ESTIMATION IN INTERVAL CENSORED DATA

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ABSTRACT

ESTIMATION IN INTERVAL CENSORED DATA

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Interval censored failure time data occur in many areas including medicine, economics, zoology, psychology, sociology and engineering. In such studies, the variable of interest is often not exactly observed, but known to fall within some interval. In this thesis, the likelihood functions for fixed and random interval censored data are obtained. Modified Maximum Likelihood and Copula Methods are utilized for the estimation of unknown parameters. Bivariate interval censored data are also considered as a generalization in this work. To estimate the association between two random variables, we focus on the situation where they follow a copula model. To check the accuracy and efficiency of the methods, some numerical studies are conducted.

Keywords: interval censored data, estimation, copula, bivariate distributions, random

ÖZ

ARALIKLI SANSÜRLÜ VERİLERDE ÇIKARIM

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Aralıklı sansürlü veri, tıp ekonomi, psikoloji, sosyoloji ve mühendislik gibi alanlarda yapılan çalışmalarda yer almaktadır. Bu tip veriler iki nokta arasında gözlemlenmekte, fakat gerçek değerleri tam olarak bilinememektedir. Bu tezde, sabit ve rasgele aralıklı sansürlü veriler için en çok olabilirlik fonksiyonları elde edilmiştir. Uyarlanmış En Çok olabilirlik ve Copula yöntemleri bilinmeyen parametrelerin tahmin edicilerini bulmak için kullanılmıştır. İki değişkenli aralıklı sansürlü veriler de bu çalışmada daha genel bir durum olarak ele alınmış, değişkenler arasındaki ilişkinin modellenmesi için Copula yöntemi önerilmiştir. Monte Carlo simulasyon modeli kullanılarak geliştirilen yöntemlerin tutarlılıkları ve etkinlikleri incelenmiştir.

Anahtar Kelimeler: aralıklı sansürlü veri, çıkarım, iki değişkenli dağılımlar, rasgele

To my parents, İsmihan and Sitare Bayramoğlu and Gülnar

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ARE	Asymptotic relative efficiency
cdf	Cumulative density function
EM	Expectation maximization algorithm
FI	Fisher information
ICM	Iterative convex minorant
iid	Independenly identically distributed
MLE	Maximum likelihood estimation/estimator
MMLE	Modified maximum likelihood estimation/estimator
MSE	Mean squared error
NPMLE	Nonparametric maximum likelihood estimation/estimator
pdf	Probability density function
se	Standard error/standard deviation of a point estimator
Var	Variance
L	Likelihood function
$P(A)$	Probability of event A
$X \sim F$	X has distribution F
$X \sim f$	X has density f
$E(X)$	Expected value of random variable X
$I(\theta)$	Fisher information about θ
$X_1, \cdots, X_n$	Data
n	Sample size
$\xrightarrow{d}$	Convergence in distribution
$\xrightarrow{d}$	Convergence in probability
$X_n = o_p(a_n)$	$\frac{X_n}{a_n} \xrightarrow{p} 0$

CHAPTER 1

INTRODUCTION AND LITERATURE REVIEW

Censoring is one of the most important concepts in recent literature where an observation of interest is incomplete and observed only when it falls into a certain range. Censored data can be considered different from missing data, as we get some information from the censored observations while missing observations provide no information about the variable of interest. Depending on the relationship between the variable of interest and the censoring point there are various types of censoring mechanisms. Basically, these are known as right censoring, left censoring and interval censoring.

1.1 Right Censoring and Left Censoring

In right censoring, known as the most common form of censoring, a data point is above a certain threshold where its exact value is unknown. In other words, subjects are examined for a certain period, no event has yet occurred when the study comes to an end. Consider patients, for example, in a clinical trial where the effects of medicine treatment on Gastrointestinal Carcinoid Tumors are studied for 10 years. Those patients who have no change in their tumor size by the end of the study are right censored.

In left censoring, a data point is a below a certain threshold where its exact value is unknown. In this case, the event of interest has already occurred prior to beginning the study. The age at which teenagers begin to drink alcohol is an example for the left censored data.

1.2 Interval Censoring

Interval censoring is used to indicate a type of incomplete data where the study subjects cannot be observed continuously. Therefore, the variable of interest is not known exactly but is only known to lie between two values. In his well known study, Turnbull [44] defined interval censored observation as a "union of several nonoverlapping windows or intervals".

Interval censored data has applications in many areas including medicine, epidemiology, economics, agronomy, zoology, psychology, sociology, demography, management, reliability and engineering. Typical examples of interval censoring mostly arise in medical and health studies. For example, an individual in the Gastrointestinal Carcinoid Tumor study, may skip some of his/her pre-scheduled appointments. This results in a gap in his/her examination times and he/she returns with a changed tumor size. Another example can be the time from HIV infection to AIDS diagnosis. For a HIV positive individual, the HIV infection time can only be determined by observing the individual's past. Thus, HIV infection time is not known exactly, but is known to lie between the last HIV negative and first HIV positive tests [39].

In most applications, the random variable of interest is the time to an event, such as a death or a disease. However, it can be any random variable representing an inspection point between $-\infty$ and ∞. To this point, our random variable of interest can be defined in a way that starts from $-\infty$ or a known point.

Many studies have been conducted under the assumption of normality for the interval censored data. Swan [40] obtained the maximum likelihood estimates of unknown parameters for interval censored data where the variable of interest is normally distributed. Similarly, Peto [31] assumed that the data from annual surveys on sexual maturity development of girls was normally distributed and proposed an approach where he maximized the log-likelihood by a Newton Raphson algorithm. Ren [35] provided goodness-of-fit tests for normally distributed interval censored data. In an another study [34], he proposed a procedure to construct "the empirical likelihood ratio confidence interval for the mean using a resampling method" which was applied to the interval censored data under normality assumption. Norwood et al.[29]

developed a "willingness to pay" method to estimate the demand for livestock manure under the assumption of normality. Cook and McDonald [10] pointed at an estimation procedure for the problem of distributional misspecification for cases with interval censoring. In their example, they assumed that the variable of interest has a normal distribution.

In literature, several studies have focused on interval censoring. These studies consider nonparametric, semi-parametric as well as parametric approaches. Nonparametric methods are only reasonable when no assumption is made about the distribution function of the variable of interest. However, no extrapolation of study results is possible for the further research. Parametric modeling is known to have some advantages over nonparametric modeling. For example, parametric methods help in finding the influence of covariates on variable of interest. However, it is quite difficult to deal with likelihood functions in order to obtain closed-form solutions of maximum likelihood estimators. Since likelihood equations for interval censored data structure do not admit explicit solutions when the random variable of interest is assumed to come from a specific probability density function, researchers prefer to use iterative algorithms to obtain maximum likelihood estimates of unknown parameters.

Finkelstein and Wolfe [13] assumed no parametric model for unobservable failure time T for interval censored data and came up with a semi parametric model. In an another study, Finkelstein [12] proposed a proportional hazards model in case of interval censoring. Self and Grossman [37] considered linear rank statistics for testing the differences between groups when we have a interval censored data. Flygare et.al [14] provided an iterative solution for finding roots of likelihood equation where they considered two parameter Weibull distribution for the failure time T. Turnbull [44] proposed and algorithm to obtain the nonparametric estimator of the function for the analysis of the censored and truncated data. This algorithm can be used for problem of analysis of interval censored data. Gentleman and Geyer [15] used convex optimization technique to maximize the likelihood function. Groeneboom and Wellner [17] proposed convex minorant algorithm to obtain the nonparametric maximum likelihood estimator and their algorithm was shown to converge faster than the Turnbull's self consistency algorithm. Wei et al.[46] considered partial likelihood estimators for regression parameters under the assumption of "the working independence". Cai and Prentice [6] considered the same problem and developed methods for the estimation of regression parameters. Huang [19] provided "efficient estimation" for the proportional hazards model. Asymptotic variance of MLE of the regression parameter was also considered. More recently, Goggins and Finkelstein [16] and Kim and Xue [22] studied the maximum likelihood method for multivariate interval-censored data. Kooperberg et.al [23] proposed spline-based method of estimating an unknown density function. Bebchuk and Betensky [2] considered "multiple imputation approach" in case of interval censoring. In another study, Betensky et al. [3] used a local likelihood method. Cai and Betensky [7] considered "hazard regression" for interval censored data with piecewise linear spline. Bogaerts et al. [4] considered multivariate interval censored data and applied accelerated failure time model.

1.3 Motivation and Summary of Work

As can be seen from the previous part, a number of parametric and nonparametric methods have been proposed to model interval censored data. The common problem while using parametric approaches is that it is not possible to obtain closed form solutions from the likelihood equations. This is one part of our motivation and we propose a parametric method that provides closed form solutions of estimators for unknown parameters in case of fixed interval censoring.

Consideration of random intervals in similar structures results in even more complicated and difficult inference problems. This problem has not been enough addressed in the literature and has been a motivation throughout the thesis. Random interval censoring with parametric distributions leads to bivariate structures between random inspection points. To handle difficulties arising from this dependency structure, we utilize the Modified Maximum Likelihood (MML) estimation due to Tiku [41] and Tiku and Suresh [43]. A special type of copula model is also considered for interval censoring problems with random intervals. Monte Carlo simulations are performed to check the accuracy of the approximations. Findings of copula approach for univariate case is also extended to bivariate interval censoring with random interavals. We also consider the estimation method of the association parameter of two dependent failure variables in case of interval censoring.

Note that failure point is considered independent of censoring points in most studies on interval censored data, in order to easily make an inference. We also have this assumption throughout the thesis.

CHAPTER 2

UNIVARIATE INTERVAL CENSORING WITH FIXED INTERVALS

In interval censored data, the only information we have for each individual is that variable of interest is observed in an interval $(L_i, R_i], i = 1, 2, ..., n$. Interval censored data that include fixed intervals with both belonging to $(-\infty, \infty)$, is referred to as case I interval censored data throughout the study. In this section, the likelihood model for case I interval censored data is presented and the proposed estimation procedure is given. Thereafter, numerical results are provided.

2.1 Analysis of Case I Interval Censored Data

Let $F_{\mu,\sigma}(x)$ be a location and scale parametric family of a distribution given as,

$$F_{\mu,\sigma}(x) = F\left(\frac{x-\mu}{\sigma}\right), \quad \mu \in (-\infty, \infty), \quad \sigma \geq 0 \tag{2.1}$$

where F is a cumulative distribution function (cdf) without parameter. Let $T_1, T_2, \cdots, T_n$ be independently identically distributed (iid) random variables with cdf $F_{\mu,\sigma}$. Exact values of T_i are not known but they are known to lie between two successive inspection points a_i and b_i $(i = 1, 2, ..., n)$.

Then, the likelihood function L based on fixed interval censored sample for $t_i \in (a_i, b_i)$ is given by

$$
\begin{aligned}
L(\mu,\sigma) &= \prod_{i=1}^{n} \{F_{\mu,\sigma}(b_i) - F_{\mu,\sigma}(a_i)\} \\
&= \prod_{i=1}^{n} \{P\{T_i \le b_i\} - P\{T_i \le a_i\}\} \\
&= \prod_{i=1}^{n} \left\{P\left\{\frac{T_i-\mu}{\sigma} \le \frac{b_i-\mu}{\sigma}\right\} - P\left\{\frac{T_i-\mu}{\sigma} \le \frac{a_i-\mu}{\sigma}\right\}\right\} \\
&= \prod_{i=1}^{n} \left\{F\left\{\frac{b_i-\mu}{\sigma}\right\} - F\left\{\frac{a_i-\mu}{\sigma}\right\}\right\} \\
&= \prod_{i=1}^{n} \{F(b_i^*) - F(a_i^*)\} \quad , \qquad (2.2)
\end{aligned}
$$

where

$$
a_i^* = \frac{a_i-\mu}{\sigma} \quad \text{and} \quad b_i^* = \frac{b_i-\mu}{\sigma} \quad .
$$

Note that T_i (for some i) might be observed either in $(-\infty, a_i)$ or (b_i, ∞). Let $\hat{\mu}$ and $\hat{\sigma}$ denote the values of μ and σ that maximize $L(\mu,\sigma)$. To determine these estimators, one needs to take first partial derivative of the log likelihood function and equate them to zero. By taking logarithms of likelihood L, we get the loglikelihood function as

$$
\begin{aligned}
lnL(\mu,\sigma) &= \sum_{i=1}^{n} ln\left[F(a_i^*) - F(a_i^*)\right] \\
&= \sum_{i=1}^{n_1} lnF(a_i^*) + \sum_{i=n_1+1}^{n_2} ln\left[F(b_i^*) - F(a_i^*)\right] + \sum_{i=n_2+1}^{n} ln\left[1 - F(b_i^*)\right] \quad . \qquad (2.3)
\end{aligned}
$$

Here, n_1 and $n - n_2$ are the numbers of observations between $(-\infty, a_i)$ and (b_i, ∞), respectively. Maximizing $lnL(\mu,\sigma)$ with respect to μ and σ, we get

$$
\begin{aligned}
\frac{\partial lnL(\mu,\sigma)}{\partial\mu} &= -\frac{n}{\sigma} + \sum_{i=1}^{n_1}\left(\frac{-1}{\sigma}\right)\frac{f(a_i^*)}{F(a_i^*)}a_i^* \\
&+ \sum_{i=n_1+1}^{n_2}\left(\frac{-1}{\sigma}\right)\frac{f(b_i^*)\,b_i^* - f(a_i^*)\,a_i^*}{F(b_i^*) - F(a_i^*)} \\
&+ \sum_{i=n_2+1}^{n}\left(\frac{-1}{\sigma}\right)\frac{f(b_i^*)}{1 - F(b_i^*)}b_i^* = 0 \qquad (2.4)
\end{aligned}
$$

and

$$\frac{\partial lnL(\mu,\sigma)}{\partial\sigma} = \sum_{i=1}^{n_1}\left(\frac{-1}{\sigma}\right)\frac{f(a_i^*)}{F(a_i^*)} + \sum_{i=n_1+1}^{n_2}\left(\frac{-1}{\sigma}\right)\frac{f(b_i^*)-f(a_i^*)}{F(b_i^*)-F(a_i^*)} + \sum_{i=n_2+1}^{n}\left(\frac{-1}{\sigma}\right)\frac{f(b_i^*)}{1-F(b_i^*)} = 0 \quad , \tag{2.5}$$

respectively. It is clear that equations (2.4) and (2.5) do not admit explicit solutions because of the terms involving the functions $g(a_i)$, $g_1(a_i, b_i)$ and $g_2(b_i)$, where

$$g(a_i) = \frac{f(a_i^*)}{F(a_i^*)} \quad , \tag{2.6}$$

$$g_1(a_i, b_i) = \frac{f(b_i^*)-f(a_i^*)}{F(b_i^*)-F(a_i^*)} \quad , \tag{2.7}$$

and

$$g_2(b_i) = \frac{f(b_i^*)}{1-F(b_i^*)} \quad . \tag{2.8}$$

A typical way to maximize $L(\mu,\sigma)$ with respect to unknown parameters μ and σ is to use self consistency algorithm proposed by Turnbull [44]. This method is an application of EM (Expectation Maximization) algorithm and iterates the likelihood equation until convergence. Although this approach is easy to implement, it has a very slow convergence rate. Since this algorithm is iterative, no closed form solution for the MLE can be obtained.

In this study, we propose Modified Maximum Likelihood method (MML) to estimate the unknown parameters in case of interval censoring. We utilize the modified maximum likelihood estimation method (Tiku, [41]) that allows us to get closed form solutions for the unknown estimators. It can be verified empirically that (Tiku, [41]) the points satisfying

$$g(x) = \frac{f(x)}{F(x)}$$

over an interval $a \leq x \leq b$ of finite length lie very close to the line

$$g\left(x\right) \cong \alpha + \beta x \quad ,$$

where

$$\beta = \frac{g\left(b\right) - g\left(a\right)}{\left(b - a\right)}$$

and

$$\alpha = g\left(a\right) - a\beta \quad .$$

By linearizing nonlinear functions in (2.6), (2.7) and (2.8) with the equation $g\left(x\right) \cong \alpha + \beta x$, we are able to simplify the ML equations for estimating the mean μ and standard deviation σ of a location - scale family distribution from an interval censored sample. Then, we consider the following linear approximations:

$$g\left(a_i\right) = \frac{f\left(a_i^*\right)}{F\left(a_i^*\right)} \cong \nu_{i1} + \upsilon_{i1} a_i^* \quad , \tag{2.9}$$

$$g_1\left(a_i, b_i\right) = \frac{f\left(b_i^*\right) - f\left(a_i^*\right)}{F\left(b_i^*\right) - F\left(a_i^*\right)} \cong \alpha_{i2} + \beta_{i2} b_i^* - \alpha_{i1} - \beta_{i1} a_i^* \tag{2.10}$$

and

$$g_2\left(b_i\right) = \frac{f\left(b_i^*\right)}{1 - F\left(b_i^*\right)} \cong \nu_{i2} + \upsilon_{i2} b_i^* \quad . \tag{2.11}$$

It is difficult to determine the values of $\left(\alpha_{i1}, \beta_{i1}, \upsilon_{i1}, \nu_{i1}\right)$ and $\left(\alpha_{i2}, \beta_{i2}, \upsilon_{i2}, \nu_{i2}\right)$ since a_i^* and b_i^* are not known. However the intervals, $\left(\frac{(a_i - h_i)}{\tilde{\sigma}}, \frac{(a_i - k_i)}{\tilde{\sigma}}\right)$ and $\left(\frac{(b_i - h_i)}{\tilde{\sigma}}, \frac{(b_i - k_i)}{\tilde{\sigma}}\right)$ are likely to contain the exact values of a_i^* and b_i^*, respectively where

$$h_i = \tilde{\mu} - l\tilde{\sigma}$$

and

$$k_i = \tilde{\mu} + l\tilde{\sigma} \quad .$$

Here, $\tilde{\mu}$ and $\tilde{\sigma}$ are the initial estimators obtained from the midpoints of the closed intervals. For the simplicity, we assume $l = 1$ in our calculations. When the substitutions are made, we get

$$\beta_{i1} = \frac{\frac{f\left(\frac{a_i-k_i}{\tilde{\sigma}}\right)}{F\left(\frac{b_i-k_i}{\tilde{\sigma}}\right)-F\left(\frac{a_i-k_i}{\tilde{\sigma}}\right)} - \frac{f\left(\frac{a_i-h_i}{\tilde{\sigma}}\right)}{F\left(\frac{b_i-h_i}{\tilde{\sigma}}\right)-F\left(\frac{a_i-h_i}{\tilde{\sigma}}\right)}}{\frac{h_i-k_i}{\tilde{\sigma}}} \quad ,$$

$$\alpha_{i1} = \frac{f\left(\frac{a_i-h_i}{\tilde{\sigma}}\right)}{F\left(\frac{b_i-h_i}{\tilde{\sigma}}\right)-F\left(\frac{a_i-h_i}{\tilde{\sigma}}\right)} - \beta_{i1}\left(\frac{a_i-h_i}{\tilde{\sigma}}\right) \quad ,$$

$$\upsilon_{i1} = \frac{\frac{f\left(\frac{a_i-k_i}{\tilde{\sigma}}\right)}{F\left(\frac{a_i-k_i}{\tilde{\sigma}}\right)} - \frac{f\left(\frac{a_i-h_i}{\tilde{\sigma}}\right)}{F\left(\frac{a_i-h_i}{\tilde{\sigma}}\right)}}{\frac{h_i-k_i}{\tilde{\sigma}}} \quad ,$$

$$\nu_{i1} = \frac{f\left(\frac{a_i-h_i}{\tilde{\sigma}}\right)}{F\left(\frac{a_i-h_i}{\tilde{\sigma}}\right)} - \upsilon_{i1}\left(\frac{a_i-h_i}{\tilde{\sigma}}\right) \quad ,$$

$$\beta_{i2} = \frac{\frac{f\left(\frac{b_i-k_i}{\tilde{\sigma}}\right)}{F\left(\frac{b_i-k_i}{\tilde{\sigma}}\right)-F\left(\frac{a_i-k_i}{\tilde{\sigma}}\right)} - \frac{f\left(\frac{b_i-h_i}{s}\right)}{F\left(\frac{b_i-h_i}{\tilde{\sigma}}\right)-F\left(\frac{a_i-h_i}{\tilde{\sigma}}\right)}}{\frac{h_i-k_i}{\tilde{\sigma}}} \quad ,$$

$$\alpha_{i2} = \frac{f\left(\frac{b_i-h_i}{\tilde{\sigma}}\right)}{F\left(\frac{b_i-h_i}{\tilde{\sigma}}\right)-F\left(\frac{a_i-h_i}{\tilde{\sigma}}\right)} - \beta_{i2}\left(\frac{b_i-h_i}{\tilde{\sigma}}\right) \quad ,$$

$$\upsilon_{i2} = \frac{\frac{f\left(\frac{b_i-k_i}{\tilde{\sigma}}\right)}{1-F\left(\frac{b_i-k_i}{\tilde{\sigma}}\right)} - \frac{f\left(\frac{b_i-h_i}{\tilde{\sigma}}\right)}{1-F\left(\frac{b_i-h_i}{\tilde{\sigma}}\right)}}{\frac{h_i-k_i}{\tilde{\sigma}}}$$

and

$$\nu_{i2} = \frac{f\left(\frac{b_i-h_i}{\tilde{\sigma}}\right)}{1-F\left(\frac{b_i-h_i}{\tilde{\sigma}}\right)} - \upsilon_{i2}\left(\frac{b_i-h_i}{\tilde{\sigma}}\right) \quad .$$

