GEOMETRIC TRACKING CONTROL OF A FULLY ACTUATED RIGID BODY AND APPLICATION TO ATTITUDE CONTROL OF A QUADROTOR UAV

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GEOMETRIC TRACKING CONTROL OF A FULLY ACTUATED RIGID BODY AND APPLICATION TO ATTITUDE CONTROL OF A QUADROTOR UAV

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ABSTRACT

GEOMETRIC TRACKING CONTROL OF A FULLY ACTUATED RIGID BODY AND APPLICATIN TO ATTITUDE CONTROL OF A QUADROTOR UAV

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This work is the study of tracking control of rigid body in a general way using a geometric approach. To achieve globally valid characteristics, it is necessary to study such a control problem in its own natural nonlinear space using differential geometric properties of the space. By linking the tracking control problem to the problem of stabilization of a single equilibrium of an error dynamics, a tracking controller in the general case of a compact Lie groups has been developed. Then, using Lassale invariance theorem convergence to one of the equilibrium points of error dynamics has been established. Behavior of the system around its equilibrium points is studied by linearizing the system, which proved the almost-global attractiveness of the desired equilibrium. The general control problem studied, in its special case of space of rotation matrices is applied to attitude control of a Quadrotor UAV. Performance of the controller is demonstrated through numerical simulations.

Keywords: Geometric Control, Tracking Control, Quadrotor

ÖZ

TAM TAHRİKLİ KATI BİR CİSMİN GEOMETRİK TAKİP KONTROLÜ VE DÖRT ROTORLU İHA NIN AÇISAL KONUM KONTTROLÜ İÇİN UYGULANMASI

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Bu çalışma katı bir cismin geometrik yaklaşım ile takip kontrolünü içermektedir. Bir kontrol probleminde elde edilen sonuçların tüm hareket uzayında geçerli olabilmesi için, problemin kendi doğal doğrusal olmayan uzayında, bu uzayın geometrik diferansiyel özelliklerinin kullanılarak incelenmesi gerekmektedir. Takip kontrol problemini hata dinamiğinin tek bir denge noktasında kararlı hale getirilme problemine bağlayarak kompakt Lie gruplarının genel bir durumu için bir takip kontrolcü tasarlanmıştır. Daha sonra Lassale değişimsizlik teoremi kullanılarak hata dinamiğinin denge noktalarından bir tanesine yakınsadığı gösterilmiştir. Sistemin denge noktaları etrafındaki davranışı doğrusallaştırma yöntemi ile incelenerek istenen denge noktasının neredeyse tüm uzayda çekici olduğu kanıtlanmıştır. Genel olarak incelenen kontrol problemi, özel bir durumu olan dönüş matrisleri uzayında kullanılarak dört rotorlu bir İHA'nın açısal konum kontrolüne uygulanmıştır. Kontrolcünün başarısı sayısal simülasyonlar ile gösterilmiştir.

Anahtar Kelimeler: Geometrik kontrol, Takip kontrol, Quadrotor

In memory of my mother

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I would like to express my gratitude to my supervisor Asst. Prof. Dr. Ali Turker Kutay for his inspiring ideas. I am also thankful to Prof. Dr. Yildiray Ozan for his invaluable comments on this work.

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$\star$	: Binary operation of group (pg.10)
$[.,.]$	: Lie bracket of vector fields or matrix commutator (pg.8)
$\langle .,. \rangle$	: Vector, covector pairing (pg.6)
$\langle\langle .,. \rangle\rangle$	: Standard inner product of $\mathbb{R}^3$
$\lvert M \rvert$	: Determinant of matrix M
$\lVert v \rVert$	: Norm of vector v
$[A]$	: matrix representative of A, a vector, linear transformation or tensor
$\hat{}$	: Isomorphism between $\mathbb{R}^3$ and $\mathfrak{so}(3)$ (pg.28)
$.^{\vee}$	: Inverse of $\hat{}$
$\nabla_X Y$	: Covariant derivative of Y with respect to X (pg.9)
$\overset{\mathbb{G}}{\nabla}$	: Levi-Chivita or Reimannian connection associated with metric $\mathbb{G}$ (pg.9)
$\overset{\mathfrak{g}}{\nabla}$	: Binary operation of Lie algebra that generates Covariant derivative on Lie group by left translation (pg.11)
$\mathbb{1}$	: Identity element of group (pg.10)
ad	: Adjoint operator of a Lie algebra (pg.11)
Ad	: Adjoint map of a Lie algebra (pg.10)
$C^{\infty}(M)$	: Set of infinit time differentiable real valued functions on M
e	: Tracking error (pg.20)
E	: Total energy of a mechanical system (pg.17)
exp	: exponential map of a Lie group (pg.11)
G	: Lie group (pg.10)
$\mathfrak{g}$	: Lie algebra of Lie group G (pg.10)
$\mathbb{G}$	: Metric tensor (pg.9)
$\mathbb{G}_{\mathbb{I}}$	: Metric generated by inner product $\mathbb{I}$ (pg.11)
$grad$	: Gradient operator (pg.10)
$Hess$	: Hessian operator (pg.22)
$\mathbb{I}$	: Inner product of Lie algebra
$I_{n\times n}$	: $n\times n$ identity matrix
J	: Inertia tensor
J_x	: Jacobin at point x

KE	: Kinetic energy of a mechanical system
$\mathscr{L}_X$	: Lie derivative with respect to X (pg.8)
L	: Control moment in x direction
L_g	: Left translation action of group by g (pg.10)
M	: Control moment in y direction
N	: Control moment in z direction
$\boldsymbol{\Omega}$	: Angular velocity vector
$\varphi^X(x, t)$	: Flow of vector field X (pg.8)
Ψ	: Configuration error function (pg.23)
q_0	: Real part of a quaternion
$\boldsymbol{q}$	: Imaginary part of a quaternion
R_g	: Right translation action of group by g (pg.10)
$SE(3)$	: Special Euclidean group
$skew$	: Skew-symmetric component of a matrix
$SO(3)$	: Special orthogonal group
sym	: Symmetric component of a matrix
Tr	: Trace operator
$\mathfrak{X}(M)$	: Set of all smooth vector fields on a manifold M (pg.8)

CHAPTER 1

INTRODUCTION

1.1 Overview

The problem of attitude stabilization and tracking of a rigid body has numerous engineering applications such as robot control, satellite orientation control and UAV. The highly nonlinear nature of problem makes the task of analysis and controller design, a challenging one. Unlike a classic control problem which the state-space of problem always considered to be a Euclidean space, the attitude of a rigid body evolves in the space of 3×3 orthogonal matrices with unit determinant (Special Orthogonal group or $SO(3)$). A parametrization of attitude as a mapping of a subset of rotation matrices to Euclidean space used to bring the equations of system in a form that classical control methods can be applied. As it mentioned these parametrization only valid in a subset of possible attitudes which is the source of singularities in attitude representation and it is proven that there exist no singularity free parametrization of attitude with three parameters [13]. These singularities obviously limit the maneuverability of system but they also lead to a bigger problem. The problem of stabilization of a dynamical system ideally would be, to manipulate the equations of system in a way that one of the equilibrium's of system becomes globally attractive which means that starting from any arbitrary initial conditions, the system returns back to the that equilibrium. Because the parametrization of attitude is only valid in a neighborhood of equilibrium point, it is not possible to globally stabilize attitude using a parametrization. Other than that, it is very often that computing the exact region of attraction is very difficult and estimation methods gives only a conservative estimate of it. Result is that using a parametrization like rotation angels, it is very difficult to analyze and understand the behavior of system for big attitude maneuvers. The problem becomes more complicated when we try to follow a trajectory using a tracking controller where the constraints imposed by limitations coming from parametrization make the task of controller design very difficult.

To resolve the limitations of attitude parametrization when arbitrary maneuvers are desired, it is common to use unit quaternions to represent the attitude. Unit quaternions are different from the attitude parametrization in the sense that the space of unit quaternions isn't a Euclidean space. In fact unit quaternions live on the four dimensional unit sphere which is a nonlinear space itself. The reason of popularity of quaternions is that every single attitude can be represented by quaternions without singularity while they impose only one algebraic constraint on differential equations of system (the unit norm), compare to six algebraic constraints one has to deal with using the rotation matrices, which is a very desirable property during numerical implementation in digital computer. The price to pay for this simplification is that the correspondence between quaternions and rotations is not one-one [13]. Actually there exist two quaternions for each rotation and that becomes the source of another problem called unwinding which is caused by motion of system from one equilibrium quaternion to its negative (see [3] and [14]).

1.2 Purpose of the thesis

The objective of this work is to study the general problem of tracking control for a system that evolves on a Lie group assuming that the dynamics of system is fully actuated and to gain an understanding of global behavior of equilibrium points of closed loop system and finally apply the results to attitude tracking control of a Quadrotor UAV

1.3 Literature survey

The question of representation of attitude is studied in [13]. An early study of tracking control for fully-actuated systems is [4] where the gradient vector fields, used to achieve an almost-global control rule. Also the notion of navigation function introduced and a navigation function for space of rotation matrices studied, and [20] contains a more comprehensive study of the same ideas using a modern notation. In [14] the rotation matrices are used to design attitude controller for rotational dynamic of rigid body and [3] contains a general study of global properties and shortcomings of such geometric controllers and it is proven that global stabilization of rigid body rotational dynamics with continuous feedback is impossible. In [2] the general tracking control problem on Lie groups is studied.

There exists a vast literate on Quadrotors in which a lot of control methods have been applied using rotation angles or quaternions as parametrization ([16],[18],[23] for example) but recently the geometric methods have been gaining more attention. In [15] and [17] the geometric approach is used to control complex maneuvers of Quadrotor.

1.4 Contributions of the Thesis

In this thesis based on the work done in [2], a tracking controller for a general class of fully-actuated mechanical systems for which configuration manifolds is a Lie group, is designed. The results are applied to rotational dynamics of a Quadrotor UAV and using the mathematical model of a commercial Quadrotor the performance of controller is demonstrated by simulations.

1.5 Contents of the Thesis

We start with a very concise review of mathematical background necessary for our development in chapter two. We also introduce in a very compact way the basic ideas of geometric mechanics which is also an essential tool to understand the theory of mechanical control systems.

The space of rotation matrices is a Lie group, an abstract mathematical object which has a very rich and well developed theory behind it. There are other control problems evolve on Lie groups, for example the problem of control of six degree of freedom rigid body evolves on a Lie group called $SE(3)$. In chapter three the tracking problem on a Lie group considered in its general setting. The tracking problem linked to the stabilization of a single equilibrium of a tracking error dynamics which is proved to evolve on the same Lie group. Based on this transformation a controller proposed and by an application of Lassale Theorem, the asymptotic stability of closed loop dynamics established. The error dynamic of a system on $SO(3)$ used for numerical simulations as an example of theoretical material of this chapter.

Finally in chapter four the controller computed in the special case space of rotation matrices which is applicable to rotational dynamic of a Quadrotor. The mathematical model of Quadrotor constructed and the tracking controller applied to that and the numerical simulation in SIMULINK environment used to study this model. The appendix contains some additional material on parametrization of attitude and the unwinding phenomenon.

CHAPTER 2

MATHEMATICAL PREREQUITS AND MECHANICS

The necessary tool for our later development in this work is the theory of differentiable manifolds. Its a wast mathematical theory started in 19^{th} century and still under development as it linked to other branches of mathematics while what we need here for our purposes is just a tiny fraction of it. These few pages of mathematics here is an effort to establish a proper notation for the rest of the work specially that the notation used by different authors sometimes can be very different and that might cause some confusion. In the next step we will use the mathematical ideas presented, to express governing equations of mechanics. We will introduce only the basic ideas in a concise and compact way, based on our needs.
Obviously our crash course here is far from being complete and sometimes even doesn't have the precision of a mathematics text, so the complete treatment of subject must be followed from excellent textbooks which some of them are listed in references ([1],[5],[7],[8],[9],[12],[21]).

2.1 Differentiable Manifolds

A topological manifold *M* is a topological space such that:

i. Each two pints have disjoint neighborhoods (Hausdorff).
ii. Has a countable basis (second countable).
iii. Locally Euclidean.

For each point on $p \in M$ there exist a homeomorphism $\varphi : u \to v$ where u is open in M with $p \in u$ and v is an open subset of $\mathbb{R}^n$. We call the pair (φ, u) a local coordinate system at p and (φ^{-1}, v) is called a local parametrization or chart of *M*.

Two charts (φ, u) and (φ', u') are called $C^{\infty}-$compatiable if the maps $\varphi' \circ \varphi^{-1}$ and $\varphi \circ \varphi'^{-1}$ are smooth (infinitely many time differentiable) maps.

A topological manifold *M* which admits a collection of charts $(\varphi_{\alpha}, u_{\alpha})$ which:

i. $M = \bigcup_{\alpha} u_{\alpha}$

ii. The charts $(\varphi_{\alpha}, u_{\alpha})$ and $(\varphi_{\beta}, u_{\beta})$ are $C^{\infty}-$compatiable for every α, β

called a differentiable (smooth) manifold and the collection of charts $(\varphi_{\alpha}, u_{\alpha})$ called an Atlas on *M*.

Take a map $F: N \rightarrow M$ between two manifolds. Rank of *F* at $p \in N$ is the rank of Jacobian of *F* at $F(p) \in M$ and *F* is called an immersion if its rank at every point equals to dimension of *N*. Plus if *F* be an injective map (each element of codomain corresponds to no more than one element of domain) then the image of *N* on *M*, *F*(*N*) called an immersed submanifold of *M*. If *F* has a continues inverse then its called an embedding.

Let $C^{\infty}(p)$ be the set of all smooth functions defined in a neighborhood of $p \in M$. We define the tangent space to *M* at *p* denoted by $T_p M$ to be the set of all mappings $X_p : C^{\infty}(p) \rightarrow \mathbb{R}$ satisfying two following properties:

i. $X_p[\alpha f + \beta g] = \alpha X_p[f] + \beta X_p[g]$

ii. $X_p[fg] = X_p[f] g[p] + f[p] X_p[g]$

This forms a vector space and elements of this vector space called tangent vectors to *M* at *p*. Also the dual of this vector space denoted by $T_p^* M$ called cotangent space of *M* at *p* and its elements are called cotangent vectors or covectors. For a chart (φ, u) on *M* and $\varphi(p) = (x^1, x^2, \ldots, x^n)$ the standard basis for tangent and cotangent spaces at *p* denoted by $\left\{ \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \ldots, \frac{\partial}{\partial x^n} \right\}$ and $\{dx^1, dx^2, \ldots, dx^n\}$ respectively.

A useful interpretation of a tangent vector can be done using the notion of tangent to a curve on manifold. Let $C(t): [a, b] \rightarrow M$ be a smooth curve and $f: M \rightarrow \mathbb{R}$ be a smooth function on *M*, then $C'(t_0) = C'_{t_0}$ becomes a tangent vector to *M* at $C(t_0)$ and derivative of *f* along C'_{t_0} defined as follows:

$$C'_{t_0}[f]=\left.\frac{d}{dt}\right|_{t=t_0}=\lim_{t\to t_0}\frac{f(C(t))-f(C(t_0))}{t-t_0}$$

We denote the vector-covector pairing by $\langle u,\alpha\rangle$ for $u\in T_pM$, $\alpha\in T_p^*M$ which is a real number. Using vector-covector pairing we can have another interpretation of derivative of a real-valued function *f*. The differential of *f* is a covector denoted by *df*. Now we can write derivative of *f* along tangent vector X_p as following:

$$\langle df, X_p\rangle=X_p[f]$$

Suppose $G:T_pM\times T_pM\to\mathbb{R}$ be a bilinear map. We define the flat map $G^\flat:T_pM\to T_p^*M$ such that $G(u,v)=\langle G^\flat(u),v\rangle$. Also the inverse of $G^\flat$ denoted by $G^\sharp:T_p^*M\to T_pM$ called the sharp map. If *G* has a matrix representation denoted by [*G*] then $G^\flat$ will have exactly the same matrix representation.

The union of all tangent (cotangent) spaces of *M* denoted by TM (T^*M) called the tangent bundle (cotangent bundle) and we have the projection map $\pi:TM\to M$, $\pi(x,v)=x$. Tangent and cotangent bundle themselves posses the structure of a differentiable manifold. Also if $(x^1,x^2,\ldots,x^n)$ be a coordinate system on *M* then $(x^1,x^2,\ldots,x^n,\dot{x}^1,\dot{x}^2,\ldots,\dot{x}^n)$ is a coordinate system on TM .

