GRAVITOMAGNETISM IN GENERAL RELATIVITY AND MASSIVE GRAVITY

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

GÖKÇEN DENIZ ÖZEN

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF MASTER OF SCIENCE
IN
PHYSICS

SEPTEMBER 2014

Approval of the thesis:

GRAVITOMAGNETISM IN GENERAL RELATIVITY AND MASSIVE GRAVITY

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ABSTRACT

GRAVITOMAGNETISM IN GENERAL RELATIVITY AND MASSIVE GRAVITY

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SEPTEMBER 2014, 61 pages

In this thesis gravitomagnetic effects are analysed in some detail. Einstein's equations for weak gravitational fields are derived. Using appropriate gauge fixings, metric perturbation is decomposed and degrees of freedom are identified. Physical degrees of freedom are chosen and it is proven that they characterize the propagation of gravitational waves. Production of gravitational waves is demonstrated as well as their effects on the polarization of test particles. Analogs of the Maxwell's equations are derived for gravity. From the analysis of the scattering amplitude, potential energy is found for massive and massless gravity theories, the appropriate spin alignment for minimum potential energy is calculated and the difference between general relativity and the massive gravity for this spin alignment is shown. In the Appendix, some useful calculations are given. Save for some details in the computations, no originality in this thesis is claimed. Somewhat standard material about weak field gravity, gauge fixings and degree of freedom counting follows closely the discussion in Chapter 7 of

Carroll's excellent book. Chapter 2 of the thesis closely follows Harris's paper "Analogy between general relativity and electromagnetism for slowly moving particles in weak gravitational fields". Chapter 3 of this thesis is a review of the paper Güllü-Tekin "Spin–spin interactions in massive gravity and higher derivative gravity theories".

Keywords: Gravitomagnetism, Linearized Gravity, Gravity Waves, Spin Alignment in Massive Gravity

ÖZ

KÜTLELİ ÇEKİM VE GENEL GÖRELİLİKTE ÇEKİMSEL MANYETİZMA

Özen, Gökçen Deniz
Yüksek Lisans, Fizik Bölümü
Tez Yöneticisi : Prof. Dr. Bayram Tekin

Eylül 2014 , 61 sayfa

Bu tez çalışmasında, çekimsel manyetik etki ayrıntılı olarak analiz edildi. Zayıf çekim alanları için Einstein denklemleri türetilmiştir. Uygun ayar dönüşümleri kullanılarak metrik tedirgemeleri ayrıştırılmış ve serbestlik dereceleri tanımlanmıştır. Bu serbestlik derecelerinin fiziksel olanları seçilmiş, ve gravitasyonel dalgaların yayılmasını tanımladıkları ispatlanmıştır. Gravitasyonel dalgaların nasıl meydana geldiği ve bu dalgaların test parçacıklarını nasıl polarize ettiği gösterilmiştir. Çekimsel alan için Maxwell denklemlerinin analogları bulunmuştur. Saçılma genliği kullanılarak, kütleli ve kütlesiz teori için potansiyel enerji bulunmuş, bu potansiyel enerjiyi minimum yapan spin uyumu hesaplanmış ve bu uyumun genel görelilik ve kütleli teori için farklı olduğu gösterilmiştir. Ek bölümde bazı yararlı hesaplar verilmiştir. Hesaplamalardaki bazı ayrıntılar dışında, bu tezde özgünlük iddia edilmemektedir. Zayıf alan kütleçekimi, ayar sabitlemeleri ve serbestlik derecesi sayımı ile ilgili oldukça standard konular, Carroll'un mükemmel kitabının 7. bölümünü yakından takip etmektedir. Bu tezin 2. bö-

lümü Harris'in "Analogy between general relativity and electromagnetism for slowly moving particles in weak gravitational fields" makalesini takip etmekte, 3. bölümü de Güllü-Tekin "Spin–spin interactions in massive gravity and higher derivative gravity theories" makalesini incelemektedir.

Anahtar Kelimeler: Çekimsel Manyetizma, Linerize Kütleçekimi, Kütleçekim Dalgaları, Kütleli Teoride Spin Uyumu

To women with great laughs

ACKNOWLEDGEMENTS

I think it is not possible to express my gratefulness in one page, but I try my best. I would like to appreciate Prof. Dr. Bayram Tekin for his guidance and limitless energy but mostly his patience because I never hesitated to knock his door throughout my thesis process. I am also grateful to him for choosing such an interesting thesis subject which excites me for my further research. I sincerely thank to Dr. İbrahim Güllü, who is like a co-supervisor for me. Without his assistance, I would be still struggling and reach nothing. I am also grateful to Prof. Dr. Atalay Karasu for encouraging me at the times I did not believe myself. He is like a wise man in the movies who says there is always hope. I also thank to Assoc. Prof. Dr. Sinan Kaan Yerli for reminding me that the dynamics of the heavens were much more attractive than the stuff I was dealing with in mathematics department.

I would like to thank Prof. Dr. Sean Carroll for allowing me to use some of the figures in his excellent book "Spacetime and Geometry".

I am grateful to my friends Elif Türkmen, Fulya Aydaş, Nazlı Tiryaki, Deniz Karaca and Oğuz Durmaz for their moral support. I also thank to Ercan Kılıçarslan and Dr. Suat Dengiz for their great contributions. I sincerely thank to Hüden Neşe for putting a smile on my face every time we talk to. She is my 'biricik dalağım' and she will always remain so.

I am grateful to my boyfriend, Yetkin Alıcı, for his motivation and understanding for the times when I was acting like a crazy woman.

Finally, I am indebted for my family for their moral support, encouragement and understanding for the times we were separated during my thesis process.

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INTRODUCTION

1.1 Introduction

"If we pick up a stone and then let it go, why does it fall to the ground ?" [1]. Newton's reply to this question was the attraction between the earth and stone. In his book Principia in 1687, he formulated the force between two masses as

$$F = \frac{-Gm_1m_2}{r^2}, \tag{1.1}$$

where G is the Newton's gravitational constant and m_1 and m_2 are the masses of the particles. Minus sign indicates that the force is attractive. In the following centuries, Newton's law (1.1) gave successful explanations of the motion of the moon and the planets [2]. The discovery of Uranus was one of the examples of this success [3]. Over the years, some irregularities in its orbit emerged. It was detected that Uranus insistently moved away from its expected Newtonian path. It was suggested that, the deviation between the calculation and the observation could be the result of the perturbation of an unknown planet [4]. Using the Newton's law, the location of this new planet, Neptune, was predicted and it was observed at that location [3, 5]. This resolution could also give an explanation about precession of Mercury's perihelion precession. It was calculated that, the observed precession was faster than the expected one according to Newton's theory. This discrepancy would exist because of small planets between Mercury and the Sun, but these planets were never observed [2]. Fortunately, there was a remedy for this trouble. The explanation came with the replacement of Newtonian theory with the Einsteinian one [5].

General Relativity (GR) is the theory of space, time and gravitation [6]. As the matter bends its vicinity; it creates curvature which is the source of gravitation. Matter is energy, hence we can say that there is a relationship between energy and the curvature. Curvature is defined by the Riemann tensor and energy momentum tensor is an attribute of matter; in other words it is a measure of energy, momentum, pressure and stress of the matter. Mathematical interpretation of these information leads us to Einstein's equation:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \tag{1.2}$$

which is the basic equation of the GR. $R_{\mu\nu}, g_{\mu\nu}, R, T_{\mu\nu}$ are Ricci tensor, metric of the space-time, Ricci scalar and the covariantly conserved energy momentum tensor, respectively. Ricci tensor is calculated from contraction of Riemann tensor; therefore, left-hand side of (1.2) represents the curvature of space time whereas the right-hand-side is the measurement of the energy and its partners. Similar to the other field equations, Einstein's equation is postulated and can not be proven by using any other principles. We can reach it by the motivation of some arguments. We can find the equation of motion by using the least principle action; therefore varying the Einstein-Hilbert action is a route to reach the Einstein's equation [7].

After Einstein's equation was proposed, it was immediately applied to the problem of Mercury's orbit. By the guidance of the Schwarzschild solution, the perihelion precession of Mercury was calculated and the answer from the calculation precisely matched the observed value [2, 3, 5, 6, 7, 14]. Also, by passing the tests such as the bending of light by the Sun and the gravitational redshift of light, it was proven that the power of GR was not limited to the Mercury's orbit.

Gravitomagnetism, which is a natural consequence of general relativity, can be described as an analogy between the equations of electromagnetism and those in general relativity, precisely between Maxwell and Einstein's field equations [8]. We know that a charge generates an electric field which is proportional to $1/r^2$, where r is the distance between the charge and the point chosen. We also know that, we will experience a magnetic field, if this charge starts to move and mathematical expression of this relationship is given by Maxwell's equations. Thanks to Newton, we have no doubt in the existence of gravitational field due to a mass. We know that almost every object in the heavens rotate around itself and revolve around another object. Therefore it will not be weird to say that like a magnetic field in electromagnetism, we can describe a gravitomagnetic field when a massive object rotates or moves. If there exists such an analogy, then it has to be supported by equations. In Chapter 1, we will show that the time component of the metric is responsible from the gravitoelectro field [6, 9]. In Chapter 2, we will derive this set of equations in general relativity, which are analogs of Maxwell equations in electromagnetism. To do this, we will use the Einstein's equation. For an isolated, slowly moving object in a weak gravitational field, we can linearize the field equations by decomposing the metric into flat metric plus a perturbation. With these linearized equations and the appropriate components of the metric perturbation, the analogous equations of electromagnetism can be derived easily in gravity [10].

Despite these successful solutions and predictions, like Newton's theory, there are some observations that GR cannot explain without recourse to additional (dark) matter and (dark) energy in the Universe [5]. The data taken from the supernova explosions point out that the Universe has an acceleration in its expansion and it leads us to the cosmological constant Λ [5, 11, 12, 13]. If GR is totally accurate, then we have to experience a dark energy component which can be represented by the cosmological constant, Λ, added to the Einstein-Hilbert action. If we compare the values of energy density ρ emerged from the experiments and the theory, we will see that a contradiction occurs between these results. The inconsistency between theory and experiments shows that GR is not the whole story, hence it must be modified [2, 5, 13, 15].

A theory of massive gravity is a way to modify the theory of general relativity by adding a mass term to the Einstein- Hilbert action. It was first studied by Pauli-Fierz in 1939, therefore the the added mass is called as "Fierz-Pauli (FP) mass" [13], which alters the interaction between two massive objects as well as the interaction between a massive one and the light. But there was a difference between this massive theory and GR in the prediction of the bending of light. As GR reduces to Newton's theory in some limits, we expect that in the limit of zero mass, results of FP theory should match with those of GR since it is a massless theory. Instead, an inconsistency arises, which is known as the van Dam–Veltman–Zakharov (vDVZ) discontinuity. We know that experiments on the bending of light coincides with GR, therefore FP theory needs a correction [15, 16]. This discontinuity can be removed if we first introduce a cosmological constant and take the limit $\frac{M_G^2}{\Lambda} \to 0$ [17]. But clearly this is not a satisfactory solution. We can describe the interaction between two massive objects by determining the $h_{\mu\nu}$. We can accomplish it by using FP theory whose equations are

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \frac{m^2}{2}(h_{\mu\nu} - g_{\mu\nu}h) = 8\pi G T_{\mu\nu}. \tag{1.3}$$

We still require that the energy-momentum tensor is covariantly conserved ($\nabla_\mu T^{\mu\nu} = 0$). Therefore, because of the Bianchi identity $\nabla_\mu(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R) = 0$, we have $m^2(\nabla_\mu h^{\mu\nu} - \nabla^\nu h) = 0$. As we can derive Einstein's equation from the Einstein-Hilbert action by using calculus of variations, we can also derive the FP equation from the FP action. One important note about the FP equation is that it necessarily is linear in $h_{\mu\nu}$ which is not a full tensor in spacetime but only a tensor with respect to the background metric $\bar{g}_{\mu\nu} = g_{\mu\nu} - h_{\mu\nu}$. Therefore in some sense FP theory is not a complete non-linear theory of massive gravity. It should be considered as a linear theory whose non-linear extension must be found. Recently, such a non-linear extension of FP massive gravity was found in [18, 19]. All non-linear extensions of FP theory eventually are built upon FP theory, hence at the linear level massive gravity is FP theory. In this thesis, gravitomagnetic effects at the linear order are studied, therefore we will not say much about the non-linear ones.

The outline of the thesis is as follows: In the next sections of this chapter linearized gravity is discussed in detail. In a weak field, metric is written as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu}$ is the metric perturbation. By using this metric, we calculate the tools to write Einstein's equations in this weak field. After that, we examine the perturbation by decomposing it into the scalar, vector and tensor components. Using the appropriate gauge transformations, we finally reach the degrees of freedom and prove that some of them are not physical, in other words, they do not represent the propagation of gravitational waves. Then gravitational

waves are studied. To find the gravitational wave solutions, we use transverse traceless gauge and find that the propagating degrees of freedom are the tensor components of the metric perturbation. By studying them, we understand the polarization of the test particles due to gravitational waves. The second chapter is devoted to gravitomagnetic effect. If there is a relation between gravity and electromagnetic theory, then we must find the analogs of Maxwell's equations in the gravitation theory. By studying the linearized field equations of gravity, we will reach this analogy. In the third chapter, FP theory is studied up to a point. Adding the mass to Einstein-Hilbert theory gives us to FP action [5]. Instead of this, we will use the scattering amplitude which tells the whole story of the interaction between two masses. We skip the calculation of the scattering amplitude and use it to find the potential for massive and the massless cases. After that, we show the spin alignments for the minimum energy configuration and prove that they are different for GR and massive gravity. Then the conclusion part comes. Also, an appendix part is added for some useful calculations.
This thesis is based on the papers by Harris "Analogy between general relativity and electromagnetism for slowly moving particles in weak gravitational fields" and by Güllü-Tekin "Spin–spin interactions in massive gravity and higher derivative gravity theories" respectively. The standard material on linearized gravity heavily depends on Carroll's excellent book [6].

1.2 Linearized Gravity

In this section, the linearization of the Einstein's equation about flat background and a gauge transformation, which ensures that the linearized theory is invariant under infinitesimal diffeomorphisms, are discussed. By linearization we mean that we are studying the weak field regime of the theory. In this weak field, we can decompose our metric into background metric and a small perturbation. By using this decomposition, linearized equations are discussed first. At the end of this part, a suitable gauge choice is introduced.

1.2.1 Linearized Einstein Theory

What we mean by weak gravitational field is that the exact spacetime metric can be written as a sum of flat background metric and a perturbation,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad \mid h_{\mu\nu} \mid << 1. \tag{1.4}$$

We assume that the perturbation $h_{\mu\nu}$ is so small that we can ignore terms higher than the first order in the relevant quantities. Smallness of a tensor quantity is somewhat ambiguous, but what we mean here is that there is a set of coordinate systems where the statement is valid. We take the flat background metric as $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. To find the inverse of the metric we take $g^{\mu\nu} = \eta^{\mu\nu} + \alpha h^{\mu\nu}$, then we have

$$\begin{aligned} g_{\mu\nu} g^{\nu\sigma} &= (\eta_{\mu\nu} + h_{\mu\nu})(\eta^{\nu\sigma} + \alpha h^{\nu\sigma}) \\ \delta^{\sigma}_{\mu} &= \delta^{\sigma}_{\mu} + \alpha \eta_{\mu\nu} h^{\nu\sigma} + \eta^{\nu\sigma} h_{\mu\nu} + \alpha h^{\nu\sigma} h_{\mu\nu}. \end{aligned} \tag{1.5}$$

Since the last term is in the second order, it vanishes at the lowest order, hence $\alpha = -1$. Therefore, we obtain the inverse of the metric as $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$ at this order. It is crucial to note that $h_{\mu\nu}$ is not a tensor because it does not transform like a tensor under general coordinate transformation. Instead, we can define it as a symmetric tensor field propagating on a flat background spacetime. Therefore, in the weak field regime the metric perturbation transforms as $h'_{\mu\nu} = \wedge^{\alpha}{}_{\mu} \wedge^{\beta}{}_{\nu} h_{\alpha\beta}$, which means that linearized gravitational theory is invariant under Lorentz transformations [20].

