DYNAMIC PRICING FOR AIRLINE REVENUE MANAGEMENT PROBLEM WITH CANCELLATION POSSIBILITY

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ABSTRACT

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WITH CANCELLATION POSSIBILITY

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In this study, dynamic pricing methods are developed for airline revenue management problem. The bookings for a particular flight are considered in two classes as restricted and flexible bookings representing whether the buyer can claim a refund in case of a cancellation. The different classes of bookings are considered for the same inventory to be sold at different prices. For pricing the restricted bookings, the principle ideas in revenue management literature are adopted to maximize revenues by managing the demand through price control and alternative mathematical models are developed. For estimating the worth of the cancellation refund claim, which is the difference between flexible booking and restricted booking prices, the risk of cancellation is considered from the risk averse buyer's point of view and corresponding pricing methods are proposed. The proposed approach for pricing the refund claim is not specific to airline sector and can also be used for similar dynamic pricing problems where the bookings services that are sold in advance are subject to cancellation.

Keywords: Revenue Management, Dynamic Pricing, Refund

ÖZ

İPTALLERİN MÜMKÜN OLDUĞU HAVAYOLLARI GELİR YÖNETİMİ PROBLEMİ İÇİN DİNAMİK FİYATLAMA

Selçuk, Ahmet Melih
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Bu çalışmada, havayolu gelir yönetimi problemi için dinamik fiyatlandırma yöntemleri geliştirilmiştir. Belli bir uçuş için yapılacak rezervasyonlar, müşterinin olası bir iptal halinde ücret iadesi hakkının bulunup bulunmamasına göre kısıtlı ve esnek rezervasyonlar olarak iki sınıf halinde ele alınmıştır. Farklı sınıftaki rezervasyonlar aynı envanterin farklı fiyatla satılan farklı türleri olarak düşünülmüştür. Kısıtlı rezervasyonları fiyatlandırmak için gelir yönetimi literatüründeki temel fikirler ele alınarak talebin fiyatlandırma yoluyla yönetimini öngören, gelirin ençoklanmasını amaçlayan matematiksel modeller geliştirilmiştir. Esnek ve kısıtlı rezervasyonlar arasındaki fiyat farkına karşılık gelen iptal halinde ücret iadesi hakkının değerinin tahmini için iptal riski, riskten kaçınan bir müşterinin bakış açısıyla ele alınmış ve fiyatlandırma yöntemleri buna göre oluşturulmuştur. ücret iade hakkının fiyatlandırılmasında önerilen ve ele alınan bakış açısı havayolu sektörüne özgü değildir ve ön satışla ayırtılan, iptali olası hizmet satışının olduğu benzer dinamik fiyatlandırma problemleri için de uygulanabilir.

Anahtar Kelimeler: Gelir Yönetimi, Dinamik Fiyatlama, Ücret adesi

To My Parents

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INTRODUCTION

Revenue management (RM) is the art of maximizing revenues by controlling the demand for a good or service through effective pricing and/or capacity allocation strategies. Basically, the process involves estimating the price-demand dynamics in time and controlling the demand over the sales horizon by changing the price in order to exploit the variations in willingness to pay for the same product among the customers and during the time horizon. Pak and Piersma (2002) define RM as the practice of increasing revenues by selling each product to the right customer at the right time for the right price emphasizing the importance of the relation between right time and right price.

In this chapter, a general introduction to revenue management is presented. The first section is on the origins of revenue management. In this part, the relevant ideas from the economic theory of pricing, pioneering work on RM and the problem environments where RM is applicable are discussed. The next section is devoted to the history of airline RM and the emergence of RM is explained. Lastly, the sector specific considerations, the objectives of airline RM and the solution approaches are discussed in the third section.

1.1 Economics of Revenue Management (RM)

Determining the price of a commodity is one of the oldest and most fundamental questions of economic theory. Theoretically, in a perfect competition environment where numerous suppliers present an identical commodity to the market, the supply-demand balance in the market dictates the price to the firms and the seller is a price taker and has no control on it. However, in real life there might be a few suppliers in the market for a commodity and/or the commodities might be differentiated by quality, brand and other specifications that in turn give suppliers the opportunity to control the price.

The pricing problem is not limited to setting a unique fixed price for a commodity. The same commodity might be offered to different customers at different prices in which case the decision maker is responsible for determining all these prices. This pricing strategy is called price discrimination since the customers who are charged higher prices are discriminated. Price discrimination enables the seller to exploit the differences in the valuation of customers for the offered commodities and increase the revenues.

The studies on price discrimination date back to early twentieth century. Pigou (1920) presents the first extensive analysis on price discrimination. In his work, he provides a classification for price discrimination according to the degree of discriminating power. First-degree (or perfect) price discrimination occurs when every customer is charged the maximum amount s/he is willing to pay; the hypothetical case of a mind-reading sales agent who knows the upmost price for every individual demand. Second-degree price discrimination involves offering alternatives or tariffs to the customers and letting them decide; like an airline offering special discounts for round trip bookings. Third-degree price discrimination occurs when the firm sorts the customers into different groups based on some identifiable characteristics (age, location, occupation etc.) and then sets a separate price for each group as in the case of student and senior discounts. Another descriptive characteristic of a price discrimination scheme is the basis of discrimination; the difference among customers that is decisive for the price they would be charged. The spatial differences (location of market), temporal differences (time of sales), income differences (customer wealth) and quality differences (commodity difference) have been addressed by Phlips (1983) as the potential grounds of price discrimination in a market. The pricing problem in RM mainly deals with the temporal differences in the sales horizon of a perishable commodity. For customers demanding the commodity at different times, different prices are given according to the inventory level, expiry date of commodity and the customer arrival time, which can be an indicator of the customers' willingness to pay. By controlling the price, the seller in turn controls the demand during the sales horizon and attempts to maximize the revenues.

Given the conceptual background of RM problem, one can infer the market conditions under which RM can be beneficial. Talluri and van Ryzin (2005) provide a list of conditions as Business Conditions Conducive to RM. Here, we mention the following crucial conditions that make RM advantageous and operationally possible.

- *Customer Heterogeneity:* The core idea of RM is to exploit the potential variations in customers' maximum willingness to pay, namely their *reservation price*. If all customers value a product identically and exhibit similar purchase behavior, then the potential to profit from the variations will be less. Thus, for the applicability of RM in any field, there must exist customer heterogeneity. Airline and hotel industries exhibit this characteristic, the customers have different reservation prices depending on when they will purchase the service and how flexible their schedules are.

- *Price Control Power:* The differences in customers' willingness to pay is not always sufficient for the firm. For the implementation of RM, the firm must have the power to control the prices. For example, if there is perfect competition in the market, the firm would be price taker and would not have the chance to set a price above the equilibrium price imposed by the market. Also, in some industries, the firms can be restricted by certain regulations and might have no chance to change their price over time.

- *Inventory Inflexibility:* Revenue management problems generally consider the sales strategies for a limited inventory that is difficult or impossible to replenish, like number of seats in a plane or rooms in a hotel. The inflexibility of inventory throughout the sales horizon is not a prerequisite for a revenue management implementation. There are studies on multiperiod inventory problems in which the inventory can be replenished and the seller makes joint inventory and pricing decisions; the ordering quantity and sales price must be determined. Nevertheless, complications about inventory replenishment motivate the seller to look for more profitable ways of pricing to make the most out of the inventory on hand.

- *Technical Requirements:* The estimation of demand response to price changes is essential for the success of RM implementations. The firm requires storage and processing of huge amounts of data to predict the demand and in certain applications the system parameters like remaining inventory, demand rate, price sensitivity of demand need to be updated very frequently. Thus, the firms can make successful RM implementations only if they have powerful algorithms and information systems to solve this problem over and over again instantly.

1.2 History of Revenue Management

The emergence and development of revenue management practice is closely connected to airline industry. Since the first business implementations and academic studies on RM are in this industry, it would not be unfair to discuss the historical development of RM together with the last fifty years of airline industry.

The development of new business practices is motivated by the shortcomings of current methods. In the air transportation sector, the seats in a certain flight are perishable commodities that are worthless after the aircraft takes off. In this respect, each empty seat is a missed opportunity to generate extra revenue and naturally carriers want to increase their *load factor*, the percentage of capacity being sold, to benefit from this opportunity. Two main symptoms regarding the empty seats are unsold seat inventory and the booked passengers who fail to show up at the time of flight, referred to hereafter as *no-shows*. In order to compensate no-shows, the decision makers came up with the idea of *overbooking*; setting the sales quantity beyond the capacity of the aircraft to generate extra revenue from the seats of no-shows. In U.S., the airlines started to use overbooking in 1960s without acknowledging it publicly. Rothstein (1985) reports that he "found much publicly available evidence that all the major airlines were deliberately overbooking".

Overbooking proved its success in counterbalancing no-shows and booking cancellations, however the inflexibility of aircraft capacity and ticket price were still constituting a potential threat for low load factor. In 1970s the price elasticity of air transport was recognized; with sufficiently low prices travellers switched from road transport to air transport. In 1978, Airline Deregulation Act set the U.S. carriers free to change their prices. Moreover, by the end of 1970s, the newly recognized demand of price elastic passengers, who may switch from other means of transport to air transport when prices are sufficiently low, motivated the low cost carriers and charters to enter the market. Hence, the price flexibility introduced by deregulation and the competition induced by opponents triggered the birth of revenue management in air transportation.

Having faced the threat of low cost carriers, major airlines were forced to develop a strategy to recapture the price sensitive leisure passengers. To benefit from this new demand potential, discount tickets to this new segment were introduced. For the *leisure travellers* who have lower reservation prices and more flexibility on their travel dates, discounts were available for round trip bookings under advance-purchase and Saturday-stay restrictions. In 1985, American Airlines made an attempt to compete with low cost carriers and launched Ultimate Super Saver discount tickets. Smith et al. (1992) notes that the Super Saver discount tickets had purchase restrictions; they had to be purchased 30 days in advance of departure, were nonrefundable, and required a seven-day minimum stay. The purchase restrictions were designated to avoid business travellers taking advantage of low fare tickets. To protect the seats for the business customers, who are expected to make their bookings later, capacity restrictions were used and the number of discount seats offered in each flight was also limited.

Meanwhile, theoretical studies on revenue management were also initiated. Littlewood (1972) studied two segment -discount and regular fee- price discrimination scheme and proved that it is optimal to continue selling discount fare tickets as long as the discounted fee is above the *displacement cost*; the expected loss of turning down a possible regular fare customer. This method is accepted as the earliest mathematical method for quantity based revenue management and the study is a milestone in the history of RM practice. Quantity based RM refers to the class of RM implementations in which the inventory- or capacity-allocation decisions are utilized for demand management. Price itself can be used as the primary tool for managing demand, this type of RM implementations are classified as price based RM.

After the success of American Airlines experience, other firms in the sector also started implementing RM. Smith et al. (1992) reports that RM implementations resulted in revenue improvements of 2% to 8% in comparison to pre-deregulation period. The technological advances and scientific progress have led to improvement of more sophisticated techniques in time. Today, RM is an essential practice for both major and low cost carriers in the air transportation sector. Moreover, RM has been utilized for pricing a variety of other commodities such as hotel rooms, rental cars, concerts and game tickets, electricity and so on.

This thesis focuses on price based RM in airline sector. The contribution of this study is twofold; firstly alternative methods are presented for pricing dynamically the single leg bookings under the assumption of no cancellation and no overbooking. The advantages and shortcomings of the proposed methods are investigated and their performances are analyzed on a comparative basis with existing dynamic pricing methods. Dynamic pricing has become popular in airline RM very recently; the earlier implementations are based on capacity allocation principle. Hence, the research on price based RM is not as extensive as quantity based RM and this is a strong motivation for us to focus on this area.

The second part is devoted to an extension allowing cancellations and the problem of pricing the *refund premium*, the additional amount a customer should pay for holding a refund claim in case the booking is cancelled. This problem particularly attracts our attention since we have not encountered any study in the literature analyzing how this price premium can be determined although it is a common practice in airline sector to charge an additional amount for the refund claim. The comparison of restricted and refundable ticket prices of major European airlines has also provided significant evidence indicating the lack of thorough quantitative approaches for refund premium pricing. With this motivation, we have developed a method for forecasting the customers' willingness to pay for holding a refundable ticket instead of a restricted one. The customer preference is modeled as a decision problem and utilizing the relevant ideas from the utility theory and the regret theory, the worth of a refund claim is estimated. Based on this estimation, we propose a pricing strategy for refund premiums. The ideas we have adopted are not specific to airline industry, thus the method can be applicable to other service industries, like entertainment or accommodation in which the sellers offer advance bookings that can be refunded in case of a cancellation.

1.3 Preliminaries of Airline Revenue Management

The term airline RM refers to a broad field of research and different solution approaches have been proposed for different system environments and problem specifications. To understand the nature of problem, sector specific demand characteristics, objectives of airlines for applying RM and market factors affecting the structure of RM methodology should be investigated.

- *Demand:* In the air travel sector, it is predominantly accepted that the time of booking is an indicator of a customer's willingness to pay. It has been observed that as the departure time of a flight approaches, the reservation prices of customers increase. This is a distinguishing characteristic of airline RM problem which describes the main tradeoff between selling the seat immediately or reserving the seat for a probable later sale at a higher price. Another common assumption about demand is that the customer arrival rate increases as the departure time approaches.

- *Objectives:* Identifying the objectives of the airline executives is a critical issue in developing a successful RM technique. Maximizing the expected total revenue is the primary target, however the company might have other tactical concerns as well. For example, maintaining the load factor at a certain level will be desirable if the airline has a market share target in terms of the total number of passengers. Another critical factor for determining the objective is risk attitude of the airline. In order to incorporate risk aversion, alternative objectives like reducing the variance of total revenue or minimizing the probability of obtaining total revenue less than a minimum acceptable amount can be utilized as in the studies due to Çetiner (2007) and Terciyanlı (2009). Barz and Waldmann (2007) employ utility functions for incorporating the risk aversion of the decision maker in the RM problem.

- *Competition:* Although, in the earlier studies on airline RM, the price posted for the flight is considered as the only factor affecting the demand intensity, recent studies argue that it is also essential to take into account the prices of other airlines offering flights on the same route. The effect of an increase in competitor price is reflected in the model as an increase the probability of sales for an individual item and the effect would be reverse in case of a decrease. The competitor prices are considered as external parameters for the pricing models developed according to this approach. There are also studies which handle the problem in a game theoretic approach; competing airlines developing and adapting their strategies in anticipation of the countermoves of their opponents. The pricing problem is modeled as a multiplayer game among rivals and the pricing strategies are evaluated accordingly (See Netessine and Shumsky, 2005). Hence, the extent of competitive structure in the market is a critical issue about the RM implementation.

- *Airline Network Structure:* There are not necessarily direct flights between any two nodes in an airline network and the travellers often take successive flights. Most airlines offer their customers connecting flights and the anticipated marginal revenue of a seat in any one of these connecting flights is not the same as that of a point-to-point flight. Thus, RM applications for such network flight structures require more sophisticated analysis than the single leg flights.

Revenue management is classified as price based RM or quantity based RM depending on the type of tactical level decision strategy. After studying the aforementioned characteristics of the RM problem under consideration, the suitable control strategy should be determined. In *capacity allocation* practice, there are different fare classes which are offered to customers until a boundary condition on the remaining inventory and/or time to expiry is reached. Talluri and van Ryzin (2005) explain this as "optimally allocating capacity of a resource to different classes of demand". In the airline example, a two fare sales policy, which controls sales of the seats in a flight with full and discount rates is an example of capacity allocation practice.

In real life implementations, reservation systems provide different mechanisms for controlling the capacity allocation. Booking limits set boundaries on the amount that could be sold for each fare class; whereas a protection level specifies the amount of capacity to reserve for a particular fare class for future sales. Thus, booking limits or protection levels are static control tools that are determined when the sales start and can be used throughout the entire sales horizon. On the other hand, bid price control is based on the idea of finding a minimum acceptable price and using it as a threshold; the requests for lower fare classes are rejected and higher fare classes are accepted. The bid price must be recalculated at each booking request, so the control mechanism is dynamic.

The idea of determining and updating the minimum acceptable price in bid price control is similarly adopted in developing price based controls, also referred to as *dynamic pricing*. In quantity based approach, the seller has to determine the price for each fare class prior to the capacity allocation and if these prices are poorly selected, desired revenue level cannot be attained no matter how efficiently the capacity is allocated to the classes. Dynamic pricing is advantageous in this respect. In dynamic pricing, price itself is the control variable and instead of determining the availability of different fare classes, the decision maker has to decide on the price to be posted.

Dynamic pricing is as old as commerce itself. Auctions, price negotiations, markdowns and other forms of dynamic pricing are utilized to increase revenues: the style goods are discounted at the end of season, special offers and markdowns are occasional for fast moving perishable goods to deplete excessive inventories, seasonal commodities have lower prices during the periods when the consumption is less. The key point for any dynamic pricing implementation is to understand the relation between demand and price. If the customers' response to price variations can be estimated, the demand can be managed effectively and it will be possible to increase the revenues. Dynamic pricing has become popular in the airline industry very recently. Due to Talluri and van Ryzin (2005) the dominance of quantity based controls in the earlier implementations of airline RM is due to the advertising and managerial constraints; managers used to publish their fares in media and tried to simplify price management process. With the old technology, updating the price would have been much slower than deciding which fare classes to sell and time is the one thing that seller does not have in airline sector since the passengers demanding different flights arrive frequently and expect the seller to post the price immediately. Since it is easier to keep track of sales and seat inventory rather than calculating the optimal price at a certain time and inventory position, in airline industry booking limit and protection level controls were preferred to dynamic pricing for quite a long time. However, with the recent advances in the information technology, it has become possible to execute complex algorithms and deliver prices to customers within seconds and now dynamic pricing is widely used in airline RM.

An important issue in pricing airline bookings is about the possibility of refund and/or rescheduling in case of cancellation. In practice, when a customer wants to reschedule his/her booking, the amount to be refunded due to cancellation of the original booking is deducted from the fee of the new booking and customer pays the difference in between. In this respect, rescheduling can be modeled as a cancellation succeeded by a new booking without loss of generality; hence we focus on cancellation refunds only. In general, the airlines offer their lowest fare restricted tickets without any possibility of cancellation and the flexible tickets, which can be rescheduled or refunded upon cancellation requests, are offered at higher prices. Accordingly, in seat inventory control applications the booking classes with higher fares are refundable and in dynamic pricing refundable tickets can be booked by paying an additional premium.

Different types of refunding policies are adopted by airline companies. Proportional refunds offer a certain percentage of the ticket fee in case of cancellation and in partial refunds the ticket fee is paid back after deducting a cancellation penalty. Full refunds is a specific case of partial refunds where the cancellation penalty is set to zero. Due to our observations, the most commonly utilized refunding policy in airline sector is partial refunding and the cancellation penalty is called the service and booking expenses, which is a considerably small amount compared to ticket price. In this study, the partial refunding policy is considered and the constant cancellation fee can be set to zero to represent the case of full refunding.

In dynamic pricing perspective, the price difference between nonrefundable and refundable bookings for the same service represents the price of the right to claim a specified portion or the entire amount spent on a service reservation in case the reservation is cancelled before the cancellation deadline. In this study, this right is named as *refund option* and is considered as a separate commodity offered for the passengers who book for the flight. Accordingly, the pricing process is treated as a two-phase problem consisting of determining the price of the service, the air travel, and the price of the refund option. Although this framework is developed for the case of airline RM, it would also be applicable for other sectors in which the prepaid service reservations, like concert or sports game tickets, hotel reservations etc., are subject to cancellation.

Evidently, refund options are desirable for the passengers due to the possibility of cancellation without significant monetary loss if none at all. On the other hand, the airlines can also utilize refunds as an opportunity to obtain extra profit and competitive advantage. Next, the conditions that motivate airlines to offer refund options are mentioned briefly.

1.3.1 Motivations for Offering Refund Options

The refund options can be considered as additional commodities for the firms performing revenue management applications. Xie and Gerstner (2007) report that, under certain conditions, offering refunds can be profitable for the firm even when no extra charge is requested for it. How does the firm profit from customer cancellations? This question is addressed below.

- *Multiple selling of limited capacity:* Customer cancellation refunds are in general partial refunds; the seller either charges a "cancellation hassle cost" and deducts this amount from the refund or offers a proportional refund in which case a certain percentage of the ticket price is refunded. These revenues collected from customers who cease to take the service would generate additional profit if the left seats could be resold. This strategy is sound in the case of any service sale with limited capacity.

- *Higher late sales price:* As mentioned for the RM applications in the airline industry, it is observed that the late coming customers are willing to pay higher for the flight. Hence, if a previously sold and cancelled ticket is resold to the late comer, the latter customer would possibly be charged a higher price. In such cases, when the reservation price of customers tends to increase during the sales horizon of a commodity, offering refunds can bring a by-profit by selling the same service at a higher price.

- *Reduced uncertainty:* The service providers generally encounter no-show situations when the advance sold services offer no refund opportunity for cancellation. The refunds give customers a motivation to inform the service provider that they cease to take the service and the capacity reserved for them could be resold.

  No-shows motivate the service provider for overbooking, which could be simply considered as selling over capacity (See Smith et al., 1992). When the overbooked capacity is below the no-shows, there is no problem for the firm. On the other hand, if the number of no-shows is less than overbooked capacity, then some service requests will be denied and the company will be faced both with legal penalties stated in the related regulations and also loss of prestige. These service denials resulting from the overbooking uncertainty are hence undesired. Offering refund options would decrease the no-shows, which in turn would decrease the uncertainty the company would be facing.

- *Fairness and Acceptance:* The major opposition against RM applications is that providing the same good/service at different prices would conflict with the customers' perception of fairness. Talluri and van Ryzin (2005) mention that in some real life implementations RM raised huge customer dissatisfaction since the pricing was interpreted as unfair. Thus, RM is in general a risky implementation and proposing refund options can legitimize it.

  The refund policies of prepaid service providers are different from that of retailers since a refund guarantee for a product aims to create confidence in the quality of the good to be sold. For example, for some products the sellers offer a money pay back guarantee for a trial period after sale in case of dissatisfaction. Nevertheless, this situation could be misused by the customers who purchase the product for using it during the trial period only and returning afterwards without any real complaint about the product. This possibility of misuse is a major opposition for offering refund guarantees for products. On the other hand, for a prepaid service, the refund opportunity would only prevent the possible loss in the situation the buyer fails to get the service and by its nature it is not open to that sort of unfair use.

With the aforementioned motivations, many airline companies offer flexible bookings in addition to the nonrefundable economy tickets. In Figure **??**, the flexible and economy ticket prices of İstanbul-London flights of two major European airlines in eight consecutive weeks are presented. In both graphs, it observed that the flexible ticket prices are constant although the economy ticket prices vary, indicating that the dynamic pricing policy applied to economy tickets but not considered for flexible tickets. Another remarkable observation is that the flexible ticket price for the same flight is substantially higher than the economy ticket, up to seven times more and it is questionable that if anyone would prefer paying a £567 premium to eliminate the cancellation risk of a £85 economy ticket.

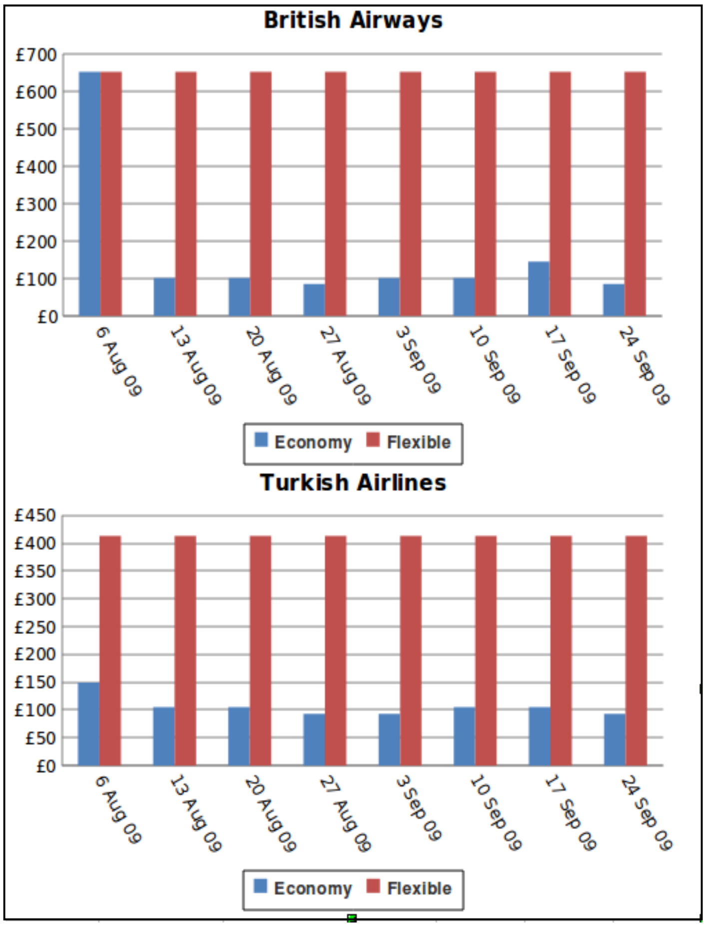

Figure 1.1: Price comparison for İstanbul-London flights

There is no doubt that these leading airline companies put great effort to find ways of effectively pricing the economy tickets with RM implementations to manage the demand; however, it seems unlikely that they implement similar sophisticated dynamic pricing policies for flexible tickets. We have repeated the same price inquiry for Turkish Airlines on August 5,2014 and observed that the pricing scheme is very similar to the case 5 years ago, flexible ticket price is fixed at £400 while the restricted booking price varies between £150 and £310 (since British Airways changed policy on refundability of economy tickets, a fair comparison is not possible). Although the excessive cost of booking refundability is less in more recent price inquiry, the policy of *keeping flexible booking price fixed while the economy ticket price changes* is still the same. Flexible tickets offer the seller the opportunity to sell refund options and generate additional revenue using the risk aversion of customers and offering these tickets at disproportionately high prices compared to economy tickets would be missing that opportunity. We believe that the anticipated cost of ticket flexibility for the firm must be estimated carefully and the pricing of refundable bookings should be studied accordingly.

1.3.2 Assumptions for Refund Options and Cancellations

In this thesis, mathematical models for pricing the restricted and flexible bookings are proposed. The valuation of refund options is critical for determining the difference between refundable and nonrefundable ticket prices. The general assumptions on air transport demand are mentioned in other studies on airline RM problem and these will be taken into account in our pricing models.

On the other hand, the refund options we are considering in this study are cancellation claims having specific properties. The following assumptions for refund options and cancellations are made parallel to the findings of prior studies on cancellations in RM and our observations on airlines selling flexible and restricted tickets. Hence, before using the refund option pricing methods that we propose, the validity of following assumptions should be verified:

- *Freedom of Choice:* Refund options are offered with a price premium and the customer decides whether it is worth to pay the additional price for holding this claim. Hence, it is assumed that, the nonrefundable and refundable tickets are available during the entire sales horizon.

- *Partial Refunding:* Only partial refunds (the total price charged at booking) are under consideration. In the formulations, a fixed cancellation hassle cost is introduced as the deductable part of amount to be refunded. This assumption is in accordance with the practices of most airline companies; the booking and service fees are nonrefundable in general. This penalty term can be set to zero if the seller implements a full refund policy. An important remark about this penalty term is that it must be less than the restricted ticket price; otherwise proposing refund options would not be logical.

- *Maturity:* Cancellation requests of refundable ticket owners are accepted (and the booking fee is refunded) until a prespecified deadline that is announced to the buyer at the time of sales. Thus, the refund option has a certain maturity date and it expires after this time point. If seller accepts all cancellation requests before the flight, this case can be modeled by setting the cancellation deadline to the time of departure $t = 0$.

- *Exogeneity of Cancellation:* The airline tickets are booked in advance and the buyers may cease to take the flight due to conflicts, changes in plans, health issues etc. We assume that the probability that the buyer ceases to take the flight is determined by those exogenous random factors and is independent of the amount to be refunded. Hence, in the airline's point of view, it is equally likely for flexible and restricted booking holders to show up at the time of flight departure.

1.4 Specifications of the Dynamic Pricing in Airline Sector

In order to designate a dynamic pricing method for determining the price of a commodity, the nature of the sales process should be studied carefully. In most cases, the seller is expected to post a price on the requests of potential buyers but the time frame for making this decision depends on the commodity to be sold. When the sale of a dozen aircrafts is considered, an airline company can wait for the manufacturer to post the price for a couple of weeks or even more whereas a passenger would not be willing to wait more than a couple of seconds for the airline to set the price for the seat in a desired flight. Thus, dynamic pricing of airline tickets requires immediate retrieval of seat inventory information, estimation of other relevant parameters and rapid determination of price with regard to all these factors.

We are interested in the pricing of the restricted (non-refundable) and the flexible (refundable) tickets and we assume that all passengers get the same level of service during the flight. In case a certain proportion of the seats are reserved for first class or business class, these seats could be treated as different commodities and can be priced independently since the service provided is not identical. Thus, within the scope of this work, the sales agent is responsible for posting the price for restricted and flexible bookings in the desired flight in at most a few seconds.

The sales is considered as a two stage process; firstly the customer decides whether it is worth to make a nonrefundable booking at the given price p. At the second stage of sales the sales agent is supposed to offer the passenger the price of the refund option q. The research on the first stage subproblem is quite new due to the difficulty of solving the dynamic pricing algorithms instantly with the older technology. After reviewing the studies on airline RM and other sectors, which are discussed in Chapter 2, we develop methods for finding the ticket price, p under certain assumptions on demand-price relations.

The second stage subproblem is determining the price of the refund option, q, for the ticket price, p, obtained in first stage. The passenger would be eligible to upgrade the nonrefundable ticket to a refundable one by purchasing the refund option. Refund option is not an actual service or product, it is a claim on a prepaid service agreement and therefore instead of a generic demand-price relationship, we consider this phase of pricing as a decision problem for the passenger involving the risk of cancellation. In case the decision problem involves uncertain outcomes, the attitude of the decision maker is essential. Different individuals can make different decisions under the same conditions due to their personal assessment of risk. The generally accepted risk attitude is risk aversion, in which case the individuals prefer certain gains or losses to risky, uncertain alternatives with higher expected returns. The risk aversion of individuals increase their willingness to pay for refund options, causing refundable ticket sales more profitable for airlines. The effect of customers' risk attitude on pricing the refund options is further analyzed in Chapter 5.

