COGNITIVE ANALYSIS OF STUDENTS' LEARNING OF TRIGONOMETRY IN DYNAMIC GEOMETRY ENVIRONMENT: A TEACHING EXPERIMENT

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submitted by **ZÜLAL ŞAHİN** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy in Secondary Science and Mathematics Education Department, Middle East Technical University** by,

Prof. Dr. Gülbin Dural Ünver ___________

Dean, Graduate School of **Natural and Applied Sciences**

Prof. Dr. Ömer Geban ___________

Head of Department, **Secondary Science and Mathematics Education**

Assoc. Prof. Dr. Ayhan Kürşat Erbaş ___________

Supervisor, **Secondary Science and Mathematics Edu. Dept. METU**

Examining Committee Members:

Prof. Dr. Ziya Argün ____________________

Secondary Science and Mathematics Edu. Dept., Gazi Uni.

Assoc. Prof. Dr. Ayhan Kürşat Erbaş ____________________

Secondary Science and Mathematics Edu. Dept., METU

Prof. Dr. Ahmet Arıkan ____________________

Secondary Science and Mathematics Edu. Dept., Gazi Uni.

Prof. Dr. Erdinç Çakıroğlu ____________________

Elementary Mathematics Edu. Dept., METU

Assoc. Prof. Dr. Bülent Çetinkaya ____________________

Secondary Science and Mathematics Edu. Dept., METU

Date: October 14, 2015

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last name : Zülal Şahin

Signature :

ABSTRACT

COGNITIVE ANALYSIS OF STUDENTS' LEARNING OF TRIGONOMETRY IN DYNAMIC GEOMETRY ENVIRONMENT: A TEACHING EXPERIMENT

Şahin, Zülal
Ph.D., Department of Secondary Science and Mathematics Education
Supervisor: Assoc. Prof. Dr. Ayhan Kürşat Erbaş

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Trigonometry is a part of mathematics in which algebra and geometry converge. Dealing with trigonometric functions at secondary level is known as a difficult task because it requires to work with right triangles, the unit circle, and graphs of trigonometric functions simultaneously. For most students, this means excessive amount of formulas unless they can establish connections among different representational systems. There is a consensus in the literature that appropriate use of technology can be effective in helping students make such connections. Dynamic geometry environments can be a useful tool in teaching trigonometry due to their opportunities that enable to construct mathematical objects within different representational systems in a dynamically-and-linked way.

The overarching purpose of this study was to design an instruction in dynamic geometry environment in order to support secondary students' *concept images* on core trigonometric functions, i.e., sine and cosine, in different representations (i.e., symbolic, circular, and graphic). The instructional sequence was designed initially through inspiring from research literature on trigonometry, historical development of trigonometry, our exploratory teaching experience, and initial interview results. And then, design of the instruction was continued through revising as a result of the on-going prospective and retrospective cognitive analysis of the data that were collected during the 17-week teaching experiment from two pairs of secondary students separately.

Students were encouraged to reason about dynamically-linked transformations of the core trigonometric functions within and between representational registers, as well as reasoning about dynamically-changed visual components referring to the core trigonometric functions. When compared with their initial serious *recognition* and *discrimination* troubles, as the study progressed, significant improvements were observed in students' *recognition* and *discrimination* abilities within and between different representational registers. The cognitive analysis of the data revealed the importance of students' constructions of well-defined *concept definition images* on foundational trigonometric concepts (i.e., angle, angle measure, trigonometric value, trigonometric functions, and periodicity) in order to recognize the same trigonometric object within different representational registers. The importance of the basic visual features' *discrimination* in comprehension of trigonometry was also revealed in this study. When the basic visual features referring to trigonometric functions (i.e., radius of the circle, position of the center, position of the reference point on the circle referring to trigonometric value) were systematically varied in the (*unit*) *circle register*, and their dynamic-and-linked oppositions in the *graphical register* were constructed, the students developed significant understandings that enabled them to discriminate the basic form of sine and cosine functions from their general forms. Finally, the findings of the study revealed that the *discrimination* ability required to be reasoning about the new situations emerging as a consequence of the changed-visual features through focusing on trigonometrically relevant objects (e.g., reference point(s), reference right triangle, radius, displacement amount and direction) rather than detailed processes (e.g., ordinate of a point, procedural definition of sine or cosine).

Keywords: Mathematics education, angles, trigonometric functions, period, recognition, discrimination, representation, representation transformations

ÖZ

DİNAMİK GEOMETRİ ORTAMINDA
ÖĞRENCİLERİN TRİGONOMETRİ ÖĞRENMELERİNİN BİLİŞSEL ANALİZİ:
BİR ÖĞRETİM DENEYİ

Şahin, Zülal
Doktora, Orta Öğretim Fen ve Matematik Alanları Eğitimi Bölümü
Tez yöneticisi: Doç. Dr. Ayhan Kürşat Erbaş

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Trigonometri matematiğin cebir ve geometriyi birleştiren bir alt alanıdır. Ortaöğretim düzeyinde trigonometrik fonksiyonlarla ilgilenmek dik üçgenler, birim çember ve trigonometrik fonksiyonların grafikleri gibi farklı gösterimlerle eş zamanlı çalışmayı gerektirdiği için zor bir konu olarak bilinir. Bu farklı gösterimler arasındaki bağlantıları kuramadıkça, pek çok öğrenci için trigonometri ezberlenmesi gereken aşırı miktardaki formüller anlamına gelir. Literatürde teknolojinin uygun kullanımının öğrencilerin bu bağlantıları kurmasında etkili olabileceğine dair ortak görüş vardır. Dinamik geometri ortamları, matematiksel nesneleri dinamik-ve-bağlantılı olarak farklı gösterimsel sistemler içinde yapılandırmayı kolaylaştıran imkanları nedeniyle trigonometri öğretiminde faydalı bir araç olabilir.

Bu çalışmanın temel amacı, ortaöğretim öğrencilerinin temel trigonometrik fonksiyonlar (sinüs ve kosinüs) üzerine kavram görüntülerini zenginleştirmek için dinamik geometri ortamında bir eğitim tasarlamaktı. Eğitimsel sıralanış başlangıçta, trigonometri araştırma literatürü, trigonometrinin tarihsel gelişimi, keşif amaçlı öğretim tecrübemiz ve ilk görüşme sonuçlarından ilham alınarak tasarlandı. Daha sonra, eğitim tasarımı, 17-haftalık öğretim deneyi boyunca, iki çift ortaöğretim öğrencisinden ayrı ayrı toplanan verilerin, sürekli devam eden geleceğe ve geçmişe dönük bilişsel analiz sonuçlarına göre gözden geçirilerek düzenlenmesiyle devam etti.

Öğrenciler, gösterimsel kayıtlar içinde ve arasında, temel trigonometrik fonksiyonların dinamik-ve-bağlantılı dönüşümleri, ve bunun yanı sıra, temel trigonometrik fonksiyonlara işaret eden dinamik olarak değişen görsel bileşenler üzerine muhakeme etmeye cesaretlendirildi. Başlangıçtaki ciddi *fark etme* ve *ayırt etme* sorunları ile karşılaştırıldığında, çalışma ilerledikçe, öğrencilerin *fark etme* ve *ayırt etme* becerilerinde önemli gelişmeler gözlemlendi. Verilerin bilişsel analizi, temel trigonometrik kavramlar üzerine (açı, açı ölçüsü, trigonometrik değer, trigonometrik fonksiyonlar ve periyodiklik) öğrencilerin iyi-tanımlı *kavram tanım görüntüleri* yapılandırmalarının, farklı gösterimsel kayıtlar içinde temsil edilen aynı trigonometrik nesneyi fark etmedeki önemini ortaya çıkardı. Ayrıca, bu çalışmada, trigonometriyi kavramada, temel görsel özelliklerin önemi ortaya çıktı. Trigonometrik fonksiyonlara işaret eden temel görsel özellikler (çemberin yarıçapı, merkezin konumu, çember üzerindeki trigonometrik değere işaret eden referans noktanın konumu) (*birim*) *çember kayıdı*nda sistematik olarak değiştirildiğinde, ve bunların dinamik-ve-bağlantılı karşılıkları *grafik kayıt*ta oluşturulduğunda, öğrenciler sinüs ve kosinüs fonksiyonlarının basit fromlarını genel formlarından ayırt etmelerine imkan veren önemli anlamalar geliştirdiler. Son olarak, çalışmanın bulguları, *ayırt etme* becerisinin, değişen görsel özellikler sonucunda ortaya çıkan yeni durumların, detaylı süreçlere odaklanmaktansa (ör., bir noktanın ordinatı, sinüs veya kosinüsün prosedürel tanımı), trigonometrik olarak ilişkili nesnelere odaklanarak (ör., referans nokta(lar), referans dik üçgen, yarıçap, yer değiştirme miktarı ve yönü) muhakeme edilmesi gerektirdiğini ortaya çıkardı.

Anahtar Kelimeler: Matematik eğitimi, açılar, trigonometrik fonksiyonlar, periyot, fark etme, ayırt etme, temsil, temsil değişimi

To my family

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INTRODUCTION

The importance of mathematics has increased day by day. Any more, there is a different view on learning and teaching of mathematics (National Council of Teachers of Mathematics [NCTM], 2000). Teaching for understanding has been emphasized on-going calls for reform in education (NCTM, 2000). Therefore, development of deep understanding of mathematical concepts is essential for mathematics education also.

Tall and Vinner (1981) differentiate a student's conception (or mental understanding) of a concept from its formal mathematical definition by using the notion of *concept image* and *concept definition*, respectively. According to them, while the term *concept image* refers to all the mental pictures and associated properties, and processes with a concept, the term *concept definition* refers the "form of words used to specify that concept" (p. 152). Students' concept images are acquired through different experiences over the years based on their prior knowledge. In many situations, it is desirable for students to evoke rich concept images (Harel, Selden & Selden, 2006).

Using formal definitions of a concept to produce examples and counter examples (Dahlberg&Housman, 1997), making connections to other concepts and converting between its multiple representations (Duval, 1999) are the ways for students to evoke rich concept images. Multiple representations, especially visual ones (such as graphs) (Goldenberg, 1988), are important factors for understanding of functions, before introducing static (symbolic) definitions (Dreyfus, 1993; Kaput, 1987). Nevertheless, students have difficulties in inferring from visualization of graphs (Goldenberg, 1988) and connecting diagram with its symbolic representation (Zazkis, Dubinsky & Dautermann, 1996). Despite of students' difficulties with visualization, it has the power in mathematical reasoning (Dreyfus, 1991) and understanding (Presmeg, 1994). Moreover, helping students acquire the ability to visualize mathematical concepts enriches their *concept images* (Harel, Selden & Selden, 2006). In this respect, visual aspects of computer technology, particularly dynamic geometry environments, can help students make connection between visual and symbolic representations of the same mathematical concept.

Dynamic geometry environments provide dynamic diagrams so that students can slightly distort to meet their expectations, which is impossible in paper-and-pencil diagrams (Laborde, Kynigos, & Strasser, 2006). Moreover, they give students the opportunity of constructing graphs of functions which can be manipulated and animated by the parameter, as well as the opportunity of dynamically linking between graphs and other representations of a situation (Mackrell, 2002); and helps them understand propositions by allowing them to perform geometric constructions with a high degree of accuracy (Hanna, 2000). However, existing technology does not guarantee learning.

Literature shows that technology usage can be effective only within appropriate teaching-learning context (Ferrara, Pratt & Robutti, 2006). The study of Jones (2002), in which published research on the use of dynamic geometry software (DGS) was categorized, stated that a range of research showed that judicious use of DGS can foster the understanding of proof. However, if used inappropriately, DGS might make things worse instead of making significant effect on students' conception (Jones, 2002). Therefore, appropriate integration of technology into mathematics courses is needed. It requires carefully designed teaching/learning activities according to not only teachers' but also students' background, the task, the mathematical context, the class context and the potentialities offered by the software (Ferrara et al., 2006).

Ubuz, Üstün and Erbaş (2009) reported that "if used appropriately" dynamic geometry environments "can serve as an important vehicle to improve student achievement in geometry and achieve a classroom culture where conjecturing, analyzing, exploring, and reasoning are daily routines" (p.148). Also, Hannafin and

Scott (2001) showed that using dynamic geometry softwares in a student-centered environment has a positive effect on the attitudes and beliefs of both teachers and students about teaching and learning geometry. In the subject design of Weeden (2002) where a dynamic geometry software was incorporated into her lessons, she observed that the classroom experience was extremely enjoyable for both students and teachers, they worked in collaboration and experienced reality of construction and properties of shape while learning to use the software; moreover, she observed that students developed a deeper understanding and a greater view of the whole picture.

Although dynamic geometry environments are often used in teaching geometry because of its crucial role of geometric and graphical representations, it can be also used in teaching trigonometry which is a visual part of mathematics converging algebra and geometry.

1.1. Trigonometry

Trigonometry is a part of mathematics in which algebra and geometry converge. Dealing with trigonometric functions is known as a difficult task since it requires to work with right triangles, the unit circle, and graphs of trigonometric functions simultaneously. Several students have trouble on coherent understanding and flexible use of trigonometric functions in different representations (Brown, 2005). To integrate trigonometric functions' meaning in any representational context, the meaning of angle measure is emphasized as the foundational cognitive idea (Thompson, 2008). However, angle measure is a problematic issue that is needed to be handled carefully in teaching of trigonometry (Akkoç, 2008; Fi, 2003; Thompson, 2008; Topçu, Kertil, Akkoç, Yılmaz &Önder, 2006). To indicate an angle measure on a visual representation, an inner arc is created. Nevertheless, without its meaning for the angle measure, this inner arc serves only as a pointer in a diagram (Thompson, 2008). Students have difficulties on clear meaning of angle measure in terms that the measured-thing is what (Thompson, 2008; Moore, 2010).

Considering measurable attributes is foundational to make sense of the angle measure (Moore, 2010; 2012; 2013; Hertel & Cullen, 2011). Measuring arc lengths *in radii* (Moore, 2010; 2012; 2014) is important idea to promote reasoning about angle measure as the meaningful numbers (Thompson, 2008). Also, there is a need to develop coherent angle measures in degrees and radians (Thompson, 2008) through merging their meanings as the proportional relation between the arc length subtended by the angle and the circle's circumference (Thompson, 2008; Moore, 2013). However, literature indicates students' troubles on the radian measure unit. The meaning of the radian measure is dominated by degree meaning and restricted only into transformations between degree and radian measures (Akkoç, 2008; Topçu, et al., 2006). At that point, π comes to light as a cognitive obstacle when reasoning about its real value in the trigonometry context (Akkoç, 2008; Fi, 2003; Topçu, et al., 2006).

The unit circle was emphasized as an important tool in the literature for strong understanding of coordinate trigonometry or periodic function trigonometry (Brown, 2005; Moore, LaForest, & Kim, 2012; Weber, 2005). Nevertheless, students have difficulties with the unit circle in terms of understanding, using and interpreting trigonometric functions (Burch, 1981; Brown, 2005; Emlek, 2007; Güntekin, 2010; Gür, 2009; Weber, 2005). Moore et al. (2012) lay emphasis on the importance of the ability to use trigonometric functions in any circular context. They reported preservice teachers' difficulties in reasoning about trigonometric functions on the non-unit circles.

Another critical aspect mentioned in the literature (Hertel & Cullen, 2011; Thompson, 2008; Weber, 2005) is based on reasoning about trigonometric functions as functions. Because trigonometric functions cannot be expressed as algebraic formulas involving arithmetical procedures, students have trouble on reasoning about them as functions (Weber, 2005). Therefore, it is attached importance to promote reasoning about trigonometric functions as functions mapping from angle measure to corresponding trigonometric value (Hertel & Cullen, 2011), as well as reasoning about angle measure as the (meaningful) numbers referring to the argument of trigonometric functions (Thompson, 2008).

Students’ difficulties on associating trigonometric functions properly with the appropriate geometric models (Brown, 2005; Weber, 2005), and their problems with the conception of prior knowledge (such as right triangles, the coordinate plane, ratio-proportion, circle) and new knowledge (such as relationship between arc length and angle measure, reference and principal angle, the unit circle) (Gür, 2009) are other obstacles for effective learning of trigonometry.

Coherent understanding and flexible use of trigonometric functions in different representations (i.e., symbolic, circular, and graphic) are expected from trigonometry instructions. In comprehension of mathematics, Duval (2006) argued the importance of *recognition* and *discrimination* tasks. *Recognition tasks* require recognition of the same object represented in two different representational registers “whose contents have very often nothing in common” (p. 112). *Discrimination tasks* require discrimination “in any semiotic representation what is mathematically relevant and what is not mathematically relevant?” (p. 115). Therefore, well-designed recognition and discrimination tasks are critical to provide students with coherent understanding and flexible use trigonometric functions in different representational registers but also provide researchers and teachers with awareness of students’ understanding.

1.2. Purpose of the Study

In spite of the difficulties with trigonometry mentioned above, there is little attention on this area in mathematics education research literature comparing with the other subject areas. Unfortunately, research literature on trigonometric functions concluded that the standard instruction, which is based on *ratio method* and *unit circle method*, did not constitute students’ strong understanding of trigonometric functions (Akkoç, 2008; Brown, 2005; Moore, 2014; Weber, 2005). This dissertation was designed to contribute *trigonometry of students* through designing an instruction to support students’ concept images on trigonometric functions in different representations (i.e., symbolic, circular, and graphic) and investigate students’ evoked concept images on trigonometric functions during the experimentation of this instruction. The instruction of the study was including a sequential *recognition* and *discrimination* tasks in the dynamic geometry environment. Instructional design process of the study started through inspiring from research literature on trigonometry, historical development of trigonometry, our exploratory teaching experience, and initial interview results; and then, continued as an on-going revision process during the experimentation to influence *students' trigonometry*.

Base of the instruction was an animated circle which was located on the coordinate plane and whose radius and center were manipulable. It was used to enrich students' *concept images* on the core trigonometric functions, i.e., sine and cosine. *Recognition* and *discrimination* of sine and cosine in the different representational registers required to deal not only with the *basic forms of core trigonometric functions*[1] but also with *their general forms*[2]. On the one hand, main focus of the recognition tasks was on promoting students to recognize the same trigonometric object that was represented within different representations' respective contents (such as, angle, trigonometric value, trigonometric function, and periodicity). On the other hand, main focus of the discrimination tasks was on promoting students to discriminate in any representational register "what is mathematically relevant, and what is not mathematically relevant" (Duval, 2006, p. 115) under the dynamic manipulation of the represented objects. In general sense, our instructional focus was on the role of coefficients on core trigonometric functions' periods and different representations. For this purpose, in the designed instruction, it was followed a way moving from their geometric representations on the (unit) circle to the graphical representations together with the symbolic representations.

A teaching experiment (Steffe & Thompson, 2000) was conducted to experience, firsthand, students' trigonometry on these recognition and discrimination tasks, as well as to test and refine our design through composing new situations not considered in the initial design to "encourage students to modify their current thinking" (p. 285). Based on these modification, some conceptual frameworks were created to

[1] The term "basic form" of core trigonometric functions refers to $y=\sin(x)$ and $y=\cos(x)$.

[2] The term "general form" of core trigonometric functions refers to $y=a\sin(bx+c)+d$ and $y=a\cos(bx+c)+d$.

analyze and enrich *students' trigonometry* related to angle, sine and cosine (as a value and as a function) and periodicity concepts. Also, some cognitive networks were produced to dissociate the content of any semiotic representation and the object represented in this representational system.

1.3. Research Questions

The overarching aim of the current study was to investigate how a dynamic geometry environment contributes secondary students' understanding of trigonometry. For this aim, in the light of the theoretical base of the designed-instruction, following main research questions and their sub-questions guided this study:

- Prior to the instruction, what *concept images* on trigonometric functions do secondary students who had just taken trigonometry have?
  - → What are the problems of students' *recognition* of foundational trigonometric concepts[3] related to trigonometric functions within any representational register?
  - → What are the problems of students' *discrimination* of trigonometric functions represented within any representational register from the respective representational registers' contents?
  - → Are there any *potential conflict factor*s in students' *concept images* on trigonometry?
- What understanding related to trigonometric functions do students develop during the experimentation of the designed-instruction in a dynamic geometry environment that emphasizes the dynamically-linked transformations of the represented objects?

[3] As foundational trigonometric concepts, *functions*, *angle*, *angle measure*, *trigonometric value*, *trigonometric functions*, and *periodicity* are determined based on trigonometry research literature, historical development of trigonometry, our exploratory teaching experience, and initial interview results.

- In what way do the dynamically-linked *conversions* of trigonometric functions between representational registers influence students' *recognition* of trigonometric functions?
- In what way do the dynamically-changed visual components referring to trigonometric functions influence students' *discrimination* of trigonometric functions?

1.4. Significance of the Study

Mathematics is known as one of the most difficult subjects. Interestingly, not only lower-achievers in mathematics but also average and even higher-achiever students have difficulties with trigonometry. Even though trigonometry is a painful subject for most students, it is an important part of mathematics in which algebra and geometry converge. On the one hand, it is a useful tool in variety real-life applications in two ways (Cooney, Beckmann, Lloyd,Wilson & Zbiek, 2010). Firstly, trigonometric functions give us the opportunity to find unknown lengths in terms of known lengths and angles. Secondly, as the natural and fundamental examples of periodic functions, trigonometric functions are used to approximate any periodic functions. For instance, in developing computer music, the sine and cosine functions are used to model sound waves. On the other hand, beyond its importance in real-life, trigonometry has also an important academic value for the students in transition from the secondary education to the tertiary education. Achievement in calculus courses in tertiary education, in which trigonometric functions are used when studying with, for example, limit, derivative and integral, requires for students to have constructed adequate *concept images* in their secondary education due to the fact that concepts are introduced in tertiary education via their formal *concept definitions.* Therefore, to provide continuity between secondary education and tertiary education, it is needed to help students construct adequate *concept images* –or, mental structures for representing and identifying corresponding concepts– during their secondary education period. Thus, the current study is significant in terms of aiming to design an instruction to support secondary students' *concept images* on trigonometric functions in different representations (i.e., symbolic, circular, and graphic), as well as to contribute *trigonometry of students*[4] through investigating their evoked concept images on trigonometric functions during the experimentation of this designed-instruction.

A concept can be acquired through converting between its multiple representations (Duval, 1999) because visualization can facilitate mathematical understanding (Presmeg, 1994). However, a lack of the ability of students to connect a diagram with its symbolic representation is the most harmful but quite common difficulty with visualization (Duval, 2006; Zazkis, Dubinsky & Dautermann, 1996). Helping students acquire the ability to visualize mathematical concepts enriches their *concept images* (Harel, Selden & Selden, 2006). There is a consensus in the literature that appropriate use of technology can be effective on students' conceptualization by providing quick visualization with a high degree of accuracy (Archavi, 2003; Arzarello, Olivero, Paola & Robutti, 2002; Mariotti, 2000; Noraini, 2007). In this respect, this study is significant in terms of its instructional design including a sequential *recognition* and *discrimination* tasks both within and between different representations of trigonometric functions (i.e., symbolic, circular, and graphic) in the dynamic geometry environment.

Multiple studies have stated positive effect of trigonometry teaching by the aid of dynamic geometry environments on students' understanding (Blacket &Tall, 1991; Choi-Koh, 2003; Hertel and Cullen, 2011; Thompson, 2007). Some studies did not focus on trigonometric functions' circular representation (Blacket &Tall, 1991; Choi-Koh, 2003); or used a realistic model of the unit circle in the dynamic geometry environment (Thompson, 2007). In this respect, it is needed to investigate the unique role of dynamically linked-representations (i.e., symbolic, circular, and graphic) of trigonometric functions on students' understanding not only in the symbolic and graphical but also most importantly circular representation. This is because historical

[4] Steffe and Thompson (2000) differentiated *students' mathematics* and *mathematics of students* phrases. On the one hand, *students' mathematics* refers to "whatever might constitute students' mathematical realities"; on the other hand, *mathematics of students* refers to "our [researchers'] interpretations of students' mathematics" (p. 268). We use *trigonometry of students* as the same meaning with *mathematics of students* notion in the trigonometry context.

origin of trigonometry was intimately tied to spherical trigonometry, and its development entailed to understand trigonometry in the circular context (Katz, 2009). Therefore, unlike some other studies in the sparse literature on trigonometry that examined students' understanding of trigonometry in a dynamic geometry environment (Blacket &Tall, 1991; Choi-Koh, 2003; Hertel and Cullen, 2011; Thompson, 2007), the instructional design of this study was followed a way moving from the (any) circular representation to the graphical representation together with the symbolic representation. In addition, this study investigated the unique role of dynamically linked-representations of trigonometric functions "toward understanding the progress students make over extended periods" (Steffe & Thompson, 2000, p. 273) through a 17-week teaching experiment. Therefore, the current study is significant due to its results that contribute the trigonometry literature.

Trigonometry was a part of the previous Turkish national high school mathematics curriculums (Talim ve Terbiye Kurulu Başkanlığı [TTKB], 2005, 2011) as a sub-learning domain at 10th grade. It was following a process starting from right triangle trigonometry through directed angles to unit circle trigonometry, and then, trigonometric functions' graphs. However, there was no mention about the general form of trigonometric functions (see *Footnote* 2) except for graphing and finding periods of them. Circular representations of trigonometric functions were restricted to the representation of the basic trigonometric functions (see *Footnote* 1) on the unit circle. Subsequently, it was changed in 2013, and trigonometry was distributed at grades 9, 11 and 12 through being revised. Although the covariation of the graphs with respect to the changed-coefficients of a trigonometric function in the general form, for example, $f(x)=a \sin (bx+c)+d$, took place as an objective of the trigonometry course at 11th and 12th grades, there was no mention again about circular representations of trigonometric functions' general forms. Thus, the current study is significant as a consequence of its suggestions regarding this issue to the curriculum designers. This may limit for students to construct *reified concept images* on trigonometric functions in their secondary education. As it is mentioned above, to provide continuity between secondary education and tertiary education, it is needed to help students construct adequate *concept images* during their secondary education period.

Trigonometric functions are the natural and fundamental examples of periodic functions. Understanding of periodicity requires dual and simultaneous reasoning about (*i*) regular intervals of DOMAIN and (*ii*) corresponding repeated values in RANGE. It is cognitively difficult process especially when reasoning about them both within and between all different representations of trigonometric functions (i.e., symbolic, graphic, and circular). Fi (2003) considers coterminal angles as the related and necessary knowledge to model periodic phenomena (e.g., Ferris wheel problem), as well as emphasize the importance of coterminal angles to generate angle measures other than the principal ones. He reported preservice mathematics teachers' inadequate knowledge of coterminal angles and the periodicity idea as the problematic parts of their understanding of trigonometric functions. However, many other studies focusing on students' understanding of trigonometric functions (Akkoç, 2008; Brown, 2005; Hertel and Cullen, 2011; Moore, 2010; 2012; Moore, LaForest, & Kim, 2012; Thompson, 2007; Topçu, Kertil, Akkoç, Yılmaz &Önder, 2006) mentioned little or nothing about the periodicity idea in a detailed way. Therefore, the current study is significant due to its results on periodicity considering different representations to contribute the trigonometry literature.

In this study, a teaching experiment (Steffe & Thompson, 2000) was conducted to experience, firsthand, *students' trigonometry* throughout recognition and discrimination tasks, as well as to test and refine our design through composing new situations not considered in the initial design to "encourage students to modify their current thinking" (p. 285). Based on these modification, some conceptual frameworks were created to analyze and enrich *students' trigonometry* related to angle, sine and cosine (as a value and as a function) and periodicity concepts. Also, some cognitive networks were produced to dissociate the content of any semiotic representation and the object represented in this representational system. It is expected that these conceptual frameworks and cognitive networks, the products of this study, guide trigonometry teachers, other researchers and educators in designing more effective tasks, as well as in analyzing *students' trigonometry*.

1.5. Definitions of Terms

What we mean by some important terms during this study is clarified for the readers as the following.

Concept image

Concept image is defined as "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (Tall & Vinner, 1981, p. 152). A learner's concept image on a mathematical concept includes all cognitive processes by which the concept are conceived.

Concept definition

Tall and Vinner (1981) define concept definition as "the form of words used to specify a concept" (p. 152). In this study, we define concept definition as the form of objects within any representational content used to specify a concept.

Potential conflict factor

Tall and Vinner (1981) describe *potential conflict factor* as "a part of concept image or concept definition which may conflict with another part of the concept image or concept definition" (p. 153). The potential conflict factors can be "conveniently considered based on the circumstances without causing any cognitive conflict. They only become *cognitive conflict factors* when evoked simultaneously" (p.154). Tall and Vinner (1981) argue the conflicting part of the *concept image* with the *formal concept definition* itself as the more serious type of the *potential conflict factor*. They assert the weak understanding of the *concept definition* as a source of the students' problems in mathematics especially when there are *potential conflict factors* between a strong *concept image* and a weak *concept definition image*.

Core trigonometric functions

There are six trigonometric functions, namely sine, cosine, tangent, cotangent, secant, and cosecant. In this study, we mean only sine and cosine as core trigonometric functions.

Basic forms of trigonometric functions

Basic forms of trigonometric functions refers to the function mapping an angle to its corresponding trigonometric value. For instance, the basic form of the sine [cosine] function can be expressed symbolically as $y = \sin(x)$ [$y = \cos(x)$].

General forms of trigonometric functions

We refer by general forms of trigonometric functions to trigonometric functions so that their symbolic expressions are linear trigonometric equations and the symbolic expressions of the angles used to determine trigonometric values are in the linear form regarding the independent variables of the functions. For example, the general form of the sine [cosine] function can be expressed symbolically as;

$$y = a \sin(bx+c)+d \qquad [y = a \cos(bx+\mathrm{c})+d]$$

Semiotic representational systems

Duval (2006) mentions representations as *semiotic representational systems* within which a sign takes its meaning in opposition to other signs and their complex associations. Semiotic representations can be considered as common tools for not only the development of a new mathematical concept but also the communication of any particular mental representations (or concept images). Therefore, natural language is a highlighted semiotic representational system.

Representational registers

Duval (2006) classifies semiotic representations into four semiotic systems as represented in *Figure 2.3* (see *Review of the Literature* chapter), which are called the *representational registers*, with respect to the nature of operations (*discursive versus non-discursive*) and the properties of processes (*multi-functional versus mono-functional*) that are used to describe a system. He defines the semiotic representations as the *register*s only which permit transformations of representations due to the fact that for understanding the thinking process in any mathematical activity, it is important to focus on the *level* of semiotic representations instead of focusing on the particular representation produced. Where, the *level* of semiotic representations means the ability to transform a mathematical object from one semiotic representation into one another.

Treatment

Transformations are crucial in the mathematical understanding. Duval (2006) separates transformations of semiotic representations in two types: *treatment* and *conversion*. Treatments refer to the transformations of representations within the same representational register.

Conversion

Conversions refer to the transformations of representations between the different representational registers (Duval, 2006).

Conversion trouble

The cognitive complexity of conversions results in the troubles of students in mathematical thinking, as well as leading them to "consider two representations of the same object in different registers as being two mathematical objects" (Duval, 2006, p. 124). These conversion troubles (or cognitive distances between registers) are observed only when tasks in which a representation within a source register is systematically varied into its converted representation in the target register (also tasks in which the roles of the source register and the target register are inverted) are given to the students.

Content of a representation

"Conversion is a representation transformation, which is more complex than treatment because any change of register first requires recognition of the same represented object between two representations whose contents have very often nothing in common." (Duval, 2006, p. 112). Content of a representation is composed of signs so that they take their meanings in opposition to other signs and their complex associations. "The content of a representation depends more on the register of the representation than on the object represented" (Duval, 2006, p. 114).

Recognition

Recognition is the one of the most important two cognitive skills (i.e., *recognition* and *discrimination*) in comprehension of mathematical concepts in a mathematical activity (Duval, 2006). It can be defined as the ability to recognize the same object represented in two different representational registers "whose contents have very often nothing in common" (Duval, 2006, p. 112).

Discrimination

Discrimination is the one of the most important two cognitive skills (i.e., *recognition* and *discrimination*) in comprehension of mathematical concepts in a mathematical activity (Duval, 2006). It can be defined as the ability to distinguish the represented object in a register from the content of the semiotic representation. It requires to be able to "discriminate in any semiotic representation what is mathematically relevant and what is not mathematically relevant?" (Duval, 2006, p.115).

Concept development process

Sfard (1991) classifies mathematical understanding as *operational conception* (dynamic, sequential and detailed) versus *structural conceptions* (static, instantaneous and integrative). However, she emphasized this distinction's dual meaning rather than dichotomy. She proposed that when learning a new mathematical concept, the first developed conception type is operational. And then, the mathematical concept's development comes true through converting the *operational conception* (various processes) into the *structural conception* (compact static whole) after a lengthy and difficult process. Through inspiring mathematical concepts' historical development processes, Sfard (1991) separates this lengthy and difficult process into three hierarchical stages: *interiorization*, *condensation* and *reification*.

Interiorization

Interiorization is the first stage of the concept development theory of Sfard (1991) that means for students becoming familiar with the processes on the mathematical object. A student in the *interiorization* stage becomes skillful in performing the processes on lower-level mathematical objects.

Condensation

Condensation is the second stage of the concept development theory of Sfard (1991) that means for students becoming skilled with seeing of the process as a condensed whole without going into details. At this stage, the processes are easily combined with other ones, lengthy sequences of operations are compressed into more manageable ones, as well as generalizations and comparisons are smooth.

Reification

Reification is the last stage of the concept development theory of Sfard (1991) that is defined as an instantaneous shift the ability to see familiar processes as a reified-object. By this object (or mathematical construct), different representations of the mathematical concept are semantically merged, which means any more fundamental properties of this reified-object in its different representations and relations among them can be investigated easily.

(Unit) circle register

In this study, throughout the teaching experiment, angles were represented on the (unit) circle located on the perpendicular coordinate system or its translated form which was the parallel displacement system. Where, the angle's vertex was the center and its initial side was the ray in the positive horizontal direction. We called this *multi-functional* and *non-discursive* representational register as the (*unit*) *circle register*. We used the *unit* term in parenthesis as a consequence of the changeable-meaning of the *unit* with respect to the assumptions of the *unit* measure.

Graphical register

Graphical representation is a visual display so as to represent the coordinated-variations of two variables on the coordinate plane. In this study, we called this *mono-functional* and *non-discursive* representational register as the *graphical register*.

Symbolic register

Symbolic representation is based on a mathematical notational system including symbols referring to numbers (e.g., 0, -1/2, π), relations (e.g., =, >, <), functional expressions (e.g., *f*, *log*, *sine*), operations (e.g., ±, √, *lim*, *dy/dx*), etc. In this study, we called this *mono-functional* and *discursive* representational register as the *symbolic register*.

Language register

Semiotic representations can be considered as common tools for not only the development of a new mathematical concept but also the communication of any particular mental representations (or concept images). Therefore, natural language is a highlighted semiotic representational system (Duval, 2006). In this study, we called this *multi-functional* and *discursive* representational register as the *language register*.

CHAPTER 2

LITERATURE REVIEW

This chapter begins with the presentation of the conceptual framework for the study. Also, this chapter includes the discussion on the relevant research literature on trigonometry under the following subheadings: "*Comprehension of Trigonometry*", "*Roots of Students' Difficulties in Trigonometry*", "*Angle versus Angle Measure*", "*Sine and Cosine*", "*Graphical Representation of Trigonometric Functions*", "*Periodicity*", "*Technology in Education*", "*Trigonometry with Technology*" and "*Summary of Literature Review*".

2.1. Conceptual Framework

The overarching purpose of this study is to examine students' *concept images* on trigonometric functions during the instruction in dynamic geometry environment (including sequential tasks which emphasize multiple semiotic representations and transformations both between and within them so that students can distinguish the represented object (sine and cosine) from its used semiotic representation). Therefore, the conceptual framework of this study includes *concept image*, *concept development process* and *semiotic representations from a cognitive point of view* aspects.

2.1.1. Concept image

Mathematics is expressed through processes including formal descriptions and representations (Sfard, 1991). However, comprehension of mathematics is more different aspect in terms of cognitive functioning underlying the diversity of mathematical processes (Duval, 2006). Tall and Vinner (1981) formulate the distinction between formally defined mathematical concepts (i.e., *concept definition*) and the cognitive processes by which they are conceived (i.e., *concept image*). They articulated the term *concept definition* as “the form of words used to specify a concept” (p. 152) and the term *concept image* as “the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes” (p. 152). The concept map in *Figure 2.1* summarizes the relations and distinction between the notion of the *concept image* and the *concept definition*.

When articulating these terms, Tall and Vinner (1981) use “concept definition” in three different terminologies (*i*) *formal concept definition*, (*ii*) *personal concept definition* and (*iii*) *concept definition image*. They use these terms; respectively, in the meaning of (*i*) the definition of a concept that is accepted by the mathematical community, (*ii*) a student’s own definition of the concept that is reconstructed from the formal concept definition by himself/herself, and (*iii*) a part of the *concept image* that is generated by the *personal concept definition* of each individual. Therefore, they argue the *concept definition image* as quite a different matter from the *formal concept definition* through considering the *concept definition image*’s different possibilities for some individuals; namely, its non-existence, being “coherently related [/unrelated] to the other parts of the *concept image*” (p. 153), as well as comprising some notions “not acknowledged by mathematicians as a part of the formal theory” (p. 153). They exemplify that,

> …a teacher may give the formal definition and work with the general notion for a short while before spending long periods in which all examples are given by formulae. In such a case, the concept image may develop into a more restricted notion, whilst the concept definition is largely inactive in the cognitive structure. Initially the student in this position can operate quite happily with his restricted notion adequate in its restricted context. He may even have been taught to respond with the correct formal definition whilst having an inappropriate concept image. Later, when he meets the concept in a broader context he may be unable to cope. (p.153)

Moreover, Tall and Vinner (1981) describe *potential conflict factor* as "a part of concept image or concept definition which may conflict with another part of the concept image or concept definition" (p. 153). The potential conflict factors can be "conveniently considered based on the circumstances without causing any cognitive conflict. They only become *cognitive conflict factors* when evoked simultaneously" (p.154). Tall and Vinner (1981) argue the conflicting part of the *concept image* with the *formal concept definition* itself as the more serious type of the *potential conflict factor*. They assert the weak understanding of the *concept definition* as a source of the students' problems in mathematics especially when there are *potential conflict factors* between a strong *concept image* and a weak *concept definition image*.

In this respect, concept development process is an absolutely important aspect to help students construct well-structured *concept image*s coherently related to the *concept definition*.

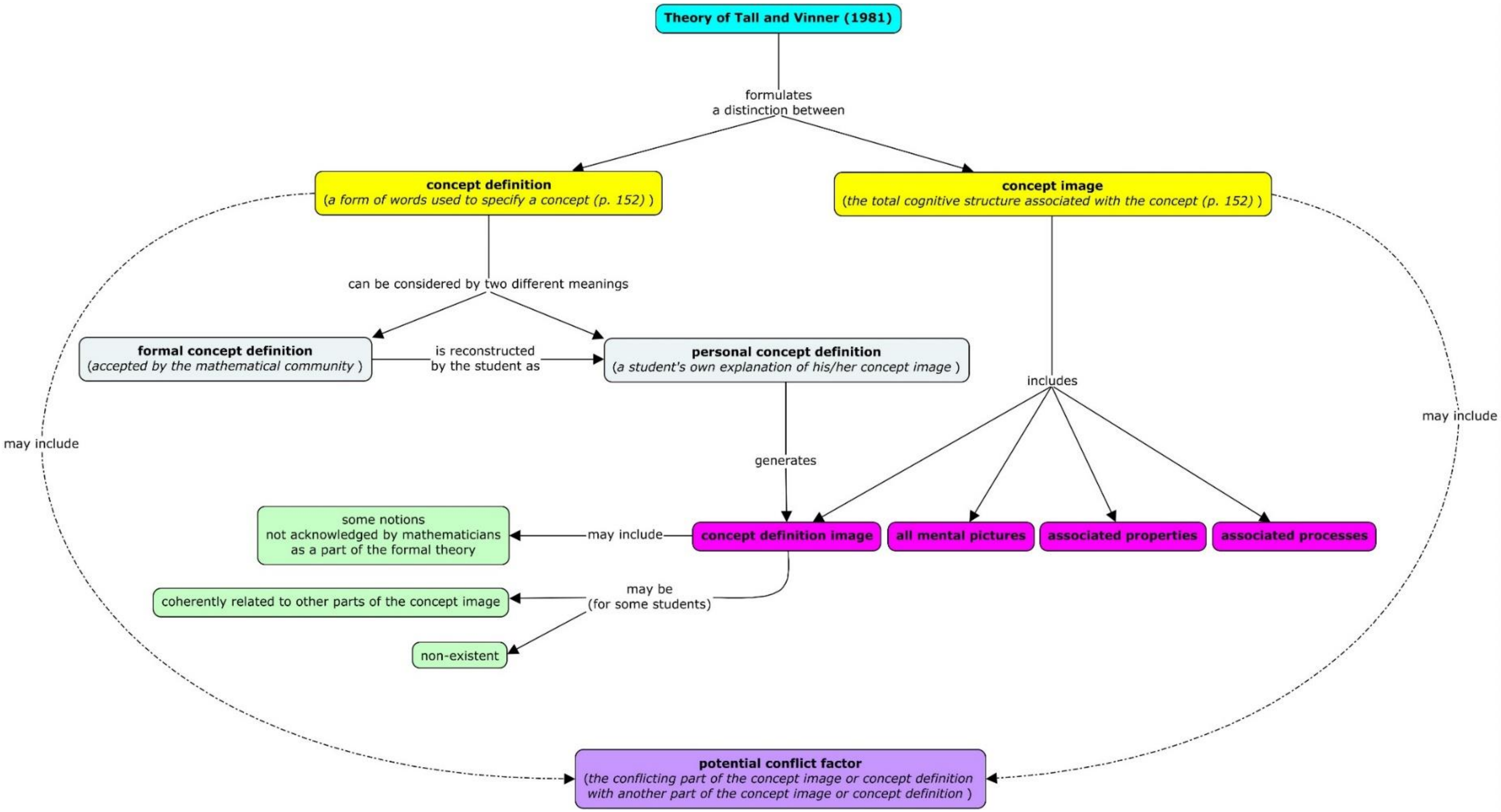


Figure 2.1. The Concept map of theory of *concept image* and *concept definition* (Tall & Vinner, 1981)

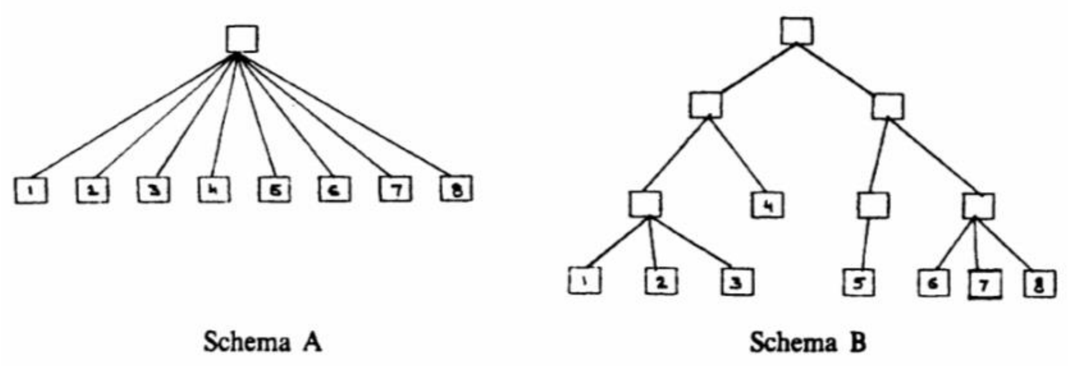
2.1.2. Concept development process

Sfard (1991) classifies mathematical understanding as *operational conception* (dynamic, sequential and detailed) versus *structural conceptions* (static, instantaneous and integrative). However, she emphasized this distinction's dual meaning rather than dichotomy. She proposed that when learning a new mathematical concept, the first developed conception type is operational. And then, the mathematical concept's development comes true through converting the *operational conception* (various processes) into the *structural conception* (compact static whole) after a lengthy and difficult process. Through inspiring mathematical concepts' historical development processes, Sfard (1991) separates this lengthy and difficult process into three hierarchical stages: *interiorization*, *condensation* and *reification*.

First of all, the *interiorization* stage means for students becoming familiar with the processes on the mathematical object. A student in the *interiorization* stage becomes skillful in performing the processes on lower-level mathematical objects.

Next, the *condensation* stage means for students becoming skilled with seeing of the process as a condensed whole without going into details. At this stage, the processes are easily combined with other ones, lengthy sequences of operations are compressed into more manageable ones, as well as generalizations and comparisons are smooth.

Finally, the *reification* stage is defined as an instantaneous shift the ability to see familiar processes as a reified-object. By this object (or mathematical construct), different representations of the mathematical concept are semantically merged, which means any more fundamental properties of this reified-object in its different representations and relations among them can be investigated easily. Sfard exemplifies the effect of reification on the *schema* of a student (in which the information is stored –or organized– in many different ways) like in *Figure 2.2*. "A *schema* is an individual's mental construction connecting related processes and objects, and appears to be somewhat similar to one's *concept image*" (Harel, Selden & Selden, 2006, p. 157).



Any information can be stored in many different schemata. For example, the two schemata pictured in this figure contain the same information (represented by (1, 2, 3, 4, 5, 6, 7, 8)). Schema A is sequential, shallow and wide. As a result of reification it can be reorganized into a deeper and narrower structure, such as Schema B. With the new organization, all cognitive processes (retrieval and storing) become much faster.

Figure 2.2. An example of different organizations of the stored information (Sfard, 1991, p. 27)

However, establishing the cognitive connections between the related processes and objects are not easy for many students. Duval (2006) argues the diversity of mathematical processes that are specific to the mathematical activities as a principal source of difficulties in comprehension of mathematics. At that point, he emphasizes “the paramount importance of semiotic representations for any mathematical activity” (p. 103). In this respect, considering semiotic representations from a cognitive point of view becomes an important issue to understand incomprehension in the learning of mathematics.

2.1.3. Semiotic representations from a cognitive point of view

Duval (2006) mentions representations as *semiotic representational systems* within which a sign takes its meaning in opposition to other signs and their complex associations. Semiotic representations can be considered as common tools for not only the development of a new mathematical concept but also the communication of any particular mental representations (or concept images). Therefore, natural language is a highlighted semiotic representational system.

Duval (2006) classifies semiotic representations into four semiotic systems as represented in *Figure 2.3*, which are called the *representation register*s, with respect to the nature of operations (discursive versus non-discursive) and the properties of processes (multi-functional versus mono-functional) that are used to describe a system. Moreover, he defines the semiotic representations as the *register*s only which permit transformations of representations due to the fact that for understanding the thinking process in any mathematical activity, it is important to focus on the *level* of semiotic representations instead of focusing on the particular representation produced. Where, the *level* of semiotic representations means the ability to transform a mathematical object from one semiotic representation into one another. Transforming one semiotic representation to another one is to be only at the level of grasping the basic properties of semiotic representations and their significance for mathematics. This means for the students to be able to distinguish the represented mathematical object from its representation registers. From this point of view, transformations are crucial in the mathematical understanding. He separates transformations of semiotic representations in two types: *treatment* and *conversion*. While *treatments* refer to the transformations of representations within the same register (curved arrows in *Figure 2.3*), *conversions* refer to the transformations of representations between the different registers (straight arrows in *Figure 2.3*).

As it is mentioned above, the mathematical understanding requires recognition of the same represented object in the different registers. This is a complex and difficult process. Duval (2006) articulates two sources of problems in this complex and difficult process. First of all is the complexity and specificity of treatments in the multifunctional registers. For example, when dealing with visualization, there are many ways of "seeing" (Duval, 2006) so that it is not easy in visual transformations of figures to see and discern from the original figure to the reconfigured-one which will make possible to establish the relation. Second source of the problems in the mathematical understanding is the conversion of representations (or change of registers); for instance, converting a cartesian graph into its corresponding equation. Duval (2006) asserts that "conversion of representations requires the cognitive DISSOCIATION of the represented object and the content of the particular semiotic representation through which it has been first introduced and used in teaching"; however, "there is a cognitive IMPOSSIBILITY OF DISSOCIATING the content of any semiotic representation and its first represented object" (p. 124). Therefore, the cognitive complexity of conversions results in the troubles of students in mathematical thinking, as well as leading them to "consider two representations of the same object in different registers as being two mathematical objects" (p. 124). These conversion troubles (or cognitive distances between registers) are observed only when tasks in which a representation within a source register is systematically varied into its converted representation in the target register (also tasks in which the roles of the source register and the target register are inverted) are given to the students. Moreover, recognition of the same mathematical object through different representations in terms of what is mathematically relevant or what is mathematically different in any representation content is a crucial cognitive condition in order to use knowledge outside of the narrow learning contexts. He articulates this issue through giving an example on the linear functions' algebraic expressions and their graphs that two graphs which seem visually alike are taken two by two and contrasted by two (or more) visual features, they are merged as if they were only one; in other words, the implicit construction of a cognitive network (like in *Figure 2.4*) for graphic representations is formed.

