin sezgileri vasıtasıyla oluşturduğu kavram görüntüsü, geometrik kavramı zihinde yapılandırma sürecinin duyusal boyutudur ve bu süreçte pratik anlamı oluşturmaya yardımcı olur (Fishbein ve Nachlieli, 1998). Fakat sezgisel yaklaşım bize her zaman doğru sonuçlar veremeyeceği için doğruluğunun kavram tanımları vasıtasıyla kontrol altına alınması gerekir (Fishbein, 1993; Kondratieva1 ve Radu, 2009). Öte yandan geometrik kavram tanımları tek başına o kavramın anlamlı öğrenildiğini garanti etmez. Örneğin bir öğrenci sorulduğunda paralelkenarın tanımını doğru şekilde yapıyor olabilir ancak eğer zihninde paralelkenar için prototip bir kavram görüntüsü oluşmuşsa kare, dikdörtgen ve eşkenar dörtgenin birer paralelkenar olduğunu kabul etmeyecektir; çünkü zihnindeki paralelkenar görüntüsü bütün açıların ya da kenarların eş olmasına izin vermeyecektir (De Villiers, 2004). Bu nedenle kavramın zihinde anlamlı yapılandırılması, kavram tanımları ve kavram görüntüleri arasında bir etkileşim ve kavramlar arasındaki ilişkilerin irdelenmesini gerektirir (De Villiers, 2004; Fishbein, 1993; Fischbein ve Nachlieli, 1998). Bu anlamda düşünüldüğünde de geometrik kavramların kapsayıcı ve hariç tutan tanımlarını birlikte inceleyip aralarındaki farkı idrak etmek geometri öğretimi ve öğrenimi açısından önemli bir zihinsel süreçtir. Dolayısıyla, bir kavramın zihinde anlamlı şekilde oluşturulması sürecinde geometrik kavram tanımları önemli role sahiptir.

Kavram oluşturma sürecindeki etkin rollerinin yanı sıra, kavram tanımlarının matematiğin bel kemiğini oluşturduğu ve matematik öğretme ve öğrenme sürecinde önemli role sahip olduğu pek çok araştırmacı tarafından kabul görmüştür (örn., De Villiers, 1998; Furinghetti ve Paola, 2002; Vinner, 1976). De Villiers'a (1998) göre tanım oluşturma süreci problem çözme, ispat yapma ya da matematiksel genellemeler yapma gibi pek çok matematik etkinliği kadar önemlidir ve hatta bu etkinliklerin de temelinde kavram tanımlarını anlama becerisi yatar. Tanımlar bir kavramı tanıtırken, onu diğer kavramlardan ayırt ederken; ispat yaparken ya da mantıksal iddiaları ortaya atarken kullanılan temel yapı taşlarıdır (Silfverberg, 2003). Yani kavram tanımları aslında matematiğe ve matematiksel teorilerin gelişmesine yol veren giriş kapılarıdır (Furinghetti ve Paola, 2002). Dilbilim açısından bakıldığında ise, tanımlar matematiksel kavramların anlamlarını iletmede en önemli araç olduklarından, öğretme ve öğrenme sürecinde yazılı ve sözel iletişimi sağlayan matematiksel dilin temelini oluştururlar (Shir ve Zaslavsky, 2001).

Poincaré (1952)'e göre tanımsal ifadenin amacı tanımlanan nesneyi diğer nesneler sınıfından ayırt edebilmektir ve bir tanımın anlaşılır olması için sadece tanımladığı nesne değil o nesneyi arasından ayırt etmek zorunda olduğumuz diğer nesnelerle birlikte incelemek gerekir; ancak o zaman aralarındaki farkı idrak etmek kolay hale gelir. Poincaré (1952) tanımlar ve sınıflandırma arasındaki bu yakın ilişkiyi açıklayarak pek çok bilim adamının bu iki konuyu bir arada çalışmasına ışık tutmuştur (örn., De Villiers ve Govender, 2002; Erez ve Yerushalmy, 2006; Fujita ve Jones, 2007; Furinghetti ve Paola, 2002). İlgili literatür incelendiğinde ise tanımlar ve sınıflandırma ilişkisinin en çok incelendiği geometri konusunun dörtgenler olduğu görülmüştür. Bu durumun pek çok nedeni vardır. Örneğin geometride dörtgenler konusu, eş tanımlar ve hiyerarşik sınıflandırma gibi konuların sözel ve görsel olarak en iyi incelenebileceği zengin bir geometrik şekiller dünyası sunar (Furinghetti ve Paola, 2002); ayrıca dinamik geometri ortamlarının yaratacağı bilişsel düşüncenin araştırılması için de dörtgenler en iyi geometri konusudur (Jones, 2000). Ama en önemlisi ise bugüne kadar yapılan çalışmalarda dörtgenler konusunun öğrencilerin geometri alanında anlamakta en çok zorlandığı konu olduğunu ve bu alandaki zorlukların henüz aşılamamış olduğunu gösteren araştırma sonuçlarına ulaşılmasıdır (Wu ve Ma, 2005). Bu nedenle yıllardır bu konuda çalışmalar yapılmasına rağmen sorunların çözümüne yönelik daha detaylı çalışmalara günümüzde de halen ihtiyaç duyulmaktadır.

Öte yandan pek çok bilim adamı, kavram tanımlarını anlamanın öğretmenlerin matematik alan bilgisinin en temel taşı olduğunu ve bu alandaki eksikliklerin onların öğretimini ve dolayısıyla da öğrencilerin öğrenimini olumsuz yönde etkileyeceği görüşünü savunur (örn., Ball, Bass, ve Hill, 2004; Zazkis ve Leikin, 2008). Alan tanımlarına hâkim olup onları etkili birer öğretim aracı olarak kullanmak; öğrenci seviyesine uygun olacak şekilde doğru tanımların seçilmesi ve doğru matematiksel dilin kullanılması öğretmenlik alan bilgisinin önemli bir parçası olarak görülmektedir (Zazkis ve Leikin, 2008). Öğretmenlerin matematiksel kavram tanımları bilgisi ve hâkimiyeti, onların kavramları öğretme yöntemi tercihlerini ve öğretme sürecindeki esnekliklerini etkileyecektir; dolayısıyla da kavramların öğrenciler tarafından zihinsel olarak doğru şekilde oluşturmasında önemli rol oynayacaktır (Winicki-Landman ve Leikin, 2001). Bu nedenle, öğretmenlerin hem kapsayıcı hem de hariç tutan tanımları, bunlar arasındaki farkları, zihinsel oluşum süreçlerini ve bu iki tanımsal olgunun kavramların sınıflandırılmasına etkisini bilmeleri, öğretimsel amaçları doğrultusunda hangisinin öğretileceğine karar verebilmeleri ve açısından önemlidir.