All in all, the contribution aimed with this work can be stated as follows:

- Despite the profound research on seat allocation practices in airline RM, there is limited work on dynamic pricing applications and this study presents a dynamic pricing methodology for airline RM and compares its performance with other credible methods in this field.
- We did not come across any study on how to determine the price difference between the refundable and nonrefundable tickets. The pricing method suggested in this work proposes a solution for this problem that is applicable not only for the airline RM problem but also for other possible applications at which presold services are subject to cancellation.
- The alternative formulations of certain parameters could be useful for airline RM problems in general. For instance, the way we define expected marginal value of a seat can be adopted in other dynamic pricing or seat allocation control methods as well. The analysis on the estimation of parameters is also important for airline personnel who are responsible for these estimations in real life implementations.

The organization of this thesis is as follows: An extensive review of literature on airline RM, dynamic pricing in RM and other fields with relevant pricing problems is given in Chapter 2. Chapter 3 briefly outlines the pricing problem we study in this thesis and the analytical models are presented. In our analysis pricing problem is disintegrated into two main parts. First part, finding the restricted (non-refundable) booking price ignoring cancellations is studied in depth in Chapter 4. In Chapter 5 our approach to the second part of the pricing problem, finding the price of refund premium for given restricted booking price is presented and mathematical models are introduced. Results of simulation studies are given in Chapter 6 and Chapter 7 covers our conclusive remarks.

CHAPTER 2

LITERATURE REVIEW

The research on airline RM problem can be classified into two groups depending on the type of control mechanism. In the first approach called "seat allocation control", the capacity is allocated to the classes with different terms and conditions and sales prices and the seller controls the availability of seats for each class throughout the sales horizon. The earlier studies in the field of airline RM are on seat allocation control. Later, price was used explicitly as the control variable over the sales horizon in airline RM. This approach called "dynamic pricing" focuses on how to determine the optimal price for a commodity depending on the inventory level and the time to expiry - the time of departure for the airline example. Although dynamic pricing strategies are proposed in this thesis, crucial ideas from seat allocation control literature have also been adopted and therefore pioneering studies on seat allocation control are summarized firstly in Section 2.1. Then, the research on dynamic pricing in airline industry and also in other sectors is reviewed in Section 2.2. Section 2.3 briefly introduces ideas in airline RM concerning ticket refundability and studies assuming possibility of cancellation. Literature review is concluded with the discussion on the similarities and differences between pricing of financial options and pricing of refund option premiums in Section 2.4.

2.1 Milestones in the Emergence of RM

The first attempt of utilizing mathematical models for seat allocation control is due to Littlewood (1972). The problem he studies is the allocation of the seats in a flight to two fare classes (discount-low and full-high fare) so that the revenue is maximized. He develops an optimal rule for accepting low fare requests at inventory level x as long as:

$$Fare_{low} \geq Fare_{high} Pr(Demand_{high} \geq x).$$

The term $Fare_{high}\,\Pr(Demand_{high} \geq x)$ expresses the opportunity cost of accepting the low fare request at inventory level x. In other words, this term is the expected marginal revenue of reserving the x^{th} seat for future high fare demand. The rule given by Littlewood (1972) is limited to the problems with two fare classes. Belobaba (1987) develops the Expected Marginal Seat Revenue (EMSRa) heuristic for multiple fare-class problem based on the same opportunity cost approach. Although its performance depends on the demand distribution, EMSRa heuristic has shown good performance in simulation studies. Belobaba (1987) further modifies the method and develops EMSRb, which is reported to have close to optimal results for finding booking limits. In this method, the expected marginal seat revenue of a fare class is obtained as $fare \times spill$, where spill is defined as the probability that the demand for a particular fare class will exceed the number of seats allocated to that class. The accuracy of spill estimation is of great importance for the optimality of seat allocation when any variation of EMSR is implemented. Belobaba and Farkas (1999) indicate this fact and study different methods for spill estimation and compare their performances. The studies of Littlewood (1972) and Belobaba (1987) had great impact on the airline RM research and therefore these pioneering studies on seat allocation control practice have been introduced in this section briefly. Noticing the significance of numerous other studies in the literature, the discussion on seat allocation control is concluded here since this thesis focuses on the dynamic pricing implementations in airline RM.

Seat allocation practices have been dominant in the field of airline RM at the beginning and dynamic pricing has been introduced to this field more recently. Yet, there are earlier studies on dynamic pricing applications in other fields. To the best of our knowledge, the study of Kincaid and Darling (1963) appears to be the earliest work on dynamic pricing. The model given by the authors mainly considers sales of a perishable stock with a specified disposal date to potential buyers arriving according to a Poisson process and two variants of seller's decision problem are studied. In the first model, the seller posts prices for each customer arrival. In a variant of the model, the seller evaluates the bids of potential buyers with accept-reject decisions. For both cases, the structural properties of optimal strategies are presented. Gallego and van Ryzin (1994) study the same generic dynamic pricing problem for the price posting case and obtain the following structural monotonicity results for optimal price to be posted by the seller:

- At any fixed time point, the optimal price decreases as the number of available items in the inventory increases.
- For any fixed number of items in the inventory, the optimal price decreases as time to disposal increases.

These results are in accordance with the monotonicity results for bid price policy in the seat allocation control.

2.2 Dynamic Pricing

Dynamic pricing has a broad scope of application area and attracted attention of researchers from different fields. It has been used in manufacturing and retail industries as well as service sector (See Talluri and van Ryzin (2005) for a detailed review of dynamic pricing literature and real-life implementations). The structure of pricing problem varies among different fields of application; while the capacity of an airplane for a particular flight or the number of available rooms of a hotel are fixed, in manufacturing and retail industries the inventories can be replenished and the seller should consider joint optimization of inventory and price. In the inventory literature, supply-price management problem has been studied under the heading of multi-period problems with variable price for different settings (finite/infinite production capacity, deterministic/stochastic demand, finite/infinite horizon etc.). The extensions of base stock policies and the conditions under which these policies are optimal have been extensively studied. Federgruen and Heching (1999) and Chen and Simchi-Levi (2006) study the optimality conditions of a base stock list price policy (s, S, p); order up to S when inventory is below s and set price p. For a detailed review on dynamic pricing studies within the context of manufacturing and retail systems, the reader is referred to Elmaghraby and Keskinocak (2003).

The possibility of inventory replenishment is a critical factor for the dynamic pricing problem. On the other hand, there are other aspects that also require careful assessment. The following classification of dynamic pricing problems is due to Elmaghraby and Keskinocak (2003):

1. Replenishment vs. No Replenishment of Inventory,
2. Dependent vs. Independent Demand Over Time,
3. Myopic vs. Strategic Customers.

The airline dynamic pricing problem has been studied so far under the assumptions that the inventory cannot be replenished and the demand rate and reservation price are both time-dependent. Talluri and van Ryzin (2005) notice that, due to the nature of air travel demand, the reservation prices of customers tend to increase as the date of flight approaches and therefore prices generally follow an increasing trend during the booking period. Thus, it is reasonable to assume that the passengers would follow myopic behavior, they would immediately buy the ticket if it is under their reservation price without any considerations about future prices. This is also the best decision for a strategic customer when the prices are expected to get higher.

Both Kincaid and Darling (1963) and Gallego and van Ryzin (1994) use a time homogenous demand model; the customer arrival rate is assumed to be a time-independent function of price, and due to the aforementioned characteristics of air travel demand, this demand model is not particularly appropriate for airline RM. In most of the RM literature, the arrival process of customers is assumed to be a Poisson process. If the customer arrival process is non-stationary, as in the case of airline RM, the time-dependence of arrivals is handled by modeling the arrivals by a Nonhomogenous Poisson process where the Poisson rate λ varies in time. Bitran and Mondschein (1997) and Bitran et al. (1998) consider Nonhomogenous Poisson arrivals to represent the temporal demand fluctuations in the dynamic pricing problem of seasonal goods.

Lin (2006) gives an important result about Poisson customer arrivals. He shows that if λ follows Gamma distribution, then the total demand follows a Negative Binomial distribution. He also refers to the extensive research on the use of Negative Binomial demand distribution in marketing science and reports evidence supporting that Negative Binomial can be a good fit in certain industries. The study of Agraval and Smith (1996) on the demand in retail industry is an example for significantly better fit of Negative Binomial than that of the Poisson or Normal distribution.

Zhao and Zheng (2000) study a more general case where both customer arrival rate and reservation price distribution are time-dependent. They indicate the conditions under which the monotonicity results obtained by Gallego and van Ryzin (1994) hold true for time-dependent reservation prices and arrival rates. In our study, the customer arrival rate and reservation price distributions are assumed to be time-dependent.

In dynamic pricing problems, the control tool of the seller for managing the demand is the price and the objective is to decide the optimal price as a function of time and other relevant factors. Alternatively, Lin (2004) defines a different decision variable to determine the *policy* of the seller: the probability of selling successfully one item to the current customer. Under the assumption that the reservation price distribution of the customer is known, the probability of sale can be represented as a function of posted price. Let v denote the probability of selling one item, p denote the posted price and the random variable P denote the reservation price of the customer. Then,

$$v = Pr(P \geq p) = 1 - F_P(p). \tag{2.1}$$

With this definition of policy, Lin uses dynamic programming formulation and derives structural properties of optimal policy using the optimality equations. The optimal expected revenue as a function of seat capacity and random demand is proven to be supermodular. Using supermodularity, he has shown the following:

- If the demand function gets stochastically larger, the optimal probability of selling one item gets smaller (correspondingly the price gets higher) at a fixed inventory level.
- The incremental contribution of one item in the inventory is decreasing in the

inventory level; that is for the same demand function, the value of an additional item increases as the inventory level decreases.

- For a given fixed demand function, probability of selling one item gets smaller as the available inventory decreases.

Lin (2004) also covers exact and approximate algorithms for deriving optimal policy for different customer arrival distributions. For obtaining an upper bound on the expected revenue, he studies the case of a clairvoyant sales agent that has the knowledge of future customer arrivals and, therefore, is expected to generate a higher revenue than the regular seller. Increased information on the future of the process ensures that the revenue estimate for the clairvoyant seller is an upper bound on the actual expected revenue and also it makes estimating the expected revenue function easier.

Talluri and van Ryzin (2005) introduce a practical linear programming (LP) formulation for dynamic pricing by maximizing the expected revenue subject to the constraint that the anticipated sales at selected price levels should be less than or equal to available capacity. The term $d(t, p(t))$ denotes the demand rate in discrete time period t when the price in this period is $p(t)$, C denotes the available capacity and $r(t, d(t, p(t)))$ denotes the corresponding expected revenue for that period. Below, a compact formulation of the optimization problem is given. Here, the price, $p(t)$, is the decision variable. This system can be reduced to a linear programming model if the demand, $d(t, p(t))$, and the expected revenue, $r(t, d(t, p(t)))$, are linear functions.

$$\max \quad \sum_{t=1}^{T} r(t, d(t, p(t))) \tag{2.2}$$

$$\text{subject to} \quad \sum_{t=1}^{T} d(t, p(t)) \leq C \tag{2.3}$$

$$d(t, p(t)) \geq 0. \tag{2.4}$$

Lin (2006) incorporates real-time demand learning into the dynamic programming formulation he gives in 2004. The formulation for the update of demand distribution in real time is provided and numerical experiments for the developed algorithms are presented to understand the effects of demand learning and update frequency. Extensions regarding the cases of batch demand, discrete price levels and time-dependent customer arrival rate are also included in this study. Şen and Zhang (2009) also facilitate demand learning for dynamic pricing of style goods. They use sales observations to refine the estimates of the customer arrival rate and the form of the demand-price relationship.

Anjos et al. (2004) propose a method for defining the price-demand relationship quantitatively and employ constrained optimization for pricing. They model the probability of selling one seat t days before flight at price $y(t)$ as a function of days to departure and price, $p(t, y(t))$. The functional forms of $p(t, y(t))$ and the time dependent demand $f(t)$ are determined empirically by examining the booking behavior of customers. Expectation values of revenue and the number of seats to be sold are obtained using price, sales probability and demand estimations similar to the LP model due to Talluri and van Ryzin (2005) in (2.2) as seen below.

$$\max \quad R = \int_0^\tau f(t)y(t)p(t, y(t))dt \tag{2.5}$$

$$\text{subject to} \quad \int_0^\tau f(t)p(t, y(t))dt \leq C. \tag{2.6}$$

The function R to be maximized is the expected revenue for a remaining capacity of C seats τ days before departure. It is also mentioned that this model was used by a major British airline company for dynamic pricing. Later, the airline company indicated the absence of competition in the model developed by Anjos et al. (2004) as a shortcoming. Currie et al. (2008) propose an improved version of this model that incorporates the competition by taking the competitor's price into account. The probability of selling one seat τ days before flight is modeled in this study as a function of number of days to departure, the ticket price and the competitor airline's ticket price and the prior model due to Anjos et al. (2004) is revised accordingly.

2.3 Cancellation and Refundable Bookings in Airline RM

The advance sales of commodities separate the purchase and consumption and create uncertainty for the buyer about the utility of actual consumption. Shugan and Xie (2000) study advance pricing of services and argue that the reservation price of buyer at purchase depends on the expected utility from the expected consumption state. They exemplify the risks about future consumption in case of buying a ticket for a concert in advance, like probable health issues, expected conflicts, mood etc. and emphasize the impact of these risks on the valuation of buyers. Similarly, booked passengers may change their mind about the travel before the flight and they are faced with the risk of paying for an unused service if the ticket is not refundable.

The uncertainty of the presence of booked passengers at the flight is considered as an opportunity to sell over the capacity by airline companies and overbooking practices are adopted accordingly. Consequently, in the airline RM literature, the studies considering no-shows and cancellations are mostly on overbooking policies (See Rothstein, 1971; Bierman and Thoman, 1975 and Bodily and Pfeifer, 1992). Alternatively, Talluri and van Ryzin (2005) mention class dependent cancellation refunds briefly. They notice that the seller can charge an extra fee for the expected refund at the time the reservation is accepted. This approach is insightful for understanding the price difference between restricted and refundable bookings.

The prices of flexible tickets are significantly higher than non-changeable, nonrefundable economy tickets. Mason (2006) attributes this difference to overbooking policies of airline companies. Increased flexibility on buyers' side induces increased uncertainty for the seller about the number of booked passengers who will show-up at the time of flight. Accordingly, overbooking becomes more risky for the airline company due to possible denied-boarding losses and to counterbalance the associated losses, airlines charge higher prices for flexible tickets.

Determination of the price difference between flexible and economy tickets is one of the major problems considered in this work. Although sales of economy and flexible tickets for the same flight at different prices is a common practice in airline sector, there is not much work in the literature about determination of this price premium. All we know is that in seat allocation control, the ticket flexibility in terms of cancellation refunding should be available to higher fare classes only. The next section covers similar pricing problems dealing with pricing of claims whose future value is uncertain to the buyer and the seller.

2.4 Pricing of Financial Options and Insurance Contracts

Insurance is a means of hedging the risks by partially or fully transferring the possible losses of an uncertain future outcome to a third party with a contract. For example, a standard automobile insurance contract covers the repair expenses, health expenses of drivers and/or passengers and the compensation for casualties or permanent disabilities in case of an accident. The insurance companies cannot charge a fixed price of such a contract for all customers since individuals carry different risks and the monetary risk they want to hedge is not identical.

Insurance pricing is in general based on classic risk theory. The probabilities reflecting the actual likelihood of loss events are used to calculate the loss expectation. Kull (2003) represents the insurance premium as follows:

$$\text{premium} = \frac{1}{1+r} E^P(X) + S(X). \tag{2.7}$$

In this formulation, X denotes the losses to be covered by the contract and $E^P(X)$ is the loss expectation under the real probability measure P. r is the risk free rate of return and the first term on the right hand side of the multiplication denotes the discounted expected loss which can be interpreted as the expected cost of the contract for the insurance company. $S(X)$ is the risk premium the customer pays to discard the risks and is the expected profit on this contract on the company's side.

In practice, the insurance companies cannot go through an extensive research for each customer to estimate the probability measure P. Instead, the customers are classified into risk groups according to certain criteria, e.g., record of previous accidents, mileage per year and driving experience are important indicators for an automobile insurance contract. Likewise, in case of life insurance, age, current and previous diseases, smoking and other addictions can be utilized for risk assessment.

A cancellation refund option is similar to an insurance contract in the sense that the passenger hedges the risk of losing the money spent on the ticket in case of a cancellation by paying an additional premium at the time of booking. The ticket refundability is useless if the passenger takes the flight, like the case of having no accidents during the coverage of insurance. Although this similarity motivates using the same ideas in refund option pricing, the airlines lack the information for classifying the customers according to their risk of cancellation so it is not likely to use this approach in practice. The only data known to the seller is time of booking, which provides very limited information about the customer. If an airline implements customer loyalty programs and stores information about the passengers, then the insurance pricing approach may be useful.

In economics and finance, the term option also refers to a claim that is useless if not exercised. A financial option is an instrument that gives the holder the right, but not the obligation to buy or sell some financial asset at a predetermined price at or before a predetermined expiry date. If we consider a call option, which gives its holder the right to buy one unit stock at price c, the value of being able to exercise this claim at a future time point t will be $max\{0, s(t) - c\}$ where $s(t)$ denotes the value of the stock in the stock market. The fundamental problem in this context is to determine the price of the option at a given time point, considering the uncertain nature of stock price.

A natural way of dealing with the option pricing problem is to consider the expectation using the real probabilities, similar to the method discussed for insurance pricing herein. However, this expected value using real probability measures approach in general leads to *arbitrage*, a transaction that offer risk-free profit. The possibility of arbitrage indicates that the price is unfair and theoretically individuals can make money out of nothing for this option price. To calculate the arbitrage-free, fair price of a future stochastic cash flow X, Black-Scholes-Merton theory of option pricing (also referred to as Black-Scholes theory) suggests calculating the expectation with respect to equivalent martingale measure Q instead of real probability measure P.

$$\text{price} = \frac{1}{1+r} E^{Q}(X). \tag{2.8}$$

Ismail (2001) explains the independence of option price from real probabilities by a two state stock/bond economy example. Accordingly, the future states of the economy at $t = T$ are defined as *Up* and *Down*, and the stock, whose value at $t = 0$ is \$100 will be worth either \$150 or \$100 with probabilities P^{Up} and $1 - P^{Up}$, respectively. On the other hand, the bond offer a risk free return for the investor at a rate of r, so \$1 invested in bond returns \$$e^{rT}$. According to this problem setting, the arbitrage-free price of call option that gives its holder the right to buy the stock at \$100 at $t = 0$ is as follows:

$$\text{Option price} = 100 \times (1 - e^{-rT}) \text{ provided that } e^{-rT} > \frac{100}{150}. \tag{2.9}$$

The price of the option is the same for $P^{Up} = 0.001$ or $P^{Up} = 0.999$, so the actual probabilities about the future state of the economy is irrelevant in financial option pricing. In our case, the booking cancellation options are not means of investment but they are solely intended for avoiding the losses in case the pre-booked flight is not taken and therefore the price of this claim must be directly related to the future state of the booking, whether it will be cancelled or not. Since purchasing a flight ticket is not an actual investment, it is not reasonable to determine the price of a refund option disregarding the real probabilities and using such a comparison with payoffs of alternative investment strategies.

The solution approaches proposed for insurance premium valuation and financial option pricing problems are not applicable for the pricing of airline booking cancellation options. Thus, a new approach based on utility functions in decision theory is devised in this work to estimate the behavior of customer reservation price for the booking cancellation option. Once the reservation prices are estimated, the option price is determined with the objective of expected total revenue maximization, which is the main objective in RM.

CHAPTER 3

PROPOSED ANALYTICAL MODELS

The existing dynamic pricing models developed for airline RM problem basically deal with dynamic pricing of a perishable commodity (seats on a particular flight) considering the nonhomogeneous nature of reservation prices of customers over time. The systems allowing cancellations and overbooking are generally avoided in these existing studies since such extensions amplify the complexity of the problem. That is, the revenue maximization problem with a single decision variable, the booking price, is considered in the literature.

In this thesis, a two dimensional revenue optimization problem is considered; the seller determines the price of the restricted booking and also the price of the refund option. Throughout this thesis, the dynamic pricing problem is studied for single leg flights. The network considerations like pricing of connected flights can be revisited by working with the results obtained for single leg problem.

Having mentioned the relevant work in the literature in the previous chapter, the mathematical models developed for simulating the real-life system and the approximations considered for reducing the immense complexity are discussed in this chapter.

3.1 Optimization of Nonrefundable and Refundable Ticket Prices

The pricing problem under our consideration deals with finding the optimal prices for *restricted* and *refundable* bookings for a particular flight. We define our decision variables p as the price of a restricted ticket and q as the premium that should be paid in addition to the restricted ticket price to get a refundable ticket.

As the name implies, dynamic pricing takes into account certain dynamical factors; factors that are subject to change throughout the sales horizon. Therefore, the optimal values of p and q depend on the *state* of the sales process. Since the single leg flights are considered in this work, we restrict our attention to the sale of the seats on a flight between a specific origin-destination pair at a given date. The state of this process can be represented by the triplet (s, t, n) denoting the inventory level (the number of remaining available seats), s, time to departure, t, and the number of refundable tickets sold so far, n. The variable t is defined in a backward fashion such that $t = T$ denotes the start of sales horizon and $t = 0$ denotes the time of flight. The initial seat inventory level (the value of s at $t = T$) is denoted by S.

The objective of the seller at a given state (s, t, n) is to maximize the expected revenue that could be generated through the remaining sales horizon. Accordingly, we define $v_t(s, n)$ as the optimal value function of the Dynamic Programming (DP) formulation we give in this section. $v_t(s, n)$ is the optimal expected revenue-to-go when the current state is (s, t, n) where s is the number of seats available for the flight (seat inventory), t is the time to departure and n is the number of refundable bookings sold so far. The optimization is considered for the decision variables p and q. p is the price of a restricted ticket and q is the additional charge for a restricted-to-refundable booking upgrade. The decision variables in the sales process are the two prices to be posted by the firm, but naturally there are other elements that determine the expected revenue-to-go. The sales would occur if P_t is greater than the posted price p. P_t is reservation price of the restricted booking, i.e., the maximum price that the customer arriving at time t is willing to pay for nonrefundable booking. In airline RM, the passengers' willingness to pay changes along the sales horizon; accordingly P_t is considered to be a time dependent random variable.

The sale of a refundable ticket is considered in two steps: sale of a restricted ticket with price p and the sale of the refund option for that passenger at price q. The sale of refund option would depend on $Q_t(p)$ which is the reservation price of the customer for refund option at time t, the additional amount that the customer arriving at time t is willing to pay for the refund option. Similar to P_t, $Q_t(p)$ is considered as a time-dependent random variable since the passengers' arriving at different times in the sales horizon have differences in valuation as mentioned before. Besides, the refund option reservation price depends on the price of the restricted ticket posted by the seller. Refundability of a reservation will have greater value as the ticket price gets higher and this fact is taken into consideration in the estimation of $Q_t(p)$. Keeping this dependence in mind, we will represent the refund option reservation price as Q_t in our formulations for the sake of notational convenience.

Remark 3.1.1 *McGill and van Ryzin (1999) note that the low-before-high fare booking arrival pattern is prominent in the seat inventory control literature. For instance, in the simplest two-class example, customers are classified as (early-coming) leisure and (late-coming) business customers and it is assumed that leisure customers demand lower fare class tickets whereas business customers demand higher fare class. Under the strict low-before-high fare arrival assumption, the customers of successive fare classes are assumed to arrive in non-overlapping time intervals; hence, we can consider an upward shift in the customer reservation prices at the end of each time interval throughout the sales horizon. In the models we propose in this thesis, we do not restrict ourselves to this demand characteristic of the airline RM problem. As a special case, we can consider the temporal change in the restricted ticket reservation price, P_t, assuming that the reservation price tend to increase stochastically as we move forward along the sales horizon; i.e. $P_{t+1} \leq_{st} P_t$. Thus, the proposed demand frame allows us to consider a gradual increase in the willingness to pay for the air travel service instead of classifying the customers into different segments.*$\square$

In the formulation we present, a probable sales transaction is considered in a small time period $[t, t-\epsilon]$, over which customer arrivals occur due to (Nonhomogeneous) Poisson arrival process. When ϵ is sufficiently small, we could assume that at most one customer could arrive over this interval with probability $\rho_t = \epsilon\lambda_t$, where λ_t denotes the time-variant rate parameter of Nonhomogenous Poisson arrival process. The recursive DP formulation obtained under these assumptions is presented next.

$$v_t(s,n) = \rho_t \max_{p,q} \left\{ \begin{array}{l} Pr(P_t < p)[\Delta_s v_{t-1}(s,n) + \Delta_n v_{t-1}(s-1,n)] \\ +Pr(P_t \geq p, Q_t < q)\Delta_n v_{t-1}(s-1,n) \\ +Pr(P_t \geq p)p + Pr(P_t \geq p, Q_t \geq q)q \end{array} \right\} \tag{3.1}$$
$$+(1-\rho_t)[\Delta_s v_{t-1}(s,n) + \Delta_n v_{t-1}(s-1,n)] + v_{t-1}(s-1,n+1),$$

$$v_0(s,n) = 0 \text{ for all } (s,n),$$
$$v_t(0,n) = 0 \text{ for all } (t,n),$$

where $\Delta_s v_t(s,n) = v_t(s,n) - v_t(s-1,n)$ and $\Delta_n v_t(s,n) = v_t(s,n) - v_t(s,n+1)$.

Derivation. The recursion along the time for the expected revenue-to-go function is obtained by evaluating every possible state at the time point $t-1$. We begin our investigation by considering all scenarios regarding the customer arrival and formulate the corresponding expected revenue-to-go functions $v_{t-1}(.,.)$.

Case 1: No customer arrives or the arriving customer does not purchase the ticket since the reservation price is less than the announced price for the restricted ticket; the revenue-to-go function after the transaction is $v_{t-1}(s,n)$.

Case 2: The customer purchases a nonrefundable ticket; the customer's reservation is greater than the announced restricted ticket price and the reservation price of the refund is less than the refund premium, the revenue-to-go function after the transaction is $v_{t-1}(s-1,n)$.

Case 3: The customer purchases a refundable ticket, the customer's reservation prices of both restricted ticket and refund option are larger than the respective prices announced by the seller; the revenue-to-go function after the transaction is $v_{t-1}(s-1,n+1)$.

According to this analysis, the following recursive formulation is obtained:

$$v_t(s,n) = \max_{p,q} \left\{ \begin{array}{l} [\rho_t Pr(P_t < p) + (1-\lambda_t)]v_{t-1}(s,n) \\ +\rho_t Pr(P_t \geq p, Q_t < q)[p + v_{t-1}(s-1,n)] \\ +\rho_t Pr(P_t \geq p, Q_t \geq q)[p + q + v_{t-1}(s-1,n+1)] \end{array} \right\}. \tag{3.2}$$

The boundary conditions follow trivially; when there is no remaining seats for sale or when the sales horizon ends with unsold seats on hand, the expected revenue-to-go equals 0. The terms in (3.1) are rearranged to obtain the form in (3.1).□

The recursive formulation in (3.1) is rewritten in terms of difference terms $\Delta_s v_{t-1}(s, n)$ and $\Delta_n v_{t-1}(s-1, n)$ since these could be useful for understanding and analyzing the behavior of marginal contribution of an additional seat to revenue and the expected future loss in revenue due to selling an extra refund option.

3.2 Ignoring Cancellation: Finding Nonrefundable Ticket Price

The optimization problem in (3.1) needs to be solved in the two dimensional decision space (p,q). Evidently, this problem is much more complex compared to single variable dynamic pricing problems studied in the literature; example formulations with single decision variable are given by Lin (2004) and Anjos et al. (2004).

The possibility of cancellation is mostly disregarded in the airline RM literature and the problem is studied under the assumption of no cancellations. Similarly, if the ticket refundability and cancellations are not considered in our model, the formulation in (3.1) reduces to the following:

$$\begin{aligned} v_t(s) &= \rho_t \max_p \{Pr(P_t < p)\Delta_s v_{t-1}(s) + Pr(P_t \geq p)p\} \\ &\quad + (1-\rho_t)\Delta_s v_{t-1}(s) + v_{t-1}(s-1) \\ &= \rho_t \max_p \{Pr(P_t \geq p)[p - \Delta_s v_{t-1}(s)]\} \\ &\quad + \Delta_s v_{t-1}(s) + (1-\rho_t)\Delta_s v_{t-1}(s) + v_{t-1}(s-1), \end{aligned} \tag{3.3}$$

where $\Delta_s v_t(s) = v_t(s) - v_t(s-1)$.

This reduced formulation is particularly useful for proving certain structural properties regarding the restricted ticket price p, and the expected revenue-to-go, $v_t(s)$. Kincaid and Darling (1963) develop the continuous time version of this formulation (albeit taking the reservation price distribution and the Poisson arrival rate time independent) and conclude that it is possible to find the closed form solution of the optimal price only if the reservation price, P_t, follows exponential distribution.

Rearranging the terms, we obtain the optimality equation in (3.2). The recursive model obtained with the boundary conditions is referred to as the Dynamic Pricing model with the acronym DP.

$$\begin{aligned} &v_t(s) - v_{t-1}(s) = \rho_t \max_p \{Pr(P_t \geq p)[p - \Delta_s v_{t-1}(s)]\}. \\ &v_t(0) = 0 \\ &v_0(s) = 0. \end{aligned} \tag{3.4}$$

$[p - \Delta_s v_{t-1}(s)]$ can be interpreted as the net earning in case of sale; the difference between the ticket price obtained in case of sale and the expected future contribution of not selling the ticket to the expected revenue-to-go. If the seat is sold in the current period, we start the next period, $t-1$, with $s-1$ seats instead of s seats. Note that $\Delta_s v_{t-1}(s)$ is independent of p, which is the ticket price announced by the seller at time t; $\Delta_s v_{t-1}(s)$ is determined by the ticket price posted in the beginning of the next period $[t-1, t-2]$. $\Delta_s v_{t-1}(s)$ is the expected amount lost by selling a ticket (seat "s") in period t. In other words, $\Delta_s v_{t-1}(s)$ is the expected amount we earn if we do not sell the seat "s" in period t. The left-hand-side of (3.2) is nonnegative (with a longer sales horizon, the seller cannot do worse); hence, it can be directly observed that the optimal price must be larger than the marginal value of the seat $\Delta_s v_{t-1}(s)$. In the quantity-based RM literature, comparison of p and $\Delta_s v_{t-1}(s)$ tells us what to do ("sell" or "do not sell") as in the case of Littlewood's rule, static and dynamic single-leg models.

The discrete-time models are studied in depth in the forthcoming parts of this thesis. Yet at this point, we shift our focus to the analysis of a compatible continuous-time problem. We compare the univariate discrete-time model studied so far with the continuous time dynamic pricing model studied in the seminal work of Kincaid and Darling (1963). Noticing the similarity of models, we extend their results on optimal pricing policies.