Then, the partial derivatives are obtained as

$$\begin{aligned}\frac{\partial lnL(\mu,\sigma)}{\partial\mu} &= \sigma\left[\sum_{i=1}^{n_1}\nu_{i1} + \sum_{i=n_1+1}^{n_2}(\alpha_{i2}-\alpha_{i1}) - \sum_{i=n_2+1}^{n}\nu_{i2}\right] \\ &- \mu\left[\sum_{i=1}^{n_1}\upsilon_{i1} + \sum_{i=n_1+1}^{n_2}(\beta_{i2}-\beta_{i1}) - \sum_{i=n_2+1}^{n}\upsilon_{i2}\right] \\ &- \left[\sum_{i=1}^{n_1}\upsilon_{i1}a_i + \sum_{i=n_1+1}^{n_2}(\beta_{i2}b_i-\beta_{i1}a_i) - \sum_{i=n_2+1}^{n}\upsilon_{i2}b_i\right] = 0 \qquad (2.12)\end{aligned}$$

and

$$\frac{\partial lnL(\mu,\sigma)}{\partial\sigma} = -\frac{n}{\sigma} - \frac{1}{\sigma}\left[\sum_{i=1}^{n_1}\left(\nu_{i1} + \upsilon_{i1}\left(\frac{a_i-\mu}{\sigma}\right)\right)\left(\frac{a_i-\mu}{\sigma}\right)\right]$$
$$-\frac{1}{\sigma}\left[\sum_{i=n_1+1}^{n_2}\left(\alpha_{i2} + \beta_{i2}\left(\frac{b_i-\mu}{\sigma}\right)\right)\left(\frac{b_i-\mu}{\sigma}\right)\right]$$
$$+\frac{1}{\sigma}\left[\sum_{i=n_1+1}^{n_2}\left(\alpha_{i1} + \beta_{i1}\left(\frac{a_i-\mu}{\sigma}\right)\right)\left(\frac{a_i-\mu}{\sigma}\right)\right]$$
$$-\frac{1}{\sigma}\left[\sum_{i=n_2+1}^{n}\left(\nu_{i2} + \upsilon_{i2}\left(\frac{b_i-\mu}{\sigma}\right)\right)\left(\frac{b_i-\mu}{\sigma}\right)\right] = 0 \quad . \tag{2.13}$$

As a result, the MMLE's are

$$\hat{\mu} = A + B\tilde{\sigma} \tag{2.14}$$

where

$$A = \frac{\sum_{i=1}^{n_1}\upsilon_{i1}a_i + \sum_{i=n_1+1}^{n_2}(\beta_{i2}b_i - \beta_{i1}a_i) - \sum_{i=n_2+1}^{n}\upsilon_{i2}b_i}{\sum_{i=1}^{n_1}\upsilon_{i1} + \sum_{i=n_1+1}^{n_2}(\beta_{i2} - \beta_{i1}) - \sum_{i=n_2+1}^{n}\upsilon_{i2}} \quad ,$$

$$B = \frac{\sum_{i=1}^{n_1}\nu_{i1} + \sum_{i=n_1+1}^{n_2}(\alpha_{i2} - \alpha_{i1}) - \sum_{i=n_2+1}^{n}\nu_{i2}}{\sum_{i=1}^{n_1}\upsilon_{i1} + \sum_{i=n_1+1}^{n_2}(\beta_{i2} - \beta_{i1}) - \sum_{i=n_2+1}^{n}\upsilon_{i2}}$$

and

$$\hat{\sigma} = \frac{-C + \sqrt{C^2 + 4nE}}{2n} \tag{2.15}$$

where

$$C = \sigma\left[\sum_{i=1}^{n_1}\nu_{i1}(a_i - A) - \sum_{i=n_1+1}^{n_2}(\alpha_{i2}(b_i - A) - \alpha_{i1}(a_i - A)) + \sum_{i=n_2+1}^{n}\nu_{i2}(b_i - A)\right]$$

and

$$E = \sigma \left[\sum_{i=1}^{n_1} \nu_{i1} \left(a_i - A\right)^2 - \sum_{i=n_1+1}^{n_2} \left(\alpha_{i2} \left(b_i - A\right)^2 - \alpha_{i1} \left(a_i - A\right)^2 \right) + \sum_{i=n_2+1}^{n} \nu_{i2} \left(b_i - A\right)^2 \right] \quad .$$

A proof is available in Tiku et. al [1] that under some regularity conditions, MMLE's have exactly the same asymptotic properties as ML estimators. See also Cohen [9] where ML estimators are obtained iteratively. As it is seen, MMLE's have closed form solutions and are easier to calculate.

Remark: For interval censoring with fixed and random intervals, it is quite difficult to find a related data set since an important assumption needs to be held. It is necessary to verify the parametric distributional structure of an interval censored data set by conducting an appropriate goodness-of-fit. This needs some further investigations on powerful goodness-of-fit tests. In fact, this area has started becoming more popular as a variety of interval-censored data sets have been encountered in real life applications. In order to be able to use a correct data set, a very powerful goodness-of-fit test, which may not be available in the literature, is required. This is itself a new research problem and causes us to proceed with simulated data sets rather than real life examples.

2.2 Simulation of Interval Censored Data

It is not possible to observe the subjects continuously in interval censoring. As a result, interval points a_i and b_i can be certain predetermined points in real life applications. To illustrate how the method developed above can be applied to interval censored data, we shall firstly deal with how to generate interval censored data.

While there are many studies about analyzing interval censored data, there are very little on the methods of simulating them. Lawless and Babineau [24] described the estimation process of the inspection points from a real interval censored data set. Lee [25] proposed a method to estimate survival function $S(t)$ for interval censored data. In his study, $S(t)$ is assumed to be unknown and therefore, has to be estimated by simulating interval censored data from a real data set. However, in some studies, the inference about a real data set is not the main concern. The aim can be to develop a new method for analyzing interval censored data and check the performance by simulation. This is also the motivation for our study.

To simulate interval censored data, we need (t_i, a_i, b_i) where (a_i, b_i) is defined as a set of inspection points. It is also assumed that the subjects are under control at these points. A variable of interest, T_i, can be generated via simulation from its distribution function. For the simulation of the (a_i, b_i), we use a method proposed by Kiani and Arasan [21] which is explained in the next section.

2.2.1 Algorithm of the Simulation

Let T be a nonnegative random variable of interest and assume $T_i \in (a_i, b_i)$ with $P\left(a_i \leq b_i\right) = 1$; $i = 1, 2, , \cdots, n$. To obtain a_i and b_i, a set of examination points $P = \{p_1, p_2, \cdots, p_k\}$, is generated. Also, a subject attendance probability q to each p_j is defined; $0 \leq q \leq 1$ and $j = 1, 2, 3, \cdots, k$. Depending on the values of q, the following interpretations can be made:

- $q = 1$: subjects attend all of the p_j's
- $q = 0$: subjects do not attend any of the p_j's
- $0 \leq q \leq 1$: subjects attend some of the p_j's and miss others.

In this setup, each subject i ($i = 1, 2, 3, \cdots, n$) has its own set of actual examination points $A_i = \{a_{i1}, a_{i2}, \cdots, a_{im_i}\}$ ($A_i \in P$) with the following assumptions:

- There are k known potential inspection points.
- In order that the process starts, all subjects attend to the first inspection point p_1.
- Subjects attend to the test with probability q.

To generate (a_i, b_i), we propose the following procedure similar to that of Kiani et.al [21]:

(i) Generate $u_i \sim Uniform\,(0, 1)$
(ii) Define an indicator function

$$I = \begin{cases} 1 & \text{if subject attends the } i^{th} \text{ inspection point } p_i \, (u_i \leq q) \\ 0 & \text{if subject does not attend the } i^{th} \text{ inspection point } p_i \, (u_i \geq q) \end{cases}$$

(iii) Steps i and ii should be run for each p_j; $j = 2, ..., k$.
(iv) We will obtain vector of attendance for all k members of P and it will direct us to a set of actual examination points A_i. For example, for the first subject if $P = \{1, 1, 0, 1, 0, 0, 1, 1\}$ then, $A_1 = \{a_{11}, a_{12}, a_{13}, a_{14}, a_{15}\} = \{p_1, p_2, p_4, p_7, p_8\}$.
(v) a_i is the largest member of A_i which is less than t_i and b_i is the smallest member of A_i which is more than t_i for $i = 1, 2, \cdots, n$.

It is clear that

- $t_i \leq a_i \rightarrow$ the observation is in left half open intervals
- $t_i \leq b_i \rightarrow$ the observation is in right half open intervals
- $a_i \leq t_i \leq b_i \rightarrow$ the observation is in closed interval.

2.2.1.1 Simulation Results

To illustrate the concept, a Monte Carlo simulation with $S = 10,000$ repetitions is conducted under normal distribution with mean zero and variance one. Inspection points are generated according to the algorithm given in previous section. Using the data, the MML estimators are obtained iteratively as in Tiku and Stewart [42]: $\hat{\mu}$ and $\hat{\sigma}$ are calculated from (2.14) and (2.15) by using $\tilde{\mu}$ and $\tilde{\sigma}$ obtained from the midpoints of the closed intervals. In the first iteration, $\tilde{\mu}$ and $\tilde{\sigma}$ are replaced by $\hat{\mu}$ and $\hat{\sigma}$ and a new pair of estimates $(\hat{\mu}, \hat{\sigma})$ calculated from (2.14) and (2.15).

Mean Square Error (MSE) values are also calculated to see the difference between the estimators and their estimated values on the basis of bias. After computing $\hat{\mu_1}, \cdots \hat{\mu_s}$ and $\hat{\sigma_1}, \cdots \hat{\sigma_s}$, values in a simulation with $10,000$ runs, we compute the MSE's as

$$MSE(\hat{\mu}) = \frac{1}{S} \sum_{s=1}^{S} (\hat{\mu_s} - \mu)^2 \qquad (2.16)$$

and

$$MSE(\hat{\sigma}) = \frac{1}{S}\sum_{s=1}^{S}(\hat{\sigma_s} - \sigma)^2 \quad . \tag{2.17}$$

Table 2.1 shows the estimated parameters for different attendance probabilities (q) and study periods (k). The proposed approach is easy to implement and has a very fast convergence rate. It can be seen from Table 2.1 that the mean length of intervals decreases as subject attendance probability q increases. Thus, smaller intervals contains more information about the data on the basis of actual failure points t_i. Hence, the mean square error (MSE) is smaller. It can also be observed from the Table 2.1 that simulated MSE values decrease as we increase study period k, subject attendance probability q and sample size n. Simulated MSE values are small enough to conclude that simulated data are produced from a stable and well designed simulation process.

Table 2.1: Values of estimates for various study periods (k) and attendance probabilities (q) ; $T \sim N(0,1)$

sample size	q	k	$\hat{\mu}$	$\hat{\sigma}$	$MSE(\hat{\mu})$	$MSE(\hat{\sigma})$
n=50	0.6	12	0.0592	0.8302	0.0441	0.0631
		36	0.0361	0.8314	0.0353	0.0452
	0.8	12	0.0311	0.8401	0.0311	0.0531
		36	-0.0201	0.8421	0.0232	0.0362
n=100	0.6	12	0.0411	0.8413	0.0112	0.0231
		36	0.0359	0.8445	0.0101	0.0132
	0.8	12	0.0241	0.8463	0.0104	0.0162
		36	0.0203	0.8511	0.0091	0.0131
n=200	0.6	12	0.0383	0.8413	0.0092	0.0097
		36	0.0317	0.8514	0.0083	0.0079
	0.8	12	0.0072	0.9043	0.0084	0.0073
		36	0.034	0.9351	0.0062	0.0071
n=500	0.6	12	0.0314	0.8637	0.0032	0.0044
		36	0.0226	0.8993	0.0013	0.0026
	0.8	12	0.0054	0.9296	0.0016	0.0031
		36	0.0023	0.9382	0.0009	0.0014
n=1000	0.6	12	0.0119	0.9257	0.0008	0.0011
		36	0.0186	0.9288	0.0006	0.0008
	0.8	12	0.0021	0.9599	0.0003	0.0002
		36	0.0011	0.9803	0.0003	0.0001

CHAPTER 3

UNIVARIATE INTERVAL CENSORING WITH RANDOM INTERVALS

Case II interval censored data arise when we only know that T, a nonnegative random variable representing the failure time of a subject, has occurred within a random interval. In this chapter, we consider two different models for case II interval censored data. In section 3.1, we describe Model I based on two examination points. We introduce the copula approach to write the likelihood function. Section 3.2 describes Model II and related inference is made assuming ordered examination points.

3.1 Analysis of Case II Interval Censored Data with Two Examination Points

Let $T_1, T_2, \cdots, T_n$ be iid random variables with cdf F_T. Let also T_i denote the failure time of interest for subject i ($i = 1, 2, \cdots, n$). Suppose that interval censored data on T_i's are observed in a random interval (U_i, V_i) ($i = 1, 2, \cdots, n$) with indicator functions

$$\varDelta_1 = I\left(T \leq U\right)$$

and

$$\varDelta_2 = I\left(U \leq T \leq V\right) \quad .$$

We have joint density of the $U = x_1, V = x_2, \varDelta_1 = \delta$ and $\varDelta_2 = 1 - \delta$ as follows:

$$P\left\{U=x_1,V=x_2,\Delta_1=\delta,\Delta_2=1-\delta\right\}=\begin{cases} g\left(x_1,x_2\right)P\left(\Delta_1=\delta,\Delta_2=1-\delta\right), & \delta=1\\ g\left(x_1,x_2\right)P\left(\Delta_1=\delta,\Delta_2=1-\delta\right), & \delta=0\end{cases}$$

$$\begin{aligned} &= g\left(x_1,x_2\right)P\left(T<x_1\right)\delta+g\left(x_1,x_2\right)P\left(x_1<T<x_2\right)\left(1-\delta\right)\\ &= g\left(x_1,x_2\right)F_T\left(x_1\right)\delta+g\left(x_1,x_2\right)\left\{F_T\left(x_2\right)-F_T\left(x_1\right)\right\}\left(1-\delta\right) \quad . \end{aligned} \tag{3.1}$$

where $g\left(x_1,x_2\right)$ is the joint density of U and V.

In this section, copula method is proposed for the estimation of a failure time data. Copulas can be understood as bivariate or multivariate joint distributions with uniform $(0,1)$ marginal distributions. They are dependency functions that are very useful in applications of multivariate distribution models in many areas where the knowledge about the structure of dependency between random variables is required. The basic theorem in theory of copulas is the well known Sklar's Theorem [28].

Sklars Theorem: Let X and Y be random variables with joint distribution function H and marginal distribution functions F and G, respectively. Then, there exists a copula C such that

$$H(x,y)=C(F(x),G(y)) \tag{3.2}$$

for all x, y in R. If F and G are continuous, then C is unique. Conversely, if C is a copula and F and G are distribution functions, then the function H defined by (3.2) is a joint distribution function with marginals F and G.

It is clear that if the joint distribution function of random variables X and Y is

$$H(x,y)=C(F(x),G(y)),$$

and

$$c(u,v)=\frac{\partial^2 C(u,v)}{\partial u\partial v},$$

then the joint probability density function of random variables X and Y is

$$f(x, y) = c(F(x), G(y))f(x)g(y)$$

where $F(x), , G(y)$ and $f(x), g(y)$ are the corresponding marginal distribution functions and probability density functions of X and Y, respectively.

Copula is a useful tool for expressing the joint distribution of random variables as a functional of marginal distribution functions. Copulas can be interpreted as dependency functions and help us to measure the dependence between random variables. Copulas are also very appropriate models for estimating the parameters of distributions. The procedure allows us to use methods of estimation of parameters for marginal distributions in the first step and then estimate the parameters of copula in the second step. Such a two step method is referred as "inference functions for margins" in the literature (see Joe and Xu [20]).

The family of Archimedian Copulas is very common in applications beacuse of its analytical form expressed in terms of so called "generating" functions as follows:

$$C_\alpha\left(u, v\right) = \chi_\alpha\left\{\chi_\alpha^{-1}(u) + \chi_\alpha^{-1}(v)\right\}, \quad 0 \leq u, v \leq 1,$$

where $0 \leq \chi_\alpha \leq 1$, $\chi'_\alpha < 0$, $\chi''_\alpha > 0$. Here $\chi'_\alpha(u) = d\chi_\alpha(u)/du$ and $\chi''_\alpha(u) = d\chi'_\alpha(u)/du$.

Archimedian copulas are appealing in studies on censoring because it allows for flexibility and keeps the model mathematically tractable. For example, taking $\chi_\alpha(u) = (1+u)^{1/(1-\alpha)}$, the Laplace transformation of a gamma distribution, we have

$$C_\alpha\left(u, v\right) = (u^{(1-\alpha)} + v^{(1-\alpha)} - 1)^{1/(1-\alpha)}, \quad \alpha > 1,$$

which is referred to as the Clayton family [8] which has a very simple form and easy to apply.

We write the likelihood function by using copula in our model as

$$
\begin{aligned}
&f_{U_1,..,U_n,V_1,..,V_n,\Delta_1,\Delta_2}(u_1,u_2,..,u_n;v_1,v_2,..,v_n;\delta,1-\delta)\\
&=\prod_{i=1}^{n}\Big\{f\left(u_i,v_i\right)F_T\left(u_i\right)\delta_i+f\left(u_i,v_i\right)\left\{F_T\left(v_i\right)-F_T\left(u_i\right)\right\}\left(1-\delta_i\right)\Big\}\\
&=\prod_{i=1}^{n}c\Big(F_U\left(u_i\right),F_V\left(v_i\right)\Big)f_U\left(u_i\right)f_V\left(u_i\right)F_T\left(u_i\right)\delta_i\\
&+c\Big(F_U\left(u_i\right),F_V\left(v_i\right)\Big)f_U\left(u_i\right)f_V\left(u_i\right)\Big\{F_T\left(v_i\right)-F_T\left(u_i\right)\Big\}(1-\delta_i)\\
&=\prod_{i=1}^{n}\Bigg[c\Big(F_U\left(u_i\right),F_V\left(v_i\right)\Big)f_U\left(u_i\right)f_V\left(u_i\right)\\
&\times\Big[F_T\left(u_i\right)\delta_i+\left\{F_T\left(v_i\right)-F_T\left(u_i\right)\right\}\left(1-\delta_i\right)\Big]\Bigg]\quad .
\end{aligned}
\tag{3.3}
$$

It appears that copulas describe the dependence structure of the model. All the information about the dependency is contained in the copula function. Thus, the choice of appropriate copula and the value of its dependence parameter are very important.

Let $X_1, X_2, ..., X_n$ be i.i.d. random variables with continuous distribution function F. Let $U = \min(X_1, X_2, ..., X_n)$ and $V = \max(X_1, X_2, ..., X_n)$. Schmitz [36] derived a copula of the joint distribution of U and V as follows:

$$
C_n\left(u,v\right)=\begin{cases} v-(v^{\frac{1}{n}}+(1-u)^{\frac{1}{n}}-1)^n, & 1-(1-u)^{\frac{1}{n}}<v^{\frac{1}{n}}\\ v, & 1-(1-u)^{\frac{1}{n}}\geq v^{\frac{1}{n}}\end{cases}
$$

We propose in this study to use the copula of the minimum and maximum (Schmitz [36]) of n iid random variables. Since we have the condition $U < V$ for our random intervals, this copula, belonging to Clayton family, is very appropriate for our model.

The pdf of min-max copula is

$$
c_n\left(u,v\right)=\begin{cases} \frac{(v^{\frac{1}{n}}+u^{\frac{1}{n}}-1)^n v^{\frac{1}{n}}(1-u)^{\frac{1}{n}}}{v(v^{\frac{1}{n}}+(1-u)^{\frac{1}{n}}-1)^2(1-u)}\frac{n-1}{n}, & 1-(1-u)^{\frac{1}{n}}<v^{\frac{1}{n}}\\ 0, & otherwise\end{cases}.
$$

Note that n plays a role only as a parameter of the copula $C_n(u, v)$, and is used for modeling of dependence between two random variables U and V. To illustrate the concept, assume, for example, $n = 2$. Then,

$$C_2(u,v) = \begin{cases} v - (\sqrt{v} + \sqrt{(1-u)} - 1)^2, & 1 - \sqrt{(1-u)} < \sqrt{v} \\ v, & otherwise \end{cases}$$

and

$$c_2(u,v) = \begin{cases} \frac{1}{\sqrt{v(1-u)}}, & 1 - \sqrt{(1-u)} < \sqrt{v} \\ 0, & otherwise \end{cases} .$$

To estimate the copula parameter n, we also propose a simple graphical procedure; see the graphs of $C_n(u, v)$ and $c_n(u, v)$ for $n = 2, 10$ and $n = 20$ in Figure 3.1, 3.2, 3.3.

