

MODELING CORRELATION STRUCTURE FOR COLLATERALIZED DEBT
OBLIGATIONS AND DETERMINING THE UNDERLYING CREDIT DEFAULT
SWAP SPREAD EQUATIONS

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DEFAULT SWAP SPREAD EQUATIONS**

submitted by **DENİZ İLALAN** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy in Department of Financial Mathematics, Middle East Technical University** by,

Prof. Dr. Bülent Karasözen
Director, Graduate School of **Applied Mathematics**

Assoc. Prof. Dr. Sevtap Kestel
Head of Department, **Financial Mathematics**

Assoc. Prof. Dr. Azize Hayfavi
Supervisor, **Financial Mathematics, METU**

Assoc. Prof. Dr. Tolga Omay
Co-supervisor, **Department of Banking and Insurance,
Cankaya University**

Examining Committee Members:

Assist. Prof. Dr. Seza Danişođlu
Business Administration, METU

Prof. Dr. Gerhard Wilhelm Weber
Financial Mathematics, METU

Assoc. Prof. Dr. Azize Hayfavi
Financial Mathematics, METU

Assoc. Prof. Dr. Ece Ceylan Karadađlı
Banking and Finance, Cankaya University

Assoc. Prof. Dr. Kasırđa Yıldırak
Actuarial Sciences, Hacettepe University

Date: _____

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last Name: DENİZ İLALAN

Signature :

ABSTRACT

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İlalan, Deniz

Ph.D., Department of Financial Mathematics

Supervisor : Assoc. Prof. Dr. Azize Hayfavi

Co-Supervisor : Assoc. Prof. Dr. Tolga Omay

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Pricing complex financial derivatives such as collateralized debt obligations (CDOs) is considered as the main reason triggering the 2008 financial crisis. The correlation structure related to the credit risks involved in a portfolio for pricing issues have been tried to overcome via a Gaussian copula framework first introduced by David Li. This approach regards the correlation among the credit risks as normally distributed, enabling us to derive analytical solutions. However, despite its simplicity, this Gaussian copula approach is far from reality, which caused mispricing of the tranches of CDOs. This phenomenon is called the correlation smile. Firstly, this thesis approaches the correlation smile issue by considering a Lévy copula framework. When this is introduced to pricing equations we saw that the correlation smile is “corrected”. Thus, we came up with a more accurate model of pricing the above mentioned tranches. The second part of the thesis aims to model the Itraxx 125 CDS spreads for different sectors which comprise the CDO. Here, we introduce an autocorrelation one process together with finite number of Fourier series terms. Introduction of Fourier series to estimate the dynamics of the process is not done in an ad-hoc manner or as done before in dealing with seasonality. Here the moving average is transformed to a “moving and fluctuating” average by the help of Fourier series. The rationale behind this “moving and fluctuating” averaging technique is due to its capability in removing high frequency structures like breaks, spikes and stochastic volatility. Instead of adding jump structures to the

model which makes the parameter estimation quite cumbersome, our model in discrete time can easily be transformed to a well-known mean reverting continuous time process. Moreover, our alternative model is a quite powerful and accurate forecasting technique.

Keywords: Collateralized Debt Obligation (CDO), Credit Default Swap (CDS), Copula, Fourier Series, Forecasting

ÖZ

TEMİNATLI BORÇ YÜKÜMLÜLÜKLERİ İÇİN KORELASYON YAPISININ MODELLENMESİ VE BUNA TEMEL TEŞKİL EDEN KREDİ TEMERRÜT TAKASI PRİM DENKLEMLERİNİN BELİRLENMESİ

İlalan, Deniz

Doktora, Finansal Matematik Bölümü

Tez Yöneticisi : Doç. Dr. Azize Hayfavi

Ortak Tez Yöneticisi : Doç. Dr. Tolga Omay

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Teminatlı borç yükümlülüğü (TBY) gibi karmaşık finansal türev ürün fiyatlamasının 2008 krizini tetikleyen ana unsur olduğu düşünülmektedir. Fiyatlama problemi, portföy içerisindeki temerrüt risklerinin birbirleriyle olan korelasyon yapısının ilk kez David Li tarafından önerilen Gauss bağlantısı sayesinde aşılmaya çalışılmıştır. Bu yaklaşım temerrüt korelasyonlarının normal dağıldığını varsaymak suretiyle analitik çözümler bulmamıza olanak tanımaktadır. Ancak basitliğine rağmen Gauss bağlantısı yaklaşımı gerçekten oldukça uzak olduğundan ötürü TBY'ne ait bölmelerin yanlış fiyatlanmasına sebep olmuştur. Buna korelasyon gülümsemesi adı verilmektedir. Bu tezde ilk olarak Lévy bağlantısı kavramı kullanılmak suretiyle bu korelasyon gülümsemesine yaklaşılmaktadır. Bunun fiyatlama denklemlerine dahil edildiğinde korelasyon gülümsemesinin düzeldiğini gördük. Dolayısıyla yukarıda bahsi geçen kısımların daha doğru fiyatlamasını yapan bir modelle karşı karşıyayız. Tezin ikinci kısmı ise TBY'ni oluşturan Itraxx isimli 125 adet kredi temerrüt takasının (KTT) primlerini modellemeyi amaçlamaktadır. Burada birinci dereceden öz ilintili ile beraber sonlu sayıda Fourier serisi terimini içeren bir süreç ortaya konulmaktadır. Süreç dinamiklerini tahmin etmede Fourier serisi, geçici bir nedenle ya da bundan öncesinde olduğu gibi mevsimselliği açıklama amacıyla kullanılmamıştır. Burada hareketli ortalama, Fourier serisi yardımıyla “hareketli ve dalgalı” ortalamaya dönüştürülmüştür. Bu “hareketli ve dalgalı” ortalamanın altında yatan mantık kırılma, ani yükseliş ve rassal oynaklık gibi yüksek frekanslı yapıların yakalanabilmesidir. Parametre tahminini güçleştiren

sıçrama yapılarını modele koymak yerine, süreksiz zamandaki bu modelimiz sürekli zamanda oldukça iyi bilinen bir ortalamaya geri dönen sürece dönüştürülebilmektedir. Ayrıca, önerdiğimiz bu alternatif model oldukça güçlü ve doğru bir kestirim yöntemidir.

Anahtar Kelimeler: Teminatlı Borç Yükümlülüğü (TBY), Kredi Temerrüt Takası (KTT), Bağlantı, Fourier Serisi, Kestirim

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CHAPTER 1

INTRODUCTION

Credit derivatives market drew quite a lot of attention beginning from 1998 and experienced a vast growth until the financial market crisis in 2008. The asset price bubble in 2008 caused a shrinkage, however, could not be able to eliminate huge trading amounts of these products.

Typically, a credit derivative is a financial asset which regards possibility of the default of a certain underlying instrument. A widely used credit derivative is a credit default swap (CDS). In a CDS agreement, the protection buyer is insured against the default of an asset backed loan via predetermined periodic payments to the protection seller just like an ordinary insurance contract.

On the other hand, a collateralized debt obligation (CDO) is another extensively used credit derivative which is usually regarded as the main instrument triggering the 2008 world financial crisis. CDO is a portfolio of credit derivatives which consists of multiple tranches, each having a different risk and return profile.

The structure of a CDO is as follows: Regional banks assign certain loans to public usually collateralized with mortgage. These loans are paid with periodic predetermined installments. An investment bank acts as a gatherer of these loans in a pool. When these loans are collected, the investment bank arranges certain tranches as senior, mezzanine and junior (equity). Whenever the pool is filled with payments the senior tranche is firstly paid, then mezzanine and finally the junior tranche gets paid. In that sense junior tranche is the most vulnerable among them. If defaults happen there will be no flow of payment from the loan pool to the junior tranche. Thus junior tranche is attractive to risk lover investors. Since more risk means more return, the junior tranche owners are promised to be paid higher return than other tranches. The return structure gradually decreases from junior to senior tranches. The investment banks were rated by the trusted rating agencies where even one day prior to sub prime meltdown Lehman Brothers was rated top by three big ratings namely Standards & Poors, Fitch and Moody's. Although these ratings placed Lehman Brothers to a position of "as reliable as the US government", still the senior tranches were paying interest which was considerably above the government T-bill rates. This was one of the attractiveness of these products.

At the beginning, the credit pools were consisting of actual loans given to public. The

loans were basically mortgage based loans, credit cards etc. Later on, the structure became a Ponzi scheme. Since the investment banks were top rated there was no cut in the demand for CDO tranches. Therefore, in order to meet higher demands, banks began to offer loans to low paying ability customers. This rendered the quality of the loan pool to diminish, thus the number of defaults to increase.

Meanwhile, the banks secured themselves via taking collateral as houses for mortgage based loans. Up to sub prime meltdown the real estate prices perform a steady growth for nearly a decade, therefore the banks safely gave those loans to low paying ability clients.

The sub prime meltdown not only raised questions regarding the credibility of investment banks, it also seriously undermined the reputation of credit rating agencies and FED. After the collapse, in order to boost up the economy, FED decided to follow some unorthodox monetary policies as quantitative easing. As a butterfly effect, major central banks throughout the world began to pump money into economies. However, this did not go as expected, since the newly created money flows to financial markets instead of real economies. Nowadays, we are still facing the problem of excess liquidity and decreasing growth in the entire world. Yet, after these loose monetary policies, we began to witness new records in stock exchanges which renders the trading of this complex derivatives to increase even further.[43]

In addition to all these, there was another quite important issue which was not taken seriously by the investment banks due to possible computation complexities. The pricing of these CDOs is not obvious in the sense that the expected time of defaults of the loan payers has to be taken into account. Moreover, it is known that during recession or depression periods, the default correlation among different sectors in the economy tends to rise. A typical CDO consists of 125 loans, so in order to calculate the price of this CDO we have to know the correlation structure of these loans which corresponds to a variance covariance matrix of 7750 cells!

It was the first attempt introduced by David Li that the correlation structure can be represented by a common factor “ a ” yielding a single factor Gaussian copula framework enabling the practitioners to compute the price of these complex financial derivatives. However, this computation technique inherited a major drawback which came to surface at the financial crisis.

The Gaussian copula approach rendered the tranches to be mispriced, thus triggered some inefficiencies in the market. Although several attempts are proposed to overcome this issue, practitioners still use this technique just as they do to compute the prices of options via Black-Scholes model, keeping in mind the volatility smile phenomenon. Here, we face another smile surface, namely the correlation smile.

First, the Gaussian copula which is presented by David Li is summarized in the sense that it is the benchmark for pricing these complex derivatives throughout the literature. Therefore, in order to grasp the idea of Li, basic preliminaries regarding copulas are given. Then, the major drawback of this pricing technique is considered. Finally, our contribution to this framework via Lévy copula is explained.

Later, we took into account the CDS spreads for different sectors which comprise the CDO. Our Itraxx 125 data has seven different sectors on which the CDSs are written. We used an averaging approach due to high cross correlation of the series. In this framework, we identified CDS spread as an auto regressive process together with Fourier series terms. The key issue here lies inside the usage of Fourier series, not for capturing the seasonality or smooth breaks as done in the literature before. Introduction of Fourier series not only grasps the moving average, but also catches the points beyond the modulus of continuity bound of the Brownian motion. In that sense we introduce a “moving and fluctuating average”. In fact, this averaging technique has the capability of capturing high frequency structures like sharp breaks, spikes and stochastic volatility. Our technique is not data specific. It can be applied to any arbitrary time series which exhibits stationarity. Finally, the forecast power of our model is demonstrated.

The thesis is organized as follows:

Chapter 1: Introduction to default based derivatives and description of the overall CDO market including the economical effects. Objective of the study and our contributions to the literature.

Chapter 2: Description of the CDO pricing model proposed by David Li. Brief explanation of copulas and basic theorems. Presentation of large homogeneous portfolio approach and CDO pricing technique with single factor Gaussian copula framework.

Chapter 3: Numerical methods for pricing CDO tranches with Gaussian copula.

Chapter 4: Explanation of Lévy copulas and our alternative model. Presentation of numerical techniques and demonstration of a better pricing. Overcoming the correlation smile phenomenon.

Chapter 5: Time series models and statistical tests which used in our model for pricing the CDS spreads.

Chapter 6: Analysis of our model with a literature survey in this field.

Chapter 7: Application of our model to a real CDS spread data.

Chapter 8: Forecasting CDS spreads with our model.

Chapter 9: Summary and concluding remarks.

CHAPTER 2

PRICING CDO TRANCHES VIA GAUSSIAN COPULA FRAMEWORK

The technique applied by the practitioners in order to price the CDO tranches is introduced by David Li [36, 37] which takes into account the Gaussian copula as a benchmark. Therefore, we begin by introducing copulas and related theorems. For detailed explanations, theorems and proofs see [14, 15, 16, 24, 41, 44, 48, 52, 53, 54].

2.1 Copulas

2.1.1 Basic Definitions and Preliminaries

Definition 2.1. Let X and Y be random variables with distribution functions and a joint distribution function. For each pair of real numbers (x, y) , three numbers can be associated, $F(x)$, $G(y)$ and $H(x, y)$, each lying on the interval $[0, 1]$. In other words, each pair of (x, y) is carried to a point $(F(x), G(y))$ in the unit square, and this ordered pair is connected to a number $H(x, y)$ in $[0, 1]$. This correspondence, which gives the value of the joint distribution function to each ordered pair of values of individual functions, is called a copula.

Generally, the copula of an n -dimensional random vector is an n -tuple characterizing the dependence structure, independent from margins. The pair copula and margins gives another explanation of the law of a random vector.

Definition 2.2. Let S_1 and S_2 be nonempty subsets of $\bar{\mathbb{R}}$ of where $\bar{\mathbb{R}}$ is the extended real line. Let H be a two-place real function such that its domain, $\text{Dom}H = S_1 \times S_2$. Let $B = [x_1, x_2] \times [y_1, y_2]$ be a rectangle all of whose vertices are in $\text{Dom}H$. Then H -volume of B is

$$V_H(B) = H(x_2, y_2) - H(x_2, y_1) - H(x_1, y_2) + H(x_1, y_1).$$

Definition 2.3. A two-place real function H is two-increasing if $V_H(B) \geq 0$ for all rectangles B whose vertices are in $\text{Dom}H$.

Definition 2.4. We say that the function H from $S_1 \times S_2$ into \mathbb{R} is grounded if

$$H(x, a_2) = 0 = H(a_1, y)$$

for all (x, y) in $S_1 \times S_2$.

Definition 2.5. A two-dimensional sub copula is a function C^* with the following properties:

- $\text{Dom}C^* = S_1 \times S_2$, where S_1 and S_2 are subsets of $I=[0,1]$.
- C^* is grounded and two-increasing.
- For every u in S_1 and every v in S_2 , $C^*(u, 1) = u$ and $C^*(1, v) = v$.

Since for every (u, v) in $\text{Dom}C^*$, we have

$$0 \leq C^*(u, v) \leq 1,$$

it follows that

$$\text{Ran}C^* \subset I = [0, 1].$$

Definition 2.6. A two-dimensional copula is a two sub copula C whose domain is unit square.

Theorem 2.1. Let C^* be a sub copula. Then for every (u, v) in $\text{Dom}C^*$

$$\max(u + v - 1, 0) \leq C^*(u, v) \leq \min(u, v).$$

Proof. Let (u, v) be any point in $\text{Dom}C^*$. Now

$$C^*(u, v) \leq C^*(u, 1) = u,$$

and

$$C^*(u, v) \leq C^*(1, v) = v;$$

therefore,

$$C^*(u, v) \leq \min(u, v).$$

On the other hand since

$$C^*(u_2, v_2) - C^*(u_2, v_1) - C^*(u_1, v_2) + C^*(u_1, v_1) \geq 0,$$

setting $u_2 = v_2 = 1$, $u_1 = u$, $v_1 = v$, we have

$$C^*(u, v) \geq \max(u + v - 1, 0).$$

□

These upper and lower bounds are known as Frechet-Hoeffding copula bounds.

Theorem 2.2. Let C^* be a sub copula. Then for every $(u_1, u_2), (v_1, v_2)$ in $\text{Dom}C^*$ we have

$$|C^*(u_2, v_2) - C^*(u_1, v_1)| \leq |(u_2 - u_1)| + |v_2 - v_1|.$$

Definition 2.7. A distribution function F is a function with domain $\bar{\mathbb{R}}$ such that

- F is non-decreasing,
- $F(-\infty) = 0$ and $F(\infty) = 1$.

Definition 2.8. A joint distribution function H is a function with domain $\bar{\mathbb{R}}^2$ such that

- H is two-increasing,
- $H(x, -\infty) = H(-\infty, y) = 0$ and $H(\infty, \infty) = 1$.

2.1.2 Sklar's Theorem

Theorem 2.3. Let H be a joint distribution function with boundaries F and G . Then there is a copula C such that for all x, y in $\bar{\mathbb{R}}$,

$$H(x, y) = C(F(x), G(y)).$$

If F and G are continuous, then C is unique; otherwise, C is uniquely determined on $\text{Ran}F \times \text{Ran}G$. On the other hand, if C is a copula and F and G are distribution functions, then the function H defined by

$$H(x, y) = C(F(x), G(y))$$

is a joint distribution function with margins F and G .

For the proof of Sklar's Theorem we consider the following:

Lemma 2.4. Let H be a joint distribution function with margins F and G . Then there exists a unique sub copula C^* such that for all x, y in ,

$$H(x, y) = C^*(F(x), G(y)).$$

Proof. The joint distribution H with $S_1 = S_2 = \bar{\mathbb{R}}$ for any points (x_1, y_1) and (x_2, y_2) in $\bar{\mathbb{R}}^2$ satisfies

$$|H(x_2, y_2) - H(x_1, y_1)| \leq |F(x_2) - F(x_1)| + |G(y_2) - G(y_1)|.$$

If $F(x_1) = F(x_2)$ and $G(y_1) = G(y_2)$, then $H(x_1, y_1) = H(x_2, y_2)$. Hence C^* is well defined. Thus the set

$$\{(F(x), G(y), H(x, y)), x, y \in \bar{\mathbb{R}}\}$$

defines a two-place real function whose domain is $\text{Ran}F \times \text{Ran}G$. Note that for each u in $\text{Ran}F$, there is an x in $\bar{\mathbb{R}}$ such that $F(x) = u$ hence

$$C^*(F(x), G(\infty)) = H(x, \infty) = F(x) = u.$$

Similarly

$$C^*(F(\infty), G(y)) = H(\infty, y) = G(y) = v.$$

Since H is a joint distribution function it is two increasing and grounded. Therefore C^* is a sub copula. \square

Lemma 2.5. *Let C^* be a sub copula. Then there exists a copula C such that*

$$C(u, v) = C^*(u, v),$$

for all (u, v) in $\text{Dom}C^$. That is any sub copula can be extended to a copula usually in a non-unique manner.*

Proof. (Sklar) It directly follows from the above lemmas. In addition, if F and G are continuous, then $\text{Ran}F = \text{Ran}G = I$. \square

2.1.3 Multivariate Copulas

The definitions and theorems in the previous sections were for two dimensions. They can be extended to higher dimensions as well. For this section it will be sufficient to give the definition of a rectangle in n dimensions and the corresponding Sklar's theorem.

Definition 2.9. Let S_1, S_2, \dots, S_n be nonempty subsets of $\bar{\mathbb{R}}$, and H be an n -tuple real valued function such that

$$\text{Dom}H = S_1 \times S_2 \times \dots \times S_n.$$

Let $B = [0, 1]$ an n -box all of whose vertices are in $\text{Dom}H$. The H volume of B is given by

$$V_H(B) = \sum \text{sgn}(c)H(c),$$

where sum is taken over all vertices c of B , and $\text{sgn}(c)$ is given by

$$\text{sgn}(c) = \begin{cases} 1 & , \text{if } c_k = a_k \text{ for an even number of } k\text{'s,} \\ -1 & , \text{if } c_k = a_k \text{ for an odd number of } k\text{'s.} \end{cases}$$

Theorem 2.6. *Let H be an n -dimensional distribution function with margins F_1, F_2, \dots, F_n . Then there is an n -copula C such that for all x in $\bar{\mathbb{R}}^n$,*

$$H(x_1, x_2, \dots, x_n) = C(F(x_1), F(x_2), \dots, F(x_n)).$$

The uniqueness property satisfied by the continuity of the distribution functions in two dimensions is also valid for n -dimensional case, where all the distributions should all together be continuous. Otherwise, the copula is uniquely given in the Cartesian product of range of individual distribution functions as before.

2.1.4 Expected Tranche Loss of a CDO

Let us consider a synthetic CDO, a CDO formed by CDSs. A protection seller of a synthetic CDO gets intermittent payments from a protection buyer, acting as a security for certain losses of subordinated tranches. The losses that the protection seller has to reimburse are determined via a reference credit portfolio.