3.2.1 Closed Form Solution

The model we are investigating is based on the assumption that a single customer request can be received and handled during a very small time period $[t, t-\epsilon]$. Assuming that the time interval is infinitesimal (in other words, $\epsilon \to 0^+$), the model can be presented in continuous time fashion as follows:

$$v_t(s) - v_{t-\epsilon}(s) = \epsilon\lambda_t \max_p \{Pr(P_t \geq p)(p - \Delta v_{t-\epsilon}(s))\}. \tag{3.5}$$

Kincaid and Darling (1963) study a particular reservation price distribution $P_t \sim EXP(1)$, where $EXP(1)$ stands for the exponential distribution with parameter 1. For this demand setting, they derive a closed form representation for the optimal price function, p^*_{st}. Their model ignores the customer arrival rate and its time dependence ($\lambda_t = 1$) and they restrict the demand function by fixing the exponential distribution's mean.

In this part it is assumed that the reservation price exponential, $P_t \sim EXP(\alpha)$ with an arbitrary α in an attempt to find an exact solution to the model provided in (3.5) for a particular demand setting. In this case, the complementary cdf term $Pr(P_t \geq p) = e^{-\alpha p}$ and the optimality equation for p^* is;

$$p^*_{st} = arg \max_p \{e^{-\alpha p}(p - \Delta v_{t-\epsilon}(s))\}. \tag{3.6}$$

Notice that the term $\Delta v_{t-\epsilon}(s)$ stands for the difference in optimal value function in s at time $(t - \epsilon)$; when the current customer leaves the system. Hence this difference is independent of price p set at the current state (s, t). Accordingly, differentiation with respect to p yields the first order optimality equation as follows:

$$-\alpha e^{-\alpha p^*_{st}}(p^*_{st} - \Delta v_{t-\epsilon}(s)) + e^{-\alpha p^*_{st}} = 0. \tag{3.7}$$

Rearranging (3.7), we obtain $-\alpha e^{-\alpha p^*_{st}}(1 - \alpha p^*_{st} + \alpha\Delta v_{t-\epsilon}(s)) = 0$. Since α and $e^{-\alpha p^*_{st}}$ cannot be 0, we have $1 - \alpha p^*_{st} + \alpha\Delta v_{t-\epsilon}(s)) = 0$ yielding $p^*_{st} = \Delta v_{t-\epsilon}(s) + \frac{1}{\alpha}$. Having obtained optimal price, the recursive formulation in (3.5) can be reduced as follows:

$$v_t(s) - v_{t-\epsilon}(s) = \epsilon\lambda_t \left[e^{-\alpha(\Delta v_{t-\epsilon}(s) + \frac{1}{\alpha})} \frac{1}{\alpha} \right]. \tag{3.8}$$

Rearranging the terms and taking the right-limit as ϵ goes to 0 we obtain the following optimality equation:

$$\lim_{\epsilon \to 0^+} \frac{v_t(s) - v_{t-\epsilon}(s)}{\epsilon} = \frac{dv_t(s)}{dt} = \frac{\lambda_t}{\alpha e} e^{-\alpha\Delta v_t(s)}. \tag{3.9}$$

Proposition 3.2.1 gives a closed form solution for the optimal value function, $v_t(s)$, for the mentioned exponential demand behavior. This provides a solution to the pricing problem also since the corresponding optimal price, p^*_{st}, is defined as $v_t(s) - v_t(s - 1) + \frac{1}{\alpha}$.

Proposition 3.2.1 *Let $P_t \sim EXP(\alpha)$ and let the revenue-to-go function, $v_t(s)$ satisfy the ODE given in (3.9) with boundary conditions $v_t(0) = 0$ and $v_0(s) = 0$. Then, the closed form representation of revenue-to-go function is as follows:*

$$v_t(s) = \frac{1}{\alpha} \ln\left(\frac{(\Lambda_t)^s}{s!e^s} + \frac{(\Lambda_t)^{s-1}}{(s-1)!e^{s-1}} + \cdots \frac{\Lambda_t}{e} + 1\right), \tag{3.10}$$

where $\Lambda_t = \int \lambda_t \mathrm{d}t$ assuming that Λ_t is well-defined for every t.

Proof. *The proof is by induction on s.*

(s=1) *Using the boundary condition $v_t(0) = 0$, we have $\Delta v_t(1) = v_t(1)$. The optimality equation becomes,*

$$\frac{dv_t(1)}{dt} = \frac{\lambda_t}{\alpha e} e^{-\alpha v_t(1)}. \tag{3.11}$$

To find the solution of the differential equation given in (3.11), we rearrange the equation as follows:

$$\alpha e^{\alpha v_t(1)} dv_t(1) = \frac{\lambda_t}{e} dt. \tag{3.12}$$

Integrating both sides we obtain;

$$e^{\alpha v_t(1)} = \frac{\Lambda_t}{e} + c. \tag{3.13}$$

At $t = 0$, the sales horizon ends so Λ_t becomes 0. Using the boundary condition $v_0(1) = 0$, we obtain $c = 1$. Equation (3.13) gives $v_t(1) = \frac{1}{\alpha} \ln\left(\frac{\Lambda_t}{e} + 1\right)$.

(s=k) *Assume that $v_t(k)$ satisfy the following equality:*

$$v_t(k) = \frac{1}{\alpha} \ln\left(\frac{(\Lambda_t)^k}{k!e^k} + \frac{(\Lambda_t)^{k-1}}{(k-1!)e^{k-1}} + \cdots \frac{\Lambda_t}{e} + 1\right). \tag{3.14}$$

(s=k+1) *The optimality equation at time t and with $(k+1)$ seats on inventory is given in (3.15).*

$$\frac{dv_t(k+1)}{dt} = \frac{\lambda_t}{\alpha e} e^{-\alpha \Delta v_t(k+1)}. \tag{3.15}$$

To find the solution of the differential equation given in (3.15), we rearrange the equation as follows:

$$\alpha e^{\alpha v_t(k+1)} dv_t(k+1) = \frac{\lambda_t}{e} e^{\alpha v_t(k)} dt. \tag{3.16}$$

Using the assumption in the previous step of induction regarding $v_t(k)$, (3.16) is rewritten as below.

$$\alpha e^{\alpha v_t(k+1)} dv_t(k+1) = \left(\lambda_t \frac{(\Lambda_t)^k}{k! e^{k+1}} + \lambda_t \frac{(\Lambda_t)^{k-1}}{(k-1)! e^k} + \cdots \lambda_t \frac{\Lambda_t}{e^2} + \lambda_t \frac{1}{e}\right) dt. \quad (3.17)$$

Notice that $\int \lambda_t (\Lambda_t)^i \mathrm{d}t = \frac{(\Lambda_t)^{i+1}}{i+1}$ for every positive integer i. Hence, integration of both sides in (3.17) yields the following equation:

$$e^{\alpha v_t(k+1)} = \frac{(\Lambda_t)^{k+1}}{(k+1)! e^{k+1}} + \lambda_t \frac{(\Lambda_t)^k}{k! e^k} + \cdots \lambda_t \frac{{\Lambda_t}^2}{2! e^2} + \frac{\lambda_t}{e} + c. \quad (3.18)$$

Inserting boundary condition $v_0(k+1) = 0$, we have $c = 1$. We obtain $v_t(k+1)$ as in (3.18), which completes the induction.

$$v_t(k+1) = \frac{1}{\alpha} \ln\left(\frac{(\Lambda_t)^{k+1}}{(k+1)! e^{k+1}} + \lambda_t \frac{(\Lambda_t)^k}{k! e^k} + \cdots \lambda_t \frac{{\Lambda_t}^2}{2! e^2} + \frac{\lambda_t}{e} + 1\right). \quad (3.19)$$

■

3.2.2 Structural Results

The continuous time model is offering a solution for a particular price-demand relation whereas the discrete time model is promising for the investigation of further characteristic properties on the behavior of marginal seat revenue $\Delta_s v_t(s)$ disregarding the relation between price and demand. For a different system formulation of dynamic pricing problem, Lin (2004) investigates the structure of the optimal expected revenue function. In this study, for the optimal revenue-to-go function $J(s, p_K)$, which is defined recursively as a function of the number of items in the inventory s and the probability mass function p_K for the number of future customers, it is shown that the inequality in (3.20) is satisfied for every positive s.

$$J(s+1, p_K) - J(s, p_K) \leq J(s, p_K) - J(s-1, p_K). \quad (3.20)$$

Talluri and van Ryzin (2005) also prove that marginal seat revenue function is a nondecreasing function of time to departure and nonincreasing function of remaining seat inventory under the assumption that the marginal revenue (which is a counterpart of $Pr(P_t \geq p)p$ in our formulation) is a convex function of the price p. We provide a more general result by eliminating this assumption related to the reservation price distribution. Lemma 3.2.2 below shows that these two conditions for $\Delta_s v_t(s)$ (being nondecreasing in t and nonincreasing in s) are equivalent to one another for our formulation. Lemma 3.2.2, the proof of which is structurally the same as the proof of Talluri and van Ryzin (2005), shows that $\Delta_s v_t(s)$ is nonincreasing in s. The combined result of the two lemmas allows us to deduce the same results for marginal seat revenue function as the results of Talluri and van Ryzin (2005) without making any assumptions on the distribution of P_t.

Lemma 3.2.2 $\Delta_s v_t(s)$ *is nondecreasing in* t *if and only if* $\Delta_s v_t(s)$ *is nonincreasing in* s.

Proof.

• In the first part of the proof, we want to show that if $\Delta_s v_t(s)$ is nondecreasing in t, then $\Delta_s v_t(s)$ is nonincreasing in s. The rearranged model for the case of no refund option can be written for inventory positions $s+1$ and s as follows:

$$\begin{aligned} v_t(s+1) - v_{t-1}(s+1) &= \rho_t \max_p \{Pr(P_t \geq p)[p - \Delta_s v_{t-1}(s+1)]\}, \\ v_t(s) - v_{t-1}(s) &= \rho_t \max_p \{Pr(P_t \geq p)[p - \Delta_s v_{t-1}(s)]\}. \end{aligned} \tag{3.21}$$

Taking the difference of the two equations side by side, we obtain

$$\begin{aligned} \Delta_s v_t(s+1) - \Delta_s v_{t-1}(s+1) = \rho_t(&\max_p \{Pr(P_t \geq p)[p - \Delta_s v_{t-1}(s+1)]\} \\ &- \max_p \{Pr(P_t \geq p)[p - \Delta_s v_{t-1}(s)]\}). \end{aligned} \tag{3.22}$$

Let $p^*_{s+1,t} = arg \max_p \{Pr(P_t \geq p)[p - \Delta_s v_{t-1}(s+1)]\}$ denote the optimum restricted ticket price for the reduced formulation. It is obvious that the optimal price is determined by both the remaining inventory and time to departure. Naturally, the optimal price for a certain state would be suboptimal at a different state of the sales process.

If we substitute $p^*_{s+1,t}$ in both terms to be maximized in p, which is optimal for the first and suboptimal for the latter maximization in (3.22), the following inequality is obtained:

$$\Delta_s v_t(s+1) - \Delta_s v_{t-1}(s+1) \leq \rho_t Pr(P_t \geq p^*_{s+1,t})[\Delta_s v_{t-1}(s) - \Delta_s v_{t-1}(s+1)] \tag{3.23}$$

Since $\Delta_s v_t(s)$ is nondecreasing in t, the left-hand-side of the inequality (3.23) is nonnegative. Therefore, $\Delta_s v_{t-1}(s) - \Delta_s v_{t-1}(s+1) \geq 0$, meaning that $\Delta_s v_t(s)$ is nonincreasing in s.

• In the second part of the proof, we want to show that if $\Delta_s v_t(s)$ is nonincreasing in s, then $\Delta_s v_t(s)$ is nondecreasing in t. Recall (3.22) we have obtained in the first part of the proof.

Let $p^*_{s,t} = \underset{p}{argmax}\{Pr(P_t \geq p)[p - \Delta_s v_{t-1}(s)]\}$. If we substitute $p^*_{s,t}$ in both terms to be maximized in p, on the right hand side of (3.22), the following inequality is obtained:

$$\Delta_s v_t(s+1) - \Delta_s v_{t-1}(s+1) \quad \geq \quad \rho_t Pr(P_t \geq p^*_{s,t})[\Delta_s v_{t-1}(s) - \Delta_s v_{t-1}(s+1)].$$

Since $\Delta_s v_t(s)$ is nonincreasing in s, the right-hand-side of the equation is nonnegative. Therefore, $\Delta_s v_t(s+1) - \Delta_s v_{t-1}(s+1) \geq 0$, meaning that $\Delta_s v_t(s)$ is nondecreasing in t. ■

Lemma 3.2.2 shows the equivalence of the two conditions using the optimality equations derived for inventory positions s and $s+1$. Lemma 3.2.3 below uses induction over time to show that the marginal seat revenue is nonincreasing in s.

Lemma 3.2.3 $\Delta_s v_t(s)$ *is nonincreasing in s.*

Proof. (due to Talluri and van Ryzin 2005, Proposition 5.2) The proof is given by induction on t.

• $\Delta_s v_0(s+1) \leq \Delta_s v_0(s)$ holds true for every $s \in [1, S-1]$ since $v_0(s+1) = v_0(s) = 0$ by boundary condition.

• Assume $\Delta_s v_{t-1}(s+1) \leq \Delta_s v_{t-1}(s)$ for every $s \in [1, S-1]$.

• Denoting the optimal price for the inventory position $(s+i)$ at time t by $p^*_{s+i,t}$ and the corresponding sales probability term $Pr(P_t \geq p^*_{s+i,t})$ by $z_{s+i,t}$, the following equalities for $\Delta_s v_t(s+2)$ and $\Delta_s v_t(s+1)$ are obtained.

$$\begin{aligned}\Delta_s v_t(s+2) &= \rho_t \left(z_{s+2,t}[p^*_{s+2,t} - \Delta_s v_{t-1}(s+2)] - z_{s+1,t}[p^*_{s+1,t} - \Delta_s v_{t-1}(s+1)]\right) \\ &\quad +\Delta_s v_{t-1}(s+2) \\ \Delta_s v_t(s+1) &= \rho_t \left(z_{s+1,t}[p^*_{s+1,t} - \Delta_s v_{t-1}(s+1)] - z_{s,t}[p^*_{s,t} - \Delta_s v_{t-1}(s)]\right) \\ &\quad +\Delta_s v_{t-1}(s+1)\end{aligned}$$

Since $p^*_{s+1,t}$ is the maximizer for the function $Pr(P_t \geq p)[p - \Delta_s v_{t-1}(s+1)]$, $p^*_{s,t}$ and $p^*_{s+2,t}$ would be suboptimal values and substituting them in these equations would yield the following inequalities:

$$\begin{aligned}\Delta_s v_t(s+2) &\leq \rho_t \left(z_{s+2,t}[p^*_{s+2,t} - \Delta_s v_{t-1}(s+2)] - z_{s+2,t}[p^*_{s+2,t} - \Delta_s v_{t-1}(s+1)]\right) \\ &\quad +\Delta_s v_{t-1}(s+2) \\ &= \rho_t z_{s+2,t}\, [\Delta_s v_{t-1}(s+1) - \Delta_s v_{t-1}(s+2)] + \Delta_s v_{t-1}(s+2) \\ \Delta_s v_t(s+1) &\geq \rho_t \left(z_{s,t}[p^*_{s,t} - \Delta_s v_{t-1}(s+1)] - z_{s,t}[p^*_{s,t} - \Delta_s v_{t-1}(s)]\right) \\ &\quad +\Delta_s v_{t-1}(s+1) \\ &= \rho_t z_{s,t}\, [\Delta_s v_{t-1}(s) - \Delta_s v_{t-1}(s+1)] + \Delta_s v_{t-1}(s+1).\end{aligned}$$

Taking the difference,

$$\begin{aligned}\Delta_s v_t(s+2) - \Delta_s v_t(s+1) &\leq (1 - \rho_t z_{s+2,t})\Delta_s v_{t-1}(s+2) + \rho_t z_{s+2,t}\Delta_s v_{t-1}(s+1) \\ &\quad (1 - \rho_t z_{s,t})\Delta_s v_{t-1}(s+1) - \rho_t z_{s,t}\Delta_s v_{t-1}(s).\end{aligned}$$

Rearranging the terms on the right hand side above, we obtain

$$\begin{aligned}\Delta_s v_t(s+2) - \Delta_s v_t(s+1) &\leq (1 - \rho_t z_{s+2,t})\left(\Delta_s v_{t-1}(s+2) - \Delta_s v_{t-1}(s+1)\right) + \\ &\quad \rho_t z_{s,t}\left(\Delta_s v_{t-1}(s+1) - \Delta_s v_{t-1}(s)\right). \qquad (3.24)\end{aligned}$$

In (3.24), ρ_t, $z_{s,t}$ and $z_{s+2,t}$ are all probability values. So, the terms $1 - \rho_t z_{s+2,t}$ and $z_{s,t}$ are nonnegative. Using the induction assumption, both $\Delta_s v_{t-1}(s+2) - \Delta_s v_{t-1}(s+1)$ and $\Delta_s v_{t-1}(s+1) - \Delta_s v_{t-1}(s)$ are less than or equal to zero. Thus, we conclude $\Delta_s v_t(s+2) - \Delta_s v_t(s+1) \leq 0$, meaning that $\Delta_s v_t(s)$ is nonincreasing in s. ■

The structural characteristics of $\Delta_s v_t(s)$ provide valuable insight about the behavior of the optimal expected revenue-to-go function. With this formulation, it is also possible to make inferences about the optimal price, p^*_{st}. The following lemmas present behavioral facts about the optimal price. It is implicitly assumed in the lemmas that the customer reservation price, P_t, has a continuous cumulative probability distribution function.

Lemma 3.2.4 *Let p^*_{st} be a maximizer of the function $Pr(P_t \geq p)[p - \Delta_s v_{t-1}(s)]$. Then, the marginal revenue function, $Pr(P_t \geq p)p$, is a nonincreasing function of price at $p = p^*_{st}$.*

Proof. *(Proof by Contradiction) Assume $Pr(P_t \geq p)p$ is increasing at $p = p^*_{st}$. Then, there exists $\epsilon > 0$ such that*

$$Pr(P_t \geq p^*_{st} + \epsilon)(p^*_{st} + \epsilon) \geq Pr(P_t \geq p^*_{st})p^*_{st}. \tag{3.25}$$

By definition of complementary cumulative distribution function of P_t, we also have

$$Pr(P_t \geq p^*_{st} + \epsilon) \leq Pr(P_t \geq p^*_{st}). \tag{3.26}$$

Since $\Delta_s v_{t-1}(s)$ is a nonnegative quantity for any given (s, t), we have

$$Pr(P_t \geq p^*_{st} + \epsilon)\Delta_s v_{t-1}(s) \leq Pr(P_t \geq p^*_{st})\Delta_s v_{t-1}(s). \tag{3.27}$$

Taking the difference of (3.26) and (3.27), we obtain

$$Pr(P_t \geq p^*_{st} + \epsilon)(p^*_{st} + \epsilon - \Delta_s v_{t-1}(s)) \geq Pr(P_t \geq p^*_{st})(p^*_{st} - \Delta_s v_{t-1}(s)). \tag{3.28}$$

which contradicts the assumption that p^*_{st} is the maximizer. By contradiction, we conclude that $Pr(P_t \geq p)p$ is nonincreasing at $p = p^*_{st}$. ■

Proposition 3.2.5 *Let p^*_{st} be defined as in Lemma 3.2.4. Then, $p^*_{st} \geq p^*_{s+1,t}$.*

Proof. *It has been shown that, $\Delta_s v_{t-1}(s)$ is nonincreasing in s. Hence, we can define $\Delta_s v_{t-1}(s + 1) = \Delta_s v_{t-1}(s) - \delta$ where δ is nonnegative.*

(Proof by Contradiction) Assume that $p^*_{st} < p^*_{s+1,t}$. By optimality of p^*_{st}, we have the following inequality:

$$Pr(P_t \geq p^*_{st})(p^*_{st} - \Delta_s v_{t-1}(s)) \geq Pr(P_t \geq p^*_{s+1,t})(p^*_{s+1,t} - \Delta_s v_{t-1}(s)). \tag{3.29}$$

Since $\delta \geq 0$ and $p^*_{st} < p^*_{s+1,t}$, we also have

$$\delta Pr(P_t \geq p^*_{st}) \geq \delta Pr(P_t \geq p^*_{s+1,t}). \tag{3.30}$$

Addition of (3.29) and (3.30) yields,

$$Pr(P_t \geq p^*_{st})(p^*_{st} - \Delta_s v_{t-1}(s+1)) \geq Pr(P_t \geq p^*_{s+1,t})(p^*_{s+1,t} - \Delta_s v_{t-1}(s+1)), \qquad (3.31)$$

which contradict the optimality of $p^*_{s+1,t}$ for $Pr(P_t \geq p)[p - \Delta_s v_{t-1}(s+1)]$. Hence, $p^*_{st} \geq p^*_{s+1,t}$ and we conclude that the optimal restricted ticket price is decreasing in remaining seat inventory s. ■

Having shown that the optimal price decreases in the seat inventory s, the consequent idea is to examine its behavior in time to departure, t. However, under the assumption of time dependent reservation price, P_t, such a monotonicity result cannot be observed.

Proposition 3.2.5 states that p^*_{st} is nondecreasing in t if the time dependence of reservation price distribution is ignored. However, as noted in Remark 3.1.1, in airline RM the reservation prices tend to increase as time to departure decreases. With this motivation, an exemplary case opposing the monotonicity result in Lemma 3.2.6 under the assumption of $P_{t+1} \leq_{st} P_t$ is also presented in Counterexample 3.2.7.

Proposition 3.2.6 *Let p^*_{st} be defined as in Lemma 3.2.2 and the reservation price distribution be time independent; i.e. $f_{P_t}(p) = f_{P_{t+1}}(p)$. Then, $p^*_{st} \leq p^*_{s,t+1}$*

Proof. *In order to prove the lemma, it is required to show that $p^*_{st} \leq p^*_{s,t+1}$ under the assumption that the reservation price distribution is time independent. Let $\hat{p} \leq p^*_{st}$. By optimality of p^*_{st}, we have*

$$Pr(P_t \geq p^*_{st})(p^*_{st} - \Delta v_{t-1}(s)) \geq Pr(P_t \geq \hat{p})(\hat{p} - \Delta v_{t-1}(s)). \qquad (3.32)$$

$\hat{p} \leq p^*_{st}$ implies $0 < Pr(P_t \geq p^*_{st}) \leq Pr(P_t \geq \hat{p})$. Since marginal seat revenue is nondecreasing in t, we have $\Delta v_t(s) - \Delta v_{t-1}(s) \geq 0$. Then,

$$Pr(P_t \geq p^*_{st})(\Delta v_t(s) - \Delta v_{t-1}(s)) \leq Pr(P_t \geq \hat{p})(\Delta v_t(s) - \Delta v_{t-1}(s)). \qquad (3.33)$$

Taking the difference (3.2.6)-(3.2.6), we obtain:

$$Pr(P_t \geq p^*_{st})(p^*_{st} - \Delta v_t(s)) \geq Pr(P_t \geq \hat{p})(\hat{p} - \Delta v_t(s)). \qquad (3.34)$$

Hence, if reservation price distribution is independent of time to departure, then $\hat{p}$, which is less than p^*_{st}, cannot yield a better value for the function to be maximized at state (s,t+1); $Pr(P_{t+1} \geq p)(p - \Delta v_{t-1}(s))$. Thus, $p^*_{s,t+1}$ cannot be less then p^*_{st}. ■

Counterexample 3.2.7 *Consider the following system state:*

- $\rho_1 = \rho_2 = 0.05$,
- $s = 1$ *seats remaining* $t = 2$ *periods before departure,*
- *Reservation prices are* $P_1 \sim U[110, 130]$ *and* $P_2 \sim U[100, 120]$, *where* U *denotes the continuous uniform distribution.*

For any ticket price p, we have $Pr(P_1 \leq p) \leq Pr(P_2 \leq p)$, *hence* $P_2 \leq_{st} P_1$. *The optimality equation for* $v_1(1)$ *yields the following:*

$$v_1(1) = \rho_1 \max_p \{Pr(P_1 \geq p)[p - \Delta_s v_0(1)]\} + v_0(1). \tag{3.35}$$

The maximization for the uniform distribution gives p for all $p \leq 110$; *so, p values less than* 110 *are suboptimal and can be disregarded. For* $p \geq 130$, *the maximization term equals* 0; *so, p values greater than* 130 *can also be excluded. The terms* $\Delta_s v_0(1)$ *and* $v_0(1)$ *are both* 0 *by the boundary condition definition and they cancel out. Hence, the optimality equation can be rewritten as follows:*

$$v_1(1) = 0.05 \max_{110 \leq p \leq 130} \left\{ \frac{(130 - p)p}{20} \right\}. \tag{3.36}$$

Solving (3.2.7) in p, $p^*_{1,1} = 110$ *and* $v_1(1) = 5.5$ *are obtained.*

Likewise, the optimality equation for $v_2(1)$ *can be formulated as follows:*

$$v_2(1) - 5.5 = 0.05 \max_{100 \leq p \leq 120} \left\{ \frac{(120 - p)(p - 5.5)}{20} \right\}. \tag{3.37}$$

Accordingly, $p^*_{1,2} = 100$ *and* $v_2(1) = 10.225$ *are obtained. Therefore, it is possible to observe the case* $p^*_{s,t+1} < p^*_{st}$ *for* $s = 1$ *and* $t = 1$ *in case* $P_{t+1} \leq_{st} P_t$.$\square$

Section 3.1 studies the problem of determination of optimal prices, (p^*, q^*), at a given state (s, t, n). So far in Section 3.2, we study a modified problem by disregarding the possibility of booking cancellations at all and ignored q. Yet, it is possible to interpret this modified univariate problem as a restriction of the original bivariate problem with a given arbitrarily large $\overline{q}$ value that no customer would be willing to pay for, i.e., $Pr(Q_t \geq \overline{q}) = 0$ for all t. Lemma 3.2.2 investigates the relationship between the restricted ticket prices obtained by these two different formulations.

Lemma 3.2.8 *Let p^*_{st} be defined as in Lemma 3.2.2 and let (p^*, q^*) denote the optimal solution of the bivariate problem in (3.1). Then, at any given state (s, t, n), $p^*_{st} \geq p^*$.*

Proof. *Consider $\hat{p} < p^*$. By optimality of p^*, we have*

$$\begin{aligned} &Pr(P_t \geq \hat{p})[\hat{p} - \Delta_s v_{t-1}(s,n)] + Pr(P_t \geq \hat{p})Pr(Q_t \geq q^*)[q^* - \Delta_n v_{t-1}(s-1,n)] \leq \\ &Pr(P_t \geq p^*)[p^* - \Delta_s v_{t-1}(s,n)] + Pr(P_t \geq p^*)Pr(Q_t \geq q^*)[q^* - \Delta_n v_{t-1}(s-1,n)] \end{aligned} \tag{3.38}$$

Since $\hat{p} < p^*$, for complementary cumulative distributions we have $Pr(P_t \geq \hat{p}) \geq Pr(P_t \geq p^*)$. Hence, for optimum refund option value q^*, the following inequality follows:

$$Pr(P_t \geq \hat{p})Pr(Q_t \geq q^*)[q^* - \Delta_n v_{t-1}(s-1,n)] \geq Pr(P_t \geq p^*)Pr(Q_t \geq q^*)[q^* - \Delta_n v_{t-1}(s-1,n)] \tag{3.39}$$

Taking the difference of (3.2.8) and (3.2.8), we obtain the following inequality:

$$Pr(P_t \geq \hat{p})[\hat{p} - \Delta_s v_{t-1}(s,n)] \leq Pr(P_t \geq p^*)[p^* - \Delta_s v_{t-1}(s,n)]. \tag{3.40}$$

(3.2.8) shows that any restricted ticket price, $\hat{p}$, which is less than the optimum p^* obtained by the bivariate model, cannot provide a better result for the maximization term of the univariate problem. Therefore, we conclude that, the optimal solution of the univariate problem, p^*_{st}, must be greater than or equal to p^*. ■

3.3 Determining Refund Premium Given Restricted Ticket Price

The reduced model studied in Section (3.2) simplifies the problem to finding the optimal restricted ticket price, p^*_{st}, disregarding the gains and losses due to ticket refundability. Assuming that the restricted ticket price obtained by ignoring the cancellation can provide a close approximation to the true optimum, a secondary subproblem can be reformulated for a given restricted ticket price. This approach does not only serve as a complementary to the pricing method in Section (3.2), but also requires particular attention due to the fact that every airline company has a current methodology to determine the restricted ticket price and might be unwilling to change that method; yet, a refund premium calculation method for given restricted ticket price could still be integrated. Let $v_t(s, n|p)$ denote the value function for a given value of ticket price

p posted at time point t. Accordingly, the problem can be formulated as follows:

$$v_t(s,n|p) = \max_q \left\{ \begin{array}{l} [\rho_t \Pr(P_t < p) + (1-\lambda_t)]\, v_{t-1}(s,n) \\ +\rho_t \Pr(P_t \geq p) \Pr(Q_t < q|P_t \geq p)\, [p + v_{t-1}(s-1,n)] \\ +\rho_t \Pr(P_t \geq p) \Pr(Q_t \geq q|P_t \geq p)\, [p + q + v_{t-1}(s-1,n+1)] \end{array} \right\}. \tag{3.41}$$

Noticing that for given p value the maximization is in terms of q only, (3.3) can be rewritten as below:

$$\begin{aligned} v_t(s,n|p) - v_{t-1}(s,n) =& \rho_t \Pr(P_t \geq p)\{[p - \Delta_s v_{t-1}(s,n)] \\ & + \max_q \{\Pr(Q_t \geq q|P_t \geq p)[q - \Delta_n v_{t-1}(s-1,n)]\}\}, \end{aligned} \tag{3.42}$$

where

$\Delta_s v_{t-1}(s,n) = v_{t-1}(s,n) - v_{t-1}(s-1,n),$

$\Delta_n v_{t-1}(s-1,n) = v_{t-1}(s-1,n) - v_{t-1}(s-1,n+1).$

Thus, for given restricted ticket price, the problem of determining the price of the refund option, $q_{st}(p)$, can be solved by finding the solution of the maximization given in (3.3). In this respect, obtaining the probability distribution of refund option reservation price, Q_t, conditioned on the value of p is a critical issue. Also, the estimation of marginal decrease in revenue due to selling an extra refund option, $\Delta_n v_{t-1}(s-1,n)$, requires intention.