We want to find the linearized Einstein's equation in this weak gravitational field. To do this, we have to compute the Christoffel symbols first, which are

$$\begin{aligned} \Gamma^{\rho}_{\mu\nu} &= \frac{1}{2} g^{\rho\lambda} \left(\partial_{\mu} g_{\lambda\nu} + \partial_{\nu} g_{\lambda\mu} - \partial_{\lambda} g_{\mu\nu} \right), \\ &= \frac{1}{2} (\eta^{\rho\lambda} - h^{\rho\lambda}) \left[\partial_{\mu}(\eta_{\nu\lambda} + h_{\nu\lambda}) + \partial_{\nu}(\eta_{\mu\lambda} + h_{\mu\lambda}) - \partial_{\lambda}(\eta_{\mu\nu} + h_{\mu\nu}) \right]. \end{aligned} \tag{1.6}$$

Multiplication of $h_{\mu\nu}$ to itself is the second order in perturbation. Then by neglecting all the second terms in (1.6) we get

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} \eta^{\rho\lambda} \left(\partial_{\mu} h_{\nu\lambda} + \partial_{\nu} h_{\lambda\mu} - \partial_{\lambda} h_{\mu\nu} \right), \tag{1.7}$$

from which we conclude that Γ^2 has the second order terms. Therefore only the derivative of the connection coefficient appears in the Riemann tensor, which is

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= g_{\mu\lambda} R^{\lambda}_{\nu\rho\sigma} \\ &= \eta_{\mu\lambda}(\partial_\rho \Gamma^{\lambda}_{\nu\sigma} - \partial_\sigma \Gamma^{\lambda}_{\nu\rho}). \end{aligned} \tag{1.8}$$

By inserting (1.7) in (1.8), one is left with the linearized Riemann tensor

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}(\partial_\rho \partial_\nu h_{\mu\sigma} + \partial_\mu \partial_\sigma h_{\nu\rho} - \partial_\mu \partial_\rho h_{\nu\sigma} - \partial_\sigma \partial_\nu h_{\mu\rho}). \tag{1.9}$$

Contracting Riemann tensor over the indices μ and ρ with the flat metric gives us the linearized Ricci tensor as

$$R_{\mu\nu} = \frac{1}{2}(\partial_\lambda \partial_\mu h^{\lambda}_{\nu} + \partial_\nu \partial_\rho h^{\rho}_{\mu} - \partial_\nu \partial_\mu h - \Box h_{\mu\nu}), \tag{1.10}$$

where we have defined $h = \eta^{\mu\nu} h_{\mu\nu} = h^{\mu}{}_{\mu}$ and the d'Alambertian operator as $\Box \equiv \partial_\mu \partial^\mu = -\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2$. Contracting the linearized Ricci tensor gives us the linearized Ricci scalar,

$$R = \eta^{\mu\nu} R_{\mu\nu} = \partial_\mu \partial_\nu h^{\mu\nu} - \Box h. \tag{1.11}$$

Putting all these objects together gives the Einstein's tensor as

$$\begin{aligned} G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \\ &= \frac{1}{2}(\partial_\lambda \partial_\mu h^{\lambda}_{\nu} + \partial_\nu \partial_\rho h^{\rho}_{\mu} - \partial_\nu \partial_\mu h - \Box h_{\mu\nu}) \\ &- \frac{1}{2}(\eta_{\mu\nu} + h_{\mu\nu})(\partial_\mu \partial_\nu h^{\mu\nu} - \Box h). \end{aligned} \tag{1.12}$$

If we omit all the second order terms in (1.12), we finally obtain the linearized Einstein's tensor as,

$$G_{\mu\nu} = \frac{1}{2}(\partial_\lambda \partial_\mu h^{\lambda}_{\nu} + \partial_\nu \partial_\rho h^{\rho}_{\mu} - \partial_\nu \partial_\mu h - \Box h_{\mu\nu} - \eta_{\mu\nu} \partial_\rho \partial_\lambda h^{\rho\lambda} + \eta_{\mu\nu} \Box h). \tag{1.13}$$

Therefore the linearized Einstein's equation in the weak gravitational field is $G_{\mu\nu} = 8\pi G T_{\mu\nu}$, where $G_{\mu\nu}$ is given by (1.13) and $T_{\mu\nu}$ is the energy momentum tensor.

1.2.2 Gauge Transformation

When Einstein formulated General Relativity, coordinate invariance (implicitly) played a major role: Namely, physics should not depend on the choice of the coordinates. For example, a measurable quantity in the x^μ coordinates should be left invariant or should transform as a tensor under a change of coordinates

$$x^\mu \longrightarrow x'^\mu = f^\mu(x^\mu). \tag{1.14}$$

Specifically the metric transforms as a $(0,2)$ rank tensor as

$$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x). \tag{1.15}$$

We can now ask what is left from this coordinate transformation when we decompose the metric as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \tag{1.16}$$

Let us assume a small coordinate transformation of the form $x'^\mu = x^\mu - \epsilon^\mu(x)$, where ϵ^μ is small. Then

$$\begin{aligned} g'_{\mu\nu}(x') &= (\delta^\alpha_\mu + \partial_\mu \epsilon^\alpha)(\delta^\beta_\nu + \partial_\nu \epsilon^\beta) g_{\alpha\beta} \\ &= g_{\mu\nu}(x) + \partial_\nu \epsilon^\beta g_{\mu\beta} + \partial_\mu \epsilon^\alpha g_{\alpha\nu} + O(\epsilon^2) \\ &= g_{\mu\nu}(x) + \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu. \end{aligned} \tag{1.17}$$

Expanding the left hand side as $g'_{\mu\nu}(x-\epsilon) = g'_{\mu\nu}(x) - \epsilon^\alpha \partial_\alpha g_{\mu\nu}(x)$ and writing $g'_{\mu\nu}(x) = \eta_{\mu\nu} + h'_{\mu\nu}(x)$, we get at this order how $h_{\mu\nu}(x)$ transforms

$$h'_{\mu\nu}(x) = h_{\mu\nu}(x) + \partial_\mu \epsilon_\nu(x) + \partial_\nu \epsilon_\mu(x). \tag{1.18}$$

In what follows we define $\epsilon_\mu = \epsilon \xi_\mu$. This formula is called a gauge transformation along a vector field ξ_μ for weak gravitational fields. It shows how the metric perturbations in different coordinate systems differ from each other. Now, let us verify that curvature of a spacetime is invariant under this gauge transformation. To do this, we define the Riemann tensor in the new coordinate system as $R'_{\mu\nu\rho\sigma}$, which is

$$\begin{aligned} R'_{\mu\nu\rho\sigma} &= \frac{1}{2}(\partial_\rho \partial_\nu h'_{\mu\sigma} + \partial_\mu \partial_\sigma h'_{\nu\rho} - \partial_\mu \partial_\rho h'_{\nu\sigma} - \partial_\sigma \partial_\nu h'_{\mu\rho}) \\ &= \frac{1}{2}[\partial_\rho \partial_\nu (h_{\mu\sigma} + 2\epsilon \partial_{(\mu}\xi_{\sigma)}) + \partial_\mu \partial_\sigma (h_{\rho\nu} + 2\epsilon \partial_{(\rho}\xi_{\nu)}) \\ &\quad - \partial_\mu \partial_\rho (h_{\sigma\nu} + 2\epsilon \partial_{(\sigma}\xi_{\nu)}) - \partial_\sigma \partial_\nu (h_{\mu\rho} + 2\epsilon \partial_{(\mu}\xi_{\rho)})], \end{aligned} \tag{1.19}$$

where $h'_{\mu\nu}$ is obtained by transforming $h_{\mu\nu}$ according to (1.18). Therefore the change in the Riemann tensor, $R'_{\mu\nu\rho\sigma} - R_{\mu\nu\rho\sigma} = \delta R_{\mu\nu\rho\sigma}$, is

$$\begin{aligned}\delta R_{\mu\nu\rho\sigma} &= \partial_\rho \partial_\nu \epsilon \partial_{(\mu} \xi_{\sigma)} + \partial_\sigma \partial_\mu \epsilon \partial_{(\rho} \xi_{\nu)} - \partial_\rho \partial_\mu \epsilon \partial_{(\sigma} \xi_{\nu)} - \partial_\sigma \partial_\nu \epsilon \partial_{(\mu} \xi_{\rho)} \\ &= 0. \end{aligned} \tag{1.20}$$

This result tells us that different metric perturbations which are related to (1.18) have the same curvature, therefore the same physical situation [21].

1.3 Degrees of Freedom

The existence of gauge transformations suggest that not all ten components of the symmetric tensor $h_{\mu\nu}$ are physical or true degrees of freedom. In this section, we obtain these degrees of freedom using algebraic and additional decompositions. With the former we decompose the metric perturbation into scalar, vector and tensor pieces, which transform into themselves under spatial transformations. Then by the latter, we decompose the elements of these pieces such that none of them can be further decomposed, namely they become irreducible representations of the rotation group. Therefore we obtain the irreducible components of $h_{\mu\nu}$, hence the ten degrees of freedom. We rewrite the Einstein's equation in terms of the components of $h_{\mu\nu}$, from which we can pick the physical degrees of freedom. We also introduce some examples of gauges which can be suitable for different circumstances. We examine this section in three parts. In the first part, we introduce the algebraic decomposition and identify the physical degrees of freedom. After that we obtain the Einstein's equation for different gauges which will be handy in the following sections. In the last part, we introduce additional decomposition and obtain all degrees of freedom which can not be decomposed any other small pieces. We use the notations and conventions of Caroll's book [6].

1.3.1 Components of The Metric Perturbation

As we have mentioned above, we decompose the metric perturbation $h_{\mu\nu}$ into three parts: The 00 component is a spatial scalar, the $0i$ components are spatial three vectors and the ij components are the symmetric spatial tensors, which are

$$\begin{aligned} h_{00} &\equiv -2\Phi \\ h_{0i} &\equiv w_i \\ h_{ij} &\equiv 2s_{ij} - 2\Psi\delta_{ij}. \end{aligned} \tag{1.21}$$

Where Ψ contains the trace of h_{ij} and s_{ij} is the strain tensor which is traceless. Hence it can be found as

$$\begin{aligned} \Psi &= -\frac{1}{6}\delta^{ij}h_{ij} \\ s_{ij} &= \frac{1}{2}(h_{ij} - \frac{1}{3}\delta^{kl}h_{kl}\delta_{ij}). \end{aligned} \tag{1.22}$$

We know that the full metric is defined as,

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = (\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu. \tag{1.23}$$

If we use the decomposition of metric and put the corresponding components into the equation , we obtain

$$ds^2 = -(1+2\Phi)dt^2 + w_i(dtdx^i + dx^i dt) + ((1-2\Psi)\delta_{ij} + 2s_{ij})\,dx^i dx^j. \tag{1.24}$$

If we examine the geodesic equation, we can find the fields which determine the motion of the test particles moving in gravitational field. In the previous section, we derived the Christoffel symbols. If we use the corresponding perturbation components, we obtain

$$\Gamma^0_{00} = \frac{1}{2}\eta^{0\lambda}(\partial_0 h_{0\lambda} + \partial_0 h_{0\lambda} - \partial_\lambda h_{00}). \tag{1.25}$$

It survives only for $\lambda = 0$, otherwise $\eta^{0\lambda} = 0$, which means that only h_{00} appears in the equation. Since $h_{00} \equiv -2\Phi$, we get

$$\Gamma^0_{00} = \partial_0\Phi. \tag{1.26}$$

For $\mu, \sigma, \rho = i, 0, 0$, the Christoffel connection becomes,

$$\Gamma^i_{00} = \frac{1}{2}\eta^{i\lambda}(\partial_0 h_{0\lambda} + \partial_0 h_{0\lambda} - \partial_\lambda h_{00}), \tag{1.27}$$

which survives only for $\lambda = i$. With the related components of the perturbation, it reads

$$\Gamma^i_{00} = \partial_i\Phi + \partial_0\omega_i. \tag{1.28}$$

From (1.7), the $0j0$ component of the connection coefficient is,

$$\Gamma^0_{j0} = \frac{1}{2}\eta^{0\lambda}(\partial_j h_{0\lambda} + \partial_0 h_{j\lambda} - \partial_\lambda h_{0j}). \tag{1.29}$$

As it is noticed easily, h_{00} is the only survivor then one has

$$\Gamma^0_{j0} = \partial_j \Phi. \tag{1.30}$$

If we carry on this procedure, we obtain the rest of the connection coefficients as,

$$\begin{aligned}
\Gamma^i_{j0} &= \partial_{[j}\omega_{i]} + \frac{1}{2}\partial_0 h_{ij} \\
\Gamma^0_{jk} &= -\partial_{(j}\omega_{k)} + \frac{1}{2}\partial_0 h_{jk} \\
\Gamma^i_{jk} &= \partial_{(j}h_{k)i} - \frac{1}{2}\partial_i h_{jk},
\end{aligned} \tag{1.31}$$

where $A_{[i}B_{j]} = \frac{1}{2}(A_iB_j - A_jB_i)$ and $A_{(i}B_{j)} = \frac{1}{2}(A_iB_j + A_jB_i)$. These equations will be very useful in computing the geodesic equation.
In an inertial frame, components of the four momentum are

$$\begin{aligned}
p^0 &= \frac{dx^0}{d\lambda} = E \\
p^i &= \frac{dx^i}{d\lambda} = Ev^i.
\end{aligned} \tag{1.32}$$

If we rewrite the geodesic equation in terms of the components of momentum, we obtain

$$\frac{dp^\mu}{d\lambda} + \Gamma^\mu_{\rho\sigma}p^\rho p^\sigma = 0. \tag{1.33}$$

If we manipulate the first term, we get $\frac{dp^\mu}{d\lambda} = \frac{dt}{d\lambda}\frac{dp^\mu}{dt}$. Plugging this result into (1.33) gives

$$\frac{dp^\mu}{d\lambda} = -\Gamma^\mu_{\rho\sigma}\frac{p^\rho p^\sigma}{E}. \tag{1.34}$$

For $\mu = 0$, (1.34) describes the evolution of the energy (or power),

$$\frac{dE}{dt} = -\frac{1}{E}(\Gamma^0_{0\sigma}p^0p^\sigma + \Gamma^0_{i\sigma}p^ip^\sigma) \tag{1.35}$$

with the related Christoffel symbols and components of the momentum tensor, this equation becomes,

$$\frac{dE}{dt} = -E[\partial_0\Phi + 2(\partial_k\Phi)v^k - (\partial_{(j}\omega_{k)} - \frac{1}{2}\partial_0 h_{jk})v^jv^k]. \tag{1.36}$$

For $\mu = i$, (1.34) describes the spatial components of the geodesic equation, which are

$$\frac{dp^i}{dt} = -\Gamma^i_{\rho\sigma}\frac{p^\rho p^\sigma}{E}. \tag{1.37}$$

Inserting the appropriate terms gives us,

$$\frac{dp^i}{dt} = -E[\partial_i\Phi + \partial_0\omega_i + 2(\partial_{[i}\omega_{j]} + \partial_0 h_{ij})v^j + (\partial_{(j}h_{k)i} - \frac{1}{2}\partial_i h_{jk})v^j v^k]. \tag{1.38}$$

Let us define the gravitoelectric and the gravitomagnetic three vector fields as

$$\begin{aligned} G^i &= -\partial_i\Phi - \partial_0\omega_i \\ H^i &= (\vec{\nabla}\times\vec{\omega})^i = \epsilon^{ijk}\partial_j\omega_k. \end{aligned} \tag{1.39}$$

Then (1.38) becomes,

$$\frac{dp^i}{dt} = E[G^i + (\vec{v}\times\vec{H})^i - 2(\partial_0 h_{ij})v^j - (\partial_{(j}h_{k)i} - \frac{1}{2}\partial_i h_{jk})v^j v^k]. \tag{1.40}$$

It is easily noticed that (1.40) looks like the Lorentz force in electromagnetism, which shows that the motion of a charged particle is affected by both electric and magnetic fields. As we see from (1.39), we have similar expressions in weak gravitational fields. Therefore we infer from (1.40) tells that the motion of a particle in a weak gravitational field is determined by the gravitoelectric field as well as gravitomagnetic one. Let us continue with the linearized Einstein's tensor. As we know the components of $h_{\mu\nu}$, we can rewrite (1.13) in terms of them. We follow the same procedure used in the previous section. As we have already determined the Christoffel symbols, we carry on the discussion with the Riemann tensor. If we put the appropriate components of $h_{\mu\nu}$ into (1.9), we find the components of the Riemann tensor as

$$\begin{aligned} R_{0j0l} &= \partial_j\partial_l\Phi + \partial_0\partial_{(j}\omega_{l)} - \frac{1}{2}\partial_0\partial_0 h_{jl} \\ R_{0jkl} &= \partial_j\partial_{[k}\omega_{l]} - \partial_0\partial_{[k}h_{l]i} \\ R_{ijkl} &= \partial_j\partial_{[k}h_{l]i} - \partial_i\partial_{[k}h_{l]j}. \end{aligned} \tag{1.41}$$