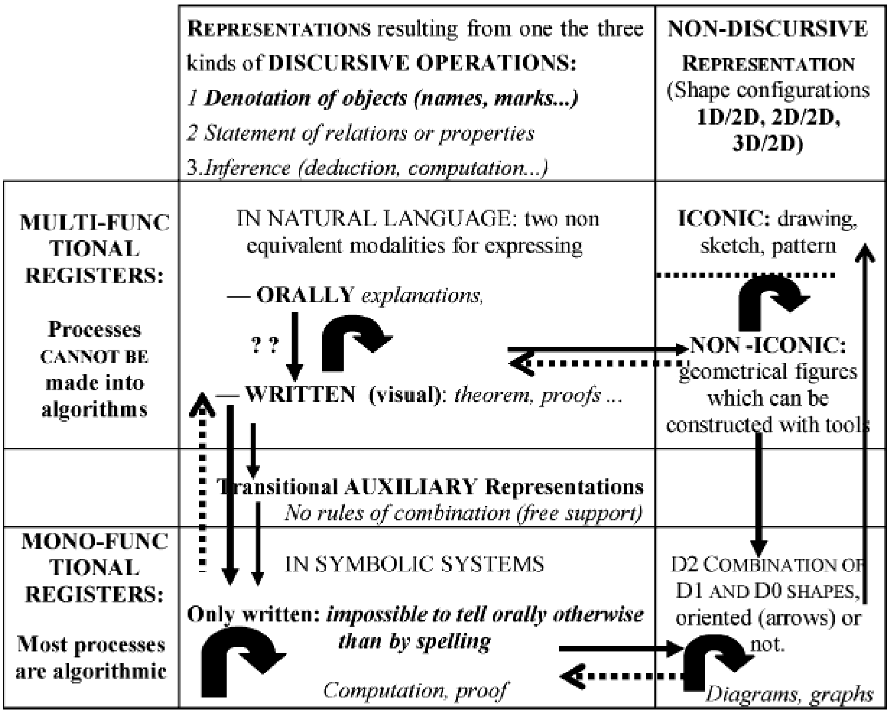


Figure 2.3. Classification of the semiotic representations into the four representational registers (Duval, 2006, p.110)

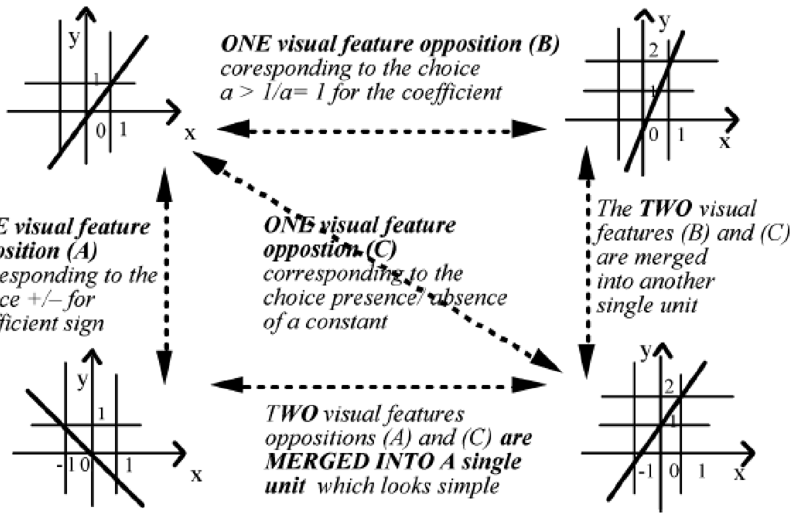


Figure 2.4. A cognitive network for any graphic representation discrimination (Duval, 2006, p.125)

To sum up, Duval (2006) emphasizes two main cognitive skills in comprehension of mathematical concepts: *recognition* and *discrimination*. When articulating these terms, he stresses the dissociation of the *content of a representational register* and the *mathematical object represented* in this register. He articulates *recognition* and *discrimination* terms, respectively, via following two perfect questions: “When facing two representations from two different registers, how can one recognize the same object represented within their respective content? … how can a student discriminate in any semiotic representation what is mathematically relevant and what is not mathematically relevant?” (p. 115).

2.2. Comprehension of Trigonometry

Trigonometry is a part of mathematics in which algebra and geometry converge. It can be introduced traditionally in two ways: the *ratio method* and the *unit circle method*. On the one hand, in the ratio method, trigonometric functions are defined as the ratios of the lengths of sides from right angled triangles for the intended acute angle. On the other hand, in the unit circle method, a circle with 1-unit radius is used to define cosine and sine functions respectively as a horizontal and vertical component of a point on the circle with respect to the intended angle. It is expected from students to give meaning trigonometric functions coherently in both the right triangle context and the unit circle context. However, interpretations of these definitions by students (i.e., *concept definition image*) constitute trigonometric functions’ meanings for each student.

Palmer (1980) determined that both methods were effective on helping students to learn basic concepts of trigonometric facts; but Burch (1981) described students’ difficulties with the unit circle (as cited in Thompson, 2007). The study of Kendal and Stacey (1996) comparing the successes of each of these methods revealed that students who were taught the ratio method performed much better than those who were taught the unit circle method (as cited in Steer, Antioneta & Eaton, 2009). However, the researchers still advocate the use of both methods because the unit circle is appropriate for periodic nature of trigonometric functions (Thompson, 2007). Unfortunately, research literature on trigonometric functions concluded that the standard instruction, which is based on these two methods, did not constitute students' strong understanding of trigonometric functions (Akkoç, 2008; Brown, 2005; Moore, 2014; Weber, 2005).

2.3. Roots of Students' Difficulties in Trigonometry

As the root of students' difficulties in trigonometry, Thompson (2008) mentioned the right triangle trigonometry and the periodic functions' trigonometry as two unrelated trigonometries of elementary and secondary school mathematics. In the right triangle trigonometry, sine and cosine are taught as ratios; and then, mentioned on exercises especially on some special angle triangles, such as 30-60-90 and 45-45-90 degree triangles, to solve some missing side. However, angle measure is not considered in many students' understanding as the argument of trigonometric functions. In the right triangle trigonometry, Thompson (2008) claimed students' consideration of whole triangle as the argument instead of the angle measure. Therefore, in comprehension of trigonometry, he stressed the importance of mentioning the arguments of trigonometric functions as the (meaningful) numbers referring to the angle measure instead of the letters to name of angles, i.e., θ.

Many researchers (Akkoç, 2008; Fi, 2003; Thompson, 2008; Topçu, Kertil, Akkoç, Yılmaz &Önder, 2006) emphasized the angle measure as problematic for thinking about trigonometric functions. Akkoç (2008), Topçu et al. (2006) and Fi (2003) concluded that preservice and/or inservice secondary mathematics teachers' understanding on the meaning of the radian measure unit was constrained by their understanding of the degree measure unit, and based only on the transformations between the degree and the radian measure units without any other meaning of the radian measure. Topçu et al. (2006) characterized the equation $\frac{D}{180} = \frac{R}{\pi}$ as a possible source of this understanding based on the participants' consideration of this equation as a cognitive unit during the interviews. Akkoç (2008) and Fi (2006) proposed the presence of π in the radian measures as another source of the participants' difficulty with the radian measure unit due to their consideration of π as the unit for the radian measure. This difficulty led to some misinterpretations of the preservice teachers; for example, consideration of 1 radian as equal to 180 degree (Fi, 2006) and consideration of the real value of π in the trigonometry context as equal to 180 (Akkoç, 2008).

In comprehension of trigonometry, Thompson (2008) stressed the need to develop coherent angle measures in degrees and radians through merging their meanings as the proportional relation between the arc length subtended by the angle and the circle's circumference. Indeed, Moore (2013) examined three precalculus students' angle measure understanding during a conducted-teaching experiment including sequential tasks that require establishing multiplicative relationship among a subtended arc length, a circle's circumference and a circle's radius. His study concluded that these quantitative relationships fortified students' transition abilities between the measurement units (e.g., an arc length measure in feet) and angle measure units (degree and radian) based on the invariant meaning of the proportional relation between the arc length and the circle's circumference (in terms of the radius as a measurement unit) (see *Figure 2.5*).

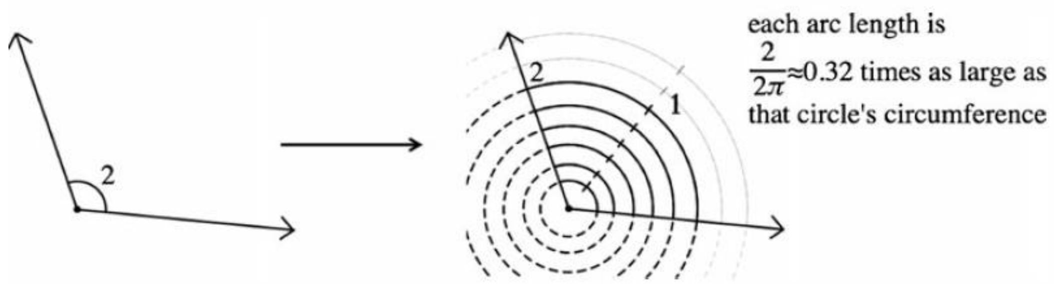


Figure 2.5. An arc length image of angle measure that involves equivalence of arcs (Moore, 2013, p. 228)

Thompson (2008; 2011) mentioned the inner arc of an acute angle of a right triangle only as a pointer in a diagram (of the textbooks). Moreover, he emphasized the lack of clear meaning of angle measure in terms that the measured-thing is what. This idea was also highlighted in the study of Moore (2010) that examined precalculus students' ways of understanding the trigonometric functions from the viewpoint of the students' constructions of *quantitative covariation* throughout a teaching experiment which was designed to support their understanding of the angle measure and the radius as a measurement unit. Moore (2010) resulted that prior to the teaching experiment, when describing the meaning of angle measure, students' conceptions did not include a measurement process involving measurable attribute (e.g., subtended arc length and circumference). Results from the teaching experiment (Moore, 2010) indicated that in conceptualizing angles and their measures, students' construction of the meaning of an angle as a measurable attribute and the radius as a unit of measurement was an important idea. Moreover, this idea of an angle measure was mentioned as foundational to leverage the reasoning abilities for learning and using the sine and cosine functions as far as students' reasoning about its covariational change with sine and cosine was supported.

2.4. Angle versus Angle Measure

Angle and angle measure are two different concepts that is needed to be dissociated from each other (Argün, Arıkan, Bulut, & Halıcıoğlu, 2014). Angle is a shape constituted by two rays with a common endpoint. However, defining the angle measure is more complex than defining the angle. Several aspects become a current issue to define angle measure (e.g., angle measure axioms, angle measure units, angle measure processes and procedures, etc.).

There are two different angle measure axioms (Argün et al., 2014). One of them is that there exists for each angle a unique real number between 0 and 180 so that it corresponds the degree measure of the angle. This axiom gives the straight angle with 180 degree, otherwise, it considers the measure of the *interior openness*[5] of an angle. The other angle measure axiom is that Birkhoff's angle measure postulate that proposes the angle measure with respect to modulo 360. This axiom enables us to define the reflex angles.

There are two commonly used angle measure units: *degree* and *radian*. The degree is the more familiar unit of angle measure. For example, in the geometry context, angles are mentioned with their degree measures. Akkoç (2008) stated this situation as the possible cause of students' strong concept images of degree which dominated their concept images of radian. On the other hand, the radian is the unit of angle measure that is mostly used in the trigonometry context. A complete revolution corresponds to 2π in radians and 360 in degrees. This situation produces the proportional equation $\frac{D}{360} = \frac{R}{2\pi}$ between an angle's degree and radian measures through considering the same ratio between a subtended arc angle and the whole circle's angle both in degrees and in radians (*Figure 2.6*). Topçu et al. (2006) argued this equation as a possible source of the restricted understanding of the meaning of the radian measure only into the transformations between degree and radian measures. As a consequence of incorrect interpretations of this equation, misunderstanding of π in the trigonometry context arose. Many researchers (Akkoç, 2008; Fi, 2003; Topçu, et al., 2006) reported the confusion about π as a real number in the trigonometry context. They stressed two main troubles about π in the trigonometry context; (*i*) reasoning about the real value of π as equal to 180 (*ii*) reasoning about π as a radian measure unit. It means that strong understanding of the meaning of the radian requires beyond the symbolic transformation from the degree measure. The study of Moore (2013) emphasized the positive effect of students' construction of the invariant meaning of the proportional relation between the arc length and the circle's circumference on their transition abilities between the measurement units (e.g., an arc length measure in feet) and angle measure units (degree and radian) based on the proportional quantitative

[5] *Interior openness* refers to one of two regions separated by the angle's rays that its measure is less than 180 degree.

relationships. This result stresses the importance of the mentioning relationships together with the visual ideas such as arcs subtended by the angle and the whole circle.

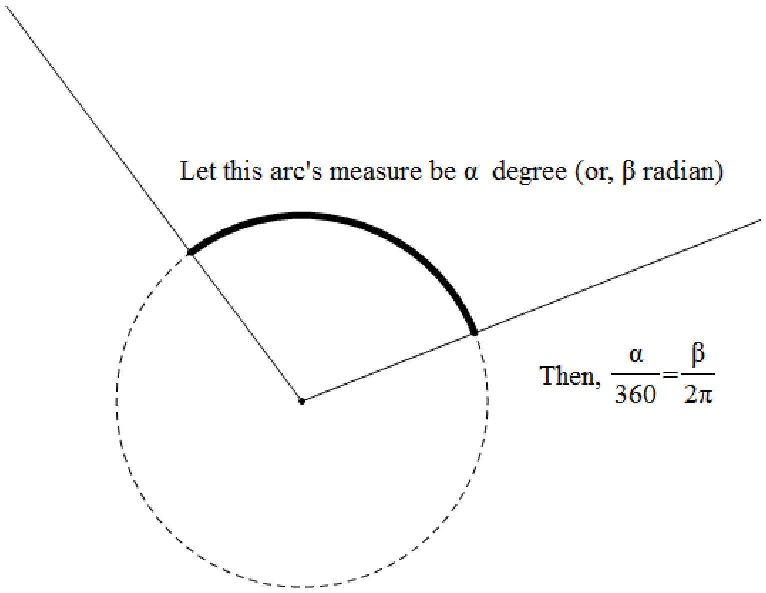


Figure 2.6. Proportional equality between degree and radian measures of the same angle

Tall and Vinner (1981) assert the weak understanding of the concept definition as a source of the students' problems in mathematics. At that point, the definition of angle measure has a key role in students' comprehension of angle measure concept. It is expected from students to give meaning the angle measure coherently both in degree and radian measure units in comprehension of trigonometric functions. There are several different definitions of angle measure in textbooks based on the different ideas; for example, rotation amount with respect to the complete revolution on the circle centered at the vertex of the angle, visual procedures with a protractor, and magnitude of the unit measure (1 degree or 1 radian) through multiplicative relationships (Moore, 2012). When indicating an angle's measure on a visual representation, generally an inner arc was created. Thompson (2008) argued this inner arc only as a pointer in a diagram, and emphasized the lack of clear meaning of angle measure in terms that the measured-thing is what. This idea emphasizes the importance of the making sense of this inner arc in terms of its meaning for the angle measure beyond indicating the measured-openness from two ones that are separated by an angle.

Although an angle has a *static* figure, the angle measure idea requires an important cognitive condition, i.e., *seeing* this *static* structure from a *dynamic* point of view. Especially, this cognitive condition is absolutely important for moving from the right triangle trigonometry to the unit circle trigonometry through considering the angle as a continuous variable through directed rotations and then their geometric representations. Many of the studies in the literature (Brown, 2005; Fi, 2003; Mitchelmore & White, 1996; Moore, 2010) emphasized the importance of thinking about angles in terms of rotations in comprehension of trigonometric functions in different representations, as well as using in the different contexts. Mitchelmore and White (1996) characterized children's perceptions of angles as static rather than as dynamic turning. Fi (2003) reported preservice mathematics teachers' weak understanding of the rotational angles in terms of reasoning and using the coterminal angles in the problem solving context despite of their strong understanding of the *counterclockwise* and *clockwise* rotation and determination of the size of the angle of a rotation. Consideration of the meaning of the coterminal angles is foundational to understand the periodic nature of the trigonometric functions.

2.5. Sine and Cosine

Dealing with trigonometric functions is known as a difficult task since it requires to work with right triangles, the unit circle, and graphs of trigonometric functions simultaneously. Because trigonometric functions cannot be expressed as algebraic formulas involving arithmetical procedures, students have trouble on reasoning about them as functions (Weber, 2005). In comprehension of these functions, there is a need to associate them properly with the appropriate geometric models (Brown, 2005; Weber, 2005).

Weber (2005) examined two groups of undergraduate students' understanding of the basic trigonometric functions (i.e., sine and cosine), one of whom received the standard instruction and the other received an experimental instruction based on Gray and Tall's (1994) the notion of *procept* theory of learning. He reported that students who received standard instruction had a trouble in constructing mentally or physically the geometric models to deal with trigonometric functions. In other words, they were unable to relate trigonometric functions flexibly with the appropriate geometric models instead of their own prototypical geometric models. For instance, many students needed to see an appropriately labelled triangle to complete the tasks including approximation to sin(θ) for specific values of θ (e.g., 40º, 170º and 140º).

Relevant result was reported in the study of Brown (2005) that investigated honor secondary school students' understanding at the end of their work with the core trigonometric functions moving from the right triangle to the unit circle on the coordinate plane. She modeled students understanding and recognition of the concepts and representations of the foundational ideas of the coordinate trigonometry. She argued that students understood trigonometry often through the geometric figures they drew and their ways of reasoning about the meaning of sine and cosine in these geometric models. She exemplified some drawings (*Figure 2.7*) of four different students' reasoning about an angle in the second quadrant based on its cosine value on the coordinate plane. She interpreted first three of them (from Dave, Sara, and Jim) together with the interview data, and resulted their compartmentalized understanding of three views of the coordinate trigonometry (i.e., *right triangle*[6], *distance*[7] and *coordinate*[8]). Brown (2005) categorized students' usage of these three views of the coordinate trigonometry into three; (*i*) having integrated all three (e.g., Michael's integrated understanding –see Michael's response in *Figure 2.7*), (*ii*) strong orientation only one view (e.g., Dave's strong orientation on *right triangle* view, and Jim's strong

[6] For example, sine of an angle (mentioned via rotation in the standard position on the coordinate plane) is defined as a ratio of the opposite side to radius in *right triangle view* of the coordinate trigonometry.

[7] For example, sine of an angle in the standard position is defined as a ratio of the directed vertical distance (of the terminal point of the rotation) to the radius in *distance view* of the coordinate trigonometry.

[8] For example, sine of an angle in the standard position is defined as the *y*-coordinate of the terminal point of the rotation on the unit circle in *coordinate view* of the coordinate trigonometry.

orientation of *coordinate* view –see their response in *Figure 2.7*), (*iii*) using only two views (e.g., Sara’s consideration of initially *distance* view, and then, *right triangle* view but incorrectly in terms of angle’s standard position –see Sara’s response in *Figure 2.7*). Brown (2005) characterized many of the students’ usage as the strong orientation only one view of the coordinate trigonometry among *right triangle*, *distance* and *coordinate*. She defined the conception of rotation angle and unit as the cognitive obstacles that affect students’ understanding of sine and cosine especially on connecting their unit circle representation to their graphical representations.

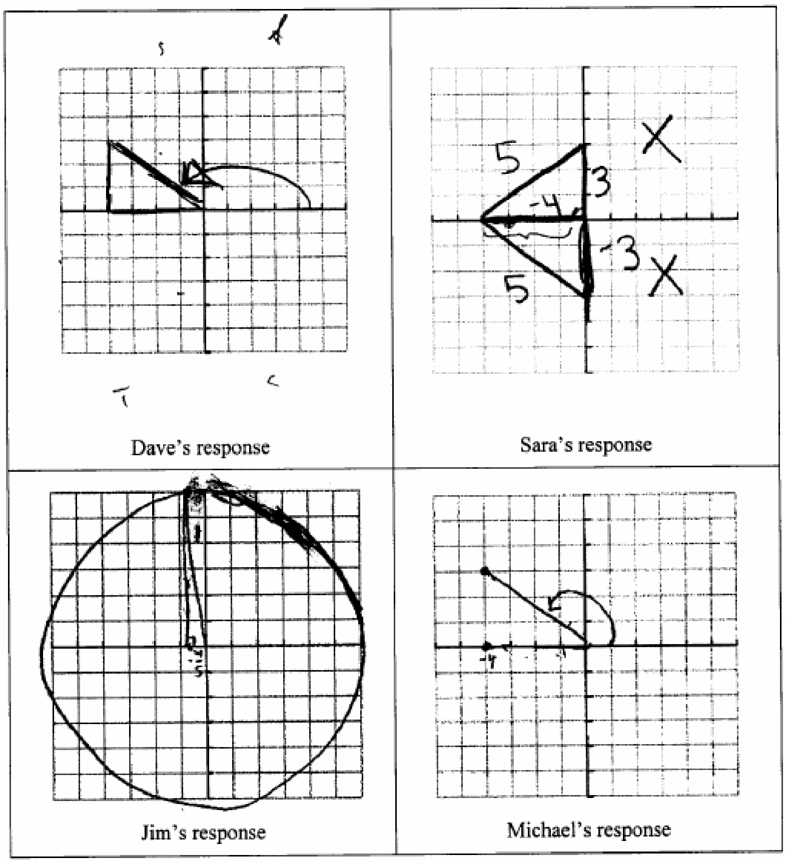


Figure 2.7. Four students’ drawings of angles whose cosine is -4/5 (Brown, 2005, p. 228)

Integrating all of three visual views of the coordinate trigonometry (right triangle, distance and coordinate) requires to overcome the disconnection between the right triangle trigonometry and the periodic function trigonometry. Thompson (2008) emphasized that "triangle trigonometry would draw from the meaning of angle measure… and would also draw from similarity –that similar right triangles have same ratios" (p. 36), as well as "periodic functions would draw from the meaning of angle measure… and would additionally highlight how one must think of varying an angle so as to systematically vary its measure" (p. 36). He emphasizes the meaning of angle measure as a common reference both for triangle trigonometry and periodic functions' trigonometry. This idea brings up two main angle concepts for discussion; namely, *reference angle* and *coterminal angle* that produce the systematic behaviors of trigonometric functions. Systematic variation of the angle measure provokes reasoning about systematic covariation of trigonometric functions. Kaput (1992) argued that making variation is needed for understanding of "invariance" which is the very important aspect of mathematical thinking. At that point, the importance of the meaning of the reference angle and the coterminal angles becomes a current issue in comprehension of trigonometric functions especially in the graphical representations. Fi (2003) reported the confusion about reference angles and coterminal angles especially in using to simplify trigonometric expressions.

If the historical development of trigonometry is concerned, it is clearly seen that trigonometry did not arise from an abstract need. On the contrary, beginning of trigonometry was intimately tied to astronomy (Katz, 2009). Dealing quantitatively with important aspects of astronomy such as the positions of stars and planets required spherical trigonometry, and its development entailed to understand plane trigonometry. The basic element in early plane trigonometry was to calculate the chord length subtending a given arc in a circle with fixed radius (Brummelen, 2009; Katz, 2009; Sanford, 1958), after then, the trigonometric "chord" function was modified to "sine" function (Katz, 2009). It was the fifteenth century when the sine of an arc was defined as the half chord of double the arc (Sanford, 1958). According to this fact, even if there is probably no best way to teaching of trigonometry, in the light of its historical development we can conclude that teaching activities should include dealing with a circle when studying trigonometric functions.

The unit circle was emphasized as an important tool in the literature for strong understanding of coordinate trigonometry or periodic function trigonometry (Brown, 2005; Moore, LaForest, & Kim, 2012; Weber, 2005). Nevertheless, students have difficulties with the unit circle in terms of understanding, using and interpreting trigonometric functions (Burch, 1981; Brown, 2005; Emlek, 2007; Güntekin, 2010; Gür, 2009; Weber, 2005). Moore, LaForest, and Kim (2012) lay emphasis on the importance of the ability to use trigonometric functions in any circular context. They reported preservice teachers' difficulties in reasoning about trigonometric functions on the non-unit circles. For instance, when encountered a non-unit circle context, students tried to relate the given circle to the unit circle through mentioning the unit circle apart from the given circle, instead of considering its radius as a unit. Moore, LaForest, and Kim (2012) concluded that provoking students to consider units with different measures promoted their unit circle understandings from their restricted understanding of the unit circle notion with "one" radius to any given circle through considering its radius as "one" unit.

2.6. Graphical Representation of Trigonometric Functions

Graphs represent the coordinated-variations of two variables on the coordinate plane. Despite of their static configurations, they represent dynamic variation of the rate of change between two variables. Duval (2006) considers graphs in the *monofunctional registers* (i.e., in which most processes are algorithmic) and *non-discursive representation* (i.e., including shape configuration) category (see *Figure 2.3* on page 27). Producing a graph requires *conversion* of a mathematical object [or function] from one register into the *monofunctional* and *non-discursive* register: for instance, "passing from the algebraic notation for an equation to its graphic representation" (Duval, 2006, p. 112). Duval (2006) argued students' changing performances based on reversing the direction of the conversion. He exemplified this idea focusing students' performances on the linear functions' graphs and their symbolic expressions. Whereas constructing linear functions' graphs does not cause any trouble for students, when only the direction of the conversion changes, i.e., seeing the rule from its graphic representation is expected, students' performances decrease dramatically. Duval (2006) entitled the second task as *recognition task* due to its requirement of *recognition* of the same object represented in two different registers "whose contents have very often nothing in common" (p. 112).

Mathematical activities including simultaneous variation of two variable are considered as critical cognitive supports for students' constructions and interpretations of graphs (Moore, 2012; Oehrtman, Carlson, & Thompson, 2008). Moore (2012) analysed three students' different interpretations of the sine and cosine graphs. First interpretation of sine and cosine graphs was not based on the collection of the ordered pairs whose values change in tandem, rather than "representing the "top half" and "bottom half" of circles" (p. 84). Moore (2012) attributed this interpretation to incomprehension of angle measure as a varying measure. Second interpretation on the Ferris wheel problem (asking to reason about the variation of the vertical distance with respect to the travelled arc length) was based on describing the vertical distance in terms of its directed-behaviour and the rate of change, initially on the circular-diagram, and then, on the created-graph. Moore (2012) attributed this interpretation to comprehension of angle measure as rooted in reasoning about arcs. However, it may be due to the realistic Ferris wheel context. Thompson (2007) resulted that "the use of Ferris wheel problem produced a distinct advantage concerning the connection between the unit circle and the graphing of trigonometric functions" (p. 205). Third interpretation, having emerged when constructing the graph of $y=\sin(3x)$, was based on the determination of some ordered pairs corresponding their x-values, e.g., 0, $\pi/2$ and π. Moore (2012) interpreted this graphing procedure as arising from "a pointwise focus and then *filling* in variation between these points" (p. 89) instead of a continuum variation of values. Considering these three different interpretation of sine function together with the *conversion* notion of Duval (2006), it is obvious that these interpretations maybe emerge as a consequence of the *conversion*s from the different source registers to the same target register (graphical representation). Therefore, students changing performances may be due to their troubles in *recognition* of the same object in the different representational registers (i.e., symbolic, circular, graphic) whose contents are different.

Two main critical ideas are emphasized in the literature to provide students with better understanding the connection of the unit circle to graphs of trigonometric functions: (*i*) meaning of *angle measure* in terms of rotations on the unit circle and (*ii*) *directed distance* approach on the unit circle and its connection to graphs of trigonometric function (Brown, 2005; Hertel & Cullen, 2011; Moore, 2012; 2014). From more inclusive point of view, the coordinated-variations of directed distance with respect to arc length *in radii* rather than any other measurement units is emphasized to generate coherent understanding of trigonometric functions (Moore, 2010; 2012; 2014). In addition, for instructions of sine and cosine, it is suggested that a right triangle be embedded on the unit circle representation together with directed lengths of its legs (Brown, 2005); and trigonometric function be interpreted as functions mapping "one measurable quantity, angle, to another measurable quantity, directed length" (Hertel & Cullen, 2011, p. 1401). Furthermore, the study of Thompson (2007) revealed the importance of the appropriate real-life situations to both teaching concept and students' lives for facilitating their understanding of the connections between the unit circle and the graphing of trigonometric functions.

2.7. Periodicity

Trigonometric functions are valuable functions for mathematics instruction due to the fact that they are natural and fundamental examples of periodic functions, and used to approximate any periodic functions. This is the reason why we, as educators, emphasize persistently the importance of learning the periodic functions' trigonometry in spite of students' difficulties with the unit circle in terms of understanding, using and interpreting (Burch, 1981; Brown, 2005; Emlek, 2007; Güntekin, 2010; Gür, 2009; Kendal & Stacey, 1996; Weber, 2005). Consideration of the meaning of coterminal angles on the unit circle approach is foundational to understand the periodic nature of the trigonometric functions. Fi (2003) considers coterminal angles as the related and necessary knowledge to model periodic phenomena (e.g., Ferris wheel problem), as well as emphasize the importance of coterminal angles to generate angle measures other than the principal ones. He reported preservice mathematics teachers' inadequate knowledge of coterminal angles and the periodicity idea as the problematic parts of their understanding of trigonometric functions.

Many other studies in the literature (Brown, 2005; Moore, 2010; Thompson, 2007) that focused on students' understanding of trigonometric functions did not investigate the periodicity phenomena, rather than focused on students' understanding of trigonometric functions in the four quadrant for the angles with principal measures.

2.8. Technology in Education

Appropriate integration of technology, which offers multiple-representation and dynamicity opportunities, into trigonometry courses may enable us to cope with these difficulties by facilitating to establish connections among symbolic, geometric (on the unit circle) and graphic representations of trigonometric functions. Technology enables students to do routine procedures quickly and accurately, to explore topics in more depth (Garofalo, Drier, Harper, Timmerman, & Shockey, 2000) and to obtain simultaneous connections between multiple representations (Ferrara, Pratt & Robutti, 2006); thus, allowing more time for conceptualizing and modeling.

There are several computer software programs to support and facilitate learning environments. Dynamic Geometry softwares (DGS) are one of the technologies used for teaching and learning of mathematics. They provide dynamic diagrams that students can slightly distort to meet their expectations, which is impossible in paper-and-pencil diagrams (Laborde, Kynigos, & Strasser, 2006), and give students the opportunity of constructing graphs of functions which can be manipulated and animated by the parameter, as well as the opportunity of dynamically linking between graphs and other representations of a situation (Mackrell, 2002); and helps them understand propositions by allowing them to perform geometric constructions with a high degree of accuracy (Hanna, 2000).

However, existing technology does not guarantee learning. Literature shows that technology usage can be effective only within appropriate teaching-learning context (Ferrara et al., 2006). The study of Jones (2002), in which published research on the use of dynamic geometry software was categorized, stated that a range of research showed that judicious use of DGS can foster the understanding of proof. However, if used inappropriately, DGS might make things worse instead of making significant effect on students' learning (Jones, 2002). Therefore, appropriate integration of technology (particularly, of DGS) into mathematics courses is needed. It requires carefully designed teaching/learning activities according to not only teachers' but also students' background, the task, the mathematical context, the class context and the potentialities offered by the software (Ferrara et al., 2006). In designing teaching activities, the most important role belongs to teachers. The teacher must decide if, when, and how technology will be used. In order to use technologies knowledgeably, intelligently, mathematically, confidently, and appropriately, teachers should be aware of the different roles of them, think clearly about their classroom goals, consider particular needs of particular students, and choose technologies to further those goals (Goldenberg, 2000).

The experimental study of Ubuz, Üstün and Erbaş (2009) showed that "if used appropriately" dynamic geometry environments "can serve as an important vehicle to improve student achievement in geometry and achieve a classroom culture where conjecturing, analysing, exploring, and reasoning are daily routines" (p.148). Also, Hannafin and Scott (2001) showed that using dynamic geometry softwares in a student-centered environment has a positive effect on the attitudes and beliefs of both teachers and students about teaching and learning geometry. In the subject design of Weeden (2002) where a dynamic geometry software was incorporated into her lessons, she observed that the classroom experience was extremely enjoyable for both students and teachers, they worked in collaboration and experienced reality of construction and properties of shape while learning to use the software; moreover, she observed that students developed a deeper understanding and a greater view of the whole picture.

2.9. Trigonometry with Technology

There is some research examined the effect of using a dynamic geometry environment on students' understanding of trigonometry (Blackett & Tall, 1991), on a 10th-grade student's learning experience and analytical thought process (Choi-Koh, 2003), on secondary school students' thinking concerning the unit circle and the graphing of trigonometric functions through animated real-life models (Thompson, 2007), and on preservice secondary mathematics teachers' understanding of trigonometric functions through their directed length interpretations (Hertel & Cullen, 2011).

Firstly, Blackett and Tall (1991) examined an experimental trigonometry course including dynamic computer-based tasks with a simple piece of software designed linking numerical input for lengths and angles to a visual display of a right triangle when compared to the traditional trigonometry instruction. Each group had 1-hour-10-minute four lessons and 35-minute four lessons in a two week period, followed by an immediate post-test and a second post-test eight weeks later. Research findings showed that the experimental students significantly outperformed than the control students on a post-test in both standard and non-standard tasks. More specifically, they stated that "experimental boys improved more than control boys and experimental girls improved more than control girls. … control girls improved less than the control boys, experimental girls improved *more* than the experimental boys" (p. 6). These findings about gender differences related to mathematics performance are unlike empirical evidence in the literature which shows that although females perform at least as well as males in mathematics performance in the early years, as they get teenage, differences in abilities emerge as biased to males especially on high cognitive level tasks (Fennema, 1974; Fennema & Carpenter, 1981; Halpern, 1986; Stage, Kreinberg, Eccles & Becker, 1985) and that males have more computer interests and more self-confidence in their ability to use computers than female students (Chen, 1986; Koohang, 1989; Shashaani, 1993). Also, Blackett and Tall (1991) argued that the computer representation enabled students to explore the relationship between numerical and geometric data in an interactive manner. However, where, the geometric representation of trigonometric functions by the software was tied to the ratio method instead of the unit circle method.

Secondly, Choi-Koh (2003) examined a 10th grade student's learning experience and analytical thought process while using the graphing calculator (Casio 9850 Plus) to study trigonometry on the task the role of coefficients in the equation $y = a\sin(bx + c) + d$ by establishing connection between graphical and symbolic representations. In this context, the geometric representation of trigonometric functions on the circle was not mentioned. The student in this study was an average-mathematics-performance student who had not taken trigonometry previously but had used a computer and a calculator. The researcher provided the student with the exploration-based learning environment during the study which was conducted in six weeks. Data collected through observations and clinical interviews. This qualitative case study concluded that the student's reasoning and thinking moved from the intuitive level (observing) to the operative level (explaining, abstracting, systematizing), and finally, to the applicative level (inductive generalizing, making formulas, reflecting), and his learning attitudes moved from passive (i.e., lack of intend to work creatively and voluntarily) to more interested and motivated by the aid of his voluntary use of graphing calculator.

Thirdly, the study of Thompson (2007) examined the role of a contextual realistic problem, Ferris wheel, as an instructional starting point on secondary school students' understanding trigonometric functions from *Realistic Mathematics Education* theoretical approach through conducting an action research. In this study, the instruction started a unit of trigonometry by modelling the motion of an animated Ferris wheel in Geometer's Sketchpad. This sketch showed the moving Ferris wheel tracing out the sine wave dynamically. Its static version was like *Figure 2.8*. Results of the study revealed that Ferris wheel problem, which models trigonometric functions via circular motion, leveraged students understanding in terms of connections between the unit circle and graphs of trigonometric functions, as well as supported students' comprehension of the general behaviour of trigonometric graphs regarding their periods and magnitudes.

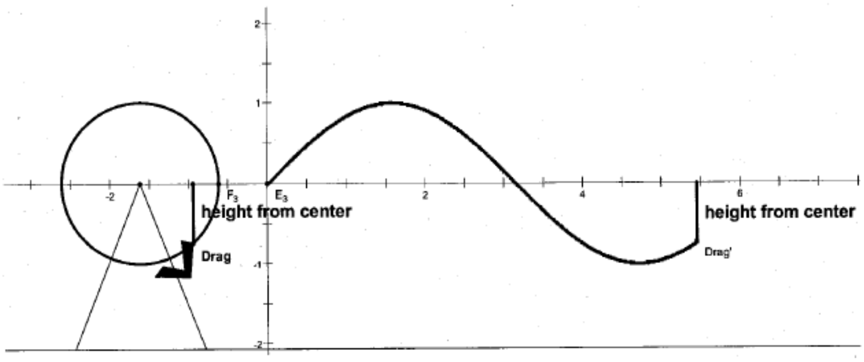


Figure 2.8. Static version linking the Ferris wheel to sine wave (Thompson, 2007, p. 144).

Finally, Hertel and Cullen (2011) designed an instructional sequence including activities based on the directed length interpretations of trigonometric functions in a dynamic geometry environment in order to generate robust and connected understanding of trigonometric functions, and then analysed this instruction's effect on preservice secondary mathematics teachers' understanding and problem solving strategies related to the trigonometry content. In designing the activities, they attached importance to promote reasoning about trigonometric functions as functions mapping from angle measure to corresponding directed length considering these two measures as measurable quantities. Each episode of the instruction included four cognitive steps. Researchers articulated these cognitive steps for the first episode as follow;

(*i*) exploring the "chord" function (see chord CD in *Figure 2.9*) inspiring from the sine function's historical origin, i.e., calculating the "chord" length (for more information about the historical development of sine, see the heading of *Sine and Cosine* in Review of the Literature Chapter), (*ii*) exploring the connection between angle measure and directed length focusing on the dynamic covariation of angle measure (principal angle BAC in *Figure 2.9*) and directed length (chord CD),

(*iii*) reasoning about mapping of this angle measure to a point, D', on the *x*-axis, and then, construction of the corresponding chord so as the point D to coincide with the point D' on the *x*-axis (*Figure 2.9*); the constructed graph in that way corresponds to y=2sin(x) function; and

(*iv*) considering the sine function on the standard unit circle representation as the half of the directed length of the chord CD (*Figure 2.9*).

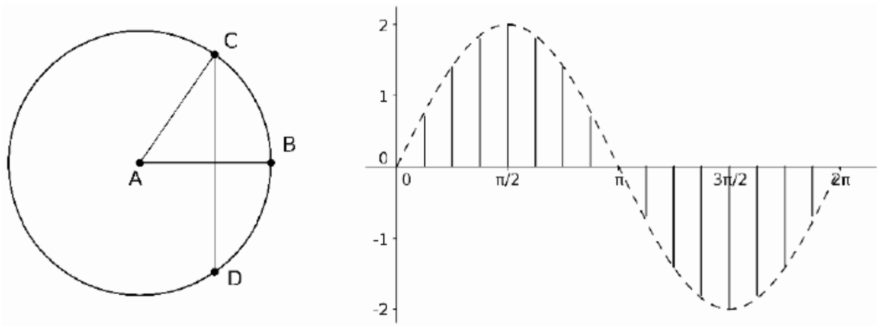


Figure 2.9. The chord function (Hertel & Cullen, 2011, p. 1404)

Researchers designed a paper-pencil assessment instrument based on the *APOS* (*action*, *process*, *object*, and *schema*) theoretical lens to assess participants' directed length interpretations. They reported statistically significant improvement from pretest to post test scores stemming from both the directed length interpretation and the use of dynamic geometry environment. They argued that dynamic geometry environment enabled students quickly, easily and efficiently "to construct figures, identify specific attributes, quantify these attributes, and analyze the relationships between these quantities by graphing them in the coordinate plane" (Hertel & Cullen, 2011, p. 1406). Finally, researchers resulted the variation of students' problem solving strategies from pretest to posttest responses based on the qualitative analysis with respect to three strategies (ratio, directed length, graph). The results revealed that while many students used ratio strategies on the pretest response, they used ratio and directed length strategies in a more flexible way on the post test. Furthermore, correct responses in all three strategies increased from pretest to posttest. They interpreted this result as the integration of the directed length approach into students' trigonometry schema.

2.10. Summary of Literature Review

To sum up, in the light of the overview of the related literature mentioned above, it is clearly seen that there are some important aspects needed to be critically concerned in trigonometry education. First of all is the importance of comprehension of trigonometric functions as functions (Hertel & Cullen, 2011; Weber, 2005). Because trigonometric functions cannot be expressed as algebraic formulas involving arithmetical procedures, most students have trouble on reasoning about them as functions (Weber, 2005). Therefore, it is attached importance to promote reasoning about trigonometric functions as functions mapping from angle measure to corresponding trigonometric value (Hertel & Cullen, 2011), as well as reasoning about angle measure as the (meaningful) numbers referring to the argument of trigonometric functions (Thompson, 2008).

In comprehension of trigonometric functions, there is a need to associate them properly with the appropriate geometric models (Brown, 2005; Weber, 2005). Two common geometric representations exist to model trigonometric functions visually: *right triangle representation* and *unit circle representation*. However, students' comprehensions of the right triangle trigonometry and the unit circle trigonometry are fragmented and unrelated (Thompson, 2008). Coherent understanding and flexible use of trigonometric functions require integration of different representations of trigonometric functions (Brown, 2005). To integrate trigonometric functions' meaning in any representational context, the meaning of angle measure is emphasized as the foundational cognitive idea (Thompson, 2008). However, angle measure is a problematic issue that is needed to be handled carefully in teaching of trigonometry (Akkoç, 2008; Fi, 2003; Thompson, 2008; Topçu, Kertil, Akkoç, Yılmaz &Önder, 2006).

The meaning of angle measure brings *angle* and *angle measure* concepts up for discussion. Angle and angle measure are two different concepts that is needed to be dissociated from each other (Argün, Arıkan, Bulut, & Halıcıoğlu, 2014). Angle is a shape constituted by two rays with a common endpoint. However, defining the angle measure is more complex than defining the angle. Several aspects become a current issue to define angle measure (e.g., angle measure axioms, angle measure units, angle measure processes and procedures, etc.). To indicate an angle measure on a visual representation, an inner arc is created. Nevertheless, without its meaning for the angle measure, this inner arc serves only as a pointer in a diagram (Thompson, 2008). There is a need of clear meaning of angle measure in terms that the measured-thing is what (Thompson, 2008; Moore, 2010). Thus, constructing the meaning of an angle as a measurable attribute is an important cognitive idea (Moore, 2010).

Although an angle has a *static* figure, the angle measure idea requires another important cognitive condition, i.e., *seeing* this *static* structure from a *dynamic* point of view. The importance of thinking about angles in terms of rotations is emphasized in comprehension of trigonometric functions in different representations (Brown, 2005; Fi, 2003; Mitchelmore & White, 1996; Moore, 2010). In addition, there is a need to develop coherent angle measures in degrees and radians (Thompson, 2008) through merging their meanings as the proportional relation between the arc length subtended by the angle and the circle's circumference (Thompson, 2008; Moore, 2013) in terms of the radius as a measurement unit (Moore, 2013; Moore, LaForest, & Kim, 2012).

However, literature indicates students' troubles on the radian measure unit. The meaning of the radian measure is dominated by degree meaning and restricted only into transformations between degree and radian measures (Akkoç, 2008; Topçu, et al., 2006). Two critical troubles related to radian measure emerge when students reason about π as a real number in the trigonometry context (Akkoç, 2008; Fi, 2003; Topçu, et al., 2006): (*i*) reasoning about the real value of π as equal to 180 (*ii*) reasoning about π as a radian measure unit. Once again, it is attached importance to promote reasoning about angle measure as the meaningful numbers (Thompson, 2008) through measuring arc lengths *in radii* (Moore, 2010; 2012; 2014).

In addition, the importance of the ability to use trigonometric functions in any circular context is emphasized because provoking students to consider units with different measures promotes their unit circle understandings from their restricted understanding of the unit circle notion with “one” radius to any given circle through considering its radius as “one” unit (Moore, LaForest, & Kim, 2012).

Finally, the coordinated-variation of the directed distance (referring to a trigonometric value) (Brown, 2005; Hertel & Cullen, 2011) with respect to arc length *in radii* rather than any other measurement units is emphasized to generate coherent understanding of trigonometric functions (Moore, 2010; 2012; 2014). Comprehension of simultaneous variation of two variable (Oehrtman, Carlson, & Thompson, 2008), angle measure as a varying measure (Moore, 2012), and thinking about *angle measure* in terms of rotations on the unit circle (Brown, 2005; Fi, 2003; Mitchelmore & White, 1996; Moore, 2010) are important ideas to construct and interpret graphs of trigonometric functions; as well as the unit circle as an important tool (Brown, 2005; Moore, LaForest, & Kim, 2012; Weber, 2005) and consideration of the meaning of coterminal angles on the unit circle (Fi, 2003) are mentioned as foundational to understand the periodic nature of the trigonometric functions.

To sum up, coherent understanding and flexible use of trigonometric functions in different representations (i.e., symbolic, circular, and graphic) are expected from trigonometry instructions. In comprehension of mathematics, Duval (2006) argued the importance of (*i*) *recognition tasks* that require recognition of the same object represented in two different representational registers “whose contents have very often nothing in common” (p. 112), and (*ii*) *discrimination tasks* that require discrimination “in any semiotic representation what is mathematically relevant and what is not mathematically relevant?” (p. 115). Therefore, well-designed recognition and discrimination tasks are critical to provide students with coherent understanding and flexible use trigonometric functions in different representational registers but also provide researchers (or teachers) with awareness of students’ understanding.

Recognition and discrimination of trigonometric functions require many dynamic cognitive processes, such as reasoning about simultaneous variation of two variable, i.e., angle measure and corresponding trigonometric value, (Oehrtman et al., 2008), considering of angle measure as a varying measure (Moore, 2012), and thinking about angles in terms of rotations (Brown, 2005; Fi, 2003; Mitchelmore & White, 1996; Moore, 2010). However, these processes are not easy in static paper-and-pencil environment. Therefore, the last and the most important aspect is the need to ease for students' *seeing* the different representations and to establish connections between them. In this respect, multiple-representation and dynamicity opportunities of new technologies enable us to achieve this much more easily, quickly and truly than the static paper-and-pencil environment.

For example, Geometer's Sketchpad (GSP) (Jackiw, 2006) can be used for teaching and learning of trigonometry because of the potential of providing dynamic diagrams that students can slightly distort to meet their expectations, which is impossible in paper-and-pencil diagrams (Laborde, Kynigos & Strasser, 2006). That is, making variation –which is needed to be understood "invariance" that is the very important aspect of mathematical thinking (Kaput, 1992)– opportunity of GSP can facilitate students' *recognition* and *discrimination* of trigonometric functions in different contents of different representational registers. Moreover, in GSP environment, it is possible to construct three different representations of trigonometric functions (i.e., symbolic, circular, and graphic), as well as to measure some important attributes related to trigonometry (e.g., arc length, arc angle, radius, directed length, etc.). As far as these potentialities of GSP are concerned, appropriate use of GSP may be useful to foster students' comprehension of trigonometry. However, its appropriate integration into trigonometry courses requires to carefully design teaching/learning activities and the instructional sequence.

Accordingly, the overarching purpose of this study is to design an instruction[9] including a sequential tasks in dynamic geometry environment with GSP in order to help students enrich their concept images on the core trigonometric functions'[10] basic

[9] Designing of the instruction started through inspiring from research literature on trigonometry, historical development of trigonometry, our exploratory teaching experience, and initial interview results; and then, continued as an on-going process during the experimentation to influence students' trigonometry.

[10] The term "core trigonometric functions" refers to sine and cosine functions.

forms[11] and general forms[12]; and then, is to construct *living* models of students' concept images through gaining experiences on *students' trigonometry* during the conducted-*teaching experiment* based on the *recognition* and *discrimination* tasks of this designed-instruction.

[11] The term "basic form" refers to, for example, $y = sin(x)$ and $y = \cos(x)$.

[12] The term "general form" refers to, for example, $y = a\sin(bx + c) + d$ and $y = a\cos(bx + c) + d$.

CHAPTER 3

METHODOLOGY

In order to investigate the research questions of this study, a teaching experiment (Steffe and Thompson, 2000) was conducted. The teaching experiment is a technique designed to understand students' conceptions in the context of mathematics teaching (Cobb & Steffe, 1983, Hunting, 1983; Steffe, 1991). Steffe and Thompson (2000) assert that "without the experiences afforded by teaching, there would be no basis for coming to understand the powerful mathematical concepts and operations students construct…" (p. 267).

3.1. Teaching Experiment

In order to explain students' mathematical learning and development in the context of teaching, experimental designs can be used. Nevertheless, classical experimental methodologies are not adequate for addressing how students learn specific mathematical concepts (Steffe & Thompson, 2000). In this respect, the *teaching experiment*, which was derived from Piaget's *clinical interview* (which is used to understand students' current knowledge), is a useful technique. But a teaching experiment is more than a clinical interview in terms of the experimentation that is included in a teaching experiment to influence students' mathematical knowledge.

For researchers, knowing how to interact with the students and what to be the outcomes of this interaction is an essential point to be able to conduct a teaching experiment. Steffe and Thompson (2000) stated that "in their attempts to learn students' mathematics, the researchers create situations [not considered in the initial design of a teaching experiment] and ways of interacting with students [both in a responsive-and-intuitive way, as well as in an analytical way] that encourage the students to modify their current thinking" (p. 285), and "meanings lie behind students' language and actions" (p. 276). They recommend for the researchers who plan to conduct a teaching experiment to experience the interaction in an exploratory teaching with similar students who would be involved in the teaching experiment before attempting to conduct a teaching experiment. Interaction is a central issue of a teaching experiment. Steffe and Thompson (2000) consider two different types of interaction with students in a teaching experiment, namely, *responsive-and-intuitive,* and *analytical*. The aim of responsive-and-intuitive interaction is for researchers to explore students' reasoning. On the other hand, when researchers have some hypotheses about students' reasoning, then, in order to test these hypotheses, they turn to analytical interactions from responsive-and-intuitive ones.

A sequence of teaching episodes is included in a teaching experiment. Each teaching episode includes (at least) a teacher-researcher (myself), one or more students (in this study, two pairs of students participated in the separate teaching experiments), a witness or an observer in order to help the teacher-researcher both understand and interpret students' actions (a trigonometry teacher working at a cram school) and a recording device (to "provide insight into the students' actions and interactions that were not available to the teacher-researcher when the interactions took place") (Steffe & Thompson, 2000, p.293). During the each teaching episode of the teaching experiment, students' independent contributions to the interactions (Steffe & Thompson, 2000) and their reasoning (Ackermann, 1995) are the focus.

It is another important point that a teaching experiment is conducted not only to test hypotheses but also continually generate them (Steffe & Thompson, 2000). Primarily, before a teaching experiment, the teacher-researcher formulates major research hypotheses to test (*in this study, an instruction including sequential tasks in dynamic geometry environment with Geometer's Sketchpad (GSP) –designed through inspiring the research literature on trigonometry, the historical development of trigonometry, our exploratory teaching experience and initial interview results– helps* *students enrich their concept images on trigonometric functions*[13]) so as to guide the initial selection of the students and the teacher-researcher's overall general intentions. However, it is recommended for the teacher-researcher to forget these major hypotheses during the each episode of the teaching experiment for the purpose of adapting to constraints[14] encountered in the interaction with the students. In this way, the teacher-researcher generates new hypotheses which allow make possible the formulation of new situations of learning. And then, each student is tested on the continually generated hypotheses through the teaching experiment by focusing on the student's independent contributions to the interactions and their reasoning during teaching episodes with the goal of the possible greatest progress.

In the light of the recommendations mentioned above, as a seven-year experienced mathematics teacher and as a researcher with the experience of the interaction with a student who had just taken trigonometry (who was in the similar condition with those in the teaching experiment[15]) in an exploratory teaching of the sine function with *Geometer's Sketchpad* (GSP) dynamic geometry environment (Şahin, Ubuz & Erbaş, 2010), I decided to conduct a teaching experiment in GSP environment to investigate how students' concept images on trigonometric functions change by the help of the designed-instruction.

3.2. Geometer's Sketchpad

Geometer's Sketchpad (GSP) (Jackiw, 2006) is a dynamic geometry environment that has the potential for students to construct simultaneously the geometric (on the circle), algebraic and graphical representations of trigonometric functions (see *Table 3.1*) with dynamic linkage. In addition to its ease to learn use,

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[13] From this point forward, trigonometric functions refer to the core trigonometric functions sine and cosine.

[14] Steffe and Thompson (2000) use the "constraint" term with a dual meaning in researchers' imputations to students' mathematical understandings: (1) constrained by the students' language and actions, (2) constrained by the students' mistakes that persist despite researchers' best efforts to eliminate them.

[15] From this point forward, the teaching experiment refers to the conducted teaching experiment in this study.

GSP was used in the teaching experiment because of its possible actions listed in *Table 3.2*.

Table 3.1. An example from different representations of the "sine" function

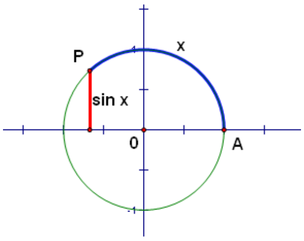
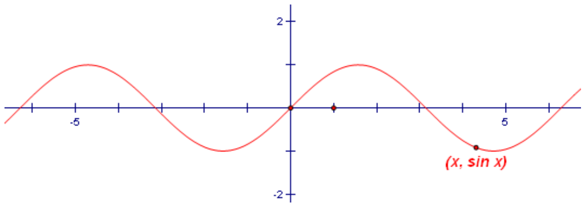
Geometric Representation	Algebraic Representation	Graphical Representation
	$f(x) = \sin(x)$	

Table 3.2. Possible actions with GSP for teaching trigonometry

Preferences	To change angle measure units from radian to degree.
Action Buttons	To animate selected objects in an intended way and speed.
Trace	To obtain simultaneous dynamic graphical representations of trigonometric functions from their geometric representations on the circle.
Construct	To construct arc on the circle, point of graph, perpendicular line, segment, etc.
Rotate	To construct the geometric representations of sin (bx+c) and cos(bx+c) (where b, c in R) for obtaining their graphical ones.
Reflect	To construct reference right triangles in all quadrants for determining absolute values of trigonometric functions through associating right triangle trigonometry.
Measure	To measure abscissa and ordinate of the point on the circle, arc length, arc angle, angle, etc.
Calculate	To calculate, for example, radian measure by dividing arc length to radius; and sine and cosine values of an angle so as to compare their predictions.
Graph	To plot points, such as (π,0), plot selected points as (x,y) and plot trigonometric functions' graph from their symbolic representations.
Drag/Drop	To see the varying and unvarying components for students in their own control through dragging and dropping selected objects in an intended way and speed.
Display	To change color and line width of selected objects to form similar or related constructed structures so as to become more meaningful at first glance.

3.3. Participants and Selection

The teaching experiment was conducted separately with two pairs of 11th grade students in the fall semester of 2012 who had just taken trigonometry in the spring semester at 10th grade. They were selected through the following three steps. Firstly, from two Anatolian High Schools[16] in Ankara in Turkey, with respect to the recommendations of the school boards and mathematics teachers, two successful 10th grades in the spring semester taking mainly mathematics and science courses were determined to observe during trigonometry teaching. After then, according to the classroom observations (from May 2, 2012 to May 31, 2012) and classroom trigonometry teachers' recommendations, ten 10th grade trigonometry students were selected based on the ability to share and communicate their ideas, work well together as pairs, having high performance in trigonometry[17] and their willingness to participate the study. Six of these students were from one school and four of them from the other. Finally, among these ten students, four students were selected as the participants of the teaching experiment by the purposive sampling through one-to-one clinical interviews on eight open-ended questions (see Appendix A) prepared to understand students' current knowledge, reasoning abilities, as well as essential mistakes on the concepts related to trigonometry (such as function, unit circle, right triangle, sine and cosine, coordinate plane, angle measure units, degree, radian, π, graphing, and periodicity).

They were the students, from the same school, who were at best (but not adequate) reasoning related to trigonometry among all ten recommended "good" trigonometry students from two schools. But they had similar essential mistakes with those of other six students. The purpose of selecting these four students was to conduct the teaching experiment in an aimed way without wasting much more time than a few weeks for their preparation with enough prior principal knowledge to learn trigonometric functions instead of dealing for several weeks with completing deficient

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[16] The schools are four-year public high schools with 9 to 12 graders who are admitted according to the basis of their performances on the national exam with average top 2.3% and 8.4 % ranking.

[17] It means students' high performances on trigonometry exams including questions mainly based on computational skills they took at their schools.

and missing prior knowledge. Three of them were female (Cemre, Defne and Ebru) and one was male (Zafer) –all names of the students changed with pseudonyms. These four selected students were grouped as two pairs (Cemre with Zafer and Defne with Ebru) taking account of their explanation styles (or thinking aloud styles) so as to create a more effective discussion environment by means of assigning each group to both one uncommunicative student (Ebru and Zafer) who has been making no effort to explain asked questions if they have no answer and one communicative student (Defne and Cemre) who has been making an effort to explain asked questions even if they have no answer.

3.4. Initial Interviews

In order not only to select the participants of the teaching experiment but also to identify their current *concept images* on trigonometric functions and (if exists) *potential conflict factors,* an approximately 75-minute task-based individual interview was conducted with each of ten 10th grade trigonometry students. Because they were not adults, parental permissions (Appendix B) were taken for all about their participation of the study. These students were taken from the class to the library of their school on a day at the end of the semester (approximately one week after their last trigonometry exams) from June 1, 2012 to June 8, 2012 (see *Table 3.3*). At the end of the semester, students had just completed the trigonometry course that aims students' understanding and reasoning on the trigonometry topics listed in *Table 3.4*.

Interview tasks were composed of eight open-ended questions (see Appendix A) designed as an eight-page question booklet (each page included only one question and enough blank to use as a worksheet). Beginning of the interview, the question booklet and a pencil were given to the students. However, the interview was beyond the answering only these questions. It allowed the researcher to scrutinize students' concept images on their current knowledge and reasoning abilities related to trigonometry mentioned in *Table 3.5* through catching the occasions of provoking students to critically think aloud.

Table 3.3. Interview dates for ten students from two school

Students	from the school*…	on the day
S1(*Zafer*)	A	
S2	B	1 June 2012
S3	B	
S4 (*Cemre*)	A	
S5 (*Defne*)	A	4 June 2012
S6 (*Ebru*)	A	
S7	B	
S8	B	5 June 2012
S9	B	
S10	B	8 June 2012

*Students from the school A took the last mathematics exams including trigonometry on 1 June 2012, others did on 25 May 2012.

Each interview's video records were gotten with two digital cameras one of whom captured the interactions between the interviewee-student and the teacher-researcher (myself), and the other focused on the question booklet. Analysis results of the initial interviews were presented in Chapter 4 through exemplifying from reasoning of the students in order to establish implications for the instruction of the teaching experiment as well as to illustrate their current knowledge and reasoning abilities related to core trigonometric functions taking account of their different representations. Although initial individual interviews were conducted with a total of ten 10th graders from two schools (who had just taken trigonometry), analysis results were presented from the four students[18] who were the participants of the study (Cemre, Defne, Ebru and Zafer –all names are pseudonyms).They were at best (but not adequate) reasoning related to trigonometry among all ten students. Nevertheless, they had similar *troubles* on reasoning about trigonometric ideas with those of other six students. These troubles arose mainly from transformations of semiotic representations, or cognitive distances between the (*unit*) *circle register,* the *graphical register* and the *symbolic register*.

[18] From this point forward, students refer to the participants of this study.

Table 3.4. 10th grade trigonometry topics participants of this study had just completed before the initial interviews

Topics*	Sub-learning topics
Right triangle trigonometry	• definition of trigonometric ratios • 30, 45 and 60 degree angles' trigonometric ratios by using the right triangles • relation of trigonometric ratios between complementary angles • finding a trigonometric ratio when one another trigonometric ratios is known
Directed angles	• unit circle • directed angle • directed arc • transformations between angle measure units • principal measure
Trigonometric functions	• domain and range of trigonometric functions defined on the unit circle • trigonometric identities • determination of trigonometric values by using trigonometric value table • transformation of a trigonometric value of ($k\pi/2 \pm \theta$) into a trigonometric value of θ (k in Z)
Graphs of trigonometric functions	• period and periodic function • finding periods of the symbolically given trigonometric functions • graphing basic trigonometric functions through plotting some critical ordered pairs on it)
Inverse trigonometric functions	• inverse trigonometric function
Trigonometric relation in the right triangle context	• sine and cosine theorems • triangle area formulas based on sine and cosine theorems
Addition and subtraction formulas	• determining trigonometric ratios of addition and subtraction of two numbers • finding half angle formulas • finding trigonometric transformations formulas
Trigonometric equations	• solving trigonometric equations

*These topics were the sub-domains of 10th grade trigonometry curriculum in Turkey in 2012 (Talim ve Terbiye Kurulu Başkanlığı [TTKB], 2005, 2011). It was subsequently changed in 2013 (TTKB, 2013).

Table 3.5. Table of specification of initial interview tasks

	Mathematical Content	*Purpose is to understand…*	*Source*
Q1	functions	students' (personal) concept definition.	
Q2	functions and relations	students' concept image on functions versus the mathematical definition.	Adapted from Lisa L. Clement, Alba Thompson, and Patrick Thompson (as cited in Clement (2001))
Q3	functions and relations	students' concept images on functions versus the mathematical definition.	
Q4	unit circle	students' (personal) concept definition.	
Q5	sine and cosine	students' (personal) concept definition (in each representational register).	
Q6	coordinate plane, ordered pair, cosine function, π	students' concept images on angle, angle measure units (degree and radian), trigonometric functions, π, domain and range of trigonometric functions.	Adapted from Topçu, Kertil, Akkoç, Yılmaz, and Önder (2006)
Q7	sine function, trigonometric equations	students' concept images on trigonometric equations and their solutions, value of trigonometric functions, domain and range of trigonometric functions, and positive/negative angle measure.	
Q8	graphs, periodicity	students' concept images on graphs of trigonometric functions, periodicity (in the *graphical register*).	Adapted from Thompson (2007)

3.5. Instructional Design of This Study

Our principal of the instruction in the teaching experiment is that making *variation* is needed for understanding of *invariance* which is the very important aspect of mathematical thinking (Kaput, 1992). Therefore, we used GSP throughout the instruction in order to create situations for students to provoke their thinking via dynamic manipulations (i.e., dragging and dropping) which help them to see variation and to determine invariances.

In the first seven tasks of the instruction (see *Table 3.6*), in order for students to become familiar with GSP usage, they studied on pre-constructed GSP files. They were constructed to facilitate students' conceptions of prior knowledge related to trigonometry and fundamental properties of trigonometric functions by the teacher-researcher (myself) and reviewed by at least two mathematics educators with extensive experience on technology in mathematics education, especially on GSP. Examples from their static versions were presented in *Figure 5.1*, *Figure 5.2* and *Figure 5.4* for Task 1, *Figure 5.6* and *Figure 5.10* for Task 2, *Figure 5.13*, *Figure 5.15* and *Figure 5.16* for Task 3, *Figure 5.17* for Task 4, *Figure 5.19* for Task 5, and *Figure 5.24* for Task 6 and Task 7]. On the other hand, in the following nine tasks of the instruction (see *Table 3.7*), each episode began with students' re-construction of a circle on the coordinate plane with GSP (whose radius and center are manipulable) and continued with the manipulation of this circle for reasoning with respect to the teacher-researcher's directions. Totally seventeen tasks (see *Table 3.6*, *Table 3.7* and Appendix C) were prepared for the instruction (so as to be one task per episode). Instructional sequence was designed taking into account some implications from the research literature on trigonometry, historical development of trigonometry, our exploratory teaching experience and initial interview results.

Firstly, before dealing with trigonometric functions, we need certain prior knowledge such as angular measurement systems more based on measuring in radii (Moore, 2012), perpendicular coordinate system, and directed arcs and angles (with the direction *counterclockwise* or *clockwise* like in the history). In addition, before dealing with trigonometric functions, we need certain prior experiences the notion of *right triangles* [in terms of the properties of similar right triangles' ratios of the lengths of corresponding sides (Thompson, 2008), as well as integration of the right triangles onto the (unit) circle], *circles* (in terms of radii, arc angles, arc lengths (Moore, 2010; 2012) and chord lengths (Hertel & Cullen, 2011)), and *functions* [in terms of their meanings (Hertel & Cullen, 2011), different representations including graphs, rules and relations between inputs and outputs etc. (Weber, 2005)]. Gür (2009) emphasized that students' problems with prior knowledge related to trigonometry are obstacles in trigonometry learning. Therefore, students' preparation with enough principal prior knowledge is one of the most important aspects in understanding of trigonometric functions. However, it is not enough because how the knowledge was stored (see *Figure 2.2*) is another absolutely important aspect in the mathematical understanding

(Sfard, 1991). In this respect, it may be a useful way in the mathematical understanding to continue to add new concepts to the old ones (which have been truly and easily understood by the students) through associating new ones with the old ones and motivating them to learn new ones.

Considering not only the importance of the unit circle for understanding the periodic nature of trigonometric functions (Brown, 2005; Fi, 2003; Thompson, 2007; Thompson, 2008; Weber, 2005) but also students' difficulties with the unit circle in terms of understanding, using and interpreting trigonometric functions (Burch, 1981; Emlek, 2007; Güntekin, 2010; Gür, 2009; Weber, 2005) and students' better performances in the ratio method in the right triangle context compared to the unit circle context (Kendal & Stacey, 1996), the instructional sequence was designed from the right triangle context to the unit circle context. Our starting point was the *Right Triangle Trigonometry* [Task 1]. Next, after the need "angles" in the right triangle context, *Angle Measure Units* [Task 2] were mentioned as mainly based on measuring in radii (Moore, 2012) and arc lengths. And then, angles in the unit circle context were the focus in terms of *Principal Angle and Reference Angle* [Task 3] through highlighting π as a real number when being discussed in the trigonometry context (Akkoç, 2008; Fi, 2003; Topçu, Kertil, Akkoç, Yılmaz &Önder, 2006). Then, after discussions on functions in terms of their meanings, different representations including graphs, rules and relations between inputs and outputs etc., trigonometric functions were defined as functions on the *Unit Circle* (*sine and cosine*) [Task 4] which was integrated with the dynamic reference right triangle with 1-unit hypothesis (see *Figure 5.17*). Afterwards, in order to understand students' difficulties about trigonometric functions' definitions, *Periodic Nature of sine and cosine* [Task 5] were preferred to handle in the following episode. Then, regarding the crucial role of multiple representations of a concept, especially visual ones, such as graphs (Goldenberg, 1988), and transformations among them on enriching students' concept images (Duval, 1999), *Relations among Different Representations* (*of sine*) [Task 6] and *Relations among Different Representations* (*of cosine*) [Task 7] were mentioned in the separate episodes according to our exploratory teaching experience (Şahin, Ubuz & Erbaş, 2010) which indicated that the knowledge on sine and cosine functions should be associated just after being stored in separate *schemas* in their own right including all cognitive processes (i.e., all mental pictures and associated properties, and processes regarding sine/cosine function) in all different representations. Moreover, sine function was preferred to be first introduced with cosine function in the light of the historical development of the trigonometric functions in which the basic element was the trigonometric "chord" function (Katz, 2009) that was turned in the fifteenth century into sine of an arc which was defined as the half chord of double the arc (Sanford, 1958). Afterwards, respectively, *Role of the Coefficient a for* $y=a.\sin(x)$ *Function in the Different Representations* [Task 8], *Role of the Coefficient d for* $y=\sin(x)+d$ *Function in the Different Representations* [Task 9], *Role of the coefficient c for* $y=\sin(x+c)$ *Function in the Different Representations* [Task 10], *Role of the Coefficient b for* $y=\sin(bx)$ *Function in the Different Representations* [Task 11] were discussed where the tasks' sequencing was determined from easy to difficult according to our exploratory teaching experience. After the all tasks on sine function were completed in an intended way, *Relation between sine and cosine Functions* [Task 12] was discussed considering all different representational registers. We preferred to discuss this relation in Task 12 instead of doing after Task 7 because constructed rich (or well-structured) concept images on sine function have the potential to ease to construct well-structures concept images on cosine function through establishing the relations and similarities of cognitive processes of sine with those of cosine. From this point of view, after then, *Role of the Coefficient a for* $y=a.\cos(x)$ *Function in the Different Representations* [Task 13], *Role of the Coefficient d for* $y=\cos(x)+d$ *Function in the Different Representations* [Task 14], *Role of the Coefficient c for* $y=\cos(x+c)$ *Function in the Different Representations* [Task 15], *Role of the Coefficient b for* $y=\cos(bx)$ *Function in the Different Representations* [Task 16] were discussed through establishing the relations and similarities of cognitive processes of sine with those of cosine. Finally, a modeling task with Ferris wheel (Appendix C) was preferred to apply in order to understand students' way of thinking and conceptualization steps (Lesh & English, 2005; Lesh & Sriraman, 2005) on sine and cosine functions. All these tasks were clarified in depth with regard to the cognitive base of the instruction under the following heading.

Table 3.6. Overview of the tasks in the first part of the instruction in the teaching experiment

One task per episode	Theme of Episodes	Objectives (*After the episode, it is expected that students should be able to …*)
Task 1	*Right Triangle Trigonometry*	→ interpret “*invariance*” components when making *variation* on a right triangle’s acute angles and lengths of sides → determine proportional ratios of similar right triangles → establish relation between opposite [adjacent] side’s length of a right triangle with 1 unit hypotenuse and sine [cosine] of an angle → realize the relationship of trigonometric ratios for complementary angles
Task 2	*Angle Measure Units*	→ discriminate the difference between what an angle is and what an angle measure is → investigate angles greater than 90 degrees or π/2 radians → identify directed angles and arcs → calculate the arc length, angle measure and radius → determine relationship among arc length, radius and angle measure → establish relations between angle-measure units (degree and radian)
Task 3	*Principal Angle and Reference Angle*	→ model angles greater than 360 degrees or 2π radians → decide principal angle of an angle → decide reference angle of an angle → determine the position of an arbitrary angle (i.e., either greater than 360 degrees or 2π radian, or principal angle and reference angle) on the unit circle → determine the position of a negative angle on the unit circle
Task 4	*Unit Circle (sine and cosine)*	→ identify the sine and cosine functions by the unit circle integrated with the reference right triangle with 1-unit hypothesis → predict the value of the sine and cosine functions by using the coordinate axes as a ruler → transfer sine and cosine ratios of an arbitrary acute angle, θ, in a right triangle to the value of the sine and cosine functions of an angle whose reference angles are the same, θ, or that are “$\theta \pm k\pi$” (where k is an integer)
Task 5	*Periodic nature of the sine and cosine*	→ interpret “*invariance*” components on the sine and cosine functions when making *variation* on the reference right triangle integrated with the unit circle both within a quadrant and between quadrants → interpret the periodic nature of the sine and cosine functions → determine the period of the sine and cosine functions
Task 6	*Relations among different representations (sine)*	→ construct graphs of sine function through associating its definition on the unit circle as an ordered pair and via taking “*trace*” advantage of GSP → differentiate the abscissa of an ordered pair between on the sine graph and on the unit circle → convert coordinates of an ordered pair on sine graph to the appropriate components on the unit circle
Task 7	*Relations among different representations (cosine)*	→ construct graphs of cosine function through associating its definition on the unit circle as an ordered pair and via taking “*trace*” advantage of GSP → differentiate the ordinate of an ordered pair between on the cosine graph and on the unit circle → convert coordinates of an ordered pair on cosine graph to the appropriate components on the unit circle

Table 3.7. Overview of the tasks in the second part of the instruction in the teaching experiment

One task per episode	Theme of Episodes	Objectives (*After the episode, it is expected that students should be able to …*)
Task 8	*Role of the coefficient a for* y=a.sin(x) *function in the different representations*	→ interpret "*in/variance*" components of y=a.sin(x) function in the different representational registers when making *variation* on the radius of a circle whose center is located on the Origin
Task 9	*Role of the coefficient d for* y=sin(x)+d *function in the different representations*	→ interpret "*in/variance*" components of y=sin(x)+d function in the different representational registers when making *variation* of the position of the unit circle's center
Task 10	*Role of the coefficient c for* y=sin(x+c) *function in the different representations*	→ interpret "*in/variance*" components of y=sin(x+c) function in the different representational registers when making *variation* of the position of the point on the unit circle with the fixed-angle
Task 11	*Role of the coefficient b for* y=sin(bx) *function in the different representations*	→ interpret "*in/variance*" components of y=sin(bx) function in the different representational registers when making *variation* of the position of the point on the unit circle with the fixed-angle
Task 12	*Relation between sine and cosine functions*	→ establish the relation between the sine and cosine functions in the different representations as the similar mathematical objects (like a condensed whole) without going into details → recognize familiar processes of the sine and cosine functions in different representational registers → transfer the sine function to the cosine function in the different representational registers, or vice
Task 13	*Role of the coefficient a for* y=a.cos(x) *function in the different representations*	→ interpret "*in/variance*" components of y=a.cos(x) function in the different representational registers when making *variation* on the radius of a circle whose center is located on the Origin
Task 14	*Role of the coefficient d for* y=cos(x)+d *function in the different representations*	→ interpret "*in/variance*" components of y=cos(x)+d function in the different representational registers when making *variation* of the position of the unit circle's center
Task 15	*Role of the coefficient c for* y=cos(x+c) *function in the different representations*	→ interpret "*in/variance*" components of y=cos(x+c) function in the different representational registers when making *variation* of the position of the point on the unit circle with the fixed-angle
Task 16	*Role of the coefficient b for* y=cos(bx) *function in the different representations*	→ interpret "*in/variance*" components of y=cos(bx) function in the different representational registers when making *variation* of the position of the point on the unit circle with the fixed-angle

3.6. Cognitive Base of Designed-Instruction of This Study

As it is mentioned above, not only students' preparation with enough knowledge on trigonometric functions but also how the knowledge was stored are absolutely important aspects in the mathematical understanding (Sfard, 1991). Sfard (1991) classifies mathematical understanding as *operational conception* (dynamic, sequential and detailed) versus *structural conceptions* (static, instantaneous and integrative). However, she emphasized this distinction's dual meaning rather than dichotomy. She proposed that when learning a new mathematical concept, the first developed conception type is operational. And then, the mathematical concept's development comes true through converting the *operational conception* (various processes) into the *structural conception* (compact static whole) after a lengthy and difficult process. Through inspiring mathematical concepts' historical development processes, Sfard (1991) separates this lengthy and difficult process into three hierarchical stages: *interiorization*, *condensation* and *reification*. First of all, the *interiorization* stage means for students becoming familiar with the processes on the mathematical object. Next, the *condensation* stage means for students becoming skilled with seeing of the process as a condensed whole without going into details. At this stage, the processes are easily combined with other ones, as well as generalizations and comparisons are smooth. Finally, the *reification* stage is defined as an instantaneous shift the ability to see familiar processes as a reified-object. By this object (or mathematical construct), different representations of the mathematical concept are semantically merged, which means any more fundamental properties of this reified-object in its different representations and relations among them can be investigated easily. Sfard (1991) indicates the effect of reification on the *schema* of a student as a reorganization into a deeper and narrower structure so that "cognitive processes become faster" (p. 27).

A *schema* is an individual's mental construction connecting related processes and objects, and appears to be somewhat similar to one's *concept image*" (Harel, Selden & Selden, 2006, p. 157). From this point of view, in the first part of the instruction in the teaching experiment, it was intended to help students construct well-structured *cognitive schemas* (or *concept images*) on both the necessary background information (such as similar right triangles, trigonometric ratios, angle measure units, directed angle, principal angle, reference angle, unit circle, π, coordinate plane) and the principal information about the basic forms[19] of trigonometric functions (such as sine and cosine as functions, periodic nature of sine and cosine, relations among different representations of the sine function, relations among different representations of the cosine function, relations between sine and cosine functions) through reorganizing their current *cognitive schemas* (or *concept images*). This part of the instruction consisted of the seven tasks clarified in *Table 3.6*. There were two main aims of these tasks. First of all was to prepare students so as to be in at least the *condensation* stage in understanding of angle measure units, right triangles, the unit circle, directed angles, principal angles, reference angles, functions and inferring from visualization of graphs. It means for students becoming skilled with seeing of the process as a condensed whole without going into details; for example, *seeing* angles as a *variable* apart from their measure, *seeing* all similar right triangles as "similar" in terms of their trigonometric ratios regardless of their lengths of sides, *seeing* the lengths of the legs of a right triangle with 1-unit hypothesis the same as the respective trigonometric value (sine or cosine), *seeing* the *reference right triangle*[20] on the unit circle in any quadrant, etc. For this purpose, students were provoked for *operational thinking* through focusing on measures. For example, when dynamically manipulating the figures, students were encouraged to determine the variance and invariance measures of an arbitrary right triangle's acute angles or sides, the variance and invariance measures of the similar right triangles' corresponding angles, sides, proportional sides, similarity ratios and trigonometric ratios, the variance and invariance measures of trigonometric ratios of complementary angles, the variance and invariance measures of an arbitrary angle through changing the preference of the angle measure unit from degrees to radians as well as directed degrees, the variance and

[19] See *Definitions of Terms* heading at the end of *Introduction* chapter.

[20] In this study, we mention the term *reference right triangle* as a right triangle in the (*unit*) *circle register* so that its vertexes are the points *P* on the circle, its center and the intersection point of the horizontal axis and the perpendicular line drawn from the point *P* to the horizontal axis (see *Figure 5.17, Figure 6.1,* and *Figure 6.22*).

invariance measures related to the radius and arc length of a circle whose center is located on the vertex of an angle, as well as the variance and invariance measures of an angle on the unit circle's any quadrants and its principal, reference, coterminal and directed values. The second aim of the first part was for students to become familiar with the processes on the trigonometric functions' basic forms. In this part of the instruction, mainly students' *operational conception*s of trigonometric functions were regarded instead of *structural conceptions* because this part corresponded to the *interioization* stage in understanding of general forms[21] of trigonometric functions. *Operational conception* of trigonometric functions refers to determination of the value of an angle measure (for an arbitrary angle measure unit preference), its corresponding trigonometric value, signs of these values with respect to the quadrants of the unit circle through focusing on the value of the coordinates of the point on the unit circle; moreover, those of the corresponding point on the graph. However, this part corresponded to at least the *condensation* stage in understanding of the basic forms of trigonometric functions. It means for students becoming skilled with seeing of the processes on and transformations within and between different representational registers for the basic form of trigonometric functions (see *Table 3.1*).

Duval (2006) mentioned representations as *semiotic representational systems* within which a sign takes its meaning in opposition to other signs and their complex associations. Semiotic representations can be considered as common tools for not only the development of a new mathematical concept but also the communication of particular mental representations (or *concept images*). Therefore, natural language is a highlighted semiotic representational system. Duval classified semiotic representations into four semiotic systems (see *Figure 2.3* in *Review of the Literature* chapter), which are called the *representation register*s, with respect to the nature of operations (*discursive* versus *non-discursive*) and the properties of processes (*multi-functional* versus *mono-functional*) that are used to describe a system. Moreover, he defines the semiotic representations as the *register*s only which permit transformations of representations due to the fact that for understanding the thinking process in any

[21] See *Definitions of Terms* heading at the end of *Introduction* chapter.

mathematical activity, it is important to focus on the *level* of semiotic representations instead of focusing on the particular representation produced. Where, the *level* of semiotic representations means the ability to transform a mathematical object from one semiotic representation to one another. Transforming one semiotic representation to another one is to be only at the level of grasping the basic properties of semiotic representations and their significance for mathematics. This means for the students to be able to distinguish the represented mathematical object from its representation registers. From this point of view, transformations are crucial in the mathematical understanding. Duval (2006) separates transformations of semiotic representations in two types: *treatment* and *conversion*. While *treatments* refer to the transformations of representations within the same register, *conversions* refer to the transformation of representations between the different registers.

As it is mentioned above, the mathematical understanding requires recognition of the same represented object in the different registers. This is a complex and difficult process. Duval (2006) articulates two sources of problems in this complex and difficult process. First of all is the complexity and specificity of treatments in the multifunctional registers. For example, when dealing with visualization, there are many ways of "seeing" (Duval, 1995) so that it is not easy in visual transformations of figures to see and discern from the original figure to the reconfigured-one which will make possible to establish the relation. Second source of the problems in the mathematical understanding is the conversion of representations (or change of registers), for instance, converting a Cartesian graph into its corresponding equation. Duval (2006) asserts that conversion of representations requires *recognition* of the same object represented in two different representational registers and *discrimination* of the represented object from the content of the semiotic representation. *Recognition* and *discrimination* are cognitively complex but the most important two cognitive skills in comprehension of mathematical concepts in a mathematical activity in order to use knowledge outside of the narrow learning contexts (Duval, 2006).

From this point of view, in the second and main part of the designed-instruction in the teaching experiment, it was intended to help students enrich their concept images on the general forms of the trigonometric functions providing them to construct the *reified-cognitive networks* for the different representational registers of the trigonometric functions without ignoring the role of language (or articulation –either orally or written). We determined four registers to represent trigonometric functions, namely, (*unit*) *circle register*[22], *graphical register*, *symbolic register* and *language register*. Our design was based on helping students enrich their concept images through giving them opportunities in GSP environment to dynamically (or in a systematically-varied way) compare and contrast visual features of the different representations of trigonometric functions via focusing on what is mathematically relevant or what is mathematically different in any representational register's content. This part of the instruction consisted of the nine tasks clarified in *Table 3.7*.

The overarching aim of these tasks was to prepare students so as to be in at least the condensation stage in understanding of general forms of trigonometric functions. It means for students at least becoming skilled with distinguishing the represented trigonometric functions from the contents of the representational registers, or at best seeing trigonometric functions as a reified-object so that all representational registers are semantically merged. In other words, any more fundamental properties of this reified-object in their different representations and relations among them can be investigated easily. In this part of the instruction, students' *structural conception*s of trigonometric functions (as a reified-object) were intended through *operational* way of thinking (that is, through dynamic manipulations, i.e., dragging and dropping, to compare and contrast visual features of the different representations of trigonometric functions in a systematically-varied way as well as under the cover of possible actions with GSP mentioned in *Table 3.2*).

In the light of the historical development of trigonometric functions, at the beginning of the each episode of the teaching experiment as from the sixth episode to the sixteenth, trigonometric functions were <u>first</u> introduced and used in the (*unit*) *circle register*. And then, it was continued the *graphical register* and ended the *symbolic*

[22] In this study, it was preferred to use the *unit* term in parenthesis as a consequence of the changeable-meaning of the *unit* with respect to the assumptions of the *unit* measure as well as the importance of conception of trigonometric functions on any circular representations either *unit-circle* or *non-unit circle*.

register without ignoring the *language register* in all other registers. And finally, the order of the registers was inverted. In other words, each episode in itself included *conversion* tasks that were simultaneous transformations among different registers through comparing and contrasting visual features (for example, the circle's radius and position of its center, magnitudes and periods of graphs, coordinates of the points on both the circle and graphs, arcs) in different representational registers of trigonometric functions in a systematically-varied way and changing the roles of the source register and the target register. Where, our aim was by the aid of GSP to provide students with the cognitive DISSOCIATION of trigonometric functions and the content of the particular semiotic representation. Furthermore, from an episode to the following one, *treatment* tasks in all registers were done through dynamic manipulations on the radius and center of an animated-circle constructed in GSP which was located on the coordinate plane. Our way in discussions during these treatments was to compare and contrast visual features of general trigonometric functions in any registers with those of basic forms of trigonometric functions that were their first represented forms. Where, our aim was to overcome by the aid of GSP the cognitive IMPOSSIBILITY OF DISSOCIATING (Duval, 2006) the content of any semiotic representation and a mathematical object's first represented form in respective register through construction of *reified-cognitive networks* within the (*unit*) *circle register* and the *graphical register*.

Finally, the last part of the instruction was designed to see students' conversion troubles. Duval (2006) assert that conversion troubles (or cognitive distances between registers) are observed only when tasks in which a representation within a source register is systematically varied to its converted representation in the target register are given to the students. At this point, modeling activities that include real-life situations can be used as a valuable guide in order to see students' way of thinking and possible conceptualization steps. From this point of view, a modeling task with Ferris wheel (see Appendix C) which was developed in the scope of the research project (with Grant no: 110K250) supported by The Scientific and Technological Research Council of Turkey (TUBITAK) was used to see students' conversion troubles. This task required to model functions which can be used to calculate, throughout the turning, some instantaneous numerical data about a car on the Ferris wheel (which 36 cars would be placed on at equal intervals, established on a platform 4 meters in height, in a 140-meter diameter and complete a full-round in 30 minutes), namely, its ground clearance, the distance to the point gotten in the car, how much time remains to complete a full round. There were two main aims of us in this modeling task. First of all was to see students' abilities to transform their knowledge on trigonometric functions in the (*unit*) *circle register* (located on coordinate plane) to the any *circle* context because of the importance of making sense of trigonometric functions' geometric representations on a circle apart from the coordinate plane to fortify *reified-structural understanding* of trigonometric functions like an object independent from its representational registers. Second aim of the modeling task was to see students' conversion troubles when not only converting a representation within the (*unit*) *circle register* to its converted representation in the *symbolic register* but also transforming the relations which is originated by the basic forms of the sine or cosine functions in the (*unit*) *circle register*, to those in the *symbolic register*.

To sum up, the intent of the instructional sequence was to help students construct *reified-structural understanding* of trigonometric functions as an object independent from its representational registers through enriching their *concept images* on all representational registers of trigonometric functions including transformations both between and within them. Detailed descriptions of the procedures for each task are presented under the following heading.

3.7. Experimentation Procedures

3.7.1. Familiarity with GSP

Because the first episode of the teaching experiment was the first encounter with GSP and its language (English) was different from the native language of the students (Turkish), the each menu of GSP and their options were introduced to the students before the implementation of Task 1. It took approximately 40 minutes. We aimed to provide for students with familiarity on GSP usage through encouraging them to use the some options of GSP, especially, under the *Edit*, *Display*, *Construct*, *Transform*, *Measure* and *Graph* menus.

Firstly, after a blank page in GSP was opened, it was asked from each student to construct geometric objects, for example, line segments and triangles, by using the tools on the vertical ribbon (such as *Selection Arrow Tool*, *Point Tool*, *Compass Tool*, *Straightedge Tool* and *Text Tool*). Next, before introducing *Construct* menu, how to select an object on GSP file was mentioned. Then, for example, a segment between the selected two points, point on it and midpoint of it was constructed by using respectively *segment*, *point on object* and *midpoint* commands from *Construct* menu. Where, when and why the commands function was discussed. For example, while the *point on object* command does not function when only two points are selected, it functions when a segment is selected.

After then, it was asked from each student to drag and drop the selected parts of the constructed figures such as dragging and dropping of the endpoints, midpoint and points on a segment. If they cannot manipulate some object (for example, midpoint on a segment), the reasons for this were discussed. In order for students to become familiar with the some frequently used-terms in English in GSP environment throughout the teaching experiment (such as *segment*, *circle*, *angle*, *degree*, *radian*, *radius*, *arc length*, *length* etc.), when speaking about them, it was preferred to use these terms in English throughout the teaching experiment even though the experiment was conducted in the Turkish language.

Afterwards, *undo*, *redo* and *preference* commands under the *Edit* menu, and *line width*, *color*, *hide objects*, *show all hidden* commands under the *Display* menu were introduced in terms of their functions through applying them as well as seeing their effects. Next, we turned again the *Construct* menu in order to provide students with the occasions to see the importance of the selection order of the objects. In construction of a ray, the selection order of two points [for instance, A before B (in contrast B before A)] affects the endpoint of the ray [A (in contrast B)].

Then, the *Measure* menu was introduced through an example on a constructed-circle via measuring its radius, circumference and area by using the *radius, circumference* and *area* active commands under the *Measure* menu. Besides, the awareness of the simultaneously change on measures with the manipulation of this circle through dragging and dropping was provided for students. Afterwards, by using these obtained measures, some calculations had students done.

Finally, *define coordinate system*, *plot points*, *plot new function* commands under the *Graph* menu were mentioned on the examples which were given by the students. After provided with familiarity on GSP usage, students were studied on pre-constructed GSP files on our main topic for Task 1. In the following six episodes, they continued to study on pre-constructed GSP files until they became familiar with GSP usage. On the other hand, the following nine episodes began with students' re-construction of a circle on the coordinate plane by GSP with respect to the teacher-researcher's directions (see *Instructional Design of This Study* heading for more details).

3.7.2. Implementation of the modeling task

Although students were studied previous tasks separately as two pairs, throughout this task, all of four students were worked as a group on a mathematical modeling task with Ferris wheel (see *Cognitive Base of Designed-Instruction* heading for its detailed description) since using small groups including three or four students were recommended in the mathematical modeling tasks' implementation process in order to develop, describe, explain, manipulate the model and control important conceptual systems (Lesh & Yoon, 2004). It was asked from students firstly to read carefully the problem, next, to summarize their understandings on it, and then, to propose their initial thoughts individually for the solution approaches and strategies in about 15 minutes. Finally, students were encouraged to think more critically about how to model the asked instantaneous numerical data about a car on the Ferris wheel (such as, its ground clearance, the distance to the point gotten in the car, etc.). During this process, it was important for us to avoid making any judgment either true or false. Instead, we tried to guide them to provoke their thinking and reasoning via questioning (e.g., how do you ensure that their model would function truly for an arbitrary position of the car on the Ferris wheel?). In the implementation process, in case they need to use GSP, it was provided for students.

3.7.3. Role of the teacher-researcher

As a teacher-researcher, throughout the seventeen-week instruction, my role was a learning partner who provided help for students when dealing with GSP files. During the instruction, I encouraged students to form conjectures (right or wrong) about trigonometric functions, guided them by asking high-level questions (see Protocols in Chapters 5, 6 and 7) to produce deeper learning than recall or recognition questions, and had them think aloud. At the end of acquisition of each big idea in trigonometry, I had them summarize their findings, and asked them to describe what to learn and how to learn it.

3.7.4. Role of the witness-teacher

Throughout the teaching experiment, a mathematics teacher attended each episode as a witness-teacher. She had a six-year experience at a cram school on students' trigonometry from each level. At the same time, she was a graduate student of mathematics education. The teacher-researcher selected her as the witness-teacher mostly based on her subject matter knowledge and skills on trigonometry, especially, on the trigonometric formulas and rules memorized by students to solve trigonometric problems easily and quickly. It was due to receive support both understand and interpret students' languages throughout the teaching experiment especially those indicating trigonometric formulas and rules. In addition, she was selected based on some other reasons; such as, her interest in technology usage in mathematics teaching, her available time schedule and her willingness to attend the episodes of the teaching experiment as the witness-teacher.

Her main role was to observe and provide feedback on the interaction with students after the teaching episodes. She was as a passive observer during the episodes due to her unwillingness to appear in the records. The teacher-researcher met her about 15 minutes before each teaching episode to give information about the instructional goals. And then, at the end of the episodes, discussions were done on significant things related to students' understandings, as well as her perspective on students' ways and means of operating for alternative interpretations. These discussions were important for not only the preparation of subsequent episodes but also the validation of the teacher-researcher's decisions.

3.7.5. Research setting and data collection

Except for the last episode of the teaching experiment (a modeling task with Ferris wheel), each teaching episode was conducted separately with two pairs of students from 24 September 2012 to 9 February 2013. The meeting date of each episode (see *Table 3.8*) was determined according to the pair's preference. If their suitable time was on the week day, the meeting took place on the library of their school after the school courses. If their suitable time was at the weekend, the meeting took place on the teacher-researcher's study room at home. After the meetings, students were leaved their homes. Although two different meeting places (the school library and the teacher-researcher's study room) existed, it was tried to provide the same research setting for both places as in *Figure 3.1*. During the episodes of their teaching experiment, each pair of students studied on the notebook computer on a long table with two connected mousses in order to give each of them the opportunity to control and manipulate the constructions. There were two digital cameras so that one of them could capture the interactions among the pair of students and myself (*Figure 3.2*), and the other could record computer screen (*Figure 3.3*).

Table 3.8. Meeting dates of episodes for each pair

	GROUP 1 (*Cemre & Zafer*)	GROUP 2 (*Defne & Ebru*)
Task 1	*29 September 2012*	*24 September 2012*
Task 2	*8 October 2012*	*10 October 2012*
Task 3	*15 October 2012*	*17 October 2012*
Task 4	*20 October 2012*	*22 October 2012*
Task 5	*3 November 2012*	*29 October 2012*
Task 6	*10 November 2012*	*9 November 2012*
Task 7	*17 November 2012*	*16 November 2012*
Task 8	*24 November 2012*	*30 November 2012*
Task 9	*1 December 2012*	*7 December 2012*
Task 10	*15 December 2012*	*14 December 2012*
Task 11	*28 December 2012*	*21 December 2012*
Task 12	*5 January 2013*	*4 January 2013*
Task 13	*19 January 2013*	*11 January 2013*
Task 14	*30 January 2013*	*18 January 2013*
Task 15	*2 February 2013*	*25 January 2013*
Task 16	*9 February 2013*	*1 February 2013*
Task 17	*11 February 2013*	

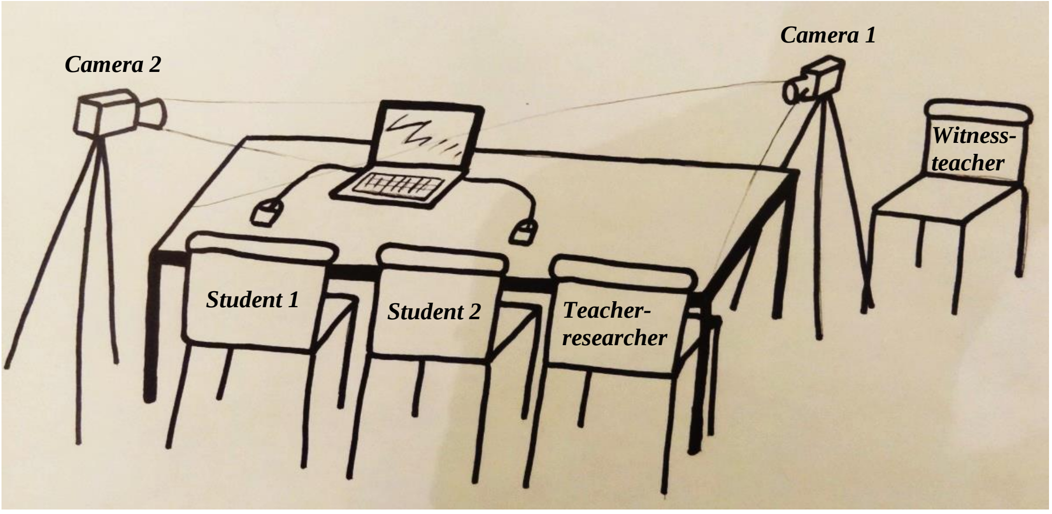


Figure 3.1. Research setting throughout the teaching experiment

Figure 3.2. An example of the Camera 1's snapshot from Task 11 with Group1

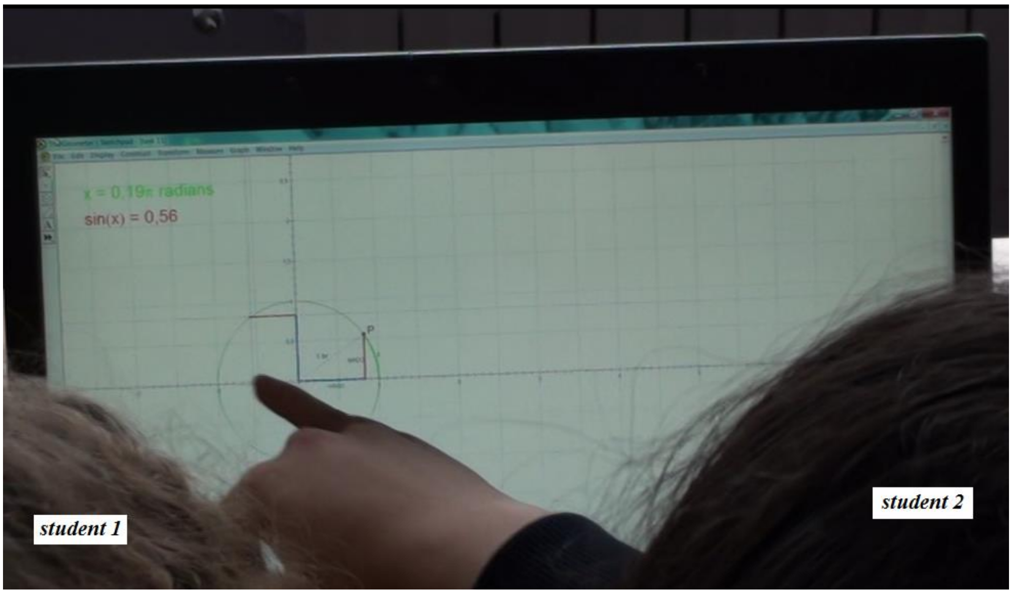


Figure 3.3. An example of the Camera 2's snapshot from Task 12 with Group2

Multiple sources of data were collected during the teaching experiment through field notes from direct observations of both I (as a teacher-researcher) and the witness-teacher, worksheets participants produced, video recordings with two digital cameras. These records were used for retrospective analysis both between episodes to prepare subsequent episodes (Steffe & Thompson, 2000) and at the end of the teaching experiment to build and revise the models of students' concept images on trigonometric functions.

3.8. Analysis of Data

3.8.1. Analysis of Initial Interviews

Analysis of the initial interview data started with viewing video-records and students' worksheets simultaneously to determine the related-parts to the first main research question of the study and its sub-questions (see *Research Questions* heading of *Introduction* chapter). Next, these parts were transcribed so as to represent both students' language and actions including gestures. After then, transcriptions re-analyzed to determine the initial model of students' concept images on trigonometric functions prior to the teaching experiment. This modeling process was cyclic. It included re-reading, re-viewing, re-organizing and re-thinking about data.

3.8.2. Analysis of Teaching Experiment

Both prospective and retrospective analyses of video-record were made in chronological order to model developments of students' *concept images* on trigonometry. On the one hand, cognitive analyses of data were conducted in an on-going way between episodes to prepare subsequent episodes in order to modify students' *concept images* as well as my role as a teacher-researcher when interacting with students in order to guide my future interactions. On the other hand, at the end of the teaching experiment, retrospective analyses were conducted in chronological order to model students' concept images to represent their developments. Students' concept images were modeled based on cognitive analyses of students' mathematical actions and languages. Duval's (2006) *recognition* and *discrimination* cognitive skills and Sfard's (1991) *interiorization*, *condensation* and *reification* classification provided us with a theoretical lens for these cognitive analyses.

On-going prospective analyses started with on the days each episode was conducted. Initially, field notes were read. Next, video records were viewed to understand (*i*) "what students did", "how they did" and "why they did", and (*ii*) "how was the interaction". And then, (if needed) new situations were designed and composed in the following episode(s) to modify students' *concept images* (These situations were articulated with their reasons in detail together with *mathematics of students* in Chapter 5).

On the other hand, at the end of the teaching experiment, all episodes' video-records were viewed for each pair correspondingly. However, this process required to view 34 different –approximately 90-minute– episodes' video-records from two pairs of students. Therefore, these video-records' cognitive analyses were conducted through separating them into three parts corresponding to three main parts of the instructional design of the teaching experiment (see *Instructional Design of This Study* and *Cognitive Base of Designed-Instruction of This Study* headings in this chapter).

Video records of the first seven episodes of the teaching experiment from each pair were viewed in chronological order correspondingly to determine important parts referring to (*i*) students' *concept definition images* related to angle and trigonometric functions, as well as (*ii*) students' *recognition* of the basic form of trigonometric functions within and between different representational registers.

Video-records of following nine episodes were also viewed in chronological order correspondingly from two pairs to determine important parts referring to (*i*) students' *recognition* of the general form of trigonometric functions within and between different representational registers, and (*ii*) students' *discrimination* of trigonometric functions represented within any representational register from their respective representational registers' contents.

Finally, video records of Modeling Task [i.e., 17th episode] were viewed correspondingly from two pairs to determine important parts referring to (*i*) students' abilities to transform their knowledge on trigonometric functions in the (*unit*) *circle register* (located on coordinate plane) to the any circular context, and (*ii*) students' conversion troubles between the (*unit*) *circle register* and the *symbolic register* in a problem solving context.

After determination of the important parts as it is mentioned above, all these parts were transcribed so as to represent both students' language and actions including gestures. And then, transcriptions were analyzed line-by-line to model development of students' *concept images* on trigonometric functions during the teaching experiment through comparing and contrasting with eachother. This modeling process was cyclic. It included re-reading, re-viewing, re-organizing and re-thinking about data.

3.9. Ethical Aspects

Throughout the study, the ethical aspects were considered. Moreover, as an important ethical aspect, confidentiality of research data was provided in two ways: by exchanging names of the students with pseudonyms, and by keeping those safe so that it could be prevented, except the teacher-researcher and the witness-teacher, everyone from reaching them.

CHAPTER 4

RESULTS FROM INITIAL INTERVIEWS

In this chapter, students' understanding on *foundational concepts* related to trigonometric functions is presented from cognitive analyses results of the initial interviews. As foundational trigonometric concepts, *functions*[23], *angle*[24], *angle measure*[25], *trigonometric value*[26], *trigonometric functions* and *periodicity*[27] were determined based on trigonometry research literature, historical development of trigonometry, our exploratory teaching experience, and initial interview results. The aim of this chapter is to provide the initial models of students' concept images on trigonometric functions prior to the teaching experiment. This chapter represents *trigonometry of students* prior to the teaching experiment in terms of their *recognition* problems on trigonometric functions within any representational register, *discrimination* problems between trigonometric functions represented within any representational register and the respective representational registers' contents, as well as *potential conflict factor*s in their *concept images* on trigonometry.