An (*r,s*) tensor on a manifold *M* at *p* is a multilinear map such that:

$$t_p:\underbrace{T_p^*M\times\ldots\times T_p^*M}_{r}\times\underbrace{T_pM\times\ldots\times T_pM}_{s}\to\mathbb{R}$$

Let $F:M\to N$ be a smooth map, then *F* induces a linear map $F_{*p}:T_pM\to T_{F(p)}N$ at $p\in M$ between tangent spaces of *M* and *N* called the differential of *F.* If $X_p\in T_pM$ be a tangent vector of *M,* then $F_{*p}(X_p)$ called the pushforward of X_p becomes a tangent vector of *N* at point *F*(*p*) and for a real valued function $f\in C^\infty(N)$ we have the following equation:

$$F_{*p}(X_p)[f]=X_p[f\circ F]$$

If $C(t):[a,b]\to M$ be a smooth curve on *M* starting at $C(0)=p$ with tangent vector $C'(0)=X_p$ then we have:

$$F_{*p}(X_p)=\left.\frac{d}{dt}\right|_{t=0}(F\circ C)(t)$$

Now let $Y_{F(p)} \in T_{F(p)} N$, then the map $F_p^*: T_{F(p)} N \to T_p M$ defines pullback of Y:

$$(F^* Y)_p = (F^{-1})_{*F(p)} Y_{F(p)}$$

Also for a $(0,k)$ tensor t on N, its pullback under F defined as below:

$$(F^* t)_p (X_p, \ldots, Y_p) = t_{F(p)} (F_{*p} X_p, \ldots, F_{*p} Y_p)$$

A vector field is an assignment of a tangent vector to each point of a manifold. A smooth vector field X, is the one takes the following form in every coordinates:

$$X = X^1 \frac{\partial}{\partial x^1} + X^2 \frac{\partial}{\partial x^2} + \ldots + X^n \frac{\partial}{\partial x^n}$$

where $X^1, \ldots, X^n$ are smooth functions of $p \in M$ at each point. The set of all smooth vector fields on a manifold M denoted by $\mathfrak{X}(M)$. Each vector field X associates to a system of n first order differential equations on M which their solutions are called integral curves of X. We define a function $\varphi : M \times \mathbb{R} \to M$ such that $\varphi^X(x,t) = \varphi_t^X(x) = \varphi_x^X(t)$ be the integral curve of X starting at $x \in M$ at time t and call it the flow of vector field X (we omit the superscript X when there is no chance of confusion). The flow of a vector field has the following fundamental properties:

i. $\varphi(x,0) = x$

ii. $\varphi(x, t+s) = \varphi_{t+s}(x) = \varphi(\varphi(x,t), s) = \varphi(\varphi_t(x), s) = \varphi_s(\varphi_t(x)) = \varphi_s \circ \varphi_t(x)$

Let f be a real valued function on a smooth manifold M. The derivative of f along integral curves of a vector field X, called the Lie derivative f with respect to X:

$$(\mathscr{L}_X f)(x) = \lim_{t \to 0} \frac{f \circ \varphi_t - f}{t}(x) = X_x[f]$$

Note that $X_x[f]$ means the derivative of f along X at x. For a curve $\gamma(t): \mathbb{R} \to M$, its time derivative $\gamma'(t)$ is a vector field on $\gamma(t)$ and we have:

$$\left. \frac{d}{dt} \right|_{t=t_0} f(\gamma(t)) = \mathscr{L}_{\gamma'(t_0)} f$$

Using the definition of Lie derivative for real valued functions we can define Lie derivative of a vector field. If X and Y are two vector fields on a manifold M, the Lie derivative of Y with respect to X at $x \in M$ is:

$$(\mathscr{L}_X Y)(f) = \lim_{t \to 0} \frac{(Yf) \circ \varphi_t - Yf}{t}(x) \quad , \quad p \in M$$

Where $Yf(x) = (\mathscr{L}_Y f)(x)$. Lie derivative of two vector fields is another vector field itself on M denoted by $[X,Y]$ called Lie bracket of X,Y. Let $X, Y \in \mathfrak{X}(M)$ and $f, g \in C^\infty(M)$ then the following holds:

i. $[X, Y] = -[Y, X]$

ii. $[X+Y, Z] = [X, Z] + [Y, Z]$

iii. $[fX, gY] = fg[X, Y] + f(\mathscr{L}_X g)Y - g(\mathscr{L}_Y f)X$

iv. $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

Let M be a smooth manifold. An affine connection on M assigns to each vector field X of M an operator $\nabla_X : \mathfrak{X}(M) \to \mathfrak{X}(M)$ with following properties:

i. $\nabla_X(aY + bZ) = a\nabla_X Y + b\nabla_X Z$

ii. $\nabla_X(fY) = f\nabla_X Y + (Xf)Y$

iii. $\nabla_{aX+bY} Z = a\nabla_X Z + b\nabla_Y Z$

iv. $\nabla_{fX}(Y) = f\nabla_X Y$

where $\nabla_X Y$ called covariant derivative of Y with respect to X.

For a curve $\gamma(t) : \mathbb{R} \to M$ the integral curves of following differential equation called the geodesics of affine connection ∇:

$$\nabla_{\gamma'(t)} \gamma'(t) = 0$$

An (r,s) tensor field on M is an assignment to each point $p \in M$ an (r,s) tensor. A Riemannian metric $\mathbb{G}$ on manifold M is a (0,2) tensor field on M which is positive-definite and symmetric which means that for $X_p, Y_p \in T_p M$ we have:

i. $\mathbb{G}(X_p, Y_p) > 0$

ii. $\mathbb{G}(X_p, Y_p) = \mathbb{G}(Y_p, X_p)$

Manifold M endowed with a Reimannian metric called a Reimannian manifold. Such a manifold has the property that the metric is an inner product on tangent space.

It's proven that for any Reimannian manifold $(M, \mathbb{G})$ there exist a unique connection called Reimannian or Levi-Civita connection $\overset{\mathbb{G}}{\nabla}$ with following properties:

i. $\overset{\mathbb{G}}{\nabla} \mathbb{G} = 0$

ii. $[X, Y]=\overset{\mathbb{G}}{\nabla}_X Y-\overset{\mathbb{G}}{\nabla}_Y X$

Using the notion of a metric it is possible to define the gradient of a real valued function. Let $(M, \mathbb{G})$ be a Reimannian manifold and $f \in C^{\infty}(M)$ then gradient of f is a vector field defined as:

$$grad\ f=\mathbb{G}^{\sharp}(df)$$

2.2 Lie Groups and Lie algebras

A set *G,* which is closed with a binary operation $\star$ is a group if:

i. $a \star (b \star c)=(a \star b) \star c$ for $a, b, c \in G$

ii. There exist $\mathbb{1} \in G$ called Identity element of group, such that $a \star \mathbb{1}=\mathbb{1} \star a=a$ for $a \in G$

iii. There exist a $a^{-1} \in G$ for every $a \in G$ called the inverse of a where $a \star a^{-1}=a^{-1} \star a=\mathbb{1}$

A Lie Group is a group which is also a smooth manifold at the same time where group operation and inversion are smooth maps.

Tangent space at Identity element $T_{\mathbb{1}} G$ of a Lie group *G*, called Lie Algebra of *G* and denoted by $\mathfrak{g}$. An important example of Lie groups is the space of orthogonal matrices with unit determinant with matrix multiplication as group operation denoted by *SO*(3) and its Lie algebra which is the space of skew-symmetric matrices, denoted by $\mathfrak{so}(3)$.

Let $(G, \star)$ be a Lie group and $\mathfrak{g}$ be its Lie algebra. For $g, h \in G$ and $\zeta, \eta \in \mathfrak{g}$ we have the following definitions:

i. For every group element the maps $R_g, L_g: G \rightarrow G$ where $L_g(h)=g \star h$ and $R_g(h)=h \star g$ are called left and right translations respectively.

ii. The Conjugation map $I_g: G \rightarrow G$ defined as $I_g(h)=g \star h \star g^{-1}$.

iii. The map $Ad_g: \mathfrak{g} \rightarrow \mathfrak{g}$ defined as $Ad_g=(I_g)_{*\mathbb{1}}$ called the adjoint map.

iv. A smooth vector field X on G called left-invariant (right-invariant) if it is L_g-related $(R_g-\text{related})$ to itself which means that $L_g^{*} X=X \quad (R_g^{*} X=X)$.

v. The vector fields $\zeta_L=(L_g)_{*\mathbb{1}}\zeta$ and $\zeta_R=(R_g)_{*\mathbb{1}}\zeta$ are called left and right-invariant vector fields generated by $\zeta\in\mathfrak{g}$.

vi. We define the adjoint operator $ad_\zeta:\mathfrak{g}\to\mathfrak{g}$ as $ad_\zeta\eta=[\zeta_L,\eta_L](\mathbb{1})$.

vii. The exponential map $\exp:\mathfrak{g}\to G$ defined by $\exp(\zeta)=\varphi_1^{\zeta_L}(\mathbb{1})$, which means the flow generated by ζ_L at $t=1$ and identity.

viii. An affine connection ∇ on G is left-invariant if for every left-invariant vector fields X and Y, $\nabla_X Y$ be a left invariant vector field.

A Lie algebra is a vector space V with a binary operation called bracket which following properties hold for every $u,v,w\in V$:

i. $[u,v]=-[v,u]$

ii. $[u,[v,w]]+[v,[w,u]]+[w,[u,v]]=0$

The set of left-invariant vector fields of a Lie group G form a Lie algebra which Lie algebra bracket is the Lie bracket defined on vector fields. It is proven that this Lie algebra is isomorph to the Lie algebra of G defined above and we have :

$$[\zeta,\eta]=ad_\zeta\eta=[\zeta_L,\eta_L](\mathbb{1})$$

For a Lie group G and its Lie algebra $\mathfrak{g}$ the exponential map has the following properties :

i. For $\zeta\in\mathfrak{g}$ we have: $\exp(t\zeta)=\varphi_t^{\zeta_L}(\mathbb{1})$.

ii. $L_g\circ\exp(t\zeta)=\varphi_t^{\zeta_L}(g)$ and $R_g\circ\exp(t\zeta)=\varphi_t^{\zeta_R}(g)$ for $g\in G$ and $\zeta\in\mathfrak{g}$.

Every inner product $\mathbb{I}$ of Lie algebra induces a left-invariant Reimannian metric on Lie group G denoted by $\mathbb{G}_{\mathbb{I}}$. Also there exist a bilinear map $\overset{\mathfrak{g}}{\nabla}:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ such that $\overset{\mathbb{G}}{\nabla}_{\zeta_L}\eta_L=\left(\overset{\mathfrak{g}}{\nabla}_\zeta\eta\right)_L$ and the following equation holds:

$$\overset{\mathfrak{g}}{\nabla}_\zeta\eta=\frac{1}{2}[\zeta,\eta]-\frac{1}{2}\mathbb{I}^\sharp\left(ad_\zeta^*\mathbb{I}^\flat(\eta)+ad_\eta^*\mathbb{I}^\flat(\zeta)\right)$$

Now we present relations we will use latter in our development:

Lemma 2.1. ([12] pg.175) Let $(G,\star)$ be a Lie group. We define two maps $\mu:G\times G\to G$ and $i:G\to G$ called multiplication and inversion maps respectively, such that: $\mu(g,h)=g\star h$, $i(g)=g^{-1}$. Then the following relations hold for their tangent maps:

$$\mu_{*(g,h)}(X_g, Y_h) = (R_h)_{*g} X_g + (L_g)_{*h} Y_h$$

$$i_{*g} X = -(R_{g^{-1}})_{*\mathbb{1}} (L_{g^{-1}})_{*g} X$$

Proof: Take a curve $(C_1(t), C_2(t)) = C: I \to G \times G$ which $(C_1(0), C_2(0)) = (g, h)$ and $(C_1'(0), C_2'(0)) = (X_g, 0)$. Let $f \in C^{\infty}(G)$ be an arbitrary real valued function. Then for the tangent map $\mu_{*(g,h)}: T_g G \times T_h G \to T_{g \star h} G$ we have:

$$\mu_{*(g,h)}(X_g, 0)[f] = \left.\frac{d}{dt}\right|_{t=0} (f \circ \mu \circ C)(t)$$

$$= \lim_{t \to 0} \frac{(f \circ \mu \circ C)(t) - (f \circ \mu \circ C)(0)}{t}$$

$$= \lim_{t \to 0} \frac{f\left(C_1(t) \star C_2(t)\right) - f(g \star h)}{t}$$

$$= \lim_{t \to 0} \frac{f\left(C_1(t) \star h\right) - f(g \star h)}{t}$$

$$= \lim_{t \to 0} \frac{f\left(R_h(C_1(t))\right) - f(R_h(g))}{t}$$

$$= \left.\frac{d}{dt}\right|_{t=0} (f \circ R_h \circ C_1)(t)$$

$$= (R_h)_{*g} X_g[f]$$

For another curve $C: I \to G \times G$ with $C(0) = (g, h)$ and $C'(0) = (0, Y_h)$ we can apply the same procedure to get:

$$\mu_{*(g,h)}(0, Y_h) = (L_g)_{*h} Y_h$$

If we add these two together from linearity of tangent map we have the result:

$$\mu_{*(g,h)}(X_g, Y_h) = (R_h)_{*g} X_g + (L_g)_{*h} Y_h$$

For second part we define the curve $C: I \to G$ which $C(0) = g$ and $C'(0) = X_g \in T_g G$, we have:

$$\mu((i \circ C), C) = \mathbb{1} \quad \Rightarrow \quad \left.\frac{d}{dt}\right|_{t=0} \mu((i \circ C), C) = 0$$

$$\Rightarrow \quad \mu_{*(g^{-1}, g)}\left(\left.\frac{d}{dt}\right|_{t=0} (i \circ C)(t), \left.\frac{d}{dt}\right|_{t=0} C(t) \right) = 0$$

$$\Rightarrow \quad \mu_{*(g^{-1}, g)}\left(i_{*g} X, X\right) = 0$$

$$\Rightarrow \quad \left(R_g\right)_{*g^{-1}} i_{*g} X + \left(L_{g^{-1}}\right)_{*g} X = 0$$

$$\Rightarrow \quad \left(R_{g^{-1}}\right)_{*\mathbb{1}} \left(R_g\right)_{*g^{-1}} i_{*g} X = -\left(R_{g^{-1}}\right)_{*\mathbb{1}} \left(L_{g^{-1}}\right)_{*g} X$$

$$\Rightarrow \quad \left(R_{g^{-1}} \circ R_g\right)_{*g^{-1}} i_{*g} X = -\left(R_{g^{-1}}\right)_{*\mathbb{1}} \left(L_{g^{-1}}\right)_{*g} X$$

Thus we have the following for differential of inversion map:

$$i_{*g} X = -(R_{g^{-1}})_{*\mathbb{1}} (L_{g^{-1}})_{*g} X \qquad \blacksquare$$

Lemma 2.2. ([1] pg.307)Let G be a Lie group with $\eta, \zeta \in \mathfrak{g}$ two vectors in its Lie algebra. Take two curves $\gamma : I \to G$ and $v : I \to \mathfrak{g}$ on G and its Lie algebra respectively where $\gamma'(t) = \left(L_{\gamma(t)}\right)_{*\mathbb{1}} v(t)$ then the following equations are true:

i. $\left.\dfrac{d}{dt}\right|_{t=0} Ad_{\exp(\zeta t)} \eta = ad_{\zeta} \eta$

ii. $\dfrac{d}{dt} \gamma^{-1}(t) = -\left(R_{\gamma^{-1}(t)}\right)_{*\mathbb{1}} v(t)$

iii. $\dfrac{d}{dt} Ad_{\gamma(t)} \eta = Ad_{\gamma(t)} [v(t), \eta]$

Proof: Take the flow of left-invariant vector field generated by ζ, then we have $\varphi_t^{\zeta_L}(g) = L_g\left(\exp(\zeta t)\right)$. From properties of flow and exponential map we have:

$$\varphi_t^{\zeta_L}(g)^{-1} \circ \varphi_t^{\zeta_L}(g) = g \;\; \Rightarrow \;\; \varphi_t^{\zeta_L}(g)^{-1}\left(L_g \circ \exp(\zeta t)\right) = g \;\; \Rightarrow \;\; \varphi_t^{\zeta_L}(g)^{-1} = R_{\exp(-\zeta t)}$$

now we can apply the definition of adjoint operator and Lie derivative:

$$ad_{\zeta} \eta = [\zeta_L, \eta_L](\mathbb{1}) = \left(\mathscr{L}_{\zeta_L} \eta_L\right)(\mathbb{1}) = \lim_{t \to 0} \frac{\left(\varphi_t^{\zeta_L}\right)^* \eta_L - \eta_L}{t}(\mathbb{1})$$

$$= \lim_{t \to 0} \frac{\left[\left(\varphi_t^{\zeta_L}\right)^* \eta_L\right]_{\mathbb{1}} - \eta}{t}$$

$$= \lim_{t \to 0} \frac{\left(\varphi_t^{\zeta_L}\right)^{-1}_{*\exp(\zeta t)} (\eta_L)_{\exp(\zeta t)} - \eta}{t}$$

$$= \lim_{t \to 0} \frac{\left(R_{\exp(-\zeta t)}\right)^{-1}_{*\exp(\zeta t)} (\eta_L)_{\exp(\zeta t)} - \eta}{t}$$

$$= \lim_{t \to 0} \frac{\left(R_{\exp(-\zeta t)}\right)^{-1}_{*\exp(\zeta t)} \left(L_{\exp(\zeta t)}\right)_{*\mathbb{1}} \eta - \eta}{t}$$

$$= \lim_{t \to 0} \frac{\left[R_{\exp(-\zeta t)} L_{\exp(\zeta t)} \right]_{*\mathbb{1}} \eta - \eta}{t}$$

$$= \lim_{t \to 0} \frac{Ad_{\exp(\zeta t)} \eta - \eta}{t} = \left. \frac{d}{dt} \right|_{t=0} Ad_{\exp(\zeta t)} \eta$$

To prove (ii , iii) we can apply lemma 1.1:

$$\begin{aligned}
\frac{d}{dt} \gamma^{-1}(t) &= \frac{d}{dt} i(\gamma(t)) \\
&= -\left(R_{\gamma^{-1}(t)}\right)_{*\mathbb{1}} \left(L_{\gamma^{-1}(t)}\right)_{*\gamma(t)} \gamma'(t) \\
&= -\left(R_{\gamma^{-1}(t)}\right)_{*\mathbb{1}} \left(L_{\gamma^{-1}(t)}\right)_{*\gamma(t)} \left(L_{\gamma(t)}\right)_{*\mathbb{1}} v(t) \\
&= -\left(R_{\gamma^{-1}(t)}\right)_{*\mathbb{1}} \left(L_{\gamma^{-1}(t)} \circ L_{\gamma(t)}\right)_{*\mathbb{1}} v(t) \\
&= -\left(R_{\gamma^{-1}(t)}\right)_{*\mathbb{1}} v(t)
\end{aligned}$$

and finally for (iii) :

$$\begin{aligned}
\left. \frac{d}{dt} \right|_{t=t_0} \gamma^{-1}(t_0) \gamma(t) &= \mu_{*\left(\gamma^{-1}(t_0),\, \gamma(t_0)\right)} (0, \gamma(t_0)) \\
&= (L_{\gamma^{-1}(t_0)})_{*\gamma(t_0)} \gamma'(t_0) \\
&= (L_{\gamma^{-1}(t_0)})_{*\gamma(t_0)} \left(L_{\gamma(t_0)}\right)_{*\mathbb{1}} v(t_0) \\
&= v(t_0) \\
&= \left. \frac{d}{dt} \right|_{t=0} \exp(t v(t_0))
\end{aligned}$$

Note that we used the following relation:

$$\left. \frac{d}{dt} \right|_{t=t_0} Ad_{\gamma(t)} \eta = \left. \frac{d}{dt} \right|_{t=t_0} Ad_{\gamma(t_0)} \left(Ad_{\gamma^{-1}(t_0) \gamma(t)} \eta \right) = Ad_{\gamma(t_0)} \left. \frac{d}{dt} \right|_{t=t_0} \left(Ad_{\gamma^{-1}(t_0) \gamma(t)} \eta \right)$$

knowing that $\gamma^{-1}(t_0) \gamma(t) \big|_{t=t_0} = \exp(t v(t_0)) \big|_{t=0} = \mathbb{1}$ and using part (i) result follows:

$$\left. \frac{d}{dt} \right|_{t=t_0} \gamma^{-1}(t_0) \gamma(t) \eta = \left. \frac{d}{dt} \right|_{t=0} \exp(t v(t_0)) \eta = ad_{v(t_0)} \eta$$

2.3 Simple Mechanical Systems

Historically the geometric mechanics dates back to the end of nineteenth century and work of Poincare on three-body problem where he decided that the classical framework for mechanics, based on theory of ordinary differential equations is not enough to answer some of the important questions like stability of solar system. His pioneering work then continued and perfected during 20^{th} century and took its current shape, what we call geometric mechanics.

We know that the every orientation of a single rigid body can be expressed as rotation of body-fixed frame with respect to a spetial frame which means that every possible orientation of a rigid body in space is an element of *SO*(3). In a problem that we only consider the orientation of rigid body, we say that our configuration manifold is *SO*(3). If other than orientation, position of body frame with respect to special frame is also important to us, every possible configuration of rigid body can be expressed as a combination of a 3-dimensional vector and the orientation. In the other words the configuration manifold of a rigid body with six degree of freedom would be $SO(3)\times\mathbb{R}^3$. Because $SO(3)$ and $\mathbb{R}^3$ are both smooth manifolds their products also will be a smooth manifold. There is a nice way to write the elements of this configuration manifold in matrix form. We define:

$$SE(3)=\left\{\begin{bmatrix} R & r \\ 0_{1\times 3} & 1 \end{bmatrix} \middle| R\in SO(3), r\in\mathbb{R}^3\right\}$$

called the Special Euclidean Group. It can be proved that *SE*(3) itself form a Lie group with matrix multiplication and its lie algebra is:

$$\mathfrak{se}(3)=\left\{\begin{bmatrix} \hat{\omega} & v \\ 0_{1\times 3} & 1 \end{bmatrix} \middle| \hat{\omega}\in\mathfrak{so}(3), v\in\mathbb{R}^3\right\}$$

In the two cases above the configuration manifolds were Lie groups. This is not true in general for every problem we study. Consider combination of *n* rigid bodies with their own body-fixed frames and a spetial frame which the orientation and spatial position of each rigid body is important to us. The configuration manifold of such a problem is the space of all possible configuration of these rigid bodies which is:

$$\left(SO(3)\times\mathbb{R}\right)^n$$

As product of smooth manifolds (Lie Groups) this configuration manifold is a smooth manifold (Lie Group). In real cases often happens that the permissible configurations are evolve not in whole of this configuration space but only in a subspace of it, which might not be a Lie group.

Consider the kinetic energy of a single rigid body with mass *m* and inertia tensor *J:*

$$KE=\frac{1}{2}\begin{bmatrix}\hat{\omega}^T & v^T\end{bmatrix}\begin{bmatrix}J & 0\\ 0 & mI_{3\times3}\end{bmatrix}\begin{bmatrix}\omega\\ v\end{bmatrix}=\frac{1}{2}\langle\langle\omega, J\omega\rangle\rangle+\frac{1}{2}m\langle\langle v, v\rangle\rangle$$

where $\langle\langle .,.\rangle\rangle$ is the standard inner product of $\mathbb{R}^3$. The matrix composed of inertia tensor and mass of rigid body in relation above is positive definite and defines an inner product on *TSE*(3) or a Reimannian metric on *SE*(3). This is true for general case, where the total kinetic energy of system (sum of kinetic energy of every single rigid body associated to the system) defines a Reimannian metric on the configuration manifold called the kinetic energy metric and we denote it by $\mathbb{G}$. In the other words if *Q* is the configuration manifold and $u\in TQ$, then we have:

$$KE(p)=\frac{1}{2}\mathbb{G}(u,u)$$

Now in our system of *n* rigid bodies consider that the *a-th* rigid body is subject to force and torques f_a and τ_a while it has linear and angular velocities v_a and ω_a . We define a covector called the Lagrangian force $F_a\in T^*Q$ that satisfies the following equation:

$$\langle F_a, u_a\rangle=\langle\langle f_a, v_a\rangle\rangle+\langle\langle\tau_a, \omega_a\rangle\rangle$$

Note that this Lagrangian force is a mathematical notion which is different from the force in its Newtonian meaning. Then total external force acting on a system of *n* rigid bodies defined as:

$$F=\sum_{i=1}^{n}F_a$$

Now consider a system of *n* rigid body with configuration manifold *Q,* the kinetic energy metric $\mathbb{G}$ and total external forces *F.* Let $V:Q\to\mathbb{R}$ be a potential function on *Q* giving rise to a potential force field *gard V.* Also take a coordinate system on *TQ* as $(q^1, q^2, \ldots, q^n, \dot{q}^1, \dot{q}^2, \ldots, \dot{q}^n)$ with $(F_1, F_2, \ldots, F_n)$ the coordinate expression of *F.* Define a real valued function $L:Q\to\mathbb{R}$ called Lagrangian as follows:

$$L(q, v_q) = KE(v_q) - V(q)$$

Such a system called a simple mechanical system and following Euler-Lagrange equation governs the motion of system:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = F_i$$

If $\gamma : I \to Q$ be a solution of his equation then $\gamma(t)$ also a solution to the following equation:

$$\overset{\mathbb{G}}{\nabla}_{\gamma'(t)} \gamma'(t) = -grad(V) + \mathbb{G}^{\sharp}(F)$$

For the special case of an unforced system with no external force and no potential field, the Lagrangian becomes the kinetic energy of system and solutions of unforced Euler-Lagrange equation are kinetic energy minimizers. In this case the geometric equivalent of Euler-Lagrange equation becomes:

$$\overset{\mathbb{G}}{\nabla}_{\gamma'(t)} \gamma'(t) = 0$$

Which is the differential equation of geodesics associated with Reimannian metric $\mathbb{G}$. This makes sense because geodesics are minimizers of length and in this case kinetic energy is a measure of length on configuration manifold.

Now take the following quantity called the total energy of system:

$$E = \frac{1}{2}\mathbb{G}(\gamma'(t), \gamma'(t)) + V(\gamma(t))$$

its time derivative proved to be:

$$\frac{dE}{dt} = \langle F, \gamma'(t) \rangle$$

A covector field F called a dissipative force if $\langle F, \gamma'(t) \rangle \leq 0$. For a simple mechanical system with dissipative forces the total energy is non increasing along the solutions curves of equations of system.

Consider a simple mechanical system with a configuration manifold which is a Lie group G. Let the kinetic energy metric $\mathbb{G}_{\mathbb{I}}$ be left-invariant (which is the case in most real situations). Then there exist an inner product $\mathbb{I}$ on Lie algebra $\mathfrak{g}$ that generates the metric by left translation. Let $\{f^i\}_{i=0,\dots,n}$ be a basis for $\mathfrak{g}^*$ and consider a set of scalar functions $\{u_i(t)\}_{i=1,\dots,n}$, then every control force at point $\gamma(t)$ can be written as

$$\sum_{i=1}^{n} u_i(t)\left(L_{\gamma^{-1}(t)}\right)_{*\gamma(t)} f^i$$

Now if $\gamma(t): I \rightarrow G$ be a solution to equation of system then the governing equations becomes:

$$\overset{\mathbb{G}}{\nabla}_{\gamma'(t)}\gamma'(t) = -grad(V) + \sum_{i=1}^{n} u_i(t)\mathbb{G}_{\mathbb{I}}^{\sharp}\left(\left(L_{\gamma(t)}\right)_{\mathbb{1}}^{*} f^i\right)$$

Also if $v(t): I \rightarrow \mathfrak{g}$ be a curve on $\mathfrak{g}$ then this equation can be expressed on Lie algebra as follows:

$$\gamma'(t) = \left(L_{\gamma(t)}\right)_{*\mathbb{1}} v(t)$$

$$v'(t) = \mathbb{I}^{\sharp}\left(ad_{v(t)}^{*}\mathbb{I}^{\flat}(v(t))\right) - \left(L_{\gamma(t)}\right)_{\mathbb{1}}^{*} grad(V) + \sum_{i=1}^{n} u_i(t)\mathbb{I}^{\sharp}\left(f^i\right)$$

when the configuration manifold of system is *SE*(3) or *SO*(3) then the $v(t)$ represents the velocities resolved in body-fixed frame.

Let *G* be an *n*-dimensional space which means that at each point and obviously in $\mathbb{1}$ its tangent space is isomorphic to $\mathbb{R}^n$. Equations of a simple mechanical system above are a set of *2n* first order differential equations and their solutions are two curves $\gamma(t) \in G$ and $v(t) \in \mathfrak{g}$. So the solution of equations of system at each point are the elements of the tangent bundle $\left(\gamma(t), \left(L_{\gamma(t)}\right)_{*\mathbb{1}} v(t)\right) \in TG$. In the other words the state space of mechanical control system above is *TG*.

CHAPTER 3

TRACKING CONTROL ON LIE GROUPS

In this chapter based on the theory developed in [2], we will consider the problem of trajectory tracking for a mechanical system whose configuration manifold is a Lie group. This general approach allows the results to be applicable to any control problem when the configuration manifold is a Lie group. Particularly problems of tracking control for three degree of freedom rigid body (orientation control) and six degree of freedom (orientation and position) are the same problems on two different Lie groups which our results can be applied.

3.1 Statement of Problem

Consider the following mechanical system on a Lie group $(G,\star)$ and its Lie algebra $\mathfrak{g}$:

$$\begin{aligned} \gamma'(t) &= \left(L_{\gamma(t)}\right)_{*\mathbb{1}} v(t) \\ v'(t) &= \mathbb{I}^{\sharp}\left(ad^{*}_{v(t)}\mathbb{I}^{\flat}(v(t))\right) - \left(L_{\gamma(t)}\right)^{*}_{\mathbb{1}} grad\, V + \sum_{i=1}^{n} u_i(t)\,\mathbb{I}^{\sharp}\left(f^i\right) \end{aligned} \tag{3.1}$$

where

i. $\gamma(t): I \to G$ is a curve on G representing the motion of system.

ii. $v(t): I \to \mathfrak{g}$ is another curve on $\mathfrak{g}$ which is related to $\gamma(t)$ by first equation. In case of rotational motion of rigid body, $v(t)$ represents the angular velocities resolved in body-fixed frame.

iii. $L_{\gamma(t)}$ represents the left translation action of the group.

iv. $\mathbb{I}$ is the inner product of $\mathfrak{g}$ that generates the kinetic energy metric $\mathbb{G}_{\mathbb{I}}$ on G. Every inner product on a vector space associates to a positive definite matrix, for example in case of 3-DOF rigid body the matrix representation of $\mathbb{I}$ is the inertia tensor.

v. $\{f_i\}$ is the set of n covectors forming a base for $\mathfrak{g}^*$ and the fact that n equals to dimension of G, guarantees that the control system is fully actuated and again in case of 3-DOF rigid body they associate to three unit vectors of body-fixed frame.

vi. $\{u_i\}$ is the set of controls which is a set of n scalar functions.

vii. V is the potential field like Earths gravity.

Consider an arbitrary curve $\gamma_d(t): I \to G$ and suppose there exist another curve $v_d(t): I \to \mathfrak{g}$ such that:

$$\gamma'_d(t) = \left(L_{\gamma_d(t)}\right)_{*\mathbb{1}} v_d(t) \tag{3.2}$$

Note that this relation put no constraint on the problem, since its a relation resulted from geometry of space. To make it more clear, its similar to say that we have a rigid body that its motion is fully controlled and it can perform arbitrary maneuvers while it respects the kinematic rules of motion.

We want our mechanical system described above, tracks down the motion of mechanical system whose trajectory is $\gamma_d(t)$. It means that starting from any initial conditions, $\gamma(t)$ converges to $\gamma_d(t)$ and the error between two trajectories remains non-increasing at any time.

3.2 Reformulation of problem and tracking error dynamics

We define a tracking error function as following relation:

$$e(t) = \gamma_d(t) \star \gamma^{-1}(t) \tag{3.3}$$

Notice that the group structure of G guarantees that always $e(t) \in G$. Now we use the differential of multiplication map in Lemma 2.1 to compute time derivative of e:

$$\left.\frac{d}{dt}\right|_{t=t_0} e(t) = \mu_{*\left(\gamma_d(t_0),\, \gamma^{-1}(t_0)\right)}\left(\left.\frac{d}{dt}\right|_{t=t_0} \gamma_d(t),\, \left.\frac{d}{dt}\right|_{t=t_0} \gamma^{-1}(t)\right)$$

$$= \left(R_{\gamma^{-1}(t_0)}\right)_{*\gamma_d(t_0)} \left.\frac{d}{dt}\right|_{t=t_0} \gamma_d(t) + \left(L_{\gamma_d(t_0)}\right)_{*\gamma^{-1}(t_0)} \left.\frac{d}{dt}\right|_{t=t_0} \gamma^{-1}(t)$$

Applying Lemma2.2 and from (3,1),(3,2) we have:

$$=\left(R_{\gamma^{-1}(t_0)}\right)_{*\gamma_d(t_0)}\left(L_{\gamma_d(t_0)}\right)_{*1} v_d(t_0)-\left(L_{\gamma_d(t_0)}\right)_{*\gamma^{-1}(t_0)}\left(R_{\gamma^{-1}(t_0)}\right)_{*1} v(t_0)$$

$$=\left(R_{\gamma^{-1}(t_0)}\circ L_{\gamma_d(t_0)}\right)_{*1} v_d(t_0)-\left(L_{\gamma_d(t_0)}\circ R_{\gamma^{-1}(t_0)}\right)_{*1} v(t_0)$$

$$=\left(L_{\gamma_d(t_0)}\right)_{*\gamma^{-1}(t_0)}\left(R_{\gamma^{-1}(t_0)}\right)_{*1}\left(v_d(t_0)-v(t_0)\right)$$

$$=\left(L_{\gamma_d(t_0)}\right)_{*\gamma^{-1}(t_0)}\left(L_{\gamma^{-1}(t_0)}\circ L_{\gamma(t_0)}\right)_{*\gamma^{-1}(t_0)}\left(R_{\gamma^{-1}(t_0)}\right)_{*1}\left(v_d(t_0)-v(t_0)\right)$$

$$=\left(L_{\gamma_d(t_0)}\right)_{*\gamma^{-1}(t_0)}\left(L_{\gamma^{-1}(t_0)}\right)_{*1}\left(L_{\gamma(t_0)}\right)_{*\gamma^{-1}(t_0)}\left(R_{\gamma^{-1}(t_0)}\right)_{*1}\left(v_d(t_0)-v(t_0)\right)$$

$$=\left(L_{\gamma_d(t_0)}\circ L_{\gamma^{-1}(t_0)}\right)_{*1}\left(L_{\gamma(t_0)}\circ R_{\gamma^{-1}(t_0)}\right)_{*1}\left(v_d(t_0)-v(t_0)\right)$$

$$=\left(L_{\gamma_d(t_0)\star\gamma^{-1}(t_0)}\right)_{*1} Ad_{\gamma(t_0)}\left(v_d(t_0)-v(t_0)\right)$$

$$=\left(L_{e(t_0)}\right)_{*1} Ad_{\gamma(t_0)}\left(v_d(t_0)-v(t_0)\right)$$

new we define $\eta: I\rightarrow \mathfrak{g}$ such that:

$$\eta(t)=Ad_{\gamma(t)}\left(v_d(t)-v(t)\right) \tag{3.4}$$

which gives the final result:

$$e'(t)=\left(L_{e(t)}\right)_{*1}\eta(t)$$

This is an interesting result because it indicates that like $\gamma(t)$ and $\gamma_d(t)$ the derivative of tracking error we defined, related to a curve on $\mathfrak{g}$ via pushforward of left translation action. This is a necessary condition for every simple mechanical system on a Lie group to satisfy that equation and because the second equation of (3.1) is a controlled equation and it is possible to manipulate the right hand side of it by choosing proper control functions, it is a sufficient condition too meaning that the dynamic of our error function has the form of a simple mechanical system. To complete our computation of error dynamics we need to calculate time derivative of η :

$$\left.\frac{d}{dt}\right|_{t=t_0}\eta(t)=\left.\frac{d}{dt}\right|_{t=t_0} Ad_{\gamma(t)}\left(v_d(t_0)-v(t_0)\right)+Ad_{\gamma(t_0)}\left.\frac{d}{dt}\right|_{t=t_0}\left(v_d(t)-v(t)\right)$$

By an application of Lemma 2.2 we have:

$$=Ad_{\gamma(t_0)}\left[v(t_0),v_d(t_0)-v(t_0)\right]+Ad_{\gamma(t_0)}\left(v'_d(t_0)-v'(t_0)\right)$$

$$=Ad_{\gamma(t_0)}\left[v(t_0),v_d(t_0)\right]+Ad_{\gamma(t_0)}\left(v'_d(t_0)-v'(t_0)\right)$$

$$= Ad_{\gamma(t_0)}\left(v'_d(t_0) - v'(t_0) + \left[v(t_0), v_d(t_0)\right]\right)$$

Finally the equations of error dynamics take the following shape:

$$\begin{aligned} e'(t) &= \left(L_{e(t)}\right)_{*\mathbb{1}} \eta(t) \\ \eta'(t) &= Ad_{\gamma(t)}\left(v'_d(t) - v'(t) + \left[v(t), v_d(t)\right]\right) \end{aligned} \tag{3.5}$$

The state space of this system is the *TG* and states of system are in the form of pairs $(e(t), e'(t))$. Equating the first of error equations above to zero results in $\eta(t)=0$ and from (3.4) we get $v_d(t)=v(t)$ and again form (3.2) and (3.1) $\gamma_d(t)=\gamma(t)$ or $e(t)=\mathbb{1}$. In the other words $(\mathbb{1}, 0)$ is an equilibrium of error dynamics and when error dynamics converges to this equilibrium point, in our mechanical system, the trajectory of system converges to the reference trajectory. So the problem of trajectory tracking on *G* boils down to problem of asymptotically stabilize the $(\mathbb{1}, 0)$ equilibrium of system (3.5).

We propose the following feedback law:

$$\sum_{i=1}^{n} u_i(t)\mathbb{I}^{\sharp}\left(f^i\right) = -\mathbb{I}^{\sharp}\left(ad^*_{v(t)}\mathbb{I}^{\flat}(v(t))\right) + \left(L_{\gamma(t)}\right)^*_{\mathbb{1}} grad\, V + v'_d(t) + \left[v(t), v_d(t)\right] - Ad_{\gamma^{-1}(t)}u \tag{3.6}$$

We will design the vector $u \in \mathfrak{g}$ to achieve desired characteristics for error dynamics but before that we need to introduce the notion of configuration error functions.

3.3 Basic Morse theory and configuration error functions

To continue with the construction of our feedback law we need some basic facts from Morse theory. Let $\Psi : M \rightarrow \mathbb{R}$ be a function defined on a manifold *M* which $(x^1, \ldots, x^n)$ is a coordinate system on it. We define its Hessian as following:

$$Hess\,\Psi = \begin{pmatrix} \frac{\partial^2 \Psi}{\partial x^1 \partial x^1} & \frac{\partial^2 \Psi}{\partial x^1 \partial x^2} & \cdots & \frac{\partial^2 \Psi}{\partial x^1 \partial x^n} \\ \frac{\partial^2 \Psi}{\partial x^2 \partial x^1} & \frac{\partial^2 \Psi}{\partial x^2 \partial x^2} & \cdots & \frac{\partial^2 \Psi}{\partial x^2 \partial x^n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 \Psi}{\partial x^n \partial x^1} & \frac{\partial^2 \Psi}{\partial x^n \partial x^2} & \cdots & \frac{\partial^2 \Psi}{\partial x^n \partial x^n} \end{pmatrix}$$

It can be proved that the rank of this matrix is independent of choice of coordinates. The set of all points of M where $d\,\Psi=0$ are called critical points of Ψ and they are the set of extermum of Ψ . Let x_0 be a critical point of Ψ and $Hess\,\Psi(x_0)$ be a non-singular matrix, then we have the following:

i. If $Hess\,\Psi(x_0)$ be a positive definite matrix (negative definite matrix) then x_0 is a minimum (maximum) value of Ψ .
ii. If $Hess\,\Psi(x_0)$ has a combination of positive and negative eigenvalues then x_0 is a saddle point and the number of its negative eigenvalues called the index of critical point x_0.

A real valued function on a manifold M with non-singular Hessian at critical points (or non-degenerate critical points) called a Morse function. Its been proven that Morse functions exist on any compact connected manifold. Also critical point of a Morse function are isolated points and each Morse function has only finite number of them. A Morse function with minimum number of critical points called a perfect Morse function.

An infinitely differentiable Morse function, bounded from below and with unique minimum at identity called a configuration error function.

3.4 Local properties of closed loop tracking error dynamics

Let Ψ be configuration error function on G and consider its gradient vector field. In feedback law (3.6) we choose following u:

$$u=-k\,\eta-\left(L_{e(t)^{-1}}\right)_{*e(t)} grad\,\Psi$$

By closing the loop the tracking error dynamics (3.5) takes the following shape:

$$\begin{aligned} e'(t)&=\left(L_{e(t)}\right)_{*\mathbb{1}}\eta(t) \\ \eta'(t)&=-k\,\eta(t)-\left(L_{e(t)^{-1}}\right)_{*e(t)} grad\,\Psi(e(t)) \end{aligned} \tag{3.7}$$

If we equate left hand side of this system to zero from first equation we get $\eta=0$ and from second equation we get $grad\,\Psi(\gamma(t))=0$ which means that equilibrium points of this system are the pairs $(e,\eta)=(c_i,0)$ where c_i is critical point of Ψ .

Lyapunov's indirect method makes the link between that local behavior of an equilibrium and its linearization. If the linearization of system around an equilibrium possess only negative (positive) eigenvalues, then that equilibrium is a stable node (unstable node) and if it has a mix of positive and negative eigenvalues, then the equilibrium is a saddle point.

To linearize (3.7) first we need to construct a coordinate system on *G*. It is known that at identity the exponential map is a local diffeomorphism meaning that it is infinitely differentiable and has a continuous inverse making it a chart around identity. Now consider an arbitrary point on Lie group *G*, knowing that left translation and inversion maps are smooth, the functions $\exp \circ L_{g^{-1}}$ and $L_g \circ \exp^{-1}$ are diffeomorphisms which defines a chart in a neighborhood of *g*. In what follows we will use bold face fonts for coordinate expression of vectors which are column vectors.

Let $\{\partial / \partial x^i\}_{i=1,\dots,n}$ be the standard base for $T_{\mathbb{1}} G \simeq \mathfrak{g}$, then the set $\{(L_{e_0})_{*\mathbb{1}} \partial / \partial x^i\}_{i=1,\dots,n}$ becomes a base for $T_{e_0} G$. Now let $\boldsymbol{x} = (x^1, \dots, x^n)^T$ be the coordinates of a point $e \in G$ in the chart we defined above around e_0 where $\boldsymbol{e_0} = (0, \dots, 0)^T$ and $\boldsymbol{\eta} = (\eta^1, \dots, \eta^n)^T$ be the exponential coordinates of η. In a vicinity of e_0 we can make the approximation that $\dot{\boldsymbol{e}} \approx (\dot{x}^1, \dots, \dot{x}^n)^T$ and first dynamic equation of (3.7) can be written in coordinates:

$$\dot{x}^i (L_{e_0})_{*\mathbb{1}} \frac{\partial}{\partial x^i} \approx \eta^i (L_{e_0})_{*\mathbb{1}} \frac{\partial}{\partial x^i}$$

Note that we used summation convention here in up and down indices. So the linear form of first equation becomes $\dot{\boldsymbol{x}} = \boldsymbol{\eta}$ where $e = L_{e_0} \circ \exp(\boldsymbol{x})$.

Because we are interested in matrix Lie groups like *SO*(3), *SE*(3) or their subgroups we can derive this linearization using exponential map of a matrix Lie group. Let *G* be a matrix Lie group, then around a point e_0 from first equation of (3.7) we have:

$$e(t) = e_0 \exp(\boldsymbol{x}(t)) = e_0 \left(I + \sum_{k=1}^{\infty} \frac{1}{k!} \boldsymbol{x}(t)^k \right)$$

where $\boldsymbol{x}$ is the coordinates of *e* we defined above. Now we have to differentiate this equation. Using the expression for differential of matrix exponential we have:

$$e(t)\boldsymbol{\eta}=e_0\frac{d}{dt}\exp(\boldsymbol{x}(t))$$
$$=e_0\exp(\boldsymbol{x}(t))\left(\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k\,!}ad_x^{k-1}\dot{\boldsymbol{x}}\right)\approx e(t)\dot{\boldsymbol{x}}$$

Which is the same result for linearization of first dynamic equation.

For the second equation, first let write the gradient of Ψ in the coordinates we defined above:

$$grad\,\Psi(\boldsymbol{x})=\frac{\partial\Psi(\boldsymbol{x})}{\partial x^i}(L_e)_{*\mathbb{1}}\frac{\partial}{\partial x^i}$$

With this expression of gradient our dynamic equitation takes the following shape in coordinates:

$$\dot{\eta}^i\frac{\partial}{\partial x^i}=\left(-k\eta^i-\frac{\partial\Psi(\boldsymbol{x})}{\partial x^i}\right)\frac{\partial}{\partial x^i}$$

This coordinate representation of second equation, is a dynamic equation on a linear space and we can linearize it by simply taking its Jacobian. Let us place our origin at one of the equilibrium points of system which we proved to be the points with zero gradient. Taking Jacobian leads to:

$$\dot{\boldsymbol{\eta}}=-k\,I_{n\times n}\boldsymbol{\eta}-\left[\left.\frac{\partial^2\Psi(\boldsymbol{x})}{\partial x^i\partial x^j}\right|_{x=0}\right]\boldsymbol{x}$$

note that $\boldsymbol{\eta}$ and $\boldsymbol{x}$ are column vectors and the matrix of second derivatives in nothing but Hessian of Ψ at origin. Result of this discussion is that around its equilibrium points and in the coordinates we defined, system (3.7) can be approximated by a linear system $\dot{\boldsymbol{z}}=A\boldsymbol{z}$ where $\boldsymbol{z}=(\boldsymbol{x}^{\boldsymbol{T}},\boldsymbol{\eta}^{\boldsymbol{T}})^T$ and A has the following matrix description:

$$A=\begin{pmatrix}0 & I_{n\times n}\\ -Hess\,\Psi(e_0) & -kI_{n\times n}\end{pmatrix}$$

Now we study the eigenvalues of this matrix. Let λ be an eigenvalue of this matrix. Using the fact that the matrices at the second raw are commute we have:

$$\begin{vmatrix}\lambda I_{n\times n} & -I_{n\times n}\\ Hess\,\Psi(e_0) & (\lambda+k)I_{n\times n}\end{vmatrix}=\left|\lambda(\lambda+k)I_{n\times n}+Hess\,\Psi(e_0)\right|=0$$

Now let h be an eigenvalue of $Hess\,\Psi(e_0)$, the characteristic polynomial becomes:

$$\left| -hI_{n\times n} + Hess\,\Psi(e_0) \right| = 0$$

comparing to the characteristic polynomial of A matrix above, we can see that each eigenvalue h of $Hess\,\Psi(e_0)$ generates two eigenvalues of A according the equation below:

$$\lambda(\lambda+k)=-h \;\Rightarrow\; \lambda_1=\frac{-k+\sqrt{k^2-4h}}{2}, \lambda_2=\frac{-k-\sqrt{k^2-4h}}{2} \tag{3.8}$$

We have chosen Ψ to be a configuration error function which means that it is a Morse function with a unique minimum at identity. We also know that equilibrium points of system (3.7) are critical points of Ψ, which makes the identity, an equilibrium with positive definite Hessian matrix. Positive definite hessian means that every eigenvalue *h,* is a positive value and because *k* is also a positive value, from (3.8) all the eigenvalues of A become negative values at identity. In the other words the pair $(\mathbb{1}, 0)$ is a stable equilibrium of error dynamics.

All the other critical points of Ψ are saddle or maximum points which means that they are critical points with a Hessian with at least one negative eigenvalue. For each negative *h,* equation (3.8) generates one positive and one negative eigenvalue for A, which makes all the other equilibrium points saddles.

3.5 Global behavior of equilibrium points

Now we study the attractiveness of the stable equilibrium in a global seance. We want to determine what happens to trajectories of (3.7) if we start from an arbitrary point on configuration manifold and with an arbitrary initial velocities? We also proved that there exists a number of saddle points, which poses the question of how their stable and unstable manifolds behave?

It is proven that on a compact manifold there exist no globally symbiotically stable equilibrium[3]. Fortunately almost-global attractiveness on a dense subset of configuration manifold is achievable and we will prove it in this section. Also we will show that the stable manifold of a saddle point is a nowhere dense subset of configuration manifold.

We start with the following Lyapunov function:

$$\psi(e(t),\eta(t)) = \Psi(e(t)) + \frac{1}{2}\mathbb{I}(\eta(t),\eta(t)) \tag{3.9}$$

Lets compute its time derivative:

$$\begin{aligned}
\mathscr{L}_{e'(t)}\psi(e(t),\eta(t)) &= \frac{d}{dt}\psi(e(t),\eta(t)) \\
&= \langle d\,\Psi(e(t)), e'(t)\rangle + \frac{1}{2}\frac{d}{dt}\mathbb{I}(\eta(t),\eta(t)) \\
&= \mathbb{G}_{\mathbb{I}}\left(\mathbb{G}^{\sharp} d\,\Psi(e(t)), e'(t)\right) + \frac{1}{2}\mathbb{I}(\eta'(t),\eta(t)) + \frac{1}{2}\mathbb{I}(\eta(t),\eta'(t)) \\
&= \mathbb{G}_{\mathbb{I}}\left(grad\,\Psi(e(t)), e'(t)\right) + \mathbb{I}(\eta'(t),\eta(t)) \\
&= \mathbb{G}_{\mathbb{I}}\left(-\left(L_{e(t)}\right)_{*\mathbb{1}}(\eta'(t)+k\eta(t)), \left(L_{e(t)}\right)_{*\mathbb{1}}\eta(t)\right) + \mathbb{I}(\eta'(t),\eta(t)) \\
&= \mathbb{I}(-\eta'(t)-k\eta(t),\eta(t)) + \mathbb{I}(\eta'(t),\eta(t)) \\
&= -k\,\mathbb{I}(\eta(t),\eta(t))
\end{aligned}$$

remember that $\mathbb{I}$ is an inner product on $\mathfrak{g}$, so we proved the following:

$$\mathscr{L}_{e'(t)}\psi = -k\|\eta(t)\|^2 \leq 0 \tag{3.10}$$

The negative time derivative shows that ψ is non-increasing along solutions of (3.7) and vanishes at $\eta=0$ which previously we proved that, corresponds to equilibrium points of system.

At this point let state LaSalle invariance theorem from [1]:

LaSalle Invariance Theorem: *For $X \in \mathfrak{X}(M)$, let $A \subset M$ be compact and positively X-invariant. Let $\psi \in C^{\infty}(M)$ satisfy $\mathscr{L}_X \psi \leq 0$ in A and let B be the largest positively X-invariant set contained in $\{x \in A \mid \mathscr{L}_X \psi = 0\}$. Then the following statements hold:*

i. *each integral curve of X with initial condition in A approaches B as $t \to +\infty$.*
ii. *if B consists of a finite number of isolated points then each integral curve of X with initial condition in A converges to a point of B as $t \to +\infty$.*

The consequence of LaSalle's theorem is that on a compact configuration manifold (which is the case in most of practical applications) trajectories of system (3.7) starting from any initial conditions, converge to one of the equilibrium points of system.

A subset of a topological space called dense, if the union of its limit points and itself generate the whole space. For example the open interval $(0,1)$ is not dense in $\mathbb{R}$ because $(0,1)\cup\{0,1\}\neq\mathbb{R}$ but it is dense in close interval $[0,1]$.

According to stable manifold theorem [22], at every equilibrium point of system (3,7) there exists immersed submanifolds called stable and unstable manifolds which their dimension are equal to number of negative and positive eigenvalues of linearization at that point respectively. These manifolds represent attracting and repelling sets of each equilibrium and one consequence of the fact that our configuration error function is a Morse function is that they are and their union is nowhere dense and complement of this nowhere dense set is open and dense. In the other words, our unique stable equilibrium is attractive everywhere on the configuration manifold except on a nowhere dense set which makes it almost-globally asymptotically stable.

3.6 Error dynamics on *SO*(3)

At this section we study the error dynamics in a special case when the configuration manifold is the space of rotation matrices. Space of 3×3 orthogonal matrices with unit determinant is a Lie group called special orthogonal group or *SO*(3) which represent the rigid rotations of three dimensional space. The Lie algebra of *SO*(3) denoted by $\mathfrak{so}(3)$, is the space of skew-symmetric matrices. Since every skew-symmetric matrix can be represented by three real numbers we define the map $\hat{\cdot}:\mathbb{R}^3\rightarrow\mathfrak{so}(3)$ as follows:

$$v=\begin{pmatrix}x\\y\\z\end{pmatrix}\Leftrightarrow\hat{v}=\begin{pmatrix}0&-z&y\\z&0&-x\\-y&x&0\end{pmatrix}$$

The $\hat{\cdot}$ and its inverse denoted by $.^{\vee}:\mathfrak{so}(3)\rightarrow\mathbb{R}^3$ are continues maps which establish an isomorphism between $\mathbb{R}^3$ and 3×3 skew-symmetric matrices. We introduce a configuration error function on *SO*(3) as follows:

$$\Psi(e)=\frac{1}{2}Tr\left(G(I_{3\times 3}-e)\right)$$

Note that e is rotation matrix, G is a positive-definite matrix and Tr represents the trace operator of a matrix. If v_i be an eigenvector of G then the critical points of this function are the rotation matrices $\exp(l\pi v_i)$ where l is a whole number and at each point e and each critical point e_0 in standard basis of cotangent space (the one left-translated from Lie algebra) we have:

$$d\Psi(e)=\frac{1}{2}(Ge-e^T G)^\vee \ , Hess\,\Psi(e_0)=\frac{1}{2}(Tr(Ge_0)I_{3\times3}-Ge_0)$$

We will prove these in the next chapter. Also left and right translation actions of group, becomes left and right multiplication by rotation matrices in $SO(3)$.

Our error dynamic equations take the following shape:

$$\begin{aligned}&\dot{e}(t)=e.\hat{\eta}(t)\\&\dot{\eta}(t)=-k\,\eta(t)-\frac{1}{2}(G.e-e^T.G)^\vee\end{aligned} \quad , \quad e(t)\in SO(3), \hat{\eta}(t)\in\mathfrak{so}(3)$$

First equation here is a differential equation on Lie group which has to be taken into account during numerical computations. It means that algebraic constraints defining orthogonality needs to be taken into account in order to get valid results from numerical computations which in this case means a system of nine differential equations coupled with six algebraic constraints. A solution to reduce the complexity of numerical computations is to use the unit quaternions.

Consider the kinematic equation $\dot{R}=R\Omega$ where R is a rotation matrix and $\Omega=[\omega_1, \omega_2, \omega_3]^T$ and let $q=(q_0, q_1, q_2, q_3)^T$ be the unit quaternion. Then q represents the following rotation matrix:

$$R=\begin{pmatrix} q_0^2+q_1^2-q_2^2-q_3^2 & 2(q_1q_2-q_0q_3) & 2(q_1q_3+q_0q_2)\\ 2(q_1q_2+q_0q_3) & q_0^2-q_1^2+q_2^2-q_3^2 & 2(q_2q_3-q_0q_1)\\ 2(q_1q_3-q_0q_2) & 2(q_2q_3+q_0q_1) & q_0^2-q_1^2-q_2^2+q_3^2\end{pmatrix} \tag{3.11}$$

computing its time derivative and equating it to $R\Omega$ gives the quaternion version of kinematic equation:

$$\begin{pmatrix}\dot{q}_0\\ \dot{q}_1\\ \dot{q}_2\\ \dot{q}_3\end{pmatrix}=\frac{1}{2}\begin{pmatrix} q_0 & -q_1 & -q_2 & -q_3\\ q_1 & q_0 & -q_3 & q_2\\ q_2 & q_3 & q_0 & -q_1\\ q_3 & -q_2 & q_1 & q_0\end{pmatrix}\begin{pmatrix}0\\ \omega_1\\ \omega_2\\ \omega_3\end{pmatrix} \tag{3.12}$$

So we have a set of four differential equations with one algebraic constraint (the unit norm), which is a huge advantage in terms of computational effort.

Figure 1 indicates the graphical version of tracking error dynamic equations in

Simulink environment.

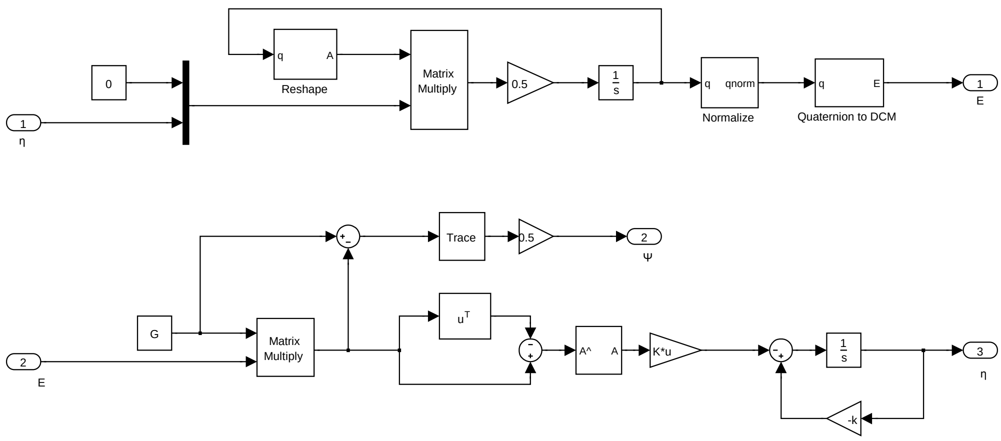

Figure 1: Tracking error dynamics equations in Simulink

Take the following set of parameters:

$$G=\begin{pmatrix} 25 & 0 & -5\sqrt{3} \\ 0 & 20 & 0 \\ -5\sqrt{3} & 0 & 15 \end{pmatrix},\ k=5$$

Consider the motion of a particle in earths gravity field subject to a dissipative force like air friction. The second equation of tracking error dynamics has two components, a gradient vector which can be compared to the gradient of height function representing the potential energy and a dissipative part which drains out the kinetic energy from system, driving it to the equilibrium points. So the parameter k determines the damping speed of velocities and the other parameter G controls how much energy stores in the system as we go away from desired equilibrium, to return it back to that point.

The first simulation is the case when the system starts from an arbitrary configuration with relatively large initial velocities. The dissipative part of

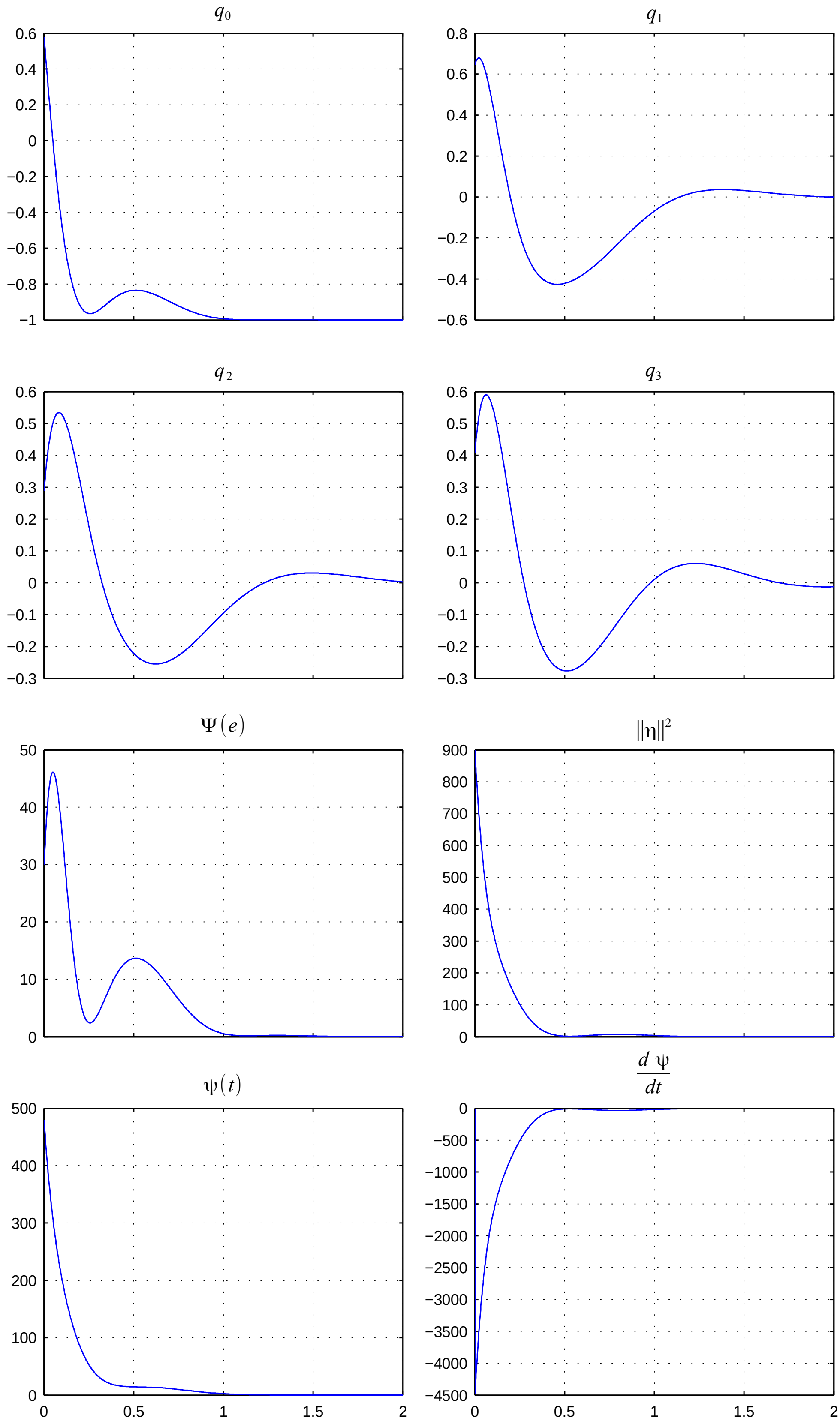

Figure 2:System starts with a big initial angular velocity

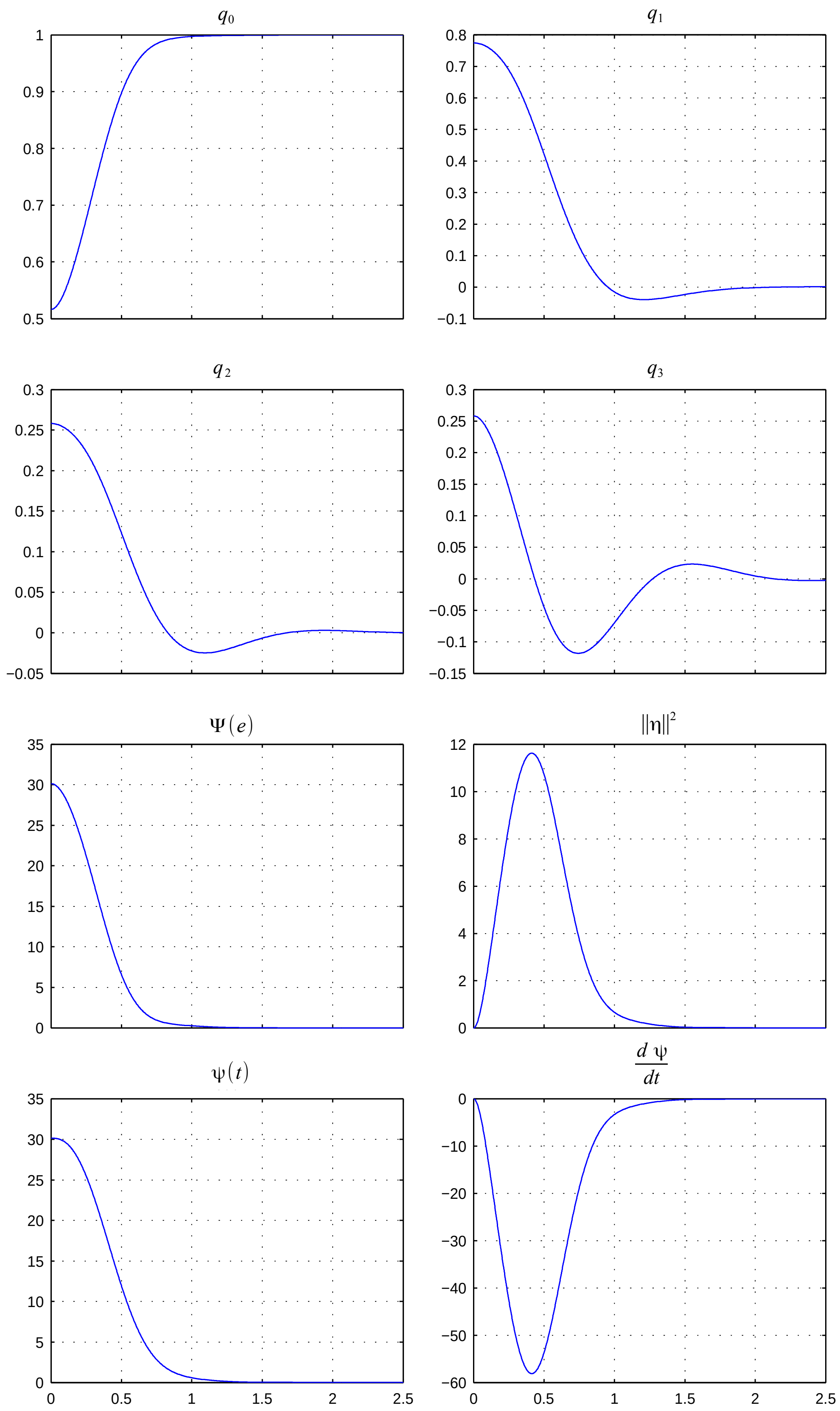

Figure 3: System starts with zero initial angular velocity

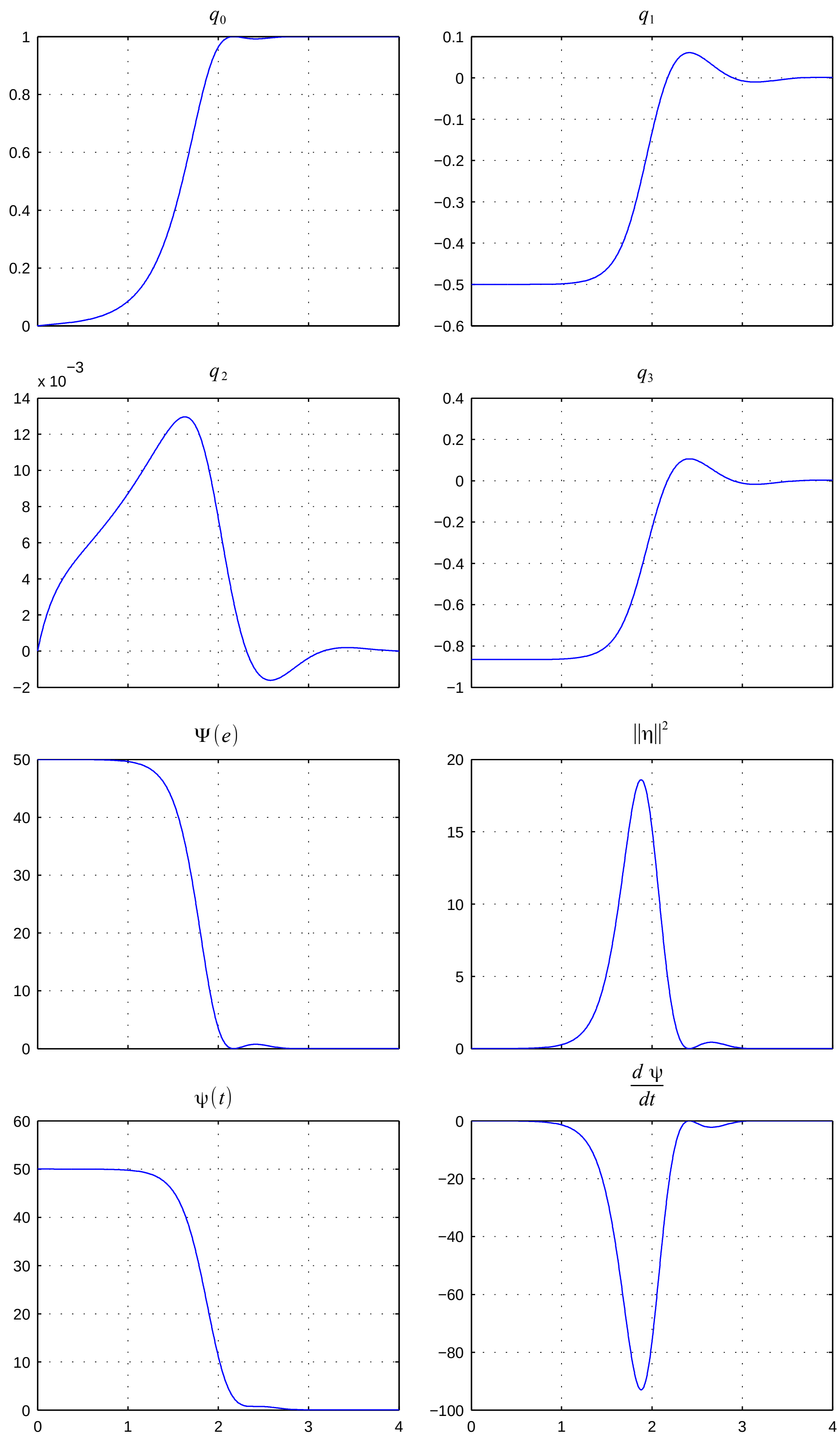

Figure 4: System starts from a saddle point

controller reduces the velocities until system comes to rest at desired equilibrium (Figure 2).

The second simulation is the case when system starts from an initial configuration with zero initial velocities. The configuration error generates a big gradient vector (like the potential energy stored in gravity field) which increases the velocity until the system converges to its equilibrium point (Figure 3).

In final simulation the system starts from one of its saddle points with a very tiny velocity. Without that tiny velocity we expect the system to remain in that saddle point forever, while a small deviation from the saddle point caused by the initial velocity push the system into the attractive set of stable equilibrium and after that the trajectories of system flows towards the desired point (Figure 4).

CHAPTER 4

ATTITUDE TRACKING CONTROL OF QUADROTOR

In this section we apply the controller of chapter 3 to the problem of attitude tracking control of a Quadrotor. The Quadrotors have been the subject of research in recent years and there exist a wast literature on modeling, estimation and control of them. Other than the practical applications, the control problem of Quadrotor is an interesting one on its own specially that availability of them in reduced prices made it possible to test complicated rigid body control algorithms in a real implementation which is an inspiration to consider the problem of controlling complicated maneuvers.

The base of most of the work done for controlling the Quadrotor would be the application of rotation angels or quaternions to model the kinematic and then using linear control theory or Lyapunov methods to design stabilizing controllers ([16], [18] for example) which as it explained in introduction, this parametrization can lead to some problems during big attitude maneuvers. The theory we developed in previous chapter encompasses the attitude control of a Quadrotor as a special case which is the study of details of it is a task for current chapter.

4.1 Mathematical model of Quadrotor

Figure 5 illustrates a simplified model of Quadrotor. Orientation of a vector expressed in body frame (fixed in center of mass) is related to its orientation expressed in inertial frame by a 3×3 rotation matrix R. So R determines orientation of body frame with respect to inertial frame or we can say that configuration space of problem is the space of 3×3 orthogonal matrices.

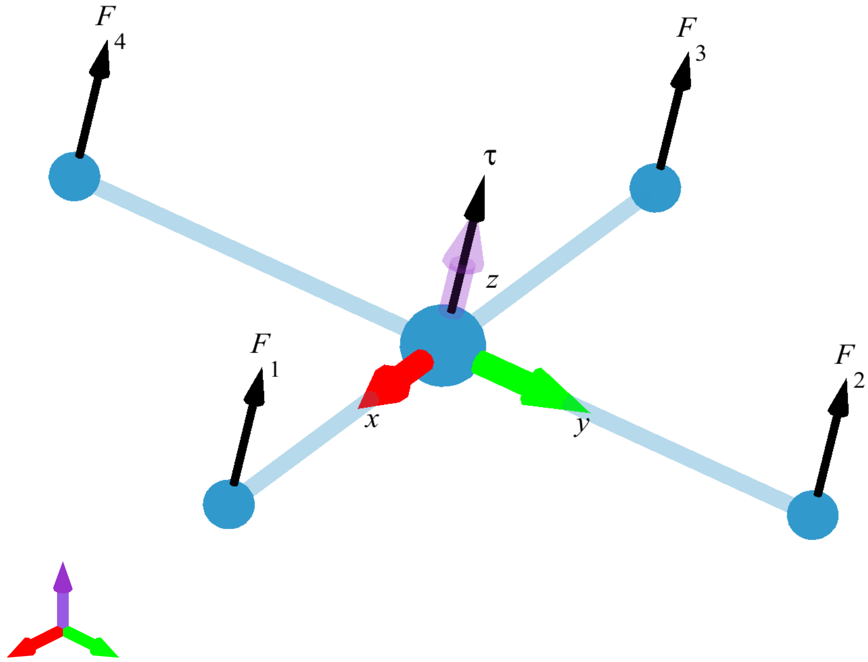

Figure 5: A simple model of Quadrotor

Forces F_1, F_2, F_3, F_4 generated by rotation of propellers give us the control torques in x and y directions. Also the rotation of propeller in the air, produces a reactive force which is tend to rotate the body around z axis and if we add four of them generated by four propellers, it gives us the control torque in z direction. Assume that two opposite propellers rotates in same direction and the two other rotates in opposite direction. Let $u_x(t) f_x, u_y(t) f_y, u_z(t) f_z$ be the three control torques generated in this way where $u_a(t)$'s are real valued functions called controls and f_a's are three unit vectors in *x,y,z* direction of body frame, then they relate to control forces with the following relation:

$$\begin{pmatrix} u_x(t) \\ u_y(t) \\ u_z(t) \end{pmatrix} = \begin{pmatrix} d & 0 & -d & 0 \\ 0 & d & 0 & -d \\ k & -k & k & -k \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{pmatrix}$$

where d is the distance from center of propeller to the center of mass of whole body and k is a constant related to aerodynamical properties of propellers. Also because we placed our body frame in center of mass forces of earths gravity generates no torques on rigid body so when we are dealing with attitude of rigid body we can ignore the effect of gravity.

We know that the configuration manifold of the problem is the space of rotation matrices denoted by $SO(3)$ which is a Lie group and its lie algebra, $\mathfrak{so}(3)$ is the space of skew-symmetric matrices. Now let $\mathbb{I}$ be the inertia tensor of rigid body which we denote its matrix representation by $[\mathbb{I}]$. The kinetic energy of system becomes:

$$KE=\frac{1}{2}\mathbb{I}(\Omega,\Omega)=\frac{1}{2}[\Omega]^T[\mathbb{I}][\Omega]$$

where $[\Omega]$ is the coordinate representation of tangent vector $\Omega\in\mathfrak{so}(3)$. Since $[\mathbb{I}]$ is a positive definite matrix, it defines an inner product on tangent space and that generates the kinetic energy metric $\mathbb{G}_{\mathbb{I}}$ on $SO(3)$ which has the same matrix representation as $\mathbb{I}$ and its inverse becomes $[\mathbb{G}_{\mathbb{I}}^{\sharp}]=[\mathbb{I}^{\sharp}]=[\mathbb{I}]^{-1}$. Using the isomorphism between $\mathfrak{so}(3)$ and $\mathbb{R}^3$ with $\hat{\cdot}$ we can write matrix version of left and right translation and also adjoint operator and adjoint map. For $\Omega_1,\Omega_2\in\mathfrak{so}(3),\alpha_1,\alpha_2\in\mathfrak{so}^*(3)$ and $R\in SO(3)$ we have the following relations:

i. $ad_{\Omega_1}\Omega_2=\hat{\Omega}_1.\Omega_2=\Omega_1\times\Omega_2$,
ii. $ad^*_{\alpha_1}\alpha_2=\hat{\alpha}_2.\alpha_1=\alpha_2\times\alpha_1$
iii. $Ad_R\Omega_1=R.\Omega_1$
iv. $(L_R)_*\Omega_1=R.\hat{\Omega}_1$

note that *A.B* means the matrix multiplication of *A* and *B* which we will write it as *AB* in the rest of this chapter and $\times$ is the cross product of vectors.

Now we can use equation (3.1) of a general mechanical system on a Lie group to write the equations of Quadrotor. Let $R(t)$ be the rotation matrix representing the attitude at time *t,* then there exist a tangent vector $\Omega(t)$ called the body angular velocity and we define it as $\hat{\Omega}(t)=R(t)\dot{R}(t)$ which actually is the physical vector of angular velocity expressed in body frame and the following equations describe the motion of system:

$$\dot{R}(t)=R(t)\hat{\Omega}(t) \tag{4.1}$$

$$\dot{\Omega}(t)=J^{-1}(J\Omega(t)\times\Omega(t))+J^{-1}U \tag{4.2}$$

where $U=\left(u_x, u_y, u_z\right)^T$ and $J=[\mathbb{I}]$. These are the familiar equations of rotational dynamics of a rigid body which came out of our general equations of mechanical system. Now we can apply the theory of chapter 3 to tracking control of this model.

4.2 A configuration error function on *SO*(3)

In this section we study the configuration error function we introduced in chapter 3 with following equation:

$$\Psi(R)=\frac{1}{2}Tr(G(I_{3\times3}-R)) \tag{4.3}$$

where G is symmetric positive definite matrix. To compute $grad\,\Psi$ remember the chart we defined on a Lie group at each point using the exponential function and left translation as $R=R_0\exp(\hat{x})$ and let denote by J_0 the Jacobian at zero, then:

$$\begin{aligned}
[grad\,\Psi(R_0)]^T &= J_0\Psi(x)\\
&=\frac{1}{2}J_0\left(Tr\left(G-GR_0-GR_0\sum_{k=1}^{\infty}\frac{1}{k\,!}\hat{x}^k\right)\right)\\
&=-\frac{1}{2}J_0\left(Tr\left(GR_0\hat{x}\right)\right)-\frac{1}{2}J_0\left(Tr\left(GR_0\sum_{k=2}^{\infty}\frac{1}{k\,!}\hat{x}^k\right)\right)
\end{aligned}$$

The second term composed of second or higher order terms and its Jacobian at zero vanishes. Some direct computation shows the following result:

$$grad\,\Psi(R)=\frac{1}{2}(GR-R^TG)^{\vee}=skew(GR)^{\vee} \tag{4.4}$$

Note that this equation is the expression of a vector on tangent space at R in the base that is left-transelated from Lie algebra, so we have:

$$(L_{R^{-1}})_{*R}\,grad\,\Psi(R)=skew(GR)^{\vee}$$

Now we compute the quaternion version of these equalities. Let R represents a rotation with angle θ around a unit vector $\boldsymbol{n}$ as the axis of rotation, then the unit quaternion $(q_0, q_1, q_2, q_3)=(q_0, \boldsymbol{q})$ represents the same rotation and we have:

$$R = I + 2q_0\hat{\boldsymbol{q}} + 2\hat{\boldsymbol{q}}^2 \ , \ q_0 = \cos\left(\frac{\theta}{2}\right) \ , \ \boldsymbol{q} = \sin\left(\frac{\theta}{2}\right)\boldsymbol{n}$$

note that for simplicity we dropped the subscript from *I*. Now we can compute quaternoin version of Ψ and $\dot{\Psi}$ as follows:

$$\Psi = -\frac{1}{2}Tr(G(2q_0\hat{\boldsymbol{q}} + 2\hat{\boldsymbol{q}}^2))$$

$$= -Tr(G\hat{\boldsymbol{q}}^2) = \boldsymbol{q}^T(Tr(G)I - G)\boldsymbol{q}$$

$$\dot{\Psi} = \dot{\boldsymbol{q}}^T(Tr(G)I - G)\boldsymbol{q} + \boldsymbol{q}^T(Tr(G)I - G)\dot{\boldsymbol{q}}$$

using (3.12) and some computation:

$$\dot{\Psi} = (q_0 I - \hat{\boldsymbol{q}})(Tr(G)I - G)\boldsymbol{q}\,\Omega$$

which means that:

$$d\Psi = (q_0 I - \hat{\boldsymbol{q}})(Tr(G)I - G)\boldsymbol{q}$$

Obviously $\boldsymbol{q}=0$ gives us one critical point but to compute the other critical points of Ψ we need to equate this to zero which results:

$$\hat{\boldsymbol{q}}(Tr(G)I - G)\boldsymbol{q} = q_0(Tr(G)I - G)\boldsymbol{q}$$

This means that q_0 is an eigenvalue of $\hat{\boldsymbol{q}}$ with $(Tr(G)I - G)\boldsymbol{q}$ as its associated eigenvector. Since $\hat{\boldsymbol{q}}$ is a skew-symmetric matrix, the only real eigenvalue of it is zero and its associated eigenvector is $\boldsymbol{q}$ so for some real scalar *k*:

$$(Tr(G)I - G)\boldsymbol{q} = k\boldsymbol{q} \quad \Rightarrow \quad G\boldsymbol{q} = (Tr(G) - k)\boldsymbol{q}$$

which means that $\boldsymbol{q}$ is an eigenvector of *G* too. So the rest of critical points of Ψ are associate with the quaternions $(0, v_i)$ which v_i is a unit eigenvector of *G*. In the other words the expression $\exp(l\pi\hat{v}_i)$, $l \in \mathbb{Z}$ generates all of four critical points of Ψ in the form of rotation matrices and since no Morse function on *SO*(3) can have less than four critical points, Ψ is a perfect Morse function.

As an example; if *G* is chosen to be diagonal with three distinct positive eigenvalues, then the following three vectors form an orthonormal set of eigenvectors of *G*:

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} , \ v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} , \ v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and from the discussion above, the following four rotation matrices generated by the exponential formula, are the set of critical points of Ψ :

$$\left\{\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix},\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix},\begin{pmatrix}-1&0&0\\0&1&0\\0&0&-1\end{pmatrix}\right\}$$

Other than identity the three other rotation matrices correspond to 180 degree rotations around *x,y* and *z* axes.

Now we compute the Hessian of Ψ at its critical points. First observe that according to (4.2) at a critical point R_0, $GR_0 = R_0^T G$. Using Rodriguez formula and in coordinates defined by $R = R_0 \exp(\hat{x})$ we have:

$$\Psi(x) = \frac{1}{2} Tr\left(G - GR_0 - GR_0 \frac{\sin\|x\|}{\|x\|}\hat{x} - GR_0 \frac{1-\cos\|x\|}{\|x\|^2}\hat{x}^2\right)$$

$$= \frac{1}{2} Tr\left(G - GR_0\right) - \frac{1-\cos\|x\|}{2\|x\|^2} Tr\left(GR_0 \hat{x}^2\right)$$

$$= \frac{1}{2} Tr\left(G - GR_0\right) + \frac{1-\cos\|x\|}{2\|x\|^2} x^T \left(Tr(GR_0) I - GR_0\right) x$$

Taking Jacobian two times at origin gives the result:

$$Hess\,\Psi(R_0) = \frac{1}{2}\left(Tr(GR_0) I - GR_0\right) \tag{4.5}$$

4.3 The tracking controller on *SO*(3)

Now we can construct the tracking controller for the special case when the configuration manifold is the *SO*(3). Suppose we want to track the motion of rigid body with the kinematic equation $\dot{R}_d = R_d \hat{\Omega}_d$. The tracking error becomes $e = R_d R^T$ and from (3.4) and (3.5), $\eta = R(\Omega_d - \Omega)$ and $\dot{\eta} = R(\dot{\Omega}_d - \dot{\Omega} + \Omega \times \Omega_d)$ and finally for the system described by equations (4.1) and (4.2) the feedback law (3.6) takes the following shape:

$$J^{-1} U = -J^{-1}(J\Omega \times \Omega) + \dot{\Omega}_d + \Omega \times \Omega_d + k(\Omega_d - \Omega) + R^T skew(GR_d R^T)^{\vee} \tag{4.6}$$

where $U = (u_x, u_y, u_z)^T$ and note that we took $u = -kR(\Omega_d - \Omega) - skew(Ge)^{\vee}$ in (3.6). Figure 6 is an illustration of this controller in Simulink.

Using the same parameters for *G* and *k* as in our previous example in chapter 3, Figure 7 illustrates how the taking controller successfully tracks down a reference

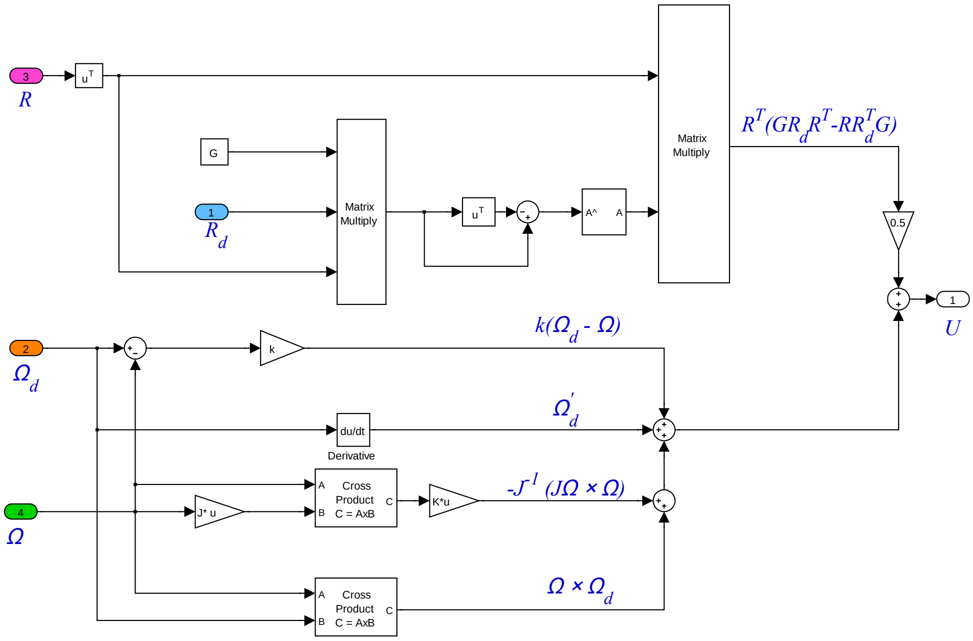

Figure 6: Tracking controller in Simulink

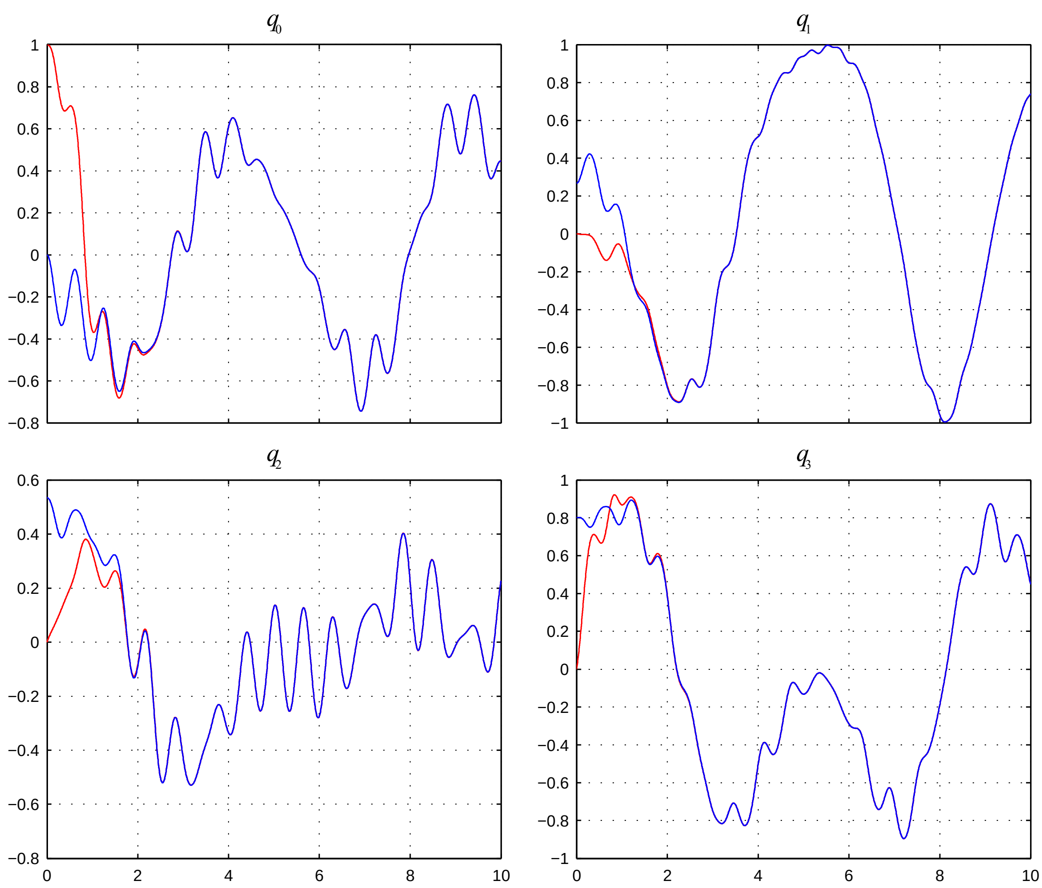

Figure 7: Tracking a reference trajectory (red)

trajectory.
Now let study how the tracking controller can stabilize a certain attitude in response to external forces. Take the following inertia matrix of a typical Quadrotor and parameters:

$$J=\begin{pmatrix} 5.6\times10^{-3} & 0 & 0 \\ 0 & 5.6\times10^{-3} & 0 \\ 0 & 0 & 8.1\times10^{-3} \end{pmatrix} kg.m^2 \;,\; G=\begin{pmatrix} 45 & 0 & -5\sqrt{3} \\ 0 & 40 & 0 \\ -5\sqrt{3} & 0 & 45 \end{pmatrix} \;,\; k=6$$

consider the case when Quadrotor is hovering with a certain heading angle and then an external force causes an unwanted deviation in attitude.
Simulation results shows that despite a big change in attitude which caused the craft to flip over the tracking controller successfully maintained stability and returned the Quadrotor back to the original attitude (Figure 8).

4.4 Application to a real life Quadrotor

So far we considered our Quadrotor and control system to be a perfect rigid body without limitations in control forces and moments. To make more real simulations we need to consider the imperfections of a real life implementation of such flying object.
Generally a Quadrotor has two separate parts of algorithms running in its on-board computer in real time, an estimation algorithm and control algorithms. The estimation algorithm is a Kalman filter or a nonlinear observer implemented as computer code that reads the unprocessed and noisy data form on-board sensors, filter out the noise and other undesired contamination and compute the attitude in the form of rotation angels or quaternions. However this attitude estimation is not perfect and the computations always deviate from the real attitude to some extent which in our simulations we will model it by multiplying the attitude with a bounded random signal. Also because computing each sample of attitude takes some time, there exist a delay between the real attitude and output of estimation algorithm which might have significant effect on the control algorithm.
Another issue we have to consider to achieve a more realistic simulation is the dynamic of motor and propeller. Most of quadrotrs use brushless DC motors with an electronic drive system to control the speed and this motor connected to a propeller

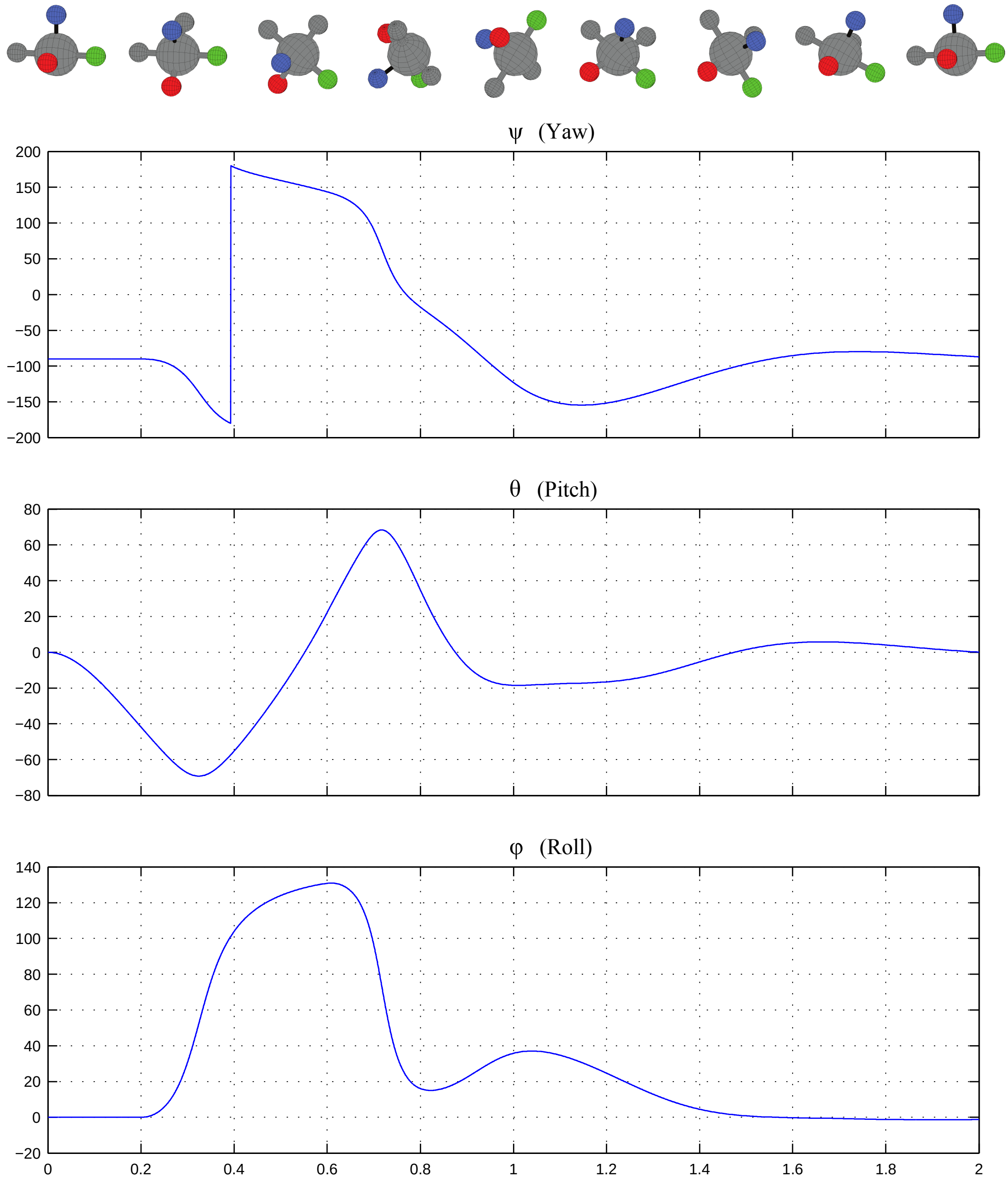

Figure 8: Rotation angles of system in response to external input

which its shape optimized to generate maximum trust in interaction with surrounding air at each rotational speed. The whole system of motor and propellers can be considered as one actuator system.

In [23] the nonlinear and adaptive control of a commercial Quadrotor studied. The following table contains the parameters of such a Quadrotor:

Takeoff weight	480 *g*
Distance between motor axes (*r*)	34 *cm*
Thrust per Motor	0.05-3.5 *N*
Moment of inertia	$I_{xx}, I_{yy}=5.6\times10^{-3}; I_{zz}=8.1\times10^{-3}(kg.m^2)$
k_m	0.016 (*m*)
k_n	$5.7\times10^{-8}(N/rpm^2)$

Parameters of sample Quadrotor

The trust of each motor *F* related to square of rotational speed with k_n. Also the propellers interact with air to produce a moment in body frame *z* direction which is proportional to *F* with k_m.

$$F=k_n.n^2 \quad , \quad M=k_m.F$$

Assuming that total trust T_t equally distributed among propellers, we can write the relation between the trust of each motor and control moments in matrix form:

$$\begin{pmatrix} L \\ M \\ N \\ T_t \end{pmatrix} = \begin{pmatrix} 0 & -0.17 & 0 & 0.17 \\ 0.17 & 0 & -0.17 & 0 \\ -0.016 & 0.016 & -0.016 & 0.016 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{pmatrix}$$

$$\begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{pmatrix} = \begin{pmatrix} 0 & 2.941 & -15.16 & 0.25 \\ -2.941 & 0 & 15.16 & 0.25 \\ 0 & -2.941 & -15.16 & 0.25 \\ 2.941 & 0 & 15.16 & 0.25 \end{pmatrix} \begin{pmatrix} L \\ M \\ N \\ T_t \end{pmatrix}$$

In [23], based on testbed data from manufacturer dynamic behavior of motor and propeller actuator modeled as ilustrated in Figure 9. This is a first order estimate of

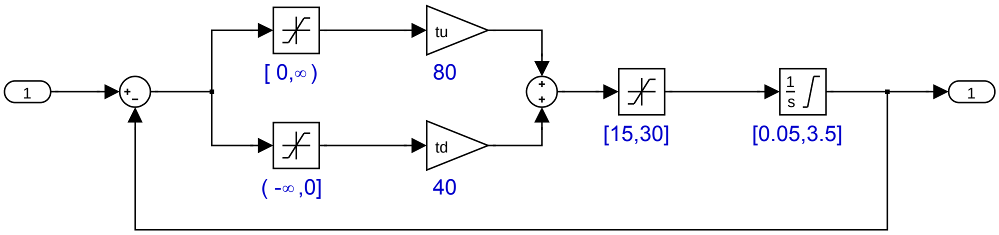

Figure 9: Approximation of motor and propeller dynamic as one actuator unit

actuator dynamic, with two different time constants for step-up (125 *ms*) and step-down (250 *ms*) inputs. Also the limitations of maximum and minimum trust considered using saturation blocks.

The control rule we introduced in (4.6) depend on an exact value of inertia tensor which is not achievable by physical measurements and no mater how accurate our instruments are, there exist a certain measure of uncertainty in the values we know and sensitivity of our control system to such uncertainties is a measure of robustness of system. To simulate such uncertainties let assume that we measure the following inertia tensor with experiments:

$$J=\begin{pmatrix} 4.0 & 0.3 & 0.5 \\ 0.3 & 4.0 & 0.6 \\ 0.5 & 0.6 & 7.0 \end{pmatrix}\times 10^{-3} \qquad kg\,.\,m^2$$

which is the inertia tensor we will set for our controller.

Figure 10 indicates the closed loop system using the tracking controller and the model of Quadrotor we described. We choose the following parameters for our tracking controller:

$$G=\begin{pmatrix} 70 & 0 & 0 \\ 0 & 80 & 0 \\ 0 & 0 & 20 \end{pmatrix},\ k=30$$

Also a conservative estimate of 10 *ms* considered for delay in estimation system.

The limitations of control moments causes a slower convergence but finally the tracking controller tracks down the reference trajectory. Also we expect that the time constants of actuator dynamic limits the band width of tracking controller and

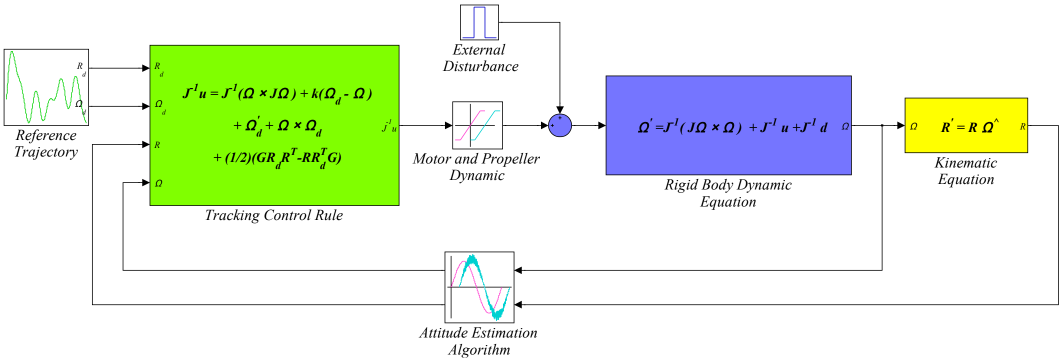

Figure 10: Tracking controller applied to model of Quadrotor

affects its ability to track very fast trajectories. Simulation results indicates a successful tracking for trajectories with frequency content, bounded to 8 Hz.

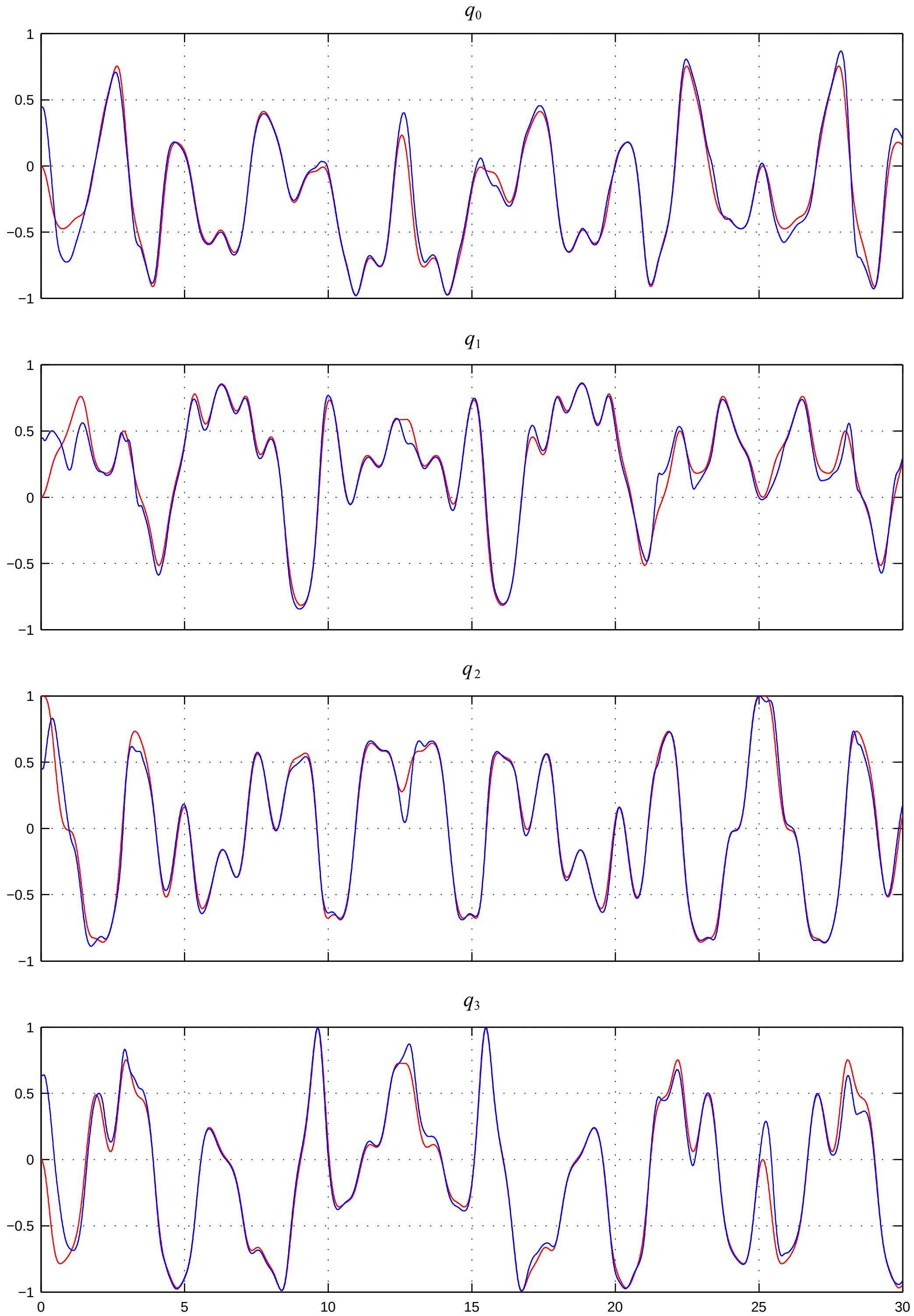

Figure 11: Quaternions representing attitude (blue) and reference trajectory (red)

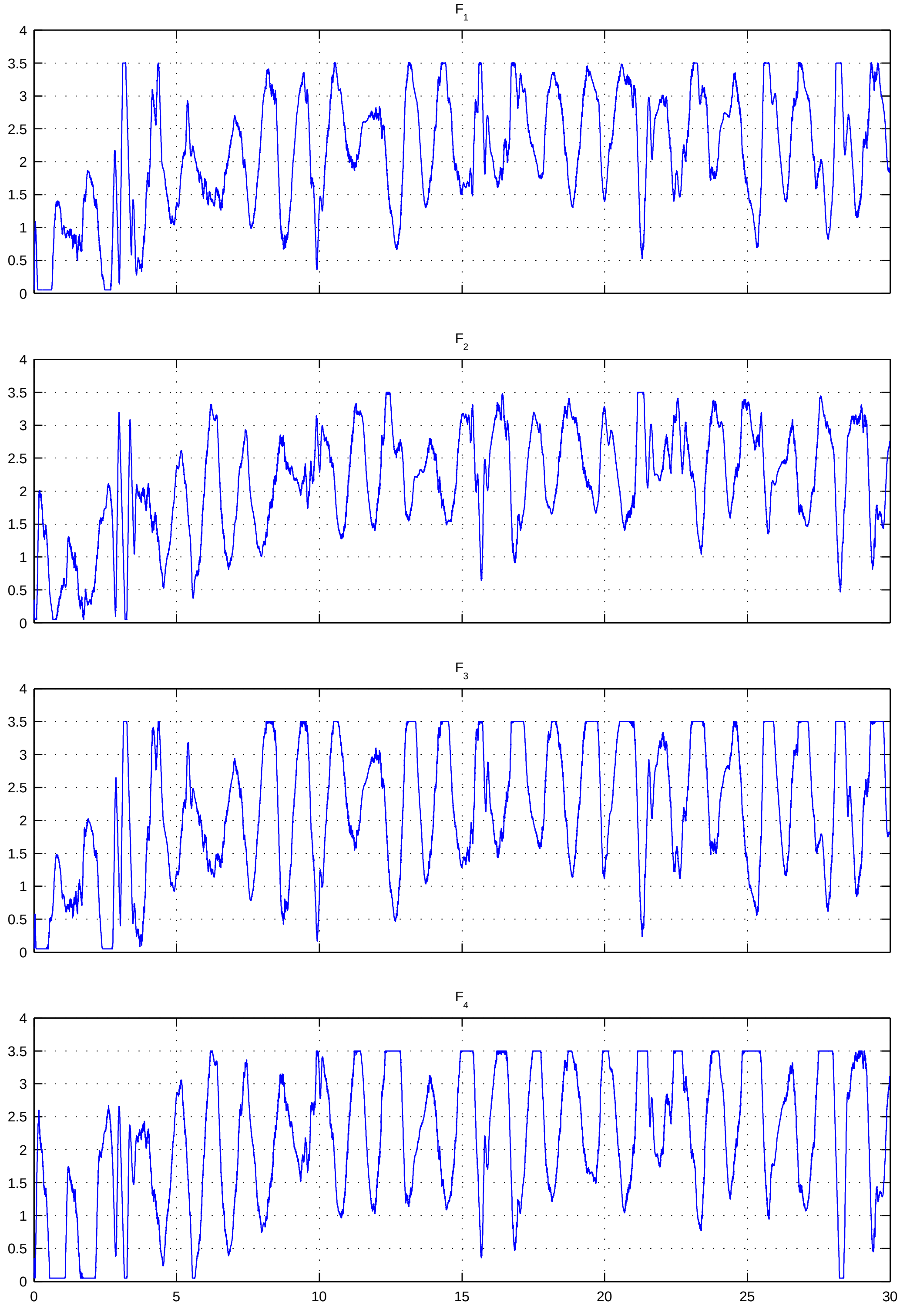

Figure 12: Control forces generated by motors

The next simulation indicates the capability of controller to maintain stability in response to disturbances.

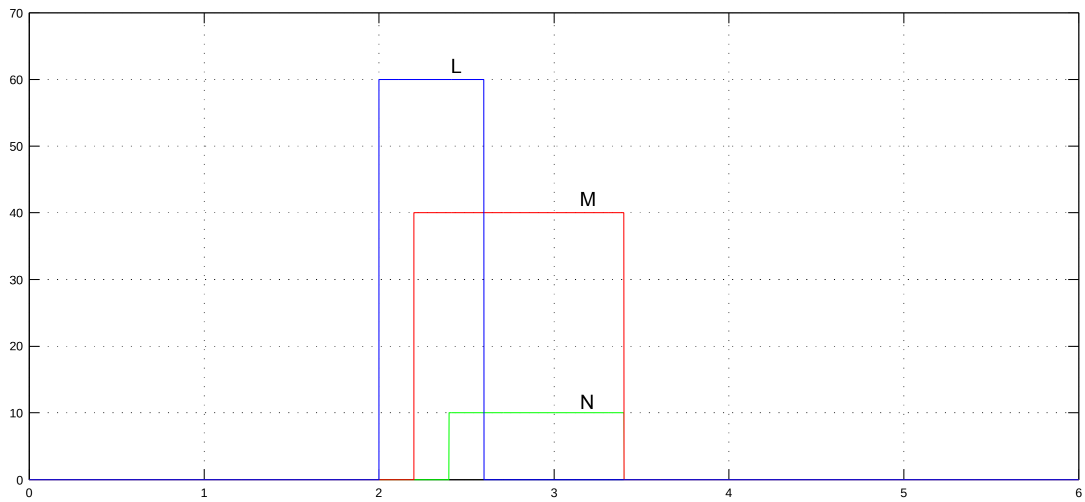

Figure 13: External disturbance moments

Three pulse functions as indicated in Figure 13 acting on Quadrotor which is in stable flight in a certain attitude and cause a fast undesired maneuvers which is tend to destabilize the system. The singularity-free nature of tracking controller resulted in a very rigid stability of original attitude and return the Quadrotor back to it after three flip-overs.