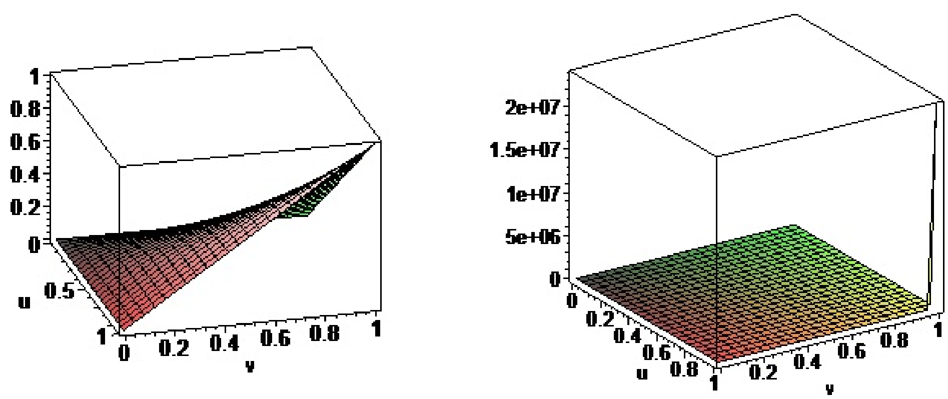

Figure 3.1: Graphs of $C_2(u, v)$ and $c_2(u, v)$

It can be observed from the graphs that if one uses the copula of joint distribution of extreme order statistics U and V then it would be better to take large values of n, which effects stability of probability density function about $u = 1$ and $v = 1$.

Now consider, the joint pdf of U, V, Δ_1andΔ_2 which we use in the likelihood function (3.3). Assume that the copula of the joint distribution of U and V is $C_2(u, v)$. Then, the likelihood function is

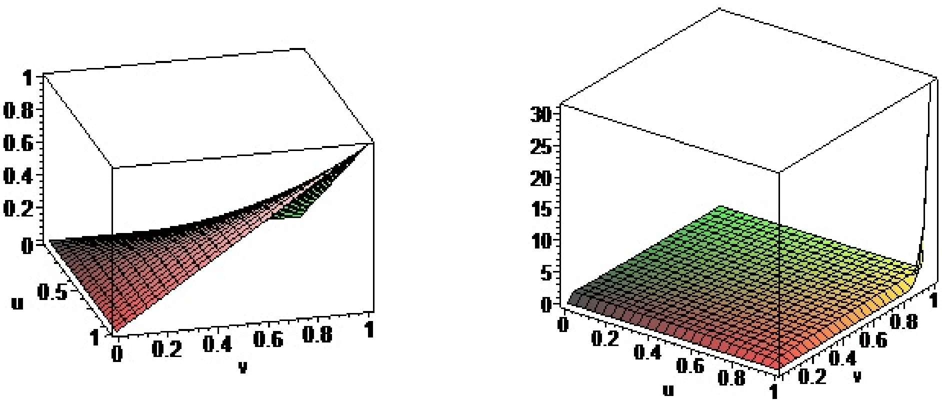

Figure 3.2: Graphs of $C_{10}(u,v)$ and $c_{10}(u,v)$

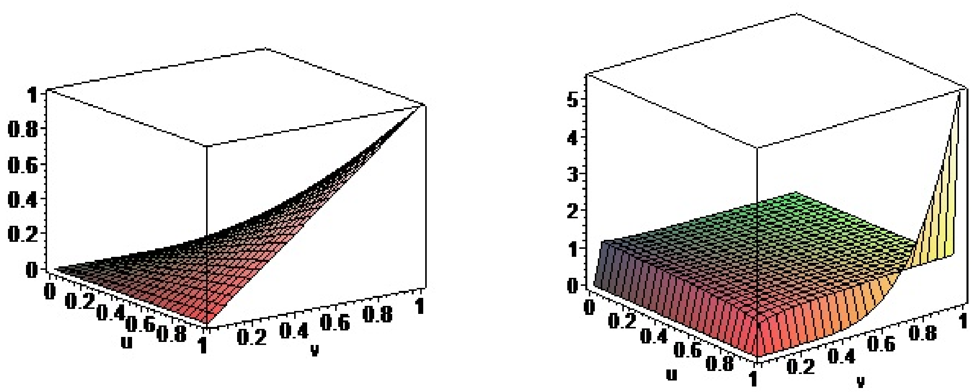

Figure 3.3: Graphs of $C_{20}(u,v)$ and $c_{20}(u,v)$

$$
\begin{aligned}
f_{U,V,\Delta_1,\Delta_2}(u,v,\delta) &= c(F_U(u),F_V(v))f_U(u)f_V(v)F_T(u)\delta \\
&+c(F_U(u),F_V(v))f_U(u)f_V(v)[F_T(v)-F_T(u)](1-\delta). \qquad (3.4)
\end{aligned}
$$

For the exponential distribution function with parameter $\lambda = 1$, we provide the graphs of likelihood function, denoted by $f^*_{U,V,\Delta_1,\Delta_2}(u,v,\delta)$; see Figures 3.4, 3.5 and 3.6.

Graphical analysis shows if one uses the maximum likelihood method for estimating parameters of distributions of U, V an T using observed data $U_1, U_2, \cdots, U_n$, $V_1, V_2, \cdots, V_n$ and δ, the proper model is seem to be copula model based on extreme order statistics with large values of n. Graphical representations show that, if $n > 10$

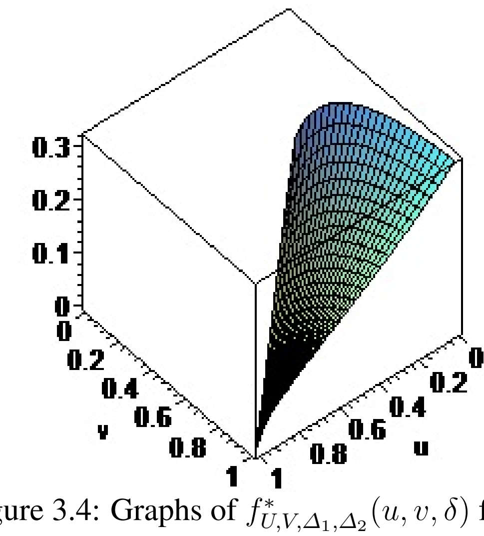

Figure 3.4: Graphs of $f^{*}_{U,V,\Delta_1,\Delta_2}(u,v,\delta)$ for $n=2$

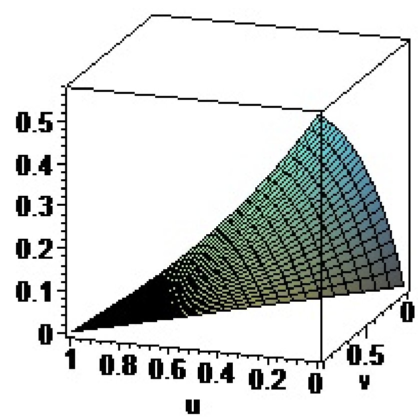

Figure 3.5: Graphs of $f^{*}_{U,V,\Delta_1,\Delta_2}(u,v,\delta)$ for $n=10$

the likelihood function becomes more regular in the sense that the corresponding derivatives can be calculated.

Figure 3.7 also shows the value of $f^{*}_{U,V,\Delta_1,\Delta_2}(u,v,\delta)$ for different values of n; u and v being fixed.

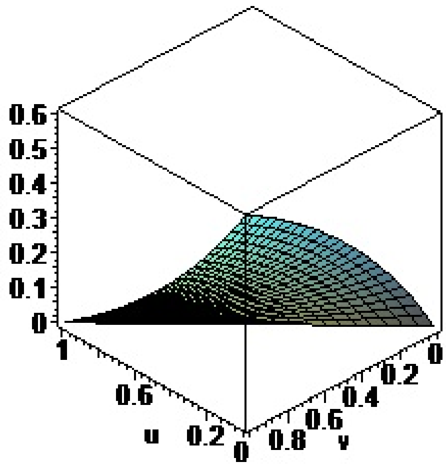

Figure 3.6: Graphs of $f^*_{U,V,\Delta_1,\Delta_2}(u, v, \delta)$ for $n = 20$

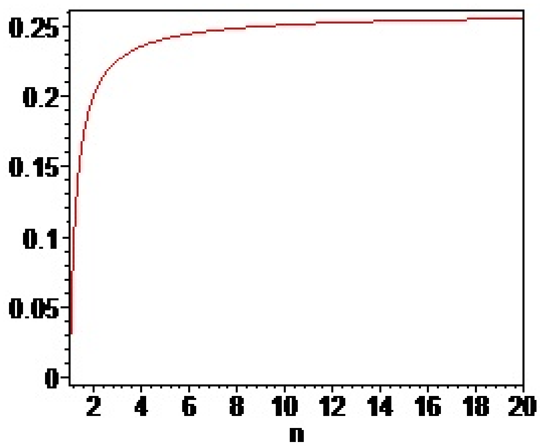

Figure 3.7: Value of $f^*_{U,V,\Delta_1,\Delta_2}(u, v, \delta)$ for different values of n; u and v fixed

3.2 Analysis of Case II Interval Censored Data with Ordered Examination Points

Consider n independent and identical components or patients put on a test. Let $Y_i = (Y_{i1}, Y_{i2}, ..., Y_{in_i})$ be ordered examination times for patient i where,

$$0 < Y_{i1} < Y_{i2} < ..... < Y_{in_i} < \infty, \quad i = 1, 2, ..., k. \tag{3.5}$$

If T_i is defined to be the i^{th} patient's unobservable failure time, then this model is called as general interval censoring scheme.

Assume that, the failure occurred before the first examination time. Then let,

$$\delta_{io} = \begin{cases} 1 & \text{if} \quad T_i < Y_{i1} \\ 0 & \text{if} \quad o.w. \end{cases} .$$

Now, consider the failure occurred between a pair of examination times (Y_{iL}, Y_{iR}) where Y_{iL} is the last examination time preceding T_i and Y_{iR} is the first examination time following T_i. Then let,

$$\delta_{i1} = \begin{cases} 1 & \text{if} \quad Y_{i1} < T_i < Y_{i2} \\ 0 & \text{if} \quad o.w. \end{cases}$$

$$\delta_{i2} = \begin{cases} 1 & \text{if} \quad Y_{i2} < T_i < Y_{i3} \\ 0 & \text{if} \quad o.w. \end{cases}$$

.

.

.

$$\delta_{ir-1} = \begin{cases} 1 & \text{if} \quad Y_{ir-1} < T_i < Y_{ir} \\ 0 & \text{if} \quad o.w. \end{cases}$$

.

.

.

$$\delta_{in_i-1} = \begin{cases} 1 & \text{if} \quad Y_{in_i-1} < T_i < Y_{in_i} \\ 0 & \text{if} \quad o.w. \end{cases}$$

If we consider that the failure did not occur, then define

$$\delta_{in_i} = \begin{cases} 1 & \text{if} \quad T_i > Y_{in_i} \\ 0 & \text{if} \quad o.w. \end{cases} .$$

Suppose that the ordered examination times for patient i follows an exponential distribution with parameter λ. Exponential distribution is one of the commonly used model used to describe the properties of the life time distributions. The life testing experiments often deal with interval censored samples and our goal is to estimate the parameters involved in an exponential distribution.

Then, the pdf of the lifetime of the component T and ordered examination time Y takes the following forms, respectively:

$$f(t) = \theta e^{-\theta t} \tag{3.6}$$

and

$$g(y) = \lambda e^{-\lambda y} \quad . \tag{3.7}$$

The likelihood function based on the random interval censored sample can be written as

$$\begin{aligned} L(\theta) = \prod_{i=1}^{k} \Big[F(Y_{i1})\Big]^{\delta_{i0}} \Big[F(Y_{i2}) - F(Y_{i2})\Big]^{\delta_{i1}} \Big[F(Y_{i3}) - F(Y_{i2})\Big]^{\delta_{i2}} \cdots \\ \cdots\cdots \Big[F(Y_{ir}) - F(Y_{ir-1})\Big]^{\delta_{ir-1}} \cdots\cdots \\ \cdots\cdots \Big[F(Y_{in_i}) - F(Y_{in_i-1})\Big]^{\delta_{in_i-1}} \Big[1 - F(Y_{in_i})\Big]^{\delta_{in_i}} n! \\ \cdot n! g(y_{i1}) g(y_{i1}) \cdots g(y_{ir-1}) \cdots g(y_{in_i}) \quad . \end{aligned} \tag{3.8}$$

The first partial derivative of the log likelihood function is

$$\begin{aligned} \frac{\partial L(\theta)}{\partial \theta} = \sum_{i=1}^{k} \Bigg\{ \delta_{i0} ln(1 - e^{-\theta y_{i1}}) + \delta_{i1} ln(e^{-\theta y_{i1}} - e^{-\theta y_{i2}}) + \cdots\cdots \\ \cdots\cdots + \delta_{ir-1} ln(e^{-\theta y_{ir-1}} - e^{-\theta y_{ir}}) + \cdots\cdots \\ \cdots\cdots + \delta_{in_{i-1}} ln(e^{-\theta y_{in_{i-1}}} - e^{-\theta y_{in_i}}) + \delta_{in_i} ln(e^{-\theta y_{in_i}}) \\ + ln(n!) + ln(\lambda^{n_i}) - \lambda y_{i1} - \lambda y_{i2} - \cdots\cdots - \lambda y_{in_i} \Bigg\} \end{aligned} \tag{3.9}$$

$$
\begin{aligned}
\frac{\partial L(\theta)}{\partial \theta} = \sum_{i=1}^{k} \Bigg\{ & \delta_{i0} \frac{e^{\theta y_{i1}} y_{i1}}{(1 - e^{-\theta y_{i1}})} + \delta_{i1} \frac{e^{\theta y_{i2}} y_{i2} - e^{\theta y_{i1}} y_{i1}}{(e^{-\theta y_{i1}} - e^{-\theta y_{i2}})} + \cdots\cdots \\
& \cdots\cdots + \delta_{ir-1} \frac{e^{\theta y_{ir}} y_{ir} - e^{\theta y_{ir-1}} y_{ir-1}}{(e^{-\theta y_{ir-1}} - e^{-\theta y_{ir}})} + \cdots\cdots \\
& \cdots\cdots + \delta_{in_{i-1}} \frac{e^{\theta y_{in_i}} y_{in_i} - e^{\theta y_{in_{i-1}}} y_{in_{i-1}}}{(e^{-\theta y_{in_{i-1}}} - e^{-\theta y_{in_i}})} - \delta_{in_i} \frac{e^{\theta y_{in_i}} y_{in_i}}{(e^{-\theta y_{in_i}})} \Bigg\} \quad .
\end{aligned}
\tag{3.10}
$$

By using the above likelihood, we can not obtain a closed form for the expression of the ML equation and numerical integration is needed to evaluate it. Turnbull [44] derived self-consistency equations for a very general censoring scheme which includes interval censoring as a special case. This yields an EM algorithm for computing the NPMLE. Groeneboom's Iterative Convex Minorant algorithm can also be used for computing the NPMLE. As suggested in Groeneboom and Wellner [17], the Iterative Convex Minorant (ICM) algorithm is considerably faster than the EM algorithm, especially when the sample size is large. Finkelstein [12] and Rabinowitz et al. [33] considered estimation in the proportional hazards model and in the linear regression model for general interval censoring, respectively. Large sample properties of their estimators are, however, unknown.

To solve this problem, let,

$$
g_0\left(z_{(i)1}\right) = \frac{e^{-z_{(i)1}}}{1 - e^{-z_{(i)1}}} \quad ,
$$

$$
g_1\left(z_{(i)1}, z_{(i)2}\right) = \frac{e^{-z_{(i)2}}}{e^{-z_{(i)1}} - e^{-z_{(i)2}}} \quad ,
$$

$$
g_2\left(z_{(i)1}, z_{(i)2}\right) = \frac{e^{-z_{(i)1}}}{e^{-z_{(i)1}} - e^{-z_{(i)2}}} \quad ,
$$

$$
g_{r-1}\left(z_{(i)r-1}, z_{(i)r}\right) = \frac{e^{-z_{(i)r}}}{e^{-z_{(i)r-1}} - e^{-z_{(i)r}}}
$$

and

$$
g_r\left(z_{(i)r-1}, z_{(i)r}\right) = \frac{e^{-z_{(i)r-1}}}{e^{-z_{(i)r-1}} - e^{-z_{(i)r}}} \quad .
$$

Consider the standardized exponential distribution with pdf

$$f(z) = e^{-z}, \quad 0 \leq z \leq \infty \quad .$$

with the expected value of the r^{th} order statistic (in a sample of size n)

$$\mu_{r:n} = \sum_{i=1}^{r} \frac{1}{(n-i+1)} \quad , \quad r = 1, 2, \cdots, n \quad .$$

Since complete sums are invariant to ordering [1], we can write

$$\frac{\partial lnL}{\partial \theta} = \sum_{i=1}^{n} g(z_{(i)}) = 0, \quad z_{(i)} = \theta t_{(i)} \quad . \tag{3.11}$$

Realizing that the function $g(z)$ is almost linear in a small interval $a \leq z \leq b$ ([1]) and $z_{(i)}$ is located in the vicinity of $w_{(i)} = \mu_{r:n}$ at any rate for large n, we obtain the following linear approximations of $g(z_{(i)})$ by using the first three terms of a Taylor series. Then,

$$\begin{aligned} g_0\left(z_{(i)1}\right) &= \frac{e^{-z_{(i)1}}}{1-e^{-z_{(i)1}}} \\ &= g_0\left(w_{(i)1}\right) + (z_{(i)1} - w_{(i)1})\frac{\partial g_0\left(z_{(i)1}\right)}{\partial z}|_{z_{(i)1}=w_{(i)1}} \\ &= \alpha_{i0} + \beta_{i0} z_{(i)1} \end{aligned} \tag{3.12}$$

$$\begin{aligned} g_1\left(z_{(i)1}, z_{(i)2}\right) &= \frac{e^{-z_{(i)2}}}{e^{-z_{(i)1}} - e^{-z_{(i)2}}} \\ &= g_1\left(w_{(i)1}, w_{(i)2}\right) + (z_{(i)2} - w_{(i)2})\frac{\partial g_1\left(z_{(i)1}, z_{(i)2}\right)}{\partial z_{(i)2}}|_{w_{(i)1}, w_{(i)2}} \\ &+ (z_{(i)1} - w_{(i)1})\frac{\partial g_1\left(z_{(i)1}, z_{(i)2}\right)}{\partial z_{(i)1}}|_{w_{(i)1}, w_{(i)2}} \\ &= \alpha_{i1} + \beta_{i1} z_{(i)2} + \gamma_{i1} z_{(i)1} \end{aligned} \tag{3.13}$$

$$
\begin{aligned}
g_2\left(z_{(i)1}, z_{(i)2}\right) &= \frac{e^{-z_{(i)2}}}{e^{-z_{(i)1}} - e^{-z_{(i)2}}} \\
&= g_1\left(w_{(i)1}, w_{(i)2}\right) + (z_{(i)2} - w_{(i)2}) \frac{\partial g_1\left(z_{(i)1}, z_{(i)2}\right)}{\partial z_{(i)2}}|_{w_{(i)1}, w_{(i)2}} \\
&+ (z_{(i)1} - w_{(i)1}) \frac{\partial g_1\left(z_{(i)1}, z_{(i)2}\right)}{\partial z_{(i)1}}|_{w_{(i)1}, w_{(i)2}} \\
&= \alpha_{i2} + \beta_{i2} z_{(i)2} + \gamma_{i2} z_{(i)1}
\end{aligned} \tag{3.14}
$$

.

.

.

$$
\begin{aligned}
g_{r-1}\left(z_{(i)r-1}, z_{(i)r}\right) &= \frac{e^{-z_{(i)r}}}{e^{-z_{(i)r-1}} - e^{-z_{(i)r}}} \\
&= g_{r-1}\left(w_{(i)r-1}, w_{(i)r}\right) + (w_{(i)r} - t_{(i)r}) \frac{\partial g_{r-1}\left(z_{(i)r-1}, z_{(i)r}\right)}{\partial z_{(i)r}}|_{w_{(i)r-1}, w_{(i)r}} \\
&+ (z_{(i)r-1} - w_{(i)r-1}) \frac{\partial g_{r-1}\left(z_{(i)r-1}, z_{(i)r}\right)}{\partial z_{(i)r-1}}|_{w_{(i)r-1}, w_{(i)r}} \\
&= \alpha_{i2r-3} + \beta_{i2r-3} z_{(i)r} + \gamma_{i2r-3} z_{(i)r-1}
\end{aligned} \tag{3.15}
$$

$$
\begin{aligned}
g_r\left(z_{(i)r-1}, z_{(i)r}\right) &= \frac{e^{-z_{(i)r-1}}}{e^{-z_{(i)r-1}} - e^{-z_{(i)r}}} \\
&= g_r\left(w_{(i)r-1}, w_{(i)r}\right) + (z_{(i)r} - w_{(i)r}) \frac{\partial g_r\left(z_{(i)r-1}, z_{(i)r}\right)}{\partial z_{(i)r}}|_{w_{(i)r-1}, w_{(i)r}} \\
&+ (z_{(i)r-1} - w_{(i)r-1}) \frac{\partial g_r\left(z_{(i)r-1}, z_{(i)r}\right)}{\partial z_{(i)r-1}}|_{w_{(i)r-1}, w_{(i)r}} \\
&= \alpha_{i2r-2} + \beta_{i2r-2} z_{(i)r} + \gamma_{i2r-2} z_{(i)r-1}
\end{aligned} \tag{3.16}
$$

.

.