Up to this point, we concentrate on simplifying the dynamic pricing problem. The immense complexity of the original formulation, which is a consequence of high dimensionality of the state space and nonlinearity of optimal expected revenue-to-go function, is one of the motivations for these efforts. The nature of the airline RM problem restricts the duration of the optimization procedure; the sales agent requires an instantaneous price information and simplification of the problem can reduce the solution time to the desired level. So far, the state space has been reduced and inferences have been made for finding p^*_{st}, the restricted ticket price ignoring the cancellation possibility, and $q_{st}(p)$, the optimal refund option price at time t for a given restricted ticket price, p, instead of simultaneously determining optimal p and q as a function of the state variables. Hence, noticing that simultaneous optimization methods can provide better prices at the expense of considerably longer solution time (simultaneous optimization may even be infeasible), we set our solution strategy as sequential optimization of the decision variables p and q. The first subproblem is to find p^*_{st} ignoring the effects of cancellation on the revenue. Chapter 4 covers the proposed methods for this reduced problem. The latter subproblem is to determine $q_{st}(p)$ for the given p and it is investigated in detail in Chapter 5.

CHAPTER 4

DETERMINING THE RESTRICTED TICKET PRICE

As mentioned in Section 3.1, the optimization problem is treated in two steps. In the first step, the cancellation possibility is not taken into consideration and the restricted ticket price is calculated for a system that does not allow cancellations. This section deals with the first subproblem introduced in Section 3.1.

In Section 3.1, it is argued that for a given value of the restricted ticket price, p, finding the corresponding optimal value of q is relatively easier than finding the optimal values of both variables simultaneously. The second subproblem is for determining refund option price, q, for a given fixed p. In the first subproblem, the cancellation possibility is disregarded and all tickets are assumed nonrefundable. Consequently, the refund premium, q, and the number of refundable tickets sold, n are removed from the formulation. Furthermore, the gain and loss terms attributed to refund options and cancellation possibility that complicate the expected revenue-to-go function vanished. Thus, for estimating the optimal value of p only, we can come up with more tractable dynamic pricing models.

For solving the first subproblem, we need to know the probability distribution of the reservation price, P_t. Working with reservation price distributions, we study the demand-price relation at an individual level instead of the classical demand-price models formulated in aggregate quantities. Section 4.1 investigates the optimal price and revenue-to-go functions for different reservation price distributions.

4.1 Demand-Price Relationship

In the reduced formulation of the bivariate problem discussed in Section 3.2, the optimality equation is given in terms of $v_{t-1}(s)$. Upon a customer arrival in state (s,t), two possible situations may occur; either the ticket is sold for p and the expected revenue-to-go becomes $v_{t-1}(s-1)$ or the customer does not purchase the ticket and the inventory remains unchanged. The optimality equation is,

$$v_t(s) = \lambda_t \max_p \{\Pr(P_t \geq p)(p + v_{t-1}(s-1)) + \Pr(P_t < p)v_{t-1}(s)\} + (1-\lambda_t)v_{t-1}(s). \quad (4.1)$$

The probability of sales is given by the complementary cumulative distribution of customer reservation price. Under the assumption that the reservation price P_t has a bounded, continuous pdf on a bounded support, probability of sales, $\Pr(P_t \geq p)$, and sales price, p can be expressed in terms of each other. Noticing this one-to-one correspondence, Lin (2004) formulates the dynamic pricing problem by using the probability of sales as the decision variable instead of the price. We denote the probability of sales by z and refer to it as the *sales incentive* in this study. Note that throughout this thesis, sales incentive is alternatively denoted as $z(p)$ to emphasize the one-to-one correspondence with restricted ticket price and as z_{st} to remind the dependence of sales incentive value on the state variables.

Remark 4.1.1 *Notice that in our analysis the individual demand is disintegrated in terms of probability of sales, z, and probability of customer arrival, λ_t. As mentioned previously, it is assumed in airline RM that the customer reservation price is a time-variant random variable; hence, z is time-variant similar to λ_t. Yet, it will be referred to as z without any explicit time index in our formulations for notational convenience.*

While arrival probability, λ_t, is independent of the ticket price, p, sales incentive is determined by p so we reflect the effect of price on demand with z. Hence, the classical price-demand formulations studied in economic literature can be adopted in our models by the definition of z. In classical market models, demand curves represent the aggregate quantities that could be sold at given price levels. At the individual level, we replace the sales quantity with the sales incentive by rescaling the demand function so that it takes values on the interval $[0,1]$.

Let $F_{P_t}(p)$ denote the cumulative distribution function of the time dependent random variable P_t. The seller should decide on the optimal value of the sales incentive z for revenue maximization. Corresponding optimal price, p^*_{st}, is the optimal restricted ticket price to be posted for a customer to arrive in state (s, t). The optimality equation is written below with respect to the decision variable z:

$$v_t(s) = \lambda_t \max_{z \in [0,1]} \left\{ z \left[p - \Delta v_{t-1}(s) \right] \mid p = F^{-1}_{P_t}(1 - z) \right\}. \tag{4.2}$$

Let $\Gamma(z) = -z\Delta v_{t-1}(s) + zp + v_{t-1}(s)$. For the optimal z^* value, we have $\dfrac{d\Gamma}{dz} = 0$ as the first order (necessary) optimality condition. The derivative is given as follows:

$$\frac{d\Gamma}{dz} = -\Delta v_{t-1}(s) + p + z\frac{dp}{dz}. \tag{4.3}$$

Notice that in the differentiation with respect to z, the derivative of $\Delta v_{t-1}(s)$ equals 0 since the sales incentive at time t does not have an influence on the marginal value function at time $t-1$. Evaluating $\dfrac{d\Gamma}{dz}$ at $z = z^*_{st}$, we obtain the following first order (necessary) optimality condition:

$$-\Delta v_{t-1}(s) + p^*_{st} + z^*_{st}\frac{dp}{dz}\Big|_{z=z^*_{st}} = 0, \tag{4.4}$$

where $p^*_{st} = F^{-1}_{P_t}(1 - z^*_{st})$.

4.1.1 Demand Models in Microeconomics

Having obtained the first order condition for the optimal sales incentive, we should determine the reservation price distribution through demand-price relationship at this stage. In this thesis, we will consider two commonly used demand functions in the economic literature, linear and isoelastic demand curves to formulate corresponding sales incentive functions in terms of p.

Linear demand curves represent equal marginal decrease in demand for the same amount of increase in price at every price level and the quantity demanded can be written as a function of price as $q(p) = a - bp$ (for $a, b > 0$ and $p \in [0, \frac{a}{b}]$). Isoelastic demand, on the other hand, is formulated as $q(p) = ap^{-b}$ (for $a > 0$, $b > 1$), representing demand curves on which the price elasticity of demand $\left(\dfrac{dq}{dp} \times \dfrac{p}{q}\right)$ is constant. Both demand curves are depicted in Figure 4.1.

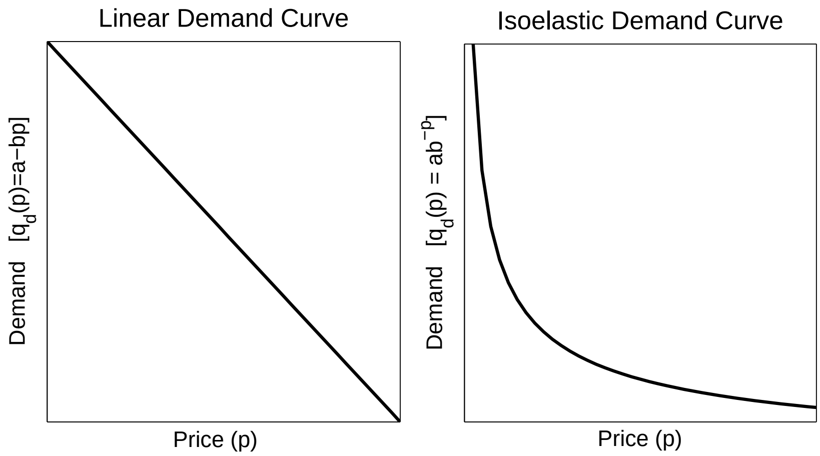

Figure 4.1: Linear and Isoelastic demand curves

The first order (necessary) optimality condition derived for the optimal sales price is $p^*_{st} = \Delta_s v_{t-1}(s) - z^*_{st} \frac{dp}{dz}|_{z=z^*_{st}}$. In analogy with the given demand models, this optimality equation is reinvestigated and the following results are obtained:

- Consider the linear relationship: $z(p) = a - bp$. This relation could be defined for $p \in [\frac{a-1}{b}, \frac{a}{b}]$ so that $z \in [0, 1]$. Keeping the time dependence of the reservation price distribution in mind, we would like to denote the distribution parameters as a_t and b_t. According to linear sales incentive formulation, $\frac{dp}{dz} = -\frac{1}{b_t}$ and substituting it in the first order optimality equation, we obtain (4.5).

$$p = \frac{1}{2}\left(\Delta_s v_{t-1}(s) + \frac{a_t}{b_t}\right). \tag{4.5}$$

  For the linear demand case, $p = \frac{a_t - z}{b_t}$ and the value function to be maximized in z is $\Gamma(z) = -\frac{1}{b_t}z^2 + (\frac{a_t}{b_t} - \Delta_s v_{t-1}(s))z + v_{t-1}(s)$. Since $\Gamma(z)$ is a concave function of z, the second order optimality condition is also satisfied for solution of (4.5), if exists. If there exists no solution for $p \in [\frac{a-1}{b}, \frac{a}{b}]$, then $\Gamma(z)$ would be maximized at one of the endpoints of the interval depending on the value of $\Delta_s v_{t-1}(s)$.

  $z(p)$ is the complementary cumulative distribution value of reservation price and accordingly reservation price follows uniform distribution in this case; $f_{P_t}(p) = b_t$ over the support $[\frac{a_t-1}{b_t}, \frac{a_t}{b_t}]$. To preserve the (stochastically) decreasing structure of reservation prices in time to departure, b_t should be nondecreasing and a_t should be nonincreasing in t to guarantee that $P_t \geq_{st} P_{t+1}$.

- Consider the isoelastic relationship: $z(p) = a_t p^{-b_t}$ where $a_t > 0$, $b_t > 1$ and sales incentive is defined for $p \geq a_t^{1/b_t}$. The derivative term in terms of time-dependent parameters could be written as $\frac{dp}{dz} = -\frac{1}{a_t b_t} p^{b_t+1}$ and the corresponding first order optimality equation is as below:

$$p^*_{st} = \Delta_s v_{t-1}(s) \frac{b_t}{b_t - 1}. \tag{4.6}$$

For the isoelastic demand definition, $\Gamma(z)$ could be expressed in terms terms of p as $a_t p^{-b_t}(p - \Delta v_{t-1}(s))$. The first derivative with respect to p is found as follows:

$$\begin{aligned}
\frac{d}{dp} a_t p^{-b_t}(p - \Delta v_{t-1}(s)) &= a_t p^{-b_t - 1}(b_t \Delta v_{t-1}(s) - p(b_t - 1)) \\
&= a_t(b_t - 1)p^{-b_t - 1}(\Delta v_{t-1}(s)\frac{b}{b-1} - p).
\end{aligned} \tag{4.7}$$

Then, the first derivative is negative for $p > \Delta v_{t-1}(s)\frac{b_t}{b_t-1}$ and positive for $p < \Delta v_{t-1}(s)\frac{b_t}{b_t-1}$. The second order derivative evaluated at $p = \Delta v_{t-1}(s)\frac{b_t}{b_t-1}$ is as given below:

$$\frac{d^2}{dp^2} a_t p^{-b_t}(p - \Delta v_{t-1}(s))|_{p=\Delta v_{t-1}(s)\frac{b_t}{b_t-1}} = -a_t b_t \Delta v_{t-1}(s)(\Delta v_{t-1}(s)\frac{b_t}{b_t - 1})^{-b_t - 2} \leq 0 \tag{4.8}$$

(4.8) implies that the second order optimality condition is also satisfied. Thus, $p^* = \Delta v_{t-1}(s)\frac{b_t}{b_t-1}$ is the unique maximizer provided that $\Delta v_{t-1}(s)\frac{b_t}{b_t-1} \geq a_t^{1/b_t}$. If $\Delta v_{t-1}(s)\frac{b_t}{b_t-1} < a_t^{1/b_t}$, then the function is decreasing on the support and the optimal price is $p^* = a_t^{1/b_t}$.

The pdf of the reservation price for the isoelastic sales incentive formulation is $f_{P_t}(p) = a_t b_t p^{-b_t-1}$. Similar to linear sales incentive formulation, b_t should be nondecreasing and a_t should be nonincreasing in t to preserve the desired temporal stochastic ordering of reservation prices.

In the linear sales incentive case, $\frac{a}{b} > p^*$ due to the definition of p values in the domain and, hence, $p^* > \Delta_s v_{t-1}(s)$. Similarly, in the isoelastic sales incentive case, $p^* > \Delta_s v_{t-1}(s)$, because $\frac{b}{b-1}$ is always larger than 1. Working with the optimality equation, we have noticed in Chapter 3 that the optimal price, p^*, should always be larger than the marginal value of a seat, $\Delta_s v_{t-1}(s)$ and our observations for particular demand-price relationships discussed here are parallel to this finding.

4.1.2 Logarithmic Sales Incentive Function

Up to this point, we work with the linear and isoelastic sales incentive formulations derived from the corresponding demand models in economic literature. In Section 3, we also present the case of exponential sales incentive model and the derivation of the optimal pricing policy under that assumption. At this point we would like to introduce another formulation for z, which gives a solution to the dynamic pricing problem similar to another problem in the inventory literature.

In the definition of pdf derived from isoelastic sales incentive function, $f_{P_t}(p) = abp^{-b-1}$, the density function is inversely proportional to p^{b+1} where $b > 1$. We similarly define a pdf inversely proportional to p, as $f_{P_t}(p) = \frac{a_t}{p}$ over a closed interval whose upper and lower bounds are $p_{up}(t)$ and $p_{low}(t)$, respectively. Corresponding sales incentive function is $z(p) = 1 - F_{P_t}(p) = a_t\,(\ln(p) - \ln(p_{low}(t)))$. With the definition of price bounds, we require $z(p_{low}(t)) = 1$ and $z(p_{up}(t)) = 0$ and accordingly we can write a_t in terms of these price bounds. The sales incentive function could be stated as a function of p as given below:

$$z(p) = \frac{\ln(p_{up}(t)) - \ln(p)}{\ln(p_{up}(t)) - \ln(p_{low}(t))}. \tag{4.9}$$

Similar to the distribution parameters in linear and isoelastic sales incentive formulations, $p_{low}(t)$ and $p_{up}(t)$ values are time dependent. In particular, $p_{low}(t)$ and $p_{up}(t)$ should increase as we move forward over the sales horizon, i.e. they both should be nonincreasing functions of t. In the first graph of Figure 4.2, an exemplary scenario for explaining the change of these price bounds over the sales horizon is depicted. Keeping the time dependence of these parameters in mind, from this point on we will not use time reference in their notation for the sake of notation simplicity. Figure 4.2 also presents a depiction of the behavior of z as a function of p. In the second graph, the interval $[p_{low}, p_{up}]$ is defined as the range of possible prices for the seller. Price levels below p_{low} are not rational for the seller since there is a higher price for which the probability of sales is also 1. Likewise, prices above p_{up} are out of our interest since at these levels the sales will be impossible.

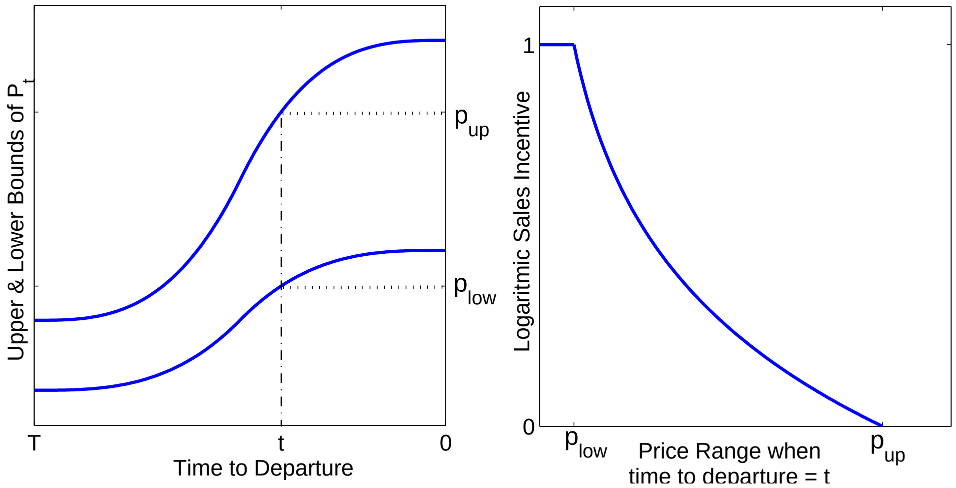

Figure 4.2: Change of reservation price bounds & Sales Incentive

As a consequence of one-to-one correspondence on the interval $[p_{low}, p_{up}]$, we could also represent price as a function of sales incentive as $p(z) = p_{low}^{z} p_{up}^{1-z}$. Further properties regarding to this sales incentive formulation are given in Lemma 4.1.2.

Lemma 4.1.2 *Let price-sales incentive relation follow $p(z) = p_{low}^{z} p_{up}^{1-z}$ for $z \in [0, 1]$. Then;*

1. $\dfrac{dp}{dz} = -\kappa p$ *holds where* $\kappa = \ln(\dfrac{p_{up}}{p_{low}})$.
2. *The optimal sales incentive value, z^*, satisfies the following equation:*

$$z^* = \frac{z^* p^*}{\kappa z^* p^* + \Delta v_{t-1}(s)}. \tag{4.10}$$

3. *If $p_{low} < \dfrac{p_{up}}{e}$, then the prices in the interval $[p_{low}, \dfrac{p_{up}}{e}]$ are suboptimal.*
4. *The sales incentive value, which satisfies (4.10), is the unique maximizer of $\Gamma(z)$.*

Proof.

- The first statement of the lemma is straightforward from the definition of p as a function of z.

- For the second part, we refer to the (necessary) optimality condition for z^* given in (4.4). Letting $\frac{dp}{dz} = -\kappa p$, the equation in (4.4) can be rewritten as follows:

$$-\Delta v_{t-1}(s) + p^* - z^* \kappa p^* = 0. \tag{4.11}$$

  Taking the negative terms in (4.11) to the right hand side and multiplying both sides by the factor $\frac{z^*}{\kappa z^* p^* + \Delta v_{t-1}(s)}$, the solution of (4.11) for z^* can be given as in (4.10).

- Third statement of the lemma gives a restriction of potential optimal prices on the interval $[p_{low}, p_{up}]$ and in order to prove it, we refer to Lemma 3.2.4, which states that the marginal revenue function, zp, is nonincreasing in price at $p = p^*$. When $z = \frac{\ln(p_{up}) - \ln(p)}{\ln(p_{up}) - \ln(p_{low})}$, the derivative of marginal revenue function is found as below:

$$\frac{d}{dp}\Pr(P_t \geq p)p = \frac{\ln(p_{up}) - \ln(p) - 1}{\ln(p_{up}) - \ln(p_{low})} = \frac{1}{\ln(p_{up}) - \ln(p_{low})}\ln(\frac{p_{up}}{p\mathrm{e}}) \tag{4.12}$$

  The derivative term in (4.12) is positive when $\frac{p_{up}}{p\mathrm{e}} \geq 1$ and in this case marginal revenue function turns out to be increasing. Hence, for the given logarithmic sales incentive formulation, the interval on which the optimal price will exist is $[\max\left\{p_{low}, \frac{p_{up}}{\mathrm{e}}\right\}, p_{up}]$.

- To prove the last statement of lemma, we consider the second order (sufficient) optimality condition in terms of z as below:

$$\begin{aligned} \frac{d^2\Gamma}{dz^2} &= 2\frac{dp}{dz} + z\frac{d^2p}{dz^2} \\ &= \kappa p(z\kappa - 2). \end{aligned} \tag{4.13}$$

  Rearranging (4.11), the solution of first order condition could be rewritten as $z = \frac{p - \Delta v_{t-1}(s)}{\kappa p}$. Corresponding second order condition is as follows:

$$\frac{d^2\Gamma}{dz^2}\Big|_{z=\frac{p-\Delta v_{t-1}(s)}{\kappa p}} = -\kappa p(1 + \frac{\Delta v_{t-1}(s)}{p}) < 0. \tag{4.14}$$

  Hence, if there exists a solution to the first order optimality condition $z = \frac{p - \Delta v_{t-1}(s)}{\kappa p}$, then that value of z is a local maximizer of $\Gamma(z)$. Next, we should prove the uniqueness of this maximizer.

Rearranging the terms, the first order optimality condition can be written as $p(1-\kappa z) = \Delta v_{t-1}(s)$ where both p and $(1-\kappa z)$ are decreasing in z and $\Delta v_{t-1}(s)$ is independent of the value of z for given s and t values. Having shown the monotonicity of $p(1-\kappa z)$ for all z in $[0,1]$, we conclude that there may exist at most one $z \in [0,1]$ that satisfies the first order optimality condition. Therefore, the local maximizer of $\Gamma(z)$ is the optimal sales incentive value on the interval $[0,1]$.

■

The logarithmic sales incentive formulation gives a nice equality which is also sufficient for finding the optimal price and an elegant formulation in terms of the lower and upper bound of customer reservation price. However, the primary motivation for the investigation of this particular model is the structure of the optimal solution and the rationale of this choice is explained in Remark 4.1.3.

Remark 4.1.3 *For $\kappa = 1$, the fraction in (4.20) resembles the solution of the newsboy problem in the inventory theory as shown in (4.15). (Note that restricting $\kappa = 1$ is equivalent to assuming that the relation $p_{up} = \mathrm{e}p_{low}$ holds for the price upper and lower bounds.)*

$$p^* = F_{P_t}^{-1}\left(\frac{\Delta v_{t-1}(s)}{z^* p^* + \Delta v_{t-1}(s)}\right). \tag{4.15}$$

For the optimal solution of the newsboy problem, the cumulative probability of the demand at the optimal inventory level equals the ratio $\frac{c_u}{c_u + c_o}$ where c_u and c_o denote unit underage and overage costs, respectively. The optimal price in our formulation is also evaluated by taking the inverse of the cumulative density function of reservation price for a similar ratio. In the newsboy problem, underage cost is the marginal revenue of an additional item that would have been sold to satisfy the excess demand, the difference between sales price and the purchase price. Similarly, in (4.15), $\Delta_s v_{t-1}(s) = v_{t-1}(s) - v_{t-1}(s-1)$ is the marginal value of an extra seat. Overage cost, on the other hand, refers to the marginal loss due to each item in the inventory that remains in the inventory after the sales, the difference between the purchase price and the salvage value. In the airline example, the inventory is fixed and an overage situation corresponds to having unsold seats at the time of departure. The term $z^* p^*$ in our formula is the marginal revenue that can be gained from selling one seat at *the current time instant. Hence, it can be interpreted as an expected opportunity loss corresponding to not selling the seat immediately and is analogous to the overage cost in the newsboy problem in this respect.*

The findings for all four $z - p$ relationships investigated so far are tabulated in Table 4.1. Notice that, only in the exponential case we keep α constant over sales horizon since the closed-form solution of the p^* cannot be obtained otherwise. For the other distributions, the parameters can be time-variant and the results will still be valid.

Table 4.1: Summary for analyzed demand-sales incentive relationships

Demand Type	z(p)	Parameters	$f_{P_t}(p)$	p^*
Exponential	$e^{-\alpha p}$	$\alpha > 0$	$\alpha e^{-\alpha p}$	$\Delta v_{t-1}(s) + \dfrac{1}{\alpha}$
Logarithmic	$\dfrac{\ln(\frac{p}{p_{up}})}{\ln(\frac{p_{low}}{p_{up}})}$	$p_{up} > p_{low} > 0$	$\dfrac{1}{p\left(\ln(\frac{p_{up}}{p_{low}})\right)}$	$\dfrac{\Delta v_{t-1}(s)}{1 - z^* \ln(\frac{p_{up}}{p_{low}})}$
Linear	a-bp	$a, b > 0$	b	$\dfrac{1}{2}\left(\Delta v_{t-1}(s) + \dfrac{a}{b}\right)$
Isoelastic	ap^{-b}	$a > 0, b > 1$	abp^{-b-1}	$\Delta v_{t-1}(s)\dfrac{b}{b-1}$

Remark 4.1.4 *For any given reservation price distribution, it is assumed that the distribution parameters are time variant with regard to the change in customers' willingness to pay. Along with the reservation price distribution, F_{P_t}, the rate parameter of the Nonhomogenous Poisson distributed customer arrivals is time dependent. Hence, the seller needs to know the values of these parameters at each unit time period in the remainder of the sales horizon in order to determine the price for the current customer with the proposed DP model. Considering the possibility that the seller may not accurately estimate the demand parameters at every future time point, we have also studied an approximate solution for the optimality equation (4.11). To avoid the*

requirement for recursive solution, an approximation for the marginal value function, $\Delta v_{t-1}(s)$, is proposed.

The approximate models developed in this thesis are based on intuition inferred from the business concepts in the RM literature and discussed in this respect. Therefore, the error bounds in these approximations are not studied herein.

4.2 Approximating Marginal Value Function

The term $\Delta_s v_{t-1}(s)$, is the opportunity cost of selling the s^{th} available seat in state (s,t). At state $(s, t-1)$, the s^{th} seat can generate additional profit only if the seller could sell all $s-1$ seats before departure and there still remains demand. Otherwise, an additional empty seat on the flight has no contribution to the total revenue. Although this simplistic perspective overlooks the interdependence of pricing strategies and seat inventory, it provides valuable insight for approximating $\Delta_s v_{t-1}(s)$.

In this part, we consider an approximation for the marginal value function by assuming that the seller could anticipate the probability of selling all s tickets at a state (s,t) and the price of the last ticket that would be charged to a last-minute customer, p_{fin}.

$SP(s,t)$: stock-out probability (the probability that all of the seats will be sold before departure; that is, the probability that the demand during the t time periods before departure is greater than s).

p_{fin} : final price the seller would post at the end of sales horizon.

$\Delta_s v_t(s)$: The approximation we consider is $\Delta_s v_{t-1}(s) \approx SP(s, t-1)p_{fin} + 0(1 - SP(s, t-1))$ for the s^{th} seat at time t.

Evidently, the maximum ticket price and the stock-out probability depend on the policy of the seller during the sales process. However, the previous sales records for similar flights can be useful for estimating the values of p_{fin} and $SP(s,t)$. Remark 4.2.1 outlines the methods in airline RM literature that employs stock-out probability or similar figures in the estimation of expected incremental seat revenue. The behavior of the stock-out probability $SP(s,t)$ is investigated as a function of the state variables in Remark 4.2.2.

Remark 4.2.1 *The probability of stock-out during the sales horizon, $SP(s,t)$, is an important figure in the estimation of the marginal value function, $\Delta v_{t-1}(s)$, in the current state, (s,t). Littlewood (1972) describes the decision rule for the 2-class problem as "keep low fare open as long as the probability that the demand for high-fare seats exceeds the remaining seat inventory is less than the ratio of low-fare to high-fare". Belobaba (1989) similarly defines the probability of spill for the n-class case, $\overline{P}(S_i)$, as the probability of receiving more than S_i requests for fare class i. The expected marginal seat revenue for fare class i, $EMSR(S_i)$, is found by multiplying the probability of spill and the fare of fare class i. EMSR values serve as thresholds to determine protection levels of the fare classes in seat inventory control.*

Belobaba and Farkas (1999) point out the importance of spill estimation for success of EMSR heuristics or similar techniques in airline RM. In their work, they employ (1) a total demand function aggregating the fare classes, (2) multiple fare class demand representation with the assumption of lower fare passengers booking before higher fare and (3) multiple period multiple class demand representation. The adequacy of each demand representation and the impact on the yield are investigated on a comparative basis. Within a similar context to spill, we employ the stock-out probability $SP(s,t)$ in our formulations. In the dynamic pricing model adopted here, there are no fare classes and the price itself is the control variable. The marginal value of a seat is regarded as a function of the state of the system (s,t) and is approximated as the maximum price anticipated for the sales horizon, p_{fin}, multiplied with the probability of stock-out, $SP(s,t)$. Here, the price p of the seat is not used but p_{fin} is used. In this work, we use the term stock-out probability instead of spill since the term spill is used to refer to both the amount of passenger demand above the capacity and the probability of demand being above the capacity. $\square$

Remark 4.2.2 *Estimating the stock-out probability $SP(s,t)$ for a certain inventory level in a given state can be possible by assuming a functional form in terms of the state variables (s,t). The following properties should be satisfied for $SP(s,t)$.*

- *$SP(s,t)$ should be a well defined function for non-negative integer values of* *seat inventory s and for positive values of t and its value should be between* 0 *and* 1 *on this domain. Note that* $s \in [0, S]$ *and* $t \in [0, T]$.

- *If there are unsold seats,* $SP(s,t)$ *will tend to* 0 *as the flight approaches.*

  $\lim_{t \to 0^+} SP(s,t) = 0$ *for* $s > 0$.

- *When inventory depletes before the departure, the probability of stock-out (which has already occurred) should be equal to* 1.

  $SP(0,t) = 1$ *for* $t > 0$.

- $SP(s,t)$ *should be nonincreasing in s; the lower the seat inventory is, the higher the probability of stock-out gets.*

- $SP(s,t)$ *should be nondecreasing in t; the more time the sales agent has before the departure, the higher the probability of stock-out gets.*$\square$

4.2.1 Parametric Estimation of $SP(s,t)$

It is reasonable to use the historical stock-out data to estimate the functional behavior of $SP(s,t)$. The statistical methodology of parameter estimation, using a parametric or nonparametric method is a critical issue for the decision makers at this point.

A plausible functional relation that satisfies the conditions in Remark 4.2.2 is $SP(s,t) = 1 - \left(\frac{s}{S+\omega}\right)^{\frac{\Lambda_{[0,t]}}{s}}$, where $\omega > s - S$ and $\Lambda_{[0,t]} = \int_0^t \lambda_\tau d\tau > 0$. The approximate dynamic pricing model based on this parametric stockout probability estimation is abbreviated as $Para - M$. Condition on ω assures that $SP(s,t) \in (0,1)$ for $s \in [0,S]$ and positivity of $\Lambda_{[0,t]}$ is required for meeting the monotonicity properties of $SP(s,t)$ explained above. $\Lambda_{[0,t]}$ is introduced in Section 3.2.1 also and denotes the expected number of customer arrivals from time t until the flight. The value of ω does not have such a direct interpretation and it should be estimated according to the demand characteristics of the flight. This formulation relies on the following two fundamental observations about the stock-out probability.

1. The Future: While we are considering the nature of depletion of the remaining seats, the anticipation of the demand in the remainder of the sales horizon

must be taken into account. In our model, we propose representing the impact of anticipated demand intensity by the ratio of expected future customers to remaining seats. This ratio is $\frac{\Lambda_{[0,t]}}{s}$, where $\Lambda_{[0,t]}$ is the time-weighted average arrival rate of customers. The ratio can simply be interpreted as a demand-to-supply ratio.