Tanımlar matematiksel, dilbilimsel ve pedagojik yönleri olan ve pek çok açıdan matematik öğretimindeki hayati önemi yadsınamayacak önemli yapı taşlarıdır. Buna rağmen, yapılan çalışmalar sadece öğrencilerin değil, sağlam alan bilgisine sahip olduğu düşünülen matematik öğretmenlerinin de matematiksel kavram tanımlarını anlamada, oluşturmada, değerlendirmede ve bu tanımları kavramlar arası ilişkileri açıklamada kullanmada zorluk yaşadığını göstermiştir (De Villiers, 1998; De Villiers ve Govender, 2002; Erez ve Yerushalmy, 2006; Fujita ve Jones, 2007; Furinghetti ve Paola, 2002; Jones, 2000; Shir ve Zaslavsky, 2001). Bu nedenle de memnun edici olmayan bu veriler, öğretmen eğitim programlarının yeterliliği ve etkililiği konusundaki endişelerin ve soru işaretlerinin doğmasına sebep olmuştur. Gerekli ve etkili önlemlerin öğretmenler öğretmenlik mesleğine başlamadan daha üniversite aşamasında alınması gerekir ki, bu da ancak öğretmen eğitimi programlarının kalitesini arttırmakla mümkün olabilir. Öğrencilerin öğrenme düzeylerini arttırabilmek için öğretmen adayları onlara gerekli ve yeterli bilgi, beceri ve nitelik kazandıracak etkili eğitim programlarıyla desteklenmelidirler. Eğitimleri sırasında, tanımları ve tanımların kavramları öğrenme ve öğretme sürecindeki rolü üzerine düşünmeye teşvik edilerek ve tanımlar yerine, tanım yapma sürecini öğrenecekleri uygulamalarda aktif rol almaları sağlanarak zorlukların belli derecede üstesinden gelinebileceği düşünülmektedir.

Öte yandan, şimdiye kadar yapılan çalışmaların bireylerin tanımlarla ilgili yaşadığı zorlukların altında yatan sebeplerin ve tanım yapma sürecindeki düşünme süreçlerinin detaylı incelenmesinde yetersiz kaldığı ve konunun daha detaylı irdelenerek çözüm yolları üretilmesine ihtiyaç olduğu görülmüştür. Dinamik geometrinin bu alandaki problemleri çözme konusunda ne derece etkili olduğunu araştıran bazı çalışmalar da yapılmıştır ama bazı çalışmalarda dinamik öğrenme ortamları bir dereceye kadar etkili bulunurken (Jones, 2000; Furinghetti ve Paola, 2002) bazıları ise içeriksel faktörler nedeniyle dinamik ortamların etkililiği konusunda net bir bulgu elde edememişlerdir (De Villiers ve Govender, 2002; Erez ve Yerushalmy, 2007). Bu nedenle de bireylerin dinamik geometri ortamındaki tanım oluşturma ve tanımları dörtgenleri sınıflandırmada kullanma sürecindeki zihinsel süreçlerinin daha detaylı incelenmesine gereksinim vardır. Bu alandaki problemlerin üstesinden gelmeye katkı sağlamak amacıyla bu çalışma, ilköğretim matematik öğretmen adaylarının tanımları ve aralarındaki ilişkiler aracılığıyla kavramları anlamlı oluşturmadaki düşünme sürecini ve bu çalışma kapsamında hazırlanan dinamik geometri destekli öğretim materyalinin düşünme düzeylerini daha üst seviyelere taşımada ve zorlukların üstesinden gelmede ne derece etkili olabileceğini niteliksel detaylı verilerle incelemek üzere uygulanmıştır. Bu çalışmada şu sorulara cevap aranmıştır:

1. Dinamik geometri destekli klinik mülâkatların uygulanmasından önce ve sonra, ilköğretim matematik öğretmen adaylarının, tanımlar ve onların öğrenme ve öğretme sürecindeki rolü hakkındaki algıları nelerdir?
2. Dinamik geometri destekli klinik mülâkatların uygulanmasından önce ve sonra, ilköğretim matematik öğretmen adaylarının, tanımların "minimum, eşdeğer, kapsayıcı ve hariç tutan" yapıları hakkındaki anlayışları nelerdir?
3. İlköğretim matematik öğretmen adayları Geometer's Sketchpad destekli etkinliklerin olduğu bir öğrenme ortamında tanım yapma ve sınıflandırma yoluyla dörtgen kavramlarını anlayışlarını nasıl geliştirirler?
4. Dinamik geometri destekli öğrenme etkinlikleri ilköğretim matematik öğretmen adaylarının dörtgenleri tanımlama, değerlendirme ve sınıflandırma becerilerini nasıl etkiler?

Çalışmanın Önemi

Bu çalışma kapsamında hazırlanan teknoloji destekli öğretim materyalinin, öğretmen adaylarının kavramsal tanımları anlamadaki zorluklarının üstesinden gelmede etkili olduğu takdirde, öğretmen eğitimi programlarında etkili bir öğretim aracı olarak yarar sağlaması ve böylelikle bu programlarının kalitesini arttırmada rol oynaması beklenmektedir. Ayrıca öğretmen adaylarının hazırlanan etkinlikler yardımıyla tanımların rolünü etkili bir şekilde anlamalarının öğretimsel kararlarını ve dolayısıyla öğrencilerinin öğrenmesini de olumlu yönde etkileyeceği düşünülmektedir. Hem öğretmenler hem öğrenciler matematiksel tanımları etkili şekilde kavrayıp kullanabilirlerse diğer matematiksel etkinliklerde karşılaştıkları sorunlar da bir dereceye kadar üstesinden gelinebilir hale gelecektir.

LİTERATÜR TARAMASI

Tarif ve Tanım Arasındaki Fark

Tanım (definition) ve tarif (description) arasındaki farka değinmek gerekir ki bu ikisi farklı beceriler gerektiren süreçlerdir (Favailli ve Romanelli, 2006). Bir kavrama ait özelliklerin hepsini listelemekle o kavram tanımlanmış olmaz sadece tarif edilmiş olur (De Villiers 1996, 1998). Bir kavramın matematiksel tanımını oluşturmak ise özellikler arasında mantıksal çıkarımlar yaparak kavramı ifade edebilecek gerekli ve yeterli tanımlayıcı özellikleri ayırt etmek gibi üst düzey düşünme becerisi gerektiren zihinsel bir süreçtir (De Villiers ve Govender, 2002; Favailli ve Romanelli, 2006; Fujita ve Jones, 2007). Bu beceri ise van Hiele'nin üçüncü seviye geometrik düşünme becerisine işaret eder (De Villiers, 1996, 1998). Örneğin paralelkenar "iki çift karşılıklı kenarı eş ve paralel olan, iki çift karşılıklı açıları eş olan ve köşegenleri birbirini ortalayan dörtgendir" şeklinde tanımlanırsa bu matematiksel bir tanım değil, pek çok özelliğin ardı ardına listelendiği bir paralelkenar tarifi (description) olur (Favilli & Romanelli, 2006). Paralelkenarın matematiksel tanımını yaparken ise karşılıklı açılarının eş olması özelliğinin karşılıklı kenarlarının paralel olmasının bir sonucu olduğu ya da karşılıklı kenarlarının paralel olmasının karşılıklı açılarının eş olması özelliğinin bir sonucu olduğu gibi mantıksal çıkarımların yapılmasını gerektirir. Aynı şekilde karşılıklı kenarların eş olması ve köşegenlerin birbirini ortalaması özellikleri de birbirinin sonucu olan özelliklerdir. Bu gibi çıkarımların ardından paralelkenar için gerek ve yeter tanımlayıcı şartlar belirlenerek "iki çift karşılıklı kenarı paralel olan dörtgen" ya da "iki çift karşılıklı kenarı eş olan dörtgen" şeklinde matematiksel olarak eş değer tanımları yapılabilir. Bu tanımlarda kullanılan tanımlayıcı özelliklerden yola çıkarak paralelkenara ait diğer bütün özellikler mantıksal çıkarım yoluyla bulunabilir.