.

$$
\begin{aligned}
g_{n_i-1}\left(z_{(i)n_i-1}, z_{(i)n_i}\right) &= \frac{e^{-z_{(i)n_i}}}{e^{-z_{(i)n_i-1}} - e^{-z_{(i)n_i}}} \\
&= g_{n_i-1}\left(w_{(i)n_i-1}, w_{(i)n_i}\right) \\
&+ (z_{(i)n_i} - w_{(i)n_i})\frac{\partial g_{n_i-1}\left(z_{(i)n_i-1}, z_{(i)n_i}\right)}{\partial z_{(i)n_i}}|_{w_{(i)n_i-1}, w_{(i)n_i}} \\
&+ (z_{(i)n_i-1} - w_{(i)n_i-1})\frac{\partial g_{n_i-1}\left(z_{(i)n_i-1}, z_{(i)n_i}\right)}{\partial z_{(i)n_i-1}}|_{w_{(i)n_i-1}, w_{(i)n_i}} \\
&= \alpha_{i2n_i-2} + \beta_{i2n_i-2} z_{(i)n_i} + \gamma_{i2n_i-2} z_{(i)n_i-1} \quad .
\end{aligned}
\tag{3.17}
$$

Incorporating (3.12 - 3.17) in (3.10), the MMLE is obtained as

$$
\hat{\theta} = \frac{\sum_{i=1}^{k}\left\{\delta_{i0}Y_{i1}\alpha_{i0} + \sum_{j=1}^{n_i-1}\delta_{ij}\left(Y_{ij}\alpha_{i(2j-1)} - Y_{ij-1}\alpha_{i(2j)}\right) + \delta_{in_i}Y_{in_i}\right\}}{\sum_{i=1}^{k}\left\{\delta_{i0}Y_{i1}^2\beta_{i0} + \sum_{j=1}^{n_i-1}\delta_{ij}\beta_{i(2j-1)}\left(Y_{ij-1} - Y_{ij}\right)^2\right\}} \quad . \tag{3.18}
$$

As can be seen, it is a closed form solution and there is no need to carry out a numerical study.

3.2.1 Fisher Information Number

In this section, we study Fisher Information (FI) for Case II interval censored samples from exponential distribution. There are four main purposes for studying the FI in censored samples:

- To obtain the asymptotic variance of MLE
  - It is known that MLE's are asymptoticaly efficient under regularity conditions. We can find the asymptotic variance of MLE using the FI if the MLE exists.
- To determine the optimal sample size for life testing experiments
  - One can compare $\frac{I_{1,\ldots,r:m}}{E(X_{r:m})}$ for $1 \leq r \leq m \leq n$ and call this quantity FI per unit time for the life testing experiment. The quantity measures which

censored sampling mechanism is more efficient in terms of the amount of FI acquired per unit time during the experiment. The censored sample with more FI in less duration is assumed to have the better performance in life testing experiment.

- To evaluate MLEs from large censored samples
  - Let us denote, the MLE's $\hat{\theta}_r$ and $\hat{\theta}_n$ from interval censored sample and MLE from complete sample respectively. For $r/n \to p$ as $n \uparrow \infty$ where $0 < p < 1$, the asymptotic relative efficiency (ARE) is given by

$$ARE(\hat{\theta}_r, \hat{\theta}_n) = \frac{I_p(\theta)}{I(\theta; X, Y)}$$

    where $I_p(\theta)$ and $I(\theta; X, Y)$ are FI numbers for a censored and complete samples, respectively.

- To evaluate the relative efficiencies of unbiased estimators in finite censored sample
  - Cramer Lower Bound provided by the Fisher Information measure can be used to examine the finite sample efficiencies of unbiased estimators based on censored samples.

A few studies have focused on Fisher information on censored data in the presence of right censoring since the censoring mechanism for interval censoring is much more complicated than that of right censoring. Escobar and Meeker [11] shows how to compute the "Fisher information matrix and the asymptotic covariance matrix for maximum likelihood estimators" for a wide class of parametric models that include combinations of censoring, truncation and explanatory variables. Ortega et al.[30] gave influence diagnostics for the Weibull case with censored data. They state that "it is not possible to compute Fisher information matrix". Qian [32] considers the "three parameter exponentiated Weibull family under type II censoring". He proposes an algorithm for computing the maximum likelihood estimator and derives the Fisher information matrix.

It is known that $\hat{\theta}$ is approximately distributed $N\left(\theta, I^{-1}(\theta)\right)$ for large n where $I^{-1}(\theta)$ is the variance covariance matrix of the unknown parameter θ; $I(\theta)$ being Fisher information. In the case of single parameter θ, the Fisher information number is given by

$$I(\theta) = -E\Big(\frac{\partial^2 lnL(X,\theta)}{\partial\theta^2}\Big) \quad . \tag{3.19}$$

The variance of $\hat{\theta}$ for Model II, can be estimated by the observed Fisher information number. For this, one needs the second partial derivative. Second partial derivative of the log likelihood function can be found as

$$\begin{aligned}\frac{\partial^2 lnL(X,\theta)}{\partial\theta^2} = \sum_{i=1}^{k}\Bigg\{ & -\delta_{i0}\frac{y_{i1}^2 e^{-\theta y_{i1}}}{(1-e^{-\theta y_{i1}})^2} - \delta_{i1}\frac{(y_{i2}-y_{i1})^2 e^{-\theta y_{i2}} e^{-\theta y_{i1}}}{(e^{-\theta y_{i1}} - e^{-\theta y_{i1}})^2} - \cdots \\ & \cdots - \delta_{ir-1}\frac{(y_{ir}-y_{ir-1})^2 e^{-\theta y_{ir}} e^{-\theta y_{ir-1}}}{(e^{-\theta y_{ir-1}} - e^{-\theta y_{ir}})^2} - \cdots \\ & \cdots - \delta_{in_i-1}\frac{(y_{in_i}-y_{in_i-1})^2 e^{-\theta y_{in_i}} e^{-\theta y_{in_i-1}}}{(e^{-\theta y_{in_i-1}} - e^{-\theta y_{in_i}})^2} \Bigg\}.\end{aligned}$$

Fisher information number for a single parameter is then obtained as

$$I(\theta) = -E\Big(\frac{\partial^2 lnL(X,\theta)}{\partial\theta^2}\Big) \tag{3.20}$$

where

$$
\begin{aligned}
E\Big(\frac{\partial^2 lnL(X,\theta)}{\partial\theta^2}\Big) &= \sum_{i=1}^{k} E\Big(\frac{\partial^2 lnL(X,\theta)}{\partial\theta^2} \mid (Y_{i1}, Y_{i2}, ..., Y_{in_i})\Big) \\
&= \sum_{i=1}^{k} \int \delta_{i0} \frac{y_{i1}^2 e^{-\theta y_{i1}}}{(1-e^{-\theta y_{i1}})^2} g(y_{i(1)}) dy_{i1} \\
&+ \sum_{i=1}^{k} \iint \delta_{i1} \frac{(y_{i2}-y_{i1})^2 e^{-\theta y_{i2}} e^{-\theta y_{i1}}}{(e^{-\theta y_{i1}}-e^{-\theta y_{i1}})^2} g(y_{i(1)}, y_{i(2)}) dy_{i1} dy_{i2} \\
&\quad . \\
&\quad . \\
&\quad . \\
&+ \sum_{i=1}^{k} \iint \delta_{ir-1} \frac{(y_{ir}-y_{ir-1})^2 e^{-\theta y_{ir}} e^{-\theta y_{ir-1}}}{(e^{-\theta y_{ir-1}}-e^{-\theta y_{ir}})^2} g(y_{i(r-1)}, y_{i(r)}) dy_{ir-1} dy_{ir} \\
&\quad . \\
&\quad . \\
&\quad . \\
&+ \sum_{i=1}^{k} \iint \delta_{in_i-1} \frac{(y_{in_i}-y_{in_i-1})^2 e^{-\theta y_{in_i}} e^{-\theta y_{in_i-1}}}{(e^{-\theta y_{in_i-1}}-e^{-\theta y_{in_i}})^2} g(y_{i(n_i-1)}, y_{i(n_i)}) dy_{in_i-1} dy_{in_i} \quad .
\end{aligned}
$$

Let $n_1 = n_2 = \cdots = n_k = n$.

$$
\begin{aligned}
&E\Big(\frac{\partial^2 lnL(X,\theta)}{\partial\theta^2}\Big)\\
&= k\left\{
\begin{array}{c}
\displaystyle\int \delta_{i0}\frac{y_{i1}^2 e^{-\theta y_{i1}}}{(1-e^{-\theta y_{i1}})^2} g(y_{i(1)})dy_{i1}\\
\displaystyle + \iint \delta_{i1}\frac{(y_{i2}-y_{i1})^2 e^{-\theta y_{i2}} e^{-\theta y_{i1}}}{(e^{-\theta y_{i1}}-e^{-\theta y_{i1}})^2} g(y_{i(1)}, y_{i(2)})dy_{i1}dy_{i2}\\
.\\
.\\
.\\
\displaystyle + \iint \delta_{ir-1}\frac{(y_{ir}-y_{ir-1})^2 e^{-\theta y_{ir}} e^{-\theta y_{ir-1}}}{(e^{-\theta y_{ir-1}}-e^{-\theta y_{ir}})^2} g(y_{i(r-1)}, y_{i(r)})dy_{ir-1}dy_{ir}\\
.\\
.\\
.\\
\displaystyle + \iint \delta_{in_i-1}\frac{(y_{in_i}-y_{in_i-1})^2 e^{-\theta y_{in_i}} e^{-\theta y_{in_i-1}}}{(e^{-\theta y_{in_i-1}}-e^{-\theta y_{in_i}})^2} g(y_{i(n_i-1)}, y_{i(n_i)})dy_{in_i-1}dy_{in_i}
\end{array}
\right\}.
\end{aligned}
$$

$$
\begin{aligned}
E\Big(\frac{\partial^2 lnL(X,\theta)}{\partial\theta^2}\Big) & \\
= \quad k & \left\{
\begin{array}{c}
\displaystyle\int F(y_{i1})\frac{y_{i1}^2 e^{-\theta y_{i1}}}{(1-e^{-\theta y_{i1}})^2} g(y_{i(1)})dy_{i1} \\[2ex]
\displaystyle+\iint \left[[F(y_{i2})-F(y_{i1})]\,\frac{(y_{i2}-y_{i1})^2 e^{-\theta y_{i2}} e^{-\theta y_{i1}}}{(e^{-\theta y_{i1}}-e^{-\theta y_{i1}})^2}\right] \\
\times g(y_{i(1)}, y_{i(2)})dy_{i1}dy_{i2} \\
. \\
. \\
. \\
\displaystyle+\iint \left[[F(y_{ir})-F(y_{ir-1})]\,\frac{(y_{ir}-y_{ir-1})^2 e^{-\theta y_{ir}} e^{-\theta y_{ir-1}}}{(e^{-\theta y_{ir-1}}-e^{-\theta y_{ir}})^2}\right] \\
\times g(y_{i(r-1)}, y_{i(r)})dy_{ir-1}dy_{ir} \\
. \\
. \\
. \\
\displaystyle+\iint \left[[F(y_{in_i})-F(y_{in_i-1})]\,\frac{(y_{in_i}-y_{in_i-1})^2 e^{-\theta y_{in_i}} e^{-\theta y_{in_i-1}}}{(e^{-\theta y_{in_i-1}}-e^{-\theta y_{in_i}})^2}\right] \\
\times g(y_{i(n_i-1)}, y_{i(n_i)})dy_{in_i-1}dy_{in_i}
\end{array}
\right\}
\end{aligned}
$$

where $g(y_{i(1)})$ is the probability function of the first order statistics and $g(y_{i(r-1)}, y_{i(r)})$ is the joint probability density function of the $(r-1)^{th}$ and r^{th} order statistics.

Let $y_{i1} = y_1, y_{i2} = y_2, \cdots, y_{in_i} = y_n$. Then,

$$E\Big(\frac{\partial^2 lnL(X,\theta)}{\partial\theta^2}\Big) = k\left\{\begin{array}{c} \int F(y_1)\frac{y_1^2 e^{-\theta y_1}}{(1-e^{-\theta y_1})^2} g(y_{(1)})dy_1 \\ + \iint \left[[F(y_2) - F(y_1)]\,\frac{(y_2-y_1)^2 e^{-\theta y_2} e^{-\theta y_1}}{(e^{-\theta y_1} - e^{-\theta y_1})^2}\right] \\ \times g(y_{(1)}, y_{(2)})dy_1 dy_2 \\ . \\ . \\ . \\ + \iint \left[[F(y_r) - F(y_{r-1})]\,\frac{(y_r-y_{r-1})^2 e^{-\theta y_r} e^{-\theta y_{r-1}}}{(e^{-\theta y_{r-1}} - e^{-\theta y_r})^2}\right] \\ \times g(y_{(r-1)}, y_{(r)})dy_{r-1} dy_r \\ . \\ . \\ . \\ + \iint \left[[F(y_n) - F(y_{n-1})]\,\frac{(y_n-y_{n-1})^2 e^{-\theta y_n} e^{-\theta y_{n-1}}}{(e^{-\theta y_{n-1}} - e^{-\theta y_n})^2}\right] \\ \times g(y_{(n-1)}, y_{(n)})dy_{n-1} dy_n \end{array}\right\}.$$

Then, $E\Big(\frac{\partial^2 lnL(X,\theta)}{\partial\theta^2}\Big)$ takes the following form:

$$E\Big(\frac{\partial^2 lnL(X,\theta)}{\partial\theta^2}\Big) = k\left\{\begin{array}{c} \int F(y_1)\frac{y_1^2 e^{-\theta y_1}}{(1-e^{-\theta y_1})^2} g(y_{(1)})dy_1 \\ + \sum_{i=1}^{n-1} \iint \left[[F(y_{i+1}) - F(y_i)]\,\frac{(y_{i+1}-y_i)^2 e^{-\theta y_{i+1}} e^{-\theta y_i}}{(e^{-\theta y_i} - e^{-\theta y_{i+1}})^2}\right] \\ \times g(y_{(i)}, y_{(i+1)})dy_i dy_{i+1} \end{array}\right\}.$$

For the simplicity, let us assume $y_{i+1} = y_2$ and $y_i = y_1$. Bu using the joint probability density function of the two order statistics we obtain

$$E\Big(\frac{\partial^2 lnL(X,\theta)}{\partial\theta^2}\Big) = k\left\{\begin{array}{c} \displaystyle\int F(y_1)\frac{y_1^2 e^{-\theta y_1}}{(1-e^{-\theta y_1})^2} n[1-G_{y_1}]^{n-1} g(y_1)dy_1 \\ \\ +n(n-1)\sum_{i=1}^{n-1}\displaystyle\iint [[F(y_2)-F(y_1)]\,\frac{(y_2-y_1)^2 e^{-\theta y_2}e^{-\theta y_1}}{(e^{-\theta y_2}-e^{-\theta y_1})^2} \\ \displaystyle\binom{n-2}{i-1}\times[G_{y_1}]^{i-1}[1-G_{y_2}]^{n-i-1}]g(y_{(i)},y_{(i+1)})dy_1dy_2 \end{array}\right\}$$

$$= k\left\{\begin{array}{c} \displaystyle\int F(y_1)\frac{y_1^2 e^{-\theta y_1}}{(1-e^{-\theta y_1^2})} n[1-G_{y_1}]^{n-1} g(y_1)dy_1 \\ \\ +n(n-1)\sum_{i=1}^{n-1}\displaystyle\iint [F(y_2)-F(y_1)]\,\frac{(y_2-y_1)^2 e^{-\theta y_2}e^{-\theta y_1}}{(e^{-\theta y_2}-e^{-\theta y_1})^2}\binom{n-2}{i-1} \\ \times[G_{y_1}]^{i-1}[1-G_{y_2}]^{n-i-1} g(y_1)g(y_2)dy_1dy_2 \end{array}\right\}.$$

Let $i-1=j$, then,

$$E\Big(\frac{\partial^2 lnL(X,\theta)}{\partial\theta^2}\Big) = k\left\{\begin{array}{c} \displaystyle\int_0^\infty (1-e^{-\theta y_1^2})\frac{y_1^2 e^{-\theta y_1}}{(1-e^{-\theta y_1^2})} n[e^{-\lambda y_1}]^{n-1}\lambda e^{-\lambda y_1}dy_1 \\ \\ +n(n-1)\displaystyle\iint [F(y_2)-F(y_1)]\,\frac{(y_2-y_1)^2 e^{-\theta y_2}e^{-\theta y_1}}{(e^{-\theta y_2}-e^{-\theta y_1})^2} g(y_1)g(y_2) \\ \times\Big\{\sum_{j=0}^{n-2}\displaystyle\binom{n-2}{j}[G_{y_1}]^{j}[1-G_{y_2}]^{n-2-j}\Big\}\,dy_1dy_2 \end{array}\right\}$$

$$= k\left\{\begin{array}{c} \displaystyle\int_0^\infty (1-e^{-\theta y_1^2})\frac{y_1^2 e^{-\theta y_1}}{(1-e^{-\theta y_1^2})} n[e^{-\lambda y_1}]^{n-1}\lambda e^{-\lambda y_1}dy_1 \\ \\ +n(n-1)\displaystyle\iint \left(e^{-\theta y_2}-e^{-\theta y_1}\right)\frac{(y_2-y_1)^2 e^{-\theta y_2}e^{-\theta y_1}}{(e^{-\theta y_2}-e^{-\theta y_1})^2}\lambda^2 e^{-\theta y_1}e^{-\theta y_2} \\ \cdot[e^{-\theta y_2}+1-e^{-\theta y_1}]^{n-2}dy_1dy_2 \end{array}\right\}$$

$$k\left\{\begin{array}{c} n\lambda\displaystyle\int_0^\infty (1-e^{-\theta y_1^2})\frac{y_1^2 e^{-\theta y_1}}{(1-e^{-\theta y_1^2})} n[e^{-\lambda y_1}]^{n-1}\lambda e^{-\lambda y_1}dy_1 \\ \\ +n(n-1)\lambda^2\displaystyle\iint \left(e^{-\theta y_2}-e^{-\theta y_1}\right)\frac{(y_2-y_1)^2 e^{-\theta y_2}e^{-\theta y_1}}{(e^{-\theta y_2}-e^{-\theta y_1})^2} e^{-\theta(y_1+y_2)} \\ \cdot[e^{-\theta y_2}+1-e^{-\theta y_1}]^{n-2}dy_1dy_2 \end{array}\right\}$$

$$= k \left\{ \begin{array}{c} n\lambda \int_0^\infty y_1^2 e^{(-\theta+n\lambda)y_1} \frac{1}{(1-e^{-\theta y_1})} dy_1 \\ \\ +n(n-1)\lambda^2 \iint \left(e^{-\theta y_2} - e^{-\theta y_1}\right) \frac{(y_2-y_1)^2 e^{-\theta y_2} e^{-\theta y_1}}{(e^{-\theta y_2} - e^{-\theta y_1})^2} e^{-\theta(y_1+y_2)} \\ \times [e^{-\theta y_2} + 1 - e^{-\theta y_1}]^{n-2} dy_1 dy_2 \end{array} \right\}$$

$$= k \left\{ \begin{array}{c} n\lambda \int_0^\infty y_1^2 e^{(-\theta+n\lambda)y_1} \left\{ \sum_{j=0}^{\infty} (e^{-\theta y_1})^j \right\} dy_1 \\ \\ +n(n-1)\lambda^2 \iint \left(e^{-\theta y_2} - e^{-\theta y_1}\right) \frac{(y_2-y_1)^2 e^{-\theta y_2} e^{-\theta y_1}}{(e^{-\theta y_2} - e^{-\theta y_1})^2} e^{-\theta(y_1+y_2)} \\ \times [e^{-\theta y_2} + 1 - e^{-\theta y_1}]^{n-2} dy_1 dy_2 \end{array} \right\}$$

$$= k \left\{ \begin{array}{c} n\lambda \sum_{j=0}^{\infty} \int_0^\infty y_1^2 e^{(-\theta+n\lambda+j\theta)y_1} dy_1 \\ \\ +n(n-1)\lambda^2 \sum_{k=0}^{\infty} \left\{ \sum_{j=0}^{n-2} \binom{n-2}{j} \sum_{l=0}^{j} \frac{(-1)^{j+1}}{(2\lambda+\theta+j\lambda)} \binom{j}{l} \frac{1}{(\theta+\lambda+l\theta+\lambda l)^3} \right\} \end{array} \right\}$$

$$= k \left\{ \begin{array}{c} n\lambda \sum_{j=0}^{\infty} \frac{1}{(\theta+n\lambda+j\theta)^3} \\ \\ +n(n-1)\lambda^2 \sum_{k=0}^{\infty} \left\{ \sum_{j=0}^{n-2} \binom{n-2}{j} \sum_{l=0}^{j} \frac{(-1)^{j+1}}{(2\lambda+\theta+j\lambda)} \binom{j}{l} \frac{1}{(\theta+\lambda+l\theta+\lambda l)^3} \right\} \end{array} \right\}.$$

By using power series in our derivation, we obtain Fisher information number as,

$$I(\theta) = k \Bigg[n\lambda \sum_{j=0}^{\infty} \frac{1}{(\theta+n\lambda+j\theta)^3} + n(n-1)\lambda^2 \sum_{l=0}^{\infty} \left\{ \sum_{j=0}^{n-2} \binom{n-2}{j} \sum_{l=0}^{j} \frac{(-1)^{j+1}}{(2\lambda+\theta+j\lambda)} \binom{j}{l} \frac{1}{(\theta+\lambda+l\theta+\lambda l)^3} \right\} \Bigg]. \tag{3.21}$$

The equation has a very simple form. Thus, $I(\theta)$ can be obtained using any statistical software for different values of θ.

3.2.2 Simulation Results

To illustrate the concept, a Monte Carlo simulation with $10,000$ repetitions is conducted under exponential distribution with parameter $\theta = 1$ where $\hat{\theta}$ is calculated from equation (3.18). Table 3.1 shows the estimated parameters for different attendance probabilities (q) and study periods (n). The proposed approach is easy to implement and has a very fast convergence rate. We can clearly see from Table 3.1 that the simulated variance and the simulated MSE values decrease as we increase study period n, subject attendance probability q and sample size k.