2. The Past: The impact of sales history should also be reflected in the stock-out probability function. The percentage of remaining tickets, $\frac{s}{S}$, gives us an idea about the sales trend for the flight so far; smaller values imply higher possibility of stock-out and vice versa. However, in our formulation, this ratio is not used directly to enable calibration. Instead of $\frac{s}{S}$, the ratio $\frac{s}{S+\omega}$ is adopted. In order to assure that $SP(s,t)$ takes values on the interval $[0,1]$, we restrict $\omega > s - S$.

The positivity of $\Lambda_{[0,t]}$ and ω being larger than $s - S$ guarantee that the derivative with respect to time given in (4.16) is positive. That is, the stock-out probability increases as the time to departure increases for a given s value.

$$\frac{\partial SP}{\partial t} = -\left(\frac{s}{S+\omega}\right)^{\frac{\Lambda_{[0,t]}}{s}} \left(\frac{\lambda_t}{s}\right) \ln\left(\frac{s}{S+\omega_1}\right) > 0. \tag{4.16}$$

We also expect the stock-out probability to be higher for smaller s values. The partial derivative given in (4.17) shows that $SP(s,t)$ is a nonincreasing function of s for given t:

$$\frac{\partial SP}{\partial s} = -\left(\frac{s}{S+\omega}\right)^{\frac{\Lambda_{[0,t]}}{s}} \left(\frac{\Lambda_{[0,t]}}{s^2}\right)\left(1 - \ln\frac{s}{S+\omega_1}\right) < 0 \tag{4.17}$$

The parameter ω, is introduced to the formulation to allow the stock-out probability for the initial inventory level to be greater than 0. For finding this parameter, we can use an estimate for the stock-out probability value at a reference state. In this respect, we can refer to the expertise of sales people for an estimate for the stockout probability at the beginning of sales (when time to departure is T and S seats are available for sale). Let $\eta = SP(S,T)$ denote this estimate. Using this estimate obtained by insights of experts from the historical sales records, ω_1 can be obtained as follows:

$$\eta = SP(S,T) = 1 - \left(\frac{S}{S+\omega_1}\right)^{\frac{\Lambda_{[0,T]}}{S}} \Rightarrow \omega_1 = S\left((1-\eta)^{-\frac{S}{\Lambda_{[0,T]}}}\right) - S. \tag{4.18}$$

Example 4.2.3 *Consider a 30-days ahead flight for which the seller intends to use the parametric estimation method for the given type of stock-out probability function. Let the parameter setting for the problem be given as below:*

- *The daily intensity of customer arrivals increase from* 5 *arrivals per day to* 40 *arrivals per day* ($\lambda_t = 5^t 40^{T-t}$).
- $S = 200$ *seats are available for sales.*
- $SP(200, 30) = 0.6$ *is seller's estimate for the probability of selling all seats before the flight.*

According to the given λ_t, *demand-to-supply ratio (i.e. expected number of customer arrivals per remaining seat) is found as* $\frac{\Lambda_{[0,T]}}{S} = 2.52$ *and the other parameter* $\omega =$ 87.5. *Hence, the seller could utilize the stock-out probability function* $SP(s,t) = 1 - \left(\frac{s}{287.5}\right)^{\frac{\Lambda_{[0,T]}}{s}}$. $\Lambda_{[0,t]}$ *is a time dependent parameter so it would be recalculated as the sales proceeds. For this scenario, the stockout probability is depicted below as a function of seat inventory s for fixed values of t in Figure 4.3.*

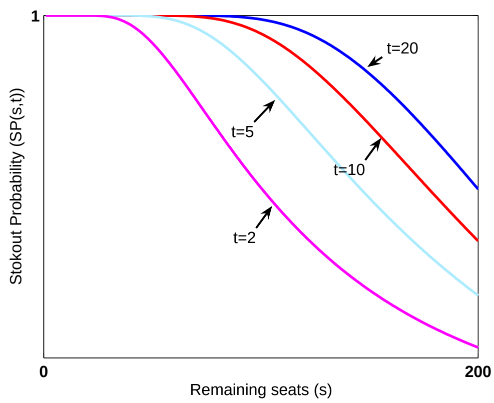

Figure 4.3: Stockout probability curves at given dates

Estimation of the stock-out probability at the initial inventory-time state, $SP(S,T)$, using the previous sales records requires attention. First of all, sales records need to be classified into groups so that the demand characteristics are similar within each group; low season statistics would not be appropriate for high season flights and vice versa. Similarly, the historical records of a flight would not be reliable if the company made a promotional, non-standard mark-down during the sales horizon since such campaigns can also deviate the demand from its normal pattern.

It is reasonable to assume that the stock-out probability for the flight at the reference state can be estimated using the stock-out probabilities of *similar* flights at *similar* states. Yet, the stock-out probabilities of previous flights is not a directly observable parameter and this brings forth the problem of estimating the stock-out probability of a flight at a given state after observing its sales record. This estimation needs to be done considering the following criteria.

Let the same inventory-time position, (s,t), be observed for two different flights and let $SP_1(s,t)$ and $SP_2(s,t)$ denote the stock-out probability estimates of these two flights at state (s,t).

- If the first airplane was sold out and the second one departed with empty seats, then $SP_1(s,t) > SP_2(s,t)$ should hold.

- If both airplanes were sold out but the seats of the first one depleted before the second, then $SP_1(s,t) > SP_2(s,t)$ should hold.

- If both airplanes departed with empty seats but the number of empty seats of the first one is less than the second, then $SP_1(s,t) > SP_2(s,t)$ should hold.

To sum up, it is a sophisticated task to implement empirical methods for the estimation of stock-out probability at a given state. For the previous flights, the sales records must be classified according to the demand patterns and the knowledge of the number of empty seats at the time of departure and the time of stock-out (if observed) should be recorded for all these flights.

In case the parametric method is not favored due to aforementioned difficulties, an alternative analytical approach can also be applicable. Section 4.2.2 covers an alternative for estimating the stock-out probability without using a functional approximation or any empirical data.

4.2.2 Predictive Estimation of $SP(s,t)$

The stock-out probability at a given state, $SP(s,t)$, is a critical figure in the estimation of optimal sales incentive value, z^*. In the dynamic pricing problem, this figure needs to be re-estimated at each customer arrival. Thus, defining $SP(s,t)$ as a function of remaining inventory and time to departure is a reasonable solution for this estimation problem. We propose a methodology for determining $SP(s,t)$ using expert opinion for function fitting. The functional form of the stock-out function is restricted so that it satisfies the required set of conditions for any given flight. Then, the parameters that introduce the demand characteristics of the sales process under consideration are determined and the final form of the stock-out function is obtained.

The stock-out probability would have been easily calculated if the number of customers to arrive until the end of sales horizon and the probability of purchasing for each customer were known. In this respect, the stock-out event is investigated in this section under the following assumption:

> **Assumption:** At the state (s,t), the seller predicts constant sales incentive for all remaining customers.

Recall that $z^* = Pr(P_t \geq p^*_{st})$. The seller determines z^* according to the expectations on future demand. If the demand realizations turn out to be parallel to expectations, z^* can be a reasonable estimate for the average future sales incentive. Notice that keeping the probability of sales constant does not imply fixing the price for the remaining the sales horizon. The same sales incentive value would correspond to higher prices as the reservation prices of customers increase towards the departure. Also notice that the actual pricing process will continue on a dynamic basis and this assumption on constant z^* is about the estimation of $SP(s,t)$ only.

Let D_t be a random variable denoting the number of customers to arrive when the time to departure is t. Of those customers, those who have a higher reservation price than the ticket price posted by the airline will purchase the tickets and vice versa. Hence, the customer arrivals can be classified into two groups as *purchases* referring to the arrival of customers who accept the posted price and *refusals* referring to the arrival of customers who do not accept the price posted by the seller. The stockout probability could then be defined in terms of prospective accept/reject decisions of customers to arrive until the end of sales horizon. At a given state (s, t), we define the probability of stockout as below:

K_t: number of customers arriving in $[0, t]$ to accept the price offer.

$D_t - K_t$: number of customers arriving in $[0, t]$ to reject the price offer.

$$Pr(K_t = k|D_t) = \begin{cases} \binom{D_t}{k}(z^*_{st})^k(1 - z^*_{st})^{D_t - k} & \text{if } 0 \leq k \leq D_t, \\ 0 & \text{otherwise.} \end{cases}$$

Stockout probability: Probability that number of arriving customers to accept the price offer is at least s; $SP(s, t) = Pr(K_t \geq s)$.

$$\begin{aligned} Pr(K_t \geq s) &= \sum_{k=s}^{\infty} Pr(K_t = k) \\ &= \sum_{k=s}^{\infty}\sum_{d=0}^{\infty} Pr(K_t = k|D_t = d)Pr(D_t = d) \\ &= \sum_{k=s}^{\infty}\sum_{d=k}^{\infty} Pr(K_t = k|D_t = d)Pr(D_t = d) \\ &= \sum_{k=s}^{\infty}\sum_{d=k}^{\infty}\binom{d}{k}(z^*_{st})^k(1 - z^*_{st})^{d-k}Pr(D_t = d). \end{aligned}$$

The customer arrivals are assumed to follow a Nonhomogenous Poisson Process. Accordingly, D_t follows cumulative distribution function for Nonhomogenous Poisson Distribution for given time variant rate λ_t.

An alternative formulation for stockout probability could be given as below:

$$Pr(K_t \geq s) = \sum_{d=s}^{\infty} Pr(K_t \geq s|D_t = d)Pr(D_t = d).$$

If we assume that the outcomes of individual customer decisions are Bernoulli trials, then $Pr(K_t \geq s|D_t = d)$ denotes the probability of observing at least s successes in d trials. This probability is equal to the probability of observing the s^{th} success at or before the d^{th} trial or equivalently observing at most $d - s$ failures until the s^{th} success. Let N_{st} denote the number of customers that arrive but do not purchase the ticket until the remaining s tickets are all successfully sold. This random variable is defined for $D_t \geq s$ such that $N_{st}|D_t$ takes values in $\{0, 1, ..., D_t - s\}$. Accordingly, tickets would be sold out after the arrival of $(N_{st} + s)^{th}$ customer.

According to these definitions and the assumption for constant z^*, we have the following observations:

1. The random variable N_{st}, conditioned on D_t, follows Negative Binomial distribution $NegBin(s, z^*_{st})$.

2. The stock-out probability for a given number of customers to arrive, $D_t = d$, is equal to $Pr(N_{st} < D_t - s|D_t = d) = Pr(D_t - N_{st} > D_t - (D_t - s)|D_t = d) = Pr(D_t - N_{st} > s|D_t = d)$. $D_t - N_{st}$ denotes the number of customers who purchase ticket in the last t periods when the current state is s.

Evaluating $Pr(N_{st} < D_t - s) = \sum_{d=s}^{\infty} Pr(N_{st} < d - s)Pr(D_t = d)$ is computationally difficult especially for large values of d. In an attempt to simplify the calculation, we use the approximation $Pr(N_{st} < D_t - s) \approx Pr(N_{st} < E[D_t] - s)$. Notice that the expected number of customers to arrive, $E[D_t] = \Lambda_{[0,t]}$ defined in Section 4.2.1. For large values of $E[D_t]$, we employ continuous distribution approximations to Negative Binomial distribution. Vose (2008) mentions that Normal approximation is a good approximation to Binomial distribution if number of successes $(E[D_t] - s)$ is greater than 50 and probability of success (z^*) is not very close to 0 $\left(NegBin(s, z) \approx Normal(\frac{s}{z} - s, \sqrt{\frac{s(1-z)}{z^2}})\right)$. For small values of z^*, Gamma distribution provides a good approximation $\left(NegBin(s, z) \approx Gamma(s, \frac{1}{z})\right)$.

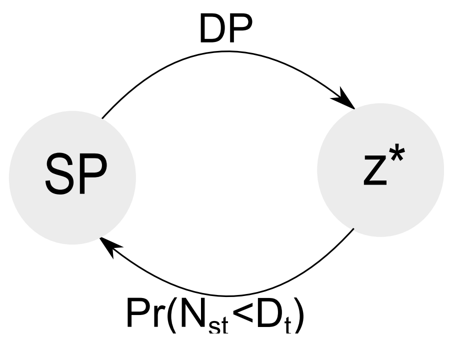

Figure 4.4: Relationship between z^* and $SP(s, t)$

Hence, for a given value of z^*, it is possible to find $SP(s,t)$. Moreover, using the DP model, z^* can be calculated when the stock-out probability is known. We know that z^* is a probability value, meaning that it can take values on the interval $[0,1]$. Using $NegBin(s,z)$, we can calculate $SP(s,t)$ values for every value of $z = \{0.01, 0.02, ..., 1\}$ at a precision level of 10^{-2}. Likewise, $SP(s,t) \in [0,1]$ and for every $SP(s,t) = \{0.01, 0.02, ..., 1\}$, z^* can be found using DP. At optimality, the value of stock-out probability for given sales incentive and the sales incentive obtained by the stock-out probability should be the same. That is, the optimal value z^* can be found by the following algorithm based on direct enumeration. The modeling approach described in the algorithm is referred to as Predictive Modeling approach with the acronym Pred-M.

Step 1. Define A as a matrix whose first row contains all possible sales incentive values at a precision level of 10^{-2}, $[0.01, 0.02, ..., 1]$, and second row contains $SP(z|s,t)$ values found by $SP(s,t) = Pr(NP_{s,t} < D_t - s)$ for the corresponding z values in the first row.

Step 2. Define B as a matrix whose second row contains possible stock-out probability values at a precision level of 10^{-2}, $[0.01, 0.02, ..., 1]$ and first row contains $z^*|SP(s,t)$ values obtained by DP for corresponding $SP(s,t)$ values in the second row.

Step 3. Define $\Delta_{ij} = \max\{|A(1,i) - B(1,j)|, |A(2,i) - B(2,j)|\}$; the maximum of the difference between the $[z; SP]$ tuples at i^{th} column of A and j^{th} column of B.

Step 4. Find (i,j) for which Δ_{ij} is minimum (where the tuples $[z; SP(z|s,t)]$ and $[z * |SP(s,t); SP]$ are closest to each other). $(A(1,i) + B(1,j))/2$ is the optimal sales incentive value.

Example 4.2.4 *Consider the problem setting in Example 4.2.3;* $S = 200$ *seats are available with* $T = 30$ *days to flight and time dependent daily customer arrival frequency follows* $\lambda_t = 5^t 40^{T-t}$. *According to the definition of* λ_t, 505 *potential buyers are expected to show up for* 200 *seats before the plane departs.*

The predictive method requires estimation of p_{fin}, *the maximum ticket price the seller could gain on a last minute sales and the time-dependent sales incentive as a function* *of price, $z(p)$. For this particular example, we choose logarithmic sales incentive with $p_{up} = 250$ and $p_{low} = 100$ at $t = 30$ days and assume $p_{fin} = 400$. The values of p_{up}, p_{low} and p_{fin} are selected arbitrarily and the algorithm works reliably with same computational effort.*

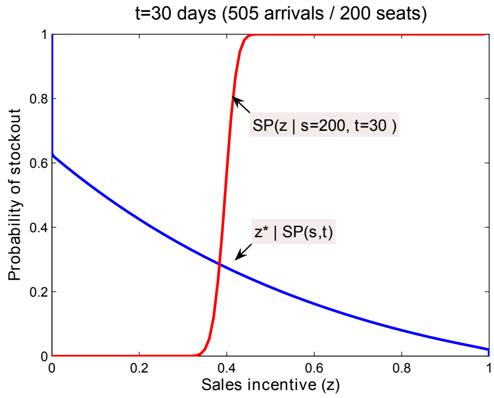

Figure 4.5: Predictive estimation of $SP(s,t)$ and z^* at $t = 30$ days

According to the proposed methodology, the point of intersection in Figure 4.5 gives the optimal value of sales incentive together with the corresponding stockout probability. Notice that the stockout probability curve, $SP(z|s,t)$, is increases rapidly from 0 to 1 over a small interval. Since we consider a large number of arrivals when $t = 30$ days, relatively small rise in probability results in a huge increase in expected number of seats to be sold and therefore the variance is quite small.

In this case, the stockout probability is not represented in the form of a function of state variables so the procedure needs to be repeated at each customer arrival. Now, let us consider a later stage during sales when $t = 2$ days. For given λ_t, 75 customers are expected to show up in the last two days and we assume $s = 30$ seats are left unsold so the demand-to-supply ratio is 2.5. Finally, we assume price upper and lower bounds for the reservation price distribution are $p_{low} = 250$ and $p_{up} = 500$. Respective findings are depicted in Figure 4.6.

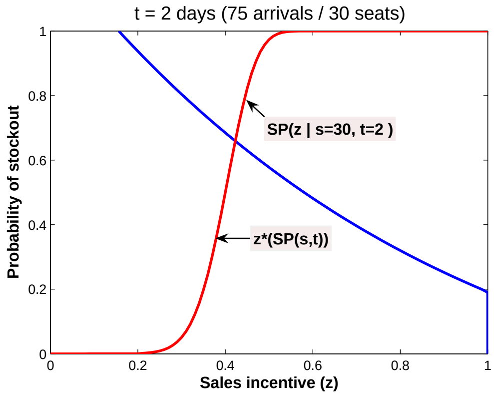

Figure 4.6: Predictive estimation of $SP(s,t)$ and z^* at $t = 2$ days

4.3 Nonrecursive Dynamic Pricing Model

Using the approximation $\Delta_s v_t(s) = v_t(s) - v_t(s-1) \approx SP(s,t-1)p_{fin}$, the recursive Dynamic Pricing model (DP) for the first subproblem to determine p is obtained.

DP :

$$z^*_{st} = arg \max_{z\in[0,1]} \left\{ -zSP(s,t-1)p_{fin} + zp + v_{t-1}(s) \mid p = F^{-1}_{P_t}(1-z) \right\}. \tag{4.19}$$

The terms $SP(s,t-1)$ and $v_{t-1}(s)$ are independent of z which determines the price at time t. Besides, the price p is defined by the inverse cumulative density function of the reservation price, P_t. Assume that the cumulative density function is differentiable. Thus, the function to be maximized in z in (4.19) is differentiable in z, implying that the maximum exists in the interval $[0,1]$.

We assume that the upper and lower bounds on price, $p_{up}(t)$ and $p_{low}(t)$, are given parameters for time t. The upper bound corresponds to the price limit, above this limit the customer will not purchase the ticket, so $Pr(P_t \geq p_{up}(t)) = 0$. With a similar interpretation, $Pr(P_t \geq p_{low}(t)) = 1$. Hence, the sales incentive, $z = Pr(P_t \geq p)$, is a nonincreasing function of price, p, on the interval $[p_{low}(t), p_{up}(t)]$. That is, the term $\dfrac{dp}{dz}$ should be negative or zero on the same interval. The following lemma gives a formulation of the optimal value of sales incentive, z, for a particular form of the function $p(z)$. The analysis in the lemma is made for a given t.

Using the value of z^* in (4.10) in $p(z^*) = F_{P_t}^{-1}(1 - z^*)$, the following equation is obtained for the optimal price p at state (s,t):

$$p(z^*) = F_{P_t}^{-1}\left(\frac{(\kappa - 1)z^* p(z^*) + SP(s,t-1)p_{fin}}{\kappa z^* p(z^*) + SP(s,t-1)p_{fin}}\right). \tag{4.20}$$

Solving (4.10) is not straightforward since the optimal value of the probability of sales, z^*, is expressed in terms of z^*. Although an explicit formulation has not been obtained, numerical methods like search algorithms can be used for finding the solution of (4.10).

The following bisection algorithm is given to find z^* numerically for the case of logarithmic demand. This algorithm should be used throughout the sales horizon at each customer arrival; thus the state variables (s,t) and the state dependent parameters $(p_{up}, p_{low}, SP(s,t))$ should be updated accordingly. Remember that for the logarithmic sales incentive case, the third statement of Lemma 4.1.2 offers a mathematical condition for suboptimal value elimination before starting a search type algorithm. Hence, this condition is also taken into consideration and the following algorithm is devised:

- `Initialization:` Set $\kappa = \ln \dfrac{p_{up}}{p_{low}}$. Set the lower and upper limits of search interval for z^*: $z_{low} = 0$, $z_{up} = \min(1, 1/\kappa)$. Enter the parameters p_{up} and p_{low} for time t, p_{fin}, $SP(s,t-1)$ and *prec*.
- `Step 1:` If $SP(s,t-1)p_{fin} \geq p_{up}$, STOP with $z^* = 0$. Otherwise, go to Step 2.
- `Step 2:` Set $\gamma = (z_{low} + z_{up})/2$. Calculate ϵ as follows: $\epsilon = -SP(s,t-1)p_{fin} + (1 - \kappa\gamma)\, p_{up}^{1-\gamma} p_{low}^{\gamma}$. Go to Step 3.
- `Step 3:` If $|\epsilon| \leq prec$, STOP with $z^* = \gamma$. If $\epsilon < -prec$, set $z_{up} = \gamma$. If $\epsilon > prec$, set $z_{low} = \gamma$. Go back to Step 2.

The parameter *prec* defined in the initialization stage is the precision given by the user for stopping the algorithm. A generic bisection algorithm is defined in Steps 2 and 3 using the optimality equation given in (4.10). The condition $SP(s,t-1)p_{fin} \geq p_{up}$ implies that the expected revenue of keeping one more seat for later sales is larger than the maximum price the current customer is willing to pay. In this case, the optimal decision is not to sell the seat to the customer at all and the assignment $z^* = 0$ at Step 1 represents this decision.

CHAPTER 5

DETERMINING THE REFUND PREMIUM

The original dynamic pricing formulation in Section 3.1 is for finding the optimal prices for the flight and the option simultaneously. On the aforementioned grounds, we handle the pricing problem in two consecutive steps. Having proposed two methods for determining the ticket price p disregarding the option in Chapter 4, the methodology developed for pricing the refund option for a given restricted ticket price is presented here.

For a fixed p value, it is shown in (3.3) that the optimal value of option price can be obtained as follows:

$$q_{s,t} = \arg\max_{q} \left\{ \Pr(Q_t \geq q | P_t \geq p)(q - \Delta_n v_{t-1}(s-1, n)) \right\}. \tag{5.1}$$

The only information about the customer that can be useful for pricing the refund option is the arrival time. The seller does not know anything specific about the customer that can indicate his/her willingness to pay for refund option. Also, there exists no opportunity to negotiate to learn about the customer's reservation price. Hence, for maximizing the expected revenue, the airline company needs to devise a method to estimate the probability distribution of the maximum price that a customer is willing to pay for the refund option. Recall that this maximum price is denoted by Q_t. The estimation of the reservation price of a commodity based on the previous sales records is practically impossible since the seller is not able to know how much the customers are willing to pay before or after the sale; hence, the seller never knows how much it is worth to a customer to have a refund option. Thus, the probability distribution of Q_t must be interpreted in terms of the probability distribution of another variable that is *observable* to the seller.

In customers' point of view, the worth of a refund option is the utility of avoiding the risk of losing the money spent for the unused ticket in case of cancellation. The cancellation risk can be predicted by the seller using cancellation and no-shows statistics of the previous flights having similar characteristics. Thus, assuming that the seller can estimate the probability distribution of random variable C_t, the probability that the customer who purchases a refundable ticket at time t will cancel the booking, it is possible to estimate the reservation price of refund option if the relationship between C_t and Q_t can be formulated mathematically.

The variables C_t and Q_t are closely related to each other. The higher the risk of cancellation is, the higher the worth of refund option will be. To formulate a mathematical relation between C_t and Q_t, the decision making perspective of the customer must be modeled in a quantifiable way. Hence, our objective at this step is to obtain the probability distribution of Q_t using the probability distribution of C_t.

The term $\Delta_n v_{t-1}(s-1, n)$ is the expected net change in revenue-to-go due to making the current booking refundable. This cost term is considered in this work as the difference between the amount to be refunded to the customer at the cancellation instant (immediate loss) and the expected return from the reselling of the unit capacity that is available for sale after the cancellations (future gain). Subramanian et al (1999) study a Markov Decision Process for airline RM problem with cancellations. They firstly define two different classes of events, cancellation and purchase requests for each fare class and devise optimality equations accordingly. Then, noticing the computational difficulties, they transform the optimality equations such that "all expected costs (caused by cancellations and no-shows) are assessed at the instant of admission (booking of a seat) along with the reward (payment of the fare)". In this part, we adopt a similar methodology for introducing the effect of cancellations into our formulations.

In our formulations, the amount refunded to a refundable ticket holder in case of a cancellation request is full fare minus the cancellation penalty, $p + q - m$. The full refunding strategy can be analyzed by setting the cancellation penalty to 0. Unlike the losses due to cancellation, the gains due to reselling depend on the time of cancellation request. Thus, to find the increase in expected revenue to go due to an increase in seat inventory with cancellation (mathematically, to approximate $\Delta_n v(s-1,t)$), we consider the change in revenue at the time of cancellation denoted by τ and come up with the corresponding expectation in (5.2). $\hat{s}_\tau$ and $\hat{n}_\tau$ refer to the (anticipated) values of state variables s and n at $t=\tau$ and $f_{T_c}(\tau)$ denotes the pdf of the time of cancellation (denoted by T_c) given that cancellation would occur. The cancellation deadline, t_d, denotes the time point in the sales horizon at which cancellation claims expire.

$$\Delta_n v_{t-1}(s-1,n|p,q) = E[C_t]\left[(p+q-m) - \int_{t_d}^{t} f_{T_c}(\tau)\Delta_s v_\tau(\hat{s}_\tau,\hat{n}_\tau)d\tau\right]. \tag{5.2}$$

Similar to the the marginal seat revenue approximation in Section 4.2, we define $\Delta_s v_\tau(\hat{s}_\tau,\hat{n}_\tau) \approx SP(\hat{s}_\tau,\tau)p_{max}$. Notice that, on the left-hand-side of the equation, the marginal seat revenue term is defined in (s,t,n) to represent the effect of outstanding cancellation claims. However, we keep the stockout probability estimation function in terms of (s,t) in order to preserve consistency with the estimation methods defined in Chapter 4.

Let $ECI(s-1,t,n,p,q)$ (Expected Cancellation Increment) refer to the increment in the value of the function given in (5.2) under this approximation. Noticing that the integral term in (5.2) is independent of the value of q, we can denote expected incremental cancellation cost in the form of $ECI(s-1,t,n,p,q) = E[C_t]q + A$ where $A = E[C_t](p-m-\int_{t_d}^{t} f_{T_c}(\tau)\Delta_s v_\tau(\hat{s}_\tau,\hat{n}_\tau)d\tau)$ is constant in q.

The primary problem in the estimation of $ECI(s-1,t,n,p,q)$ is the uncertainty of the cancellation time, τ; with probability C_t the customer returns with a cancellation claim but the time of this request affects the value of the additional seat which becomes available for sale once the reservation is cancelled. The expectation formulated in (5.2) contains $f_{T_c}(\tau)$, the probability distribution of cancellation time given that the current booking will be cancelled. Yet, this information is not sufficient because the seat inventory at the time of cancellation, τ, cannot be known as well. Depletion of seat inventory depends not only on the demand characteristic but also the pricing policy of the seller, so, it is impossible to derive a formulation of $\hat{s}_\tau$ analytically. Likewise, the future state of number of refundable bookings, $\hat{n}_\tau$, is determined by demand, pricing policy and cancellation claims used; making it difficult to estimate.

An alternative approximate approach to expected loss due to cancellation can be useful for estimating an analogous term that we define as $ECL(s-1,t,n,p,q)$, Expected

Cancellation Loss. The amount refunded to the customer, $p + q - m$, is lost if (1) the ticket is cancelled during the sales horizon and (2) sales ends before this additional seat is resold, which means stockout does not occur. The estimation of the cancellation probability, C_t, which is necessary for finding the loss expectation, is also required for the determination of the pdf of the refund option reservation price, Q_t. Similarly, the stockout probability, $SP(s-1,t)$ is already estimated in the restricted price determination in Chapter 4. So, the same formulations can be used in this part as well. Accordingly, without requiring additional knowledge regarding the sales process or the current customer, the expected cancellation loss can be approximated as follows:

$$ECL(s-1,t,n,p,q) \approx E[C_t](1 - SP(s-1,t))(p+q-m). \tag{5.3}$$

The stock-out probability term on right-hand side of (5.3) is assumed to be independent of n. This definition is equivalent to assuming that the expected cancellation loss of a customer booking a refundable ticket at time t is independent of the number of refund claims sold (and not used) until time t. Noticing that this assumption may decrease the accuracy of ECL estimation, it is necessary to keep the dimensionality at a manageable level.

Recall (5.1); for finding the optimal refund option price for a given p, together with the expected cancellation loss, the probability distribution of refund option reservation price, Q_t must be known. In this respect, in Section 5.1 the relationship between cancellation probability of the customer, C_t, and the reservation price of that customer for the refund claim, Q_t, is investigated.

5.1 Decision Making Under Gains and Opportunity Losses

In the first stage of the dynamic pricing problem, the seller decides on the restricted ticket price according to the revenue maximization principle. The airline company processes individual customer requests hundreds of times during the sales horizon for each flight and this sales process is experienced for every flight. So, the overall revenue of the airline company will converge to the total expected revenue by the Law of Large Numbers after hundreds of flights. Thus, the seller is assumed to be risk neutral and the uncertainties regarding the total revenue are not taken into consideration. On the other hand, a restricted ticket buyer faces the risk of spending money on an unused service in case of cancellation and is expected to be risk-averse against this probable loss. In order to estimate the maximum amount the customer is willing to pay for a restricted-to-refundable ticket upgrade, in this section the buyers will be considered as risk-averse decision makers choosing between the nonrefundable and refundable ticket according to their own utility function.

Remark 5.1.1 *The risk aversion of individuals could be an opportunity to exploit for the corporations. The insurance companies sell a great number of insurance policies at prices above the expected loss utilizing risk aversion. Similarly, financial derivatives are proposed to investors who want to reduce the uncertainty in their future gains and brokering parties benefit from risk aversion in this case. In their seminal work on Prospect Theory, Kahneman and Tversky (1979) define loss aversion as the stronger risk aversion of people when losses are under consideration than they have in situations considering gains. Thus, as long as the customers are expecting probable future losses, selling contracts that would avoid the risks can be a very profitable business practice if the price is determined carefully.*

Selling refundable tickets with a price premium is similar to selling insurance policies in the sense that in both cases the tendency of customers to prefer paying a premium to avoid uncertain future losses is exploited. For the customer, the loss in case of a cancellation depends on whether the booking is refundable or not; the final loss for nonrefundable ticket will be higher with probability C_t. This difference in utility rationalizes the difference in prices of restricted and refundable tickets that is denoted by q.