Bir kavrama ait olan pek çok gerek koşul ve bu koşullar arasında da o kavramı karakterize etmeye yetecek pek çok yeter koşul olabilir. Bu nedenle kavramı tanımlamaya yetecek gerekli ve yeterli özellikleri içeren matematiksel olarak eş değer birden fazla tanım oluşturmak mümkündür. Tanımda kullanılan gerek ve yeter özellik dışında kalan diğer tanımlayıcı özellikler ise ispatlanması gereken teoremler olurlar. Örneğin dikdörtgen "3 dik açısı olan dörtgen şeklinde tanımlanırsa dördüncü iç açının da dik açı olacağı da ispatlanabilir bir teorem olur. Yani gerektiğinden fazla özellik içerecek şekilde yapılan bir tanım, örneğin "dikdörtgen dört iç açısının ölçüsü 90° olan dörtgendir," içerisinde bir tanım ve en az bir tane de ispatlanabilir bir teoremi barındıran ekonomik olmayan bir tanım olur (Van Dormolen ve Zaslavsky, 2003). Sonuç olarak tanım yapmadaki en önemli bilişsel beceri, o kavramın gerekli ve yeterli tanımlayıcı özelliklerini diğer özellikleri arasından ayırt edebilmektir.

Kapsayıcı ve Hariç Tutan Tanımlar

Usiskin ve Griffin (2008) kapsayıcı (inclusive) tanımlar ve hariç tutan (exclusive) tanımlar olmak üzere iki çeşit tanımdan bahseder. Bir tanım diğer tanımın içerdiğini kasıtlı olarak içermiyorsa bu tanıma dışlayan(exclusive) tanım eğer diğerinin içeriğini de kapsıyorsa bu tanıma da kapsayan (inclusive) tanım demişlerdir. Literatürde yamuğun tanımı konusunda farklı görüşler yer almaktadır. Bazı ekoller yamuğun "yalnız bir çift kenarı paralel olan dörtgen" şeklindeki hariç tutan tanımını doğru kabul ederken bazıları da "en az bir çift kenarı paralel olan dörtgen" şeklindeki kapsayıcı tanımını doğru kabul ederler. Hariç tutan tanım paralelkenarları ve yamuğu iki ayrı dörtgen sınıfı olarak ayırır ve yamuğu tek başına bir dörtgen sınıfı olarak tanımlar. Öte yandan kapsayıcı yamuk tanımı bütün paralelkenarı da içine alan, paralelkenarların da özel yamuklar olduğunu ifade eden bir tanımdır. Yani yamuğun bu iki tanımı arasındaki farklılık dörtgenlerin farklı sınıflandırılmasına neden olur.

Eğer dörtgen kavramlarının her biri diğerleriyle ilişkisi göz ardı edilerek tek başına bir kavram olarak tanımlanırsa dörtgenler birbirinden ayrık kavramlar olarak sınıflandırılırlar; aralarındaki kapsama ilişkileri düşünülerek tanımlanırsa hiyerarşik olarak sınıflandırılırlar (Usiskin ve Griffin, 2008). Kavramların birbirinden soyutlanarak değil de kapsayıcı şekilde tanımlanması literatürde daha çok destek görür. Fujita ve Jones (2007)'a göre kapsayıcı tanımların sonucu olarak ortaya çıkan hiyerarşik ilişkiler daha ekonomik ve fonksiyoneldir çünkü bir kavram için doğru olan bir özellik o kavramın kapsadığı diğer kavramlar için de doğru olacaktır ve tekrar ele alınmalarına veya ispatlanmalarına gerek kalmayacaktır. Ayrıca hiyerarşik ilişkileri irdelemek kavramlar arasındaki geçişli, asimetrik ve zıt asimetrik ilişkilerin de anlaşılmasını sağlayacaktır

(Fujita ve Jones, 2007). Örneğin kare özel bir dikdörtgense ve dikdörtgen de özel bir ikizkenar yamuksa kare de özel bir ikizkenar yamuktur, bu da kavramlar arasındaki geçişlilik ilişkisini ifade eder. Asimetri ilişkisini ifade edecek olursak, kare özel bir dikdörtgendir ama dikdörtgen özel bir kare değildir. Öte yandan dikdörtgenin bütün özellikleri karede de varken karenin bütün özellikleri dikdörtgende yoktur, bu da kare ve dikdörtgen ile bunların özellikleri arasındaki zıt asimetrik ilişkiyi açıklar (Schwarz ve Hershkowitz, 1999).

De Villiers (1994)'a göre kapsayıcı (inclusive) ya da hariç tutan (exclusive) tanımların her ikisi de kabul görmüş ve her ikisi de matematiğin farklı alanlarında eşit şekilde kullanılmışlardır; yani her ikisi de doğru kabul edilir. Hangisinin kullanılacağı eğitimsel amaçlara ya da kişisel tercihlere göre değişebilir; ancak kapsayıcı tanımların ve geometrik nesneler arasındaki kapsayıcı ilişkilerin irdelenmesi son dönemlerde daha çok kabul görmektedir. De Villiers (1994)'e göre kapsayıcı ilişkileri göz önünde bulundurmak "daha ekonomik tanımların ve teoremlerin oluşturulmasını sağlar; tümden gelimsel sistematiğin ve daha özel kavramların özelliklerinin elde edilmesini kolaylaştırır; problem çözme sırasında kolaylık sağlayan zihinsel şemanın oluşumunu sağlar; çok daha fazla alternatif tanım üretilmesine olanak sağlar" (p.15). Özetle, kapsayıcı tanımların öğrenilmesi hariç tutan tanımların öğrenilmesine göre zihinsel düşünme kalitesini arttırma, daha üst düzey düşünme becerisi kazandırma gibi avantajları nedeniyle daha ön plandadır.

Hariç tutan tanımlarda her bir kavram diğer kavramlardan soyutlanarak ve aralarındaki ilişki göz önünde bulundurulmadan tanımlandığı için zihinde prototip (tek tip) kavram şekillerinin oluşmasına ve dolayısıyla da bireylerin bu kavramlar arasındaki ilişkileri anlayamamasına neden olabilir (Kondratieval ve Radu, 2009, Schwarz ve Hershkowitz, 1999,). Yani birey bir kavrama ait zihninde oluşturduğu tek tip şekle dayanarak yaptığı akıl yürütmesinde, o kavramın özelliklerini taşıyan ama görsel olarak farklı gelen başka bir kavramı onun bir örneği olarak kabul etmeyebilir (Schwarz ve Hershkowitz, 1999). Mesela, karenin bütün kenarlarının eş olması, onun zihninde oluşturduğu dikdörtgen şekline uymadığı için kareyi bir dikdörtgen olarak kabul etmeyebilir. Dolayısıyla hariç tutan tanımların öğretimi, prototip kavramsal şekillerin oluşmasına neden olarak kavramlar arası ilişkilerin anlaşılmasını zorlaştırabileceği için biraz daha dezavantajlı görülmektedir. Ancak De Villiers (1994)'ın da ifade ettiği gibi öğretimsel amaçlar doğrultusunda hariç tutan tanımların kullanılması bazı durumlarda daha uygun olabilir. Örneğin, geometrik kavramlar arasındaki ilişkileri anlayabilecek düzeyde zihinsel becerileri henüz gelişmemiş küçük yaştaki öğrencilere başlangıç olarak kavramların tek tek öğretilmesi daha uygun olabilir.

Bu çalışmada dörtgen kavramlarının hem kapsayıcı (inclusive) ve hem de hariç tutan (exclusive) tanımları ve bu tanımlara dayalı dörtgen sınıflandırmaları bir arada ele alınmıştır.