Contracting on two of the indices, one gets

$$R_{\mu\nu} = \eta^{\rho\sigma}R_{\mu\rho\nu\sigma} = -R_{\mu0\nu0} + \delta^{ij}R_{\mu i\nu j}, \tag{1.42}$$

from which we reach the components of the Ricci tensor, which are

$$
\begin{aligned}
R_{00} &= \vec{\nabla}^2\Phi + \partial_0\partial_k\omega^k + 3\partial_0^2\Psi \\
R_{0j} &= \frac{1}{2}\partial_j\partial_k\omega^k - \frac{1}{2}\vec{\nabla}^2\omega_j + 2\partial_0\partial_j\Psi + \partial_0\partial_k s^k{}_j \\
R_{ij} &= -\partial_i\partial_j(\Phi - \Psi) - \partial_0\partial_{(i}\omega_{j)} + \Box\Psi\delta_{ij} - \Box s_{ij} + 2\partial_k\partial_{(i}s_{j)^k},
\end{aligned} \tag{1.43}
$$

where we have defined $\vec{\nabla}^2 = \delta^{ij}\partial_i\partial_j$ in three dimensional flat space. If we express the Ricci scalar as $R = \eta^{\rho\sigma}R_{\rho\sigma}$, we can rewrite the Einstein's tensor as

$$
\begin{aligned}
G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}R_{\rho\sigma} \\
&= R_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}R_{00} - \frac{1}{2}\eta_{\mu\nu}\eta^{ij}R_{ij}.
\end{aligned} \tag{1.44}
$$

With (1.21) and (1.43), the components of (1.44) become

$$
\begin{aligned}
G_{00} &= 2\vec{\nabla}^2\Psi + \partial_k\partial_l s^{kl} \\
G_{0j} &= -\frac{1}{2}\vec{\nabla}^2\omega_j + \frac{1}{2}\partial_j\partial_k\omega^k + 2\partial_0\partial_j\Psi + \partial_0\partial_k s^k{}_j \\
G_{ij} &= (\delta_{ij}\vec{\nabla}^2 - \partial_i\partial_j)(\Phi - \Psi) + \delta_{ij}\partial_0\partial_k\omega^k - \partial_0\partial_{(i}\omega_{j)} + 2\delta_{ij}\partial_0^2\Psi \\
&\quad - \Box s_{ij} + 2\partial_k\partial_{(i}s_{j)^k} - \delta_{ij}\partial_k\partial_l s^{kl}.
\end{aligned} \tag{1.45}
$$

As we have mentioned in the beginning of this section, not all ten components are physical degrees of freedom. We can show this by using (1.45) in Einstein's equation. Let us begin by equating the first equation of (1.45) to $8\pi G T_{00}$, which gives us

$$
\vec{\nabla}^2\Psi = 4\pi G T_{00} - \frac{1}{2}\partial_k\partial_l s^{kl}. \tag{1.46}
$$

It is seen that (1.46) is not a wave equation therefore Ψ does not propagate. We can determine it in terms of T_{00} and s^{kl} for any time. In other words, we solve (1.46) for $\Psi(t, \vec{x})$ using the three dimensional Green's function.

Next, if we use the second equation of (1.45) in the $0j$ component of Einstein's equation and raise the index of ω_j, we obtain

$$
(\delta_{jk}\vec{\nabla}^2 - \partial_j\partial_k)\omega^k = -16\pi G T_{0j} + 4\partial_0\partial_j\Psi + 2\partial_0\partial_k s^k{}_j, \tag{1.47}
$$

which does not depend on time either, so ω_j is not a propagating degree of freedom. Similar to Ψ, it is determined by T_{0j} and the strain tensor s^{ij}. Finally, the ij component of the Einstein's equation gives

$$(\delta_{ij}\vec{\nabla}^2 - \partial_i\partial_j)\Phi = 8\pi G T_{ij} + (\delta_{ij}\vec{\nabla}^2 - \partial_i\partial_j - 2\delta_{ij}\partial_0^2)\Psi - \delta_{ij}\partial_0\partial_k\omega^k$$
$$+ \partial_0\partial_{(i}\omega_{j)} + \Box s_{ij} - 2\partial_k\partial_{(i}s_{j)^k} - \delta_{ij}\partial_k\partial_l s^{jl}, \qquad (1.48)$$

which shows that the above explanations are valid also for Φ, so we conclude that it is not a propagating degree of freedom, either.

As we understand from the above discussions, scalar and vector components of $h_{\mu\nu}$ are determined in terms of the strain and the energy momentum tensors; therefore they are not true degrees of freedom. Propagating degrees of freedom come only from the tensor piece of $h_{\mu\nu}$. By propagating degrees of freedom, we mean that they contain all the information about the gravitational radiation. At the end of this section, we will describe these degrees of freedom.

1.3.2 Gauge Transformations

In the previous section, we obtained the linear gauge transformation which leaves the Riemann tensor invariant. Under this transformation, components of the metric perturbation change as,

$$\begin{aligned}
\Phi &\to \Phi + \partial_0\xi^0 \\
\omega_i &\to \omega_i + \partial_0\xi^i - \partial_i\xi^0 \\
\Psi &\to \Psi - \frac{1}{3}\partial_i\xi^i \\
s_{ij} &\to s_{ij} + \partial_{(i}\xi_{j)} - \frac{1}{3}\partial_k\xi^k\delta_{ij}.
\end{aligned} \qquad (1.49)$$

In electromagnetism, there are some gauges which fit for different cases. The same situation is valid also in gravity. By constructing some analogies with electromagnetism, we define analogous gauges in gravity.

Let us start with the "transverse gauge", which looks like the Coulomb gauge, $\partial_i A^i$. We first assume that the strain tensor satisfies this, which we mean the equation

$$\partial_i s^{ij} = 0. \qquad (1.50)$$

If we manipulate the last equation of (1.49) according to (1.50), we get

$$\vec{\nabla}^2\xi^j + \frac{1}{3}\partial_j\partial_i\xi^i = -2\partial_i s^{ij}. \qquad (1.51)$$

Here, there is no boundary condition given; instead of specifying a solution we are interested in whether it exists or not. We know that the solution of (1.51) is written in terms of Green's function, so we can choose such ξ^j that it satisfies (1.50). Still we have to determine ξ^0. To do this we assume that the condition on (1.50) is also valid for ω^i, by which we mean

$$\partial_i \omega^i = 0, \tag{1.52}$$

which leads us to

$$\vec{\nabla}^2 \xi^0 = \partial_i \omega^i + \partial_0 \partial_i \xi^i. \tag{1.53}$$

Similarly, we can choose ξ^0 which satisfies the above equation. As we can determine ξ^μ from the corresponding differential equations, we define the transverse gauge by (1.50) and (1.52). If we use these conditions on (1.45), the components of the Einstein's equation for the transverse gauge becomes,

$$\begin{aligned}
G_{00} &= 2\vec{\nabla}^2 \Phi = 8\pi G T_{00} \\
G_{0j} &= -\frac{1}{2}\vec{\nabla}^2 \omega_j + 2\partial_0 \partial_j \Psi = 8\pi G T_{0j} \\
G_{ij} &= (\delta_{ij}\vec{\nabla}^2 - \partial_i \partial_j)(\Phi - \Psi) - \partial_0 \partial_{(i}\omega_{j)} + 2\delta_{ij}\partial_0^2 \Psi - \Box s_{ij} = 8\pi G T_{ij}.
\end{aligned} \tag{1.54}$$

We use these equations to determine the gravitational waves in the following section.

Let us continue with the "synchronous gauge", which looks like the temporal gauge, $A^0 = 0$. We assume that it is valid for the scalar potential Φ, which means

$$\Phi = 0, \tag{1.55}$$

If we use it in (1.49), we get

$$\partial_0 \xi^0 = -\Phi. \tag{1.56}$$

Just as the first example, we can find some ξ^0 which satisfies (1.56) by direct integration; therefore if we find ξ^i then we are done. If all the vector components vanish,

$$\omega^i = 0. \tag{1.57}$$

If we use it in the second equation of (1.49), we obtain

$$\partial_0 \xi^i = -\omega^i + \partial_i \xi^0. \tag{1.58}$$

As we have already determined ξ^0, we can also choose ξ^i which satisfies (1.58). Again, since we can determine ξ^μ by differential equations above therefore the conditions (1.55) and (1.57) together define the synchronous gauge.
"Lorenz gauge" is the last example of the gauges we discuss, which is defined as

$$\partial_\mu h^\mu{}_\nu - \frac{1}{2}\partial_\nu h = 0. \tag{1.59}$$

As we show in the last section of this chapter, this gauge is used to calculate the production of gravitational waves.

1.3.3 Further Reduction

The decomposition of the metric perturbation into scalar, vector and tensor components is known as algebraic decomposition. By additional decomposition, we can determine the physical degrees of freedom more directly. It is based on the idea that a vector field $\vec{\omega}$ can be decomposed into transverse $\vec{\omega_\perp}$ and longitudinal $\vec{\omega_\parallel}$ parts:

$$\omega^i = \omega^i_\perp + \omega^i_\parallel, \tag{1.60}$$

where a transverse vector is divergenceless and a longitudinal vector is curl-free, which are described by

$$\partial_i \omega^i_\perp = 0 \ \text{ and } \ \epsilon^{ijk}\partial_j \omega_{\parallel k} = 0, \tag{1.61}$$

respectively. Since the divergence of curl is zero, then we can represent the transverse part as a curl of some other vector ξ^i, which means

$$\omega^i_\perp = \epsilon^{ijk}\partial_j \xi_k. \tag{1.62}$$

Similarly, a longitudinal vector is the divergence of a scalar λ,

$$\omega_{\parallel i} = \partial_i \lambda. \tag{1.63}$$

It is clear that λ represents one degree of freedom. If we take the divergence of (1.62), to satisfy the first equation of (1.61), we have to set

$$\partial_j \xi^j = 0, \tag{1.64}$$

which means that ξ^j is transverse. Although ξ^j represents three vectors, from (1.62) and (1.64) we can find only two of them. So one of them is determined by the other two. Therefore with the scalar λ, the vector field ω_i has three degrees of freedom.

If we apply the similar procedure for the strain tensor, we get

$$s^{ij} = s^{ij}_{\perp} + s^{ij}_{S} + s^{ij}_{\|}, \tag{1.65}$$

where the terms in the right hand side of (1.65) are known as transverse, solenoidal and longitudinal parts, respectively. Transverse part is divergenceless, i.e. $\partial_i s^{ij}_{\perp} = 0$. Divergence of the solenoidal part is a transverse vector, which is divergenceless and is written as

$$\begin{aligned} \partial_i s^{ij}_S &= s^{ij}_{\perp} \\ \partial_j \partial_i s^{ij}_S &= \partial_j s^{ij}_{\perp} = 0, \end{aligned} \tag{1.66}$$

and the divergence of the longitudinal vector is again a longitudinal vector, which is curl-free,

$$\begin{aligned} \partial_i s^{ij}_{\|} &= s^{ij}_{\|} \\ \epsilon^{jkl} \partial_i \partial_k s^{i}_{\|j} &= 0. \end{aligned} \tag{1.67}$$

The curl of the gradient is zero, therefore longitudinal part can be expressed in terms of a scalar θ, and the solenoidal part can be derived from a transverse vector ζ^i as

$$\begin{aligned} s^{ij}_{\|} &= (\partial_i \partial_j - \frac{1}{3}\delta_{ij} \vec{\nabla}^2)\theta \\ s_{sij} &= \partial(_i \zeta_j), \end{aligned} \tag{1.68}$$

where $\partial_i \zeta^i = 0$ and the round bracket represents symmetrization.

It is clear that the scalar θ describes one degree of freedom. The explanation for ξ^i is also valid for ζ^j, which means that ζ^j represents two degrees of freedom. Therefore there are two degrees of freedom left which the transverse traceless tensor $s_\perp^{ij}$ represents. We finally describe the ten components of $h_{\mu\nu}$ in terms of four scalars $(\Phi, \Psi, \lambda, \theta)$ with one degree of freedom each; two vectors (η^i, ζ^j) with two degrees of freedom each and one transverse traceless tensor $s_\perp^{ij}$ with two degrees of freedom, which are the physical ones.

1.4 Gravitational Waves

In this section, we examine the gravitational radiation by using the propagating degrees of freedom we found in section 1.3. We prefer to study in vacuum to neglect the effects of the source, therefore all components of the energy momentum tensor vanish. If we use this in (1.54), we obtain the 00 component of the Einstein's tensor as

$$G_{00} = 2\nabla^2\Psi = 0. \tag{1.69}$$

from which we get $\Psi = 0$, assuming that $\Psi = 0$ at infinity, this follows. If we use it, we obtain the $0j$ components as

$$G_{0j} = -1/2\nabla^2\omega_j = 0, \tag{1.70}$$

which again imply $w_j = 0$. Finally, with the condition $\Psi = w_j = 0$, the ij components become

$$G_{ij} = (\delta_{ij}\nabla^2 - \partial_i\partial_j)\Phi - \Box s_{ij} = 0. \tag{1.71}$$

We know that s_{ij} is traceless, so if we take the trace of (1.71), we obtain

$$\begin{aligned} \delta^{ij}(\delta_{ij}\nabla^2 - \partial_i\partial_j)\Phi = 0 \\ \nabla^2\Phi = 0, \end{aligned} \tag{1.72}$$

which gives us $\Phi = 0$. If we plug it into (1.71), the ij equation becomes

$$\Box s_{ij} = 0, \tag{1.73}$$

which emerges a wave equation for s_{ij}. Instead of solving this, we prefer to continue with $h_{\mu\nu}$ in which Ψ, Φ and w_i vanish and s_{ij} is transverse. This form of $h_{\mu\nu}$ is known as the "transverse traceless gauge". Under this gauge, (1.21) can be rewritten as

$$\begin{aligned} h_{00}^{TT} &= h_{0i}^{TT} = 0 \\ h_{ij}^{TT} &= 2s_{ij}. \end{aligned} \tag{1.74}$$

We therefore write the transverse traceless gauge $h_{\mu\nu}^{TT}$ in the matrix form, which is

$$h_{\mu\nu}^{TT} = 2\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & s_{11} & s_{12} & s_{13} \\ 0 & s_{21} & s_{22} & s_{23} \\ 0 & s_{31} & s_{32} & s_{33} \end{bmatrix}. \tag{1.75}$$

It is easy to see that with the above matrix, (1.73) implies,

$$\Box h_{\mu\nu}^{TT} = 0, \tag{1.76}$$

which is the wave equation whose solutions help us to understand the characterization of the gravitational waves. Before finding these solutions, if we dig $h_{\mu\nu}^{TT}$ more, we reach the following results: As it is understood from (1.75), $h_{\mu\nu}^{TT}$ is purely spatial,

$$h_{0\nu}^{TT} = 0. \tag{1.77}$$

If we take the trace of (1.77), we get

$$\eta^{\mu\nu} h_{\mu\nu}^{TT} = 0. \tag{1.78}$$

Since s_{ij} is a transverse traceless tensor, this leads us to

$$\partial_\mu h_{TT}^{\mu\nu} = 0. \tag{1.79}$$

Now we are ready to solve the wave equation in (1.76). We know that the solutions of this kind of equations are the plane waves, therefore we have

$$h_{\mu\nu}^{TT} = C_{\mu\nu} e^{ik_\sigma x^\sigma}, \tag{1.80}$$

where $C_{\mu\nu}$ represents the symmetric amplitude matrix with (complex) constant components and k_σ represents the constant (real) wave-four vector. We take the real part of (1.80) because we are interested in the physical solutions [20, 22]. We can find the components of the amplitude matrix $C_{\mu\nu}$ from (1.80). For $\mu = 0$, it becomes

$$h_{0\nu}^{TT} = C_{0\nu} e^{ik_\sigma x^\sigma}. \tag{1.81}$$

which is true for all $k_\sigma x^\sigma$, therefore with (1.77) we reach that $C_{\mu\nu}$ is purely spatial, which is

$$C_{0\nu} = 0. \tag{1.82}$$

If we take the trace of (1.80) and use (1.78), we reach that $C_{\mu\nu}$ is also traceless, which is

$$\eta^{\mu\nu} C_{\mu\nu} = 0. \tag{1.83}$$

We introduce (1.80) as the solution of the wave equation; hence it has to satisfy (1.76), which gives

$$-k^\sigma k_\sigma h_{\mu\nu}^{TT} = 0. \tag{1.84}$$

From (1.75), it is seen that all components of $h_{\mu\nu}^{TT}$ are not zero, so above is true for

$$k^\sigma k_\sigma = 0, \tag{1.85}$$

which indicates that (1.80) can be accepted as a solution of the wave equation if the wave vector is null; that is the wave vector is propagating at the speed of light.

