QUANTUM GROUPS, R-MATRICES AND FACTORIZATION

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ABSTRACT

QUANTUM GROUPS, R-MATRICES AND FACTORIZATION

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R-matrices are solutions of the Yang-Baxter equation. They give rise to link invariants. Quantum groups can be used to obtain R-matrices. Roughly speaking, Drinfeld's quantum double corresponds to LU-decomposition. We proved a partial result concerning factorization of the quantum group $M_{p,q}(n)$ into simpler pieces to ease the computations.

Keywords: R-matrix, quantum group, knot theory

ÖZ

KUANTUM GRUPLARI, R-MATRİSLERİ VE FAKTORİZASYON

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R-matrisleri Yang-Baxter denkleminin çözümleridir. R-matrisleri kullanılarak düğüm değişmezleri elde edilir. Kuantum grupları kullanılarak R-matris elde edilebilmektedir. Drinfeld'in quantum çift metodu LU-parçalamasına denk gelmektedir. Hesaplamaları kolaylaştırmak için, $M_{p,q}(n)$ kuantum grubunu daha basit parçalara ayırmak hakkında kısmi bir sonuç ispatladık.

Anahtar Kelimeler: R-matris, kuantum grup, düğüm teorisi

To my family

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TABLE OF CONTENTS

LIST OF FIGURES

FIGURES

LIST OF ABBREVIATIONS

PBW	Poincaré-Birkhoff-Witt
TQFT	Topological quantum field theory
YBE	Yang-Baxter equation
FRT	Faddeev-Reshetikhin-Takhtadjian

CHAPTER 1

INTRODUCTION

Quantum groups are noncommutative, noncocommutative Hopf algebras with some additional structure on them. The initial study of quantum groups was conducted by Drinfeld and Jimbo independently in the mid-1980's ([9], [11], [12]). The term quantum group is first used by Drinfeld ([10]). Quantum groups are studied in many different fields of mathematics including mathematical physics, knot theory, quantum integrable systems, noncommutative geometry, and representation theory.

The foundational study of Hopf algebras was conducted by Heinz Hopf in 1940's. The name Hopf algebra was given by Armand Borel in 1953. Inspired by the work of Jean Dieudonné, Pierre Cartier in 1956 gave the first formal definition of Hopf algebra with the name hyperalgebra. However, contemporary definition of Hopf algebra was given by Bertram Kostant in 1966. Besides these names, main contributers to the topic are John Milnor, John C. Moore around 1960's, Moss Sweedler around 1970's, Vladimir Drinfeld, Michio Jimbo around 1980's ([2]).

Our motivation of studying quantum groups was to find R-matrices. An automorphism R of $V \otimes V$ is called an R-matrix if it is a solution of the Yang-Baxter equation (YBE):

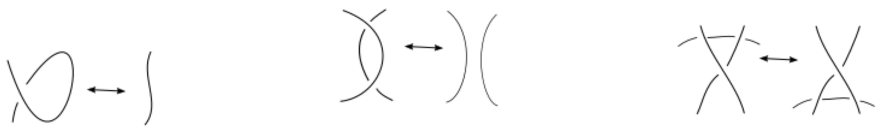
$$(R \otimes id)(id \otimes R)(R \otimes id) = (id \otimes R)(R \otimes id)(id \otimes R)$$

which holds in the automorphism group of $V \otimes V \otimes V$, where V is a vector space. R-matrices can be obtained from quantum groups (and vice versa). R-matrices give rise to knot invariants. Furthermore, some knot invariants such as the Jones polynomial, and HOMFLY polynomial can be obtained from R-matrices.

A link of n components is the image of a smooth (or piecewise smooth) embedding of disjoint unions of n S^1's into $\mathbb{R}^3$ (or S^3). A knot is a link of one component. Two knots (or links) are considered to be the same if there is an isotopy of the ambient space taking one to the other. A knot diagram is a projection of the given knot to a plane with finitely many double points. Double points are called crossings. The following theorem of Reidemeister reduces the space isotopy to plane isotopy:

Theorem 1.0.1. *Two link diagrams belong to isotopic links if and only if one can be obtained from the other by a finite sequence of Reidemeister moves (figure 1.1) and plane isotopies.*

Proof can be found in [22].



(a) Reidemeister Move I (b) Reidemeister Move II (c) Reidemeister Move III

Figure 1.1: Reidemeister moves

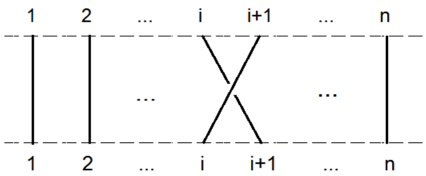
Although Reidemeister's theorem suggests a way to distinguish links, it is not always possible to prove that two links are the same using Reiemeister moves, and impossible to prove that they are different. Therefore we need algebraic expressions which are invariant under isotopy of links. We call them link invariants. Linking number, genus, fundamental group, polynomial invariants (Alexander polynomial, Jones polynomial, HOMFLY polynomial), homological invariants (knot Floer homology, Khovanov homology), Vassiliev invariants (finite type invariants), quantum invariants are some examples of link invariants.

Let n be a nonnegative integer and $\alpha_i = (i, 0, 0)$, $\beta_i = (i, 0, 1)$ for $i \in \{1, 2, ..., n\}$. A braid on n strands is the union of n pairwise non-intersecting monotonic in z direction curves connecting one of α_i to one of β_j. The closure of a braid is defined by connecting each α_i to β_i with unknotted curves. The isotopy classes of braids on n strands, B_n, form a group where multiplication is given by rescaling and concatenation. Let b_i be the braid shown in the figure 1.2a. It is obvious that the set $\{b_1, b_2, ..., b_{n-1}\}$ generates B_n. Artin gave a presentation of B_n in the following theorem ([3]).

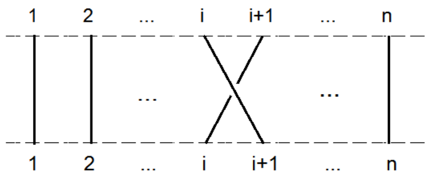
Theorem 1.0.2. *The braid group B_n is generated by $\{b_1, b_2, ..., b_{n-1}\}$ subject to the relations*

$$b_i b_j = b_j b_i \quad \textit{if} \quad |i-j| \geq 2 \tag{1.1}$$

$$b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1} \quad \textit{for} \quad 1 \leq i \leq n-2 \tag{1.2}$$



(a) b_i



(b) b_i^{-1}

Figure 1.2: Elementary braids

Burau gave a representation of braid groups in 1936 ([8]). The question whether this representation is faithful for all n, occupied mathematicians for a long time. In 1991 Moody, in 1993 Long and Paton and in 1999 Bigelow proved that Burau representation is not faithful for $n \geq 9$, $n \geq 6$, $n \geq 5$, respectively ([21], [18], [6]). The natural question is, are there any faithful representations? Lawrence gave a representation in 1990 ([17]) and Krammer proved that this representation (called Lawrence-Krammer representation) is faithful for $n = 4$ ([15]) and then extended his proof for all n ([16]) using algebraic methods. By considering B_n as the mapping class group of an n-punctured disk, Bigelow proved that the Lawrence-Krammer representation is faithful for all n in 2001 ([7]). But linearity of mapping class groups in general is still an open question.

Braids are closely related to knot theory because the closure of a braid is a link. Moreover the following theorem of Alexander proved in 1923 ([1]) shows that the converse is also true.

Theorem 1.0.3. *Every link is the closure of some braid.*

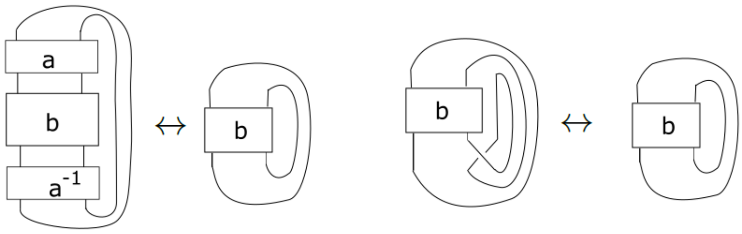
Let $a, b \in B_n$. The first and second Markov moves are defined as follows:

1. $b \leftrightarrow aba^{-1}$
2. $b \leftrightarrow bb_n^{\pm 1}$

Note that $b_n \notin B_n$. Here we identify $b \in B_n$ with its image under the inclusion $B_n \hookrightarrow B_{n+1}$.

The following theorem of Markov determines when the closures of braids give rise to the same link ([20]).

Theorem 1.0.4. *Two braid closures belong to isotopic links if and only if one can be obtained from the other by a finite sequence of Markov moves (figure 1.3).*



(a) First Markov move (b) Second Markov move

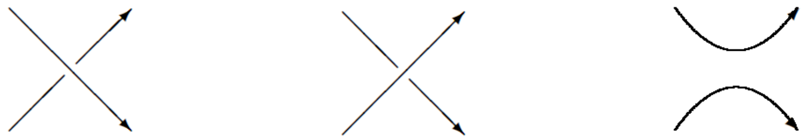
Figure 1.3: Geometric illustration of Markov moves

Since the closure of a braid is a link, we can construct link invariants using the braid group. If we use an R-matrix representation of the braid groups (i.e. sending b_i to $id^{\otimes i-1} \otimes R \otimes id^{\otimes n-i-1}$, where R is an R-matrix) the braid relation (1.2) is automatically satisfied. The trace of the resulting matrix corresponding to a given a link is invariant under Markov moves ([27]). Hence this trace is the link invariant. This is actually a topological quantum field theory (TQFT). The definition of TQFT is given by Atiyah in 1989 ([4], [5]). It is basically a process of defining a functor from category of cobordisms to category of vector spaces.

The Jones polynomial is an oriented link invariant defined as a Laurent polynomial in $\sqrt{t}$. Vaughan Jones discovered the Jones polynomial originally using von Neumann algebras in 1984 ([13]) which brought him the Fields Medal in 1990. It can also be defined by skein relations:

1. $V_{\circlearrowright} = 1$
2. $\frac{1}{t}V_{L_+} - tV_{L_-} = (\sqrt{t} - \frac{1}{\sqrt{t}})V_{L_0}$

where $\circlearrowright$ is the unknot and L_+, L_-, L_0 are identical outside of a neighbourhood containing only a fixed crossing and are as in the figure 1.4 inside the neighbourhood.



(a) L_+ (b) L_- (c) L_0

Figure 1.4: Positive crossing, negative crossing and 0-smoothing of a crossing

Another way of defining the Jones polynomial is by using a functor from the category $\mathcal{T}$ of isotopy classes of tangles to the category $\mathcal{V}$ of vector spaces with the help of R-matrices ([27], [28], [23], [26]).

Let m, n be nonnegative integers. A tangle L of type (m, n) is the union of finitely many piecewise smooth oriented curves in $\mathbb{R}^2 \times [0, 1]$ such that L intersects the boundary plane $\mathbb{R}^2 \times \{0\}$ transversally at m points and the boundary plane $\mathbb{R}^2 \times \{1\}$ transversally at n points. Note that a tangle of type $(0, 0)$ is a link in $\mathbb{R}^3$.

The objects of $\mathcal{T}$ are finite sequences of $\pm$ signs and the empty set and the morphisms of $\mathcal{T}$ are the tangles connecting these sequences. A functor $\mathcal{F}$ from $\mathcal{T}$ to $\mathcal{V}$ maps a tangle to a linear transformation. If R is an R-matrix and we let $\mathcal{F}(\times) = R$, then the Reidemeister move III (figure 1.1c) is automatically satisfied since R satisfies the YBE.

Every orientable 3-manifold can be obtained by a surgery of S^3 along a link in S^3. Thus invariants of links are candidates for giving rise to invariants of 3-manifolds. Using quantum invariants, one can obtain a 3-manifold invariant. However, to get a 3-manifold invariant we need to make some special choices such as setting the quantization parameter q to be a root of unity ([24], [30], [29]).

Drinfeld's quantum double (see Section 2.3) can be thought as the analogue of LU-decomposition in linear algebra of a quantum group and it is a process which enables us to obtain an R-matrix. Marc Rosso decomposed $U_h sl(n+1)$ using Drinfeld's quantum double and found a formula for the universal R-matrix of $U_h sl(n+1)$ ([25]). Our aim is to factorize the bialgebra $M_{p,q}(n)$ into simpler pieces and our future hope is to get R-matrices and new link invariants with this process.

In the second chapter we will introduce Hopf algebras and mention some important topics on Hopf algebras such as the Faddeev-Reshetikhin-Takhtadjian (FRT) construction and Drinfeld's quantum double. In the third chapter we give examples of some well-known bialgebras and quantum groups. In the fourth chapter we use a similar method to [25] to find a Poincaré-Birkhoff-Witt theorem for $U_qgl(n)$. In the fifth chapter we prove the duality between the Hopf algebra $U_qgl(n)$ and the bialgebra $M_q(n)$. The last chapter is dedicated to the factorization of the bialgebra $M_{p,q}(n)$.

CHAPTER 2

QUANTUM GROUPS

2.1 Hopf Algebras and R-Matrices

We will follow the notation in [14]. Let $\Bbbk$ be a field. All tensors will be over $\Bbbk$ and linear maps are $\Bbbk$-linear throughout the text.

Definition 2.1.1. *Let A be a vector space over $\Bbbk$ and $\mu : A \otimes A \to A$ and $\eta : \Bbbk \to A$ be linear maps. The triple (A, μ, η) is said to be an algebra if the following diagrams commute:*

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\mu\otimes id"] \arrow[d, "id\otimes\mu"'] & A \otimes A \arrow[d, "\mu"] \\ A \otimes A \arrow[r, "\mu"'] & A \end{tikzcd} \qquad \begin{tikzcd} \Bbbk \otimes A \arrow[r, "\eta\otimes id"] \arrow[dr, "\cong"'] & A \otimes A \arrow[d, "\mu"] & A \otimes \Bbbk \arrow[l, "id\otimes\eta"'] \arrow[dl, "\cong"] \\ & A & \end{tikzcd}$$

Definition 2.1.2. *Let (A, μ_A, η_A) and (B, μ_B, η_B) be algebras. A linear map ϕ is called an algebra morphism if the following diagrams commute:*

$$\begin{tikzcd} A \otimes A \arrow[r, "\phi\otimes\phi"] \arrow[d, "\mu_A"'] & B \otimes B \arrow[d, "\mu_B"] \\ A \arrow[r, "\phi"'] & B \end{tikzcd} \qquad \begin{tikzcd} \Bbbk \arrow[r, "\eta_A"] \arrow[dr, "\eta_B"'] & A \arrow[d, "\phi"] \\ & B \end{tikzcd}$$

Definition 2.1.3. *Let A be a vector space over $\Bbbk$ and $\Delta : A \to A \otimes A$ and $\varepsilon : A \to \Bbbk$ be linear maps. The triple (A, Δ, ε) is said to be a coalgebra if the following diagrams commute:*

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & A \otimes A \arrow[d, "id\otimes\Delta"] \\ A \otimes A \arrow[r, "\Delta\otimes id"'] & A \otimes A \otimes A \end{tikzcd} \qquad \begin{tikzcd} \Bbbk \otimes A & A \otimes A \arrow[l, "\varepsilon\otimes id"'] \arrow[r, "id\otimes\varepsilon"] & A \otimes \Bbbk \\ & A \arrow[ul, "\cong"] \arrow[u, "\Delta"] \arrow[ur, "\cong"'] & \end{tikzcd}$$

Notation 2.1.4. *(Sweedler's sigma notation) In order avoid the complexity of index notation we write*

$$\Delta(x) = \sum_{(x)} x' \otimes x''$$

for any $x \in A$.

Definition 2.1.5. *Let $(A, \Delta_A, \varepsilon_A)$ and $(B, \Delta_B, \varepsilon_B)$ be coalgebras. A linear map ϕ is called a coalgebra morphism if the following diagrams commute:*

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "\Delta_A"'] & B \arrow[d, "\Delta_B"] \\ A \otimes A \arrow[r, "\phi\otimes\phi"'] & B \otimes B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\varepsilon_A"] \arrow[d, "\phi"'] & \Bbbk \\ B \arrow[ur, "\varepsilon_B"'] & \end{tikzcd}$$

If (A, μ, η) is an algebra then so is $(A \otimes A, \mu \otimes \mu, \eta \otimes \eta)$. Similarly, if (A, Δ, ε) is a coalgebra then so is $(A \otimes A, (id \otimes \tau \otimes id) \circ (\Delta \otimes \Delta), \varepsilon \otimes \varepsilon)$, where $\tau(a \otimes b) = b \otimes a$.

Definition 2.1.6. *Let (A, μ, η) be an algebra and (A, Δ, ε) is a coalgebra. The quintuple $(A, \mu, \eta, \Delta, \varepsilon)$ is said to be a bialgebra if the maps μ and η are morphisms of coalgebras or equivalently, the maps Δ and ε are morphisms of algebras.*

Definition 2.1.7. *Let (A, μ, η) be an algebra and (C, Δ, ε) be a coalgebra. For $f, g \in Hom(C, A)$ we define $f * g$, the convolution of f and g, to be the composition of maps*

$$C \xrightarrow{\Delta} C \otimes C \xrightarrow{f\otimes g} A \otimes A \xrightarrow{\mu} A$$

If $A = C$ then $*$ is naturally defined on $End(A)$.

Definition 2.1.8. *Let $(H, \mu, \eta, \Delta, \varepsilon)$ be a bialgebra. An endomorphism S of H is called an antipode for the bialgebra H if*

$$S * id_H = id_H * S = \eta \circ \varepsilon \tag{2.1}$$

A Hopf algebra is a bialgebra with an antipode.

Remark 2.1.9. *The equation* (2.1) *implies the following*

$$\sum_{(x)} S(x')x'' = \sum_{(x)} x'S(x'') = \varepsilon(x)1$$

for all $x \in H$.

Proposition 2.1.10. *Let* $(H, \mu, \eta, \Delta, \varepsilon, S)$ *be Hopf algebra. Then* S *is an algebra antimorphism and coalgebra antimorphism, that is, it satisfies*

$$S(xy) = S(y)S(x) \qquad S(1) = 1$$

$$\sum_{(S(x))} S(x)' \otimes S(x)'' = \sum_{(x)} S(x'') \otimes S(x') \qquad \varepsilon(S(x)) = \varepsilon(x)$$

for every $x, y \in H$.

Proof. Let $x, y \in H$. The map S is an algebra antimorphism, since:

$$\begin{aligned}
S(xy) &= \sum_{(x)(y)} S(x'\varepsilon(x'')y'\varepsilon(y'')) \\
&= \sum_{(x)(y)} S(x'y')x''\varepsilon(y'')S(x''') \\
&= \sum_{(x)(y)} S(x'y')x''y''S(y''')S(x''') \\
&= \sum_{(x)(y)} S((xy)')(xy)''S(y''')S(x''') \\
&= \sum_{(x)(y)} \varepsilon(x'y')S(y'')S(x'') \\
&= \sum_{(x)(y)} S(\varepsilon(y')y'')S(\varepsilon(x')x'') \\
&= S(y)S(x),
\end{aligned}$$

$$S(1) = 1S(1) = \sum_{(1)} 1'S(1'') = \varepsilon(1)1 = 1.$$

The map S is a calgebra antimorphism, since:

$$\begin{aligned}
\varepsilon(S(x)) &= \sum_{(x)} \varepsilon(S(x'\varepsilon(x''))) = \sum_{(x)} \varepsilon(S(x'))\varepsilon(x'') \\
&= \sum_{(x)} \varepsilon(S(x')x'') = \varepsilon(\varepsilon(x)1) = \varepsilon(x),
\end{aligned}$$

$$\sum_{(S(x))} S(x)' \otimes S(x)'' = \Delta(S(x)) = \sum_{(x)} \Delta(\varepsilon(x'')S(x')) = \sum_{(x)} \varepsilon(x'')S(x')' \otimes S(x')'',$$

$$\varepsilon(x)1 \otimes 1 = \Delta(\varepsilon(x)1) = \sum_{(x)} \Delta(S(x')x'') = \sum_{(x)} (S(x')x'')' \otimes (S(x')x'')'',$$

so we have,

$$\begin{aligned}
\sum_{(x)} S(x'') \otimes S(x') &= \sum_{(x)} S(x''') \otimes S(\varepsilon(x')x'') \\
&= \sum_{(x)} (S(x')x'')'S(x'''') \otimes (S(x')x'')''S(x''') \\
&= \sum_{(x)} S(x')'x''S(x''''') \otimes S(x')''x'''S(x'''') \\
&= \sum_{(x)} S(x')'x''S(x'''') \otimes S(x')''\varepsilon(x''') \\
&= \sum_{(x)} S(x')'x''S(x''') \otimes S(x')'' \\
&= \sum_{(x)} S(x')'\varepsilon(x'') \otimes S(x')'' \\
&= \sum_{(x)} S(x)' \otimes S(x)''.
\end{aligned}$$

□

Definition 2.1.11. *A bialgebra $(H, \mu, \eta, \Delta, \varepsilon)$ is called quasi-cocommutative if there exists an invertible element R of the algebra $H \otimes H$ such that for all $x \in H$ we have*

$$\Delta^{op}(x) = R\Delta(x)R^{-1}.$$

Here $\Delta^{op} = \tau_{H,H} \circ \Delta$ where $\tau_{H,H}(h_1 \otimes h_2) = h_2 \otimes h_1$. R is called the universal R-matrix of the bialgebra H. A Hopf algebra is quasi-cocommutative if its underlying bialgebra is quasi-cocommutative.

Notation 2.1.12. *If $R = \sum_i r_i \otimes s_i$ then we denote by R_{12}, R_{13}, R_{23} the elements*

$$R_{12} = \sum_i r_i \otimes s_i \otimes 1$$

$$R_{13} = \sum_i r_i \otimes 1 \otimes s_i$$

$$R_{23} = \sum_i 1 \otimes r_i \otimes s_i$$

Definition 2.1.13. *A quasi-cocommutative bialgebra* $(H, \mu, \eta, \Delta, \varepsilon, R)$ *or a quasi-cocommutative Hopf algebra* $(H, \mu, \eta, \Delta, \varepsilon, S, R)$ *is braided if the universal R-matrix satisfies the following relations:*

$$(\Delta \otimes id_H)(R) = R_{13}R_{23}$$

$$(id_H \otimes \Delta)(R) = R_{13}R_{12}.$$

Theorem 2.1.14. *The universal R-matrix of a braided Hopf algebra* $(H, \mu, \eta, \Delta, \varepsilon, R)$ *satisfies the equation*

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

Proof.

$$\begin{aligned} R_{12}R_{13}R_{23} &= R_{12}(\Delta \otimes id_H)(R) \\ &= (\Delta^{op} \otimes id_H)(R)R_{12} \\ &= (\tau_{H,H} \otimes id_H)(\Delta \otimes id_H)(R)R_{12} \\ &= (\tau_{H,H} \otimes id_H)(R_{13}R_{23})R_{12} \\ &= R_{23}R_{13}R_{12} \end{aligned}$$

□

Remark 2.1.15. *Theorem 2.1.14 implies that the universal R-matrix* $R = \sum_i r_i \otimes s_i$ *satisfies*

$$\sum_{i,j,k} r_k r_j \otimes s_k r_i \otimes s_j s_i = \sum_{i,j,k} r_j r_i \otimes r_k s_i \otimes s_k s_j \tag{2.2}$$

Definition 2.1.16. *A cobraided bialgebra* $(H, \mu, \eta, \Delta, \varepsilon, r)$ *is a bialgebra H together with a linear form r on* $H \otimes H$ *satisfying the conditions*

(i) there exists a linear form $\overline{r}$ *on* $H \otimes H$ *such that*

$$r * \overline{r} = \overline{r} * r = \varepsilon$$

(ii) we have

$$\mu^{op} = r * \mu * \overline{r}$$

(iii) and

$$r(\mu \otimes id_H) = r_{13} * r_{23} \quad and \quad r(id_H \otimes \mu) = r_{13} * r_{12}$$

where the linear forms r_{12}, r_{23} and r_{13} are defined by

$$r_{12} = r \otimes \varepsilon, \quad r_{23} = \varepsilon \otimes r, \quad r_{13} = (\varepsilon \otimes r)(\tau_{H,H} \otimes id_H).$$

The linear form r is called the universal R-form of H. A Hopf algebra is cobraided if the underlying bialgebra is.

Definition 2.1.17. *Let (A, μ, η) be an algebra. An A-module is a pair (V, μ_V) where V is a vector space and $\mu_V : A \otimes V \to V$ is a linear map such that the following diagrams commute:*

$$\begin{tikzcd}
A \otimes A \otimes V \arrow[r, "\mu\otimes id"] \arrow[d, "id\otimes\mu_V"'] & A \otimes V \arrow[d, "\mu_V"] \\
A \otimes V \arrow[r, "\mu_V"'] & V
\end{tikzcd}
\qquad
\begin{tikzcd}
\Bbbk \otimes V \arrow[r, "\eta\otimes id"] \arrow[dr, "\cong"'] & A \otimes V \arrow[d, "\mu_V"] \\
 & V
\end{tikzcd}$$

Definition 2.1.18. *Let (V, μ_V) and (W, μ_W) be two modules of an algebra A. A linear map ϕ is called a morphism of A-modules if the following diagram commutes:*

$$\begin{tikzcd}
A \otimes V \arrow[r, "id\otimes\phi"] \arrow[d, "\mu_V"'] & A \otimes W \arrow[d, "\mu_W"] \\
V \arrow[r, "\phi"'] & W
\end{tikzcd}$$

If U and V are A-modules, where A is an algebra, then $U \otimes V$ is an $A \otimes A$ module by

$$(a \otimes a')(u \otimes v) = au \otimes a'v$$

where $a, a' \in A$, $u \in U$ and $v \in V$. Moreover, if A is a bialgebra then the algebra morphism Δ enables us to equip $U \otimes V$ with an A-module structure by

$$a(u \otimes v) = \Delta(a)(u \otimes v) = \sum_{(a)} a'u \otimes a''v$$

where $a \in A$, $u \in U$ and $v \in V$.

Definition 2.1.19. *Let (A, Δ, ε) be a coalgebra. An A-comodule is a pair (V, Δ_V) where V is a vector space and $\Delta_V : V \to A \otimes V$ is a linear map such that the following diagrams commute:*

$$\begin{tikzcd} V \arrow[r, "\Delta_V"] \arrow[d, "\Delta_V"'] & A \otimes V \arrow[d, "id\otimes\Delta_V"] \\ A \otimes V \arrow[r, "\Delta\otimes id"'] & A \otimes A \otimes V \end{tikzcd} \qquad \begin{tikzcd} \Bbbk \otimes V & A \otimes V \arrow[l, "\varepsilon\otimes id"'] \\ & V \arrow[ul, "\cong"] \arrow[u, "\Delta_V"'] \end{tikzcd}$$

Notation 2.1.20. *By convention we write*

$$\Delta_V(x) = \sum_{(x)} x_A \otimes x_V$$

for any $x \in V$.

Definition 2.1.21. *Let (V, Δ_V) and (W, Δ_W) be comodules of a coalgebra A. A linear map ϕ is called a morphism of A-comodules if the following diagram commutes:*

$$\begin{tikzcd} V \arrow[r, "\phi"] \arrow[d, "\Delta_V"'] & W \arrow[d, "\Delta_W"] \\ A \otimes V \arrow[r, "id\otimes\phi"'] & A \otimes W \end{tikzcd}$$

Definition 2.1.22. *Let V be a vector space. An automorphism c of $V \otimes V$ is called an R-matrix if it satisfies the Yang-Baxter equation*

$$(c \otimes id_V)(id_V \otimes c)(c \otimes id_V) = (id_V \otimes c)(c \otimes id_V)(id_V \otimes c)$$

which holds in the automorphism group of $V \otimes V \otimes V$

Let

$$c(v_i \otimes v_j) = \sum_{k,l} c^{k\,l}_{i\,j} v_k \otimes v_l.$$

Then one has

$$(c \otimes id_V)(v_i \otimes v_j \otimes v_k) = \sum_{l,m,n} c^{l\,m}_{i\,j} \delta_{kn} v_l \otimes v_m \otimes v_n$$
$$(id_V \otimes c)(v_i \otimes v_j \otimes v_k) = \sum_{l,m,n} \delta_{il} c^{m\,n}_{j\,k} v_l \otimes v_m \otimes v_n.$$

Then c satisfies the Yang-Baxter equation if and only if it satisfies the following equality for all i, j, k, l, m, n:

$$\sum_{p,q,r,s,t,u} (c^{p\,q}_{i\,j}\delta_{kr})(\delta_{ps}c^{t\,u}_{q\,r})(c^{l\,m}_{s\,t}\delta_{un})(v_l \otimes v_m \otimes v_n) =$$
$$\sum_{p,q,r,s,t,u} (\delta_{ip}c^{q\,r}_{j\,k})(c^{s\,t}_{p\,q}\delta_{ru})(\delta_{sl}c^{m\,n}_{t\,u})(v_l \otimes v_m \otimes v_n)$$

which is equivalent to

$$\sum_{p,q,t} c^{p\,q}_{i\,j} c^{t\,n}_{q\,k} c^{l\,m}_{p\,t} = \sum_{q,r,t} c^{q\,r}_{j\,k} c^{l\,t}_{i\,q} c^{m\,n}_{t\,r} \tag{2.3}$$

Lemma 2.1.23. *Let $(H, \mu, \eta, \Delta, \varepsilon, R)$ be a braided bialgebra and V be an H-module. The automorphism $c^R_{V,V}$ of $V \otimes V$ defined by*

$$c^R_{V,V}(v \otimes w) = \tau_{V,V}[R(v \otimes w)]$$

is an R-matrix.

Proof. Let $x \in H$, $v, w \in V$. The map $c^R_{V,V}$ is H-linear:

$$\begin{aligned} c^R_{V,V}(x(v \otimes w)) &= c^R_{V,V}(\Delta(x)(v \otimes w)) \\ &= \tau_{V,V}(R\Delta(x)(v \otimes w)) \\ &= \tau_{V,V}(\Delta^{op}(x)R(v \otimes w)) \\ &= \Delta(x)\tau_{V,V}(R(v \otimes w)) \\ &= xc^R_{V,V}(v \otimes w) \end{aligned}$$

The map $c^R_{V,V}$ is an automorphism with inverse given by

$$(c^R_{V,V})^{-1}(v \otimes w) = R^{-1}(w \otimes v).$$

Let us check this:

$$\begin{aligned} c^R_{V,V}((c^R_{V,V})^{-1}(v \otimes w)) &= c^R_{V,V}(R^{-1}(w \otimes v)) \\ &= \tau_{V,V}(RR^{-1}(w \otimes v)) \\ &= v \otimes w \end{aligned}$$

$$\begin{aligned} (c^R_{V,V})^{-1}(c^R_{V,V}(v \otimes w)) &= (c^R_{V,V})^{-1}(\tau_{V,V}(R(v \otimes w))) \\ &= R^{-1}R(v \otimes w)) \\ &= v \otimes w \end{aligned}$$

Let $R = \sum_i r_i \otimes s_i$, $u \otimes v \otimes w \in V \otimes V \otimes V$, $c^R_{V,V} = c$

$$\begin{aligned} (c \otimes id_V)(id_V \otimes c)(c \otimes id_V)(u \otimes v \otimes w) &= \sum_i (c \otimes id_V)(id_V \otimes c)(s_i v \otimes r_i u \otimes w) \\ &= \sum_{i,j} (c \otimes id_V)(s_i v \otimes s_j w \otimes r_j r_i u) \\ &= \sum_{i,j,k} (s_k s_j w \otimes r_k s_i v \otimes r_j r_i u) \end{aligned}$$

$$
\begin{aligned}
(id_V \otimes c)(c \otimes id_V)(id_V \otimes c)(u \otimes v \otimes w) &= \sum_i (id_V \otimes c)(c \otimes id_V)(u \otimes s_i w \otimes r_i v) \\
&= \sum_{i,j} (id_V \otimes c)(s_j s_i w \otimes r_j u \otimes r_i v) \\
&= \sum_{i,j,k} (s_j s_i w \otimes s_k r_i v \otimes r_k r_j u)
\end{aligned}
$$

These two are equal in view of the equation (2.2), hence $c^R_{V,V}$ is a solution of the Yang-Baxter equation. □

Similarly, let $(H, \mu, \eta, \Delta, \varepsilon, r)$ be a cobraided bialgebra and V be an H-comodule. The automorphism $c^r_{V,V}$ of $V \otimes V$ defined by

$$c^r_{V,V} = (r \otimes id_{V \otimes V}) \circ (id_H \otimes \tau_{V,H} \otimes id_V) \circ (\Delta_V \otimes \Delta_V) \circ \tau_{V,V}$$

is an R-matrix, where Δ_V is the coaction map.

2.2 The Faddeev-Reshetikhin-Takhtadjian (FRT) Construction

Theorem 2.2.1. *Let V be a vector space and c be an automorphism of $V \otimes V$ satisfying the Yang-Baxter equation. There exists a cobraided bialgebra $A(c)$ together with a linear map $\Delta_V : V \to A(c) \otimes V$ such that*

(i) the map Δ_V equips V with the structure of a comodule over $A(c)$,

(ii) the map c becomes a comodule map with respect to this structure,

(iii) there exists a unique linear form r on $A(c) \otimes A(c)$ turning $A(c)$ into a cobraided bialgebra such that $c^r_{V,V} = c$.

(iv) the bialgebra $A(c)$ is unique up to isomorphism

Proof. First, let us define $A(c)$ as an algebra. Choose a basis $\{v_1, v_2, ...v_n\}$ for the vector space V. Let the coefficients $c^{p\,q}_{i\,j}$ be defined by

$$c(v_i \otimes v_j) = \sum_{1 \le p,q \le n} c^{p\,q}_{i\,j} v_p \otimes v_q.$$

Let $\Bbbk\{t_i^j\} = \Bbbk\{t_i^j | i, j \in \{1, 2, ..., n\}\}$ be the free algebra generated by $\{t_i^j | i, j \in \{1, 2, ..., n\}\}$ over $\Bbbk$. Consider the two-sided ideal $I(c)$ of $\Bbbk\{t_i^j\}$ generated by the elements

$$C_{i\,j}^{p\,q} = \sum_{1 \le a,b \le n} c_{i\,j}^{a\,b} t_a^p t_b^q - \sum_{1 \le a,b \le n} t_i^a t_j^b c_{a\,b}^{p\,q}.$$

The algebra $A(c)$ is quotient of the free algebra $\Bbbk\{t_i^j\}$ by the two-sided ideal $I(c)$.

Next, we put a bialgebra structure on the algebra $A(c)$. Define coproduct and counit on the generators as follows:

$$\Delta(t_i^j) = \sum_{l=1}^{n} t_i^l \otimes t_l^j$$
$$\varepsilon(t_i^j) = \delta_{ij}$$

where δ_{ij} is the Kronecker delta and extend these maps to $A(c)$ as algebra maps.