Literatürdeki Çalışmalardan Örnekler

Fujita ve Jones (2007), öğretmen adaylarının tanım bilgisini ve dörtgenleri sınıflandırmadaki becerisini anket çalışmasıyla inceledikleri bir çalışma yapmışlardır. Ankette 2 soru yer almış, bunlardan ilkinde katılımcılardan bazı dörtgenler arasındaki ilişkileri belirlemeleri istenirken diğer soruda bir deltoid tanımı verilmiş ve but anıma göre paralelkenar, kare, dikdörtgen ve yamuğu tanımlamaları istenmiştir. Çalışmanın sonunda, öğretmen adaylarının bu konudaki geometrik düşünme becerilerinin van Hiele düşünme düzeylerine göre 1 ya da 2. seviye gibi düşük seviyelerde kaldığını bulmuştur. Ayrıca, katılımcıların çoğunun, yamuk dışındaki dörtgenlerin hemen hepsini doğru çizebilmelerine rağmen, bu dörtgenleri tanımlamada başarısız oldukları görülmüştür. Bir başka çalışmada ise öğretmen adaylarının bilinen en basit dörtgen kavramı olan kareyi bile tanımlamada başarısız olduğu ve verilen kare tanımlarından hangilerinin doğru tanım olduğu konusunda anlaşmazlık yaşadıkları görülmüştür (Shir ve Zaslavsky, 2001).

Bir diğer çalışmada ise De Villiers ve Govender (2002), öğretmen adaylarının tanımları anlayışlarını ve sketcpad ortamında tanım yapma becerilerinin gelişimini 18 öğretmen adayıyla mülâkat yaparak araştırmışlardır. İlk mülâkatta katılımcılardan eşkenar dörtgen tanımı yapmaları ve verilen eşkenar dörgen tanımlarının doğruluğunu değerlendirmeleri istenmiştir. İlk mülâkatın sonuçları, öğretmen adaylarının tanımların doğasını, sadece gerekli ve yeterli özellikleri içermesi gerektiğini bilmediklerini; yaptıkları tanımların genellikle ekonomik olmayan, bazen de yanlış tanımlar olduğunu göstermiştir. İkinci mülâkatta ise ilk mülâkatta verilen eşkenar dörtgen tanımlarını sketchpad ortamında incelemeleri istenmiştir ve bu kez katılımcıların hemen hepsinin doğru tanımları belirleyebildiği görülmüştür. Sonuncu mülâkatta ise, doğru fakat ekonomik olmayan bir eşkenar dörtgen tanımı verilerek tanımı değerlendirmeleri istendiğinde, katılımcıların çoğunluğunun ekonomik olmayan bir tanım olduğunu tespit edebildikleri ancak tanımı ekonomik hale getirmeleri istendiğinde başarısız oldukları görülmüştür. Bu bulgu üzerine araştırmacılar, katılımcıların sketcpad ortamındaki başarısını, daha önceki sketchpad deneyimlerinden elde ettikleri üstün kullanma becerilerine bağlamışlar ve elde edilen verilerin onların gerçek zihinsel sürecini yansıtmadığı yargısına varmışlardır.

YÖNTEM

Araştırma Modeli

Çalışmanın amacı ilköğretim matematik öğretmen adaylarının geometrik kavramları tanımlama ve sınıflandırmadaki zihinsel süreçlerini detaylı olarak incelemek olduğundan bir çok veri toplama aracından elde edilen detaylı veriler aracılığıyla bir olguyu derinlemesine inceleyen niteliksel araştırma türü olan durum incelemesi yapılmıştır. Birden fazla katılımcı, "ilköğretim matematik öğretmen adaylarının geometrik kavramları tanımlama ve sınıflandırmadaki zihinsel süreçleri" olgusunu daha detaylı olarak anlamak için araç olarak kulanıldığından bu araştırma türü, araçsal çoklu durum incelemesi olarak adlandırılmaktadır (Stake, 2005).

Katılımcılar

Araştırmanın katılımcıları amaca yönelik olarak Orta Doğu Teknik Üniversitesi, Eğitim Fakültesi, İlköğretim Matematik Eğitimi Programı'nda eğitim gören son sınıf öğrencileri arasından seçilen 5 ilköğretim matematik öğretmen adayıdır. Katılımcılar, kayıtlı oldukları program kapsamında teknolojiyle alakalı dersler almışlardır. Bu derslerde Geometer's Sketchpad Programı'na (GSP) yüzeysel olarak değinilmiş, fakat katılımcılar aktif olarak programı kullanmamışlardır. Katılımcıları seçmede GSP seviyesinin kriter olarak kullanılması literatüre dayanmaktadır. Bazı çalışmalarda, katılımcıların programı üst seviyede kullanma yetisinin gerçek zihinsel süreçleri hakkında yanıltıcı bilgi elde edilmesine yol açtığı görülürken (eg., De Villiers & Govender, 2002), bazı çalışmalarda ise bu yetinin gerekli seviyede olmamasının araştırma sürecini sekteye uğrattığı görülmüştür (eg., Erez & Yerushalmy, 2007). Bu nedenle bu araştırmada katılımcılara süreci etkilemeyecek seviyede GSP kullanım yeterliliği kazandırmak amaçlanmış ve bunun için de araştırmanın başında, katılımcıların herbiriyle bire bir GSP öğretme çalışması yapılmıştır.

Pilot Çalışma

Çalışma kapsamında geliştirilen veri araçlarının işlerliğini görmek ve gerekli düzeltme ve iyileştirmeleri yapmak amacıyla Orta Doğu Teknik Üniversitesi Eğitim Fakültesi İlköğretim Matematik Öğretmenliği Bölümü'nden yeni mezun olan 2 gönüllü öğretmen adayı ile 2010-2011 Güz döneminde pilot çalışma yapılmıştır. Pilot çalışma sırasıyla şu aşamaları içermiştir:

- ***GSP Öğretimi ve "Dörtgenler" Testinin Uygulanması.*** Eğitim Fakültesi İlköğretim Bölümü seminer odasında katılımcıların herbiriyle birebir ve yaklaşık 2-2,5 saat süren GSP çalışması yapıldı. Çalışma sırasında katılımcı ve araştırmacı için iki ayrı dizüstü bilgisayarı kullanıldı. GSP çalışmasının ardından aynı gün "Dörtgenler I" testi katılımcılara uygulandı.
- ***Başlangıç Mülâkatının Uygulanması.*** GSP Öğretimi ve "Dörtgenler I" testinin uygulanmasından 1 hafta sonra başlangıç mülâkatı, ilk klinik mülâkatın başlamasından 15-20 dakika önce Bilgi İşlem Dairesi Başkanlığı İnsan Bilgisayar Etkileşimi (İBE) Laboratuvarı'nda uygulanarak video ve ses kayıtları alındı.
- ***Klinik Mülâkatlar (1, 2, 3, 4).*** Pilot çalışmada hangisinin daha çok veri sağladığını test etmek amacıyla klinik mülâkatlar araştırmacı tarafından Concurrent Thinking Aloud Clinical Interview (CTACI) and Retrospective Thinking Aloud Clinical Interview (RTACI) olmak üzere iki farklı formatta hazırlanarak her biri bir katılımcıya uygulandı. RTACI formatında katılımcı ilk olarak etkinlikleri tek başına tamamladıktan sonra mülâkata alınarak düşünce süreçleri irdelenirken CTACI formatında ise katılımcı etkinlikleri mülâkat sırasında tamamladı. Her bir katılımcıyla 4 mülâkat farklı günlerde ve İBE Laboratuvarı'nda uygulanarak video, ses ve ekran kayıtları alındı.
- ***Bitiş Mülâkatının Uygulanması.*** 4. ve son klinik mülâkatın uygulanmasının ardından 15-20 dakika süreyle bitiş mülâkatı yapılarak video ve ses kayıtları alındı.
- ***"Dörtgenler II" Testinin Uygulanması.*** Son mülâkatın tamamlanmasının hemen ardından "Dörtgenler II" testi katılımcılara uygulandı.