The pricing of a synthetic CDO works similar as the pricing of ordinary CDSs since a synthetic CDO consists of CDSs. To calculate the loss of a certain tranche say from K_1 to K_2 with $0 \leq K_1 \leq K_2 \leq 1$ of the reference portfolio, let us make further assumptions and introduce some new notations.

Assume that the spread payment dates are discrete, $t_1 \leq t_2 \leq \dots \leq t_n = T$ where T denotes the maturity of the synthetic CDO. Actually, this makes quite sense since the payment due dates are also discrete in real life. Some further relevant notations are as follows

- s denotes the yearly spread payments made by the protection buyer.
- $L_{K_1 K_2}^R(t)$ denotes the loss of tranche K_1, K_2 up to time t , taking into consideration possibility of recoveries. Usually models assume zero recovery, which is the assumption here; however, conventionally recovery rate is taken as 40%. Moreover, there are models taking into account stochastic recovery rates which are beyond the aim of this work.
- Short term interest rate $r(t)$ is given, considered to be constant and free from tranche loss. Again, there are studies considering stochastic interest rate which according to us is not the main item in pricing synthetic CDOs.

Now, utilizing from Girsanov's Theorem, consider the risk neutral measure and denote it with Q . Indicate expectation of the above mentioned tranche loss under this new measure by $\mathbb{E}_Q[L_{K_1 K_2}^R(t)]$ or simply by $\mathbb{E}[L_{K_1 K_2}^R(t)]$. The discount factor is

$$\mathbb{E}_Q[\exp(-\int_{t_0}^{t_1} r(u)du)]$$

or in short $D(t_0, t_1)$.

The valuation of a synthetic CDO very much looks like the valuation of a simple swap. One first has to evaluate the present value of spread payments, then evaluate discounted value of protection payments taking into account the expectation of defaults, and finally equalize them.

The value of the premium leg is computed as follows

$$\begin{aligned} \text{Premium Leg} &= \sum_{i=1}^n \Delta t_i \cdot s \cdot \mathbb{E}_Q \left[(1 - L_{K_1 K_2}^R(t)) \exp \left(- \int_{t_0}^{t_i} r(u)du \right) \right] \\ &= \sum_{i=1}^n \Delta t_i \cdot s \cdot [(1 - \mathbb{E}L_{K_1 K_2}^R(t))D(t_0, t_i)] \end{aligned}$$

where $\Delta t_i = t_i - t_{i-1}$.

Protection payments are made in case of a default. For simplicity, integration is avoided by discretization. Therefore we have the following:

$$\begin{aligned}
\text{Protection Leg} &= \mathbb{E}_Q \left[\int_{t_0}^{t_1} \exp \left(- \int_{t_0}^s r(u) du \right) dL_{K_1 K_2}^R(s) \right] \\
&\approx \sum_{i=1}^n \mathbb{E}_Q \exp \left[\left(- \int_{t_0}^{t_i} r(u) du \right) (L_{K_1 K_2}^R(t_i) - L_{K_1 K_2}^R(t_{i-1})) \right] \\
&= \sum_{i=1}^n ((\mathbb{E}L_{K_1 K_2}^R(t_i) - \mathbb{E}L_{K_1 K_2}^R(t_{i-1})) \cdot D(t_0, t_i)).
\end{aligned}$$

Equalizing the premium and protection leg yields

$$s = \frac{\sum_{i=1}^n (\mathbb{E}L_{K_1 K_2}^R(t_i) - \mathbb{E}L_{K_1 K_2}^R(t_{i-1})) \cdot D(t_0, t_i)}{\sum_{i=1}^n \Delta t_i \cdot s \cdot [(1 - \mathbb{E}L_{K_1 K_2}^R(t)) D(t_0, t_i)]}.$$

In case a loss occurs in the portfolio denoted by $L_{\text{portfolio}}^R(t)$, we can calculate the corresponding percentage loss as follows

$$L_{K_1 K_2}^R(t) = \frac{\min \left(\left(L_{\text{portfolio}}^R(t), K_2 \right) - K_1 \right)^+}{K_2 - K_1}.$$

If the discrete distribution of the aggregate loss of the reference portfolio up to time t is known (considering recoveries) and can take only m possible values then,

$$L_{\text{portfolio}}^R(t) = \left[L_{\text{portfolio}}^{R,k}(t) \quad \text{with risk neutral probability} \quad F^R(t, k) \right]_{k=1, \dots, m}.$$

Thus, we have the following

$$\begin{aligned}
\mathbb{E}L_{K_1 K_2}^R(t) &= \mathbb{E}_Q \left[\frac{\min \left(\left(L_{\text{portfolio}}^R(t), K_2 \right) - K_1 \right)^+}{K_2 - K_1} \right] \\
&= \frac{1}{K_2 - K_1} \sum_{k=1}^m \left[\frac{\min \left(\left(L_{\text{portfolio}}^{R,k}(t), K_2 \right) - K_1 \right)^+}{K_2 - K_1} \right].
\end{aligned}$$

If the loss distribution function of the portfolio is continuous, then

$$\begin{aligned}
\mathbb{E}L_{K_1K_2}^R(t) &= \frac{1}{K_2 - K_1} \int_{K_1}^1 (\min(x, K_2) - K_1) dF^R(t, x) \\
&= \frac{1}{K_2 - K_1} \left(\int_{K_1}^{K_2} (x - K_1) dF^R(t, x) + \int_{K_2}^1 (K_2 - K_1) dF^R(t, x) \right) \\
&= \frac{1}{K_2 - K_1} \left(\int_{K_1}^1 (x - K_1) dF^R(t, x) + \int_{K_2}^1 (x - K_1) dF^R(t, x) \right) \\
&\quad + \int_{K_2}^1 (K_2 - K_1) dF^R(t, x) \\
&= \frac{1}{K_2 - K_1} \left(\int_{K_1}^1 (x - K_1) dF^R(t, x) + \int_{K_2}^1 (x - K_2) dF^R(t, x) \right).
\end{aligned}$$

It can be realized that it is crucial in pricing of a CDO tranche to obtain the loss distribution function. The benchmark model for this, which will be presented below is one factor Gaussian copula model first brought in by David Li [37].

2.1.5 Large Homogeneous Portfolio Approximation and Gaussian Copula Approach

Let τ_i be a random variable denoting time to default of a firm i from the reference portfolio. Instead of focusing the firms default times one by one, this framework considers the default correlations among the firms for pricing. The Gaussian copula approach allows us to price CDO tranches without taking into account the marginal distribution of the firms. Let $A_i(t)$ be the standardized asset return of firm i up to time which is assumed to be of following form

$$A_i(t) = a_i M(t) + \sqrt{1 - a_i^2} X_i(t), \quad (2.1)$$

where $M(t)$ is the market risk and $X_i(t)$ is the firm based (idiosyncratic) risk, both standard normally distributed. (This is a commonly used convention since asset returns are almost normally distributed as the Black-Scholes option pricing formula proclaims). The above equation renders $A_i(t)$ to be standard normally distributed as well. If we denote the distribution function of default time τ_i by Q_i , then the issuer said to be defaulted before time t when

$$\phi[A|i(t)] \leq Q|i(t),$$

or, equivalently,

$$A_i(t) \leq \phi^{-1}(Q_i(t)) =: C_i(t).$$

Here, $Q_i(t) = Q[\tau_i \leq t]$, where Q is the risk neutral probability implied from observable market prices. (In fact, as we shall see later, in the Gaussian copula frameworks

the whole asset return structure of the firms reduce to a single component, which is derived from the existing market prices, enabling us to price the tranches of CDOs). According to (2.1) the i^{th} issuer defaults up to time t , when

$$X_i(t) = \frac{C_i(t) - a_i M(t)}{\sqrt{1 - a_i^2}},$$

Then probability that i^{th} issuer defaults up to time t , given $M(t)$ becomes,

$$p_i(t|M) \leq \phi \left(\frac{C_i(t) - a_i M(t)}{\sqrt{1 - a_i^2}} \right).$$

2.1.6 Loss Distribution of the Large Homogeneous Portfolio Under One Factor Gaussian Model

Just like the idea of central limit theorem, here the aim is to derive analytical results for pricing the CDO tranches for the limiting cases. The related assumptions for the credit issuers now have the same:

- Portfolio weights,
- Default probability $Q(t)$,
- Recovery rate R ,
- Correlation to the sensitivity of the market.

$$p(t|M) \leq \phi \left(\frac{C(t) - aM(t)}{\sqrt{1 - a^2}} \right),$$

Proposition 2.7. *For any p and x in $(0,1]$ the following holds:*

$$\lim_{m \rightarrow \infty} \sum_{k=0}^{\lfloor mx \rfloor} \binom{m}{k} p^k (1-p)^{(m-k)} = \begin{cases} 0 & , \text{if } x < p, \\ 1 & , \text{if } x > p. \end{cases}$$

Lemma 2.8. *For the case of no recovery for all assets forming the portfolio, the loss distribution of an infinitely large homogeneous portfolio with asset returns follows a one factor Gaussian copula model as*

$$A_i(t) = aM(t) + \sqrt{1 - a^2} X_i(t),$$

$$F_\infty(t, x) = \phi \left(\frac{\sqrt{1 - a^2} \phi^{-1} - C(t)}{a} \right),$$

where $x \in [0, 1]$ is the percentage loss of the portfolio.

Under assumptions of large homogeneous portfolio model, the integrals can be computed analytically as the following lemma asserts:

Lemma 2.9. *In the large homogeneous portfolio model, expected loss at time t of mezzanine tranche absorbing losses from K_1 to K_2 percent of the overall portfolio in case of zero recovery is:*

$$\mathbb{E}L_{K_1K_2}(t) = \frac{\phi_2(-\phi^{-1}(K_1, C(t), \rho)) - \phi_2(-\phi^{-1}(K_2, C(t), \rho))}{K_2 - K_1},$$

where ϕ_2 is bivariate normal distribution function with variance covariance matrix

$$\rho = \begin{pmatrix} 1 & -\sqrt{1-a^2} \\ -\sqrt{1-a^2} & 1 \end{pmatrix}$$

CHAPTER 3

NUMERICAL APPROXIMATION OF TRANCHE PRICING WITH GAUSSIAN COPULA

The Gaussian copula approach involves some improper integrals which has to be computed numerically. Moreover, the correlation coefficient “ a ” involved in the variance covariance matrix should be calculated via an “implied correlation” from the existing market prices of CDO’s. However, having obtained “ a ” will only enables us to price the other tranches instantly. Here the aim is to offer another theoretical model which explains the correlation surfaces better than the traditional approach. Simulation yields the following (for details see [38])

3.1 Canonical Copula Form

A direct application of Sklar’s Theorem yields

$$\begin{aligned} f(x_1, x_2, \dots, x_n) &= \frac{\partial^n [C(F_1(x_1)), \dots, C(F_n(x_n))]}{\partial F_1(x_1), \dots, \partial F_n(x_n)} \times \prod_{i=1}^n f_i(x_i) \quad (3.1) \\ &= c(F_1(x_1), \dots, F_n(x_n)) \times \prod_{i=1}^n f_i(x_i) \end{aligned}$$

where $c(F_1(x_1), \dots, F_n(x_n)) = f(x_1, x_2, \dots, x_n) / \prod_{i=1}^n f_i(x_i)$

3.2 Exact Maximum Likelihood Method

Let Θ be the parameter space and θ be the k -dimensional vector of parameters to be estimated. Let $LKL(\Theta)$ and $lkl(\theta)$ be, respectively, likelihood and log-likelihood function for observation at time t . Define the log-likelihood function $lkl(\theta)$ as follows

$$lkl(\theta) = \sum_{i=1}^T lkl_t(\theta)$$

Taking into account the canonical form described in (3.1) we have

$$\text{lkl}(\theta) = \sum_{t=1}^T \ln c(F_1(x_1), \dots, (F_n(x_n))) + \sum_{t=1}^T \sum_{n=1}^N \ln f_n(x_n^t)$$

The maximum likelihood estimator $\hat{\theta}$ is defined as $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2, \dots, \hat{\theta}_k)$. For the Gaussian copula let $\Theta = \{V : V \in \mathbb{R}^{N \times N}\}$ denote parameter space with V being a symmetric and positive definite matrix. The application of canonical form yields

$$\text{lkl}^{\text{gaussian}}(\theta) = -\frac{T}{2}V - \frac{1}{2} \sum_{t=1}^T \xi_t'(V^{-1} - I)\xi_t. \quad (3.2)$$

Assuming that the log likelihood function is differentiable with respect θ and the solution of (3.2) $\partial/\partial\theta = 0$ defines a global maximum, the maximum likelihood estimator can be recovered as

$$\frac{\partial}{\partial V^{-1}} \text{lkl}^{\text{gaussian}}(\theta) = \frac{T}{2}V - \frac{1}{2} \sum_{t=1}^T \xi_t'\xi_t \Rightarrow \hat{V} = \frac{1}{T} \sum_{t=1}^T \xi_t'\xi_t$$

3.3 CDO Tranche Pricing

Let A and B the attachment and detachment points respectively. If we denote the reached loss of reference portfolio at time t as $L(t)$ the loss of the tranche can be given as:

$$L^{A,B}(t) = (L(t) - A)I_{\{A,B\}}L(t) + (B - A)I_{\{B, \sum_{i=1}^n N_i\}}L(t).$$

Just like a regular swap analysis a fair price of a CDO tranche is given by

$$s_{A,B}^* = \frac{E^* \left(\left[\int_0^T D(0,t) dL^{A,B}(t) \right] \right)}{E^* \left[\alpha \sum_{i=1}^n D(0,t_i) \min\{\max[B - L(t_i), 0], B - A\} \right]},$$

where D denotes the discount factor, the numerator being the expected loss (default leg) and the denominator being the installments (premium leg).

3.4 Simulation Results with Matlab

The Matlab code given in the appendix is divided into 3 parts:

- Generating default times via Gaussian copula by the help of log likelihood estimation.

- Equalizing the default and premium legs.
- Calculating the tranche spreads for different values of correlation and recovery rates.

For 125 loan payers, 1 year of maturity and 1000 simulations, we have Figure 3.1, Figure 3.2 and Figure 3.3

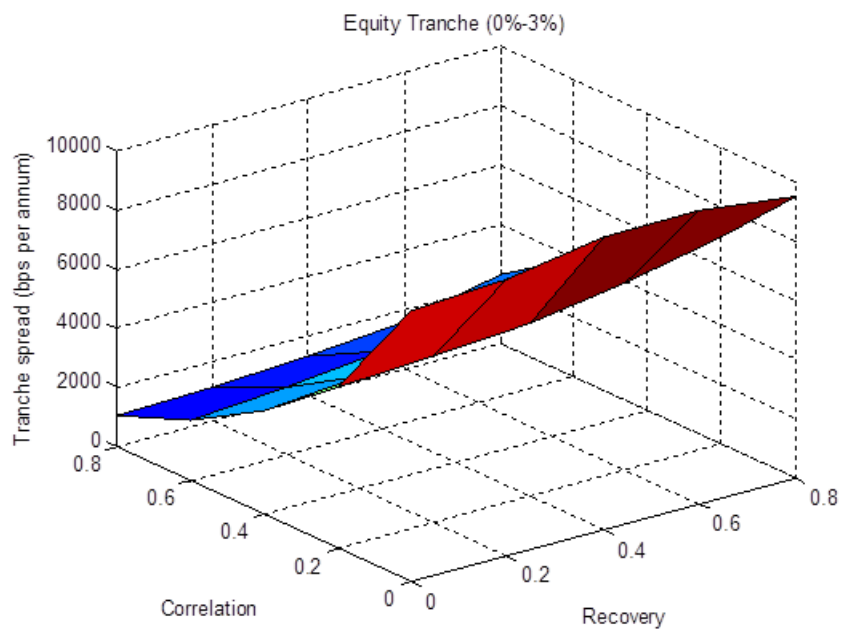


Figure 3.1: Equity tranche spread modeled with Gaussian copula for different recovery rates.

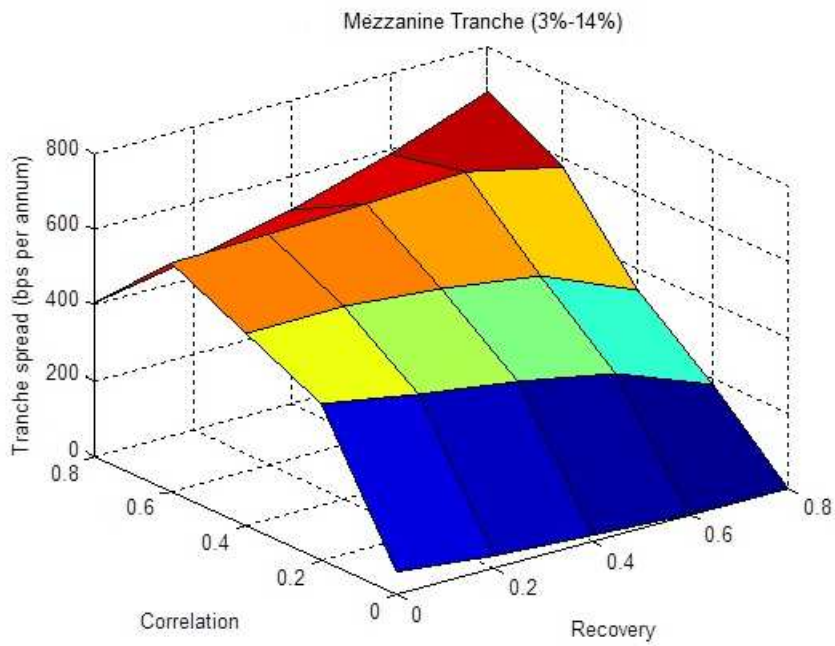


Figure 3.2: Mezzanine tranche spread modeled with Gaussian copula for different recovery rates.

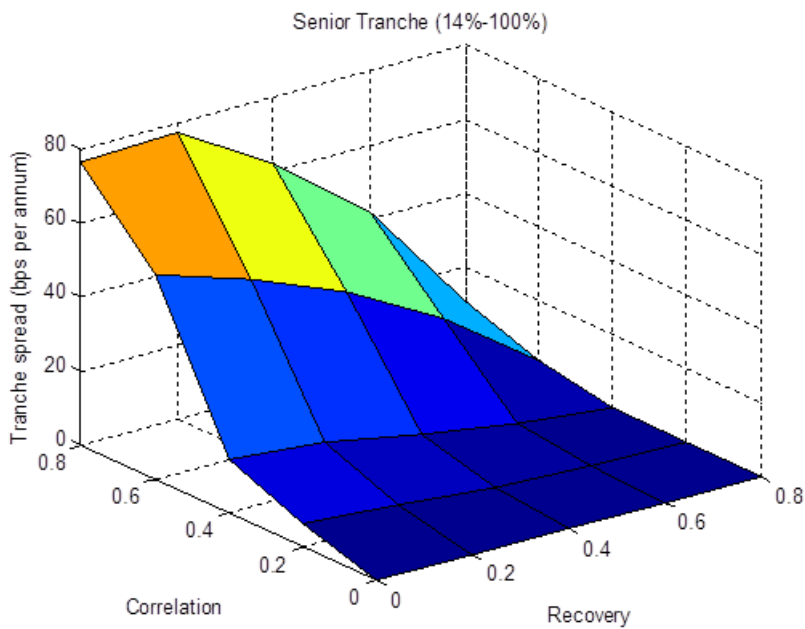


Figure 3.3: Senior tranche spread modeled with Gaussian copula for different recovery rates.

CHAPTER 4

INTRODUCING LÉVY COPULA TO OVERCOME THE CORRELATION SMILE PROBLEM

4.1 Lévy Copulas

Definition 4.1. A stochastic process (X_t) on \mathbb{R}^d with $X_0 = 0$ is called a Lévy process if:

- For $n \geq 1$ and $0 \leq t_0 < t_1 < \dots < t_n \leq T$, the random variables

$$X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \dots, X_{t_n} - X_{t_{n-1}}$$

are independent.

- Distribution of $X_{t+s} - X_t$ does not depend on t .
- For every $t \in [0, 1]$ and $\epsilon > 0$, $\lim_{s \rightarrow t} \Pr[X_s - X_t] > \epsilon = 0$
- $\exists \Omega_0 \in F$ with $P[\Omega_0] = 1$ such that $\forall \omega \in \Omega, X_t(\omega)$ is right continuous and has left limits in $t \in [0, 1]$.

This definition renders a Lévy process (X_t) to exhibit discontinuities (jumps). Characteristics of the jumps are given by jump measure μ defined on $[0, T] \times \mathbb{R}^n$ by

$$\mu(A) = \#\{(t, \Delta X_t) \in A\}.$$

For every measurable set $A \subset \mathbb{R}^n$, $\mu([t_1, t_2] \times A)$ counts the number of jumps with sizes not exceeding A between t_1 and t_2 . On the other hand, Lévy measure ν is defined as

$$\nu(A) = E[\#\{t \in [0, 1] : \Delta X_t \neq 0, \Delta X_t \in A\}], A \in B(\mathbb{R}^d)$$

indicates expected number of jumps, per unit time within the set A .

Definition 4.2. Let (X_t) be a Lévy process on \mathbb{R}^d with Lévy measure ν . The tail integral of ν is a function $U : (\mathbb{R}^d - 0) \rightarrow \mathbb{R}$ defined as follows:

$$U(x_1, \dots, x_d) = \nu \left(\prod_{i=1}^d I(x_i) \right) \left(\prod_{i=1}^d \text{sqn}(x_i) \right)$$

where

$$I(x) = \begin{cases} [x, \infty) & , \text{if } x > 0 \\ (-\infty, x] & , \text{if } x < 0 \end{cases}$$

Definition 4.3. A function $F : \mathbb{R}_\infty^d \rightarrow \mathbb{R}_\infty$ is called a Lévy d -copula function (or Lévy copula), if

- $F(u_1, \dots, u_d) \neq 0$ for $(u_1, \dots, u_d) \neq (\infty, \dots, \infty)$.
- $F(u_1, \dots, u_d) = 0$ if $u_i = 0$ for at least one $i \in \{1, \dots, d\}$.
- F is d -increasing.
- $F_i(u) = u$ for any $i \in \{1, \dots, d\}$, $u \in \mathbb{R}$.

Theorem 4.1 (Generalized Sklar's Theorem). *Let ν be a Lévy measure on \mathbb{R}^d . Then there exists a Lévy copula F such that tail integrals of ν satisfy*

$$U_I((X_i)_{i \in I}) = F_I((U_i)_{i \in I})$$

for any non-empty $I \subset \{1, \dots, d\}$ and any $(x_i)_{i \in I} \in \mathbb{R}^I$. Conversely, if F is a d -dimensional Lévy copula and ν_1, \dots, ν_d are Lévy measures on \mathbb{R} with tail integrals U_1, \dots, U_d , then there exists a unique Lévy measure \mathbb{R}^d with one-dimensional tail integrals U_1, \dots, U_d .

Theorem 4.2. *Let (X_i) be a Lévy process on \mathbb{R}^d . Then the independence copula is given by*

$$F_\perp(u_1, \dots, u_d) = \sum_{i=1}^d u_i \prod_{j \neq i} I_\infty(u_j).$$

Definition 4.4. Define

$$S_+ = \{x \in \mathbb{R}^d : \text{sgn}(x_1) = \dots = \text{sgn}(x_d)\}$$

and

$$S_- = \{x \in \mathbb{R}^2 : \text{sgn}(x_1) \neq \text{sgn}(x_2)\}.$$

Let X_t be a Lévy process on \mathbb{R}^d . Its jumps are considered completely positively dependent if there is an increasing set $D \subset S$ such that $\Delta X_t \in D$, $t \geq 0$. For $d = 2$, jumps of X_t are completely negatively dependent if there is a decreasing set D of S_- such that $\Delta X_t \in D$, $t \geq 0$.

Theorem 4.3. *Let (X_t) be a Lévy process on \mathbb{R}^d whose Lévy measure is supported by an ordered set $D \subset S$. The complete positive dependence Lévy copula is:*

$$F_{||}(u_1, \dots, u_d) = \min(|u_1|, \dots, |u_d|)I_S + (u_1, \dots, u_d) \times \prod_{i=1}^d \text{sgn}(u_i).$$

If $d = 2$, the complete negative dependence Lévy copula is

$$F_{|}(u_1, u_2) = \min(|u_1|, |u_2|)I_S + (u_1, u_2).$$

Conversely, if $F_{||}$ or $F_{|}$ is a Lévy copula of (X_t) , then the Lévy measure is supported by a strictly ordered subset $D \subset S$. If, in addition, tail integrals U_i of X^i are continuous and satisfy

$$\lim_{x \rightarrow 0} U_i(x) = \infty, i = 1, \dots, d,$$

then jumps of X_t are completely dependent (for further discussion see [51]).

4.2 Modeling of CDO Tranches via Lévy Copula

Our Itraxx 125 data which is to be widely explored in the subsequent chapters indicate that the seven sectors on which the CDSs are written comprising the synthetic CDO have almost the same correlation structure during times of crises. That is, all sectors jump at the same time. Actually, this is not data specific in the sense that the CDS spreads have tendency to move together during crisis periods. For instance, if there is a negative financial shock then we expect the possibility of default to rise for all possible companies, even for the countries. Therefore, for modeling purposes, we took into account a perfectly dependent Lévy copula which has the following form. The reader can find extra information in [3, 5, 6, 11, 21, 25, 30, 57].

$$F_{||} = \min \sum_{i=1}^7 (|u_i|) I_S + (u_i)_{\{i=1, \dots, 7\}} \prod_{i=1}^7 \text{sgn}(u_i).$$

Now, again for 125 loan payers, 1 year of maturity and 1000 simulations, we have Figure 4.1, Figure 4.2 and Figure 4.3 as given below

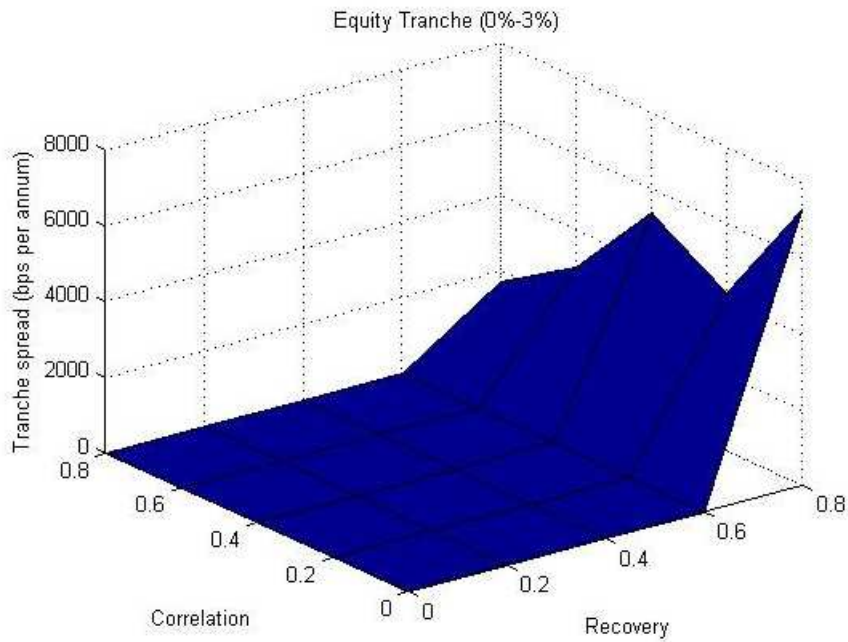


Figure 4.1: Equity tranche spread modeled with Lévy copula for different recovery rates.

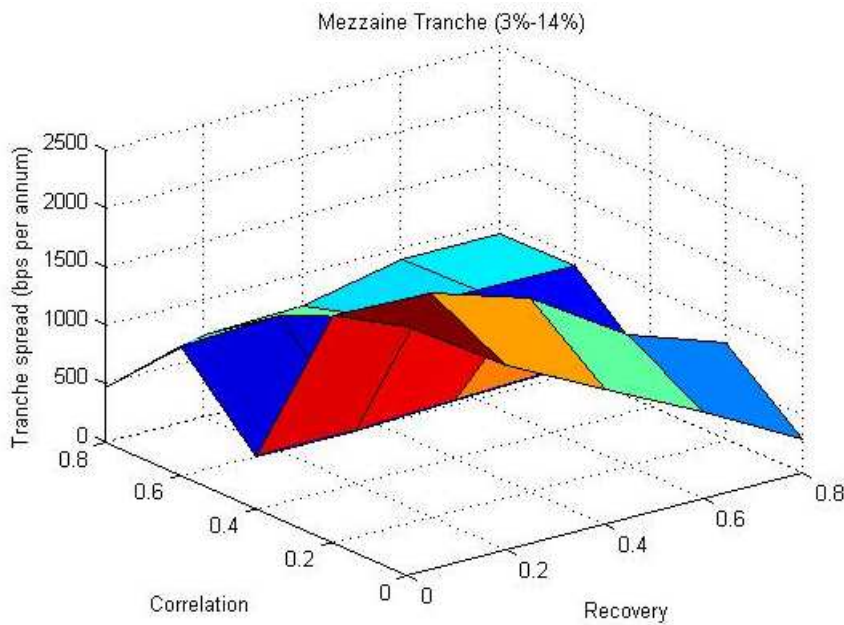


Figure 4.2: Mezzanine tranche spread modeled with Lévy copula for different recovery rates.

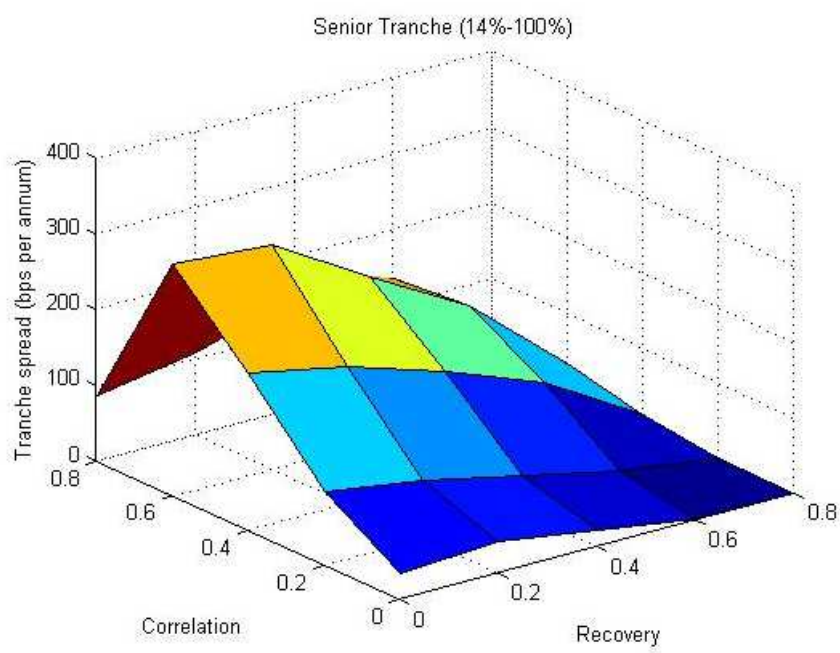


Figure 4.3: Senior tranche spread modeled with Lévy copula for different recovery rates.

With the Gaussian copula framework we confront the “correlation smile” problem. This phenomenon occurs due to the inconsistency of single parameter correlation structure. When the correlation parameter is derived from the market prices, i.e. implied correlation, senior tranches imply high correlation whereas junior and part of mezzanine tranches imply low correlation.

Comparing Figure 3.1 and Figure 4.1, we see that for junior tranche when correlation parameter gets higher so does the expected tranche loss and the associated fair spread. Gaussian copula framework implies a higher correlation and, thus, a higher fair spread. On the contrary our model gives lower spread everywhere.

For senior tranche the case is different when we look at Figure 3.3 and Figure 4.3. Since senior tranche investors are affected only when losses in the collateral go beyond most of the pool notional, many defaults should occur in order for this to happen. However, in practice higher correlation implies lower spread since the probability of huge number of defaults is quite unlikely. Gaussian copula framework implies lower correlation and thus lower fair spread. Our model, on the other hand, gives a higher spread.

Finally, looking at Figure 3.2 and Figure 4.2 we witness for mezzanine tranche, Gaussian copula model gives higher and sometime later lower correlation. Our model gives lower spread at the beginning and later higher spread which is consistent for correcting the correlation smile phenomenon.

CHAPTER 5

TIME SERIES CONCEPTS RELATED TO OUR CDS SPREAD PRICING MODEL

5.1 Introduction

In this chapter we introduce the notions related to time series which have become the ingredients of our model. In order to make forecasting we need stationarity of the relevant series. Therefore, first we mention stationarity and related concepts. Later, we give brief explanations of widely used time series modeling techniques and their relation not only among each other but also with stochastic differential equations. Statistical testing procedures applied to these models are taken into account in the subsequent sections. Definitions and theorems are taken from [17, 34, 39, 55].

5.2 Some Basic Concepts

Definition 5.1 (Strong Stationarity). Let $(z_{t_1}, z_{t_2}, \dots, z_{t_n})$ be random variables such that $(z((\omega, t)|t = 0, \pm 1, \pm 2, \dots, \pm n))$ and let F be an n dimensional joint distribution function i.e.

$$F_{z_{t_1}, z_{t_2}, \dots, z_{t_n}}(r_1, r_2, \dots, r_n) = (P(\omega)|z(\omega, t_1), \dots, z(\omega, t_n) \leq r_n).$$

The time series $(z_{t_i}, i = 0, 1, \dots, n)$ is said to be strongly stationary if

$$F_{z_{t_1}, z_{t_2}, \dots, z_{t_n}}(\cdot) = F_{z_{t_1+k}, z_{t_2+k}, \dots, z_{t_n+k}}(\cdot).$$

Definition 5.2 (Weak Stationarity). The time series $(z_{t_i}, i = 0, 1, \dots, n)$ is said to be weakly stationary if:

- $\mathbb{E}(z_t) = \mu, \mu \in \mathbb{R}, \forall t,$
- $\mathbb{E}(z_t^2) = \sigma^2, \sigma^2 \in \mathbb{R}, \forall t,$
- $\text{Cov}(z_t, z_k) = \text{Cov}(z_{t+l}, z_{k+l}), \forall t, k, l.$

Definition 5.3 (Autocorrelation Function). Let X and Y be two random variables. The correlation is defined as

$$\rho_{X,Y} = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}.$$

Let (Z_t) be a stationary process the autocorrelation function (ACF) is defined as

$$\rho_k = \frac{\text{Cov}(z_t, z_{t-k})}{\sqrt{\text{Var}(z_t)\text{Var}(z_{t-k})}} = \frac{\text{Cov}(z_t, z_{t-k})}{\text{Var}(z_t)} = \frac{\gamma_k}{\gamma_0}.$$

Definition 5.4 (Partial Autocorrelation Function). Let (Z_t) be a stationary process. The partial autocorrelation function (PACF) defined by

$$\text{Corr} = (Z_{t+k}, Z_t) | Z_{t+k-1}, Z_{t+k-2}, \dots, Z_{t+1}$$

is a conditional correlation which basically eliminates the intervening values between Z_{t+k} and Z_t .

Definition 5.5 (Weak Stationarity). The time series $(z_t, i = 0, 1, \dots, n)$ is called white noise if

- $\mathbb{E}(z_t) = 0 \forall t,$
- $\mathbb{E}(z_t^2) = \sigma^2, \sigma^2 \in \mathbb{R}, \forall t,$
- $\text{Cov}(z_t, z_{t+k}) = 0, \forall t, k.$

Definition 5.6 (Cross Correlation). Given two stationary time series (X_t) and (Y_t) the cross covariance function is defined as

$$\gamma_{xy}(h) = \mathbb{E}(x_{t+h} - \mu_x)(y_t - \mu_y)$$

where μ_x and μ_y are the means of the series (X_t) and (Y_t) respectively. The cross correlation is defined as

$$\rho_{xy}(h) = \frac{\gamma_{xy}}{\sqrt{\gamma_x(0)\gamma_y(0)}}$$

5.3 Main Time Series Models

5.3.1 Autoregressive (AR) Model

Consider the time series

$$y_t = \alpha_0 + \alpha_1 y_{t-1} + \epsilon_t$$

where ϵ_t is white noise with constant σ^2 variance. This is called an autoregressive one ($AR(1)$) model. It can easily be extended to $AR(p)$ model as:

$$y_t = \alpha_0 + \alpha_1 y_{t-1} + \dots + \alpha_p y_{t-p} + \epsilon_t.$$

Assuming weak stationarity we have,

$$\mathbb{E}(y_t) = \mu, \quad \text{Var}(y_t) = \gamma_0, \quad \text{Cov}(y_t, y_{t-1}) = \gamma_t,$$

where μ, γ_0 and γ_t are constants. The mean of $AR(1)$ model can be computed under stationarity by

$$\mathbb{E}(y_t) = \mu = \frac{\alpha_0}{1 - \alpha_1}.$$

Thus for the existence of mean we must have $\alpha_1 \neq 1$. Moreover mean is zero if and only if $\alpha_0 = 0$. Considering $\alpha_0 = 0$ and letting $\alpha_1 = \alpha$ the variance can be computed as follows

$$\begin{aligned} \gamma_0 &= \text{Var}(y_t) = \mathbb{E}[y_t - \mathbb{E}(y_t)]^2 = \mathbb{E}[y_t y_t] = \mathbb{E}[y_t]^2 \\ &= \mathbb{E}[\epsilon_t + \alpha\epsilon_{t-1} + \alpha^2\epsilon_{t-2} + \dots]^2 + \alpha^2\sigma^2 + \alpha^4\sigma^2 + \dots = \frac{\sigma^2}{1 - \alpha^2}. \end{aligned}$$

Here we took into account the stationarity properties. In order for the variance to be bounded and non-negative we must have $\alpha^2 < 1$. In summary for weak stationarity of the above $AR(1)$ model, $|\alpha| < 1$ should hold.

5.3.2 Relationship Between OU-Process and $AR(1)$ Model

The Ornstein-Uhlenbeck (OU) stochastic process s_t satisfies the following SDE,

$$ds_t = \theta(\mu - s_t)dt + dB_t$$

where $\theta, \mu, \sigma \in \mathbb{R}^+$ and B_t is the standard Brownian motion.

The autoregressive p , ($AR(p)$) model is defined by the following difference equation

$$S_t = \sum_{i=1}^p \lambda_i S_{t-i} + \epsilon_t$$

where λ_i are real constants and ϵ_t is the white noise. Now consider the following $AR(1)$ model

$$s_{t+1} = \theta(\mu - s_t)(t_{i+1} - t_i) + \sigma(B_{t_{i+1}} + B_{t_i}),$$

Now

$$B_{t_{i+1}} + B_{t_i} = \epsilon_t \sqrt{(t_{i+1} - t_i)}$$

where ϵ_t is standard normally distributed. Then we have

$$s_{t+1} = \theta(\mu - s_t)(t_{i+1} - t_i) + \sigma\epsilon_t \sqrt{(t_{i+1} - t_i)},$$

which is Euler-Maryuama discretization of OU process at times $t_{i+1} - t_i, i \in \mathbb{N}$.

5.3.3 Moving Average (MA) Model

A moving average one ($MA(1)$) model is defined by

$$y_t = \alpha_0 + \beta\epsilon_{t-1} + \epsilon_t$$

where ϵ_t is a white noise. We can shift the model without altering any characteristics thus letting $\mu = 0$ makes no difference. The mean is trivially zero. The variance is computed as follows: The general $MA(p)$ process is,

$$\begin{aligned} \gamma_0 &= \text{Var}(y_t) = \mathbb{E}[y_t - \mathbb{E}(y_t)]^2 = \mathbb{E}[y_t y_t] = \mathbb{E}[y_t]^2 = \mathbb{E}[\epsilon_t + \beta\epsilon_{t-1} + \epsilon_t + \beta\epsilon_{t-1}] \\ &= \beta^2 \mathbb{E}\epsilon_{t-1}^2 + \beta \mathbb{E}\epsilon_{t-1}\epsilon_t + \beta \mathbb{E}\epsilon_t\epsilon_{t-1} + \mathbb{E}\epsilon_t^2 = \sigma^2(1 + \beta^2). \end{aligned}$$

The general $MA(p)$ process is

$$y_t = \mu + \beta_1\epsilon_{t-1} + \beta_2\epsilon_{t-2} + \dots + \beta_p\epsilon_{t-p} + \epsilon_t, \epsilon_t \sim N(0, \sigma^2).$$

5.3.4 Relationship Between $AR(1)$ and $MA(\infty)$ Model

Consider the $AR(1)$ model stated as

$$y_t = \alpha_0 + \alpha_1 \cdot y_{t-1} + \epsilon_t.$$

Recursive substitution yields the following

$$y_t = \alpha^p y_{t-p} + \alpha^{p-1} \epsilon_{t-p+1} + \dots + \alpha^2 \epsilon_{t-2} + \alpha \epsilon_{t-1} + \epsilon_t.$$

If $|\alpha| < 1$, then

$$\lim_{p \rightarrow \infty} \alpha^p y_{t-p} = 0$$

which gives us

$$y_t = \sum_{i=0}^{\infty} \alpha^i \epsilon_{t-i}.$$

The last expression is precisely an $MA(\infty)$ process. When $|\alpha| < 1$ we say that the $AR(1)$ process is “invertible”.

5.3.5 Autoregressive Moving Average (ARMA) Model

An autoregressive moving average one ($ARMA(1)$) model is defined by

$$y_t = \lambda + \alpha y_{t-1} + \beta \epsilon_{t-1} + \epsilon_t.$$

Now

$$\mathbb{E}(y_t) = \lambda + \alpha\mathbb{E}(y_{t-1}) + \beta\mathbb{E}(\epsilon_{t-1}) + \mathbb{E}(\epsilon_t).$$

From covariance stationarity we have

$$\mathbb{E}(y_t) = \mathbb{E}(y_{t-1}) = \mu,$$

thus,

$$\mu = \lambda + \alpha\mu \Rightarrow \mu = \frac{\lambda}{1 - \alpha}.$$

The variance is computed as follows:

$$\begin{aligned} y_t &= \mu(1 - \alpha) + \alpha y_{t-1} + \beta\epsilon_{t-1} + \epsilon_t, \\ \Rightarrow y_t - \mu &= \alpha(y_{t-1} - \mu) + \beta\epsilon_{t-1} + \epsilon_t, \\ \Rightarrow \mathbb{E}(y_t - \mu)^2 &= \mathbb{E}(\alpha(y_{t-1} - \mu) + \beta\epsilon_{t-1} + \epsilon_t)^2, \\ \Rightarrow \gamma_0 &= \alpha^2\gamma_0 + 2\alpha\beta\sigma^2 + \sigma^2 + \beta^2\sigma^2, \\ \Rightarrow \gamma_0 - \alpha^2\gamma_0 &= 2\alpha\beta\sigma^2 + \sigma^2 + \beta^2\sigma^2, \\ \Rightarrow \gamma_0 &= \frac{1 + 2\alpha\beta^2\sigma^2}{(1 - \alpha^2)}. \end{aligned}$$

5.3.6 Unit Root

Consider an $AR(p)$ process of form

$$y_t = a_0 + a_1y_{t-1} + \dots + a_py_{t-p} + \epsilon_t.$$

We can assume $a_0 = 0$. The process is said to exhibit a unit root if

$$\sum_{i=1}^p a_i = 1.$$

For an $AR(1)$ process of form

$$y_t = a_0 + a_1y_{t-1} + \epsilon_t$$

has a unit root when $a_1 = 1$. In fact this corresponds to the random walk which in the limiting case becomes the Brownian motion. If a process exhibits unit root then it is non stationary.

5.4 Akaike Information Criterion

Definition 5.7. If X is a discrete random variable and $f(x)$ is the value of its pdf at x then the entropy of X is

$$H(x) = - \sum_{x \in X} f(x) \log_a f(x).$$

If X is a continuous random variable and $f(x)$ is the value of its pdf at x then the entropy of X is:

$$h(x) = - \int_X f(x) \log_a f(x) dx.$$

For value of a (the basis of logarithm) 2, e and 10 are commonly used.

Definition 5.8. If X and Y are discrete random variables and $f(x, y)$ is the value of their pdf at (x, y) then the joint entropy of X and Y is

$$H(X, Y) = - \sum_{x \in X} \sum_{y \in Y} f(x, y) \ln f(x, y).$$

Definition 5.9. If X and Y are discrete random variables, $f(x, y)$ and $f(y|x)$ are the value of their probability and joint probability distribution at (x, y) then the conditional entropy of Y given YX is:

$$H(Y|X) = - \sum_{x \in X} \sum_{y \in Y} f(x, y) \ln f(y|x).$$

These definitions can easily be transformed to continuous case via integral.

Definition 5.10 (Kullback-Leibler Divergence). For two probability distributions $f(x)$, $g(x)$ and for a discrete random variable X the Kullback-Leibler Divergence is defined as:

$$D(f||g) = \sum_{x \in X} f(x) \ln \frac{f(x)}{g(x)}.$$

It is a measure for comparison of entropy of two distributions over same random variable.

Theorem 5.1. *Normal distribution has the largest entropy amongst all random variables of equal variance.*

Proof. Let $g(x) \sim N(\mu, \sigma^2)$. Let $f(x)$ be an arbitrary pdf with the same variance. The differential entropy (entropy for continuous random variables) is translation invariant since $h(X + c) = h(X)$, $c \in \mathbb{R}$. Therefore we can assume $\mathbb{E}[f(x)] = \mu$. Now consider the Kullback-Liebler divergence between the above mentioned distributions

$$\begin{aligned} 0 &\leq D(f||g) = \int_{-\infty}^{\infty} f(x) \ln \left(\frac{f(x)}{g(x)} \right) dx = \int_{-\infty}^{\infty} f(x) \ln(f(x)) dx \\ &= - \int_{-\infty}^{\infty} f(x) \ln(g(x)) dx = -h(f) - \int_{-\infty}^{\infty} f(x) \ln(g(x)) dx \end{aligned}$$

The last term can be expressed as follows

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x) \ln(g(x)) dx &= \int_{-\infty}^{\infty} f(x) \ln \left(\frac{1}{\sqrt{2\pi\sigma^2}} \times e^{-\frac{(x-\mu)^2}{2\sigma^2}} \right) dx \\
&= \int_{-\infty}^{\infty} f(x) \ln \left(\frac{1}{\sqrt{2\pi\sigma^2}} \right) dx \ln(e) \int_{-\infty}^{\infty} f(x) \left(-\frac{(x-\mu)^2}{2\sigma^2} \right) dx \\
&= -\frac{1}{2} \ln(2\pi\sigma^2) - \ln(e) \frac{\sigma^2}{2\sigma^2} = -\frac{1}{2} (\ln(2\pi\sigma^2) + \ln(e)) = -\frac{1}{2} \ln(2\pi\sigma^2) = h(g).
\end{aligned}$$

Thus we have $h(g) - h(f) \geq 0$, and from the third property we get $h(g) - h(f) = 0$ if and only if $f(x) = g(x)$. \square

Definition 5.11 (Maximum Likelihood Estimation). Let X_i , $i = 1, 2, \dots, n$ be random variables with a joint density

$$f_{\theta}(x_1, x_2, \dots, x_n) = f(x_1, x_2, \dots, x_n | \theta),$$

where θ denotes true state of nature. The maximum likelihood principle is finding $\hat{\theta}$ as the estimator of θ which makes the observed data most probable. If X_i , $i = 1, 2, \dots, n$ are i.i.d., then the likelihood function becomes

$$\prod_{i=1}^n f(x_i | \theta).$$

Since the structure of the maximization problem is preserved under strictly increasing transformations, for convention, usually the function to be maximized becomes

$$\sum_{i=1}^n \ln(f(x_i | \theta)).$$

Definition 5.12 (Akaike Information Criterion). [1] The Kullback-Leibler divergence for continuous random variables is

$$D(f||g) = \int_{-\infty}^{\infty} f(x) \ln f(x) dx - \int_{-\infty}^{\infty} f(x) \ln g(x|\theta) dx.$$

This in fact can be expressed as

$$D(f||g) = \mathbb{E}_f [\ln f(x)] - \mathbb{E}_f \ln [g(x|\theta)],$$

where expectations are taken with respect to true state f .

$$\int_{-\infty}^{\infty} f(x) \ln f(x) dx = c,$$

where c is a constant from the definition of entropy although the true distribution function is not known.

Now the only expression to be estimated is

$$\mathbb{E}_f(\ln [g(x|\theta)])$$

which is the main work of Akaike's Information Criterion (AIC) [1]. This estimator is found as

$$\text{AIC}(k) = -2\text{lkl}(\hat{\theta}) + 2k.$$

where lkl is log likelihood for model, $\hat{\theta}$ is maximum likelihood estimate of the parameters and k denotes number of variables in the model.

Intuitively, minimum AIC value gives us the preferred model among the candidates for a given data. The value which minimizes the AIC gives us the lag length[12].

5.5 Statistical Tests

In this sub-section we briefly give the statistical test utilized in our model. For the test of stationarity, unit root tests are given. Later the Box-Jenkins methodology is introduced for the determination of p and q coefficients in the underlying $ARMA(p, q)$ process. Finally Jarque-Bera normality test is given for the diagnostic check of residuals.

5.5.1 Unit Root Tests (Linear)

This subsection analyses basic unit root tests in a linear framework.

5.5.1.1 Dickey Fuller Test

In order to test whether

$$y_t = \alpha_1 y_{t-1} + \varepsilon_t$$

follows a random walk i.e. has unit root we set the null hypothesis $H_0 : \alpha_1 = 1$ versus alternative hypothesis $H_1 : \alpha_1 < 1$. The unit root testing procedure offered by Dickey and Fuller [13] takes into account t -statistics ratio of the least-squares estimation of α_1 . The least square estimation of mean and variance becomes:

$$\hat{\alpha}_1 = \frac{\sum_{t=1}^T y_{t-1} y_t}{\sum_{t=1}^T y_{t-1}^2}, \quad \hat{\sigma}_e^2 = \frac{\sum_{t=1}^T (y_t - \hat{\alpha}_1 y_{t-1})^2}{T - 1},$$

where T is the sample size. The t -statistics ratio is:

$$DF = \frac{\hat{\alpha}_1 - 1}{\text{std.dev.}(\hat{\alpha}_1)} = \frac{\sum_{t=1}^T y_{t-1} e_t}{\hat{\sigma}_e \sqrt{\sum_{t=1}^T y_{t-1}^2}}.$$

If the sample size is increased then DF in the limit is a function of standard Wiener process. In case of $\alpha_1 = 1$ then it converges to another nonstandard asymptotic distribution. Critical values are computed via simulation and presence of unit root is determined.[13]

5.5.1.2 Augmented Dickey Fuller Test

Wide usage of $AR(p)$ processes lead to testing the unit root for lagged models. Augmented Dickey Fuller Test (ADF) verifies the existence of unit root in an $AR(p)$ framework by performing hypothesis testing $H_0 : \beta = 1$ versus $H_1 : \beta < 1$ for the regression

$$y_t = \mu + \beta t + \gamma y_{t-1} + \sum_{i=1}^{p-1} \alpha_i \Delta y_{t-i} + \varepsilon_t, \quad \varepsilon_t \sim N(0, \sigma^2),$$

where, μ, β are constants and Δ is the difference operator. The ADF t -statistics is given by

$$ADF = \frac{\hat{\beta} - 1}{\text{std.dev.}(\hat{\beta})};$$

here $\hat{\beta}$, is the least-squares estimator of β .

5.5.2 Unit Root Tests (Non-Linear)

This section analyzes more general unit root tests. First, tests with threshold are introduced. Then, more general models are taken into consideration including non-linear trends together with smooth changes also included within those trends.

5.5.2.1 Enders and Granger (EG) TAR Type Test

Enders and Granger[18] use threshold autoregressive (TAR) models to offer tests of the null hypothesis of a unit root allowing under the alternative hypothesis for stationary asymmetric threshold to a constant expectation or deterministic linear trend.

$$y_t - y_{t-1} = \alpha + \beta_1 I_t y_{t-1} + \beta_2 (1 - I_t) y_{t-1} + \sum_{j=1}^k \hat{\delta}_j \Delta y_{t-j} + \hat{\lambda}_t, \quad (5.1)$$

$$I_t = \begin{cases} 1 & , \text{if } y_{t-1} > 0 \\ -1 & , \text{if } y_{t-1} < 0 \end{cases}$$

and $\hat{\lambda}_t$ is a white noise. if $\beta_1 = \beta_2 = 0$ in (5.1) then y_t exhibits a unit root, while if $\beta_1 = \beta_2 < 0$, y_t is a stationary TAR process with symmetric change, and if $\beta_1 < 0, \beta_2 < 0$ and $\beta_1 \neq \beta_2$, y_t is a stationary TAR process showing asymmetric alteration. Enders and Granger[18] suggest testing for whether y_t has a unit root using the F -statistic for testing $\beta_1 = \beta_2 = 0$ in (5.1) and/or the most significant of the t -statistics from those for testing $\beta_1 = 0$ and $\beta_2 = 0$.

5.5.2.2 Leybourne Newbold and Vougos (LNV) Smooth Break Test

Leybourne, Newbold and Vougas [35] build up unit root tests where the alternative is stationary around a linear trend with a smooth break, and give their small sample properties. Let y_t be process with changing trend function together by smooth transition on the time domain $t = 1, 2, \dots, T$.

$$y_t = \alpha + \alpha_2 S_t(\gamma, \tau) + \varepsilon_t \quad (5.2)$$

$$y_t = \alpha + \beta_1 t + \alpha_2 S_t(\gamma, \tau) + \varepsilon_t \quad (5.3)$$

$$y_t = \alpha + \beta_1 t + \alpha_2 S_t(\gamma, \tau) + \beta_2 t S_t(\gamma, \tau) + \varepsilon_t \quad (5.4)$$

where ε_t is a zero mean stationary process and $S_t(\gamma, \tau)$ is the logistic smooth transition function, defined by:

$$S_t(\gamma, \tau) = [1 + \exp \{-\gamma(t - \tau T)\}]^{-1}, \gamma_i > 0.$$

In this strategy, structural change is modeled as smooth shift between different regimes rather than an instant structural break. The transition function $S_t(\gamma, \tau)$ is continuous and bounded between 0 and 1. Thus the STR model can be regarded as regime-switching model which renders for two regimes, connected with extreme values of the shift function, $S_t(\gamma, \tau) = 0$ and $S_t(\gamma, \tau) = 1$, while shift from one regime to other is stable. Here, γ determines smoothness of the transition. Two regimes are associated with small and large values of the transition variable $s_t = t$ relative to threshold $c = \tau$. For the large values of γ , $S_t(\gamma, \tau)$ passes through the interval (0,1) very quickly, and as γ reaches $+\infty$ this function changes value from 0 to 1 in an instant manner at time $t = \tau T$. Therefore, if it is assumed that ε_t is a white noise, then (5.2) is a stationary process around a mean which changes from initial value α_i to final value $\alpha_i + \alpha_2$. Leybourne et al.[35] also give similar conditions for models stated in (5.2) and (5.3). In these specifications no change and one instantaneous structural change are limiting cases whereas this specification is more general which covers gradual structural

changes as well. Leybourne et al.[35] establish the hypotheses for unit root testing based on (5.2), (5.3) and (5.4) as follows

H_0 : Unit Root (Linear Non stationary)

H_a : Stationary (Stationary around smoothly changing trend and intercept)

Following Leybourne et al.[35] the test statistics proposed here is calculated with a two-step procedure

Step 1 Usage of a nonlinear least-squares (NLS) algorithm, estimates only the predictable component of the ideal model and calculate the NLS residuals

$$\text{Model 1 : } \hat{\varepsilon}_t = y_t - \hat{\alpha} - \hat{\alpha}_2 S_t(\gamma, \tau),$$

$$\text{Model 2 : } \hat{\varepsilon}_t = y_t - \hat{\alpha} - \hat{\beta}_1 t - \hat{\alpha}_2 S_t(\gamma, \tau),$$

$$\text{Model 3 : } \hat{\varepsilon}_t = y_t - \hat{\alpha} + \hat{\beta}_1 t + \hat{\alpha}_2 S_t(\gamma, \tau) + \hat{\beta}_2 t S_t(\gamma, \tau).$$

Step 2 Work out the ADF statistic, the t ratio connected with $\hat{\beta}_1$ in the ordinary least-squares (OLS) regression.

5.5.2.3 Kapetanios Shin and Snell (KSS) ESTAR Test

Let y_t follow a single variable exponential smooth transition autoregressive (ESTAR) model of level 1

$$y_t = \beta y_{t-1} + \gamma y_{t-1} [1 - \exp(-\theta y_{t-d}^2)] + \varepsilon_t,$$

which after reparameterising can be written suitably as

$$\Delta y_t = \varphi y_{t-1} + \gamma y_{t-1} [1 - \exp(-\theta y_{t-d}^2)] + \varepsilon_t,$$

where $\varphi = \beta - 1$.

Overall stationarity of the process y_t can be found by testing null hypothesis $H_0 : \theta = 0$ against alternative $H_1 : \theta > 0$. However, testing null hypothesis straightly is not possible since γ is not known under value of the null. To solve this difficulty, Kapetanios et al.[31] use the method of Luukkonen et al.[40] to restore the transition function by its appropriate Taylor approximation to get a t -type test statistic. By using Taylor approximation, the following supplementary regression is obtained

$$\Delta y_t = \delta y_{t-d}^3 + e_t,$$

where e_t contains original shocks ε_t as well as the error term arising from Taylor approximation. The test statistic for $\delta = 0$ against $\delta < 0$ is $t_{NL} = \hat{\delta}/\text{s.e.}(\hat{\delta})$, where $\hat{\delta}$ is the OLS estimate and $\text{s.e.}(\hat{\delta})$ is the standard error of $\hat{\delta}$.

In a wide framework when errors are serially correlated, the auxiliary regression is improved by p^{th} order lag of dependent variable. The extended model can be seen as

follows

$$\Delta y_t = \sum_{j=1}^p \rho_j \Delta y_{t-j} + \delta y_{t-d}^3 + e_t,$$

From here one can obtain the t -statistics for $\delta = 0$ against $\delta < 0$ as:

$$t = \frac{\hat{\delta}}{\text{st.dev}(\hat{\delta})} \xrightarrow{d} \frac{\left\{ \frac{1}{4} W(1)^4 - \frac{3}{2} \int_0^1 W(r)^2 dr \right\}}{\sqrt{\int_0^1 W(r)^6 dr}},$$

where $W(r)$ is a standard Wiener process defined on $r \in [0, 1]$.

5.5.2.4 Sollis Smooth Break with TAR Type Test

Sollis [49] developed a unit root test by combining Enders and Granger [18] and Leybourne, Newbold and Vougas [35]. Enders and Granger [18] used threshold autoregressive (TAR) models to propose tests of the null hypothesis of a unit root that let under alternative hypothesis for stationary asymmetric change to a constant mean or a deterministic linear trend. In Sollis [49], the EG tests are generalized to the case of a nonlinear trend. In particular, smooth transition technique employed by Leybourne, Newbold and Vougas [35], is together with a TAR method of EG to build up unit-root tests that permits under the alternative hypothesis a transition between deterministic linear trends, about which stationary asymmetric change might happen. The resemblance of the EG and LNV techniques for testing for existence of a unit root in original series both eliminating deterministic parts of the data before unit root test claims that models used by these authors might be united. Let y_t be generated by Model 1, Model 2 or Model 3 with $\hat{\epsilon}_t$ formed by the following TAR model. Combining (5.1) and Model 1, Model 2 or Model 3, Sollis test is obtained by

$$\Delta \hat{\epsilon}_t = \alpha + \beta_1 I_t \hat{\epsilon}_{t-1} + \beta_2 (1 - I_t) \hat{\epsilon}_{t-1} + \sum_{j=1}^k \hat{\delta}_j \Delta \hat{\epsilon}_{t-j} + \hat{\lambda}_t, \quad (5.5)$$

where

$$I_t = \begin{cases} 1 & , \text{if } \hat{\epsilon}_{t-1} > 0 \\ -1 & , \text{if } \hat{\epsilon}_{t-1} < 0 \end{cases}$$

and $\hat{\lambda}_t$ is a white noise. Thus y_t is a smooth transition TAR (ST-TAR) process. Without taking into consideration of which model from Model 1, Model 2 or Model 3 is used to explain the deterministic parts of y_t , if $\beta_1 = \beta_2 = 0$ in (5.5) then $\hat{\epsilon}_t$ and thus y_t exhibits a unit root, while if $\beta_1 = \beta_2 < 0$, y_t is a stationary ST-TAR process with symmetric, and if $\beta_1 < 0$, $\beta_2 < 0$ and $\beta_1 \neq \beta_2$, y_t is a stationary ST-TAR process showing asymmetric change. The testing if y_t has a unit root using the F -statistic for

the test of $\beta_1 = \beta_2 = 0$ in (5.5) and/or the highest significant of the t -statistics from those for testing $\beta_1 = 0$ and $\beta_2 = 0$.

$H_0 : \beta_i = 0$, for all i , (Linear Non stationary)

$H_0 : \beta_i < 0$, for some i . (Stationary around nonlinear trend and intercept)

5.5.2.5 Omay and Yildirim (OY) Smooth Break with ESTAR Test

This test by Omay and Yildirim [46, 56] is for unit root null hypothesis by mixing KSS and LNV techniques make use of exponential smooth transition autoregressive (ESTAR) models for offering tests of null hypothesis of a unit root that allow under alternative hypothesis for stationary non-linear shift towards a constant expectation. Here, the KSS tests are generalized to the case of non-linear trend. In this approach, again a transition function is considered which is exactly the same as described in LNV framework.

The hypotheses for unit root testing based on (5.2), (5.3) and (5.4) become

H_0 : Unit Root (Linear and nonstationary)

H_a : Stationary (Nonlinear and stationary around smoothly changing trend and intercept)

Following Leybourne et al.[35] , the test statistics proposed here are calculated with a two-step procedure:

Step 1 Using a nonlinear least-squares (NLS) algorithm, estimate only a deterministic component of the preferred model and compute the NLS residuals as done in LNV setup.

Step 2 Calculate the KSS statistic, the t ratio associated with $\hat{\rho}_i$ in the ordinary least-squares (OLS) regression

$$\Delta \hat{\varepsilon}_t = \hat{\rho} \hat{\varepsilon}_t^3 + \sum_{j=1}^k \hat{\delta}_j \Delta \hat{\varepsilon}_{t-j} + \hat{\eta}_t.$$

$H_0 : \beta_i = 0$, for all i , (Linear Non stationary)

$H_0 : \beta_i < 0$, for some i , (Nonlinear and stationary around nonlinear trend and intercept)

5.5.2.6 Box-Jenkins Test

This test proposed by Box and Jenkins [10] focuses on the plot of the ACF and PACF of the time series. The ACF and PACF of $ARMA(p, q)$ process has typical characteristics

related to lag operator. Thus the test is applied by looking at the ACF and PACF of the series. The use of ACF enables us to identify the true data generating process as stationary including whether the series exhibit a certain seasonality or not. However, as mentioned earlier one should also have the inevitability condition. Moreover, the Box-Jenkins testing involves a diagnostic checking to make sure that the residuals from the estimated model are a white noise. AIC described in Section 5.2.7 is used to identify p and q . We can define an “identification procedure” for Box-Jenkins test as follows

- **Check for Stationarity via ACF**

Let’s consider an $AR(1)$ process without drift i.e.

$$\begin{aligned} y_t &= \varphi_1 y_{t-1} + \varepsilon_t = \varphi_1(\varphi_1 y_{t-2} + \varepsilon_{t-1}) + \varepsilon_t \\ &= \varepsilon_t + \varphi_1 \varepsilon_{t-1} + \varphi_1^2 \varepsilon_{t-2} + \varphi_1^3 \varepsilon_{t-3} + \cdots + \varphi_1^t y_0. \end{aligned}$$

One can let $y_0 = 0$ and have $\mathbb{E}(y_t) = 0$ without loss of generality. For large values of t ,

$$\text{Var}(y_t) = \frac{\sigma^2}{(1 - \varphi_1^2)},$$

if $\|\phi_1\| < 1$ and

$$\text{Cov}(y_t, y_{t-s}) = \frac{\varphi_1^s \sigma^2}{(1 - \varphi_1^2)} = \varphi_1^s \text{Var}(y_t).$$

If we consider an $AR(2)$ process without drift i.e.

$$y_t = \varphi_1 y_{t-1} + \varphi_2 y_{t-2} + \varepsilon_t$$

for stationarity we need to have $\varphi_1 + \varphi_2 < 1$, $\varphi_2 - \varphi_1 < 1$ and $\varphi_2 < 1$.

Thus, we can infer that certain restrictions on the parameters should be satisfied in order to meet stationarity criterion.

Now consider an $MA(1)$ process without drift, i.e.,

$$y_t = \varepsilon_t - \theta_1 \varepsilon_{t-1}.$$

Then, we have

$$\mathbb{E}(y_t) = 0, \quad \text{Var}(y_t) = \sigma^2(1 + \theta_1^2)$$

and

$$\text{Cov}(y_t, y_{t-s}) = \begin{cases} -\theta_1 \sigma^2 & , \text{if } s = 1 \\ -0 & , \text{otherwise} \end{cases}$$

For an $MA(2)$ process described as

$$y_t = \varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2}$$

we similarly have $\mathbb{E}(y_t) = 0$, $\text{Var}(y_t) = \sigma^2(1 + \theta_1^2 + \theta_2^2)$ and

$$\text{Cov}(y_t y_{t-1}) = \begin{cases} -\theta_1 \sigma^2 (1 - \theta_2) & , \text{if } s = 1 \\ -\theta_2 \sigma^2 & , \text{if } s = 2 \\ 0 & , \text{otherwise} \end{cases}$$

Generally speaking, moving average processes are stationary but not necessarily invertible. From the above discussions we can say that the ACF of a stationary $AR(k)$ process decreases to zero when k increases. On the other hand, the ACF of a $MA(1)$ process cuts-off after 1, i.e., the number of lag operation, always leading to stationarity. For the time series $(y_i), i = 1, \dots, n$, the sample auto correlation at lag k is given by

$$\hat{\rho}_k = r_k = \frac{\sum_{t=1}^{n-k} (Y_t - \bar{Y})(Y_{t-k} - \bar{Y})}{\sum_{t=1}^n (Y_t - \bar{Y})^2}, k = 0, 1, 2, \dots,$$

where $\bar{y} = \sum_{t=1}^n y_t/n$ is the sample mean. For vast sample sizes $\hat{\rho}_k$ is normally distributed with mean ρ_k and variance

$$\text{Var}(\hat{\rho}_k) \approx \frac{1}{n} (1 + 2\rho_1^2 + 2\rho_2^2 + \dots + 2\rho_m^2)$$

due to Bartlett. Thus if the linear connection between time series observations divided by a lag of k time units is denoted by r_k , then the standard deviation S_{r_k} becomes

$$s_{r_k} = \sqrt{\frac{1 + 2 \sum_{j=1}^{k-1} r_j^2}{n}}.$$

We finally have the t -statistics $t_{r_k} = r_k/s_{r_k}$ which helps us to agree on whether the series in question has stationarity. If $|t_{r_k}| > 2$ then r_k is considered to be statistically large thus we have non-stationarity. If the series is not stationary, it can be transformed into a stationary series by the help of the difference operator. In practice, we almost never go beyond second order difference since real data involve at most second degree of non-stationarity.

- **Identify the Model by ACF and PACF**

In order to choose a model, that is to find p and q for the $ARMA(p, q)$ process in addition to ACF, one should also take into account the PACF. There are different types of behaviors for ACF and PACF according to the model. For an $AR(p)$ model the ACF dies down and PACF cuts off after lag p . The roles are reversed for $MA(p)$ model. In an $ARMA(p, q)$ framework both ACF and PACF dies down. The decay process can be exponential, sine wave or a mixture of both according to the lag. The sample autocorrelation function at lag k is

$$\text{Cov}(y_t y_{t-1}) = \begin{cases} r_1 & , \text{if } k = 1 \\ \frac{\left(r_k - \sum_{j=1}^{k-1} r_{k-1,j} \cdot r_{k-j} \right)}{\left(1 - \sum_{j=1}^{k-1} r_{k-1,j} \cdot r_k \right)} & , \text{if } k = 2, 3, \dots \end{cases}$$

where

$$r_{kj} = r_{k-1,j} - r_{kk} r_{k-1,k-j}$$

for $j = 1, 2, \dots, k-1$. r_{kk} can be thought as the sample auto correlation of time series separated by a lag of k units with the intervening outcomes are eliminated. The standard error of r_{kk} is $s_{r_k} = \sqrt{1/n}$ with t-statistics $t_{r_{kk}} = (r_{kk})/(s_{r_{kk}})$. Similar argument applies in determination of stationarity. If $|t_{r_{kk}}| > 2$ for any k , then r_{kk} is considered to be statistically large.

- **Estimate the Parameters**

Although least-squares can be used, since MA processes are involved, maximum likelihood method is also applicable for parameter estimations. To test whether a drift term is included in the model,

$$\left| \frac{\bar{z}}{s_z/\sqrt{n}} \right| > 2$$

should hold. Here, \bar{z} is the mean, s_z is the standard deviation and n is the size for the sample. Now, we have identified p and q .

- **Diagnostic Checking**

Check whether the estimated coefficients are statistically significant under t -statistics. If $ARMA(p, q)$ fits the model then the residuals should be uncorrelated. In that sense, we should look at the portmanteau test statistics.

$$Q^*(k) = (n-d)(n-d+2) \sum_{l=1}^k \frac{r_l^2(e)}{n-d-l} \sim \chi_{(k-p-q)}^2$$

- **Select the Model with the Minimum Entropy**

When p and q are determined we have 3 possible candidates to fit the data. Those are $ARMA(p, q)$, $AR(p)$ and $MA(q)$. The one with the minimum Akaike Information Criterion defined by:

$$\text{AIC} = -2 \ln(\text{LKL}) + 2k,$$

where LKL is the likelihood function and k is the number of parameters that should be estimated is our model.

5.5.3 Jarque-Bera Normality Test

This test proposed by Jarque and Bera (1980)[27, 28, 29] mainly concentrates on the skewness and kurtosis of the sample data. In order for the sample to be generated from a normal distribution one should expect the skewness and kurtosis to be equal to zero. In Jarque-Bera test having a kurtosis of zero is equivalent to have it less than 3. The Jarque-Bera test statistics is defined as follows

$$\text{JB} = \frac{n}{6} \left(S^2 + \frac{(K - 3)^2}{4} \right),$$

where n is number of observations, S and K denote skewness and kurtosis given by

$$\mathbf{S} = \frac{\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^3}{\left(\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2 \right)^{3/2}}, \quad \mathbf{K} = \frac{\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^4}{\left(\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2 \right)^2}.$$

\bar{x} being the sample mean.

For sample sizes of 2000 or larger, this test statistic asymptotically converges to a Chi-squared distribution with 2 degrees of freedom (normality is rejected if test statistic exceeds Chi-squared value).

CHAPTER 6

OUR CDS SPREAD MODEL

6.1 Motivation

Enders and Lee[20] constructed a unit root test by usage of Fourier series in order to approximate smooth breaks. Their approach comes from the fact that exact break time cannot be known a priori. Therefore, a slight modification of Dickey-Fuller test has been proposed by the following manner

$$y_t = d(t) + \rho y_{t-1} + \gamma t + \varepsilon_t.$$

$d(t)$ is defined by using the Fourier expansion

$$d(t) = \alpha_0 + \sum_{k=1}^n \alpha_k \sin\left(\frac{2\pi kt}{T}\right) + \sum_{k=1}^n \beta_k \cos\left(\frac{2\pi kt}{T}\right) ; \quad n \leq \frac{T}{2}$$

where n is frequency and T is number of observations. Becker, Enders and Hurn [7] showed that structural changes are caught in low frequencies. On the other hand, as pointed out by Enders and Lee [20], Becker et al. [7, 8] and Enders and Jones[19], higher frequencies of the Fourier terms approximates the stochastic components of the series. Therefore, by using the low frequencies of Fourier terms, we modeled the structural breaks (pure jumps) and similar issues in deterministic components and with high frequencies we modeled the other stochastic components of the series such as seasonality, stochastic volatility, spikes, etc. Hence, by estimating the series in hand, this approach leads us to obtain a better forecasting equation.

6.2 Our Model

6.2.1 The Algorithm

Our model takes into account very general time series. Any non-linearity, structural breaks (i.e., pure jumps), any mean reverting breaks (i.e., spikes, volatility clustering,

stochastic volatility, etc.), can be considered. Moreover, we let the time series be periodic in the entire real line, thus we do not confine ourselves in a bounded domain. For forecasting purposes, our algorithm applies only to stationary series. An identification procedure algorithm is defined as follows

Step 1 Apply the linear unit root tests to series defined in Section 5.5.1. If the test fails then employ more general non linear unit root tests in Section 5.5.2. These tests also identify the structure of the series in hand with respect to structural break in deterministic components, linearity or non linearity in the stochastic component. Notice that by employing these newly proposed tests, the generality allowed in the testing procedure increases. If the series are stationary, i.e., one of the tests show that there is no unit root then follow Step 2.

Step 2 Consider the time dependent function [22]

$$f(t) = \sum_{k=1}^n \left(\alpha_k \sin \left(\frac{2\pi kt}{T} \right) + \beta_k \cos \left(\frac{2\pi kt}{T} \right) \right).$$

Evaluate the function for $k = 1$ and estimate the parameters a_1 and b_1 with least-squares. Then demean the original series by this moving and fluctuating average.

Step 3 Use Box-Jenkins test to determine p and q in an $ARMA(p, q)$ framework. Make diagnostic check for the residuals with Jarque-Bera test. If normality is satisfied then stop the procedure. Otherwise, go back to Step 2, increase the frequency by one and demean the original series with this new moving and fluctuating average.

Step 4 If the normality is achieved for the residuals term from $ARMA(p, q)$ model then the model is ready for forecasting and the transformation from stochastic difference equation space to stochastic differential equation space by using Hull and White [23] SDE. Otherwise return to step 2.

Once the process is terminated we end up with the following model

$$y_t = f(t) + \varepsilon_t + \sum_{i=1}^p \varphi_i y_{t-i} + \sum_{i=1}^q \theta_i \varepsilon_{t-i}.$$

However, notice that since our series exhibit stationarity, moving average model can be transformed to an $AR(1)$ process by the ergodicity condition. Moreover, the Fourier expansion has the capability of mimicking the moving average regardless of the lag. Therefore, under stationarity, we end up with the following expansion as the forthcoming theorem asserts,

$$y_t = f(t) + \alpha y_{t-1} + \varepsilon_t. \quad (6.1)$$

Theorem 6.1. *Any stationary time series can be represented by an $AR(1)$ process together with a Fourier series (i.e. can be approximated by a partial Fourier sum).*

Proof. Subtract $AR(1)$ process from any type of series y_t which includes structural break, non linearity or any type of anomalies in the data generating process.

$$y_t - \sum_{i=1}^p \varphi_i y_{t-i} + \varepsilon_t = f(t).$$

By Dirichlet conditions we can approximate the nonlinear function $f(t)$ as stated below:

$$f(t) = \sum_{k=1}^{T/2} \alpha_k \sin\left(\frac{2\pi kt}{T}\right) + \beta_k \cos\left(\frac{2\pi kt}{T}\right).$$

From the definition of $AR(1)$, we know that it is stationary and the residual term is $\varepsilon_t \sim N(0, \sigma^2)$. \square

Our aim is to represent the series with a mean reverting process which fluctuates around our moving and fluctuating average together with normally distributed residual terms. The normality of the residual terms renders us to transform our process in discrete time to continuous time in an easy and intuitive manner, as will be explained below.

6.2.2 Transformation to Continuous Time

From the discussion in Section 5.5.2 (6.1) can be transformed to the following Hull and White Model

$$dy(t) = (f(t) - y(t))dt + \sigma dW(t) \quad (6.2)$$

where $\sigma > 0$ and $W(t)$ is a standard Wiener process.

The solution of (6.2) is

$$\begin{aligned} & dy(t) = (f(t) - y(t))dt + \sigma dW(t), \\ \Rightarrow & d(e^t y(t)) = e^t y(t)dt + e^t dy(t) = e^t y(t)dt + e^t ((f(t) - y(t))dt + \sigma dW(t)), \\ & = e^t y(t)dt + e^t f(t)dt - e^t y(t)dt + \sigma e^t dW(t) = e^t f(t)dt + \sigma e^t dW(t) \\ \Rightarrow & e^t y(t) = y(0) + \int_0^t e^s f(s)ds + \sigma \int_0^t e^s dW(s), \\ \Rightarrow & y(t) = e^{-t} y(0) + \int_0^t e^{t-s} f(s)ds + \sigma \int_0^t e^{t-s} dW(s). \end{aligned}$$

For further discussion in SDEs see [2, 9, 23, 24, 32, 33, 42, 45, 50].

CHAPTER 7

APPLICATION OF THE MODEL TO THE CDS SPREAD DATA

7.1 Description of the Data

Our data is obtained from Itraxx 125 index, which is the product name for the family of CDS index products including the regions of Europe, Australia, Japan and non-Japan Asia. Itraxx 125 forms a huge sector of the whole credit derivatives. It involves credit default swap (CDS) premiums for loan payers from seven different sectors namely automotive, industry, consumer, energy, financial (senior), financial (sub) and non-financial with a maturity of 10 years. Unlike a CDS traded over the counter the data, we are dealing forms a CDS index which is completely standardized credit security traded in an organized liquid market. [4, 26, 47, 58] take into account the determination of CDS spreads. Our approach, briefly explained in Chapter 6 is a completely new approach. The data are daily and quite rich in number (more than 2000 observations). In the plots the horizontal denotes the day and the vertical axis denotes the CDS spread premium in terms of bps. The Figures 7.1,7.2,7.3,7.4,7.5,7.6,7.7 below are time series plotting of the sector based CDS spreads beginning from 6.22.2004 for 2040 days. Figure 7.8 is the depiction of their simple average.

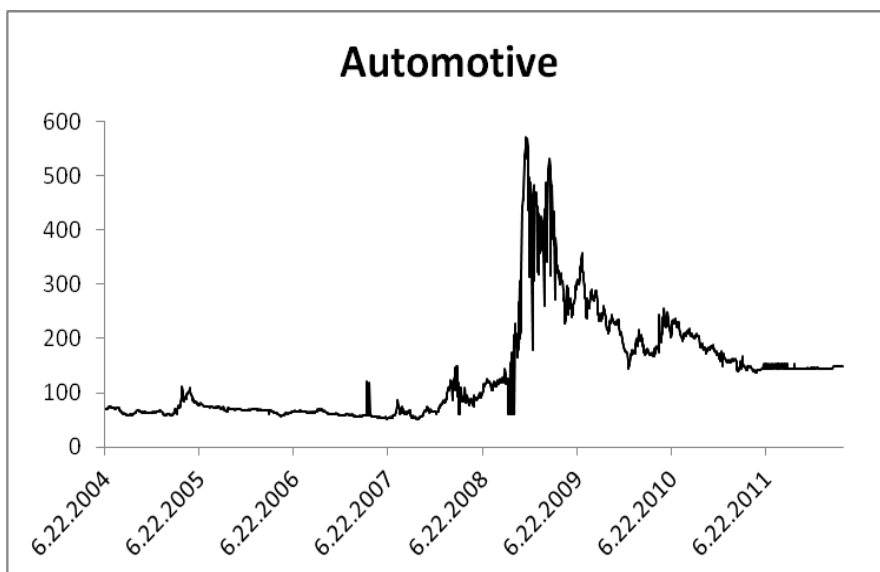


Figure 7.1: ITraxx 125 CDS spread for automotive sector.

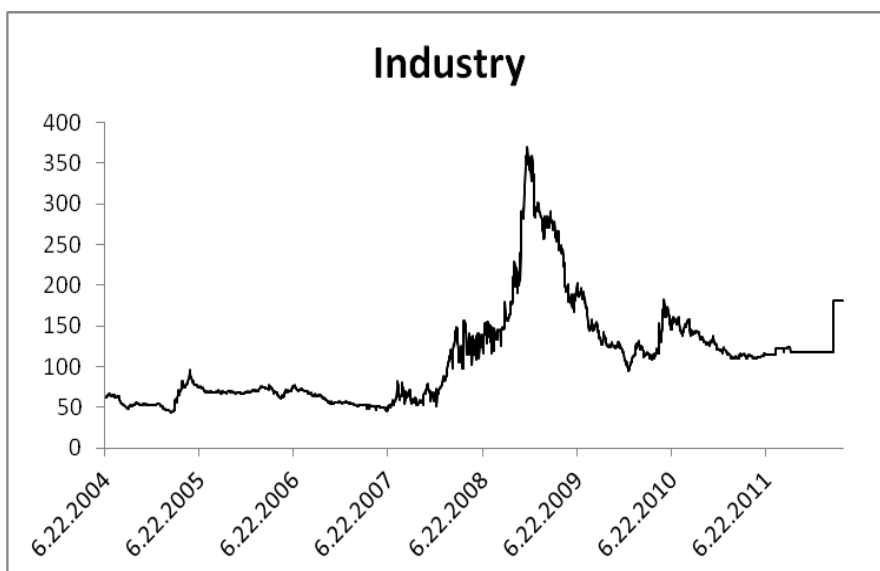


Figure 7.2: ITraxx 125 CDS spread for industry sector.

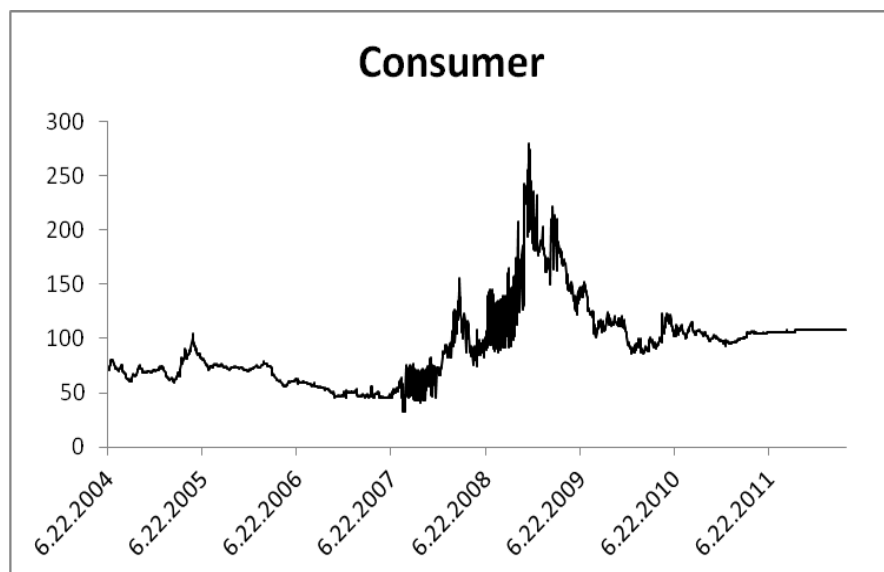


Figure 7.3: ITraxx 125 CDS spread for consumer sector.

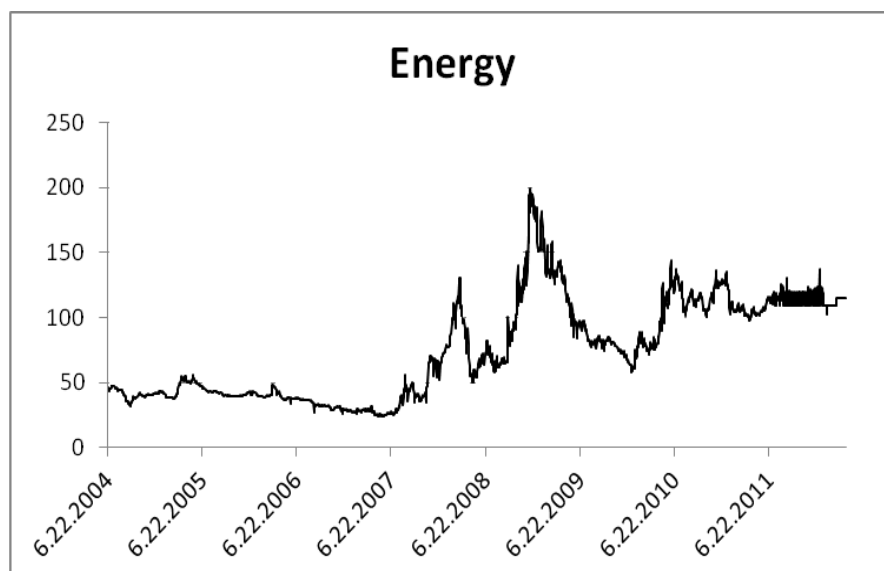


Figure 7.4: ITraxx 125 CDS spread for energy sector.

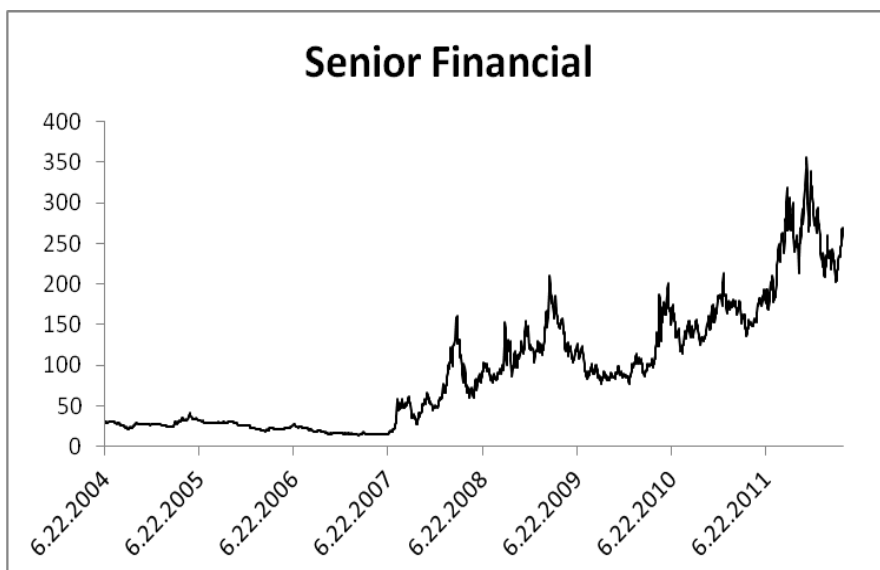


Figure 7.5: ITraxx 125 CDS spread for financial senior sector.

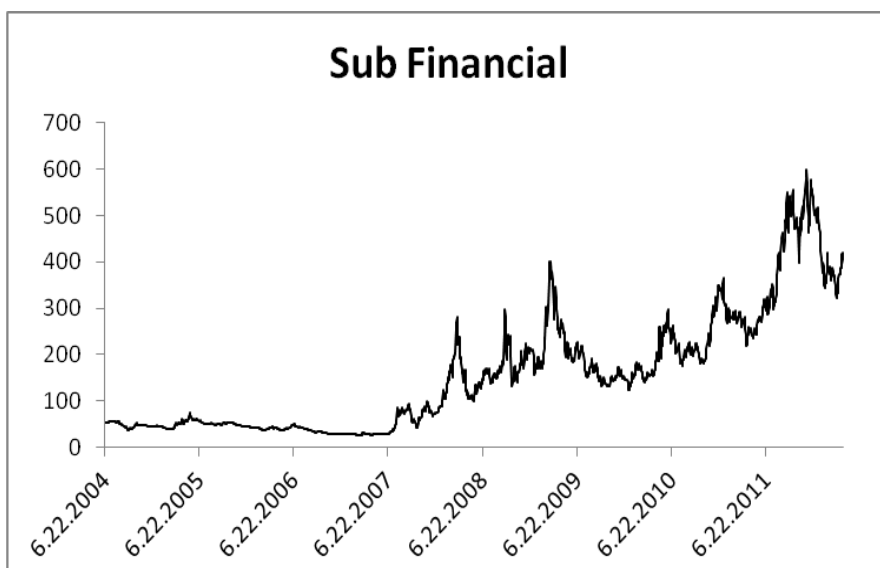


Figure 7.6: ITraxx 125 CDS spread for financial sub sector.

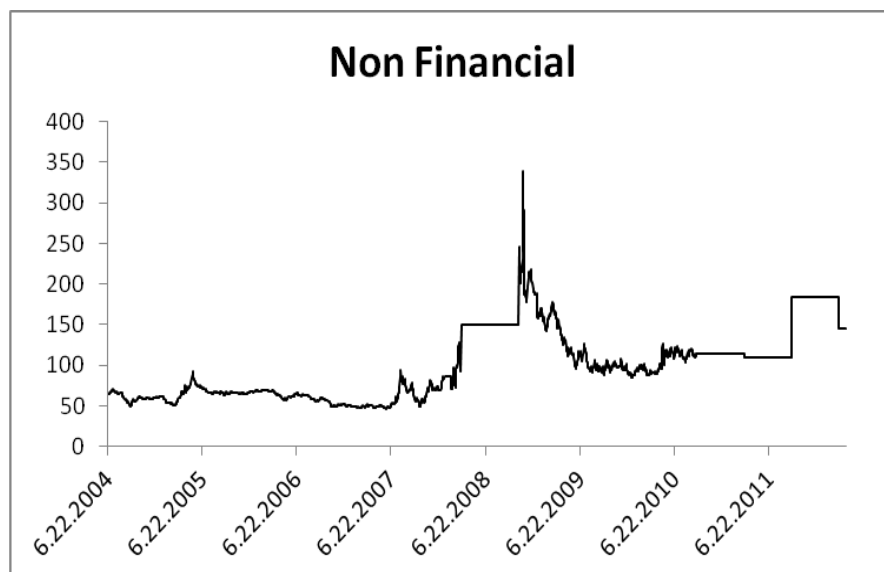


Figure 7.7: ITraxx 125 CDS spread for non financial sector.

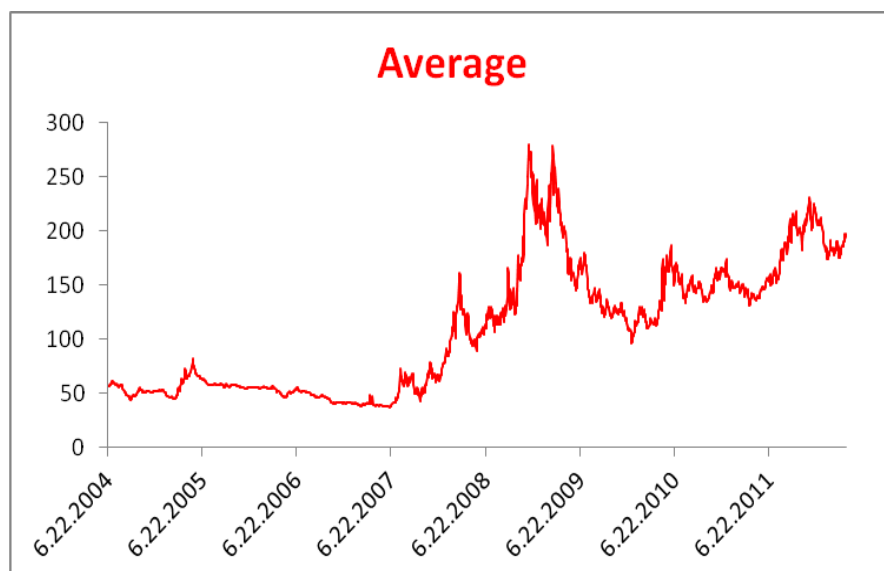


Figure 7.8: Simple average of the seven sectors comprising Itraxx 125 index.

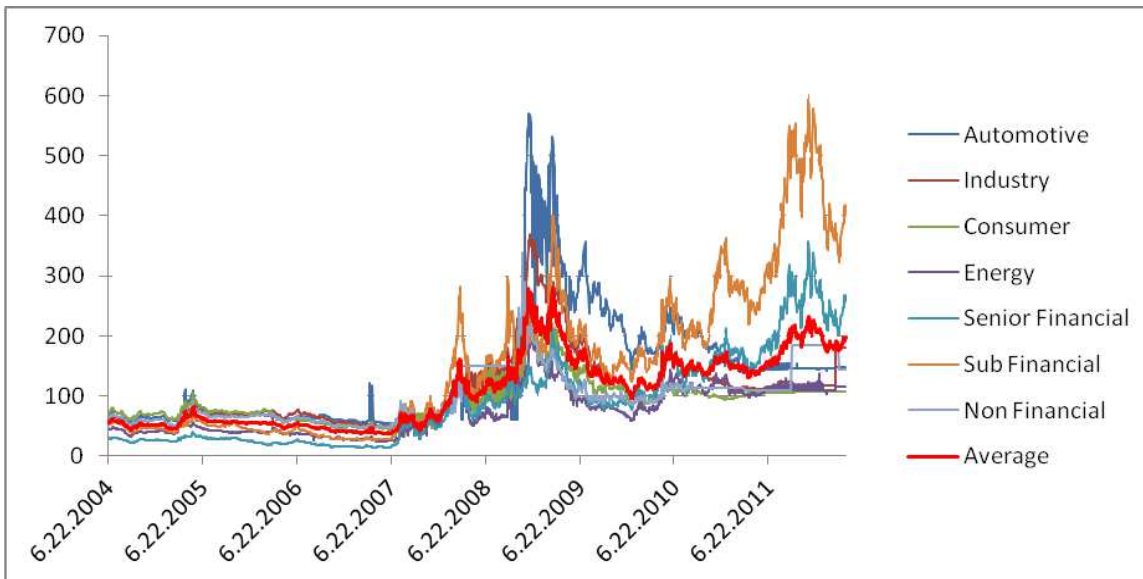


Figure 7.9: ITraxx 125 CDS spreads for seven sectors and their average.

Looking at Figure 7.9, we can see certain dates in which the behavior of the series exhibit “jumps”. Another way of saying that is, there are times where the movement cannot be explained by Brownian motion.

One can also realize a strong correlation among sectors by looking at the graphs. It is also likely that the mean has changed after some time. Moreover, at a glance, it seems probable that certain mean reverting type structures occurred.

The major structural break corresponds to the 2008 financial crisis. CDSs exhibit similar structural break during times of crises despite of the sector. Towards the end, we see high volatility particularly for financial sub and financial senior sectors. This is mainly due to the unorthodox expansionary monetary policies namely quantitative easing applied by FED. Major central banks throughout the world began to follow loose monetary policies in order to boost up economies. However, this further diminished the credibility of the financial sector. This is the reason for high CDS spreads pertaining to financial sector. Since, financial sub sector is more vulnerable to shocks than financial senior sector; its CDS spread was highest. CDSs written on non-financial sector is the least liquid market compared to other CDSs. Therefore there are times when lack of demand is at stage. Due to this reason we sometimes see some constant and then sharp movements.

7.2 Analysis of the Data

We took into account the cross correlation among the series given in Definition 5.6. The chart below gives the correlation coefficients of the means of CDS spreads comprising the CDO:

Table 7.1: Cross correlation of the series (correlation of the means).

	Automotive	Industry	Consumer	Energy	Senior Fin.	Sub Fin.	Non Fin.
Automotive	1.000	0.905	0.872	0.785	0.506	0.486	0.610
Industry	0.905	1.000	0.942	0.842	0.544	0.522	0.768
Consumer	0.872	0.942	1.000	0.837	0.573	0.559	0.774
Energy	0.785	0.842	0.837	1.000	0.833	0.803	0.809
Senior Fin.	0.506	0.544	0.573	0.833	1.000	0.992	0.782
Sub Fin.	0.486	0.522	0.559	0.803	0.992	1.000	0.774
Non Fin.	0.610	0.768	0.774	0.809	0.780	0.774	1.000

Having examined the high correlation coefficients among the means, we take the average of the series and considered it as a “representative”. These findings also support our Lévy copula framework. Now we follow Step 1 and apply certain unit root tests to the data for checking stationarity. The data passes KSS and OY tests as seen below. Moreover it almost passes EL test for $k = 3$ which is 3 Fourier series terms. When Fourier terms are increased beyond 3, EL test is also passed.

Table 7.2: Unit root test results.

	ADF	EG F	EG t	KSS	LNV	Solis F	Solis t	OY	EL
Model 1	-1.155	0.783	-0.389	-3.726*	-3.045	4.955	-3.021	-4.919*	-1.719
Model 2					-2.989	4.096	-2.762	-5.024*	-2.861
Model 3				-3.831*	-3.047	4.956	-3.034	-5.035*	-2.878

Table 7.3: Critical values for unit root tests.

		10%	5%	1%
ADF		-2.580	-2.890	-3.510
EG	EG F	3.740	4.560	6.410
	EG t	5.150	6.080	8.120
LNV	Model 1	-3.797	-4.103	-4.685
	Model 2	-4.277	-4.565	-5.141
	Model 3	-4.552	-4.825	-5.420
KSS	Model 1	-1.920	-2.220	-2.820
	Model 2	-2.660	-2.930	-3.480
	Model 3	-3.130	-3.400	-3.930
Solis F	Model 1	7.706	8.845	11.620
	Model 2	9.995	11.309	13.733
	Model 3	11.234	12.558	15.392
Solis t	Model 1	-3.119	-3.356	-3.820
	Model 2	-3.495	-3.731	-4.159
	Model 3	-3.671	-3.897	-4.368
OY	Model 1	-3.489	-3.801	-4.412
	Model 2	-3.879	-4.180	-4.757
	Model 3	-4.053	-4.370	-4.969

After achieving stationarity, we move to Step 2 and then Step 3. From the unit root tests, we also identified the structure of the series as well, due to the reason that the alternative hypothesis of the series included testable hypothesis such as nonlinear stationarity, structural break and stationarity and nonlinearity around a structural break. These identifications are helpful for our modeling, because these structures are well approximated by Fourier series as mentioned above. The Matlab code given in the appendix shows that our data can be represented by an auto regressive 1 process having a normally distributed error term with constant variance together with a finite number of Fourier terms.

Here are graphical representations of the moving and fluctuating average obtained by different number of Fourier series terms given in 7.10,7.11,7.12 and 7.13.

In 7.14,7.15,7.16 and 7.17 below are some graphs of the variance of residuals after de-meaning certain number of Fourier terms. Notice that although the variance decreases gradually, normality is achieved at 997th iteration with the inclusion of an $AR(1)$ process.

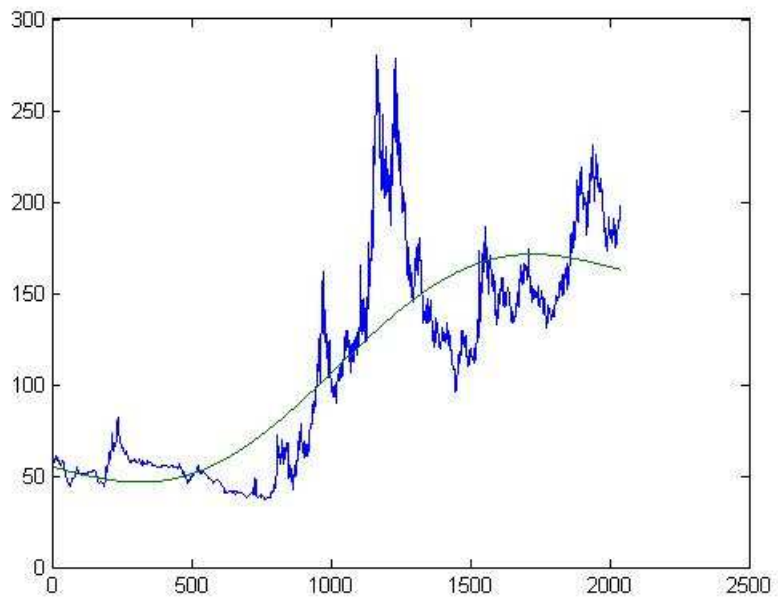


Figure 7.10: Moving and fluctuating average with first Fourier terms.

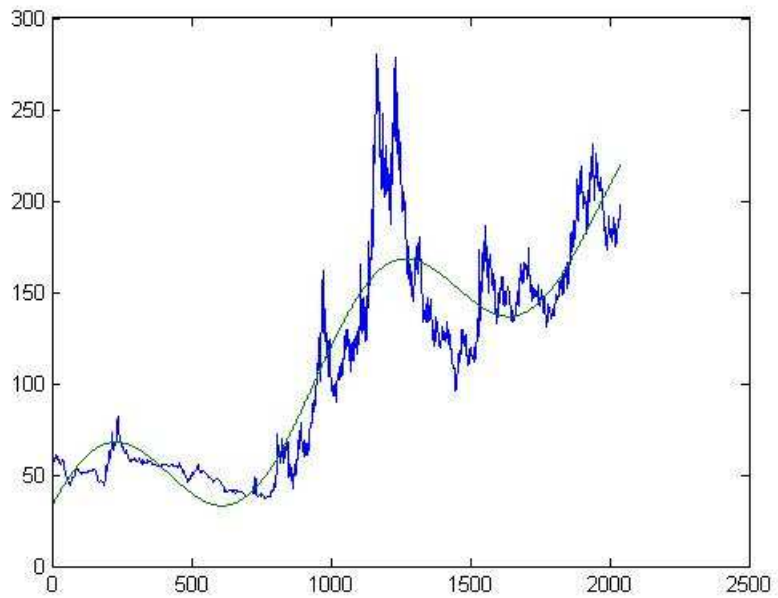


Figure 7.11: Moving and fluctuating average with first and second Fourier terms.

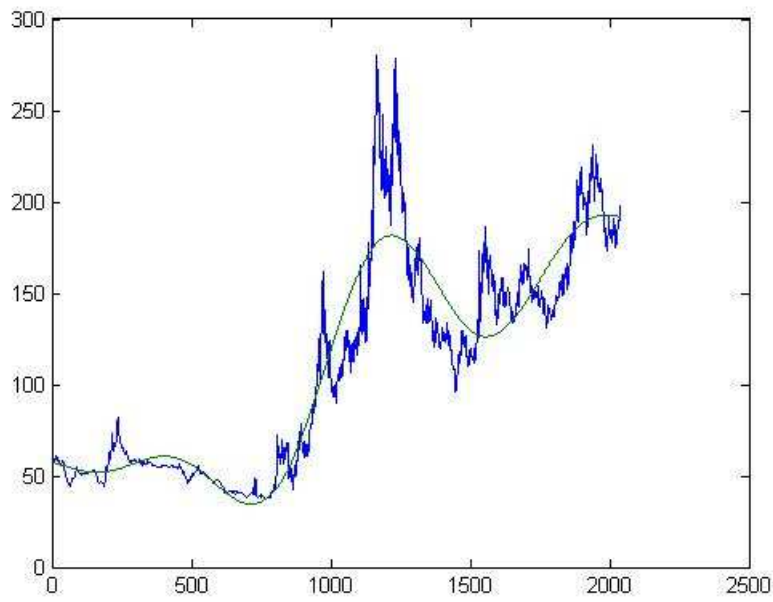


Figure 7.12: Moving and fluctuating average with first, second and third Fourier terms.

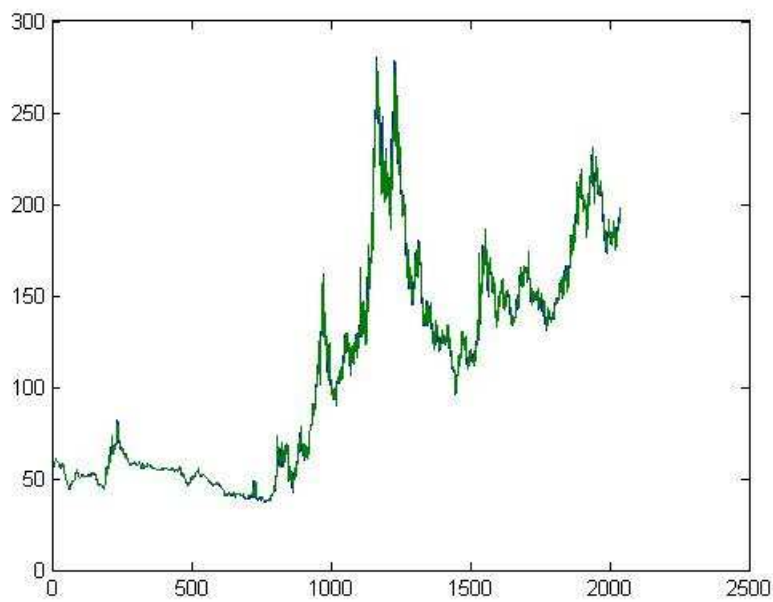


Figure 7.13: Moving and fluctuating average with 997 Fourier terms together with $AR(1)$ process.

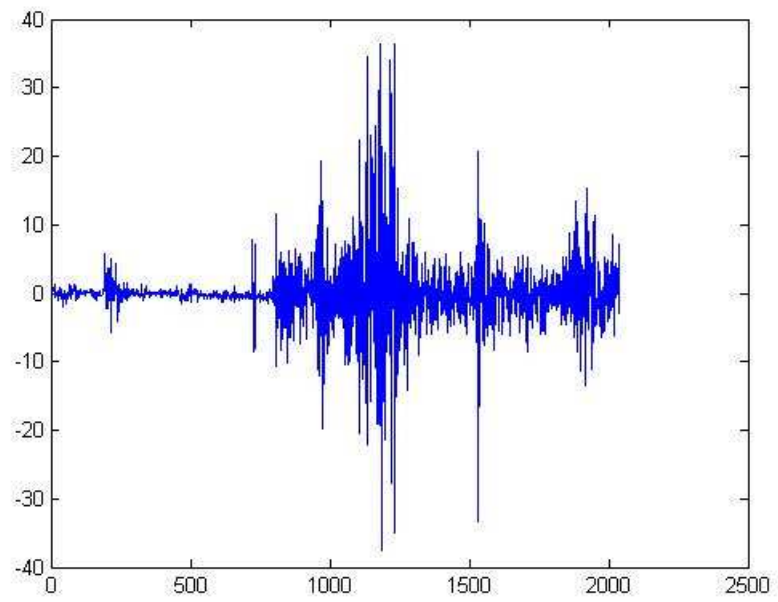


Figure 7.14: Residuals when first Fourier terms are demeaned.

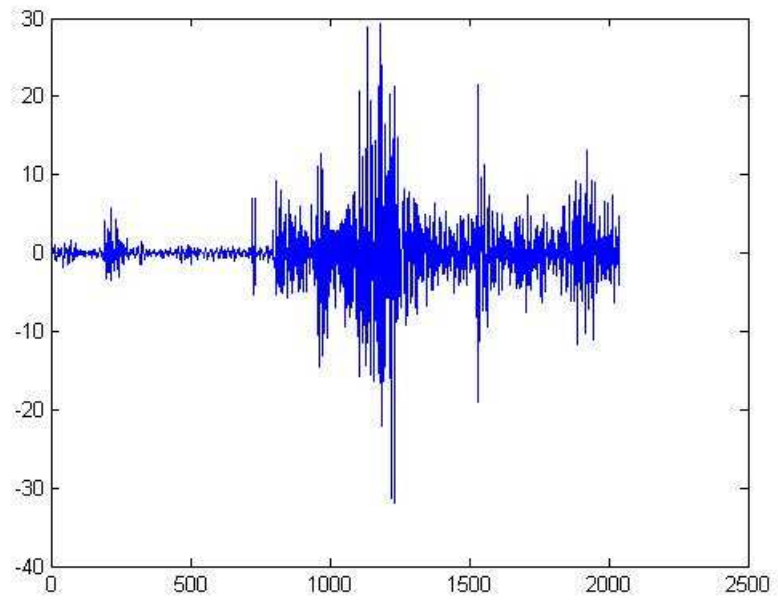


Figure 7.15: Residuals when first 100 Fourier terms are demeaned.

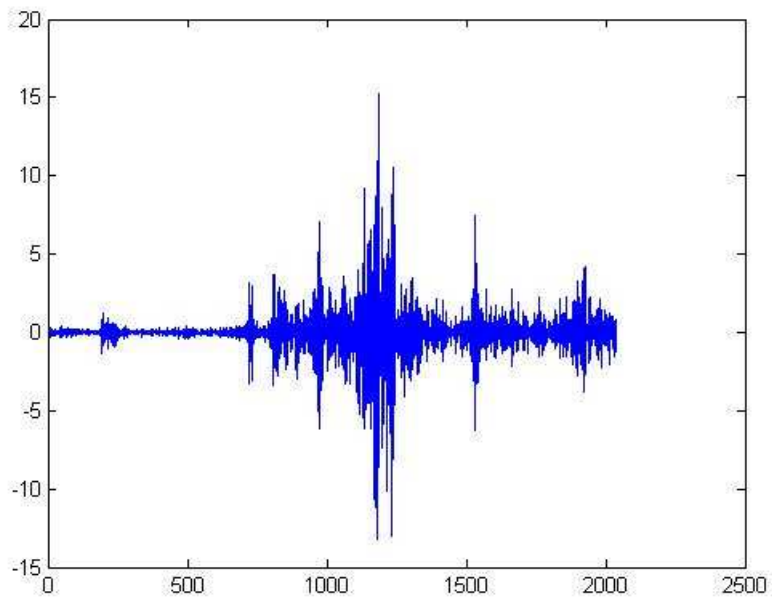


Figure 7.16: Residuals when first 500 Fourier terms are demeaned.

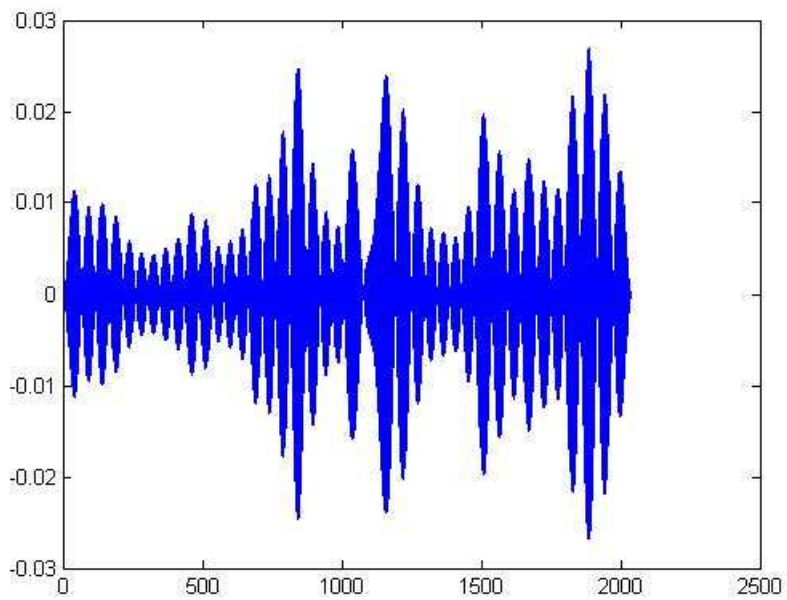


Figure 7.17: Residuals when first 997 Fourier terms are demeaned.

Table 7.4: Variances of the Residuals

# of Fourier terms	Variance	Normality
1	20,926	No
100	14,516	No
500	2,651	No
997	0,06	Yes

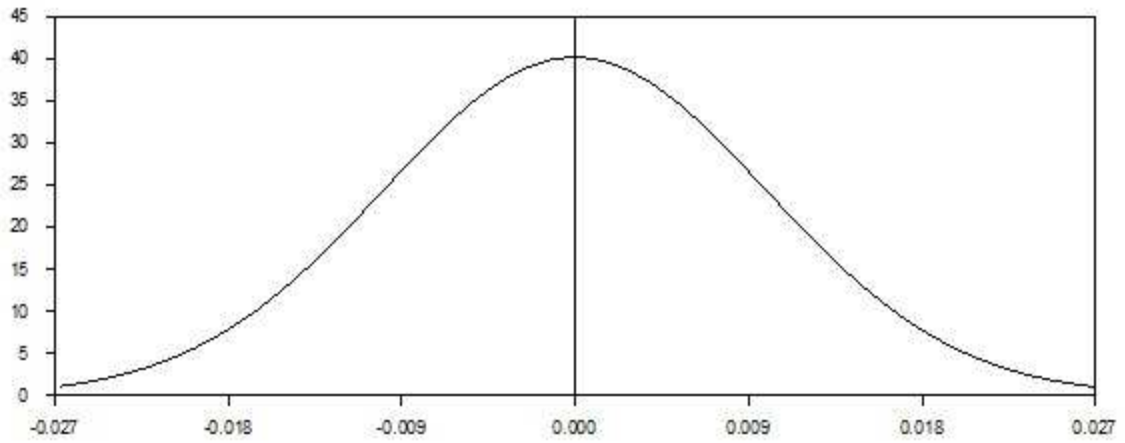


Figure 7.18: Pdf of residuals when 997 Fourier terms are demeaned.

CHAPTER 8

FORECASTING OF THE CDS SPREADS WITH OUR MODEL

8.1 Properties of Forecasts

Consider our model presented in Section 6.2 which is

$$y_t = f(t) + \alpha y_{t-1} + \varepsilon_t, \varepsilon_t \sim N(0, \sigma^2).$$

Iteration of one period yields

$$y_{t+1} = f(t) + \alpha y_t + \varepsilon_{t+1}.$$

Since $f(t)$ and α are estimated, we are able to forecast y_{t+1} given information available at t as

$$\mathbb{E}(y_{t+1}|y_t, y_{t-1}, \dots, \varepsilon_t, \varepsilon_{t-1}, \dots) = f(t) + \alpha y_t.$$

Similarly since

$$y_{t+2} = f(t) + \alpha y_{t+1} + \varepsilon_{t+2}$$

the forecasting function becomes

$$\begin{aligned} & \mathbb{E}(y_{t+2}|y_t, y_{t-1}, \dots, \varepsilon_t, \varepsilon_{t-1}, \dots) \\ &= f(t) + \alpha \mathbb{E}(y_{t+1}|y_t, y_{t-1}, \dots, \varepsilon_t, \varepsilon_{t-1}, \dots) \\ &= f(t) + \alpha(f(t) + \alpha y_t). \end{aligned}$$

Further iteration gives the following forecast function

$$\mathbb{E}(y_{t+j}|y_t, y_{t-1}, \dots, \varepsilon_t, \varepsilon_{t-1}, \dots) = f(t)(1 + \alpha + \alpha^2 + \dots + \alpha^{j-1}) + \alpha^j y_t.$$

Now since

$$\lim_{j \rightarrow \infty} \mathbb{E}(y_{t+j}|y_t, y_{t-1}, \dots, \varepsilon_t, \varepsilon_{t-1}, \dots) = (f(t))/(1 - \alpha) + \lim_{j \rightarrow \infty} \alpha^j y_t$$

we can conclude that the conditional forecast converges to unconditional mean if the process is a stationary ARMA type. If the stationarity of the series are not obtained the forecast function diverges or unidentified in the case of $\alpha \leq 1$

$$\lim_{j \rightarrow \infty} \mathbb{E}(y_{t+j} | y_t, y_{t-1}, \dots, \varepsilon_t, \varepsilon_{t-1}, \dots) = \frac{f(t)}{1 - \alpha} + \lim_{j \rightarrow \infty} \alpha^j y_t = \infty. \quad (8.1)$$

In next subsection, we apply our moving and fluctuating averaging technique to the CDS data extensively discussed in Chapter 7.

8.2 Forecasting with Our Model

Let us consider the CDS data presented in Chapter 7, which is the average of the seven sectors comprising the CDO. Now, we wish to compare the accuracy and power of our model for this particular data set. First we take a simple $AR(1)$ process, then an $AR(1)$ process together with first two Fourier terms and finally our new technique which is $AR(1)$ process together with 997 Fourier terms. We take the 1800th data point as a basis and try to forecast the remaining data up to 2040. The results are shown in Figure 8.1, Figure 8.2 and Figure 8.3

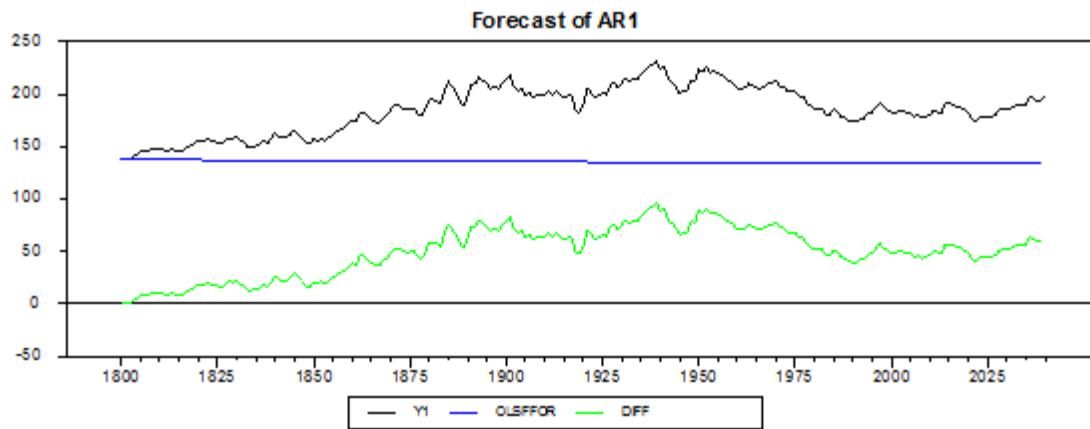


Figure 8.1: Forecasting with an $AR(1)$ process.

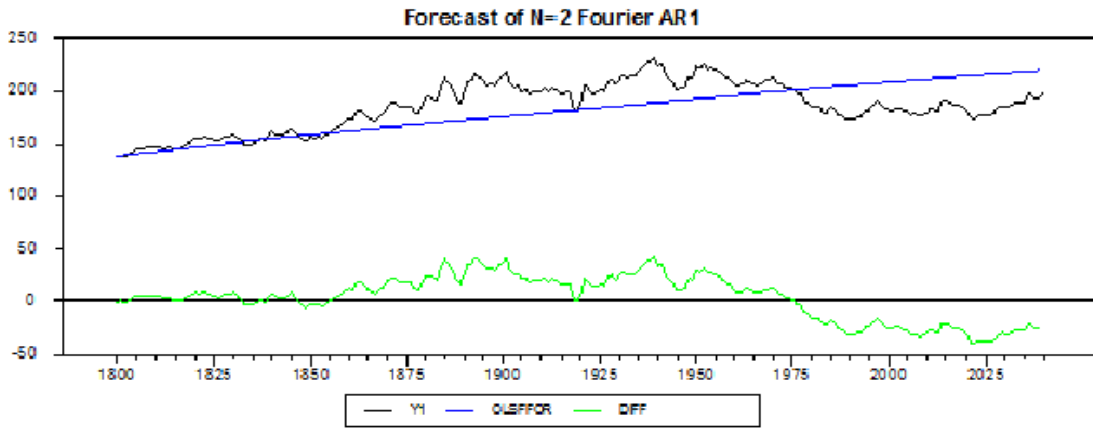


Figure 8.2: Forecasting with an $AR(1)$ process together with the first and second Fourier terms.

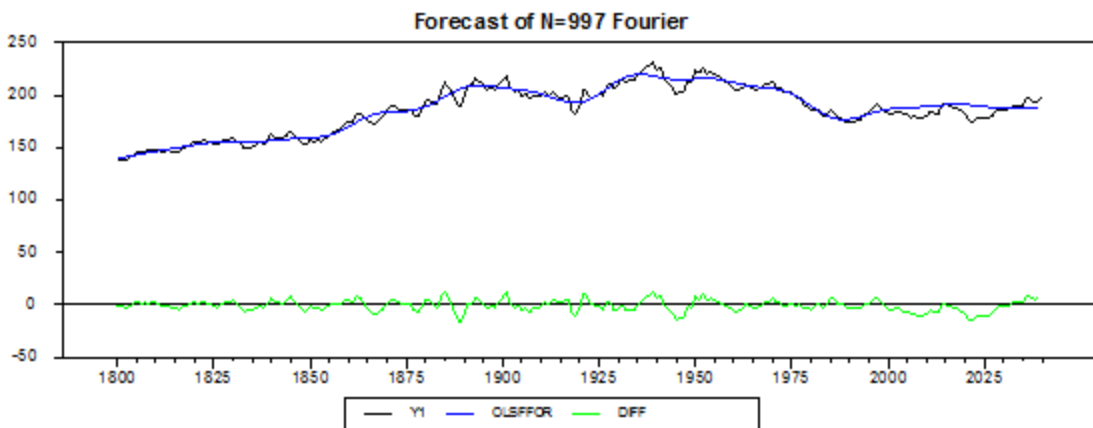


Figure 8.3: Forecasting with an $AR(1)$ process together with 997 Fourier terms.

Looking at Figure 8.1, Figure 8.2 and Figure 8.3 above, we can deduce the following for the original and demeaned data

1. The $AR(1)$ process is too far from accuracy. Discrepancies from real data are inherited into the model and moreover they get wider as time index increases as shown in (8.1)
2. The $AR(1)$ model combined with the first two terms of Fourier series, k is definitely better in forecasting than the sole $AR(1)$ process. A mean reverting structure is at stage. However, the oscillations around the mean are still very high. Therefore, we proceed with the algorithm that we mention in Section 6.2.1.
3. The final model, obtained by usage of our proposed algorithm have better forecasting power than the above two models. The obtained model is the $AR(1)$ process with 997 Fourier terms $k=1, \dots, 997$ clearly has high accuracy. The variance around the mean is very low showing us that we can use this model for forecasting purposes. On the other hand, the predicted variables are oscillating around the moving and fluctuating average with a very small variance, demonstrating that we obtained a robust model for forecasting the CDS spreads.

In the time series literature, the forecasting power of the model is used for model selection, as well. Depending on these findings, we can conclude that we have a well specified and concrete model.

CHAPTER 9

CONCLUSION

The synthetic CDO market, comprised of CDSs drew quite a lot of attention due to high returns compared to sure gains. However, the complexity of this derivative and problems in its pricing caused the 2008 sub prime crisis which had a contagious effect. Here, the default correlation is assumed to be normally correlated, rendering us to extract implied correlation parameter from the existing CDO prices. However, due to this confining assumption, in this thesis, first we aim to correct this mispricing. This is done by the introduction of the correlation of jumps among sectors which can be regarded as the crisis periods. By this new technique, we were able to obtain better results compared to the existing pricing still applied by practitioners. The phenomenon, known as the correlation smile under the pricing via Gaussian copula is corrected in our model by the introduction of a perfectly dependent Lévy copula.

Later we analyzed the underlying CDS spread processes of the CDO. Here, by the introduction of our new “moving and fluctuating” average technique comprised of a partial Fourier sum, we not only introduced a new technique applicable to a specific data, but also a general model extendable to any stationary time series. The power of this new technique is the allowance of the modeling in discrete time to be transformed into a well known and analytically tractable continuous time stochastic differential equation. Thus, instead of an ad-hoc stochastic differential equation including jump, spike and/or stochastic volatility components where the parameter estimation is hard and usually non-solvable, our technique exhibits much easy analytic solutions. Here, the crucial thing and the major contribution is trying to find a finite sum of Fourier series terms where the distribution of the residuals are normal. Therefore, we not only tried to avoid usage of known filtering techniques but also precluded any over fitting issues. Moreover, instead of direct computation, we estimated the Fourier coefficients, which is a complete different approach.

Another contribution is the forecasting with our new methodology. We saw that our moving and fluctuating average together with an $AR(1)$ process is quite powerful in forecasting compared to other processes.

Throughout the model, the key is the inclusion of $AR(1)$ process in the sense that usage of only partial Fourier sums is not sufficient to achieve normality of the residuals. Having established the normality of the residual terms, we transform this discrete time process to a continuous stochastic differential equation and thus forming a bridge

between discrete and continuous world.

Moreover, we constructed a model which is not data specific. One can form an $AR(1)$ process with finite number of Fourier series terms for any arbitrary data. The only assumption we need here is stationarity which is quite plausible in the sense that accurate forecasting can only be done with it.

We only included original computer codes in the appendix. For the testing of unit root, selection of lag, determination of the Fourier coefficients and their significance and graphical representations, various Matlab and Winrats codes are used. Part of them are in package form and can be found anywhere. Most are already included in Matlab, Winrats and Eviews as specific codes. Even one can find some of them in Excel. However, if still requested, those codes can also be given.

As mentioned before, we proposed a new technique which is not data specific and has a very high forecasting power. We believe that our model will give good forecast results when applied to other time series as well. Our model gives us the flexibility to transform any data in discrete time, into a well known stochastic differential equation. As a future study, one can generate data from stochastic differential equations, including jumps and stochastic volatility, then use our technique to express it in the form of Hull and White model and compare the simulation results.

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APPENDIX A

Normality Test Results

Table A.1: Normality test results

k	JB value	
	without AR	with AR
1	1696.66	25697.19
2	1666.95	25372.82
3	1321.40	24951.07
4	971.77	24689.43
5	1137.25	24274.09
6	2753.32	23566.25
7	2530.65	23411.05
8	2218.51	22819.04
9	2396.25	22346.14
10	1797.96	22162.81
...
100	6416.25	15734.38
101	6506.51	15761.95
102	6632.51	15796.36
103	6791.96	15924.30
104	6957.53	16119.95
105	6992.36	16295.49
106	7425.07	16873.11
107	7593.39	17124.48
108	7692.78	17178.45
109	7848.88	17229.13
...
200	19314.07	21934.58
201	18782.85	21264.90
202	18381.90	20582.26
203	18102.82	20172.59
204	18049.30	20110.50
205	18874.58	19953.77
206	17639.35	19704.45
207	17904.22	19903.35

k	JB value	
	without AR	with AR
300	20397.13	17252.27
301	20617.44	17588.55
302	20924.96	18037.44
303	20959.53	18067.64
304	21273.48	18217.59
305	21604.30	18431.70
306	21609.17	18436.46
307	21743.30	18516.88
308	21771.01	18594.76
309	21804.50	18660.68
...
400	37748.76	36605.39
401	37616.82	36347.54
402	37744.45	36635.47
403	37622.41	36356.59
404	37538.81	36123.09
405	37173.60	35709.88
406	37619.54	36209.12
407	38365.10	36890.89
408	38244.38	36689.74
409	39293.48	37867.14
...
500	26192.25	27396.68
501	25999.09	26987.15
502	25641.55	26387.46
503	25222.65	25629.53
504	24894.70	24930.87
505	24615.60	24268.13
506	24533.73	23849.34
507	24410.54	23377.97
508	24106.60	22923.76
509	23732.27	22319.84

k	JB value	
	without AR	with AR
600	21203.67	19701.37
601	21967.24	20718.63
602	22665.59	21975.61
603	23169.25	22469.14
604	24218.92	24076.64
605	24000.55	23647.69
606	23854.41	23268.31
607	23869.78	23202.91
608	23954.91	23306.35
609	24283.94	23678.35
...
700	21590.62	16043.39
701	21101.79	15746.76
702	20607.89	15267.12
703	20362.54	15186.81
704	20019.22	14803.39
705	19726.43	14552.50
706	19570.10	14352.26
707	19553.72	14170.40
708	19411.10	13929.74
709	19235.04	13602.34
...
800	25598.18	6691.66
801	25269.18	6503.55
802	24884.61	6302.83
803	24456.19	6069.20
804	24370.50	6031.54
805	24419.89	6145.07
806	24482.00	6095.74
807	25286.86	6187.15
808	25959.22	6304.02
809	27634.50	6938.01

k	JB value	
	without AR	with AR
900	14502.42	7966.29
901	14519.84	7970.06
902	14392.66	7901.58
903	14302.42	7894.53
904	14254.62	8025.81
905	14083.39	8012.09
906	14028.36	8122.11
907	13789.01	7833.52
908	13746.58	7678.11
909	13607.99	7474.80
...
990	587.79	373.21
991	526.53	324.38
992	378.43	167.16
993	307.79	158.50
994	211.39	124.87
995	95.15	32.08
996	108.95	27.71
997	62.27	1.65

APPENDIX B

Sine Fourier Coefficients and Their Significance

Table B.1: Sine Fourier Coefficients and Their Significance

Term	Coefficient	t-statistics	Significance
1	-41.96	-2.80	0.01
2	49.23	6.53	0.00
3	-38.74	-7.77	0.00
4	15.34	4.08	0.00
5	-10.49	-3.50	0.00
6	-22.73	-9.12	0.00
7	8.55	3.98	0.00
8	-11.73	-6.27	0.00
9	11.76	7.02	0.00
10	-5.76	-3.85	0.00
...
100	-1.36	-9.66	0.00
101	0.05	0.34	0.74
102	-0.52	-3.73	0.00
103	-0.88	-6.43	0.00
104	-0.53	-3.91	0.00
105	0.38	2.80	0.01
106	-1.99	-15.06	0.00
107	0.52	3.98	0.00
108	-0.33	-2.50	0.02
109	-0.60	-4.67	0.00
...
200	0.21	3.59	0.00
201	0.45	7.53	0.00
202	-0.35	-6.00	0.00
203	0.48	8.27	0.00
204	-0.30	-5.20	0.00
205	0.43	7.55	0.00
206	-0.05	-0.97	0.34
207	-0.40	-7.10	0.00

Term	Coefficient	t-statistics	Significance
300	-0.27	-9.80	0.00
301	0.44	15.91	0.00
302	0.13	4.57	0.00
303	-0.03	-1.04	0.30
304	-0.39	-14.29	0.00
305	0.06	2.36	0.02
306	-0.01	-0.37	0.71
307	0.11	4.07	0.00
308	0.00	0.08	0.94
309	-0.01	-0.37	0.71
...
400	0.08	7.18	0.00
401	-0.18	-16.61	0.00
402	-0.25	-23.53	0.00
403	-0.08	-7.81	0.00
404	-0.20	-19.59	0.00
405	0.15	2.80	0.01
406	-1.99	14.17	0.00
407	-0.01	-1.16	0.25
408	-0.05	-4.50	0.00
409	0.28	28.14	0.00
...
500	0.04	14.87	0.00
501	0.11	38.94	0.00
502	-0.32	-119.53	0.00
503	0.16	68.48	0.00
504	-0.25	-108.48	0.00
505	0.20	89.05	0.00
506	-0.07	-32.32	0.00
507	-0.08	-34.70	0.00
508	0.16	69.87	0.00
509	-0.12	-53.20	0.00

Term	Coefficient	t-statistics	Significance
600	-0.01	-2.54	0.01
601	0.07	12.91	0.00
602	-0.16	-29.56	0.00
603	0.12	23.37	0.00
604	-0.17	-31.76	0.00
605	0.03	5.99	0.00
606	-0.17	-29.63	0.00
607	0.01	1.32	0.19
608	0.06	10.52	0.00
609	-0.16	-27.44	0.00
...
700	-0.07	-9.33	0.00
701	0.24	37.07	0.00
702	-0.19	-25.59	0.00
703	0.16	21.56	0.00
704	-0.04	-6.25	0.00
705	0.00	0.39	0.70
706	0.17	24.23	0.00
707	-0.20	-27.79	0.00
708	0.14	18.50	0.00
709	0.00	0.40	0.69
...
800	-0.01	-2.13	0.04
801	-0.03	-3.87	0.00
802	0.07	10.14	0.00
803	-0.01	-2.22	0.00
804	0.03	4.53	0.00
805	0.04	6.23	0.00
806	0.01	2.24	0.03
807	0.00	0.51	0.62
808	-0.03	-4.47	0.00
809	0.06	8.50	0.00

Term	Coefficient	t-statistics	Significance
900	0.01	2.36	0.02
901	0.00	0.61	0.54
902	0.01	2.97	0.00
903	0.00	-0.42	0.68
904	-0.01	-1.29	0.20
905	0.02	4.94	0.00
906	-0.01	-2.52	0.02
907	0.01	2.81	0.01
908	0.01	1.81	0.08
909	-0.01	-2.25	0.03
...
990	0.00	0.40	0.69
991	0.00	2.06	0.05
992	0.00	-0.77	0.45
993	0.01	2.34	0.02
994	0.00	-1.64	0.11
995	0.01	3.16	0.00
996	0.00	-0.41	0.69
997	0.01	2.18	0.04

APPENDIX C

Cosine Fourier Coefficients and Their Significance

Table C.1: Cosine Fourier Coefficients and Their Significance

Term	Coefficient	t-statistics	Significance
1	0.37	8.12	0.00
2	35.04	307.10	0.00
3	-3.78	-76.39	0.00
4	-6.51	-120.91	0.00
5	22.63	283.31	0.00
6	-15.69	-214.91	0.00
7	-0.70	-14.96	0.00
8	-2.66	-55.22	0.00
9	-6.41	-119.06	0.00
10	1.31	28.69	0.00
...
100	0.17	3.69	0.00
101	-0.18	-4.08	0.00
102	0.24	5.37	0.00
103	1.13	25.26	0.00
104	-1.03	-22.58	0.00
105	1.53	34.21	0.01
106	-0.48	-10.66	0.00
107	-0.92	-20.16	0.00
108	-0.34	-7.54	0.00
109	-0.33	-7.34	0.00
...
200	-0.95	-22.14	0.00
201	0.77	18.42	0.00
202	-0.95	-22.46	0.00
203	0.38	8.97	0.00
204	-0.63	-14.98	0.00
205	0.21	5.04	0.00
206	0.37	8.98	0.00
207	0.35	8.32	0.00

Term	Coefficient	t-statistics	Significance
300	-0.25	-6.83	0.00
301	-0.50	-13.30	0.00
302	0.47	12.75	0.00
303	0.03	0.83	0.41
304	0.22	6.12	0.00
305	-0.46	-12.28	0.00
306	0.00	-0.09	0.93
307	-0.14	-3.74	0.00
308	-0.15	-3.98	0.00
309	0.09	2.50	0.02
...
400	-0.21	-6.69	0.00
401	0.38	12.26	0.00
402	-0.37	-12.10	0.00
403	-0.36	-11.74	0.00
404	-0.07	2.16	0.04
405	-0.15	-4.93	0.00
406	-0.01	0.45	0.66
407	0.08	2.63	0.01
408	-0.07	-2.40	0.02
409	-0.01	-0.33	0.75
...
500	-0.21	10.65	0.00
501	0.36	15.28	0.00
502	-0.12	-5.23	0.00
503	0.01	0.61	0.55
504	0.04	1.62	0.11
505	-0.25	-10.58	0.00
506	0.24	1.62	0.00
507	-0.20	-8.60	0.00
508	-0.04	-1.76	0.09
509	0.10	4.10	0.00

Term	Coefficient	t-statistics	Significance
600	-0.01	-0.59	0.56
601	0.16	9.34	0.00
602	-0.02	-1.18	0.24
603	0.09	5.54	0.00
604	0.09	5.47	0.00
605	0.10	5.94	0.00
606	-0.06	-3.55	0.00
607	-0.06	-3.86	0.00
608	0.04	2.63	0.01
609	-0.13	-8.01	0.00
...
700	-0.16	-15.23	0.00
701	0.08	7.59	0.00
702	0.05	4.45	0.00
703	-0.14	-13.01	0.00
704	0.12	11.88	0.00
705	-0.14	-13.89	0.00
706	0.12	11.39	0.00
707	0.07	6.63	0.00
708	-0.13	-8.58	0.00
709	0.16	16.38	0.00
...
800	-0.05	-8.79	0.00
801	-0.01	-1.55	0.13
802	-0.07	-11.56	0.00
803	0.05	8.36	0.00
804	-0.07	-13.53	0.00
805	0.06	10.35	0.00
806	-0.02	-2.99	0.00
807	0.08	15.11	0.00
808	-0.05	-10.40	0.00
809	0.07	12.62	0.00

Term	Coefficient	t-statistics	Significance
900	-0.02	-8.46	0.00
901	0.00	0.68	0.50
902	-0.01	-3.64	0.00
903	0.02	6.77	0.00
904	-0.04	-13.44	0.00
905	0.01	5.28	0.00
906	-0.01	-4.92	0.00
907	-0.01	-3.37	0.00
908	0.01	2.77	0.01
909	-0.01	-4.32	0.00
...
990	0.00	-1.69	0.10
991	0.00	0.41	0.68
992	0.00	-1.61	0.12
993	0.00	0.32	0.75
994	0.00	-1.13	0.26
995	-0.01	-2.89	0.01
996	0.00	2.24	0.03
997	-0.01	-2.58	0.01

APPENDIX D

Matlab Codes

```
function Gaussian_Copula = Gauss(n,r)
R=ones(n,n);
i=1:n;
j=1:n;
if i~=j
R(i,j)=r;
end

h=0;

for i=0.025:.025:.975
    h=h+1;
    k=0;
    for j=0.025:.025:.975
        X=[i;j];
        k=k+1;
        U=norminv(X);
        block1=1/(det(R)^0.5);
        block2=-0.5*U'*(inv(R)-ones(n,n))*U;
        gauss_grid(h,k)=block1*exp(block2);
    end
end
surf(gauss_grid)

end
```

```
function [ PV_def, PV_premium ]=cash_flow(expiry,def_time,rec,zc_rate,...
capital,C,D)

% Computation of Default and Premium Legs of a CDO
% expiry : CDO maturity
% def_time : simulated default time
% zc_rate : constant zero coupon rate
% capital : notional amount
```

```

% C: attachment
% D: detachment

PV_def=0;
PV_premium=0;
num=size(def_time,2);
loss=zeros(num,1); % default for each credit
tot_loss=0; %cumulative portfolio loss
periodic_loss=zeros(expiry,1); % accumulated loss at each payment
out_capital=zeros(expiry,1); % outstanding tranche capital
fee=zeros(expiry,1);
total_fee=0;
indicator=0;
c=0;
% calculate total loss in k th simulation
for i=1:num
    if def_time(1,i)<expiry % there certainly is a loss when simulated
        % default time is less than CDO maturity
        loss(i)=(1-rec)*capital;
        tot_loss=tot_loss+loss(i); % sum of individual losses
    end
end

% % DEFAULT LEG SIMULATION % %

% if loss is below the attachment point C there's no default payment
if tot_loss<C
    PV_def=0;
% if loss is above C and below D there's default payment
elseif tot_loss>C & tot_loss<D
    for i=1:num
        if def_time(1,i)<expiry
            indicator=indicator+loss(i); %cumulative loss is memorized
            if indicator>C % tranche begins to absorb losses above C%
                if c==0
                    disc_fact_def=0;
                    r=zc_rate;
                    disc_fact_def=(1+r)^(-def_time(1,i)); % discount factor
                    % at default
                    PV_def=PV_def+(indicator-C)*disc_fact_def;
                    % only the loss exceeding C is absorbed
                    c=1;
                else
                    disc_fact_def=0;
                    r=zc_rate;
                    disc_fact_def=(1+r)^(-def_time(1,i));
                    PV_def=PV_def+loss(i)*disc_fact_def;
                end
            end
        end
    end
end
end

```

```

end
% if portfolio loss is above D, tranche absorb losses only up to D
elseif tot_loss>D
for i=1:num
    if def_time(1,i)<expiry
        indicator=indicator+loss(i);
        if indicator>C & indicator<D % if the losses are in C-D range
            if c==0
                disc_fact_def=0;
                r=zc_rate;
                disc_fact_def=(1+r)^(-def_time(1,i));
                PV_def=PV_def+(indicator-C)*disc_fact_def;
                c=1;
            else
                disc_fact_def=0;
                r=zc_rate;
                disc_fact_def=(1+r)^(-def_time(1,i));
                PV_def=PV_def+loss(i)*disc_fact_def;
            end
        elseif indicator>D
            if c==1
                disc_fact_def=0;
                r=zc_rate;
                disc_fact_def=(1+r)^(-def_time(1,i));
                absorbed_loss=D-(indicator-loss(i));
                PV_def=PV_def+(absorbed_loss*disc_fact_def);
                c=2;
            end
        end
    end
end
end
end
end

% % PREMIUM LEG SIMULATION % %
for i=1:expiry
    periodic_loss(i)=0;
    for j=1:num
        if def_time(i,j)<i
            % calculated the accumulated portfolio losses
            periodic_loss(i)=periodic_loss(i)+(1-rec)*capital;
        end
    end
    out_capital(i)=min(max(D-periodic_loss(i),0),D-C);
    % outstanding capital at each payment
    fee(i)=((1+zc_rate)^(-i))*out_capital(i);
    PV_premium=PV_premium+fee(i);
end
end

```

```

_____ CDO Tranches _____
function [ price_eq,price_mezz,price_sen ] = CDO_tranche(ref_ent,T,k )
% fair price of tranches of a synthetic CDO
% ref_ent : # of reference entities
% T : maturity
% k : # of simulations

tic
% initialize a vector of zeros for each tranche
CDO_P1=zeros(5,5);
CDO_P2=zeros(5,5);
CDO_P3=zeros(5,5);
% initialize a vector of zeros for different recoveries
Recovery =zeros(ref_ent,5);
hazard=zeros(5,1);
% obligors spread is set to 150 bps
spread=150/10000;

% hazard rate=spread/(1-recovery)
% calculate the corresponding hazard rate

for rec_cycle=1:5
    Recovery(:,rec_cycle)=(.2*rec_cycle)-.2;
% Recovery(:,rec_cycle)=0.4;
% recovery is from 0% to 80%
    hazard(rec_cycle)=spread/(1-Recovery(1,rec_cycle));
end

ZC=0.05; % constant interest rate

Amount=zeros(ref_ent,1); % vector of notional amount for each credit
Amount(:)=100; % each credit has 100 units of notional amount
C=zeros(3,1);
D=zeros(3,1);
% fix three attachment (0%,3%,14%) and detachment (3%,14%,100%) points
C(1)=(0/100)*sum(Amount);
D(1)=(3/100)*sum(Amount);
C(2)=(3/100)*sum(Amount);
D(2)=(14/100)*sum(Amount);
C(3)=(14/100)*sum(Amount);
D(3)=(100/100)*sum(Amount);

time=zeros(ref_ent,1);
index=zeros(ref_ent,1);

R=[0:.2:0.8]; % constant pairwise correlation from 0% to 80%
% correlation loop
for R_cycle=1:5
    for xx=1:ref_ent
        for yy=1:xx
            if xx==yy
                corr(xx,yy)=1;
            else
                corr(xx,yy)=R(R_cycle); % populate correlation matrix
            end
        end
    end
end

```

```

        corr(yy,xx)=R(R_cycle);
    end
end
end

def_t=gaussian_time(corr,k,ref_ent);
% generate default times with gaussian copula and constant hazard rate

S_fees=zeros(5,3);
% dummy variable for memorizing the simulated payment leg
S_default=zeros(5,3);
% dummy variable for memorizing the simulated default leg
M_fees=zeros(5,3);
M_default=zeros(5,3);
% variable memorizing the payment leg for each loop of recovery&corr

for n=1:k % simulation loop begins
    for rec_cycle=1:5 % recovery loop begins
        [time,index]=sort(def_t(n,:));
        % sort vector of default times
        tau=[time./hazard(rec_cycle);index];
%generate vector of default times by dividing the corr. hazard rate
        for u=1:3 % start the loop for each tranche
            recovery=0;
            fees=0;
            % calculate the default and premium legs
            [default,fees]=
cash_flow(T,tau,Recovery(1,rec_cycle),ZC,Amount(1),C(u),D(u));
            S_fees(rec_cycle,u)=S_fees(rec_cycle,u)+fees;
            S_default(rec_cycle,u)=S_default(rec_cycle,u)+default;
        end
    end
end
for u=1:3
    for rec_cycle=1:5
        M_fees(rec_cycle,u)=S_fees(rec_cycle,u)/k;
        M_default(rec_cycle,u)=S_default(rec_cycle,u)/k
        % average default leg
    end
end

for rec_cycle=1:5
    CDO_P1(R_cycle,rec_cycle)=(M_default(rec_cycle,1)/
(M_fees(rec_cycle,1)))*10000; % spread for 0%-3% tranche
    CDO_P2(R_cycle,rec_cycle)=(M_default(rec_cycle,2)/
(M_fees(rec_cycle,2)))*10000; % spread for 3%-14% tranche
    CDO_P3(R_cycle,rec_cycle)=(M_default(rec_cycle,3)/
(M_fees(rec_cycle,3)))*10000; % spread for 14%-100% tranche
end
end

price_eq=CDO_P1;
price_mezz=CDO_P2;

```

```
price_sen=CDO_P3;

figure(1)
surf((0:.2:.8),R,price_eq);
title('Equity Tranche (0%-3%)')
xlabel('Recovery');
ylabel('Correlation');
zlabel('Tranche spread (bps per annum)');

figure(2)
surf((0:.2:.8),R,price_mezz);
title('Mezzaine Tranche (3%-14%)')
xlabel('Recovery');
ylabel('Correlation');
zlabel('Tranche spread (bps per annum)');

figure(3)
surf((0:.2:.8),R,price_sen);
title('Senior Tranche (14%-100%)')
xlabel('Recovery');
ylabel('Correlation');
zlabel('Tranche spread (bps per annum)');

toc
```