[23] Reasoning about trigonometric functions as functions is critical in understanding of trigonometric functions (Hertel & Cullen, 2011; Thompson, 2008; Weber, 2005).

[24] Angle and angle measure are two different concepts that is needed to be dissociated from each other (Argün, Arıkan, Bulut, & Halıcıoğlu, 2014).

[25] Reasoning about angle measure as the (meaningful) numbers referring to the argument of trigonometric functions (Thompson, 2008) is important in understanding of trigonometric functions.

[26] Trigonometric values correspond to the outputs of trigonometric functions.

[27] Trgionometric functions are natural and fundamental examples of periodic functions. Understanding of periodicity requires dual and simultaneous reasoning about (*i*) regular intervals of DOMAIN and (*ii*) corresponding repeated values in RANGE.

4.1. Students' Initial Concept Images on Trigonometric Functions Prior to Teaching Experiment

4.1.1. On functions

It was observed that in determination of functionality, all of the students had some troubles that may influence their reasoning on trigonometric functions. Firstly, when trying to define the function concept in Question 1 (Q1), it was observed that in the *symbolic register*, their concept images on functionality were restricted to being polynomial functions (see [Zafer] Protocol 1 and *Table 4.1* as examples), which implies that their reasoning about trigonometric functions was not as functions in the *symbolic register*.

[Zafer] Protocol 1

Zafer: When saying a function, an equation comes to my mind…
Researcher: What kind of equations comes?
Zafer: What kind of equations… consisting of variables, degrees and constant terms comes to my mind.
Researcher: Can you give an example?
Zafer: Like that (*writing his function example in Table 4.1*)
Researcher: Why this is a function?
Zafer: Why… it has degrees (*pointing exponents in the example with his pencil*)… a variable (*pointing terms with x*)… so it is a function.

Table 4.1. Students' exemplifications of the function in the *symbolic register*

Example of Cemre	$f(x) = ax + bx + c$
Example of Defne	$f(x) = ax^2 + bx + c$
Example of Ebru	$f(x) = 10x + 20$
Example of Zafer	$f(x) = 3x^3 + 12x^2 + x + 6$

Secondly, their concept images on functionality included some *conversion* troubles arising in the *multi-functional and non-discursive register* from usage of the associated processes for being a function within the *graphical register*. For example, in Q2 when determining whether the location of the caterpillar on the paper with respect to time could be a function or not, Zafer transferred his knowledge on the vertical-line test (corresponding to the circumstance for being a function that each input in the domain must be related to exactly one output in the range) within the *graphical register* (or the *mono-functional and non-discursive register*) into the *multi-functional and non-discursive register*. He constructed a vertical-line on the caterpillar's path (*Figure 4.1*) and then explained that "it [the caterpillar's location on the paper with respect to time] is not a function… because… when drawn a straight line like that… It is passing through two different points (*concretizing two intersection points of the path with the vertical line*)". Thus, it appears for Zafer to unable to dissociate different meanings of the curve within these two different representational registers.

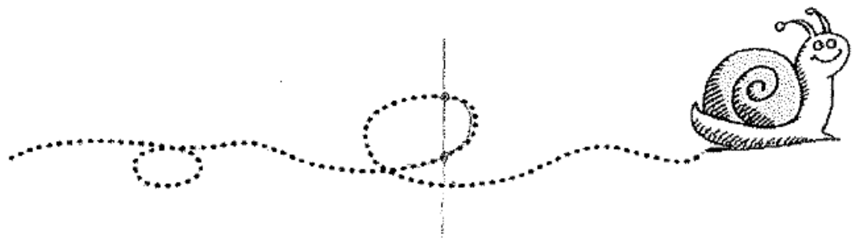


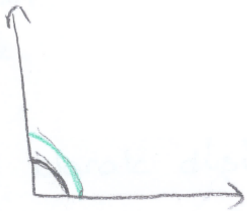
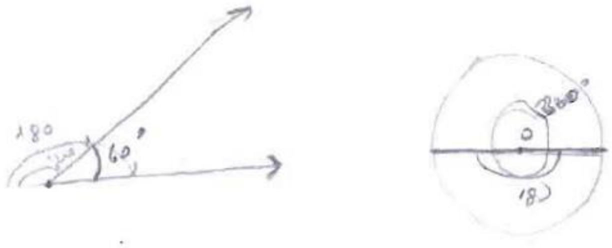
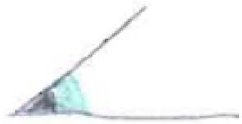
Figure 4.1. Zafer's transformation of the vertical-line test within the *graphical register* into the *multi-functional and non-discursive register*

All these troubles related to functionality mentioned above arise from their dominated concept images on functions by polynomial functions in the *symbolic register* and their dominated mental images on functions within the *multi-functional and non-discursive register* by the *graphical register*.

4.1.2. On definition of angles

When trying to define “angle” concept, all students expressed an angle as “[a thing] between two rays with common initial point (*constructing an angle like in Table 4.2*)” but they did not determine the thing was what. When the researcher insisted on asking them to determine the thing was what, they expressed by some concepts such as area (Defne), space (Zafer, Ebru) and distance (Cemre). However, they were confused on what to be measured when measuring an angle (see [Cemre] Protocol 1 as an example). Only Defne articulated the angle measure through taking the straight and complete angles, namely 180 and 360 in degrees as references (*Table 4.2*). None of others, in spite of their circular constructions inside of the angle [such as arcs and arc sectors], established any relations of these circular constructions with the angle measure (lines 4-14 in [Cemre] Protocol 1; lines 3-12 in [Ebru] Protocol 1). However, they reasoned intuitively that the measure of an angle did not change with respect to change of circular constructions in terms of radii (lines 6-8 in [Cemre] Protocol 1; lines 8-10 in [Ebru] Protocol 1). Consequently, even if students’ concept definition images on angles included an intuitive relation between angle’s openness and measure, they also included a trouble on what was the measured part of an angle in determination of its measure.

Table 4.2. Students' constructions of angle*

Cemre's construction	
Defne's construction	
Ebru's construction	

* *Zafer preferred to articulate the angle by his construction in the first quadrant of the unit circle like in Table 4.3, instead of constructing new one.*

[Cemre] Protocol 1

Researcher: What is the angle?
Cemre: It is a thing between two lines… type of degrees… we measure… A thing but I didn't express what. (*After 6-second pause*) for example, here is an angle (*constructing rays of an angle like in Table 4.2*)… distance between two intersecting lines (*drawing the black arc*)...
Researcher: Well. What about this distance (*drawing the green arc*)? Does it mean the same angle?
Cemre: Yes.
Researcher: But their distances are not same. How they are referring to the same angle?
Cemre: They are the same in degrees.
Researcher: Then, angle should be different from "distance", is it?
Cemre: Yes.
Researcher: Well, what is the angle?
Cemre: (*Smiling and stopping speaking and looking to the angle she constructed*)

[Ebru] Protocol 1

Researcher: What is the angle?
Ebru: I mean… an angle is a thing like a space between two lines (*constructing an angle and the black arc sector like in Table 4.2*)… That is, an angle is here (*concretizing the arc sector*)…
Researcher: Is the angle here you constructed by black color (*pointing the black arc sector*)?
Ebru: Yes.
Researcher: Well. If I construct here (*constructing green arc sector*), is it a different angle?
Ebru: No, they shouldn't be different (*pausing without speaking and looking to the researcher*).
Researcher: But these parts are not same (*pointing black and green arc sectors*). How they are referring to the same angle?
Ebru: (*Smiling*) I don't know...

4.1.3. On angle measures

It was observed that in articulation of angles within the (*unit*) *circle register*, all of the students had some troubles that may influence their reasoning on trigonometric functions. Firstly, in Q4, their reasoning on angles within the (*unit*) *circle register* was restricted to those in the first quadrant, only in the positive direction, as well as in very specific instances in degrees [such as 30^{o} and 45^{o}] and radians [such as π/2, π, 3π/2 and 2π]. Except Zafer, all of them started to articulate angles in the (*unit*) *circle register* with the construction of examples in the first quadrant in very specific instances in degrees [such as 30^{o} and 45^{o}] after expressing angles corresponding to the axes in radians [such as π/2, π, 3π/2 and 2π] (see *Table 4.3* and [Defne] Protocol 1 as examples).

[Defne] Protocol 1

Defne: We called here as π/2 (*pointing the intersection point of the unit circle with the positive part of the y-axis and writing π/2*), here as π (*pointing the intersection point of the unit circle with the negative part of the x-axis, and writing π*), here as 3π/2 (*pointing the intersection point of the unit circle with the negative part of the y-axis, and writing 3π/2*), and here as 2π (*pointing the intersection point of the unit circle with the positive part of the x-axis, and writing 2π*).

Researcher: Ok.

Defne: Um… When an angle takes a value, (*drawing a line segment from the origin to the unit circle in the first quadrant, concretizing the point on the circle*), such as 45 degrees, it may be here (*drawing a curved arrow inside of the circle next to the origin and writing 45°*).

However, even if Zafer considered angles in each quadrant of the unit circle, he reasoned about angles within all other quadrants based on α angle in the first quadrant such as $\pi-\alpha$, $\pi+\alpha$ and $2\pi-\alpha$ (lines 1-3 and 14-16 in [Zafer] Protocol 2) but within the *symbolic register* (lines 18-20 and 22-25 in [Zafer] Protocol 2) without considering their symmetrical relations within the (*unit*) *circle register* (see *Table 4.3*).

[Zafer] Protocol 2

Zafer: We know angles are there. We called here as $\pi-\alpha$ (*writing* $\pi-\alpha$ *into the second quadrant of the unit circle he constructed*), here as $\pi+\alpha$ (*writing* $\pi+\alpha$ *into the third quadrant*), here as $2\pi-\alpha$ (*writing* $2\pi-\alpha$ *into the fourth quadrant*).

Researcher: What do you mean with $\pi-\alpha$?

Zafer: The second quadrant… I mean here are the first quadrant, second quadrant, third quadrant and fourth quadrant (*writing numbers with roman numerals*). Where, we find [the position of] an angle in the second quadrant taking advantage of $\pi-\alpha$. Taking account of this (*rotating his pencil around the* $\pi-\alpha$ *symbol*), we can transform an angle here (*pointing the second quadrant with his index finger*) to that in the first quadrant (*pointing the first quadrant with his index finger*). …or transform an angle from the first quadrant to the second quadrant (*pointing respectively the first and second quadrants with his index finger*).

Researcher: Well, where is α?

Zafer: α angle is like that (*drawing a line segment from the origin to the unit circle in the first quadrant, concretizing the point on the circle, writing* α *next to this point*) …so be it 30 (*writing 30 inside of the angle*).

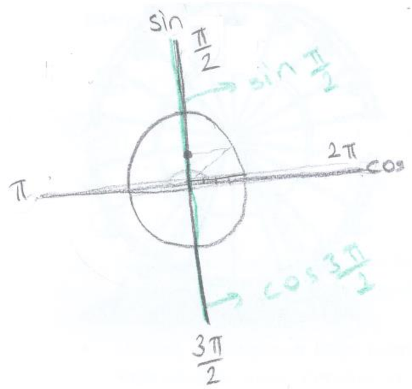
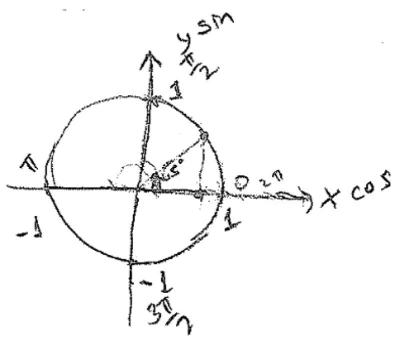
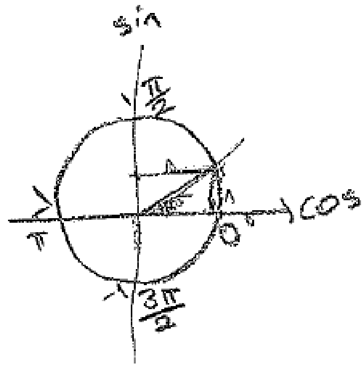
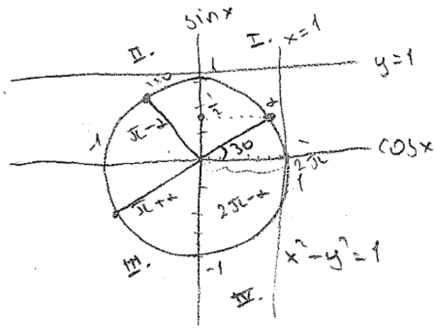
Researcher: Ok. According to α, can you construct $\pi-\alpha$?

Zafer: Well, $\pi-\alpha$ is like that (*drawing a segment from the origin to the unit circle in the second quadrant*) …so π minus 30… …that is, 150 degrees (*concretizing the point on the circle, writing 150 next to this point*)

Researcher: Ok. Where is $\pi+\alpha$?

Zafer: $\pi+\alpha$ is here (*pointing the third quadrant*) between 180 and 270. That is, 180 [plus 30], it is 210 and I can draw it like that (*drawing a segment from the origin to the unit circle in the third quadrant)* and 150 degrees (*concretizing the point on the circle, writing 150 next to this point*).

Table 4.3. Students' constructions to articulate meaning of the unit circle

Construction of Cemre	
Construction of Defne	
Construction of Ebru	
Construction of Zafer	

Secondly, it was observed that in spite of their reasoning that the same π notation must be refer to the same objects (line 6 in [Ebru] Protocol 2), within the *symbolic register,* all of the students dissociated π in and out of the trigonometry context in terms of its real value (lines 13-15 in [Zafer] Protocol 3; and lines 8-11 in [Ebru] Protocol 2). In other words, they treated π in the *symbolic register* as if it had two different number values: "180" in the trigonometry context (lines 8-15 in [Zafer] Protocol 3; lines 1-2 in [Ebru] Protocol 2; and lines 10-11 in [Defne] Protocol 2) and about 3.14 out of the trigonometry context (lines 10-12 in [Zafer] Protocol 3; lines 3-5 in [Ebru] Protocol 2; and lines 1-2 in [Defne] Protocol 2). Therefore, within the *symbolic register*, "π" had a potential to become a *cognitive conflict factor* in and out of the trigonometry context when considered simultaneously.

[Zafer] Protocol 3

Researcher: Well, you mentioned "degree", "alpha" and "π"… Does the angle measure have a unit?
Zafer: Angle measure's unit… is degree…
Researcher: For example, is here (*pointing the point corresponding to α on his construction for unit circle*) corresponding to 30 degrees?
Zafer: Yes, 30 degrees.
Researcher: Well, what about here (*pointing the point corresponding to π–α*)?
Zafer: It is π minus 30… that is, 180 minus 30, there is 150 degrees.
Researcher: What is π in here?
Zafer: π is equal to 180.
Researcher: Is it different from π that you use when calculating circumferences of circles by the formula 2πr?
Zafer: Um… There [in the circumference formula], π is 3.14, but here [in trigonometry], π is… (*After a 5-second pause*) We take π here as 180, but there as 3.14… No, there is no relation between them, they are different things.

[Ebru] Protocol 2

Researcher: Well, you mentioned π/2, π, 3π/2 and 2π… Can you explain π in more detail?
Ebru: π is 180 (*smiling*)…
Researcher: Is it different from π that you use when calculating circumferences of circles by the formula 2πr?
Ebru: It was approximately 3.14… 180 and 3.14 (*uttering in a low voice*)… (*After a 12-second pause*) I think they should be same things but why we write 180 for 3.14…

Researcher: Why do you think they are same?
Ebru: Eventually, both are π… …why we express them by the same notation if they were different (*a 7-second pause*)… But, how and why one is 180 and the other is 3.14? I don't know… (*After a 11-second pause*) but I surmise… …they are as if different…

[Defne] Protocol 2

Researcher: What do you think about cos(3.14)?
Defne: Well, then we find the value of cosine on 3.14…
Researcher: Ok. What is it approximately?
Defne: We think 3.14 as an angle… When saying cos(3.14), we should find the cosine value corresponding to a 3.14-degree angle… cos(3.14)… (*after a 7-second pause, dragging her pencil in the first quadrant near the x-axis from the origin to the right side on her construction for the unit circle*) Then, it [3.14 as an angle] is very small value… So, its cosine should be near the 1…
Researcher: Ok. What about cos(π)?
Defne: Then, you are asking the value of cosine of 180 (*pointing the point on the unit circle corresponding to the π-radian angle*), it is minus 1…

Thirdly, when more critiques on π were done not only in the *symbolic register* but also in the *graphical register*, as well as in and out of the trigonometry context on Q6 and Q8, it was observed that students considered π in different representational registers as being different mathematical objects which may correspond to different real values (lines 28-42 in [Zafer] Protocol 7). In spite of their reasoning about the real value of π within the *symbolic register* as 180 in the trigonometry context and as about 3.14 out of the trigonometry context (mentioned above), when dealing with *conversion* activities (i.e., change of registers), some major troubles based on π were observed in students' reasoning.

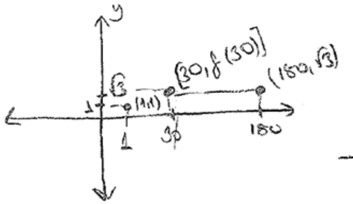
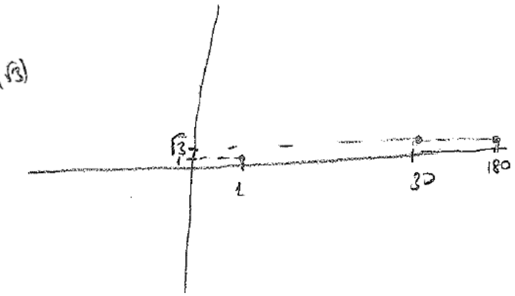
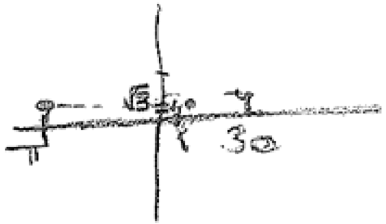
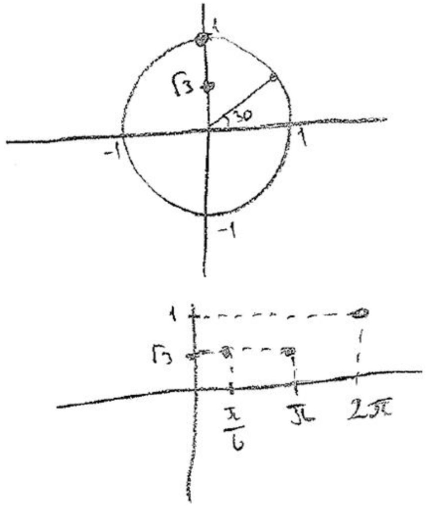
First, on the one hand, when trying to convert the ordered pair $(\pi, f\left(\frac{\pi}{6}\right))$ within the *symbolic register* into the *graphical register* (see *Table 4.4*), Cemre and Defne reasoned about its abscissa, π, through transferring their reasoning on π as 180 within the *symbolic register* in the trigonometry context to that out of the trigonometry context not only within the *symbolic register* (*Table 4.4*) but also within the *graphical register* (*Table 4.5*). For example, before constructing the point corresponding to $(\pi, f\left(\frac{\pi}{6}\right))$ on the coordinate plane, after her determination of the value of its ordinate as $\sqrt{3}$, Cemre said that "…π to $\sqrt{3}$, that is, 180 to $\sqrt{3}$ (*writing her solution like in Table 4.4*)" and then constructed the point as if $(180, \sqrt{3})$ within the *graphical register*. On the other hand, Ebru and Zafer reasoned about π, respectively, as if $\pi = -1$ and $2\pi = 1$ within the *graphical register* (*Table 4.5*) through considering the points on the unit circle corresponding to π and 2π as angle measures in radians within the *unit circle register*. For example, when constructing the point corresponding to $(\pi, \sqrt{3})$ on the coordinate plane, Ebru reasoned that "…where is π on the *x*-axis… (*After 8-second pause*) if we consider π in here, that is, minus one to zero like in the unit circle (*pointing a position on the left side of the x-axis regarding the origin with her pen*)… I think it should be here (*concretizing the position*)" and then constructed the point within the *graphical register* like in *Table 4.5*. In a similar way, not only when constructing the ordered pairs in Q6 on the coordinate plane (*Table 4.5*; and lines 10-12 in [Zafer] Protocol 4) but also when determining the position of a real number on the *x*-axis scaled with numbers regarding π (lines 19-42 in [Zafer] Protocol 7), Zafer reasoned that "…2π corresponds to 1 on the *x*-axis (*pointing the intersection point of the unit circle with the x-axis's positive side which he constructed like in Table 4.3*)".

Table 4.4. Students' operations in the symbolic register on Q6

*ordered pairs**	*Operation of Cemre*	*Operation of Defne*	*Operation of Ebru*	*Operation of Zafer*
$(30, f(30))$	$f(30) = 2\cos 30$ $f(30) = 2 \cdot \frac{\sqrt{3}}{2}$ $f(30) = \sqrt{3}$	$f(30) = 2\cos 30 \Rightarrow \sqrt{3}$ $(30, \sqrt{3})$	$2 \cdot \cos 30$ $= 2 \cdot \frac{\sqrt{3}}{2}$ $[30, \sqrt{3}]$	$2\cos 30 = \sqrt{3}$, $(30, \sqrt{3})$
$(\pi, f(\frac{\pi}{6}))$	$180, f(\frac{180}{6})$ $(180, f(30)$ $(180, \sqrt{3})$	$f(\frac{\pi}{6}) = f(30) = 2 \cdot \cos 30 \Rightarrow$ $(180, \sqrt{3})$	$f(x) = 2 \cdot \cos(\frac{\pi}{6})$ $[\pi, \sqrt{3}]$	$2\cos 30 = \sqrt{3}$, $(\pi, \sqrt{3})$
$(1, f(\frac{\pi}{3}))$	$f(\frac{180}{3}) = 2\cos 60$ $f(60) = 2 \cdot \frac{1}{2}$	$f(\frac{\pi}{3}) = f(60) = 2 \cdot \cos 60 =$ $(1, 1)$	$f(x) = 2 \cdot \cos(\frac{\pi}{3})$ $f(x) = 1$ $[1, 1]$	$2\cos 60 = 1$, $(1, 1)$

*Where, $f: R \rightarrow R$, $f(x) = 2\cos(x)$

Table 4.5. Students’ reasoning in the graphical register on Q6

Construction of Cemre	*Construction of Defne*	*Construction of Ebru*	*Construction of Zafer*
			

Second, in spite of her reasoning on π as 180 within the *graphical register* out of the trigonometry context, a systematic variation of Defne's reasoning on π was observed as a consequence of the change of the source register from *symbolic* to *graphical* and as a consequence of the variation on the scaling of the *x*-axis from numbers regarding π to the real values. That is, when trying to convert the graph in Q8 into the *symbolic register*, Defne reasoned about the origin corresponding to "zero" on the *x*-axis as if it was equal to 2π within the *graphical register* (*Figure 4.2*) through considering the equivalence between 0 and 2π as angle measures in radians within the *unit circle register* (*Figure 4.3*) (lines 29-41 in [Defne] Protocol 5).

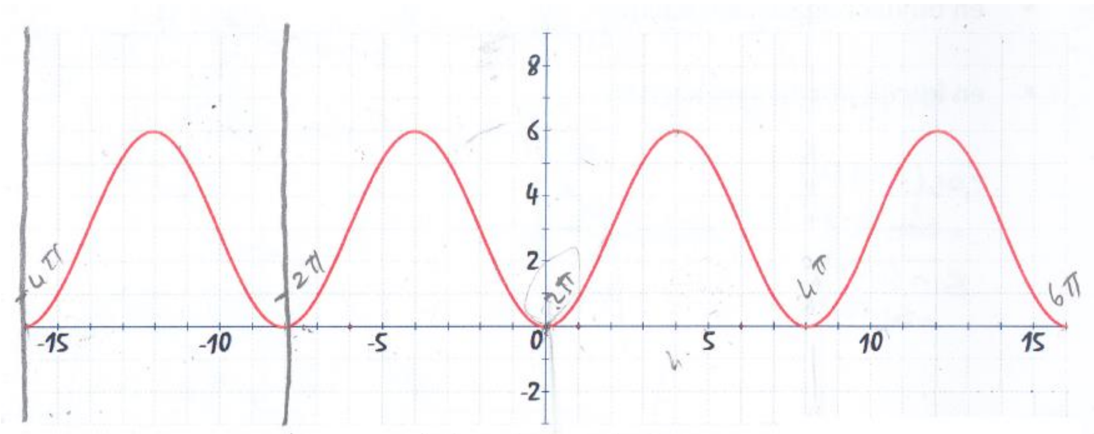


Figure 4.2. Defne's reasoning about the repetition of the graph in Q8

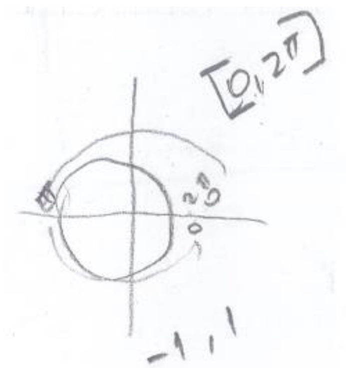


Figure 4.3. Defne's reasoning about the domain and range of sine and cosine in Q8

Third, in Q6 when determining ordinates of the given ordered pairs in the *symbolic register* (see *Table 4.4*) that were defined by the cosine function, all of the students accepted the argument as an angle in radians only if π notation exists in the expression, otherwise as an angle in degrees. In other words, while their "seeing" of 30 in the expression $f(30) = 2cos(30)$ was in degrees (lines 1 and 4 in [Cemre] Protocol 2; and lines 1-4 in [Ebru] Protocol 3), their "seeing" of $\frac{\pi}{3}$ and $\frac{\pi}{6}$ in the expressions $f\left(\frac{\pi}{3}\right) = 2cos(\frac{\pi}{3})$ and $f\left(\frac{\pi}{6}\right) = 2cos(\frac{\pi}{6})$ were in radians (lines 10 and 17 in [Cemre] Protocol 2; and lines 22-24 and 30-32 in [Ebru] Protocol 3). Besides, Ebru and Zafer got confused about whether 30 in the *x*-component was an angle in degrees or not because of their assumption on 30 in the function $[f(30) = 2cos(30)]$ as degrees. Firstly, Ebru expressed her confusion about whether 30 in the *x*-component should be assumed either as an angle in degrees or as a number (lines 4-19 in [Ebru] Protocol 3). Secondly, Zafer also considered 30 in the *x*-component as an angle in degrees in the *symbolic register* (line 10 in [Zafer] Protocol 4) and tried to show $(30, \sqrt{3})$ on the (*unit*) *circle register* instead of on the *graphical register* (*Table 4.5*). However, since he did not feel a satisfaction about this thinking process (lines 6-9 in [Zafer] Protocol 4), he preferred to consider 30 as if it was equal to π/6 through transforming 30 as if an angle in degrees to the corresponding value, π/6, in radians in the *symbolic register* (lines 9-12 in [Zafer] Protocol 4), and then transferred this reasoning within the *graphical register* apart from the unit circle (*Table 4.5*).

[Cemre] Protocol 2

Cemre: Where, I want firstly to find the value of cosine for 30, and then, to show on the coordinate plane it (*pointing the ordered pair (30, f(30)) with her pen*)
Researcher: Ok.
Cemre: cos(30) is $\sqrt{3}/2$ (*uttering in a low voice*)… it should be multiplied with 2… That is, it is asked from me to show 30 to $\sqrt{3}$ on the coordinate plane (*writing her solution for the ordered pair like in Table 4.4 and constructing a coordinate plane*) …so be here 30 (*drawing a point on the x-axis*), here $\sqrt{3}$ (*drawing a point on the y-axis*) and so here 30 to $\sqrt{3}$ (*constructing point corresponding to the ordered pair* $(30, \sqrt{3})$ *and writing* $(30, \sqrt{3})$ *near this point*).

Cemre: Well, cos(π/6), that is, cos(30)… and then, again it (*pointing the ordinate*) is $\sqrt{3}$… π to $\sqrt{3}$, that is, 180 to $\sqrt{3}$ (*writing her solution like in Table 4.4*)

Researcher: I want to show it [the ordered pair] also on the same coordinate plane.

Cemre: Ok. …so be here 180 (*drawing a point on the x-axis right side of the 30*), here $\sqrt{3}$ (*pointing the point on the y-axis she constructed for previous ordered pair*) and so here 180 to $\sqrt{3}$ (*constructing point corresponding to the ordered pair* $(180, \sqrt{3})$ *and writing* $(180, \sqrt{3})$ *near this point*).

Cemre: Finally, cos(π/3)… …180 divided by 30, that is, cos(60) (*uttering in a low voice*)… 1/2, when multiplying 2, it (*pointing the ordinate*) is 1… that is, 1 to 1 (*constructing the point corresponding to the ordered pair* $(1,1)$ *and writing* $(1,1)$ *near this point*).

[Ebru] Protocol 3

Ebru: Well, I will write 30 for *x* (*pointing respectively 30 in f(30) and x symbol in the f(x)*)… that is, 2cos(30) (*spelling and writing 2cos(30)*)… The value of cos(30) is $\sqrt{3}/2$… two times $\sqrt{3}/2$… that is, $\sqrt{3}$… 30 to $\sqrt{3}$ (*writing her solution for the ordered pair like in Table 4.4 and constructing a coordinate plane*). Now, I don't know how to assume 30 (*pointing 30 in the abscissa*) as degree or number…

Researcher: How do you understand whether it is degree or number?

Ebru: If we take it as degree (*pointing 30 in the function*), we should take it also as degree (*pointing 30 in the abscissa*)… I think taking one as number one as degree is absurd…

Researcher: Ok.

Ebru: Then, if 30 is degree (*pointing abscissa*)… (*Turning her glance from the paper, after a 4-second pause*) if here (*pointing the ordinate*) represents cosine… …but *y*-component must be related to sine… it is also absurd… I am confused… (*After 6-second pause*) …umm, I surmise… I show it [$(30, \sqrt{3})$] like that (*constructing the point corresponding to the ordered pair* $(30, \sqrt{3})$ *in Table 4.5*).

Researcher: Why do you assume 30 in the abscissa as a number and 30 in function as degrees? You said that "taking one as number one as degree is absurd".

Ebru: I know… But if I took it as degree, I would do nothing, and I would be more confused… For this reason, I preferred to take it as a number.

Researcher: Ok. I want to show other ordered pairs also on the same coordinate plane.

Ebru: Well, here (*pointing the second ordered pair in Q6*), π to f(π/6) (*uttering through pointing the abscissa and ordinate respectively*)… …two times cos(π/6), that is, 2cos(30) again it (*pointing the ordinate*) is $\sqrt{3}$… π to $\sqrt{3}$ (*writing her solution for the ordered pair like in Table 4.4*). (*Turning her coordinate plane*) …where is π on the *x*-axis… (*After 8-second pause*) if we consider π in here, that is, -1 to 0 like in the unit circle (*pointing a position on the left side of the x-axis regarding the origin with her pen*)… I think it should be here (*concretizing the position*)…

Researcher: Ok.

Ebru: Here (*pointing the third ordered pair in Q6*), 1 to f(π/3) (*uttering through pointing the abscissa and ordinate respectively*)… …two times cos(π/3), that is, 2cos(60)… It is 1… 1 to 1 (*writing her solution for the ordered pair like in Table 4.4 and constructing the point corresponding (1,1) like in Table 4.5*).

[Zafer] Protocol 4

Zafer: (*After completing all ordinates' solutions in Q6 within the symbolic register through uttering like that of Cemre and Ebru, when trying to show* $(30, \sqrt{3})$ *in the graphical register, Zafer constructed a coordinate plane with unit circle like in Table 4.5*) Now, we think 30 degrees like that (*drawing a line segment from the origin to the unit circle in the first quadrant, concretizing the point on the circle*)… $\sqrt{3}$ (*uttering* $\sqrt{3}$ *and pausing 6 seconds*)… $\sqrt{3}$ is the *y*-value… If we take here as $\sqrt{3}$ (*concretizing a point on the y-axis between 0 and 1 near the same height with the point he constructed on the circle to refer 30 degrees like in Table 4.5*)…Umm... How can I think… (*After a 6-second pause*) if we consider apart from unit circle, we can take 30 as π/6 like that (*constructing a new coordinate plane below his previous construction with unit circle and showing* $(\pi/6, \sqrt{3})$ *on it like in Table 4.5*)… …π to $\sqrt{3}$ like that... and 1… (*after 5-second pause*) corresponds to 2π on the *x*-axis, so, 1 to 1 is like that (*showing them like in Table 4.5*).

All these troubles mentioned above imply that within the *symbolic register*, students' conceptions of the relation between radians and degrees measure units as if it was a computational equality without the degree and radian notations [such as π=180 and $\frac{\pi}{6} = 30$] (lines 10 and 17 in [Cemre] Protocol 2; lines 10-11 in [Defne] Protocol 2; lines 20-22 and 30-32 in [Ebru] Protocol 3; and line 10 in [Zafer] Protocol 4) instead of a proportional equality with degree and radian notations [such as πR=180^o and $\frac{\pi}{6}$ R=30^o]. Moreover, their troubles in reasoning on π within the *graphical register* arise from their dominated concept images on π by the trigonometry context. In other words, in making sense of π within the *graphical register*, although Cemre and Defne preferred to transfer their reasoning on π within the *symbolic register*, Ebru and Zafer preferred to transfer their reasoning on π within the (*unit*) *circle register*. However, Cemre and Defne were unable to dissociate in the *symbolic register* the meaning of proportional equality between π in radians and 180 in degrees from computational equality (lines 10-11 in [Cemre] Protocol 2; and *Table 4.5*). Conversely, Ebru and Zafer were unable to dissociate two meanings of the intersection points of the unit circle with the *x*-axis [i.e., the points (-1,0) and (1,0)] as an **abscissa** corresponding to these points within the *graphical register* and as an **angle** in radians corresponding to these points within the *unit circle register,* which arose from their mental images on angles as points on the circle instead of corresponding arcs. Besides, when trying to convert her reasoning on π within the (*unit*) *circle register* into the *graphical register*, Defne was unable to dissociate the meaning of the equivalence between "zero" and 2π in radians from the equality between them within the *graphical register*, which arose from her dominated concept image on principal angles (lines 40-41 in [Defne] Protocol 5) despite of her mental image on angles as dynamic turning (*Figure 4.3*, lines 8-11 in [Defne] Protocol 1, and lines 29-41 in [Defne] Protocol 5).

To sum up, students' reasoning on π in different representational registers indicates that "coordinate plane" became a *cognitive conflict factor* when the position of π on the *x*-axis is considered simultaneously within the *graphical register* and the (*unit*) *circle register*.

4.1.4. On definition of trigonometric functions

It was observed that when trying to define sine and cosine in Q5, none of the students mentioned them as functions (e.g., lines 1-6 in [Cemre] Protocol 3). Instead, they defined sine and cosine firstly as values obtained from calculations of the ratios in the right triangle context [respectively, as opposite side/hypotenuse and adjacent side/hypotenuse] (lines 8-13 in [Cemre] Protocol 3; and *Table 4.6*). However, they were not aware that these ratios for an angle in a right triangle were the same as those in all similar right triangles (e.g., lines 5-7 in [Ebru] Protocol 4). This unawareness arose from their reasoning about sine and cosine within the right triangle context as calculations instead of ratios obtained from proportions in the similar right triangles.

[Cemre] Protocol 3

Researcher: What comes to your mind when thinking about sine or cosine?

Cemre: sine and cosine… conversion formulas come, the unit circle comes… degrees come… π comes… right triangle comes… what else? (*After 7-second pause*) nothing else comes to my mind…

Researcher: Ok. Does function comes?

Cemre: No, function doesn't come to my mind now…

Researcher: How to define sine and cosine?

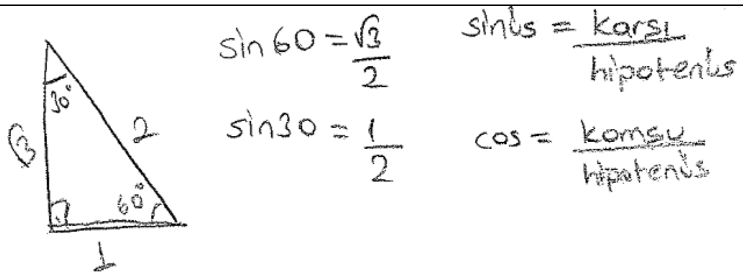
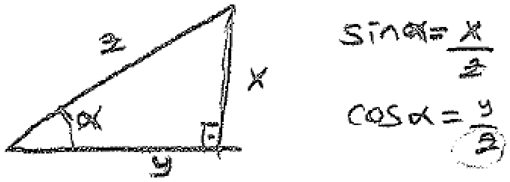
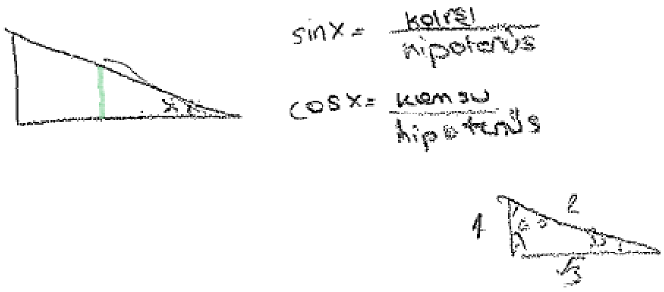
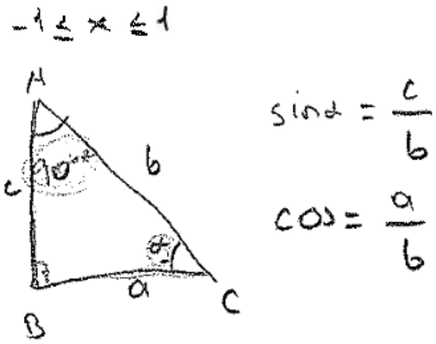
Cemre: In a right triangle, sine is opposite/hypotenuse and cosine is adjacent/hypotenuse (*drawing a right triangle with specific angle measures in degrees 30^o-60^o-90^o and writing these ratios like in Table 4.6*) If the opposite side of the 30 is 1, then that of 60 is $\sqrt{3}$, that of 90 is 2 (*writing these values corresponding sides of her construction of right triangle through uttering*)… So sin60 is $\sqrt{3}/2$ and sin30 is 1/2…

[Ebru] Protocol 4

Researcher: For example, I take this small right triangle (*drawing the green segment in the right triangle that Ebru constructed like in Table 4.6*), again is opposite side/hypotenuse or sine of this angle (*pointing x angle in Ebru's construction*) same?

Ebru: (*After a 6-second pause*) I think they are different… …because hypotenuses [lengths] of big triangle and small triangle are different, then their sine should differ (*pointing the sine formula she wrote*)…

Table 4.6. Students' definitions of sine and cosine

Cemre	
Defne	
Ebru	
Zafer	

Secondly, when articulating the unit circle, all students expressed the *x*-axis as the cosine axis and *y*-axis as the sine axis without awareness of the correlation between *x*-axis [*y*-axis] and cosine [sine] (see *Table 4.3* and [Defne] Protocol 3 as examples). They preferred to define sine and cosine for an angle but in the first quadrant by a set of geometric procedures within the (*unit*) *circle register* (e.g., lines 1-6 in [Zafer] Protocol 5; lines 3-7 in [Cemre] Protocol 4 and *Table 4.3*). However, Cemre had a trouble to transfer these geometric procedures to sine and cosine for the angles corresponding to the axes such as π/2 and 3π/2 in radians (lines 8-15 in [Cemre] Protocol 4).

[Defne] Protocol 3

Defne: Here is cosine axis (*dragging her pen left and right on the x-axis*) and here is sine axis (*dragging her pen up and down on the y-axis*)
Researcher: Why?
Defne: (*After 5-second pause*) is there any reason for this? Daresay… I don't know why these axes correspond to sine and cosine…

[Zafer] Protocol 5

Zafer: On the unit circle, sine takes values between -1 to 1 (*pointing -1 and 1 on the y-axis simultaneously with his right hand's thumb and index fingers*) or it takes values on the diameter. For example, here (*pointing the point on the unit circle which corresponded to α on his construction in Table 4.3*) sine takes a value (*drawing a perpendicular-dashed line segment from that point to the y-axis*) here (*concretizing the intersection point on the y-axis*).
Researcher: All right. You said the y-axis was the sine axis. Why does it take values only between -1 and 1?
Zafer: Because it takes values on the diameter… …inside the unit circle (*dragging his pen up and down on the y-axis between -1 and 1*)… …for the rest, -1 and 1 comes from here…
Researcher: If the radius was 2-unit, then where would it take values?
Zafer: I think, then, sine would be defined inside the circle… …between -2 and 2 (*dragging his pen up and down on the y-axis approximately between -2 and 2*).
Researcher: Then, would it be again sin(*x*)?
Zafer: Would it be again sin(*x*) (*uttering in a low voice*)… Yes, it would be again sin(*x*).

[Cemre] Protocol 4

Researcher: You said the unit circle comes to your mind when thinking about sine for example. Can you explain?
Cemre: Ok. For example, (*turning to her construction of unit circle in Q4 in Table 4.3*) an angle (*drawing a line segment from the origin to the unit circle in the first quadrant without concretizing*), such as 45 degrees, then, its sine is here (*drawing*

a perpendicular line segment to the y-axis and concretizing the intersection point on the y-axis).
Researcher: Well. You said also, π comes to your mind. Can you explain how to find sin of π/2 from the unit circle, for example?
Cemre: Umm… sin(π/2) is here (*drawing the positive part of the y-axis from the origin to up with the green pencil*)
Researcher: Ok. What about cosin of 3π/2?
Cemre: cos(3π/2) (*uttering in a low voice*)… (*After 11-second pause*) cos(3π/2) is here (*drawing the negative part of the y-axis from the origin to down with the green pencil*).

Thirdly, Ebru preferred to convert her definition of sine and cosine as a set of geometric procedures within the (*unit*) *circle register* into the *symbolic register* as coordinates of a point on the unit circle [respectively, *y*-component and *x*-component] (*Figure 4.4*). However, it was observed that she made an over-generalization of this concept definition within the *symbolic register* to an arbitrary ordered pair. In other words, she thought an arbitrary ordered pair's abscissa as cosine and ordinate as sine. For instance, when converting the ordered pairs in the Q6 within the *symbolic register* to the corresponding points in the *graphical register,* she confused about their ordinates that were defined by a function with respect to cosine as a consequence of her over-generalized reasoning about the ordinate of an ordered-pair as sine (lines 12-14 in [Ebru] Protocol 3). It may due to her consideration of the whole *x*-axis as the cosine axis and the whole *y*-axis as the sine axis without awareness of the correlation between the *x*-axis [*y*-axis] and cosine [sine].

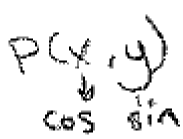


Figure 4.4. Ebru's conversions about definition of sine and cosine from the (*unit*) *circle register* into the *symbolic register*

Finally, when defining sine and cosine on the unit circle, all students' mental images related to sine [cosine] of an angle were the point on the *y*-axis [*x*-axis] instead of the reference right triangle's opposite leg [adjacent leg] within the (*unit*) *circle register* (lines 1-6 in [Zafer] Protocol 5; lines 3-7 in [Cemre] Protocol 4 and *Table 4.3*). This mental image led students to encounter troubles in reasoning about coordinates of a point on a *non-unit* circle (lines 7-16 in [Zafer] Protocol 5). This implies students' compartmentalized concept definition images within the right triangle context and the unit circle context.

4.1.5. On values of trigonometric functions

All students reasoned that sine and cosine takes their values between -1 and 1 (*Figure 4.3*, Zafer's definition in *Table 4.6*, line 1 in [Zafer] Protocol 5, lines 33-35 in [Defne] Protocol 5). However, it was observed that all of them faced some troubles about ranges of sine and cosine as a consequence of their inabilities to dissociate ranges of sine and cosine functions' basic forms and those of their general forms. Firstly, when converting the ordered pairs in Q6 within the *symbolic register* to the corresponding points in the *graphical register*, Zafer got confused about the positions of their ordinates [which were defined by the cosine function like in *Table 4.4*] on the *y*-axis within the *graphical register* as a consequence of his transformation of his reasoning about the range of cosine function's basic form within the *unit circle register* (lines 1-6 in [Zafer] Protocol 5) onto reasoning about a general form of cosine function (lines 4-9 in [Zafer] Protocol 4), firstly, within the (*unit*) *circle register*, and then, within the *graphical register* (see *Construction of Zafer* in *Table 4.5*). In other words, he determined the position of $\sqrt{3}$ on the *y*-axis as if it was smaller than 1 as a consequence of his reasoning that cosine takes their values on the diameter without considering $\sqrt{3}$ as the value of $2\cos(\frac{\pi}{6})$.

Secondly, when trying to convert the function within the *graphical register* to the *symbolic register* in Q8, although Defne, Ebru and Zafer brought sine and/or cosine functions to their mind considering to the shape of the graph independent from the coordinate plane (line 1 in [Ebru] Protocol 6; line 11 in [Defne] Protocol 5; and lines 4-6 in [Zafer] Protocol 7), they were unable to reason about the function in the *symbolic register* as a consequence of their failure to dissociate the content of the trigonometric functions' graphs in the general form and those in the basic form [which were their first represented forms] in terms of "range". That is, they got confused about the range of the function given by the graph in Q8 concerning whether the range of a function with respect to sine (or cosine) can include only positive values, as well as the greater values than 1 considering the range of the basic forms of the sine and cosine functions within the *graphical register* (lines 11, 16-19 in [Defne] Protocol 5; lines 1-4 in [Ebru] Protocol 6; and lines 4-6 in [Zafer] Protocol 7).

In addition to their troubles about ranges of sine and cosine functions mentioned above, students encountered some other troubles when trying to determine the angle measures corresponding to $-\frac{\sqrt{3}}{2}$ under the $y = sin(x)$ function in Q7. Firstly, a systematic variation of students' performances was observed as a consequence of the variation on the sign of the trigonometric value from positive to negative. Initially, all of them preferred to reason without seeing the negative sign of $-\frac{\sqrt{3}}{2}$, and determined x as 60. Where, while Zafer determined 60 [degrees] verbally according to his memorization in the *symbolic register,* and then, transferred it in radians as $\pi/3$ in written (lines 1-2 in [Zafer] Protocol 6), others preferred to construct a right triangle in determination of 60 [degrees] (*Table 4.7*; lines 1-2 in [Cemre] Protocol 5; line 1 in [Defne] Protocol 4; and line 1 in [Ebru] Protocol 5). However, in determination process for $-\frac{\sqrt{3}}{2}$, except Defne (see *Defne's construction* in *Table 4.8*), all others encountered some troubles arising from difficulties in *conversion* of a trigonometric equation given in the *symbolic register* into the corresponding construction within the (*unit*) *circle register.*

For example, despite of her true explanation about the quadrants in which sine takes negative values (line 2 in [Ebru] Protocol 5), Ebru could construct neither angles' position within these quadrants not their sine values (see *Ebru's construction* in the *Table 4.8*), which may arise from her restricted concept definition image on sine into the first quadrant within the (*unit*) *circle register* (see *On definition of trigonometric functions* heading). As second example, after his reasoning in the *symbolic register* (lines 1-8 in [Zafer] Protocol 6), Zafer constructed three angles within the (*unit*) *circle register*, respectively, –π/3, π/3 and 2π/3 but based only on one dimension [angle measure] instead of two dimensions [angle measure and its sine value], and without considering their symmetrical relations (see *Zafer's construction* in *Table 4.8*). As a result of his unsymmetrical-one dimension construction, his reasoning restricted to only the remembered rules in the *symbolic register* in determining the values corresponding to the negative sine value ([Zafer] Protocol 6). Even though Cemre, as third example, constructed a symmetrical structure in the third quadrant with that in the first quadrant (lines 6-19 in [Cemre] Protocol 5), due to her being unable to remember that sine took also negative values for angles in the fourth quadrant (lines 2-4 and 35-37 in [Cemre] Protocol 5), she could not construct the reflection of the trigonometric equation within the fourth quadrant (see *Cemre's construction* in *Table 4.8*).

Secondly, when trying to determine the real x values corresponding to $-\frac{\sqrt{3}}{2}$ under the $y = sin(x)$ function in Q7, a systematic variation of students' performances was observed as a consequence of the variation on the sign of the angle value from positive to negative. In other words, they encountered troubles in *treatment* of the angles with the same principal angle as both positive and negative within the (*unit*) *circle register*. For an initial example, even if Zafer was seemed to be able to construct a negative angle –π/3 within the (*unit*) *circle register* (see *Zafer's construction* in *Table 4.8*), his determination of –π/3 was based on his reasoning with respect to the rules within the *symbolic register* (lines 1-4 in [Zafer] Protocol 6), as well as his construction of it within the (*unit*) *circle register* was as if a negative angles' prototypical example. Besides, he was unable to reason about this angle's positive equivalent forms, which may arise from his concept image of angles as static rather than as dynamic turning. For another example, although Cemre and Defne determined and constructed 240 [degrees] within the (*unit*) *circle register* as a positive value providing $sin(x) = -\frac{\sqrt{3}}{2}$ equation through considering angles as dynamic turning (lines 6-19 in [Cemre] Protocol 5; and lines 11-16 in [Defne] Protocol 4), they were unable to reason about this angle's negative equivalent forms as a conclusion of either their reasoning about negative angles based solely on the memorized-rules without any reasons within the *symbolic register* (lines 18-27 in [Defne] Protocol 4) or their restricted concept images on angle measures only to principal measures (see *On angle measures* heading).