Pilot çalışma sonrası elde edilen bütün ses ve video kayıtları tek tek yazıya geçirilmiş ve diğer verilerle birlikte (ekran kayıtları, uygulanan testler, her türlü yazılı materyal, araştırmacının alan notları vs.) organize edilerek bilgisayar dosyası haline getirilmişlerdir. Bütün analizler aslına uygun olacak şekilde yapılmış ve bunun için de veriler defalarca gözden geçirilerek doğru yorumların ve çıkarımların yapılmasına çalışılmıştır.

Bu çalışmanın amaçları doğrultusunda, katılımcıların düşünme süreçlerini bire bir takip edip gerektiğinde düşünce süreclerini tamamlayıcı sorular sormak gerektiğinden CTACI formatının bu amaç için daha avantajlı olduğu görülmüştür. Çünkü RTACI formatında katılımcıların etkinlik üzerinde çalışırkenki düşünce süreçlerini, daha sonraki mülâkat sürecinde unutmuş olmalarının söz konusu olduğu görülmüştür. Ayrıca, katılımcıların verilen etkinlikler üzerinde çalıştıkları esnada yaşayacakları teknik bir probleme anlık müdahale etmek CTACI'da mümkünken, RTACI'da bu avantaj yoktur. RTACI formatı ise katılımcıya kendi başına istediği kadar düşünme imkanı tanıması bakımından CTACI'ya göre daha avantajlıdır; ancak katılımcılara istedikleri kadar içsel düşünme süresi tanınarak CTACI'nın bu dezavantajının minimuma indirilebileceği düşünülmüştür. Öte yandan, RTACI formatında araştırmacının takip soruları hazırlaması için yeteri kadar süresi olmasına karşın CTACI formatında ise araştırmacının sorularını katılımcıların cevaplarından hemen sonra anlık olarak oluşturabilmesi gerekmektedir. Öğrencilerin verebileceği olası cevaplar düşünülerek takip sorularının önceden hazırlanması ve araştırmacının katılımcı cevaplarıyla gereği gibi başedebilmek için yeterli donanım ve bilgiye sahip olmasının bu dezavantajı avantaja dönüştüreceği düşünülmüştür. Her iki mülâkat formatının avantaj ve dezavantajları göz önünde bulundurulduğunda, CTACI formatının dezavantajlarının büyük ölçüde üstesinden gelinebileceği ve araştırma sorularına yönelik daha etkili veri sağlayacağı düşünülmüştür.

Asıl çalışmadan önce uygulanan bu pilot çalışma, soruların ve etkinliklerin eksik yönlerinin tespitinde ve gerekli geliştirmelerin ve düzeltmelerin yapılmasının yanında araştırmacının mülâkat yapma, gözlemleme, veri toplama, analiz ve yorum yapma becerilerini geliştirmesi açısından da çok faydalı olmuştur.

Veri Toplama Süreci ve Veri Toplama Araçları

Çalışmanın olgusunu tüm yönleriyle anlayabilmek için Creswell, Hanson, Clark, and Morales (2007)'in de önerdiği gibi veri toplama sürecinde çok sayıda veri toplama aracı kullanılmıştır. Klinik mülâkatların ses ve video kayıtlarına ek olarak niteliksel testler, mülâkatlar, araştırmacının uygulama esnası ve sonrasında aldığı notlar ve katılımcıların her türlü yazılı kayıtları very toplama kaynağı olarak kullanılmıştır. Asıl çalışmanın verileri pilot çalışmayla aynı süreç izlenerek 2010-2011 Bahar döneminde toplanmıştır. Veri toplama süreci ve veri kaynaklarıyla ilgili detaylı bilgi takip eden bölümlerde verilmiştir.

1. **Geometer's Sketchpad Öğretim Seansı**

Süreç, katılımcıların GSP fonksiyonlarını istenilen düzeyde kullanabilmeleri için gerekli bilgi ve beceriyi kendilerine kazandırmayı amaçlayan bire bir GSP öğretme seansı ile başladı ve bu seansta vurgulanan en önemli nokta çizmek ve inşa etmek arasındaki fark oldu. Yaklaşık 2 saat süren GSP öğretme seansı her bir katılımcıyla bire bir olarak Orta Doğu Teknik Üniversitesi, Eğitim Fakültesi'ndeki seminer odasında, araştırmacı ve katılımcı için birer dizüstü bilgisayar kullanılarak gerçekleştirildi. Seans, GSP menülerinin fonksiyonlarının tanıtılmasıyla başlayıp çizmek ve inşa etmek arasındaki farkı vurgulayan pratiklerle devam etti. Yapılan pratikler, klinik mülâkatlar sırasında geometrik şekil inşa etme sürecinde ihtiyaç duyulabilecek temel bilgileri içerecek ve toplanacak verileri etkilemeyecek şekilde düzenlendi. Sürecin başından sonuna kadar katılımcılar kendi bilgisayar ekranlarında aktif olarak çalıştılar.

2. **Dörtgenler I Testi**

Literatürdeki bulgular doğrultusunda araştırmacı tarafından geliştirilen ve altı adet açık uçlu soru ve alt sorulardan oluşan Dörtgenler I testi, GSP öğretim seansının hemen bitiminde aynı seminer odasında uygulandı. Testin amacı, ilköğretim matematik öğretmen adaylarının klinik mülâkatlar öncesi, dörtgen tanımları oluşturma, değerlendirme ve dörtgenler arasındaki ilişkileri anlayışlarını belirlemektir. Testin içerik geçerliliğ, Eğitim Fakültesindeki birbirinden bağımsız uzmanların, soruları ilgili amaçla eşleştirmeleri yoluyla sağlanmaya çalışılmıştır. Ayrıca, format açısından dilin ve yönergelerin doğru kullanılıp kullanılmadığı, alakasız bilginin olup olmadığı ve testin fiziksel görüntüsü de aynı uzmanlar tarafından değerlendirildi.

3. **Başlangıç Mülâkatı**

GSP öğretme seansı ve Dörtgenler I testinin uygulanmasından 1 hafta sonra, kendi öğrenim hayatlarında tanımlarla olan tecrübelerini, tanımların öğrenme ve öğretme sürecindeki rolüne ilişkin bakış açılarını, tanım yapma konusunda kendilerine olan güvenlerini ve iyi bir tanımın nasıl olması gerektiğine dair düşüncelerini açığa çıkarmak için katılımcılarla, 7 açık uçlu sorudan oluşan ve yaklaşık 20 dakika süren bir mülâkat yapıldı. Ses ve video kayıtları alınan mülâkatın hemen ardından da ilk klinik mülâkata geçildi.

4. Klinik Mülâkatlar (1, 2, 3, 4)

Her bir katılımcıyla bireysel olarak uygulanan ve her biri yaklaşık 2 saat süren 4 adet klinik mülâkat araştırmacı tarafından literatürdeki eksiklikler ve öneriler dikkate alınarak geliştirilmiştir. Klinik mülâkatlar, genel amacı katılımcıların dörtgen tanımlarını anlama, oluşturma, değerlendirme ve bu tanımların dörtgenleri sınıflandırmadaki rolünü anlama sürecini ve bu süreçteki gelişimlerini ölçmeyi amaçlayan dinamik geometri destekli etkinliklerden oluşmaktadır. Bazı etkinliklerin uyarlanıp geliştirilmesinde De Villiers ve Govender (2002), De Villiers (2009)'in çalışmalarından faydalanılmıştır.