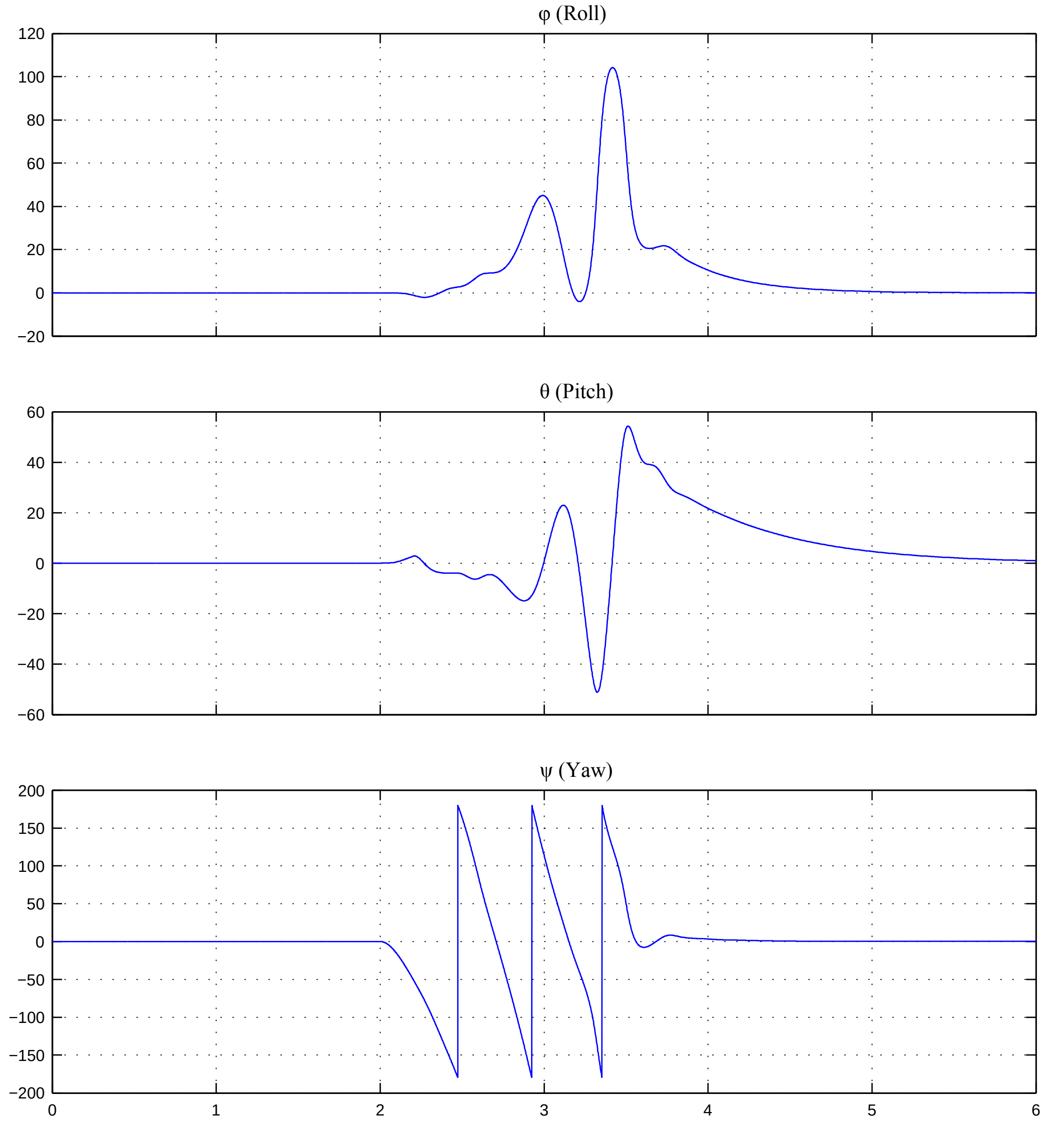

Figure 14: Rotation angels in response to external input

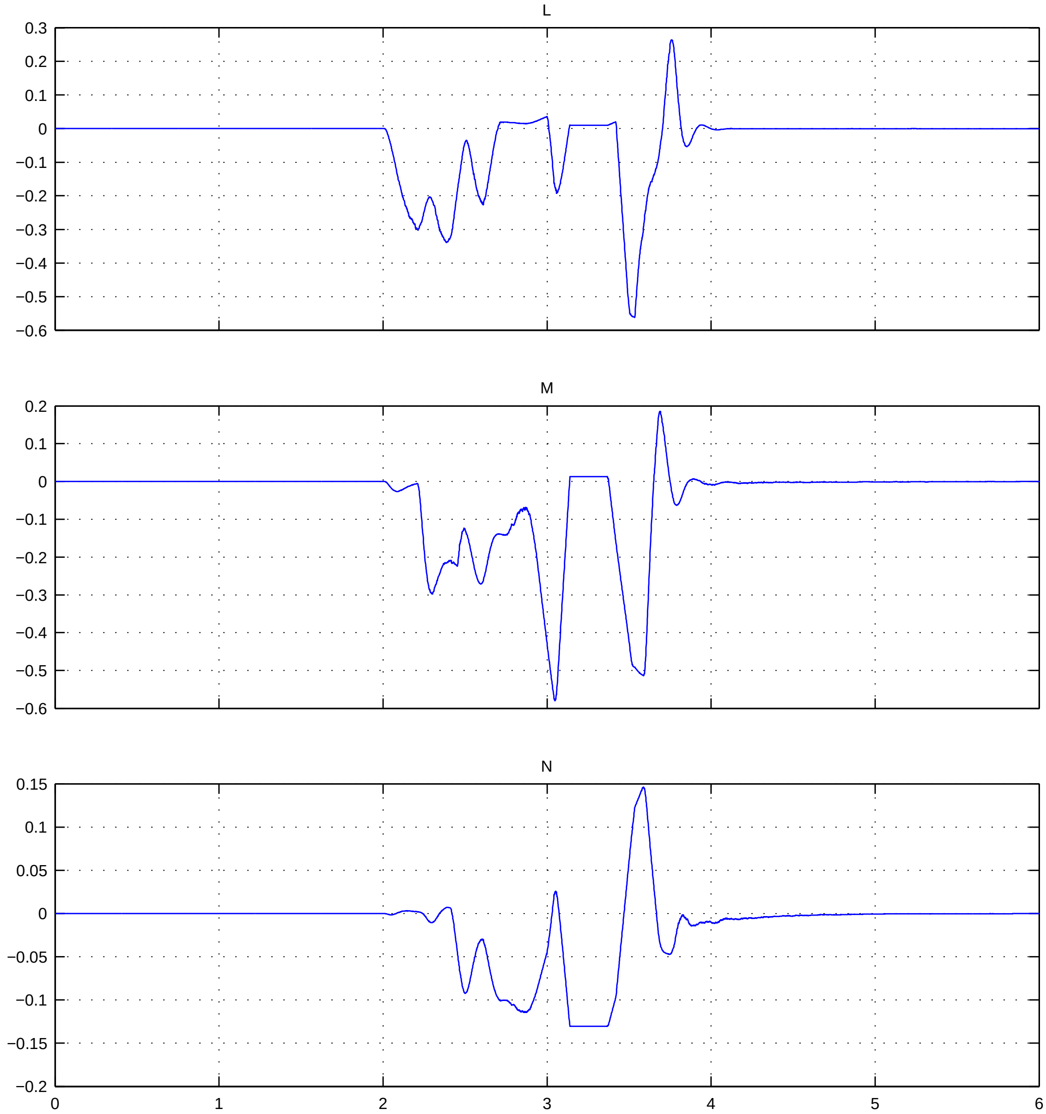

Figure 15: Control moments in response to external input

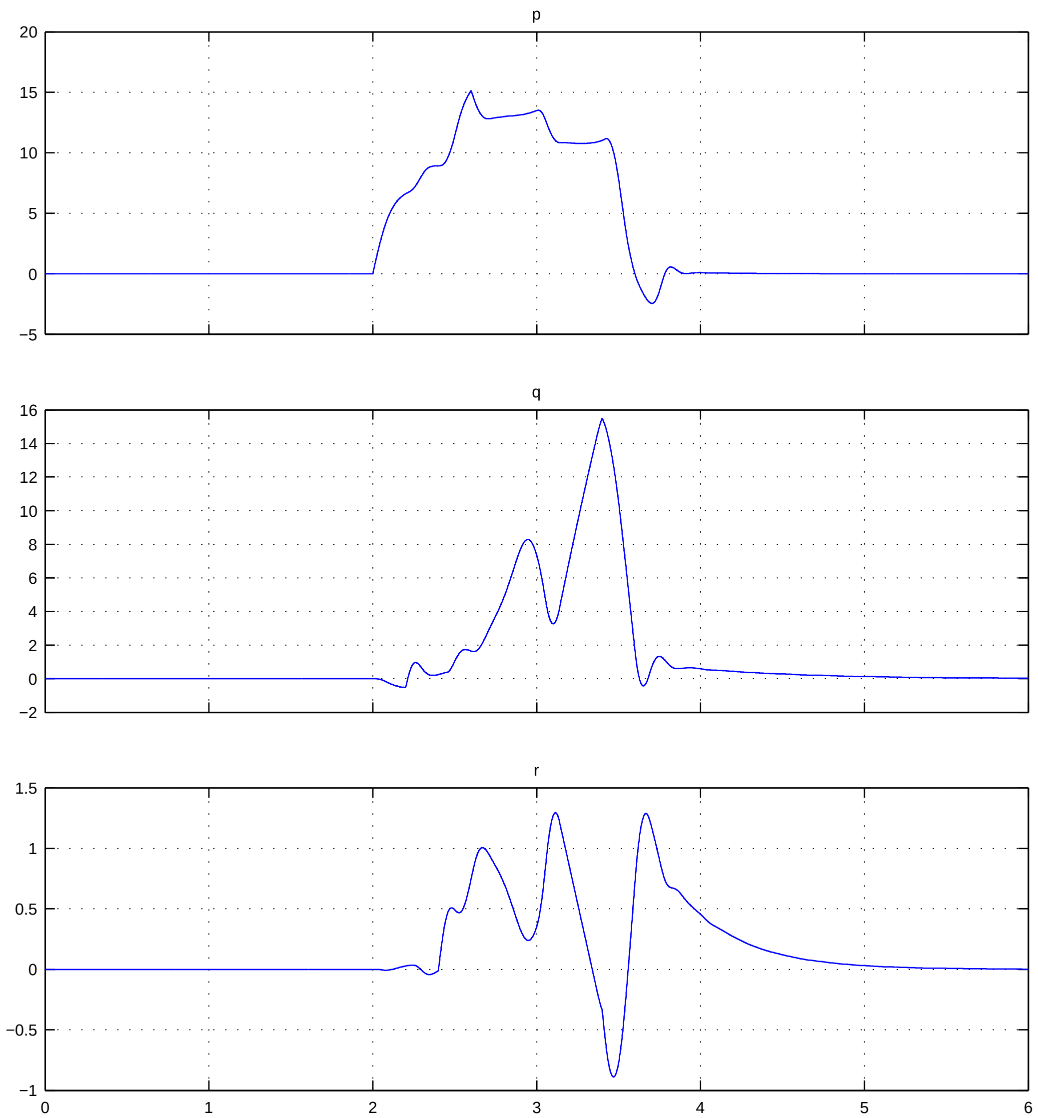

Figure 16: Angular velocities in response to external input

CHAPTER 5

CONCLUSIONS

This work seeks to answer the question of how one can define a feedback rule that globally stabilize the attitude of a rigid body. To achieve global properties no representation of attitude allowed and one has to work with the rotation matrices directly. These rotation matrices belong to class of mathematical structures called Lie groups, so instead of studying only rotation matrices we turned to the general problem of tracking control on a Lie group. We defined a tracking error function on a general Lie group and proved that the dynamic of this tracking error evolves on the same Lie group and the tracking problem reduces to stabilizing one equilibrium point of this error dynamic. We introduced the configuration error function, to be a Morse function which posses the property that the nature of its critical points completely determined by their hessian matrix. To stabilize the desired equilibrium of error dynamic we chose a feedback rule to be the gradient of the configuration error function and by linearizing equations of error dynamic around these equilibrium points we proved that the nature of extermum of configuration error function on being a minimum, maximum of saddle point determines the dynamic behavior of equilibrium points of error dynamics in the sense that weather they are stable, unstable of saddle points. Using LaSalle invariance theorem on a compact Lie group we proved that on a dense subset of configuration manifold the trajectories of system converge to the desired equilibrium point which is the best possible result considering that global asymptotic stability in not possible in a non-Euclidean space.

We computed the control rule in a special case of space of rotation matrices to be able to apply it to rotational dynamic of a Quadrotor UAV. Numerical simulations of the ideal case without considering actuator dynamics or any other imperfection of a real system, proved the validity of mathematical computations. Finally based on previous works, a model of real Quadrotor used for more real simulations. Other than limitations imposed by delays of attitude estimation system or time constants of actuator dynamics, the tracking controller perform its task perfectly in tracking reference trajectory and stabilizing the rotational dynamics.

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APPENDIX A

ATTITUDE PARAMETRIZATION AND CONTROL

In this appendix we review some results regarding attitude parametrization and their shortcomings when being used in controller design to achieve globally attractive stabilizers.

The rigid body rotations in three denominational space in their geometric meaning form a group and this group of rigid rotations has a representation as a matrix group of 3×3 orthogonal matrices with unit determinant called special orthogonal group or $SO(3)$. This means that there exist a one-one correspondence between physical rotations of three dimensional Euclidean space and rotation matrices. The space of rotation matrices also poses the structure of a three dimensional smooth manifold. This means that at each point on this manifold, the tangent space has constant rank of three or in the other words, at each point the angular velocity would be a three dimensional vector. It also means that each point on this manifold has a neighborhood around it and there exist a smooth function with continuous inverse which maps that neighborhood to three dimensional Euclidean space (a chart). In other words, three is the smallest number of independent variables that one needs to represent the attitude locally.

Now let us ask the question if these three numbers are enough to identify every single attitude? If the answer is positive it means that we can find a three dimensional parametrization of attitude without singularity because singularities of representations, like rotation angles are coming from the fact that those smooth functions we mentioned above, are not well-defined at some points. It is proven that such functions can not be found [13], or it is not possible to embed $SO(3)$ in three dimensional Euclidean space. It is also proven that the smallest n that $SO(3)$ can be embedded in $\mathbb{R}^n$ is five [13], which means that five is the smallest number of parameters that we need for a unique and singularity-free representation of attitude. It is also proven that we need at least four overlapping charts to cover entire $SO(3)$ [25].

Unit quaternions among the common attitude parametrization are an exception in the sense that they do not reside in an Euclidean space so they do not fit into our discussion above. They provide a singularity-free parametrization of attitude but the mapping from unit quaternions to $SO(3)$ is not one-one and there exist two quaternions for each physical rotation. This issue needs to be taken into account to analyze global behavior of system when quaternions are involved in equations of dynamical system subject to study. Since for every rotation q and $-q$ represent the same attitude, our dynamical system can be stable at one of them and unstable at the other. Starting from an initial condition very close to unstable one the quaternion dynamical system has a natural motion towards the stable point. In physical system since we started from a point very close to desired attitude it is expected that system with a small maneuver comes to rest at equilibrium point but the motion of system in quaternion space can cause a huge maneuver of physical system before it ends up at a point very close to the starting point [13]. By Lyapunov's definition of stability this means the unsuitability of physical system while the quaternion system might be asymptotically stable. As an example, consider the following system with quaternion PD feedback from [16].

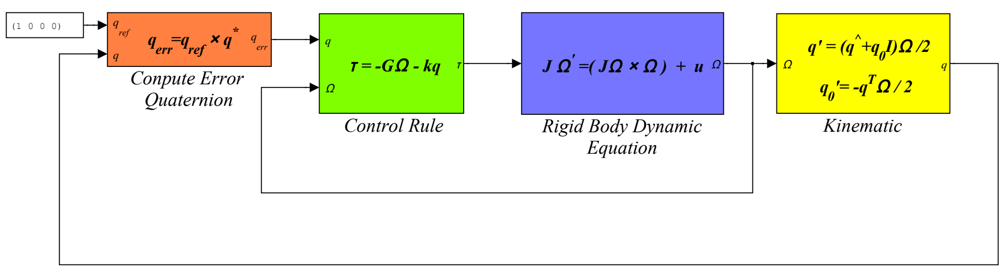

Figure 17: Quaternion PD controller

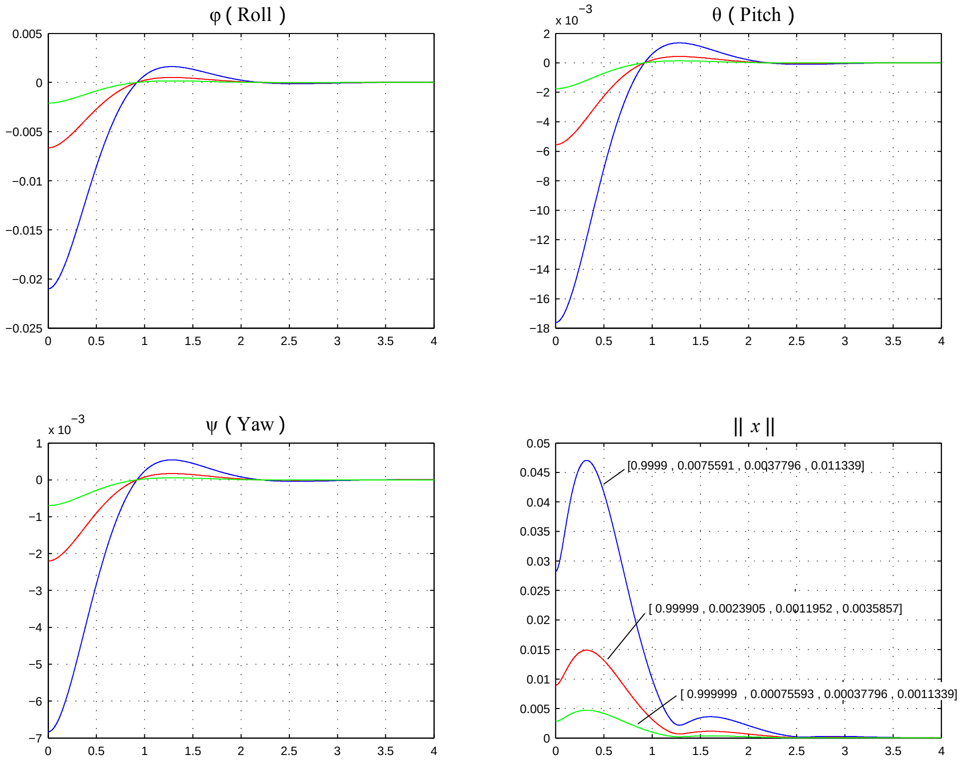

Figure 18: Starting from positive quaternions

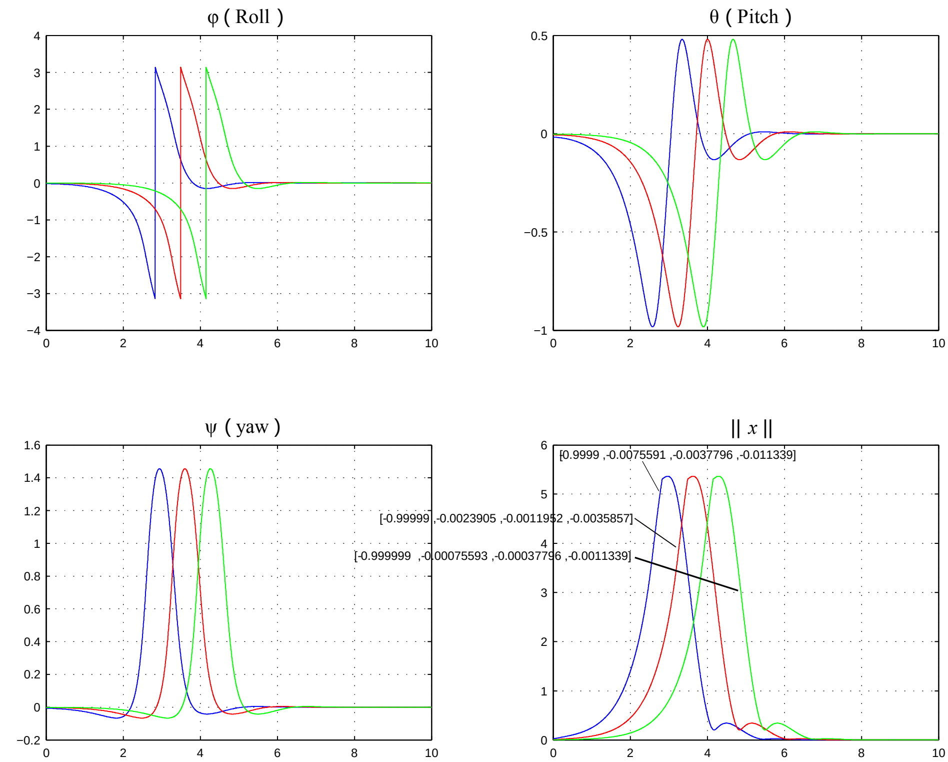

Figure 19: Starting from negative quaternions

The article claims to prove global asymptotic stability (Theorem 2). Let us see what it means for physical system. To have a measure of closeness to equilibrium point let us consider the norm of system state $x=[\varphi, \theta, \psi, p, q, r]^T$ which is composed of rotation angels and angular velocities.

As it can clearly be seen in Figure 18, as we get closer to the origin,the maximum of norm of state vector gets smaller as well. Actually according to definition of Lyapunov stability, we can make the norm of state vector arbitrarily small by getting close enough to origin. Now let us consider the negative of the same initial quaternions which in physical system correspond to exactly the same attitudes. As it is indicated in Figure 19 no matter how close we get to origin the maximum of state vector norm, remains greater than a certain value. It means that the system is unstable according to Lyapunov definition of stability.

Result of this discussion is that the two-one correspondence between quaternions and rotations of rigid body leads to unwinding phenomenon and to make any claim about global stability properties of a control system involving quaternions, one needs to consider this effect.