The decision making model based on maximizing the expected worth of final gains could be reasonable under the assumption of risk neutrality of the passengers. However, the refund options, which are developed for eliminating the risk of cancellations, are preferred by risk averse passengers and the decision process should be modeled accordingly. Hence, the decision making models based on expected utility maximization will be employed in our analysis.

According to the decreasing marginal utility assumption accepted for risk averse in-

dividuals, the utility function $g(\alpha)$ should be an increasing concave function mapping positive monetary gains to positive utility values. These requirements on the functional form of $g(\alpha)$ can be described mathematically as follows:

- $g(\alpha) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$,
- $\dfrac{dg}{d\alpha} \geq 0 \; \forall \alpha \geq 0$,
- $\dfrac{d^2 g}{d\alpha^2} \leq 0 \; \forall \alpha \geq 0$. $\square$

Understanding the customers' willingness to pay for the refund option is a critical issue for determining its price. The optimal value of q cannot be found unless the cumulative distribution function of the reservation price, $F_{Q_t}(q)$, and the expected incremental cancellation cost, $EIC(s, t, n, p, q)$, are estimated. The reservation price, Q_t, represents a price threshold for the customer's decision problem of purchasing the refund option or not at time t. In this respect, the estimation of reservation price distribution is connected to the seller's decision problem of determining q.

Having decided to book for a certain flight with a given price p, the customer should decide whether the risk of missing the flight is worth paying the additional premium for holding a refundable ticket which would prevent monetary loss in this case. The decision problem of the passenger for choosing between restricted and refundable tickets is depicted in the decision tree in Figure 5.1. In the decision tree, the probability that a restricted (nonrefundable) ticket owner does not take the flight and the probability that a refundable ticket owner cancels the booking are both denoted by C_t. Assuming that these two events have the same probability of occurrence is consistent with the assumption of exogeneity of cancellation (cancellation being independent of having the refund option) discussed in Chapter 1.

The difficulty of the aforementioned net gain approach is twofold. The first problem is to estimate the monetary value that the customer attributes to receiving the air travel service. The seller knows the price paid for that service, p; however, it could only be a lower limit on the customer reservation price for the pre-booked service. The second problem regards co-existence of gains and losses in the same utility function. In their seminal work, Kahneman and Tversky (1979) show that the utility function of an in-

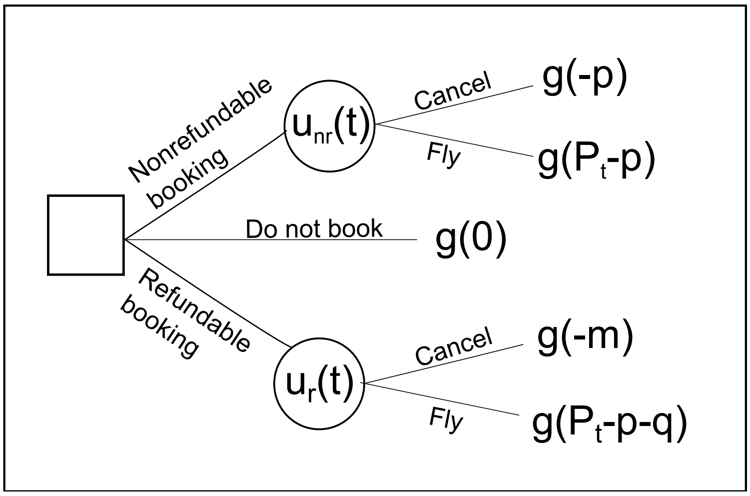

Figure 5.1: Passenger's decision - utility based on gains

dividual is structurally different for gains and losses and demonstrate the phenomena of loss-aversion, which they describe as preference of avoiding losses to acquiring gains.

Notice that the reservation price of the refund option is the value of q for which the customer is indifferent between refundable and nonrefundable tickets. For each outcome (take or miss the flight), service-related utilities are identical for nonrefundable and refundable booking decisions. Furthermore, the cancellation probability is assumed to be independent of having the refund option since we assume that the *willing customer* would like to cancel the booking only due to exogenous factors, like health problems, conflicts, etc.

It is noted by Bell (1982) and Loomes and Sugden (1982) that the utility functions defined with the final gain may fail to represent the decision makers' behavior and they presented experiments that contradict the assumptions of expected utility maximization principle. On the other hand, they show that utility functions could still be insightful with an alternative metric; the post-decision regret that the decision maker would have if the chosen alternative is not the best alternative for the final outcome. Thus, minimization of anticipated (dis)utility of regret is considered as an alternative objective in decision making process and regret is employed together with final gains while defining the utility function of the decision maker. With this motivation, in Section 5.2 the decision problem is reinvestigated with this extended formulation of utility.

5.2 Regret and Negative Utility

In the decision theory, regret is defined as the difference between the payoff that would have been obtained if the best course of action had been chosen for a particular outcome and the payoff of the selected alternative for the same outcome. In this respect, incorporation of regret makes it possible to define utility functions that evaluate the opportunity loss of an alternative together with its gain. A formal definition of regret is given below.

Definition 5.2.1 *(Loomes and Sugden 1982) Regret is a measure of how much better the decision maker's position would have been if s/he had chosen an alternative course of action for the given outcome.*

To the best of our knowledge, the first studies on regret in decision making under uncertainty are in early 1980s. Loomes and Sugden (1982) present the Regret Theory explaining the concept of regret broadly for single-criterion and multi-criteria decision making and discussing the legitimacy of regret minimization in decision making. Bell (1982) also studies incorporation of regret into expected utility theory and notices that the existence of a sense of loss via inclusion of regret in the utility function formulation makes the expected utility maximization a better descriptive model. Both papers give examples of the paradoxical behaviors that contradict the axioms of expected utility on gains. Yet, these behaviors of the decision makers are consistent with the desire to avoid post-decision regret.

5.2.1 Mathematical Modeling

In refund option pricing problem considered in this study, the customer has three choices (do not book, nonrefundable booking or nonrefundable booking) and for both outcomes (flying or not) determining the regret is quite straightforward. The gains are defined in the same way as depicted in Figure 5.1. The regret associated with either of restricted and flexible booking alternatives can also be described in monetary terms easily as seen in Figure 5.2. The (dis)utility values defined in terms of the regret are denoted as ζ_{nr} and ζ_r for nonrefundable and refundable booking alternatives, respectively. Notice that the third option, *Do not book*, is omitted in the decision tree. Since the customers' willingness to pay for the refund option, q, is related to the tradeoff between the nonrefundable and refundable booking, we restrict our attention to these two alternatives.

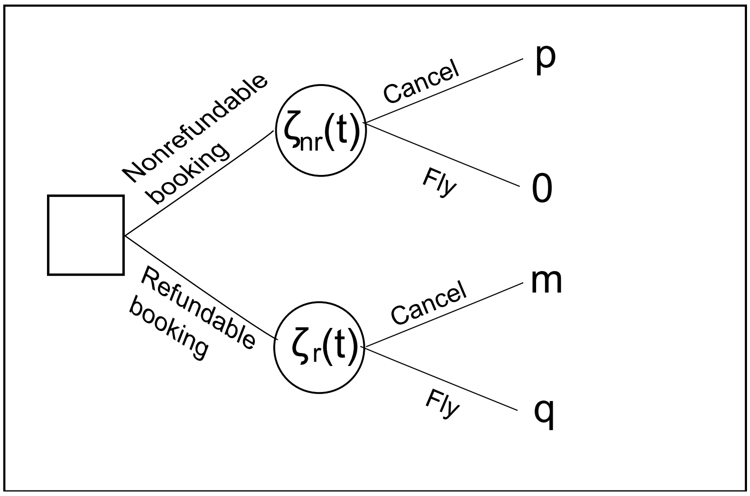

Figure 5.2: Passenger's decision - regrets of decision-outcome tuples

Consider the case the decision is to get the restricted booking without refund option: if the customer ceases to receive the service, then the post-decision regret in monetary terms would be equal to p since the best alternative would have been not purchasing the ticket. On the other hand, if the customer receives the booked service, then restricted booking would be the best course of action and regret would be 0. The alternative decision is to get the refundable ticket by paying an additional amount q: in this case, the regret in case of a cancellation would be equal to m, and when the customer receives the service, the regret would be equal to q. In the formulations below, we work with conditional random variables for given C_t. Below, we define the random variables $U_{nr}(t)$ and $U_r(t)$ as the utilities of the monetary regret for nonrefundable and refundable ticket owners, respectively. ζ is increasing in regret denoted by a negative quantity in the following formulations.

$$U_{nr}(t)|C_t = \begin{cases} \zeta(-p) & \text{with probability } C_t, \\ \zeta(0) & \text{with probability } 1 - C_t. \end{cases}$$

$$U_r(t)|C_t = \begin{cases} \zeta(-m) & \text{with probability } C_t, \\ \zeta(-q) & \text{with probability } 1 - C_t. \end{cases}$$

$$E\left(U_{nr}(t)|C_t\right) = (1 - C_t)\zeta(0) + C_t\zeta(-p), \qquad E\left(U_r(t)|C_t\right) = (1 - C_t)\zeta(-q) + C_t\zeta(-m) \tag{5.4}$$

Recalling the expected (dis)utility values for nonrefundable and refundable booking options in Figure 5.2 and using (5.7), $\zeta_{nr}(t)$ and $\zeta_r(t)$ are found as below:

$$\zeta_{nr}(t) = E(E(U_{nr}(t)|C_t)) = (1 - E(C_t))\zeta(0) + E(C_t)\zeta(-p)$$
$$\zeta_r(t) = E(E(U_r(t)|C_t])) = (1 - E(C_t))\zeta(-q) + E(C_t)\zeta(-m).$$

Remark 5.2.2 *From historical sales realization records, it is possible for the seller to observe for every booking whether the customer cancels the booking or not. So, distribution parameters of the cancellation probability C_t can be estimated for current customers using past data. On the other hand, estimation of the customer reservation price, P_t, is a more complicated task. In historical sales records, the seller's knowledge is limited to the accept/reject decisions of the customers for the prices offered to them and their maximum willingness to pay for the offered service is unknown. Thus, the utility based analysis is considered impractical due to necessity of reservation price estimation and in this work we will assume that the seller could only estimate the distribution of cancellation probability, C_t.* □

5.2.2 Solution Approach

The regret of each decision must be converted into utility terms. Howard (1988) notices that "exponential utility curves satisfactorily treat a wide range of individual and corporate risk preference". In addition, Kirkwood (2004) shows that in most cases an exponential utility function with appropriate parameters provides a very good approximation for general utility functions. Bouakiz and Sobel (1992) and Barz and Waldman (2007) also utilize exponential utility functions to incorporate risk aversion for decision making situations involving loss. They quantify the (negative) utility to be maximized as $\zeta(\theta) = -e^{-\beta\theta}$ where $\theta < 0$ denotes the loss in monetary terms and β

is the positive risk aversion coefficient. In the reference cited above, it is noted that larger values of β represents greater risk aversion. Figure 5.3 illustrates the change in risk aversion due to increase in value of risk aversion coefficient, which appears as the increased concavity of the utility function.

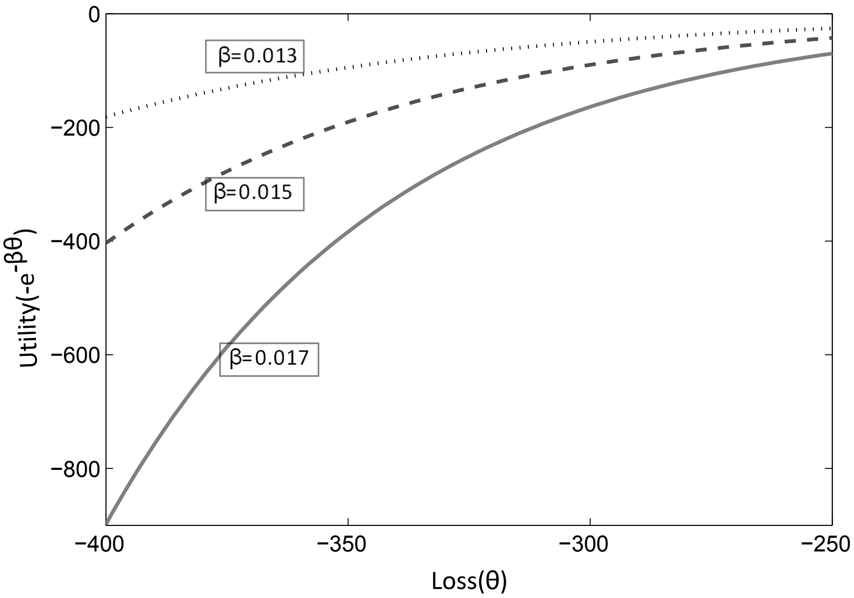

Figure 5.3: The utility function for different degrees of risk aversion

Understanding the customers' willingness to pay for booking refundability and estimating the future revenue impacts of selling an additional refund claim are the two critical issues for determining refund option price. Let P_t and Q_t denote the customer's reservation price for base service and cancellation refund option, respectively, where subscript t denotes the dependence to the time between booking and service. Assuming that the seller has an estimate of expected incremental cancellation cost, $\Delta_n v_{t-1}(s-1, n)$, which is a function of remaining inventory, s, and time to expiry, t, for given price setting. We consider the following two alternative refund option pricing policies.

- **Point Estimate.** Let the seller have a point estimate for the customer's refund option reservation price, $\hat{q}_t$. Then, the pricing policy imposed by revenue maximization principle is elementary; offer refund option price $\hat{q}_t$ provided that $\hat{q}_t > \Delta_n v_{t-1}(s-1, n)$.

- **Probability Distribution Estimation.** Let the seller have an estimation for the cdf of customer's refund option reservation price, $Pr(Q_t \geq q|P_t \geq p)$. Then, the refund option price could be determined according to the maximization formulated in (5.5).

$$q^*_{st}(p) = \arg\max_q \{Pr(Q_t \geq q|P_t \geq p)(q - \Delta_n v_{t-1}(s-1, n))\}. \tag{5.5}$$

The regret of each decision should be converted into utility terms using utility functions defined for losses. Bouakiz and Sobel (1992) and Barz and Waldman (2007) utilize exponential utility functions to incorporate risk aversion for decision making situations involving loss. They quantify the (negative) utility to be maximized as $\zeta(\theta) = -e^{-\beta\theta}$ where $\theta < 0$ denotes the loss in monetary terms and β is the positive risk aversion coefficient. It is noted that larger values of β represents greater risk aversion.

Respecting the popularity in the literature and considering its analytical advantages, we prefer exponential utility functions. In this study, we consider the following (dis)utility function: $\zeta(\theta) = a - be^{-\beta\theta}$ where $a \geq b > 0$ and $\theta < 0$. Next, we model the customers' choice between nonrefundable and refundable bookings as a decision problem and show that a point estimate $\hat{q}_t$ for refund option price obtained accordingly at time t. Lemma 5.2.3 presents a neat formulation of the point estimate of refund option reservation price, $\hat{q}_t$. Lemma 5.2.4 gives an estimate of the probability distribution of reservation price, Q_t.

Lemma 5.2.3 *Let $\zeta(\theta) = a - be^{-\beta\theta}$ where $a \geq b > 0$ and $\theta < 0$. Then, for given p, m and $C_t = c$, the breakeven refund option price $\hat{q}_t$ is*

$$\hat{q}_t = \frac{1}{\beta} \ln\left(\frac{c(e^{\beta p} - e^{\beta m})}{1-c} + 1\right). \tag{5.6}$$

Proof. Proof. Using the expressions given in (5.7) for the considered utility function, we have

$$E\left(U_{nr}(t)|C_t = c\right) = (1-c)(a - be^{-\beta\times 0}) + c(a - be^{-\beta(-p)}) \tag{5.7}$$

$$E\left(U_{nr}(t)|C_t = c\right) = (1-c)(a - be^{-\beta(-q)}) + c(a - be^{-\beta(-m)}). \tag{5.8}$$

$E\left(U_{nr}(t)|C_t = c\right)$ above is constant in q and $E\left(U_{nr}(t)|C_t = c\right)$ is concave decreasing in q. Since the breakeven refund option price $\hat{q}_t$ is given by the q value for which these two curves intersect, solving

$$E\left(U_{nr}(t)|C_t = c\right) = E\left(U_{nr}(t)|C_t = c\right)$$

$\hat{q}_t$ is obtained as in (5.6). ■

Figure 5.4 presents a graphical demonstration of Lemma 5.2.3 for a given parameter setting. Observe that refundable booking has greater (dis)utility value (i.e., more preferable) than nonrefundable booking while option price is less than the breakeven refund option price. Therefore, it is the maximum price the customer will be willing to pay for a restricted to flexible booking upgrade. The breakeven price determined by four parameters is increasing in cancellation probability, c, and restricted booking price, p, and decreasing in cancellation penalty, m.

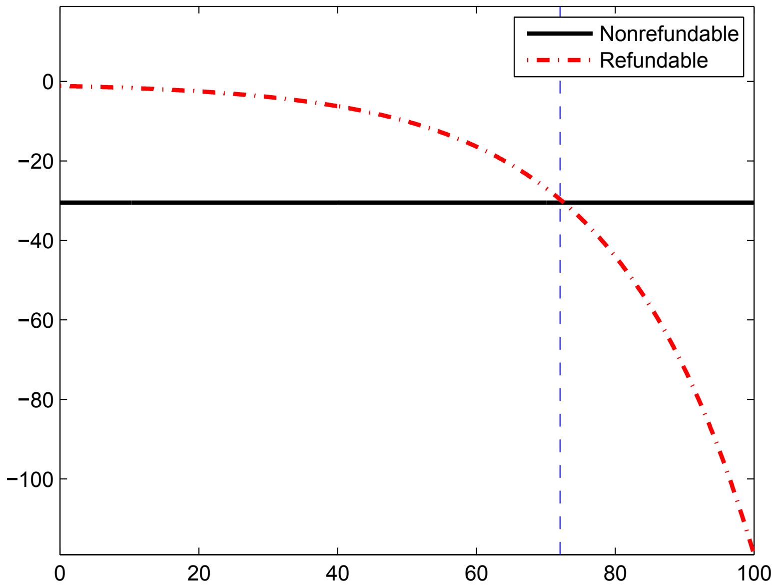

Figure 5.4: Breakeven q when $c = 0.2, \beta = 0.05$, $p = 100$ and $m = 10$

The breakeven refund option price presented in Lemma 5.2.3 defines the customer's maximum willingness to pay for refund option given a fixed cancellation probability value, $C_t = c$. Assuming that the cancellation probability is a time-variant random variable, the reservation price of refund option would be a random variable as well.

Let $\phi(c) = \dfrac{1}{\beta}\ln\left(\dfrac{c(e^{\beta p} - e^{\beta m})}{1-c} + 1\right)$ be defined as a monotone increasing function for $c \in [0, 1)$. Then, using Lemma 5.2.3, we can obtain the probability distribution of

Q_t from the distribution of C_t (recall Remark 5.2.2). The relationship between the two random variables is given in Lemma 5.2.4. Proof is based on relating the cumulative distributions of two random variables when one is derived from the other with a monotone increasing function. A compact proof is provided here for C_t and Q_t; however, Ross (2008) presents a more comprehensive result on general monotone functions.

Lemma 5.2.4 *Let the function $\phi(c)$ be defined such that*

$$\phi(c) = \frac{1}{\beta}\ln\left(\frac{c(e^{\beta p} - e^{\beta m})}{(1-c)} + 1\right). \tag{5.9}$$

Let the ticket price, p, be larger than cancellation penalty, m. Then, $F_{C_t} = F_{Q_t} \circ \phi$ and $F_{Q_t} = F_{C_t} \circ \phi^{-1}$ where F_{C_t} and F_{Q_t} denote the cumulative density functions of random variables C_t and Q_t, respectively.

Proof. The first derivative of $\phi(c)$ with respect to c below

$$\frac{d\phi(c)}{dc} = \frac{1}{\beta}\left(\frac{1-c}{c(e^{\beta p} - e^{\beta m}) + (1-c)}\right)\left(\frac{e^{\beta p} - e^{\beta m}}{(1-c)^2}\right)$$

is positive for $p > m$. (Note that the ticket price p being greater than cancellation penalty m assures the positivity of the term $e^{\beta p} - e^{\beta m}$.) Hence, ϕ is monotone increasing on its domain and it is invertible (the inverse function ϕ^{-1} is well defined).

The monotonicity of function ϕ assures existence of the inverse function ϕ^{-1}. Through a few algebraic operations, we obtain ϕ^{-1} as follows:

$$\phi^{-1}(Q_t) = \frac{e^{\beta Q_t} - 1}{e^{\beta p} + e^{\beta Q_t} - e^{\beta m} - 1}.$$

Since ϕ is a monotonically increasing function, it preserves the ordering on its domain. That is, $\phi(c) < \phi(c')$ if and only if $c < c'$. Accordingly, we have the following equality for every c:

$$F_{C_t}(c) = Pr(C_t < c) = Pr(\phi(C_t) < \phi(c)) = Pr(Q_t < \phi(c)) = F_{Q_t}(\phi(c)).$$

Therefore, we have $F_{C_t} = F_{Q_t} \circ \phi$ and $F_{Q_t} = F_{C_t} \circ \phi^{-1}$. ■

Therefore, the seller might transform the cumulative distribution function of cancellation probability C_t to the cumulative distribution function of refund option reservation price for the (dis)utility function considered in Lemma 5.2.3.

Table 5.1: Break-even option price values (p = 200 and m = 10)

β / $\mathbf{C_t = c}$	**0.001**	**0.003**	**0.005**	**0.010**	**0.030**	**0.050**
0.020	4.304	5.342	6.691	12.066	73.994	122.207
0.040	8.768	10.818	13.431	23.256	95.886	136.459
0.060	13.400	16.432	20.223	33.726	109.445	144.982
0.080	18.212	22.191	27.073	43.595	119.417	151.162
0.100	23.212	28.102	33.986	52.957	127.385	156.062
0.120	28.413	34.172	40.967	61.890	134.076	160.157
0.140	33.828	40.407	48.020	70.456	139.884	163.698
0.160	39.468	46.817	55.151	78.706	145.046	166.839
0.180	45.350	53.410	62.367	86.684	149.719	169.676
0.200	51.490	60.194	69.673	94.428	154.008	172.276
0.220	57.903	67.181	77.077	101.971	157.992	174.688
0.240	64.610	74.381	84.586	109.339	161.727	176.948
0.260	71.631	81.805	92.206	116.560	165.258	179.082
0.280	78.989	89.466	99.947	123.656	168.619	181.112
0.300	86.709	97.379	107.817	130.646	171.838	183.055
0.320	94.819	105.556	115.825	137.552	174.938	184.925
0.340	103.349	114.016	123.982	144.389	177.939	186.734
0.360	112.333	122.775	132.300	151.176	180.857	188.493
0.380	121.809	131.853	140.789	157.927	183.705	190.209
0.400	131.819	141.271	149.465	164.659	186.497	191.891
0.420	142.409	151.053	158.340	171.387	189.243	193.544
0.440	153.633	161.224	167.432	178.126	191.955	195.176
0.460	165.549	171.814	176.757	184.892	194.641	196.793
0.480	178.225	182.855	186.337	191.700	197.310	198.399
0.500	191.738	194.382	196.192	198.566	199.971	199.999

The rows and columns of Table 5.1 display the change of cancellation option reservation price $q_t = \phi(c)$ with respect increase in risk aversion coefficient β and cancellation probability $C_t = c$, respectively. The monotonicity of $\phi(c)$ in c, which has been shown in the proof of Lemma 5.2.4, is observed along the columns. $\phi(c)$ values are also increasing along the rows; reservation prices increase in risk aversion coefficient β. For different values of p and m tuples, the same pattern is observed if the cancellation probability is less than 0.5 (if cancellation is less likely than taking the flight). Conjecture 5.2.5 is given based on these observations:

Conjecture 5.2.5 *Assume that for a passenger who booked on a flight, it is more probable to take the flight than not taking it ($C_t < 0.5$). In this case, the reservation price of the customer found by the breakeven analysis, Q_t, increases as the coefficient β increases.*

It has been noted that the increase in β value corresponds to increased risk aversion. Thus, Conjecture 5.2.5 can be restated as follows: If $C_t < 0.5$, the maximum price a customer is willing to pay for the cancellation refund option increases as the customer's risk aversion increases.

In order to prove the conjecture, one needs to show that the derivative term $\dfrac{d\phi}{d\beta}$ is positive for cancellation probability values less than 0.5. Conjecture 5.2.5 states results for particular values of C_t. A more general condition can be given on the distributions of random variables C_t and Q_t rather than specific values. Lemma 5.2.6 states a necessary condition that leads to stochastic ordering of reservation prices corresponding to different degrees of risk aversion.

Lemma 5.2.6 *Let the random variable $C_t < 0.5$ and let $\dfrac{d\phi^{-1}}{d\beta} \leq 0$ hold for any (p, m) tuple such that $m < p$. Let the functions ϕ_1 and ϕ_2 be defined as in (5.9) with risk aversion coefficients β_1 and β_2 respectively such that $\beta_1 \leq \beta_2$. Then, $Q_t^1 = \phi_1(C_t)$ is stochastically smaller than $Q_t^2 = \phi_2(C_t)$.*

Proof. By Lemma 5.2.4, we know that $F_{Q_t^1}(q) = F_{C_t} \circ \phi_1^{-1}(q)$ and $F_{Q_t^2}(q) = F_{C_t} \circ \phi_2^{-1}(q)$. Since $\dfrac{d\phi^{-1}}{d\beta} < 0, \beta_1 \leq \beta_2$ implies $\phi_1^{-1}(q) \geq \phi_2^{-1}(q)$ for every value of q.

The cumulative density function of cancellation probability C_t is a non-decreasing function; hence the following inequality holds for every q:

$$F_{Q_t^1}(q) = F_{C_t} \circ \phi_1^{-1}(q) \geq F_{C_t} \circ \phi_2^{-1}(q) = F_{Q_t^2}(q). \tag{5.10}$$

The inequality $F_{Q_t^1}(q) > F_{Q_t^2}(q)$ indicates that Q_t^1 is stochastically smaller than Q_t^2. ■

Remember that the second stage subproblem deals with the maximization problem

$$q_{s,t}(p) = \arg\max_q \{\Pr(Q_t \geq q | \Pr(P_t \geq p))(q - \Delta_n v_{t-1}(s-1, n|p, q))\}.$$

So far, we have formulated a concrete method of estimating the cdf of Q_t using the cdf of C_t. The further discussions on the effect of customer's degree of risk aversion on this relation is made in order to validate our findings. It is natural to expect that more risk averse individuals would be more willing to pay for eliminating their risks and for a reasonable range of cancellation probability values ($C_t < 0.5$) this is observed in our calculations. In Chapter 6, this methodology will be implemented for finding the refund premium in the simulation studies.

CHAPTER 6

RESULTS AND DISCUSSION

In this study, the airline dynamic pricing problem is studied in two consecutive stages. For the first stage subproblem, the *DP* model is investigated in detail on a theoretical basis so far. In Chapter 6, we firstly intend to evaluate the adequacy of the pricing scheme of the proposed model and compare it with an alternative dynamic pricing model. Then, the revenue generation and load factor performance of *DP* implementation are assessed through simulation studies.

The second stage subproblem considers pricing the refund premiums for given base (nonrefundable) ticket prices and a concrete methodology is introduced in Chapter 5. The revenue contribution of options for ticket refundability and proposed pricing method are also analyzed with sales simulations that allow cancellations.

6.1 Parameter Setting

In order to simulate a realistic sales process for the airline RM problem, we have merged our observations on the real life booking reservation systems with the theoretical basis of dynamic pricing. The parameters that we require to model the temporal nature of demand and to calculate the prices with the proposed pricing algorithms are classified into three main groups. With the setting we present here, we first construct a base scenario for the simulation runs and alterations in these parameters are also made to understand the impact of changes in individual factors.

a. Sales parameters: In order to define the state space, the primary decision to make is to decide the length of the sales horizon and the seat inventory. Although typical sales horizons vary between 6 to 12 months, "early sales" are mostly managed by strategic marketing campaigns and special promotions. Dynamic pricing is most effectively implemented at the operational level in the last 30-days period, during which more 75% of the tickets are sold. We assume there are 100 seats available for sales at the beginning of last 30-days period ($S = 100$ seats; $T = 30$ days is considered in the base scenario, simulation results for different values of S are also included in Appendix 2). Another parameter determined at the strategic level and applied to all customers is the cancellation penalty, m. The typical ticket price for our demand setting varies between $\$50 - 300$ and in real life implementations we have observed relatively low hassle cost for refundable tickets, thus we assume $m = 10$.

b. Temporal demand parameters: The distribution of reservation price, P_t, and customer arrival rate, λ_t are the two components of time-variant disaggregated demand we introduced in our formulations. The reservation price distribution we study in the simulations is the logarithmic distribution, which is explicitly formulated in Section 4.1.2. This distribution is defined on a bounded support, so we modeled the temporal shift in distribution as an increase in lower and upper bounds as time to departure approaches. Parallel to our observations from the annual sales data from a major European airline, we have set the reservation price to range between $(29 - 149)$ when $t = 30$ days to departure and between $(199 - 299)$ at the end of booking horizon. The increase of lower and upper reservation price bounds is piecewise linear in time; as $p_{low}(t)$ increases from 29 to 199 along the sales horizon, 50% of this increase is modeled with linear increase at constant rate and the other 50% with jumps at predetermined time points in the sales horizon. The Nonhomogeneous Poisson arrival rate λ_t follows a convex increase along the sales horizon, assumed to start with 3 arrivals/day at the beginning of sales horizon and 20 arrivals/day at the end. Note that this parameter setting for λ_t is considered as the base demand scenario and alternatives are also studied where applicable.

c. Empirically determined parameters: Similar to reservation prices, the cancellation probability is assumed to be a time dependent random variable. The real life data in most airlines' online reservation systems is limited to sales realization reports; hence, detailed records of customer cancellations or customer preferences between refundable and restricted tickets are not available. Due to lack of further information regarding the distribution of cancellation probability, we consider a uniform distribution whose bounds vary in time. One particular restriction regarding the cancellation probability, C_t is that it remains less that 0.5 in all simulation scenarios.

Customer risk aversion is used together with probability of cancellation in refund option pricing calculations. Observing the exponential utility functions, we have introduced three alternative customer risk aversion parameter values as 0.02/0.04/0.08/0.16 to model different levels of risk aversion among airline customers. Also, in stockout probability estimation, an initial estimate for stockout probability is required. We have little knowledge regarding this parameter and thus assumed initial probability to be 0.5.