5. Dörtgenler II Testi

Araştırmacı tarafından hazırlanan ve 6 açık uçlu soru ile alt sorulardan oluşan test, Dörtgenler I testindeki sorulardan farklı ama aynı amaçlara yönelik olarak hazırlanmış soruları içermektedir. Testteki soruların büyük kısmı araştırmacı tarafından geliştirilmiş sadece 1 soru De Villiers (2009)'dan alınarak teste uyarlanmıştır.

Dörtgen II testinin geçerliliği de Dörtgen I testinde olduğu gibi, Eğitim Fakültesi'nden birbirinden bağımsız uzmanların karışık şekilde verilen test sorularıyla soruların amaçlarını eşleştirmeleri suretiyle sağlanmaya çalışılmıştır. Bunun dışında uzmanlar testleri kullanılan dil ve yönergelerin açıklığı, gereksiz bilgi bulunup bulunmaması, düzen ve görünüş açısından da değerlendirmişlerdir.

6. Bitiş Mülâkatı

Klinik mülâkatların sonuncusunun ardından, katılımcıların bu çalışmadaki dörgen kavramlarıyla ve GSP öğrenme ortamıyla olan tecrübeleri hakkındaki genel görüşlerini; GSP etkinliklerinin olumlu ve olumsuz yönleri hakkındaki ve sınıf ortamında öğrenmeyle GSP ortamında dörtgenleri öğrenme arasındaki farka dair görüşlerini; tanım yapma süreciyle ilgili çalışma kapsamında öğrendiklerini ve olduysa tanım yapma becerilerindeki gelişmeleri; tanım oluşturmanın öğretme ve öğrenme sürecinde mathematiksel bir etkinlik olarak kullanılıp kullanılamayacağına dair görüşlerini belirlemek amacıyla 12 açık uçlu sorunun yer aldığı bir mülâkat yapılmıştır.

7. Saha Notları

Saha notları, veri toplama sürecinde araştırmacının işittiği, gördüğü, deneyimlediği, düşündüğü her türlü şeyi yazdığı yazılı kayıtlardır (Fraenkel & Wallen, 2006). Bu çalışmada her türlü etkinliğin, davranışın, katılımcıların yüz ifadelerinin, çalışma sırasında meydana gelen özel bir durumun, çalışma alanının ve kullanılan materyallerin durumunun tanımlandığı betimleyici yazılı kayıtlara ek olarak çalışma süreciyle ilgili problemler, çalışma sonuçlarını etkileyebilecek olası etkenler, herhangi bir terslik durumuyla ilgili endişeler gibi gözlemlenen durumlar hakkında araştırmacının kendi düşüncesini ve yorumunu yansıtan kayıtlar da araştırmacı tarafından not alınmıştır.

8. Katılımcıların Yazılı Notları

Katılımcıların testler ve mülâkatlar sırasında kullandıkları her türlü yazılı kağıt veri toplama aracı olarak kullanılmıştır. Ayrıca, mülâkatlar sırasında katılımcılar GSP etkinlikleri üzerinde çalışırken kaydedilen ekran kayıtları da veri olarak kullanılmıştır.

ANALİZ

Tüm nitel çalışmalarda olduğu gibi bu çalışmada da veri analizi, veri toplama süreciyle eş zamanlı olarak gelişti. Daha sistematik bir yol izlemek için Creswell'in (2007) data analiz süreçleri takip edildi. Ilk aşama olarak ses ve görüntü kayıtları alınan bütün klinik mülâkatlar yazıya aktarıldı ve diğer veri kaynaklarıyla birlikte bilgisayar dosyaları olarak organize edildi. Bir sonraki aşamada mülâkatların yazıya aktarılan kayıtları, çalışma alanı notları, katılımcıların yazılı kayıtları, ses, video ve ekran kayıtları tekrar tekrar okunup incelenerek büyük boyutlardaki veriyi daha anlamlı hale getirebilmek adına kategoriler oluşturuldu. Daha sonar analiz devam ederken, bu kategoriler altında da daha özel kategoriler belirlendi. Bu kategoriler ışığında her bir durum detaylı olarak anlatıldı ve durumlar arasındaki göze çarpan benzerlikler ve farklılıklar tespit edildi. Daha sonra elde edilen bulguları yorumlanarak anlamlandırdı ve son olarak da genel bir çerçeve sunuldu.

BULGULAR

Bu bölümde araştıma sorularını cevaplamaya yönelik bulgular ele alınmıştır.

Tanımlar ve Tanımların Öğrenme ve Öğretme Sürecindeki Rolü Hakkındaki Algıları

Çalışmanın başında yapılan mülâkatın bulgularına göre tüm katılımcılar tanımların öğretme ve öğrenme sürecinde önemli role sahip olduğunu düşünmelerine karşın kendi tanım yapma becerilerine güvenmemektedirler. Kendilerini tanım yapma konusunda yeterli görmedikleri gibi iyi bir tanım nasıl olmalıdır, neleri içermelidir konusunda bilgi sahibi de değildirler. Katılımcıların hemen hepsi, tanımda kavrama ait tüm özellikler sıralanırsa, o tanımın iyi bir tanım olacağını düşünmektedirler ki bu düşüncenin yaygın olduğu literatür bulgularında da mevcuttur. Katılımcıların hemen hepsinin ortak yakınmasının ise, eğitim hayatları boyunca tanımların dersin başında tahtaya ya da deftere yazılıp orada kalmaları, tanımlar üzerine düşünmeye teşvik edilmemeleri; tanım yapma konusundaki yetersizliklerinin de bundan kaynaklandığını düşündükleri görüldü. Öğretmen adayları, tanımlar üzerinde düşünmeye ancak üniversitede aldıkları derslerde başladıklarını ifade ettiler. Kendi öğretmenlik hayatlarında da kendi tanımlarını oluşturmak yerine ders kitaplarındaki hazır tanımları kulllanacaklarını düşündüklerini belirttiler ki bunun da tanım yapamayacaklarına olan güvensizliklerini destekler nitelikte olduğu görüldü.

Tanımların "Minimum, Eşdeğer, Kapsayıcı Ve Hariç Tutan" Yapıları Hakkındaki Anlayışları

Klinik mülâkatlardan önce uygulanan testin sonuçları da öğretmen adaylarının tanım yapma konusundaki güvensizliklerini ve iyi bir tanımın nasıl olması gerektiğini bilmedikleri bulgusunu desteklemiştir. Yaptıkları tanımların, literaturdeki anlamıyla tanım değil de tanımlanan kavrama ait özelliklerin uzun bir listesini içeren tasvir olduğu görülmüştür. Bildikleri bütün özellikleri listelemelerinin nedeninin ise özellikler arasında muhakeme yaparak aralarından yeterli tanımlayıcı özellikleri belirleyememeleri olduğu tespit edilmiştir.

Öğretmen adaylarının dörtgenler arasındaki kapsayıcı ilişkileri belli bir ölçüde bildikleri görülmüş ancak dörtgenlerin kapsayıcı tanımlarını oluşturmaları istendiğinde başarısız oldukları görülmüştür. Bu da bildikleri şeyi doğru matematiksel dili kullanarak ifade etmede güçlük çektiklerinin göstergesidir, yani öğretmen adaylarının doğru matematiksel dili kullanmakta zorluk çektikleri görülmüştür. Kavramların özellikleriyle ilgili bilgi eksiklikleri ya da yanlış bilgileri de tanım yaparken doğru ifadeleri kullanmalarına ya da verilen bir tanımın doğruluğunu değerlendirmelerine engel olmuştur. Örneğin, eşkenar dörtgenin simetri özelliği konusundaki bilgi eksikliği neticesinde "eşkenar dörtgen birbirini dik kesen köşegenleri simetri eksenleri olan dörgendir" tanımını, doğru ve minimum özellik içermesine ragmen doğru değerlendiremedikleri görülmüştür.