Final trouble, which students encountered when trying to determine the real values corresponding to $-\frac{\sqrt{3}}{2}$ under the $y = sin(x)$ function in Q7, was their dominated concept images on the sign of the sine value with respect to the quadrants by the memorized-rules without any reasons within the *symbolic register*. In other words, it was observed that they were unable to recognize when to use, why to use and how to use about these rules. For example, the identities between sine and cosine [such as, $sin\left(x + \frac{\pi}{2}\right) = \cos(x)$ and $sin\left(x + \frac{3\pi}{2}\right) = -\cos(x)$] caused Zafer's reasoning about the sine of $2\pi/3$ in the second quadrant as negative (lines 10-14 in [Zafer] Protocol 6), Ebru's confusion about reasoning on sine of an angle in the third quadrant with respect to the angle in the first quadrant whose reference angles were the same (lines 3-11 in [Ebru] Protocol 5), as well as Cemre's unsureness about her solution despite of true reasoning within the (*unit*) *circle register* (lines 17-24 in [Cemre] Protocol 5).

Table 4.7. Students' initial reasoning in Q7

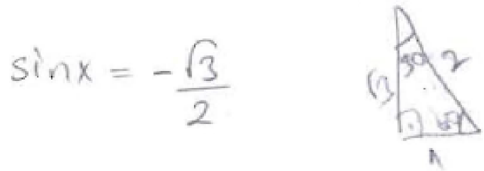
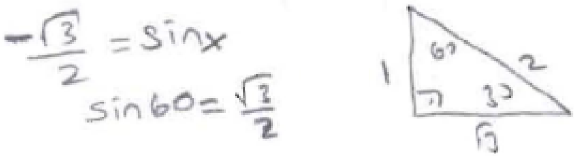
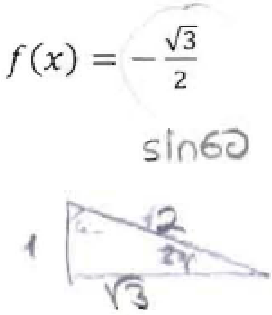
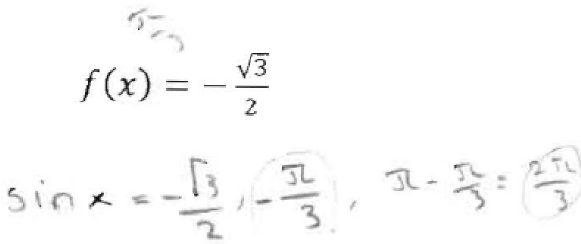
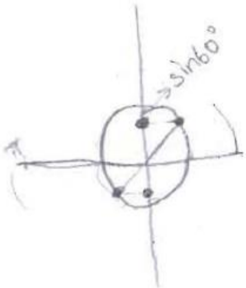
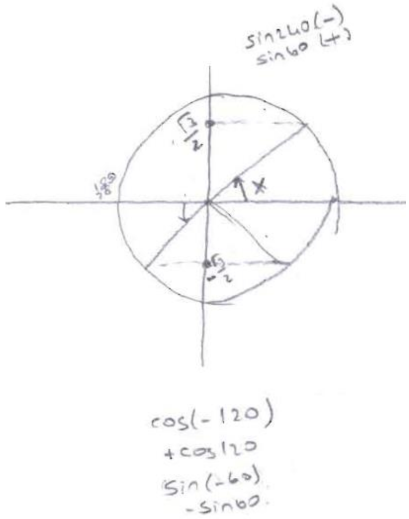
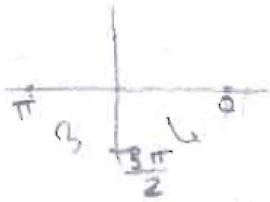
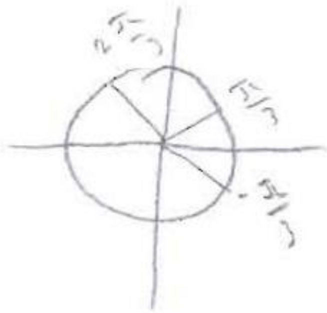
Cemre's initial way of reasoning	
Defne's initial way of reasoning	
Ebru's initial way of reasoning	
Zafer's initial way of reasoning	

Table 4.8. Students' second step reasoning in Q7

Cemre's construction	
Defne's construction	
Ebru's construction	
Zafer's construction	

[Cemre] Protocol 5

Cemre: Now, I think how to do about this minus… …If it is sin(60), sin(60) isn't $-\sqrt{3}/2$, it is $\sqrt{3}/2$. (*After 14-second pause*) sin(60) is in the first quadrant, and in the first quadrant sign of sine is positive… In the third quadrant, negative… Is it 240 degrees?

Researcher: How do you find 240 degrees?

Cemre: If we think on the unit circle (*drawing the unit circle like in Table 4.8*), I am here as sin(60) (*constructing an angle on the unit circle and concretizing the projection point on the y-axis of the point on the unit circle corresponding the 60-degree angle*). Because of the negative sign, I should be in the third quadrant (*putting her pencil on the circle's third quadrant*). We assume here as 180 (*putting her pencil on the intersection point with the negative x-axis*). Since here (*pointing the point on the circle's first quadrant which she constructed as 60 degrees*) is 60, 180 plus 60, that is, 240 (*drawing the line segment from the point on the circle's first quadrant through the origin to a point on the circle's third quadrant*). That is, here is 60 (*turning her pencil from the positive x-axis to the 60 degrees in the positive direction*), here is also 60 (*turning her pencil from the negative x-axis to the 60 degrees in the positive direction*), here is 180 (*putting her pencil on the intersection point with the negative x-axis*), 180 plus 60, that is 240… But I am not sure if it is correct...

Researcher: Why?

Cemre: Well, I am not sure if I am correct to adding 180 to 60… Umm, 240 aren't familiar to me… Also, sine and cosine changed their names as cosine and sine when an angle exceeding π/2 and 3π/2, but here [in this solution] we didn't change sine as cosine.

Researcher: Why do the names of sine and cosine change when an angle exceeding π/2 and 3π/2?

Cemre: (*After 6-second pause*) I don't know the reason.

Researcher: Ok. Well, are there any other *x*-values to provide the equation?

Cemre: When I think as cos(30), cos(30) is equal to sin(60).

Researcher: You seek the solution for this equation (*pointing her equation in Table 4.7*), do you?

Cemre: (*Pausing 11 seconds*)

Researcher: Ok. In which quadrants does sine take negative values?

Cemre: Here, all [trigonometric functions] are positive (*pointing the first quadrant in her pencil*), here are negative (*pointing the third quadrant in her pencil and turning her view from the question booklet to upward*)… Umm... I try to remember, I had memorized them. (*After 5-second pause*) I cannot do anything about this question.

[Defne] Protocol 4

Defne: Now, for being$\sqrt{3}/2$, it should be sin(60). But as a result of negative sign… (*After 5-second pause*) for being negative [sine value], sine in second quadrant is positive, in first quadrant is positive (*uttering through turning her view from the question booklet to upward*), then, it should be in third and fourth. Oh no… I am confused… (*After 11-second pause, she drawn a unit circle*) $\sqrt{3}/2$ is like that (*constructing an angle in the first quadrant so as to correspond whose sine to* $\sqrt{3}/2$). $-\sqrt{3}/2$ could be here and also here, (*constructing angles symmetrically in the third and fourth quadrants so as to correspond whose sine to* $-\sqrt{3}/2$) They come up to same point (*dragging her pencil from left to right on the chord between the points on the circle corresponding to the angles in third and fourth quadrants and pointing* $-\sqrt{3}/2$ *on the y-axis*). Then here is 180 (*writing 180 near the intersection point of the circle with the negative x-axis*) and also 60 degrees comes from here (*drawing a curved arrow in the positive direction in third quadrant*) that is, 240… That is, for being negative [$-\sqrt{3}/2$], *x* should be 240 (*writing it symbolically up-right side of the unit circle like in Table 4.8*). So, the lowest positive *x* is 240.

Researcher: Ok. What about the highest negative *x*?

Defne: Negative *x* (*stopping speaking and looking to me*).

Researcher: Have you ever heard about negative angles?

Defne: Yes. For example, cos(-120) absorb negative sign and it becomes plus cos120, but sin(-60) gives negative sign outside and it becomes –sin60 (*writing them like in Table 4.8*). So an angle has never been negative, we have taken the negative sign outside the sine.

Researcher: (*Pointing her writing -60 in the sin(-60)*) You wrote -60 here. So, an angle become negative, does it?

Defne: Yes, but we take it outside like that (*pointing the negative sign of the –sin(60)*), so the angle is not negative anymore (*pointing the 60 under the sine*).

[Ebru] Protocol 5

Ebru: $\sqrt{3}/2$ is equal to sin60 from here (*pointing her construction 30-60-90 right triangle*). But the negative sign exists here. Sine is negative in third and fourth [quadrants]. 120 is in the second quadrant, 180 is not appropriate. I am confused… (*After 8-second pause, constructing a coordinate plane without unit circle like in Table 4.8*) integer multiples of 60, it can be 120, or 180… Do we need to change sine as cosine for transferring sin(60) into the third quadrant?

Researcher: Why do we need this change?

Ebru: I don't know why, but we changed sine and cosine when an angle exceeding 90 and 270. For example,$\sin(\frac{\pi}{2}+x)$ is equal to $\cos(x)$.

Researcher: But in this question there is not such a term related to π/2 or 3π/2.

Ebru: Yes. (*After 7-second pause*) I don't want to think about this question.

[Zafer] Protocol 6

Zafer: How to do… sin(*x*) is equal to $-\sqrt{3}/2$ (*writing it as an equation like in Table 4.7*). Where, our angle is 60. 60, so, is π/3, and even –π/3 because sine gives the negative sign outside, yes, it is right (*writing it near the equation and getting it into the circle*). Therefore, the highest negative value is –π/3. Besides, π minus π/3, that is, 2π/3 is another value. (*Turning the question and reading in a low voice*). I

methinks that the highest negative value is this (*getting π/3 again into the circle*), the lowest positive real value… …what the lowest positive real value is… …umm… (*After 7-second pause*) now, if we take π/3, it isn't $-\sqrt{3}/2$. We have to obtain negative… (*Constructing a unit circle and angles, respectively, –π/3 in the fourth quadrant, π/3 in the first quadrant and 2π/3 in the second quadrant*) I methinks 2π/3 is the lowest positive real value.
Researcher: Is the value of sine for 2π/3 is negative?
Zafer: Well, sine changes as cosine after π/2 (*pointing the intersection point of the unit circle with the positive y-axis*), and cosine is negative in the second quadrant.

To sum up, students' inabilities to dissociate the ranges of trigonometric functions' basic forms from those of their general forms, the content of the trigonometric functions' graphs in the general form and those in the basic form, as well as students' inabilities to associate the meanings of trigonometric value and trigonometric identities both between and within different representational registers were cognitive obstacles in students' concept images on trigonometric functions.

4.1.6. On periodicity

When trying to convert the function within the *graphical register* to the *symbolic register* in Q8, although all of the students reasoned about the function (which was given by the graph in Q8) to be repeating within the *graphical register* (lines 1-5 in [Cemre] Protocol 6; lines 1-4 in [Defne] Protocol 5; line 6 in [Ebru] Protocol 6; and line 55 in [Zafer] Protocol 7), none of them was able to reason about the period of the function truly in the *symbolic register*. Initially, Cemre got confused about the meaning of the period in the *symbolic register* as a consequence of her reasoning about the period as the number of repetitions (lines 5-8 in [Cemre] Protocol 6). Secondly, Defne got confused about the period of the function concerning whether the period of a function with respect to sine (or cosine) could be different from 2π as a consequence of her transformation of her reasoning about the period of the basic forms of the sine (or cosine) functions as 2π onto her reasoning about that of their general forms within the *graphical register* (lines 5-15 in [Defne] Protocol 5). Next, Ebru got confused about the meaning of the period in the *graphical register* as a consequence of her reasoning about the period in the *symbolic register* as the abscissa of the end point on the right side of the graph (lines 6-14 in [Ebru] Protocol 6). Finally, Zafer got confused about the meaning of the period in the *symbolic register* despite of his reasoning about the period truly in the *graphical register* as regular intervals in which the function repeats its values (lines 55-63 in [Zafer] Protocol 7).

[Cemre] Protocol 6

Cemre: Graphs' ascent indicates its increase (*dragging her pen on the graph from the origin to the first peak point up-right ward*), its descent indicates its decrease (*dragging her pen on the graph from the first peak point to the lowest point down-right ward*). It is a continuously increasing and decreasing graph (*drawing sine wave on the air with her pencil*). That is, it repeats…

Researcher: What is its period of repetition?

Cemre: (After 6-second pause) I don't remember how to find its period… Is it the repetition number?

Researcher: Do you remember how you find trigonometric functions period?

Cemre: I don't remember exactly, but we were dividing 2π by a number.

Researcher: What is the aim of division?

Cemre: I methinks that this is a rule… (*After 7-second pause*) I don't know the reason…

[Defne] Protocol 5

Defne: Well, this is a repeated function's graph… If we separate here (*putting her left hand's index finger on the up side of the y-axis and dragging her pen on the y-axis from up to down*) and here (*dragging a vertical line cutting the x-axis on 8 and then erasing it*)… It repeats…

Researcher: What is the period of the repetition?

Defne: I guess… …it should be 2π… (*After 7-second pause*) ok, alright, let us think its [the graph's] repeated part is here (*dragging her index fingers from up to down respectively on the y-axis and the line corresponding to x=8*) or here (*dragging the vertical lines corresponding to x=-16 and x=-8 in Figure 4.2*).

Researcher: Ok. Then, what is the period?

Defne: 2π… …because this graph is resemble to sine and cosine, and their periods are 2π... Umm… (*After pausing 7-second through looking to the graph's left side regarding the y-axis*) hmm, ok, it repeats from zero (*pointing the origin with her pen*) to 2π (*pointing the point corresponding to -8 on the x-axis*)… Yes, I am sure it [period] is 2π…

Researcher: Do you think this is a function related to sine or cosine?

Defne: Umm… Actually, I don't think so… If numbers were not so (*dragging her pen from 6 to 0 on the y-axis up and down*) I would think it [the function given by the graph] as one of sine or cosine… (*After pausing 13-second through looking to the graph*) in the first place, those are the values of π, we were thinking them as π (*dragging her pen on the x-axis from left to right*)

Researcher: But here, there is no value regarding π.

Defne: Yeah, π is not mentioned here (*stopping speaking and looking to me*).

Researcher: What is your opinion, if mentioned, π should be where?

Defne: Well, those would be the values of π (*dragging her pen on the x-axis from left to right*), also those would be functions' values with respect to the values of π (*dragging her pen from 6 to 0 on the y-axis up and down*). (*Pausing during 8-second and looking to the graph*).

Researcher: In this graph, scaling on the *x*-axis is like 5, 10, 15… If you try to show the position of π on the *x*-axis, where do you put it?

Defne: Umm… This graph looks like sine and cosine… They take their values between zero and 2π (*pointing respectively the origin and the point 8 on the x-axis with his pen*)… (*Drawing Figure 4.3*) they take their values between -1 and 1 for angles between zero and 2π (*drawing a curved arrow in the positive direction through turning a full round from zero radian to 2π radian Figure 4.3*)… That is, zero and 2π (*looking to her construction in Figure 4.3*)… That is, zero and 2π are same… So here is 2π (*putting her pen on the origin and writing 2π on the graph near the origin*).

Researcher: Is 2π equal to zero?

Defne: Yeah, equal, because they refer the same position (*dragging her pen on the x-axis from the origin to the right*)…

[Ebru] Protocol 6

Ebru: It [graph] little resembles to sine but it is not sine… Because sine goes to negatives also (*dragging her index finger in a curved way like sine wave below the x-axis*)… I mean sine is between -1 and 1, but this is between zero and 6 (*pointing zero and 6 on the y-axis with her index finger*)

Researcher: To which function can this graph belong?

Ebru: (*Pausing 5-second through looking to the graph*) this repeats… (*Pausing during 8-second and looking to the graph*).

Researcher: What do you think about its period?

Ebru: Sixteen…

Researcher: How do you determine this value?

Ebru: Umm… Last point that sine can go to is 2π (*holding her hands upright and parallel to indicate an interval*)… So, also here, that is 16, (*pointing the right-end point of the graph corresponding 16 on the x-axis in Figure 4.5, and writing 16 here*) is the last point of the graph… I think the period is 16…

[Zafer] Protocol 7

Zafer: (*Writing as ordered pairs maximum and minimum points of the graph's right side with respect to the y-axis in Figure 4.6, and pausing 7 seconds*)

Researcher: Does it familiar to you?

Zafer: In fact, it resembles sine or cosine… but it is not sine or cosine because they take negative values also. (*Stopping to speak and looking to the graph during 14 seconds*)

Researcher: Can you draw sine graph for me?

Zafer: Ok. (*Constructing a table like in Figure 4.7*) Because period of sine is 2π, we take these values up to 2π (*creating x values*). $\sin(0)$ is zero, $\sin(\pi/2)$ is 1, $\sin(\pi)$ is 0, $\sin(3\pi/2)$ is minus 1, and $\sin(2\pi)$ is zero (*filling the first raw in Figure 4.7*) I can draw it like that (*constructing the coordinate plane and showing ordered (x, sin(x)) pairs from the table on it, and then, combining them with a curve like in Figure 4.8*).

Researcher: Does this give you a clue about the graph in the question?

Zafer: Here (*pointing the table*) we gave values for *x* and obtained corresponding sine values with respect to them. Here also (*pointing the graph*), in this function, we write 4 for *x*, it [function] give us 6, f(12) is equal to 6, and f(8) is equal to zero (*writing f(4)=6, f(12)=6 and f(8)=0 below the graph like Figure 4.9*)

Researcher: What does sine function give for 4?

Zafer: (*Waiting without speaking 5 seconds*)

Researcher: Where is the position of 4 here (*pointing his constructed x-axis in Figure 4.8*)?

Zafer: 4 is out of the unit circle, so it may be around here (*pointing a position right side of the 2π on the x-axis*)

Researcher: Well, what value does sine function take for 4?

Zafer: It takes zero?

Researcher: How?

Zafer: If we extend the line like that (*extending positive x-axis towards right side*), around here (*concretizing a point on the x-axis which is approximately same distance but opposite direction with the origin to the point corresponding 2π, and writing 4π below this point like in Figure 4.8*) is 4π, that is, 4, therefore, $\sin(4\pi)$ is zero.

Researcher: Is 4 is equal to 4π?

Zafer: I thought 4π as 4.

Researcher: Instead of thinking 4π as 4, do you think about exact position of 4?

Zafer: Umm, how can I think? Because 2π corresponds to 1 on the *x*-axis (*pointing the intersection point of the unit circle with the x-axis's positive side which he constructed like in Table 4.3*), then, I methinks 4 is around here (*pointing a point on the x-axis between 2π and 4π and writing 4 below the point*).

Researcher: Is 2π equal to 1?

Zafer: If we take π as 180… (*After 5-second pause*) but, on the unit circle, 2π corresponds to 1… It seems to me that where [on the *x*-axis] 180 is very big… So, 4 should be around here (*pointing again 4 on the x-axis*)…

Researcher: Ok. Now, just for you maybe catch a hint to solve the question. You can draw $y=\sin(x)$ function's graph. Can you draw $y=2\sin(x)$ function's graph for me?

Zafer: Ok. Then, we get these values twice. So, zero, two, zero, minus two and zero (*filling the second raw of the table in Figure 4.7*). I can draw it like that (*showing* *new ordered pairs corresponding to (x, 2sin(x)), and then, combining them with a curve like in Figure 4.8 by navy blue color*).

Researcher: Does it give you an idea about how can you change a form of the graph related to sine?

Zafer: Well, we extend twice the graph for *y* values but *x* values are still same. But it is not give me a hint about this graph.

Researcher: Ok. When constructing the *x* values on the table, you said that the period of sine is 2π. Do you think about this graph belongs to a periodic function?

Zafer: In fact, it repeats. So, a period exists.

Researcher: What is it?

Zafer: Its period is… Here is the repeated part (*circling the first wave of the graph between [0,8]*). That is it repeats between zero and eight (*pointing his pen these points on the x-axis*).

Researcher: Then, what is its period?

Zafer: Between zero and eight.

Researcher: Period is a number or an interval?

Zafer: An interval like that (*writing closed interval [0,8] like in Figure 4.9*).

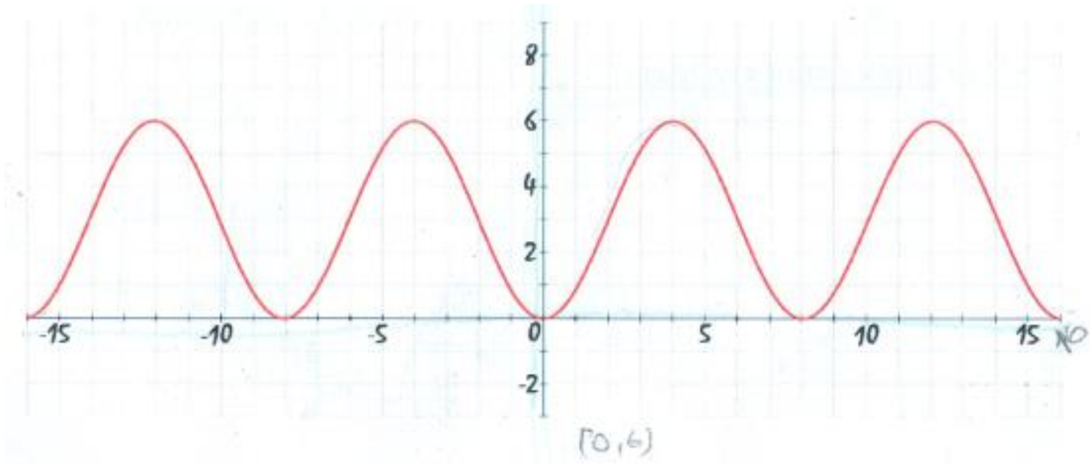


Figure 4.5. Ebru's reasoning about the repetition of the graph in Q8

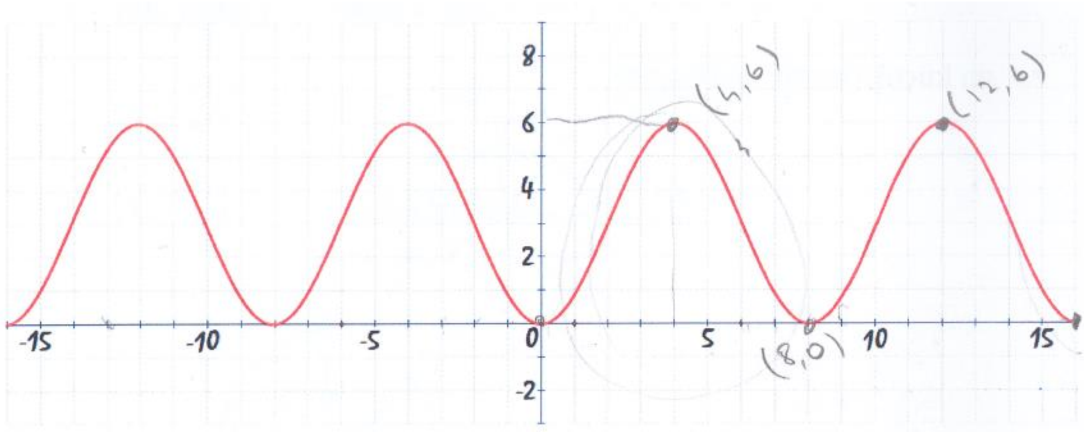
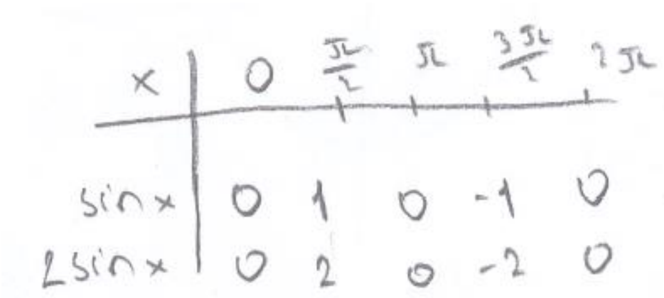


Figure 4.6. Zafer’s reasoning about the repetition of the graph in Q8



x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
sin x	0	1	0	-1	0
2sin x	0	2	0	-2	0

Figure 4.7. Zafer’s construction of *y*=sin(*x*) and *y*=2sin(*x*) in the table

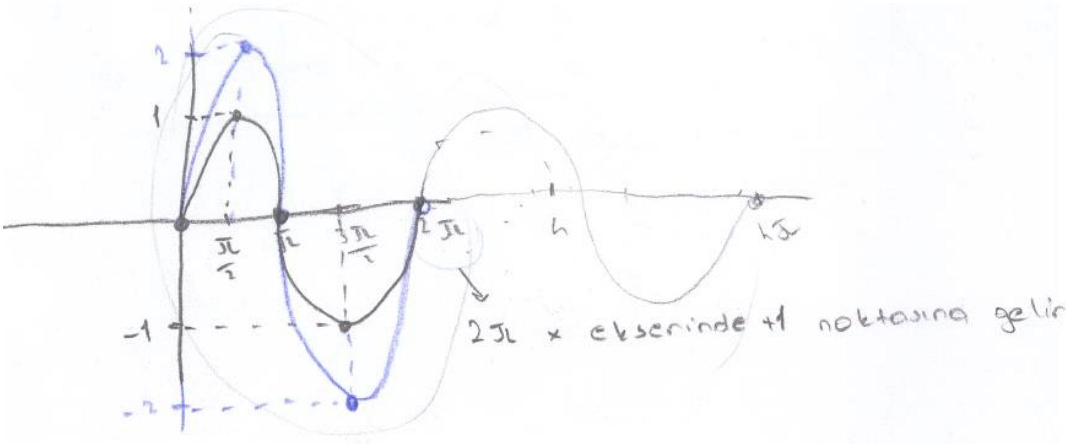


Figure 4.8. Zafer’s construction of *y=sin(x)* and *y=2sin(x)* functions’ graphs

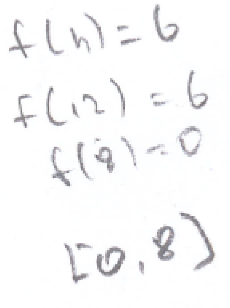


Figure 4.9. Zafer’s reasoning in the *symbolic register* about the function in Q8

4.2. Summary of Students’ Initial Concept Images

Cognitive analyses of the initial interviews revealed that students’ concept images included critical troubles on *foundational concepts* related to trigonometric functions. To begin with, their concept [definition] images on functionality were restricted to being polynomial functions in the *symbolic register*, as well as their mental images on functions’ visual representations were dominated by the *graphical register*.

In addition, students’ concept definition images on angles included an intuitive relation between openness and measure of an angle. However, they also included a trouble on what was the measured part to determine an angle’s measure. That is to say, in spite of their circular constructions inside of an angle [such as arcs and arc sectors], they were unable to associate these circular constructions with angle measure. Nevertheless, they reasoned about angle measure intuitively not to vary with respect to variations of radii of these circular constructions. Except Defne, other students’ mental images on angles were *static* instead of *dynamic turning*.

Another trouble on students’ concept images related to angle measure was based on radian measure unit. None of the students was able to reason about the meaning of radian apart from degree measures. That is to say, the meaning of radian measure was dominated by degree meaning and restricted only into transformations between degree and radian measures within the *symbolic register*. Their transformations between degree and radian implied that students' understanding of the relation between degree and radian was as if a computational equality without the degree and radian notations [such as π=180 and $\frac{\pi}{6} = 30$] instead of a proportional equality with degree and radian notations [such as πR=180° and $\frac{\pi}{6}$ R=30°]. As a consequence of this understanding, π became a source of troubles on students' concept images.

On the one hand, it was observed that in spite of their reasoning that the same π notation must be refer to the same object within the *symbolic register,* all students dissociated π in and out of the trigonometry context in terms of its real value. In other words, they treated π in the *symbolic register* as if it had two different real values; i.e., "180" in the trigonometry context and about 3.14 out of the trigonometry context. Therefore, within the *symbolic register*, "π" had a potential to become a *cognitive conflict factor* in and out of the trigonometry context when considered simultaneously.

On the other hand, many other troubles of students were observed based on π as a consequence of the change of the source register from *symbolic* to *graphical* and as a consequence of the variation on the scaling of the *x*-axis from numbers regarding π to the real values. All students' reasoning on π in the *graphical register* was constrained by the trigonometry context. In making sense of π within the *graphical register*, Cemre and Defne preferred to transfer their reasoning on π within the *symbolic register* in the trigonometry context (i.e., π as equal to 180) to that out of the trigonometry context. That is, they located π in the *graphical register* on 180 on the *x*-axis. Conversely, Ebru and Zafer preferred to transfer their reasoning on π within the (*unit*) *circle register* and considered respectively *π=-1* and *2π=1* in the *graphical register*, respectively. That is to say, they were unable to dissociate two meanings of the intersection points of the unit circle with the *x*-axis [i.e., the points (-1,0) and (1,0)] as an **abscissa** corresponding to these points within the *graphical register* and as an **angle** in radians corresponding to these points within the *unit circle register*. This arose from their mental images on angles as points on the unit circle instead of corresponding arcs. Besides, when trying to convert her reasoning on π within the (*unit*) *circle register* into the *graphical register,* Defne considered origin as 2π on the *x*-axis based on their same positions when referring to angles in radians within the (*unit*) *circle register*. It means that she was unable to dissociate the meaning of the equivalence between “zero” and 2π in radians from the equality between them within the *graphical register*, which arose from her dominated concept image on principal angles despite of her mental image on angles as dynamic turning. Consequently, students’ reasoning on π in different representational registers indicates that “coordinate plane” became a *cognitive conflict factor* when the position of π on the *x*-axis is considered simultaneously within the *graphical register* and the (*unit*) *circle register*.

Students’ initial attempts to define sine [cosine] were based on the right triangle context. They defined sine [cosine] as ratio of opposite [adjacent] side to hypotenuse. However, none of the students was aware that this ratio for an angle in a right triangle were the same as that in all similar right triangles. This unawareness arose from their reasoning about sine and cosine within the right triangle context as calculations instead of ratios obtained from proportions in the similar right triangles.

Next, when trying to define sine [cosine] in the (*unit*) *circle register,* they preferred to stay in the first quadrant and define by a set of geometric procedures including drawing an angle in the first quadrant, concretizing its reference point on the unit circle, drawing a dashed-perpendicular line segment from this point to the *y*-axis [*x*-axis], and concretizing the intersection point of this segment with the *y*-axis [*x*-axis]. However, Cemre had a trouble to transfer these geometric procedures to sine and cosine for the angles corresponding to the axes such as π/2 and 3π/2 in radians. It may arise from their mental images related to sine [cosine] of an angle that were the point on the *y*-axis [*x*-axis] instead of the reference right triangle’s opposite leg [adjacent leg] within the (*unit*) *circle register*. Furthermore, Ebru preferred to convert her definition of sine [cosine] as a set of geometric procedures within the (*unit*) *circle register* into the *symbolic register* as ordinate [abscissa] of a point on the unit circle. Unfortunately, she thought an arbitrary ordered-pair’s ordinate [abscissa] as sine [cosine] through making an overgeneralization of the ordered-pair definitions of sine and cosine. It may due to their consideration of the whole *x*-axis [*y*-axis] as the cosine [sine] axis without awareness of the correlation between the *x*-axis [*y*-axis] and cosine

[sine]. Students' mental images on sine [cosine] as a point on the *y*-axis [*x*-axis] as well as their consideration of whole *y*-axis [*x*-axis] as sine [cosine] axis led them to encounter troubles in reasoning about ordinate [abscissa] of a point on a *non-unit* circle as sine [cosine].

Finally, students had troubles in reasoning about angles with the same principal measure through positive and negative equivalent measures as a consequence of their reasoning about negative angles based solely on the memorized-rules without any reasons within the *symbolic register* as well as their restricted concept images on angle measures only to principal measures. Moreover, their concept images on trigonometric functions were based only on their basic forms. They had troubles on reasoning about the general forms of trigonometric functions based on their ranges and values especially within the (*unit*) *circle register* and the *graphical register*. That is to say, they were unable to dissociate, for examples, a sinusoidal graph from the graph of $y=\sin(x)$ within the *graphical register,* as well as the ordinate [abscissa] of a point on the *non-unit circle* from the ordinate [abscissa] of a point on the *unit circle* in terms of sine [cosine] within the (*unit*) *circle register*. Furthermore, students' concept images on the period concept included crucial troubles within the different representational registers as a consequence of their problematic concept definition images on the periodicity. None of the students was able to appropriately associate the meaning of the repetition in the *graphical register* with the meaning of the period in the *symbolic register*.

CHAPTER 5

RESULTS FROM TEACHING EXPERIMENT: PART 1

In this chapter, developments of students' understanding on *foundational concepts* related to trigonometric functions (see Chapter 4) is presented from cognitive analyses results of the teaching experiment's first part from each pair [Cemre&Zafer and Defne&Ebru] so as to show the variation of students' concept images related to trigonometric functions' basic forms with respect to their prior ones as a result of the instruction in the GSP environment. The aim of this chapter is to provide the living models of students' concept images on trigonometric functions' basic forms during the first part of the teaching experiment (see *Instructional Design of This Study* sub-heading in Chapter 3 for detailed description of first part of the designed-instruction). This part of the study represents *trigonometry of students* during the first 7 episodes of the teaching experiment in terms of the effect of the dynamically-linked *conversions* of trigonometric functions between representational registers on students' *recognition*, students' *cognitive obstacles* during the experimentation of *recognition* tasks of the designed-instruction, as well as the role of the dynamic and linked representations on students' overcome *cognitive conflict factors* and *obstacles*.

5.1. Development of Students' Concept Images on Trigonometric Functions (Basic Forms[28]) throughout Teaching Experiment

5.1.1. Development of students' concept definition images

Regarding the initial interview results, the researcher determined that students' concept definition images related to trigonometric functions were included two major troubles. First of all was on trigonometric functions' definitions. Students' concept definition images on trigonometric functions had compartmentalized in the unit circle context and right triangle context, as well as troubled in terms of their well-definedness in each context. Second major trouble was on angle definition. While students' concept definition images on angles included an intuitive relation between angles' openness and measure, they also included some troubles arising from unawareness about what were the measured part of an angle and the unit of measurement in determination of its measure. Therefore, the first four tasks were designed to provide students with integrated and well-defined concept definition images on trigonometric functions in both contexts, as well as to provide meaningful objects and units to measure an angle but after the awareness about the importance of angles for trigonometric functions.

5.1.1.1. On trigonometric functions

In the first task, after provided with familiarity on GSP usage, the major focus in terms of trigonometric functions was to provide students with well-defined concept definition images on sine and cosine in the right triangle context through considering trigonometric ratios on the similar right triangles. For this reason, initially, it was spoken about "similarity" on the similar right triangles (*Figure 5.1*) in order to observe (if exist) students' troubles on it because their concept images on *similarity* were not investigated throughout the initial interviews. When discussing the similar right

[28] See *Definitions of Terms* heading at the end of *Introduction* chapter.

triangles, students correctly associated their equal angles and proportional corresponding sides in the *language register* (e.g., lines 1-15 in [Defne & Ebru] Protocol 1), as well as truly expressed in the *symbolic register* three ratios between proportional corresponding side lengths so that these ratios were equal to the similarity ratio (e.g., lines 17-30 in [Defne & Ebru] Protocol 1). Moreover, they generated these ratios in GSP environment, and reasoned coherently about them under the variation of the similar right triangles' acute angles and side lengths (e.g., lines 37-54 in [Defne & Ebru] Protocol 1). However, none of the students mentioned the equality between the ratio of two arbitrary sides from one right triangle and the ratio of their corresponding sides from the other [in other words, the equality of trigonometric ratios between the similar right triangles] (e.g., lines 1-11 in [Cemre & Zafer] Protocol 1) until the researcher's recommendation for them to consider the ratios of two arbitrary sides and that of their correspondences between similar right triangles (e.g., lines 12-14 in [Cemre & Zafer] Protocol 1). When having calculated with GSP an arbitrary ratio of two sides of one of the similar right triangles and that of their corresponding sides of the other triangle, all students realized the equalities of these both (e.g., lines 15-19 and 27-30 in [Cemre & Zafer] Protocol 1). Besides, it was not difficult for them by the aid of the drag-and-drop option of GSP to realize both remaining equal to each other under the variations, as well as their remaining equal to the corresponding ratio of any similar right triangle under all variations except for angle measures (e.g., lines 20-26 in [Cemre & Zafer] Protocol 1). However, none of the students associated these ratios with trigonometric ratios unless the researcher's recommendation for them to investigate such ratios on the similar right triangles whose side lengths were labeled as opposite, adjacent and hypotenuse regarding a specified angle (e.g., lines 31-37 in [Cemre & Zafer] Protocol 1). When the ratios appeared on the screen with labels such as adjacent/hypotenuse, hypotenuse/adjacent, adjacent/opposite, etc. all students associated these ratios with trigonometric counterparts (e.g., lines 38-48 in [Cemre & Zafer] Protocol 1). Moreover, when investigating the trigonometric ratios via making dynamic manipulations on similar right triangles by the aid of the drag-and-drop option of GSP, it was observed that all of the students were able to reason coherently about the variation of trigonometric ratios (e.g., lines 49-59 in [Cemre & Zafer] Protocol 1).

However, while Cemre and Zafer were able to generalize this reasoning as the factors affecting the trigonometric ratios apart from GSP (lines 60-63 in [Cemre & Zafer] Protocol 1), Defne and Ebru were not until considering some sine or cosine values of the same angle on the right triangles with different specific hypotenuses (lines 4-14 in [Defne & Ebru] Protocol 2). After enough trials, Defne and Ebru were also able to reach a generalization about the variation of trigonometric ratios (e.g., lines 15-18 in [Defne & Ebru] Protocol 2). At this point, as a consequence of their compartmentalized concept definition images in the right triangle context and the unit circle context that were revealed throughout the initial interviews, the researcher determined that students would encounter troubles on associating legs of a 1-unit-hypotenuse right triangle with sine or cosine. So, she encouraged them to continue their investigations on the right triangle with 1-unit hypotenuse hoping to merge their concept definition images in both contexts. Because GSP did not give the opportunity to obtain exactly a 1 unit hypotenuse through dragging the point *D* (*Figure 5.2*), none of the students was able to exactly realize the equality between the measure of *ED* and the ratio of opposite/hypotenuse [i.e., sin(*C*)]. So, the researcher asked them to consider this case (in which the hypotenuse was 1) on the ratio of opposite/hypotenuse in the *symbolic register* (e.g., lines 19-20 in [Defne & Ebru] Protocol 2). Through focusing this ratio as an equation, initially, one student in each group, Cemre and Defne, reasoned the opposite [side] as equal to the value of the ratio (e.g., line 21 in [Defne & Ebru] Protocol 2). And then, when the researcher changed the value of the[sine] ratio through manipulating the angle, Ebru and then Defne (similarly, Zafer and then Cemre) reasoned the opposite side as equal to sine (e.g., lines 23-34 in [Defne & Ebru] Protocol 2). However, their reasoning like that was in the *symbolic register*. Thus, the researcher determined to scrutinize students' abilities to convert this reasoning in the *symbolic register* onto the unit circle through drawing on a paper a unit circle integrated with a right triangle whose hypotenuse was 1 (*Figure 5.3*). All of the students were able to convert this reasoning in the *symbolic register* into the (*unit*) *circle register* (e.g., lines 35-45 in [Defne & Ebru] Protocol 2), which emerged as a result of teaching experiment (e.g., lines 46-53 in [Defne & Ebru] Protocol 2). In the similar way, they reasoned easily about the adjacent side was equal to the cosine in the (*unit*) *circle register*. After then, the researcher predicted that students would encounter troubles on giving meaning trigonometric ratios as a transition from an angle measure to its corresponding trigonometric value. So, at the end of Task 1, she encouraged students to interpret sine and cosine for both acute angles of the same right triangle hoping to fortify their perceptions about trigonometric ratios as relations between angles (as inputs) and trigonometric ratios (as outputs). Where, initially, students interpreted sine of the angle *C* (*Figure 5.4*) as "sine is ratio of opposite to hypotenuse… …that is, *AB* divided by *AC*". And then, when asked them to interpret "*AB* divided by *AC*" in terms of angle *A*, they interpreted that "…*AB* is adjacent to angle *A*... …so, it is cosine of [angle] *A*". Where, students' language implies that they started thinking about sine and cosine of an angle algebraically rather than arithmetically. At this point, the researcher determined that all of the students were able to correctly define sine and cosine of an acute angle in both contexts (*right triangle* and *unit circle*), as well as merge their compartmentalized concept definition images on sine [cosine] value of an angle in the first quadrant within the (*unit*) *circle register*. Henceforward, the subsequent progress of students' concept images on sine and cosine that emerged as a result of the teaching experiment was presented in terms of the different representational registers under the following respective headings.

[Defne & Ebru] Protocol 1

Researcher: What do you think about these two triangles (*dragging her index finger on the right triangles ABC and EDC on the screen like in Figure 5.1*)?
Defne: They are right… and similar…
Researcher: Why?
Ebru: Their angles are same (*pointing angles A and E, and then B and D respectively from big and small right triangles*)…
Defne: Uh-huh (*nodding her head up and down*).
Researcher: Well, what does similar mean?
Defne: That is, this side (*pointing the AC segment*) is similar to this side (*pointing the EC segment*)… this side (*pointing the BC segment*) is similar to this (*pointing the DC segment*)… …and this (*pointing the AB segment*) is similar to this (*pointing the DE segment*)
Ebru: Well, in the big triangle and in the small triangle, sides, opposite to equal angles, are proportional to each other…
Defne: Uh-huh (*nodding her head up and down*).
Researcher: What does "proportionality" mean?

Ebru: Well… …when they are similar, a similarity ratio exists…
Researcher: What is similarity ratio?
Ebru: For example, the ratio of… …the opposite side of this angle (*pointing the angle A in the big triangle, and then dragging her index finger on BC segment*)… to this angle's opposite side (*pointing the angle E in the small triangle, and then dragging her index finger on DC segment*)…
Defne: When considering these ratios, we are starting always from, for example, big triangle… That is, AC divided by EC is equal to BC divided by DC… as well as equal to AB divided by ED (*pointing the corresponding sides on the screen*).
Ebru: Uh-huh (*nodding her head up and down*). We can consider them also in the reverse direction…
Defne: Ok. In which order we start [to find the similarity ratio]… …others should be taken in the same order...
Ebru: Yeah…
Researcher: Can you calculate these ratios by GSP?
Defne: Ok. The ratio of EC to (*pointing the EC line segment through mouse movement on the figure like in Figure 5.1*)…
Ebru: …to AC…
Defne: Yeah. That is, EC divided by AC… …is equal to ED divided by AB… …and also equal to DC divided by BC…
(*They calculated these ratios after measuring the right triangles side lengths in GSP, respectively.*)
Defne: Yes, it is true. All these ratios are same...
Ebru: Uh-huh…
Researcher: What are the changing components when you are dragging the point D?
Defne: (*Dragging the point D left and right*) in small triangle, side lengths are changing and these ratios are changing (*pointing these measures on the screen*)….
Ebru: Yeah, similarity ratios…
Researcher: Ok. What about the points A and C?
Defne: Now (*dragging the point A up and down*), in both triangles, because this line segment does not move, its measure doesn't change (*pointing the BC line segment through mouse movement*) and right angles don't change… …other than these, all measures change.
Ebru: Similarity ratios don't change also…
Defne: Yes. Now (*dragging the point C left and right*), measures of AB and ED don't change (*pointing the measures of AB and ED line segments*), so, similarity ratios (*pointing ratios*) don't change… …other than these, all measures change.
Ebru: (*Nodding her head up and down*).

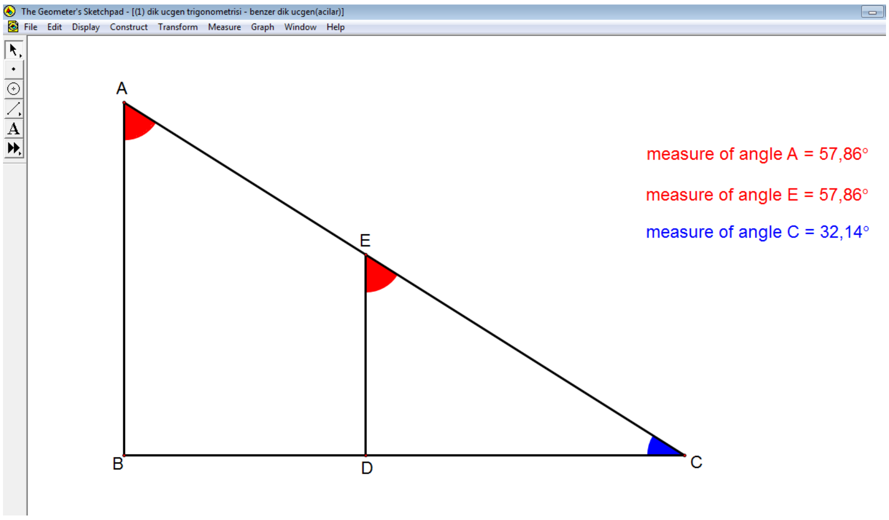


Figure 5.1. Manipulable similar right triangles

[Cemre & Zafer] Protocol 1

Researcher: (*After similar conversations about the proportional ratios in the similar right triangles like in [Defne & Ebru] Protocol 1*) Ok. Are there some other proportional ratios obtained from similar right triangles?

Zafer: We can take them in reverse order… That is, instead of EC divided by AC (*pointing the ratio EC/AC on the screen*), we can take AC divided by EC (*pointing respectively AC and EC on the ratio EC/AC*)… …instead of ED divided by AB (*pointing the ratio ED/AB on the screen*), we can take AB divided by ED (pointing respectively AB and ED on *the ratio ED/AB*) and like that…

Cemre: Yes (*Nodding her head up and down*).

Researcher: Ok. What else?

Cemre&Zafer: (*Looking to the screen, and then, each other without speaking*)

Researcher: For example, if you found the ratio between two arbitrary sides of one of the similar right triangles… …and also the ratio between their corresponding sides of the other triangle, would these ratios are also equal?

Cemre: (*After 6-second pause*) let's find (*opening the calculate option of the measure menu of GSP and calculating respectively the ratios AC/BC and EC/DC, these are corresponding to* $sec(\hat{C})$).

Zafer: Their results are same.

Cemre: Yes, they are same.

Researcher: Well, when dragging the points D, A and C, can you control them?

Cemre: They are still same (*dragging the point D left and right*)… …now, they are changing (*dragging the point A up and down*)…

Zafer: …but they are still equal to each other.

Cemre: Yes. Now, both are again changing, but always equal (*dragging the point C left and right*)...

Zafer: Uh-huh (*nodding his head up and down*).

Researcher: Zafer, can you calculate some other proportional ratios different from these (*pointing the ratios Cemre calculated*)?

Zafer: Ok (*calculating respectively the ratios AB/BC and ED/DC, these are corresponding to* $tan(\hat{C})$). These are also equal to each other.

Cemre: Our similarity ratios were same (*pointing the results of the ratios EC/AC, ED/AB and DC/BC*)… …but these ratios are different (*pointing on the screen her and Zafer's results –which were corresponded to* $sec(\hat{C})$ *and* $tan(\hat{C})$*, respectively*), why it happened? Third [ratio] would be different from them also.

Zafer: (*Looking to the screen without speaking*)

Researcher: Ok. Let's discuss this issue on another GSP file (*opening the GSP file whose static version was like in Figure 5.2*)…

(*They calculated some ratios on both triangles such as adjacent/hypotenuse, hypotenuse/adjacent, adjacent/opposite, etc.*)

Cemre: (*After ratios appeared on the screen with labels such as adjacent/hypotenuse, hypotenuse/adjacent, adjacent/opposite*) ah! Adjacent divided by hypotenuse is cosine (*pointing the label of the ratio and looking to Zafer*).

Zafer: (*Nodding his head up and down*).

Researcher: Well, was cosine in small triangle equal to cosine in big triangle for the angle C (*figuring respectively the small and big right triangles on the screen*)?

Cemre: Yes, we found this from small triangle and this from big triangle (*pointing two ratios labelled as adjacent/hypotenuse on the screen*)… They are same.

Zafer: Yes.

Researcher: Ok. What are the changing components when you are dragging the point D?

Cemre: (*Dragging the point D left and right*) lengths of small triangle changed… …similarity ratios changed… …but these ratios didn't change (*pointing trigonometric ratios*).

Zafer: Yes.

Researcher: Well, what about the points A and C?

Zafer: (*Dragging the point A up and down and the point C left and right*) then these ratios changed (*pointing trigonometric ratios*) but they are still equal to each other (*pointing two ratios adjacent/hypotenuse [hypotenuse/adjacent and adjacent/opposite] obtained from two similar right triangles*).

Cemre: Uh-huh (*nodding her head up and down*).

Zafer: That is, their changes are dependent only on [the angle] C.

Cemre: Yes. That is, they don't change when lengths change, they change only when angle changes.

Zafer: Yes.

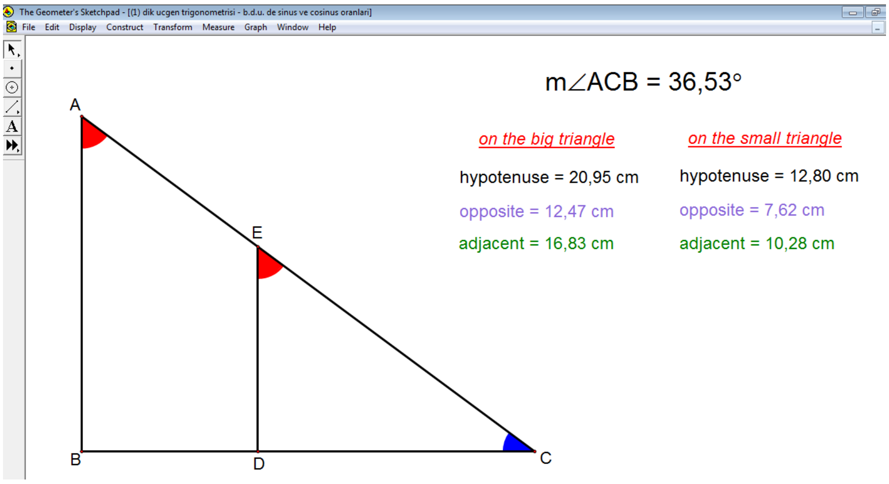


Figure 5.2. Similar right triangles whose side lengths were labeled as opposite, adjacent and hypotenuse regarding the angle *C*

[Defne & Ebru] Protocol 2

Defne: (*Pointing the ratios with labels such as adjacent/hypotenuse, hypotenuse/adjacent, adjacent/opposite*) ah, these ratios from similarity produce sine and cosine, don't? Ah! Of course (*laughing*)...

Researcher: Well, if the hypotenuse is 15, for example, what do you think about sine of this angle (*pointing the angle C*)?

Defne: Sine is the ratio of opposite to hypotenuse, that is, these are sine (*pointing two ratios opposite/hypotenuse from similar right triangles on the screen*)…

Ebru: (*Dragging the point D so that the small right triangle's hypotenuse would be approximately 15cm*)

Defne&Ebru: 0.53 (*pointing the value of the opposite/hypotenuse from small triangle*)…

Researcher: Ok. What about sine of the same angle when the hypotenuse is 4?

Defne: (*Dragging the point D so that the small right triangle's hypotenuse would be approximately 4cm*) 0.53…

Ebru: Uh-huh (*nodding her head up-and down*)…

Researcher: What about the 1-cm hypotenuse?

Defne: …would it be same? …we aren't changing angle…

Ebru: (*Dragging the point D so that the small right triangle's hypotenuse would be approximately 1cm*) yes, it is same, 0.53.

Researcher: Well, what do you think if the hypotenuse is 1 here (*pointing the denominator of the ratio opposite/hypotenuse on the screen*)?

Defne: Then, opposite divided by 1 is equal to 0.53, that is, opposite is 0.53…

Ebru: (*Looking to the screen without speaking*)

Researcher: Ok. If we change the angle (*dragging the point A up and down*), then, this ratio is changing also (*pointing the ratio opposite/hypotenuse*). So, what about the opposite side when the hypotenuse is 1 here (*pointing the denominator of the ratio opposite/hypotenuse on the screen*)?

Ebru: Is it sine?
Defne: Hang on… …is it always like that? When here is 1, this ratio is equal to opposite (*pointing denominator and numerator of the ratio opposite/hypotenuse respectively*), then, this (*pointing "opposite" in the ratio opposite/hypotenuse*) is equal to this (*pointing the numeric value of the ratio opposite/hypotenuse*)… …this is sine (*pointing the numeric value of the ratio opposite/hypotenuse*)… so, opposite is sine… Ah, yes!
Ebru: Uh-huh (*nodding her head up and down*).
Researcher: Well. If we construct a right triangle on the unit circle like that (*figuring a unit circle with a right triangle on its first quadrant like in Figure 5.3*), since this right triangle's hypotenuse is 1, what should the opposite side be?
Ebru: Yes, it should be sine (*nodding her head up and down*).
Defne: Ah! It must be sine (*pointing the line segment AB on Figure 5.3*)… …yes…
Ebru: That is, it [the relation between y-component and sine] comes from here (*smiling*)…
Defne: Yes… …because if hypotenuse is 1, then this opposite (*circling the AB line segment on Figure 5.3*) is sine… …ah, yes!
Ebru: Then, here on *y*-axis (*figuring with her index finger the projection of the point A on the y-axis*) gives sine.
Defne: I had known here is 1 (*pointing the AC line segment on Figure 5.3*) here is sine (*pointing the AB line segment on Figure 5.3*) but I hadn't known about their relation (*pointing the AB and AC line segments*)…
Ebru: Me too… I had never thought so…
Defne: Especially, I had never thought to associate sine with similarity…
Ebru: Me too…
Defne: I surprised that this 1 was such interesting thing (*pointing the hypotenuse on Figure 5.3*)… I had never thought like that…

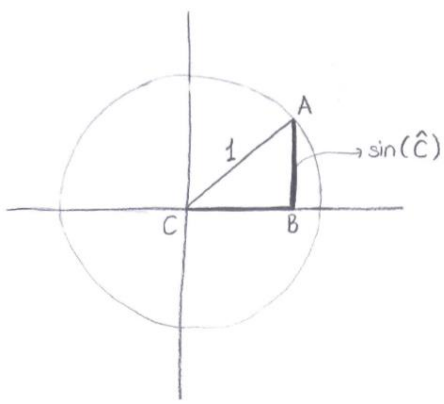


Figure 5.3. Teacher-researcher's construction on paper to associate the definition of "sine" on the right triangle with that on the unit circle

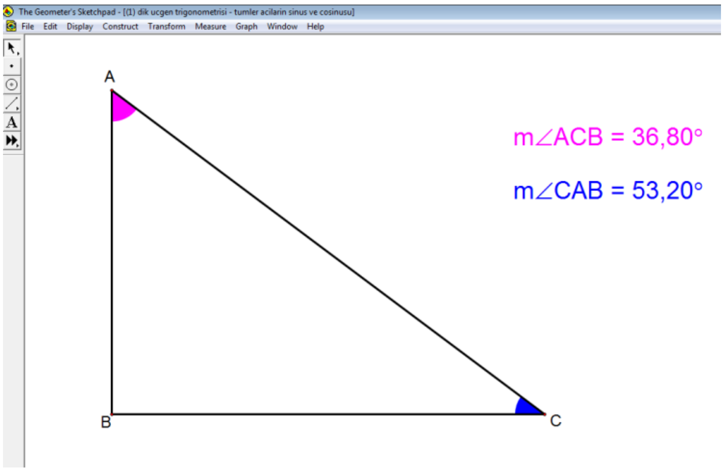
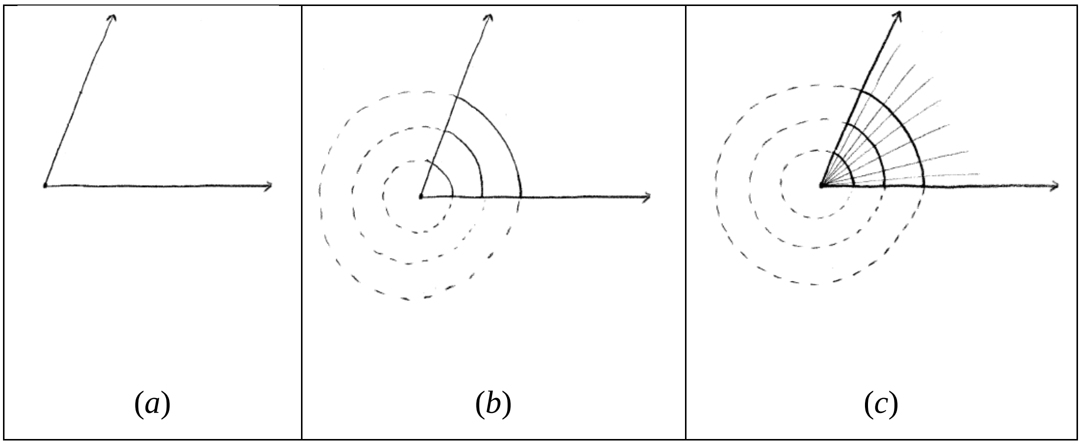


Figure 5.4. Right triangle to discuss the relation between sine and cosine ratios of complementary angles

5.1.1.2. On angles

In the second task, students were introduced "angle" as the figure formed by two rays with the common initial point through drawing on a paper (*Figure 5.5* (*a*)), and then, were explained the definition of "angle measure" in degrees as the number of equal arcs in the angle when the circumference of the circle (centered at the vertex of the angle) is divided into 360 equal arcs by rays (*Figure 5.5* (*b* and *c*)).



(*a*)	(*b*)	(*c*)

Figure 5.5. Teacher-researcher's constructions to articulate "angle" and "angle measure in degrees" on paper

After introducing "angle" and "angle measure in degrees" like mentioned above, it was observed that students reasoned on *Figure 5.5* (*c*) about sameness of the number of equal-arcs remaining in an angle regarding circles with different radii (e.g., [Cemre & Zafer] Protocol 2). However, the researcher predicted that in determination of the angle measure, students would encounter troubles on selecting the appropriate arc from two ones of circles separated by the angle. Thus, in the second task, she encouraged them to investigate the variation in the angle measure when manipulating its openness in GSP hoping to observe these troubles.

[Cemre & Zafer] Protocol 2

Researcher: Do you think the number of equal arcs of all circles for this angle would be same (*pointing Figure 5.5(c)*)?

Zafer: Yes, because those (*pointing the rays in the angle*) separate each arc (*pointing the arcs in the angle from three different circles*) into the same number [equal] arcs.

Cemre: Yes, they [the number of equal arcs in the angle] are same… …actually, we had done in our elementary school when learning fractions… All those [rays in the angle] get away from center (*dragging her pen on the rays in the angle through starting the vertex point*). So, they [rays] separate them (*rotating her pencil around three arc sectors in the angle*) into equal pieces in the same number (*rotating her pen around sub-arc sectors of the biggest arc sector in the angle*).

When investigating the variation on angle's measure through manipulating its openness in GSP environment under "degrees" preferences as the angle measure unit, it was observed that while Defne focused coherently on the same up-part from two ones of the plane separated by the angle (lines 11-19 in [Defne & Ebru] Protocol 3), others changed their focuses from one part to the other when their focused-part turned from the obtuse angle to the reflex angle (e.g., lines 20-22 and 34-38 in [Defne & Ebru] Protocol 3). These actions imply that while Defne's seeing of an angle was as a dynamic turning, others' seeing of an angle was as a static interior region separated by its rays. However, none of them considered an angle's measure through paying attention on the direction (lines 3-10 in [Defne & Ebru] Protocol 3). So, the researcher predicted that students would encounter troubles when dealing with directed angles.

Thus, she encouraged them to investigate the variation of the angle measure when changing angle measure units from *degrees* to *radians,* as well as *directed degrees* hoping to observe these troubles.

[Defne & Ebru] Protocol 3

Researcher: Can you construct an angle in GSP?
Defne: (*Constucting an angle like in Figure 5.6*)
Researcher: Well, now please measure it.
Ebru: (*Measure it through selecting three points –which define the intended angle– in an order so as to be two rays' initial points in the middle*)
Researcher: What do you think about measure if select in reverse order these points?
Ebru: It would be the same.
Researcher: Can you measure it?
Defne: (*Ebru measure the angle through selecting points in reverse order, and Defne looking to the screen*) same [result] happened.
Researcher: Well, can you look at the maximum value of an angle measure by dragging the point *C*?
Defne: 360 (*without investigating the variation on angle measure through dragging the point C, and then starting to drag point C*)
Ebru: It was not 360, it was 180 (*seeing angle measure decreasing from 180 after the straight angle*).
Defne: Why? In my opinion, it must go up to 360. When we are turning it like that (*pointing the point A with her index finger and turning a full round in counterclockwise*), is a circle formed… This circle is 360 degrees.
Ebru: But an angle should be between two lines (*pointing the small part from two parts of the plane separated by the angle*), and an angle between two lines could be at most 180 degrees.
Defne: Then, why does the complete angle exist?
Researcher: Why? (*None of Defne and Ebru was speaking through 20 seconds*)
Researcher: Ok. Just a moment ago, Defne mentioned a circle. If we construct a circle like that (*constructing a circle so that its center could be the vertex of the angle like in Figure 5.7*), where is the measured arc for the angle by GSP. Can you point it me?
Ebru: (*Pointing the small arc which was the below one*)
Defne: Yes.
Researcher: What about now (*dragging the point C so as the small arc to remain above*)
Defne: Here (*dragging her index finger on the above arc from left to right*)…
Ebru: Above one…
Researcher: Why does GSP give angle measures considering sometimes above sometimes below arcs?
Ebru: Because an angle should be between two lines (*pointing the small part from two parts of the plane separated by the angle*), and an angle between two lines could be at most 180 degrees.

Researcher: Is this also an angle between two rays (*pointing the angle corresponding to the arc corresponding to the reflex angle*)?
Defne: Certainly.
Ebru: (*not answering and being quiet*).

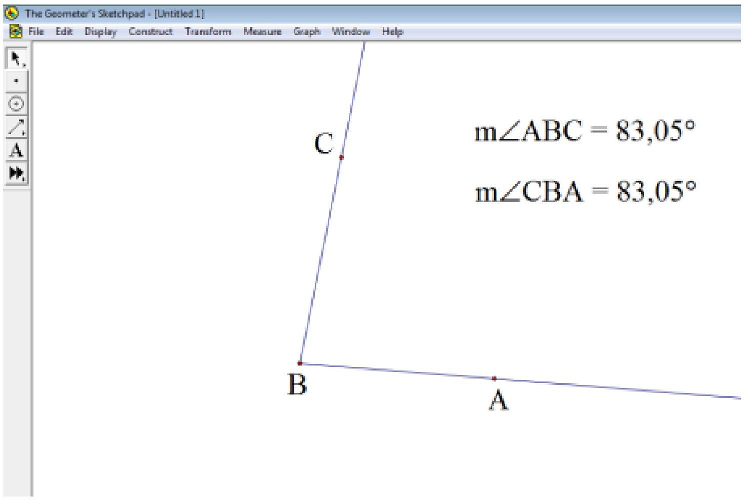


Figure 5.6. Construction of an angle and its measure in degrees in GSP

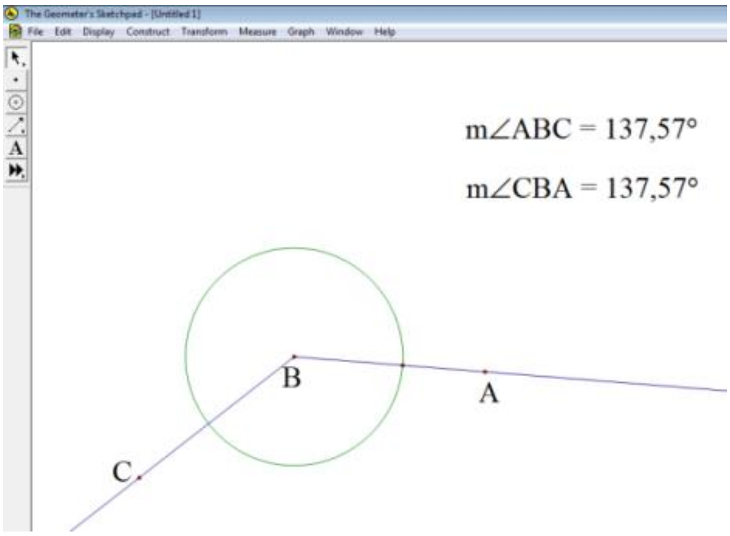


Figure 5.7. Construction of a circle centered on the vertex of the angle

When investigating the variation on angle's measure through manipulating its openness in GSP environment[29] under "radians" and then "directed degrees" as the angle measure unit, it was observed that initially, none of the students could explain any reason why the angle measure changed from positive to negative (e.g., lines 3-13 in [Cemre & Zafer] Protocol 3). After then, according to their investigations in GSP, they reasoned about the sign of the angle measure as negative [positive] if the position of the angle's interior region was down [up] regarding the horizontal straight angle without mentioning direction (e.g., lines 12-17 in [Cemre & Zafer] Protocol 3). While this reasoning prompted Cemre to associate signs of angle measures with the *y*-axis of the coordinate plane (lines 18-25 in [Cemre & Zafer] Protocol 3) instead of mentioning direction, it prompted other students to associate negative measure of an angle with the direction. Although they considered the counterclockwise direction as positive and the clockwise direction as negative, two different points of view having emerged when articulating the direction. While Zafer and Ebru fixed the initial side of the angle as the starting point of rotation and changed the direction of rotation (lines 26-34 in [Cemre & Zafer] Protocol 3; lines 8-13 in [Defne & Ebru] Protocol 4), Defne fixed her focus on the same up-arc but changed the initial side of the angle as the starting point of rotation (lines 2-6 in [Defne & Ebru] Protocol 4). Defne's actions imply that she could consider the same angle as both negative and positive but only through focusing on the same [up] arc between the angle's sides and changing her point of view between its initial side and terminal side. Ebru's actions imply that her seeing of an angle was either as negative or as positive through focusing on both the angle's same initial side and its interior region (lines 8-13 in [Defne & Ebru] Protocol 4). Zafer's actions imply that despite of his seeing the same angle as both negative and positive, his consideration of them was only for those in the fourth quadrant of the unit circle as if a prototypical example of angles with negative measures (lines 31-35 in [Cemre & Zafer] Protocol 3). The researcher determined that all students had troubles on seeing an arbitrary angle with four possible directed-angle measures (i.e., m<ABC in counter/clockwise direction, m<CBA in counter/clockwise direction) as a

[29] GSP's angle measure defaults give the measure of the angle's interior region regarding three angle measure units (degrees, radians and directed degrees) unless measuring an arc's angle.

consequence of their unawareness of three key components defining an angle: *initial side*, *terminal side* and *direction*. So, she encouraged them to investigate the variation in the angle measures through considering the angle measures of the circle's two arcs separated by the angle hoping them recognize these key components.

[Defne & Ebru] Protocol 4

Researcher: When would an angle be negative?

Defne: When we were going from here (*pointing the point A in Figure 5.8)* to here (*turning her pen on the screen up to the point C in the counterclockwise direction*), it [angle measure] would be positive… and from here (*pointing the point C*) to here (*turning her pen on the screen up to the point A in the clockwise direction*), it would be negative, would not be like that?

Researcher: Would not be like that Ebru?

Ebru: An angle was positive when we were going through starting from here (*pointing the point A in Figure 5.8*) and continuing like that (*turning her pencil on the screen up to the point C in the counterclockwise direction*)… …and negative when going through like that (*dragging the point C down in the counterclockwise direction like in Figure 5.9 and turning her pencil on the screen from the point A to the point C in the clockwise direction*).

[Cemre & Zafer] Protocol 3

Researcher: (*After change of the angle measure unit preference from degrees to radians*) well, now we can discuss about change of angle measure in radians.

Cemre: (*Looking to the screen when dragging the point C to see the variation on the angle measure*) Ah (*dragging the point C up* [*like in Figure 5.8*] *and down* [*like in Figure 5.9*] *regarding the position in which the angle measure turned from positive to negative*)! When going there, it [angle measure] happens positive (*dragging the point C upward*), there, negative (*dragging the point C downward*). Why is it happening (*looking at Zafer*)?

Zafer: (*Nodding his head up and down*)

Cemre: Why is it happening?

Zafer: (*Looking to the screen without speaking*)

Cemre: Here is the negative region (*bringing down the point C through dragging*), here is positive region (*bringing up the point C through dragging*).

Researcher: What does the negative region for angles mean?

Zafer: Negative region for angles means… …after 180 degrees, that is, below part of 180 (*indicating a horizontal line with his right hand and then reflecting his hand from up to down regarding this line without drawing*)…

Cemre: When we think with respect to the coordinate plane, there are quadrants, third and fourth ones called as negative quadrants. So, angles' measures in these quadrants are negative

Researcher: Why?

Cemre: Because there is a number line (*indicating a vertical line with her right hand's index finger*). We think the middle point of it as zero (*putting her thumb and index fingertips closed to each other*), up part of it as positive numbers (*indicating a ray upward*), down part of it as negative numbers (*indicating a ray downward*)

Zafer: Well, we have a coordinate plane. When going through starting from here (*indicating a point with his right hand's index finger*) to continue through in this direction (*turning his finger so as to figure a half-arc from this point in the counterclockwise direction*), we are positive angles. That is, if we come here through starting here (*indicating again the starting point and turning his finger so as to figure an arc greater than three quarter-arc*), then this angle happens an angle greater than 270, but if we start as negative (*turning his index finger in the clockwise direction so as to figure an arc smaller than a quarter-arc*), then this angle is equal to minus 70 or so.

Cemre: Hmm… That is, minus sign was indicating the direction… …hmm...

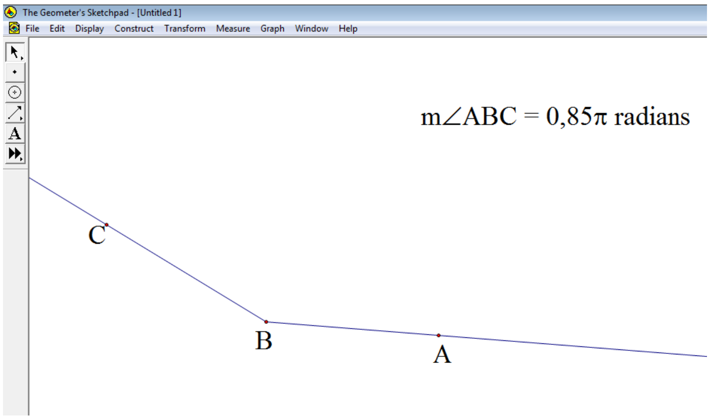


Figure 5.8. Construction of an angle with positive measure in radians in GSP

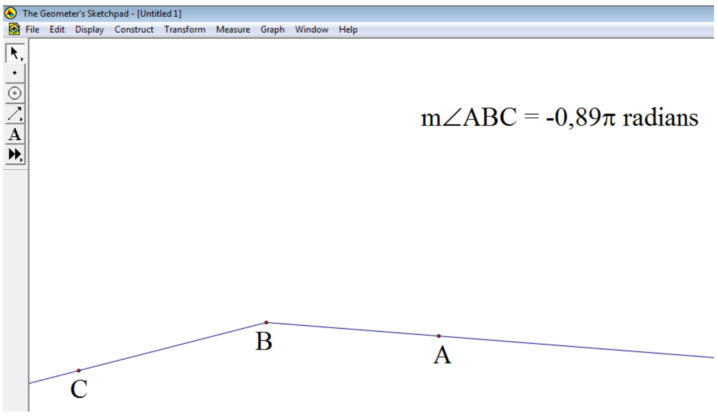


Figure 5.9. Construction of an angle with negative measure in radians in GSP

When discussing the related measures and the reasons of their mathematical relevance under the variation of angle measures, initially, only the angle measures of ABC and two arcs were considered (*Figure 5.10*). In this step, all of the students associated the measure of the angle ABC given by GSP with the appropriate arc's angle measure[30] through articulating signs' meaning in m<ABC with the appropriate *initial side* and *direction* (e.g., lines 1-13 in [Cemre & Zafer] Protocol 4). Moreover, they interpreted the angle ABC with a positive [negative] angle measure as its negative [positive] equivalence (e.g., lines 14-35 in [Cemre & Zafer] Protocol 4). At this point, the researcher predicted that students would encounter troubles as a consequence of the change of the *initial side* for the *direction*. Thus, she encouraged them to investigate the related measures and the reasons of their mathematical relevance through considering the angle CBA. As a result of their reasoning on directed angles mentioned above, it was not difficult for students to recognize the role of the *initial side* of a directed-angle. All these discussions mentioned above were also done for angle measures in radians, and same reasoning on *direction* and *initial side* was observed. After then, regarding the initial interview results, the researcher predicted that students would encounter troubles not only on association of the "radian" measure with the arc length and radius, but also on considering π notation in the angle measure in radians as a real number, i.e. approximately 3.14. So, at the end of Task 2, she asked students to think about the meaning of "radian" measure unit and "π" notation in the angle measure in radians.

[Cemre & Zafer] Protocol 4

Zafer: (*When m<ABC is positive*) these are same (*pointing the measures of the ABC angle and the red arc DE on the screen like up-one in Figure 5.10*). Now, arc angle was still positive, other did negative (*dragging the point C so as the ABC angle's measure to turn from positive to negative*).

Cemre: So, it measured this part (*pointing the blue arc on the screen like down-one in Figure 5.10*) Just because this (*pointing the red arc's angle measure*) was measured from this side (*figuring the red arc with her index finger in the counterclockwise direction*), this (*pointing the ABC angle's measure*) was

[30] GSP's arc construction defaults allow only constructing an arc from the first selected point to the second one on a circle in the positive direction; therefore, it does not give directed arc angle measure.

measured from this side (*figuring the blue arc with her index finger in the clockwise direction*). That is, an angle measured from this side is positive (*turning her index finger in the counterclockwise direction through starting the point D*)… …from this side is negative (*turning her index finger in the clockwise direction through starting the point D*).

Zafer: Yeah. (*Dragging the point C during 8-second in case only for the ABC angle's negative measure*) their sum is always 360 (*pointing the measures of the ABC angle and the red arc DE*).

Cemre: Can I look? (*Dragging the point C starting from negative cases*) hmm, yes… …but here (d*ragging the point C in case the ABC angle's positive measure*), not 360, their sum is 360 (*pointing the measures of the ABC angle and the blue arc ED*).

Zafer: Because these arcs constitute a circle (*figuring respectively the red and blue arcs in the counterclockwise direction*). A circle corresponds to 360 in degrees.

Researcher: What would the ABC angle's measure be if it was considered in the clockwise direction (*in case the negative measure for the angle ABC on the screen*)?

Cemre: Then, we consider it from this side (*figuring the blue arc with her index finger in the clockwise direction*), that is, we measure this arc (*pointing blue arc*), its measure is 212 comma 09…

Zafer: But it should be negative…

Cemre: Of course.

Researcher: Well. What would the ABC angle's measure be if it was considered in the counterclockwise direction (*in case the positive measure for the angle ABC on the screen*)?

Zafer: Then, it would be positive… We determine its value considering this red arc (*figuring the red arc with his index finger in the counterclockwise direction*).

Cemre: …that is, 233 comma 66 (*pointing the red arc's angle measure on the screen*). I have understood. It had been actually very easy. I had never thought about angles like that (*looking to the researcher*),

Zafer: (*Smiling*) me too…

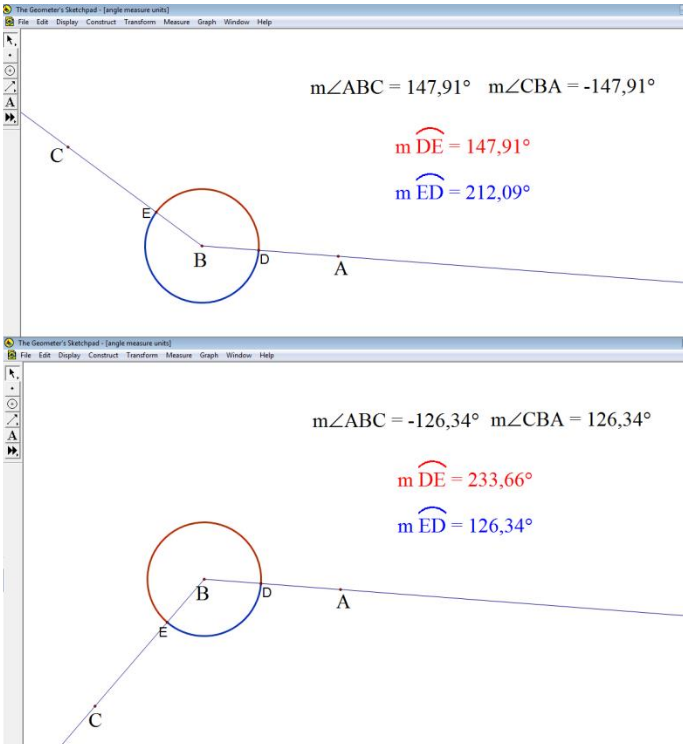


Figure 5.10. Constructions of an angle with two arcs to make sense of the direction

When discussing the related measures (such as arc lengths, arc angles, radius and angle measure) and the reasons of their mathematical relevance under the variation of an angle's openness, as well as the radius of the circle (centered on the vertex of the angle) in GSP environment[31], it was observed that initially, none of the students was able to associate an angle's measure in "radians" with the arc length and radius. That is, when making variation on angles' openness and radii, firstly, they interpreted the variation on given measures separately, such as change in radius, angle measure, or arc length (e.g., lines 1-5 in [Cemre & Zafer] Protocol 5). And then, one student in each group, Cemre and Defne, turned their focus from the variation of measures separately to the relation of these variations. That is, Cemre and Defne, after restricting

[31] While GSP's angle measure defaults give the angle measure in radians with π notation instead of considering it as a real number, its calculation defaults show a selected angle measure in radians on "*New Calculation*" window as the equal real value through considering π with its real value.

the variation into the case in which the angle measure was fixed, focused on the relation between variations of two measures [arc length and radius] through specified radii by natural numbers, and then, reasoned their proportionality (e.g., lines 5-18 in [Cemre & Zafer] Protocol 5). However, none of the students was able to associate these ratios with the angle measure in radians (e.g., lines 19-33 in [Cemre & Zafer] Protocol 5) until the researcher's recommendation for them to consider π in the angle measure (e.g., lines 34-37 in [Cemre & Zafer] Protocol 5). Where, because of students' problematic reasoning about the real value of π in and out the trigonometry context that was revealed throughout the initial interviews, the researcher encouraged them to use GSP's calculate option when considering π in the angle measure (e.g., lines 36-39 in [Cemre & Zafer] Protocol 5). And then, all students noticed that the angle measure in radians corresponded to the ratio of the (appropriate) arc length by the radius (e.g., lines 40-63 in [Cemre & Zafer] Protocol 5). At this point, it was observed a distinct shift on students' reasoning about the radian measure unit. That is to say, students had just started to associate arc lengths [when the radius was 1] with the angle measures (e.g., lines 64-76 in [Cemre & Zafer] Protocol 5), as well as associated π as the angle measure in radians with its meaning as a real number (i.e., approximately 3.14) and its meaning as an angle in degrees (i.e., 180 in degrees) (e.g., lines 40-50 and 72-78 in [Cemre & Zafer] Protocol 5). However, only Zafer associated these critiques on the angle measure in radians with the unit circle. Therefore, at the beginning of Task 3[32], the researcher determined to scrutinize their reasoning on the radian measure unit on the unit circle, but it was not difficult for students to transfer their reasoning onto the unit circle. Since students' this reasoning about the radian measure unit and π was inconsistent with their prior ones revealed throughout the initial interviews, the researcher scrutinized their reasoning also in some following tasks, for example in Task 6, 7, 8 and 13; and observed again students' coherent reasoning about the radian measure unit and π like mentioned above (e.g., lines 47-76 in [Cemre & Zafer]

[32] From this point forward, throughout the teaching experiment, angles were represented on the (unit) circle located on the perpendicular coordinate system or its translated form which was the parallel displacement system. Where, the angle's vertex was the center and its initial side was the ray in the positive horizontal direction. We called this *multi-functional and non-discursive representational register* as the (*unit*) *circle register*.

Protocol 11; lines 18-44 in [Defne & Ebru] Protocol 11; lines 79-95 in [Cemre & Zafer] Protocol 13). At this point, the researcher determined that all of the students were able to define an angle with two measures (i.e., with the highest negative and the lowest positive values) in each angle measure units within the (*unit*) *circle register*. Henceforward, the subsequent progress of students' concept images on angles that emerged as a result of the teaching experiment was presented in terms of the different representational registers under the following respective headings.

[Cemre & Zafer] Protocol 5

Researcher: Well, if we return to our starting point, to what is the angle measure related?
Cemre: Is it arc length?
Zafer: But now it is changing (*dragging the point D in Figure 5.11*).
Cemre: Uh-huh… …radius is also changing (*pointing the measure r with her index finger*), but angle [measure] did not change (*pointing m<ABC*). (*Dragging the point D so as to obtain the radius measures with natural numbers like in Figure 5.11 and try to calculate ratios of the length of arc DE to radius*) about 5 [length of arc DE] and 2 [radius] (*looking to the screen like in Figure 5.11(b) and calculating the ratio of 5 to 2*), that is, approximately, two and half… …now, 10 and 4 (*looking to the screen like in Figure 5.11(c) and calculating the ratio of 10 to 4*), again two and half… Ok, the ratio of this (*pointing the length of the arc DE*) to this (*pointing the radius*) are always approximately 2.5, are there (*looking to Zafer*)?
Zafer: How?
Cemre: Look, for example, the ratio of 2.57 to 1 (*dragging the point D so as to obtain the measures like in Figure 5.11(a), and pointing the arc length of DE and radius on the screen*), or of 5.13 to 2 (*dragging the point D so as to obtain the measures like in Figure 5.11(b), and pointing the arc length of DE and radius on the screen*).
Zafer: Yeah (C*alculating the ratios (length DE)/r and (length ED)/r like in Figure 5.12*).
Cemre: This (*pointing the ratio (length DE)/r*) did not change, and this (*pointing the ratio (length ED)/r*) also did not change (*changing the radius through dragging the point D*)…
Zafer: (*Nodding his head up and down*). So, change the angle ['s openness].
Cemre: Ok (*changing the angle's openness through dragging point C*), both [ratios] are changing now.
Zafer: Yes, but now again they are same (*changing the radius through dragging the point D*).
Researcher: In this case, is there any other stable measure?
Cemre: Yes, this is also stable (*pointing angle measure*)…
Zafer: (*Nodding his head up and down*).
Researcher: Is there any relation of this angle measure with these ratios (*pointing with her index finger these measures on the screen*)?
Cemre&Zafer: (*Looking to the screen without speaking*)

Researcher: Well, you can consider π in the angle measure.
Cemre&Zafer: (*Looking to the screen without speaking*)
Researcher: There is π here (*pointing π notation in the angle measure*), there is not π here (*pointing ratios of arc lengths to radius*).
Cemre: We multiply 0.88 with π…
Researcher: Can you multiple them by the GSP's calculate option?
Cemre: (*Calculating this multiplication*) ah, result comes out the same as this (*pointing the ratio lengthDE/r on the screen like up-one in Figure 5.12*). …but, as what does it [GSP] take π? If we accept this (*pointing the numeric part of the angle measure*) about 0.9… …this (*pointing the ratio lengthDE/r*) is about its three times…
Zafer: (*Opening the calculate option of GSP and selecting only π value*) ok, it takes π as 3.14…
Cemre: Ah!… This π (*pointing the π notation in the angle measure*) was 3.14…
Zafer: Uh-huh… (*After 6-second pause*) then, the division of this (*pointing the ratio lengthDE/r*) by π or 3.14 should give this (*pointing the numeric value of the angle measure*)… (*Calculating this division like in Figure 5.12*) yes...
Cemre: (*Dragging the point C to manipulate the angle's measure*) yes…
Researcher: What about now (*dragging the point C so as the given angle measure to be negative*)?
Cemre: Then, the division of this (*pointing the ratio lengthED/r*) by π should be 84 (*pointing the numeric value of the angle measure on the screen like down-one in Figure 5.12*)… (*Calculating this division*), yes...
Zafer: (*Smiling and nodding his head up and down*). Then, an angle's radian measure is… …of the division of arc length inside of an angle by radius… …the value with respect to π…
Cemre: (*Nodding her head up and down*)…
Researcher: Well, what can you say about these ratios if the radius is 1 (*pointing them on the screen*)?
Cemre: Then, these ratios are equal to arc lengths. So, angle's measure is equal to arc length.
Zafer: (*Dragging the point D to obtain the radius with 1 cm length*) uh-huh (*nodding his head up and down*). That is, on the unit circle, arc length is the angle measure in radians…
Cemre: It is interesting… I have never thought angle measure in radians is related to arc length…
Zafer: …but of the circle whose radius is 1…
Cemre: Of course, because it [radian measure] corresponds to these ratios (*pointing them on the screen*).
Zafer: (*After 4-second pause*) that is to say, it is the reason of π is 180 in degrees… I had memorized π as equal to 180… I had thought this π (*pointing the π notation of the angle measure in radians*) was different from π we had learned as 3.14 before. However, they have been equal to each other… …π has referred to 180 degrees because straight line imply half-circumference.
Cemre: Exactly. I hadn't thought the relation of π with half arc length of a circle with 1 radius.

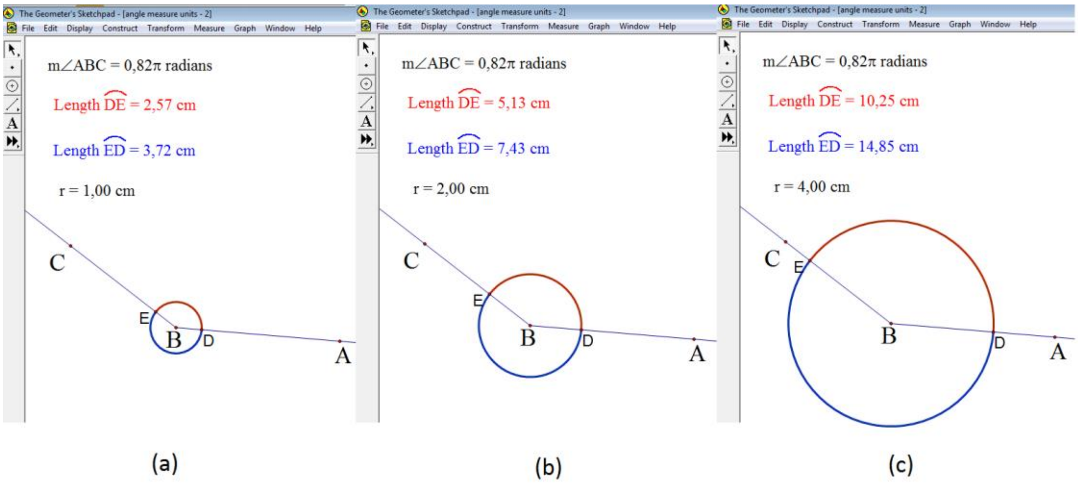


Figure 5.11. Ways of operating to establish relation between arc lengths and arc angles

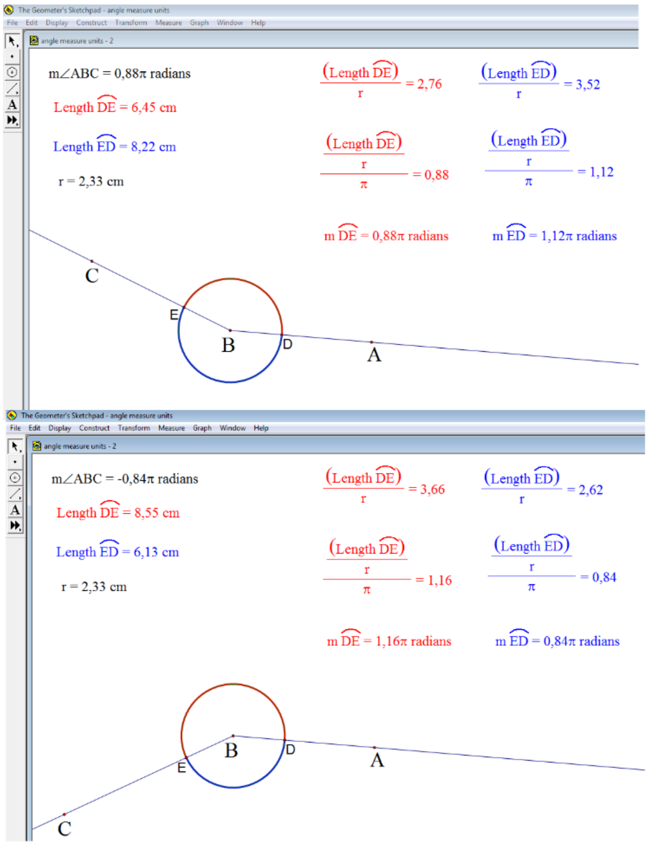


Figure 5.12. Manipulable calculations of the ratios of arc lengths to the radius by the dynamically-connected constructions

5.1.2. Development of Students' Concept Images Rooted in (Unit) Circle Register and Shaped between (Unit) Circle Register and Symbolic Register

5.1.2.1. On angles

At the beginning of Task 3, students were able to define an angle with two measures (i.e., with the highest negative and the lowest positive values) in each angle measure units within the (*unit*) *circle register* (e.g., lines 1-11 in [Defne & Ebru] Protocol 5), but not with the infinitely many [negative or positive] equivalent measures (e.g., lines 12-13 in [Defne & Ebru] Protocol 5). Thus, the researcher encouraged them to identify an angle within the (*unit*) *circle register* through dynamic directed turnings regarding its absolute measure greater than 360 degrees or 2π radians in the *symbolic register* (e.g., lines 1-3, 14 and 29 in [Cemre & Zafer] Protocol 6), as well as an angle's measure within the *symbolic register* regarding dynamic directed turnings in the (*unit*) *circle register* (e.g., lines 14-17, 40, 43-46 and 52-53 in [Defne & Ebru] Protocol 5).

When discussing an angle's measure within the *symbolic register* regarding the dynamic directed turnings in the (*unit*) *circle register*, it was observed that none of the students encountered troubles on expressing an angle's measure as an additive operation between the corresponding measure to the *principal turning* from the initial side to the terminal side and the corresponding measure to the *full-rounds* with the appropriate signs regarding the directions of turnings (e.g., lines 18-24, 41-42, 47-51 and 54-55 in [Defne & Ebru] Protocol 5). Moreover, they were able to express an angle's measure corresponding to the *piecewise principal turning* as an additive operation with the appropriate sign regarding the direction of turning (e.g., lines 1-30 in [Defne & Ebru] Protocol 6).

Reversely, when discussing the terminal sides (regarding the position of the point P) of the angles within the (*unit*) *circle register* whose absolute measures were greater than 360 degrees or 2π radians, it was observed that none of the students encountered troubles on defining them regarding two-step turning in any directions: (1) *principal turning* from the initial side to the terminal side (2) some *full-rounds*. Initially, they preferred to consider the needed full-rounds in the (*unit*) *circle register* (e.g., lines 6-8 and 20-23 in [Cemre & Zafer] Protocol 6; lines 25-36 in [Defne & Ebru] Protocol 5), and then, the *principal turning* from the initial side to the terminal side of the intended angle measure (e.g., lines 8-10, 15-18 and 23-25 in [Cemre & Zafer] Protocol 6). However, their initial attempts to identify these angles' terminal sides were based on the determination of the appropriate quadrant of the unit circle (e.g., lines 15-17, 20 and 30-34 in [Cemre & Zafer] Protocol 6) until the researcher's insistence for them on determination of its exact position regarding the point *P* on the unit circle (e.g., lines 19 and 41 in [Cemre & Zafer] Protocol 6). When trying to identify the exact position of these angles' terminal sides, it was observed that all of the students easily reasoned about the *principal turning* through dragging the point *P* in the appropriate direction up to the measure of the alpha angle appeared with the intended principal measure on the screen (e.g., lines 26-28 and 42-43 in [Cemre & Zafer] Protocol 6). However, it was observed only on Cemre and Zafer that they also reasoned about the *principal turning* from the initial side to the terminal side through considering this turning in two steps regarding the closest coordinate axes: (1) turning from the initial side to the closest coordinate axis in the same direction as the *principal turning* (2) turning from this coordinate axis to the terminal side in the direction so that the way of turning would be the shorter arc (lines 26-28, 36-39 and 54-70 in [Cemre & Zafer] Protocol 6). This reasoning prompted a distinct shift on their reasoning about an angle in the (*unit*) *circle register*. That is to say, Cemre and Zafer began to associate an angle within the (*unit*) *circle register* with its complementary and/or supplementary parts in any quadrant (e.g., lines 56-62 and 66-70 in [Cemre & Zafer] Protocol 6). Also, Defne and Ebru reached this level of reasoning when the researcher provoked them to identify an angle in any quadrant within the (*unit*) *circle register* at least in three different ways without considering *full-round turnings* (e.g., lines 31-52 in [Defne & Ebru] Protocol 6). When having reached this level of reasoning, students mentioned the additive operations regarding the reference angle in the *symbolic* *register* [i.e., π±α and ±α] as the most convenient forms to consider an angle in the (*unit*) *circle register* (e.g., lines 53-58 in [Defne & Ebru] Protocol 6). However, their articulations revealed that none of the students preferred them according to their awareness on the reference angle's role. Instead, their preferences were based on the problematic part of their concept images on the angle measures [in the form π/2±α and 3π/2±α] in terms of the memorized-rules without any reasons within the *symbolic register* (e.g., lines 59-64 in [Defne & Ebru] Protocol 6) that were revealed also throughout the initial interviews. Despite of their unawareness on the reference angle's role, all students were able to identify an angle via not only the *two-step principal turning* regarding the reference angle in the (*unit*) *circle register* but also the additive operations regarding the reference angle in the *symbolic register*. Since students' this level of reasoning on angles was inconsistent with their prior ones revealed throughout the initial interviews, all these discussions mentioned above were also done on the another GSP page whose static version was like in *Figure 5.16*, and same reasoning was observed.

At the end of Task 3, all students were able to reason easily about an angle in the (*unit*) *circle register* with the infinitely many negative and positive equivalent measures in the *symbolic register* [including the highest negative and lowest positive measures, coterminal measures greater than 360 degrees or 2π radians, as well as additive operations regarding its reference angle] but also infinitely many equivalent measures in the *symbolic register* as the same structure in the (*unit*) *circle register*. Moreover, they were able to associate an angle in any quadrant within the (*unit*) *circle register* with its complementary and/or supplementary parts. This kind of reasoning about angles was vital for trigonometric functions' coherent conceptions in the (*unit*) *circle register* through leading for students to (*a*) extent trigonometric functions' domain from the acute angles over the obtuse and reflex angles in the (*unit*) *circle register*, and then, to the real number set (R) in the *symbolic register*, (*b*) transfer their integrated conceptions on trigonometric functions in the first quadrant in the (*unit*) *circle register* with the right triangle context into the any other quadrants by means of the reference right triangle, (*c*) associate sine [cosine] of an angle in any quadrant with sine [cosine] of this angle's reference angle in the first quadrant, (*d*) reason about periodic nature of trigonometric functions.

At this point, the researcher predicted that students would encounter troubles on transferring their integrated concept definition images on sine and cosine [in the *right triangle* context and the *unit circle* context] within the first quadrant of the (*unit*) *circle register* into the any other quadrant of the (*unit*) *circle register* as an again merged concept definition images in both contexts. So, the following task, Task 4, was designed to observe students these troubles.

[Defne & Ebru] Protocol 5

Researcher: How can you identify this angle's measure (*dragging her index finger on the positive x-axis from right to left, and then, from the origin to the point P on the screen like in Figure 5.13(b)*)?

Defne: We come 159 degrees from here to here in positive direction (*dragging an arc in the counterclockwise direction with her index finger from the intersection point of the unit circle with the positive x-axis to the point P*), but comes from here to here in negative direction (*dragging an arc in the clockwise direction with her index finger from the intersection point of the unit circle with the positive x-axis to the point P*), that is, 159 minus 360… what is it…

Ebru: …minus 201

Defne: Yes.

Researcher: Ok. What else?

Defne&Ebru: (*Looking to the screen without speaking*)

Researcher: Well, if I turned from here to here (*dragging an arc on the unit circle in the counterclockwise direction with her index finger from the point 1on the x-axis to the point P*), and then, a full-round like that (*dragging a circle through beginning from the point P in the counterclockwise direction*), what degrees would I turn?

Defne&Ebru: 360 plus 159 (*figuring with their index fingers a full-round from the intersection point of the unit circle with the positive x-axis, and then, an arc from the intersection point to the point P in counterclockwise direction*)…

Ebru: 519 degrees…

Researcher: Why do you think of 360?

Defne: Because one full-round refers to 360 degrees.

Ebru: Uh-huh (*nodding her head up and down*).

Researcher: Ok. Where would the new position of the point P on the unit circle be for the 519-degree angle?

Defne: Isn't it the same position (*pointing the point P on the unit circle, and looking to Ebru*)?

Ebru: Yes.

Researcher: Why?

Defne: Because we turned for 519, a full-round (*dragging a circle in the counterclockwise direction through beginning from the point 1 on the x-axis*) and 159 degrees (*dragging an arc on the unit circle in the counterclockwise direction with her index finger from the point 1 on the x-axis to the point P on the screen like in Figure 5.13(b)*)… …so, we are still there (*pointing the point P*).

Ebru: Yes.

(*The researcher constructed rotated position of the point 1 on the x-axis* about the origin *by “360+alpha” degrees in the positive direction like in Figure 5.14 in order for students to compare the position of 360+alpha degrees with that if alpha degrees*)

Researcher: Ok, what about two full-round?

Defne: Then, it would be two times 360 plus 159…

Ebru: Yes (*nodding her head up and down*).

Researcher: Ok. What about if I turned from here to here (*dragging an arc in the positive direction with her index finger from the intersection point of the unit circle with the positive x-axis to the point P*), and then, a full-round like that (*dragging a circle through beginning from the point P in the clockwise direction*)?

Defne: 159 (*figuring an arc from the intersection point of the unit circle with the positive x-axis to the point P in the positive direction with her index finger*)… …minus 360 (*figuring a circle beginning from the point P in the clockwise direction with her index finger*)…

Ebru: Uh-huh (*nodding her head up and down*).

Researcher: Well, what about two full-rounds (*dragging with her index finger two full-rounds through beginning from the point P in the clockwise direction*)?

Defne: Then, it would be 159 minus two times 360…

Ebru: (*Nodding her head up and down*).

Researcher: How many measures can I find for the angle corresponding to the point P?

Defne: As many as we want…

Ebru: Infinite…

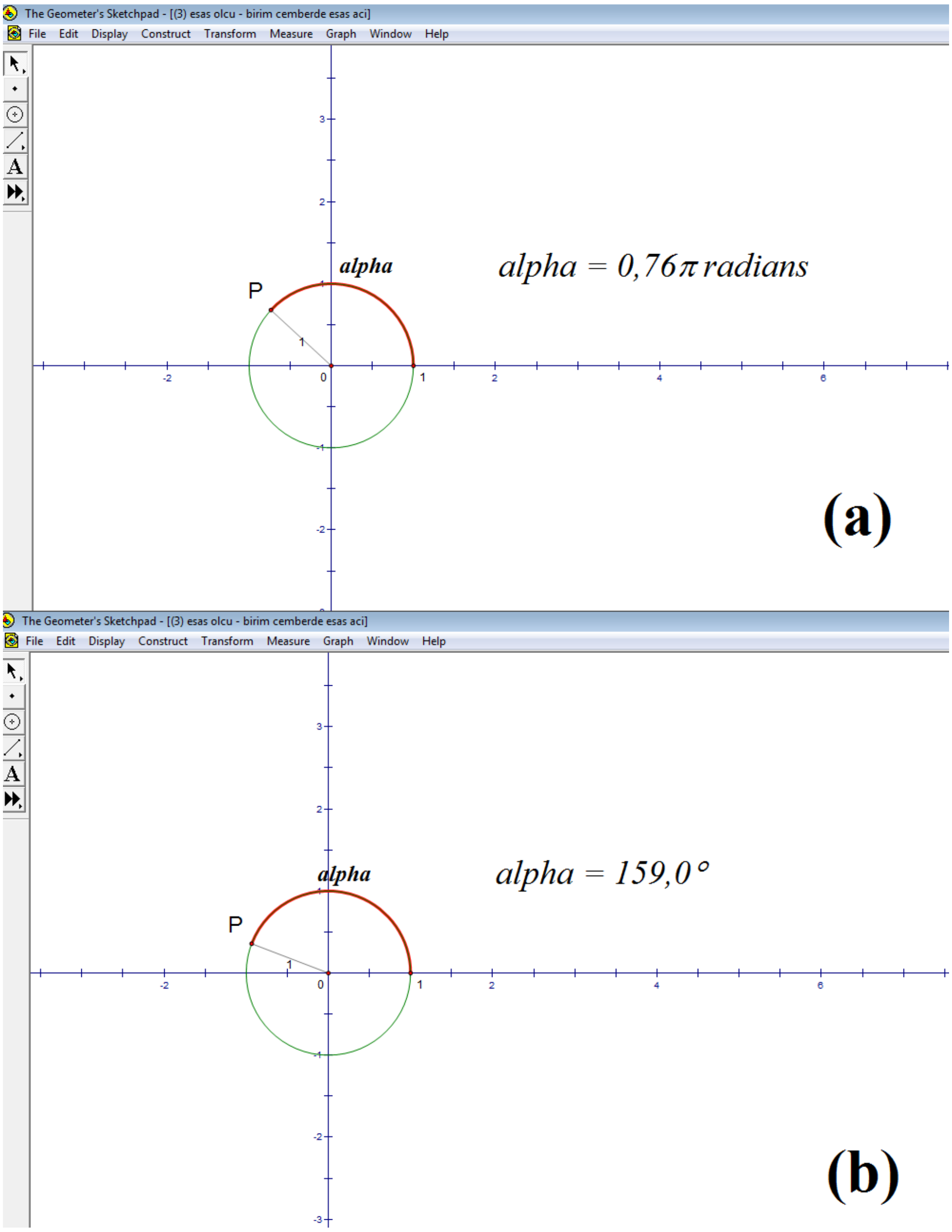


Figure 5.13. An angle on the (*unit*) *circle register* and its principal measure in the *symbolic register*

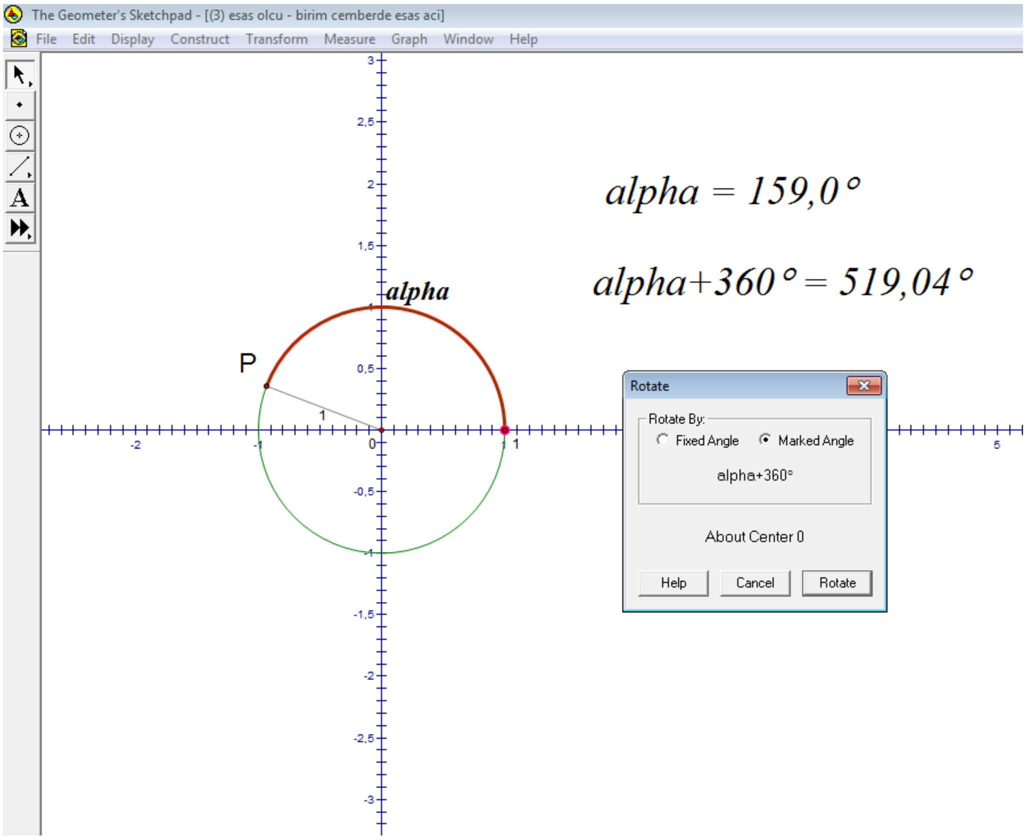


Figure 5.14. An example of the dynamic construction of angles with the same principal angle by GSP

[Cemre & Zafer] Protocol 6

Researcher: Well, if I ask you where the position of the point P on the unit circle is for the 2.5π-radian angle, what do you think (*pointing the point P on the screen like in Figure 5.13(a)*)?

Zafer: 2.5π… …is here (*pointing the positive y-axis on the screen with his index finger through dragging up and down*).

Cemre: It [terminal side] would turn as much as 2π (*figuring a circle in the counterclockwise direction with her index finger through starting from a point which was as if the far right point of this circle*), and then, as much as half [π] (*figuring a quarter arc from the same starting point in the counterclockwise direction with her index finger*).

Zafer: It would be 90 degrees there (*pointing the intersection point of the unit circle with the positive y-axis*)…

Cemre: Yes. I think so, too.

Researcher: Ok. What about 2.3π radians?

Zafer: It would be between these two (*pointing simultaneously on the screen the point 1 on the y-axis with his right hands index finger and the point 1 on the x-axis with his right hands middle finger*).

Cemre: Uh-huh.

Researcher: Ok. But I want you to show 2.3π radians' exact position on the unit circle.

Cemre: It would be smaller than 90 degrees…

Zafer: Now, it [the measure] became 2π up to here (*dragging the point P so as to complete exactly one full round in the counterclockwise direction starting from 0 radians*)… …and 0.3 [π] is happening in here (*continuing to drag the point P in the counterclockwise direction and dropping it when the measure of the alpha angle appeared as 0.3π radians on the screen*).

Cemre: When we think here as 0.5 [π] (*bringing the point P on the positive y-axis*), 0.3 [π] is here somewhere (*dragging it in the clockwise direction up to the measure of the alpha angle appeared as 0.3π radians on the screen*).

Researcher: Ok. What about 5.3π radians?

Cemre: 1.3 [π], so it would be here somewhere (*pointing the third quadrant of the unit circle with the cursor*).

Zafer: It would be between these two (*pointing simultaneously on the screen the point -1 on the x-axis with his right hands index finger and the point -1 on the y-axis with his right hands middle finger*).

Researcher: How did you determine?

Zafer: Firstly, we subtract 4π, and then, it would be 1.3π… Up to here (*pointing the point -1 on the x-axis*), it [the angle measure] is π, therefore, it [1.3π] would be between these two (*pointing with his index finger on the screen respectively the point -1 on the x-axis and the point -1 on the y-axis*).

Cemre: Yes.

Researcher: Ok. But I want you to show again its exact position on the unit circle.

Cemre: (*Bringing the point P in the third quadrant through dragging so as the alpha measure to be approximately 1.3π radians*) it is there.

Zafer: Yes.

Cemre: They [5.3π radians and 1.3π radians] mean same thing [position] on the unit circle.

Zafer: Yes.

Researcher: Well. We determined that two different angle measures, that is, 5.3π and 1.3π in radians corresponded to the same position on the unit circle. Are there such any other measures corresponding to the point P?

Zafer: We can consider negatively, that is, from here to here (*figuring an arc with his index finger in clockwise direction from the point 1 on the x-axis to the point P on the screen*).

Cemre: …through the opposite direction…

Researcher: So, what is the measure?

Zafer: If we think starting from here (*pointing the point 1 on the x-axis with the cursor*), up to here it [arc length] is 0.5 [π] (*figuring a quarter arc with the cursor in the clockwise direction in the fourth guardant on the unit circle*), and then here is approximately 0.6, 0.7 [π] (*figuring an arc in the clockwise direction from the point -1 on the y-axis to the point P on the screen*)… …it would be 0.7 [π] because when adding 0.3 [π] more, it would be here (*figuring an arc in the clockwise direction from the point P to the point -1 on the x-axis on the screen*).

Cemre: We look for a measure… …so that it could show the same position [on the unit circle]… …with 1.3π radians, don't we?
Researcher: Definitely yes.
Cemre: This in the positive direction (*figuring an arc with her left hand's index finger in the counterclockwise direction through starting from a point which was as if the far right point of this circle*) and this in the negative direction (*figuring an arc with her right hand's index finger in the clockwise direction through starting from again the same initial point*), these two complete each other to 2π…
Zafer: Yes, we can find these values through subtracting one of them from 2π…
Cemre: Uh-huh.
Researcher: Good. What else? Are there such any other measures corresponding to the point P?
Zafer: Minus 2.7π… …minus 4.7π, for example…
Cemre: Yes, minus 6.7 [π]…
Researcher: Then, how many measures can I find for the angle corresponding to the point P?
Zafer: A lot…
Cemre: Yes…

[Defne & Ebru] Protocol 6

Researcher: Without considering full-rounds, how can you interpret this angle, on the unit circle (*dragging her index finger on the positive x-axis from right to left, and then, from the origin to the point B on the screen like in Figure 5.15*)?
Defne: Here (*dragging an arc with her index finger on the unit circle from the point 1 on the x-axis to the point B in the counterclockwise direction*) and here (*dragging an arc with her index finger on the unit circle from the point 1 on the x-axis to the point B in the clockwise direction*)…
Researcher: Ok. How can you express these angles?
Ebru: Alpha plus two times beta…
Defne: Yes…
Researcher: What is its sign?
Defne&Ebru: Plus…
Researcher: Why?
Defne: Because it is in the positive direction…
Ebru: Yes.
Researcher: Ok. What else?
Ebru: Minus three alphas… …minus two betas…
Defne: No, it should be plus two betas…
Researcher: Determine this together.
Ebru: Look! Three alpha exist (*pointing yellow parts on the screen like in Figure 5.15*), and two beta (*pointing blue parts on the screen like in Figure 5.15*), don't?
Defne: Yes. So, should be… …minus three alphas plus two betas…
Ebru: No, it should be minus two betas…
Researcher: Please write your answers on the paper.
(*Both wrote* $-(3 \propto +2\beta)$ *on the paper*)
Defne&Ebru: (*When seeing their same writings, they were smiling.*)

Researcher: Ebru, is your expression just a moment ago as "minus three alphas minus two betas"… …its un-parenthesized version?

Ebru: Yes.

Defne: Hmm. Yes, yes… I understand.

Researcher: Ok. You expressed the angle corresponding the point B in two ways, from here to here (*dragging her index finger regarding the point B on the up arc in the counterclockwise direction on the screen like in Figure 5.15*) and here to here (*dragging her index finger regarding the point B on the down arc in the clockwise direction*) through adding corresponding alphas and betas. What else? How can you express it?

Defne: (*After 5-second pause*) this (*dragging an arc with her index finger on the unit circle from the point 1 on the x-axis to the point B in the counterclockwise direction on the screen like Figure 5.15*) means that this minus this (*dragging two arcs with her index finger on the unit circle firstly from the point 1 on the x-axis to the point -1 on the x-axis in the counterclockwise direction, and then, from the point -1 on the x-axis to the point B in the clockwise direction on the screen*)… That is, π minus alpha…

Researcher: What else?

Defne: We can go from $3\pi/2$ (*putting her pen on the point E in Figure 5.15*) to here (*figuring an arc from the point E to the point B in the clockwise direction*).

Researcher: Express it.

Defne: $3\pi/2$ minus… …two alphas plus beta…

Researcher: Is this in parenthesis?

Defne: Yes.

Researcher: Ok. What else?

Ebru: Then, we can say $\pi/2$ plus beta…

Researcher: Good… …Well, which ones of these expressions are more preferable, your first expressions or last ones?

Ebru: Last ones…

Defne: For example, I prefer to say π minus alpha.

Ebru: Yes… (*After 5-second pause*) in fact, also "$\pi/2$ plus beta" can be preferable… …both [($\pi/2 +\beta$) and ($\pi-\alpha$)] are in the same ease.

Defne: But if we say $\pi/2$ plus beta, then names change. So, "π minus alpha" is more reasonable [than ($\pi/2 +\beta$)].

Ebru: But now, we are speaking only on angles.

Researcher: Defne, what change if we say $\pi/2$ plus beta?

Defne: Then, sine changes as cosine and cosine changes as sine after $\pi/2$ (*pointing the point 1 on the y-axis*).

(*Similar discussions were done on the points D and F on the GSP page like in Figure 5.15, as well as the points A, B and C on the GSP page like in Figure 5.16*).

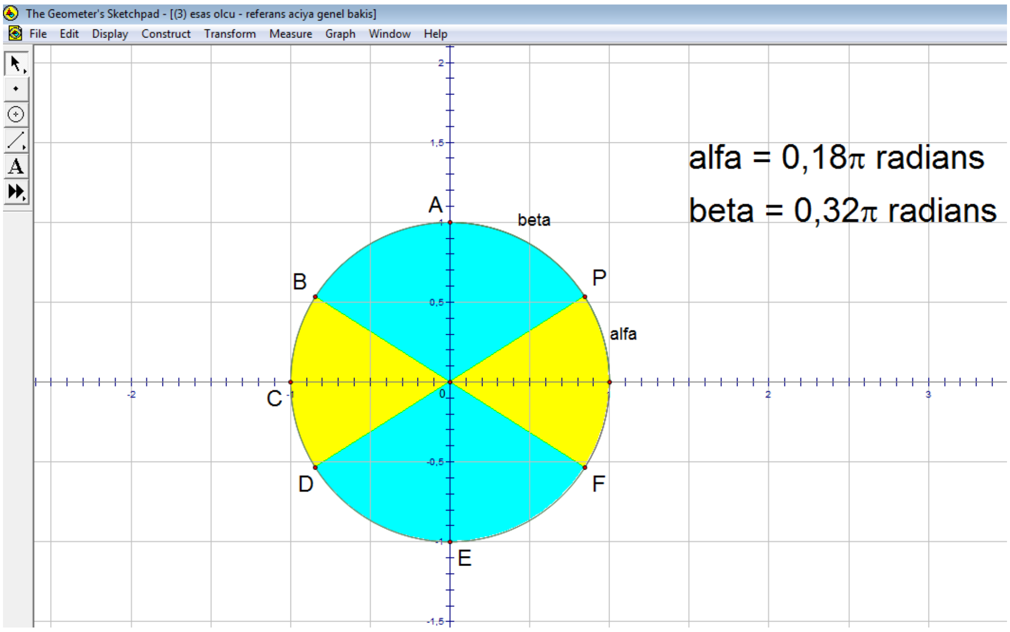


Figure 5.15. Angles in each quadrant with the same reference angle

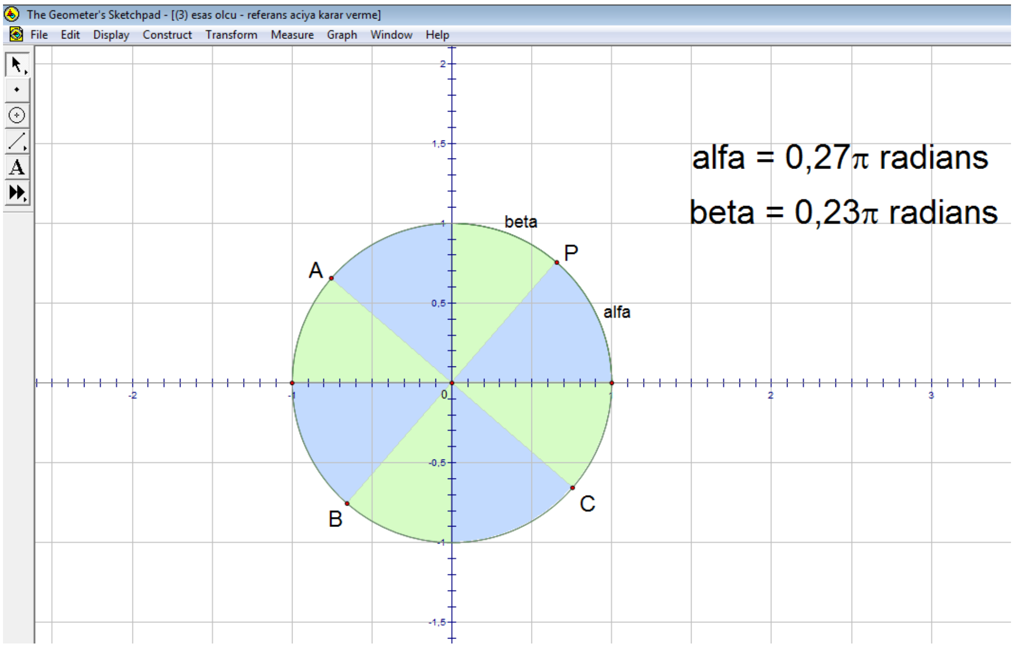


Figure 5.16. Angles in each quadrant with complementary reference angle

5.1.2.2. On trigonometric functions

As a result of students' problematic concept images on functionality that were revealed throughout the initial interviews, at the beginning of Task 4, the researcher defined the function concept as a relation between two sets (i.e., domain and range) with the property that each input in the domain must be related to exactly one output in the range. Then, she asked students to discuss whether the relation mapping the green arc into the red [blue] line segment was a function or not (*Figure 5.17*) without mentioning the sine [cosine] term. Where, it was observed that all of the students were able to reason about the functionality of this visual relation between the green arc and the red [blue] line segment in the (*unit*) *circle register* through investigating the variation on these two visual objects (e.g., [Defne & Ebru] Protocol 7) by the aid of the *drag-and-drop* option of GSP.

At this point, the researcher predicted that students would associate the measure of the red [blue] segment with some of three measures: *length* of the red [blue] segment, *sine* [*cosine*] of the alpha, and *ordinate* [*abscissa*] of the point *P* (*Figure 5.17*). Where, to provide students' concentration only on the directed-measure idea, the researcher preferred to measure length of the red [blue] segment in the unit of coordinate system on preconstructed GSP page (*Figure 5.17*). Before mentioning the definition of sine [cosine] as a function, she encouraged students to investigate the variation among these three measures when manipulating the position of the point *P* regarding the quadrants (e.g., lines 1-12, 20-21, 31 and 38 in [Cemre & Zafer] Protocol 7) hoping for students to construct well-structured cognitive networks among them. Initially, it was observed that when the point *P* was in the first quadrant, students reasoned about the equality of these three measures (e.g., lines 1-19 in [Cemre & Zafer] Protocol 7). However, when manipulating the position of the point *P* from the first quadrant to another quadrant, it was observed that all students focused only on the variation of signs (e.g., lines 20-26, 31-43 in [Cemre & Zafer] Protocol 7) instead of the relation between these three measures. Although considering dynamically linked measures, all students were able to reason easily and truly about the signs of the sine [cosine] values in each quadrant, they were unable to articulate why sine [cosine] took negative values in the third and fourth [second and third] quadrants (e.g., lines 27-30 in [Cemre & Zafer] Protocol 7).

At this point, the researcher predicted that this was due to their restricted concept definition images on the domain of sine [cosine] into the first quadrant in the (*unit*) *circle register*, which had been revealed throughout the initial interviews. Thus, she encouraged them to reason about the underlying causes of the variation in signs of the values of sine [cosine] regarding the quadrants (e.g., lines 1-19 in [Defne & Ebru] Protocol 8). Where, except Ebru, all other students reasoned about the variation on signs through associating sine [cosine] with the ordinate [abscissa] of the point *P* (e.g., lines 6-19 in [Defne & Ebru] Protocol 8) through considering *y*-axis [*x*-axis] as the sine [cosine] axis without mentioning the reference right triangle (e.g., lines 20-28 in [Defne & Ebru] Protocol 8). Their actions imply that their mental images related to sine [cosine] of an angle was including a set of geometric procedures based on the description of the ordinate [abscissa] of the point *P* (e.g., lines 12-16 in [Defne & Ebru] Protocol 8). On the other hand, Ebru's different point of view emerged on the definition of sine [cosine] based on the *directed-opposite* [*adjacent*] *length* of the reference right triangle regarding the quadrants within the (*unit*) *circle register* (e.g., lines 4-5 in [Defne & Ebru] Protocol 8), which was not revealed until that time. Moreover, in the following tasks, for example Task 5, 6 and 7, her language indicated that she reasoned in the same way about sine [cosine] of an angle in any quadrant within the (*unit*) *circle register*. This implies that anymore Ebru's mental image related to sine [cosine] of an angle just included a reference right triangle in each quadrant with the *directed-opposite* [*adjacent*] *side*. This mental image prompted a distinct shift on Ebru's reasoning about sine [cosine] of an angle in any quadrant within the (*unit*) *circle register*. That is to say, Ebru began to reason quickly and accurately about the signs of sine [cosine] regarding the quadrants within the (*unit*) *circle register* without trying to remember the memorized-rules in this regard, as well as the values of sine [cosine] for the angles corresponding to the axes such as 0 and $3\pi/2$ in radians (e.g., lines 50-72 in [Defne & Ebru] Protocol 8). Moreover, she began to dynamic reasoning about the change of the values of sine [cosine] with respect to the change of the angles in any quadrant (e.g., lines 71-72 in [Defne & Ebru] Protocol 8). However, other students did not rich this kind of mental image related to sine [cosine] of an angle within the (*unit*) *circle register* throughout Task 4. Therefore, in order to ease for them to develop such a mental image associating sine [cosine] of an angle with the *directed-opposite* [*adjacent*] *side* of the reference right triangle in any quadrant within the (*unit*) *circle register*, the researcher preferred in the following task, Task 5, to study on the unit circle integrated with a reference right triangle whose opposite [adjacent] leg was labeled as sine [cosine] (*Figure 5.19*).

At the beginning of Task 5, she asked students to interpret the value of sine [cosine] of the alpha angle without seeing the dynamically linked sin(alpha) [cos(alpha)] measure on the screen (e.g., lines 1-4 in [Cemre & Zafer] Protocol 8). Where, up to that time even though all students were able to interpret sine [cosine] of an angle as the ordinate [abscissa] of the point *P* both in the (*unit*) *circle register* and in the *language register*, it was interestingly observed that they were unable to convert their interpretations arithmetically into the *symbolic register*. That is, none of the students was aware of the determination of the sine [cosine] value of an angle in the *symbolic register* by using the coordinate axes as a signed-ruler in the (*unit*) *circle register* (e.g., lines 5-10 in [Cemre & Zafer] Protocol 8). They were only able to discuss the sine [cosine] values in terms of the upper and lower bounds based on the memorized rules (e.g., lines 11-20 in [Cemre & Zafer] Protocol 8). Students' language implies that their concept images on the values of sine [cosine] in the *symbolic register* were restricted to the memorized exact values of sine [cosine] at the special angles without any reasons. At this point, the researcher determined to provoke students to focus the red [blue] line segment in terms of its magnitude based on the unit of coordinate axes. For this purpose, she encouraged them to compare two visual objects, namely, the red line segment and the radius line segment on the positive *y*-axis in terms of their magnitudes through modeling these two objects with the real-life situations (e.g., lines 21-27 in [Cemre & Zafer] Protocol 8). Where, it was observed that all of the students were able to quickly and accurately compare these two visual objects in terms of their magnitudes (e.g., lines 21-44 in [Cemre & Zafer] Protocol 8) considering their signs (e.g., lines 45-46 in [Cemre & Zafer] Protocol 8).

After then, the researcher asked students to estimate the exact value referring to the *directed-red line segment* (e.g., line 47-48 in [Cemre & Zafer] Protocol 8). Students' estimations were almost same with the sine values of the mentioned angles in any quadrant (e.g., lines 47-63 in [Cemre & Zafer] Protocol 8). When the researcher unveiled the sin(alpha) measure on the screen which was given by GSP as the dynamically linked measure to the alpha measure, all students were surprised and excited (e.g., lines 64-68 in [Cemre & Zafer] Protocol 8). This implies that each student had just been able to convert the geometric procedures in their concept definition images on sine within the (*unit*) *circle register* into the values of sine within the *symbolic register*. Furthermore, when similar discussions were done in each quadrant for at least three different points, as well as for cosine, it was observed again that students were able to make quite closed predictions to the signed values of sine and cosine of the alpha angle which was constructed and measured in GSP so as to indicate the principal angle and the principal measure in radians.

At that point, the researcher determined that all of the students were able to transform the reference right triangle from the first quadrant into the other quadrants within the (*unit*) *circle register* thereby associating its opposite [adjacent] side regarding the reference angle with the sine [cosine] value of the principal angle. Moreover, they were able to convert the opposite [adjacent] side of the reference right triangle in any quadrant within the (*unit*) *circle register* into the value of sine [cosine] corresponding to the principal angle within the *symbolic register* thereby associating the directed-opposite [adjacent] side with the sign of the value of sine [cosine]. Besides, in the absence of the reference right triangle regarding the position of the point *P* on the axes, they were able to reason about sine [cosine] of these angles again as the **limit case** of the directed-opposite [adjacent] side; and they were able to convert the existing {non-existing} directed-opposite [adjacent] side within the (*unit*) *circle register* into ±1 {zero} as the value of sine [cosine] corresponding to these angles within the *symbolic register* through considering directions (e.g., lines 55-72 in [Defne & Ebru] Protocol 8). Furthermore, they started to reason about sine [cosine] as a function of the (principal) angle within the (*unit*) *circle register* (e.g., lines 76-83 in

[Cemre & Zafer] Protocol 8) on the contrary to their initial reasoning prior to the teaching experiment.

However, their reasoning like that was in the (*unit*) *circle register*. The researcher predicted that they would encounter troubles on converting this reasoning from the (*unit*) *circle register* into the *symbolic register*. Thus, hoping to observe these troubles, she encouraged students to discuss the sine [cosine] function within the *symbolic register* based on a sine [cosine] value for an arbitrary angle which was the first one that came to their mind (e.g., lines 1-2 in [Cemre & Zafer] Protocol 9). Although up to that time all discussions on the sine [cosine] function were done based on the angles in "radians" in GSP environment, it was observed that all students expressed a sine value of a real number, such as 30, as an angle in "degrees" that came first to their mind but without stating clearly their "degree" preference as the angle measure unit (e.g., lines 3-5 in [Cemre & Zafer] Protocol 9) as a consequence of their dominated concept images on angle measure units by "degrees" and their restricted concept images on sine [cosine] values by the memorized rules that was revealed throughout the initial interviews. At that point, the researcher determined to scrutinize students' awareness of the differentiation between the angle measure units for the domain of the sine [cosine] function in the *symbolic register*. So, she provoked students to reason about the different output of GSP as the sine [cosine] value of this real number when the angle measure preference of GSP in radians (e.g., lines 6-8 in [Cemre & Zafer] Protocol 9). Where, although all students were aware that the difference between two outputs (GSP's and theirs) arose from the difference between angle measure units (e.g., lines 9-37 in [Cemre & Zafer] Protocol 9), they did not articulate how the sine [cosine] value of this real number corresponded to the appeared output on the screen as a result of the calculation by GSP when the angle measure was in radians (e.g., lines 9-10 and 38-45 in [Cemre & Zafer] Protocol 9). It may arise from GSP's angle measure defaults in radians which produce an angle measure with π notation. So, the researcher encouraged them to reason about an angle's measure in radians both with and without π notation through taking the advantage of GSP's calculate option. Where, it was observed that all students were able to reason about a real number without π notation, for example 30, as an angle measure in radians through transforming it into the symbolic form with π notation (e.g., lines 46-80 in [Cemre & Zafer] Protocol 9) as well as considering π with its approximate real value, i.e., 3.14. It was the point that students had just started to reason about a real number as an angle measure with two different ways through considering angle measure units. In other words, they were able to convert a real number in the *symbolic register* into two different angles within the (*unit*) *circle register* through considering two angle measure units, i.e., degrees and radians. Conversely, they were able to convert a *static angle structure* within the (*unit*) *circle register* into two different real numbers as its principal measure in the *symbolic register* through considering different angle measure units. This means that anymore all students were aware of the importance of the preference of the angle measure unit[33] for the real value of an angle measure, as well as for the value of sine [cosine].

However, the researcher determined that students would encounter troubles on extension of this reasoning between a real number in the *symbolic register* and a *static angle structure* within the (*unit*) *circle register* into the reasoning between the real number set (R) in the *symbolic register* and the *dynamic angle structures* in the (*unit*) *circle register*. Thus, hoping to observe these troubles, she encouraged them to discuss about the relation between measures of the angles with the **same** *static structure* but **different** *dynamic structures* in the (*unit*) *circle register*. Where, based on these angles' **same** *static structure* producing the **same** sine [cosine] value within the (*unit*) *circle register*, Defne and Ebru initially reasoned about their measures as if equal to each other in the *symbolic register* (lines 1-15 in [Defne & Ebru] Protocol 9). After the researcher's provocation of them to discuss this equality as an equation on the paper and pencil environment (lines 16-17 in [Defne & Ebru] Protocol 9), they tried to focus on what was mathematically same in this respect; and reasoned about their difference in terms of measures in the *symbolic register*, as well as their sameness in terms of *static angle structures* in the (*unit*) *circle register* (lines 18-24 in [Defne & Ebru] Protocol 9). On the other hand, it was observed that based on the corresponding arc lengths to the angles with the **same** *static structure* but **different** *dynamic*

[33] From this task forward, throughout the teaching experiment, the radian measure unit was used as the angle measure unit.

structures in the (*unit*) *circle register*, Zafer reasoned that they produced different angle measures in the *symbolic register* (lines 1-14 in [Cemre & Zafer] Protocol 10). Where, Cemre's language and actions imply that she also reasoned in the same way but through being affected by Zafer's reasoning (lines 10-22 in [Cemre & Zafer] Protocol 10). It was the point that students had just started to reason about each *dynamic angle structure* with the **same** *static structure* as a different angle in the (*unit*) *circle register* with a different measure in the *symbolic register* thereby associating them with the same sine [cosine] value. Moreover, they were aware of the repetition of the sine [cosine] values regarding the equivalent but not equal angles.

This kind of reasoning on the equivalent but not equal angles was vital for the recognition of the sine [cosine] function's periodic nature in all representational registers. It was because extending the domain set of the sine [cosine] function from the principal measures into the real number set (R) and seeing the patterns among them under the sine [cosine] function were the base of the periodicity. In other words, the recognition of periodicity got easier through assigning different angles with the same principal measure into the same sine [cosine] value rather than merging the equivalent but not equal angles into the principal measure, and then assigning this principal measure into its sine [cosine] value. Moreover, this kind of reasoning was also important to convert sine [cosine] from the (*unit*) *circle register* into the *graphical register*. So, in Task 6 [Task 7], the researcher preferred to encourage students to discuss the sine [cosine] function in a more detailed way in the *graphical register* as well. Henceforward, the subsequent progress of students' concept images on the sine and cosine functions that emerged as a result of the teaching experiment was presented under the following respective headings in terms of all different representational registers through considering students transformation abilities regarding *angles* as the domain of trigonometric functions, *trigonometric values* as the ranges of trigonometric functions and *periodicity* as the pattern based on behaviors of trigonometric functions both between and within the registers.

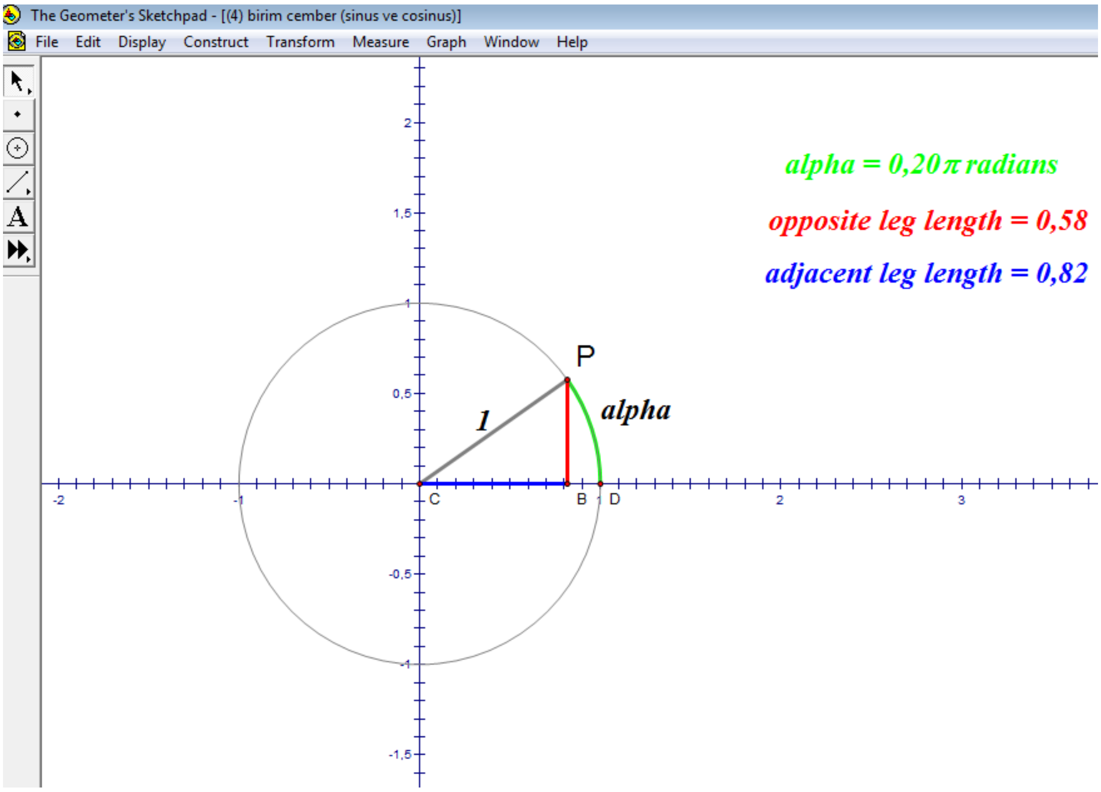


Figure 5.17. Reference right triangle and principal arc on the unit circle

[Defne & Ebru] Protocol 7

Researcher: Well, I am mapping this green arc (*dragging her index finger on the green arc in the counterclockwise direction on the screen like in Figure 5.17*) into this red segment (*dragging her index finger on the red segment up-and-down*). Is this mapping a function, or not?
Defne&Ebru: (*Looking to the screen without speaking*)
Researcher: Ok. For each green arc, can we find a red segment?
Ebru: (*Dragging the point P on the circle in the counterclockwise direction*) yes.
Researcher: Well, for each green arc, can we find two different red segments?
Ebru: No… No… only one.
Defne: (*Nodding her head up and down*).
Ebru: When green [arc] is increasing in size, each green [arc] corresponds to only one red [segment].
Defne: Uh-huh.
Researcher: Ok. Then, is this relation mapping the green arc into the red segment a function?
Defne&Ebru: Yes.
(*Similar discussions were done on the relation mapping the green arc into the blue segment*).

[Cemre & Zafer] Protocol 7

Researcher: You know about the coordinate system. I want you to find coordinates of the point P by using the measure menu.

Zafer: They [coordinates] would be the same as sin(alpha) and cos(alpha)…

Cemre: (*Measuring separately abscissa and ordinate of the point P when it is in the first quadrant of the unit circle*) *x* [abscissa] is same with this (*pointing respectively x_P and adjacent leg length on the screen like in Figure 5.18*) *y* is same with this (*pointing respectively y_P and opposite leg length on the screen*).

Researcher: Zafer, please find sine and cosine values of the alpha angle.

Zafer: (*Calculating sin(alpha) and cos(alpha) values by calculate option of GSP, at this time the point P is still in the first quadrant*) Yes, they are same.

Cemre: Uh-huh.

Researcher: Please put the same measures alongside.

Cemre: (*When the point P is in the first quadrant*) sine is same with this (*putting sin(alpha) on the right side of the opposite leg length*).

Zafer: …*y* is also same with them.

Cemre: Uh-huh (*putting y_P on the left side of the opposite leg length*). These three ones also are same each other (*putting x_P, adjacent leg length and cos(alpha) alongside like in Figure 5.18*).

Zafer: (*Nodding his head up and down*).

Researcher: Well, see the variation in all these measures when dragging the point *P* (*pointing separately two measurement rows on the screen like in Figure 5.18*).

Zafer: (*Dragging the point P on the unit circle from the first quadrant to the second quadrant in the counterclockwise direction*).

Cemre: (*When the P is turned into the second quadrant*) *x* happened minus.

Zafer: Because *x* passed in the minus region.

Cemre: Cosine also happened minus… …because cosine is minus in the second quadrant.

Researcher: Why is minus in the second quadrant?

Cemre: This is so (*smiling*)… I don't know its reason.

Researcher: What about you, Zafer?

Zafer: We know it as a rule…

Researcher: Ok. Then, we continue to drag the point P and to look the variation.

Cemre: (*Dragging the point P on the unit circle from the second quadrant to the third quadrant in the counterclockwise direction*) …in the third quadrant, both happened minus (*pointing sin(alpha) and cos(alpha) measures on the screen*)…

Zafer: Lengths are always plus (*pointing the opposite leg length and the adjacent leg length on the screen*)…

Cemre: Uh-huh (*nodding his head up and down*).

Researcher: Ok. What about fourth quadrant?

Zafer: (*Dragging the point P on the unit circle from the third quadrant to the fourth quadrant in the counterclockwise direction*) …in the fourth quadrant, sine is minus, cosine is plus (*pointing sin(alpha) and cos(alpha) measures on the screen*)… so do *y* and *x* (*pointing y_P andx_P measures on the screen*)…

Cemre: Uh-huh (*nodding her head up and down*).

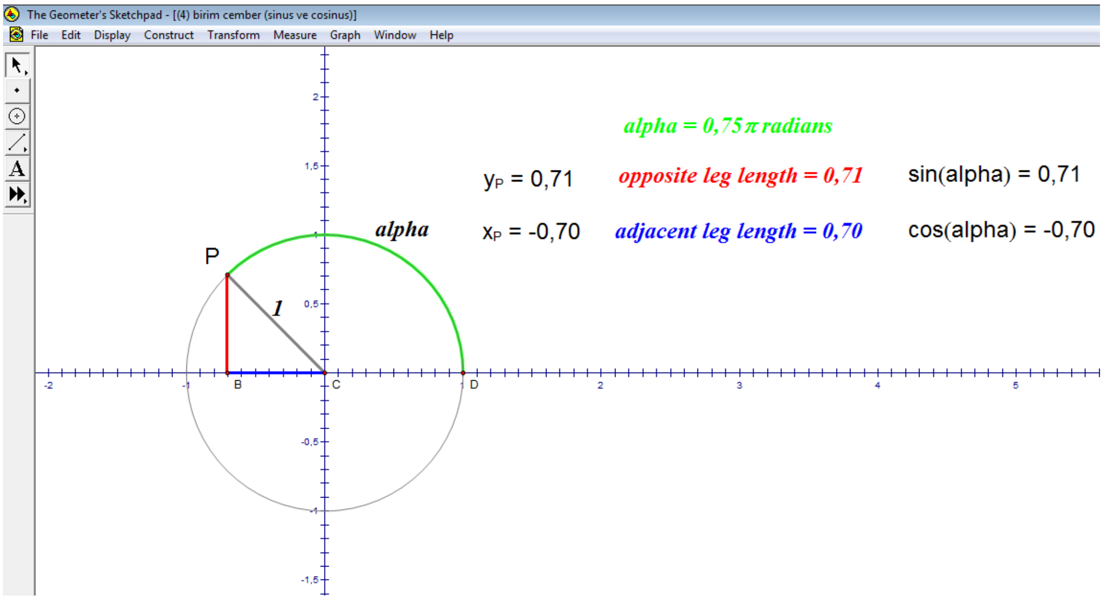


Figure 5.18. Relations among ordinate [abscissa] of *P*, length of the red [blue] segment, sine [cosine] of alpha

[Defne & Ebru] Protocol 8

Researcher: What would be the reason of this variation? What do you think?

Ebru: This is due to trigonometric ratios…

Researcher: How?

Ebru: That is, due to… …cosine is adjacent [side length] divided by 1 [hypotenuse]… and its value [adjacent side's length] is minus.

Defne: Because for an angle we specify on the unit circle… …the values of sine or cosine are minus… …regarding the region in which the point is…

Researcher: Good. This is true but why this is so? For example, why is sine plus in the second quadrant, and minus in the third quadrant? Can we assume for example, in the first quadrant all of them would be minus, in the second quadrant while one of them would be minus, other would be plus, etc.

Defne: For example, here (*pointing the point P on the screen which is in the third quadrant at that time*), *x*-axis is minus (*pointing a perpendicular line segment from the point P to the x-axis*), *y*-axis also minus (*pointing a perpendicular line segment from the point P to the y-axis*)… then both of them [sine and cosine] are minus here.

Researcher: Ok. I understand the coordinates of *P* in here (*pointing the point P on the screen which is in the third quadrant at that time*) are minus. But why sine and cosine are minus here?

Defne: Because *x*-axis is the cosine axis and *y*-axis is the sine axis…

Researcher: Can you point to the cosine axis on the screen?

Defne: Here (*pointing the whole x-axis*).

Researcher: What is the highest value that cosine takes?

Defne: Cosine takes highest 1 as the value.

Researcher: So, why the whole *x*-axis is cosine axis?
Ebru: It takes its values regarding the unit circle.
Defne: Yes, so, [it takes values] between -1 and +1. If the circle has 2-unit radius, then it [cosine] would take values between -2 and +2.
Ebru: But we know as a rule… …they [cosine’s values] must be between -1 and 1.
Researcher: Well, we talked about sine and cosine in the first task. How is found cosine of an angle on the right triangle?
Defne&Ebru: The ratio of adjacent side by hypotenuse…
Researcher: Well, in a right triangle, which side has the highest length?
Defne: Hypotenuse…
Ebru: Yes.
Researcher: Then, can the ratio of adjacent side by hypotenuse be greater than 1?
Defne: No…
Researcher: Then, can we say if the radius was 2, then the cosine would be between -2 and 2?
Defne: Hmm, you’re right. It [that cosine is abscissa] is true only on the unit circle… …yes.
Ebru: Uh-huh.
Researcher: Look this right triangle (*drawing the reference right triangle with her index finger on the screen when it is in the first quadrant*). If you were asked about sine of this acute angle (*pointing the alpha angle of the right triangle*), what would you say?
Defne: (*Looking to the screen*) red [opposite side] divided by grey [hypotenuse]… grey is 1, so, [sine of alpha is] red…
Ebru: (*Looking to the screen*) yes.
Researcher: What about cosine of the same angle?
Defne: Blue is the cosine (*pointing the blue segment from left to the right with her index finger on the screen*)…
Ebru: Uh-huh.
Researcher: Ok. What is sin(0)?
Defne & Ebru: sin(0) is zero…
Defne: (*Dragging the point P so as to be on the intersection point of the unit circle with the positive x-axis*).
Researcher: How do you define sine of zero? There is no right triangle here.
Defne: Anymore, we consider it with the ordinate of *P*.
Researcher: If you consider directly the red segment [opposite side], is there a red segment at that time (*when the point P is on the (1,0) point*)?
Ebru: Red segment doesn’t exist. Then, sin(0) is zero.
Defne: Uh-huh.
Researcher: Well, what about the blue segment at that time (*when the point P is on the (1,0) point*)?
Defne: Blue segment exist… …and it is 1 [unit length]. So, cos(0) is one.
Ebru: Yes.
Researcher: Ok. What about sin(3π/2)?
Defne: (*Dragging the point P in the clockwise direction from (1,0) to (0,-1) on the unit circle*).
Ebru: [sin(3π/2) is] minus one. When we are going to 3π/2, red segment is going towards minus 1.

Researcher: Ok. How do you determine sine value of an angle?

Defne: On the [reference] right triangle… …due to its correspondence to the opposite side, we say red segment as sine (*dragging her index finger on the red segment up and down*)…

Ebru: Since on the [reference] right triangle (*pointing the reference right triangle on the screen*) red one indicates sine, when we bring it somewhere [in any quadrant] on the unit circle… …we see that red segments' length is plus or minus.

[Cemre & Zafer] Protocol 8

Researcher: I will hide sine and cosine values on the screen (*hiding sin(alpha) and cos(alpha) values with a serviette so as to appear only the unit circle on the screenshot of the GSP page like in Figure 5.19*). Now, I would want you to find these values.

Zafer: Ah! That is, it… …zero point so and so would we find (*looking to the researcher when raising his eyebrows and smiling*)?

Cemre: (*Laughing*) no…

Researcher: If I asked you to find a value of sine or cosine like zero point so and so… …This would not be easy, wouldn't it?

Cemre&Zafer: Yes, of course (*smiling*).

Researcher: So, I would not want you to do this. I want you to discuss only the sine value for the alpha angle (*pointing the green arc on the screen like in Figure 5.19*)… …greater or smaller than one, or equal to one?

Cemre&Zafer: Hmm…

Zafer: (*Turning his view from the screen to the upward*) it [sin(alpha)] is smaller than 1… …because it could not be greater than 1.

Cemre: Exactly…

Researcher: Why?

Zafer: Because it is between minus 1 and plus 1.

Cemre: Uh-huh (*nodding her head up and down*).

Researcher: Ok. Well, we suppose that you stay in this building (*pointing the radius line segment on the positive y-axis on the screen with his index finger through dragging up and down*), and I stay in this one (*pointing the red line segment on the screen with his index finger through dragging up and down*)… You are in a building with this height (*putting her thumb and index fingers respectively zero and 1 points on the y-axis*)… I am in one with this height (*putting her thumb and index fingers end points of the red line segment*)… …which of them is higher?

Cemre&Zafer: (*Immediately*) ours…

Researcher: Then, how is mine regarding 1 (*the point P in the first quadrant at that time*)?

Cemre&Zafer: Smaller…

Researcher: Ok, what about now (*dragging the point P in the counterclockwise direction but at that time it is still in the first quadrant*)?

Cemre&Zafer: Smaller [than 1]…

Researcher: Ok, what about now (*dragging the point P in the counterclockwise direction but at that time it is still in the first quadrant but too close to the y-axis*)?

Cemre: Close to [1]…

Zafer: But still smaller [than 1]…

Researcher: Ok, what about now (*dragging the point P in the counterclockwise direction but at that time it is in the second quadrant*)?
Cemre: Small…
Zafer: Still smaller [than 1]…
Researcher: Ok, what about now (*dragging the point P in the counterclockwise direction but at that time it is in the third quadrant*)?
Cemre: Small (*smiling*)…
Zafer: Anymore, you are in minuses (*smiling*).
Cemre: Yes (*smiling*).
Researcher: Ok. I ask you (*dragging the point P in the second quadrant*)… What is the height of the building I stay?
Cemre&Zafer: (*Coming close to the screen*)
Zafer: Zero point…
Cemre: …seven or so…
Zafer: Yes.
Cemre: If we do like that (*pointing the point P and dragging a perpendicular line to the y-axis*)…
Researcher: Do you want to draw a perpendicular line to the *y*-axis?
Cemre: Yes.
Researcher: (*Constructing the perpendicular line to the y-axis*).
Zafer: It [the length of the red segment] is between 0.6 and 0.7…
Cemre: It is closer to the 0.6…
Researcher: What can be it?
Zafer: 0.63…
Cemre: (*Smiling*) I say 0.62…
Researcher: (*Unveiling the sin(alpha) measure on the screen*).
Cemre&Zafer: (*When seeing that sin(alpha) value measured by GSP is too close to their predictions*) Ah (*laughing*)!
Cemre: Oh! I don’t believe! We said sine of zero point 78 [π radian] (*smiling*)!
Researcher: Yes.
Zafer: Great!
Researcher: Ok. If we want to write this, what do you write?
Zafer: (*Looking to the screen*) sine of… …zero point 78π…
Cemre: …is equal to zero point 62…
Zafer: Uh-huh.
(*Similar discussions were done in each quadrant at least three different points, as well as cosine; students made too closed predictions of sine and cosine*.)
…
Researcher: Well, is this relation between the angle and corresponding sine… …a function?
Cemre&Zafer: Yes.
Researcher: Why?
Cemre: Because each angle (*dragging her index finger on the green arc in the counterclockwise direction on the screen*)… …gives [only] one sine (*dragging her index finger on the red segment up-and-down*).
Zafer: Uh-huh (*nodding his head up and down*).

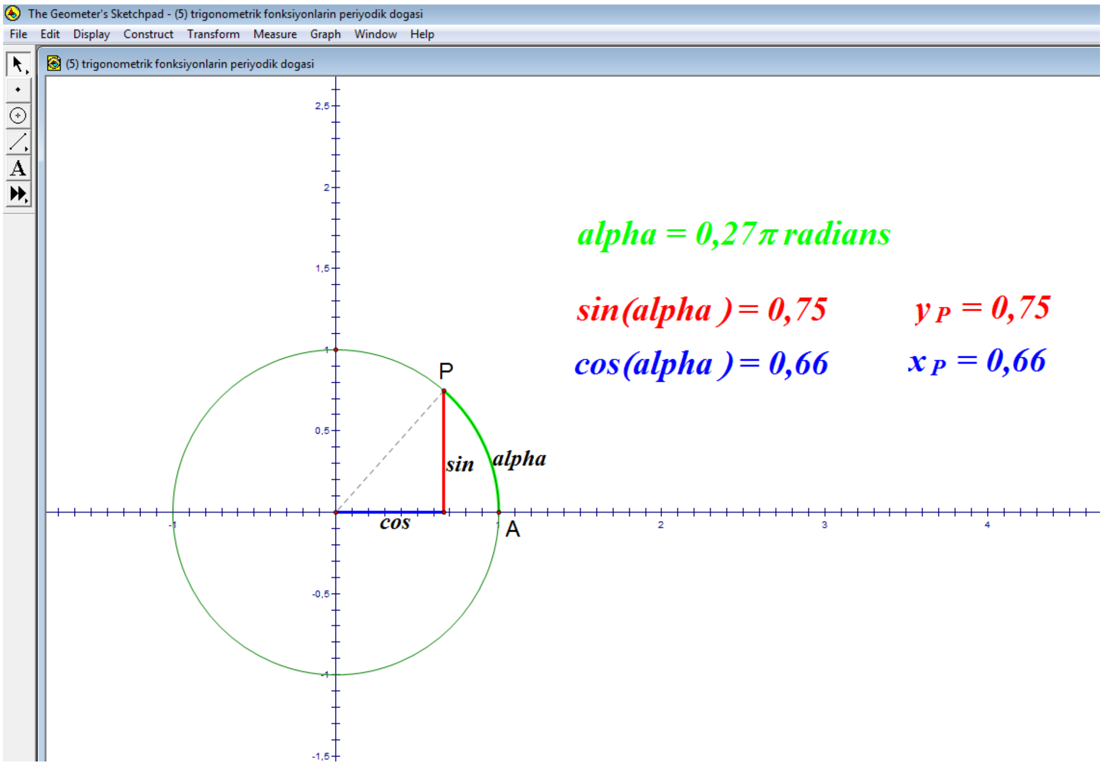


Figure 5.19. Reference right triangle with legs entitled by the related trigonometric expression

[Cemre & Zafer] Protocol 9

Researcher: You defined sine and cosine as functions of the angle... Is there any sine value for whatever an angle you remember?