Table 3.1: Values of estimates for various study periods (n) and attendance probabilities (q); $T \sim Exp(1)$

sample size	q	n	$\hat{\theta}$	$MSE(\hat{\theta})$	$Var(\hat{\theta})$	$(I(\hat{\theta}))^{-1}$
k=50	0.6	12	0.825	0.0571	0.0251	0.0266
		36	0.839	0.0553	0.0270	0.0278
	0.8	12	0.855	0.0501	0.0233	0.0246
		36	0.861	0.0483	0.0219	0.0237
k=100	0.6	12	0.862	0.0479	0.0189	0.0208
		36	0.870	0.0476	0.0182	0.0196
	0.8	12	0.863	0.0469	0.0187	0.0194
		36	0.872	0.0461	0.0177	0.0189
k=200	0.6	12	0.887	0.0433	0.0132	0.0143
		36	0.890	0.0424	0.0128	0.0136
	0.8	12	0.894	0.0421	0.0124	0.0133
		36	0.899	0.0419	0.0120	0.0128
k=500	0.6	12	0.903	0.0316	0.0119	0.0124
		36	0.909	0.0269	0.0116	0.0119
	0.8	12	0.911	0.0251	0.0115	0.0117
		36	0.916	0.0229	0.0103	0.0108
k=1000	0.6	12	0.920	0.0207	0.0086	0.0093
		36	0.931	0.0119	0.0078	0.0085
	0.8	12	0.945	0.0108	0.0074	0.0078
		36	0.964	0.0102	0.0066	0.0073

3.3 Convergence Rate of the MMLE

Since the estimator $\hat{\theta}$ in (3.18) is obtained by using MML method, we know its consistency and asymptotic normality from the previous works; see Tiku [41], Tiku and Akkaya [1] and Senoglu and Tiku [38]. To discuss the rate of convergence of the estimator, write

$$\lim_{k\to\infty} \sup_{-\infty<x<\infty} P\left\{\frac{\hat{\theta}_k - \theta}{se(\hat{\theta}_k)} \leq x\right\} = N_{(0,1)}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-t^2}{2}} dt \tag{3.22}$$

where $\hat{\theta} \equiv \hat{\theta}_k$ is the MLE and $se(\hat{\theta}_k) = \sqrt{Var(\hat{\theta}_k)}$. That is,

$$\frac{\hat{\theta}_k - \theta}{se(\hat{\theta}_k)} = \zeta_k \tag{3.23}$$

where $\zeta_k \xrightarrow{d} \zeta$, and ζ is a random variable having the standard normal distribution; see Wasserman [45].

It is known that

$$se(\hat{\theta}) = \sqrt{\frac{1}{I_k(\hat{\theta}_k)}}. \tag{3.24}$$

We can write from (3.21) that

$$I_k(\hat{\theta}_k) = kC \tag{3.25}$$

where k is the sample size and C is a constant with respect to k. By applying the result given in (3.22), we conclude that

$$\frac{\hat{\theta}_k - \theta}{\sqrt{\frac{1}{I_k(\hat{\theta}_k)}}} = \zeta_k \xrightarrow{d} \zeta \tag{3.26}$$

where ζ is a random variable having the standard normal distribution. Then, it is straightforward to write

$$\frac{\hat{\theta}_k - \theta}{\sqrt{\frac{1}{kC}}} = \zeta_k \xrightarrow{d} \zeta \quad . \tag{3.27}$$

It follows that,

$$k^{\alpha}(\hat{\theta}_k - \theta) = \zeta k^{(\alpha - \frac{1}{2})} \to 0 \quad \text{when} \quad k \to \infty \tag{3.28}$$

where $0 < \alpha < \frac{1}{2}$. In other words, one can easily denote the convergence rate as

$$\hat{\theta}_k - \theta = o_p(k^{-\alpha}), \quad 0 < \alpha < \frac{1}{2} \tag{3.29}$$

since

$$\frac{(\hat{\theta}_k - \theta)}{k^{-\alpha}} \xrightarrow{p} 0 \quad \text{when,} \quad k \to \infty; \quad 0 < \alpha < \frac{1}{2} \quad . \tag{3.30}$$

This means that the MMLE converges to the true value but has a smaller convergence rate compared to the regular MLE. In fact, convergence rate of the ML estimators can be slower than the usual $\sqrt{k}$ convergence rate for censored data. Detailed discussion can be found in Sun [39].

CHAPTER 4

EXTENDING THE THEORY TO BIVARIATE INTERVAL CENSORING WITH RANDOM INTERVALS

Bivariate interval censored failure time data can be seen mostly in clinical examination tests where failure time is known to lie within an interval instead of being observed directly and each subject has bivariate events [18]. In the literature, many authors have analyzed bivariate failure time data in case of right censoring. Examples of such studies are Cai and Kim [5], Cai and Prentice [6], Li and Lagakos [26], Lin and Ying [27]. The main difficulty of analyzing bivariate interval censored data is due to the "correlation structure" between two related variable of interest. In this chapter, "estimation of the association parameter" between two related failure variables is discussed [39].

4.1 Estimation of the Association Parameter

Let T_1 and T_2 be two correlated failure times. The association of T_1 and T_2 is the main concern for studies on analysis of bivariate failure time data. A number of approaches have been presented for the analysis of the association for right censored data. Copula model is one of the common approaches for modelling the joint distribution of the T_1 and T_2. The advantage of using copula method is that it provides a convenient way to express the joint distribution of two or more random variables. One can find detailed information about these approaches in Sun [39].

In this study, the association between T_1 and T_2 is modeled using the copula method. The proposed model is more general than Sun [39] for making inference about the association in interval censoring.

4.1.1 The Copula Model and the Likelihood Function

Let $F_1(t)$ and $F_2(t)$ be the marginal distribution functions of T_1 and T_2, respectively and $F(t_1, t_2)$ be their joint distribution function. A copula model assumes that $F(t_1, t_2)$ can be expressed as

$$F(t_1, t_2) = C_\alpha(F_1(t_1), F_2(t_2))$$

where C_α is a distribution function on the unit square and $\alpha \in R$ is a global association parameter. Assume that there are two pairs of random variables $(X^{(1)}, X^{(2)})$ and $(Y^{(1)}, Y^{(2)})$ showing the inspection points for T_1 and T_2, respectively. The exact values of T_1 and T_2 are not known but they fall into some intervals $(X^{(1)}, X^{(2)})$ and $(Y^{(1)}, Y^{(2)})$.

Let us define the indicator functions as

$$\Delta_1^x = I\left(T_1 < X^{(1)})\right) = \begin{cases} 1 & \text{if} \quad T_1 < X^{(1)} \\ 0 & \text{if} \quad o.w. \end{cases} ,$$

$$\Delta_2^x = I\left(X^{(1)}) < T_1 < X^{(2)}\right) = \begin{cases} 1 & \text{if} \quad X^{(1)}) < T_1 < X^{(2)} \\ 0 & \text{if} \quad o.w. \end{cases} ,$$

$$\Delta_3^x = I\left(T_1 > X^{(2)}\right) = \begin{cases} 1 & \text{if} \quad T_1 > X^{(2)} \\ 0 & \text{if} \quad o.w. \end{cases} = 1 - \Delta_1^x - \Delta_2^x \quad ,$$

$$\Delta_1^y = I\left(T_2 < Y^{(1)})\right) = \begin{cases} 1 & \text{if} \quad T_2 < Y^{(1)} \\ 0 & \text{if} \quad o.w. \end{cases} ,$$

$$\Delta_2^y = I\left(Y^{(1)} < T_2 < Y^{(2)}\right) = \begin{cases} 1 & \text{if} \quad Y^{(1)}) < T_2 < Y^{(2)} \\ 0 & \text{if} \quad o.w. \end{cases}$$

and

$$\Delta_3^y = I\left(T_2 > Y^{(2)}\right) = \begin{cases} 1 & \text{if} \quad T_2 > Y^{(2)} \\ 0 & \text{if} \quad o.w. \end{cases} = 1 - \Delta_1^y - \Delta_2^y$$

The we have the following conditions

$$L\left(\theta | x^{(1)}, x^{(2)}, y^{(1)}, y^{(2)}, \delta_1^x, \delta_1^x, \delta_2^x, \delta_3^x, \delta_1^y, \delta_2^y, \delta_3^y\right) =$$

$$= \begin{cases} f(x^{(1)}, x^{(2)}, y^{(1)}), y^{(2)})P\left\{T_1 < x^{(1)}, T_2 < y^{(1)}\right\}, & \delta_1^x = 1, \delta_1^y = 1 \\ f(x^{(1)}, x^{(2)}, y^{(1)}, y^{(2)})P\left\{T_1 < x^{(1)}, y^{(1)} < T_2 < y^{(2)}\right\}, & \delta_1^x = 1, \delta_2^y = 1 \\ f(x^{(1)}, x^{(2)}, y^{(1)}, y^{(2)})P\left\{T_1 < x^{(1)}, T_2 > y^{(2)}\right\}, & \delta_1^x = 1, \delta_3^y = 1 \\ f(x^{(1)}, x^{(2)}, y^{(1)}, y^{(2)})P\left\{x^{(1)} < T_1 < x^{(2)}, T_2 < y^{(1)}\right\}, & \delta_2^x = 1, \delta_2^y = 1 \\ f(x^{(1)}, x^{(2)}, y^{(1)}, y^{(2)})P\left\{x^{(1)} < T_1 < x^{(2)}, y^{(1)} < T_2 < y^{(2)}\right\}, & \delta_2^x = 1, \delta_3^y = 1 \\ f(x^{(1)}, x^{(2)}, y^{(1)}, y^{(2)})P\left\{x^{(1)} < T_1 < x^{(2)}, T_2 > y^{(2)}\right\}, & \delta_2^x = 1, \delta_3^y = 1 \\ f(x^{(1)}, x^{(2)}, y^{(1)}), y^{(2)})P\left\{T_1 > x^{(1)}, T_2 < y^{(1)}\right\}, & \delta_3^x = 1, \delta_1^y = 1 \\ f(x^{(1)}, x^{(2)}, y^{(1)}), y^{(2)})P\left\{T_1 > x^{(1)}, y^{(1)} < T_2 < y^{(2)}\right\}, & \delta_3^x = 1, \delta_2^y = 1 \\ f(x^{(1)}, x^{(2)}, y^{(1)}), y^{(2)})P\left\{T_1 > x^{(1)}, T_2 < y^{(1)}\right\}, & \delta_3^x = 1, \delta_3^y = 1 \end{cases}$$

The likelihood function is guven by

$$
\begin{aligned}
&L\left(\theta|x^{(1)},x^{(2)},y^{(1)},y^{(2)},\delta_1^x,\delta_1^x,\delta_2^x,\delta_3^x,\delta_1^y,\delta_2^y,\delta_3^y\right) = \\
&= \prod_{i=1}^{n}\Bigg\{ f(x_i^{(1)},x_i^{(2)},y_i^{(1)},y_i^{(2)})F_T\left(x_i^{(1)},y_i^{(1)}\right)\times\delta_{1i}^x\delta_{1i}^y \\
&+ f(x_i^{(1)},x_i^{(2)},y_i^{(1)},y_i^{(2)})\left\{F_T\left(x_i^{(1)},y_i^{(2)}\right)-F_T\left(x_i^{(1)},y_i^{(1)}\right)\right\}\times\delta_{1i}^x\delta_{2i}^y \\
&+ f(x_i^{(1)},x_i^{(2)},y_i^{(1)},y_i^{(2)})\left\{F_{T_1}\left(x_i^{(1)}\right)-F_T\left(x_i^{(1)},y_i^{(2)}\right)\right\} \\
&\times\delta_{1i}^x(1-\delta_{1i}^y-\delta_{2i}^y) \\
&+ f(x_i^{(1)},x_i^{(2)},y_i^{(1)},y_i^{(2)})\left\{F_T\left(x_i^{(1)},y_i^{(1)}\right)-F_T\left(x_i^{(2)},y_i^{(1)}\right)\right\} \\
&\times\delta_{2i}^x\delta_{1i}^y \\
&+ f(x_i^{(1)},x_i^{(2)},y_i^{(1)},y_i^{(2)})\Big\{F_T\left(x_i^{(1)},y_i^{(1)}\right)-F_T\left(x_i^{(1)},y_i^{(2)}\right) \\
&- F_T\left(x_i^{(2)},y_i^{(1)}\right)-F_T\left(x_i^{(2)},y_i^{(2)}\right)\Big\}\delta_{2i}^x\delta_{2i}^y \\
&+ f(x_i^{(1)},x_i^{(2)},y_i^{(1)},y_i^{(2)})\Big\{F_{T_1}\left(x_i^{(2)}\right)-F_T\left(x_i^{(2)},y_i^{(2)}\right)- \\
&- F_{T_1}\left(x_i^{(1)}\right)-F_T\left(x_i^{(1)},y_i^{(2)}\right)\Big\}\delta_{2i}^x(1-\delta_{1i}^y-\delta_{2i}^y) \\
&+ f(x_i^{(1)},x_i^{(2)},y_i^{(1)},y_i^{(2)})\left\{F_{T_2}\left(y_i^{(1)}\right)-F_T\left(x_i^{(2)},y_i^{(2)}\right)\right\} \\
&\times(1-\delta_{1i}^x-\delta_{2i}^x)\delta_{1i}^y \\
&+ f(x_i^{(1)},x_i^{(2)},y_i^{(1)},y_i^{(2)})\Big\{F_{T_2}\left(y_i^{(2)}\right)-F_T\left(x_i^{(2)},y_i^{(2)}\right)- \\
&- F_{T_2}\left(y_i^{(1)}\right)-F_T\left(x_i^{(2)},y_i^{(1)}\right)\Big\} \\
&\times(1-\delta_{1i}^x-\delta_{2i}^x)\delta_{2i}^y \\
&+ f(x_i^{(1)},x_i^{(2)},y_i^{(1)},y_i^{(2)})\left\{1-F_T\left(x_i^{(2)},y_i^{(2)}\right)\right\} \\
&\times(1-\delta_{1i}^x-\delta_{2i}^x)(1-\delta_{1i}^y-\delta_{2i}^y)\Bigg\} \quad . \qquad (4.1)
\end{aligned}
$$

Let,

$$F_{11}(\alpha,x,y)=F_T\left(x_i^{(1)},y_i^{(1)}\right)=C\Big(F_{T_1}(x_i^{(1)}),F_{T_2}(y_i^{(1)})\Big) \quad ,$$

$$
\begin{aligned}
F_{12}(\alpha,x,y) &= F_T\left(x_i^{(1)},y_i^{(2)}\right) - F_T\left(x_i^{(1)},y_i^{(1)}\right) \\
&= C\Big(F_{T_1}(x_i^{(1)}),F_{T_2}(y_i^{(2)})\Big) - C\Big(F_{T_1}(x_i^{(1)}),F_{T_2}(y_i^{(1)})\Big) \quad ,
\end{aligned}
$$

$$
\begin{aligned}
F_{13}(\alpha,x,y) &= F_{T_1}\left(x_i^{(1)}\right) - F_T\left(x_i^{(1)},y_i^{(2)}\right) \\
&= F_{T_1}\left(x_i^{(1)}\right) - C\Big(F_{T_1}(x_i^{(1)}),F_{T_2}(y_i^{(2)})\Big) \quad ,
\end{aligned}
$$

$$
\begin{aligned}
F_{21}(\alpha,x,y) &= F_T\left(x_i^{(1)},y_i^{(1)}\right) - F_T\left(x_i^{(2)},y_i^{(1)}\right) \\
&= C\Big(F_{T_1}(x_i^{(1)}),F_{T_2}(y_i^{(1)})\Big) - C\Big(F_{T_1}(x_i^{(2)}),F_{T_2}(y_i^{(1)})\Big) \quad ,
\end{aligned}
$$

$$
\begin{aligned}
F_{22}(\alpha,x,y) &= F_T\left(x_i^{(1)},y_i^{(1)}\right) - F_T\left(x_i^{(1)},y_i^{(2)}\right) - F_T\left(x_i^{(2)},y_i^{(1)}\right) + F_T\left(x_i^{(2)},y_i^{(2)}\right) \\
&= C\Big(F_{T_1}(x_i^{(1)}),F_{T_2}(y_i^{(1)})\Big) - C\Big(F_{T_1}(x_i^{(1)}),F_{T_2}(y_i^{(2)})\Big) \\
&\quad - C\Big(F_{T_1}(x_i^{(2)}),F_{T_2}(y_i^{(1)})\Big) - C\Big(F_{T_1}(x_i^{(2)}),F_{T_2}(y_i^{(2)})\Big) \quad ,
\end{aligned}
$$

$$
\begin{aligned}
F_{23}(\alpha,x,y) &= F_{T_1}\left(x_i^{(2)}\right) - F_T\left(x_i^{(2)},y_i^{(2)}\right) - F_{T_1}\left(x_i^{(1)}\right) + F_T\left(x_i^{(2)},y_i^{(2)}\right) \\
&= F_{T_1}\left(x_i^{(2)}\right) - C\Big(F_{T_1}(x_i^{(2)}),F_{T_2}(y_i^{(2)})\Big) \\
&\quad - F_{T_1}\left(x_i^{(1)}\right) - C\Big(F_{T_1}(x_i^{(1)}),F_{T_2}(y_i^{(2)})\Big) \quad ,
\end{aligned}
$$

$$
\begin{aligned}
F_{32}(\alpha,x,y) &= F_{T_2}\left(y_i^{(2)}\right) - F_T\left(x_i^{(2)},y_i^{(2)}\right) \\
&= F_{T_2}\left(y_i^{(2)}\right) - C\Big(F_{T_1}(x_i^{(2)}),F_{T_2}(y_i^{(2)})\Big) \quad ,
\end{aligned}
$$

$$
\begin{aligned}
F_{32}(\alpha,x,y) &= F_{T_2}\left(y_i^{(2)}\right) - F_T\left(x_i^{(2)},y_i^{(2)}\right) - F_{T_2}\left(y_i^{(1)}\right) + F_T\left(x_i^{(2)},y_i^{(1)}\right)\\
&= F_{T_2}\left(y_i^{(1)}\right) - C\Big(F_{T_1}(x_i^{(2)}),F_{T_2}(y_i^{(2)})\Big)\\
&- F_{T_2}\left(y_i^{(1)}\right) + C\Big(F_{T_1}(x_i^{(2)}),F_{T_2}(y_i^{(1)})
\end{aligned}
$$

and

$$
\begin{aligned}
F_{33}(\alpha,x,y) &= 1 - F_T\left(x_i^{(2)},y_i^{(2)}\right)\\
&= 1 - C\Big(F_{T_1}(x_i^{(2)}),F_{T_2}(y_i^{(1)}) \quad .
\end{aligned}
$$

Substituting F_{11}, F_{12}, F_{13}, F_{21}, F_{22}, F_{23}, F_{31}, F_{32} and F_{33} in (4.1) we get

$$
\begin{aligned}
&L\left(\theta|x^{(1)},x^{(2)},y^{(1)},y^{(2)},\delta_1^{(x)},\delta_1^{(x)},\delta_2^{(x)},\delta_3^{(x)},\delta_1^{(y)},\delta_2^{(y)},\delta_3^{(y)}\right) =\\
&= \prod_{i=1}^{n}\Bigg\{ f(x_i^{(1)},x_i^{(2)},y_i^{(1)},y_i^{(2)})F_{11}(\alpha,\mathbf{x},\mathbf{y})\delta_{1i}^{(x)}\delta_{1i}^{(y)} + f(x_i^{(1)},x_i^{(2)},y_i^{(1)},y_i^{(2)})F_{12}(\alpha,\mathbf{x},\mathbf{y})\delta_{1i}^{(x)}\delta_{2i}^{(y)}\\
&\qquad + f(x_i^{(1)},x_i^{(2)},y_i^{(1)},y_i^{(2)})F_{13}(\alpha,\mathbf{x},\mathbf{y})\delta_{1i}^{(x)}(1-\delta_{1i}^{(y)}-\delta_{2i}^{(y)})\\
&+ f(x_i^{(1)},x_i^{(2)},y_i^{(1)},y_i^{(2)})F_{21}(\alpha,\mathbf{x},\mathbf{y})\delta_{2i}^{(x)}\delta_{1i}^{(y)} + f(x_i^{(1)},x_i^{(2)},y_i^{(1)},y_i^{(2)})F_{22}(\alpha,\mathbf{x},\mathbf{y})\delta_{2i}^{(x)}\delta_{2i}^{(y)}\\
&\qquad + f(x_i^{(1)},x_i^{(2)},y_i^{(1)},y_i^{(2)})F_{23}(\alpha,\mathbf{x},\mathbf{y})\delta_{2i}^{(x)}(1-\delta_{1i}^{(y)}-\delta_{2i}^{(y)})\\
&\qquad + f(x_i^{(1)},x_i^{(2)},y_i^{(1)},y_i^{(2)})F_{31}(\alpha,\mathbf{x},\mathbf{y})(1-\delta_{1i}^{(x)}-\delta_{2i}^{(x)})\delta_{1i}^{(y)}\\
&\qquad + f(x_i^{(1)},x_i^{(2)},y_i^{(1)},y_i^{(2)})F_{32}(\alpha,\mathbf{x},\mathbf{y})(1-\delta_{1i}^{(x)}-\delta_{2i}^{(x)})\delta_{2i}^{(y)}\\
&\qquad + f(x_i^{(1)},x_i^{(2)},y_i^{(1)},y_i^{(2)})F_{33}(\alpha,\mathbf{x},\mathbf{y})\delta_{2i}^{(x)}(1-\delta_{1i}^{(y)}-\delta_{2i}^{(y)})\Bigg\} \quad . \qquad (4.2)
\end{aligned}
$$