Tanım Yapma ve Dörgenler Arasındaki İlişkileri Açıklama Yoluyla Dörtgen Kavramlarının Öğrenilmesi Süreci

Klinik mülâkatların başlangıcında, öğretmen adaylarının dörtgenlerin özellikleriyle ilgili problemleri olduğu tespit edildi. Özelliklerle ilgili bilgi eksikliği, yanlış bilgi ve özellikleri doğru matematiksel dili kullanarak ifade edememe gibi problemleri olduğu görüldü. Bu aşamada dinamik GSP figürü üzerinde çalışmak, bilmedikleri özellikleri keşfetmelerini, yanlış bildiklerinin farkına varıp düzeltmelerini sağladı. Yani GSP figürlerinin dinamik olarak hareket ettirilmesi durumunda inşa edilen kritik özelliklerini koruması, hareket ettirme sonucu ölçüleri değişen ama korunan kritik tanımsal kenar, açı, köşegen, simetri özelliklerinin öğretmen adayları tarafından keşfedilmesini sağladı. Fakat bu aşamada GSP figürleri, dörtgene ait ölçüleri ve bu ölçülerdeki değişimleri göstermek suretiyle pasif bir rol oynadı; ölçülerdeki değişimleri gözlemleyip korunan tanımsal özellikleri keşfetmek ve bu özellikleri doğru şekilde ifade edebilmek öğretmen adaylarının işiydi. Fakat, doğru özellikleri keşfetmelerine rağmen bu özellikleri matematiksel dil ile ifade etmede zorlandıkları görüldü. Örneğin, deltoidin simetri özelliğini söylerken "deltoid AB doğru parçasına göre simetriktir" demenin katılımcılar için kolay olduğu, ancak bu özelliği matematiksel dili daha genel kullanarak "deltoid açıortay olan köşegene göre simetriktir" demenin daha zor olduğu görüldü. Öğretmen adaylarının ifade etmekte en çok zorlandıkları özellik ise ikizkenar yamuğun köşegen özelliği oldu.

Özellikleri doğru şekilde ifade edememeleri ya da matematiksel terimleri yanlış kullanmaları, öğretmen adaylarının tanımlarda da istenen anlamı verememesine neden oldu. Örneğin, ikizkenar yamuğu tanımlarken en az bir çift kenarının eş olduğunu söylemeleri ama bu eş kenarların karşılıklı mı yoksa komşu kenarlar mı olduğunu tanımda belirtmemelerinin yaptıkları tanımı anlamsızlaştırdığı görüldü. Bu gibi durumlarda düşünmeye zorlandıklarında tek bir kelimenin eksikliğinin, varlığının ya da yanlış kullanımının bile tanımı nasıl anlamsız hale getirebildiğinin farkına vardılar.

Klinik mülâkatlar öncesinde uygulanan testte dörtgenler arasındaki kapsayıcı ilişkiler konusunda farkındalıkları olduğu görülse de daha detaylı irdelendiğinde dörgenler arasındaki ilişkileri açıklamada zorluk yaşadıkları görüldü. En ilginç bulgu ise, en çok aşina olunan dörtgenler olan paralalelkenarlar arasındaki hiyerarşik ilişkiyi hemen hiçbir katılımcının doğru gösterememesi oldu. Öğretmen adaylarının çoğu eşkenar dörtgen ve dikdörtgen arasındaki hiyerarşik ilişkiyi açıklayamadı. Eşkenar dörtgenin özel bir dikdörtgen ya da ta tersi dikdörtgenin özel bir eşkenar dörgen olduğunu düşündüler. Bu ilişkileri anlayamamalarının nedenleriinin ise kafalarında oluşturdukları prototip imajlardan ve dörtgenler ile özellikleri arasındaki zıt kapsayıcı ilişkiyi anlayamamaları olduğu tespit edildi.Örneğin öğretmen adaylarından biri eşkenar dörtgeni hep bildiğimiz eşkenar dörtgen olarak düşündüğünü, görünüş olarak farklı olduklarından eşkenar dörrtgenin özel bir deltoid olabileceğini düşünmediğini söyledi. Bu da sadece şekilsel değerlendirmeler yaptıklarına ve bu şekilsel değerlendirmelerin dörtgenler arasındaki ilişkileri anlayamamalarına dair bir bulguydu. Bu noktada figürü hareket ettirip diğer dörtgenlere dönüştürme etkinliği katılımcıların prototipler ve zıt kapsayıcı ilişkilerle ilgili problemlerinin üstesinden gelmelerinde yardımcı oldu. Dinamik dörtgenin bütün kritik özellikleri hareket ettirme sırasında korunduğundan dönüştürülebildiği bütün dörgenlerin dinamik dörtgenin özelliklerine sahip olması katılımcıların, hareket ettirilen dörtgenin özel durumları olan dörtgenleri keşfetmelerini sağladı. Sonrasında ise özellikleri arasında çıkarımlar yaparak dönüştürülen dörtgene neden dönüştürülebildiğinin arkasındaki gerekçeyi kolaylıkla açıklayabildikleri görüldü.

Dörtgenlerin özelliklerini keşfedip onları doğru şekilde ifade edebildikten ve dörtgenler arasındaki hiyerarşik ilişkileri keşfettikten sonra dörtgenleri ve temsil ettikleri dörtgenler sınıflarını tanımlamaları istendiğinde ise çalışmanın başında da olduğu gibi öğretmen adaylarının tanım yerine tasvir yaptıkları, yeterli tanımlayıcı özellikleri belirlemek yoluyla minimum bilgi içeren tanımlar yapamadıkları görüldü. Sadece tanımlarında kullandıkları özellikleri kullanarak tanımlanan dörtgenin dinamik figürünü inşa etmeleri ise tanımlarında yer alan fazladan bilgileri ya da eksik bilgileri farketmelerini sağladı. Tanımda yer alan hangi özelliğin o dörtgeni oluşturmak için gereksiz olduğunu, hangi özelliklerin yeterli tanımlayıcı özellikler olduğunu, başka bilgi eklemek gerekip gerekmediğini bu süreçte sorguladılar ve sadece gerekli ve yeterli tanımlayıcı özellikleri tespit ederek tasvirlerini büyük ölçüde tanım haline getirebildiler. Tanım yapmak yerine verilen tanımları değerlendirmeleri istendiğinde ise ilk başta sadece tanımda verilen bilgilerin doğru olup olmadığını control ettikleri, ama verilen bilgilerin yeterli tanımlayıcı özellikler olup olmadıklarını control etmedikleri görüldü. Bu bilişsel süreçte de, tanımları GSP figür inşaasıyla test edip verilen özellikleri irdelemeleri, doğru sonuçlara varmalarını sağladı. Bu süreçler tekrarlandıkça öğretmen adaylarının özellikler arasında çıkarım yapabilme, gerekli ve yeterli tanımlayıcı özellikleri akıl yürütme yoluyla bulma becerilerinin geliştiği ve ilerleyen aşamalarda GSP testine ihtiyaç duymadan minimum özellik içeren tanımlar yapabildikleri, verilen tanımların doğruluğunu değerlendirebildikleri görüldü.