Finally if we take the divergence of (1.80) and use (1.79), we get

$$(ik_\mu) C^{\mu\nu} e^{ik_\sigma x^\sigma} = 0, \tag{1.86}$$

which implies

$$C^{\mu\nu} k_\mu = 0, \tag{1.87}$$

which shows that the wave vector and $C^{\mu\nu}$ are orthogonal.

To be more specific on the wave vector, let us set its timelike component to its angular frequency and choose the direction of propagation in the z direction

$$k^\mu = (\omega, 0, 0, k) = (\omega, 0, 0, \omega), \tag{1.88}$$

where $k = \omega$ because k^μ is null. The conditions on (1.77), (1.87) and (1.88) together imply

$$C_{3\nu} = 0. \tag{1.89}$$

Therefore we conclude that the non-vanishing components of $C_{\mu\nu}$ are only $C_{11}, C_{12}, C_{12}, C_{22}$. But $C_{\mu\nu}$, is traceless and symmetric, therefore we can write it in the matrix form as

$$C_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & C_{11} & C_{12} & 0 \\ 0 & C_{21} & -C_{11} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{1.90}$$

Hence under this gauge, these two components characterize the plane wave propagating in the z direction.

Gravitational waves may have some physical effects on the test particles, which are initially at rest. It is not enough to consider the path of a single particle only, because it stays stationary in the transverse traceless gauge, regardless of the wave's propagation. To obtain a coordinate independent measure of its effects, we consider the relative motion on the test particles [6, 20]. To de this, we examine the equation of the geodesic deviation, which is

$$\frac{D^2}{d\tau^2} S^\mu = R^\mu_{\nu\rho\sigma} U^\nu U^\rho S^\sigma, \tag{1.91}$$

where U^μ is the four-velocity of these particles and S^μ is the separation vector. We want to write the right hand side of (1.91) to the first order in $h^{TT}_{\mu\nu}$. If we study with slowly moving particles, we can write U^μ as a sum of unit vector with timelike component and higher order terms in $h^{TT}_{\mu\nu}$, which we neglect. Therefore U^μ can be expressed by

$$U^\mu = (1, 0, 0, 0), \tag{1.92}$$

which implies that we only need $R_{\mu 00\sigma}$. In transverse traceless gauge, (1.9) turns to

$$R_{\mu 00\sigma} = \frac{1}{2}(\partial_0\partial_0 h^{TT}_{\mu\sigma} + \partial_\mu\partial_\sigma h^{TT}_{00} - \partial_0\partial_\sigma h^{TT}_{\mu 0} - \partial_0\partial_\mu h^{TT}_{0\sigma}). \tag{1.93}$$

If we use (1.75) and contract (1.93), we rewrite the Riemann tensor as

$$\begin{aligned} R^{\mu}_{00\sigma} &= \eta^{\mu\lambda} R_{\lambda 00\sigma} \\ &= \frac{1}{2}\frac{\partial^2}{\partial t^2} h^{TT\mu}_{\sigma}. \end{aligned} \tag{1.94}$$

We know $\tau = t$ for slowly moving particles. So with the above equation, one can obtain (1.91) as

$$\frac{\partial^2}{\partial t^2} S^{\mu} = \frac{1}{2} S^{\sigma} \frac{\partial^2}{\partial t^2} h^{TT\mu}_{\sigma}. \tag{1.95}$$

We have chosen that the wave is passing in the z direction. For $\mu = 3$, (1.95) vanishes so we infer that only S^1 and S^2, which are perpendicular to the travelling wave, will be affected in the presence of the wave. Therefore the gravitational wave is transverse in both its mathematical formulation and physical effects [20]. As we have mentioned before, $C_{\mu\nu}$ represents the characterization of the wave. We can rename its components as

$$h_+ = C_{11} \;\; and \;\; h_\times = C_{12}, \tag{1.96}$$

so we can replace (1.90) by,

$$C_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{1.97}$$

Let us begin with discussing the effects of h_+ only, by setting $h_\times = 0$. For $\mu = 1$, (1.95) becomes

$$\frac{\partial^2}{\partial t^2} S^1 = \frac{1}{2} S^1 h^{TT}_{11} + \frac{1}{2} S^2 h^{TT}_{12}. \tag{1.98}$$

For $h_\times = h_{12} = 0$, if we use (1.80), (1.98) is rewritten as

$$\frac{\partial^2}{\partial t^2} S^1 = \frac{1}{2} S^1 (h_+ e^{ik_\sigma x^\sigma}). \tag{1.99}$$

If we apply the same procedure for $\mu = 2$, we obtain

$$\frac{\partial^2}{\partial t^2} S^2 = -\frac{1}{2} S^2 (h_+ e^{ik_\sigma x^\sigma}). \tag{1.100}$$

Solutions of the differential equations in (1.99) and (1.100)are found as

$$S^1 = (1 + \frac{1}{2}h_+ e^{ik_\sigma x^\sigma})S^1(0),$$
$$S^2 = (1 - \frac{1}{2}h_+ e^{ik_\sigma x^\sigma})S^2(0). \tag{1.101}$$

Thus, particles with initial separation in the x direction will oscillate in the same direction; similar to those with initial y separation. Consider these particles form a circle in the xy plane and for some time we have $h_+ > 0$. Then (1.101) shows that particles oscillate in the x direction move apart, while those in y direction comes closer; which means that the circle formed by particles will be squashed. Later h_+ becomes zero so the circle turns to its original form. After h_+ becomes negative, the procedure for positive h_+ will be reversed [23]. This is known as $+$ polarization, which is shown in the Figure 1.1. If we examine the case where

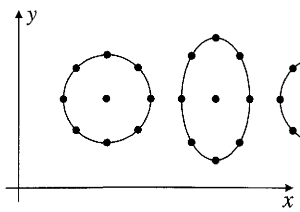

Figure 1.1: The effect of a gravitational wave with $+$ polarization.

$h_+ = 0$ but $h_\times \neq 0$, we get

$$\frac{\partial^2}{\partial t^2}S^1 = \frac{1}{2}S^2(h_\times e^{ik_\sigma x^\sigma}),$$
$$\frac{\partial^2}{\partial t^2}S^2 = \frac{1}{2}S^1(h_\times e^{ik_\sigma x^\sigma}). \tag{1.102}$$

which yield the solutions

$$S^1 = S^1(0) + \frac{1}{2}h_\times e^{ik_\sigma x^\sigma}S^2(0),$$
$$S^2 = S^2(0) + \frac{1}{2}h_\times e^{ik_\sigma x^\sigma}S^1(0). \tag{1.103}$$

It seems that these solutions are rotated. Hence the test particles oscillate in the same forms but in the rotated axis [23]. This is known as $\times$ polarization, which is shown in the Figure 1.2.

As it is understood from the figures, h_+ and $h_\times$ represent the plus and the cross polarizations, which are the independent modes of linear polarization of

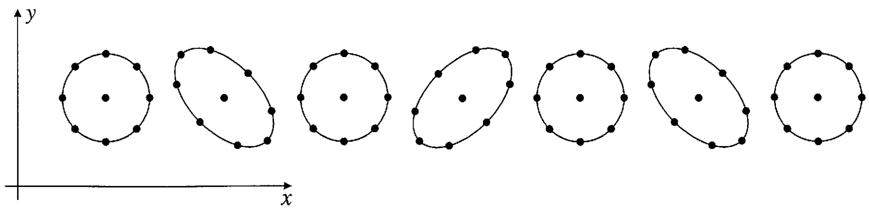

Figure 1.2: The effect of a gravitational wave with $\times$ polarization.

the gravitational waves. As we have mentioned in section 1.3, h_+ and $h_\times$ are the propagating degrees of freedom, which are obtained from $s_\perp^{ij}$ and we have shown that they characterize the gravitational radiation.
We can also describe the modes of right and left handed polarizations as

$$\begin{aligned} h_R &= \frac{1}{\sqrt{2}}(h_+ + ih_\times), \\ h_L &= \frac{1}{\sqrt{2}}(h_+ - ih_\times). \end{aligned} \tag{1.104}$$

The effect of a pure h_R and h_L wave is seen in Figure 1.3.

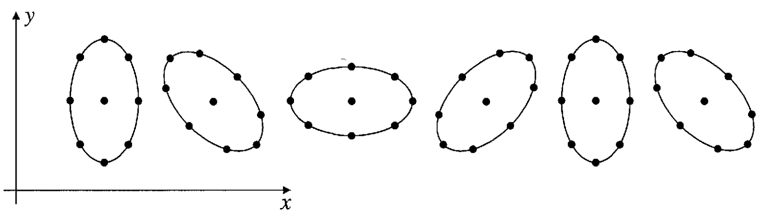

Figure 1.3: The effect of a gravitational wave with R polarization that rotates test particles in a right handed sense.

1.5 Gravitational Wave Production

In electromagnetism, electromagnetic radiation originates from the accelerated charges. In gravity, there are analogous waves which are generated from the accelerated massive objects. Although we expect that this radiation rises from the dipole term, because of the conservation of momentum we see that the gravitational radiation is proportional to the second derivative of the quadrupole moment tensor; which is the scope of this section [22]. For this purpose, we couple Einstein's equation to matter, which means that the energy momentum tensor does not vanish anymore. It implies that the scalar and vector components as well as the strain tensor will appear in the solutions of this equation. Therefore gravitational radiation can not be in the transverse traceless form. Instead, we introduce the trace-reversed perturbation, which reduces to $h_{\mu\nu}^{TT}$ far from the sources. If we plug it into the linearized field equations and solve them under the Lorenz gauge, we obtain a wave equation whose solutions can be written in terms of Green's function. If we apply Fourier transform to these solutions, we will see that the spatial components of the trace-reversed perturbation contains the gravitational radiation. We examine this section in two parts. In the first part we introduce the trace-reversed perturbation and its properties. By using Lorenz gauge, we solve the Einstein's equation in terms of the new perturbation. In the second part, we apply Fourier transform to these solutions to show that the gravitational radiation produced by an isolated massive object is proportional to the second derivative of the quadrupole moment tensor of the energy density.

1.5.1 Einstein's Equation in the Presence of Matter

Let us begin by introducing the trace-reversed perturbation,

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu}. \tag{1.105}$$

It is a reasonable name for this perturbation, since

$$\bar{h} = -h. \tag{1.106}$$

We have shown that, in vacuum gravitational waves are in transverse traceless form. Hence trace-reversed perturbation has to reduce to this form away from the source. If we remove the trace of (1.105) and take the transverse of what is left, we obtain

$$\bar{h}_{\mu\nu}^{TT} = h_{\mu\nu}^{TT}. \tag{1.107}$$

Under a gauge transformation defined in (1.18), we transform (1.105) as

$$\bar{h}'_{\mu\nu} = h'_{\mu\nu} - \frac{1}{2}h'\eta_{\mu\nu}, \tag{1.108}$$

in which we need to find what h' is, since it is trivial to obtain $h'_{\mu\nu}$ by using (1.18). If we take the trace of $h'_{\mu\nu}$, we get

$$h' = h + 2\partial_\lambda \xi^\lambda. \tag{1.109}$$

If we put the result of $h'_{\mu\nu}$ and (1.109) into (1.108), we obtain

$$\bar{h}'_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)} - \partial_\lambda \xi^\lambda \eta_{\mu\nu}. \tag{1.110}$$

As it is easily seen that, the first two terms of the right hand side of the above equation give $\bar{h}_{\mu\nu}$, therefore (1.110) becomes,

$$\bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} + 2\partial(\mu\xi_\nu) - \partial_\lambda \xi^\lambda \eta^{\mu\nu}. \tag{1.111}$$

If we take the partial derivatives of each side and use the identities $\eta_{\mu\lambda}\xi^\lambda = \xi_\mu$ and $\eta_{\mu\nu}\partial^\mu = \partial_\nu$, we get

$$\partial^\mu \bar{h}'_{\mu\nu} = \partial^\mu \bar{h}_{\mu\nu} + \Box\xi_\nu. \tag{1.112}$$

If we introduce ξ_μ as a gauge parameter, which satisfies

$$\Box\xi_\mu = -\partial_\lambda \bar{h}^\lambda{}_\mu. \tag{1.113}$$

Therefore we have the Lorenz gauge as

$$\partial^\mu \bar{h}_{\mu\nu} = 0, \tag{1.114}$$

which is the analog of the $\partial_\mu A^\mu = 0$ gauge in electromagnetic theory. As (1.114) shows the trace-reversed perturbation is transverse. For convenience, we drop the primed notation in the rest of the section.

We should note that, the original perturbation $h_{\mu\nu}$ is not transverse under the Lorenz gauge. To see this, we take the divergence of both sides of (1.105), which gives

$$\partial^\mu \bar{h}_{\mu\nu} = \partial_\mu h^{\mu\nu} - \frac{1}{2}\partial_\mu h\eta^{\mu\nu}. \tag{1.115}$$

Left hand side of the above equation vanishes because of (1.114), hence we obtain (1.115) as

$$\partial_\mu h^{\mu\nu} = \frac{1}{2}\partial_\mu h\eta^{\mu\nu}. \tag{1.116}$$

As it is seen from (1.13), we have expressed $G_{\mu\nu}$ in terms of the original perturbation. So if we rewrite (1.105) as

$$h_{\mu\nu} = \bar{h}_{\mu\nu} + \frac{1}{2} h \eta_{\mu\nu} \tag{1.117}$$

and plug it into (1.13), under the Lorenz gauge we obtain the Einstein's tensor as

$$G_{\mu\nu} = -\frac{1}{2} \Box \bar{h}_{\mu\nu}. \tag{1.118}$$

Therefore, the Einstein's equation takes the form

$$\Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}, \tag{1.119}$$

whose solution is written in terms of Green's function. If the Green's function satisfies

$$\Box_x G(x^\sigma - y^\sigma) = \delta^{(4)}(x^\sigma - y^\sigma), \tag{1.120}$$

then we can write the general solution of (1.119) as

$$\bar{h}_{\mu\nu} = -16\pi G \int G(x^\sigma - y^\sigma) T_{\mu\nu}(y^\sigma) d^4y. \tag{1.121}$$

We should note that solutions of (1.121) can be advanced or retarded [24]. We are seeking for the effects of waves which propagate forward so we study with the retarded Green's function, which is given by

$$G(x^\sigma - y^\sigma) = -\frac{1}{4\pi|\vec{x} - \vec{y}|} \delta[|\vec{x} - \vec{y}| - (x^0 - y^0)] \theta(x^0 - y^0), \tag{1.122}$$

where the θ function is 1 when $x^0 > y^0$, otherwise it is zero. If we put (1.122) into (1.121) and take the integral with respect to y^0, we get

$$\bar{h}_{\mu\nu}(t, \vec{x}) = 4G \int \frac{1}{|\vec{x} - \vec{y}|} T_{\mu\nu}(t - |\vec{x} - \vec{y}|, \vec{y}) d^3y, \tag{1.123}$$

where $t = x^0$ and $t_r = t - |\vec{x} - \vec{y}|$ is called as retarded time. As it is seen from (1.123), the gravitational radiation, $\bar{h}_{\mu\nu}$, can be thought as a sum of the effects of the sources, $T_{\mu\nu}$, at the points $(t_r, \vec{x} - \vec{y})$. Here, $\vec{x}$ represents the points where $\bar{h}_{\mu\nu}$ is determined and $\vec{y}$ represents the points where the source is located from which we conclude that $|\vec{x} - \vec{y}|$ is the distance between them [22].