We need to show that these maps are well-defined and the diagrams of the Definition 2.1.3 commute, i.e.,

$$(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta, \tag{2.4}$$
$$(\varepsilon \otimes id)\Delta = (id \otimes \varepsilon)\Delta = id \tag{2.5}$$

Let us show that ε and Δ are well-defined:

$$\begin{aligned}
\varepsilon(C_{i\,j}^{p\,q}) &= \sum_{1 \le a,b \le n} c_{i\,j}^{a\,b} \varepsilon(t_a^p)\varepsilon(t_b^q) - \sum_{1 \le a,b \le n} \varepsilon(t_i^a)\varepsilon(t_j^b) c_{a\,b}^{p\,q} \\
&= \sum_{1 \le a,b \le n} c_{i\,j}^{a\,b} \delta_{ap}\delta_{bq} - \sum_{1 \le a,b \le n} \delta_{ia}\delta_{jb} c_{a\,b}^{p\,q} \\
&= c_{i\,j}^{p\,q} - c_{i\,j}^{p\,q} \\
&= 0
\end{aligned}$$

$$
\begin{aligned}
\Delta(C^{p\,q}_{i\,j}) &= \sum_{1\le a,b\le n} c^{a\,b}_{i\,j}\Delta(t^p_a)\Delta(t^q_b) - \sum_{1\le a,b\le n} \Delta(t^a_i)\Delta(t^b_j)c^{p\,q}_{a\,b}\\
&= \sum_{a,b=1}^{n}\sum_{l,m=1}^{n} c^{a\,b}_{i\,j}(t^l_a\otimes t^p_l)(t^m_b\otimes t^q_m) - \sum_{a,b=1}^{n}\sum_{l,m=1}^{n}(t^l_i\otimes t^a_l)(t^m_j\otimes t^b_m)c^{p\,q}_{a\,b}\\
&= \sum_{l,m=1}^{n}\sum_{a,b=1}^{n} c^{a\,b}_{i\,j}t^l_a t^m_b\otimes t^p_l t^q_m - \sum_{l,m=1}^{n}\sum_{a,b=1}^{n} t^l_i t^m_j\otimes t^a_l t^b_m c^{p\,q}_{a\,b}\\
&= \sum_{l,m=1}^{n}(C^{l\,m}_{i\,j} + \sum_{a,b=1}^{n} t^a_i t^b_j c^{l\,m}_{a\,b})\otimes t^p_l t^q_m - \sum_{l,m=1}^{n} t^l_i t^m_j \otimes (\sum_{a,b=1}^{n} c^{a\,b}_{l\,m}t^p_a t^q_b - C^{p\,q}_{l\,m})\\
&= \sum_{l,m=1}^{n} C^{l\,m}_{i\,j}\otimes t^p_l t^q_m + \sum_{a,b,l,m=1}^{n} t^a_i t^b_j c^{l\,m}_{a\,b}\otimes t^p_l t^q_m\\
&\quad - \sum_{a,b,l,m=1}^{n} t^l_i t^m_j\otimes c^{a\,b}_{l\,m}t^p_a t^q_b + \sum_{l,m=1}^{n} t^l_i t^m_j\otimes C^{p\,q}_{l\,m}\\
&= \sum_{l,m=1}^{n} C^{l\,m}_{i\,j}\otimes t^p_l t^q_m + \sum_{l,m=1}^{n} t^l_i t^m_j \otimes C^{p\,q}_{l\,m}
\end{aligned}
$$

The elements $C^{l\,m}_{i\,j}$ and $C^{p\,q}_{l\,m}$ are in the ideal $I(c)$, which means $\Delta(C^{p\,q}_{i\,j}) = 0$ in $A(c)$. In order to show (2.4) and (2.5) it is enough to check these on the generators of $A(c)$. To show (2.4) apply the LHS map to $t^j_i \in A(c)$.

$$
\begin{aligned}
(\Delta\otimes id)\Delta(t^j_i) &= \sum_{l=1}^{n}(\Delta\otimes id)(t^l_i\otimes t^j_l)\\
&= \sum_{l=1}^{n}\sum_{m=1}^{n}(t^m_i\otimes t^l_m)\otimes t^j_l\\
&= \sum_{m=1}^{n}\sum_{l=1}^{n} t^m_i\otimes(t^l_m\otimes t^j_l)\\
&= \sum_{m=1}^{n}(id\otimes\Delta)(t^m_i\otimes t^j_m)\\
&= (id\otimes\Delta)\Delta(t^j_i)
\end{aligned}
$$

To show (2.5) apply $(\varepsilon\otimes id)\Delta$ and $(id\otimes\varepsilon)\Delta$ to $t^j_i \in A(c)$.

$$
\begin{aligned}
(\varepsilon\otimes id)\Delta(t^j_i) &= \sum_{l=1}^{n}(\varepsilon\otimes id)(t^l_i\otimes t^j_l)\\
&= \delta_{il}\otimes t^j_l\\
&= 1\otimes t^j_i
\end{aligned}
$$

$$
\begin{aligned}
(id \otimes \varepsilon)\Delta(t_i^j) &= \sum_{l=1}^{n}(id \otimes \varepsilon)(t_i^l \otimes t_l^j) \\
&= t_i^l \otimes \delta_{lj} \\
&= t_i^j \otimes 1
\end{aligned}
$$

Next, let us define the linear map Δ_V on the basis $\{v_1, v_2, ...v_n\}$ as follows:

$$\Delta_V(v_i) = \sum_{j=1}^{n} t_i^j \otimes v_j.$$

To prove that Δ_V endows V with a left comodule structure over the bialgebra $A(c)$, we need to show that the diagrams of Definition 2.1.19 commute, i.e., we need to show:

$$(id \otimes \Delta_V)\Delta_V = (\Delta \otimes id)\Delta_V, \tag{2.6}$$

$$(\varepsilon \otimes id)\Delta_V = id \tag{2.7}$$

Apply the LHS map of (2.6) to $v_i \in V$.

$$
\begin{aligned}
(id \otimes \Delta_V)\Delta_V(v_i) &= \sum_{j=1}^{n}(id \otimes \Delta_V)(t_i^j \otimes v_j) \\
&= \sum_{j,l=1}^{n}(t_i^j \otimes t_j^l \otimes v_l) \\
&= \sum_{l=1}^{n}(\Delta \otimes id)(t_i^l \otimes v_l) \\
&= (\Delta \otimes id)\Delta_V(v_i)
\end{aligned}
$$

Apply the LHS map of (2.7) to $v_i \in V$.

$$
\begin{aligned}
(\varepsilon \otimes id)\Delta_V(v_i) &= \sum_{j=1}^{n}(\varepsilon \otimes id)(t_i^j \otimes v_j) \\
&= \sum_{j=1}^{n}(\delta_{ij} \otimes v_j) \\
&= 1 \otimes v_i
\end{aligned}
$$

The coaction Δ_V induces a coaction $\Delta_{V\otimes V}$ of $A(c)$ on $V \otimes V$ defined by

$$\Delta_{V\otimes V}(v_i \otimes v_j) = \sum_{l,m=1}^{n} t_i^l t_j^m \otimes v_l \otimes v_m.$$

To prove that c is a comodule map, we need to show

$$\Delta_{V\otimes V} \circ c = (id \otimes c) \circ \Delta_{V\otimes V}.$$

Apply the map $\Delta_{V\otimes V} \circ c - (id \otimes c) \circ \Delta_{V\otimes V}$ to $v_i \otimes v_j \in V \otimes V$.

$$\begin{aligned}
&(\Delta_{V\otimes V} \circ c - (id \otimes c) \circ \Delta_{V\otimes V})(v_i \otimes v_j) \\
&\quad = \Delta_{V\otimes V}(c(v_i \otimes v_j)) - (id \otimes c)(\Delta_{V\otimes V}(v_i \otimes v_j)) \\
&\quad = \sum_{l,m,p,q=1}^{n} t_p^l t_q^m \otimes c_{i\,j}^{p\,q} v_l \otimes v_m - \sum_{l,m,p,q=1}^{n} t_i^p t_j^q \otimes c_{p\,q}^{l\,m} v_l \otimes v_m \\
&\quad = \sum_{l,m=1}^{n} (\sum_{p,q=1}^{n} c_{i\,j}^{p\,q} t_p^l t_q^m - t_i^p t_j^q c_{p\,q}^{l\,m}) \otimes v_l \otimes v_m \\
&\quad = \sum_{l,m=1}^{n} C_{i\,j}^{l\,m} \otimes v_l \otimes v_m \\
&\quad = 0_{A(c)\otimes V\otimes V}
\end{aligned}$$

Now let us prove the existence and uniqueness of the linear form r on $A(c) \otimes A(c)$ turning $A(c)$ into a cobraided bialgebra such that $c_{V,V}^r = c$. If such a linear form exists, then we have:

$$\begin{aligned}
c_{V,V}^r(v_i \otimes v_j) &= (r \otimes id_{V\otimes V}) \circ (id_H \otimes \tau_{V,H} \otimes id_V) \circ (\Delta_V \otimes \Delta_V) \circ \tau_{V,V}(v_i \otimes v_j) \\
&= \sum_{l,m=1}^{n} r(t_j^l \otimes t_i^m) v_l \otimes v_m \\
&= c(v_i \otimes v_j) \\
&= \sum_{l,m=1}^{n} c_{i\,j}^{l\,m} v_l \otimes v_m
\end{aligned}$$

Hence, $r(t_j^l \otimes t_i^m) = c_{i\,j}^{l\,m}$ for all i, j, l, m. Then uniqueness of r follows from Definition 2.1.16 part (iii), which restricts the way to extend the universal R-form on products of elements of the algebra $A(c)$. To prove existence we need to define r on all of $A(c) \otimes A(c)$. Define r on the generators of $\Bbbk\{t_i^j\} \otimes \Bbbk\{t_i^j\}$ by

$$\begin{aligned}
r(t_i^j \otimes 1) = r(1 \otimes t_i^j) = \varepsilon(t_i^j) = \delta_{ij} \\
r(t_j^l \otimes t_i^m) = c_{i\,j}^{l\,m}
\end{aligned}$$

and extend on $\Bbbk\{t_i^j\} \otimes \Bbbk\{t_i^j\}$ using Definition 2.1.16 part (iii). Now we need to prove that r is well-defined. It is enough to prove:

$$r(I(c) \otimes 1) = r(1 \otimes I(c)) = 0$$
$$r(C^{p\,q}_{i\,j} \otimes t^l_m) = r(t^l_m \otimes C^{p\,q}_{i\,j}) = 0$$

for all i, j, l, m, p, q.

$$r(I(c) \otimes 1) = r(1 \otimes I(c)) = \varepsilon(I(c)) = 0$$

$$\begin{aligned}
r(C^{p\,q}_{i\,j} \otimes t^l_m) &= \sum_{1 \le a,b \le n} c^{a\,b}_{i\,j} r(t^p_a t^q_b \otimes t^l_m) - \sum_{1 \le a,b \le n} r(t^a_i t^b_j \otimes t^l_m) c^{p\,q}_{a\,b} \\
&= \sum_{1 \le a,b,s \le n} c^{a\,b}_{i\,j} r(t^p_a \otimes t^s_m) r(t^q_b \otimes t^l_s) - \sum_{1 \le a,b,s \le n} r(t^a_i \otimes t^s_m) r(t^b_j \otimes t^l_s) c^{p\,q}_{a\,b} \\
&= \sum_{1 \le a,b,s \le n} c^{a\,b}_{i\,j} c^{p\,s}_{m\,a} c^{q\,l}_{s\,b} - \sum_{1 \le a,b,s \le n} c^{a\,s}_{m\,i} c^{b\,l}_{s\,j} c^{p\,q}_{a\,b} = 0
\end{aligned}$$

$$\begin{aligned}
r(t^l_m \otimes C^{p\,q}_{i\,j}) &= \sum_{1 \le a,b \le n} c^{a\,b}_{i\,j} r(t^l_m \otimes t^p_a t^q_b) - \sum_{1 \le a,b \le n} r(t^l_m \otimes t^a_i t^b_j) c^{p\,q}_{a\,b} \\
&= \sum_{1 \le a,b,s \le n} c^{a\,b}_{i\,j} r(t^s_m \otimes t^q_b) r(t^l_s \otimes t^p_a) - \sum_{1 \le a,b,s \le n} r(t^s_m \otimes t^b_j) r(t^l_s \otimes t^a_i) c^{p\,q}_{a\,b} \\
&= \sum_{1 \le a,b,s \le n} c^{a\,b}_{i\,j} c^{s\,q}_{b\,m} c^{l\,p}_{a\,s} - \sum_{1 \le a,b,s \le n} c^{s\,b}_{j\,m} c^{l\,a}_{i\,s} c^{p\,q}_{a\,b} = 0
\end{aligned}$$

These are zero because c is an R-matrix, that is, it satisfies (2.3).

Last, let us prove the universality of the algebra $A(c)$. Let (B, Δ'_V) be an algebra satisfying the conditions of Theorem 2.2.1. We need to show that there exists a unique bialgebra morphism such that the following diagram commutes:

$$\begin{tikzcd}
V \arrow[r, "\Delta_V"] \arrow[dr, "\Delta'_V"'] & A(c) \otimes V \arrow[d, "\phi \otimes id_V"] \\
& B \otimes V
\end{tikzcd} \tag{2.8}$$

There exists a family $(x^j_i)_{1 \le i,j \le n}$ of elements of B uniquely determined by

$$\Delta'_V(v_i) = \sum_{j=1}^{n} x^j_i v_j.$$

Then the equations (2.6) and (2.7) imply that

$$\Delta(x^j_i) = \sum_{l=1}^{n} x^l_i \otimes x^j_l, \quad \text{and}$$
$$\varepsilon(x^j_i) = \delta_{ij}$$

The fact that the map c is a comodule map implies

$$\Delta'_{V\otimes V} \circ c = (id \otimes c) \circ \Delta'_{V\otimes V}$$

which is equivalent to the vanishing of

$$\sum_{p,q=1}^{n} c^{p\,q}_{i\,j} x^l_p x^m_q - x^p_i x^q_j c^{l\,m}_{p\,q}$$

for every i, j, l and m. Now it is obvious that we have a bialgebra map $\phi : \Bbbk\{t^j_i\} \to B$ given by $\phi(t^j_i) = x^j_i$ which factors through $A(c)$. Let us check the commutation of the diagram 2.8:

$$\begin{aligned}
(\phi \otimes id_V)(\Delta_V(v_i)) &= \sum_j^n \phi(t^j_i) \otimes v_j \\
&= \sum_j^n x^j_i \otimes v_j \\
&= \Delta'_V(v_i)
\end{aligned}$$

Conversely, the relation $(\phi \otimes id_V)\Delta_V = \Delta'_V$ determines the map ϕ, which proves the uniqueness of ϕ. $\square$

2.3 Drinfeld's Quantum Double

We see that braided Hopf algebras provide R-matrices. The problem is now to find more such Hopf algebras. Drinfeld discovered a way to construct a braided Hopf algebra from any finite-dimensional Hopf algebra with an invertible antipode. This method is known as Drinfeld's quantum double construction.

Definition 2.3.1. *Let $(H, \mu, \eta, \Delta_H, \varepsilon_H)$ be a bialgebra and $(C, \Delta_C, \varepsilon_C)$ be a coalgebra. C is said to be a module-coalgebra over H if there exists a morphism of coalgebras $\phi : H \otimes C \to C$ inducing an H-module structure on C, that is,*

$$\begin{aligned}
(\phi \otimes \phi)\Delta_{H\otimes C} &= \Delta_C \phi \\
\varepsilon_{H\otimes C} &= \varepsilon_C \phi \\
\phi(\mu \otimes id_C) &= \phi(id_H \otimes \phi) \\
\phi(\eta \otimes id_C) &= id_C
\end{aligned}$$

Definition 2.3.2. *A pair (X, A) of bialgebras is matched if there exist linear maps $\alpha : A \otimes X \to X$ and $\beta : A \otimes X \to A$ turning X into a left module-coalgebra over A, and turning A into a right module-coalgebra over X, such that, if we set*

$$\alpha(a \otimes x) = a \cdot x \quad \textit{and} \quad \beta(a \otimes x) = a^x,$$

the following conditions are satisfied:

$$a \cdot (xy) = \sum_{(a)(x)} (a' \cdot x')(a''^{x''} \cdot y), \tag{2.9}$$

$$a \cdot 1 = \varepsilon(a)1, \tag{2.10}$$

$$(ab)^x = \sum_{(b)(x)} a^{b' \cdot x'} b''^{x''}, \tag{2.11}$$

$$1^x = \varepsilon(x)1, \tag{2.12}$$

$$\sum_{(a)(x)} a'^{x'} \otimes a'' \cdot x'' = \sum_{(a)(x)} a''^{x''} \otimes a' \cdot x' \tag{2.13}$$

for all $a, b \in A$ and $x, y \in X$.

Remark 2.3.3. *The assertions that the map α turns X into a module-coalgebra over A and the map β turns A into a right module-coalgebra over X means the following:*

$$(ab) \cdot x = a \cdot (b \cdot x),$$

$$1 \cdot x = x,$$

$$\sum_{(a \cdot x)} (a \cdot x)' \otimes (a \cdot x)'' = \sum_{(a)(x)} a' \cdot x' \otimes a'' \cdot x''$$

$$\varepsilon(a \cdot x) = \varepsilon(a)\varepsilon(x)$$

$$a^{xy} = (a^x)^y,$$

$$a^1 = a,$$

$$\sum_{(a^x)} (a^x)' \otimes (a^x)'' = \sum_{(a)(x)} a'^{x'} \otimes a''^{x''}$$

$$\varepsilon(a^x) = \varepsilon(a)\varepsilon(x)$$

Theorem 2.3.4. *Let (X, A) be a matched pair of bialgebras. There exists a unique bialgebra structure on the vector space $X \otimes A$, called the bicrossed product of X and A and denoted by $X \bowtie A$, such that its product, unit, coproduct and counit are given* *by*

$$(x \otimes a)(y \otimes b) = \sum_{(a)(y)} x(a' \cdot y') \otimes a''^{y''} b,$$
$$\eta(1) = 1 \otimes 1,$$
$$\Delta(x \otimes a) = \sum_{(a)(x)} (x' \otimes a') \otimes (x'' \otimes a''),$$
$$\varepsilon(x \otimes a) = \varepsilon(x)\varepsilon(a)$$

for all $x, y \in X$ and $a, b \in A$. Moreover, the injective maps $\iota_X : X \to X \otimes A$ and $\iota_A : A \to X \otimes A$ given by $\iota_X(x) = x \otimes 1$ and $\iota_A(a) = 1 \otimes a$ are bialgebra morphisms. We have

$$x \otimes a = (x \otimes 1)(1 \otimes a). \tag{2.14}$$

If the bialgebras X and A are Hopf algebras with antipodes S_X and S_A, respectively, then $X \bowtie A$ is a Hopf algebra with antipode S given by

$$S(x \otimes a) = \sum_{(x)(a)} S_A(a'') \cdot S_X(x'') \otimes S_A(a')^{S_X(x')}.$$

Proof. First, let us prove that $X \bowtie A$ is a bialgebra. We need to show that the diagrams of Definition 2.1.1 to show that $X \bowtie A$ is an algebra.

$$\begin{aligned}
((x \otimes a)(y \otimes b))(z \otimes c) &= \sum_{(a)(y)} (x(a' \cdot y') \otimes a''^{y''} b)(z \otimes c) \\
&= \sum_{(a)(b)(y)(z)} x(a' \cdot y')((a''^{y''} b)' \cdot z') \otimes (a''^{y''} b)''^{z''} c \\
&= \sum_{(a)(b)(y)(z)} x(a' \cdot y')((a''^{y''} b') \cdot z') \otimes (a'''^{y'''} b'')^{z''} c \\
&= \sum_{(a)(b)(y)(z)} x(a' \cdot y')((a''^{y''} \cdot b') \cdot z') \otimes (a'''^{y'''})^{b'' \cdot z''} b'''^{z'''} c
\end{aligned}$$

$$\begin{aligned}
(x \otimes a)((y \otimes b)(z \otimes c)) &= \sum_{(b)(z)} (x \otimes a)(y(b' \cdot z') \otimes b''^{z''} c) \\
&= \sum_{(a)(b)(y)(z)} x(a' \cdot (y(b' \cdot z'))') \otimes a''^{(y(b' \cdot z'))''} b''^{z''} c \\
&= \sum_{(a)(b)(y)(z)} x(a' \cdot y'(b' \cdot z')) \otimes a''^{y''(b'' \cdot z'')} b'''^{z'''} c \\
&= \sum_{(a)(b)(y)(z)} x(a' \cdot y')((a''^{y''} \cdot b') \cdot z') \otimes (a'''^{y'''})^{b'' \cdot z''} b'''^{z'''} c
\end{aligned}$$

$$
\begin{aligned}
(1 \otimes 1)(x \otimes a) &= \sum_{(x)} 1(1 \cdot x') \otimes 1^{x''} a \\
&= \sum_{(x)} x' \otimes \varepsilon(x'') a \\
&= \sum_{(x)} x' \varepsilon(x'') \otimes a \\
&= x \otimes a
\end{aligned}
$$

$$
\begin{aligned}
(x \otimes a)(1 \otimes 1) &= \sum_{(a)} x(a' \cdot 1) \otimes a''^{1} 1 \\
&= \sum_{(a)} x \varepsilon(a') \otimes a'' \\
&= \sum_{(a)} x \otimes \varepsilon(a') a'' \\
&= x \otimes a
\end{aligned}
$$

To show that $X \bowtie A$ is a coalgebra we need to show diagrams of 2.1.3 commute:

$$
\begin{aligned}
(\Delta \otimes id)\Delta(x \otimes a) &= (\Delta \otimes id) \sum_{(a)(x)} (x' \otimes a') \otimes (x'' \otimes a'') \\
&= \sum_{(a)(x)} \Delta(x' \otimes a') \otimes (x'' \otimes a'') \\
&= \sum_{(a)(x)} (x' \otimes a') \otimes (x'' \otimes a'') \otimes (x''' \otimes a'''),
\end{aligned}
$$

$$
\begin{aligned}
(id \otimes \Delta)\Delta(x \otimes a) &= (id \otimes \Delta) \sum_{(a)(x)} (x' \otimes a') \otimes (x'' \otimes a'') \\
&= \sum_{(a)(x)} (x' \otimes a') \otimes \Delta(x'' \otimes a'') \\
&= \sum_{(a)(x)} (x' \otimes a') \otimes (x'' \otimes a'') \otimes (x''' \otimes a'''),
\end{aligned}
$$

$$\begin{aligned}(\varepsilon \otimes id)\Delta(x \otimes a) &= (\varepsilon \otimes id) \sum_{(a)(x)} (x' \otimes a') \otimes (x'' \otimes a'') \\ &= \sum_{(a)(x)} \varepsilon(x' \otimes a') \otimes (x'' \otimes a'') \\ &= \sum_{(a)(x)} \varepsilon(x')\varepsilon(a') \otimes (x'' \otimes a'') \\ &= \sum_{(a)(x)} (\varepsilon(x')x'' \otimes \varepsilon(a')a'') \\ &= x \otimes a,\end{aligned}$$

$$\begin{aligned}(id \otimes \varepsilon)\Delta(x \otimes a) &= (id \otimes \varepsilon) \sum_{(a)(x)} (x' \otimes a') \otimes (x'' \otimes a'') \\ &= \sum_{(a)(x)} (x' \otimes a') \otimes \varepsilon(x'' \otimes a'') \\ &= \sum_{(a)(x)} (x' \otimes a') \otimes \varepsilon(x'')\varepsilon(a'') \\ &= \sum_{(a)(x)} (x'\varepsilon(x'') \otimes a'\varepsilon(a'')) \\ &= x \otimes a.\end{aligned}$$

Next, let us show that the map Δ is an algebra morphism.

$$\begin{aligned}\Delta((x \otimes a)(y \otimes b)) &= \sum_{(a)(y)} \Delta(x(a' \cdot y') \otimes a''^{y''} b) \\ &= \sum_{(a)(y)(x)(b)} (x(a' \cdot y'))' \otimes (a''^{y''} b)' \otimes (x(a' \cdot y'))'' \otimes (a''^{y''} b)'' \\ &= \sum_{(a)(y)(x)(b)} x'(a' \cdot y') \otimes a''^{y''} b' \otimes x''(a''' \cdot y''') \otimes a''''^{y''''} b''\end{aligned}$$

$$\begin{aligned}\Delta(x \otimes a)\Delta(y \otimes b) &= (\sum_{(a)(x)} x' \otimes a' \otimes x'' \otimes a'')(\sum_{(b)(y)} y' \otimes b' \otimes y'' \otimes b'') \\ &= \sum_{(a)(b)(x)(y)} (x' \otimes a' \otimes x'' \otimes a'')(y' \otimes b' \otimes y'' \otimes b'') \\ &= \sum_{(a)(b)(x)(y)} ((x' \otimes a')(y' \otimes b')) \otimes ((x'' \otimes a'')(y'' \otimes b'')) \\ &= \sum_{(a)(b)(x)(y)} x'(a' \cdot y') \otimes a''^{y''} b' \otimes x''(a''' \cdot y''') \otimes a''''^{y''''} b''\end{aligned}$$

Next, let us show that the map ε is an algebra morphism.

$$\begin{aligned}
\varepsilon((x\otimes a)(y\otimes b)) &= \sum_{(a)(y)} \varepsilon(x(a'\cdot y')\otimes a''^{y''}b) \\
&= \sum_{(a)(y)} \varepsilon(x(a'\cdot y'))\varepsilon(a''^{y''}b) \\
&= \sum_{(a)(y)} \varepsilon(x)\varepsilon(a')\varepsilon(y')\varepsilon(a'')\varepsilon(y'')\varepsilon(b) \\
&= \sum_{(a)(y)} \varepsilon(x)\varepsilon(a)\varepsilon(y)\varepsilon(b) \\
\varepsilon((x\otimes a)(y\otimes b)) &= \varepsilon((x\otimes a)\varepsilon(y\otimes b)) \\
&= \varepsilon(x)\varepsilon(a)\varepsilon(y)\varepsilon(b)
\end{aligned}$$

Next, let us prove that ι_X and ι_A are bialgebra morphisms and the equation (2.14) holds. Let $x, y \in X, a, b \in A$. The map ι_X is an algebra morphism:

$$\begin{aligned}
\iota_X(x)\iota_X(y) &= (x\otimes 1)(y\otimes 1) \\
&= \sum_{(y)} x(1\cdot y')\otimes 1^{y''} \\
&= \sum_{(y)} xy'\otimes \varepsilon(y'') \\
&= \sum_{(y)} xy'\varepsilon(y'')\otimes 1 \\
&= xy\otimes 1 \\
&= \iota_X(xy),
\end{aligned}$$

$$\iota_X(\eta_X(1)) = \iota_X(1) = 1\otimes 1 = \eta(1).$$

The map ι_X is a coalgebra morphism:

$$\begin{aligned}
(\iota_X\otimes\iota_X)\Delta_X(x) &= (\iota_X\otimes\iota_X)(\sum_{(x)} x'\otimes x'') \\
&= \sum_{(x)} x'\otimes 1\otimes x''\otimes 1 \\
&= \Delta(x\otimes 1) \\
&= \Delta(\iota_X(x)),
\end{aligned}$$

$$\varepsilon(\iota_X(x)) = \varepsilon(x\otimes 1) = \varepsilon(x).$$

The map ι_A is an algebra morphism:

$$\begin{aligned}
\iota_A(a)\iota_A(b) &= (1 \otimes a)(1 \otimes b) \\
&= \sum_{(a)} (a' \cdot 1) \otimes a''^{1} b \\
&= \sum_{(a)} \varepsilon(a') \otimes a'' b \\
&= \sum_{(y)} 1 \otimes \varepsilon(a') a'' b \\
&= 1 \otimes ab \\
&= \iota_A(ab),
\end{aligned}$$

$$\iota_A(\eta_A(1)) = \iota_A(1) = 1 \otimes 1 = \eta(1).$$

The map ι_A is a coalgebra morphism:

$$\begin{aligned}
(\iota_A \otimes \iota_A)\Delta_A(a) &= (\iota_A \otimes \iota_A)(\sum_{(a)} a' \otimes a'') \\
&= \sum_{(a)} 1 \otimes a' \otimes 1 \otimes a'' \\
&= \Delta(1 \otimes a) \\
&= \Delta(\iota_A(a)),
\end{aligned}$$

$$\varepsilon(\iota_A(a)) = \varepsilon(1 \otimes a) = \varepsilon(a).$$

The equation (2.14) is satisfied:

$$(x \otimes 1)(1 \otimes a) = x(1 \cdot 1) \otimes 1^1 a = x \otimes a.$$

Last, let us show that the map S is an antipode for $X \bowtie A$.

$$
\begin{aligned}
&\sum_{(x)(a)} S(x' \otimes a')(x'' \otimes a'') \\
&\quad = \sum_{(x)(a)} (S_A(a'') \cdot S_X(x'') \otimes S_A(a')^{S_X(x')})(x''' \otimes a''') \\
&\quad = \sum_{(x)(a)} S_A(a'') \cdot S_X(x'')((S_A(a')^{S_X(x')})' \cdot (x''')') \otimes (S_A(a')^{S_X(x')})''^{(x''')''} a''' \\
&\quad = \sum_{(x)(a)} S_A(a''') \cdot S_X(x''')(S_A(a'')^{S_X(x'')} \cdot x'''') \otimes S_A(a')^{S_X(x')x'''''} a'''' \\
&\quad = \sum_{(x)(a)} S_A(a'') \cdot (S_X(x'')x''') \otimes S_A(a')^{S_X(x')x''''} a''' \\
&\quad = \sum_{(x)(a)} \varepsilon(x'')S_A(a'') \cdot 1 \otimes S_A(a')^{S_X(x')x'''} a''' \\
&\quad = \sum_{(x)(a)} \varepsilon(S_A(a''))1 \otimes S_A(a')^{S_X(x')x''} a''' \\
&\quad = \sum_{(a)} \varepsilon(x)1 \otimes S_A(a')^1 a'' = \sum_{(a)} \varepsilon(x)1 \otimes S_A(a')a'' \\
&\quad = \varepsilon(x)\varepsilon(a)1 \otimes 1 = \varepsilon(x \otimes a)1 \otimes 1
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{(x)(a)} (x' \otimes a')S(x'' \otimes a'') \\
&\quad = \sum_{(x)(a)} (x' \otimes a')(S_A(a''') \cdot S_X(x''') \otimes S_A(a'')^{S_X(x'')}) \\
&\quad = \sum_{(x)(a)} x'((a')' \cdot (S_A(a''') \cdot S_X(x'''))') \otimes (a')''^{(S_A(a''') \cdot S_X(x'''))''} S_A(a'')^{S_X(x'')} \\
&\quad = \sum_{(x)(a)} x'((a'S_A(a''''')) \cdot S_X(x''''')) \otimes a''^{S_A(a'''') \cdot S_X(x'''')} S_A(a''')^{S_X(x''')} \\
&\quad = \sum_{(x)(a)} x'((a'S_A(a'''')) \cdot S_X(x''')) \otimes (a''S_A(a'''))^{S_X(x'')} \\
&\quad = \sum_{(x)(a)} x'((a'S_A(a''')) \cdot S_X(x''')) \otimes \varepsilon(a'')1^{S_X(x'')} \\
&\quad = \sum_{(x)(a)} x'((a'S_A(a'')) \cdot S_X(x''')) \otimes \varepsilon(S_X(x''))1 \\
&\quad = \sum_{(x)} \varepsilon(a)x'(1 \cdot S_X(x'')) \otimes 1 = \sum_{(x)} \varepsilon(a)x'S_X(x'') \otimes 1 \\
&\quad = \varepsilon(a)\varepsilon(x) \otimes 1 = \varepsilon(x \otimes a)1 \otimes 1
\end{aligned}
$$

$\square$

Definition 2.3.5. *Let $H = (H, \mu, \eta, \Delta, \varepsilon)$ be a bialgebra. Then the opposite algebra of H is defined by $H^{op} = (H, \mu^{op}, \eta, \Delta, \varepsilon)$ where $\mu^{op} = \tau \circ \mu$.*

Theorem 2.3.6. *Let $H = (H, \mu, \eta, \Delta, \varepsilon, S, S^{-1})$ be a finite-dimensional Hopf algebra and $X = (H^{op})^* = (H^*, \Delta^*, \varepsilon^*, (\mu^{op})^*, \eta^*, (S^{-1})^*, S^*)$ be the dual of the opposite Hopf algebra. Let $\alpha : H \otimes X \to X$ and $\beta : H \otimes X \to H$ be the linear maps given by*

$$\alpha(a \otimes f) = a \cdot f = \sum_{(a)} f(S^{-1}(a'')?a'), \quad \textit{and}$$
$$\beta(a \otimes f) = a^f = \sum_{(a)} f(S^{-1}(a''')a')a''$$

for $a \in H$ and $f \in X$, where $f(S^{-1}(a'')?a')$ is the map defined by

$$f(S^{-1}(a'')?a')(x) = f(S^{-1}(a'')xa'),$$

for all $x \in H$. Then the pair (H, X) is matched.

Proof. Before we start the proof, note that the product, coproduct, unit and counit in $X = (H^{op})^*$ are respectively given by

$$(\Delta)^*(f \otimes g)(a) = \sum_{(a)} f(a')g(a'')$$
$$(\mu)^*(f)(a \otimes b) = f(ba)$$
$$(\varepsilon)^*(1)(a) = \varepsilon(a)$$
$$(\eta)^*(f) = f(1)$$

for all $a, b \in H$ and $f, g \in X$.

First, let us prove that α endows X with a left module-coalgebra structure over H.

Let $a, b \in H$ and $f \in X$.

$$\begin{aligned}
a \cdot (b \cdot f) &= \sum_{(b)} a \cdot f(S^{-1}(b'')?b') \\
&= \sum_{(a)(b)} f(S^{-1}(b'')S^{-1}(a'')?a'b') \\
&= \sum_{(a)(b)} f(S^{-1}((ab)'')?(ab)') \\
&= (ab) \cdot f
\end{aligned}$$

$$1 \cdot f(x) = \sum_{(1)} f(S^{-1}(1'')x1') = f(x)$$

for every $x \in H$. Hence, $1 \cdot f = f$.

$$\begin{aligned}
\sum_{(a\cdot f)} (a \cdot f)' \otimes (a \cdot f)''(x \otimes y) &= (a \cdot f)(yx) \\
&= \sum_{(a)} f(S^{-1}(a'')yxa') \\
&= \sum_{(a)} f(S^{-1}(a''')y\varepsilon(a'')xa') \\
&= \sum_{(a)} f((S^{-1}(a'''')y(a'''))(S^{-1}(a'')xa')) \\
&= \sum_{(a)(f)} a' \cdot f' \otimes a'' \cdot f''(x \otimes y)
\end{aligned}$$

for all $x \otimes y \in H \otimes H$. Hence we have

$$\sum_{(a\cdot f)} (a \cdot f)' \otimes (a \cdot f)'' = \sum_{(a)(f)} a' \cdot f' \otimes a'' \cdot f''.$$

$$\eta^*(a \cdot f) = (a \cdot f)(1) = \sum_{(a)} f(S^{-1}(a'')1a') = f(\varepsilon(a)1) = \varepsilon(a)f(1) = \varepsilon(a)\varepsilon(f)$$

Next, let us prove that β endows H with a right module-coalgebra structure over X.

Let $a \in H$ and $f, g \in X$.

$$\begin{aligned}
(a^f)^g &= \sum_{(a^f)} g(S^{-1}((a^f)''')(a^f)')(a^f)'' \\
&= \sum_{(a)} f(S^{-1}(a''''')a')g(S^{-1}(a'''')a'')a''' \\
&= \sum_{(a)} f((S^{-1}(a''')a')')g((S^{-1}(a''')a')'')a'' \\
&= \sum_{(a)} (fg)(S^{-1}(a''')a')a'' \\
&= a^{fg}
\end{aligned}$$

$$a^1 = \sum_{(a)} \varepsilon(S^{-1}(a''')a')a'' = \sum_{(a)} \varepsilon(S^{-1}(a'''))\varepsilon(a')a'' = \sum_{(a)} \varepsilon(a''')\varepsilon(a')a'' = a$$

$$\begin{aligned}
\sum_{(a^f)} (a^f)' \otimes (a^f)'' &= \sum_{(a)} f(S^{-1}(a'''')a')a'' \otimes a''' \\
&= \sum_{(a)} f(S^{-1}(a''''')a')a'' \otimes \varepsilon(a''')a'''' \\
&= \sum_{(a)} f(S^{-1}(a''''')\varepsilon(a''')a')a'' \otimes a'''' \\
&= \sum_{(a)} f(S^{-1}(a'''''')a''''S^{-1}(a''')a')a'' \otimes a''''' \\
&= \sum_{(a)(f)} f'(S^{-1}(a''')a')a'' \otimes f''(S^{-1}(a'''''')a'''')a''''' \\
&= \sum_{(a)(f)} a'^{f'} \otimes a''^{f''}
\end{aligned}$$

$$\begin{aligned}
\varepsilon(a^f) &= \sum_{(a)} \varepsilon(f(S^{-1}(a''')a')a'') \\
&= \sum_{(a)} f(S^{-1}(a''')\varepsilon(a'')a') \\
&= \sum_{(a)} f(S^{-1}(a'')a') \\
&= f(\varepsilon(a)1) \\
&= \varepsilon(a)f(1) \\
&= \varepsilon(a)\varepsilon(f)
\end{aligned}$$

Last, let us show that α and β satisfies the relations of Definition 2.3.2. Let $a, b \in H$ and $f, g \in X$.

$$
\begin{aligned}
&\sum_{(a)(f)} (a' \cdot f')(a''^{f''} \cdot g)(x) \\
&\quad = \sum_{(a)(x)(f)} (a' \cdot f')(x')(a''^{f''} \cdot g)(x'') \\
&\quad = \sum_{(a)(x)(f)} f'(S^{-1}(a'')x'a')f''(S^{-1}(a''''')a''')(a'''' \cdot g)(x'') \\
&\quad = \sum_{(a)(x)(f)} f'(S^{-1}(a'')x'a')f''(S^{-1}(a'''''')a''')g(S^{-1}(a''''')x''a'''')
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{(a)(x)} f(S^{-1}(a'''''')a'''S^{-1}(a'')x'a')g(S^{-1}(a''''')x''a'''') \\
&= \sum_{(a)(x)} \varepsilon(a'')f(S^{-1}(a''''')x'a')g(S^{-1}(a'''')x''a''') \\
&= \sum_{(a)(x)} f(S^{-1}(a'''')x'a')g(S^{-1}(a''')x''a'') \\
&= \sum_{(S^{-1}(a'')xa')} f((S^{-1}(a'')xa')')g((S^{-1}(a'')xa')'') \\
&= \sum_{(a)} (fg)(S^{-1}(a'')xa') \\
&= a \cdot (fg)(x)
\end{aligned}
$$

for all $x \in H$, so we have

$$a \cdot (fg) = \sum_{(a)(f)} (a' \cdot f')(a''^{f''} \cdot g)$$

$$
\begin{aligned}
a \cdot \varepsilon(x) &= \sum_{(a)} \varepsilon(S^{-1}(a'')xa') \\
&= \sum_{(a)} \varepsilon(S^{-1}(a''))\varepsilon(x)\varepsilon(a') \\
&= \sum_{(a)} \varepsilon(a'')\varepsilon(x)\varepsilon(a') \\
&= \varepsilon(a)\varepsilon(x)
\end{aligned}
$$

for all $x \in H$, so we have $a \cdot \varepsilon = \varepsilon(a)\varepsilon$

$$\begin{aligned}
\sum_{(b)(f)} a^{b' \cdot f'} b''^{f''} &= \sum_{(b)(f)} a^{f'(S^{-1}((b')'')?(b')')} f''(S^{-1}((b'')''')(b'')')(b'')'' \\
&= \sum_{(a)(b)(f)} f'(S^{-1}(b'')S^{-1}(a''')a'b')a'' f''(S^{-1}(b''''')b''')b'''' \\
&= \sum_{(a)(b)} f(S^{-1}(b''''')b''' S^{-1}(b'')S^{-1}(a''')a'b')a''b'''' \\
&= \sum_{(a)(b)} f(S^{-1}(b'''')\varepsilon(b'')S^{-1}(a''')a'b')a''b''' \\
&= \sum_{(a)(b)} f(S^{-1}(b''')S^{-1}(a''')a'b')a''b'' \\
&= \sum_{(ab)} f(S^{-1}((ab)''')(ab)')(ab)'' \\
&= (ab)^f
\end{aligned}$$

$$1^f = f(1)1 = \varepsilon(f)1$$

$$\begin{aligned}
\sum_{(a)(f)} a'^{f'} \otimes a'' \cdot f'' &= \sum_{(a)(f)} f'(S^{-1}(a''')a')a'' \otimes f''(S^{-1}(a''''')?a'''') \\
&= \sum_{(a)} a'' \otimes f(S^{-1}(a''''')?a'''' S^{-1}(a''')a') \\
&= \sum_{(a)} a'' \otimes f(S^{-1}(a'''')?\varepsilon(a''')a') \\
&= \sum_{(a)} a'' \otimes f(S^{-1}(a''')?a')
\end{aligned}$$

$$\begin{aligned}
\sum_{(a)(f)} a''^{f''} \otimes a' \cdot f' &= \sum_{(a)(f)} f''(S^{-1}(a''''')a''')a'''' \otimes f'(S^{-1}(a'')?a') \\
&= \sum_{(a)} a'''' \otimes f(S^{-1}(a''''')a''' S^{-1}(a'')?a') \\
&= \sum_{(a)} a''' \otimes f(S^{-1}(a'''')\varepsilon(a'')?a') \\
&= \sum_{(a)} a'' \otimes f(S^{-1}(a''')?a')
\end{aligned}$$

Thus, we have

$$\sum_{(a)(f)} a'^{f'} \otimes a'' \cdot f'' = \sum_{(a)(f)} a''^{f''} \otimes a' \cdot f'$$

□

Definition 2.3.7. *The quantum double of H is defined by*

$$D(H) = X \bowtie H$$

where H is a finite-dimensional Hopf algebra with invertible antipode and $X = (H^{op})^*$.

Theorem 2.3.8. *Let $\{e_i\}_{i\in I}$ be a basis of H and $\{e^i\}_{i\in I}$ be its dual basis. $D(H)$ is a braided Hopf algebra with the universal R-matrix*

$$R = \sum_{i\in I}(1 \otimes e_i) \otimes (e^i \otimes 1).$$

Proof. We need to show

(i) R is invertible in $D(H) \otimes D(H)$,

(ii) $R\Delta(\xi \otimes x) = \Delta^{op}(\xi \otimes x)R$, for all $\xi \in X$ and $x \in H$,

(iii) $(\Delta \otimes id)(R) = R_{13}R_{23}$ and $(id \otimes \Delta)(R) = R_{13}R_{12}$.

Remark 2.3.9. *Since $\{e_i\}_{i\in I}$ is a basis of H and $\{e^i\}_{i\in I}$ is its dual basis, we have*

$$a = \sum_{i\in I} e^i(a)e_i$$

$$\sum_{(a)} a' \otimes a'' = \sum_{(e_i),i\in I} e^i(a)\, e_i' \otimes e_i''$$

$$\sum_{(a)} a' \otimes a'' \otimes a''' = \sum_{(e_i),i\in I} e^i(a)\, e_i' \otimes e_i'' \otimes e_i'''$$

for all $a \in H$.

(i) Our claim is

$$R^{-1} = \sum_{i\in I}(1 \otimes e_i) \otimes ((e^i \circ S) \otimes 1).$$

Let $A = a \otimes f \otimes b \otimes g$ be an element in $H \otimes X \otimes H \otimes X$. Pairing this element with RR^{-1} using the duality between H and X, we get

$$
\begin{aligned}
< RR^{-1}, A > &= \sum_{i,j\in I} < (1\otimes e_i e_j)\otimes (e^i(e^j\circ S)\otimes 1), a\otimes f\otimes b\otimes g > \\
&= \varepsilon(a)g(1)\sum_{i,j\in I} f(e_ie_j)(e^i(e^j\circ S))(b) \\
&= \varepsilon(a)g(1)\sum_{(b)}\sum_{i,j\in I} f(e_ie_j)e^i(b')e^j(S(b'')) \\
&= \varepsilon(a)g(1)\sum_{(b)} f\bigg(\sum_{i\in I} e^i(b')\sum_{j\in I} e^j(S(b''))e_ie_j\bigg) \\
&= \varepsilon(a)g(1)\sum_{(b)} f(b'S(b'')) \\
&= \varepsilon(a)g(1)\varepsilon(b)f(1) \\
&=< 1\otimes 1\otimes 1\otimes 1, A >
\end{aligned}
$$

So, $RR^{-1} = 1\otimes 1\otimes 1\otimes 1$. Similarly, pairing A with $R^{-1}R$, we get

$$
\begin{aligned}
< R^{-1}R, A > &= \sum_{i,j\in I} < (1\otimes e_i e_j)\otimes ((e^i\circ S)e^j\otimes 1), a\otimes f\otimes b\otimes g > \\
&= \varepsilon(a)g(1)\sum_{i,j\in I} f(e_ie_j)((e^i\circ S)e^j)(b) \\
&= \varepsilon(a)g(1)\sum_{(b)} f\bigg(\sum_{i\in I} e^i(S(b'))\sum_{j\in I} e^j(b'')e_ie_j\bigg) \\
&= \varepsilon(a)g(1)\sum_{(b)} f(S(b')b'') \\
&= \varepsilon(a)g(1)\varepsilon(b)f(1) \\
&=< 1\otimes 1\otimes 1\otimes 1, A >
\end{aligned}
$$

So, $R^{-1}R = 1\otimes 1\otimes 1\otimes 1$.

(ii) Now let us check that $R\Delta(\xi\otimes x) = \Delta^{op}(\xi\otimes x)R$, for all $\xi\in X$ and $x\in H$.

Evaluating the LHS and the RHS on $A = a \otimes f \otimes b \otimes g$, we get

$$\begin{aligned}
< R\Delta(\xi \otimes x), A > &= \sum_{(x)(\xi),i\in I} < (1 \otimes e_i)(\xi' \otimes x') \otimes (e^i \otimes 1)(\xi'' \otimes x''), A > \\
&= \sum_{(x)(\xi)(e_i),i\in I} < \xi'(S^{-1}(e_i''')?e_i') \otimes e_i''x' \otimes e^i\xi'' \otimes x'', A > \\
&= \sum_{(x)(\xi)(e_i)(b),i\in I} \xi'(S^{-1}(e_i''')ae_i')f(e_i''x')e^i(b')\xi''(b'')g(x'') \\
&= \sum_{(x)(e_i)(b),i\in I} \xi(b''S^{-1}(e_i''')ae_i')f(e_i''x')e^i(b')g(x'') \\
&= \sum_{(x)(b)} \xi(b''''S^{-1}(b''')ab')f(b''x')g(x'') \\
&= \sum_{(x)(b)} \varepsilon(b''')\xi(ab')f(b''x')g(x'') \\
&= \sum_{(x)(b)} \xi(ab')f(b''x')g(x'')
\end{aligned}$$

$$\begin{aligned}
< \Delta^{op}(\xi \otimes x)R, A > &= \sum_{(x)(\xi),i\in I} < (\xi'' \otimes x'')(1 \otimes e_i) \otimes (\xi' \otimes x')(e^i \otimes 1), A > \\
&= \sum_{(x)(\xi),i\in I} < \xi'' \otimes x''''e_i \otimes \xi'e^i(S^{-1}(x''')?x') \otimes x'', A > \\
&= \sum_{(x)(b)(\xi),i\in I} \xi''(a)f(x''''e_i)\xi'(b')e^i(S^{-1}(x''')b''x')g(x'') \\
&= \sum_{(x)(b),i\in I} \xi(ab')f(x''''e^i(S^{-1}(x''')b''x')e_i)g(x'') \\
&= \sum_{(x)(b)} \xi(ab')f(x''''S^{-1}(x''')b''x')g(x'') \\
&= \sum_{(x)(b)} \varepsilon(x''')\xi(ab')f(b''x')g(x'') \\
&= \sum_{(x)(b)} \xi(ab')f(b''x')g(x'') \\
&=< R\Delta(\xi \otimes x), A >
\end{aligned}$$

(iii) Now let us show that $(\Delta \otimes id)(R) = R_{13}R_{23}$ which is equivalent to

$$\sum_{(e_i),i\in I} 1 \otimes e_i' \otimes 1 \otimes e_i'' \otimes e^i \otimes 1 = \sum_{i,j\in I} 1 \otimes e_i \otimes 1 \otimes e_j \otimes e^ie^j \otimes 1.$$

Let us evaluate the LHS and the RHS on $B = a \otimes f \otimes b \otimes g \otimes c \otimes h$, we get

$$
\begin{aligned}
< \sum_{(e_i), i \in I} 1 \otimes e_i' \otimes 1 \otimes e_i'' \otimes e^i \otimes 1, B > &= \sum_{(e_i), i \in I} \varepsilon(a)\varepsilon(b)h(1)f(e_i')g(e_i'')e^i(c) \\
&= \sum_{(c)} \varepsilon(a)\varepsilon(b)h(1)f(c')g(c'')
\end{aligned}
$$

$$
\begin{aligned}
&< \sum_{i,j \in I} 1 \otimes e_i \otimes 1 \otimes e_j \otimes e^i e^j \otimes 1, B > \\
&\qquad = \sum_{i,j \in I} \varepsilon(a)\varepsilon(b)h(1)f(e_i)g(e_j)(e^i e^j)(c) \\
&\qquad = \sum_{(c), i,j \in I} \varepsilon(a)\varepsilon(b)h(1)f(e_i)g(e_j)e^i(c')e^j(c'') \\
&\qquad = \sum_{(c), i,j \in I} \varepsilon(a)\varepsilon(b)h(1)f(e^i(c')e_i)g(e^j(c'')e_j) \\
&\qquad = \sum_{(c)} \varepsilon(a)\varepsilon(b)h(1)f(c')g(c'')
\end{aligned}
$$

Similarly, $(id \otimes \Delta)(R) = R_{13}R_{12}$ is equivalent to

$$
\sum_{(e_i), i \in I} 1 \otimes e_i \otimes e^{i'} \otimes 1 \otimes e^{i''} \otimes 1 = \sum_{i,j \in I} 1 \otimes e_i e_j \otimes e^j \otimes 1 \otimes e^i \otimes 1.
$$

$$
\begin{aligned}
&< \sum_{(e^i), i \in I} 1 \otimes e_i \otimes e^{i'} \otimes 1 \otimes e^{i''} \otimes 1, B > \\
&\qquad = \sum_{(e^i), i \in I} \varepsilon(a)h(1)f(e_i)g(1)e^{i'}(b)e^{i''}(c) \\
&\qquad = \sum_{i \in I} \varepsilon(a)h(1)f(e_i)g(1)e^i(cb) \\
&\qquad = \varepsilon(a)h(1)f(cb)g(1)
\end{aligned}
$$

$$
\begin{aligned}
< \sum_{i,j \in I} 1 \otimes e_i e_j \otimes e^j \otimes 1 \otimes e^i \otimes 1, B > &= \sum_{i,j \in I} \varepsilon(a)h(1)g(1)f(e_i e_j)e^i(c)e^j(b) \\
&= \sum_{i,j \in I} \varepsilon(a)h(1)g(1)f(e^i(c)e_i e^j(b)e_j) \\
&= \varepsilon(a)h(1)f(cb)g(1)
\end{aligned}
$$

$\square$

CHAPTER 3

EXAMPLES

In this chapter, we introduce the bialgebra structure on the matrix algebras $M_{p,q}(n)$ and $M_q(n)$ and the Hopf algebra structure on the quantized universal enveloping algebra $U_q gl(n)$ of the Lie algebra $gl(n)$.

3.1 Bialgebra Structure of $\mathbf{M_{p,q}(n)}$

Definition 3.1.1. *Let p and q be nonzero elements of a field K and $M_{p,q}(n) = K\{a_{ij}|i,j \in \{1,2,...,n\}\}/I$ be the quotient of the free algebra generated by the generators $\{a_{ij}|i,j \in \{1,2,...,n\}\}$ over K by the two-sided ideal I generated by the relations*

$$a_{il}a_{ik} = pa_{ik}a_{il}, \tag{3.1}$$

$$a_{jk}a_{ik} = qa_{ik}a_{jk}, \tag{3.2}$$

$$a_{jk}a_{il} = p^{-1}qa_{il}a_{jk}, \tag{3.3}$$

$$a_{jl}a_{ik} - a_{ik}a_{jl} = (p - q^{-1})a_{jk}a_{il} \tag{3.4}$$

whenever $j > i$ and $l > k$.

Define coproduct and counit on the generators as follows:

$$\Delta(a_{ij}) = \sum_{k=1}^{n} a_{ik} \otimes a_{kj}$$

$$\varepsilon(a_{ij}) = \delta_{ij}$$

where δ_{ij} is the Kronecker delta and extend these maps to $M_{p,q}(n)$ as algebra maps.

Lemma 3.1.2. *The maps Δ and ε are well-defined on $M_{p,q}(n)$.*

Proof. Assume $j > i$ and $l > k$. First let us verify that the map ε is well-defined.

$$\begin{aligned}
\varepsilon(a_{il}a_{ik} - pa_{ik}a_{il}) &= \varepsilon(a_{il})\varepsilon(a_{ik}) - p\varepsilon(a_{ik})\varepsilon(a_{il}) = 0,\\
\varepsilon(a_{jk}a_{ik} - qa_{ik}a_{jk}) &= \varepsilon(a_{jk})\varepsilon(a_{ik}) - q\varepsilon(a_{ik})\varepsilon(a_{jk}) = 0,\\
\varepsilon(a_{jk}a_{il} - p^{-1}qa_{il}a_{jk}) &= \varepsilon(a_{jk})\varepsilon(a_{il}) - p^{-1}q\varepsilon(a_{il})\varepsilon(a_{jk}) = 0,\\
\varepsilon(a_{jl}a_{ik} - a_{ik}a_{jl}) &= \varepsilon(a_{jl})\varepsilon(a_{ik}) - \varepsilon(a_{ik})\varepsilon(a_{jl}) = 0\\
\varepsilon((p - q^{-1})a_{jk}a_{il}) &= \varepsilon(a_{jk})\varepsilon(a_{il}) = 0.
\end{aligned}$$

Similarly, we check that the map Δ is well-defined. First, show that Δ preserves the relation (3.1), i.e.

$$\Delta(a_{il}a_{ik}) = \Delta(pa_{ik}a_{il}).$$

$$\begin{aligned}
&\Delta(a_{il}a_{ik})\\
&= \Delta(a_{il})\Delta(a_{ik}) = (\sum_{r=1}^{n} a_{ir} \otimes a_{rl})(\sum_{s=1}^{n} a_{is} \otimes a_{sk}) = \sum_{r,s=1}^{n}(a_{ir}a_{is} \otimes a_{rl}a_{sk})\\
&= \sum_{1=r<s\leq n}^{n}(a_{ir}a_{is} \otimes a_{rl}a_{sk}) + \sum_{1=r=s\leq n}^{n}(a_{ir}a_{is} \otimes a_{rl}a_{sk}) + \sum_{1=s<r\leq n}^{n}(a_{ir}a_{is} \otimes a_{rl}a_{sk})\\
&= \sum_{s<r}(a_{is}a_{ir} \otimes a_{sl}a_{rk}) + \sum_{r=s}(a_{is}a_{ir} \otimes pa_{sk}a_{rl})\\
&\quad + \sum_{s<r}(pa_{is}a_{ir} \otimes (a_{sk}a_{rl} + (p - q^{-1})a_{rk}a_{sl}))\\
&= \sum_{s<r}(a_{is}a_{ir} \otimes pq^{-1}a_{rk}a_{sl}) + \sum_{r=s}(a_{is}a_{ir} \otimes pa_{sk}a_{rl})\\
&\quad + \sum_{s<r}(pa_{is}a_{ir} \otimes a_{sk}a_{rl}) + \sum_{s<r}(pa_{is}a_{ir} \otimes pa_{rk}a_{sl}) - \sum_{s<r}(pa_{is}a_{ir} \otimes q^{-1}a_{rk}a_{sl})\\
&= p\sum_{r=s}(a_{is}a_{ir} \otimes a_{sk}a_{rl}) + p\sum_{s<r}(a_{is}a_{ir} \otimes a_{sk}a_{rl}) + p\sum_{r<s}(pa_{ir}a_{is} \otimes a_{sk}a_{rl})\\
&= p\sum_{r=s}(a_{is}a_{ir} \otimes a_{sk}a_{rl}) + p\sum_{s<r}(a_{is}a_{ir} \otimes a_{sk}a_{rl}) + p\sum_{r<s}(a_{is}a_{ir} \otimes a_{sk}a_{rl})\\
&= p\sum_{r,s=1}^{n}(a_{is}a_{ir} \otimes a_{sk}a_{rl}) = p(\sum_{s=1}^{n} a_{is} \otimes a_{sk})(\sum_{r=1}^{n} a_{ir} \otimes a_{rl}) = p\Delta(a_{ik})\Delta(a_{il})\\
&= \Delta(pa_{ik}a_{il}).
\end{aligned}$$

Second, show that Δ preserves the relation (3.2), i.e.

$$\Delta(a_{jk}a_{ik}) = \Delta(qa_{ik}a_{jk}).$$

$$\begin{aligned}
&\Delta(a_{jk}a_{ik}) \\
&= \Delta(a_{jk})\Delta(a_{ik}) = (\sum_{r=1}^{n} a_{jr} \otimes a_{rk})(\sum_{s=1}^{n} a_{is} \otimes a_{sk}) = \sum_{r,s=1}^{n}(a_{jr}a_{is} \otimes a_{rk}a_{sk}) \\
&= \sum_{1=r<s\leq n}^{n} (a_{jr}a_{is} \otimes a_{rk}a_{sk}) + \sum_{1=r=s\leq n}^{n} (a_{jr}a_{is} \otimes a_{rk}a_{sk}) \\
&\quad + \sum_{1=s<r\leq n}^{n} (a_{jr}a_{is} \otimes a_{rk}a_{sk}) \\
&= \sum_{s<r}(a_{js}a_{ir} \otimes a_{sk}a_{rk}) + \sum_{r=s}(qa_{is}a_{jr} \otimes a_{sk}a_{rk}) \\
&\quad + \sum_{s<r}((a_{is}a_{jr} + (p - q^{-1})a_{js}a_{ir}) \otimes qa_{sk}a_{rk}) \\
&= \sum_{s<r}(a_{js}a_{ir} \otimes a_{sk}a_{rk}) + \sum_{r=s}(qa_{is}a_{jr} \otimes a_{sk}a_{rk}) \\
&\quad + \sum_{s<r}(a_{is}a_{jr} \otimes qa_{sk}a_{rk}) + \sum_{s<r}(pa_{js}a_{ir} \otimes qa_{sk}a_{rk}) - \sum_{s<r}(q^{-1}a_{js}a_{ir} \otimes qa_{sk}a_{rk}) \\
&= q\sum_{r=s}(a_{is}a_{jr} \otimes a_{sk}a_{rk}) + q\sum_{s<r}(a_{is}a_{jr} \otimes a_{sk}a_{rk}) + q\sum_{r<s}(pa_{jr}a_{is} \otimes a_{rk}a_{sk}) \\
&= q\sum_{r=s}(a_{is}a_{jr} \otimes a_{sk}a_{rk}) + q\sum_{s<r}(a_{is}a_{jr} \otimes a_{sk}a_{rk}) \\
&\quad + q\sum_{r<s}(pp^{-1}qa_{is}a_{jr} \otimes q^{-1}a_{sk}a_{rk}) \\
&= q\sum_{r,s=1}^{n}(a_{is}a_{jr} \otimes a_{sk}a_{rk}) = q(\sum_{s=1}^{n} a_{is} \otimes a_{sk})(\sum_{r=1}^{n} a_{jr} \otimes a_{rk}) = q\Delta(a_{ik})\Delta(a_{jk}) \\
&= \Delta(qa_{ik}a_{jk})
\end{aligned}$$

Third, show that Δ preserves the relation (3.3), i.e.

$$\Delta(a_{jk}a_{il}) = \Delta(p^{-1}qa_{il}a_{jk}).$$

$$\begin{aligned}
&\Delta(a_{jk}a_{il}) \\
&= \Delta(a_{jk})\Delta(a_{il}) = (\sum_{r=1}^{n} a_{jr} \otimes a_{rk})(\sum_{s=1}^{n} a_{is} \otimes a_{sl}) = \sum_{r,s=1}^{n}(a_{jr}a_{is} \otimes a_{rk}a_{sl})
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{1=r<s\leq n}^{n}(a_{jr}a_{is}\otimes a_{rk}a_{sl}) + \sum_{1=r=s\leq n}^{n}(a_{jr}a_{is}\otimes a_{rk}a_{sl}) + \sum_{1=s<r\leq n}^{n}(a_{jr}a_{is}\otimes a_{rk}a_{sl}) \\
&= \sum_{r<s}(p^{-1}qa_{is}a_{jr}\otimes(a_{sl}a_{rk}-(p-q^{-1})a_{sk}a_{rl})) + \sum_{r=s}(qa_{is}a_{jr}\otimes p^{-1}a_{sl}a_{rk}) \\
&\quad + \sum_{s<r}((a_{is}a_{jr}+(p-q^{-1})a_{js}a_{ir})\otimes p^{-1}qa_{sl}a_{rk}) \\
&= \sum_{r<s}(p^{-1}qa_{is}a_{jr}\otimes a_{sl}a_{rk}) + \sum_{r=s}(qa_{is}a_{jr}\otimes p^{-1}a_{sl}a_{rk}) + \sum_{s<r}(a_{is}a_{jr}\otimes p^{-1}qa_{sl}a_{rk}) \\
&\quad - \sum_{r<s}(p^{-1}qa_{is}a_{jr}\otimes(p-q^{-1})a_{sk}a_{rl}) + \sum_{s<r}((p-q^{-1})a_{js}a_{ir}\otimes p^{-1}qa_{sl}a_{rk}) \\
&= p^{-1}q(\sum_{r<s}(a_{is}a_{jr}\otimes a_{sl}a_{rk}) + \sum_{r=s}(a_{is}a_{jr}\otimes a_{sl}a_{rk}) + \sum_{s<r}(a_{is}a_{jr}\otimes a_{sl}a_{rk})) \\
&\quad + p^{-1}q(p-q^{-1})(-\sum_{r<s}(a_{is}a_{jr}\otimes a_{sk}a_{rl}) + \sum_{r<s}(a_{jr}a_{is}\otimes a_{rl}a_{sk})) \\
&= p^{-1}q\sum_{r,s=1}^{n}(a_{is}a_{jr}\otimes a_{sl}a_{rk}) \\
&\quad + p^{-1}q(p-q^{-1})(-\sum_{r<s}(a_{is}a_{jr}\otimes a_{sk}a_{rl}) + \sum_{r<s}(p^{-1}qa_{is}a_{jr}\otimes pq^{-1}a_{sk}a_{rl})) \\
&= p^{-1}q\sum_{r,s=1}^{n}(a_{is}a_{jr}\otimes a_{sl}a_{rk}) = p^{-1}q(\sum_{s=1}^{n}a_{is}\otimes a_{sl})(\sum_{r=1}^{n}a_{jr}\otimes a_{rk}) \\
&= p^{-1}q\Delta(a_{il})\Delta(a_{jk}) \\
&= \Delta(p^{-1}qa_{il}a_{jk})
\end{aligned}
$$

Last, show that Δ preserves the relation (3.4), i.e.

$$\Delta(a_{jl}a_{ik}-a_{ik}a_{jl}) = \Delta((p-q^{-1})a_{jk}a_{il}).$$

$$
\begin{aligned}
&\Delta(a_{jl}a_{ik}-a_{ik}a_{jl}) \\
&= \Delta(a_{jl})\Delta(a_{ik}) - \Delta(a_{ik})\Delta(a_{jl}) \\
&= (\sum_{r=1}^{n}a_{jr}\otimes a_{rl})(\sum_{s=1}^{n}a_{is}\otimes a_{sk}) - (\sum_{s=1}^{n}a_{is}\otimes a_{sk})(\sum_{r=1}^{n}a_{jr}\otimes a_{rl}) \\
&= \sum_{r,s=1}^{n}(a_{jr}a_{is}\otimes a_{rl}a_{sk}) - \sum_{r,s=1}^{n}(a_{is}a_{jr}\otimes a_{sk}a_{rl})
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{1=r<s\leq n}^{n} (a_{jr}a_{is} \otimes a_{rl}a_{sk}) + \sum_{1=r=s\leq n}^{n} (a_{jr}a_{is} \otimes a_{rl}a_{sk}) \\
&\quad + \sum_{1=s<r\leq n}^{n} (a_{jr}a_{is} \otimes a_{rl}a_{sk}) - \sum_{1=r<s\leq n}^{n} (a_{is}a_{jr} \otimes a_{sk}a_{rl}) \\
&\quad - \sum_{1=r=s\leq n}^{n} (a_{is}a_{jr} \otimes a_{sk}a_{rl}) - \sum_{1=s<r\leq n}^{n} (a_{is}a_{jr} \otimes a_{sk}a_{rl}) \\
&= \sum_{r<s}(a_{jr}a_{is} \otimes a_{rl}a_{sk}) + \sum_{r=s}(a_{jr}a_{is} \otimes a_{sl}a_{rk}) + \sum_{s<r}(a_{jr}a_{is} \otimes a_{rl}a_{sk}) \\
&\quad - \sum_{r<s}(pq^{-1}a_{jr}a_{is} \otimes p^{-1}qa_{rl}a_{sk}) - \sum_{r=s}(q^{-1}a_{jr}a_{is} \otimes a_{rk}a_{sl}) \\
&\quad - \sum_{s<r}((a_{jr}a_{is} - (p-q^{-1})a_{js}a_{ir}) \otimes (a_{rl}a_{sk} - (p-q^{-1})a_{rk}a_{sl})) \\
&= \sum_{r=s}(a_{jr}a_{is} \otimes pa_{rk}a_{sl}) - \sum_{r=s}(q^{-1}a_{jr}a_{is} \otimes a_{rk}a_{sl}) \\
&\quad + \sum_{s<r}(a_{jr}a_{is} \otimes (p-q^{-1})a_{rk}a_{sl}) + \sum_{r<s}((p-q^{-1})a_{jr}a_{is} \otimes a_{sl}a_{rk}) \\
&\quad - \sum_{r<s}((p-q^{-1})a_{jr}a_{is} \otimes (p-q^{-1})a_{sk}a_{rl}) \\
&= (p-q^{-1})\sum_{r=s}(a_{jr}a_{is} \otimes a_{rk}a_{sl}) + (p-q^{-1})\sum_{s<r}(a_{jr}a_{is} \otimes a_{rk}a_{sl}) \\
&\quad + (p-q^{-1})\sum_{r<s}(a_{jr}a_{is} \otimes a_{rk}a_{sl}) + \sum_{r<s}((p-q^{-1})a_{jr}a_{is} \otimes (p-q^{-1})a_{sk}a_{rl}) \\
&\quad - \sum_{r<s}((p-q^{-1})a_{jr}a_{is} \otimes (p-q^{-1})a_{sk}a_{rl}) \\
&= (p-q^{-1})\sum_{r,s=1}^{n}(a_{jr}a_{is} \otimes a_{rk}a_{sl}) = (p-q^{-1})(\sum_{r=1}^{n} a_{jr} \otimes a_{rk})(\sum_{s=1}^{n} a_{is} \otimes a_{sl}) \\
&= (p-q^{-1})\Delta(a_{jk})\Delta(a_{il}) = \Delta((p-q^{-1})a_{jk}a_{il})
\end{aligned}
$$

□

Lemma 3.1.3. *The algebra $M_{p,q}(n)$ is a bialgebra with the above coproduct and counit.*

Proof. It is enough to show $M_{p,q}(n)$ is a coalgebra. So we need to show the following maps are equal:

$$(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta, \tag{3.5}$$

$$(\varepsilon \otimes id)\Delta = (id \otimes \varepsilon)\Delta = id \tag{3.6}$$

It suffices to check these on generators. To show (3.5) apply the LHS map to $a_{ij} \in M_{p,q}(n)$.

$$\begin{aligned}(\Delta \otimes id)\Delta(a_{ij}) &= \sum_{k=1}^{n}(\Delta \otimes id)(a_{ik} \otimes a_{kj}) \\ &= \sum_{k=1}^{n}\sum_{l=1}^{n}(a_{il} \otimes a_{lk}) \otimes a_{kj} \\ &= \sum_{l=1}^{n}\sum_{k=1}^{n} a_{il} \otimes (a_{lk} \otimes a_{kj}) \\ &= \sum_{l=1}^{n}(id \otimes \Delta)(a_{il} \otimes a_{lj}) \\ &= (id \otimes \Delta)\Delta(a_{ij})\end{aligned}$$

To show (3.6) apply $(\varepsilon \otimes id)\Delta$ to $a_{ij} \in M_{p,q}(n)$ and $(id \otimes \varepsilon)\Delta$ to $a_{ij} \in M_{p,q}(n)$.

$$\begin{aligned}(\varepsilon \otimes id)\Delta(a_{ij}) &= \sum_{k=1}^{n}(\varepsilon \otimes id)(a_{ik} \otimes a_{kj}) \\ &= \delta_{ik} \otimes a_{kj} \\ &= 1 \otimes a_{ij} \\ (id \otimes \varepsilon)\Delta(a_{ij}) &= \sum_{k=1}^{n}(id \otimes \varepsilon)(a_{ik} \otimes a_{kj}) \\ &= a_{ik} \otimes \delta_{kj} \\ &= a_{ij} \otimes 1\end{aligned}$$

□

3.2 The Bialgebra Structure of $\mathbf{M_q(n)}$

Definition 3.2.1. *Let q be a nonzero element of a field K and $M_q(n) = K\{a_{ij}|i,j \in \{1,2,...,n\}\}/I$ be the quotient of the free algebra generated by the generators $\{a_{ij}|i,j \in \{1,2,...,n\}\}$ over K by the two-sided ideal I generated by the relations*

$$a_{il}a_{ik} = qa_{ik}a_{il}, \qquad a_{jk}a_{il} = a_{il}a_{jk},$$
$$a_{jk}a_{ik} = qa_{ik}a_{jk}, \qquad a_{jl}a_{ik} - a_{ik}a_{jl} = (q - q^{-1})a_{jk}a_{il}$$

whenever $j > i$ *and* $l > k$.

Define coproduct and counit on the generators as follows:

$$\Delta(a_{ij}) = \sum_{k=1}^{n} a_{ik} \otimes a_{kj} \qquad\qquad \varepsilon(a_{ij}) = \delta_{ij}$$

where δ_{ij} is the Kronecker delta and extend these maps to $M_q(n)$ as algebra maps.

Lemma 3.2.2. *The algebra* $M_q(n)$ *a bialgebra with the above coproduct and counit.*

Proof. In the Definition 3.1.1, if $p = q$ we get the bialgebra $M_q(n)$. □

3.3 Hopf Algebra Structure of $\mathbf{U_q gl(n)}$

Definition 3.3.1. *Let* $U_q gl(n)$ *be the algebra generated by* e_i, f_i, k_j, k_j^{-1}*, for* $i = 1, 2, ..., n-1$ *and* $j = 1, 2, ..., n$ *with the following relations:*

$$k_i k_j = k_j k_i, \tag{3.7}$$

$$k_i k_i^{-1} = k_i^{-1} k_i = 1, \tag{3.8}$$

$$k_i e_j k_i^{-1} = q^{\delta_{i,j} - \delta_{i,j+1}} e_j, \tag{3.9}$$

$$k_i f_j k_i^{-1} = q^{-\delta_{i,j} + \delta_{i,j+1}} f_j, \tag{3.10}$$

$$e_i f_j - f_j e_i = \delta_{i,j} \frac{k_i k_{i+1}^{-1} - k_i^{-1} k_{i+1}}{q - q^{-1}}, \tag{3.11}$$

$$e_i e_j = e_j e_i, f_i f_j = f_j f_i, \ if \ \ |i - j| \geq 2, \tag{3.12}$$

$$e_i^2 e_{i\pm1} + e_{i\pm1} e_i^2 = (q + q^{-1}) e_i e_{i\pm1} e_i, \tag{3.13}$$

$$f_i^2 f_{i\pm1} + f_{i\pm1} f_i^2 = (q + q^{-1}) f_i f_{i\pm1} f_i. \tag{3.14}$$

Define coproduct, counit and antipode on the generators as follows:

$$\begin{aligned}
\Delta(k_i^{\pm1}) &= k_i^{\pm1} \otimes k_i^{\pm1}, & \varepsilon(k_i^{\pm1}) &= 1, & S(k_i) &= k_i^{-1}, \\
\Delta(e_i) &= e_i \otimes k_i k_{i+1}^{-1} + 1 \otimes e_i, & \varepsilon(e_i) &= 0, & S(e_i) &= -e_i k_i^{-1} k_{i+1}, \\
\Delta(f_i) &= f_i \otimes 1 + k_i^{-1} k_{i+1} \otimes f_i, & \varepsilon(f_i) &= 0, & S(f_i) &= -k_i k_{i+1}^{-1} f_i.
\end{aligned}$$

and extend Δ and ε on $U_q gl(n)$ as algebra homomorphisms and S as an algebra antihomomorphism.

Lemma 3.3.2. *The maps Δ, ε and S are well-defined on $U_q gl(n)$.*

Proof. We need to show that the maps Δ, ε and S preserve the relations of Definition 3.3.1. Let us start with Δ and show that Δ preserves the relations (3.7)-(3.14):

$$\begin{aligned}\Delta(k_i k_j - k_j k_i) &= \Delta(k_i)\Delta(k_j) - \Delta(k_j)\Delta(k_i)\\ &= (k_i \otimes k_i)(k_j \otimes k_j) - (k_j \otimes k_j)(k_i \otimes k_i)\\ &= (k_i k_j \otimes k_i k_j) - (k_j k_i \otimes k_j k_i) = 0,\end{aligned}$$

$$\begin{aligned}\Delta(k_i k_i^{-1} - k_i^{-1} k_i) &= \Delta(k_i)\Delta(k_i^{-1}) - \Delta(k_i^{-1})\Delta(k_i)\\ &= (k_i \otimes k_i)(k_i^{-1} \otimes k_i^{-1}) - (k_i^{-1} \otimes k_i^{-1})(k_i \otimes k_i)\\ &= (k_i k_i^{-1} \otimes k_i k_i^{-1}) - (k_i^{-1} k_i \otimes k_i^{-1} k_i) = 0,\end{aligned}$$

$$\begin{aligned}\Delta(k_i e_j k_i^{-1}) &= \Delta(k_i)\Delta(e_j)\Delta(k_i^{-1})\\ &= (k_i \otimes k_i)(e_j \otimes k_j k_{j+1}^{-1} + 1 \otimes e_j)(k_i^{-1} \otimes k_i^{-1})\\ &= (k_i e_j k_i^{-1} \otimes k_i k_j k_{j+1}^{-1} k_i^{-1}) + (k_i k_i^{-1} \otimes k_i e_j k_i^{-1})\\ &= q^{\delta_{i,j} - \delta_{i,j+1}} e_j \otimes k_j k_{j+1}^{-1} + 1 \otimes q^{\delta_{i,j} - \delta_{i,j+1}} e_j\\ &= q^{\delta_{i,j} - \delta_{i,j+1}} (e_j \otimes k_j k_{j+1}^{-1} + 1 \otimes e_j)\\ &= q^{\delta_{i,j} - \delta_{i,j+1}} \Delta(e_j),\end{aligned}$$

$$\begin{aligned}\Delta(k_i f_j k_i^{-1}) &= \Delta(k_i)\Delta(f_j)\Delta(k_i^{-1})\\ &= (k_i \otimes k_i)(f_j \otimes 1 + k_j^{-1} k_{j+1} \otimes f_j)(k_i^{-1} \otimes k_i^{-1})\\ &= (k_i f_j k_i^{-1} \otimes k_i k_i^{-1}) + (k_i k_j^{-1} k_{j+1} k_i^{-1} \otimes k_i f_j k_i^{-1})\\ &= (q^{-\delta_{i,j} + \delta_{i,j+1}} f_j \otimes 1) + (k_j^{-1} k_{j+1} \otimes q^{-\delta_{i,j} + \delta_{i,j+1}} f_j)\\ &= q^{-\delta_{i,j} + \delta_{i,j+1}} (f_j \otimes 1 + k_j^{-1} k_{j+1} \otimes f_j)\\ &= q^{-\delta_{i,j} + \delta_{i,j+1}} \Delta(f_j),\end{aligned}$$

$$
\begin{aligned}
&\Delta(e_if_j - f_je_i)\\
&\quad = \Delta(e_i)\Delta(f_j) - \Delta(f_j)\Delta(e_i)\\
&\quad = (e_i \otimes k_ik_{i+1}^{-1} + 1 \otimes e_i)(f_j \otimes 1 + k_j^{-1}k_{j+1} \otimes f_j)\\
&\quad\ - (f_j \otimes 1 + k_j^{-1}k_{j+1} \otimes f_j)(e_i \otimes k_ik_{i+1}^{-1} + 1 \otimes e_i)\\
&\quad = e_if_j \otimes k_ik_{i+1}^{-1} + f_j \otimes e_i + e_ik_j^{-1}k_{j+1} \otimes k_ik_{i+1}^{-1}f_j + k_j^{-1}k_{j+1} \otimes e_if_j\\
&\quad\ - f_je_i \otimes k_ik_{i+1}^{-1} - k_j^{-1}k_{j+1}e_i \otimes f_jk_ik_{i+1}^{-1} - f_j \otimes e_i - k_j^{-1}k_{j+1} \otimes f_je_i\\
&\quad = (e_if_j - f_je_i) \otimes k_ik_{i+1}^{-1} + q^{\delta_{i,j}-\delta_{j,i+1}}q^{\delta_{i+1,j+1}-\delta_{i,j+1}}k_j^{-1}k_{j+1}e_i \otimes k_ik_{i+1}^{-1}f_j\\
&\quad\ + k_j^{-1}k_{j+1} \otimes (e_if_j - f_je_i) - q^{\delta_{i,j}-\delta_{i,j+1}}q^{-\delta_{i+1,j}+\delta_{i+1,j+1}}k_j^{-1}k_{j+1}e_i \otimes k_ik_{i+1}^{-1}f_j\\
&\quad = \delta_{i,j}\frac{k_ik_{i+1}^{-1} - k_i^{-1}k_{i+1}}{q-q^{-1}} \otimes k_ik_{i+1}^{-1} + k_j^{-1}k_{j+1} \otimes \delta_{i,j}\frac{k_ik_{i+1}^{-1} - k_i^{-1}k_{i+1}}{q-q^{-1}}\\
&\quad = \frac{\delta_{i,j}}{q-q^{-1}}(k_ik_{i+1}^{-1} \otimes k_ik_{i+1}^{-1} - k_i^{-1}k_{i+1} \otimes k_ik_{i+1}^{-1}\\
&\quad\ + k_j^{-1}k_{j+1} \otimes k_ik_{i+1}^{-1} - k_j^{-1}k_{j+1} \otimes k_i^{-1}k_{i+1})\\
&\quad = \frac{\delta_{i,j}}{q-q^{-1}}((k_i \otimes k_i)(k_{i+1}^{-1} \otimes k_{i+1}^{-1}) - (k_i^{-1} \otimes k_i^{-1})(k_{i+1} \otimes k_{i+1}))\\
&\quad = \frac{\delta_{i,j}}{q-q^{-1}}(\Delta(k_i)\Delta(k_{i+1}^{-1}) - \Delta(k_i^{-1})\Delta(k_{i+1}))\\
&\quad = \Delta(\delta_{i,j}\frac{k_ik_{i+1}^{-1} - k_i^{-1}k_{i+1}}{q-q^{-1}}).
\end{aligned}
$$

Assume $|i-j| \geq 2$.

$$
\begin{aligned}
\Delta(e_ie_j - e_je_i) &= \Delta(e_i)\Delta(e_j) - \Delta(e_j)\Delta(e_i)\\
&= (e_i \otimes k_ik_{i+1}^{-1} + 1 \otimes e_i)(e_j \otimes k_jk_{j+1}^{-1} + 1 \otimes e_j)\\
&\ - (e_j \otimes k_jk_{j+1}^{-1} + 1 \otimes e_j)(e_i \otimes k_ik_{i+1}^{-1} + 1 \otimes e_i)\\
&= e_ie_j \otimes k_ik_{i+1}^{-1}k_jk_{j+1}^{-1} + e_i \otimes k_ik_{i+1}^{-1}e_j + e_j \otimes e_ik_jk_{j+1}^{-1} + 1 \otimes e_ie_j\\
&\ - e_je_i \otimes k_jk_{j+1}^{-1}k_ik_{i+1}^{-1} - e_j \otimes k_jk_{j+1}^{-1}e_i - e_i \otimes e_jk_ik_{i+1}^{-1} - 1 \otimes e_je_i\\
&= (e_ie_j - e_je_i) \otimes k_ik_{i+1}^{-1}k_jk_{j+1}^{-1} + e_i \otimes k_ik_{i+1}^{-1}e_j + e_j \otimes k_jk_{j+1}^{-1}e_i\\
&\ + 1 \otimes (e_ie_j - e_je_i) - e_j \otimes k_jk_{j+1}^{-1}e_i - e_i \otimes k_ik_{i+1}^{-1}e_j\\
&= 0,
\end{aligned}
$$

$$
\begin{aligned}
&\Delta(f_if_j - f_jf_i)\\
&\quad = \Delta(f_i)\Delta(f_j) - \Delta(f_j)\Delta(f_i)\\
&\quad = (f_i \otimes 1 + k_i^{-1}k_{i+1} \otimes f_i)(f_j \otimes 1 + k_j^{-1}k_{j+1} \otimes f_j)\\
&\quad\ - (f_j \otimes 1 + k_j^{-1}k_{j+1} \otimes f_j)(f_i \otimes 1 + k_i^{-1}k_{i+1} \otimes f_i)\\
&\quad = f_if_j \otimes 1 + f_ik_j^{-1}k_{j+1} \otimes f_j + k_i^{-1}k_{i+1}f_j \otimes f_i + k_i^{-1}k_{i+1}k_j^{-1}k_{j+1} \otimes f_if_j\\
&\quad\ - f_jf_i \otimes 1 - f_jk_i^{-1}k_{i+1} \otimes f_i - k_j^{-1}k_{j+1}f_i \otimes f_j - k_j^{-1}k_{j+1}k_i^{-1}k_{i+1} \otimes f_jf_i\\
&\quad = (f_if_j - f_jf_i) \otimes 1 + k_j^{-1}k_{j+1}f_i \otimes f_j + k_i^{-1}k_{i+1}f_j \otimes f_i\\
&\quad\ + k_i^{-1}k_{i+1}k_j^{-1}k_{j+1} \otimes (f_if_j - f_jf_i) - k_i^{-1}k_{i+1}f_j \otimes f_i - k_j^{-1}k_{j+1}f_i \otimes f_j\\
&\quad = 0,
\end{aligned}
$$

$$
\begin{aligned}
&\Delta(e_i^2e_{i\pm1} + e_{i\pm1}e_i^2)\\
&\quad = \Delta(e_i)^2\Delta(e_{i\pm1}) + \Delta(e_{i\pm1})\Delta(e_i)^2\\
&\quad = (e_i \otimes k_ik_{i+1}^{-1} + 1 \otimes e_i)^2(e_{i\pm1} \otimes k_{i\pm1}k_{i+1\pm1}^{-1} + 1 \otimes e_{i\pm1})\\
&\quad\ + (e_{i\pm1} \otimes k_{i\pm1}k_{i+1\pm1}^{-1} + 1 \otimes e_{i\pm1})(e_i \otimes k_ik_{i+1}^{-1} + 1 \otimes e_i)^2\\
&\quad = e_i^2e_{i\pm1} \otimes k_i^2k_{i+1}^{-2}k_{i\pm1}k_{i+1\pm1}^{-1} + e_ie_{i\pm1} \otimes k_ik_{i+1}^{-1}e_ik_{i\pm1}k_{i+1\pm1}^{-1}\\
&\quad\ + e_ie_{i\pm1} \otimes e_ik_ik_{i+1}^{-1}k_{i\pm1}k_{i+1\pm1}^{-1} + e_{i\pm1} \otimes e_i^2k_{i\pm1}k_{i+1\pm1}^{-1}\\
&\quad\ + e_i^2 \otimes k_i^2k_{i+1}^{-2}e_{i\pm1} + e_i \otimes k_ik_{i+1}^{-1}e_ie_{i\pm1} + e_i \otimes e_ik_ik_{i+1}^{-1}e_{i\pm1}\\
&\quad\ + 1 \otimes e_i^2e_{i\pm1} + e_{i\pm1}e_i^2 \otimes k_{i\pm1}k_{i+1\pm1}^{-1}k_i^2k_{i+1}^{-2}\\
&\quad\ + e_{i\pm1}e_i \otimes k_{i\pm1}k_{i+1\pm1}^{-1}k_ik_{i+1}^{-1}e_i + e_{i\pm1}e_i \otimes k_{i\pm1}k_{i+1\pm1}^{-1}e_ik_ik_{i+1}^{-1}\\
&\quad\ + e_{i\pm1} \otimes k_{i\pm1}k_{i+1\pm1}^{-1}e_i^2 + e_i^2 \otimes e_{i\pm1}k_i^2k_{i+1}^{-2}\\
&\quad\ + e_i \otimes e_{i\pm1}k_ik_{i+1}^{-1}e_i + e_i \otimes e_{i\pm1}e_ik_ik_{i+1}^{-1} + 1 \otimes e_{i\pm1}e_i^2\\
&\quad = (e_i^2e_{i\pm1} + e_{i\pm1}e_i^2) \otimes k_ik_{i+1}^{-1}k_{i\pm1}k_{i+1\pm1}^{-1}k_ik_{i+1}^{-1} + 1 \otimes (e_i^2e_{i\pm1} + e_{i\pm1}e_i^2)\\
&\quad\ + (q+q^{-1})e_ie_{i\pm1} \otimes k_ik_{i+1}^{-1}k_{i\pm1}k_{i+1\pm1}^{-1}e_i + (q+q^{-1})e_{i\pm1} \otimes e_ik_{i\pm1}k_{i+1\pm1}^{-1}e_i\\
&\quad\ + (q+q^{-1})e_i^2 \otimes k_ik_{i+1}^{-1}e_{i\pm1}k_ik_{i+1}^{-1} + (q+q^{-1})e_i \otimes e_ie_{i\pm1}k_ik_{i+1}^{-1}\\
&\quad\ + (q+q^{-1})e_{i\pm1}e_i \otimes e_ik_{i\pm1}k_{i+1\pm1}^{-1}k_ik_{i+1}^{-1} + (q+q^{-1})e_i \otimes k_ik_{i+1}^{-1}e_{i\pm1}e_i\\
&\quad = (q+q^{-1})(e_i \otimes k_ik_{i+1}^{-1} + 1 \otimes e_i)(e_{i\pm1} \otimes k_{i\pm1}k_{i+1\pm1}^{-1}\\
&\quad\ + 1 \otimes e_{i\pm1})(e_i \otimes k_ik_{i+1}^{-1} + 1 \otimes e_i)\\
&\quad = \Delta((q+q^{-1})e_ie_{i\pm1}e_i),
\end{aligned}
$$

$$
\begin{aligned}
&\Delta(f_i^2 f_{i\pm1} + f_{i\pm1} f_i^2)\\
&\quad= \Delta(f_i)^2\Delta(f_{i\pm1}) + \Delta(f_{i\pm1})\Delta(f_i)^2\\
&\quad= (f_i \otimes 1 + k_i^{-1}k_{i+1} \otimes f_i)^2(f_{i\pm1} \otimes 1 + k_{i\pm1}^{-1}k_{i+1\pm1} \otimes f_{i\pm1})\\
&\qquad+ (f_{i\pm1} \otimes 1 + k_{i\pm1}^{-1}k_{i+1\pm1} \otimes f_{i\pm1})(f_i \otimes 1 + k_i^{-1}k_{i+1} \otimes f_i)^2\\
&\quad= f_i^2 f_{i\pm1} \otimes 1 + f_i^2 k_{i\pm1}^{-1}k_{i+1\pm1} \otimes f_{i\pm1}\\
&\qquad+ f_i k_i^{-1}k_{i+1} f_{i\pm1} \otimes f_i + f_i k_i^{-1}k_{i+1}k_{i\pm1}^{-1}k_{i+1\pm1} \otimes f_i f_{i\pm1}\\
&\qquad+ k_i^{-1}k_{i+1} f_i f_{i\pm1} \otimes f_i + k_i^{-1}k_{i+1} f_i k_{i\pm1}^{-1}k_{i+1\pm1} \otimes f_i f_{i\pm1}\\
&\qquad+ k_i^{-2}k_{i+1}^2 f_{i\pm1} \otimes f_i^2 + k_i^{-2}k_{i+1}^2 k_{i\pm1}^{-1}k_{i+1\pm1} \otimes f_i^2 f_{i\pm1} + f_{i\pm1} f_i^2 \otimes 1\\
&\qquad+ k_{i\pm1}^{-1}k_{i+1\pm1} f_i^2 \otimes f_{i\pm1} + f_{i\pm1} f_i k_i^{-1}k_{i+1} \otimes f_i\\
&\qquad+ k_{i\pm1}^{-1}k_{i+1\pm1} f_i k_i^{-1}k_{i+1} \otimes f_{i\pm1} f_i + f_{i\pm1} k_i^{-1}k_{i+1} f_i \otimes f_i\\
&\qquad+ k_{i\pm1}^{-1}k_{i+1\pm1} k_i^{-1}k_{i+1} f_i \otimes f_{i\pm1} f_i + f_{i\pm1} k_i^{-2}k_{i+1}^2 \otimes f_i^2\\
&\qquad+ k_{i\pm1}^{-1}k_{i+1\pm1} k_i^{-2}k_{i+1}^2 \otimes f_{i\pm1} f_i^2\\
&\quad= (f_i^2 f_{i\pm1} + f_{i\pm1} f_i^2) \otimes 1 + (q+q^{-1}) f_i k_{i\pm1}^{-1}k_{i+1\pm1} f_i \otimes f_{i\pm1}\\
&\qquad+ (q+q^{-1}) k_i^{-1}k_{i+1} f_{i\pm1} f_i \otimes f_i + (q+q^{-1}) k_i^{-1}k_{i+1}k_{i\pm1}^{-1}k_{i+1\pm1} f_i \otimes f_i f_{i\pm1}\\
&\qquad+ (q+q^{-1}) f_i f_{i\pm1} k_i^{-1}k_{i+1} \otimes f_i + (q+q^{-1}) f_i k_{i\pm1}^{-1}k_{i+1\pm1} k_i^{-1}k_{i+1} \otimes f_{i\pm1} f_i\\
&\qquad+ (q+q^{-1}) k_i^{-1}k_{i+1} f_{i\pm1} k_i^{-1}k_{i+1} \otimes f_i^2\\
&\qquad+ k_i^{-1}k_{i+1}k_{i\pm1}^{-1}k_{i+1\pm1}k_i^{-1}k_{i+1} \otimes (f_i^2 f_{i\pm1} + f_{i\pm1} f_i^2)\\
&\quad= (q+q^{-1})(f_i \otimes 1 + k_i^{-1}k_{i+1} \otimes f_i)(f_{i\pm1} \otimes 1\\
&\qquad+ k_{i\pm1}^{-1}k_{i+1\pm1} \otimes f_{i\pm1})(f_i \otimes 1 + k_i^{-1}k_{i+1} \otimes f_i)\\
&\quad= \Delta((q+q^{-1}) f_i f_{i\pm1} f_i).
\end{aligned}
$$

Next, let us show that ε preserves the relations (3.7)-(3.14):

$$
\begin{aligned}
\varepsilon(k_i k_j - k_j k_i) &= \varepsilon(k_i)\varepsilon(k_j) - \varepsilon(k_j)\varepsilon(k_i) = 0,\\
\varepsilon(k_i k_i^{-1} - k_i^{-1} k_i) &= \varepsilon(k_i)\varepsilon(k_i^{-1}) - \varepsilon(k_i^{-1})\varepsilon(k_i) = 0,\\
\varepsilon(k_i e_j k_i^{-1} - q^{\delta_{i,j}-\delta_{i,j+1}} e_j) &= \varepsilon(k_i)\varepsilon(e_j)\varepsilon(k_i^{-1}) - q^{\delta_{i,j}-\delta_{i,j+1}}\varepsilon(e_j) = 0,\\
\varepsilon(k_i f_j k_i^{-1} - q^{-\delta_{i,j}+\delta_{i,j+1}} f_j) &= \varepsilon(k_i)\varepsilon(f_j)\varepsilon(k_i^{-1}) - q^{-\delta_{i,j}+\delta_{i,j+1}}\varepsilon(f_j) = 0,\\
\varepsilon(e_i f_j - f_j e_i) &= \varepsilon(e_i)\varepsilon(f_j) - \varepsilon(f_j)\varepsilon(e_i) = 0,\\
\varepsilon(\delta_{i,j}\frac{k_i k_{i+1}^{-1} - k_i^{-1}k_{i+1}}{q-q^{-1}}) &= \frac{\delta_{i,j}}{q-q^{-1}}(\varepsilon(k_i)\varepsilon(k_{i+1}^{-1}) - \varepsilon(k_i^{-1})\varepsilon(k_{i+1})) = 0,
\end{aligned}
$$

$$\varepsilon(e_ie_j - e_je_i) = \varepsilon(e_i)\varepsilon(e_j) - \varepsilon(e_j)\varepsilon(e_i) = 0,$$
$$\varepsilon(f_if_j - f_jf_i) = \varepsilon(f_i)\varepsilon(f_j) - \varepsilon(f_j)\varepsilon(f_i) = 0,$$
$$\varepsilon(e_i^2e_{i\pm1} + e_{i\pm1}e_i^2) = \varepsilon(e_i)^2\varepsilon(e_{i\pm1}) + \varepsilon(e_{i\pm1})\varepsilon(e_i)^2 = 0,$$
$$\varepsilon((q+q^{-1})e_ie_{i\pm1}e_i) = (q+q^{-1})\varepsilon(e_i)\varepsilon(e_{i\pm1})\varepsilon(e_i) = 0,$$
$$\varepsilon(f_i^2f_{i\pm1} + f_{i\pm1}f_i^2) = \varepsilon(f_i)^2\varepsilon(f_{i\pm1}) + \varepsilon(f_{i\pm1})\varepsilon(f_i)^2 = 0,$$
$$\varepsilon((q+q^{-1})f_if_{i\pm1}f_i) = (q+q^{-1})\varepsilon(f_i)\varepsilon(f_{i\pm1})\varepsilon(f_i).$$

Last, let us show that S preserves the relations (3.7)-(3.14):

$$S(k_ik_j - k_jk_i) = S(k_j)S(k_i) - S(k_i)S(k_j) = k_j^{-1}k_i^{-1} - k_i^{-1}k_j^{-1} = 0$$
$$S(k_ik_i^{-1}) = S(k_i^{-1})S(k_i) = k_ik_i^{-1} = 1 = S(1)$$
$$S(k_i^{-1}k_i) = S(k_i)S(k_i^{-1}) = k_i^{-1}k_i = 1 = S(1)$$
$$S(k_ie_jk_i^{-1}) = S(k_i^{-1})S(e_j)S(k_i) = -k_ie_jk_i^{-1}k_j^{-1}k_{j+1} = -q^{\delta_{i,j}-\delta_{i,j+1}}e_jk_j^{-1}k_{j+1}$$
$$S(q^{\delta_{i,j}-\delta_{i,j+1}}e_j) = q^{\delta_{i,j}-\delta_{i,j+1}}S(e_j) = -q^{\delta_{i,j}-\delta_{i,j+1}}e_jk_j^{-1}k_{j+1}$$
$$S(k_if_jk_i^{-1}) = S(k_i^{-1})S(f_j)S(k_i) = -k_jk_{j+1}^{-1}k_if_jk_i^{-1} = -q^{-\delta_{i,j}+\delta_{i,j+1}}k_jk_{j+1}^{-1}f_j$$
$$S(q^{-\delta_{i,j}+\delta_{i,j+1}}f_j) = q^{-\delta_{i,j}+\delta_{i,j+1}}S(f_j) = -q^{-\delta_{i,j}+\delta_{i,j+1}}k_jk_{j+1}^{-1}f_j,$$

$$\begin{aligned}
S(e_if_j - f_je_i) &= S(f_j)S(e_i) - S(e_i)S(f_j)\\
&= k_jk_{j+1}^{-1}f_je_ik_i^{-1}k_{i+1} - e_ik_i^{-1}k_{i+1}k_jk_{j+1}^{-1}f_j\\
&= q^{2-\delta_{i,j}+\delta_{i,j+1}-\delta_{i+1,j+1}+\delta_{i+1,j}}(f_je_i - e_if_j)\\
&= -q^{2-\delta_{i,j}+\delta_{i,j+1}-\delta_{i+1,j+1}+\delta_{i+1,j}}\frac{\delta_{i,j}}{q-q^{-1}}(k_ik_{i+1}^{-1} - k_i^{-1}k_{i+1})\\
&= \frac{\delta_{i,j}}{q-q^{-1}}(k_i^{-1}k_{i+1} - k_ik_{i+1}^{-1})\\
S(\delta_{i,j}\frac{k_ik_{i+1}^{-1} - k_i^{-1}k_{i+1}}{q-q^{-1}}) &= \frac{\delta_{i,j}}{q-q^{-1}}(S(k_{i+1}^{-1})S(k_i) - S(k_{i+1})S(k_i^{-1}))\\
&= \frac{\delta_{i,j}}{q-q^{-1}}(k_{i+1}k_i^{-1} - k_{i+1}^{-1}k_i),
\end{aligned}$$

$$\begin{aligned}
S(e_ie_j - e_je_i) &= S(e_j)S(e_i) - S(e_i)S(e_j)\\
&= e_jk_j^{-1}k_{j+1}e_ik_i^{-1}k_{i+1} - e_ik_i^{-1}k_{i+1}e_jk_j^{-1}k_{j+1}\\
&= e_ie_jk_i^{-1}k_{i+1}k_j^{-1}k_{j+1} - e_ie_jk_i^{-1}k_{i+1}k_j^{-1}k_{j+1},\ if\ \ |i-j| \geq 2\\
&= 0,
\end{aligned}$$

$$
\begin{aligned}
S(f_i f_j - f_j f_i) &= S(f_j)S(f_i) - S(f_i)S(f_j) \\
&= k_j k_{j+1}^{-1} f_j k_i k_{i+1}^{-1} f_i - k_i k_{i+1}^{-1} f_i k_j k_{j+1}^{-1} f_j \\
&= f_i f_j k_i k_{i+1}^{-1} k_j k_{j+1}^{-1} - f_i f_j k_i k_{i+1}^{-1} k_j k_{j+1}^{-1},\ if\ \ |i-j| \geq 2 \\
&= 0,
\end{aligned}
$$

$$
\begin{aligned}
S(e_i^2 e_{i\pm1} + e_{i\pm1} e_i^2) &= S(e_{i\pm1})S(e_i)^2 + S(e_i^2)S(e_{i\pm1}) \\
&= -e_{i\pm1} k_{i\pm1}^{-1} k_{i+1\pm1} e_i k_i^{-1} k_{i+1} e_i k_i^{-1} k_{i+1} \\
&\quad - e_i k_i^{-1} k_{i+1} e_i k_i^{-1} k_{i+1} e_{i\pm1} k_{i\pm1}^{-1} k_{i+1\pm1} \\
&= -e_{i\pm1} e_i^2 k_i^{-2} k_{i+1}^2 k_{i\pm1}^{-1} k_{i+1\pm1} - e_i^2 e_{i\pm1} k_i^{-2} k_{i+1}^2 k_{i\pm1}^{-1} k_{i+1\pm1} \\
&= -(e_{i\pm1} e_i^2 + e_i^2 e_{i\pm1}) k_i^{-2} k_{i+1}^2 k_{i\pm1}^{-1} k_{i+1\pm1} \\
&= -(q+q^{-1}) e_i e_{i\pm1} e_i k_i^{-2} k_{i+1}^2 k_{i\pm1}^{-1} k_{i+1\pm1} \\
S((q+q^{-1}) e_i e_{i\pm1} e_i) &= (q+q^{-1}) S(e_i) S(e_{i\pm1}) S(e_i) \\
&= -(q+q^{-1}) e_i k_i^{-1} k_{i+1} e_{i\pm1} k_{i\pm1}^{-1} k_{i+1\pm1} e_i k_i^{-1} k_{i+1} \\
&= -(q+q^{-1}) e_i e_{i\pm1} e_i k_i^{-2} k_{i+1}^2 k_{i\pm1}^{-1} k_{i+1\pm1},
\end{aligned}
$$

$$
\begin{aligned}
S(f_i^2 f_{i\pm1} + f_{i\pm1} f_i^2) &= S(f_{i\pm1})S(f_i)^2 + S(f_i)^2 S(f_{i\pm1}) \\
&= -k_{i\pm1} k_{i+1\pm1}^{-1} f_{i\pm1} k_i k_{i+1}^{-1} f_i k_i k_{i+1}^{-1} f_i \\
&\quad - k_i k_{i+1}^{-1} f_i k_i k_{i+1}^{-1} f_i k_{i\pm1} k_{i+1\pm1}^{-1} f_{i\pm1} \\
&= -k_{i\pm1} k_{i+1\pm1}^{-1} k_i^2 k_{i+1}^{-2} f_{i\pm1} f_i^2 - k_{i\pm1} k_{i+1\pm1}^{-1} k_i^2 k_{i+1}^{-2} f_i^2 f_{i\pm1} \\
&= -k_{i\pm1} k_{i+1\pm1}^{-1} k_i^2 k_{i+1}^{-2} (f_{i\pm1} f_i^2 + f_i^2 f_{i\pm1}) \\
&= -(q+q^{-1}) k_{i\pm1} k_{i+1\pm1}^{-1} k_i^2 k_{i+1}^{-2} f_i f_{i\pm1} f_i \\
S((q+q^{-1}) f_i f_{i\pm1} f_i) &= (q+q^{-1}) S(f_i) S(f_{i\pm1}) S(f_i) \\
&= (q+q^{-1}) k_i k_{i+1}^{-1} f_i k_{i\pm1} k_{i+1\pm1}^{-1} f_{i\pm1} k_i k_{i+1}^{-1} f_i \\
&= (q+q^{-1}) k_{i\pm1} k_{i+1\pm1}^{-1} k_i^2 k_{i+1}^{-2} f_i f_{i\pm1} f_i.
\end{aligned}
$$

□

Lemma 3.3.3. *$U_q gl(n)$ is a Hopf algebra with the above coproduct, counit and antipode.*

Proof. It is enough to show $U_q gl(n)$ is a coalgebra, and S is an antipode. So we need to show the following maps are equal:

$$(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta, \tag{3.15}$$

$$(\varepsilon \otimes id)\Delta = (id \otimes \varepsilon)\Delta = id \tag{3.16}$$

$$\mu(S \otimes id_H)\Delta = \mu(id_H \otimes S)\Delta = \eta\varepsilon \tag{3.17}$$

To show the equation (3.15) apply the LHS map to each generator e_i, f_i, k_j of $U_q gl(n)$:

$$\begin{aligned}
(\Delta \otimes id)\Delta(e_i) &= (\Delta \otimes id)(e_i \otimes k_i k_{i+1}^{-1} + 1 \otimes e_i)\\
&= e_i \otimes k_i k_{i+1}^{-1} \otimes k_i k_{i+1}^{-1} + 1 \otimes e_i \otimes k_i k_{i+1}^{-1} + 1 \otimes 1 \otimes e_i\\
&= (id \otimes \Delta)(e_i \otimes k_i k_{i+1}^{-1} + 1 \otimes e_i)\\
&= (id \otimes \Delta)\Delta(e_i)\\
(\Delta \otimes id)\Delta(f_i) &= (\Delta \otimes id)(f_i \otimes 1 + k_i^{-1} k_{i+1} \otimes f_i)\\
&= f_i \otimes 1 \otimes 1 + k_i^{-1} k_{i+1} \otimes f_i \otimes 1 + k_i^{-1} k_{i+1} \otimes k_i^{-1} k_{i+1} \otimes f_i\\
&= (id \otimes \Delta)(f_i \otimes 1 + k_i^{-1} k_{i+1} \otimes f_i)\\
&= (id \otimes \Delta)\Delta(f_i)\\
(\Delta \otimes id)\Delta(k_i) &= (\Delta \otimes id)(k_i \otimes k_i)\\
&= k_i \otimes k_i \otimes k_i\\
&= (id \otimes \Delta)\Delta(k_i)
\end{aligned}$$

To check the equation (3.16) is satisfied, apply $(\varepsilon \otimes id)\Delta$ and $(id \otimes \varepsilon)\Delta$ to each generator e_i, f_i, k_j of $U_q gl(n)$:

$$\begin{aligned}
(\varepsilon \otimes id)\Delta(e_i) &= (\varepsilon \otimes id)(e_i \otimes k_i k_{i+1}^{-1} + 1 \otimes e_i)\\
&= 0 \otimes k_i k_{i+1}^{-1} + 1 \otimes e_i\\
&= 1 \otimes e_i\\
(id \otimes \varepsilon)\Delta(e_i) &= (id \otimes \varepsilon)(e_i \otimes k_i k_{i+1}^{-1} + 1 \otimes e_i)\\
&= e_i \otimes 1 + 1 \otimes 0\\
&= e_i \otimes 1
\end{aligned}$$

$$
\begin{aligned}
(\varepsilon \otimes id)\Delta(f_i) &= (\varepsilon \otimes id)(f_i \otimes 1 + k_i^{-1}k_{i+1} \otimes f_i) \\
&= 0 \otimes 1 + 1 \otimes f_i \\
&= 1 \otimes f_i \\
(id \otimes \varepsilon)\Delta(f_i) &= (id \otimes \varepsilon)(f_i \otimes 1 + k_i^{-1}k_{i+1} \otimes f_i) \\
&= f_i \otimes 1 + k_i^{-1}k_{i+1} \otimes 0 \\
&= f_i \otimes 1
\end{aligned}
$$

$$
\begin{aligned}
(\varepsilon \otimes id)\Delta(k_i) &= (\varepsilon \otimes id)(k_i \otimes k_i) \\
&= 1 \otimes k_i \\
(id \otimes \varepsilon)\Delta(k_i) &= (id \otimes \varepsilon)(k_i \otimes k_i) \\
&= k_i \otimes 1
\end{aligned}
$$

Demonstrate the equation (3.17) by applying $\mu(S \otimes id_H)\Delta$, $\mu(id_H \otimes S)\Delta$ and $\eta\varepsilon$ on each generator e_i, f_i, k_j of $U_q gl(n)$:

$$
\begin{aligned}
\mu(S \otimes id_H)\Delta(e_i) &= \mu(S \otimes id_H)(e_i \otimes k_i k_{i+1}^{-1} + 1 \otimes e_i) \\
&= \mu(-e_i k_i^{-1}k_{i+1} \otimes k_i k_{i+1}^{-1} + 1 \otimes e_i) \\
&= -e_i + e_i = 0, \\
\mu(id_H \otimes S)\Delta(e_i) &= \mu(id_H \otimes S)(e_i \otimes k_i k_{i+1}^{-1} + 1 \otimes e_i) \\
&= \mu(e_i \otimes k_{i+1}k_i^{-1} + 1 \otimes -e_i k_i^{-1}k_{i+1}) \\
&= e_i k_{i+1}k_i^{-1} - e_i k_{i+1}k_i^{-1} = 0 \\
\eta\varepsilon(e_i) &= \eta(0) = 0,
\end{aligned}
$$

$$
\begin{aligned}
\mu(S \otimes id_H)\Delta(f_i) &= \mu(S \otimes id_H)(f_i \otimes 1 + k_i^{-1}k_{i+1} \otimes f_i) \\
&= \mu(-k_i k_{i+1}^{-1} f_i \otimes 1 + k_{i+1}^{-1}k_i \otimes f_i) \\
&= -k_i k_{i+1}^{-1} f_i + k_i k_{i+1}^{-1} f_i = 0 \\
\mu(id_H \otimes S)\Delta(f_i) &= \mu(id_H \otimes S)(f_i \otimes 1 + k_i^{-1}k_{i+1} \otimes f_i) \\
&= \mu(f_i \otimes 1 + k_i^{-1}k_{i+1} \otimes -k_i k_{i+1}^{-1} f_i) \\
&= f_i - f_i = 0 \\
\eta\varepsilon(f_i) &= \eta(0) = 0
\end{aligned}
$$

$$
\begin{aligned}
\mu(S \otimes id_H)\Delta(k_i) &= \mu(S \otimes id_H)(k_i \otimes k_i) \\
&= \mu(k_i^{-1} \otimes k_i) = 1 \\
\mu(id_H \otimes S)\Delta(k_i) &= \mu(id_H \otimes S)(k_i \otimes k_i) \\
&= \mu(k_i \otimes k_i^{-1}) = 1 \\
\eta\varepsilon(k_i) &= \eta(1) = 1
\end{aligned}
$$

□

CHAPTER 4

A POINCARÉ-BIRKHOFF-WITT THEOREM FOR $\mathbf{U_q gl(n)}$

In this chapter, we give a Poincaré-Birkhoff-Witt (P.B.W.) basis for the Hopf algebra $U_q gl(n)$ (see Definition 3.3.1). We follow the methods in [25].

4.1 P.B.W. Basis of $\mathbf{U_q gl(n)}$

Let $\alpha_1, \alpha_2, ..., \alpha_{n-1}$ be the fundamental roots and Φ^+ be the set of positive roots of $gl(n)$. Let $i \leq j$. Each positive root α can be written as $\alpha = \alpha(i, j+1) = \alpha_i + \alpha_{i+1} + ... + \alpha_j$. Let $\gamma = \alpha - \alpha_j$ and $\beta = \alpha - \alpha_i$ where $i \neq j$. Then define by induction

$$e_\alpha = \begin{cases} e_\gamma e_j - q e_j e_\gamma & \text{if } i \neq j, \\ e_i & \text{if } i = j. \end{cases}$$

$$f_\alpha = \begin{cases} f_i f_\beta - q^{-1} f_\beta f_i & \text{if } i \neq j, \\ f_i & \text{if } i = j. \end{cases}$$

We order the elements as follows:

$$e_{\alpha(i,j)} < e_{\alpha(k,l)} \text{if } i > k \text{ or } (i = k \text{ and } j > l),$$
$$f_{\alpha(i,j)} < f_{\alpha(k,l)} \text{if } i < k \text{ or } (i = k \text{ and } j < l).$$

Example 4.1.1. *For $n = 4$ the positive roots are*

$$\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \alpha_2, \alpha_2 + \alpha_3, \alpha_3.$$

The ordering of the elements e_α is

$$\alpha(3, 4) < \alpha(2, 4) < \alpha(2, 3) < \alpha(1, 4) < \alpha(1, 3) < \alpha(1, 2).$$

The ordering of the elements f_α is

$$\alpha(1,2) < \alpha(1,3) < \alpha(1,4) < \alpha(2,3) < \alpha(2,4) < \alpha(3,4).$$

Define adjoint representations $ad : U_q gl(n) \to End(U_q gl(n))$ and $ad' : U_q gl(n) \to End(U_q gl(n))$ by

$$\begin{aligned} ad &= (L \otimes R)(S \otimes id)\Delta \text{ and} \\ ad' &= (L \otimes R)(id \otimes S)\Delta. \end{aligned}$$

where L and R are the left and right multiplication, respectively.

In particular, we have

$$\begin{aligned} ad(e_i)(x) &= xe_i - e_i k_i^{-1} k_{i+1} x k_{i+1}^{-1} k_i, \\ ad'(f_i)(x) &= f_i x - k_i^{-1} k_{i+1} x k_{i+1}^{-1} k_i f_i. \end{aligned}$$

Using ad and ad' we can redefine e_α and f_α by

$$\begin{aligned} e_\alpha &= e_\gamma e_j - q e_j e_\gamma \\ &= e_\gamma e_j - e_j k_j^{-1} k_{j+1} e_\gamma k_{j+1}^{-1} k_j \\ &= ad(e_j)(e_\gamma), \\ f_\alpha &= f_i f_\beta - q^{-1} f_\beta f_i \\ &= f_i f_\beta - k_i^{-1} k_{i+1} f_\beta k_{i+1}^{-1} k_i f_i \\ &= ad'(f_i)(f_\beta), \end{aligned}$$

where $i < j$, and α, β, γ are as above.

4.2 Commutation of e_α and e_β

Lemma 4.2.1. *The map ad is an algebra antihomomorphism and the map ad' is an algebra homomorphism.*

Proof. Let $x, y, z \in U_q gl(n)$. We verify the first assertion.

The map ad is linear:

$$\begin{aligned} ad(x+y)(z) &= (L \otimes R)(S \otimes id)\Delta(x+y)(z) \\ &= (L \otimes R)(S(x') \otimes x'' + S(y') \otimes y'')(z) \\ &= S(x')zx'' + S(y')zy'' \\ &= ad(x)(z) + ad(y)(z). \end{aligned}$$

The element $ad(1)$ acts as identity on $U_q gl(n)$:

$$\begin{aligned} ad(1)(z) &= (L \otimes R)(S \otimes id)\Delta(1)(z) \\ &= (L \otimes R)(S(1) \otimes 1)(z) \\ &= z. \end{aligned}$$

The map ad reverses the order of product:

$$\begin{aligned} ad(xy)(z) &= (L \otimes R)(S \otimes id)\Delta(xy)(z) \\ &= S(x'y')zx''y'' \\ &= S(y')S(x')zx''y'' \\ &= ad(y)(ad(x)(z)). \end{aligned}$$

The map ad' is linear:

$$\begin{aligned} ad'(x+y)(z) &= (L \otimes R)(id \otimes S)\Delta(x+y)(z) \\ &= (L \otimes R)(x' \otimes S(x'') + y' \otimes S(y''))(z) \\ &= x'zS(x'') + y'zS(y'') \\ &= ad'(x)(z) + ad'(y)(z). \end{aligned}$$

The element $ad'(1)$ acts as identity on $U_q gl(n)$:

$$\begin{aligned} ad'(1)(z) &= (L \otimes R)(id \otimes S)\Delta(1)(z) \\ &= (L \otimes R)(1 \otimes S(1))(z) \\ &= z. \end{aligned}$$

The map ad' preserves the product:

$$\begin{aligned} ad'(xy)(z) &= (L\otimes R)(id\otimes S)\Delta(xy)(z) \\ &= x'y'zS(x''y'') \\ &= x'y'zS(y'')S(x'') \\ &= ad'(x)(ad'(y)(z)). \end{aligned}$$

This proves that the map ad is an algebra antihomomorphism and the map ad' is an algebra homomorphism. $\square$

Corollary 4.2.2. *We have the following properties:*

$$(ad(e_i))^2 ad(e_{i\pm1}) + ad(e_{i\pm1})(ad(e_i))^2 = (q+q^{-1})ad(e_i)ad(e_{i\pm1})ad(e_i),$$
$$(ad'(f_i))^2 ad'(f_{i\pm1}) + ad'(f_{i\pm1})(ad'(f_i))^2 = (q+q^{-1})ad'(f_i)ad'(f_{i\pm1})ad'(f_i).$$

Lemma 4.2.3. *Suppose $x, y \in U_q gl(n)$ with the following commutation relations:*

$$\begin{aligned} k_i^{-1}k_{i+1}xk_{i+1}^{-1}k_i &= q^a x \\ k_i^{-1}k_{i+1}yk_{i+1}^{-1}k_i &= q^b y \end{aligned}$$

for some $a, b \in \mathbb{Z}$. Then

$$\begin{aligned} ad(e_i)(xy) &= q^b ad(e_i)(x)y + xad(e_i)(y), \\ ad'(f_i)(xy) &= ad'(f_i)(x)y + q^a xad'(f_i)(y). \end{aligned}$$

Proof. Prove the first relation:

$$\begin{aligned} &q^b ad(e_i)(x)y + xad(e_i)(y) \\ &\qquad = q^b xe_iy - q^b e_ik_i^{-1}k_{i+1}xk_{i+1}^{-1}k_iy + xye_i - xe_ik_i^{-1}k_{i+1}yk_{i+1}^{-1}k_i \\ &\qquad = q^b xe_iy - q^a q^b e_ixy + xye_i - q^b xe_iy \\ &\qquad = -q^a q^b e_ixy + xye_i \\ &\qquad = xye_i - e_ik_i^{-1}k_{i+1}xyk_{i+1}^{-1}k_i \\ &\qquad = ad(e_i)(xy). \end{aligned}$$

Prove the second relation:

$$\begin{aligned}
&ad'(f_i)(x)y + q^a x ad'(f_i)(y)\\
&\qquad\qquad = f_i xy - k_i^{-1}k_{i+1}xk_{i+1}^{-1}k_i f_i y + q^a x f_i y - q^a x k_i^{-1}k_{i+1}yk_{i+1}^{-1}k_i f_i\\
&\qquad\qquad = f_i xy - q^a x f_i y + q^a x f_i y - q^a q^b xy f_i\\
&\qquad\qquad = f_i xy - q^a q^b xy f_i\\
&\qquad\qquad = f_i xy - k_i^{-1}k_{i+1}xyk_{i+1}^{-1}k_i f_i\\
&\qquad\qquad = ad'(f_i)(xy).
\end{aligned}$$

□

Lemma 4.2.4. *Let* $\alpha = \alpha_i + ... + \alpha_j$ *and* $s \in \{1, 2, ..., i-2, i+1, i+2, ..., j-1, j+2, ..., n-1\}$. *Then*

1. $ad(e_s)(e_\alpha) = e_\alpha e_s - e_s e_\alpha = 0$,
2. $ad(e_j)(e_\alpha) = e_\alpha e_j - q^{-1}e_j e_\alpha = 0$,
3. $e_i e_\alpha - q^{-1}e_\alpha e_i = 0$.

Proof. First, let us prove the first assertion. We have $k_s^{-1}k_{s+1}e_\alpha k_{s+1}^{-1}k_s = e_\alpha$, so

$$ad(e_s)(e_\alpha) = e_\alpha e_s - e_s k_s^{-1}k_{s+1}e_\alpha k_{s+1}^{-1}k_s = e_\alpha e_s - e_s e_\alpha$$

and

$$\begin{aligned}
&ad(e_s)(e_\alpha)\\
&\quad = ad(e_s)ad(e_j)ad(e_{j-1})...ad(e_{s+2})ad(e_{s+1})ad(e_s)(e_{\alpha_i+...+\alpha_{s-1}})\\
&\quad = ad(e_j)ad(e_{j-1})...ad(e_{s+2})ad(e_s)ad(e_{s+1})ad(e_s)(e_{\alpha_i+...+\alpha_{s-1}})\\
&\quad = (q+q^{-1})^{-1}ad(e_j)ad(e_{j-1})...ad(e_{s+2})((ad(e_s))^2 ad(e_{s+1})\\
&\qquad + ad(e_{s+1})(ad(e_s))^2)(e_{\alpha_i+...+\alpha_{s-1}})\\
&\quad = (q+q^{-1})^{-1}ad(e_j)ad(e_{j-1})...ad(e_{s+2})ad(e_{s+1})(ad(e_s))^2(e_{\alpha_i+...+\alpha_{s-1}})\\
&\quad = (q+q^{-1})^{-1}ad(e_j)ad(e_{j-1})...ad(e_{s+1})(ad(e_s))^2 ad(e_{s-1})(e_{\alpha_i+...+\alpha_{s-2}})\\
&\quad = (q+q^{-1})^{-1}ad(e_j)ad(e_{j-1})...ad(e_{s+1})((q+q^{-1})ad(e_s)ad(e_{s-1})ad(e_s)\\
&\qquad - ad(e_{s-1})(ad(e_s))^2)(e_{\alpha_i+...+\alpha_{s-2}}) = 0.
\end{aligned}$$

The second assertion holds, since:

$$\begin{aligned}
ad(e_j)(e_\alpha) &= e_\alpha e_j - e_j k_j^{-1} k_{j+1} e_\alpha k_{j+1}^{-1} k_j \\
&= e_\alpha e_j - q^{-1} e_j e_\alpha \\
ad(e_j)(e_\alpha) &= ad(e_j) ad(e_j) ad(e_{j-1})(e_{\alpha_i + \ldots + \alpha_{j-2}}) \\
&= ((q+q^{-1}) ad(e_j) ad(e_{j-1}) ad(e_j) - ad(e_{j-1})(ad(e_j))^2)(e_{\alpha_i + \ldots + \alpha_{j-2}}) \\
&= 0.
\end{aligned}$$

The third assertion is true, since:

$$\begin{aligned}
e_i e_\alpha - q^{-1} e_\alpha e_i &= ad(e_j) ad(e_{j-1}) \ldots ad(e_{i+2})(e_i e_{\alpha_i + \alpha_{i+1}} - q^{-1} e_{\alpha_i + \alpha_{i+1}} e_i) \\
&= ad(e_j) ad(e_{j-1}) \ldots ad(e_{i+2})(e_i^2 e_{i+1} - (q+q^{-1}) e_i e_{i+1} e_i + e_{i+1} e_i^2) \\
&= 0.
\end{aligned}$$

□

Lemma 4.2.5. *Let* $\alpha = \alpha_i + \ldots + \alpha_j$ *and* $s \in \{1, 2, \ldots, i-2, i+1, i+2, \ldots, j-1, j+2, \ldots, n-1\}$*. Then*

1. $ad'(f_s)(f_\alpha) = f_s f_\alpha - f_\alpha f_s = 0$,

2. $ad'(f_i)(f_\alpha) = f_i f_\alpha - q f_\alpha f_i = 0$,

3. $f_\alpha f_j - q f_j f_\alpha = 0$.

Proof. Let us prove the first assertion. We have $k_s^{-1} k_{s+1} f_\alpha k_{s+1}^{-1} k_s = f_\alpha$, so

$$\begin{aligned}
ad'(f_s)(f_\alpha) &= f_s f_\alpha - k_s^{-1} k_{s+1} f_\alpha k_{s+1}^{-1} k_s f_s \\
&= f_s f_\alpha - f_\alpha f_s,
\end{aligned}$$

and

$$
\begin{aligned}
&ad'(f_s)(f_\alpha)\\
&\quad = ad'(f_s)ad'(f_i)ad'(f_{i+1})...ad'(f_{s-2})ad'(f_{s-1})ad'(f_s)(f_{\alpha_{s+1}+...+\alpha_j})\\
&\quad = ad'(f_i)ad'(f_{i+1})...ad'(f_{s-2})ad'(f_s)ad'(f_{s-1})ad'(f_s)(f_{\alpha_{s+1}+...+\alpha_j})\\
&\quad = (q+q^{-1})^{-1}ad'(f_i)ad'(f_{i+1})...ad'(f_{s-2})((ad'(f_s))^2ad'(f_{s-1})\\
&\qquad + ad'(f_{s-1})(ad'(f_s))^2)(f_{\alpha_{s+1}+...+\alpha_j})\\
&\quad = (q+q^{-1})^{-1}ad'(f_i)ad'(f_{i+1})...ad'(f_{s-2})(ad'(f_{s-1})(ad'(f_s))^2)(f_{\alpha_{s+1}+...+\alpha_j})\\
&\quad = (q+q^{-1})^{-1}ad'(f_i)ad'(f_{i+1})...ad'(f_{s-1})(ad'(f_s))^2ad'(f_{\alpha_{s+1}})(f_{\alpha_{s+2}+...+\alpha_j})\\
&\quad = (q+q^{-1})^{-1}ad'(f_i)ad'(f_{i+1})...ad'(f_{s-1})((q+q^{-1})ad'(f_s)ad'(f_{s+1})ad'(f_s)\\
&\qquad - ad'(f_{\alpha_{s+1}})(ad'(f_s))^2)(f_{\alpha_{s+2}+...+\alpha_j})\\
&\quad = 0.
\end{aligned}
$$

The second assertion holds, since:

$$
\begin{aligned}
ad'(f_i)(f_\alpha) &= f_if_\alpha - k_i^{-1}k_{i+1}f_\alpha k_{i+1}^{-1}k_if_i\\
&= f_if_\alpha - qf_\alpha f_i\\
ad'(f_i)(f_\alpha) &= ad'(f_i)ad'(f_i)ad'(f_{i+1})(f_{\alpha_{i+2}+...+\alpha_j})\\
&= ((q+q^{-1})ad'(f_i)ad'(f_{i+1})ad'(f_i) - ad'(f_{i+1})ad'(f_i))^2)(f_{\alpha_{i+2}+...+\alpha_j})\\
&= 0.
\end{aligned}
$$

The third assertion holds, since:

$$
\begin{aligned}
f_\alpha f_j - qf_jf_\alpha &= ad'(f_i)ad'(f_{i+1})...ad'(f_{j-2})(f_{\alpha_{j-1}+\alpha_j}f_j - qf_jf_{\alpha_{j-1}+\alpha_j})\\
&= ad'(f_i)ad'(f_{i+1})...ad'(f_{j-2})(f_i^2f_{i-1} - (q+q^{-1})f_if_{i-1}f_i + f_{i-1}f_i^2)\\
&= 0.
\end{aligned}
$$

□

Lemma 4.2.6. *Let $\alpha = \alpha_i + ... + \alpha_j$ and $\gamma = \alpha - \alpha_j$ and $\beta = \alpha - \alpha_i$. Then, we have*

$$
e_\alpha = \begin{cases} e_ie_\beta - qe_\beta e_i & \text{if } i \neq j,\\ e_i & \text{if } i = j.\end{cases}
$$

$$
f_\alpha = \begin{cases} f_\gamma f_j - q^{-1}f_jf_\gamma & \text{if } i \neq j,\\ f_i & \text{if } i = j.\end{cases}
$$

Proof. We prove by induction on the length $l = j - i$ of the root α.

If $l = 1$ then $\alpha = \alpha_i + \alpha_{i+1}$, $\beta = \alpha_{i+1}$ and $\gamma = \alpha_i$, so

$$\begin{aligned} e_\alpha &= e_i e_{i+1} - q e_{i+1} e_i \\ &= e_i e_\beta - q e_\beta e_i, \end{aligned}$$

and

$$\begin{aligned} f_\alpha &= f_i f_{i+1} - q^{-1} f_{i+1} f_i \\ &= f_\gamma f_{i+1} - q^{-1} f_{i+1} f_\gamma. \end{aligned}$$

Assume the assertion is true for $j - i = l - 1$. For $j - i = l \geq 2$, $\alpha = \alpha_i + ... + \alpha_{i+l}$ and $\beta = \alpha_{i+1} + ... + \alpha_{i+l}$. Let $\gamma = \alpha_i + ... + \alpha_{i+l-1}$ and $\alpha' = \alpha_{i+1} + ... + \alpha_{i+l-1}$. Then we have:

$$\begin{aligned} e_\alpha &= e_\gamma e_{i+l} - q e_{i+l} e_\gamma \\ &= e_i e_{\alpha'} e_{i+l} - q e_{\alpha'} e_i e_{i+l} - q e_{i+l} e_i e_{\alpha'} + q^2 e_{i+l} e_{\alpha'} e_i \\ &= e_i e_{\alpha'} e_{i+l} - q e_{\alpha'} e_{i+l} e_i - q e_i e_{i+l} e_{\alpha'} + q^2 e_{i+l} e_{\alpha'} e_i \\ &= e_i (e_{\alpha'} e_{i+l} - q e_{i+l} e_{\alpha'}) - q (e_{\alpha'} e_{i+l} - q e_{i+l} e_{\alpha'}) e_i \\ &= e_i e_\beta - q e_\beta e_i \end{aligned}$$

and

$$\begin{aligned} f_\alpha &= f_i f_\beta - q^{-1} f_\beta f_i \\ &= f_i f_{\alpha'} f_{i+l} - q^{-1} f_i f_{i+l} f_{\alpha'} - q^{-1} f_{\alpha'} f_{i+l} f_i + q^{-2} f_{i+l} f_{\alpha'} f_i \\ &= f_i f_{\alpha'} f_{i+l} - q^{-1} f_{i+l} f_i f_{\alpha'} - q^{-1} f_{\alpha'} f_i f_{i+l} + q^{-2} f_{i+l} f_{\alpha'} f_i \\ &= (f_i f_{\alpha'} - q^{-1} f_{\alpha'} f_i) f_{i+l} - q^{-1} f_{i+l} (f_i f_{\alpha'} - q^{-1} f_{\alpha'} f_i) \\ &= f_\gamma f_{i+l} - q^{-1} f_{i+l} f_\gamma \end{aligned}$$

where the second equalities of both equations hold by the induction hypothesis. $\square$

Proposition 4.2.7. *Let $\alpha = \alpha_i + ... + \alpha_j$, $\beta = \alpha_p + ... + \alpha_r$ and $i \leq p$. Then,*

$$e_\alpha e_\beta = \begin{cases} e_\beta e_\alpha & \textit{if} \;\; p \geq j+2 \\ q e_\beta e_\alpha + e_{\alpha+\beta} & \textit{if} \;\; p = j+1 \\ q^{-1} e_\beta e_\alpha & \textit{if} \;\; p = i \textit{ and } r \geq j+1 \\ e_\beta e_\alpha & \textit{if} \;\; i < p < j \textit{ and } r < j \\ q^{-1} e_\beta e_\alpha & \textit{if} \;\; i < p \leq j \textit{ and } r = j \\ e_\beta e_\alpha - (q - q^{-1}) e_{\alpha_p + ... + \alpha_j} e_{\alpha_i + ... + \alpha_r} & \textit{if} \;\; i < p \leq j \textit{ and } r \geq j+1 \end{cases}$$

Proof. If $p \geq j+2$ then $e_{\alpha_i}, ...e_{\alpha_j}$ commute with e_β, so we have

$$e_\alpha e_\beta = e_\beta e_\alpha$$

If $p = j+1$, let $\gamma = \alpha - \alpha_j$. Then, $e_\alpha = e_\gamma e_j - q e_j e_\gamma$, and $e_\gamma e_\beta = e_\beta e_\gamma$.

$$e_\alpha e_\beta = e_\gamma e_j e_\beta - q e_j e_\beta e_\gamma$$
$$e_\beta e_\alpha = e_\gamma e_\beta e_j - q e_\beta e_j e_\gamma$$

Hence,

$$\begin{aligned} e_\alpha e_\beta - q e_\beta e_\alpha &= e_\gamma e_j e_\beta - q e_j e_\beta e_\gamma - q e_\gamma e_\beta e_j + q^2 e_\beta e_j e_\gamma \\ &= e_\gamma (e_j e_\beta - q e_\beta e_j) - q(e_j e_\beta - q e_\beta e_j) e_\gamma \\ &= e_\gamma e_{\alpha_j + \beta} - q e_{\alpha_j + \beta} e_\gamma \end{aligned}$$

Continuing in this way we get

$$e_\alpha e_\beta - q e_\beta e_\alpha = e_{\alpha+\beta}$$

For the case $p \leq j$, let $\gamma = \alpha_p + ... + \alpha_j$, $\alpha' = \alpha - \gamma$ and $\beta' = \beta - \gamma$. If $r = j$, $\beta = \gamma$. We will see the commutation relation between e_α and e_γ, and commutation relation between e_α and e_β if $r < j$ in the course of the proof, so assume $r \geq j+1$.

$$e_\alpha e_\beta = ad(e_r)...ad(e_{j+2})(e_\alpha e_{\gamma + \alpha_{j+1}})$$
$$e_\beta e_\alpha = ad(e_r)...ad(e_{j+2})(e_{\gamma + \alpha_{j+1}} e_\alpha)$$

$$k_{j+1}^{-1} k_{j+2} e_\alpha k_{j+2}^{-1} k_{j+1} = q \; e_\alpha$$
$$k_{j+1}^{-1} k_{j+2} e_\gamma k_{j+2}^{-1} k_{j+1} = q \; e_\gamma$$

So, we have

$$\begin{aligned}
ad(e_{j+1})(e_\alpha e_\gamma) &= q\ ad(e_{j+1})(e_\alpha)e_\gamma + e_\alpha ad(e_{j+1})(e_\gamma)\\
&= q\ e_{\alpha+\alpha_{j+1}}e_\gamma + e_\alpha e_{\gamma+\alpha_{j+1}}\\
ad(e_{j+1})(e_\gamma e_\alpha) &= q\ ad(e_{j+1})(e_\gamma)e_\alpha + e_\gamma ad(e_{j+1})(e_\alpha)\\
&= q\ e_{\gamma+\alpha_{j+1}}e_\alpha + e_\gamma e_{\alpha+\alpha_{j+1}}
\end{aligned}$$

If $p > i$, by Lemma 4.2.4, $e_p, e_{p+1}, ..., e_j$ commute with $e_{\alpha+\alpha_{j+1}}$, so

$$e_{\alpha+\alpha_{j+1}}e_\gamma = e_\gamma e_{\alpha+\alpha_{j+1}}.$$

Let $\gamma' = \gamma - \alpha_j$. Then in the same way, $e_\alpha e_{\gamma'} = e_{\gamma'}e_\alpha$.

$$\begin{aligned}
ad(e_j)(e_\alpha e_{\gamma'}) &= q\ ad(e_j)(e_\alpha)e_{\gamma'} + e_\alpha ad(e_j)(e_{\gamma'})\\
&= e_\alpha e_\gamma\\
ad(e_j)(e_{\gamma'}e_\alpha) &= q^{-1}\ ad(e_j)(e_{\gamma'})e_\alpha + e_{\gamma'}ad(e_j)(e_\alpha)\\
&= q^{-1}\ e_\gamma e_\alpha
\end{aligned}$$

Hence, $e_\alpha e_\gamma - q^{-1}\ e_\gamma e_\alpha = 0$, and

$$\begin{aligned}
0 &= ad(e_{j+1})(e_\alpha e_\gamma) - q^{-1}ad(e_{j+1})(e_\gamma e_\alpha)\\
&= q\ e_{\alpha+\alpha_{j+1}}e_\gamma + e_\alpha e_{\gamma+\alpha_{j+1}}\\
&\quad - q^{-1}(q\ e_{\gamma+\alpha_{j+1}}e_\alpha + e_\gamma e_{\alpha+\alpha_{j+1}})\\
&= (q-q^{-1})e_\gamma e_{\alpha+\alpha_{j+1}} + e_\alpha e_{\gamma+\alpha_{j+1}} - e_{\gamma+\alpha_{j+1}}e_\alpha\\
&= ad(e_r)...ad(e_{j+2})((q-q^{-1})e_\gamma e_{\alpha+\alpha_{j+1}} + e_\alpha e_{\gamma+\alpha_{j+1}} - e_{\gamma+\alpha_{j+1}}e_\alpha)\\
&= (q-q^{-1})ad(e_r)...ad(e_{j+2})(e_\gamma e_{\alpha+\alpha_{j+1}}) + ad(e_r)...ad(e_{j+2})(e_\alpha e_{\gamma+\alpha_{j+1}})\\
&\quad - ad(e_r)...ad(e_{j+2})(e_{\gamma+\alpha_{j+1}}e_\alpha)\\
&= (q-q^{-1})e_\gamma ad(e_r)...ad(e_{j+2})(e_{\alpha+\alpha_{j+1}}) + e_\alpha e_\beta - e_\beta e_\alpha\\
&= (q-q^{-1})e_\gamma e_{\alpha_i+...+\alpha_r} + e_\alpha e_\beta - e_\beta e_\alpha.
\end{aligned}$$

If $p = i$, then $\alpha = \gamma$ and $e_{p+1}, e_{p+2}, ..., e_j$ commute with $e_{\alpha+\alpha_{j+1}} = e_{\gamma+\alpha_{j+1}}$, so we have

$$\begin{aligned}
e_{\gamma+\alpha_{j+1}}e_\gamma &= ad(e_j)...ad(e_{p+1})(e_{\gamma+\alpha_{j+1}}e_p)\\
e_\gamma e_{\gamma+\alpha_{j+1}} &= ad(e_j)...ad(e_{p+1})(e_p e_{\gamma+\alpha_{j+1}})
\end{aligned}$$

By Lemma 4.2.4, $e_pe_{\gamma+\alpha_{j+1}} - q^{-1}e_{\gamma+\alpha_{j+1}}e_p = 0$ so

$$\begin{aligned}
0 &= e_pe_{\gamma+\alpha_{j+1}} - q^{-1}e_{\gamma+\alpha_{j+1}}e_p \\
&= ad(e_j)...ad(e_{p+1})(e_pe_{\gamma+\alpha_{j+1}} - q^{-1}e_{\gamma+\alpha_{j+1}}e_p) \\
&= e_\gamma e_{\gamma+\alpha_{j+1}} - q^{-1}e_{\gamma+\alpha_{j+1}}e_\gamma \\
&= ad(e_r)...ad(e_{j+2})(e_\gamma e_{\gamma+\alpha_{j+1}} - q^{-1}e_{\gamma+\alpha_{j+1}}e_\gamma) \\
&= e_\gamma e_\beta - q^{-1}e_\beta e_\gamma \\
&= e_\alpha e_\beta - q^{-1}e_\beta e_\alpha
\end{aligned}$$

□

Proposition 4.2.8. *Let $\alpha = \alpha_i + ... + \alpha_j$, $\beta = \alpha_p + ... + \alpha_r$ and $i \le p$. Then,*

$$f_\beta f_\alpha = \begin{cases} f_\alpha f_\beta & \text{if } p \ge j+2 \\ q(f_\alpha f_\beta - f_{\alpha+\beta}) & \text{if } p = j+1 \\ q^{-1}f_\alpha f_\beta & \text{if } p = i \text{ and } r \ge j+1 \\ f_\alpha f_\beta & \text{if } i < p < j \text{ and } r < j \\ q^{-1}f_\alpha f_\beta & \text{if } i < p \le j \text{ and } r = j \\ f_\alpha f_\beta - (q-q^{-1})f_{\alpha_i+...+\alpha_r}f_{\alpha_p+...+\alpha_j} & \text{if } i < p \le j \text{ and } r \ge j+1 \end{cases}$$

Proof. If $p \ge j+2$ then $f_{\alpha_i}, ...f_{\alpha_j}$ commute with f_β, so we have

$$f_\alpha f_\beta = f_\beta f_\alpha$$

If $p = j+1$, let $\gamma = \alpha - \alpha_j$. Then, $f_\alpha = f_\gamma f_j - q^{-1}f_jf_\gamma$, and $f_\gamma f_\beta = f_\beta f_\gamma$.

$$\begin{aligned}
f_\alpha f_\beta &= f_\gamma f_j f_\beta - q^{-1}f_jf_\beta f_\gamma \\
f_\beta f_\alpha &= f_\gamma f_\beta f_j - q^{-1}f_\beta f_j f_\gamma
\end{aligned}$$

Hence,

$$\begin{aligned}
f_\alpha f_\beta - q^{-1}f_\beta f_\alpha &= f_\gamma f_j f_\beta - q^{-1}f_jf_\beta f_\gamma - q^{-1}f_\gamma f_\beta f_j + q^{-2}f_\beta f_j f_\gamma \\
&= f_\gamma(f_jf_\beta - q^{-1}f_\beta f_j) - q^{-1}(f_jf_\beta - q^{-1}f_\beta f_j)f_\gamma \\
&= f_\gamma f_{\alpha_j+\beta} - q^{-1}f_{\alpha_j+\beta}f_\gamma
\end{aligned}$$

Continuing in this way we get

$$f_\alpha f_\beta - q^{-1}f_\beta f_\alpha = f_{\alpha+\beta}$$

For the case $p \leq j$, let $\gamma = \alpha_p + ... + \alpha_j$, $\alpha' = \alpha - \gamma$ and $\beta' = \beta - \gamma$. If $p = i$, $\alpha = \gamma$. We will see the commutation relation between f_β and f_γ in the course of the proof, so assume $p \geq i+1$.

$$f_\alpha f_\beta = ad'(f_i)...ad'(f_{p-2})(f_{\alpha_{p-1}+\gamma} f_\beta)$$
$$f_\beta f_\alpha = ad'(f_i)...ad'(f_{p-2})(f_\beta f_{\alpha_{p-1}+\gamma})$$

$$k_{p-1}^{-1} k_p f_\beta k_p^{-1} k_{p-1} = q^{-1}\ f_\beta$$
$$k_{p-1}^{-1} k_p f_\gamma k_p^{-1} k_{p-1} = q^{-1}\ f_\gamma$$

So, we have

$$\begin{aligned} ad'(f_{p-1})(f_\beta f_\gamma) &= ad'(f_{p-1})(f_\beta) f_\gamma + q^{-1} f_\beta ad'(f_{p-1})(f_\gamma) \\ &= f_{\alpha_{p-1}+\beta} f_\gamma + q^{-1} f_\beta f_{\alpha_{p-1}+\gamma} \\ ad'(f_{p-1})(f_\gamma f_\beta) &= ad'(f_{p-1})(f_\gamma) f_\beta + q^{-1} f_\gamma ad'(f_{p-1})(f_\beta) \\ &= f_{\alpha_{p-1}+\gamma} f_\beta + q^{-1} f_\gamma f_{\alpha_{p-1}+\beta} \end{aligned}$$

If $r > j$, by Lemma 4.2.5, $f_p, f_{p+1}, ..., f_j$ commute with $f_{\alpha_{p-1}+\beta}$, so

$$f_{\alpha_{p-1}+\beta} f_\gamma = f_\gamma f_{\alpha_{p-1}+\beta}.$$

Let $\gamma' = \gamma - \alpha_p$. Then in the same way, $f_\beta f_{\gamma'} = f_{\gamma'} f_\beta$.

$$\begin{aligned} ad'(f_p)(f_\beta f_{\gamma'}) &= ad'(f_p)(f_\beta) f_{\gamma'} + q f_\beta ad'(f_p)(f_{\gamma'}) \\ &= q f_\beta f_\gamma \\ ad'(f_p)(f_{\gamma'} f_\beta) &= ad'(f_p)(f_{\gamma'}) f_\beta + q f_{\gamma'} ad'(f_p)(f_\beta) \\ &= f_\gamma f_\beta \end{aligned}$$

Hence, $f_\gamma f_\beta - q f_\beta f_\gamma = 0$, and

$$\begin{aligned}
0 &= ad'(f_{p-1})(f_\gamma f_\beta) - q ad'(f_{p-1})(f_\beta f_\gamma) \\
&= f_{\alpha_{p-1}+\gamma} f_\beta + q^{-1} f_\gamma f_{\alpha_{p-1}+\beta} \\
&\quad - q(f_{\alpha_{p-1}+\beta} f_\gamma + q^{-1} f_\beta f_{\alpha_{p-1}+\gamma}) \\
&= (q^{-1} - q) f_\gamma f_{\alpha_{p-1}+\beta} + f_{\alpha_{p-1}+\gamma} f_\beta - f_\beta f_{\alpha_{p-1}+\gamma} \\
&= ad'(f_i)...ad'(f_{p-2})((q^{-1} - q) f_\gamma f_{\alpha_{p-1}+\beta} + f_{\alpha_{p-1}+\gamma} f_\beta - f_\beta f_{\alpha_{p-1}+\gamma}) \\
&= (q^{-1} - q) ad'(f_i)...ad'(f_{p-2})(f_\gamma f_{\alpha_{p-1}+\beta}) + ad'(f_i)...ad'(f_{p-2})(f_{\alpha_{p-1}+\gamma} f_\beta) \\
&\quad - ad'(f_i)...ad'(f_{p-2})(f_\beta f_{\alpha_{p-1}+\gamma}) \\
&= (q^{-1} - q) f_\gamma ad'(f_i)...ad'(f_{p-2})(f_{\alpha_{p-1}+\beta}) + f_\alpha f_\beta - f_\beta f_\alpha \\
&= (q^{-1} - q) f_\gamma (f_{\alpha_i+...+\alpha_r}) + f_\alpha f_\beta - f_\beta f_\alpha.
\end{aligned}$$

If $r = j$, then $\beta = \gamma$ and $f_p, f_{p+1}, ..., f_{j-1}$ commute with $f_{\alpha_{p-1}+\beta} = f_{\alpha_{p-1}+\gamma}$, so we have

$$\begin{aligned}
f_{\alpha_{p-1}+\gamma} f_\gamma &= ad'(f_p)...ad'(f_{j-1})(f_{\alpha_{p-1}+\gamma} f_j) \\
f_\gamma f_{\alpha_{p-1}+\gamma} &= ad'(f_p)...ad'(f_{j-1})(f_{\alpha_{p-1}+\gamma} f_j)
\end{aligned}$$

By Lemma 4.2.5, $f_{\alpha_{p-1}+\gamma} f_j - q f_j f_{\alpha_{p-1}+\gamma} = 0$ so

$$\begin{aligned}
0 &= f_{\alpha_{p-1}+\gamma} f_j - q f_j f_{\alpha_{p-1}+\gamma} \\
&= ad'(f_p)...ad'(f_{j-1})(f_{\alpha_{p-1}+\gamma} f_j - q f_j f_{\alpha_{p-1}+\gamma}) \\
&= f_{\alpha_{p-1}+\gamma} f_\gamma - q f_\gamma f_{\alpha_{p-1}+\gamma} \\
&= ad'(f_i)...ad'(f_{p-2})(f_{\alpha_{p-1}+\gamma} f_\gamma - q f_\gamma f_{\alpha_{p-1}+\gamma}) \\
&= f_\alpha f_\gamma - q f_\gamma e_\alpha \\
&= f_\alpha f_\beta - q f_\beta e_\alpha
\end{aligned}$$

□

Lemma 4.2.9. *Let* $\alpha = \alpha_i + ... + \alpha_j$. *Then,*

$$\Delta(e_\alpha) = e_\alpha \otimes k_i k_{j+1}^{-1} + (q^{-1} - q) \sum_{s=i}^{j-1} (e_{\alpha_i+...+\alpha_s} \otimes e_{\alpha_{s+1}+...+\alpha_j} k_i k_{s+1}^{-1}) + 1 \otimes e_\alpha$$

Proof. Do induction on the length $l = j - i + 1$ of the root α. The case $l = 1$ is trivial. Assume the result for $l \geq 1$. For $l+1$,

$$
\begin{aligned}
&\Delta(e_{\alpha+\alpha_{j+1}}) \\
&= \Delta(e_\alpha)\Delta(e_{j+1}) - q\Delta(e_{j+1})\Delta(e_\alpha) \\
&= (e_\alpha \otimes k_i k_{j+1}^{-1} + (q^{-1}-q)\sum_{s=i}^{j-1}(e_{\alpha_i+\ldots+\alpha_s} \otimes e_{\alpha_{s+1}+\ldots+\alpha_j} k_i k_{s+1}^{-1}) + 1 \otimes e_\alpha) \\
&\quad (e_{j+1} \otimes k_{j+1}k_{j+2}^{-1} + 1 \otimes e_{j+1}) - q(e_{j+1} \otimes k_{j+1}k_{j+2}^{-1} + 1 \otimes e_{j+1})(e_\alpha \otimes k_i k_{j+1}^{-1} \\
&\quad + (q^{-1}-q)\sum_{s=i}^{j-1}(e_{\alpha_i+\ldots+\alpha_s} \otimes e_{\alpha_{s+1}+\ldots+\alpha_j} k_i k_{s+1}^{-1}) + 1 \otimes e_\alpha) \\
&= e_\alpha e_{j+1} \otimes k_i k_{j+2}^{-1} + e_{j+1} \otimes e_\alpha k_{j+1} k_{j+2}^{-1} + e_\alpha \otimes k_i k_{j+1}^{-1} e_{j+1} + 1 \otimes e_\alpha e_{j+1} \\
&\quad + (q^{-1}-q)\sum_{s=i}^{j-1}(e_{\alpha_i+\ldots+\alpha_s} e_{j+1} \otimes e_{\alpha_{s+1}+\ldots+\alpha_j} k_i k_{s+1}^{-1} k_{j+1} k_{j+2}^{-1}) \\
&\quad + (q^{-1}-q)\sum_{s=i}^{j-1}(e_{\alpha_i+\ldots+\alpha_s} \otimes e_{\alpha_{s+1}+\ldots+\alpha_j} k_i k_{s+1}^{-1} e_{j+1}) \\
&\quad - q e_{j+1} e_\alpha \otimes k_i k_{j+2}^{-1} - q e_{j+1} \otimes k_{j+1} k_{j+2}^{-1} e_\alpha - q e_\alpha \otimes e_{j+1} k_i k_{j+1}^{-1} - q \otimes e_{j+1} e_\alpha \\
&\quad - q(q^{-1}-q)\sum_{s=i}^{j-1}(e_{j+1} e_{\alpha_i+\ldots+\alpha_s} \otimes k_{j+1} k_{j+2}^{-1} e_{\alpha_{s+1}+\ldots+\alpha_j} k_i k_{s+1}^{-1}) \\
&\quad - q(q^{-1}-q)\sum_{s=i}^{j-1}(e_{\alpha_i+\ldots+\alpha_s} \otimes e_{j+1} e_{\alpha_{s+1}+\ldots+\alpha_j} k_i k_{s+1}^{-1}) \\
&= e_{\alpha+\alpha_{j+1}} \otimes k_i k_{j+2}^{-1} + e_{j+1} \otimes e_\alpha k_{j+1} k_{j+2}^{-1} + q^{-1} e_\alpha \otimes e_{j+1} k_i k_{j+1}^{-1} + 1 \otimes e_{\alpha+\alpha_{j+1}} \\
&\quad + (q^{-1}-q)\sum_{s=i}^{j-1}(e_{\alpha_i+\ldots+\alpha_s} e_{j+1} \otimes e_{\alpha_{s+1}+\ldots+\alpha_j} k_i k_{s+1}^{-1} k_{j+1} k_{j+2}^{-1}) \\
&\quad + (q^{-1}-q)\sum_{s=i}^{j-1}(e_{\alpha_i+\ldots+\alpha_s} \otimes e_{\alpha_{s+1}+\ldots+\alpha_j} e_{j+1} k_i k_{s+1}^{-1}) \\
&\quad - q q^{-1} e_{j+1} \otimes e_\alpha k_{j+1} k_{j+2}^{-1} - q e_\alpha \otimes e_{j+1} k_i k_{j+1}^{-1} \\
&\quad - q^{-1} q (q^{-1}-q)\sum_{s=i}^{j-1}(e_{\alpha_i+\ldots+\alpha_s} e_{j+1} \otimes e_{\alpha_{s+1}+\ldots+\alpha_j} k_{j+1} k_{j+2}^{-1} k_i k_{s+1}^{-1}) \\
&\quad - q(q^{-1}-q)\sum_{s=i}^{j-1}(e_{\alpha_i+\ldots+\alpha_s} \otimes e_{j+1} e_{\alpha_{s+1}+\ldots+\alpha_j} k_i k_{s+1}^{-1}) \\
&= e_{\alpha+\alpha_{j+1}} \otimes k_i k_{j+2}^{-1} + (q^{-1}-q) e_\alpha \otimes e_{j+1} k_i k_{j+1}^{-1} + 1 \otimes e_{\alpha+\alpha_{j+1}} \\
&\quad + (q^{-1}-q)\sum_{s=i}^{j-1}(e_{\alpha_i+\ldots+\alpha_s} \otimes (e_{\alpha_{s+1}+\ldots+\alpha_j} e_{j+1} - q e_{j+1} e_{\alpha_{s+1}+\ldots+\alpha_j}) k_i k_{s+1}^{-1})
\end{aligned}
$$

$$= e_{\alpha+\alpha_{j+1}} \otimes k_i k_{j+2}^{-1} + 1 \otimes e_{\alpha+\alpha_{j+1}}$$
$$+ (q^{-1} - q) \sum_{s=i}^{j} (e_{\alpha_i+...+\alpha_s} \otimes e_{\alpha_{s+1}+...+\alpha_{j+1}} k_i k_{s+1}^{-1})$$

□

Lemma 4.2.10. *Let* $\alpha = \alpha_i + ... + \alpha_j$. *Then,*

$$\Delta(f_\alpha) = f_\alpha \otimes 1 + (1 - q^{-2}) \sum_{s=i}^{j-1} (f_{\alpha_i+...+\alpha_s} k_{s+1}^{-1} k_{j+1} \otimes f_{\alpha_{s+1}+...+\alpha_j}) + k_i^{-1} k_{j+1} \otimes f_\alpha$$

Proof. Do induction on the length $l = j - i + 1$ of the root α. The case $l = 1$ is trivial. Assume the result for $l \geq 1$. For $l + 1$,

$$\Delta(f_{\alpha+\alpha_{j+1}})$$
$$= \Delta(f_\alpha)\Delta(f_{j+1}) - q^{-1}\Delta(f_{j+1})\Delta(f_\alpha)$$
$$= (f_\alpha \otimes 1 + (1 - q^{-2}) \sum_{s=i}^{j-1} (f_{\alpha_i+...+\alpha_s} k_{s+1}^{-1} k_{j+1} \otimes f_{\alpha_{s+1}+...+\alpha_j})$$
$$+ k_i^{-1} k_{j+1} \otimes f_\alpha)(f_{j+1} \otimes 1 + k_{j+1}^{-1} k_{j+2} \otimes f_{j+1})$$
$$- q^{-1}(f_{j+1} \otimes 1 + k_{j+1}^{-1} k_{j+2} \otimes f_{j+1})(f_\alpha \otimes 1$$
$$+ (1 - q^{-2}) \sum_{s=i}^{j-1} (f_{\alpha_i+...+\alpha_s} k_{s+1}^{-1} k_{j+1} \otimes f_{\alpha_{s+1}+...+\alpha_j}) + k_i^{-1} k_{j+1} \otimes f_\alpha)$$
$$= f_\alpha f_{j+1} \otimes 1 + f_\alpha k_{j+1}^{-1} k_{j+2} \otimes f_{j+1} + k_i^{-1} k_{j+1} f_{j+1} \otimes f_\alpha + k_i^{-1} k_{j+2} \otimes f_\alpha f_{j+1}$$
$$+ (1 - q^{-2}) \sum_{s=i}^{j-1} (f_{\alpha_i+...+\alpha_s} k_{s+1}^{-1} k_{j+1} f_{j+1} \otimes f_{\alpha_{s+1}+...+\alpha_j})$$
$$+ (1 - q^{-2}) \sum_{s=i}^{j-1} (f_{\alpha_i+...+\alpha_s} k_{s+1}^{-1} k_{j+2} \otimes f_{\alpha_{s+1}+...+\alpha_j} f_{j+1})$$
$$- q^{-1} f_{j+1} f_\alpha \otimes 1 - q^{-1} f_{j+1} k_i^{-1} k_{j+1} \otimes f_\alpha$$
$$- q^{-1} k_{j+1}^{-1} k_{j+2} f_\alpha \otimes f_{j+1} - q^{-1} k_i^{-1} k_{j+2} \otimes f_{j+1} f_\alpha$$
$$- q^{-1}(1 - q^{-2}) \sum_{s=i}^{j-1} (f_{j+1} f_{\alpha_i+...+\alpha_s} k_{s+1}^{-1} k_{j+1} \otimes f_{\alpha_{s+1}+...+\alpha_j})$$
$$- q^{-1}(1 - q^{-2}) \sum_{s=i}^{j-1} (k_{j+1}^{-1} k_{j+2} f_{\alpha_i+...+\alpha_s} k_{s+1}^{-1} k_{j+1} \otimes f_{j+1} f_{\alpha_{s+1}+...+\alpha_j})$$

$$= f_{\alpha+\alpha_{j+1}} \otimes 1 + f_\alpha k_{j+1}^{-1} k_{j+2} \otimes f_{j+1} + q^{-1} f_{j+1} k_i^{-1} k_{j+1} \otimes f_\alpha + k_i^{-1} k_{j+2} \otimes f_{\alpha+\alpha_{j+1}}$$
$$+ q^{-1}(1-q^{-2}) \sum_{s=i}^{j-1} (f_{j+1} f_{\alpha_i+\ldots+\alpha_s} k_{s+1}^{-1} k_{j+1} \otimes f_{\alpha_{s+1}+\ldots+\alpha_j})$$
$$+ (1-q^{-2}) \sum_{s=i}^{j-1} (f_{\alpha_i+\ldots+\alpha_s} k_{s+1}^{-1} k_{j+2} \otimes f_{\alpha_{s+1}+\ldots+\alpha_j} f_{j+1})$$
$$- q^{-1} f_{j+1} k_i^{-1} k_{j+1} \otimes f_\alpha - q^{-2} f_\alpha k_{j+1}^{-1} k_{j+2} \otimes f_{j+1}$$
$$- q^{-1}(1-q^{-2}) \sum_{s=i}^{j-1} (f_{j+1} f_{\alpha_i+\ldots+\alpha_s} k_{s+1}^{-1} k_{j+1} \otimes f_{\alpha_{s+1}+\ldots+\alpha_j})$$
$$- q^{-1}(1-q^{-2}) \sum_{s=i}^{j-1} (f_{\alpha_i+\ldots+\alpha_s} k_{s+1}^{-1} k_{j+2} \otimes f_{j+1} f_{\alpha_{s+1}+\ldots+\alpha_j})$$
$$= f_{\alpha+\alpha_{j+1}} \otimes 1 + (1-q^{-2}) f_\alpha k_{j+1}^{-1} k_{j+2} \otimes f_{j+1} + k_i^{-1} k_{j+2} \otimes f_{\alpha+\alpha_{j+1}}$$
$$+ (1-q^{-2}) \sum_{s=i}^{j-1} (f_{\alpha_i+\ldots+\alpha_s} k_{s+1}^{-1} k_{j+2} \otimes (f_{\alpha_{s+1}+\ldots+\alpha_j} f_{j+1} - q^{-1} f_{j+1} f_{\alpha_{s+1}+\ldots+\alpha_j}))$$
$$= f_{\alpha+\alpha_{j+1}} \otimes 1 + k_i^{-1} k_{j+2} \otimes f_{\alpha+\alpha_{j+1}}$$
$$+ (1-q^{-2}) \sum_{s=i}^{j} (f_{\alpha_i+\ldots+\alpha_s} k_{s+1}^{-1} k_{j+2} \otimes f_{\alpha_{s+1}+\ldots+\alpha_{j+1}})$$

□

Definition 4.2.11. *With $U_q^+ gl(n)$, $U_q^- gl(n)$, $U_q^0 gl(n)$, denote the algebras generated by e_i, f_i, k_j respectively, where $i = 1, 2, ..., n-1$, $j = 1, 2, ..., n$.*

Proposition 4.2.12. *1. The set*

$$B^0 = \{\prod_i^n k_i^{c_i} : c_i \in \mathbb{Z}\}$$

is a basis for $U_q^0 gl(n)$.

2. The set

$$B^+ = \{\prod_{\alpha\in\Phi^+} e_\alpha^{c_\alpha} : c_\alpha \in \mathbb{N}\},$$

is a basis for $U_q^+ gl(n)$. The set of positive roots Φ^+ is ordered with respect to e_α's.

3. The set

$$B^- = \{\prod_{\alpha\in\Phi^+} f_\alpha^{c_\alpha} : c_\alpha \in \mathbb{N}\},$$

is a basis for $U_q^- gl(n)$. The set of positive roots Φ^+ is ordered with respect to f_α's.

Hence the set $B = B^- \otimes B^0 \otimes B^+$ is a basis for $U_q gl(n)$.

Proof. Part 1: Trivial.

Part 2: Let $\alpha(1) < \alpha(2) < ... < \alpha(N)$, where $\alpha(i)$'s are all the positive roots. The algebra $U_q^+ gl(n)$ is generated by e_i's so non-ordered monomials in e_α's generate $U_q^+ gl(n)$. To prove that the ordered set generate $U_q^+ gl(n)$, it is enough to prove that each $e_{\alpha(i_1)} e_{\alpha(i_2)} ... e_{\alpha(i_j)}$ is a linear combination of the ordered elements $e_{\alpha(1)}^{c_{\alpha(1)}} e_{\alpha(2)}^{c_{\alpha(2)}} ... e_{\alpha(N)}^{c_{\alpha(N)}}$, with $c_{\alpha(1)} + c_{\alpha(2)} + ... + c_{\alpha(N)} \leq j$. We will do a double induction on j and for a fixed j on i_1.

The case $j = 1$ is obvious. Assume the assertion is true for j. To prove that it is true for $j+1$, we need to show that $e_{\alpha(i_1)} e_{\alpha(i_2)} ... e_{\alpha(i_{j+1})}$ is linear combination of the ordered elements with $c_{\alpha(1)} + c_{\alpha(2)} + ... + c_{\alpha(N)} \leq j + 1$. We will now do induction on i_1.

Applying the induction hypothesis to $e_{\alpha(i_2)} e_{\alpha(i_3)} ... e_{\alpha(i_{j+1})}$, $e_{\alpha(i_1)} e_{\alpha(i_2)} ... e_{\alpha(i_{j+1})}$ is a linear combination of elements

$$e_{\alpha(i_1)} e_{\alpha(s)}^{c_{\alpha(s)}} e_{\alpha(s+1)}^{c_{\alpha(s+1)}} ... e_{\alpha(N)}^{c_{\alpha(N)}} \text{ with } c_{\alpha(s)} + c_{\alpha(s+1)} + ... + c_{\alpha(N)} \leq j.$$

If $i_1 = 1$, the result is clear. Assume the result till i_1.

If $i_1 \leq s$, we are done. If $i_1 > s$, we have

$$e_{\alpha(i_1)} e_{\alpha(s)}^{c_{\alpha(s)}} e_{\alpha(s+1)}^{c_{\alpha(s+1)}} ... e_{\alpha(N)}^{c_{\alpha(N)}} = e_{\alpha(i_1)} e_{\alpha(s)} e_{\alpha(s)}^{c_{\alpha(s)}-1} e_{\alpha(s+1)}^{c_{\alpha(s+1)}} ... e_{\alpha(N)}^{c_{\alpha(N)}}$$

Using the commutation relations for $e_{\alpha(i_1)} e_{\alpha(s)}$, we have three possibilities: for some non-zero coefficients λ and μ

$$e_{\alpha(i_1)} e_{\alpha(s)} = \begin{cases} \lambda e_{\alpha(s)} e_{\alpha(i_1)} \\ \lambda e_{\alpha(s)} e_{\alpha(i_1)} + \mu e_{\alpha(i_1)+\alpha(s)} \\ \lambda e_{\alpha(s)} e_{\alpha(i_1)} + \mu e_\gamma e_{\alpha'+\gamma+\beta'} \quad \text{where } \alpha(i_1) = \alpha' + \gamma, \alpha(s) = \gamma + \beta' \end{cases}$$

and we have $\alpha(s) < \gamma < \alpha' + \gamma + \beta' < \alpha(i_1)$.

Then,
$e_{\alpha(i_1)}e_{\alpha(s)}e_{\alpha(s)}^{c_{\alpha(s)}-1}e_{\alpha(s+1)}^{c_{\alpha(s+1)}}...e_{\alpha(N)}^{c_{\alpha(N)}}$

$$= \begin{cases} \lambda e_{\alpha(s)}e_{\alpha(i_1)}e_{\alpha(s)}^{c_{\alpha(s)}-1}e_{\alpha(s+1)}^{c_{\alpha(s+1)}}...e_{\alpha(N)}^{c_{\alpha(N)}} \\ \lambda e_{\alpha(s)}e_{\alpha(i_1)}e_{\alpha(s)}^{c_{\alpha(s)}-1}e_{\alpha(s+1)}^{c_{\alpha(s+1)}}...e_{\alpha(N)}^{c_{\alpha(N)}} + \mu e_{\alpha(i_1)+\alpha(s)}e_{\alpha(s)}^{c_{\alpha(s)}-1}e_{\alpha(s+1)}^{c_{\alpha(s+1)}}...e_{\alpha(N)}^{c_{\alpha(N)}} \\ \lambda e_{\alpha(s)}e_{\alpha(i_1)}e_{\alpha(s)}^{c_{\alpha(s)}-1}e_{\alpha(s+1)}^{c_{\alpha(s+1)}}...e_{\alpha(N)}^{c_{\alpha(N)}} + \mu e_{\gamma}e_{\alpha'+\gamma+\beta'}e_{\alpha(s)}^{c_{\alpha(s)}-1}e_{\alpha(s+1)}^{c_{\alpha(s+1)}}...e_{\alpha(N)}^{c_{\alpha(N)}} \end{cases}$$

where $c_{\alpha(s)} - 1 + c_{\alpha(s+1)} + ... + c_{\alpha(N)} \leq j-1$.

For the second term of second case, apply induction hypothesis for j, as there are j terms. For all of the other cases, apply induction hypothesis for j to reorder $e_{\alpha(i_1)}e_{\alpha(s)}^{c_{\alpha(s)}-1}e_{\alpha(s+1)}^{c_{\alpha(s+1)}}...e_{\alpha(N)}^{c_{\alpha(N)}}$ as a linear combination of monomials with at most j terms, then apply induction hypothesis for i_1, as $\alpha(s) < \alpha(i_1)$ and $\gamma < \alpha(i_1)$. This proves B^+ spans $U_q^+gl(n)$.

Let Q be the root lattice, then $U_q^+gl(n)$, $U_qgl(n)$, $U_qgl(n)\otimes U_qgl(n)$ are $Q\times Q$ graded. To prove B^+ is linearly independent, we will use Q-degree.

Recall that, if $\alpha = \alpha_i + ... + \alpha_j$, then

$$\Delta(e_\alpha) = e_\alpha \otimes k_ik_{j+1}^{-1} + (q^{-1}-q)\sum_{s=i}^{j-1}(e_{\alpha_i+...+\alpha_s}\otimes e_{\alpha_{s+1}+...+\alpha_j}k_ik_{s+1}^{-1}) + 1\otimes e_\alpha$$

So we have, $\Delta : U_q^+gl(n) \to U_q^+gl(n)\otimes U_qgl(n)$ and Δ preserves the Q-degree.

$\Delta(e_\alpha)$ has a component of bidegree $(\alpha_i, \alpha-\alpha_i)$ if and only if α is of the form $\alpha_i + ...$.

In this case, the component of bidegree $(m\alpha_i, m(\alpha-\alpha_i))$ of $\Delta(e_\alpha^m)$ is a scalar multiple of $e_i^m \otimes (e_{\alpha-\alpha_i})^m K^m$, and the component of bidegree $((m+m_1+...+m_r)\alpha_i, ...)$ of $\Delta((e_{\alpha_i+...+\alpha_{i+r}})^{m_r}...(e_{\alpha_i+\alpha_{i+1}})^{m_1}e_{\alpha_i}^m)$ is a scalar multiple of $e_i^{(m+m_1+...+m_r)} \otimes (e_{\alpha_{i+1}+...+\alpha_{i+r}})^{m_r}...(e_{\alpha_{i+1}})^{m_1}K^{(m+m_1+...+m_r)}$, where $K = k_ik_{i+1}^{-1}$. The right hand side of the tensor is already ordered. More generally, the component of bidegree $(p\alpha_i, ...)$ of $\Delta(e_{\alpha(1)}^{c_{\alpha(1)}}e_{\alpha(2)}^{c_{\alpha(2)}}...e_{\alpha(N)}^{c_{\alpha(N)}})$ with p maximal is a scalar multiple of

$$e_i^p \otimes e_{\alpha(1)'}^{c_{\alpha(1)}}e_{\alpha(2)'}^{c_{\alpha(2)}}...e_{\alpha(N)'}^{c_{\alpha(N)}}K',$$

where $K' = \sum c_{\alpha(j)}k_ik_{i+1}^{-1}$ and

$$\alpha(j)' = \begin{cases} \alpha(j) - \alpha_i & \text{if} \quad \alpha(j) = \alpha_i + ... \\ \alpha(j) & \text{otherwise} \end{cases}$$

To reorder $e_{\alpha(1)'}^{c_{\alpha(1)}} e_{\alpha(2)'}^{c_{\alpha(2)}} ... e_{\alpha(N)'}^{c_{\alpha(N)}}$, we should commute vectors of the form $e_{\alpha_{i+1}+...}$, since the other vectors are already ordered. These commutations are of type $e_{\beta_1} e_{\beta_2} = c e_{\beta_2} e_{\beta_1}$ for a non-zero c. So the component is a scalar multiple of

$$e_i^p \otimes e_{\alpha(1)}^{c'_{\alpha(1)}} e_{\alpha(2)}^{c'_{\alpha(2)}} ... e_{\alpha(N)}^{c'_{\alpha(N)}} K'.$$

Now consider a linear relation between $e_{\alpha(1)}^{c_{\alpha(1)}} e_{\alpha(2)}^{c_{\alpha(2)}} ... e_{\alpha(N)}^{c_{\alpha(N)}}$. We can assume that all of the monomials have the same Q-degree, i.e. $\sum c_{\alpha(j)} \alpha(j)$ is fixed. We will do induction on this degree to prove that the relation is trivial.

$e_{\alpha(j)}$ is non-zero, so if Q-degree is $\alpha(j)$ then relation is trivial.

Consider the biggest integer i such that there appears a $e_{\alpha_i+...}$ with a non-zero exponent among the monomials of the relation. Let p be the largest total exponent at which all $e_{\alpha_i+...}$ appear. If we apply Δ, only the monomials in which this total exponent is exactly p will have a component of degree $(p\alpha_i, ...)$. So apply Δ to the relation. On the right of the tensor, we obtain a new relation between the monomials $e_{\alpha(1)}^{c'_{\alpha(1)}} e_{\alpha(2)}^{c'_{\alpha(2)}} ... e_{\alpha(N)}^{c'_{\alpha(N)}} K'$, and if $e_{\alpha(1)}^{c_{\alpha(1)}} e_{\alpha(2)}^{c_{\alpha(2)}} ... e_{\alpha(N)}^{c_{\alpha(N)}}$ are pairwise distinct, so are $e_{\alpha(1)}^{c'_{\alpha(1)}} e_{\alpha(2)}^{c'_{\alpha(2)}} ... e_{\alpha(N)}^{c'_{\alpha(N)}} K'$. The new relation is trivial by the induction hypothesis, since the Q-degree of $e_{\alpha(1)}^{c'_{\alpha(1)}} e_{\alpha(2)}^{c'_{\alpha(2)}} ... e_{\alpha(N)}^{c'_{\alpha(N)}} K'$ is strictly smaller than the Q-degree of $e_{\alpha(1)}^{c_{\alpha(1)}} e_{\alpha(2)}^{c_{\alpha(2)}} ... e_{\alpha(N)}^{c_{\alpha(N)}}$. The coefficients of this relation is non-zero multiples of the initial one, so the coefficients of all the monomials in which $e_{\alpha_i+...}$ appears with exponent p should be zero, which contradicts the choice of p. Thus the initial relation is trivial, which means the set B^+ is linearly independent.

Part 3: Similar to part 2.

$\square$

CHAPTER 5

DUALITY BETWEEN $\mathbf{U_q gl(n)}$ AND $\mathbf{M_q(n)}$

In this chapter, we study the duality between $U_q gl(n)$ and $M_q(n)$. In the first section, we introduce the notion of duality between bialgebras. Second section proves that $U_q gl(n)$ and $M_q(n)$ are in duality in full detail.

5.1 Duality Between Bialgebras

Definition 5.1.1. *Let $(U, \mu, \eta, \Delta, \varepsilon)$ and $(H, \mu, \eta, \Delta, \varepsilon)$ be bialgebras and $<,>$ be a bilinear form on $U \times H$. We say that the bilinear form realizes a duality between U and H, or that the bialgebras U and H are in duality if we have*

$$< uv, x > = \sum_{(x)} < u, x' >< v, x'' >, \tag{5.1}$$

$$< u, xy > = \sum_{(u)} < u', x >< u'', y >, \tag{5.2}$$

$$< 1, x > = \varepsilon(x), \tag{5.3}$$

$$< u, 1 > = \varepsilon(u) \tag{5.4}$$

for all $u, v \in U$ and $x, y \in H$.

Moreover, if U and H are Hopf algebras with antipode S, then they are said to be in duality if the underlying bialgebras are in duality and we have

$$< S(u), x > =< u, S(x) >$$

for all $u \in U$ and $x \in H$.

Let ϕ be the linear map from U to the dual vector space H^* and ψ be the linear map from H to the dual vector space U^* defined by

$$\phi(u)(x) =< u, x > \text{ and } \psi(x)(u) =< u, x > .$$

Proposition 5.1.2. *With the above notation, the relations* (5.1), (5.3) *and* (5.2), (5.4) *of Definition 5.1.1 are equivalent to ϕ and ψ being algebra homomorphisms respectively.*

Proof. The map ϕ is an algebra morphism if we have

$$\phi(1) = 1_{H^*} = \varepsilon$$
$$\phi(uv) = \phi(u)\phi(v)$$

Thus, the map ϕ being an algebra morphism is equivalent to

$$\varepsilon(x) = \phi(1)(x) =< 1, x >$$
$$< uv, x > = \phi(uv)(x) = \sum_{(x)} \phi(u)(x')\phi(v)(x'') = \sum_{(x)} < u, x' >< v, x'' >$$

Similarly, ψ being an algebra morphism is equivalent to

$$\varepsilon(u) = \psi(1)(u) =< u, 1 >$$
$$< u, xy > = \psi(xy)(u) = \psi(x)(u')\psi(y)(u'') = \sum_{(u)} < u', x >< u'', y >$$

□

5.2 Duality Between $\mathbf{U_q gl(n)}$ and $\mathbf{M_q(n)}$

Recall the bialgebra $M_q(n)$ from Section 3.1 and the Hopf algebra $U = U_q gl(n)$ from Section 3.3. To establish the duality between $M_q(n)$ and $U_q gl(n)$ we will construct an algebra map ψ from $M_q(n)$ to the dual algebra $U_q^* gl(n)$. Consider the representation ρ defined on the generators by

$$\rho(e_i) = E_{i,i+1},$$
$$\rho(f_i) = E_{i+1,i},$$
$$\rho(k_i) = D_i,$$

where E_{ij} denotes the elementary matrix, i.e. the ijth entry of E_{ij} is 1, and all the other entries are 0, and D_i denotes the diagonal matrix

$$D_i = E_{11} + E_{22} + ... + qE_{i,i} + E_{i+1,i+1} + E_{i+2,i+2} + ... + E_{nn}.$$

If u is an element of $U_q gl(n)$ using the P.B.W. basis given in the Chapter 4, we have

$$\rho(u) = \begin{pmatrix} A_{11}(u) & A_{12}(u) & \dots & A_{1n}(u) \\ A_{21}(u) & A_{22}(u) & \dots & A_{2n}(u) \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1}(u) & A_{n2}(u) & \dots & A_{nn}(u) \end{pmatrix}$$

Let $\psi : H = M_q(n) \to U^* = U_q gl(n)^*$ be the algebra morphism defined on the generators by $\psi(a_{ij}) = A_{ij}$.

Our aim is to show that the algebra morphism ψ is well-defined and the bilinear form $\psi(x)(u) =< u, x >$ realizes a duality between $M_q(n)$ and $U_q gl(n)$.

Let $\alpha(1) < \alpha(2) < ... < \alpha(N)$ be all the positive roots. By the Proposition 4.2.12, $u \in U$ can be written in the form

$$u = f_{\alpha(1)}^{c_1} ... f_{\alpha(N)}^{c_N} k_1^{s_1} ... k_n^{s_n} e_{\alpha(1)}^{d_1} ... e_{\alpha(N)}^{d_N}$$

where $c_i, d_i \in \mathbb{N}$, $s_i \in \mathbb{Z}$.

Lemma 5.2.1. *Let $\alpha = \alpha(i, j+1) = \alpha_i + \alpha_{i+1} + ... + \alpha_j$ be a positive root as defined in Section 4.1.*

1. $\rho(e_\alpha) = E_{i,j+1}$
2. $\rho(f_\alpha) = (-q)^{i-j} E_{j+1,i}$

Proof. Proof will be by induction on $l = j - i$.

1. If $l = 1$, then $\alpha = \alpha_i + \alpha_{i+1}$ and $e_\alpha = e_i e_{i+1} - q e_{i+1} e_i$ by definition of e_α. Thus we have

$$\begin{aligned} \rho(e_\alpha) &= E_{i,i+1} E_{i+1,i+2} - q E_{i+1,i+2} E_{i,i+1} \\ &= E_{i,i+2} \end{aligned}$$

Assume the result is true for $l = j - i$. Then for $l = j - i + 1$ we have

$$\begin{aligned}\rho(e_{\alpha+\alpha_{j+1}}) &= \rho(e_\alpha e_{j+1} - q e_{j+1} e_\alpha) \\ &= E_{i,j+1}E_{j+1,j+2} - qE_{j+1,j+2}E_{i,j+1} \\ &= E_{i,j+2}\end{aligned}$$

2. If $l = 1$, $f_\alpha = f_i f_{i+1} - q^{-1} f_{i+1} f_i$ by Lemma 4.2.6 and we have

$$\begin{aligned}\rho(f_\alpha) &= E_{i+1,i}E_{i+2,i+1} - q^{-1}E_{i+2,i+1}E_{i+1,i} \\ &= -q^{-1}E_{i+2,i}\end{aligned}$$

Suppose the result is true for $l = j - i$. Then for $l = j - i + 1$ we have

$$\begin{aligned}\rho(f_{\alpha+\alpha_{j+1}}) &= \rho(f_\alpha f_{j+1} - q^{-1} f_{j+1} f_\alpha) \\ &= (-q)^{i-j}(E_{j+1,i}E_{j+2,j+1} - q^{-1}E_{j+2,j+1}E_{j+1,i}) \\ &= (-q)^{i-j-1}E_{j+2,i}\end{aligned}$$

□

Lemma 5.2.2. *Let* $u = f_{\alpha(1)}^{c_1} ... f_{\alpha(N)}^{c_N} k_1^{s_1} ... k_n^{s_n} e_{\alpha(1)}^{d_1} ... e_{\alpha(N)}^{d_N} \in U$. *If* $c_i > 1$ *or* $d_i > 1$ *for some* i *or* $c_i \geq c_j \geq 1$ *or* $d_i \geq d_j \geq 1$ *for any pair* i, j *then* $\rho(u) = 0$

Proof. Assume $c_i > 1$ for some i, and $\alpha(i) = \alpha(k, l)$. By Lemma 5.2.1, we have

$$\rho(f_{\alpha(i)}^{c_i}) = \rho(f_{\alpha(i)})^{c_i} = (-q)^{k-l+1}E_{l,k}^{c_i} = 0.$$

Similarly, if $d_i > 1$ for some i, and $\alpha(i) = \alpha(k, l)$, we have

$$\rho(e_{\alpha(i)}^{d_i}) = \rho(e_{\alpha(i)})^{d_i} = E_{k,l}^{d_i} = 0.$$

Now suppose $c_i = c_j = 1$ and $\alpha(i) = \alpha(k, l) < \alpha(j) = \alpha(s, t)$. Then by the order for f_α given in Section 4.1, we have $k < s$ or $k = s$ and $l < t$, so that $\delta_{k,t} = 0$. Thus,

$$\rho(f_{\alpha(i)} f_{\alpha(j)}) = \rho(f_{\alpha(i)})\rho(f_{\alpha(j)}) = \lambda E_{l,k}E_{t,s} = \lambda \delta_{k,t} E_{l,s} = 0$$

where $\lambda = (-q)^{k+s-l-t+2}$

Similarly, suppose $d_i = d_j = 1$ and $\alpha(i) = \alpha(k, l) < \alpha(j) = \alpha(s, t)$. By the order for e_α given in Section 4.1, we have $k > s$ or $k = s$ and $l > t$, so that $\delta_{l,s} = 0$. Thus,

$$\rho(e_{\alpha(i)} e_{\alpha(j)}) = \rho(e_{\alpha(i)})\rho(e_{\alpha(j)}) = E_{k,l}E_{s,t} = \delta_{l,s} E_{k,t} = 0$$

$\square$

Corollary 5.2.3. *Let $\alpha = \alpha(i,j)$, $\beta = \alpha(r,t)$ and $s_1, s_2, ..., s_n \in \mathbb{Z}$. Then only possible nonzero values of ρ can be the following:*

$$\rho(k_1^{s_1}...k_n^{s_n}) = D(q^{s_1}, q^{s_2}, ..., q^{s_n}) \qquad \rho(k_1^{s_1}...k_n^{s_n} e_\beta) = q^{s_r} E_{r,t}$$
$$\rho(f_\alpha k_1^{s_1}...k_n^{s_n}) = q^{s_i}(-q)^{i-j+1} E_{j,i} \qquad \rho(f_\alpha k_1^{s_1}...k_n^{s_n} e_\beta) = q^{s_i}(-q)^{i-j+1}\delta_{i,r} E_{j,t}$$

where D is the diagonal matrix with the given entries.

The product of elements $\alpha, \beta \in U^*$ is given by

$$(\alpha\beta)(u) = \sum_{(u)} \alpha(u')\beta(u'').$$

where $u \in U$.

Thus, we need to determine the form of elements $(\rho\otimes\rho)\big(\Delta(u)\big)$ to observe the structure of product in U^*. Since both Δ and ρ (and so is $\rho\otimes\rho$) are algebra morphisms, we have

$$(\rho\otimes\rho)\big(\Delta(uv)\big) = \big[(\rho\otimes\rho)\big(\Delta(u)\big)\big]\big[(\rho\otimes\rho)\big(\Delta(v)\big)\big].$$

Let $\alpha = \alpha(i,j) < \beta = \alpha(r,t)$. By the Lemma 4.2.9 and Lemma 4.2.10, we have

$$\Delta(f_\alpha) = f_\alpha \otimes 1 + (1-q^{-2}) \sum_{m=i}^{j-2} (f_{\alpha(i,m+1)} k_{m+1}^{-1} k_j \otimes f_{\alpha(m+1,j)}) + k_i^{-1} k_j \otimes f_\alpha$$
$$\Delta(e_\alpha) = e_\alpha \otimes k_i k_j^{-1} + (q^{-1}-q) \sum_{m=i}^{j-2} (e_{\alpha(i,m+1)} \otimes e_{\alpha(m+1,j)} k_i k_{m+1}^{-1}) + 1 \otimes e_\alpha$$

Denote $(\rho \otimes \rho)(\Delta) = \overline{\rho}$. Then, we have

$$\begin{aligned}
\overline{\rho}(f_\alpha) &= (-q)^{i-j+1} E_{j,i} \otimes I + D_i^{-1} D_j \otimes (-q)^{i-j+1} E_{j,i} \\
&\quad + (1-q^{-2}) \sum_{m=i}^{j-2} ((-q)^{i-m} E_{m+1,i} D_{m+1}^{-1} D_j \otimes (-q)^{m-j+2} E_{j,m+1}) \\
&= (-q)^{i-j+1} \big(E_{j,i} \otimes I + D_i^{-1} D_j \otimes E_{j,i} + (q^{-1}-q) \sum_{m=i}^{j-2} E_{m+1,i} \otimes E_{j,m+1} \big) \\
\overline{\rho}(e_\alpha) &= E_{i,j} \otimes D_i D_j^{-1} + I \otimes E_{i,j} \\
&\quad + (q^{-1}-q) \sum_{m=i}^{j-2} E_{i,m+1} \otimes E_{m+1,j} D_i D_{m+1}^{-1} \\
&= E_{i,j} \otimes D_i D_j^{-1} + I \otimes E_{i,j} + (q^{-1}-q) \sum_{m=i}^{j-2} E_{i,m+1} \otimes E_{m+1,j}
\end{aligned}$$

$$\begin{aligned}
\overline{\rho}(f_\alpha^2) &= q^{2(i-j+1)} \big(E_{j,i} D_i^{-1} D_j \otimes E_{j,i} + D_i^{-1} D_j E_{j,i} \otimes E_{j,i} \big) \\
&= q^{2(i-j+1)} (q+q^{-1}) E_{j,i} \otimes E_{j,i} \\
\overline{\rho}(e_\alpha^2) &= E_{i,j} \otimes D_i D_j^{-1} E_{i,j} + E_{i,j} \otimes E_{i,j} D_i D_j^{-1} \\
&= (q+q^{-1}) E_{i,j} \otimes E_{i,j}
\end{aligned}$$

$$\begin{aligned}
\overline{\rho}(f_\alpha f_\beta) &= \lambda \big[E_{j,i} D_r^{-1} D_t \otimes E_{t,r} + D_i^{-1} D_j E_{t,r} \otimes E_{j,i} + (q^{-1}-q) E_{t,i} D_r^{-1} D_t \otimes E_{j,r} \big] \\
&= \lambda \big[q^{-\delta_{i,r}} E_{j,i} \otimes E_{t,r} + q^{\delta_{j,t}} E_{t,r} \otimes E_{j,i} + (q^{-1}-q) q^{-\delta_{i,r}} E_{t,i} \otimes E_{j,r} \big]
\end{aligned}$$

where $\lambda = (-q)^{i+r-j-t+2}$. The third summand appears if $i+1 \le t \le j-1$.

$$\begin{aligned}
\overline{\rho}(e_\alpha e_\beta) &= E_{i,j} \otimes D_i D_j^{-1} E_{r,t} + E_{r,t} \otimes E_{i,j} D_r D_t^{-1} + (q^{-1}-q) E_{r,j} \otimes E_{i,t} \\
&= q^{\delta_{i,r}} E_{i,j} \otimes E_{r,t} + q^{-\delta_{j,t}} E_{r,t} \otimes E_{i,j} + (q^{-1}-q) E_{r,j} \otimes E_{i,t}
\end{aligned}$$

The third summand appears if $r+1 \le j \le t-1$.

If $\alpha \le \beta \le \gamma$, we have

$$(\rho \otimes \rho)(\Delta(e_\alpha e_\beta e_\gamma)) = 0, \qquad (\rho \otimes \rho)(\Delta(f_\alpha f_\beta f_\gamma)) = 0.$$

Corollary 5.2.4. *Let $K = k_1^{s_1} ... k_n^{s_n}$, and $\alpha \le \beta$, $\gamma \le \delta$. Then $\overline{\rho}(u) = (\rho \otimes \rho)(\Delta)(u)$ may be nonzero only if u is one of the elements*

$$K, Ke_\gamma, Ke_\gamma e_\delta, f_\alpha K, f_\alpha f_\beta K, f_\alpha K e_\gamma, f_\alpha f_\beta K e_\gamma, f_\alpha K e_\gamma e_\delta, f_\alpha f_\beta K e_\gamma e_\delta.$$

Lemma 5.2.5. *The algebra morphism $\psi : H = M_q(n) \to U^* = U_q gl(n)^*$ defined at the beginning of this section is well-defined.*

Proof. To show that ψ is well-defined, we need to show

$$A_{ad}A_{ac}(u) = qA_{ac}A_{ad}(u) \tag{5.5}$$

$$A_{bc}A_{ac}(u) = qA_{ac}A_{bc}(u) \tag{5.6}$$

$$A_{bc}A_{ad}(u) = A_{ad}A_{bc}(u) \tag{5.7}$$

$$A_{bd}A_{ac}(u) - A_{ac}A_{bd}(u) = (q - q^{-1})A_{ad}A_{bc}(u) \tag{5.8}$$

for every $b > a$, $d > c$, $u \in U$. Let $K = k_1^{s_1}...k_n^{s_n}$, $\alpha = \alpha(i,j) \leq \beta = \alpha(r,t)$ and $\gamma = \alpha(h,l) \leq \delta = \alpha(p,s)$. By Corollary 5.2.4, we need to consider the following cases:

(i) If $u = K = k_1^{s_1}...k_n^{s_n}$, we have $\Delta(K) = K \otimes K$, so that $\overline{\rho}(K) = D \otimes D$ where $D = D(q^{s_1}, q^{s_2}, ..., q^{s_n})$, and all products evaluated on u vanish except

$$(A_{aa}A_{dd})(u) = (A_{dd}A_{aa})(u) = q^{s_a+s_d},$$

so that

$$(A_{aa}A_{dd} - A_{dd}A_{aa})(u) = 0 = A_{ad}A_{da}(u).$$

Hence, the relations (5.5)-(5.8) are all satisfied for $u = K$.

(ii) If $u = f_\alpha K$, we have

$$\begin{aligned}\overline{\rho}(f_\alpha K) &= \lambda\big(E_{j,i}D \otimes D + D_i^{-1}D_jD \otimes E_{j,i}D \\ &\quad + (q^{-1} - q)\sum_{m=i}^{j-2} E_{m+1,i}D \otimes E_{j,m+1}D\big) \\ &= \lambda q^{s_i}\big(E_{j,i} \otimes D + D_i^{-1}D_jD \otimes E_{j,i} \\ &\quad + (q^{-1} - q)\sum_{m=i}^{j-2} q^{s_{m+1}}E_{m+1,i} \otimes E_{j,m+1}\big)\end{aligned}$$

where $\lambda = (-q)^{i-j+1}$. All products evaluated on u vanish except the cases:

(a) If $i < a = d < j, b = j, c = i$

$$\begin{aligned} A_{ji}A_{aa}(u) = A_{aa}A_{ji}(u) &= \lambda q^{s_i+s_a} \\ (A_{ja}A_{ai} - A_{ai}A_{ja})(u) &= 0 - \lambda(q^{-1}-q)q^{s_i}q^{s_a} \\ (q-q^{-1})A_{aa}A_{ji}(u) &= (q-q^{-1})\lambda q^{s_i+s_a} \end{aligned}$$

(b) If $a = c = i, b = d = j$

$$\begin{aligned} A_{jj}A_{ji}(u) = qA_{ji}A_{jj}(u) &= \lambda q^{s_i+s_j+1} \\ A_{ji}A_{ii}(u) = qA_{ii}A_{ji}(u) &= \lambda q^{2s_i} \end{aligned}$$

(iii) If $u = f_\alpha f_\beta K$, we have

$$\begin{aligned} \overline{\rho}(f_\alpha f_\beta K) &= \lambda\big(q^{-\delta_{i,r}}E_{j,i}D \otimes E_{t,r}D + q^{\delta_{j,t}}E_{t,r}D \otimes E_{j,i}D \\ &\quad + (q^{-1}-q)q^{-\delta_{i,r}}E_{t,i}D \otimes E_{j,r}D\big) \\ &= \lambda q^{s_i+s_r}\big(q^{-\delta_{i,r}}E_{j,i} \otimes E_{t,r} + q^{\delta_{j,t}}E_{t,r} \otimes E_{j,i} \\ &\quad + (q^{-1}-q)q^{-\delta_{i,r}}E_{t,i} \otimes E_{j,r}\big) \end{aligned}$$

where $\lambda = (-q)^{i+r-j-t+2}$. The third summand appears if $i+1 \leq t \leq j-1$. If $\alpha = \beta$, all products evaluated on u vanish, so assume $\alpha < \beta$. We have four cases:

(a) If $i = r$ and $j < t$, then third summand does not appear and all products evaluated on u vanish except

$$A_{ti}A_{ji}(u) = qA_{ji}A_{ti}(u) = \lambda q^{2s_i}$$

(b) If $i < r$ and $j = t$, then third summand does not appear and all products evaluated on u vanish except

$$A_{jr}A_{ji}(u) = qA_{ji}A_{jr}(u) = \lambda q^{s_i+s_r+1}$$

(c) If $i < r$ and $j < t$, then third summand does not appear and all products evaluated on u vanish except

$$(A_{tr}A_{ji} - A_{ji}A_{tr})(u) = \lambda q^{s_i+s_r} - \lambda q^{s_i+s_r} = (q-q^{-1})A_{ti}A_{jr}(u)$$

(d) If $i < r$ and $i+1 \leq t \leq j-1$, then all products evaluated on u vanish except

$$A_{ji}A_{tr}(u) = A_{tr}A_{ji}(u) = \lambda q^{2s_i+s_r}$$
$$(A_{jr}A_{ti} - A_{ti}A_{jr})(u) = (q-q^{-1})\lambda q^{s_i+s_r} = (q-q^{-1})A_{ji}A_{tr}(u)$$

(iv) If $u = Ke_\gamma$, we have

$$\begin{aligned}\overline{\rho}(Ke_\gamma) &= DE_{h,l} \otimes DD_hD_l^{-1} + D \otimes DE_{h,l}\\ &\quad + (q^{-1}-q)\sum_{m=h}^{l-2} DE_{h,m+1} \otimes DE_{m+1,l}\\ &= q^{s_h}(E_{h,l} \otimes DD_hD_l^{-1} + D \otimes E_{h,l}\\ &\quad + (q^{-1}-q)\sum_{m=h}^{l-2} q^{s_{m+1}}E_{h,m+1} \otimes E_{m+1,l})\end{aligned}$$

All products evaluated on u vanish except the cases

(a) If $h < b = c < l, a = h, d = l$

$$(A_{bl}A_{hb} - A_{hb}A_{bl})(u) = 0 - (q^{-1}-q)q^{s_h}q^{s_b}$$
$$(q-q^{-1})A_{bb}A_{hl}(u) = (q-q^{-1})q^{s_h+s_b}$$
$$A_{bb}A_{hl}(u) = A_{hl}A_{bb}(u) = q^{s_h+s_b}$$

(b) If $a = c = h, b = d = l$

$$A_{ll}A_{hl}(u) = qA_{hl}A_{ll}(u) = q^{s_h+s_l}$$
$$A_{hl}A_{hh}(u) = qA_{hh}A_{hl}(u) = q^{2s_h+1}$$

(v) If $u = Ke_\gamma e_\delta$, we have

$$\begin{aligned}\overline{\rho}(Ke_\gamma e_\delta) &= q^{\delta_{h,p}}DE_{h,l} \otimes DE_{p,s} + q^{-\delta_{l,s}}DE_{p,s} \otimes DE_{h,l}\\ &\quad + (q^{-1}-q)DE_{p,l} \otimes DE_{h,s}\\ &= q^{s_h+s_p}(q^{\delta_{h,p}}E_{h,l} \otimes E_{p,s} + q^{-\delta_{l,s}}E_{p,s} \otimes E_{h,l}\\ &\quad + (q^{-1}-q)E_{p,l} \otimes E_{h,s})\end{aligned}$$

The third summand appears if $p+1 \leq l \leq s-1$. If $\gamma = \delta$, all products evaluated on u vanish, so assume $\gamma < \delta$. We have the following cases:

(a) If $h = p$ and $l > s$, then third summand does not appear and all products evaluated on u vanish except

$$A_{hl}A_{hs}(u) = qA_{hs}A_{hl}(u) = q^{2s_h+1}$$

(b) If $h > p$ and $s = l$, then third summand does not appear and all products evaluated on u vanish except

$$A_{hl}A_{pl}(u) = qA_{pl}A_{hl}(u) = q^{s_h+s_p}$$

(c) If $h > p$ and $l > s$, then third summand does not appear and all products evaluated on u vanish except

$$(A_{hl}A_{ps} - A_{ps}A_{hl})(u) = q^{s_h+s_p} - q^{s_h+s_p} = (q - q^{-1})A_{hs}A_{pl}(u)$$

(d) If $h > p$ and $p + 1 \leq l \leq s - 1$, then all products evaluated on u vanish except

$$\begin{aligned} A_{hl}A_{ps}(u) &= A_{ps}A_{hl}(u) = q^{s_h+s_p} \\ (A_{hs}A_{pl} - A_{pl}A_{hs})(u) &= (q - q^{-1})q^{s_h+s_p} = (q - q^{-1})A_{ji}A_{tr}(u) \end{aligned}$$

(vi) If $u = f_\alpha K e_\gamma$, we have

$$\begin{aligned} \overline{\rho}(f_\alpha K e_\gamma) = \lambda\big(&E_{j,i} \otimes D + D_i^{-1}D_jD \otimes E_{j,i} \\ &+ (q^{-1} - q)\sum_{m=i}^{j-2} q^{s_{m+1}}E_{m+1,i} \otimes E_{j,m+1}\big)\big(E_{h,l} \otimes D_hD_l^{-1} \\ &+ I \otimes E_{h,l} + (q^{-1} - q)\sum_{m=h}^{l-2} E_{h,m+1} \otimes E_{m+1,l}\big) \end{aligned}$$

where $\lambda = (-q)^{i-j+1}q^{s_i}$. We have the following cases:

(a) If $h = i$ and $l = j$ then

$$\begin{aligned} \overline{\rho}(f_\alpha K e_\gamma) = \lambda\big(&E_{j,j} \otimes DD_iD_j^{-1} + D_i^{-1}D_jD \otimes E_{j,j} \\ &+ (q^{-1} - q)\sum_{m=i}^{j-2} q^{s_{m+1}}(E_{m+1,j} \otimes E_{j,m+1} + E_{j,m+1} \otimes E_{m+1,j}) \\ &+ (q^{-1} - q)^2\sum_{m=i}^{j-2} q^{s_{m+1}}E_{m+1,m+1} \otimes E_{j,j} \\ &+ q^{s_i}(E_{j,i} \otimes E_{i,j} + E_{i,j} \otimes E_{j,i})\big) \end{aligned}$$

and all products evaluated on u vanish except that

$$\begin{aligned}
A_{ja}A_{aj}(u) &= A_{aj}A_{ja}(u) = q^{s_a}\lambda(q^{-1}+q) \\
(A_{jj}A_{aa}(u) - A_{aa}A_{jj})(u) &= -q^{s_a}\lambda(q^{-1}+q)^2 = (q-q^{-1})A_{ja}A_{aj}(u) \\
A_{ji}A_{ij}(u) &= A_{ij}A_{ji}(u) = q^{s_i}\lambda \\
(A_{jj}A_{ii}(u) - A_{ii}A_{jj})(u) &= q^{s_i}\lambda(q-q^{-1}) = (q-q^{-1})A_{ja}A_{aj}(u)
\end{aligned}$$

where $i+1 \leq a \leq j-1$.

(b) If $h = i$ and $l < j$ then

$$\begin{aligned}
\overline{\rho}(f_\alpha K e_\gamma) = \lambda\big(& E_{j,l} \otimes DD_iD_l^{-1} + D_i^{-1}D_jD \otimes E_{j,l} \\
&+ (q^{-1}-q)\sum_{m=i}^{j-2} q^{s_{m+1}-\delta_{l,m+1}} E_{m+1,l} \otimes E_{j,m+1} \\
&+ (q^{-1}-q)\sum_{m=i}^{l-2} q^{s_{m+1}} E_{j,m+1} \otimes E_{m+1,l} \\
&+ (q^{-1}-q)^2\sum_{m=i}^{l-2} q^{s_{m+1}} E_{m+1,m+1} \otimes E_{j,l} \\
&+ q^{s_i}(E_{j,i} \otimes E_{i,l} + E_{i,l} \otimes E_{j,i})\big)
\end{aligned}$$

and all products evaluated on u vanish except that

$$\begin{aligned}
A_{ja}A_{al}(u) &= A_{al}A_{ja}(u) = q^{s_a}\lambda(q^{-1}+q) \\
(A_{jl}A_{aa}(u) - A_{aa}A_{jl})(u) &= -q^{s_a}\lambda(q^{-1}+q)^2 = (q-q^{-1})A_{ja}A_{al}(u) \\
A_{jl}A_{bb}(u) &= A_{bb}A_{jl}(u) = q^{s_b}\lambda \\
(A_{jb}A_{bl}(u) - A_{bl}A_{jb})(u) &= -q^{s_b}\lambda(q^{-1}+q) = (q-q^{-1})A_{ja}A_{aj}(u) \\
A_{ji}A_{il}(u) &= A_{il}A_{ji}(u) = q^{s_i}\lambda \\
(A_{jl}A_{ii}(u) - A_{ii}A_{jl})(u) &= q^{s_i}\lambda(q-q^{-1}) = (q-q^{-1})A_{ja}A_{aj}(u) \\
A_{jl}A_{ll}(u) &= qA_{ll}A_{jl}(u) = q^{s_l-1}\lambda
\end{aligned}$$

where $i+1 \leq a \leq l-1$, $l+1 \leq b \leq j-1$.

(c) If $h = i$ and $l > j$ then

$$\begin{aligned}\overline{\rho}(f_\alpha K e_\gamma) = \lambda\big(&E_{j,l} \otimes DD_iD_l^{-1} + D_i^{-1}D_jD \otimes E_{j,l}\\&+ (q^{-1}-q)\sum_{m=i}^{j-2} q^{s_{m+1}} E_{m+1,l} \otimes E_{j,m+1}\\&+ (q^{-1}-q)\sum_{m=i}^{l-2} q^{s_{m+1}} E_{j,m+1} \otimes E_{m+1,l}\\&+ (q^{-1}-q)^2\sum_{m=i}^{j-2} q^{s_{m+1}} E_{m+1,m+1} \otimes E_{j,l}\\&+ q^{s_i}(E_{j,i} \otimes E_{i,l} + E_{i,l} \otimes E_{j,i})\big)\end{aligned}$$

and all products evaluated on u vanish except that

$$\begin{aligned}A_{ja}A_{al}(u) = A_{al}A_{ja}(u) &= q^{s_a}\lambda(q^{-1}+q)\\(A_{jl}A_{aa}(u) - A_{aa}A_{jl})(u) &= -q^{s_a}\lambda(q^{-1}+q)^2 = (q-q^{-1})A_{ja}A_{al}(u)\\A_{jl}A_{bb}(u) = A_{bb}A_{jl}(u) &= q^{s_b}\lambda\\(A_{bl}A_{jb}(u) - A_{jb}A_{bl})(u) &= -q^{s_b}\lambda(q^{-1}+q) = (q-q^{-1})A_{ja}A_{aj}(u)\\A_{ji}A_{il}(u) = A_{il}A_{ji}(u) &= q^{s_i}\lambda\\(A_{jl}A_{ii}(u) - A_{ii}A_{jl})(u) &= q^{s_i}\lambda(q-q^{-1}) = (q-q^{-1})A_{ja}A_{aj}(u)\\A_{jl}A_{jj}(u) = qA_{jj}A_{jl}(u) &= q^{s_j}\lambda\end{aligned}$$

where $i+1 \leq a \leq j-1$, $j+1 \leq b \leq l-1$.

(d) If $h < i$ and $i < l$ then

$$\overline{\rho}(f_\alpha K e_\gamma) = \lambda q^{s_h}(E_{j,i} \otimes E_{h,l} + E_{h,l} \otimes E_{j,i} + (q^{-1}-q)E_{h,i} \otimes E_{j,l})$$

and all products evaluated on u vanish except that

$$\begin{aligned}A_{ji}A_{hl}(u) = A_{hl}A_{ji}(u) &= q^{s_h}\lambda\\(A_{jl}A_{hi}(u) - A_{hi}A_{jl})(u) &= -q^{s_h}\lambda(q^{-1}+q) = (q-q^{-1})A_{ji}A_{hl}(u).\end{aligned}$$

(e) If $h < i$ and $i = l$ then

$$\overline{\rho}(f_\alpha K e_\gamma) = \lambda q^{s_h}(E_{j,i} \otimes E_{h,i} + q^{-1}E_{h,i} \otimes E_{j,i})$$

and all products evaluated on u vanish except that

$$A_{ji}A_{hi}(u) = qA_{hi}A_{ji}(u) = \lambda q^{s_h}.$$

(f) If $h < i$ and $i > l$ then

$$\overline{\rho}(f_\alpha K e_\gamma) = \lambda q^{s_h}(E_{j,i} \otimes E_{h,l} + E_{h,l} \otimes E_{j,i})$$

and all products evaluated on u vanish except that

$$(A_{ji}A_{hl} - A_{hl}A_{ji})(u) = 0 = (q^{-1} + q)A_{jl}A_{hi}.$$

(g) If $i < h$ and $h < j$ then

$$\overline{\rho}(f_\alpha K e_\gamma) = \lambda q^{s_h}(E_{j,i} \otimes E_{h,l} + E_{h,l} \otimes E_{j,i} + (q^{-1} - q)E_{h,i} \otimes E_{j,l})$$

and all products evaluated on u vanish except that

$$A_{ji}A_{hl}(u) = A_{hl}A_{ji}(u) = q^{s_h}\lambda$$
$$(A_{jl}A_{hi}(u) - A_{hi}A_{jl})(u) = -q^{s_h}\lambda(q^{-1} + q) = (q - q^{-1})A_{ji}A_{hl}(u).$$

(h) If $i < h$ and $h = j$ then

$$\overline{\rho}(f_\alpha K e_\gamma) = \lambda q^{s_h}(E_{h,i} \otimes E_{h,l} + qE_{h,l} \otimes E_{h,i})$$

and all products evaluated on u vanish except that

$$A_{hl}A_{hi}(u) = qA_{hi}A_{hl}(u) = \lambda q^{s_h+1}.$$

(i) If $i < h$ and $h > j$ then

$$\overline{\rho}(f_\alpha K e_\gamma) = \lambda q^{s_h}(E_{j,i} \otimes E_{h,l} + E_{h,l} \otimes E_{j,i})$$

and all products evaluated on u vanish except that

$$(A_{hl}A_{ji} - A_{ji}A_{hl})(u) = 0 = (q^{-1} + q)A_{hi}A_{jl}.$$

(vii) If $u = f_\alpha f_\beta K e_\gamma$, we have

$$\begin{aligned}\overline{\rho}(f_\alpha f_\beta K e_\gamma) = \lambda\big(&q^{-\delta_{i,r}}E_{j,i} \otimes E_{t,r} + q^{\delta_{j,t}}E_{t,r} \otimes E_{j,i} \\ &+ (q^{-1} - q)q^{-\delta_{i,r}}E_{t,i} \otimes E_{j,r}\big)\big(E_{h,l} \otimes D_h D_l^{-1} \\ &+ I \otimes E_{h,l} + (q^{-1} - q)\sum_{m=h}^{l-2} E_{h,m+1} \otimes E_{m+1,l}\big)\end{aligned}$$

where $\lambda = (-q)^{i+r-j-t+2}q^{s_i+s_r}$. The third summand appears if $i + 1 \leq t \leq j - 1$. We have the following cases:

(a) If $\alpha = \beta$, then we have

$$\overline{\rho}(f_\alpha f_\beta Ke\gamma) = \lambda(q^{-1}+q)\big(E_{j,i}E_{h,l} \otimes E_{j,i}D_hD_l^{-1} + E_{j,i} \otimes E_{j,i}E_{h,l} + (q^{-1}-q)\sum_{m=h}^{l-2} E_{j,i}E_{h,m+1} \otimes E_{j,i}E_{m+1,l}\big)$$

which is zero if $i \neq h$. If $h = i$ we have

$$\overline{\rho}(f_\alpha f_\beta Ke\gamma) = \lambda(q^{-1}+q)\big(qE_{j,l} \otimes E_{j,i} + E_{j,i} \otimes E_{j,l}\big)$$

and all products evaluated on u vanish except that

$$A_{jl}A_{ji}(u) = qA_{ji}A_{jl}(u) = q\lambda(q^{-1}+q)$$

Suppose $\alpha < \beta$ for the below cases.

(b) If $i = r$ and $j < t$, then $\overline{\rho}(f_\alpha f_\beta Ke\gamma) \neq 0$ only if $h = i$, where

$$\overline{\rho}(f_\alpha f_\beta Ke\gamma) = \lambda(E_{j,l} \otimes E_{t,i} + q^{-1}E_{j,i} \otimes E_{t,l} + qE_{t,l} \otimes E_{j,i} + E_{t,i} \otimes E_{j,l})$$

and all products evaluated on u vanish except

$$A_{ti}A_{jl}(u) = A_{jl}A_{ti}(u) = \lambda$$
$$(A_{tl}A_{ji} - A_{ji}A_{tl})(u) = \lambda(q - q^{-1}) = (q - q^{-1})A_{jl}A_{ti}(u)$$

(c) If $i < r$ and $j = t$, then $\overline{\rho}(f_\alpha f_\beta Ke\gamma) \neq 0$ only if $h = i$ or $h = r$. If $h = i$ and $r \leq l - 1$ we have

$$\overline{\rho}(f_\alpha f_\beta Ke\gamma) = \lambda(E_{j,l} \otimes E_{jr} + q^{-1}E_{j,r} \otimes E_{j,l})$$

and all products evaluated on u vanish except

$$A_{jl}A_{jr}(u) = qA_{jr}A_{jl}(u) = \lambda.$$

If $h = i$ and $r = l$ we have

$$\overline{\rho}(f_\alpha f_\beta Ke\gamma) = \lambda(q^{-1}+q)E_{j,r} \otimes E_{j,r}$$

and all products evaluated on u vanish.

If $h = i$ and $r > l$ we have

$$\overline{\rho}(f_\alpha f_\beta Ke\gamma) = \lambda(E_{j,l} \otimes E_{j,r} + qE_{j,r} \otimes E_{j,l})$$

and all products evaluated on u vanish except

$$A_{jr}A_{jl}(u) = qA_{jl}A_{jr}(u) = q\lambda.$$

If $h = r$ we have

$$\overline{\rho}(f_\alpha f_\beta Ke\gamma) = \lambda(E_{j,i} \otimes E_{j,l} + qE_{j,l} \otimes E_{j,i})$$

and all products evaluated on u vanish except

$$A_{jl}A_{ji}(u) = qA_{ji}A_{jl}(u) = q\lambda.$$

(d) If $i < r$ and $j < t$, then $\overline{\rho}(f_\alpha f_\beta Ke\gamma) \neq 0$ only if $h = i$ or $h = r$. If $h = i$ and $r \leq l - 1$ we have

$$\overline{\rho}(f_\alpha f_\beta Ke\gamma) = \lambda(E_{j,l} \otimes E_{t,r} + E_{t,r} \otimes E_{j,l} + (q^{-1} - q)E_{j,r} \otimes E_{t,l})$$

and all products evaluated on u vanish except

$$A_{jl}A_{tr}(u) = A_{tr}A_{jl}(u) = \lambda$$
$$(A_{tl}A_{jr} - A_{jr}A_{tl})(u) = (q - q^{-1})\lambda = (q - q^{-1})A_{tr}A_{jl}(u).$$

If $h = i$ and $r = l$ we have

$$\overline{\rho}(f_\alpha f_\beta Ke\gamma) = \lambda(q^{-1}E_{j,l} \otimes E_{t,l} + E_{t,l} \otimes E_{j,l})$$

and all products evaluated on u vanish except

$$A_{tl}A_{jl}(u) = qA_{jl}A_{tl}(u) = \lambda.$$

If $h = i$ and $r > l$ we have

$$\overline{\rho}(f_\alpha f_\beta Ke\gamma) = \lambda(E_{j,l} \otimes E_{t,r} + E_{t,r} \otimes E_{j,l})$$

and all products evaluated on u vanish except

$$(A_{tr}A_{jl} - A_{jl}A_{tr})(u) = 0 = (q - q^{-1})A_{tl}A_{jr}(u).$$

If $h = r$ we have

$$\overline{\rho}(f_\alpha f_\beta Ke\gamma) = \lambda(E_{j,i} \otimes E_{t,l} + E_{t,l} \otimes E_{j,i})$$

and all products evaluated on u vanish except

$$(A_{tl}A_{ji} - A_{ji}A_{tl})(u) = 0 = (q - q^{-1})A_{ti}A_{jl}(u).$$

(e) If $i < r$ and $i+1 \le t \le j-1$, then $\overline{\rho}(f_\alpha f_\beta K e\gamma) \neq 0$ only if $h = i$ or $h = r$. If $h = i$ and $r \le l-1$ we have

$$\overline{\rho}(f_\alpha f_\beta K e\gamma) = \lambda(E_{j,l} \otimes E_{t,r} + (1 + (q^{-1} - q)^2)E_{t,r} \otimes E_{j,l} \\ + (q^{-1} - q)E_{j,r} \otimes E_{t,l} + (q^{-1} - q)E_{t,l} \otimes E_{j,r})$$

and all products evaluated on u vanish except

$$A_{jr}A_{tl}(u) = A_{tl}A_{jr}(u) = \lambda(q^{-1} - q)$$
$$(A_{jl}A_{tr} - A_{tr}A_{jl})(u) = -(q - q^{-1})^2\lambda = (q - q^{-1})A_{tl}A_{jr}(u).$$

If $h = i$ and $r = l$ we have

$$\overline{\rho}(f_\alpha f_\beta K e\gamma) = \lambda(q^{-1}E_{j,l} \otimes E_{t,l} + q^{-2}E_{t,l} \otimes E_{j,l})$$

and all products evaluated on u vanish except

$$A_{jl}A_{tl}(u) = qA_{tl}A_{jl}(u) = q^{-1}\lambda.$$

If $h = i$ and $r > l$ we have

$$\overline{\rho}(f_\alpha f_\beta K e\gamma) = \lambda(E_{j,l} \otimes E_{t,r} + E_{t,r} \otimes E_{j,l} + (q^{-1} - q)E_{t,l} \otimes E_{j,r})$$

and all products evaluated on u vanish except

$$A_{jl}A_{tr}(u) = A_{tr}A_{jl}(u) = \lambda$$
$$(A_{jr}A_{tl} - A_{tl}A_{jr})(u) = 0 - (q^{-1} - q)\lambda = (q - q^{-1})A_{tl}A_{jr}(u).$$

If $h = r$ we have

$$\overline{\rho}(f_\alpha f_\beta K e\gamma) = \lambda(E_{j,i} \otimes E_{t,l} + E_{t,l} \otimes E_{j,i} + (q^{-1} - q)E_{t,i} \otimes E_{j,l})$$

and all products evaluated on u vanish except

$$A_{ji}A_{tl}(u) = A_{tl}A_{ji}(u) = \lambda$$
$$(A_{jl}A_{ti} - A_{ti}A_{jl})(u) = 0 - (q^{-1} - q)\lambda = (q - q^{-1})A_{tl}A_{jr}(u).$$

(viii) If $u = f_\alpha K e_\gamma e_\delta$, say $\alpha = \alpha(i,j)$ and $\gamma = \alpha(h,l) \le \delta = \alpha(p,s)$, then we have

$$\overline{\rho}(f_\alpha K e_\gamma e_\delta) = \lambda(E_{j,i} \otimes I + D_i^{-1}D_j \otimes E_{j,i} + (q^{-1} - q)\sum_{m=i}^{j-2} E_{m+1,i} \otimes E_{j,m+1}) \\ (q^{\delta_{h,p}}E_{h,l} \otimes E_{p,s} + q^{-\delta_{l,s}}E_{p,s} \otimes E_{h,l} + (q^{-1} - q)E_{p,l} \otimes E_{h,s})$$

where $\lambda = (-q)^{i-j+1}q^{s_h+s_p}$The third summand appears if $p+1 \leq l \leq s-1$. We have the following cases: If $\gamma = \delta$, all products evaluated on u vanish, so assume $\gamma < \delta$. We have the following cases:

(a) If $\gamma = \delta$, we have

$$\overline{\rho}(f_\alpha K e_\gamma e_\delta) = \lambda(q+q^{-1})(E_{j,i}E_{h,l} \otimes E_{h,l} + D_i^{-1}D_j E_{h,l} \otimes E_{j,i}E_{h,l} + (q^{-1}-q)\sum_{m=i}^{j-2} E_{m+1,i}E_{h,l} \otimes E_{j,m+1}E_{h,l})$$

which is zero if $i \neq h$. If $i = h$, then

$$\overline{\rho}(f_\alpha K e_\gamma e_\delta) = \lambda(q+q^{-1})(E_{j,l} \otimes E_{h,l} + q^{-1}E_{h,l} \otimes E_{j,l})$$

and all products evaluated on u vanish except

$$A_{jl}A_{hl}(u) = qA_{hl}A_{jl}(u) = \lambda(q+q^{-1})$$

Assume $\gamma < \delta$ for the below cases.

(b) If $h = p$ and $l > s$, then $\overline{\rho}(f_\alpha K e_\gamma e_\delta) \neq 0$ only if $i = h$, where

$$\overline{\rho}(f_\alpha K e_\gamma e_\delta) = \lambda(qE_{j,l} \otimes E_{h,s} + E_{j,s} \otimes E_{h,l} + q^{-1}E_{h,s} \otimes E_{j,l} + E_{h,l} \otimes E_{j,s})$$

and all products evaluated on u vanish except

$$A_{js}A_{hl}(u) = A_{hl}A_{js}(u) = \lambda$$
$$(A_{jl}A_{hs} - A_{hs}A_{jl})(u) = (q-q^{-1})\lambda = (q-q^{-1})A_{js}A_{hl}(u)$$

(c) If $h > p$ and $s = l$, then $\overline{\rho}(f_\alpha K e_\gamma e_\delta) \neq 0$ only if $i = h$ or $i = p$. If $i = h$ then

$$\overline{\rho}(f_\alpha K e_\gamma e_\delta) = \lambda(E_{j,l} \otimes E_{p,l} + q^{-1}E_{p,l} \otimes E_{j,l})$$

and all products evaluated on u vanish except

$$A_{jl}A_{pl}(u) = qA_{pl}A_{jl}(u) = \lambda$$

If $i = p$ and $j < h$ then

$$\overline{\rho}(f_\alpha K e_\gamma e_\delta) = \lambda(q^{-1}E_{j,l} \otimes E_{h,l} + E_{h,l} \otimes E_{j,l})$$

and all products evaluated on u vanish except

$$A_{hl}A_{jl}(u) = qA_{jl}A_{hl}(u) = \lambda$$

If $i = p$ and $j = h$ then

$$\overline{\rho}(f_\alpha K e_\gamma e_\delta) = \lambda(q^{-1} + q)E_{j,l} \otimes E_{j,l}$$

and all products evaluated on u vanish.

If $i = p$ and $h \leq j - 1$ then

$$\overline{\rho}(f_\alpha K e_\gamma e_\delta) = \lambda(q^{-1}E_{j,l} \otimes E_{h,l} + q^{-2}E_{h,l} \otimes E_{j,l})$$

and all products evaluated on u vanish except

$$A_{jl}A_{hl}(u) = qA_{hl}A_{jl}(u) = q^{-1}\lambda$$

(d) If $h > p$ and $l > s$, then $\overline{\rho}(f_\alpha K e_\gamma e_\delta) \neq 0$ only if $i = h$ or $i = p$. If $i = h$ then

$$\overline{\rho}(f_\alpha K e_\gamma e_\delta) = \lambda(E_{j,l} \otimes E_{p,s} + E_{p,s} \otimes E_{j,l})$$

and all products evaluated on u vanish except

$$(A_{jl}A_{ps} - A_{ps}A_{jl})(u) = 0 = (q - q^{-1})A_{js}A_{pl}(u)$$

If $i = p$ and $j < h$ then

$$\overline{\rho}(f_\alpha K e_\gamma e_\delta) = \lambda(E_{j,s} \otimes E_{h,l} + E_{h,l} \otimes E_{j,s})$$

and all products evaluated on u vanish except

$$(A_{hl}A_{js} - A_{js}A_{hl})(u) = 0 = (q - q^{-1})A_{hs}A_{jl}$$

If $i = p$ and $j = h$ then

$$\overline{\rho}(f_\alpha K e_\gamma e_\delta) = \lambda(E_{j,s} \otimes E_{j,l} + qE_{j,l} \otimes E_{j,s})$$

and all products evaluated on u vanish except

$$A_{jl}A_{js}(u) = qA_{js}A_{jl} = q\lambda$$

If $i = p$ and $h \leq j - 1$ then

$$\overline{\rho}(f_\alpha K e_\gamma e_\delta) = \lambda(E_{j,s} \otimes E_{h,l} + E_{h,l} \otimes E_{j,s} + (q^{-1} - q)E_{h,s} \otimes E_{j,l})$$

and all products evaluated on u vanish except

$$A_{js}A_{hl}(u) = A_{hl}A_{js}(u) = \lambda$$
$$(A_{jl}A_{hs} - A_{hs}A_{jl})(u) = 0 - (q^{-1} - q)\lambda = (q - q^{-1})A_{js}A_{hl}(u)$$

(e) If $h > p$ and $p + 1 \leq l \leq s - 1$, then $\overline{\rho}(f_\alpha K e_\gamma e_\delta) \neq 0$ only if $i = h$ or $i = p$. If $i = h$ then

$$\overline{\rho}(f_\alpha K e_\gamma e_\delta) = \lambda(E_{j,l} \otimes E_{p,s} + E_{p,s} \otimes E_{j,l} + (q^{-1} - q)E_{p,l} \otimes E_{j,s})$$

and all products evaluated on u vanish except

$$A_{jl}A_{ps}(u) = A_{ps}A_{jl}(u) = \lambda$$
$$(A_{js}A_{pl} - A_{pl}A_{js})(u) = -(q^{-1} - q)\lambda = (q - q^{-1})A_{ps}A_{jl}(u)$$

If $i = p$ and $j < h$ then

$$\overline{\rho}(f_\alpha K e_\gamma e_\delta) = \lambda(E_{j,s} \otimes E_{h,l} + E_{h,l} \otimes E_{j,s} + (q^{-1} - q)E_{j,l} \otimes E_{h,s})$$

and all products evaluated on u vanish except

$$A_{hl}A_{js}(u) = A_{js}A_{hl}(u) = \lambda$$
$$(A_{hs}A_{jl} - A_{jl}A_{hs})(u) = -(q^{-1} - q)\lambda = (q - q^{-1})A_{js}A_{hl}$$

If $i = p$ and $j = h$ then

$$\overline{\rho}(f_\alpha K e_\gamma e_\delta) = \lambda(E_{j,s} \otimes E_{j,l} + q^{-1}E_{j,l} \otimes E_{j,s})$$

and all products evaluated on u vanish except

$$A_{js}A_{jl}(u) = qA_{jl}A_{js} = \lambda$$

If $i = p$ and $h \leq j - 1$ then

$$\begin{aligned}\overline{\rho}(f_\alpha K e_\gamma e_\delta) = \lambda(&E_{j,s} \otimes E_{h,l} + ((q^{-1} - q)^2 + 1)E_{h,l} \otimes E_{j,s} \\ &+ (q^{-1} - q)E_{h,s} \otimes E_{j,l} + E_{j,l} \otimes E_{h,s})\end{aligned}$$

and all products evaluated on u vanish except

$$A_{jl}A_{hs}(u) = A_{hs}A_{jl}(u) = \lambda(q^{-1} - q)$$
$$(A_{js}A_{hl} - A_{hl}A_{js})(u) = -(q^{-1} - q)^2\lambda = (q - q^{-1})A_{hs}A_{jl}(u)$$

(ix) If $u = f_\alpha f_\beta K e_\gamma e_\delta$, we have

$$\begin{aligned}
&\overline{\rho}(f_\alpha f_\beta K e_\gamma e_\delta) \\
&\qquad = \lambda(q^{-\delta_{i,r}} E_{j,i} \otimes E_{t,r} + q^{\delta_{j,t}} E_{t,r} \otimes E_{j,i} + (q^{-1} - q) q^{-\delta_{i,r}} E_{t,i} \otimes E_{j,r}) \\
&\qquad\quad (q^{\delta_{h,p}} E_{h,l} \otimes E_{p,s} + q^{-\delta_{l,s}} E_{p,s} \otimes E_{h,l} + (q^{-1} - q) E_{p,l} \otimes E_{h,s})
\end{aligned}$$

where $\lambda = (-q)^{i+r-j-t+2} q^{s_i+s_r}$. The third summand of the first factor appears if $i + 1 \le t \le j - 1$ and the third summand of the second factor appears if $p + 1 \le l \le s - 1$.

If $\alpha = \beta$ and $\gamma = \delta$ then $\overline{\rho}(f_\alpha f_\beta K e_\gamma e_\delta) \neq 0$ only if $\alpha = \gamma$ and we have

$$\overline{\rho}(f_\alpha f_\beta K e_\gamma e_\delta) = \lambda(q + q^{-1})^2 E_{j,j} \otimes E_{j,j}$$

and all products evaluated on u vanish.

If $\alpha = \beta$ and $\gamma < \delta$ then $\overline{\rho}(f_\alpha f_\beta K e_\gamma e_\delta) \neq 0$ only if $i = h = p$, so $l > s$ and we have

$$\overline{\rho}(f_\alpha f_\beta K e_\gamma e_\delta) = \lambda(q + q^{-1})(q E_{j,l} \otimes E_{j,s} + E_{j,s} \otimes E_{j,l})$$

and all products evaluated on u vanish except

$$A_{jl} A_{js}(u) = q A_{js} A_{jl}(u) = q\lambda(q^{-1} + q)$$

If $\alpha < \beta$ and $\gamma = \delta$ then $\overline{\rho}(f_\alpha f_\beta K e_\gamma e_\delta) \neq 0$ only if $i = r = h$, so $j < t$ and we have

$$\overline{\rho}(f_\alpha f_\beta K e_\gamma e_\delta) = \lambda(q + q^{-1})(q^{-1} E_{j,l} \otimes E_{t,l} + E_{t,l} \otimes E_{j,l})$$

and all products evaluated on u vanish except

$$A_{tl} A_{jl}(u) = q A_{jl} A_{tl}(u) = \lambda(q^{-1} + q)$$

Thus, assume $\alpha < \beta$ and $\gamma < \delta$.

If $i = r$ and $j < t$, then $h = p = i$, so $l > s$ and

$$\overline{\rho}(f_\alpha f_\beta K e_\gamma e_\delta) = \lambda(E_{j,l} \otimes E_{t,s} + E_{t,s} \otimes E_{j,l} + q^{-1} E_{j,s} \otimes E_{t,l} + q E_{t,l} \otimes E_{j,s})$$

and all products evaluated on u vanish except

$$\begin{gathered}
A_{ts} A_{jl}(u) = A_{jl} A_{ts}(u) = \lambda \\
(A_{tl} A_{js} - A_{js} A_{tl})(u) = \lambda(q - q^{-1}) = (q - q^{-1}) A_{ts} A_{jl}(u)
\end{gathered}$$

If $i < r$ and $j < t$, then $p = i$ and $r = h$. We have three cases:

(a) If $l > s$,

$$\overline{\rho}(f_\alpha f_\beta K e_\gamma e_\delta) = \lambda(E_{j,s} \otimes E_{t,l} + E_{t,l} \otimes E_{j,s})$$

and all products evaluated on u vanish except

$$(A_{tl}A_{js} - A_{js}A_{tl})(u) = 0 = (q - q^{-1})A_{ts}A_{jl}(u)$$

(b) If $l = s$,

$$\overline{\rho}(f_\alpha f_\beta K e_\gamma e_\delta) = \lambda(q^{-1}E_{j,l} \otimes E_{t,l} + E_{t,l} \otimes E_{j,l})$$

and all products evaluated on u vanish except

$$A_{tl}A_{jl}(u) = qA_{jl}A_{tl}(u) = \lambda$$

(c) If $l < s$,

$$\overline{\rho}(f_\alpha f_\beta K e_\gamma e_\delta) = \lambda(E_{j,s} \otimes E_{t,l} + E_{t,l} \otimes E_{j,s} + (q^{-1} - q)E_{j,l} \otimes E_{t,s})$$

and all products evaluated on u vanish except

$$A_{tl}A_{js}(u) = A_{js}A_{tl}(u) = \lambda$$
$$(A_{ts}A_{jl} - A_{jl}A_{ts})(u) = \lambda(q - q^{-1}) = (q - q^{-1})A_{ts}A_{jl}(u)$$

If $i < r$ and $j = t$, then $p = i$ and $r = h$. We have three cases:

(a) If $l > s$,

$$\overline{\rho}(f_\alpha f_\beta K e_\gamma e_\delta) = \lambda(E_{j,s} \otimes E_{j,l} + qE_{j,l} \otimes E_{j,s})$$

and all products evaluated on u vanish except

$$A_{jl}A_{js}(u) = qA_{js}A_{jl}(u) = q\lambda$$

(b) If $l = s$,

$$\overline{\rho}(f_\alpha f_\beta K e_\gamma e_\delta) = \lambda(q^{-1} + q)E_{j,l} \otimes E_{j,l}$$

and all products evaluated on u vanish.

(c) If $l < s$,

$$\overline{\rho}(f_\alpha f_\beta K e_\gamma e_\delta) = \lambda(E_{j,s} \otimes E_{j,l} + q^{-1} E_{j,l} \otimes E_{j,s})$$

and all products evaluated on u vanish except

$$A_{js} A_{jl}(u) = q A_{jl} A_{js}(u) = \lambda$$

If $i < r$ and $j > t$, then $p = i$ and $r = h$. We have three cases:

(a) If $l > s$,

$$\overline{\rho}(f_\alpha f_\beta K e_\gamma e_\delta) = \lambda(E_{j,s} \otimes E_{t,l} + E_{t,l} \otimes E_{j,s} + (q^{-1} - q) E_{t,s} \otimes E_{j,l})$$

and all products evaluated on u vanish except

$$A_{js} A_{tl}(u) = A_{tl} A_{js}(u) = \lambda$$
$$(A_{jl} A_{ts} - A_{ts} A_{jl})(u) = -(q^{-1} - q)\lambda = (q - q^{-1}) A_{js} A_{tl}(u)$$

(b) If $l = s$,

$$\overline{\rho}(f_\alpha f_\beta K e_\gamma e_\delta) = \lambda(q^{-1} E_{j,l} \otimes E_{t,l} + q^{-2} E_{t,l} \otimes E_{j,l})$$

and all products evaluated on u vanish except

$$A_{jl} A_{tl}(u) = q A_{tl} A_{jl}(u) = q^{-1}\lambda$$

(c) If $l < s$,

$$\begin{aligned}\overline{\rho}(f_\alpha f_\beta K e_\gamma e_\delta) = \lambda(&E_{j,s} \otimes E_{t,l} + ((q^{-1} - q)^2 + 1) E_{t,l} \otimes E_{j,s} \\ &+ (q^{-1} - q)(E_{j,l} \otimes E_{t,s} + E_{t,s} \otimes E_{j,l}))\end{aligned}$$

and all products evaluated on u vanish except

$$A_{jl} A_{ts}(u) = A_{ts} A_{jl}(u) = (q^{-1} - q)\lambda$$
$$(A_{js} A_{tl} - A_{tl} A_{js})(u) = -(q^{-1} - q)^2\lambda = (q - q^{-1}) A_{jl} A_{ts}(u)$$

Hence, all relations (5.5)-(5.8) are satisfied. □

Now, we are ready to prove the following theorem.

Theorem 5.2.6. *The bilinear form $< u, x >= \psi(x)(u)$ realizes a duality between the bialgebras $U_q gl(n)$ and $M_q(n)$.*

Proof. We saw that the algebra morphism ψ is well-defined by Lemma 5.2.5. Hence, it is enough to show that the bilinear form $< u, x >= \psi(x)(u)$ satisfies the relations (5.1) and (5.3) of Definition 5.1.1. Let us show bilinear form $< u, x >= \psi(x)(u)$ satisfies the relation (5.1). The identity $\rho(1) = I_n$ gives:

$$\big(< 1, a_{ij} > \big) = \big(A_{ij}(1)\big) = \rho(1) = I_n = \big(\varepsilon(a_{ij})\big)$$

By the relation (5.2) and (5.4) we have

$$< 1, xy > =< 1, x >< 1, y >,$$
$$< 1, 1 > = \varepsilon(1) = 1$$

So $x \mapsto< 1, x >$ is an algebra morphism. Since both $x \mapsto< 1, x >$ and ε are algebra morphisms and they coincide on the generators of $M_q(n)$, they are equal.

To prove relation (5.3), let $C(x)$ be the following condition on an element x of $M_q(n)$:

For any pair $(u, v) \in U$, we have

$$< uv, x > = \sum_{(x)} < u, x' >< v, x'' > .$$

The relation (5.4) gives

$$< uv, 1 >= \varepsilon(uv) = \varepsilon(u)\varepsilon(v) =< u, 1 >< v, 1 > .$$

Hence, $C(1)$ is satisfied. Now, we will prove that condition $C(a_{ij})$ holds for all $1 \leq i, j \leq n$. Using the identities $\rho(uv) = \rho(u)\rho(v)$ and $\Delta(\big(a_{ij}\big)) = \big(a_{ij}\big) \otimes \big(a_{ij}\big)$, we get

$$\big(\psi(a_{ij})(uv)\big) = \big(A_{ij}(uv)\big) = \rho(uv) = \big(< uv, a_{ij} > \big) = \rho(u)\rho(v)$$
$$= \big(< u, a_{ij} > \big)\big(< v, a_{ij} > \big) = \big(\sum_{(a_{ij})} < u, a'_{ij} >< v, a''_{ij} >\big)$$

The last equation holds because $\Delta(a_{ij})$ is given by matrix multiplication. Now, the following lemmas will complete the proof. $\square$

Lemma 5.2.7. *If conditions $C(x)$ and $C(y)$ hold, then so does $C(\lambda x + y)$ for any constant λ.*

Proof. Using the identity

$$\Delta(\lambda x + y) = \lambda \Delta(x) + \Delta(y) = \lambda \sum_{(x)} x' \otimes x'' + \sum_{(y)} y' \otimes y''$$

and $<,>$ being bilinear, we have

$$\begin{aligned} < uv, \lambda x + y > &= \lambda < uv, x > + < uv, y > \\ &= \lambda \sum_{(x)} < u, x' >< v, x'' > + \sum_{(y)} < u, y' >< v, y'' > \\ &= \sum_{(\lambda x + y)} < u, (\lambda x + y)' >< v, (\lambda x + y)'' > \end{aligned}$$

□

Lemma 5.2.8. *If conditions $C(x)$ and $C(y)$ hold, then so does $C(xy)$.*

Proof. Using relation (5.2), and conditions $C(x)$ and $C(y)$, we have

$$\begin{aligned} < uv, xy > &= \sum_{(uv)} < (uv)', x >< (uv)'', y > \\ &= \sum_{(u)(v)} < u'v', x >< u''v'', y > \\ &= \sum_{(u)(v)(x)(y)} < u', x' >< v', x'' >< u'', y' >< v'', y'' > . \end{aligned}$$

They also yield

$$\begin{aligned} \sum_{(xy)} < u, (xy)' >< v, (xy)'' > &= \sum_{(x)(y)} < u, x'y' >< v, x''y'' > \\ &= \sum_{(u)(v)(x)(y)} < u', x' >< u'', y' >< v', x'' >< v'', y'' > . \\ &=< uv, xy > . \end{aligned}$$

□

CHAPTER 6

FACTORIZATION

This chapter is the heart of the thesis where our results on factorization are presented.

We begin the chapter by introducing a new algebra $R_{p,q}(n)$. This algebra can be thought of the quantized analogue of functions on the double diagonal matrices. In our main result, the Factorization Theorem, we prove that there is an algebra map $\phi : M_{p,q}(n) \to R_{p,q}(n)$ under which the elements a_{ij} get "factorized". The rest of the chapter is devoted to the proof of this theorem.

6.1 The Algebra $\mathbf{R_{p,q}(n)}$

Definition 6.1.1. *Let $R_{p,q}(n) = K\{x_i^{(k)}, y_i^{(k)} | 1 \leq k \leq n-1, 1 \leq i \leq 2n-1\}/J$ be the quotient of the free algebra over K generated by the generators $\{x_i^{(k)}, y_i^{(k)} | k \in \{1, 2, ..., n-1\}, i \in \{1, 2, ..., 2n-1\}\}$ by the two-sided ideal J generated by the relations*

$$\begin{aligned}
x_{2i}^{(k)} x_{2i-1}^{(k)} &= p x_{2i-1}^{(k)} x_{2i}^{(k)}, & y_{2i+1}^{(k)} y_{2i}^{(k)} &= p y_{2i}^{(k)} y_{2i+1}^{(k)}, \\
x_{2i+1}^{(k)} x_{2i}^{(k)} &= q x_{2i}^{(k)} x_{2i+1}^{(k)}, & y_{2i}^{(k)} y_{2i-1}^{(k)} &= q y_{2i-1}^{(k)} y_{2i}^{(k)}, \\
x_i^{(k)} x_j^{(k)} &= x_j^{(k)} x_i^{(k)}, & y_i^{(k)} y_j^{(k)} &= y_j^{(k)} y_i^{(k)}, \\
x_i^{(k_1)} x_l^{(k_2)} &= x_l^{(k_2)} x_i^{(k_1)}, & y_i^{(k_1)} y_l^{(k_2)} &= y_l^{(k_2)} y_i^{(k_1)}, \\
x_i^{(k_3)} y_l^{(k_4)} &= y_l^{(k_4)} x_i^{(k_3)}
\end{aligned}$$

for every $i, j, k, l, k_1, k_2, k_3, k_4$ where $k_1 \neq k_2$, $|j - i| \geq 2$.

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

$$X^{(k)} = \begin{pmatrix} x_1^{(k)} & x_2^{(k)} & & & \\ & x_3^{(k)} & x_4^{(k)} & & \\ & & \ddots & \ddots & \\ & & & & x_{2n-2}^{(k)} \\ & & & & x_{2n-1}^{(k)} \end{pmatrix}$$

$$Y^{(k)} = \begin{pmatrix} y_1^{(k)} & & & & \\ y_2^{(k)} & y_3^{(k)} & & & \\ & y_4^{(k)} & y_5^{(k)} & & \\ & & \ddots & \ddots & \\ & & & y_{2n-2}^{(k)} & y_{2n-1}^{(k)} \end{pmatrix}$$

6.2 The Factorization Theorem

Theorem 6.2.1. *The map $\phi : M_{p,q}(n) \to R_{p,q}(n)$ mapping a_{ij} to $\hat{a}_{ij}$, where $\hat{a}_{ij}$ is the ijth entry of the matrix $\hat{A} = X^{(1)}X^{(2)}...X^{(n-1)}Y^{(1)}Y^{(2)}...Y^{(n-1)}$, is well-defined, i.e. the entries of $\hat{A} = \left(\hat{a}_{ij}\right)$ satisfy relations*

$$\begin{aligned} \hat{a}_{il}\hat{a}_{ik} &= p\hat{a}_{ik}\hat{a}_{il}, \\ \hat{a}_{jk}\hat{a}_{ik} &= q\hat{a}_{ik}\hat{a}_{jk}, \\ \hat{a}_{jk}\hat{a}_{il} &= p^{-1}q\hat{a}_{il}\hat{a}_{jk}, \\ \hat{a}_{jl}\hat{a}_{ik} &= \hat{a}_{ik}\hat{a}_{jl} + (p - q^{-1})\hat{a}_{jk}\hat{a}_{il} \end{aligned}$$

whenever $j > i$ and $l > k$.

6.3 The proof of the Main Theorem

To prove the theorem we will use infinite double diagonal matrices and the following lemmas.

Definition 6.3.1. *Let $R_{p,q}(\infty) = K\{x_i^{(k)} | k \in \mathbb{Z}_+, i \in \mathbb{Z}\}/J_\infty$ be the quotient of the free algebra over K generated by the generators $\{x_i^{(k)} | k \in \mathbb{Z}_+, i \in \mathbb{Z}\}$ by the two-sided ideal J_∞ generated by the relations*

$$\begin{aligned} x_{2i}^{(k)} x_{2i-1}^{(k)} &= p x_{2i-1}^{(k)} x_{2i}^{(k)}, \\ x_{2i+1}^{(k)} x_{2i}^{(k)} &= q x_{2i}^{(k)} x_{2i+1}^{(k)}, \\ x_i^{(k)} x_j^{(k)} &= x_j^{(k)} x_i^{(k)}, \\ x_i^{(k_1)} x_l^{(k_2)} &= x_l^{(k_2)} x_i^{(k_1)}, \end{aligned}$$

for every i, j, k, l, k_1, k_2 where $k_1 \neq k_2$, $|j-i| \geq 2$.

Let $\tilde{X}^{(k)}$ denote the double diagonal infinite matrix with entries from $R_{p,q}(\infty)$ and iith and $ii+1$th entries given by

$$\begin{aligned} (\tilde{X}^{(k)})_{ii} &= x_{2i-1}^{(k)}, \\ (\tilde{X}^{(k)})_{ii+1} &= x_{2i}^{(k)}, \\ (\tilde{X}^{(k)})_{ij} &= 0 \end{aligned}$$

for every $i, j \in \mathbb{Z}$, where $|j-i| \geq 2$.

Lemma 6.3.2.

$$(\tilde{X}^{(1)} \tilde{X}^{(2)} ... \tilde{X}^{(n)})_{ij} = \begin{cases} x_{2i-1}^{(1)} x_{2i-1}^{(2)} ... x_{2i-1}^{(n)} & \text{if } i = j \\ \displaystyle\sum_{k_{j-i}=j-i}^{n} ... \sum_{k_2=2}^{k_3-1} \sum_{k_1=1}^{k_2-1} \omega & \text{if } 0 < j-i \leq n \\ 0 & \text{otherwise} \end{cases}$$

where $\omega = x_{2i-1}^{(1)} x_{2i-1}^{(2)} ... x_{2i-1}^{(k_1-1)} x_{2i}^{(k_1)} x_{2i+1}^{(k_1+1)} ... x_{2j-3}^{(k_{j-i}-1)} x_{2(j-1)}^{(k_{j-i})} x_{2j-1}^{(k_{j-i}+1)} ... x_{2j-1}^{(n-1)} x_{2j-1}^{(n)}$.

Remark 6.3.3. *Note that with this notation we mean, exactly $j-i$ many terms with even index appears in each summand.*

Proof. Do induction on n. If $n = 2$

$$(\tilde{X}^{(1)} \tilde{X}^{(2)})_{ij} = \begin{cases} 0 & \text{if } i > j \\ x_{2i-1}^{(1)} x_{2j-1}^{(2)} & \text{if } i = j \\ x_{2i-1}^{(1)} x_{2(j-1)}^{(2)} + x_{2i}^{(1)} x_{2j-1}^{(2)} & \text{if } j = i+1 \\ x_{2i}^{(1)} x_{2(j-1)}^{(2)} & \text{if } j = i+2 \\ 0 & \text{if } j-i \geq 3 \end{cases}$$

Assuming the result for $n-1$, we have

$$\begin{aligned}
(\tilde{X}^{(1)}\tilde{X}^{(2)}...\tilde{X}^{(n)})_{ij} &= (\tilde{X}^{(1)}\tilde{X}^{(2)}...\tilde{X}^{(n-1)})_{ij-1}(\tilde{X}^{(n)})_{j-1j} \\
&\quad + (\tilde{X}^{(1)}\tilde{X}^{(2)}...\tilde{X}^{(n-1)})_{ij}(\tilde{X}^{(n)})_{jj} \\
&= \sum_{k_{j-i-1}=j-i-1}^{n-1} ... \sum_{k_1=1}^{k_2-1} x_{2i-1}^{(1)}...x_{2i}^{(k_1)}...x_{2(j-2)}^{(k_{j-i-1})}...x_{2j-3}^{(n-1)}x_{2j-2}^{(n)} \\
&\quad + \sum_{k_{j-i}=j-i}^{n-1} ... \sum_{k_1=1}^{k_2-1} x_{2i-1}^{(1)}...x_{2i}^{(k_1)}...x_{2(j-1)}^{(k_{j-i})}...x_{2j-1}^{(n-1)}x_{2j-1}^{(n)} \qquad (6.1) \\
&= \sum_{k_{j-i}=j-i}^{n} ... \sum_{k_1=1}^{k_2-1} x_{2i-1}^{(1)}...x_{2i}^{(k_1)}...x_{2(j-1)}^{(k_{j-i})}...x_{2j-1}^{(n)} \qquad (6.2)
\end{aligned}$$

Note that last equality holds because $k_{j-i}=n$ summand of (6.2) is the first summand of (6.1). If $j-i>n$ (or $j-i<0$) then $j-1-i>n-1$ (or $j-1-i<-1$), which means $(\tilde{X}^{(1)}\tilde{X}^{(2)}...\tilde{X}^{(n-1)})_{ij-1}=0$ and $(\tilde{X}^{(1)}\tilde{X}^{(2)}...\tilde{X}^{(n-1)})_{ij}=0$. So we get $(\tilde{X}^{(1)}\tilde{X}^{(2)}...\tilde{X}^{(n-1)}\tilde{X}^{(n)})_{ij}=0$. □

Let $\tilde{Y}^{(k)}$ denote the double diagonal infinite matrix with entries form $R_q(\infty)$ and iith and $i+1i$th entries given by

$$\begin{aligned}
(\tilde{Y}^{(k)})_{ii} &= y_{2i-1}^{(k)}, \\
(\tilde{Y}^{(k)})_{i+1i} &= y_{2i}^{(k)}, \\
(\tilde{Y}^{(k)})_{ij} &= 0
\end{aligned}$$

for every $i,j\in\mathbb{Z}$, where $|j-i|\geq 2$.

Lemma 6.3.4.

$$(\tilde{Y}^{(1)}\tilde{Y}^{(2)}...\tilde{Y}^{(n)})_{ij} = \begin{cases} y_{2i-1}^{(1)}y_{2i-1}^{(2)}...y_{2i-1}^{(n)} & \textit{if } i=j \\ \displaystyle\sum_{k_{i-j}=i-j}^{n} ... \sum_{k_2=2}^{k_3-1}\sum_{k_1=1}^{k_2-1}\omega' & \textit{if } 0<i-j\leq n \\ 0 & \textit{otherwise} \end{cases}$$

where $\omega' = y_{2i-1}^{(1)}y_{2i-1}^{(2)}...y_{2i-1}^{(k_1-1)}y_{2i-2}^{(k_1)}y_{2i-3}^{(k_1+1)}...y_{2j+1}^{(k_{j-i}-1)}y_{2j}^{(k_{j-i})}y_{2j-1}^{(k_{j-i}+1)}...y_{2j-1}^{(n-1)}y_{2j-1}^{(n)}$.

Remark 6.3.5. *Note that with this notation we mean, exactly $i-j$ many terms with even index appears in each summand.*

Proof. Do induction on n. If $n = 2$

$$(\tilde{Y}^{(1)}\tilde{Y}^{(2)})_{ij} = \begin{cases} 0 & \text{if } j > i \\ y^{(1)}_{2i-1}y^{(2)}_{2j-1} & \text{if } i = j \\ y^{(1)}_{2i-2}y^{(2)}_{2j-1} + y^{(1)}_{2i-1}y^{(2)}_{2j} & \text{if } i = j+1 \\ y^{(1)}_{2i}y^{(2)}_{2j} & \text{if } i = j+2 \\ 0 & \text{if } i-j \geq 3 \end{cases}$$

Assuming the result for $n-1$, we have

$$\begin{aligned}
(\tilde{Y}^{(1)}\tilde{Y}^{(2)}...\tilde{Y}^{(n)})_{ij} &= (\tilde{Y}^{(1)}\tilde{Y}^{(2)}...\tilde{Y}^{(n-1)})_{ij+1}(\tilde{Y}^{(n)})_{j+1j} \\
&\quad + (\tilde{Y}^{(1)}\tilde{Y}^{(2)}...\tilde{Y}^{(n-1)})_{ij}(\tilde{Y}^{(n)})_{jj} \\
&= \sum_{k_{i-j-1}=i-j-1}^{n-1} ... \sum_{k_1=1}^{k_2-1} y^{(1)}_{2i-1}...y^{(k_1)}_{2i-2}...y^{(k_{i-j-1})}_{2(j+1)}...y^{(n-1)}_{2j+1}\,y^{(n)}_{2j} \\
&\quad + \sum_{k_{i-j}=i-j}^{n-1} ... \sum_{k_1=1}^{k_2-1} y^{(1)}_{2i-1}...y^{(k_1)}_{2i-2}...y^{(k_{i-j})}_{2j}...y^{(n-1)}_{2j-1}\,y^{(n)}_{2j-1} \qquad (6.3) \\
&= \sum_{k_{i-j}=i-j}^{n} ... \sum_{k_1=1}^{k_2-1} y^{(1)}_{2i-1}...y^{(k_1)}_{2i}...y^{(k_{i-j})}_{2j}...y^{(n)}_{2j-1} \qquad (6.4)
\end{aligned}$$

Note that last equality holds because $k_{i-j} = n$ summand of (6.4) is the first summand of (6.3).

If $i-j > n$ (or $i-j < 0$) then $i-1-j > n-1$ (or $i-1-j < -1$), which means $(\tilde{Y}^{(1)}\tilde{Y}^{(2)}...\tilde{Y}^{(n-1)})_{ij+1} = 0$ and $(\tilde{Y}^{(1)}\tilde{Y}^{(2)}...\tilde{Y}^{(n-1)})_{ij} = 0$. So we get $(\tilde{Y}^{(1)}\tilde{Y}^{(2)}...\tilde{Y}^{(n-1)}\tilde{Y}^{(n)})_{ij} = 0$. $\square$

Lemma 6.3.6. $(X^{(1)}X^{(2)}...X^{(n-1)}Y^{(1)}Y^{(2)}...Y^{(n-1)})_{ij} = \sum_{\substack{i\leq l \\ j\leq l}}^{n} X_{il}Y_{lj}$

where

$$X_{il} = \sum_{k_{l-i}=l-i}^{n} ... \sum_{k_1=1}^{k_2-1} x^{(1)}_{2i-1}...x^{(k_1)}_{2i}...x^{(k_{l-i})}_{2(l-1)}...x^{(n)}_{2l-1} \quad \textit{and}$$

$$Y_{lj} = \sum_{k_{l-j}=l-j}^{n} ... \sum_{k_1=1}^{k_2-1} y^{(1)}_{2l-1}...y^{(k_1)}_{2l}...y^{(k_{l-j})}_{2j}...y^{(n)}_{2j-1}$$

Lemma 6.3.7. *The following relations hold for* $\tilde{A}^{(n)} = \tilde{X}^{(1)}\tilde{X}^{(2)}...\tilde{X}^{(n)}$*:*

$$\begin{aligned}
\tilde{A}^{(n)}_{ad}\tilde{A}^{(n)}_{ac} &= p\tilde{A}^{(n)}_{ac}\tilde{A}^{(n)}_{ad},\\
\tilde{A}^{(n)}_{bc}\tilde{A}^{(n)}_{ac} &= q\tilde{A}^{(n)}_{ac}\tilde{A}^{(n)}_{bc},\\
\tilde{A}^{(n)}_{bc}\tilde{A}^{(n)}_{ad} &= p^{-1}q\tilde{A}^{(n)}_{ad}\tilde{A}^{(n)}_{bc},\\
\tilde{A}^{(n)}_{bd}\tilde{A}^{(n)}_{ac} &= \tilde{A}^{(n)}_{ac}\tilde{A}^{(n)}_{bd} + (p-q^{-1})\tilde{A}^{(n)}_{bc}\tilde{A}^{(n)}_{ad}
\end{aligned}$$

if $d > c$ *and* $b > a$*.*

Proof. Proof will be by induction on n. The relations hold in $\tilde{A}^{(1)} = \tilde{X}^{(1)}$ by definition. Consider $\tilde{A}^{(n)} = \tilde{X}^{(1)}\tilde{X}^{(2)}...\tilde{X}^{(n)}$.

First, denoting $\tilde{A}'^{(n-1)} = \tilde{X}^{(2)}...\tilde{X}^{(n)}$, note that

$$\begin{aligned}
\tilde{A}^{(n)}_{ij} &= \tilde{A}^{(n-1)}_{ij-1}x^{(n)}_{2j-2} + \tilde{A}^{(n-1)}_{ij}x^{(n)}_{2j-1},\\
\tilde{A}^{(n)}_{ij} &= x^{(1)}_{2i-1}\tilde{A}'^{(n-1)}_{ij} + x^{(1)}_{2i}\tilde{A}'^{(n-1)}_{i+1j}.
\end{aligned}$$

Now assume a, b, c, d are as above, and that the assertion holds for $n-1$.

$$\begin{aligned}
\tilde{A}^{(n)}_{ad}\tilde{A}^{(n)}_{ac} &= (\tilde{A}^{(n-1)}_{ad-1}x^{(n)}_{2d-2} + \tilde{A}^{(n-1)}_{ad}x^{(n)}_{2d-1})(\tilde{A}^{(n-1)}_{ac-1}x^{(n)}_{2c-2} + \tilde{A}^{(n-1)}_{ac}x^{(n)}_{2c-1})\\
&= \tilde{A}^{(n-1)}_{ad-1}x^{(n)}_{2d-2}\tilde{A}^{(n-1)}_{ac-1}x^{(n)}_{2c-2} + \tilde{A}^{(n-1)}_{ad}x^{(n)}_{2d-1}\tilde{A}^{(n-1)}_{ac-1}x^{(n)}_{2c-2}\\
&+ \tilde{A}^{(n-1)}_{ad-1}x^{(n)}_{2d-2}\tilde{A}^{(n-1)}_{ac}x^{(n)}_{2c-1} + \tilde{A}^{(n-1)}_{ad}x^{(n)}_{2d-1}\tilde{A}^{(n-1)}_{ac}x^{(n)}_{2c-1}\\
&= p\tilde{A}^{(n-1)}_{ac-1}x^{(n)}_{2c-2}\tilde{A}^{(n-1)}_{ad-1}x^{(n)}_{2d-2} + p\tilde{A}^{(n-1)}_{ac-1}x^{(n)}_{2c-2}\tilde{A}^{(n-1)}_{ad}x^{(n)}_{2d-1}\\
&+ p\tilde{A}^{(n-1)}_{ac}x^{(n)}_{2c-1}\tilde{A}^{(n-1)}_{ad-1}x^{(n)}_{2d-2} + p\tilde{A}^{(n-1)}_{ac}x^{(n)}_{2c-1}\tilde{A}^{(n-1)}_{ad}x^{(n)}_{2d-1}\\
&= p(\tilde{A}^{(n-1)}_{ac-1}x^{(n)}_{2c-2} + \tilde{A}^{(n-1)}_{ac}x^{(n)}_{2c-1})(\tilde{A}^{(n-1)}_{ad-1}x^{(n)}_{2d-2} + \tilde{A}^{(n-1)}_{ad}x^{(n)}_{2d-1})\\
&= p\tilde{A}^{(n)}_{ac}\tilde{A}^{(n)}_{ad}
\end{aligned}$$

Here,

$$\begin{aligned}
\tilde{A}^{(n-1)}_{ad-1}x^{(n)}_{2d-2}\tilde{A}^{(n-1)}_{ac-1}x^{(n)}_{2c-2} &= p\tilde{A}^{(n-1)}_{ac-1}x^{(n)}_{2c-2}\tilde{A}^{(n-1)}_{ad-1}x^{(n)}_{2d-2},\\
\tilde{A}^{(n-1)}_{ad}x^{(n)}_{2d-1}\tilde{A}^{(n-1)}_{ac-1}x^{(n)}_{2c-2} &= p\tilde{A}^{(n-1)}_{ac-1}x^{(n)}_{2c-2}\tilde{A}^{(n-1)}_{ad}x^{(n)}_{2d-1},\\
\tilde{A}^{(n-1)}_{ad}x^{(n)}_{2d-1}\tilde{A}^{(n-1)}_{ac}x^{(n)}_{2c-1} &= p\tilde{A}^{(n-1)}_{ac}x^{(n)}_{2c-1}\tilde{A}^{(n-1)}_{ad}x^{(n)}_{2d-1}
\end{aligned}$$

by induction hypothesis and the fact that $x^{(n)}_i$ and $x^{(n)}_j$ commute if $|i-j| > 1$.

If $d > c+1$ then

$$\tilde{A}^{(n-1)}_{ad-1}x^{(n)}_{2d-2}\tilde{A}^{(n-1)}_{ac}x^{(n)}_{2c-1} = p\tilde{A}^{(n-1)}_{ac}x^{(n)}_{2c-1}\tilde{A}^{(n-1)}_{ad-1}x^{(n)}_{2d-2}$$

holds by induction hypothesis. If $d = c + 1$ then

$$\begin{aligned}
\tilde{A}^{(n-1)}_{ad-1} x^{(n)}_{2d-2} \tilde{A}^{(n-1)}_{ac} x^{(n)}_{2c-1} &= \tilde{A}^{(n-1)}_{ac} x^{(n)}_{2c} \tilde{A}^{(n-1)}_{ac} x^{(n)}_{2c-1} \\
&= p\tilde{A}^{(n-1)}_{ac} x^{(n)}_{2c-1} \tilde{A}^{(n-1)}_{ac} x^{(n)}_{2c} \\
&= p\tilde{A}^{(n-1)}_{ac} x^{(n)}_{2c-1} \tilde{A}^{(n-1)}_{ad-1} x^{(n)}_{2d-2}.
\end{aligned}$$

$$\begin{aligned}
\tilde{A}^{(n)}_{bc} \tilde{A}^{(n)}_{ac} &= (x^{(1)}_{2b} \tilde{A}'^{(n-1)}_{b+1c} + x^{(1)}_{2b-1} \tilde{A}'^{(n-1)}_{bc})(x^{(1)}_{2a} \tilde{A}'^{(n-1)}_{a+1c} + x^{(1)}_{2a-1} \tilde{A}'^{(n-1)}_{ac}) \\
&= x^{(1)}_{2b} \tilde{A}'^{(n-1)}_{b+1c} x^{(1)}_{2a} \tilde{A}'^{(n-1)}_{a+1c} + x^{(1)}_{2b-1} \tilde{A}'^{(n-1)}_{bc} x^{(1)}_{2a} \tilde{A}'^{(n-1)}_{a+1c} \\
&+ x^{(1)}_{2b} \tilde{A}'^{(n-1)}_{b+1c} x^{(1)}_{2a-1} \tilde{A}'^{(n-1)}_{ac} + x^{(1)}_{2b-1} \tilde{A}'^{(n-1)}_{bc} x^{(1)}_{2a-1} \tilde{A}'^{(n-1)}_{ac} \\
&= qx^{(1)}_{2a} \tilde{A}'^{(n-1)}_{a+1c} x^{(1)}_{2b} \tilde{A}'^{(n-1)}_{b+1c} + qx^{(1)}_{2a} \tilde{A}'^{(n-1)}_{a+1c} x^{(1)}_{2b-1} \tilde{A}'^{(n-1)}_{bc} \\
&+ qx^{(1)}_{2a-1} \tilde{A}'^{(n-1)}_{ac} x^{(1)}_{2b} \tilde{A}'^{(n-1)}_{b+1c} + qx^{(1)}_{2a-1} \tilde{A}'^{(n-1)}_{ac} x^{(1)}_{2b-1} \tilde{A}'^{(n-1)}_{bc} \\
&= q(x^{(1)}_{2a} \tilde{A}'^{(n-1)}_{a+1c} + x^{(1)}_{2a-1} \tilde{A}'^{(n-1)}_{ac})(x^{(1)}_{2b} \tilde{A}'^{(n-1)}_{b+1c} + x^{(1)}_{2b-1} \tilde{A}'^{(n-1)}_{bc}) \\
&= q\tilde{A}^{(n)}_{ac} \tilde{A}^{(n)}_{bc}
\end{aligned}$$

Here,

$$\begin{aligned}
x^{(1)}_{2b} \tilde{A}'^{(n-1)}_{b+1c} x^{(1)}_{2a} \tilde{A}'^{(n-1)}_{a+1c} &= qx^{(1)}_{2a} \tilde{A}'^{(n-1)}_{a+1c} x^{(1)}_{2b} \tilde{A}'^{(n-1)}_{b+1c}, \\
x^{(1)}_{2b} \tilde{A}'^{(n-1)}_{b+1c} x^{(1)}_{2a-1} \tilde{A}'^{(n-1)}_{ac} &= qx^{(1)}_{2a-1} \tilde{A}'^{(n-1)}_{ac} x^{(1)}_{2b} \tilde{A}'^{(n-1)}_{b+1c}, \\
x^{(1)}_{2b-1} \tilde{A}'^{(n-1)}_{bc} x^{(1)}_{2a-1} \tilde{A}'^{(n-1)}_{ac} &= qx^{(1)}_{2a-1} \tilde{A}'^{(n-1)}_{ac} x^{(1)}_{2b-1} \tilde{A}'^{(n-1)}_{bc}
\end{aligned}$$

by induction hypothesis and the fact that $x^{(n)}_i$ and $x^{(n)}_j$ commute if $|i - j| > 1$.

If $b > a + 1$ then

$$x^{(1)}_{2b-1} \tilde{A}'^{(n-1)}_{bc} x^{(1)}_{2a} \tilde{A}'^{(n-1)}_{a+1c} = qx^{(1)}_{2a} \tilde{A}'^{(n-1)}_{a+1c} x^{(1)}_{2b-1} \tilde{A}'^{(n-1)}_{bc}$$

holds by induction hypothesis. If $b = a + 1$ then

$$\begin{aligned}
x^{(1)}_{2b-1} \tilde{A}'^{(n-1)}_{bc} x^{(1)}_{2a} \tilde{A}'^{(n-1)}_{a+1c} &= x^{(1)}_{2a+1} \tilde{A}'^{(n-1)}_{a+1c} x^{(1)}_{2a} \tilde{A}'^{(n-1)}_{a+1c} \\
&= qx^{(1)}_{2a} \tilde{A}'^{(n-1)}_{a+1c} x^{(1)}_{2a+1} \tilde{A}'^{(n-1)}_{a+1c} \\
&= qx^{(1)}_{2a} \tilde{A}'^{(n-1)}_{a+1c} x^{(1)}_{2b-1} \tilde{A}'^{(n-1)}_{bc}.
\end{aligned}$$

$$
\begin{aligned}
\tilde{A}^{(n)}_{bc}\tilde{A}^{(n)}_{ad} &= (\tilde{A}^{(n-1)}_{bc-1}x^{(n)}_{2c-2}+\tilde{A}^{(n-1)}_{bc}x^{(n)}_{2c-1})(\tilde{A}^{(n-1)}_{ad-1}x^{(n)}_{2d-2}+\tilde{A}^{(n-1)}_{ad}x^{(n)}_{2d-1}) \\
&= \tilde{A}^{(n-1)}_{bc-1}x^{(n)}_{2c-2}\tilde{A}^{(n-1)}_{ad-1}x^{(n)}_{2d-2}+\tilde{A}^{(n-1)}_{bc}x^{(n)}_{2c-1}\tilde{A}^{(n-1)}_{ad-1}x^{(n)}_{2d-2} \\
&+ \tilde{A}^{(n-1)}_{bc-1}x^{(n)}_{2c-2}\tilde{A}^{(n-1)}_{ad}x^{(n)}_{2d-1}+\tilde{A}^{(n-1)}_{bc}x^{(n)}_{2c-1}\tilde{A}^{(n-1)}_{ad}x^{(n)}_{2d-1} \\
&= p^{-1}q\tilde{A}^{(n-1)}_{ad-1}x^{(n)}_{2d-2}\tilde{A}^{(n-1)}_{bc-1}x^{(n)}_{2c-2}+p^{-1}q\tilde{A}^{(n-1)}_{ad-1}x^{(n)}_{2d-2}\tilde{A}^{(n-1)}_{bc}x^{(n)}_{2c-1} \\
&+ p^{-1}q\tilde{A}^{(n-1)}_{ad}x^{(n)}_{2d-1}\tilde{A}^{(n-1)}_{bc-1}x^{(n)}_{2c-2}+p^{-1}q\tilde{A}^{(n-1)}_{ad}x^{(n)}_{2d-1}\tilde{A}^{(n-1)}_{bc}x^{(n)}_{2c-1} \\
&= p^{-1}q(\tilde{A}^{(n-1)}_{ad-1}x^{(n)}_{2d-2}+\tilde{A}^{(n-1)}_{ad}x^{(n)}_{2d-1})(\tilde{A}^{(n-1)}_{bc-1}x^{(n)}_{2c-2}+\tilde{A}^{(n-1)}_{bc}x^{(n)}_{2c-1}) \\
&= p^{-1}q\tilde{A}^{(n)}_{ad}\tilde{A}^{(n)}_{bc}
\end{aligned}
$$

Here,

$$
\begin{aligned}
\tilde{A}^{(n-1)}_{bc-1}x^{(n)}_{2c-2}\tilde{A}^{(n-1)}_{ad-1}x^{(n)}_{2d-2} &= p^{-1}q\tilde{A}^{(n-1)}_{ad-1}x^{(n)}_{2d-2}\tilde{A}^{(n-1)}_{bc-1}x^{(n)}_{2c-2}, \\
\tilde{A}^{(n-1)}_{bc-1}x^{(n)}_{2c-2}\tilde{A}^{(n-1)}_{ad}x^{(n)}_{2d-1} &= p^{-1}q\tilde{A}^{(n-1)}_{ad}x^{(n)}_{2d-1}\tilde{A}^{(n-1)}_{bc-1}x^{(n)}_{2c-2}, \\
\tilde{A}^{(n-1)}_{bc}x^{(n)}_{2c-1}\tilde{A}^{(n-1)}_{ad}x^{(n)}_{2d-1} &= p^{-1}q\tilde{A}^{(n-1)}_{ad}x^{(n)}_{2d-1}\tilde{A}^{(n-1)}_{bc}x^{(n)}_{2c-1}
\end{aligned}
$$

by induction hypothesis and the fact that $x^{(n)}_i$ and $x^{(n)}_j$ commute if $|i-j|>1$.

If $d > c+1$ then

$$
\tilde{A}^{(n-1)}_{bc}x^{(n)}_{2c-1}\tilde{A}^{(n-1)}_{ad-1}x^{(n)}_{2d-2} = p^{-1}q\tilde{A}^{(n-1)}_{ad-1}x^{(n)}_{2d-2}\tilde{A}^{(n-1)}_{bc}x^{(n)}_{2c-1}
$$

holds by induction hypothesis. If $d = c+1$ then

$$
\begin{aligned}
\tilde{A}^{(n-1)}_{bc}x^{(n)}_{2c-1}\tilde{A}^{(n-1)}_{ad-1}x^{(n)}_{2d-2} &= \tilde{A}^{(n-1)}_{bc}x^{(n)}_{2c-1}\tilde{A}^{(n-1)}_{ac}x^{(n)}_{2c} \\
&= p^{-1}q\tilde{A}^{(n-1)}_{ac}x^{(n)}_{2c}\tilde{A}^{(n-1)}_{bc}x^{(n)}_{2c-1} \\
&= p^{-1}q\tilde{A}^{(n-1)}_{ad-1}x^{(n)}_{2d-2}\tilde{A}^{(n-1)}_{bc}x^{(n)}_{2c-1}.
\end{aligned}
$$

$$
\begin{aligned}
\tilde{A}^{(n)}_{bd}\tilde{A}^{(n)}_{ac} &= (\tilde{A}^{(n-1)}_{bd-1}x^{(n)}_{2d-2}+\tilde{A}^{(n-1)}_{bd}x^{(n)}_{2d-1})(\tilde{A}^{(n-1)}_{ac-1}x^{(n)}_{2c-2}+\tilde{A}^{(n-1)}_{ac}x^{(n)}_{2c-1})\\
&= \tilde{A}^{(n-1)}_{bd-1}\tilde{A}^{(n-1)}_{ac-1}x^{(n)}_{2d-2}x^{(n)}_{2c-2}+\tilde{A}^{(n-1)}_{bd}\tilde{A}^{(n-1)}_{ac-1}x^{(n)}_{2d-1}x^{(n)}_{2c-2}\\
&+ \tilde{A}^{(n-1)}_{bd-1}\tilde{A}^{(n-1)}_{ac}x^{(n)}_{2d-2}x^{(n)}_{2c-1}+\tilde{A}^{(n-1)}_{bd}\tilde{A}^{(n-1)}_{ac}x^{(n)}_{2d-1}x^{(n)}_{2c-1}\\
&= (\tilde{A}^{(n-1)}_{ac-1}\tilde{A}^{(n-1)}_{bd-1}+(p-q^{-1})\tilde{A}^{(n-1)}_{bc-1}\tilde{A}^{(n-1)}_{ad-1})x^{(n)}_{2d-2}x^{(n)}_{2c-2}\\
&+ (\tilde{A}^{(n-1)}_{ac-1}\tilde{A}^{(n-1)}_{bd}+(p-q^{-1})\tilde{A}^{(n-1)}_{bc-1}\tilde{A}^{(n-1)}_{ad})x^{(n)}_{2d-1}x^{(n)}_{2c-2}\\
&+ (\tilde{A}^{(n-1)}_{ac}\tilde{A}^{(n-1)}_{bd-1}+(p-q^{-1})\tilde{A}^{(n-1)}_{bc}\tilde{A}^{(n-1)}_{ad-1})x^{(n)}_{2d-2}x^{(n)}_{2c-1}\\
&+ (\tilde{A}^{(n-1)}_{ac}\tilde{A}^{(n-1)}_{bd}+(p-q^{-1})\tilde{A}^{(n-1)}_{bc}\tilde{A}^{(n-1)}_{ad})x^{(n)}_{2d-1}x^{(n)}_{2c-1}\\
&= \tilde{A}^{(n-1)}_{ac-1}x^{(n)}_{2c-2}\tilde{A}^{(n-1)}_{bd-1}x^{(n)}_{2d-2}+(p-q^{-1})\tilde{A}^{(n-1)}_{bc-1}x^{(n)}_{2c-2}\tilde{A}^{(n-1)}_{ad-1}x^{(n)}_{2d-2}\\
&+ \tilde{A}^{(n-1)}_{ac-1}x^{(n)}_{2c-2}\tilde{A}^{(n-1)}_{bd}x^{(n)}_{2d-1}+(p-q^{-1})\tilde{A}^{(n-1)}_{bc-1}x^{(n)}_{2c-2}\tilde{A}^{(n-1)}_{ad}x^{(n)}_{2d-1}\\
&+ \tilde{A}^{(n-1)}_{ac}x^{(n)}_{2c-1}\tilde{A}^{(n-1)}_{bd-1}x^{(n)}_{2d-2}+(p-q^{-1})\tilde{A}^{(n-1)}_{bc}x^{(n)}_{2c-1}\tilde{A}^{(n-1)}_{ad-1}x^{(n)}_{2d-2}\\
&+ \tilde{A}^{(n-1)}_{ac}x^{(n)}_{2c-1}\tilde{A}^{(n-1)}_{bd}x^{(n)}_{2d-1}+(p-q^{-1})\tilde{A}^{(n-1)}_{bc}x^{(n)}_{2c-1}\tilde{A}^{(n-1)}_{ad}x^{(n)}_{2d-1}\\
&= (\tilde{A}^{(n-1)}_{ac-1}x^{(n)}_{2c-2}+\tilde{A}^{(n-1)}_{ac}x^{(n)}_{2c-1})(\tilde{A}^{(n-1)}_{bd-1}x^{(n)}_{2d-2}+\tilde{A}^{(n-1)}_{bd}x^{(n)}_{2d-1})\\
&+ (p-q^{-1})(\tilde{A}^{(n-1)}_{bc-1}x^{(n)}_{2c-2}+\tilde{A}^{(n-1)}_{bc}x^{(n)}_{2c-1})(\tilde{A}^{(n-1)}_{ad-1}x^{(n)}_{2d-2}+\tilde{A}^{(n-1)}_{ad}x^{(n)}_{2d-1})\\
&= \tilde{A}^{(n)}_{ac}\tilde{A}^{(n)}_{bd}+(p-q^{-1})\tilde{A}^{(n)}_{bc}\tilde{A}^{(n)}_{ad}
\end{aligned}
$$

Here,

$$
\begin{aligned}
\tilde{A}^{(n-1)}_{bd-1}\tilde{A}^{(n-1)}_{ac-1} &= \tilde{A}^{(n-1)}_{ac-1}\tilde{A}^{(n-1)}_{bd-1}+(p-q^{-1})\tilde{A}^{(n-1)}_{bc-1}\tilde{A}^{(n-1)}_{ad-1},\\
\tilde{A}^{(n-1)}_{bd}\tilde{A}^{(n-1)}_{ac-1} &= \tilde{A}^{(n-1)}_{ac-1}\tilde{A}^{(n-1)}_{bd}+(p-q^{-1})\tilde{A}^{(n-1)}_{bc-1}\tilde{A}^{(n-1)}_{ad},\\
\tilde{A}^{(n-1)}_{bd}\tilde{A}^{(n-1)}_{ac} &= \tilde{A}^{(n-1)}_{ac}\tilde{A}^{(n-1)}_{bd}+(p-q^{-1})\tilde{A}^{(n-1)}_{bc}\tilde{A}^{(n-1)}_{ad}
\end{aligned}
$$

by induction hypothesis and the fact that $x^{(n)}_i$ and $x^{(n)}_j$ commute if $|i-j|>1$.

If $d>c+1$ then

$$\tilde{A}^{(n-1)}_{bd-1}\tilde{A}^{(n-1)}_{ac} = \tilde{A}^{(n-1)}_{ac}\tilde{A}^{(n-1)}_{bd-1}+(p-q^{-1})\tilde{A}^{(n-1)}_{bc}\tilde{A}^{(n-1)}_{ad-1}$$

holds by induction hypothesis. If $d=c+1$ then since $b>a$ we have

$$p^{-1}(\tilde{A}^{(n-1)}_{ac}\tilde{A}^{(n-1)}_{bc}-q^{-1}\tilde{A}^{(n-1)}_{bc}\tilde{A}^{(n-1)}_{ac})=0.$$

Hence

$$\begin{aligned}\tilde{A}^{(n-1)}_{bd-1}\tilde{A}^{(n-1)}_{ac}x^{(n)}_{2d-2}x^{(n)}_{2c-1} &= \tilde{A}^{(n-1)}_{bc}\tilde{A}^{(n-1)}_{ac}x^{(n)}_{2c}x^{(n)}_{2c-1}\\ &= (\tilde{A}^{(n-1)}_{bc}\tilde{A}^{(n-1)}_{ac} + p^{-1}\tilde{A}^{(n-1)}_{ac}\tilde{A}^{(n-1)}_{bc}\\ &\quad - p^{-1}q^{-1}\tilde{A}^{(n-1)}_{bc}\tilde{A}^{(n-1)}_{ac})px^{(n)}_{2c-1}x^{(n)}_{2c}\\ &= (p\tilde{A}^{(n-1)}_{bc}\tilde{A}^{(n-1)}_{ac} + \tilde{A}^{(n-1)}_{ac}\tilde{A}^{(n-1)}_{bc}\\ &\quad - q^{-1}\tilde{A}^{(n-1)}_{bc}\tilde{A}^{(n-1)}_{ac})x^{(n)}_{2c-1}x^{(n)}_{2c}\\ &= (\tilde{A}^{(n-1)}_{ac}\tilde{A}^{(n-1)}_{bc} + (p-q^{-1})\tilde{A}^{(n-1)}_{bc}\tilde{A}^{(n-1)}_{ac})x^{(n)}_{2c-1}x^{(n)}_{2c}\\ &= (\tilde{A}^{(n-1)}_{ac}\tilde{A}^{(n-1)}_{bd-1} + (p-q^{-1})\tilde{A}^{(n-1)}_{bc}\tilde{A}^{(n-1)}_{ad-1})x^{(n)}_{2c-1}x^{(n)}_{2d-2}.\end{aligned}$$

□

Definition 6.3.8. *Let $\bar{R}_q(\infty) = R_q(\infty)/\bar{J}$ be the quotient of the algebra $R_q(\infty)$ over K by the two-sided ideal $\bar{J}$ generated by the relations*

$$\begin{aligned}x^{(k)}_i &= \bar{x}^{(k)}_i,\\ x^{(k+n-1)}_{i+2k-1} &= \bar{y}^{(k)}_i,\\ x^{(k)}_j &= 0,\\ x^{(k+n-1)}_{j+2k-1} &= 0,\\ x^{(m)}_s &= 0\end{aligned}$$

if $1 \le k \le n-1$, $1 \le i \le 2n-1$, $j \le 0$ *or* $j \ge 2n$, $m \ge 2n-1$ *and* $\forall s \in \mathbb{Z}$.

Denote

$$\begin{aligned}\bar{X}^{(k)} &= \tilde{X}^{(k)} \ \ (mod \bar{J}),\\ \bar{Y}^{(k)} &= \tilde{X}^{(k+n-1)} \ \ (mod \bar{J})\end{aligned}$$

if $1 \le k \le n-1$.

Remark 6.3.9. *The elements of the algebra $\bar{R}_q(\infty)$ still satisfy the commutation and anti-commutation relations*

$$\begin{aligned}x^{(k)}_{i+1}x^{(k)}_i &= qx^{(k)}_i x^{(k)}_{i+1} \ \ (mod\ \bar{J}),\\ x^{(k)}_i x^{(k)}_j &= x^{(k)}_j x^{(k)}_i \ \ (mod\ \bar{J}),\\ x^{(k_1)}_i x^{(k_2)}_l &= x^{(k_2)}_l x^{(k_1)}_i \ \ (mod\ \bar{J})\end{aligned}$$

for every i, j, k, l, k_1, k_2 where $k_1 \neq k_2$, $|j-i| \geq 2$.

Thus the entries of matrix $\bar{A} = \bar{X}^{(1)}\bar{X}^{(2)}...\bar{X}^{(n-1)}\bar{Y}^{(1)}\bar{Y}^{(2)}...\bar{Y}^{(n-1)}$ also satisfy the corresponding relations, i.e. we have

$$\begin{aligned}
\bar{A}_{ad}\bar{A}_{ac} &= q\bar{A}_{ac}\bar{A}_{ad}, \\
\bar{A}_{bc}\bar{A}_{ac} &= q\bar{A}_{ac}\bar{A}_{bc}, \\
\bar{A}_{bc}\bar{A}_{ad} &= \bar{A}_{ad}\bar{A}_{bc}, \\
\bar{A}_{bd}\bar{A}_{ac} &= \bar{A}_{ac}\bar{A}_{bd} + (q - q^{-1})\bar{A}_{bc}\bar{A}_{ad}
\end{aligned}$$

if $d > c$ and $b > a$.

Remark 6.3.10. *The algebras $R_q(n)$ and $\bar{R}_q(\infty)$ are isomorphic via the following map:*

$$\begin{aligned}
x_i^{(k)} &\mapsto \bar{x}_i^{(k)}, \\
y_i^{(k)} &\mapsto \bar{y}_i^{(k)}
\end{aligned}$$

for every $1 \leq k \leq n-1$, $1 \leq i \leq 2n-1$.

Lemma 6.3.11. *The matrix $\hat{A} = X^{(1)}X^{(2)}...X^{(n-1)}Y^{(1)}Y^{(2)}...Y^{(n-1)}$ can be obtained by cutting the zero rows and columns of the infinite matrix $\bar{A}$, hence the assertion of Theorem 6.2.1 is true.*

Proof. Under the above identification

$$\begin{aligned}
x_l^{(k)} &\mapsto \bar{x}_l^{(k)}, \\
y_l^{(k)} &\mapsto \bar{y}_l^{(k)}
\end{aligned}$$

for every $1 \leq k \leq n-1$, $1 \leq l \leq 2n-1$, we have

$$\begin{aligned}
(\bar{X}^{(k)})_{ij} &= (X^{(k)})_{ij}, \\
(\bar{Y}^{(k)})_{ij} &= (Y^{(k)})_{ij}
\end{aligned}$$

if $1 \leq k \leq n-1$ and $1 \leq i, j \leq n$. Hence,

$$(\bar{A})_{ij} = (\hat{A})_{ij}$$

for every $1 \leq i, j \leq n$. Now since

$$(\bar{A})_{ij} = 0$$

for every $i, j \leq 0$ or $i, j \geq n + 1$, if we cut the zero rows and columns of $\bar{A}$ we get exactly $\hat{A}$.

Now since entries of $\bar{A}$ satisfy the relations, entries of $\hat{A}$ also satisfy the corresponding ones, which proves Theorem 6.2.1. $\square$

6.4 Conclusion

For quantum groups, Drinfeld's quantum double (see section 2.3) plays the role of LU-decomposition of linear algebra. It produces an R-matrix as a by-product.

Our goal is to construct new R-matrices by embedding the bialgebra $M_{p,q}(n)$ into some larger (bi)algebra B where the commutators between the generators of the (bi)algebra B have simpler expressions than those of $M_{p,q}(n)$. Our factorization theorem is an effort in this direction.

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