Kapsayıcı tanımlar yerine hariç tutan, yani dörtgeni özel durumlarından ayrı tutarak tek başına tanımlayan tanımlar yapmaları istendiğinde ise tanımsal ifadede zorlandıkları görüldü. İlk yaptıkları hariç tutan tanımlar, genellikle tanımlanan dörtgenin özel durumlarını da içerdi ve özel durumları tanımdan soyutlayacak özellikleri bulmak öğretmen adayları için en zor aşama oldu. Bu aşamada özellikler arasında akıl yürütmelere ek olarak karşıt örnek arama metodunu kullandılar; yani tanım, sadece tanımlamak istediği dörtgeni mi tanımlıyor yoksa istenmemesine rağmen başka dörtgenleri de kapsıyor mu diye bir düşünme süreciyle karara vardılar.

Klinik mülâkatlarda gözlemlenen bir diğer bilişsel süreç ise özel dörtgenleri dörtgen bazında değil de başka dörtgenler bazında tanımlamaktı. Yani "kare….olan bir dörtgendir" ile "kare…..olan bir eşkenar dörtgendir" arasındaki farkı anlamadaki bilişsel süreçlerini gözlemlemekti. Fakat, öğretmen adaylarının ilk yaptıkları özel dörtgen bazındaki tanımların dörtgen bazında yaptıkları tanımlardan farkı olmadığı, dolayısıyla başka bir özel dörtgen bazında tanımladıklarında o özel dörtgenin özelliklerinin de tanıma otomatik olarak eklenmiş olduğunu ve tekrardan o özelliklere tanımda yer vermemeleri gerektiği mantığını anlayamadıkları görüldü. Fakat, dörgen bazında tanımlamakla özel bir dörtgen bazında tanımlamanın farkını düşünmeye teşvik edildiklerinde bu nüansın farkına vardılar ve tanım yaparkenki süreçte doğru tanımsal özellikleri belirleyebildiler.

Bu aşamaya kadar öğretmen adayları bir dörtgen kavramı için pek çok lalternatif tanım oluşturma yollarını öğrenip uyguladılar ve çalışmanın son aşamasında da tanım yapmadaki bilişsel düşünme becerilerinin geldiği noktayı göstererek, bilinen dörtgen tanımlarını daha da genellemek ya da daha da özelleştirmek yoluyla, aşina olunmayan yeni dörtgenleri başarıyla tanımlayabildikleri ve hiyerarşiye ekleyebildikleri görüldü.

Sonuç olarak bulgular, öğretmen adaylarının tanımları, bu tanımlara ait örnekleri ve örneklerin dinamik figürlerini aynı ortamda sunan GSP destekli etkinlikler eşliğinde tanım oluşturma sürecini tecrübe ederek matematiksel olarak daha doğru tanımlar oluşturduklarını ve dörgen kavramları arasındaki ilişkileri daha anlamlı şekilde açıklayabildiklerini göstermiştir.

Geometer's Sketchpad Öğrenme Ortamının Katkısı

Analiz sonuçlarına göre literatürdeki veriler ve tavsiyeler doğrultusunda geliştirilen Geometers' Sketchpad destekli tanım oluşturma ve değerlendirme etkinliklerinin öğretmen adaylarının dörtgenlerin kritik tanımlayıcı özelliklerini belirlemede, bir tanımın matematiksel doğruluğunu değerlendirmede, dörtgenlerin özellikleri ile dörtgenler arasındaki karşıt hiyerarşik ilişkiyi anlamada ve farklı dörtgen hiyerarşilerini oluşturmada oldukça etkili olduğu bulunmuştur.

Fakat etkili bir kontrol süreci, doğru yönlendirme ve düşünmeye teşvik edici sorular olmadan bu etkinliklerin tek başına etkili olması düşünülemez. Kullanılan etkinliklerin uygunluğu, etkinliğin yapıldığı sosyal ortam, kullanılan öğretim metodu, öğreticinin gerekli bilgi, donanım ve tecrübeye sahip olması gibi faktörler bu etkinliklerin etkili olmasındaki tamamlayıcı unsurlar olduklarından dikkate alınmalıdırlar.

SONUÇ

Bu çalışma kapsamında geliştirilen Geometer's Sketchpad destekli etkinliklerin öğretmen adaylarının geometrik kavram tanımlarıyla ilgili sorunlarının üstesinden gelmede önemli derecede etkili olduğu ve öğretmen adaylarının tanım oluşturma sürecindeki bilişsel becerilerinin daha üst seviyelere çıktığı görülmüştür.

GSP'nin en etkili olduğu bilişsel süreçler ise dörtgenlere ait kritik özellikleri keşfetme, dörtgenlerin yeterli tanımlayıcı özelliklerini belirleme, dörtgenler arasındaki kapsayıcı ilişkileri açıklama, tanımlarda verilen bilginin fazla mı eksik mi olduğunu tespit etmek suretiyle doğru ve minimum bilgi içeren tanımlar oluşturabilme, verilen tanımların doğruluğunu değerlendirme olmuştur. GSP'nin yanında, karşıt örnek arama yöntemi ve özellikler arasında çıkarımlar yapma yöntemleri de bilişsel süreçlerde kullanılmıştır.

Araştırmanın sonunda, öğretmen adaylarının iyi bir tanım nasıl olmalıdır sorusuna kendinden emin cevaplar verebildikleri ve bir kavram için birden fazla alternatif tanım oluşturma becerilerine oldukça güvendikleri, bilişsel süreçlerini tasvir yapmaktan tanım yapmaya ilerlettikleri ve matematiksel dili ve terimleri daha doğru kullanabildikleri görülmüştür.

Türk Eğitim sitemindeki matematik öğretiminde tanımlar genellikle önemsenmemelerine ragmen bu çalışmadan elde edilen bulguların, tanımların öğretme sürecindeki etkin kullanımına örnek teşkil ederek tanım yapmanın önemli bir matematiksel etkinlik olduğunu vurgulayacağı beklenmektedir. Bu tarz öğretim materyallerine öğretim programlarında yer verilerek öğretmen adaylarının tanımlarla ilgili gerekli bilgi ve becerileri kazanmasının, tanımlar üzerine düşünmeye teşvik edilmek suretiyle kavramları daha anlamlı öğrenmelerinin, tanımları daha etkin şekilde öğretme süreçlerine dahil edebilmelerini sağlayacağı ve dolayısıyla öğrencilerinin öğrenimini de etkileyeceği düşünülmektedir.

Appendix P

Tez Fotokopisi İzin Formu

ENSTİTÜ

Fen Bilimleri Enstitüsü	☐
Sosyal Bilimler Enstitüsü	☒
Uygulamalı Matematik Enstitüsü	☐
Enformatik Enstitüsü	☐
Deniz Bilimleri Enstitüsü	☐

YAZARIN

Soyadı : ÖZTOPRAKÇI
Adı : SEÇİL
Bölümü : ELEMENTARY MATHEMATICS EDUCATION

TEZİN ADI (İngilizce) : PRE-SERVICE MIDDLE SCHOOL MATHEMATICS TEACHERS' UNDERSTANDING OF QUADRILATERALS THROUGH THE DEFINITIONS AND THEIR RELATIONSHIPS

TEZİN TÜRÜ : Yüksek Lisans ☐ Doktora ☒

1. Tezimin tamamından kaynak gösterilmek şartıyla fotokopi alınabilir. ☐
2. Tezimin içindekiler sayfası, özet, indeks sayfalarından ve/veya bir bölümünden kaynak gösterilmek şartıyla fotokopi alınabilir. ☐
3. Tezimden bir bir (1) yıl süreyle fotokopi alınamaz. ☒

TEZİN KÜTÜPHANEYE TESLİM TARİHİ: