PROPERTIES OF CHARMONIUMLIKE STATES

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

HIKMET ÖZŞAHIN

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF MASTER OF SCIENCE
IN
PHYSICS

SEPTEMBER 2015

Approval of the thesis:

**PROPERTIES OF CHARMONIUMLIKE STATES**

submitted by **HIKMET ÖZŞAHIN** in partial fulfillment of the requirements for the degree of **Master of Science in Physics Department, Middle East Technical University** by,

Prof. Dr. Gülbin Dural Ünver
Dean, Graduate School of **Natural and Applied Sciences** ________________

Prof. Dr. Mehmet T. Zeyrek
Head of Department, **Physics** ________________

Prof. Dr. Tahmasib Aliyev
Supervisor, **Physics Department, METU** ________________

**Examining Committee Members:**

Prof. Dr. Ali Ulvi Yılmazer
Physics Engineering Department, Ankara University ________________

Prof. Dr. Tahmasib Aliyev
Physics Department, METU ________________

Prof. Dr. Altuğ Özpineci
Physics Department, METU ________________

Prof. Dr. Osman Yılmaz
Physics Department, METU ________________

Assoc. Prof. Dr. İsmail Turan
Physics Department, METU ________________

**Date:** ________________

**I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.**

Name, Last Name: HIKMET ÖZŞAHIN

Signature :

# ABSTRACT


PROPERTIES OF CHARMONIUMLIKE STATES



Özşahin, Hikmet
M.S., Department of Physics
Supervisor : Prof. Dr. Tahmasib Aliyev



September 2015, 42 pages



In this work, hadronic properties of exotic meson candidate $Y(3940)$ and its orthogonal state are studied. These meson states are considered as a mixture of charmonium ($\chi_{c_0}$) and meson molecule. The interpolating quark currents for $\chi_{c_0}$ and $D^*D^*$ are used to construct the correlation function. QCD sum rules approach for two-point function is then used to derive OPE and phenomenological representation of the correlation function of these mesons. Next, these two representations are connected by the analytical continuity in the complex plane and the sum rules for the mass and the mixing angle are derived. Using these sum rules, properties of charmonium states are studied.



Keywords: Charmonium, Exotic Mesons, QCD Sum Rules, Mixing Angle

# ÖZ

CHARMONİUM BENZERİ PARÇACIKLARININ ÖZELLİKLERİ

Özşahin, Hikmet
Yüksek Lisans, Fizik Bölümü
Tez Yöneticisi : Prof. Dr. Tahmasib Aliyev

Eylül 2015 , 42 sayfa

Bu çalışmada $Y(3940)$ ve $Y^{'}$ mezonlarının çeşitli hadronik özellikleri incelendi. Bu parçacıklar, çarmonium ve mezon molekülünün karşımı olarak değerlendirilmiştir. $\chi_{c_0}$ ve $D^*D^*$ parçacıklarının kuark akıları kullanılarak korelasyon fonksiyonu oluşturulmuştur. İki-nokta korelasyon fonksiyonu için QCD toplam kuralları kullanılarak, bu parçacıkların korelasyon fonksiyonlarının OPE ve fenomenolojik temsilleri çıkarıldı. Daha sonra bu iki temsil, kompleks düzlemde analitik süreklilik ile birleştirilerek, bu parçacıkların kütle ve karışım açıları için toplam kuralları çıkarıldı.

Anahtar Kelimeler: Charmonium, Egzotik parçacıklar,QCD toplam kuralları, Karışım açısı

To my family

## ACKNOWLEDGMENTS

First of all, I would like to express my gratitude to my supervisor, Prof. Dr. Tahmasib Aliyev. His support and patience is one of the things that made this work possible.

I would also like thanks my family for their continuous support and encouragement throughout my education.

# TABLE OF CONTENTS

# LIST OF TABLES

TABLES

# LIST OF FIGURES

FIGURES

# CHAPTER 1

# INTRODUCTION

Quantum electrodynamics proved to be a viable theory to explain quantum phenomena of the electromagnetic interaction with exchanging of photons. This theory is based on perturbation method, since the fine structure constant is a small quantity, allowing a perturbative expansion when calculating physical observables. The same method can be applied to explain phenomena with the participation of other forces as well. In 1934, Hideki Yukawa proposed that the nuclear force was carried out by a massive particle, which he named as meson [3]. From the range of the force, he reasoned that the mass of this particle should be nearly 200 times the mass of the electron. In 1937, separate experiments by Anderson & Neddermeyer [4] and Street & Stevenson [5] discovered a particle that is close to the mass of Yukawa's prediction. However, it turned out that this particle, now called muon, doesn't interact via the nuclear force. In 1947, another particle, so called pion, predicted by Yukawa was observed by Powell and his colleagues [6, 7].

Today it's well known that hadrons are not fundamental particles, but composed of quarks and antiquarks. In the 1950s, the first models of elementary particle physics were proposed. Several early attempt to explain structure of the hadrons, such as the ones by Enrico Fermi and Chen-Ning Yang [8] and Shoichi Sakata [9] ended up successfully explaining mesons, but failed with baryons. In 1961, Gell-Mann and Ne'eman independently introduced a classification scheme for baryons and mesons, which Gell-Mann called the Eightfold Way [10, 11]. This model was grouping mesons and baryons into octets. Later on, in 1964 he

Table1.1: Isospin and flavor quantum numbers of the quarks [1]

| | $d$ | $u$ | $s$ | $c$ | $b$ | $t$ |
|---|---|---|---|---|---|---|
| $Q$ - Electric charge | $-\frac{1}{3}$ | $+\frac{2}{3}$ | $-\frac{1}{3}$ | $+\frac{2}{3}$ | $-\frac{1}{3}$ | $+\frac{2}{3}$ |
| $I$ - Isospin | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 0 | 0 | 0 |
| $I_z$ - Isospin $z$-component | $-\frac{1}{2}$ | $+\frac{1}{2}$ | 0 | 0 | 0 | 0 |
| $S$ - Strangeness | 0 | 0 | $-1$ | 0 | 0 | 0 |
| $C$ - Charm | 0 | 0 | 0 | $+1$ | 0 | 0 |
| $B$ - Bottomness | 0 | 0 | 0 | 0 | $-1$ | 0 |
| $T$ - Topness | 0 | 0 | 0 | 0 | 0 | $+1$ |

and Zweig independently proposed that protons, neutrons and all other baryons and mesons are made of electrically charged constituent particles called quarks [12, 10]. In this early model, quarks have three different flavours called up,down and strange. Based on this model, Gell-Mann and Harald Fritzsch introduced the color charge quantum number [13]. According to this, the theory explaining quarks and the strong interaction is a non-Abelian gauge theory with SU(3) color gauge and three different flavours of quarks, which also constituents an approximate SU(3) flavour symmetry. Quarks are strongly interacting fermions with spin quantum number $\frac{1}{2}$, and positive parity. Each quark has an associated flavor quantum number and antiquarks have the opposite flavor signs.

The theory of strong interaction, known as quantum chromodynamics (QCD) assumes that dynamics of the strong interaction is encoded in the following Lagrangian:

$$\mathscr{L} = \Sigma_q \bar{\psi}_{q,a}(i\gamma^\mu \delta_{ab}\partial_\mu - g_s\gamma^\mu t^C_{ab}A^C_\mu - m_q\delta_{ab})\psi_{q,b} - \frac{1}{4}F^A_{\mu\nu}F^{A\mu\nu} \tag{1.1}$$

where the summation over repeated indices is understood. The $\gamma^\mu$ matrices are the Dirac $\gamma$-matrices and $\psi_{q,a}$ are quark-field spinors with mass $m_q$ and flavor $q$. The color index $a$ runs from $a = 1$ to $a = 3$, *i.e.* quarks come in three colors, symbolically denoted as red, blue and green.
The $A^C_\mu$ indicates the gluon fields, with the index $C$ running from 1 to 8, *i.e.* there are eight different kinds of gluon. The $T^C_{ab}$ correspond to eight $3 \times 3$ matrices that are generators of the $SU(3)$ color group. They encode the fact that

a gluon's interaction with the quark changes its color charge.

The field strenth tensor $F^A_{\mu\nu}$ is given by

$$F^A_{\mu\nu} = \partial_\mu A^A_\nu - \partial_\nu A^A_\mu - g_s f_{ABC} A^B_\mu A^C_\nu \tag{1.2}$$

where $f_{ABC}$ is the structure constant of the $SU(3)_c$ group. The last term in Eq. (1.2) is so-called self interaction term. Similar term is absent in quantum electrodynamics.

In QCD, predictions for observables are expressed in terms of the scale that depends on the strong coupling constant $\alpha_s(\mu_R^2)$. Here the $\mu_R^2$ denotes the renormalization scale. When $\mu_R^2$ is taken close to the momentum transfer $Q^2$ in a given proccess, then the coupling constant indicates the effective strength of the interaction involved.
The coupling satisfies the following renormalization group equation:

$$\frac{d\alpha_s}{d\mu_R^2} = -\left\{ b_0\alpha_s^2 + b_1\alpha_s^3 + b_2\alpha_s^4 + \cdots \right\} \tag{1.3}$$

The minus sign in front of the parenthesis on the right-hand side of this equation is the origin of Asymptotic Freedom [14, 15]. The coupling constant is small at high energies (small distance) and large at low energies, *i.e.*, interaction strength which is determined by the coupling becomes weak at high energies and strong at low energies. According to this, as quarks get closer, interaction strength weakens and quarks become nearly free particles.

Since $\alpha_s$ is small at high energies, calculations for physical observables can be done perturbatively in this energy region. In the series expansion in $\alpha_s$, higher order contributions can be neglected since as the power of $\alpha_s$ increases, contribution of corresponding Feynman graph decreases. This way, one only needs to calculate finite amount of terms to obtain observables.

However, high energy behaviours corresponding to short distance interaction are different than low energy interactions which are determined by the color confinement. At long distances, perturbation theory does not work since the coupling $\alpha_s$ approaches to unity. This is the non-perturbative region of QCD. In this region, interactions are completely governed by non-perturbative QCD effects and situation is complicated since there is a lack of any reliable theory to deal with the non-perturbative region of QCD.

There are several models and methods to treat the non-perturbative QCD problem, such as the chiral perturbation theory, Lattice theory, QCD Sum rules etc. One of the most important and successful method for non-perturbative QCD is QCD Sum Rules which is first introduced by Shiftman, Vainshtein and Zakharov [16, 17]. Due to its applicability to variety of problems, QCD Sum Rules became a widely used tool in hadron physics.

In QCD Sum Rule method, a correlation function of interpolating quark currents representing the corresponding hadron is calculated in two ways: First, the correlation function is calculated in terms of quark and gluon degrees of freedom. The long distance contributions are taken into account in terms of so-called condensates, where in "normal" perturbation theory these terms are equal to zero. The operator product expansion is used to separate the short-distance and long-distance contributions. The short distance contribution is written in terms of Wilson's coefficients which encode short-distance behaviour of the theory and vacuum expectation value of gauge invariant operators [18]. The vacuum expectation values of operators like $\langle q\bar{q}\rangle$ are, by definition, zero in traditional perturbation theory. In QCD Sum Rules method, however, they encode the long-distance effects which is governed by confinement and they have to be included in the method phenomenologically whereas Wilson's coefficients can be calculated theoretically.