Remark 6.1.1 *The risk attitude of customers towards the possibility of not being reimbursed in case of booking cancellation may depend on various factors including customers wealth, character and flying frequency and such information is not always available for the seller. Hence, we assumed that customers arriving at different times along sales horizon have the same degree of risk aversion and To have a deeper understanding about the risk aversion of different customer groups, historical sales and booking records should be analyzed to study customer responses to refundable booking offers. For that purpose, loyalty programs could be utilized and the information on customers' preference between refundable and nonrefundable tickets for the past purchases could be investigated.*

There are also other parameters regarding the computational part of simulations that we have decided after experimentation with alternative values.

- Number of replications for steady-state analysis in the simulation studies is set to 500 empirically. For each case, the distribution of revenue and load factor results given by these 500 replications are tested for goodness-of-fit to Normal Distribution. Accordingly, we have obtained confidence intervals according to Normal Distribution assumption.
- In iterative numerical optimization routines developed for finding optimal sales incentive, stopping criterion for the algorithms is set to 10^{-3}. That is, the max-

imum error in optimal sales incentive values, $z^* = \Pr(P_t \geq p^*)$, is limited to 0.001.

- The discretization of continuous sales process requires a time discretization unit which is small enough so that the probability of multiple customer arrivals in a single interval is negligibly small. We set interval length to 0.5 minutes so that the possibility of multiple customer arrivals is less than 5×10^{-5} for the given parameter setting.

With the sales simulations, we intend to assess the performance of proposed pricing algorithm. We also adopt a mixed integer programming (MIP) approach to compare with DP model. In this respect, the sales and demand parameters described so far for the continuous time setting are transformed and introduced to this alternative model. The detailed description of the MIP is presented in 6.2.

6.2 Mixed Integer Programming Model for Determining Price

Talluri and van Ryzin (2005) present an IP model for discrete time dynamic pricing problem that is reviewed in Section 2.2. The aggregate demand over a given period of time is considered as a time dependent function of price; $d(t, p(t))$ where t represents the time period and $p(t)$ represents the sales price for that time period. $p(t)$ is selected from a discrete set of possible prices. Accordingly, the anticipated revenue for period t becomes $p(t) \times d(t, p(t))$. The objective is maximizing the revenue in the remainder of the sales horizon and the limited number of seats available for sale imposes the constraint on the total demand that can be satisfied. Hence, the problem is reduced to allocating the available seat capacity to the discrete time periods by controlling the demand in each period through price manipulation.

The dynamic pricing method we presented in Section 4.2 is based on the use of a continuous time setting. Then, the sales horizon is discretized into small time intervals of unit length (ϵ) and the pricing problem is restated as finding the optimal price at the beginning of each time interval. In this section, we discretize the sales horizon into larger time intervals which could possibly have unequal lengths and we refer to these intervals as *episodes*. Moreover, the price is discretized by restricting it to take values from a finite set of alternative prices. Using a model with discrete state space and discrete decision space, a variation of the Integer Programming model due to Talluri and van Ryzin (2005) is considered in this part and is referred to as Mathematical Programming (MP) model.

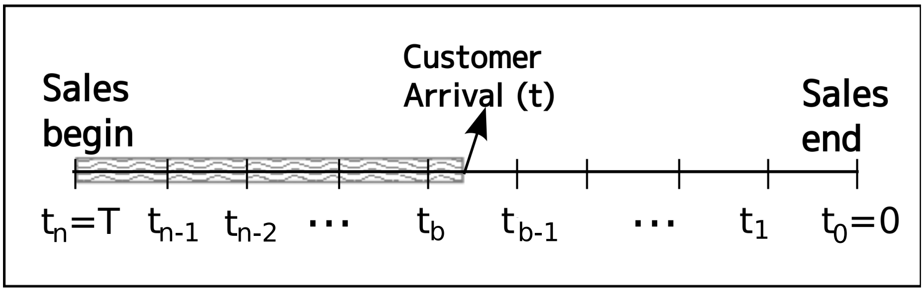

Figure 6.1: Discretization of the sales horizon

Our objective is to attain compatibility between the proposed *DP* model and the *MP* model so that the two models are comparable in sales simulations. With this perspective, we propose a reformulation of the demand in a given interval for a given price using the generally accepted notions in airline RM and previously mentioned ideas in Sections 1.3 and 2.2 in this thesis.

6.2.1 Demand Aggregation

In the *MP* model, the period-price assignments should be represented by binary variables. Correspondingly, the demand should be defined as the number of tickets that could be sold during an episode at a given price. In the notation, i is used for the discrete time index. n refers to the total number of episodes. m is the cardinality of the set of possible prices and j is used for indexing the prices in this set. The set of alternative prices is defined as $\{p_1, ..., p_m\}$. t_{i-1} and t_i denote the beginning and end points of episode i.

In *DP* reservation price distribution and arrival probability of customers are assumed to be time dependent. The reformulation of demand requires aggregation of customer arrival intensity (λ_t) and sales incentive ($1 - F^{(i)}(p)$) over episodes. Let the parameter μ_{ij} define demand-price relationship in the *MP* model, denoting the expected number of seats that can be sold in episode i when the price posted in this episode is p_j.

Accordingly, μ_{ij} could be obtained as below:

$$\mu_{ij} = \int_{t_{i-1}}^{t_i} \lambda_t (1 - F^{(i)}(p_j))dt. \tag{6.1}$$

The parameters of the reservation price distribution $F^{(i)}(p)$ are assumed constant within each episode. In order to adopt the generally accepted low-fare before high-fare customer arrival pattern for consistency, it is assumed that the reservation price in episode $i-1$ is stochastically larger than the reservation price in the preceding episode i, i.e. $F^{(i-1)}(p) \leq F^{(i)}(p)$ for every p.

6.2.2 Mixed Integer Programming Formulation

The decision variables used in the MP model are y_{ij} that denotes the number of seats that are sold in the i^{th} episode at price level j and the binary variable x_{ij} that is defined below.

$$x_{ij} = \begin{cases} 1 & \text{if the price in the } i^{\text{ th}} \text{ episode is } p_j, \\ 0 & \text{otherwise.} \end{cases}$$

At the beginning of the last τ episodes when the remaining number of available seats is equal to s, the MP model to be solved is given below. The model would give the solution for the whole sales horizon when $\tau = T$ and $s = S$.

$$\begin{aligned} \text{MP:} \quad \text{Max} \quad & \sum_{i=1}^{\tau} \sum_{j=1}^{m} p_j y_{ij} \\ \text{subject to} \quad & \sum_{i=1}^{\tau} \sum_{j=1}^{m} y_{ij} \leq S, \\ & \sum_{j=1}^{m} x_{ij} = 1 && \forall i \in [1, \tau], \\ & 0 \leq y_{ij} \leq \mu_{ij} x_{ij} && \forall i \in [1, \tau],\ j \in [1, m], \\ & x_{ij} \in \{0, 1\} && \forall i \in [1, \tau],\ j \in [1, m]. \end{aligned}$$

Along with the ticket price in each episode, solution of the MP model also provides the number of seats to be sold in each episode. Yet, the difference in the implementation of dynamic pricing from seat allocation practice is that the prices for the episodes are the only control variables that determine the seller's policy. Therefore, the dynamic pricing policy is characterized by only values of the binary variables x_{ij} representing the optimal price levels in the following episodes. Since a price update scheme is considered in this section to solve the model at the beginning of each episode, the only relevant decision variable value is that of $x_{\tau j}$ for the current episode τ.

6.2.3 Discretization into MP for Logarithmic Demand

In order to transform the *DP* model parameters into *MP* model counterparts, the first step is partitioning the sales horizon into episodes. Note that for both methods, it may not always be meaningful to develop a dynamic pricing policy for the entire sales horizon. In the airline industry, the sales horizons commonly vary between 90 days and 360 days prior to flight. Unless there are batch bookings and/or group reservations (those should be priced with other methods), majority of the seats are sold on a narrower time window before departure. The frequency plot in Figure 6.2 is depicted with the sales realization data of 150 flights of a mainstream airline during a year on a particular itinerary with no competitor having a direct flight on the same route. In the horizontal axis, the fraction of tickets sold before the last 30-days within overall sales for that flight is given. In most of the flights, the seats that are booked prior to the last 30 days constitute a very small ratio in total capacity sold. Observing this, we decide to restrict our attention to pricing policies during the last 30 days before departure.

For the numerical illustrations in this section, the 30−days sales horizon is partitioned into 5 episodes for the *MP* model. The continuous and piecewise constant versions of time-dependent model parameters are depicted in Figure 6.3. The values of these parameters are also given in Table 2. The parameter transformation methodology is summarized below:

- In *MP* model, the durations of 5 episodes are assumed to be 15 days, 6 days, 4 days, 3 days and 2 days.
- In *DP* model the reservation price distributions are assumed to be logarithmic as defined in Section 4.1.2 with linearly increasing lower and upper bounds along

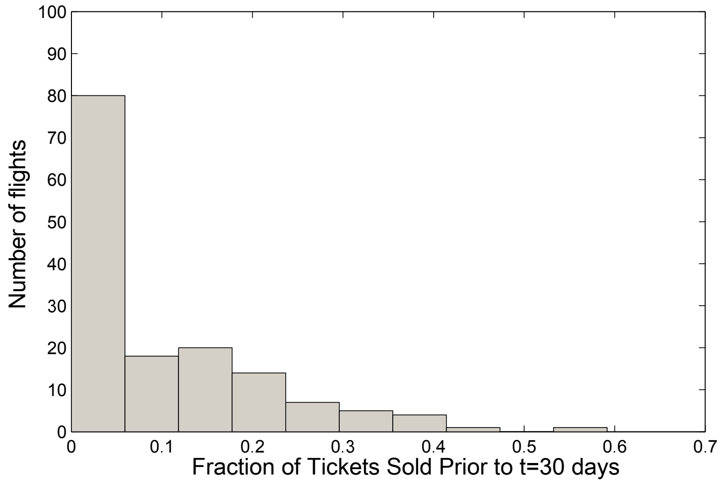

Figure 6.2: Percentage of tickets sold before the last 30 Days

time; $p_{low}(t)$ and $p_{up}(t)$, respectively. In the *MP* model, lower and upper bounds of logarithmic reservation price assumed piecewise constant: fixed within each episode to the episode average of *DP* counterpart.

- In *DP* model, daily customer arrival rate is convex increasing from 3 arrival/day to 20 arrivals/day; $\lambda_t = 3^{\frac{t}{30}}20^{1-\frac{t}{30}}$.In *MP* model, the Nonhomogenous Poisson arrival rate is also piecewise constant, λ_t being integrated between endpoints of corresponding episode.

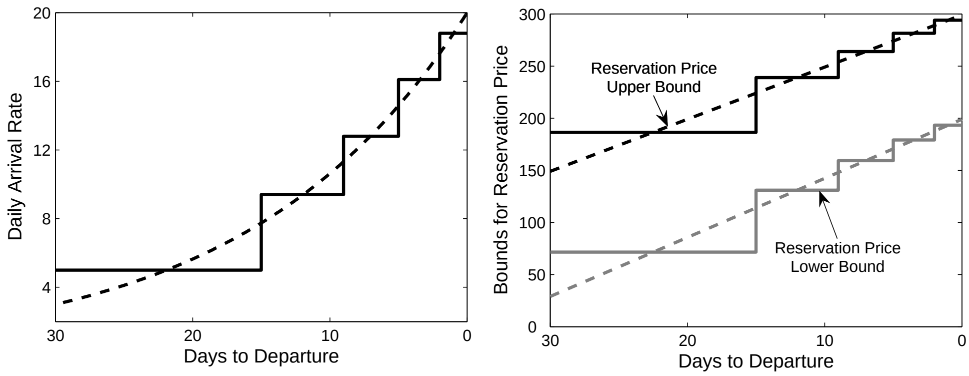

Figure 6.3: Time-dependent demand parameters for *DP* and *MP*

As seen in Table 6.1, the average daily arrival rate is found as 5 for the first episode ($t \in [15, 30]$ days) and the expected number of total arrivals during this episode is

$5\times(30-15) = 75$. Within the first episode of the *MP* model, the reservation prices are assumed to follow logarithmic behaviour with $p_{low} = 71$ and $p_{up} = 186$. If the ticket price is p_j, then the sales incentive for this fare class would be $z_j = \dfrac{\ln(p_{up}) - \ln(p_j)}{\ln(p_{up}) - \ln(p_{low})}$ and the aggregate expected number of seats that can be sold in the first episode at price p_j will be $75z_j$ in the *MP* model.

Table 6.1: Parameters for *DP* and *MP* models

Parameters	Epi. 1	Epi. 2	Epi. 3	Epi. 4	Epi. 5
Sales Horizon (in days)	[30,15]	(15,9]	(9,5]	(5,2]	(2,0]
Arrivals/Day [3 $\rightarrow$ 20]	5.0	9.4	12.9	16.1	18.8
$p_{low}(t)$: [29 $\rightarrow$ 199]	71	131	159	179	193
$p_{up}(t)$: [149 $\rightarrow$ 299]	186	239	264	281	294

Solving *MP* only once at $t = 30$ days to departure would give a static temporal price discrimination scheme. In order to use *MP* as a tool for dynamic pricing, it could be solved periodically (for instance, on a daily basis). It is also possible to resolve the model at every customer arrival at corresponding state (s, t). Notice that the demand parameters needs to be updated as well.

For *DP*, the length of a unit time interval, ϵ, is chosen as 30-seconds. *DP* gives the optimal price for all time intervals and every possible value of seat inventory as p_{st}. The *MP* model, on the other hand, works once for a particular (s, t) pair. For the numerical analysis here, *MP* is solved repeatedly by fixing s or t to understand the evolution of optimal price as a function of time to departure or remaining seat inventory. The exemplary results are depicted in Figure 6.4 and Figure 6.5 in order to see the behaviour of the optimal price in t and s by fixing one of the state variables.

In Figure 6.4 on the 30-days sales horizon, it is seen that the pricing schemes obtained by *DP* and *MP* are reasonable. The initial increasing trend is due to the increase in customer reservation prices along the sales horizon. The decrease towards the end of the sales horizon can be due to an increase in expected revenue to be obtained by selling more of the remaining seats at good prices without lowering the prices much. It is observed that the optimal prices with 10 seats are higher than those obtained for 30 seats. The most significant difference between the findings for *DP* and *MP* is due to the difference between the definition of reservation price bounds. For *MP*, p_{up} and

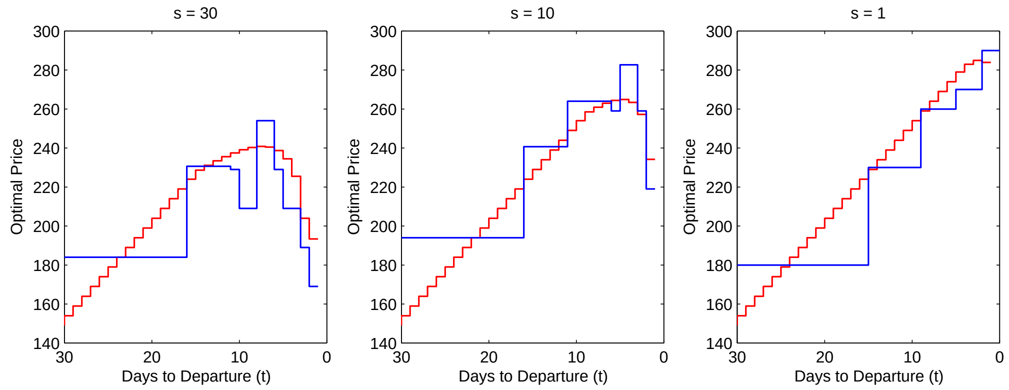

Figure 6.4: Optimal prices for seat inventory levels $s = 1, s = 10$ and $s = 30$

p_{low} are piecewise constant whereas the bounds are linearly increasing for *DP*.

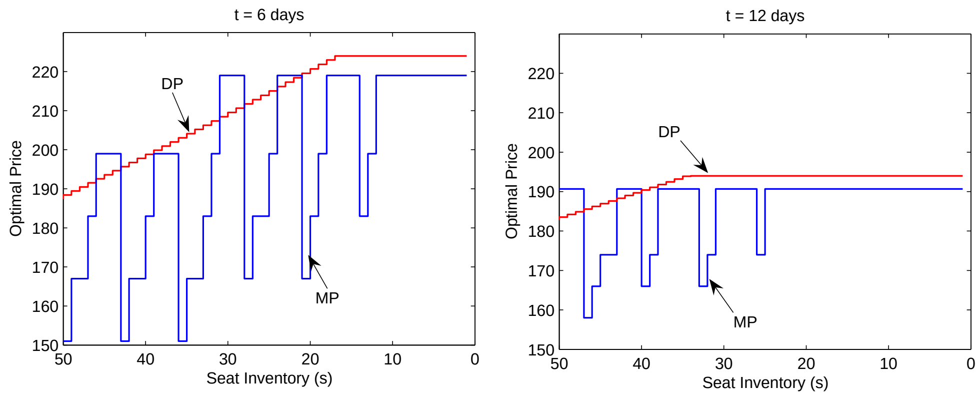

Figure 6.5: Optimal prices for $t = 6$ and $t = 12$ days to departure

In Figure 6.5, the optimal prices obtained by *DP* are decreasing in the seat inventory level as expected. However, the results are counterintuitive for the *MP* model: cyclical ups and downs are observed instead of a general trend. This phenomenon is due to the structure of the *MP* model to find an optimal seat allocation for the remaining episodes together with the optimal prices. In this respect, the optimal price for $(s+1, t)$ could be larger than that for (s, t) due to a major change in sales strategy with an additional available seat in period t. This could be noted as a shortcoming of the *MP* model.

To wrap up, *DP* could represent the continuous change in reservation price, P_t, and customer arrival probability, ρ_t better while *MP* is restricted to a piecewise constant approximation to these time-dependent parameters. For the numerical results presented in this section, both *DP* and *MP* models are solved using MATLAB. When these two models are compared in terms of computation times, *DP* significantly outperforms *MP* model. For the small-scale sample problem under consideration, CPU time to solve *DP* is less than a minute and it is sufficient to solve it once at the beginning of the sales horizon. Solving *MP* for a given (s,t) pair typically lasts 60 to 90 seconds. Note that *MP* is solved every day in the example problem.

6.3 Nonrefundable Ticket Pricing Sales Simulations

MP and *DP* are both exact models developed under the assumption that the seller has *complete* information on prospective demand; knowing at what rate the customers would arrive with what kind of reservation price distributions. Accordingly, the performances of the two 'exact' models are compared under this assumption in the simulations. These strict assumptions on sellers capability to foresee the future demand are relaxed to a certain extent in the approximate models where the seller has *incomplete* future demand information and requires an approximation for the marginal seat revenue. The performances of 'approximate' models Para-*M* and Pred-*M* defined in Chapter 4 are compared in this respect to assess which one provides a better approximation. The revenue and load-factor performances of Para-*M* and Pred-*M* are also compared with *DP* to understand the value of future demand information and the accuracy of marginal seat revenue approximations.

6.3.1 Seller Has Complete Future Demand Information

The *DP* approach to airline RM problem is constructed as a recursive formulation working backwards in time. In our simulation design, at the beginning of each run, the optimal sales price is found for every state (s,t) in the discretized state space such that for any given seat inventory - time to departure combination, the optimal prices are known before the sales begins. In each replication, random arrival times and reservation prices are generated for customers according to the given temporal demand parameters and at each customer arrival, the seller posts the precalculated price. For algorithmic details regarding the simulation setting, please refer to Appendix B.

The *MP* approach is adapted from seat allocation models; the remaining seat inventory is partitioned into the episodes in the remainder of the sales horizon together with the price to offer at each episode. For instance at the beginning of sales horizon, *MP* runs with 30−days to departure and 100 seats available for sales. The optimal policy determines to sell s_1 seats at price p_1 and next s_2 seats at price p_2 and so on. In order to represent the effect of sales realizations to the seller's pricing policy within a sales horizon, the prices found by *MP* are updated on a daily basis model during each simulation run. That is, if 2 seats are sold on first day, *MP* runs with 29−days to departure and 98 seats available for sales to find corresponding prices.

The average revenue results obtained with 500 simulation runs show that, *DP* (average revenue: $18,130$) outperforms *MP* (average revenue: $14,190$) significantly, with 28 percent higher revenue generation. Together with the structural shortcomings of the pricing policy generated by *MP* (discussed in Section 6.2.3), we conclude that *DP* model could provide more realistic and effective pricing policies in comparison to *MP*.

6.3.2 Seller Has Incomplete Future Demand Information

Recursive solution to *DP* requires marginal seat revenue, $\Delta v(s,t)$, which could be calculated only when the *seller foresees future demand*; knowing prospective customers' arrival times and reservation price distributions. With the approximate models proposed in this work, we have studied an approximation of marginal seat revenue with an estimate of stock-out probability, $SP(s,t)$. By introducing these models, we reduce the demand knowledge requirement of the seller: the reservation price distribution of current customer (P_t) and the number of prospective customers that would arrive until the departure ($\Lambda_{[0,t]}$) would be sufficient for determining the the price of the nonrefundable booking. For instance, if time to departure is 30 days and recursion is defined on discrete time intervals of 30 seconds, solution of *DP* model requires estimating the value of λ_t at 1440 consecutive intervals. On the other hand, if the seller has an estimation for the stock-out probability, price could be calculated directly without further need to future demand parameters.

- The model-based stockout probability estimation, Pred-M presumes a certain mathematical relationship between the state variables and the stock-out probability. The mathematical relation introduced in Section 4.2.1 is as below:

$$SP(s,t) = 1 - \left(\frac{s}{S+\omega}\right)^{\frac{\Lambda_{[0,t]}}{s}} \tag{6.2}$$

In order to find the value of parameter ω, we require the anticipated number of future arrivals and an initial estimate for the stockout probability at the beginning of the sales horizon; $SP(S,T)$. Hence, the pricing policy of the seller and the corresponding revenue performance are dependent on the estimated value of $SP(S,T)$. In our simulation runs, we have tested scenarios with different values of initial stockout probability estimates and corresponding revenue and load factor results are depicted in Figure 6.6. The revenue maximizing initial stockout probability estimate is $SP(S,T) = 0.2$ with an average of $15,902$. Hence, the pricing policy based on Pred-M stock-out probability approximation performs 12% worse in revenue generation than the *DP* model. Notice that load factor is around 0.85, which is also significantly less than that of exact model.

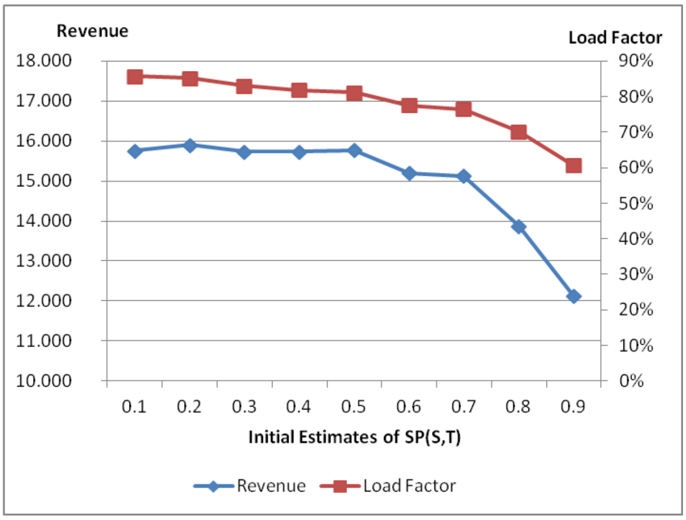

Figure 6.6: Simulation results for given initial stockout estimates

- The Pred-M stock-out probability estimation based approximate model does not require an initial parameter estimate. At each customer arrival generated in the sales simulations, the price (p) and sales incentive (z) are calculated assuming an interdependence between SP and z, as explained in detail in Section 4.2.2.

According to the simulation results, predictive model achieves an average revenue of $16,664$, which is 5% better than the revenue performance of model-based approximation. Moreover, the average load factor is 0.99 so the predictive model outperforms its model-based counterpart in both criteria.

In Table 6.2, simulation statistics for the exact pricing policy and both approximate pricing policies are summarized. For both revenue and load factor parameters, the expected values (averages) and 95% confidence interval limits are presented.

Table 6.2: Nonrefundable simulation results - medium demand

	Revenue		Load Factor	
Method	*Average*	95% CI	*Average*	95% CI
Para-M	$15,912$	$[14,057 - 17,766]$	0.85	$[0.77, 0.92]$
Pred-M	$16,664$	$[16,144 - 17,183]$	0.99	$[0.99, 1.00]$
DP	$18,069$	$[16,763 - 19,374]$	0.99	$[0.96, 1.00]$

We consider the demand scenario used in the simulation runs as base level of demand and also studied high and low demand cases. Remind that the customer arrival rate has been formulated such that it varies from $\Lambda_T = 3$ arrivals/day to $\Lambda_0 = 20$ arrivals/day along the sales horizon. Empirically determined low demand and high demand parameters used in alternative simulation scenarios are tabulated in Table 6.3. The demand-to-supply ratio, $\frac{\Lambda_{[0,T]}}{S}$ is also provided to give an idea about the aggregate demand along the sales horizon.

Table 6.3: Customer arrival rate parameters used in simulation runs

Demand Level	λ_T	λ_0	Demand-to-Supply
Low	2 arrival/day	15 arrival/day	1.93
Medium	3 arrival/day	20 arrival/day	2.68
High	4 arrival/day	30 arrival/day	3.87

The simulation results obtained for low and high demand alternatives are given as below.

Table 6.4: Nonrefundable simulation results - alternative demands

		Revenue		Load Factor	
Scenario	Method	*Average*	95% CI	*Average*	95% CI
Low	Para-M	$13,716$	$[11,355 - 16,077]$	0.75	[0.63 – 0.86]
	Pred-M	$15,213$	$[14,586 - 15,840]$	0.99	[0.98 – 1.00]
	DP	$16,116$	$[14,486 - 17,746]$	0.98	[0.91 – 1.00]
High	Para-M	$17,608$	$[15,732 - 19,483]$	0.84	[0.77 – 0.92]
	Pred-M	$18,121$	$[17,608 - 18,633]$	0.99	[0.99 – 1.00]
	DP	$20,147$	$[19,131 - 21,162]$	0.99	[0.97 – 1.00]

We performed our simulation runs with Matlab 9.0 on a PC having a 3.0 GHz processor and 4 GB RAM. CPU time results for these simulation runs are given as below:

- *DP* model calculates optimal price at every possible state in 107.3 seconds and 500 replications are completed in 12 minutes for the given parameter setting. The simulation for *DP* model is constructed such that the optimal price matrix obtained in the beginning is used in each run and not recalculated. In simulations for approximate models, restricted price is calculated in each arrival for the corresponding state.

- Para-M runs a simple routine for obtaining restricted ticket price at each customer arrival in less than 0.1 seconds. 500 replications are completed in 27 minutes.

- Pred-M requires solving a recursive formulation defined on stockout probability and sales incentive. Restricted ticket price at given state is found in 2 seconds and 500 replications are completed in approximately 17 hours.

In airline RM, online reservation systems are expected to complete price inquiries within a couple seconds. Hence, we consider all three algorithms satisfactory in terms of CPU times for posting restricted booking price.

6.3.3 Sales Price Comparisons - Single Sales Run

Revenue and load factor comparisons provide information on the sales performances of proposed methods. In order to acquire further insight regarding the pricing policies, we also compared the methods over a single simulation run. Randomly generated sales scenario is replicated and following results are obtained:

- 257 customer arrival generated for 100 seats inventory. The revenue corresponding to exact pricing $18,326$, while predictive stock-out estimation based pricing policy generates $16,927$ and model-based stock-out estimation based pricing policy generates $15,462$.

- The price record over the $30-$days sales horizon in Figure 6.7 depicts the pricing strategies of competing policies. Notice that, approximate model based on Pred-M model offers lowest prices at the beginning of the sales horizon and highest prices at the end. On the other hand, DP model starts sales horizon with the highest prices and the price increases at slower pace in comparison to other methods until the last 2 days of the sales horizon.

- The load factor of model-based approximation policy, Para-M is found as 0.83, whereas both DP and Pred-M policies are 1.00 for this simulation run. Notice that although all seats are sold with DP and Pred-M policies, the difference in pricing policies is reflected to the depletion trends of seats in sales simulations of these two methods. When we check the ratio of seats sold at $t = 10$ days to departure, we observe that only 15% of seats are sold when seller follows DP pricing policy and 32% of seats are sold in case of Pred-M.

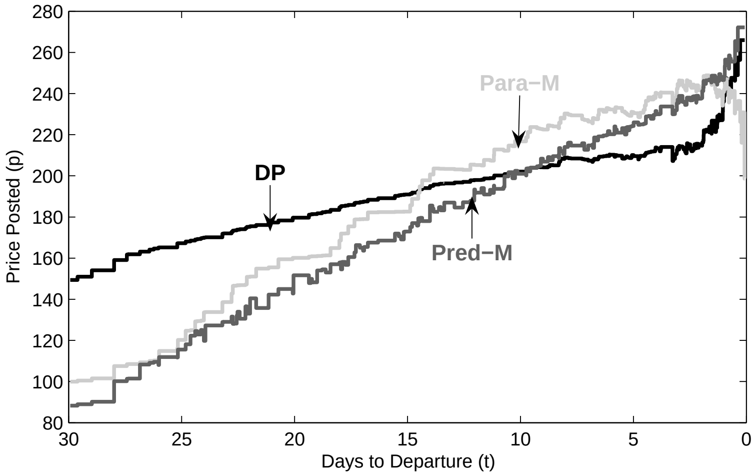

Figure 6.7: Sales price comparisons for proposed formulations

6.4 Refundable Ticket Pricing Sales Simulations

The simulation runs for investigating the impact of offering refund options requires modeling a sales process that involves two type of events during the sales horizon. Arrivals of prospective customers are modeled with respect to Nonhomogenous Poisson Arrival assumption as in nonrefundable scenarios. Customer cancellations are introduced to the simulation model as an attribute assigned to customers that purchase refundable tickets. Of those customers purchasing refundable tickets, a secondary arrival is scheduled if s/he wishes to cancel the booking. The primary assumptions considered in simulating the sales process with cancellations are enlisted below:

1. Assuming that customers decide between refundable and nonrefundable bookings as described in Section 5.2.2, the customer reservation price for refund option is calculated as described at Lemma 5.2.4.

2. When a booking request is received, the seller decides the nonrefundable ticket price using the 'exact' *DP* formulation. Approximate models are not preferred for determining the nonrefundable ticket price to see the improvement in revenue by refund option sales when the best practice is applied for finding p. On the other hand, the marginal value results found in the solution of *DP* are not used for solving the refund option pricing problem since the model is developed

for approximated $\Delta_s v_t(s)$ and $\Delta_n v_t(s)$ values.

3. It is assumed that the seller knows the probability distribution of cancellation probability, C_t for the current customer.

4. The refund options are sold to the customers until the last 48-hours of the sales horizon. We assume that the late purchasing customers would have less uncertainty from the time of booking until time of flight having very slim probabilities of booking cancellation in between.

5. For each customer who purchases refundable booking, a binary cancellation flag is assigned with respect to the individual cancellation probability at the time of sales. Those customers who would cancel their booking are also assigned a cancellation time.

6. Cancellation requests are all scheduled before the last 24-hours of the sales horizon. In real life examples, booking cancellations have similar temporal restrictions; *late* cancellation claims are partially refunded or not refunded at all.

6.4.1 Degree of Customer Risk Aversion

For different levels of customer risk aversion, the revenue generated by selling refundable tickets is investigated for medium demand rate with parameters given in Table 6.3. The results are tabulated below, with respect to revenue generated, load factor, total number of refund options sold and total number of cancellation claims made. Average and standard deviation statistics for corresponding parameter are denoted by μ and σ, respectively. The option pricing methodologies are referred to as *"Point"*,referring to determining the refund option price with the point estimate, and *"Interval"*, referring to estimation of pdf of refund option reservation price first and calculation of the optimal option price afterwards. (See Section 5.2.2 for details).

The interpretations of simulation results regarding the degree of customer risk aversion are as given below:

- The average revenue for sales simulations of nonrefundable tickets (no-cancellations)

Table 6.5: Simulation results for different degrees of customer risk aversion

		Revenue		Load Factor		Options Sold		Options Used	
Method	β	μ	σ	μ	σ	μ	σ	μ	σ
Point	0.002	18,893	821	0.98	0.02	35.3	5.5	5.9	2.4
	0.004	18,935	807	0.98	0.03	36.6	4.4	6.6	2.4
	0.008	19,193	983	0.98	0.04	36.4	3.9	6.6	2.6
	0.016	20,246	999	0.98	0.03	36.2	4.2	6.1	2.2
Interval	0.002	19,007	902	0.98	0.03	54.1	5.4	8.9	3.3
	0.004	19,234	1,095	0.97	0.04	56.2	4.8	8.2	2.5
	0.008	19,723	1,178	0.97	0.04	61.1	4.4	9.2	2.8
	0.016	21,222	1,011	0.97	0.03	67.9	4.8	9.9	3.7

case is found as $18,130$. Observing the average revenue values, we infer that selling refund options improves the revenue at all risk aversion levels for both point estimation and interval estimation of refund option price. The revenue improvement is parallel to customer risk aversion; as β increases gradually, the refund options become more and more profitable.

- When interval estimation method is applied for refund option pricing more refund options are sold and higher revenues are generated in comparison to point estimation case. Therefore, interval estimation method outperforms point estimation in our experiments.

- The load factors are quite close to 1 although significant percentage of booked customers are allowed to cancel their bookings before the flight and some of them actually use their cancellation claims. This situation is primarily due to reselling of the cancelled tickets to consecutive customers.

- The percentage of customers who prefer refundable tickets is approximately, 35% for the point estimation case and 55% – 65% for interval estimation. However, only 15% of refundable bookings are cancelled in each scenario.

To sum up, we observe that selling refund options are profitable and the increase in refund option selling probability is paralel to the degree of customer risk aversion. At all levels of risk aversion, we observe that ’Interval’ estimation model is more favorable than ’Point’ estimation model in terms of revenue generation.

6.4.2 Level of Demand Intensity

Remember the alternative demand parameters defined in Table 6.3. According to the customer arrival rate alternatives, the simulation results are obtained for low, medium and high demand parameters for both 'Point' and 'Interval' methods. Results of *DP* are also tabulated in Table 6.6 to benchmark the effect of option selling possibility on the revenue at different demand levels. The load factor results are quite straightforward; as the demand-to-supply gets higher the load factor increases for each pricing policy. When we compare the revenue results of refund option pricing alternatives, we observe that the 'Point' estimation method performs slightly better than the 'Interval' estimation methods. As the demand intensity increases, the 'Interval' estimation model performs better.

Table 6.6: Refundable simulation results - alternative demands

		Revenue		Load Factor	
Scenario	Method	*Average*	95% CI	*Average*	95% CI
Low	Point	16,831	[14,782 – 18,880]	0.97	[0.88 – 1.00]
	Interval	16,659	[14,432 – 18,886]	0.95	[0.84 – 1.00]
	DP (no *q*)	16,116	[14,486 – 17,746]	0.98	[0.91 – 1.00]
Medium	Point	18,721	[17,208 – 20,234]	0.98	[0.93 – 1.00]
	Interval	19,110	[17,305 – 20,915]	0.98	[0.92 – 1.00]
	DP(*no*q)	18,069	[16,763 – 19,374]	0.99	[0.96 – 1.00]
High	Point	20,943	[19,841 – 22,045]	0.99	[0.97 – 1.00]
	Interval	21,114	[19,985 – 22,243]	0.99	[0.97 – 1.00]
	DP (no *q*)	20,147	[19,131 – 21,162]	0.99	[0.97 – 1.00]

6.5 Further Observations on Pricing Policies

Thus far, we present alternative pricing models for determining the restricted ticket price, p, and refund option price, q, and assess each model according to its analytical limitations and sales performance in a theoretical perspective. Yet, simpler empirical observations could be deduced from sales simulations to provide generic policy rules for presented dynamic pricing problem.

The sales incentive, $z = \Pr(P_t \geq p)$ is used as a decision variable interchangeably with the sales price in our formulations. In the approximate models, we presume a relationship between z and the stock-out probability $SP(s,t)$, which was defined as a function in terms of demand-to-supply ratio, $\frac{\Lambda_{[0,t]}}{s}$ in Section 4.2.1. In this respect, on a simulation run with exact pricing model (DP), the change of sales incentive and the supply-to-demand ratio over the sales horizon is investigated. Simulation results for supply-to-demand ratio the complementary probability of sales incentive (referred to as *keep incentive* of the seller) are depicted in Figure 6.8.

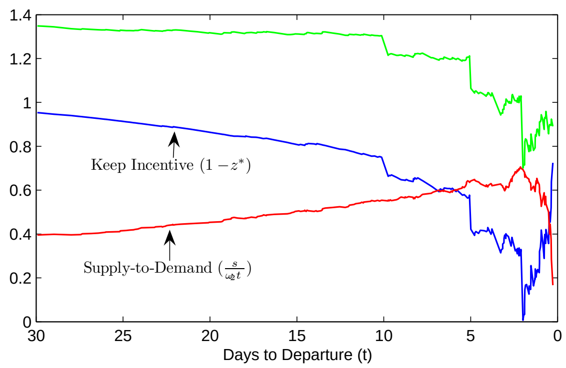

Figure 6.8: Change of sales incentive with supply-to-demand ratio over time

In accordance with our expectations, we observe negative correlation between keep incentive, $1 - z$, and supply-to-demand ratio, $\frac{s}{\Lambda_{[0,t]}}$ (lower supply-to-demand values would motivate the seller to keep more seats for late-comers who have higher willingness to pay). The sum of these two variables is also depicted in the figure and the graph resembles a piecewise constant behavior over time with the jumps observed around $t = 10$ days, $t = 5$ days and $t = 2$ days, which are the time points when the jumps on time dependent reservation price bounds are set. The relation between the two variables could be approximated as $(1 - z^* + \frac{s}{\Lambda_{[0,t]}}) = K_t$, which is equivalent to $z^* = \frac{s}{\Lambda_{[0,t]}} + (1 - K_t)$ where the value of constant K_t changes with each jump. According to these findings, we have devised a rule-based pricing policy as follows:

1. If $t \geq 10$ days, then $z^* = \frac{s}{\Lambda_{[0,t]}} - 0.3$.

2. If $t \in [5, 10)$ days, then $z^* = \frac{s}{\Lambda_{[0,t]}} - 0.2$.

3. If $t \in [2, 5)$ days, then $z^* = \frac{s}{\Lambda_{[0,t]}}$.

4. If $t < 2$ days, then $z^* = \frac{s}{\Lambda_{[0,t]}} + 0.2$.

Simulation results obtained for rule-based policy are summarized in Table 6.7. The average revenue of the rule-based policy is only 1.1% worse than that of exact pricing policy, hence we believe the policy performs quite good despite its simplicity.

Table 6.7: Simulation results - DP vs. Rule-Based Pricing Policy

	Revenue		Load Factor	
Method	*Average*	95% CI	*Average*	95% CI
Rule-Based	17,887	[17,177 – 18,596]	0.99	[0.98, 1.00]
DP	18,069	[16,763 – 19,374]	0.99	[0.96, 1.00]

Remark 6.5.1 *The rule-based policy outperformed both approximate pricing policies, Pred-M and Para-M. However, it is derived for the specific demand setting used in the simulation runs. For different customer arrival patterns, different temporal affects on reservation price distributions, this empirical rule would not be applicable. Still, we believe that the relationship between the sales incentive, z^*, and supply-to-demand ratio, $\frac{s}{\Lambda_{[0,t]}}$ is promising for developing effective dynamic pricing policies.*

CHAPTER 7

CONCLUSION AND FUTURE RESEARCH

The pricing problem studied herein is partitioned into two major components in accordance with the two main pricing decisions we considered in the problem formulation: determining restricted booking price ignoring booking cancellations/refundability and determining refund option price for given restricted booking price. In this chapter, the conclusions drawn from the simulation studies are given for these two pricing problems first. Then, potential improvements in introduced methods are discussed and we make our final remarks about future research directions.

7.1 Subproblem-1: Finding p Ignoring Cancellation Refunds

- **Exact Model:** For low, medium and high demand scenarios, it is observed that both approximate pricing policies based on stock-out probability estimation, Pred-*M* and Para-*M*, are outperformed by the *DP* model for the first subproblem. This observation is parallel to our expectation since *DP* model calculates optimal price with actual values of marginal seat revenue instead of approximating it.

- **Approximate Models:** Among the two approximate methods, Pred-*M* displays better performance than Para-*M* in terms of revenue and load factor. Notice that higher load factor implies that the seller behaves more risk averse, i.e. the seller is less inclined to take risk and keeps seats for late coming customers (who have greater willingness to pay) than selling more tickets to early arriving leisure customers. Hence, Pred-*M* appears more preferable than Para-*M* with its slightly better revenue generation performance. Also observe that, Pred-M is more robust; the standard deviation of its revenue is smaller than the revenue generated by implementation of Para-M.

Note that the performance of Para-M is dependent on the accurate estimation of parameter ω. Although in our simulations alternative values of initial stock-out probability estimate ($\eta = SP(S,T)$) are tried and we determined the best value of ω accordingly, we did not update our estimate for ω at later points in sales horizon. For real life applications, updating the parameters of presumed stockout function and/or trying other functional forms could improve the performance of pricing policies based on model-based $SP(s,t)$ estimation.

- **Value of Information:** The simulation studies show that *DP* model is the most successful method for the first subproblem, however it could only be solved when seller has complete information on time-variant parameters of disaggregated demand, namely customer arrival rate, λ_t, and restricted ticket reservation price, P_t. For the simulated demand scenarios, the difference between average revenues of *DP* method and approximate methods (Pred-M and Para-M) can be considered as the value of this information to the seller.

- **Policy Rule:** The revenue performance of the policy rule studied for the first subproblem is also noteworthy. Although it is defined on a simplistic empirical rule between sales incentive and demand-to-supply ratio, it has better revenue performance then both approximate methods. However, the parameters in the definition of this rule is specific to the given demand pattern. So, for different scenarios, similar parameters should be recalculated using simulation based methods.

7.2 Subproblem-2: Finding q for Given p

We consider our solution for the second subproblem, finding q for given p, as more contributing since the refund possibility and pricing of the refund option is studied with a new perspective in revenue management. The decision theoretic approach to customers' valuation of refund options is not specific to airline industry and it could be applied to other service industries where there is significant time lag between time of booking and time of service delivery. In real life implementations, early bookings are very common in hospitality industry and hotel reservations could be made several months in advance to benefit from price advantages. Hence, we believe that sales of cancellation refunds could be used efficiently in tourism sector.

The simulation results on revenue improvement due to selling refund options show the significance of effective pricing of q. As mentioned previously, customer risk-aversion could be exploited by the firms to improve their revenues and results show that the higher the degree of customer risk-aversion gets, the more profitable offering refund options become. Among the two alternative methods proposed for finding q, the model based on probability distribution estimation of refund option reservation price, F_{Q_t} is found more promising.

To sum up, we recommend *DP* model for finding p and *Interval*method for determining q. In case time variant demand parameters could not be estimated accurately, seller could prefer Pred-*M* for finding restricted ticket price to have a higher load factor and more robust revenue-to-go. Para-*M* can be suitable for pricing if batch arrivals of last-minute customers is probable; in which case having empty seats just before departure is advantageous for the seller.

7.3 Possible Improvements and Future Research

Parameter estimation plays a major role in the success of dynamic pricing methods proposed in this study. In this respect, we consider estimation of probability distributions of customer reservation price (F_{P_t}) and probability of cancellation (F_{C_t}) as promising areas for further research. Historical records of previous price inquiries of customers can be stored on airline reservation systems and accept/reject decisions for offered price can be utilized to infer reservation prices with a Bayesian estimation model. Cancellation statistics of previous flights and no-shows are also recorded on airline database systems and this data could be used for estimating cancellation probabilities.

In this study, parameter estimation problem is not restricted to estimation of demand

and cancellation parameters introduced in the original problem formulation. The success of approximate models developed for finding p and q are based on estimation of stockout probability, $SP(s,t)$. In the parametric estimation model we proposed, expert opinion is used for the initial value only, $\eta = SP(s,t)$. For stockout probability estimation, alternative models could be also developed to make greater use of expert opinion. We already know by definition of stockout probability that $SP(0,t) = 1$ and $SP(s,0) = 1$ for $s > 0$. Stockout probability estimates at different points in the state-space could be gathered from sector proffessionals ($\eta_i = SP(s_i,t_i)$ for $i = 1,2,...,n$) and surface fitting techniques could be applied to find an alternative functional form for stockout probability function.

Note that although the possibility of cancellation is taken into consideration, overbooking is not allowed in our formulations (this restriction allows defining the boundary condition $v_t(0) = 0$). The dynamic pricing approach studied herein is based on expected net revenue, defined as the sales probability multiplied with the difference between the sales price and the opportunity cost of an additional seat. Similarly, in case of overbooking marginal cost of selling one more seat over the capacity could be approximated. When the no-shows (customers who book for the flight and do not show up at the time of departure) exceed the overbooked capacity, the airline company does not incur any costs; however, certain penalties exist in case a customer is denied on boarding. The marginal revenue term could be approximated by using the probability of observing less than expected no-shows (similar to $SP(s,t)$) and the penalties that could be incurred in case of overbooking (similar to p_{fin}). With this approximation, a similar approach could be adopted for developing a dynamic pricing methodology for overbooked capacity.

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APPENDIX A

ADDITIONAL SIMULATION RESULTS

Table A.1: Revenue generation - 120 seats on hand at t=30 days

0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
44258	44135	43824	43635	43014	42582	41635	40821	39002
46783	46189	46196	45571	45203	44971	44498	43565	42520
48004	47742	47529	47331	46837	46606	46096	45704	44424
47996	47936	47463	47284	47234	46816	46538	45752	44433
48290	47837	47810	47463	47367	47177	46656	46035	44868
48875	48854	48607	48165	48060	47808	47218	46963	46352

Table A.2: Load factor results - 120 seats on hand at t=30 days

0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
90.9%	90.8%	90.7%	90.7%	90.4%	90.2%	89.9%	89.5%	88.8%
91.5%	91.2%	91.2%	91.0%	90.9%	90.8%	90.6%	90.3%	89.9%
91.7%	91.6%	91.5%	91.5%	91.3%	91.2%	91.0%	90.9%	90.4%
91.7%	91.6%	91.5%	91.4%	91.4%	91.3%	91.2%	90.9%	90.4%
91.8%	91.6%	91.6%	91.5%	91.5%	91.4%	91.2%	91.0%	90.5%
91.9%	91.7%	91.7%	91.6%	91.5%	91.5%	91.3%	91.2%	91.0%

Table A.3: Revenue generation - 150 seats on hand at t=30 days

0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
57022	56892	56558	56444	55884	55374	54472	53474	51362
61619	61030	60988	60368	60031	59712	59209	58190	56895
63934	63673	63491	63250	62656	62417	61824	61362	59766
64012	63902	63487	63249	63132	62757	62354	61384	59916
64450	63865	63979	63548	63451	63210	62541	61876	60452
65532	65492	65213	64731	64642	64336	63699	63423	62628

Table A.4: Load factor results - 150 seats on hand at t=30 days

0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
82.06%	81.90%	81.58%	81.57%	81.10%	80.65%	79.91%	79.08%	77.42%
83.63%	83.16%	83.16%	82.71%	82.48%	82.35%	81.92%	81.16%	80.27%
84.33%	84.08%	83.99%	83.83%	83.41%	83.27%	82.94%	82.59%	81.64%
84.26%	84.13%	83.94%	83.73%	83.72%	83.46%	83.27%	82.55%	81.61%
84.51%	84.09%	84.11%	83.82%	83.87%	83.68%	83.24%	82.78%	81.90%
84.68%	84.63%	84.48%	84.17%	84.02%	83.97%	83.52%	83.42%	82.89%

APPENDIX B

SIMULATION CODES IN MATLAB

Cancellation Allowed Simulation Run for *DP* Model

```
function [rev,empty,pricerec,salesrec,reservprec,tsold,tarrvl,canc_log]=...
SIM_exactP_intvlQ(S,Tdays,Pinit,Pfin,lambda,prec,intlength,Lin2Jump,m,beta,SPinit)
%1 sales replication

%At every possible state, optimal prices with DP is found
[Poptarry,value,time,Res_Price_Bounds]=fill_matrix_pst(S,Tdays,...
Pinit,Pfin,lambda,prec,intlength,Lin2Jump);

%Initialization of simulation parameters
reservps=zeros(1,1000);
reservqs=zeros(2,1000);
prices=zeros(1,1000);
priceqs=zeros(1,1000);
sales=zeros(1,1000);
saleqs=zeros(1,1000);
tarrvl=zeros(1,1000);
custcount=0;
canc_log=[-1 -1];

linit=lambda(1);
lfin=lambda(2);
```

```
%Nonhomogenous Customer Arrivals are generated
[tarr]=generate_arrivals(Tdays,linit,lfin);
snow=S;
rev=0;
nb_arrivals=length(tarr);
for custcount=1:nb_arrivals
tnow=tarr(custcount);
%Check: Customer arrival = Cancellation
if (tnow<canc_log(1,1))
%customer refunded
rev=rev-canc_log(1,2);
snow=snow+1;
%realized cancellation is removed from cancellation event list
canc_log(1,1)=-1;
%remaining cancellations in the event list are sorted
canc_log=flipdim(sortrows(canc_log),1);
else

if (snow>0)
intnumber=ceil(tnow*1440/intlength);

Plo=Res_Price_Bounds(1,intnumber);
Phi=Res_Price_Bounds(2,intnumber);
%seller determines price
P_opt= Poptarry(snow,intnumber);

%———Option A: Point Estimation of Q—————
%Q_opt=find_q_point(P_opt,beta,m,tnow,Tdays);
%———Option A: Point Estimation of Q—————

%———Option B: Interval Estimation of Q———
%estimate stock-out probability
```

```
SP=find_Pstockout_1(SPinit, linit, lfin, snow, S, tnow, Tdays );
Q_opt=find_q_optimal(P_opt,beta,m,tnow,Tdays,SP);
%———Option B: Interval Estimation of Q———
prices(custcount)=P_opt;
sales(custcount)=-1;
%customer arrives with a certain prob. of buying
pbuy=rand(1);
%customer arrives with a certain prob. of cancellation
C_t=0.1+0.4*(tnow/Tdays)*rand(1);

reservp=(Plo^(1-pbuy))*(Phi^pbuy);
reservq=log(1+(C_t/(1-C_t))*(exp(beta*P_opt)-exp(beta*m)))/beta;
if (tnow < 2)
reservq = 0;
end
reservps(custcount)=reservp;
reservqs(1,custcount)=reservq;
reservqs(2,custcount)=C_t;
if (reservp > P_opt)
if (reservq > Q_opt)
%customer books refundable
rev=rev+P_opt+Q_opt;
snow=snow-1;
sales(custcount)=1;
saleqs(custcount)=1;
tsold(1,custcount)=tnow;
tsold(2,custcount)=1;
%future cancellation record needed
prob_temp=rand(1);
%customer cancels with probability C_t
if (prob_temp < C_t)
%random cancelation time
t_temp=1+(tnow-1)*rand(1);
```

```
%refund amount when canceled is logged
refund=P_opt+Q_opt-m;
canc_log=[canc_log; t_temp refund];
canc_log=flipdim(sortrows(canc_log),1);
end
else
%customer books non-refundable
rev=rev+P_opt;
snow=snow-1;
sales(custcount)=1;
tsold(1,custcount)=tnow;
tsold(2,custcount)=0;
end
%sales complete(customer leaves with or without purchase)
end
end
end
empty=snow;
tarrvl=nonzeros(tarr);
pricerec=nonzeros(prices);
salesrec=nonzeros(sales);
reservprec=nonzeros(reservps);
reservqrec=reservqs;

end
```

Subroutine for solving recursive *DP* Model

```
%Function returns optimal price at every possible state; p*(s,t)
function [Poptarry,value,time,Res_Prices,lambda_arry]=...
fill_matrix_pst(S,Tdays,Pinit,Pfin,lambda,prec,intlength,Lin2Jump)
%STATEVARIABLES%
%S:seat inventory
%Tdays:time in DAYs

%DEMAND
%Pinit=[Plo(T) Phi(T)]; reservation prices at the start of sales horizon
%Pinit=[Plo(0) Phi(0)]; reservation prices at the end of sales horizon
%lambda = [linit, lfin] initial and final arrival rates

%OTHER PARAMETERS
%prec:precision parameter (error tolerance for numeric optimization)
%intlength:interval length in MINUTEs
%Lin2Jump:0=>Linear Increase, 1=>Piecewise Constant with Jumps res. price
%values between 0 and 1 yields jumps at given points.

tic
%RESERVATION PRICE JUMPS
jump_times=[10 5 2];
jump_percent=[0.2 0.3 0.5];
Pfin_linear_up=Pinit(2)+(Pfin(2)-Pinit(2))*(1-Lin2Jump);
Pfin_linear_lo=Pinit(1)+(Pfin(1)-Pinit(1))*(1-Lin2Jump);
upper_bound_jumps=jump_percent*(Pfin(2)-Pinit(2))*Lin2Jump;
lower_bound_jumps=jump_percent*(Pfin(1)-Pinit(1))*Lin2Jump;
%RESERVATION PRICE JUMPS

T=(Tdays*1440)/intlength;
```

```
Poptarry=zeros(S,T);
value=zeros(S+1,T+1);
Res_Prices=zeros(2,T);
lambda_arry=zeros(2,T);
for tnow=1:T
%Find reservation price bounds Plo-Phi

%Linear increase
Plo=(Pinit(1)*tnow+Pfin_linear_lo*(T-tnow))/T;
Phi=(Pinit(2)*tnow+Pfin_linear_up*(T-tnow))/T;
%Add jumps
if tnow <jump_times(1)*(1440/intlength)
Plo=Plo+ lower_bound_jumps(1);
Phi=Phi+ upper_bound_jumps(1);
if tnow <jump_times(2)*(1440/intlength)
Plo=Plo+ lower_bound_jumps(2);
Phi=Phi+ upper_bound_jumps(2);
if tnow <jump_times(3)*(1440/intlength)
Plo=Plo+ lower_bound_jumps(3);
Phi=Phi+ upper_bound_jumps(3);
end
end
end
Res_Prices(1,tnow)=Plo;
Res_Prices(2,tnow)=Phi;
%Find arrival rate at current time point
[lnow,lavg]=find_lambda(tnow,T,lambda(1),lambda(2));
lnow = lnow*(intlength/1440);

for snow=1:S
delta=value(snow+1,tnow)-value(snow,tnow);
Popt=find_p_log(Phi,Plo,prec,delta);
salespr=(log(Phi)-log(Popt))/(log(Phi)-log(Plo));
```

```
Poptarry(snow,tnow)=Popt;
value(snow+1,tnow+1)=value(snow+1,tnow)+lnow*salespr*(Popt-delta);
end
end
time=toc;
end
```

Subroutine Generating Customer Arrivals

```
%Function returns randomly generated Nonhomogenous Poisson arrivals
function [tarr]=generate_arrivals(T,linit,lfin)
tarr=zeros(1,round(T*lfin));
tnow=T;
i=1;
while (tnow > 0)
[tnext]=nhp_arrival(tnow, T, linit,lfin);
if tnext > 0
tarr(i)=tnext;
end
i=i+1;
tnow=tnext;
end
tarr=nonzeros(tarr)';
```

Function Returning Next Customer Arrival

```
 %Subroutine used in generate_arrivals.m
function [tnext]=nhp_arrival(tnow, T, linit,lfin)
%due to Lewis&Shedler (Lewis P.A.W., Shedler G.S., "Simulation of Nonhomoge-
nous Poisson Process by
%Thinning", Nav. Res. Logist. Quart., 26:403–413 (1979)
t=tnow;
lnow=find_lambda(t,T,linit,lfin);
lmax=lfin;
u=rand(1,2);
t=t+log(u(1))/lmax;
while (u(2)> (lnow/lmax))
u=rand(1,2);
t=t+log(u(1))/lmax;
end
tnext=t;
end
```

Function Returning Time Dependent Poisson Arrival Rate

```
%Function returns Nonhomogenous Poisson Arrival Rates
function [lnow, lavg]=find_lambda(t,T,linit,lfin)
%lambda increases exponentially along the horizon
lnow=lfin*(lfin/linit)^(-t/T);
lavg=((lfin*T)/(log(lfin/linit)*t))*(1-(lfin/linit)^(-t/T));
end
```

Stock-out Probability Estimation Routine - Para-*M*

```
%Function returns stockout probability estimate for parametric method
function SP=find_Pstockout_1(SPinit, linit, lfin, snow, S, tnow, Tdays );

[lnow, lavg]=find_lambda(Tdays,Tdays,linit,lfin);
omega1=S*((1-SPinit)^(-S/(lavg*Tdays))-1);
Lambda = lavg;

SP=1-(snow/(S+omega1))^(Lambda*tnow/snow);

end
```

Subroutine Finding Optimal Refund Option Price - Point Estimation

---

```
 function [P_opt]=p_opt_predictive(Plo,Phi,prec,P_maxx,s,d)
%both arrays are defined so that first row contains SP and second row
%contains z
SP_to_z_arry=zeros(2,100);
z_to_SP_arry=zeros(2,100);

for i=1:100
%Optimal sales incentive is obtained by approximation delta = SP*P_maxx
SP_to_z_arry(1,i)=i/100;
delta= SP_to_z_arry(1,i)*P_maxx;
[z_given_SP, P_given_SP]=find_p_log(Phi,Plo,prec,delta);
SP_to_z_arry(2,i)=z_given_SP;
%probability of stockout for given sales incentive is found by Neg. Bin
z_to_SP_arry(2,i)=i/100;
SP=SP_given_z(z_to_SP_arry(2,i),s,d);
z_to_SP_arry(1,i)=SP;
end

error=1;
for i=1:100
for j=1:100
errornew=max(abs((SP_to_z_arry(1,i)-z_to_SP_arry(1,j))),...
abs((SP_to_z_arry(2,i)-z_to_SP_arry(2,j))));
if (errornew<error)
z_index=i;
SP_index=j;
error=errornew;
end
end
```

```
end

z_opt=SP_to_z_arry(:,z_index);
SP_opt=z_to_SP_arry(:,SP_index);
P_opt=Phi^(1-z_opt(2))*Plo^z_opt(2);
```

```
 %Function returns opt. price with recursion solved for logarithmic demand
function [v,P_opt]=find_p_log(Phi,Plo,prec,delta)

%Phi: Price upper bound for LOGARITHMIC DEMAND
%Plo: Price lower bound for LOGARITHMIC DEMAND
%prec: error tolerance for numeric optimization
%delta: v_t(s) - v_{t-1}(s)

kappa=log(Phi/Plo);

if (delta>Phi)
v=0;
i=0;
else
vu=min(1,1/kappa);
vl=0;
v=(vu+vl)/2;
eps=-delta+(Phi^(1-v))*(Plo^v)*(1-kappa*v);
i=0;
while(abs(eps)>prec && i<20 )
if (eps>prec)
vl=v;
v=(vu+vl)/2;
eps=-delta+(Phi^(1-v))*(Plo^v)*(1-kappa*v);
i=i+1;
else if (eps<-prec)
vu=v;
v=(vu+vl)/2;
eps=-delta+(Phi^(1-v))*(Plo^v)*(1-kappa*v);
```

```
i=i+1;
end
end
end
end
P_opt=Plo^(v)*Phi^(1-v);
iter=i;
```

Stockout Probability for Given Sales Incentive (used in Pred-*M*)

```
function [SP]=SP_given_z(z,s,d)
if ((s<=0))
SP=1;
else if (d<s ||(z==0))
SP=0;
else
SP=nbincdf(d-s, s, z);
end
end
```

Subroutine Finding Optimal Refund Option Price - Point Estimation

---

```
 %Function returns optimal Q using point estimation method
function [Q_opt]= find_q_point(P_opt,beta,m,tnow,Tdays);

pcanc=0.1+0.4*(tnow/Tdays)*0.5;%expected value of canc. prob
Q_opt=log(1+(pcanc/(1-pcanc))*(exp(beta*P_opt)-exp(beta*m)))/beta;

end
```

Subroutine Finding Optimal Refund Option Price - Interval Estimation

---

```
 %Function returns optimal Q using interval estimation method
function [Q_opt] = find_q_optimal(P_opt,beta,m,tnow,Tdays,SP);

C_t_expected=0.1+0.4*(tnow/Tdays)*0.5;

C_t_cdf=zeros(2,101);
Q_t_cdf=zeros(2,101);
for i=0:100
C_t_cdf(1,i+1)=0.1+0.4*(tnow/Tdays)*(i/100);
C_t_cdf(2,i+1)=(i/100);

pcanc=C_t_cdf(1,i+1);
Q_t_cdf(1,i+1)=log(1+(pcanc/(1-pcanc))*(exp(beta*P_opt)-exp(beta*m)))/beta;
Q_t_cdf(2,i+1)=(i/100);
end

Q_return=zeros(1,101);
for i=0:100
ECL=C_t_expected*(1-SP)*(P_opt + Q_return(1,i+1) - m);
Q_return(1,i+1)= (1-Q_t_cdf(2,i+1))*(Q_t_cdf(1,i+1)- ECL);
end

[max_return,index]=max(Q_return);
Q_opt=Q_t_cdf(1,index);
end
```

CURRICULUM VITAE

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PROFESSIONAL EXPERIENCE

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