The simplified form of the likelihood function (4.2) is obtained as

$$L\left(\theta|x^{(1)},x^{(2)},y^{(1)},y^{(2)},\delta_1^{(x)},\delta_1^{(x)},\delta_2^{(x)},\delta_3^{(x)},\delta_1^{(y)},\delta_2^{(y)},\delta_3^{(y)}\right)=$$

$$=\prod_{i=1}^{n} f(x_i^{(1)},x_i^{(2)},y_i^{(1)},y_i^{(2)})\Bigg\{F_{11}(\alpha,\mathbf{x{,}y})\delta_{1i}^{(x)}\delta_{1i}^{(y)}$$

$$+F_{12}(\alpha,\mathbf{x{,}y})\delta_{1i}^{(x)}\delta_{2i}^{(y)}+F_{13}(\alpha,\mathbf{x{,}y})\delta_{1i}^{(x)}(1-\delta_{1i}^{(y)}-\delta_{2i}^{(y)})+F_{21}(\alpha,\mathbf{x{,}y})\delta_{2i}^{(x)}\delta_{1i}^{(y)}$$

$$+F_{22}(\alpha,\mathbf{x{,}y})\delta_{2i}^{(x)}\delta_{2i}^{(y)}+F_{23}(\alpha,\mathbf{x{,}y})\delta_{2i}^{(x)}(1-\delta_{1i}^{(y)}-\delta_{2i}^{(y)})+F_{31}(\alpha,\mathbf{x{,}y})(1-\delta_{1i}^{(x)}-\delta_{2i}^{(x)})\delta_{1i}^{(y)}$$

$$+F_{32}(\alpha,\mathbf{x{,}y})(1-\delta_{1i}^{(x)}-\delta_{2i}^{(x)})\delta_{2i}^{(y)}+F_{33}(\alpha,\mathbf{x{,}y})\delta_{2i}^{(x)}(1-\delta_{1i}^{(y)}-\delta_{2i}^{(y)})\Bigg\} \quad . \qquad (4.3)$$

Given the likelihood function (4.3), one can assume any copula functions with appropriate marginal distribution functions, which leads to various types of bivariate interval censored models. In this chapter, we will not focus on any specified copula functions as well as marginal distributions so as to let readers do this as a further application. The common way to obtain the MLE of dependency parameter α is to maximize the likelihood function

$$L\left(\theta|x^{(1)},x^{(2)},y^{(1)},y^{(2)},\delta_1^{(x)},\delta_1^{(x)},\delta_2^{(x)},\delta_3^{(x)},\delta_1^{(y)},\delta_2^{(y)},\delta_3^{(y)}\right),$$

with respect to unknown parameters of F_{T_1} and F_{T_2} and α, simultaneously. However, this method with complex mathematical expressions results in longer run times. On the other hand, it is quite simpler to deal with copula approach that allows us to separately estimate unknown parameters of specified marginal distributions and association parameter of copula. In practice, this can easily be achieved by estimating unknown parameters of marginal distributions first, followed by estimation of dependency parameter α.

CHAPTER 5

CONCLUSION AND FURTHER RESEARCH

Estimation of unknown parameters of statistical distributions based on interval censored data is one of the most important problems facing in medical and health studies, reliability and life testing studies. Interval censoring often occurs when individuals or components in a study are inspected intermittently so that variable of interest is observed to lie between successive points.

Two types of interval censoring that commonly occur in practice, are considered in this study. We have proposed using Modified Maximum Likelihood Estimation (MML) and Copula Methods for the estimation procedure of unknown parameters of variable of interest in case of interval censoring. To evaluate how accurate the approximations are and to see whether the applied method is correct or not, some numerical calculations are done for numerous attendance probabilities and study periods for interval censored data with fixed and random intervals. The results from Monte Carlo simulation runs are used to examine the MSE values of the parameter estimates. It appears that estimates are easily obtained and proposed methods seem to provide fairly accurate estimates. As a conclusion, it seems to be reasonable to use the MML and Copula model for computing the parameter estimates.

It is also important to decide the choice of the copula. In this thesis, the min-max copula is proposed for the estimation due to the structure of the model. We present a simple way of assessing the value of the dependence parameter of the min-max copula.

We also considered bivariate interval censored data. Bivariate interval censoring can occur when the outcomes are not directly observable but are detected from periodic examination points. In bivariate interval censoring, each subject may experience bivariate events. To estimate the association between two variable of interest we focus on the situation where they follow a copula model.

For skewed interval censored data structures, estimation for unknown parameters of variable of interest can be considered as a further research. This problem can also be extended to multivariate distributions. In addition to these, inference based on copula models can be an interesting research topic for both univariate and multivariate interval censoring.

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APPENDIX A

MML ESTIMATORS BASED ON FIXED INTERVAL CENSORED DATA

$$
\begin{aligned}
L(\mu,\sigma) &= \prod_{i=1}^{n}\{F_{\mu,\sigma}(b_i) - F_{\mu,\sigma}(a_i)\} \\
&= \prod_{i=1}^{n}\{P\{T_i \le b_i\} - P\{T_i \le a_i\}\} \\
&= \prod_{i=1}^{n}\left\{P\left\{\frac{T_i-\mu}{\sigma} \le \frac{b_i-\mu}{\sigma}\right\} - P\left\{\frac{T_i-\mu}{\sigma} \le \frac{a_i-\mu}{\sigma}\right\}\right\} \\
&= \prod_{i=1}^{n}\left\{F\left\{\frac{b_i-\mu}{\sigma}\right\} - F\left\{\frac{a_i-\mu}{\sigma}\right\}\right\} \\
&= \prod_{i=1}^{n}\{F(b_i^*) - F(a_i^*)\}
\end{aligned}
$$

where

$$a_i^* = \frac{a_i-\mu}{\sigma} \quad \text{and} \quad b_i^* = \frac{b_i-\mu}{\sigma} \quad .$$

On taking logarithms of likelihood L, we get

$$
\begin{aligned}
lnL(\mu,\sigma) &= \sum_{i=1}^{n} ln\left[F(a_i^*) - F(a_i^*)\right] \\
&= \sum_{i=1}^{n_1} lnF(a_i^*) + \sum_{i=n_1+1}^{n_2} ln\left[F(b_i^*) - F(a_i^*)\right] + \sum_{i=n_2+1}^{n} ln\left[1 - F(b_i^*)\right] \quad .
\end{aligned}
$$

Then,

$$lnL(\mu,\sigma) = \sum_{i=1}^{n_1} lnF\left(\frac{a_i-\mu}{\sigma}\right) + \sum_{i=n_1+1}^{n_2} ln\left[F\left(\frac{b_i-\mu}{\sigma}\right) - F\left(\frac{a_i-\mu}{\sigma}\right)\right]$$
$$+ \sum_{i=n_2+1}^{n} ln\left[1 - F\left(\frac{b_i-\mu}{\sigma}\right)\right] = 0 \quad .$$

Here, n_1 and $n - n_2$ are the numbers of observations between $(-\infty, a_i)$ and (b_i, ∞) respectively.

Maximizing $lnL(\mu,\sigma)$ with respect to μ and σ we get,

$$\frac{\partial lnL(\mu,\sigma)}{\partial\sigma} = \sum_{i=1}^{n_1}\left(\frac{-1}{\sigma}\right)\frac{f(a_i^*)}{F(a_i^*)}$$
$$+ \sum_{i=n_1+1}^{n_2}\left(\frac{-1}{\sigma}\right)\frac{f(b_i^*) - f(a_i^*)}{F(b_i^*) - F(a_i^*)}$$
$$+ \sum_{i=n_2+1}^{n}\left(\frac{-1}{\sigma}\right)\frac{f(b_i^*)}{1 - F(b_i^*)} = 0$$

and

$$\frac{\partial lnL(\mu,\sigma)}{\partial\mu} = -\frac{n}{\sigma} + \sum_{i=1}^{n_1}\left(\frac{-1}{\sigma}\right)\frac{f(a_i^*)}{F(a_i^*)}a_i^*$$
$$+ \sum_{i=n_1+1}^{n_2}\left(\frac{-1}{\sigma}\right)\frac{f(b_i^*)\,b_i^* - f(a_i^*)\,a_i^*}{F(b_i^*) - F(a_i^*)}$$
$$+ \sum_{i=n_2+1}^{n}\left(\frac{-1}{\sigma}\right)\frac{f(b_i^*)}{1 - F(b_i^*)}b_i^* = 0$$

respectively. Consider following linear approximations:

$$g(a_i) = \frac{f(a_i^*)}{F(a_i^*)} = \nu_{i1} + \upsilon_{i1}a_i^*$$
$$g_1(a_i, b_i) = \frac{f(b_i^*) - f(a_i^*)}{F(b_i^*) - F(a_i^*)} = \alpha_{i2} + \beta_{i2}b_i^* - \alpha_{i1} - \beta_{i1}a_i^*$$
$$g_2(b_i) = \frac{f(b_i^*)}{1 - F(b_i^*)} = \nu_{i2} + \upsilon_{i2}b_i^*$$

Then,

$$\begin{aligned}
\frac{\partial lnL(\mu,\sigma)}{\partial\mu} &= \sum_{i=1}^{n_1}(\nu_{i1}+\upsilon_{i1}a_i^*) + \sum_{i=n_1+1}^{n_2}(\alpha_{i2}+\beta_{i2}b_i^*-\alpha_{i1}-\beta_{i1}a_i^*) \\
&- \sum_{i=n_2+1}^{n}(\nu_{i2}+\upsilon_{i2}b_i^*) = 0 \\
&= \sum_{i=1}^{n_1}(\sigma\nu_{i1}+\upsilon_{i1}(a_i-\mu)) \\
&+ \sum_{i=n_1+1}^{n_2}(\sigma\alpha_{i2}+\beta_{i2}(b_i-\mu)-\sigma\alpha_{i1}-\beta_{i1}(a_i-\mu)) \\
&- \sum_{i=n_2+1}^{n}(\sigma\nu_{i2}+\upsilon_{i2}(b_i-\mu)) = 0 \\
&= \sigma\left[\sum_{i=1}^{n_1}\nu_{i1} + \sum_{i=n_1+1}^{n_2}(\alpha_{i2}-\alpha_{i1}) - \sum_{i=n_2+1}^{n}\nu_{i2}\right] \\
&- \mu\left[\sum_{i=n_1+1}^{n_2}\upsilon_{i1} + \sum_{i=n_1+1}^{n_2}(\beta_{i2}-\beta_{i1}) - \upsilon_{i2}\right] \\
&+ \left[\sum_{i=1}^{n_1}\upsilon_{i1}a_i + \sum_{i=n_1+1}^{n_2}(\beta_{i2}b_i-\beta_{i1}a_i) - \sum_{i=n_2+1}^{n}\upsilon_{i2}b_i\right] \quad .
\end{aligned}$$

The MML estimator $\hat{\mu}$ is obtained to be

$$\hat{\mu} = A + B\tilde{\sigma} \tag{A.1}$$

where,

$$A = \frac{\sum_{i=1}^{n_1}\upsilon_{i1}a_i + \sum_{i=n_1+1}^{n_2}(\beta_{i2}b_i-\beta_{i1}a_i) - \sum_{i=n_2+1}^{n}\upsilon_{i2}b_i}{\sum_{i=1}^{n_1}\upsilon_{i1} + \sum_{i=n_1+1}^{n_2}(\beta_{i2}-\beta_{i1}) - \sum_{i=n_2+1}^{n}\upsilon_{i2}}$$

$$B = \frac{\sum_{i=1}^{n_1}\nu_{i1} + \sum_{i=n_1+1}^{n_2}(\alpha_{i2}-\alpha_{i1}) - \sum_{i=n_2+1}^{n}\nu_{i2}}{\sum_{i=1}^{n_1}\upsilon_{i1} + \sum_{i=n_1+1}^{n_2}(\beta_{i2}-\beta_{i1}) - \sum_{i=n_2+1}^{n}\upsilon_{i2}} \quad .$$

$$
\begin{aligned}
\frac{\partial lnL(\mu,\sigma)}{\partial\sigma} &= -\frac{n}{\sigma}-\frac{1}{\sigma}\left[\sum_{i=1}^{n_1}\left(\nu_{i1}+\upsilon_{i1}\left(\frac{a_i-\mu}{\sigma}\right)\right)\left(\frac{a_i-\mu}{\sigma}\right)\right] \\
&-\frac{1}{\sigma}\left[\sum_{i=n_1+1}^{n_2}\left(\alpha_{i2}+\beta_{i2}\left(\frac{b_i-\mu}{\sigma}\right)\right)\left(\frac{b_i-\mu}{\sigma}\right)\right] \\
&+\frac{1}{\sigma}\left[\sum_{i=n_1+1}^{n_2}\left(\alpha_{i1}+\beta_{i1}\left(\frac{a_i-\mu}{\sigma}\right)\right)\left(\frac{a_i-\mu}{\sigma}\right)\right] \\
&-\frac{1}{\sigma}\left[\sum_{i=n_2+1}^{n}\left(\nu_{i2}+\upsilon_{i2}\left(\frac{b_i-\mu}{\sigma}\right)\right)\left(\frac{b_i-\mu}{\sigma}\right)\right]=0 \qquad \text{(A.2)} \\
&= n\sigma^2+\sigma\sum_{i=1}^{n_1}\nu_{i1}(a_i-A-B\sigma)+\upsilon_{i1}(a_i-A-B\sigma)^2 \\
&+\sigma\sum_{i=n_1+1}^{n_2}\alpha_{i2}(b_i-A-B\sigma)+\sum_{i=n_1+1}^{n_2}\beta_{i2}(b_i-A-B\sigma)^2 \\
&-\sigma\sum_{i=n_1+1}^{n_2}\alpha_{i1}(a_i-A-B\sigma)+\sum_{i=n_1+1}^{n_2}\beta_{i1}(a_i-A-B\sigma)^2 \\
&-\sigma\sum_{i=1}^{n_1}\nu_{i2}(b_i-A-B\sigma)+\upsilon_{i2}(b_i-A-B\sigma)^2
\end{aligned}
$$

Then the MML estimator $\hat{\sigma}$ is found to be

$$
\hat{\sigma}=\frac{-C+\sqrt{C^2+4nE}}{2n} \qquad \text{(A.3)}
$$

where

$$
C=\sigma\left[\sum_{i=1}^{n_1}\nu_{i1}(a_i-A)-\sum_{i=n_1+1}^{n_2}(\alpha_{i2}(b_i-A)-\alpha_{i1}(a_i-A))+\sum_{i=n_2+1}^{n}\nu_{i2}(b_i-A)\right]
$$

$$
E=\sigma\left[\sum_{i=1}^{n_1}\nu_{i1}(a_i-A)^2-\sum_{i=n_1+1}^{n_2}\left(\alpha_{i2}(b_i-A)^2-\alpha_{i1}(a_i-A)^2\right)+\sum_{i=n_2+1}^{n}\nu_{i2}(b_i-A)^2\right] \quad .
$$

APPENDIX B

MML ESTIMATORS BASED ON RANDOM INTERVAL CENSORED DATA AND FISHER INFORMATION NUMBER

B.1 MML ESTIMATORS

$$\begin{aligned} L(\theta) = \prod_{i=1}^{k} & \Big[F(Y_{i1})\Big]^{\delta_{i0}} \Big[F(Y_{i2}) - F(Y_{i2})\Big]^{\delta_{i1}} \Big[F(Y_{i3}) - F(Y_{i2})\Big]^{\delta_{i2}} \cdots \\ & \cdots\cdots \Big[F(Y_{ir}) - F(Y_{ir-1})\Big]^{\delta_{ir-1}} \cdots\cdots \\ & \cdots\cdots \Big[F(Y_{in_i}) - F(Y_{in_i-1})\Big]^{\delta_{in_i-1}} \Big[1 - F(Y_{in_i})\Big]^{\delta_{in_i}} n! \\ & \cdot n! g(y_{i1}) g(y_{i1}) \cdots g(y_{ir-1}) \cdots g(y_{in_i}) \quad . \end{aligned} \tag{B.1}$$

The first partial derivative of the log likelihood function is,

$$\begin{aligned} \frac{\partial L(\theta)}{\partial \theta} = \sum_{i=1}^{k} \Bigg\{ & \delta_{i0} ln(1 - e^{-\theta y_{i1}}) + \delta_{i1} ln(e^{-\theta y_{i1}} - e^{-\theta y_{i2}}) + \cdots\cdots \\ & \cdots\cdots + \delta_{ir-1} ln(e^{-\theta y_{ir-1}} - e^{-\theta y_{ir}}) + \cdots\cdots \\ & \cdots\cdots + \delta_{in_{i-1}} ln(e^{-\theta y_{in_{i-1}}} - e^{-\theta y_{in_i}}) + \delta_{in_i} ln(e^{-\theta y_{in_i}}) \\ & + ln(n!) + ln(\lambda^{n_i}) - \lambda y_{i1} - \lambda y_{i2} - \cdots\cdots - \lambda y_{in_i} \Bigg\} \end{aligned}$$

$$
\begin{aligned}
\frac{\partial L(\theta)}{\partial \theta} = \sum_{i=1}^{k} \Bigg\{ & \delta_{i0} \frac{e^{\theta y_{i1}} y_{i1}}{(1 - e^{-\theta y_{i1}})} + \delta_{i1} \frac{e^{\theta y_{i2}} y_{i2} - e^{\theta y_{i1}} y_{i1}}{(e^{-\theta y_{i1}} - e^{-\theta y_{i2}})} + \cdots\cdots \\
& \cdots\cdots + \delta_{ir-1} \frac{e^{\theta y_{ir}} y_{ir} - e^{\theta y_{ir-1}} y_{ir-1}}{(e^{-\theta y_{ir-1}} - e^{-\theta y_{ir}})} + \cdots\cdots \\
& \cdots\cdots + \delta_{in_{i-1}} \frac{e^{\theta y_{in_i}} y_{in_i} - e^{\theta y_{in_{i-1}}} y_{in_{i-1}}}{(e^{-\theta y_{in_{i-1}}} - e^{-\theta y_{in_i}})} - \delta_{in_i} \frac{e^{\theta y_{in_i}} y_{in_i}}{(e^{-\theta y_{in_i}})} \Bigg\} \quad .
\end{aligned} \tag{B.2}
$$

$$
\begin{aligned}
g_0\left(z_{(i)1}\right) &= \frac{e^{-z_{(i)1}}}{1 - e^{-z_{(i)1}}} \\
&= g_0\left(w_{(i)1}\right) + (z_{(i)1} - w_{(i)1}) \frac{\partial g_0\left(z_{(i)1}\right)}{\partial z}|_{z_{(i)1} = w_{(i)1}} \\
&= \alpha_{i0} + \beta_{i0} z_{(i)1}
\end{aligned} \tag{B.3}
$$

$$
\begin{aligned}
g_1\left(z_{(i)1}, z_{(i)2}\right) &= \frac{e^{-z_{(i)2}}}{e^{-z_{(i)1}} - e^{-z_{(i)2}}} \\
&= g_1\left(w_{(i)1}, w_{(i)2}\right) + (z_{(i)2} - w_{(i)2}) \frac{\partial g_1\left(z_{(i)1}, z_{(i)2}\right)}{\partial z_{(i)2}}|_{w_{(i)1}, w_{(i)2}} \\
&+ (z_{(i)1} - w_{(i)1}) \frac{\partial g_1\left(z_{(i)1}, z_{(i)2}\right)}{\partial z_{(i)1}}|_{w_{(i)1}, w_{(i)2}} \\
&= \alpha_{i1} + \beta_{i1} z_{(i)2} + \gamma_{i1} z_{(i)1}
\end{aligned} \tag{B.4}
$$

$$
\begin{aligned}
g_2\left(z_{(i)1}, z_{(i)2}\right) &= \frac{e^{-z_{(i)2}}}{e^{-z_{(i)1}} - e^{-z_{(i)2}}} \\
&= g_1\left(w_{(i)1}, w_{(i)2}\right) + (z_{(i)2} - w_{(i)2}) \frac{\partial g_1\left(z_{(i)1}, z_{(i)2}\right)}{\partial z_{(i)2}}|_{w_{(i)1}, w_{(i)2}} \\
&+ (z_{(i)1} - w_{(i)1}) \frac{\partial g_1\left(z_{(i)1}, z_{(i)2}\right)}{\partial z_{(i)1}}|_{w_{(i)1}, w_{(i)2}} \\
&= \alpha_{i2} + \beta_{i2} z_{(i)2} + \gamma_{i2} z_{(i)1}
\end{aligned} \tag{B.5}
$$

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.

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$$
\begin{aligned}
g_{r-1}\left(z_{(i)r-1}, z_{(i)r}\right) &= \frac{e^{-z_{(i)r}}}{e^{-z_{(i)r-1}} - e^{-z_{(i)r}}} \\
&= g_{r-1}\left(w_{(i)r-1}, w_{(i)r}\right) + (w_{(i)r} - t_{(i)r})\frac{\partial g_{r-1}\left(z_{(i)r-1}, z_{(i)r}\right)}{\partial z_{(i)r}}|_{w_{(i)r-1}, w_{(i)r}} \\
&+ (z_{(i)r-1} - w_{(i)r-1})\frac{\partial g_{r-1}\left(z_{(i)r-1}, z_{(i)r}\right)}{\partial z_{(i)r-1}}|_{w_{(i)r-1}, w_{(i)r}} \\
&= \alpha_{i2r-3} + \beta_{i2r-3} z_{(i)r} + \gamma_{i2r-3} z_{(i)r-1}
\end{aligned}
\tag{B.6}
$$

$$
\begin{aligned}
g_{r}\left(z_{(i)r-1}, z_{(i)r}\right) &= \frac{e^{-z_{(i)r-1}}}{e^{-z_{(i)r-1}} - e^{-z_{(i)r}}} \\
&= g_{r}\left(w_{(i)r-1}, w_{(i)r}\right) + (z_{(i)r} - w_{(i)r})\frac{\partial g_{r}\left(z_{(i)r-1}, z_{(i)r}\right)}{\partial z_{(i)r}}|_{w_{(i)r-1}, w_{(i)r}} \\
&+ (z_{(i)r-1} - w_{(i)r-1})\frac{\partial g_{r}\left(z_{(i)r-1}, z_{(i)r}\right)}{\partial z_{(i)r-1}}|_{w_{(i)r-1}, w_{(i)r}} \\
&= \alpha_{i2r-2} + \beta_{i2r-2} z_{(i)r} + \gamma_{i2r-2} z_{(i)r-1}
\end{aligned}
\tag{B.7}
$$

.

.

.

$$
\begin{aligned}
g_{n_i-1}\left(z_{(i)n_i-1}, z_{(i)n_i}\right) &= \frac{e^{-z_{(i)n_i}}}{e^{-z_{(i)n_i-1}} - e^{-z_{(i)n_i}}} \\
&= g_{n_i-1}\left(w_{(i)n_i-1}, w_{(i)n_i}\right) \\
&+ (z_{(i)n_i} - w_{(i)n_i})\frac{\partial g_{n_i-1}\left(z_{(i)n_i-1}, z_{(i)n_i}\right)}{\partial z_{(i)n_i}}|_{w_{(i)n_i-1}, w_{(i)n_i}} \\
&+ (z_{(i)n_i-1} - w_{(i)n_i-1})\frac{\partial g_{n_i-1}\left(z_{(i)n_i-1}, z_{(i)n_i}\right)}{\partial z_{(i)n_i-1}}|_{w_{(i)n_i-1}, w_{(i)n_i}} \\
&= \alpha_{i2n_i-2} + \beta_{i2n_i-2} z_{(i)n_i} + \gamma_{i2n_i-2} z_{(i)n_i-1} \quad .
\end{aligned}
\tag{B.8}
$$

Incorporating (B.3 - B.8) in (B.2) we get,

$$
\begin{aligned}
\frac{\partial L(\theta)}{\partial\theta} = \sum_{i=1}^{k}\Bigg\{ & \delta_{i0} z_{(i)1}\left(\alpha_{i0}+\beta_{i0} z_{(i)1}\right) \\
& + \delta_{i1}\left[z_{(i)2}\left(\alpha_{i1}+\beta_{i1} z_{(i)2}+\gamma_{i1} z_{(i)1}\right)-z_{(i)1}\left(\alpha_{i2}+\beta_{i2} z_{(i)2}+\gamma_{i2} z_{(i)1}\right)\right] \\
& + \delta_{i2}\left[z_{(i)3}\left(\alpha_{i3}+\beta_{i3} z_{(i)3}+\gamma_{i3} z_{(i)2}\right)-z_{(i)2}\left(\alpha_{i4}+\beta_{i4} z_{(i)3}+\gamma_{i4} z_{(i)2}\right)\right] \\
& \quad . \\
& \quad . \\
& \quad . \\
& + \delta_{ir-1}[z_{(i)r}\left(\alpha_{i2r-3}+\beta_{i2r-3} z_{(i)r}+\gamma_{i2r-3} z_{(i)r-1}\right) \\
& - z_{(i)r-1}\left(\alpha_{i2r-2}+\beta_{i2r-2} z_{(i)r}+\gamma_{i2r-2} z_{(i)r-1}\right)] \\
& \quad . \\
& \quad . \\
& \quad . \\
& + \delta_{in_{i-1}}[z_{(i)n_i}\left(\alpha_{i2n_i-3}+\beta_{i2n_i-3} z_{(i)n_i}+\gamma_{i2n_i-3} z_{(i)n_{i-1}}\right) \\
& \quad - z_{(i)n_{i-1}}\left(\alpha_{i2n_i-2}+\beta_{i2n_i-2} z_{(i)n_i}+\gamma_{i2n_i-2} z_{(i)n_{i-1}}\right)] \\
& + \delta_{in_i}\left[z_{(i)n_i}\left(\alpha_{i2n_i-1}+\beta_{i2n_i-1} z_{(i)n_i}\right)\right] \Bigg\}
\end{aligned}
\tag{B.9}
$$

$$
\hat{\theta} = \frac{\sum_{i=1}^{k}\left\{\delta_{i0} Y_{i1}\alpha_{i0} + \sum_{j=1}^{n_i-1}\delta_{ij}\left(Y_{ij}\alpha_{i2j-1} - Y_{ij-1}\alpha_{i2j}\right) + \delta_{in_i} Y_{in_i}\right\}}{\sum_{i=1}^{k}\left\{\delta_{i0} Y_{i1}^2\beta_{i0} + \sum_{j=1}^{n_i-1}\delta_{ij}\beta_{i2j-1}\left(Y_{ij-1} - Y_{ij}\right)^2\right\}} \tag{B.10}
$$

B.2 FISHER INFORMATION NUMBER

$$
\begin{aligned}
E\Big(\frac{\partial^2 lnL(X,\theta)}{\partial\theta^2}\Big) &= \sum_{i=1}^{k} E\Big(\frac{\partial^2 lnL(X,\theta)}{\partial\theta^2} \mid (Y_{i1}, Y_{i2}, ..., Y_{in_i})\Big) \\
&= \sum_{i=1}^{k} \int \delta_{i0} \frac{y_{i1}^2 e^{-\theta y_{i1}}}{(1-e^{-\theta y_{i1}})^2} g(y_{i(1)}) dy_{i1} \\
&+ \sum_{i=1}^{k} \iint \delta_{i1} \frac{(y_{i2}-y_{i1})^2 e^{-\theta y_{i2}} e^{-\theta y_{i1}}}{(e^{-\theta y_{i1}} - e^{-\theta y_{i1}})^2} g(y_{i(1)}, y_{i(2)}) dy_{i1} dy_{i2} \\
&. \\
&. \\
&. \\
&+ \sum_{i=1}^{k} \iint \delta_{ir-1} \frac{(y_{ir}-y_{ir-1})^2 e^{-\theta y_{ir}} e^{-\theta y_{ir-1}}}{(e^{-\theta y_{ir-1}} - e^{-\theta y_{ir}})^2} g(y_{i(r-1)}, y_{i(r)}) dy_{ir-1} dy_{ir} \\
&. \\
&. \\
&. \\
&+ \sum_{i=1}^{k} \iint \delta_{in_i-1} \frac{(y_{in_i}-y_{in_i-1})^2 e^{-\theta y_{in_i}} e^{-\theta y_{in_i-1}}}{(e^{-\theta y_{in_i-1}} - e^{-\theta y_{in_i}})^2} g(y_{i(n_i-1)}, y_{i(n_i)}) dy_{in_i-1} dy_{in_i} \quad .
\end{aligned}
$$

Let $n_1 = n_2 = \cdots = n_k = n$.

$$
\begin{aligned}
& E\Big(\frac{\partial^2 lnL(X,\theta)}{\partial\theta^2}\Big) \\
&= k\left\{\begin{array}{c}
\int \delta_{i0}\dfrac{y_{i1}^2 e^{-\theta y_{i1}}}{(1-e^{-\theta y_{i1}})^2} g(y_{i(1)}) dy_{i1} \\
+\iint \delta_{i1}\dfrac{(y_{i2}-y_{i1})^2 e^{-\theta y_{i2}} e^{-\theta y_{i1}}}{(e^{-\theta y_{i1}}-e^{-\theta y_{i1}})^2} g(y_{i(1)}, y_{i(2)}) dy_{i1} dy_{i2} \\
. \\
. \\
. \\
+\iint \delta_{ir-1}\dfrac{(y_{ir}-y_{ir-1})^2 e^{-\theta y_{ir}} e^{-\theta y_{ir-1}}}{(e^{-\theta y_{ir-1}}-e^{-\theta y_{ir}})^2} g(y_{i(r-1)}, y_{i(r)}) dy_{ir-1} dy_{ir} \\
. \\
. \\
. \\
+\iint \delta_{in_i-1}\dfrac{(y_{in_i}-y_{in_i-1})^2 e^{-\theta y_{in_i}} e^{-\theta y_{in_i-1}}}{(e^{-\theta y_{in_i-1}}-e^{-\theta y_{in_i}})^2} g(y_{i(n_i-1)}, y_{i(n_i)}) dy_{in_i-1} dy_{in_i}
\end{array}\right\}.
\end{aligned}
$$

$$
E\Big(\frac{\partial^2 lnL(X,\theta)}{\partial\theta^2}\Big)
$$

$$
= \quad k\left\{\begin{array}{c}
\displaystyle\int F(y_{i1})\frac{y_{i1}^2 e^{-\theta y_{i1}}}{(1-e^{-\theta y_{i1}})^2} g(y_{i(1)})dy_{i1} \\
\displaystyle + \iint \Bigg[[F(y_{i2})-F(y_{i1})]\,\frac{(y_{i2}-y_{i1})^2 e^{-\theta y_{i2}} e^{-\theta y_{i1}}}{(e^{-\theta y_{i1}}-e^{-\theta y_{i1}})^2}\Bigg] \\
\times g(y_{i(1)},y_{i(2)})dy_{i1}dy_{i2} \\
. \\
. \\
. \\
\displaystyle + \iint \Bigg[[F(y_{ir})-F(y_{ir-1})]\,\frac{(y_{ir}-y_{ir-1})^2 e^{-\theta y_{ir}} e^{-\theta y_{ir-1}}}{(e^{-\theta y_{ir-1}}-e^{-\theta y_{ir}})^2}\Bigg] \\
\times g(y_{i(r-1)},y_{i(r)})dy_{ir-1}dy_{ir} \\
. \\
. \\
. \\
\displaystyle + \iint \Bigg[[F(y_{in_i})-F(y_{in_i-1})]\,\frac{(y_{in_i}-y_{in_i-1})^2 e^{-\theta y_{in_i}} e^{-\theta y_{in_i-1}}}{(e^{-\theta y_{in_i-1}}-e^{-\theta y_{in_i}})^2}\Bigg] \\
\times g(y_{i(n_i-1)},y_{i(n_i)})dy_{in_i-1}dy_{in_i}
\end{array}\right\}
$$

where $g(y_{i(1)})$ is the probability function of the first order statistics and $g(y_{i(r-1)}, y_{i(r)})$ is the joint probability density function of the $(r-1)^{th}$ and r^{th} order statistics.

Let $y_{i1} = y_1, \quad y_{i2} = y_2, \cdots, y_{in_i} = y_n$. Then,

$$E\Big(\frac{\partial^2 lnL(X,\theta)}{\partial\theta^2}\Big) = k\left\{\begin{array}{c}\int F(y_1)\frac{y_1^2 e^{-\theta y_1}}{(1-e^{-\theta y_1})^2}g(y_{(1)})dy_1 \\ \\ +\iint\left[[F(y_2)-F(y_1)]\,\frac{(y_2-y_1)^2 e^{-\theta y_2}e^{-\theta y_1}}{(e^{-\theta y_1}-e^{-\theta y_1})^2}\right] \\ \times g(y_{(1)},y_{(2)})dy_1 dy_2 \\ . \\ . \\ . \\ +\iint\left[[F(y_r)-F(y_{r-1})]\,\frac{(y_r-y_{r-1})^2 e^{-\theta y_r}e^{-\theta y_{r-1}}}{(e^{-\theta y_{r-1}}-e^{-\theta y_r})^2}\right] \\ \times g(y_{(r-1)},y_{(r)})dy_{r-1}dy_r \\ . \\ . \\ . \\ +\iint\left[[F(y_n)-F(y_{n-1})]\,\frac{(y_n-y_{n-1})^2 e^{-\theta y_n}e^{-\theta y_{n-1}}}{(e^{-\theta y_{n-1}}-e^{-\theta y_n})^2}\right] \\ \times g(y_{(n-1)},y_{(n)})dy_{n-1}dy_n\end{array}\right\}.$$

Then, $E\Big(\frac{\partial^2 lnL(X,\theta)}{\partial\theta^2}\Big)$ takes the following form:

$$E\Big(\frac{\partial^2 lnL(X,\theta)}{\partial\theta^2}\Big) = k\left\{\begin{array}{c}\int F(y_1)\frac{y_1^2 e^{-\theta y_1}}{(1-e^{-\theta y_1})^2}g(y_{(1)})dy_1 \\ \\ +\sum_{i=1}^{n-1}\iint\left[[F(y_{i+1})-F(y_i)]\,\frac{(y_{i+1}-y_i)^2 e^{-\theta y_{i+1}}e^{-\theta y_i}}{(e^{-\theta y_i}-e^{-\theta y_{i+1}})^2}\right] \\ \times g(y_{(i)},y_{(i+1)})dy_i dy_{i+1}\end{array}\right\}.$$

For the simplicity, let $y_{i+1} = y_2$ and $y_i = y_1$. By using following joint probability density function of the two order statistics, binomial theorem and power series

$$
\begin{aligned}
g_{i,i+1}(y_i, y_{i+1}) &= \frac{n!}{(i-1)!(n-i-1)!}\left[G(y_i)\right]^{i-1}\left[1-G(y_{i+1})\right]^{n-i-1} g(y_i)g(y_{i+1}) \\
&= n(n-1)\binom{n-2}{i-1}\left[G(y_i)\right]^{i-1}\left[1-G(y_{i+1})\right]^{n-i-1} g(y_i)g(y_{i+1})
\end{aligned}
\tag{B.11}
$$

$$
(a+b)^n = \sum_{m=0}^{n}\binom{n}{m}a^m b^{n-m} \tag{B.12}
$$

$$
\frac{1}{1-x} = 1 + x + x^2 + \cdots\cdots \tag{B.13}
$$

we obtain,

$$
E\Big(\frac{\partial^2 lnL(X,\theta)}{\partial\theta^2}\Big) = k\left\{\begin{array}{c}
\int F(y_1)\frac{y_1^2 e^{-\theta y_1}}{(1-e^{-\theta y_1})^2} n[1-G_{y_1}]^{n-1} g(y_1)dy_1 \\
\\
+n(n-1)\sum_{i=1}^{n-1}\iint [[F(y_2)-F(y_1)]\,\frac{(y_2-y_1)^2 e^{-\theta y_2}e^{-\theta y_1}}{(e^{-\theta y_2}-e^{-\theta y_1})^2} \\
\binom{n-2}{i-1}\times[G_{y_1}]^{i-1}[1-G_{y_2}]^{n-i-1}]g(y_{(i)},y_{(i+1)})dy_1dy_2
\end{array}\right\}.
$$

$$
= k\left\{\begin{array}{c}
\int F(y_1)\frac{y_1^2 e^{-\theta y_1}}{(1-e^{-\theta y_1^2})} n[1-G_{y_1}]^{n-1} g(y_1)dy_1 \\
\\
+n(n-1)\sum_{i=1}^{n-1}\iint [F(y_2)-F(y_1)]\,\frac{(y_2-y_1)^2 e^{-\theta y_2}e^{-\theta y_1}}{(e^{-\theta y_2}-e^{-\theta y_1})^2}\binom{n-2}{i-1} \\
\times[G_{y_1}]^{i-1}[1-G_{y_2}]^{n-i-1} g(y_1)g(y_2)dy_1dy_2
\end{array}\right\}.
$$

Let $i-1=j$, then,

$$
E\Big(\frac{\partial^2 lnL(X,\theta)}{\partial\theta^2}\Big) = k\left\{\begin{array}{c}
\int_0^\infty (1-e^{-\theta y_1^2})\frac{y_1^2 e^{-\theta y_1}}{(1-e^{-\theta y_1^2})} n[e^{-\lambda y_1}]^{n-1}\lambda e^{-\lambda y_1}dy_1 \\
\\
+n(n-1)\iint [F(y_2)-F(y_1)]\,\frac{(y_2-y_1)^2 e^{-\theta y_2}e^{-\theta y_1}}{(e^{-\theta y_2}-e^{-\theta y_1})^2} g(y_1)g(y_2) \\
\times\left\{\sum_{j=0}^{n-2}\binom{n-2}{j}[G_{y_1}]^j[1-G_{y_2}]^{n-2-j}\right\}dy_1dy_2
\end{array}\right\}
$$

$$
= k \left\{ \begin{array}{c} \int_0^\infty (1-e^{-\theta y_1^2}) \frac{y_1^2 e^{-\theta y_1}}{(1-e^{-\theta y_1^2})} n[e^{-\lambda y_1}]^{n-1} \lambda e^{-\lambda y_1} dy_1 \\ \\ +n(n-1) \iint \left(e^{-\theta y_2} - e^{-\theta y_1}\right) \frac{(y_2-y_1)^2 e^{-\theta y_2} e^{-\theta y_1}}{(e^{-\theta y_2} - e^{-\theta y_1})^2} \lambda^2 e^{-\theta y_1} e^{-\theta y_2} \\ \cdot [e^{-\theta y_2} + 1 - e^{-\theta y_1}]^{n-2} dy_1 dy_2 \end{array} \right\}
$$

$$
k \left\{ \begin{array}{c} n\lambda \int_0^\infty (1-e^{-\theta y_1^2}) \frac{y_1^2 e^{-\theta y_1}}{(1-e^{-\theta y_1^2})} n[e^{-\lambda y_1}]^{n-1} \lambda e^{-\lambda y_1} dy_1 \\ \\ +n(n-1)\lambda^2 \iint \left(e^{-\theta y_2} - e^{-\theta y_1}\right) \frac{(y_2-y_1)^2 e^{-\theta y_2} e^{-\theta y_1}}{(e^{-\theta y_2} - e^{-\theta y_1})^2} e^{-\theta(y_1+y_2)} \\ \cdot [e^{-\theta y_2} + 1 - e^{-\theta y_1}]^{n-2} dy_1 dy_2 \end{array} \right\}
$$

$$
= k \left\{ \begin{array}{c} n\lambda \int_0^\infty y_1^2 e^{(-\theta+n\lambda)y_1} \frac{1}{(1-e^{-\theta y_1})} dy_1 \\ \\ +n(n-1)\lambda^2 \iint \left(e^{-\theta y_2} - e^{-\theta y_1}\right) \frac{(y_2-y_1)^2 e^{-\theta y_2} e^{-\theta y_1}}{(e^{-\theta y_2} - e^{-\theta y_1})^2} e^{-\theta(y_1+y_2)} \\ \times [e^{-\theta y_2} + 1 - e^{-\theta y_1}]^{n-2} dy_1 dy_2 \end{array} \right\}
$$

$$
= k \left\{ \begin{array}{c} n\lambda \int_0^\infty y_1^2 e^{(-\theta+n\lambda)y_1} \left\{ \sum_{j=0}^{\infty} (e^{-\theta y_1})^j \right\} dy_1 \\ \\ +n(n-1)\lambda^2 \iint \left(e^{-\theta y_2} - e^{-\theta y_1}\right) \frac{(y_2-y_1)^2 e^{-\theta y_2} e^{-\theta y_1}}{(e^{-\theta y_2} - e^{-\theta y_1})^2} e^{-\theta(y_1+y_2)} \\ \times [e^{-\theta y_2} + 1 - e^{-\theta y_1}]^{n-2} dy_1 dy_2 \end{array} \right\}
$$

$$
= k \left\{ \begin{array}{c} n\lambda \sum_{j=0}^{\infty} \int_0^\infty y_1^2 e^{(-\theta+n\lambda+j\theta)y_1} dy_1 \\ \\ +n(n-1)\lambda^2 \sum_{k=0}^{\infty} \left\{ \sum_{j=0}^{n-2} \binom{n-2}{j} \sum_{l=0}^{j} \frac{(-1)^{j+1}}{(2\lambda+\theta+j\lambda)} \binom{j}{l} \frac{1}{(\theta+\lambda+l\theta+\lambda l)^3} \right\} \end{array} \right\}
$$

$$= k \left\{ \begin{array}{c} n\lambda \sum_{j=0}^{\infty} \frac{1}{(\theta+n\lambda+j\theta)^3} \\ \\ +n(n-1)\lambda^2 \sum_{k=0}^{\infty} \left\{ \sum_{j=0}^{n-2} \binom{n-2}{j} \sum_{l=0}^{j} \frac{(-1)^{j+1}}{(2\lambda+\theta+j\lambda)} \binom{j}{l} \frac{1}{(\theta+\lambda+l\theta+\lambda l)^3} \right\} \end{array} \right\}.$$

By using power series in our derivation, we obtain Fisher information number as,

$$I(\theta) = k \Bigg[n\lambda \sum_{j=0}^{\infty} \frac{1}{(\theta + n\lambda + j\theta)^3} + n(n-1)\lambda^2 \sum_{l=0}^{\infty} \left\{ \sum_{j=0}^{n-2} \binom{n-2}{j} \sum_{l=0}^{j} \frac{(-1)^{j+1}}{(2\lambda + \theta + j\lambda)} \binom{j}{l} \frac{1}{(\theta + \lambda + l\theta + \lambda l)^3} \right\} \Bigg]. \tag{B.14}$$

APPENDIX C

MATLAB CODES FOR SIMULATION OF INTERVAL CENSORED DATA AND MODIFIED MAXIMUM LIKELIHOOD ESTIMATION

```
# SIMULATION OF INTERVAL CENSORED DATA AND
  MMLE FOR FIXED INTERVALS

function [l,r] =
data_generate_tez_531_simo_last(n, mu, sigma, q,
k, c, p_artis, simoCount)

    muler_dizisi = [];
    sigmalar_dizisi = [];

    for simoIndex=1:simoCount

        simoIndex
    i = 0;
    y = normrnd(mu, sigma, [1 n]);

    y

    dizi_enkucukler = [];
    dizi_ortadakilerL = [];
    dizi_ortadakilerR = [];
```

```
dizi_enbuyukler = [];

for i=1:n
    u = zeros(1,k);

    u = unifrnd(0, 1, [1 k]);

    p = zeros(1,k);

    p(1) = c;
    deg = c + p_artis;
    for j = 2:k
        p(j) = deg;
        deg = deg + p_artis;
    end

    I = zeros(1,k);
    AT = [];

    I(1) = 1;
    AT(1) = p(1);

    for j=2:k
        if u(j) <= q
           AT = [AT, p(j)];
        end
    end

    y

    AT

    for j=1 : length(AT)
```

```
        if y(i) < min(AT)
            dizi_enkucukler = [dizi_enkucukler ,

            min(AT)];
        elseif y(i) > max(AT)
           dizi_enbuyukler =  [dizi_enbuyukler ,

           max(AT)];
        else
            diziTempKucukler = [];
            diziTempBuyukler = [];
            for tempIndex = 1 : length(AT)
                if y(i) > AT(tempIndex)
                    diziTempKucukler =
                    [diziTempKucukler, AT(tempIndex)];
                else
                    diziTempBuyukler =
                    [diziTempBuyukler, AT(tempIndex)];
                end
            end
            dizi_ortadakilerR =
            [dizi_ortadakilerR, min(diziTempBuyukler)];
            dizi_ortadakilerL =
            [dizi_ortadakilerL, max(diziTempKucukler)];

        end
        break;
    end
end

dizi_enkucukler
dizi_ortadakilerL
dizi_ortadakilerR
```

```
    dizi_enbuyukler

ort_y = mean(y)
std_dev_y = std(y,0)

ky = 0;
hy = 0;

ky = ort_y - (std_dev_y / sqrt(n));
hy = ort_y + (std_dev_y / sqrt(n));
ky;
hy;

beta_eleman_sayisi = length(dizi_ortadakilerL);
Beta_1 = zeros( 1, beta_eleman_sayisi);

for x=1:beta_eleman_sayisi

    fli_k = normpdf((dizi_ortadakilerL(x) - ky)/std_dev_y);
    Fri_k = normcdf((dizi_ortadakilerR(x)- ky)/std_dev_y);
    Fli_k = normcdf((dizi_ortadakilerL(x) - ky)/std_dev_y);
    ust_sol = fli_k / ( Fri_k - Fli_k );

    fli_h = normpdf((dizi_ortadakilerL(x) - hy)/std_dev_y);
    Fri_h = normcdf((dizi_ortadakilerR(x)- hy)/std_dev_y);
    Fli_h = normcdf((dizi_ortadakilerL(x) - hy)/std_dev_y);
    ust_sag = fli_h / (Fri_h - Fli_h);

    alt = (hy-ky) / std_dev_y;

    Beta_1(x) = (ust_sol - ust_sag) / (alt);
end
```

```
Beta_1

alpha_eleman_sayisi = length(dizi_ortadakilerL);
Alpha_1 = zeros(1, alpha_eleman_sayisi);

for x=1:alpha_eleman_sayisi

    fli_h = normpdf((dizi_ortadakilerL(x) - hy)/std_dev_y);
    Fri_h = normcdf((dizi_ortadakilerR(x) - hy)/std_dev_y);
    Fli_h = normcdf((dizi_ortadakilerL(x) - hy)/std_dev_y);
    li_h = (dizi_ortadakilerL(x) - hy) / std_dev_y;

    Alpha_1(x) = ( fli_h / ( Fri_h - Fli_h ) )
    - ( Beta_1(x) * li_h ) ;
fli_h;
Fri_h;
Fli_h;
li_h;
end

Alpha_1

w_eleman_sayisi = length(dizi_enkucukler);
W_1 = zeros(1,w_eleman_sayisi);

for x=1:w_eleman_sayisi

    fli_k = normpdf((dizi_enkucukler(x) - ky)/std_dev_y);
    Fli_k = normcdf((dizi_enkucukler(x) - ky)/std_dev_y);
    fli_h = normpdf((dizi_enkucukler(x) - hy)/std_dev_y);
    Fli_h = normcdf((dizi_enkucukler(x) - hy)/std_dev_y);
    alt = (hy-ky) / std_dev_y;
```

```
    W_1(x) = ((fli_k / Fli_k) - (fli_h / Fli_h )) / ( alt );

end
dizi_enkucukler;
W_1

v_eleman_sayisi = length(dizi_enkucukler);
V_1 = zeros(1,v_eleman_sayisi);

for x=1:v_eleman_sayisi

    fli_h = normpdf((dizi_enkucukler(x) - hy)/std_dev_y);
    Fli_h = normcdf((dizi_enkucukler(x) - hy)/std_dev_y);
    li_h = (dizi_enkucukler(x) - hy) / std_dev_y;

    V_1(x) = (fli_h / Fli_h) - ( W_1(x) * li_h);
end

dizi_enkucukler;
V_1

beta_eleman_sayisi = length(dizi_ortadakilerR);
Beta_2 = zeros(1,beta_eleman_sayisi);

for x=1:beta_eleman_sayisi

    fri_k = normpdf((dizi_ortadakilerR(x) - ky)/std_dev_y);
    Fri_k = normcdf((dizi_ortadakilerR(x)- ky)/std_dev_y);
    Fli_k = normcdf((dizi_ortadakilerL(x) - ky)/std_dev_y);

    ust_sol = fri_k /(Fri_k - Fli_k);

    fri_h = normpdf((dizi_ortadakilerR(x) - hy)/std_dev_y);
```

```
    Fri_h = normcdf((dizi_ortadakilerR(x)- hy)/std_dev_y);
    Fli_h = normcdf((dizi_ortadakilerL(x) - hy)/std_dev_y);

    ust_sag = fri_h / (Fri_h - Fli_h);

    alt = (hy-ky) / std_dev_y;

    Beta_2(x) = (ust_sol - ust_sag) / alt;

end

 dizi_ortadakilerL;
 dizi_ortadakilerR;
 Beta_2

alpha_eleman_sayisi = length(dizi_ortadakilerR);
Alpha_2 = zeros(1,alpha_eleman_sayisi);

for x=1:alpha_eleman_sayisi

    fri_h = normpdf((dizi_ortadakilerR(x) - hy)/std_dev_y);
    Fri_h = normcdf((dizi_ortadakilerR(x)- hy)/std_dev_y);
    Fli_h = normcdf((dizi_ortadakilerL(x) - hy)/std_dev_y);
    ri_h = (dizi_ortadakilerR(x) - hy ) / std_dev_y ;

    Alpha_2(x) = (fri_h / ( Fri_h - Fli_h ))
    - (Beta_2(x) * ri_h) ;

end

  dizi_ortadakilerR;
  Alpha_2
```

```
w_eleman_sayisi = length(dizi_enbuyukler);
W_2 = zeros(1,w_eleman_sayisi);

for x=1:w_eleman_sayisi

    fri_k = normpdf((dizi_enbuyukler(x) - ky) / std_dev_y);
    Fri_k = normcdf((dizi_enbuyukler(x)- ky) / std_dev_y);
    fri_h = normpdf((dizi_enbuyukler(x) - hy) / std_dev_y);
    Fri_h = normcdf((dizi_enbuyukler(x)- hy) / std_dev_y);
    alt = (hy-ky) / std_dev_y;

    W_2(x) = ((fri_k / (1 - Fri_k))

   - (fri_h / (1 - Fri_h))) / ( alt );

end

 dizi_enbuyukler;
 W_2

v_eleman_sayisi = length(dizi_enbuyukler);
V_2 = zeros(1,v_eleman_sayisi);

for x=1:w_eleman_sayisi

    fri_h = normpdf((dizi_enbuyukler(x) - hy) / std_dev_y);
    Fri_h = normcdf((dizi_enbuyukler(x)- hy) / std_dev_y);
    ri_h = (dizi_enbuyukler(x) - hy) / std_dev_y;

    V_2(x) = (fri_h / (1 - Fri_h )) - ( W_2(x) * ri_h );

end
```

```
 V_2
 dizi_enbuyukler;

A = 0;

pay_bir = 0;
pay_iki = 0;
pay_uc = 0;
payda_bir = 0;
payda_iki = 0;
payda_uc = 0;

sol_eleman_sayisi = length(dizi_enkucukler);
orta_eleman_sayisi = length(dizi_ortadakilerR);
sag_eleman_sayisi = length(dizi_enbuyukler);

for x=1:sol_eleman_sayisi
    pay_bir = pay_bir + (V_1(x) * dizi_enkucukler(x));
end

for x=1:orta_eleman_sayisi
    pay_iki = pay_iki + ( (Beta_1(x) * dizi_ortadakilerR(x))

  - (Beta_2(x) * dizi_ortadakilerL(x))) ;
end

for x=1:sag_eleman_sayisi
    pay_uc = pay_uc + (V_2(x) * dizi_enbuyukler(x)) ;
end

for x=1:sol_eleman_sayisi
    payda_bir = payda_bir + (V_1(x)) ;
end
```

```
for x=1:orta_eleman_sayisi
    payda_iki = payda_iki + (Beta_1(x) - Beta_2(x)) ;
end

for x=1:sag_eleman_sayisi
    payda_uc = payda_uc + (V_2(x)) ;
end

A = (-pay_bir - pay_iki + pay_uc )

/ ( -payda_bir - payda_iki + payda_uc );

A

B = 0;
pay_bir = 0;
pay_iki = 0;
pay_uc = 0;
payda_bir = 0;
payda_iki = 0;
payda_uc = 0;

sol_eleman_sayisi = length(dizi_enkucukler);
orta_eleman_sayisi = length(dizi_ortadakilerR);
sag_eleman_sayisi = length(dizi_enbuyukler);


for x=1:sol_eleman_sayisi
    pay_bir = pay_bir + (W_1(x));
end
```

```
for x=1:orta_eleman_sayisi
    pay_iki = pay_iki + ( Alpha_2(x) - Alpha_1(x)) ;
end

for x=1:sag_eleman_sayisi
    pay_uc = pay_uc + (W_2(x)) ;
end

for x=1:sol_eleman_sayisi
    payda_bir = payda_bir + (V_1(x)) ;
end

for x=1:orta_eleman_sayisi
    payda_iki = payda_iki + (Beta_1(x) - Beta_2(x)) ;
end

for x=1:sag_eleman_sayisi
    payda_uc = payda_uc + (V_2(x)) ;
end

B = (- pay_bir - pay_iki + pay_uc )

/ ( - payda_bir - payda_iki + payda_uc );

B

C = 0;
pay_bir = 0;
pay_iki = 0;
pay_uc = 0;
payda_bir = 0;
payda_iki = 0;
payda_uc = 0;
```

```
sol_eleman_sayisi = length(dizi_enkucukler);
orta_eleman_sayisi = length(dizi_ortadakilerR);
sag_eleman_sayisi = length(dizi_enbuyukler);

for x=1:sol_eleman_sayisi
    pay_bir = pay_bir + (W_1(x) * ((dizi_enkucukler(x)-A)));
end

for x=1:orta_eleman_sayisi
    pay_iki = pay_iki + ((Alpha_2(x) * (dizi_ortadakilerR(x)-A))

   - ((Alpha_1(x) * (dizi_ortadakilerL(x)-A))));
end

for x=1:sag_eleman_sayisi
    pay_uc = pay_uc + (W_2(x) * (dizi_enbuyukler(x)-A)) ;
end

C = (- pay_bir - pay_iki + pay_uc);
C

E=0;
pay_bir = 0;
pay_iki = 0;
pay_uc = 0;

sol_eleman_sayisi = length(dizi_enkucukler);
orta_eleman_sayisi = length(dizi_ortadakilerR);
sag_eleman_sayisi = length(dizi_enbuyukler);

for x=1:sol_eleman_sayisi
    pay_bir = pay_bir + (V_1(x) * ((dizi_enkucukler(x)-A)^2));
```

```
end

for x=1:orta_eleman_sayisi
    pay_iki = pay_iki + ((( Beta_2(x)) *

   ((dizi_ortadakilerR(x)-A)^2))

  - ((Beta_1(x) * ((dizi_ortadakilerL(x)-A)^2))));
end

for x=1:sag_eleman_sayisi
    pay_uc = pay_uc + (V_2(x) * ((dizi_enbuyukler(x)-A)^2)) ;
end

E = (- pay_bir - pay_iki + pay_uc);

E

dizi_enkucukler;
dizi_enbuyukler;
dizi_ortadakilerR;
dizi_ortadakilerL;

sigma_hat= 0;
var_hat = 0;
mu_hat = 0;
sigma_hat1 =  (-C + sqrt((C^2) + (4*n*E)));
sigma_hat2 = (-(sqrt(2*(n)*(n-2))));
sigma_hat = sigma_hat1 / sigma_hat2 ;

mu_hat = (A + (B * sigma_hat);
mse = sum
sigma_hat
```

```
mu_hat

 muler_dizisi = [muler_dizisi, mu_hat];
 sigmalar_dizisi = [sigmalar_dizisi, sigma_hat];
 ms = [dizi, msesimo]

mse = [];
for j = 1 : length
    ms = (muler_dizisi - mu_hat)/simo
end
    mu_ort = mean( muler_dizisi )
    sig_ort = mean (sigmalar_dizisi)

end

# SIMULATION OF INTERVAL CENSORED DATA AND
  MMLE FOR RANDOM INTERVALS

function [l,r] = data_generate_tez_531_simo_teta
(n, teta, q, k, c, p_artis, simoCount)

    muler_dizisi = [];
    sigmalar_dizisi = [];

    for simoIndex=1:simoCount

        simoIndex
    i = 0;
    y = exprnd(teta, [1 n]);

    y

    dizi_enkucukler = [];
```

```
dizi_ortadakilerL = [];
dizi_ortadakilerR = [];
dizi_enbuyukler = [];

for i=1:n
    u = zeros(1,k);

    u = unifrnd(0, 1, [1 k]);

    p = zeros(1,k);

    p(1) = c;
    deg = c + p_artis;
    for j = 2:k
        p(j) = deg;
        deg = deg + p_artis;
    end

    I = zeros(1,k);
    AT = [];

    I(1) = 1;
    AT(1) = p(1);

    for j=2:k
        if u(j) <= q
           AT = [AT, p(j)];
        end
    end

    y
    AT
```

```
for j=1 : length(AT)
    if y(i) < min(AT)
        dizi_enkucukler = [dizi_enkucukler ,

         min(AT)];
    elseif y(i) > max(AT)
       dizi_enbuyukler =  [dizi_enbuyukler ,

        max(AT)];
    else
        diziTempKucukler = [];
        diziTempBuyukler = [];
        for tempIndex = 1 : length(AT)
            if y(i) > AT(tempIndex)
                diziTempKucukler = [diziTempKucukler,

                AT(tempIndex)];
            else
                diziTempBuyukler = [diziTempBuyukler,

                AT(tempIndex)];
            end
        end
        dizi_ortadakilerR = [dizi_ortadakilerR,

        min(diziTempBuyukler)];
        dizi_ortadakilerL = [dizi_ortadakilerL,

        max(diziTempKucukler)];

    end
    break;
end
```

```
    end

    dizi_enkucukler
    dizi_ortadakilerL
    dizi_ortadakilerR
    dizi_enbuyukler


v_eleman_sayisi = length(dizi_meanler);
V_2 = zeros(1,v_eleman_sayisi);


for x=1:w_eleman_sayisi

    fri_h = normpdf((dizi_enbuyukler(x) - hy) / std_dev_y);
    Fri_h = normcdf((dizi_enbuyukler(x)- hy) / std_dev_y);
    ri_h = (dizi_enbuyukler(x) - hy) / std_dev_y;
D = 0;

pay_bir = 0;
pay_iki = 0;
pay_uc = 0;
payda_bir = 0;
payda_iki = 0;
payda_uc = 0;

sol_eleman_sayisi = length(dizi_enkucukler);
orta_eleman_sayisi = length(dizi_ortadakilerR);
sag_eleman_sayisi = length(dizi_enbuyukler);

for x=1:sol_eleman_sayisi
    pay_bir = pay_bir + (V_1(x) * dizi_enkucukler(x));
end
```

```
for x=1:orta_eleman_sayisi
    pay_iki = pay_iki + ( (or_1(x) * dizi_ortadakilerR(x))

   - (or_2(x) * dizi_ortadakilerL(x))) ;
end

for x=1:sag_eleman_sayisi
    pay_uc = pay_uc + (V_2(x) * dizi_enbuyukler(x)) ;
end

for x=1:sol_eleman_sayisi
    payda_bir = payda_bir + (V_1(x)) ;
end

for x=1:orta_eleman_sayisi
    payda_iki = payda_iki + (or_1(x) - or_2(x)) ;
end

for x=1:sag_eleman_sayisi
    payda_uc = payda_uc + (V_2(x)) ;
end

D = (-pay_bir - pay_iki + pay_uc ) /

( -payda_bir - payda_iki + payda_uc );
D

Z = 0;
pay_bir = 0;
pay_iki = 0;
pay_uc = 0;
payda_bir = 0;
```

```
payda_iki = 0;
payda_uc = 0;

sol_eleman_sayisi = length(dizi_enkucukler);
orta_eleman_sayisi = length(dizi_ortadakilerR);
sag_eleman_sayisi = length(dizi_enbuyukler);

for x=1:sol_eleman_sayisi
    pay_bir = pay_bir + (W_1(x));
end

for x=1:orta_eleman_sayisi
    pay_iki = pay_iki + ( Alpha_2(x) - Alpha_1(x)) ;
end

for x=1:sag_eleman_sayisi
    pay_uc = pay_uc + (W_2(x)) ;
end

for x=1:sol_eleman_sayisi
    payda_bir = payda_bir + (V_1(x)) ;
end

for x=1:orta_eleman_sayisi
    payda_iki = payda_iki + (or_1(x) - or_2(x)) ;
end

for x=1:sag_eleman_sayisi
    payda_uc = payda_uc + (V_2(x)) ;
end

Z = (- pay_bir - pay_iki + pay_uc ) /
```

```
( - payda_bir - payda_iki + payda_uc );

Z

G = 0;
pay_bir = 0;
pay_iki = 0;
pay_uc = 0;
payda_bir = 0;
payda_iki = 0;
payda_uc = 0;

sol_eleman_sayisi = length(dizi_enkucukler);
orta_eleman_sayisi = length(dizi_ortadakilerR);
sag_eleman_sayisi = length(dizi_enbuyukler);

for x=1:sol_eleman_sayisi
    pay_bir = pay_bir + (W_1(x) * ((dizi_enkucukler(x)-A)));
end

for x=1:orta_eleman_sayisi
    pay_iki = pay_iki + ((Alpha_2(x)

    * (dizi_ortadakilerR(x)-A))

   - ((Alpha_1(x) * (dizi_ortadakilerL(x)-A))));
end

for x=1:sag_eleman_sayisi
    pay_uc = pay_uc + (W_2(x) * (dizi_enbuyukler(x)-A)) ;
end

G = (- pay_bir - pay_iki + pay_uc);
```

```
G

F=0;
pay_bir = 0;
pay_iki = 0;
pay_uc = 0;

sol_eleman_sayisi = length(dizi_enkucukler);
orta_eleman_sayisi = length(dizi_ortadakilerR);
sag_eleman_sayisi = length(dizi_enbuyukler);

for x=1:sol_eleman_sayisi
    pay_bir = pay_bir + (V_1(x)

    * ((dizi_enkucukler(x)-A)^2));
end

for x=1:orta_eleman_sayisi
    pay_iki = pay_iki + ((( or_2(x)) *

    ((dizi_ortadakilerR(x)-A)^2))

    - ((or_1(x) * ((dizi_ortadakilerL(x)-A)^2))));
end

for x=1:sag_eleman_sayisi
    pay_uc = pay_uc + (V_2(x) * ((dizi_enbuyukler(x)-A)^2)) ;
end

F = (- pay_bir - pay_iki + pay_uc);

teta_hat = 0;
teta_hat = (F + G + D)/((or * G)+ Z);
```

```
tetalar_dizisi = [tetalar_dizisi, teta_hat];
mse = [dizi, msesimo]

mse = [];
for j = 1 : length
    ms = (tetalar_dizisi - teta_hat)/simo
end
    teta = mean(tetalar_dizisi )

end
```

CURRICULUM VITAE

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EDUCATION

Degree	Institution	Year of Graduation
M.S.	M.S. Bilkent University	M.S. 2009
B.S.	B.S. METU	B.S. 2006
High School	Alparslan High School	2001

PROFESSIONAL EXPERIENCE

Year	Place	Enrollment
2009 - 2014	METU	Research Assistant
2006 - 2009	Bilkent University	Research Assistant

PUBLICATIONS

Eryilmaz S., Bayramoglu K. (2013) *Life behavior of σ-shock models for uniformly distributed interarrival times.* Statistical Papers, Vol. $55, 841-852$

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