1.5.2 Fourier Transform and The Quadrupole Moment

To have a better understanding of the gravitational waves, we should dig (1.123) more. For this purpose, let us introduce the Fourier transforms to make calculations simpler. For a a function $\Phi(t,\vec{x})$ Fourier and inverse Fourier transforms are defined as

$$\begin{aligned} \tilde{\Phi}(\omega,\vec{x}) &= \frac{1}{\sqrt{2\pi}}\int dt e^{i\omega t}\Phi(t,\vec{x}) \\ \Phi(t,\vec{x}) &= \frac{1}{\sqrt{2\pi}}\int d\omega e^{-i\omega t}\tilde{\Phi}(\omega,\vec{x}). \end{aligned} \tag{1.124}$$

Under these transformations, the metric perturbation becomes

$$\tilde{\bar{h}}_{\mu\nu}(\omega,\vec{x}) = \frac{1}{\sqrt{2\pi}}\int dt e^{i\omega t}\bar{h}_{\mu\nu}(t,\vec{x}). \tag{1.125}$$

If we use the definition of $\bar{h}_{\mu\nu}(t,\vec{x})$ and change t to t_r, we have

$$\tilde{\bar{h}}_{\mu\nu}(\omega,\vec{x}) = 4G\int d^3y e^{i\omega|\vec{x}-\vec{y}|}\frac{\tilde{T}_{\mu\nu}(\omega,\vec{y})}{|\vec{x}-\vec{y}|}. \tag{1.126}$$

We study with a source such that it is isolated, slowly moving and far from the observer. Therefore we can replace $\dfrac{e^{i\omega|\vec{x}-\vec{y}|}}{|\vec{x}-\vec{y}|}$ by $\dfrac{e^{i\omega r}}{r}$, where r is the distance between the source and the observer. Hence we write (1.126) as

$$\tilde{\bar{h}}_{\mu\nu}(\omega,\vec{x}) = 4G\frac{e^{i\omega r}}{r}\int d^3y\tilde{T}_{\mu\nu}(\omega,\vec{y}). \tag{1.127}$$

If we apply Lorenz gauge condition in Fourier space, we obtain

$$\frac{1}{\sqrt{2\pi}}\partial_\mu\int d\omega e^{-i\omega t}\tilde{\bar{h}}^{\mu\nu}(\omega,\vec{x}) = 0, \tag{1.128}$$

which implies,

$$\tilde{\bar{h}}^{0\nu} = -\frac{i}{\omega}\partial_i\tilde{\bar{h}}^{i\nu}. \tag{1.129}$$

It is easily understood that, instead of all components of $\tilde{\bar{h}}^{\mu\nu}$, it is enough to compute only the spacelike components. Here is our strategy: If we set $\mu = i$ and $\nu = j$ in (1.127) we can determine h_{ij}, therefore h^{0i}. To accomplish this, we play with the right hand side of (1.128) to have a simpler form. If it is integrated by parts, we get

$$\int d^3y(\partial_k y^i)\tilde{T}^{kj} = \int d^3y\partial_k(y^i\tilde{T}^{kj}) - \int d^3y y^i\partial_k\tilde{T}^{kj}. \tag{1.130}$$

First term is a surface integral which will vanish for an isolated source. By using the Bianchi identity, $\partial_\mu T^{\mu\nu} = 0$ in Fourier space, second term becomes,

$$-\partial_k \tilde{T}^{k\nu} = i\omega \tilde{T}^{0\nu}. \tag{1.131}$$

Also we write $\delta^i_k = \dfrac{\partial y^i}{\partial y^k} = \partial_k y^i$, hence under these circumstances (1.130) turns to

$$\int d^3y \tilde{T}^{ij} = \int d^3y i\omega \tilde{T}^{0j}. \tag{1.132}$$

If we divide $\tilde{T}^{0j}$ into its symmetric and anti symmetric parts and use the integration by parts once more, after setting the surface terms to zero we are left with,

$$\int d^3y \tilde{T}^{ij} = -\frac{i\omega}{2} \int y^i y^j (\partial_l \tilde{T}^{0l}) d^3y, \tag{1.133}$$

and by the condition stated in (1.131), above equation becomes

$$\int d^3y \tilde{T}^{ij} = -\frac{i\omega}{2} \int y^i y^j (\partial_l \tilde{T}^{00}) d^3y. \tag{1.134}$$

If we define $I_{ij}(t) = \int y^i y^j T^{00}(t, y) d^3y$ as the quadrupole moment tensor of the energy density of the source, (1.127) takes the form

$$\tilde{\bar{h}}_{ij}(\omega, \vec{x}) = -2G\omega^2 \frac{e^{i\omega r}}{r} \tilde{I}_{ij}(\omega). \tag{1.135}$$

We can also express it in Fourier space. If we use $t_r = t - r$, we obtain

$$\bar{h}_{ij} = \frac{2G}{r} \frac{d^2 I_{ij}(t_r)}{dt^2}. \tag{1.136}$$

Therefore we have accomplished to show that the gravitational radiation, $\bar{h}_{ij}$, is generated from the second derivative of the quadrupole moment tensor of an isolated source.

CHAPTER 2

GRAVITOMAGNETISM IN ANALOGY WITH ELECTROMAGNETISM

2.1 Introduction[1]

When we compare the electromagnetic theory and gravity, we see the close resemblance which starts from the basic equations of these theories. It is obvious that in both theories, force, potential and the potential energy have the same forms which only differ in the mass and the charge terms. Also, as we have proven in Chapter 1 just as the electromagnetic radiation, the gravitational waves propagate at the speed of light. Moreover in electromagnetism the dipole moment of the charge density, in gravity the quadrupole moment of the energy density give rise to these radiations. Therefore the similarities cause a natural question: It is known that a single charge produces only an electric field around itself. If the charge starts to move, it also generates a magnetic field. Is it possible to obtain such a field which occurs due to a moving mass? The answer is yes, but we must pay the price. Electromagnetic theory is linear whereas gravity is not. Therefore if we want to construct such a field, we accomplish this only in weak gravitational fields. This analogous field is called "gravitomagnetic field", which is an extra field produced by a moving mass. In this section we derive the Maxwell type equations from starting the linearized field equations, which we have already calculated in Chapter 1. Also, not the Maxwell equations but the Lorentz force describes the motion of a charged particle in electromagnetic field. So to understand the motion of a massive particle in the gravitational field,

[1] This chapter closely follows the paper [25] and expands upon some of the computations.

from starting the geodesic equation we derive the Lorentz type force in gravity. Also we examine the effect of the gravitomagnetic field on the motion of a gyroscope. In fact, for the static case there is already a precession, but changes in its angular momentum reveals that there exists a force which adds some additional terms into the precession [26]. We divide this chapter in four parts. In the first part we give motivation by introducing the equations which we use in the remaining parts of the chapter. By analogy, we introduce the gravitational analog of the electromagnetic field tensor as $f_{\alpha\beta}$, in which "gravitoelectric" and "gravitomagnetic" fields come to the stage for the first time. As we see later $f_{\alpha\beta}$ obeys the Bianchi identity from which we obtain two of the Maxwell type equations. In the second part we specialize the calculations in static fields and obtain time independent Maxwell type equations. Also we derive the Lorentz force for a charged massive particle in the presence of both electromagnetic and the gravitational field. In the third part, we generalize the equations found in section 2 and write these analogous equations in the most general case. Also we show that there is no difference between the Lorentz forces in time independent and the dependent fields because we omit the extra terms in the latter because we see that they all in second order. In the last part, we define the analogs of some quantities in electromagnetism, from which we introduce the angular momentum in linearized theory. If we examine the change of the angular momentum, we see that the gravitomagnetic field creates the force which adds extra terms to precession of a gyroscope, hence affects its motion.

2.2 Gravitomagnetic Fields

For a particle with mass m and charge e, which is moving in the presence of electromagnetic and gravitational fields, we write the equations of motion as

$$m[\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}] = eF^\mu{}_\nu\frac{dx^\nu}{d\tau}, \tag{2.1}$$

where $F^{\mu}{}_{\nu}$ is the electromagnetic field tensor whose components are

$$F^{\mu}{}_{\nu} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0. \end{bmatrix} \tag{2.2}$$

and $\dfrac{dx^{\nu}}{d\tau}$ is the four velocity; for slowly moving objects which can be defined as

$$\frac{dx_{\mu}}{d\tau} = (1, \vec{v}). \tag{2.3}$$

By lowering the index of (1.7) one can obtain

$$\begin{aligned} \Gamma_{\sigma 0 \beta} &= \frac{1}{2}(\partial_0 h_{\sigma\beta} + \partial_\beta h_{\sigma 0} - \partial_\sigma h_{0\beta}), \\ &= \frac{\partial_0 h_{\sigma\beta}}{2} - \frac{1}{2}(\partial_\sigma h_{0\beta} - \partial_\beta h_{\sigma 0}) \end{aligned} \tag{2.4}$$

Let us introduce a new quantity $f_{\sigma\beta}$ as the gravitational analog of $F_{\mu\nu}$, which is

$$f_{\sigma\beta} = \frac{\partial_\sigma h_{0\beta} - \partial_\beta h_{\sigma 0}}{2} = -f_{\beta\sigma}. \tag{2.5}$$

By using the analogy between $f_{\sigma\beta}$ and $F_{\mu\nu}$, we can write $f_{\sigma\beta}$ in its matrix form as

$$f_{\alpha\beta} = \frac{1}{2}\begin{bmatrix} 0 & -2g_x & -2g_y & -2g_z \\ 2g_x & 0 & -H_z & H_y \\ 2g_y & H_z & 0 & -H_x \\ 2g_z & -H_y & H_x & 0 \end{bmatrix}. \tag{2.6}$$

As it is easily understood, $\vec{g}$ and $\vec{H}$ are the gravitational analogs of $\vec{E}$ and $\vec{B}$ respectively.

If we differentiate (2.5) with respect to μ, σ and β separately and add the results, we obtain

$$\partial_\mu f_{\sigma\beta} + \partial_\sigma f_{\beta\mu} + \partial_\beta f_{\mu\sigma} = 0, \tag{2.7}$$

from which we understood that, similar to $F_{\mu\nu}$, $f_{\sigma\beta}$ obeys the linearized Bianchi identity.

For $\mu, \sigma, \beta = 1, 2, 3$, with the relevant components of (2.6), one can see that (2.7) takes the form

$$\vec{\nabla} \cdot \vec{H} = 0. \tag{2.8}$$

If we apply the remaining choices of the indices to (2.7), we obtain

$$2\vec{\nabla} \times \vec{g} + \frac{\partial \vec{H}}{\partial t} = 0, \tag{2.9}$$

where $\vec{g}$ is called gravitoelectric field, which is produced by a static mass, whereas $\vec{H}$ is called gravitomagnetic field, which is an additional field which is produced by the moving mass [20].

2.3 Static Fields

Before introducing the gravitational analogs of the Maxwell equations, let us focus on the case of static fields.
In static fields, it is seen that (2.4) changes to

$$\Gamma_{\sigma 0 \beta} = -f_{\sigma\beta}. \tag{2.10}$$

We can easily obtain the matrix form of $\Gamma^{\mu}_{0\beta}$ from (2.6) and (2.10), which is

$$\Gamma^{\mu}_{0\beta} = \frac{1}{2}\begin{bmatrix} 0 & -2g_x & -2g_y & -2g_z \\ 2g_x & 0 & H_z & -H_y \\ 2g_y & -H_z & 0 & H_x \\ 2g_z & H_y & -H_x & 0. \end{bmatrix} \tag{2.11}$$

To understand the motion of a charged massive particle, we turn back to (2.1). If we use (1.7), we have

$$\Gamma^{\mu}_{\alpha\beta}\frac{dx^{\alpha}}{d\tau}\frac{dx^{\beta}}{d\tau} \approx \Gamma^{\mu}_{00} + 2\Gamma^{\mu}_{0j}v^{j} \tag{2.12}$$

If we insert (2.12) to (2.1) and use the relevant components of (2.11), for $\mu = i$ one can write (2.1) as

$$m\frac{d^2x^i}{d\tau^2} + m\Gamma^{i}_{00} + 2m\Gamma^{i}_{0j}v^{j} = eF^{i}{}_{\nu}\frac{dx^{\nu}}{d\tau} \tag{2.13}$$

If we expand the summations in both sides and use the corresponding components of (2.2) we obtain

$$m\frac{d^2\vec{x}}{dt^2} = e(\vec{E} + \vec{v} \times \vec{B}) + m(\vec{g} + \vec{v} \times \vec{H}), \tag{2.14}$$

It is clear that the first part of (2.14) represents the Lorentz force in electromagnetism, which identifies the motion of a charged particle in electromagnetic field. By constructing an analogy, we can infer that the second part of (2.14) is the Lorentz type force in gravity; which shows that the motion of a particle is affected not only the gravitoelectric field $\vec{g}$ but also the gravitomagnetic field $\vec{H}$. Let us continue with Einstein's field equation, which is

$$R_{\mu\nu} = 8\pi G S_{\mu\nu}, \tag{2.15}$$

where we defined $S_{\mu\nu}$ as

$$S_{\mu\nu} = T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \tag{2.16}$$

To make calculations simpler we set pressure and other terms which are relevant to internal energies to 0 in $T_{\mu\nu}$, which finally becomes $T = \rho u^{\mu} u_{\mu}$. Therefore from (2.16), we obtain the components to the first order as

$$S_{00} = \frac{\rho}{2}. \tag{2.17}$$

and

$$S_{0i} = -\rho u^i. \tag{2.18}$$

In the weak field limit, for $\mu = 0$ the Ricci tensor becomes

$$R_{0\nu} = \partial_\nu \Gamma^{\alpha}_{0\alpha} - \partial_\alpha \Gamma^{\alpha}_{0\nu} \tag{2.19}$$

From (2.11), it is easily seen that the first term of the right hand side of (2.19) vanishes. If we also use (2.15), we get

$$R_{0\nu} = -\partial_\alpha \Gamma^{\alpha}_{0\nu} = 8\pi G S_{0\nu}. \tag{2.20}$$

If we put $\nu = 0$ and apply the suitable coordinates of (2.11) and the result of (2.17), we obtain

$$\vec{\nabla} \cdot \vec{g} = 4\pi(-G\rho), \tag{2.21}$$

and for $\nu = i$ from (2.20) we have

$$\vec{\nabla} \times \vec{H} = 4\pi(-4G\rho\vec{u}). \tag{2.22}$$

In static fields, (2.9) reduces to

$$\vec{\nabla} \times \vec{g} = 0. \tag{2.23}$$

Therefore we conclude that (2.8), (2.21), (2.22) and (2.23) together represent the Maxwell type equations for static fields in gravity. We should also point out that $(-G\rho)$ and $(-4G\rho\vec{u})$ are the analog of the electric charge density and electric current density in Maxwell equations.

2.4 Time Dependent Fields

As we have mentioned in the previous chapter, we can introduce gauge conditions to simplify the linearized field equations. Let us choose

$$\partial_\alpha h^\alpha{}_\beta - \frac{1}{2}\partial_\beta h = 0. \tag{2.24}$$

If we impose this condition to (1.10), with (2.15) we obtain

$$R_{\mu\nu} = -\frac{1}{2}\Box h_{\mu\nu} = 8\pi G S_{\mu\nu}, \tag{2.25}$$

with a little effort which turns to

$$(\partial_0^2 - \vec{\nabla}^2)h_{\mu\nu} = 16\pi G S_{\mu\nu}. \tag{2.26}$$

From (2.16) we have $S_{ij} = \rho u_i u_j \approx 0$. For $i = j$ (2.16) gives $S_{ii} = \rho/2$. From (2.17) we therefore conclude that $S_{00} = S_{ii}$. Hence by using (2.26) we infer that $h_{00} = h_{ii}$ and $h_{ij} = 0$ when $i \neq j$. If we combine these objects, we define $h_{\mu\nu}$ in its matrix form as

$$h_{\mu\nu} = \begin{bmatrix} \Phi & -A_x & -A_y & -A_z \\ -A_x & \Phi & 0 & 0 \\ -A_y & 0 & \Phi & 0 \\ -A_z & 0 & 0 & \Phi. \end{bmatrix} \tag{2.27}$$

Therefore for $\mu = \nu$, (2.26) changes to

$$(\partial_0^2 - \vec{\nabla}^2)\Phi = 8\pi G\rho \tag{2.28}$$

and

$$(\partial_0^2 - \vec{\nabla}^2)\vec{A} = 16\pi G\rho\vec{u} \tag{2.29}$$

for the other choices of the indices. If insert the corresponding components of $f_{\sigma\beta}$ and $h_{\mu\nu}$ in (2.5); for $\beta = 0$ and $\sigma = 1, 2, 3$ respectively, we obtain

$$2\vec{g} = \partial_0\vec{A} + \vec{\nabla}\Phi. \tag{2.30}$$

and for the other choices of the indices (2.5) gives

$$\vec{H} = \vec{\nabla} \cdot \vec{A}. \tag{2.31}$$

Also if we expand the summation in (2.24) and use (2.27),we obtain

$$\vec{\nabla}.\vec{A} + 2\partial_0\Phi = 0, \tag{2.32}$$

and

$$\partial_0\vec{A} = 0. \tag{2.33}$$

In the previous section, we did the calculations in static fields hence obtained the time independent Maxwell type equations in gravity. In this section we want to write the more general equations which reduces to (2.8), (2.21), (2.22) and (2.23) in static fields. Hence we need the general form of (2.22) such that it becomes valid in time dependent fields. To do this, we take the curl of (2.31),which is

$$\vec{\nabla} \times \vec{H} = \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2\vec{A}. \tag{2.34}$$

If we insert (2.33) into (2.29), we obtain

$$\vec{\nabla}^2\vec{A} = -16\pi G\rho\vec{u} \tag{2.35}$$

From (2.32), it is straightforward to obtain

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{A}) = -2\vec{\nabla}(\partial_0\Phi) \tag{2.36}$$

From the condition on (2.33), one can obtain (2.30) as

$$2\vec{g} = \vec{\nabla}\Phi. \tag{2.37}$$

If we combine these equations, we finally obtain (2.34) as

$$\vec{\nabla} \times \vec{H} - 4\frac{\partial \vec{g}}{\partial t} = 4\pi(-4G\rho\vec{u}). \tag{2.38}$$

Therefore, the most general form of the Maxwell type equations in gravity becomes

$$\begin{aligned}
\vec{\nabla} \cdot \vec{g} &= -4\pi G\rho\vec{u} \\
\vec{\nabla} \cdot \vec{H} &= 0 \\
2\vec{\nabla} \times \vec{g} + \frac{\partial \vec{H}}{\partial t} &= 0 \\
\vec{\nabla} \cdot \vec{H} - 4\frac{\partial \vec{g}}{\partial t} &= 4\pi(-4G\rho\vec{u}).
\end{aligned} \tag{2.39}$$

Before completing this section, we should determine the Lorentz type force for time dependent fields. To do this, we use (2.4) and (2.5). For $\sigma = j$ and $\beta = k$, with the appropriate conditions in (2.6) and (2.27), we have

$$\begin{aligned}
\Gamma^j_{0k} &= -\eta^{ji} f_{ik} + \frac{1}{2}\eta^{ji}\partial_0 h_{ik}, \\
&= -\epsilon_{jkl} H_l + \frac{1}{2}\partial_0 \Phi \delta_{jk}.
\end{aligned} \tag{2.40}$$

If we put this into (2.13) and use the relevant components, we obtain the Lorentz type force for a massive charged particle as

$$m\frac{d^2\vec{x}}{dt^2} = e(\vec{E} + \vec{v} \times \vec{B}) + m(\vec{g} + \vec{v} \times \vec{H}) + \frac{m}{2}\partial_0 \Phi \vec{v}. \tag{2.41}$$

If we take the ratio of time derivative term to $m\vec{g}$ and use (2.37), we have

$$v\frac{m}{2}\frac{\partial_0 \Phi}{m\vec{g}} \approx \frac{-\partial\Phi/\partial t}{\partial\Phi/\partial\vec{x}} = \vec{v}^2. \tag{2.42}$$

But we neglect the terms higher that first order, therefore we can conclude that the Lorentz type force in time dependent fields is the same as it is in static fields.

2.5 Application to Rotating Bodies

In the previous section, we have constructed an analogy between gravity and electromagnetism. We also showed that when a massive object moves, it creates an additional field called gravitomagnetic field. In the last section of this chapter and the following one, we discuss its effects. In this section we study the precession of a gyroscope due to the gravitomagnetic field. Since there exists an analogy, we start with modifying some familiar results of magnetostatics to gravity. In electromagnetic theory, we have defined the force and the torque on a magnetic dipole as

$$\begin{aligned} \vec{F} &= (\vec{m}\cdot\vec{\nabla})\vec{B} = \vec{\nabla}(\vec{m}\cdot\vec{B}) \\ \vec{N} &= \vec{m}\times\vec{B}, \end{aligned} \tag{2.43}$$

where $\vec{m}$ is the magnetic moment,

$$\vec{m} = \frac{1}{2}\int d^3x\,(\vec{x}\times\vec{J}) \tag{2.44}$$

and $\vec{J}$ is the electrical current. In gravity, for a rotating body we infer that the above results may take the form as

$$\begin{aligned} \vec{F} &= \frac{(\vec{S}\cdot\vec{\nabla})\vec{H}}{2} = \frac{\vec{\nabla}\times(\vec{S}\times\vec{H})}{2} \\ \vec{N} &= \frac{\vec{S}\times\vec{H}}{2}, \end{aligned} \tag{2.45}$$

where

$$\vec{S} = \int d^3x\,\vec{x}\times\rho\vec{u} \tag{2.46}$$

is the angular momentum. We know that the magnetic field due to a magnetic dipole is

$$\vec{B}(x) = \frac{3\hat{n}(\hat{n}\cdot\vec{m}) - \vec{m}}{r^3}, \tag{2.47}$$

Therefore we can write that the gravitomagnetic field as

$$\vec{H} = -2G\frac{(3\hat{n}(\hat{n}\cdot\vec{S}) - \vec{S})}{2r^3}. \tag{2.48}$$

Torque is the rate of change of the angular momentum, which is formulated as

$$\frac{d\vec{S}}{dt} = \frac{\vec{S}\times\vec{H}}{2}. \tag{2.49}$$

As a compact object such as the earth rotates, it creates a gravitomagnetic field which we have formulated in (2.48) and (2.46) tells us how much a gyroscope precesses in this field. To determine this, we write the relativistic equation of motion of a particle in terms of S^μ whose timelike component vanishes in the rest frame of the particle. Let us define it as

$$\frac{dS^\mu}{d\tau} + \Gamma^\mu_{\nu\rho} S^\nu \frac{dx^\rho}{d\tau} = 0. \tag{2.50}$$

It is easy to see that for $\mu = 0$ (2.47) vanishes because in the rest frame the four velocity $u^\mu = (1, \vec{0})$ and we choose the angular momentum vector as $S^\mu = (0, \vec{S})$. For $\mu = i$, with the conditions we stated above we only have

$$\frac{dS^i}{d\tau} + \Gamma^i_{j0} S^j = 0. \tag{2.51}$$

If we use (1.7) and put the corresponding components, we obtain

$$\frac{d\vec{S}}{dt} = \frac{1}{2}\vec{S} \times \vec{H}, \tag{2.52}$$

which means that (2.50) is an acceptable solution since it reduces to (2.49) in the non relativistic limit, as it has to. Therefore we play with (2.50) to determine the geodetic precession. If we rewrite it for a covariant vector S_μ, we obtain

$$\frac{dS_\mu}{d\tau} = \Gamma^\sigma_{\mu\beta} S_\sigma u^\beta, \tag{2.53}$$

where we set $S_\mu = (-S_0, \vec{S})$ and $u^\beta \approx (1, \vec{v})$. If we expand the sums and calculate $\Gamma^\sigma_{\mu\beta}$ in terms of Φ and A, the spatial part of (2.53) gives us

$$\frac{d\vec{S}}{d\tau} = \frac{1}{2}(\vec{S} \times \vec{H}) - \frac{1}{2}(2\vec{v} \cdot \vec{S}\vec{\nabla}\Phi + \vec{S}\vec{v} \cdot \vec{\nabla}\Phi - \vec{v}\vec{S} \cdot \vec{\nabla}\Phi) - \frac{1}{2}\vec{S}\partial_0\Phi, \tag{2.54}$$

which causes a trouble. Angular momentum of a gyroscope should be constant, but as we see from the above equation, it is not; which means that we reach another unacceptable solution. To solve this problem, we can define a vector S' related to $\vec{S}$ by

$$\vec{S} = (1 - \Phi/2)\vec{S'} + \vec{v}(\vec{v} \cdot \vec{S'})/2. \tag{2.55}$$

When we omit second order terms, we see that $\vec{S'}$ is a constant quantity. To obtain a similar equation to (2.50), we differentiate (2.55). If we equate this result to (2.54) and use (2.37), we obtain

$$\frac{d\vec{S'}}{dt} = \vec{S'} \times [\vec{H} - 3(\vec{v} \times \vec{g})]/2. \tag{2.56}$$

where the term $\vec{v} \times \vec{g}$ represents the geodetic precession which is caused by gravitomagnetic field.

CHAPTER 3

SPIN-SPIN INTERACTIONS IN GENERAL RELATIVITY AND MASSIVE GRAVITY

3.1 Introduction[1]

As we have seen in the previous chapter, in the weak field limit ($|h_{\mu\nu}| << 1$) and for small velocities ($v/c << 1$), the field equations of GR have a similar form as Maxwell's theory. In addition to the field equations, geodesic equation also can be recast in the form of the Lorentz force. One natural question arises: In electrodynamics, we know that electric or magnetic dipole moments interact with each other. What is the similar situation in GR and massive gravity. More concretely, we can ask the following question: Consider two massive spherically symmetric objects such as two galaxies or even compact objects such as two black holes that spin around their own axis, that interact with each other. In the weak field limit what is the force between them and how does this force change whether graviton has a mass or not? First of all, it is clear that according to Newton these two objects will not see each other's spin or their angular momentum around their own axis. The Newton's force is

$$F = -\frac{Gm_1m_2}{r^2}, \tag{3.1}$$

no matter how fast or slow these objects rotate. In General Relativity, the picture is quite different as we have seen in Chapter 2. Because, just like a spinning electric charge creates magnetic fields with which it affects other spinning or moving mass, creates gravitomagnetic fields and these fields will affect other

[1] This chapter closely follows the paper [27] and expands upon some of the computations.

spinning objects giving rise to spin-spin forces. This gravitomagnetic field was given in (2.48) which we repeat here

$$\vec{H} = -\frac{2G(3\hat{n}\hat{n}\cdot\vec{J}_1 - \vec{J}_1)}{r^3}, \tag{3.2}$$

where $\vec{J}_1$ is the spin of one of the objects that we noted above. The potential energy of our system will be

$$\begin{aligned} U_{spin-spin} &= -\frac{1}{2}\vec{J}_2\cdot\vec{H}, \\ &= \frac{3G(\hat{n}\cdot\vec{J}_1)(\hat{n}\cdot\vec{J}_2) - \vec{J}_1\vec{J}_2}{r^3}. \end{aligned} \tag{3.3}$$

We should note that, in this section we shall prove this formula from a different point of view. General Relativity at the lowest order represents the static Newton's force (3.1). Clearly the force, coming from

$$\vec{F}_{spin-spin} = -\vec{\nabla}U_{spin-spin} \tag{3.4}$$

will not be a lot in magnitude compared to the Newton's force for distances where the weak field gravity is valid. So one might mistakenly conclude that we have computed an irrelevant force but this is not correct since the spin-spin force is the strongest force that acts on the orientation of spins. Namely, Newton's force does not act on the spin orientations. The next question is the effect of introducing a small graviton mass on these two forces. Before one carries out the analogous calculations for massive gravity, one might guess that the effects of a tiny mass would be tiny. But this is indeed not correct at all: Even for the static Newtonian force, massive gravity yields

$$F = -\frac{4}{3}\frac{Gm_1m_2e^{-m_gr}}{r^2}, \tag{3.5}$$

for which the massless limit $m_g \to 0$, does not go to the Newton's force. This is called the van Dam-Veltman-Zakharov (vDVZ) discontinuity which has been a non-trivial problem to solve. For $m_g \neq 0$, the exponential decay is expected, and called the Yukawa type force. Equation (3.5) will be derived in this chapter. Having hit this surprise in massive gravity, we can ask what would the spin-spin interaction look like in massive gravity. Interestingly, this question was only asked and answered recently in [27]. It turns out that in addition to the experimental decay of the spin-spin force, massive gravity predicts a different spin orientation for interacting spinning objects. The details will be given in this chapter. But suffices it to say here that at separations $m_g r > 1.62$, massive gravity predicts parallel spin alignments with spins perpendicular to the axis joining the spinning sources. For $m_g r \leqslant 1.62$, (more correctly instead of 1.62, golden number $\frac{1+\sqrt{5}}{2} \approx 1.62$ appears) massive gravity and GR predict the same configuration that is spins are anti-parallel and they lay in the axis joining the sources. Let us note that the spin-spin potential energy for massive gravity is

$$\begin{aligned} U_{spin-spin} = &-\frac{Ge^{-m_g r}(m_g^2 r^2 + m_g r + 1)}{r^3} \\ &\times [\vec{J}_1 \cdot \vec{J}_2 - 3\frac{1 + m_g r + m_g^2 r^2/3}{m_g^2 r^2 + m_g r + 1}\vec{J}_1 \cdot \hat{r}\vec{J}_2 \cdot \hat{r}]. \end{aligned} \tag{3.6}$$

We divide this chapter in three parts: In the first two sections, we obtain the potential energy in both massless and massive cases. To do this, we write the potential energy in terms of the Green's function and the energy momentum tensor. As we see in the following two sections just the Green's functions differs in the calculations, which leads us to different potential energies in massless and the massive theories, respectively. It is easily seen that the force due to the spin-spin potential energy is small in magnitude, but it is the strongest force over the spin alignments. In the last section, we discuss these alignments for the minimum potential energy in detail. As we show, it differs in GR and the massive cases. As the calculations reveal, unlike GR, the spin configuration in massive gravity depends on the distance between the massive objects. Up to $m_g r \leqslant 1.62$, spin alignments for massive gravity and GR are the same: Spins are pointed to each other in the direction of the line joining two sources. But beyond this distance, while this configuration will not change in GR, we observe a dramatic change in massive gravity where spin directions abruptly become perpendicular to the line joining two sources.

3.2 Potential Energy Calculation in the Massless Theory

Given two conserved sources, we can define the gravitational potential energy in the weak field limit as

$$U = -\frac{4\pi G}{t}\int d^4x\, d^4x'\, T^{\mu\nu}(x)\, G_{\mu\nu\alpha\beta}(x,x')\, T^{\alpha\beta}(x'), \tag{3.7}$$

where $G_{\mu\nu\alpha\beta}(x,x')$ is the Green's function of the theory and t is a large time that does not appear at the end of the calculation [27]. The Green's function has four indices because linearized gravity equation is in the form

$$\mathcal{O}_{\alpha\beta\mu\nu}h^{\alpha\beta} = 16\pi G T^{\mu\nu}, \tag{3.8}$$

where $\mathcal{O}$ is an operator whose inverse is the Green's function. For both massive and massless gravity theories, the relevant computation was given in [28], but we can summarize the procedure as follows: We obtain the field equations by using the least action principle. More explicitly, by varying the action we can reveal the field equations which make this action minimum. If we apply this to Einstein-Hilbert action with added mass terms, we reach the FP equations. In weak gravitational fields, we linearize the FP equations around the flat background metric. If we play with these linearized equations, we obtain (3.8) which leads us to (3.7). Without giving the proof of this result, which is beyond the scope of this thesis, let us quote the final answer for massless and massive case respectively. (The proof is in [28]). In the massless case, we have

$$\begin{aligned} 4Ut &= -2\kappa T'_{\mu\nu}(\partial^2)^{-1}T^{\mu\nu} + \kappa T'(\partial^2)^{-1}T,\\ &= -2\kappa T'_{00}(\partial^2)^{-1}T^{00} - 2\kappa T'_{0i}(\partial^2)^{-1}T^{0i} - 2\kappa T'_{00}(\partial^2)^{-1}T^{00}\\ &- 2\kappa T'_{ij}(\partial^2)^{-1}T^{ij} + \kappa T'(\partial^2)^{-1}T, \end{aligned} \tag{3.9}$$

where we have dropped the integral signs to simplify the notation. The non zero components of the energy momentum tensor of the masses m_1 and m_2 take the form in 4 dimensions,

$$\begin{aligned} T_{00} &= m_2\delta^3(\vec{x}-\vec{x}_2),\\ T'_{00} &= m_1\delta^3(\vec{x}'-\vec{x}_1),\\ T^i{}_0 &= \frac{-1}{2}J_2^{b_1}\epsilon^{ib_1j}\partial_j\delta^3(\vec{x}-\vec{x}_2),\\ T'^i{}_0 &= \frac{-1}{2}J_1^{a_1}\epsilon^{ia_1k}\partial'_k\delta^3(\vec{x}'-\vec{x}_1), \end{aligned} \tag{3.10}$$

where J_1 and J_2 represent the spins of the masses. If we take the integral of both sides of (3.10), one can easily see that

$$m = \int d^3x \, T_{00}(\vec{x}),$$
$$J^i = \int d^3x \, T^i{}_0(\vec{x}). \tag{3.11}$$

We know that $T = \eta^{\mu\nu} T_{\mu\nu}$. By using (3.10), we obtain $T = -T_{00}$ and $T' = -T'_{00}$. If we combine all these objects, we rewrite (3.9) as

$$\begin{aligned} 4Ut &= -2\kappa m_1 m_2 \delta^3(\vec{x}' - \vec{x}_1)(\partial^2)^{-1}\delta^3(\vec{x} - \vec{x}_2) \\ &+ \kappa J_1^{a_1} \epsilon^{i a_1 k} \partial'_k \delta^3(\vec{x}' - \vec{x}_1)(\partial^2)^{-1}[(\frac{1}{2} J_2^{b_1} \epsilon^{i b_1 j} \partial_j \delta^3(\vec{x} - \vec{x}_2)] \\ &+ \kappa m_1 \delta^3(\vec{x}' - \vec{x}_1)(\partial^2)^{-1} m_2 \delta^3(\vec{x} - \vec{x}_2), \end{aligned} \tag{3.12}$$

where we define the Green's function in four dimensions as,

$$(\partial^2)^{-1} = G_R(x, x') = \frac{\Gamma(1/2)}{4\pi^{3/2} r}, \tag{3.13}$$

in which $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and $\frac{1}{r} = \frac{1}{|\vec{x} - \vec{x}'|}$. We start with computing the first and third terms of (3.12), whose sum gives the potential energy. If we plug (3.13) into these equations and take the integrals, after adding them we obtain

$$-\kappa m_1 m_2 \delta^3(\vec{x}' - \vec{x}_1)(\partial^2)^{-1}\delta^3(\vec{x} - \vec{x}_2) = -\frac{\kappa m_1 m_2}{4\pi} \frac{1}{|\vec{x}_1 - \vec{x}_2|}. \tag{3.14}$$

If we change $|\vec{x}_1 - \vec{x}_2|$ to r and impose $\kappa = 16\pi G$, (3.15) becomes

$$-\kappa m_1 m_2 \delta^3(\vec{x}' - \vec{x}_1)(\partial^2)^{-1}\delta^3(\vec{x} - \vec{x}_2) = -\frac{G m_1 m_2}{r}, \tag{3.15}$$

which is the potential energy as promised. Although it seems that there must be a 4 in the nominator, it disappears because of the 4 in the left hand side of (3.12). Therefore we write the final answer in the rest of the calculations.

The second term of the equation (3.12) is more tricky. For the sake of our calculation, let us label the second part by *. If we use integration by parts, we obtain,

$$\begin{aligned} * = &\frac{\kappa}{4\pi} J_1^{a_1} J_2^{b_1} \epsilon^{i a_1 k} \epsilon^{i b_1 j} [\partial'_k((\delta^3(\vec{x}' - \vec{x}_1) \frac{1}{|\vec{x} - \vec{x}'|} \partial_j \delta^3(\vec{x} - \vec{x}_2))] \\ - &\frac{\kappa}{4\pi} J_1^{a_1} J_2^{b_1} \epsilon^{i a_1 k} \epsilon^{i b_1 j} [(\partial'_k \frac{1}{|\vec{x} - \vec{x}'|} \delta^3(\vec{x}' - \vec{x}_1) \partial_j \delta^3(\vec{x} - \vec{x}_2))]. \end{aligned} \tag{3.16}$$

It is clear that first term of the equation vanishes. If we use integration by parts once more, after dropping the surface terms, we obtain

$$* = \frac{\kappa}{4\pi} J_1^{a_1} J_2^{a_1} \partial_j \partial'_j \frac{1}{|\vec{x}_1 - \vec{x}_2|} - \frac{\kappa}{4\pi} J_1^j J_2^k \partial_j \partial'_k \frac{1}{|\vec{x}_1 - \vec{x}_2|}. \tag{3.17}$$

The last thing we have to do to obtain the exact form of (3.12) is to compute the derivatives. For convenience, we describe $\frac{1}{|\vec{x}_1 - \vec{x}_2|} = [(\vec{x}_1 - \vec{x}_2)_m (\vec{x}_1 - \vec{x}_2)_m]^{-1/2}$, then the derivatives become,

$$\begin{aligned}
\frac{\partial}{\partial x_1^j} \frac{1}{|\vec{x}_1 - \vec{x}_2|} &= -\frac{(\vec{x}_1 - \vec{x}_2)_j}{|\vec{x}_1 - \vec{x}_2|^3}, \\
\frac{\partial}{\partial x_2^j} \frac{\partial}{\partial x_1^j} \frac{1}{|\vec{x}_1 - \vec{x}_2|} &= 0, \\
\frac{\partial}{\partial x_2^j} \frac{\partial}{\partial x_1^k} \frac{1}{|\vec{x}_1 - \vec{x}_2|} &= \frac{\delta_{kj}}{|\vec{x}_1 - \vec{x}_2|^3} - 3\frac{(\vec{x}_1 - \vec{x}_2)_k}{|\vec{x}_1 - \vec{x}_2|^5} \times (\vec{x}_1 - \vec{x}_2)_j.
\end{aligned} \tag{3.18}$$

Therefore, we can finally express * as,

$$\begin{aligned}
* &= -\frac{\kappa}{4pi} J_1^j J_2^k [\frac{\delta_{kj}}{|\vec{x}_1 - \vec{x}_2|^3} - 3\frac{(\vec{x}_1 - \vec{x}_2)_k}{|\vec{x}_1 - \vec{x}_2|^5} \times (\vec{x}_1 - \vec{x}_2)_j]], \\
&= -\frac{4G}{r^3}[\vec{J}_1 \cdot \vec{J}_2 - 3\vec{J}_1 \cdot \hat{r} \vec{J}_2 \cdot \hat{r}],
\end{aligned} \tag{3.19}$$

which is the spin-spin potential energy. As we discuss in the last section, even the force due to this potential energy is small in magnitude, it governs the spin orientation in massless theory. If we add (3.15) to (3.19), we obtain the potential energy for the massless theory as

$$U = -\frac{Gm_1 m_2}{r} - \frac{G}{r^3}[\vec{J}_1 \cdot \vec{J}_2 - 3\vec{J}_1 \cdot \hat{r} \vec{J}_2 \cdot \hat{r}]. \tag{3.20}$$

3.3 Potential Energy Calculation in the Massive Theory

In this section, we examine the situation which we give a mass to graviton. In this case (3.9) takes the form

$$4Ut = 2\kappa T'_{\mu\nu} \{-\partial^2 + m_g^2\}^{-1} T^{\mu\nu} + \frac{2}{3}\kappa T' \{\partial^2 - m_g^2\}^{-1} T. \tag{3.21}$$

If we insert the conditions in (3.10) to (3.21), we obtain

$$\begin{aligned}
4Ut = &-2\kappa m_1 m_2 \delta^3(\vec{x}' - \vec{x}_1)(\partial^2 - m_g^2)^{-1} \delta^3(\vec{x} - \vec{x}_2) \\
&- \kappa J_1^{a_1} \epsilon^{ia_1 k} \partial'_k \delta^3(\vec{x}' - \vec{x}_1)(\partial^2 - m_g^2)^{-1} [J_2^{b_1} \epsilon^{ib_1 j} \partial_j \delta^3(\vec{x} - \vec{x}_2)] \\
&+ \frac{2}{3}\kappa m_1 m_2 \delta^3(\vec{x}' - \vec{x}_1)(\partial^2 - m_g^2)^{-1} \delta^3(\vec{x} - \vec{x}_2),
\end{aligned} \tag{3.22}$$

where $(\partial^2 - m_g^2)^{-1}$ denotes the Green's function for the massive case. In four dimensions, it is defined as,

$$G_R(x, x') = \frac{(m_g/r)^{1/2}}{(2\pi)^{3/2}} K_{1/2}(rm_g),$$
$$= \frac{e^{-rm_g}}{4\pi r}. \tag{3.23}$$

We follow the same procedure used in the previous section. Therefore we rewrite the sum of the first and third equations of (3.22) as,

$$-\frac{4}{3}\kappa m_1 m_2 \delta^3(\vec{x}' - \vec{x}_1)(\partial^2 - m_g^2)^{-1}\delta^3(\vec{x} - \vec{x}_2) = \frac{-4Gm_1m_2}{3}\frac{e^{-rm_g}}{r}, \tag{3.24}$$

which is the Newtonian potential energy in massive theory.

Just as in the previous section, let us label the second part of (3.22) by *. After integrating by parts, it becomes

$$* = \kappa J_1^{a_1} J_2^{a_1} \frac{\partial}{\partial x_2^j}\frac{\partial}{\partial x_1^j} G_R(\vec{x}_1, \vec{x}_2) - \kappa J_1^j J_2^k \frac{\partial}{\partial x_2^j}\frac{\partial}{\partial x_1^k} G_R(\vec{x}_1, \vec{x}_2). \tag{3.25}$$

Now we are left with derivatives which are,

$$\frac{\partial}{\partial x_1^j}\frac{\partial}{\partial x_2^j}\frac{e^{-rm_g}}{r} = -\frac{e^{-rm_g}}{r^3} m_g^2 r^2,$$
$$\frac{\partial}{\partial x_1^j}\frac{\partial}{\partial x_2^k}\frac{e^{-rm_g}}{r} = \frac{e^{-rm_g}}{r^3}[rm_g\delta_{kj} + \delta_{kj}] - \frac{e^{-rm_g}}{r^3}(\vec{x}_1 - \vec{x}_2)_k(\vec{x}_1 - \vec{x}_2)_j$$
$$\times [m_g^2 + \frac{2m_g}{r} + \frac{m_g}{r} + \frac{3}{r^2}]. \tag{3.26}$$

If we put them together, we can write * as

$$* = -\frac{Ge^{-rm_g}}{4\pi r^3}(m_g^2 r^2 + m_g r + 1)$$
$$\times [\vec{J}_1 \cdot \vec{J}_2 - 3\frac{1 + m_g r + m_g^2 r^2/3}{m_g^2 r^2 + m_g r + 1}\vec{J}_1 \cdot \hat{r}\vec{J}_2 \cdot \hat{r}], \tag{3.27}$$

which is the spin-spin potential energy in massive case. Just as GR, although it seems negligible when compared to (3.5), the force rising from (3.27) governs the spin orientations in massive theory.

If we add (3.24) to (3.27), we finally obtain the potential energy for massive theory, which is

$$U = \frac{-4Gm_1m_2}{3}\frac{e^{-rm_g}}{r} - \frac{-Ge^{-rm_g}}{r^3}(m_g^2 r^2 + m_g r + 1)$$
$$\times [\vec{J}_1 \cdot \vec{J}_2 - 3\frac{1 + m_g r + m_g^2 r^2/3}{m_g^2 r^2 + m_g r + 1}\vec{J}_1 \cdot \hat{r}\vec{J}_2 \cdot \hat{r}]. \tag{3.28}$$

3.4 Spin Orientations in GR and Massive Gravity

As we have shown in the previous section, the force coming from the potential energy due to the spin spin interactions is negligible compared to the Newton's force for distances where the weak field gravity is valid. In this section we show that the spin-spin force is the strongest force that affects the orientation of spins. We also show that the minimum energy configuration changes whether graviton has a mass or not. It is clear that to minimize the potential energies in (3.20) and (3.28), we should maximize the function h, which is

$$h = \vec{J}_1 \cdot \vec{J}_2 - f(x)\vec{J}_1 \cdot \hat{r}\vec{J}_2 \cdot \hat{r}, \tag{3.29}$$

where $x = m_g r$. When we compare the potential energies in GR and in massive gravity, it is easily seen that $f(x) = 3$ and $f(x) = 3\dfrac{1+x+x^2/3}{1+x+x^2}$ respectively. We should note that, in the limits, for massive gravity $f(x) \in [3,1)$. We use the spherical coordinates, and choose the plane of $\vec{J}_1$ and $\hat{r}$ as the xy plane, and choose the direction of $\hat{r}$ as the x axis. Therefore, in this coordinate system, we have

$$\begin{aligned}
\vec{J}_1 &= J_1(\cos\psi_1 \hat{\imath} + \sin\psi_1 \hat{\jmath}), \\
\vec{J}_2 &= J_2(\cos\psi_2 \sin\theta_2 \hat{\imath} + \sin\psi_2 \sin\theta_2 \hat{\jmath} + \cos\theta_2 \hat{k}).
\end{aligned} \tag{3.30}$$

Then, the relevant scalar products are,

$$\begin{aligned}
\vec{J}_1 \cdot \hat{r} &= J_1 \cos\psi_1, \\
\vec{J}_2 \cdot \hat{r} &= J_2 \cos\psi_2 \sin\theta_2, \\
\vec{J}_1 \cdot \vec{J}_2 &= J_1 J_2(\cos\psi_1 \cos\psi_2 \sin\theta_2 + \sin\psi_1 \sin\psi_2 \sin\theta_2).
\end{aligned} \tag{3.31}$$

Therefore, (3.29) can be rewritten as,

$$\begin{aligned}
h &= J_1 J_2(\cos\psi_1 \cos\psi_2 \sin\theta_2 + \sin\psi_1 \hat{\jmath}) - f(x) J_1 \cos\psi_1 J_2 \cos\psi_2 \sin\theta_2, \\
&= J_1 J_2(\cos\psi_1 \cos\psi_2 \sin\theta_2 (1-f) + \sin\psi_1 \sin\psi_2 \sin\theta_2).
\end{aligned} \tag{3.32}$$

Let us start with θ_2. Since we are searching for the conditions which make h maximum, $\dfrac{\partial h}{\partial \theta_2} = 0$ must be satisfied. Hence,

$$J_1 J_2[(\cos\psi_1 \cos\psi_2 \cos\theta_2 (1-f) + \sin\psi_1 \sin\psi_2 \cos\theta_2] = 0, \tag{3.33}$$

which leads us that $\theta_2 = +\frac{\pi}{2}$ and $\theta_2 = -\frac{\pi}{2}$. For the maximum h, $\theta_2 = +\frac{\pi}{2}$ must be chosen. If we put it into (3.29), we obtain

$$h = J_1 J_2 (\cos\psi_1 \cos\psi_2 (1-f) + \sin\psi_1 \sin\psi_2). \tag{3.34}$$

Extremization with respect to two angles gives us,

$$\frac{\partial h}{\partial \psi_1} = -\sin\psi_1 \cos\psi_2 (1-f) + \cos\psi_1 \sin\psi_2 = 0 \tag{3.35}$$

$$\frac{\partial h}{\partial \psi_2} = -\cos\psi_1 \sin\psi_2 (1-f) + \sin\psi_1 \cos\psi_2 = 0. \tag{3.36}$$

It is clear from the above equations that our discussion depends whether f is 1 or not. Let us carry on the calculation with the first case. Then above equations become,

$$\begin{aligned} \cos\psi_1 \sin\psi_2 &= 0, \\ \sin\psi_1 \cos\psi_2 &= 0. \end{aligned} \tag{3.37}$$

We can easily conclude that the solutions of above equations are,

$$\begin{aligned} \psi_1 &= \psi_2 = 0, \\ \psi_1 &= \psi_2 = \pi/2. \end{aligned} \tag{3.38}$$

If we put them into (3.33), we obtain the followings:

$$h(\psi_1 = \psi_2 = 0) = 0. \tag{3.39}$$

$$h(\psi_1 = \psi_2 = \pi/2) = J_1 J_2. \tag{3.40}$$

We are searching for the maximum h, which is given by (3.40). Notice that, ψ_1 and ψ_2 are the angles of J_1 and J_2 with the plane, which means that J_1 and J_2 are in the same direction and perpendicular to axis joining the sources.

We continue with the second case, which is $f \neq 1$. Multiplying the first equation of (3.35) with $1-f$ and adding it to (3.36) gives us

$$-\sin\psi_1 \cos\psi_2 (1-f)^2 + \sin\psi_1 \cos\psi_2 = 0, \tag{3.41}$$

with a little effort which changes to

$$[(1-f)^2 - 1] \sin\psi_1 \cos\psi_2 = 0. \tag{3.42}$$

It is seen that we have again two cases. One is

$$(1-f)^2 - 1 = 0, \tag{3.43}$$

which shows that f can be either 0 or 2. Since we have $f \in [3, 1)$, we reach that it is 2. It means that f can be 2 or $\sin\psi_1 \cos\psi_2 = 0$ to satisfy (3.41). If we examine the first case in which we equate f to 2, we obtain the equation

$$x^2 - x - 1 = 0, \tag{3.44}$$

with the roots $x_1 = \dfrac{1+\sqrt{5}}{2}$ and $x_2 = \dfrac{1-\sqrt{5}}{2}$. Since $x_2 < 0$, it is not a physical solution, so it is ignored. At $x = x_1 \sim 1.62$, h becomes

$$\begin{aligned} h &= J_1 J_2(-\cos\psi_1 \cos\psi_2 + \sin\psi_1 \sin\psi_2), \\ &= -J_1 J_2 \cos(\psi_1 + \psi_2). \end{aligned} \tag{3.45}$$

It is easily seen that h is maximum for $\cos(\psi_1 + \psi_2) = -1$, which is satisfied for $\psi_1 + \psi_2 = \pi$. That is the same as the GR case, which is spins are anti-parallel and they lay in the axis joining the sources as seen in the figures 3.1 and 3.2.

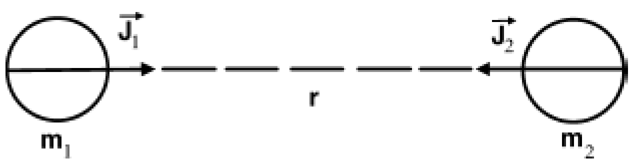

Figure 3.1: Minimum energy configuration in GR, as long as the weak field limit is applicable.

Now we can examine the second case which is $\sin\psi_1 \cos\psi_2 = 0$. For this case, we again have two possibilities which are ψ_1 can be 0 or π and ψ_2 is arbitrary; or ψ_2 can be $\pi/2$ or $3\pi/2$ and ψ_1 is arbitrary. Notice that putting $\psi_1 = 0$ or $\psi_1 = \pi$ into (3.32) only matters a minus sign which will be gone when we take the derivatives. Hence if we put $\psi_1 = 0$ into (3.32), we obtain

$$h = J_1 J_2 (1-f) \cos\psi_2. \tag{3.46}$$

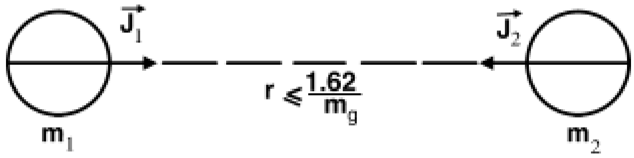

Figure 3.2: Minimum energy configuration in massive gravity for $m_g r \leq 1.62$.

If we take the derivative with respect to ψ_2, we get

$$\frac{\partial h}{\partial \psi_2} = -J_1 J_2 (1-f) \sin \psi_2 = 0. \tag{3.47}$$

Which leads us $\psi_2 = 0$ or $\psi_2 = \pi$. If we put these results separately into (3.32), we get

$$\begin{aligned} h(\psi_1 = \psi_2 = 0) &= J_1 J_2 (1-f), \\ h(\psi_1 = 0, \psi_2 = \pi) &= -J_1 J_2 (1-f). \end{aligned} \tag{3.48}$$

Since $f \in [3, 1)$, $(1-f) < 0$. Thus, h is maximum for the second equation of (3.48). The second possibility is $\psi_2 = \pi/2$, which makes

$$h = J_1 J_2 \sin \psi_1. \tag{3.49}$$

Taking the derivative with respect to ψ_1 gives us

$$\frac{\partial h}{\partial \psi_1} = J_1 J_2 \cos \psi_1 = 0, \tag{3.50}$$

which leads us $\psi_1 = \pi/2$ or $\psi_1 = 3\pi/2$. If we put these results separately into the equation (3.49), we obtain,

$$\begin{aligned} h(\psi_1 = \psi_2 = \pi/2) &= J_1 J_2, \\ h(\psi_1 = \pi/2, \psi_2 = \pi/2) &= -J_1 J_2. \end{aligned} \tag{3.51}$$

For this case, h is maximum for the second equation of (3.51). We can conclude that, when $m_g r > 1.62$, massive gravity predicts parallel spin alignments with spins perpendicular to the axis joining the spinning sources as shown in the figure 3.3

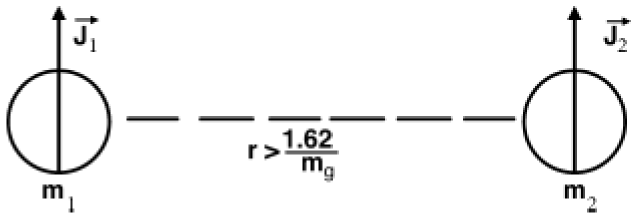

Figure 3.3: Minimum energy configuration in massive gravity for m_gr>1.62

CHAPTER 4

CONCLUSION

In this thesis, the gravitomagnetic field and its effect on the spin configurations for the minimum potential energy due to this field are discussed in both GR and the massive gravity theory. It is verified that this configuration differs in these theories. Gravitomagnetic field is the analog of the magnetic field in electromagnetism, which is a linear theory. Since GR is nonlinear, such an analogy can only be constructed in the weak gravitational fields in which we linearize the gravity.

In Chapter 1 we have derived the linearized field equations in weak field regimes. We also studied the gravitational waves and their production.

In Chapter 2, we studied the analogy between GR and electromagnetism in detail and obtain the Maxwell type equations in gravity. It is easily seen that the expressions in both theories are closely related to each other. For example, equations of force, potential and the potential energy are in the same form except for the mass and the charge are replaced with each other. Also, gravitational and the electromagnetic radiation propagate at the speed of light and the former is produced when the massive particle accelerates while the latter originates from the accelerated charges.

It is also known that a single charge produces only an electric field around itself. If the charge moves, it also generates a magnetic field. Because of the analogies we stated above, it is natural to ask whether we can find a field similar to the magnetic field in electromagnetism when the mass starts to move. Actually we can, but only in the weak gravitational field. From starting the linearized equations found in Chapter 1, we have derived the Maxwell type equations in linearized gravity. In fact these equations can be thought as a proof of the analogies between these theories. We have defined $f_{\alpha\beta}$, which is the backbone of the calculations, as the gravitational analog of the electromagnetic field tensor. Therefore, its components are the gravitoelectric and the gravitomagnetic fields instead of the electric and the magnetic fields. If we apply the Bianchi identity to $f_{\alpha\beta}$, with the proper choices of the indices we obtain two of the Maxwell type equations. We have also obtained the Lorentz type force to understand the motion of the particles in the presence of the gravitational fields. It shows that the particle's motion is described by both gravitoelectric and the gravitomagnetic fields. Then we specialize the calculations in static fields, which leads us to the time independent Maxwell type equations. In the third part of this chapter we have obtained the general forms of these equations. It is important to note that we observe no difference between the Lorentz type forces in static and time dependent fields. It follows from the fact that we omit the extra terms in the latter because they are second order.

In the last part, we have studied the effect of the gravitomagnetic field on the precession of a gyroscope. To do this, we define the analogs of some quantities in electromagnetism and we define the angular momentum using these them. From the change of the angular momentum, we see that the gravitomagnetic field produces a force which adds extra terms in the precession of a gyroscope and affects its motion.

In Chapter 3, we have studied how two spherically symmetric massive objects spinning around their own axes interact each other. In Newton's theory they are interacted by the Newton's force regardless from their velocities or angular momentums. As we have shown in Chapter 2, these spinning masses generate a gravitomagnetic field and affect each other via this field in GR. When we compute the spin spin force from the spin spin potential energy, we see that it is negligible compared to Newton's force. We have the same situation in massive gravity. Therefore we conclude that although $\vec{F}_{spin-spin}$ is negligible compared to Newton's force in GR and Yukawa type force in massive gravity, it is the strongest force which determines the spin configuration of these massive objects. To find this force, we write the potential energy in terms of Green's function. We do not give any calculations but the final answer of the potential energy formulation we have mentioned above. Therefore we obtained the following results: In GR, we write the potential energy as a sum of Newton's potential energy and $U_{spin-spin}$, which rises from the gravitomagnetic filed. It is valid also for the massive case, but this time Newton's potential energy changes to Yukawa type potential energy. As we have mentioned before, gravitomagnetic field affects the spin configuration which we study for the minimum potential energy. Calculations reveal this configuration differs in massless and the massive cases. In GR, as long as the weak field limit is acceptable the spin orientation, that is spins are anti parallel and they lay in the axis joining the sources, does not depend on the distance between these sources. On the other hand, distance plays an important role in massive gravity. At separations $m_g r > 1.62$, massive gravity predicts parallel spin alignments with spins perpendicular to the axis joining the sources. For $m_g r \leqslant 1.62$, massive gravity and GR predict the same configuration.

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CHAPTER 5

APPENDIX

HELMHOLTZ THEOREM[1]

In section 1.3, we used the Helmholtz theorem which was based on the idea that a vector field can be decomposed into a transverse and longitudinal parts. In this section, we will give the proof. Let us begin with the definition of Helmholtz theorem. A vector field V whose divergence and curl vanishes at infinity can be decomposed into the sum of the irrotational (curl-free) and the solenoidal (divergence-free) vector fields. We can express its mathematical form as

$$\vec{V} = -\vec{\nabla}\phi + \vec{\nabla} \times \vec{A}, \tag{5.1}$$

where $-\vec{\nabla}\phi$ and $\vec{\nabla} \times \vec{A}$ represent the irrotational and solenoidal vector fields respectively. We have to justify that this decomposition is always valid. Let us begin the proof by taking the divergence and curl of $\vec{V}$, which are

$$\begin{aligned} \vec{\nabla} \cdot \vec{V} &= s(\vec{r}) \\ \vec{\nabla} \times \vec{V} &= \vec{c}(\vec{r}), \end{aligned} \tag{5.2}$$

where $s(\vec{r})$ and $\vec{c}(\vec{r})$ are the functions of position. By using these functions, we can construct a scalar potential $\phi(\vec{r}_1)$ and vector potential $\vec{A}(\vec{r}_1)$ as

$$\begin{aligned} \phi(\vec{r}_1) &= \frac{1}{4\pi} \int \frac{s(\vec{r}_2)}{r_{12}} d^3x \\ \vec{A}(\vec{r}_1) &= \frac{1}{4\pi} \int \frac{\vec{c}(\vec{r}_2)}{r_{12}} d^3x. \end{aligned} \tag{5.3}$$

[1] This chapter closely follows Arfken's book [29].

It is easy to see that if $s = 0$, then $\vec{V}$ is divergence-free, therefore (5.3) implies that $\phi = 0$. Similarly if $c = 0$, then $\vec{V}$ is curl-free which implies $\vec{A} = 0$. Here, $\vec{r_1}$ with (x_1, y_1, z_1) and $\vec{r_2}$ with (x_2, y_2, z_2) denote the field and the source point respectively. We noted that $\vec{V}$ is such a vector field that its divergence (s) and curl (c) vanishes at infinity, therefore above integrals exist.
We claim that $\vec{V}$ is uniquely specified by its divergence s and curl c. The proof of this claim can be followed by the Chapter 1 of Arfken's book [29]. If we return to (5.1), we obtain

$$\vec{\nabla} \cdot \vec{V} = -\vec{\nabla} \cdot \vec{\nabla}\phi, \tag{5.4}$$

since the divergence of the curl vanishes, and

$$\vec{\nabla} \times \vec{V} = \vec{\nabla} \times (\vec{\nabla} \times \vec{A}), \tag{5.5}$$

since curl of a gradient is zero. If we can show the following equations

$$\begin{aligned} -\vec{\nabla}\phi(\vec{r_1}) &= s(\vec{r_1}) \\ \vec{\nabla} \times (\vec{\nabla} \times \vec{A}(r_1)) &= \vec{c}(\vec{r_1}), \end{aligned} \tag{5.6}$$

then we can say that $\vec{V}$ in (5.1) has the proper divergence and curl hence we are done. Let us begin with the divergence of $\vec{V}$. If we use (5.3) and (5.4), we will have

$$\vec{\nabla} \cdot \vec{V} = -\vec{\nabla} \cdot \vec{\nabla}\phi = -\frac{1}{4\pi}\vec{\nabla} \cdot \vec{\nabla} \int \frac{s(\vec{r_2})}{r_{12}} d^3x. \tag{5.7}$$

We should note that $\vec{\nabla}^2$ operates on the field of $\vec{r_1}$, and it can commute with the integral, therefore we obtain

$$\vec{\nabla} \cdot \vec{V} = -\frac{1}{4\pi} \int s(\vec{r_2})\vec{\nabla}_1^2(\frac{1}{r_{12}})\, d^3x. \tag{5.8}$$

We should be careful that our source is at $\vec{r_2}$, not at the origin, which means that the above integral takes the form

$$\begin{aligned} \vec{\nabla}_1^2(\frac{1}{r_{12}}) = \vec{\nabla}_2^2(\frac{1}{r_{12}}) &= -4\pi\delta(\vec{r_1} - \vec{r_2}) \\ &= -4\pi\delta(\vec{r_2} - \vec{r_1}). \end{aligned} \tag{5.9}$$

With the above condition, (5.8) becomes

$$\begin{aligned}\vec{\nabla}\cdot\vec{V} &= -\frac{1}{4\pi}\int s(\vec{r}_2)\vec{\nabla}_2^2(\frac{1}{r_{12}})\,d^3\\ &= -\frac{1}{4\pi}\int s(\vec{r}_2)(-4\pi)\delta(\vec{r}_2-\vec{r}_1)\,d^3x\\ &= s(\vec{r}_1).\end{aligned} \tag{5.10}$$

We finally reach that the assumed form of $\vec{V}$ and ϕ are consistent with (5.2). To complete the proof of Helmholtz's theorem, we need to show that curl of $\vec{V}$ is equal to $\vec{c}(r_1)$. We can write (5.5) as

$$\begin{aligned}\vec{\nabla}\times\vec{V} &= \vec{\nabla}\times(\vec{\nabla}\times\vec{A})\\ &= \vec{\nabla}\vec{\nabla}\cdot\vec{A}-\vec{\nabla}^2\vec{A}.\end{aligned} \tag{5.11}$$

If we use (5.3), we will obtain the first part of the above equation as

$$4\pi\vec{\nabla}\vec{\nabla}\cdot\vec{A} = \int \vec{c}(\vec{r}_2)\cdot\vec{\nabla}_1\vec{\nabla}_1(\frac{1}{r_{12}})d^3x. \tag{5.12}$$

Again, if we replace the second derivatives with respect to $\vec{r}_1$ by the second derivatives with respect to $\vec{r}_2$ and then integrate each sides of (5.12) by parts, we have

$$\begin{aligned}4\pi\vec{\nabla}\vec{\nabla}\cdot\vec{A}|_x &= \int \vec{c}(\vec{r}_2)\cdot\vec{\nabla}_2\frac{\partial}{\partial x^2}(\frac{1}{r_{12}})d^3x\\ &= \int \vec{\nabla}_2\cdot\vec{c}(\vec{r}_2)\frac{\partial}{\partial x^2}(\frac{1}{r_{12}})]d^3x - \int[\vec{\nabla}_2\cdot\vec{c}(\vec{r}_2)]\frac{\partial}{\partial x^2}(\frac{1}{r_{12}})]d^3x.\end{aligned} \tag{5.13}$$

The second integral vanishes because of (5.2). By using Gauss's theorem, we can transform the first integral to a surface integral. We noted that $\vec{c}$ would vanish for large r. If we choose such a large surface, then the first integral also vanishes, which leads us that $\vec{\nabla}\vec{\nabla}\cdot\vec{A}=0$. Therefore (5.11) turns to

$$\vec{\nabla}\times\vec{V} = -\vec{\nabla}^2\vec{A} = -\frac{1}{4\pi}\int \vec{c}(r_2)\vec{\nabla}_1^2(\frac{1}{r_{12}})d^3x. \tag{5.14}$$

Again, if we use (5.9), then the curl of $\vec{V}$ becomes,

$$\begin{aligned}\vec{\nabla}\times\vec{V} &= -\frac{1}{4\pi}\int \vec{c}(\vec{r}_2)(-4\pi)\delta(\vec{r}_2-\vec{r}_1)d^3x\\ &= \vec{c}(\vec{r}_1),\end{aligned} \tag{5.15}$$

which indicates that (5.1) and (5.3) are in harmony with (5.2). This completes the proof.