The other way of treating the correlation function is to represent it in terms of hadronic parameters. In order to do this, a complete set of intermediate state is inserted to the correlating function and then the correlation function is written in terms of hadronic parameters such as hadron mass, decay constant and *etc.* Then, these two different representations of the correlation function is matched via a dispersion relation and a sum rules for corresponding hadronic parameter

Table1.2: Meson classification according to their quantum numbers.

| Meson family | $S$ | $L$ | $J$ | $P$ | $J^P$ | $n^{2S+1}L_J$ |
|---|---|---|---|---|---|---|
| Pseudoscalar | 0 | 0 | 0 | $-$ | $0^-$ | $1^1S_0$ |
| Pseudovector | 1 | 1 | 1 | $+$ | $1^+$ | $1^1P_1$ |
| Vector | 1 | 0 | 1 | $-$ | $1^-$ | $1^3S_1$ |
| Scalar | 1 | 1 | 0 | $+$ | $0^+$ | $1^3P_0$ |
| Axialvector | 1 | 1 | 1 | $+$ | $1^+$ | $1^3P_1$ |

is obtained.

Another aspect of QCD is the so called confinement mechanism. It is well known that hadrons are strongly interaction particles and they are bound states of quarks and antiquarks. Confinement dictates that all observable hadrons to be colorless, *i.e.* they must be color singlet of the symmetry group $SU(3)_c$. Neither quarks nor gluons can be observed as free particles and the number of possible bound states are restricted by this confinement mechanism. Bound states of quarks and antiquarks can be grouped into two categories: Baryons which have $qqq$ quark structure and mesons with $q\bar{q}$ quark structure. Baryons have baryon quantum number $B = 1$ while mesons have baryon number $B = 0$. Quark antiquark pairs in mesons don't need to be the same flavor and a meson composed of quark-antiquark pair with the same color charge is called quarkonium.

Since gluons don't carry intrinsic quantum number other than $C$ parity and color charge and since color is confined, most of the quantum number of a strongly interacting particle have comes from the quantum numbers of their constituent quarks and antiquarks. Mesons, in accordance with the standard constituent quark model, are classified according to their isospin $I$, total angular momentum $J$, parity $P$ and quark content. Total angular momentum of a meson can be expressed as $J = L \oplus S$. Since quarks are fermions with spin $\frac{1}{2}$ a meson can only have a spin quantum number $S = 0, 1$. The parity for a meson is given as $P = (-1)^{1+L}$. Mesons can also be indicated according to spectroscopic notation, with their radial excitation quantum number $n$, spin multiplicity $2S + 1$, orbital angular momentum $L$ and total angular momentum $J$. Table 1.2 shows different

Table1.3: Symbols for flavorless mesons [1]

| | $J^{PC}$ | $0^{-+}, \cdots$ | $1^{+-}, \cdots$ | $1^{--}, \cdots$ | $0^{++}, \cdots$ |
|---|---|---|---|---|---|
| quark content | $^{2S+1}L_J$ | $^1(Leven)_J$ | $^1(Lodd)_J$ | $^3(Leven)_J$ | $^3(Lodd)_J$ |
| $u\bar{d}, u\bar{u} - d\bar{d}, d\bar{u}$ | $(I = 1)$ | $\pi$ | $b$ | $\rho$ | $a$ |
| $d\bar{d} + u\bar{u}$ and/or $s\bar{s}$ | $(I = 0)$ | $\eta, \eta'$ | $h, h'$ | $\omega, \phi$ | $f, f'$ |
| $c\bar{c}$ | $(I = 0)$ | $\eta_c$ | $h_c$ | $\psi^\dagger$ | $\chi_c$ |
| $b\bar{b}$ | $(I = 0)$ | $\eta_b$ | $h_b$ | $\upsilon$ | $\chi_b$ |
| $t\bar{t}$ | $(I = 0)$ | $\eta_t$ | $h_t$ | $\theta$ | $\chi_t$ |

Table1.4: Symbols for flavored mesons [1]

| antiquark $\rightarrow$<br>quark $\downarrow$ | $\bar{u}$ | $\bar{d}$ | $\bar{c}$ | $\bar{s}$ | $\bar{t}$ | $\bar{b}$ |
|---|---|---|---|---|---|---|
| $u$ | – | – | $\overline{D}^0$ | $K^+$ | $T^-$ | $B^+$ |
| $d$ | – | – | $D^-$ | $K^0$ | $T^-$ | $B^0$ |
| $c$ | $D^0$ | $D^+$ | – | $D_s^+$ | $\overline{T}_c^0$ | $B_c^+$ |
| $s$ | $K^-$ | $\overline{K}^0$ | $D_s^-$ | – | $T_s^-$ | $B_s^0$ |
| $t$ | $T^0$ | $T^+$ | $T_c^0$ | $T_s^+$ | – | $T_b^+$ |
| $b$ | $B^-$ | $\overline{B}^0$ | $B_c^-$ | $\overline{B}_s^0$ | $T_b^-$ | – |

class of mesons with their quantum numbers.

Another way of classifying mesons is to group them according to their flavor content. Flavorless mesons (quarkonia) are composed of a quark-antiquark pair of the same flavor and all their flavor quantum numbers but the isospin number are zero: $S = 0, C = 0, B = 0, T = 0$. Symbols for flavorless mesons are given in Table 1.3. Strongly decaying mesons are indicated with the mass in parenthesis.

For flavored mesons, one of the constituent quark is heavier than the other. In naming these mesons, an upper-case letter indicating the heavier quark is used. The convention that the flavor and the charge of a quark have the same sign is employed, and also $I_3$ isospin quantum number of the $u$ quark is taken as positive while for the $d$ quark it's taken as negative. With this convention, mesons have the same charge sign as its flavor. Table 1.4 shows symbols for flavored mesons.

The study of spectroscopy and decay properties of flavored mesons provides useful information about the dynamics of QCD at the low energy regions. On the experimental side, remarkable progress has been made over the last two decades. Main body of experimental data about the heavy flavored mesons came from the Beauty experiments. The B-factories like PEPII at SLAC in the USA and KEKB at KEK in Japan were designed to test the CP violation mechanism. Through these experiments, came contributions to the field of hadron spectroscopy, and especially to the area of charmonium spectroscopy [19]. Starting from the discovery of the $X(3872)$ state by Belle Collaboration [20] and others [21] , more than twenty new charmonium states have been discovered [19].

These newly discovered states might have more complex structure than those predicted by the conventional quark model pictures. Many of these states, like $Y(3930)$, $Z(3930)$, $X(3940)$, $Y(4008)$, $Z^+(4050)$, $Y(4140)$, $X(4160)$, $Z_2^+(4250)$, $Y(4360)$, $Z^+(4430)$ and $Y(4460)$ states are treated by considering them as a mixture of a charmonium state with either a tetra-quark or a meson molecule. Their masses and decay widths do not lie within the theoretical predictions from the potential models [22]. For that reason, they are considered to be a natural candidate for exotic states. Hadronic parameters of some of these $XYZ$ states are widely investigated in the framework of QCD Sum Rules method.

In this thesis, the properties of $Y(3940)$ state is investigated by considering it as a mixture of a chamonium and a meson molecule states. The state $Y(3940)$ has been observed by Belle Collaboration in the decay $B \longrightarrow (J/\psi\omega)K$, with a mass $m = 3943 \pm 11 \pm 13$ MeV and a decay width $\Gamma = 87 \pm 22 \pm 26$ MeV [23]. The discovery of $Y(3940)$ has also been confirmed by BaBar Collaboration [24] with a smaller mass of $m = 3919.4 \pm 2.2 \pm 1.6$ MeV and a width $\Gamma = 13 \pm 6 \pm 3$ MeV [25].An analysis for the states $Y(3940)$ and its orthogonal $Y'$ within the QCD sum rules method is presented in Chapter 3.

# CHAPTER 2

# QCD SUM RULES

QCD Sum Rule method has been a widely used tool for predicting observable properties of hadronic ground states. The advantage of this method is that hadrons are treated with a model independent approach in which each ground state hadron is represented in terms of their interpolating quark current, taken at large virtualities. One would start from the asymptotic side of QCD, where quark-gloun interaction can be treated perturbatively and move to larger distances step by step where hadronic states are formed.

To begin in asymptoticaly free region of QCD, quarks and gluons in the corresponding hadronic process have to be highly virtual. With this condition, the smallness of the strong coupling $\alpha_s(Q^2)$ is guaranteed and the perturbation theory can be used. Typically, highly virtual quarks and gluons in an interaction are obtained when momentum transfer $Q^2$ is at large. However, even these specially configured types of interaction are not enough to make sure that the perturbation theory alone would be sufficient. Quarks and gluons participating the interaction is confined inside the hadrons and thus non-perturbative effects still play a considerable role even though quarks and gluons are highly virtual. Hence, in studying these interaction, one needs to know dynamics of the theory at hadronic scale.

To overcome this, one can also consider processes where there is no initial or final hadronic state. Such an interactions are realized when a quark-antiquark pair is produced and annihilated in an electron-positron elastic scattering. In this scattering process , as shown in Fig. 2.1, quark-antiquark pair is produced and

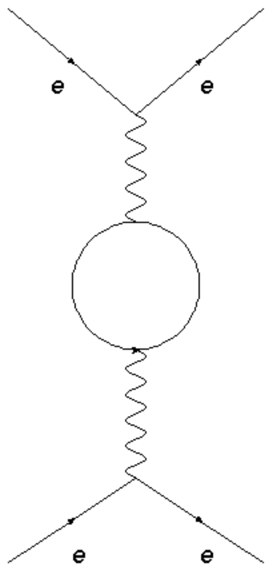


Figure 2.1: Quark-antiquark creation and annihilation by virtual photon [2]

annihilated by the virtual photon. The quark-antiquark pair in this interaction travel at short distances without non-pertubative effects. Although this process contributes a very little radiative correction to cross section for electron-electron scattering, the amplitude of quark-antiquark creation is an essential object for QCD sum rules. Using the gauge invariance, the form of the correlation function can be written as follows:

$$\Pi_{\mu\nu}(q) = i \int d^4x e^{iqx} \langle 0| T\{j_\mu(x) j_\nu(0)\} |0\rangle = (q_\mu q_\nu - g_{\mu\nu}q^2)\Pi(q^2) \tag{2.1}$$

where $j_\mu(x) = \bar{\psi}\gamma_\mu\psi$ is an interpolating quark current with the quantum numbers of the corresponding hadron and $\psi$ is fermionic field operator $\psi = u, d, s, \ldots$, $q$ is the four momentum of the virtual photon. Since there is no initial or final hadronic state in this interaction, these quark currents are associated with the QCD vacuum, $|0\rangle$.

The correlation function is an object of dual nature. At large momentum transfers, $Q^2 \equiv -q^2 >> \Lambda^2_{QCD}$, this correlation function would be solely dominated by the short-distance effects. At small momentum transfers, however, long-distance effects should also be accounted for to explain the behaviour of this function.

This dual nature of the correlation function can be used to relate hadronic observable parameters to the short-distance dynamics of the quark-gluon interaction. At one hand, the correlation function can be a short-distance object encoding the quark-gluon interaction. This representation of the correlation function is called theoretical side, or OPE side. At the other hand, the correlation function can also be related to the hadronic observables. This representation is called phenomenological side. Depending on the virtuality of quarks and gluons participating the interaction, the correlation function can be treated in either way. But the main point of QCD Sum Rules is that there is a kinematic region where both representation of the correlation function is applicable. Matching these two different representation of the correlation function, sum rules for various hadronic parameters can be obtained.

QCD sum rules had been a reliable tool for predicting various parameter in hadron spectroscopy. There are several reviews [26, 27, 28, 29, 2] which explains various aspects and application of this method. In the following sections, we briefly review the details of the traditional QCD sum rules method.

## 2.1 Theoretical Side of the Correlation Function

We begin our treatment of the correlation function by considering the theoretical representation. The quark interpolating currents inside the correlator are determined according to their quantum numbers and flavor content of the corresponding hadron. For various familiy of mesons, interpolating currents can be listed as;

$$
\begin{aligned}
j^s &= \bar{q}q(J^P = 0^+) \\
j^p &= \bar{q}\gamma^5 q(J^P = 0^-) \\
j^V_\mu &= \bar{q}\gamma_\mu q(J^P = 1^-) \\
J^A &= \bar{q}\gamma_\mu\gamma^5 q(J^P = 1^+)
\end{aligned}
$$

As mentioned before, at high virtualities the correlation function can be treated in the framework of perturbation theory and approximated by the free-quark

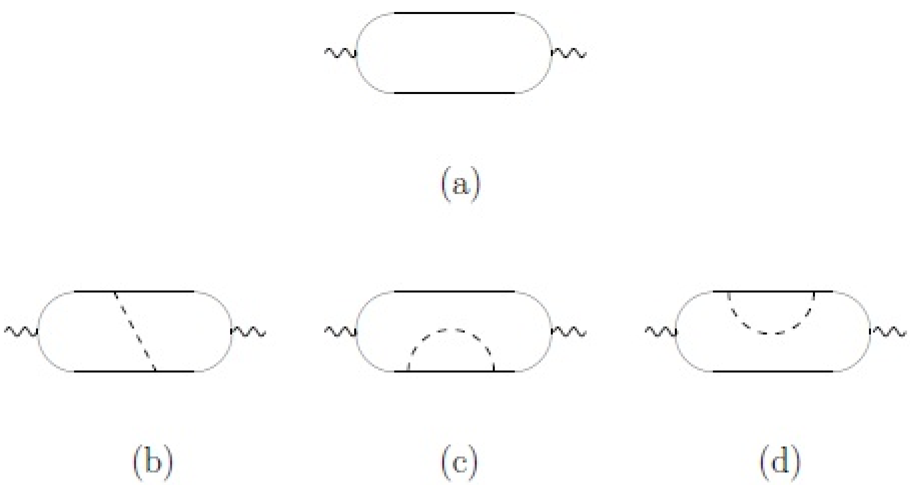


Figure 2.2: Diagrams contributing to the perturbative part of the correlation function: free-quark loop (a), radiative corrections to loop diagrams (b,c,d). [2]

diagrams, as shown in the Fig. 2.2(a). One then can proceed with the standart procedure to evaluate the correlator by considering the free-quark propagators. This perturbative contribution can be further improved by considering higher order loop diagrams, as shown in Fig. 2.2(b,c,d). Hovewer, this perturbative part of the correlator is not the only contribution. One also has to include non-perturbative effects due to soft quarks and gluons in QCD vacuum. Because of the nonlineer nature of the QCD Lagrangian, the vacuum fluctuates with these soft quarks and gluons. Since there is no consistent theory for the QCD vacuum, these effects must be included phenomelogically.

A method to account these considerations mentioned above was developed by Wilson [18]. To apply it, one has to expand the product of interpolating quark currents in a series of local operators;

$$i\int d^4x e^{iqx} \langle 0| T\{j_\mu(x) j_\nu(0)\} |0\rangle = (q_\mu q_\nu - g_{\mu\nu} q^2) \sum_d C_d(q^2) \langle O_d\rangle \tag{2.2}$$

where $\langle O_d\rangle \equiv \langle 0| O_d |0\rangle$. So we have

$$\Pi(q^2) = \sum_d C_d(q^2) \langle O_d \rangle \tag{2.3}$$

The validity of this expansion can be understood as one considers the short-distance limit $x \to 0$. The product of two closely seperated interpolating currents creates a disturbance in the field near the point 0. However, this disturbance can also be described with a local operator placed at the same point. Hence, we can obtain the form of OPE by writing this local operator in terms of basis operators [30]. Here the vacuum expectation values of the operators $\langle O_d \rangle$, also called vacuum condensates, constitude the standart basis at which we can expand the correlation function. Coefficients $C_d(q^2)$ are $c$-numbers called Wilson coefficients. In this expansion, the operators are ordered according to their canonical dimensions $d$, with the first term operator corresponds to the unit operator $O_0 = 1$. Operators in higher order terms with $d \neq 0$ correspond to diagrams with insertions of soft quarks and gluons. The long-distance and short-distance effects in the correator are thus seperated in this expansion, with the former are represented by the universal vacuum condensates while the latter are encoded by the Wilson coefficients. Note that these vacuum condensates don't depend on neither the quantum number nor the flavor content of interpolating currents. Vacuum condensate of QCD vacuum are phenomenological objects independent of the method used. The quantum number and flavor content of the correlator are rather absorbed into the Wilson coefficients.

Vacuum condensates introduced in this way are non-perturbative objects and their numerical values can be calculated by using different methods like Lattice QCD or instanton model . Since these vacuum condensates are independent of the model used, they can be extracted from one sum rule and used in others.

Some examples of these vacuum condensates are;

$$
\begin{aligned}
\langle O_3 \rangle &= \langle \bar{q} q \rangle \\
\langle O_4 \rangle &= \left\langle G^a_{\mu\nu} G^{a\mu\nu} \right\rangle \\
\langle O_5 \rangle &= \left\langle \bar{q} \sigma_{\mu\nu} \frac{\lambda^a}{2} G^{a\mu\nu} q \right\rangle \\
\left\langle O_6^{\psi} \right\rangle &= \left\langle (\bar{\psi} \Gamma_r \psi)(\bar{\psi} \Gamma_s \psi) \right\rangle \\
\left\langle O_6^{G} \right\rangle &= f_{abc} \left\langle G^a_{\mu\nu} G^{b\nu}_{\sigma} G^{c\sigma\mu} \right\rangle
\end{aligned}
\tag{2.4}
$$

Where $\Gamma_{r,s}$ are Dirac bilinears.

Note that in QCD there in no colorless, gauge and Lorentz invariant operators with dimensions $d = 1, 2$. So operator product expansion of quark currents continues with an operator of dimension $d = 3$ after the unit operator. Also vacuum condensates of operators with $d > 6$ usually add small contribution to the expansion and neglected in most of QCD sum rules applications.

Historically, operator product expansion of interpolating currents was developed to evaluate Feynman diagrams and its use in QCD sum rules is not trivial. Apart from the non-vanishing expectation values of local operators, introduction of non-perturbative terms to OPE has another effect: That these non-perturbative terms break down the expansion itself, starting at some dimension. The critical dimension $d_{crit}$ at which the expansion is badly broken depends on the interpolating currents used in the method. For vector currents these effects appear at $d > 10$ while for scalar and pseudoscalar currents the effect emerge at intermedia levels. The computational recipe for the terms in OPE is as follows: Feynman diagrams for the product of interpolating currents are cut through quark and gluon line in all possible way. Then these cut lines are annihilated into the QCD vacuum. There are also soft gluon and quark line attached to the diagrams which are also produced and annihilated by the QCD vacuum. These diagrams, which are shown in Fig. 2.3, determine the coefficients in front of the vacuum condensates.

To match the correlation function calculated from the OPE side with the phe-

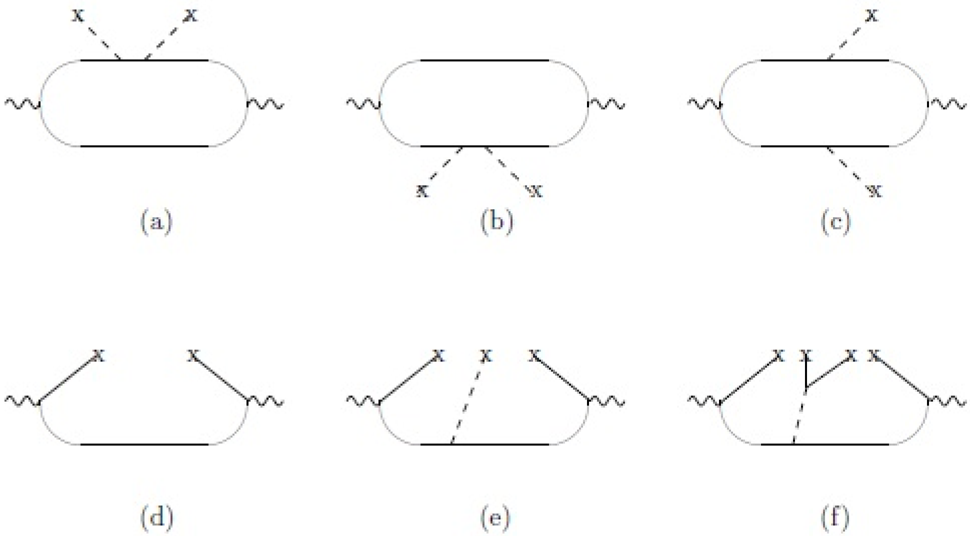

Figure 2.3: Diagrams corresponding to the vacuum condensates. [2]

nomenological representation a more convenient form is used;

$$\Pi^{OPE} = \int_0^{\infty} \frac{\rho^{OPE}(s)}{s - q^2} ds \tag{2.5}$$

Where the spectral density $\rho^{OPE}(s) = \frac{1}{\pi} Im \Pi^{OPE}(q^2)$.

## 2.2 Phenomenological Side

In this section we discuss how the correlation function can be related to physical hadronic states. Note that the correlation function is an analytical function of $q^2$. As mention before, the correlation function becomes a short-distance object at hight virtualities $q^2 << 0$. If $q^2$ is shifted from negative to positive values, the average distance between the quarks grows and the interaction gets stronger. Hence quarks start to form hadrons and long-distance effects become important.

A rigorous way to quantify the hadronic content of the correlation function $\Pi_{\mu\nu}(q^2)$ at long-distance can be obtained by using a complete set of intermediate

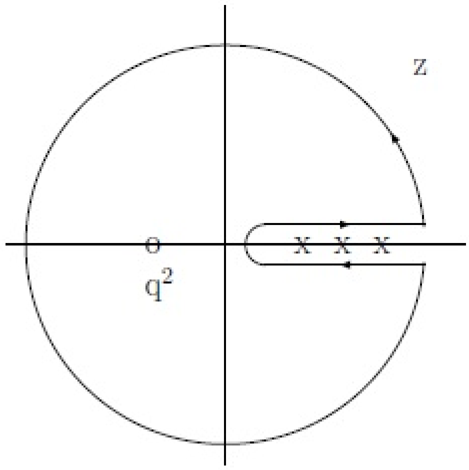


Figure 2.4: Contour in $q^2$ complex plane. Crosses indicates the hadronic threshold at $q^2 > 0$ and open point indicates the $q^2 < 0$ reference point.

hadronic states:

$$1 = \sum_n \int \frac{d^4 p_n}{(2\pi)^4} \delta(p_n^2 - m_n^2) \Theta(p_n^2) \left| n \right\rangle \left\langle n \right| \tag{2.6}$$

where summation goes over all possible hadronic states $|n >$ created by the currents. Inserting this into Eq. (2.2), we obtain a unitary relation:

$$2Im\Pi_{\mu\nu}(q) = \sum_n \langle 0| \, j \, |n\rangle \, \langle n| \, j \, |0\rangle \, d\tau_n (2\pi)^4 \delta^{(4)}(q - p_n) \tag{2.7}$$

where $d\tau_n$ denotes the phase space integration. For our purposes, we single out the ground state meson contribution at the right-hand side ;

$$\frac{1}{\pi} Im\Pi(q^2) = f_n^2 \delta(q^2 - m_n^2) + \rho^h(q^2)\Theta(q^2 - s_0^h) \tag{2.8}$$

where the decay constant $f_n$ is defined as $f_n m_n = \langle n(q)| \, |j \, |0\rangle$. $f_n$ is a parameter that is determined by the long-distance dynamics of the theory. The spectral density function $\rho^h(q^2)$ encodes the contributions from exited and continuum states. And $s_0^h$ is the threshold of the lowest-lying continuum state. Hence we obtained two different representation for the correlation function. Using the analytically of $\Pi$ throughout the both regions, we derive a dispersion relation that connects the OPE representation of the correlator to the hadronic sum we obtain in Eq. (2.7). For that purpose, we employ the Cauchy's formula, with the choosen contour shown in Fig. 2.4.

$$\begin{aligned}\Pi(q^2) &= \frac{1}{2\pi i}\oint dz \frac{\Pi(z)}{z-q^2} \\ &= \frac{1}{2\pi i}\int_{|z|=R} dz \frac{\Pi(z)}{z-q^2} + \frac{1}{2\pi i}\int_0^R dz \frac{\Pi(z+i\epsilon)-\Pi(z-i\epsilon)}{z-q^2}\end{aligned} \tag{2.9}$$

The radius $R$ of the circle can be put to infinity if the correlation function vanishes at $|z| \longrightarrow \infty$. The first term on the right-hand side of Eq. (2.9) then becomes zero. If the correlator doesn't vanish at the limit, the denominator $\frac{1}{z-q^2}$ can be expanded in power series and it's guaranteed that higher order terms $O(z^{-n})$ vanish. The remaining terms are called subtraction terms. The second integral can be replaced by an integral over the imaginary part of the correlator. Using the Schwartz reflection, $Disc\Pi(z) = \Pi(z+i\epsilon) - \Pi(z-i\epsilon) = 2iIm\Pi(z)$. After this, we obtain the dispersion relation:

$$\Pi(q^2) = \frac{1}{\pi}\int_{t_{min}}^{\infty} ds \frac{Im\Pi(s)}{s-q^2} \tag{2.10}$$

Using the hadronic sum Eq. (2.7), we obtain the phenomenological representation of the correlator $\Pi(q^2)$:

$$\Pi(q^2) = \frac{f_n^2}{m_n^2 - q^2} + \int_{s_0^h}^{\infty} ds \frac{\rho^h(s)}{s-q^2} \tag{2.11}$$

### 2.3 Borel Transformation and Quark-Hadron Duality

We have now obtained two different representation for the correlation function; Eq. (2.5) derived from the region $q^2 << 0$ whereas Eq. (2.11) obtained by considering the hadronic decomposition of the correlator. To obtain the sum rules, these two representation are matched;

$$\Pi^{(phen)}(q^2) = \Pi^{OPE}(q^2) \tag{2.12}$$

However, this form of the sum rules has little use for our purposes. Estimating the parameters of the ground state hadron is not possible since the sum rules in Eq. (2.12) are plagued by the subtraction terms and contributions of excited and continuum states. To suppress these terms in the sum rules, one considers the Borel transform defined as:

$$B_{M^2}\Pi(q^2) = lim_{-q^2,n\to\infty}\frac{(-q^2)^{(n+1)}}{n!}\left(\frac{d}{dq^2}\right)^n\Pi(q^2) \tag{2.13}$$

This transformation removes all the subtraction terms in the dispersion relation and introduce exponential suppression for excited and continuum contributions. After this transformation, the explicit form of the sum rules becomes:

$$f_n^2 e^{\frac{-m_n^2}{M^2}} + \int_{s_0^h}^{\infty}\rho^h(s)e^{\frac{-s}{M^2}}ds = \int_0^{\infty}\rho^{OPE}(s)e^{\frac{-s}{M^2}}ds \tag{2.14}$$

There is one step to finalize our derivation of QCD sum rules. One can estimate the integral over excited and continuum states at the left-hand side of Eq. (2.14) by using quark-hadron duality ansatz. This ansatz states that in the spacelike region $q^2 \to -\infty$, all excited and continuum states can safely be neglected and the limit $\Pi(q^2) \longrightarrow \Pi^{(pert)}(q^2)$ becomes valid;

$$\int_{s_0^h}^{\infty}\frac{\rho^h(s)}{s-q^2}ds = \int_{s_0}^{\infty}\frac{\rho^{OPE}}{s-q^2}ds \tag{2.15}$$

where $s_0$ is an effective threshold parameter that isn't necessarily coincide with the hadronic threshold $s_0^h$. After performing the Borel transformation, we have for the left-hand side of Eq. (2.14):

$$\int_{s_0^h}^{\infty}ds\rho^h(s)e^{\frac{-s}{M^2}} = \int_{s_0}^{\infty}ds\rho^{OPE}(s)e^{\frac{-s}{M^2}} \tag{2.16}$$

Using the quark-hadron duality Eq. (2.16), the integration over excited and continuum states at left-hand side of Eq. (2.14) can be subtracted from the right-hand side. Therefore, we obtain the final form of the sum rules:

$$f_n^2 e^{\frac{-m_n^2}{M^2}} = \int_0^{s_0}ds\rho^{OPE}e^{\frac{-s}{M^2}} \tag{2.17}$$

The Borel parameter $M^2$ introduced in the Borel transformation is an unphysical quantity and thus any prediction made by this method must be independent of the Borel parameter. Therefore one should find a region at which physical parameters don't depend on the Borel parameter $M^2$. The choice of the Borel parameter is restricted. Borel parameter $M^2$ can't be too small since the small values of $M^2$ lead to smaller suppression and excited states may become too important to be neglected. The low limit of the Borel parameter $M^2$ is decided by demanding that the highest dimension operator in OPE remains a small fraction of the sum of all terms. On the other hand, the Borel parameter can't also be too large, since this make the quark-hadron duality unapplicable. Therefore one has to choose a upper limit to the Borel parameter so that the suppressed states above $s_0$ remains small part of the dispersion integral. After determining this Borel window, one can find physical parameters by demanding independence from variance of the Borel parameter within this dictated Borel window.

# CHAPTER 3

# SUM RULES FOR CHARMONIUMLIKE STATES

In this chapter, we present the mass and the mixing angle calculation. In our treatment of $Y(3940)$ and its orthogonal state $Y^{'}$, these two states are considered as a mixture of charmonium and meson molecule with quark currents for a charmonium state:

$$j(x)_{\chi_0} = \overline{c}(x)c(x) \tag{3.1}$$

and for a $D^{*}D^{*}$ molecule;

$$j_{D^{*}D^{*}} = \{\overline{q}(x)\gamma_{\mu}c(x)\}\{\overline{c}(x)\gamma^{\mu}q(x)\} \tag{3.2}$$

Where $c(x)$ and $q(x)$ are $c$-quark and light quark ($u$,$d$,$s$) operators, respectively.

Then, the interpolating current for $Y(3940)$ state can be written as;

$$j^{(1)} = -\frac{\langle\overline{q}q\rangle}{\sqrt{2}} cos\theta\overline{c}(x)c(x) + sin\theta\{\overline{q}(x)\gamma_{\mu}c(x)\}\{\overline{c}(x)\gamma^{\mu}q(x)\} \tag{3.3}$$

whereas it orthogonal state $Y^{'}$ can be represented by;

$$j^{(2)} = \frac{\langle\overline{q}q\rangle}{\sqrt{2}} sin\theta\overline{c}(x)c(x) + cos\theta\{\overline{q}(x)\gamma_{\mu}c(x)\}\{\overline{c}(x)\gamma^{\mu}q(x)\} \tag{3.4}$$

With the factor $-\frac{\langle\overline{q}q\rangle}{\sqrt{2}}$ being the normalization constant. Using these interpolating currents, we calculate the mixing angle and mass of $Y^{'}$ state. Before continuing the calculation itself, we first derive the heavy and light quark propagators used in this calculation. The method of calculation the correlation function outlined in Chapter 2 was simple and straightforward, but inefficient. The correlation function, by its nature, is a gauge invariant quantity and any term calculated from OPE side should also be gauge invariant. However, in the traditional Feynamn diagram analysis of the Wilson's coefficients, one has to deal with additional terms that aren't gauge invariant. Although these terms cancel each other at the end, they make the calculation much more inefficient. There were several computational method proposed to overcome this [31, 32]. In this work, we use the method [33] which is based on the Fock-Schwinger gauge [34].

Since the correlation function is gauge invariant, gauge condition used in this calculation is irrelevant and any particular choice would lead the same answer for the correlation function at consideration. Hence, one can use any desired gauge condition to eliminate unwanted terms and make the calculation itself easier. In this method, the gauge condition proposed by Fock and Schwinger is used to eliminate gauge non-invariant terms;

$$(x - x_0)A^{\mu} = 0 \tag{3.5}$$

Here $A^{\mu}$ is the external field operator and $x_0$ is an arbitrary point on the Minkowski spacetime. With the choice of gauge fixing $x_0 = 0$, we have:

$$x_{\mu}A^{\mu} = 0 \tag{3.6}$$

With this gauge choice, the external gluon field operator $A^{\mu}(x)$ can be represented as [31];

$$A_{\mu} = \frac{1}{2}x^{\nu}G_{\nu\mu}(0) + \frac{1}{3.1!}x^{\nu}x^{\rho}D_{\nu}G_{\rho\mu}(0) + \cdots \tag{3.7}$$

Where $D$ denotes the covariant derivative and $G$ denotes the field-strength tensor. One can also write an analogous expansion for the fermion field as well;

$$\psi(x) = \psi(0) + x_{\mu}D^{\mu}\psi(0) + \frac{1}{2}x_{\mu}x_{\nu}D^{\mu}D^{\nu}\psi(0) + \cdots \tag{3.8}$$

We consider quarks propagating through external gluon fields and denote the propagator of the quark field in this external field as;

$$iS(x, y) = \langle 0| T\{q(x), \overline{q}(y)\} |0\rangle \tag{3.9}$$

Expanding this propagator:

$$iS(x,y) = iS^0(x-y) + g\int d^4z iS^0(x-y) i\not{A}(z) iS^0(x-y) + O(g^2) \qquad (3.10)$$

Where $S^0(x-y)$ denotes the free quark propagator. The form of this free quark propagator changes with the mass of the quark. For light quarks with $m_q << \Lambda_{QCD}$, the quark mass can be neglected or its correction can be accounted perturbatively. For heavy quarks, however, the mass cannot be taken as zero since $m_Q >> \Lambda_{QCD}$.

Calculating the quark propagators in external field, we can derive simple rules for the coefficients in OPE side of the correlation function. Using the Dirac equation,

$$(\not{D} - m)\psi = 0$$

and the identity,

$$\left\langle G^a_{\nu\mu} G^b_{\sigma\rho} \right\rangle = \frac{1}{96}\delta^{ab}(\varrho_{\mu\rho}\varrho_{\nu\sigma} - \varrho_{\mu\sigma}\varrho_{\nu\rho}\left\langle G^2 \right\rangle$$

we can write the above expressions for fermionic fields Eq.(3.8) and gluonic fields Eq.(3.7) in terms of quark and gluon condensates. This way, once the propagators are derived, correlation function can be calculated automatically with the desired sensitivity to quark and gluon condensates.

The heavy quark propagator in configuration space is obtained by performing Fourier transform of the momentum space expression [35];

$$iS^{ab}_Q(q) = \frac{i\delta^{ab}}{\not{q} - m} + \frac{i\lambda^{ab}_n}{8}\varrho_s G^n_{\mu\nu}\frac{\sigma^{\mu\nu}(\not{q}+m) + (\not{q}+m)\sigma^{\mu\nu}}{(p^2-m^2)^2} + \cdots \qquad (3.11)$$

Here is the first term is the free quark propagator. Performing the Fourier transform,

$$iS^{ab}_Q(x) = \int \frac{d^4q}{(2\pi)^4} iS^{ab}_Q(q) e^{-iq.x} \qquad (3.12)$$

one obtains:

$$iS^{ab}_Q(x) = \frac{M^2_Q\delta^{ab}}{(2\pi)^2}\left\{\frac{K_1(m_Q\sqrt{-x^2}}{\sqrt{-x^2}} - \frac{i\not{x}K_2(m_Q\sqrt{-x^2})}{x^2}\right\} + \cdots \qquad (3.13)$$

The free light quark propagator in coordinate representation can be written as:

$$S^0(x-y) = \frac{1}{2\pi^2}\frac{\not{x}-\not{y}}{(x-y)^4}$$

Putting this and Eq. (3.7) into the full propagator Eq.(3.10), we obtain [35]:

$$\begin{aligned} iS_q^{ab}(x) &= \frac{i\not{x}\delta^{ab}}{2\pi^2x^4} - \frac{\delta^{ab}}{12}\langle\overline{q}q\rangle - \frac{i\delta^{ab}\varrho_s^2x^2\not{x}}{2^5\times x^5}\langle\overline{q}q\rangle^2 + \frac{1}{32\pi^2}\varrho_s G_{\mu\nu}^{ab}\frac{\sigma^{\mu\nu}im_q\not{x}+im_q\not{x}\sigma^{\mu\nu}}{x^2} \\ &= \frac{\delta^{ab}x^2}{192}\left(1-\frac{im_q}{6}\not{x}\right)\langle\varrho_s\overline{q}\sigma Gq\rangle + \cdots \end{aligned} \tag{3.14}$$

In the next section, we use these propagators to obtain spectral densities for mass and the mixing angle.

## 3.1 Calculations

### 3.1.1 The Mixing Angle

The interpolating current for $Y(3940)$ is given as:

$$j^{(1)}(x) = cos\theta j_1(x) + sin\theta j_2(x) \tag{3.15}$$

and for its orthogonal state $Y^{'}$:

$$j^{(2)}(x) = -sin\theta j_1(x) + cos\theta j_2(x) \tag{3.16}$$

where,

$$\begin{aligned} j_1(x) &= -\frac{\langle\overline{q}q\rangle}{\sqrt{2}}\overline{c}(x)c(x) \\ j_2(x) &= \{\overline{q}(x)\gamma_\mu c(x)\}\{\overline{c}(x)\gamma^\mu q(x)\} \end{aligned}$$

and the factor $-\frac{\langle\overline{q}q\rangle}{\sqrt{2}}$ is normalization constant. The mixing angle $\theta$ can be found from the following correlation function of these currents:

$$\Pi(q) = i\int d^4xe^{iq.x}\langle 0|T\left\{j^{(1)}(x)j^{(2)}(0)\right\}|0\rangle \tag{3.17}$$

Since these two states are orthogonal to each other, we expect the correlation function to be zero. Then, the mixing angle $\theta$ can be determined by requiring that the value of $\theta$ must give $\Pi(q) = 0$. Using Eq. (3.15) and Eq. (3.16), the correlation function becomes:

$$\Pi(q) = sin\theta cos\theta(\Pi_{22} - \Pi_{11}) + cos^2\theta\Pi_{12} - sin^2\theta\Pi_{21} \tag{3.18}$$

where $\Pi_{ij} = i\int d^4xe^{iq.x}\langle 0|\, T\left\{j_i^{(1)}j_j^{(2)}\right\}|0\rangle$.

It is easy to show that the values of the mixing angle $\theta$ is given by:

$$tan2\theta = \frac{-ac \pm b\sqrt{b^2+a^2-c^2}}{-bc \mp a\sqrt{b^2+a^2-c^2}} \tag{3.19}$$

where $a = \frac{1}{2}(\Pi_{22} - \Pi_{11})$, $b = \frac{1}{2}(\Pi_{12} + \Pi_{21})$ and $c = \frac{1}{2}(\Pi_{12} - \Pi_{21})$.

This relation further simplifies if we note that $\Pi_{21} = \Pi_{12}$;

$$tan2\theta = \frac{2\Pi_{21}}{\Pi_{11} - \Pi_{22}} \tag{3.20}$$

Using the interpolating current, Eq. (3.18) becomes:

$$\begin{aligned}
\Pi(q) = isin\theta cos\theta \int d^4xe^{iq.x} \Bigg\{ \\
&\langle 0|\, T\left[\{\overline{q}(x)\gamma_\mu c(x)\}\{\overline{c}(x)\gamma^\mu q(x)\}\{\overline{q}(0)\gamma_\mu c(0)\}\{\overline{c}(0)\gamma^\mu q(0)\}\right]|0\rangle \\
&-\frac{\langle\overline{q}q\rangle^2}{2}\langle 0|\, T\left[\overline{c}(x)c(x)\overline{c}(0)c(0)\right]|0\rangle \Bigg\} \\
&- icos^2\theta\frac{\langle\overline{q}q\rangle}{\sqrt{2}}\int d^4xe^{iq.x}\langle 0|\, T\left[\overline{c}(x)c(x)\{\overline{q}(0)\gamma_\mu c(0)\}\{\overline{c}(0)\gamma^\mu q(0)\}\right]|0\rangle \\
&+ isin^2\theta\frac{\langle\overline{q}q\rangle}{\sqrt{2}}\int d^4xe^{iq.x}\langle 0|\, T\left[\{\overline{q}(x)\gamma_\mu c(x)\}\{\overline{c}(x)\gamma^\mu q(x)\}\overline{c}(0)c(0)\right]|0\rangle
\end{aligned}$$

Using the Wick's theorem to evaluate T-products, we have:

$$
\begin{aligned}
\Pi(q) = & \, 9isin\theta cos\theta \int d^4 x e^{iq.x} \left\{ tr\left[S^q(x)\gamma_\nu S^c(-x)\gamma^\mu\right] tr\left[S^c(x)\gamma^\nu S^q(-x)\gamma_\mu\right]\right\} \\
& + isin\theta cos\theta \frac{\langle \overline{q}q\rangle^2}{2} \int d^4 x e^{iq.x} tr\left[S^c(x)S^c(-x)\right] \\
& - icos^2\theta \frac{\langle \overline{q}q\rangle^2}{\sqrt{2}} \int d^4 x e^{iq.x} tr\left[S^c(x)S^c(-x)\right] \\
& + isin^2\theta \frac{\langle \overline{q}q\rangle^2}{\sqrt{2}} \int d^4 x e^{iq.x} tr\left[S^c(x)S^c(-x)\right] \qquad (3.21)
\end{aligned}
$$

where $tr$ denotes traces taken over Lorentz indices only and $S^c$ and $S^q$ are heavy and light quark propagators derived in Eq. (3.13) and Eq. (3.14). Note that the light quark operators in the third and fourth term are factored out as vacuum condensates. After calculating these integrals, we find the following spectral functions:

$$
\begin{aligned}
\rho_{11}(s) - \rho_{22}(s) = \int_{\alpha_{min}}^{\alpha_{max}} d\alpha \Bigg\{ & 18m_Q\mu^2 6(m_0^2 + 3\mu)m_q^2 + 9m_0^2\mu \\
& -12m_q\alpha(1-\alpha)\left[3\mu^2 + m_0^2(2\mu - s)\right] - 36m_0^2 m_q m_Q^2 \\
& -16\alpha(1-\alpha)(14\mu - 7s + 9m_0^2) \\
& +48(1-\alpha)m_q m_Q(3\alpha m_q - 13m_Q)m_Q \langle \overline{q}q\rangle \Bigg\}
\end{aligned}
$$

and,

$$
\rho_{12} = \rho_{21} = \frac{\langle \overline{q}q\rangle^2}{\sqrt{2}\pi^2} \int_{\alpha_{min}}^{\alpha_{max}} d\alpha \left\{m_Q^2 + \alpha(1-\alpha)s\right\} \qquad (3.22)
$$

where,

$$
\begin{aligned}
\mu &= \frac{m_Q^2}{\alpha(1-\alpha)} - s \\
\alpha_{min} &= \frac{1}{2}\left[1 - \sqrt{1 - \frac{4M_Q^2}{s}}\right] \\
\alpha_{max} &= \frac{1}{2}\left[1 + \sqrt{1 - \frac{4M_Q^2}{s}}\right] \qquad (3.23)
\end{aligned}
$$

### 3.1.2 Mass and Residue

For the mass and the residue of the $Y^{'}$ state, we use the interpolating current given in Eq.(3.4). The form of the correlation function then becomes:

$$\Pi(q) = i\int d^4xe^{iq.x}\left\langle 0\right| T\{j^{(2)}(x)j^{(2)}(0)\}\left|0\right\rangle$$
$$= isin\theta cos\theta\frac{\langle\overline{q}q\rangle}{\sqrt{2}}\int d^4xe^{iq.x}\left\{\left\langle 0\right| T\left\{\overline{c}(x)c(x)\right\}\left\{\left(\overline{q}(0)\gamma_\nu c(0)\right)\left(\overline{c}(0)\gamma^\nu q(0)\right)\right\}\left|0\right\rangle\right.$$
$$+\left\langle 0\right| T\left[\left\{\left(\overline{q}(x)\gamma_\nu c(x)\right)\left(\overline{x}(0)\gamma^\nu q(x)\right)\right\}\overline{c}(0)c(0)\right]\left|0\right\rangle\}$$
$$+icos^2\theta\int d^4xe^{iq.x}\left\langle 0\right| T\left[\left(\overline{q}(x)\gamma_\mu c(x)\right)\left(\overline{c}(x)\gamma^\mu q(x)\right)\left(\overline{q}(0)\gamma_\nu c(0)\right)\left(\overline{c}(0)\gamma^\nu q(0)\right)\right]\left|0\right\rangle$$
$$+isin^2\theta\frac{\langle\overline{q}q\rangle^2}{2}\int d^4xe^{iq.x}\left\langle 0\right| T\left[\overline{c}(x)c(x)\overline{c}(0)c(0)\right]\left|0\right\rangle \tag{3.24}$$

Using the Wick's theorem, we have:

$$\Pi(q) = 9isin\theta cos\theta\frac{\langle\overline{q}q\rangle}{\sqrt{2}}\int d^4xe^{iq.x}\left\{tr\left[S^q(x)\gamma_\nu S^c(-x)\gamma^\mu\right]tr\left[S^c(x)\gamma^\nu S^q(-x)\gamma_\mu\right]\right\}$$
$$= isin\theta cos\theta\frac{\langle\overline{q}q\rangle}{\sqrt{2}}\int d^4xe^{iq.x}tr\left[S^c(x)S^c(-x)\right]$$
$$= icos^2\theta\int d^4xe^{iq.x}tr\left[S^c(x)S^c(-x)\right]$$
$$= isin^2\theta\frac{\langle\overline{q}q\rangle^2}{\sqrt{2}}\int d^4xe^{iq.x}tr\left[S^c(x)S^c(-x)\right] \tag{3.25}$$

To derive the sum rules for the mass and the residue, we use the general form of the sum rules given in Eq. (2.17). For the residue, the sum rules formula reads:

$$\lambda^2 = e^{\frac{m^2_{Y'}}{M^2}}\int_0^{s_0} ds\rho e^{\frac{-s}{M^2}} \tag{3.26}$$

For the mass sum rules, we take the derivative of the Eq.(2.17) with respect to $1/M^2$:

$$m^2 = \frac{\int_0^{s_0} dss\rho(s)e^{-s/M^2}}{\int_0^{s_0} ds\rho(s)e^{-s/M^2}} \tag{3.27}$$

where in both Eq. (3.26) and Eq. (3.27);

$$\rho(s) = \frac{1}{\pi}Im\Pi(s) \tag{3.28}$$

Evaluating the integrals in Eq. (3.25), we have found:

$$\rho(s) = \frac{1}{384\pi^4}\int_{\alpha_{min}}^{\alpha_{max}} d\alpha \Bigg\{ \frac{3}{2} m_Q \mu^2 \langle \overline{q}q \rangle cos^2\theta - 144 m_Q^2 m_q \langle \overline{q}q \rangle \mu cos^2\theta$$
$$+ \left(3m_0^2 \left[3m_q - 8\alpha(1-\alpha)m_q\right] + 64\pi^2\alpha(1-\alpha)\langle \overline{q}q \rangle\right) \langle \overline{q}q \rangle \mu cos^2\theta$$
$$+ \left(6m_0^2 m_q \left[-6m_q^2 + 2\alpha(1-\alpha)s\right] 16\pi^2 \left[m_q(3m_q - 4m_q) - 2\alpha(1-\alpha)s\right] \langle \overline{q}q \rangle\right)$$
$$+48\pi^2 \left[m_Q^2 + \alpha(1-\alpha)(2\mu - s)\right] \langle \overline{q}q \rangle^2 sin\theta(3sin\theta - 2\sqrt{2}cos\theta)$$
$$-36\alpha(1-\alpha)m_q \langle \overline{q}q \rangle \mu^2 cos^2\theta \Bigg\}$$

where,

$$\mu = \frac{m_Q^2}{\alpha(1-\alpha)} - s$$
$$\alpha_{min} = \frac{1}{2}\left[1 - \sqrt{1 - \frac{4M_Q^2}{s}}\right]$$
$$\alpha_{max} = \frac{1}{2}\left[1 + \sqrt{1 - \frac{4M_Q^2}{s}}\right] \qquad (3.29)$$

## 3.2 Numerical Analysis

In this section, the numerical results obtained on the mass and residues of $Y'$ and the mixing angle are presented. The values of the quark masses and condensates used in the analysis are as follows: For the $c$-quark, its $\overline{MS}$ scheme mass is used $\overline{m_c}(m_c) = (1.28 \pm 0.03)GeV$ [36]. $\langle \bar{q}q \rangle (1GeV) = -(0.245^{+28}_{-19})MeV$ [37], $\langle \varrho_s^2 G^2 \rangle = (0.47)GeV$, $m_0^2 = 0.8GeV$ [38].

Beside these input parameters, QCD sum rules obtained in this work contain additional parameters. These are the Borel parameter $M^2$ and the continuum threshold $s_0$. The value of the continuum threshold $s_0$ is related to the first excited state. In both analysis, the value of the continuum threshold is taken as $\sqrt{s_0} = (4.4 \pm 0.1)GeV$. The Borel parameter $M^2$, on the other hand, an unphysical parameter and any result obtained in the calculations of observable quantities should not depend on it. To this end, one needs to find a specific region of the

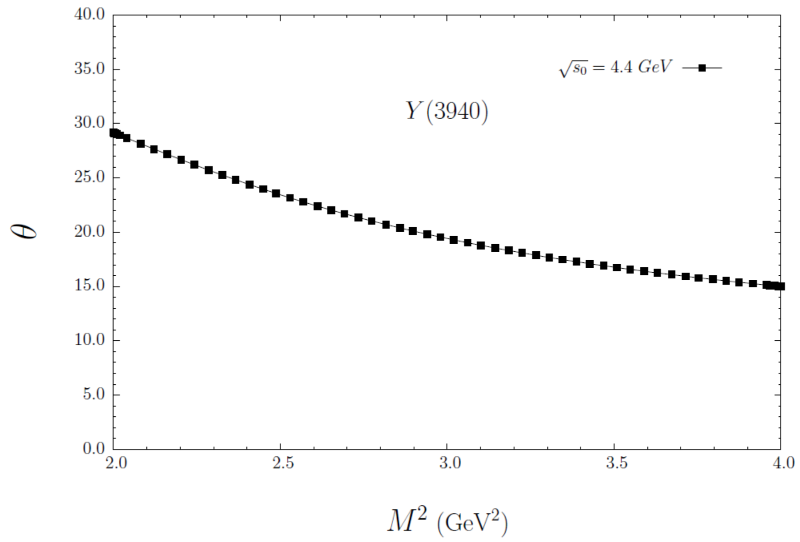


Figure 3.1: The dependence of the mixing angle on the Borel parameter $M^2$

Borel parameter $M^2$ where dependence of the results on the Borel parameter is minimum. The lower and upper limit of this Borel region are dictated by the requirement that at the lower limit, OPE series should be convergent and at the upper limit, the contribution coming from continuum states should be less than 1/3 of the contribution coming from the perturbative part, *i.e.*;

$$\frac{\int_{s_0}^{\infty} ds\rho(s)e^{-s/M^2}}{\int_{4m_c^2}^{\infty} ds\rho(s)e^{-s/M^2}} < \frac{1}{3}$$

Then, the following Borel window is found:

$$2.0 < M^2 < 4.0GeV^2 \tag{3.30}$$

In Fig. 3.1, we present the dependence of the mixing angle on the Borel parameter $M^2$ at $\sqrt{s_0} = 4.4GeV$. From this, the mixing angle is found to be $\theta = (20 \pm 2)^o$.

In Fig. 3.2 and Fig. 3.3 ,we present the dependence of the mass of $Y'$ and the residue of the state $Y'$ on the Borel parameter $M^2$. We see that the mass and residue shows very limited variation with the variation of the Borel parameter.

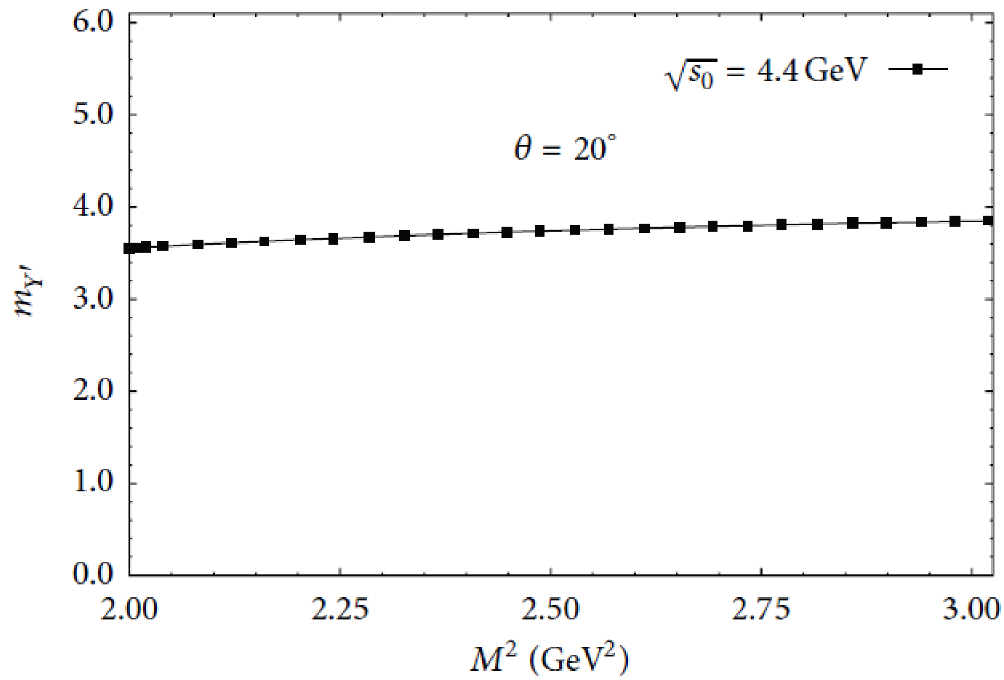


Figure 3.2: The dependence of the mass on the Borel parameter $M^2$

From these, the mass and residue values are found to be:

$$\begin{aligned} m_{Y'} &= (3.85 \pm 0.20) GeV \\ \lambda_{Y'} &= (1.9 \pm 0.4) \times 10^{-2} GeV^{-3} \end{aligned} \tag{3.31}$$

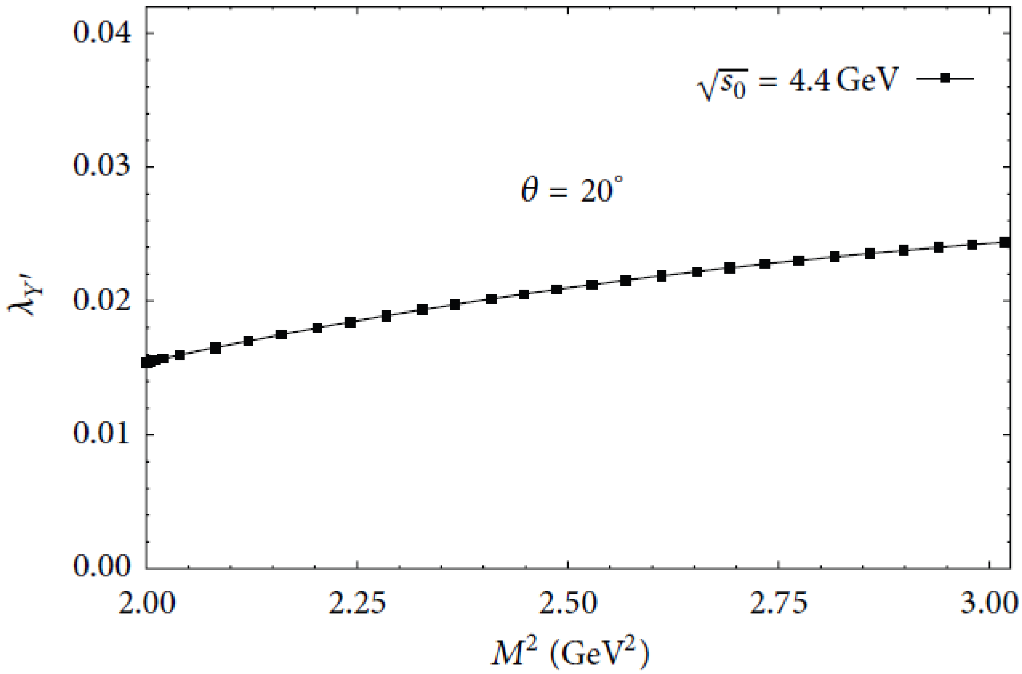


Figure 3.3: The dependence of the residue of $Y'$ on the Borel parameter $M^2$

# CHAPTER 4

# CONCLUSION

The B factories were designed to search for CP violation. But their unexpected contributions to the field of hadron spectroscopy made a revival in the subject. Starting with discovery of $X(3872)$ in 2003 by Belle Collaboration, more and more charmoniumlike state are discovered. Today, over 20 charmoniumlike states, called $XYZ$ states, are known and their structure remain an unsolved puzzle. These $XYZ$ states are not considered as an ordinary $c\bar{c}$ states since they can not be explained by the quark model.

Owing to the asymptotic freedom behaviour, perturbative methods in QCD does not always work. In these cases, one needs a non-perturbative method to calculate the observable properties of particles. Among these methods, QCD sum rules occupies a special place, which based on fundamental QCD Lagrangian.

In the first section of this thesis, QCD sum rules method is discussed and sum rules for two-point correlation function is derived. Afterwards, this sum rules is used to determine the mixing angle that appear between the interpolation current of $Y(3940)$ state and its orthogonal state $Y^{'}$. Also, using the same sum rules the mass and residue of $Y^{'}$ is determined. These states are considered as a mixture of charmonium and meson molecule $D^*D^*$ with the quantum number assignment $J^P = 0^+$.

The mass of the $Y^{'}$ state is found to be $m_{Y'} = (3.85 \pm 0.20) GeV$ while the residue $\lambda_{Y'} = (1.9 \pm 0.4) \times 10^{-2} GeV^{-3}$. From this we concluded that the mass estimation of the $Y^{'}$ state is in agreement with the experimental results.

Using the same sum rules, the mixing was estimated to be $\theta = (20 \pm 2)^o$. In [39], the mixing angle of this state was calculated and it was found if the mixing angle is choosen to be $\theta = (76 \pm 5)$, then the mass can be reproduced with an agreement with the experimental value. Our prediction on this mixing angle three times less than that one presented in [39]. With this observation one can conclude that the $D^*D^*$ is not quite right picture for describing the properties of the $Y(3940)$.

# REFERENCES


[1] K. O. et al. (Particle Data Group), "Review of particle physics," *Chin. Phys. C*, vol. 38, p. 090001, 2014.

[2] P. Colangelo and A. Khodjamirian, "Qcd sum rules, a modern perspective," *arXiv preprint hep-ph/0010175*, 2000.

[3] H. Yukawa, "On the interaction of elementary particles," *Proc. Phys. Math. Soc. Jap.*, vol. 17, no. 18, 1935.

[4] S. H. Neddermeyer and C. D. Anderson, "Note on the nature of cosmic-ray particles," *Physical Review*, vol. 51, 1937.

[5] J. C. Street and E. C. Stevenson, "New evidence for the existence of a particle of mass intermediate between the proton and electron," *Physical Review*, vol. 52, 1937.

[6] C. F. Powell and G. P. S. Occhialini, "Observations on the tracks of slow mesons in photographic emulsions," *Nature*, vol. 160, no. 4066, 1947.

[7] C. F. Powell and G. P. S. Occhialini, "Observations on the production of mesons by cosmic radiation," *Nature*, vol. 162, 1948.

[8] E. Fermi and C. N. Yang, "Are mesons elementary particles?," *Phys. Rev.*, vol. 76, no. 12, 1949.

[9] S. Sakata, "On a composite model for the new particles," *Prog. Theor. Phys*, vol. 16, no. 6, pp. 686–688, 1956.

[10] M. Gell-Mann *et al.*, "A schematic model of baryons and mesons," *Physics Letters*, vol. 8, no. 3, pp. 214–215, 1964.

[11] Y. Ne'eman, "Derivation of strong interactions from a gauge invariance," *Nuclear physics*, vol. 26, no. 2, pp. 222–229, 1961.

[12] G. Zweig, "An su(3) model for strong interaction symmetry and its breaking," tech. rep., CM-P00042884, 1964.

[13] H. Gell-Mann, M. Fritzsch and H. Leutwyler, "Advantages of the color octet gluon picture," *Physics Letters*, vol. 47B, no. 4, pp. 365–368, 1973.

[14] D. J. Gross and F. Wilczek, "Ultraviolet behavior of non-abelian gauge theories," *Physical Review Letters*, vol. 30, no. 26, p. 1343, 1973.

[15] H. D. Politzer, "Reliable perturbative results for strong interactions?," *Physical Review Letters*, vol. 30, no. 26, p. 1346, 1973.

[16] M. A. Shifman, A. Vainshtein, and V. I. Zakharov, "Qcd and resonance physics. theoretical foundations," *Nuclear Physics B*, vol. 147, no. 5, pp. 385–447, 1979.

[17] M. A. Shifman, A. Vainshtein, and V. I. Zakharov, "Qcd and resonance physics. applications," *Nuclear Physics B*, vol. 147, no. 5, pp. 448–518, 1979.

[18] K. G. Wilson, "Non-lagrangian models of current algebra," *Physical Review*, vol. 179, no. 5, p. 1499, 1969.

[19] S. Godfrey and S. L. Olsen, "The exotic xyz charmonium-like mesons," *arXiv preprint arXiv:0801.3867*, 2008.

[20] S.-K. Choi, S. Olsen, K. Abe, T. Abe, I. Adachi, B. S. Ahn, H. Aihara, K. Akai, M. Akatsu, M. Akemoto, *et al.*, "Observation of a narrow charmoniumlike state in exclusive b$\pm\rightarrow$ k$\pm\pi+$ $\pi$- j/$\psi$ decays," *Physical review letters*, vol. 91, no. 26, p. 262001, 2003.

[21] D. Acosta, J. Adelman, T. Affolder, T. Akimoto, M. Albrow, D. Ambrose, S. Amerio, D. Amidei, A. Anastassov, K. Anikeev, *et al.*, "Measurement of the j/$\psi$ meson and b-hadron production cross sections in p p$^{-}$ collisions at s= 1960 g e v," *Physical review D*, vol. 71, no. 3, p. 032001, 2005.

[22] M. Nielsen, F. S. Navarra, and S. H. Lee, "New charmonium states in qcd sum rules: a concise review," *Physics Reports*, vol. 497, no. 2, pp. 41–83, 2010.

[23] S.-K. Choi, S. Olsen, K. Abe, I. Adachi, H. Aihara, Y. Asano, S. Bahinipati, A. Bakich, Y. Ban, I. Bedny, *et al.*, "Observation of a near-threshold $\omega$ j/$\psi$ mass enhancement in exclusive b$\rightarrow$ k $\omega$ j/$\psi$ decays," *Physical review letters*, vol. 94, no. 18, p. 182002, 2005.

[24] B. Aubert, M. Bona, D. Boutigny, Y. Karyotakis, J. Lees, V. Poireau, X. Prudent, V. Tisserand, A. Zghiche, J. G. Tico, *et al.*, "Observation of y (3940)$\rightarrow$ j/$\psi$ $\omega$ in b$\rightarrow$ j/$\psi$ $\omega$ k at babar," *Physical review letters*, vol. 101, no. 8, p. 082001, 2008.

[25] D. Bernard, "Results on conventional and exotic charmonium at babar," *arXiv preprint arXiv:1311.0968*, 2013.

[26] V. Novikov, L. Okun, M. Shifman, A. Vainshtein, M. Voloshin, and V. Zakharov, "Charmonium and gluons: basic experimental facts and theoretical introduction, phys," 1978.

[27] L. J. Reinders, H. Rubinstein, and S. Yazaki, "Hadron properties from qcd sum rules," *Physics Reports*, vol. 127, no. 1, pp. 1–97, 1985.

[28] M. A. Shifman, *Vacuum structure and QCD sum rules.* Elsevier, 1992.

[29] S. Narison, "Chiral symmetry-spectral sum rules-ms-scheme and perturbative qcd," 1989.

[30] M. E. Peskin and D. V. Schroeder, *An introduction to quantum field theory.* Westview, 1995.

[31] V. Novikov, M. A. Shifman, A. Vainshtein, and V. I. Zakharov, "Calculations in external fields in quantum chromodynamics. technical review," *Fortschritte der Physik*, vol. 32, no. 11, pp. 585–622, 1984.

[32] V. Novikov, M. A. Shifman, A. Vainshtein, and V. I. Zakharov, "Operator expansion in quantum chromodynamics beyond perturbation theory," *Nuclear Physics B*, vol. 174, no. 2, pp. 378–396, 1980.

[33] M. Dubovikov and A. Smilga, "Analytical properties of the quark polarization operator in an external self-dual field," *Nuclear Physics B*, vol. 185, no. 1, pp. 109–132, 1981.

[34] J. Schwinger, "On gauge invariance and vacuum polarization," *Physical Review*, vol. 82, no. 5, p. 664, 1951.

[35] Z.-W. Huang and J. Liu, "Analytic calculation of doubly heavy hadron spectral density in coordinate space," *arXiv preprint arXiv:1205.3026*, 2012.

[36] K. Chetyrkin, J. Kühn, A. Maier, P. Maierhöfer, P. Marquard, M. Steinhauser, and C. Sturm, "Charm and bottom quark masses: an update," *Physical Review D*, vol. 80, no. 7, p. 074010, 2009.

[37] B. Ioffe, "Qcd (quantum chromodynamics) at low energies," *Progress in Particle and Nuclear Physics*, vol. 56, no. 1, pp. 232–277, 2006.

[38] V. Belyaev and B. Ioffe, "Determination of baryon and baryon-resonance masses from quantum-chromodynamics sum rules. nonstrange baryons," *Sov. Phys.-JETP (Engl. Transl.);(United States)*, vol. 56, no. 3, 1982.

[39] R. Albuquerque, J. Dias, M. Nielsen, and C. Zanetti, "Y (3940) as a mixed charmonium-molecule state," *Physical Review D*, vol. 89, no. 7, p. 076007, 2014.

[40] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions: with formulas, graphs, and mathematical tables.* No. 55, Courier Corporation, 1964.

[41] L. H. Ryder, *Quantum field theory.* Cambridge university press, 1996.

# APPENDIX A

# EVALUATION OF THE CORRELATION FUNCTION

To evaluate the integral appearing at the Eq. (3.21), we use the representations for the heavy and light quark propagators given in Eq. (3.13) and Eq. (3.14). By comparing Eq. (3.18) with the Eq. (3.21), we identify the forms of $\Pi_{ij}(q)$'s as;

$$\Pi_{11}(q) = -\frac{i\left\langle \overline{q}q\right\rangle^2}{2}\int d^4x e^{iq.x} tr\left[S^c(x)S^c(-x)\right] \tag{A.1}$$

$$\Pi_{22}(q) = i9\int d^4x e^{iq.x} tr\left[S^q(x)\gamma_\nu S^q(-x)\gamma^\mu\right] tr\left[S^c(x)\gamma^\nu S^q(-x)\gamma_\mu\right] \tag{A.2}$$

$$\Pi_{12}(q) = \Pi_{21}(q) = \frac{-i\left\langle \overline{q}q\right\rangle^2}{\sqrt{2}}\int d^4 e^{iq.x} tr\left[S^c(x)S^c(-x)\right] \tag{A.3}$$

As it's clearly seen, there are only two different kind of integrals that are needed to be evaluated. Here in this Appendix, we only present the detailed calculation of one integral, to give an example of the calculation method used in this work;

$$I_1 = -i\int d^4x e^{iq.x} tr\left[S^c(x)S^c(-x)\right] \tag{A.4}$$

With the heavy quark propagator representation Eq. (3.13), this becomes:

$$\begin{aligned} I_1 = &-\frac{iM_Q^4}{(2\pi)^4}\int d^4x e^{iq.x} tr\left\{\frac{K_1(M_Q\sqrt{-x^2}}{\sqrt{-x^2}} - \frac{i\not{x}K_2(M_Q\sqrt{-x^2})}{x^2}\right\} \\ &\left\{\frac{K_1(M_Q\sqrt{-x^2}}{\sqrt{-x^2}} + \frac{i\not{x}K_2(M_Q\sqrt{-x^2})}{x^2}\right\} \\ = &\frac{M_Q^4}{(2\pi)^4}\int d^4x e^{iq.x}\left\{-\frac{K_1^2(M_Q\sqrt{-x^2}}{x^2} + \frac{K_2^2(M_Q\sqrt{-x^2}}{x^2}\right\} \end{aligned} \tag{A.5}$$

where functions $K_1$ and $K_2$ are modified Bessel functions of the second kind. After Wick rotation where $t \to -it$, we have;

$$I_1 = -\frac{M_Q^4}{(2\pi)^4} \int d^4x_E e^{-iq_E.x_E} \left\{ \frac{K_1^2(M_Q\sqrt{x_E^2}}{x_E^2} - \frac{K_2^2(M_Q\sqrt{x_E^2}}{x_E^2} \right\} \tag{A.6}$$

Using the following integral representations for the Bessel function [40];

$$\frac{K_\nu(M_Q\sqrt{x_E^2})}{(\sqrt{x_E^2})^\nu} = \frac{1}{2}\int \frac{dt}{t^{\nu+1}} e^{-\frac{M_Q}{2}\left\{t+\frac{x_E^2}{t}\right\}} \tag{A.7}$$

We have the following form for the product of two Bessel function:

$$\frac{K_{\nu_1}K_{\nu_2}}{(\sqrt{x_E^2})^{\nu_1+\nu_2}} = \frac{1}{4}\int \frac{dtdr}{t^{\nu_1+1}r^{\nu_2+1}} e^{-\frac{m_Q}{2}(t+r)} e^{-\frac{m_Q}{2}\left(\frac{1}{t}+\frac{1}{r}\right)x_E^2}$$

Performing the following change of variable: $t \to \frac{M_Q}{2\alpha}$, $r \to \frac{M_Q}{2\beta}$:

$$\frac{K_{\nu_1}K_{\nu_2}}{(\sqrt{x_E^2})^{\nu_1+\nu_2}} = \frac{2^{\nu_1+\nu_2-2}}{m_Q^{\nu_1+\nu_2}} \int \alpha^{\nu_1-1}d\alpha\beta^{\nu_2-1}d\beta e^{-\frac{m_Q^2}{4}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)} e^{-(\alpha+\beta)e_E^2}$$

Introducing the identity $\int d\rho\delta(\rho-\rho\alpha-\rho\beta)$ and performing scale transformation $\alpha \to \rho\alpha$ and $\beta \to \rho\beta$;

$$\frac{K_{\nu_1}K_{\nu_2}}{(\sqrt{x_E^2})^{\nu_1+\nu_2}} =$$
$$= \frac{2^{\nu_1+\nu_2-2}}{m_Q^{\nu_1+\nu_2}} \int \rho^{\nu_1+\nu_2-1}d\rho\delta(1-\alpha-\beta)\alpha^{\nu_1-1}d\alpha\beta^{\nu_2-1}d\beta e^{-\frac{m_Q^2}{4\rho}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)} e^{-\rho(\alpha+\beta)x_E^2}$$

$$= \frac{2^{\nu_1+\nu_2-2}}{m_Q^{\nu_1+\nu_2}} \int \rho^{\nu_1+\nu_2-1}d\rho\alpha^{\nu_1-1}(1-\alpha)^{\nu_2-1}d\alpha e^{-\frac{m_Q^2}{4\rho}\left(\frac{1}{\alpha}+\frac{1}{1-\alpha}\right)} e^{-\rho x_E^2}$$

In a general expression of the form;

$$\int d^4x e^{-iq_E.x_E} \frac{K_{\nu_1}(m_Q\sqrt{x_E^2})K_{\nu_2}(m_Q\sqrt{x_E^2})}{\sqrt{\left(\sqrt{x_E^2}\right)^n}}$$

We have the following "master equation":

$$\int d^4xe^{-iq_E.x_E}\frac{K_{\nu_1}(m_Q\sqrt{x_E^2})K_{\nu_2}(m_Q\sqrt{x_E^2})}{\sqrt{\left(\sqrt{x_E^2}\right)^n}} =$$

$$= \frac{2^{\nu_1+\nu_2-2}}{m_Q^{\nu_1+\nu_2}}\int \rho^{\nu_1+\nu_2-1}d\rho\alpha^{\nu_1-1}(1-\alpha)^{\nu_2-1}d\alpha e^{-\frac{m_Q^2}{4\rho}\left(\frac{1}{\alpha}+\frac{1}{1-\alpha}\right)}$$
$$\times\int d^4e^{-iq_E.x_E}e^{-\rho x_E^2}\left(\sqrt{x_E^2}\right)^{\nu_1+\nu_2-n} \tag{A.8}$$

Using this form with $\nu_1=\nu_2=1\&\ n=2$ for the first term and $\nu_1=\nu_2=2$ and $n=2$ for the second term in the parenthesis in Eq. (A.6), the $I_1$ becomes:

$$I_1=\frac{-m_Q^2}{\pi^4}\int \rho d\rho d\alpha e^{-\frac{m_Q^2}{4\rho}\left(\frac{1}{\alpha}+\frac{1}{1-\alpha}\right)}\int d^4xe^{-iq_E.x_E}e^{-\rho x_E^2}\left[1-\frac{4\rho^2}{m_Q^2}\alpha(1-\alpha)x_E^2\right]$$

Now we perform the integration over $d^4x$ at the dimension $d$, and obtain:

$$I_1=-\pi^{d/2-4}m_Q^2\int \rho^{1-d/2}d\rho d\alpha e^{-\sigma/\rho}\left[1+\frac{2\rho}{m_Q^2}\alpha(1-\alpha)\left(-d+\frac{q_E^2}{2\rho}\right)\right]$$

where $\sigma=\frac{m_Q^2}{4\alpha(1-\alpha)}+\frac{q_E^2}{4}$. Performing a change of variable $u\to\sigma/\rho$;

$$I_1=\pi^{d/2-4}m_Q^2\int d\alpha\sigma^{2-d/2}\int duu^{d/2-3}e^{-u}\left[1+\frac{2\sigma}{m_Q^2u}\alpha(1-\alpha)\left(-d+\frac{q_E^2u}{2\sigma}\right)\right]$$
$$=-\pi^{d/2}m_Q^2\int d\alpha\left[\sigma^{2-d/2}\Gamma(d/2-2)\left(1+\frac{q_E^2\alpha(1-\alpha)}{m_Q^2}\right)\right.$$
$$\left.-\frac{2\alpha d(1-\alpha)}{m_Q^2}\sigma^{1-d/2}\Gamma(d/2-1)\right]$$

For $\epsilon=d-4$;

$$I_1=-\pi^{d/2-4}m_Q^2\int d\alpha\left[\sigma^{-\epsilon/2}\Gamma(\epsilon/2)\left(1+\frac{q_E^2\alpha(1-\alpha)}{m_Q^2}\right)\right.$$
$$\left.-\frac{2\alpha d(1-\alpha)}{m_Q^2}\sigma^{-1-\epsilon/2}\Gamma(\epsilon/2+1)\right]$$

Using $\Gamma(\epsilon/2) = 2/\epsilon + 1 - \gamma + O(\epsilon^2)$ [41];

$$I_1 = -\frac{m_Q^2}{\pi^2}\int d\alpha\left[\left(1-\frac{\epsilon}{2}ln\sigma\right)\left(2/\epsilon+1-\gamma+O(\epsilon^2)\right)\left(1+\frac{q_E^2\alpha(1-\alpha)}{m_Q^2}\right)\right]$$
$$= -\frac{m_Q^2}{\pi^2}\int d\alpha\,(1-ln\sigma)\left(1+\frac{q_E^2\alpha(1-\alpha)}{m_Q^2}\right)$$

Taking the imaginary part of this:

$$\frac{1}{\pi}ImI_1(q_E^2) = \frac{1}{\pi^2}\int_{\alpha_{min}}^{\alpha_{max}} d\alpha\left(m_Q^2 + q_E^2\alpha(1-\alpha)\right) \tag{A.9}$$

where,

$$\alpha_{min} = \frac{1}{2}\left[1-\sqrt{1-\frac{4M_Q^2}{q_E^2}}\right]$$
$$\alpha_{max} = \frac{1}{2}\left[1+\sqrt{1-\frac{4M_Q^2}{q_E^2}}\right]$$

Using Eq. (A.9), the spectral function $\rho_{12(s)}$ and $\rho_{21}(s)$ can be found as:

$$\rho_{12}(s) = \rho_{21}(s) = \frac{\langle\overline{q}q\rangle^2}{\sqrt{2}\pi^2}\int_{\alpha_{min}}^{\alpha_{max}} d\alpha\left(m_Q^2 + q_E^2\alpha(1-\alpha)\right) \tag{A.10}$$