Cemre: For example sine thirty [sin(30)]… is (*turning her view from screen to the upward*) 1/2, isn't it?

Zafer: Uh-huh.

Researcher: (*When the angle measure preference of GSP was in radians, calculating the sin(30)*) is 1/2 equal to minus zero point 99 (*pointing the calculation output on the screen*)…

Cemre: Just a minute! Sine thirty… …1/2… Then, this is in radians (*pointing the calculation result on the screen with her pencil*)… If we say π/6…

Zafer: …then, sine happens 1/2.

Cemre: Uh-huh.

Zafer: Let's we calculate and look (*calculating sin(π/6) by GSP measure menu*)! Yes.

Cemre: (*Writing paper sin(30)=1/2 and sin(π/6)=1/2 alongside*) Teacher! But, these are in domain (*pointing 30 and π/6 on her writing*)… …but things they go [in the range] are same (*pointing both 1/2 on her writing*), is that so?

Researcher: But look! The output of sin(30) is different from 1/2 (*pointing the expression* $sin(30) = -0.99$ *on the screen*).

Cemre: But was not sin(30) 1/2!

Researcher: What could be the reason of this?

Zafer: It [GSP] accepts a different thing as sin(30)… …may consider π… …It [30] is not in degrees we have known…. It is not 30 degrees (*looking to the expression* $sin(30) = -0.99$ *on the screen*).

Researcher: Cemre, your 30 is in degrees (*pointing 30 on her writing*)?

Cemre: Yes, sine thirty degree (*adding degree symbol to 30 on her writing*)…

Researcher: Let's we look our current angle measure preference in GSP (*opening "Preferences" window of GSP under the Edit menu*).

Cemre&Zafer: … radians (*when seeing the current angle measure preference of GSP as radian*).

Researcher: Let's we change our angle measure preference as degrees (*changing the angle measure unit of GSP from radians to degrees*).

Cemre: It [output of sin(30)] happened zero point fifty (*pointing the expression* $sin(30) = 0.50$ *on the screen which was dynamically changed regarding the angle measure preference*)…

Researcher: So, you have to identify your preference as angle measure unit…

Cemre: You are right… …yes.

Zafer: (*Nodding his head up and down*).

Researcher: Ok. Well (*changing again the angle measure unit of GSP from degrees to radians*), how did sine thirty in radians mean minus zero point 99 (*pointing the expression* $sin(30) = -0.99$ *on the screen*).

Zafer: (*After 4-second pause*) if it was 30π… It would correspond to here (*pointing the intersection point of the unit circle and the positive x-axis*)… so it shouldn't be 30π…

Researcher: What else?

Cemre&Zafer: (*Looking to the screen without speaking*)

Researcher: Let's we look how many times of π is 30 (*opening calculate menu of GSP*)… 30 divided by…

Zafer: π... (*When the output of the calculation was appeared on the screen like in Figure 5.20*) nine point fifty five…

Researcher: That is, 30 is equal to 9.55π, isn't it (*pointing respectively 30, 9.55 and π on the expression 30/π=9.55*)?

Cemre: Yes.

Zafer: Uh-huh (*nodding his head up and down*).

Researcher: Ok. Where is the position of the point P… …for the angle with the measure 9.55π radian on the unit circle? Please drag the point P on the true position.

Zafer: (*Dragging the point P in the counterclockwise direction*)

Cemre: Eight [π radian]… …1.55 [π radian]… …in the fourth quadrant.

Zafer: (*Continuing to drag the point P until the measure of the alpha angle would appear as 1.55π radian on the screen line in Figure 5.20*) here it is.

Researcher: Looking to the unit circle, can you predict the sine value at that point (*pointing the point P on the screen*)?

Cemre: It is much closed to 1…

Zafer: …minus one…

Cemre: Of course. (*After 5-second pause*) now, here (*pointing 30 in the expression* $sin(30) = -0.99$ *on the screen*) 30 is in radians, isn't it?

Researcher: Yes.

Cemre: But [sine] of 30 degree is 0.50, isn't it!

Researcher: Uh-huh.
Cemre: I have understood.
Zafer: (*Nodding his head up and down*). (*After 4-second pause*) we consider 30 as 9.55π.
Researcher: Exactly. 9.55 times 3.14 are equal to 30 (*pointing respectively 9.55, π and 30 on the expression 30/π=9.55 on the screen*)… …
Zafer: Uh-huh (*nodding his head up and down*).
Cemre: I have understood.
Zafer: …then, we don't put this symbol here (*pointing π notation in the angle measure*), do we?
Cemre: Uh-huh, π wouldn't exist.
Zafer: It would be 30 radians.
Researcher: Yes.
Cemre: Uh-huh (*nodding her head up and down*).

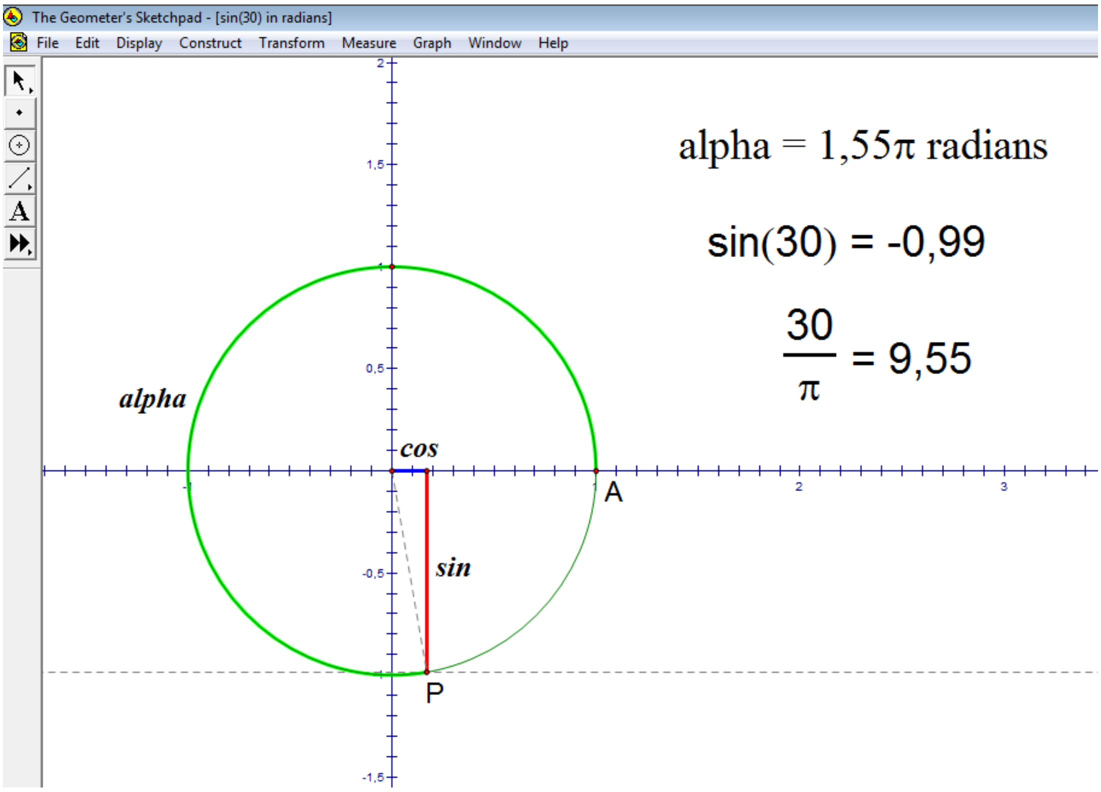


Figure 5.20. An example of a real number's consideration as an angle measure in radians

[Defne & Ebru] Protocol 9

Researcher: Can you show me… …where is zero π radian-angle?
Defne: Here (*dragging point P on the positive x-axis*).
Researcher: Well, what about 2π radian-angle?
Ebru: Same where again…
Researcher: Then, are zero π radian and 2π radian same?
Ebru: Yes.
Defne: (*Nodding her head up and down*).
Ebru: Their things are same… that is, their cosine lengths are same… …their sines values also are same…
Researcher: I don't ask you their sine and cosine. I ask only that as an angle measure, two are same. Are zero π radian and 2π radian same?
Defne&Ebru: Yes, they are same…
Researcher: What about 4π radian?
Defne: It is also same.
Ebru: Yes.
Researcher: Look (*Writing* $2\pi = 0\pi$ *on a paper*)! Then, if we divide two sides by π, 2 must be equal to zero.
Defne: Thus, they are not same!
Ebru: Then, we would say that… …they are angles corresponding to the same point on the unit circle.
Defne: Ah yes! None of them is equal, but they show the same place [on the unit circle]. In second, third rounds… …again they aren't equal to each other but due to pointing same where [on the unit circle], we assume them as equivalent.
Ebru: Uh-huh (*nodding her head up and down*).

[Cemre & Zafer] Protocol 10

Researcher: Because we are working in the unit circle, you know 0.26π radian (*pointing the alpha measure on the screen*) is this arc's length (*drawing the principal arc in the counterclockwise direction on the screen with her index finger*).
Cemre: Yes.
Zafer: Uh-huh (*nodding her head up and down*).
Researcher: If I ask you to 2.26π radian, what would you say?
Zafer: One full-round more (*pointing a full round with his index finger on the screen in the counterclockwise direction through starting from the point P*)…
Cemre: Uh-huh (*nodding her head up and down*). Their sines are same…
Researcher: What about these two angles?
Zafer: They are different things.
Cemre: (*Listening without speaking*).
Researcher: Why?
Zafer: …because their lengths are different.
Cemre: Hmm, yes…

Researcher: That is, 0.26π and 2.26π are different two elements in the domain of sine and cosine, aren't they?
Zafer: Uh-huh (*nodding his head up and down*).
Cemre: …as well as 4.26π, 6 [.26π], 8 [.26π] are like that…
Zafer: (*Nodding his head up and down*) …also 0.26π minus 2π…
Cemre: (*Nodding her head up and down*)…minus 4[π], minus 6[π]… Uh-huh… I understand.

5.1.3. Development of Students' Concept Images Rooted in (Unit) Circle Register and Shaped between Graphical Register and Others

5.1.3.1. Regarding angles

At the end of Task 3, all students were able to associate a *static angle structure* in the (*unit*) *circle register* with the infinitely many [negative or positive] equivalent measures in the *symbolic register* through considering *dynamic directed turnings*. Inversely, they were able to associate the infinitely many equivalent measures in the *symbolic register* with the **same** *static structure* in the (*unit*) *circle register*. Afterwards, at the end of Task 5, they had just started to differentiate angles with the **same** *static structure* but **different** *dynamic structures* in the (*unit*) *circle register*, as well as angles with the equivalent but not equal measures in the *symbolic register*. However, the researcher determined that students' this differentiation would need these angles' differentiated visual representation. So, in order to support students' differentiation of the equivalent but not equal angles, in Task 6 [Task 7], the researcher encouraged students to interpret angles on the sine [cosine] graph in the *graphical register*, as well.

Since Task 6 [Task 7] was the first introduction of the sine [cosine] function in the *graphical register*, in order to provide for students with the smooth transitions among the (*unit*) *circle register*, *symbolic register* and *graphical register*, the researcher preferred to start with discussions thereby restricting angles into the principal angles. So, she encouraged them to interpret angles in the *graphical register* initially as the *x*-component of the ordered pair (*alpha*, *sin*(*alpha*)) [(*alpha*, *cos*(*alpha*))] that was constructed as dynamically-linked to the alpha principal angle in the (*unit*) *circle register,* as well as dynamically-linked to the alpha principal measure in the *symbolic register* (e.g., lines 1-15 in [Cemre & Zafer] Protocol 11). Where, it was observed that only Zafer had a trouble in making sense of the *x*-component as the angle measure due to his confusion between the angle measure and its sine value in the *graphical register* (lines 27-45 in [Cemre & Zafer] Protocol 11) until the researcher's recommendation to investigate the variation of the ordered pair (lines 46-47 in [Cemre & Zafer] Protocol 11). When investigating the variation of the ordered pair, he determined the *x*-component as the angle through focusing on (*i*) the π-radian angle in the (*unit*) *circle register*, (*ii*) the corresponding abscissa to about 3.14 in the *graphical register*, and (*iii*) the meaning of π as the angle measure in radians with its meaning as a real number (i.e., approximately 3.14) in the *symbolic register* (lines 48-55 in [Cemre & Zafer] Protocol 11). This implies that on the contrary to the initial interview results anymore "coordinate plane" was not a *cognitive conflict factor* when the position of π on the *x*-axis is considered simultaneously within the *graphical register* and the (*unit*) *circle register*. Moreover, it was observed that all students reasoned on an arbitrary angle in the same way within each representational register (e.g., lines 50-76 in [Cemre & Zafer] Protocol 11; lines 11-30 in [Defne & Ebru] Protocol 11). In other words, they were able to convert a principal angle in one register into its correspondence in another register (e.g., lines 86-90 in [Cemre & Zafer] Protocol 11). For example, when asked them to reason about the sine [cosine] value of a real number smaller than 2π in the *symbolic register*, students were able to convert this real number as an angle measure in radians within the *symbolic register* into the same real number on the *x*-axis as the corresponding angle within the *graphical register* (e.g., lines 31-41 in [Defne & Ebru] Protocol 11).

But, when asked to reason about the sine value of a real number greater than 2π in the *symbolic register*, they initially preferred to reason about the corresponding angle to this real number by its principal measure (e.g., lines 42-48 in [Defne & Ebru] Protocol 11). And then, when the researcher asked them to reason about the positions of this real number and its principal measure on the *x*-axis in the *graphical register* (e.g., lines 49-50 in [Defne & Ebru] Protocol 11), students' actions imply that they were able to differentiate the equivalent but not equal angles' positions on the *x*-axis with the 2π-length intervals in the *graphical register* based on their ability to convert the **continuously-repeated full-round turnings** in the counterclockwise [clockwise] direction in the (*unit*) *circle register* into the **continuously-repeated regular intervals** on the *x*-axis in the positive [negative] direction in the *graphical register* (e.g., lines 51-83 in [Defne & Ebru] Protocol 11; lines 32-89 in [Cemre & Zafer] Protocol 12). In addition, they were able to differentiate the equivalent but not equal angles' measures in the *symbolic register* through reasoning either from the *graphical register* (e.g., lines 69-83 and 100-106 in [Defne & Ebru] Protocol 11) or from the (*unit*) *circle register* (e.g., lines 44-76 in [Cemre & Zafer] Protocol 12). It was the point that students started to reason about the real number set (R) in the *symbolic register* as the domain set of the sine [cosine] function via thinking the angle concept in the *graphical register* as a continuous and repeated variable on the *x*-axis. Moreover, they started to reason about the angles through converting the **dynamic directed arcs** in the (*unit*) *circle register* into the **dynamic directed line segments** in the *graphical register* (e.g., lines 86-90 in [Cemre & Zafer] Protocol 11). This reasoning prompted a distinct shift on their reasoning about *x*-axis as the domain set of the sine [cosine] function in the *graphical register*. That is to say, they began to distinguish the parts of the sine [cosine] graph regarding the quadrants on the *x*-axis in the *graphical register* (e.g., lines 11-23 in [Defne & Ebru] Protocol 11).

Besides, they had just started to ascribe the role of the reference angle to the right cause on the sine [cosine] values in any quadrants. In other words, they had just started to compare two different principal angles with the same sine [cosine] value through considering the angles as a continuous and independent variable of the sine [cosine] function (e.g., lines 84-92 in [Defne & Ebru] Protocol 11). Moreover, they extended their ability to express an angle in any quadrant in terms of its reference angle which was observed initially at the end of Task 3 through comparing its sine [cosine] value with that of the reference angle (e.g., lines 93-106 in [Defne & Ebru] Protocol 11). Henceforward, the subsequent progress of students' concept images on angles that emerged as a result of the teaching experiment was presented under the following respective headings in terms of students' *discrimination* of angles represented within any representational register from the respective representational registers' contents.

5.1.3.2. Regarding trigonometric values

At the end of Task 5, all students were able to associate the directed-opposite [adjacent] side of the reference right triangle with the sine [cosine] value of the corresponding angle not only inside of the quadrants but also in the limit cases within the (*unit*) *circle register*. Moreover, they were able to convert the opposite [adjacent] side of the reference right triangle in any quadrant within the (*unit*) *circle register* into the sine [cosine] value of the corresponding angle within the *symbolic register* thereby associating the direction of the opposite [adjacent] side with the sign of the value of sine [cosine]. Furthermore, they were able to associate the *y*-component [*x*-component] of the point on the unit circle with the sine [cosine] value of the corresponding angle within the (*unit*) *circle register*. However, throughout the initial interviews, since the "coordinate plane" was observed as a *cognitive conflict factor* when the *graphical register* and the (*unit*) *circle register* were considered simultaneously, the researcher predicted that students would encounter troubles on conversions of the sine and cosine functions between the (*unit*) *circle register* and the *graphical register*. So, she determined to discuss these two functions in separate tasks in order to support students' differentiation of the contents of the coordinate planes in the (*unit*) *circle register* and the *graphical register*.

On the one hand, in Task 6, she encouraged students to interpret the sine value in the *graphical register* as the *y*-component of the ordered pair (*alpha*, *sin*(*alpha*)) that was constructed as dynamically-linked to the directed-opposite side of the reference right triangle in the (*unit*) *circle register*, as well as dynamically-linked to the sine value of the alpha angle in the *symbolic register*. Where, it was observed that since the sine value was represented with the ordinate in both of the (*unit*) *circle register* and the *graphical register*, none of the students had troubles about converting the sine value between the registers (e.g., lines 77-85 in [Cemre & Zafer] Protocol 11;

lines 31-48 in [Defne & Ebru] Protocol 11). Students' actions and language imply that they were able to quickly and accurately interpret the variation of the sine values in both registers based on the variations in the visually-same-direction of the *y*-components of the point *P* in the (*unit*) *circle register* and the point *P'* in the *graphical register* (*Figure 5.22*) (e.g., lines 27-33 in [Cemre & Zafer] Protocol 11; lines 11-23 in [Defne & Ebru] Protocol 11).

On the other hand, in Task 7, when asked students to interpret the cosine value in the *graphical register* as the *y*-component of the ordered pair (*alpha*, *cos*(*alpha*)) which was constructed as dynamically-linked to cosine in the (*unit*) *circle register*, students were able to again accurately interpret the variation of the cosine values but initially through focusing only on the variation in one register. That is, they interpreted the variation of the cosine values either based only on the variation of the *x*-component of the point *P* in the (*unit*) *circle register* or based only on the variation of the *y*-component of the point *P'* in the *graphical register* (*Figure 5.23*). Moreover, each student felt the need at least once to be confirmed about his/her understanding about the conversion of the cosine value as from the *x*-component of the point *P* in the (*unit*) *circle register* as to the *y*-component of the point *P'* in the *graphical register* through expressing, for example, that "…in here (*pointing the point P on the screen like in Figure 5.23*) *x* is cosine… …and in here (*pointing the point P'*) *y* is cosine, isn't it?". At that point, the researcher constructed two line segments with the blue color corresponding to the *x*-component of the point *P* and the *y*-component of the point *P'* like in *Figure 5.23* in order to ease for students to associate the variation of the cosine values between the (*unit*) *circle register* and the *graphical register*. At that point, students started to compare the variations of these two constructions within the (*unit*) *circle register* and the *graphical register*, and to reason about their variations in the same way in terms of their signed-magnitudes and their variations in the different way in terms of their directions. From this point forward, they were able to differentiate exactly the contents of the coordinate planes in the (*unit*) *circle register* and the *graphical register* under the consideration of the sine and cosine functions. Moreover, they were able to reason about the content of the coordinate plane in the *graphical register* through considering the *x*-axis as the domain set of the sine [cosine] function (e.g., lines 16-20 in [Cemre & Zafer] Protocol 12) and the range set of the sine [cosine] function as a part of the *y*-axis which was formed dependently by the *x*-values under the sine [cosine] function (e.g., lines 77-85 in [Cemre & Zafer] Protocol 11; lines 31-48 in [Defne & Ebru] Protocol 11).

After students' differentiation of the contents of the coordinate planes in the (*unit*) *circle register* and the *graphical register*, throughout Task 6 [Task 7], the researcher encouraged them to reason in detail about the behavior of the sine [cosine] function through provoking them to convert their reasoning in a representational register into another one. Firstly, she encouraged them to interpret the shape of the sine [cosine] graph as the static representation of the dynamic variation of the sine [cosine] values regarding the angles (e.g., lines 12-13 in [Defne & Ebru] Protocol 10). Although one student in each group, Defne and Zafer, intuitively attributed the shape of the sine [cosine] graph to the cause of the circle shape of the unit circle, none of the students was able to articulate the cause clearly (e.g., lines 14-18 in [Defne & Ebru] Protocol 10). So, the researcher encouraged them to compare the rates of change of the sine [cosine] values in three consecutive equal partitions of the first quarter arc on the unit circle (e.g., lines 19-50 in [Defne & Ebru] Protocol 10) through constructing these equal arcs and corresponding change on sine [cosine] values visually by GSP (e.g., *Figure 5.25*). Where, it was observed that based on these visual parts' magnitudes, all students interpreted that the same amounts of change in angles would not cause the same amounts of change in the sine [cosine] values (e.g., lines 51-77 in [Defne & Ebru] Protocol 10).

After then, the researcher provoked them to reason about the change of the sine [cosine] values regarding the change of the angle in each quadrant without these visual constructions referring to the rate of change (e.g., lines 1,6 and 8 in [Defne & Ebru] Protocol 11) when studying on the GSP page like in *Figure 5.24*. Where, it was observed that all students were able to interpret the change of the sine [cosine] values regarding the angle change in each quadrant easily and truly (e.g., lines 1-11 in [Defne & Ebru] Protocol 11) not only through focusing on the (*unit*) *circle register* but also focusing on the *graphical register* (e.g., lines 2-4, 9-10 and 12-17 in [Defne & Ebru] Protocol 11).

Next, instead of the sine [cosine] values' change regarding the angle change, the researcher asked students to interpret the sine [cosine] value of a real number in the *symbolic register*, such as 2, 3 and 10 (e.g., lines 31, 37 and 42 in [Defne & Ebru] Protocol 11). When determining the sine [cosine] values corresponding to these real numbers, students' actions and language imply that they were able to convert the meaning of the sine [cosine] function among the registers (e.g., lines 31-48 in [Defne & Ebru] Protocol 11). For example, when determining sin(2), Defne initially determined 2 as an argument in the domain of the sine function, and its corresponding element in the range of the sine function as the sine value of 2 in the *symbolic register* (line 32 in [Defne & Ebru] Protocol 11). And then, considering arguments in the domain set of the sine function as angles, she tried to obtain the angle with the 2-radian measure in the (*unit*) *circle register* through dragging the point *P* on the unit circle (lines 32-33 in [Defne & Ebru] Protocol 11). Next, she focused on the abscissa of the point *P'* in the *graphical register*, which was the correspondence of the point *P* in the (*unit*) *circle register*, to obtain 2 as an angle measure on the *x*-axis in the *graphical register* (lines 32-33 in [Defne & Ebru] Protocol 11). Finally, she determined the ordinate of the point *P'* whose abscissa was 2 as the value of sin(2) (lines 32-34 in [Defne & Ebru] Protocol 11).

Moreover, when determining the sine [cosine] value corresponding to a real number greater than 2π in the *symbolic register*, students' actions and language imply that they were able to convert their reasoning about the repetition of the sine [cosine] values within the (*unit*) *circle register* into the *graphical register* (e.g., lines 42-56 in [Defne & Ebru] Protocol 11). In depth, they were able to convert the meaning of the sine [cosine] values of the angles with the **same** *static structure* but **different** *dynamic structures* in the (*unit*) *circle register* into the meaning of the *parallel displacement* of the point on the **principal part** of the sine [cosine] graph[34] along the *x*-axis by the 2π-

[34] The principal part of the sine [cosine] graph means the part of the sine [cosine] graph defined on the [0,2π) interval.

length line segments in the *graphical register* thereby converting the meaning of full-round turnings in the **counterclockwise [clockwise]** direction in the (*unit*) *circle register* into the meaning of the *parallel displacement* along the *x*-axis in the **positive [negative]** direction in the *graphical register* (e.g., lines 57-83 in [Defne & Ebru] Protocol 11; lines 32-89 in [Cemre & Zafer] Protocol 12). In other words, they were able to convert the dynamic variation of the sine [cosine] values regarding **continuous turnings** in the counterclockwise {clockwise} direction in the (*unit*) *circle register* into the static representation of the dynamic variation of the sine [cosine] values by associating them with the appropriate right {left} part of the sine [cosine] graph. For example, they were able to convert the second full-round turning in the counterclockwise direction in the (*unit*) *circle register* into the **right second part of the sine [cosine] graph**[35] in the *graphical register* (e.g., lines 51-55 in [Defne & Ebru] Protocol 11; lines 32-52 in [Cemre & Zafer] Protocol 12), as well as convert the first full-round turning in the clockwise direction into the **left first part of the sine [cosine] graph**[36] (e.g., lines 64-68 in [Defne & Ebru] Protocol 11; lines 1-31 in [Cemre & Zafer] Protocol 12). At that point, they started to reason about the sine [cosine] function on the real number set in the *graphical register* based on their ability to transform the principal part of the sine [cosine] graph into the other repeated parts through the parallel displacement of the principal part (e.g., lines 53-89 in [Cemre & Zafer] Protocol 12). This implies that anymore students' reasoning on the sine [cosine] values was intimately dependent on the angles instead of considering the sine [cosine] values apart from the angles through restricting them between -1 and 1 as a rule so that they were able to see the patterns on the behavior of the sine [cosine] function. In other words, discussions on the sine [cosine] values additionally within the *graphical register* provoked students to compare the different sine [cosine] values with each other regarding their angles. This reasoning on the sine [cosine] values prompted a distinct shift on students' reasoning about the patterns on the behavior of the sine

[35] The **right** *second* part of the sine [cosine] graph corresponds to the *second* full-round turning in the **counterclockwise** direction.

[36] The **left** *first* part of the sine [cosine] graph corresponds to the *first* full-round turning in the **clockwise** direction.

[cosine] function. For example, they had just started to reason about the role of the reference angle on the sine [cosine] values regarding the quadrants. In other words, students had just started to reason about the same sine [cosine] value in the range set with two different principal angles of the domain set (e.g., lines 84-106 in [Defne & Ebru] Protocol 11). Up to that time, sine and cosine were considered with their basic forms. Henceforward, when they were considered with their general forms, or when the representational contents (such as radius of the circle, position of the center, positions of the reference points on the circle referring to trigonometric value) were systematically varied, the subsequent progress of students' concept images that emerged as a result of the teaching experiment was presented in terms of students' *discrimination* of trigonometric functions represented within any register from the respective representational content under the heading: *Development of Students' Concept Images Rooted in (Unit) Circle Register and Shaped between Graphical Register and Others*.

5.1.3.3. Regarding periodicity as pattern based on behaviors of trigonometric functions

According to the initial interview results, the researcher determined that students' concept images on the period concept included crucial troubles within the different representational registers as a consequence of their problematic concept definition images on the periodicity. For example, none of the students was able to appropriately associate the meaning of the repetition in the *graphical register* with the meaning of the period in the *symbolic register* (see *On periodicity* sub-heading for detailed information in Chapter 4). Therefore, in Task 6 [Task 7], before mentioning the period concept, the researcher determined to scrutinize students' abilities to convert their reasoning in Task 5 about the repetition of the sine [cosine] values regarding the equivalent but not equal angles in the (*unit*) *circle register* into the other representational registers.

When discussing the variation of the sine [cosine] values regarding the variation of the angles under their dynamic-simultaneous manipulations in the (*unit*) *circle register*, the *graphical register* and the *symbolic register*, it was observed that students were able to reason about the repetition of the sine [cosine] values within the (*unit*) *circle register* as a consequence of full-round turnings (e.g., lines 32-52 in [Cemre & Zafer] Protocol 12). Moreover, they were able to convert the meaning of the repetition within the (*unit*) *circle register* into the *graphical register* (e.g., lines 32-52 in [Cemre & Zafer] Protocol 12), as well as within the *symbolic register* via the (*unit*) *circle register* into the *graphical register* (e.g., lines 42-56 in [Defne & Ebru] Protocol 11) as a pattern on the behavior of the sine [cosine] function. Where, students' language and actions imply that their preferences to express the critical points of the repetition of the sine [cosine] values were based on the angles, i.e., the domain set of the sine [cosine] function (e.g., lines 34-43 in [Cemre & Zafer] Protocol 12; lines 51-56 in [Defne & Ebru] Protocol 11). This preference prompted a distinct shift on students' reasoning about the repetition on the sine [cosine] function in the *graphical register*. That is to say, they began to transform the principal part of the sine [cosine] graph (*Footnote 34*) into the other repeated parts through the parallel displacement of the principal part along the x-axis by the 2π-length line segments in the *graphical register* (e.g., lines 57-83 in [Defne & Ebru] Protocol 11; lines 32-89 in [Cemre & Zafer] Protocol 12). It was the point that all students started to reason about the repetition of sine [cosine] function's basic form clearly in each representational registers.

At this point, the researcher asked students to reason about the meaning of the period concept under the consideration of the sine [cosine] function (e.g., lines 1-24 in [Defne & Ebru] Protocol 12). Where, it was observed that none of the students was able to make clear what the period means (e.g., lines 24-37 in [Defne & Ebru] Protocol 12). At that point, after emphasizing the repetition of the sine values on the regular intervals with different magnitudes (e.g., lines 12-21 and 38-46 in [Defne & Ebru] Protocol 12), the researcher defined the period concept not going into detail as the regular intervals of the domain set in which a function repeats its values in the *graphical register* and as the lengths of these regular intervals in the *symbolic register* (e.g., lines 47-63 in [Defne & Ebru] Protocol 12). It was the point that students had just started to reason about the periodicity of the sine function with more than one period (e.g., lines 55-67 in [Defne & Ebru] Protocol 12). And then, the researcher defined the *prime period* as the smallest period of a periodic function (e.g., lines 68-73 in [Defne & Ebru] Protocol 12). From this point forward, throughout the teaching experiment, the *period* term was used in the meaning of the *prime period*. Henceforward, the subsequent progress of students' concept images on the period concept that emerged as a result of the teaching experiment was presented in terms of the general forms of trigonometric functions under the heading: *Periodicity as pattern based on behaviors of trigonometric functions*.

[Cemre & Zafer] Protocol 11

Researcher: In the previous two tasks, we discussed about sine through associating with the relation between this green arc and this red line segment (*pointing respectively green arc and red line segments on the screen*).
Cemre: Yes.
Zafer: Uh-huh (*nodding his head up and down*).
Researcher: Ok. Now, I want to construct an ordered pair on the coordinate system so that its *x*-component would be the measure of this green arc and *y*-component would be the value of sine, or directed-red segment.
(*Constructing the (alpha,sin(alpha)) ordered-pair taking advantages of the plot as (x,y) option of GSP and labelling with P'*)
Researcher: (*Pointing the point P' on the screen like in Figure 5.21*) this point's *x*-component is the angle measure… …and *y*-component is the sine value of this angle.
Zafer: …*x* is angle; *y* is sine value, isn't it?
Researcher: Yes. What is the angle measure now?
Cemre: (*Pointing the dynamically-linked alpha measure on the screen*) 0.34π radian
Researcher: What about sine?
Cemre: 0.87…
Zafer: (*Looking to the screen carefully*)
Researcher: What do you say about this point (*pointing the point P' on the screen*)?
Zafer: (*Coming closer to the screen and looking carefully*) it [point *P'*] is same height with this [point *P*] (*pointing these points on the screen*).
Researcher: So, which values of them would be same?
Zafer: …*y* values…
Researcher: What else?
Cemre: Its [the P' point's] *x* component is same with angle [measure]…
Researcher: (*Constructing the segments corresponding to the P' point's x and y components within the graphical register respectively with green and red colors*

to ease for students to associate them to green arc and red line segment on the unit circle like in Figure 5.22) Well, let's look… …when the point *P* is dragging, how is acting the point *P'*?

Cemre: (*Dragging the point P in the counterclockwise direction*) both [*P* and *P'*] are going in the same way [height].

Zafer: Of course, they are going in parallel each other… But I didn't understand that… …now, *x*-component was angle, wasn't it?

Cemre: (*After 3-second pause*) yes, *x* was angle.

Zafer: That is, was this length its value (*pointing the projection point of the point P' on the x-axis*)?

Cemre: Yes.

Zafer: Then, how do they [green-segment's lengths] come up to here, that is to 5 (*at that time the point P in the fourth quadrant, and the abscissa of the point P' greater than 5*)? Were the values of sine between -1 and 1?

Cemre: (*Looking to the screen without speaking*)

Researcher: What would be the reason of this?

Cemre&Zafer: (*Looking to the screen without speaking*)

Researcher: Ok. Let's continue to look variation… …to find an explanation to this problem.

Zafer: (*Dragging the point P and looking carefully to the screen*)

Cemre: What do you think?

Zafer: (*Throughout approximately 1-minute, continuing to drag and drop the point P on GSP page like in Figure 5.22*) yes, this is angle (*pointing the projection point of the point P' on the x-axis*)… …because this is π (*pointing the green arc on the unit circle in case the point P was on the point -1 on the x-axis*)… As for that (*pointing the projection point of the point P' on the x-axis*), this is corresponding to about 3.14.

Cemre: (*Listening Zafer's articulation carefully through looking to the screen*) bring it [point *P*] here (pointing *the point 1 on the x-axis*).

Zafer: That is 2π (*dragging the point P from the fourth quadrant to the first quadrant in the counter clockwise direction*)… As for that (*pointing the projection point of the point P' on the x-axis*), there is 6 point [~6.28] … …that is, again 2π. Yes, I think this is angle (*pointing the green line segment on the x-axis*)…

Cemre: …yes, [it is about] 6.28… or, 2π.

Zafer: Yes, I have understood exactly.

Researcher: (*Bringing the point P into the second quadrant*) what about now? What is the angle measure in here (*pointing the point P*) and here (*pointing the point P'*)?

Cemre: In here (*pointing the angle measure on the screen like in Figure 5.22*), 0.76π…

Zafer: (*Nodding his head up and down*)

Researcher: What about here (*pointing the point P'*)?

Cemre: (*Counting increasing tenth parts from the point 2 towards the abscissa of the point P' on the positive x-axis*) one, two, three, and… …a bit more…

Zafer: Yeah, about 2.38…

Researcher: What do you think about these two numbers? Are they same?

Zafer: They would be same.

Cemre: Let's we multiply 0.76 with π. (*Calculating this multiplication; and reading the output result as*) 2.39… Yes, they are same.

Zafer: Uh-huh (*nodding his head up and down*).

Researcher: Well, with which element in the range set is matched by this angle in the domain set?

Cemre: …with 0.69 (*pointing to the sin(alpha) measure on the screen*)…

Researcher: What about looking to the unit circle?

Zafer: Here (*dragging his index finger from the point P to the y-axis on the dashed-line which was perpendicular to the y-axis like in Figure 5.22*), that is, 0.69...

Cemre: From here it is also same (*dragging her index finger from the point P' to the y-axis on the dashed-line which was perpendicular to the y-axis*)

Zafer: Uh-huh (*nodding his head up and down*), it is ok, now. I understand.

Cemre: I understand, too… That is, the way this [point *P*] takes on the unit circle (*dragging her index finger on the green arc on the unit circle*)… …is taken by this [point *P'*] on the *x*-axis (*dragging her index finger on the green line segment starting from the origin to rightward*).

Zafer: Right... That is, they are different representations.

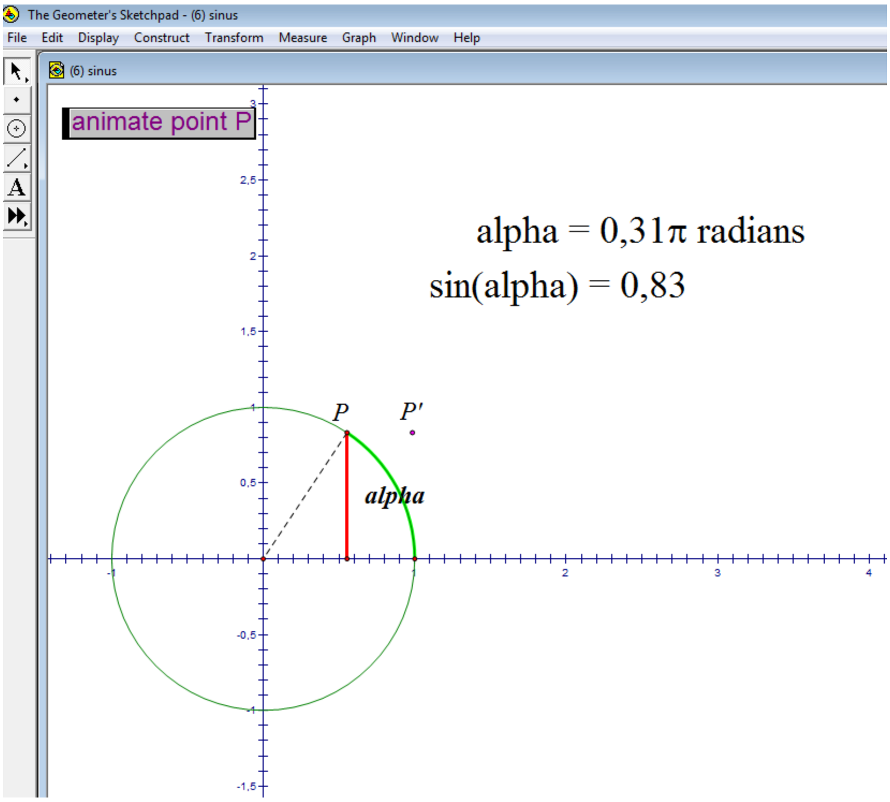


Figure 5.21. Construction of the (*alpha,sin(alpha)*) ordered pair within the *graphical regsiter*

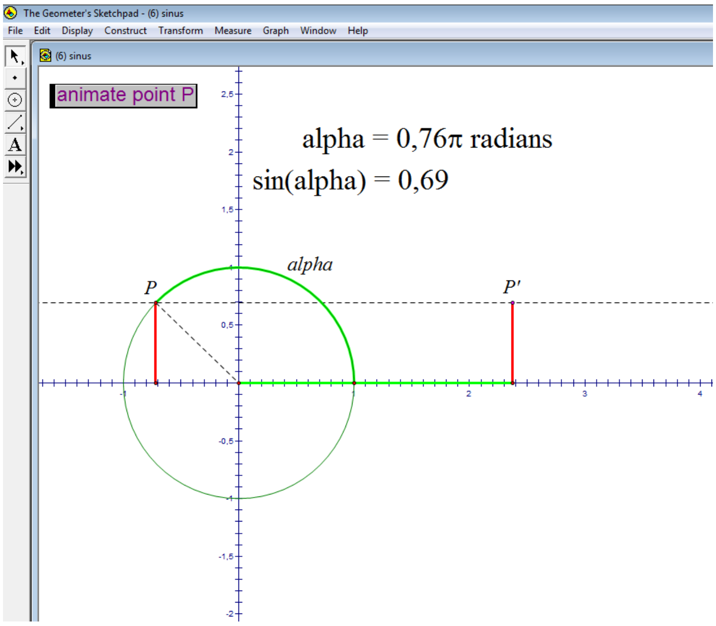


Figure 5.22. Constructions to associate relateted componets of sine between the *graphical regsiter* and the (*unit*) *circle register*

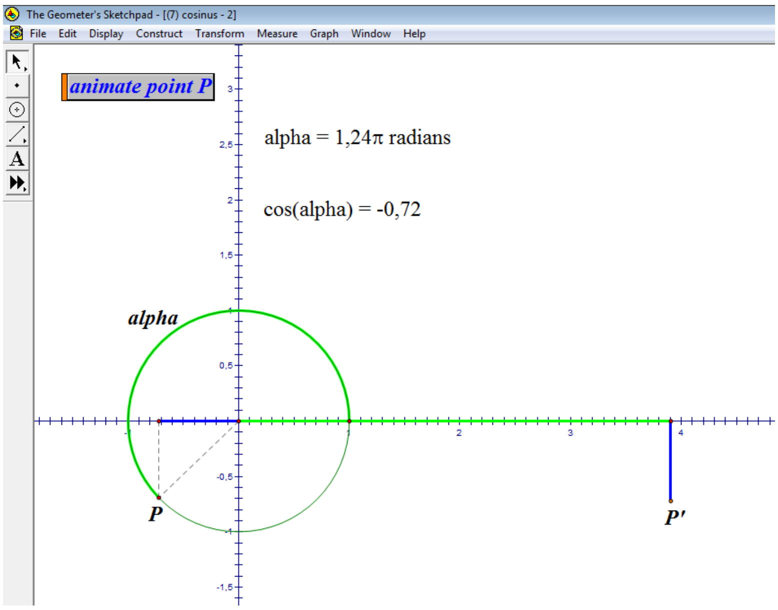


Figure 5.23. Constructions to associate relateted componets of cosine between the *graphical regsiter* and the (*unit*) *circle register*

[Defne & Ebru] Protocol 10

Researcher: (*Drawing the sine graph in the [0,2π) interval via taking the trace point (for the point P') and animate point (for the point P) advantages under the Display menu of GSP*) please discuss when the angles change, how do the corresponding sine values change?

Ebru: (*Dragging the cursor within the first quadrant in the counterclockwise direction on the unit circle*) here as the angle is increasing, this (*pointing the red line segment on the unit circle*) is increasing… Also, sine [value] will be increasing. Next, sine values, after first quadrant (*dragging the cursor within the second quadrant in the counterclockwise direction on the unit circle*) it [red segment] will go down. And then, it will turn in the minus values (*dragging the cursor from the third quadrant to the fourth quadrant*).

Researcher: Good. Why was this shape curved (*pointing the traced-sine graph in the [0,2π) interval on the screen like in Figure 5.24*)? Or why wasn't it linear?

Ebru: I don't know.

Defne: Is it due to the circle shape?

Researcher: How does the circle shape affect the shape of this graph (*pointing the traced-sine graph in the [0,2π) interval on the screen like in Figure 5.24*)?

Defne&Ebru: (*Looking to the screen without speaking*)

Researcher: (*Opening a new window in GSP; and constructing the unit circle like in Figure 5.25*) let's we divide this quarter arc (*pointing the quarter arc of the unit circle in the first quadrant*) into three equal pieces. How would be the measure of them?

Defne: 30 degree…

Ebru: …or π/6 radian…

Defne: Yes.

Researcher: (*Constructing two points on this quarter arc through rotating the point A about the center by fixed angles π/6 and 2π/6 radians, next, perpendicular lines from these points to the y-axis, and then vertical line segments separated by these perpendicular lines on the y-axis like in Figure 5.25*) Now, are these three arc same?

Defne&Ebru: Yes.

Researcher: Ok. Does each of these equal parts cause same increase in sine value?

Defne&Ebru: (*Looking to the screen without speaking*)

Researcher: Well, when the angles change from here to here (*putting her right hands' thumb and index fingertips on the point A closed to each other, and then turning her index finger from the point A to the point corresponding to the angle π/6 radians in the counterclockwise direction on the screen like in Figure 5.25*), sine values change from here to here (*putting her right hands' thumb and index fingertips on the origin closed to each other, and then dragging her index finger up-right on the pink line segment*). Is it Ok?

Defne&Ebru: Yes.

Researcher: When the angles change from here to here (*putting her right hands' thumb and index fingertips on the point corresponding to the angle π/6 radians closed to each other, and then turning her index finger from this point to the point* *corresponding to the angle 2π/6 radians in the counterclockwise direction*), sine values change from here to here (*putting her right hands' thumb and index fingertips on the down edge point of the red line segment on the y-axis, and then dragging her index finger up-right on the red line segment*). Which of two π/6-radian arcs (*pointing first two arcs on the screen*) causes greater increase in sine?

Defne: First one…

Ebru: This (*pointing the pink line segment on the y-axis*)… (*After 3-second pause*) then, this would be the least [increase in sine] (*pointing the green line segment on the y-axis in Figure 5.25*)…

Defne: Uh-huh (*nodding her head up and down*).

Researcher: Then, changes in sine values regarding angles are not proportional… I mean that… …the same amount of changes in angles don't cause the same change in sine values…

Defne: Uh-huh (*nodding their heads up and down*).

Ebru: Yes.

Researcher: So, shape of the graph isn't linear due to…

Defne: …circle shape (*smiling*)…

Ebru: (*Nodding her head up and down*)

Researcher: Look! As angles are increasing in the first quadrant, sine values are increasing (*putting her right hand's thumb and index fingertips on the point A; and then dragging her index finger on the circle within the first quadrant in the counterclockwise direction and her thumb finger on the x-axis as if her index fingers' projection point on the x-axis on the screen like in Figure 5.25*) but the increased amounts are decreasing (*pointing respectively pink, red and green line segments on the y-axis*).

Defne&Ebru: Yes.

Researcher: That is, in the first quadrant sine is increasing, but decreasing the increased amount.

Defne&Ebru: Yes.

(*In the similar way, the researcher explained the variation of sine regarding angles in the second quadrant. After then, she asked students to interpret the variation on sine regarding angles when manipulating the point P on the GSP page like in Figure 5.24*).

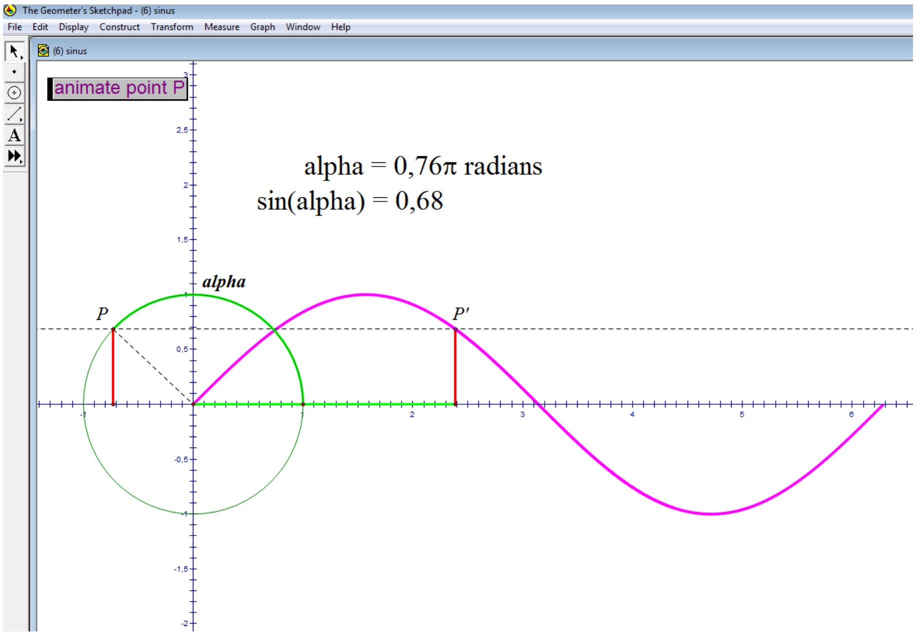


Figure 5.24. Construction of sine graph in the [0,2π) interval via taking the trace point (for the point *P'*) and animate point (for the point *P*) advantages of GSP

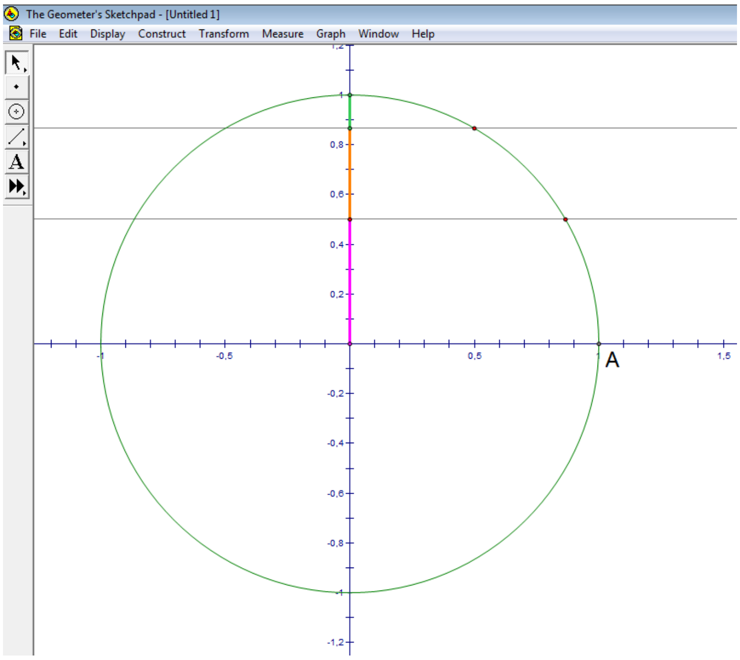


Figure 5.25. Construction the rate of change for the sine function within the *(unit) circle register*

[Defne & Ebru] Protocol 11

Researcher: Ok, now. How does sine change in the first quadrant?
Defne: (*Dragging the quarter arc in the first quadrant with her index finger in the counterclockwise direction on the screen like in Figure 5.24*) as going to the up-right, it [sine] is increasing, but increased amount is decreasing.
Ebru: That is, decreasingly increase…
Researcher: What about in the second quadrant Ebru?
Ebru: Decreasing… …but increasingly decrease…
Researcher: In the third quadrant Defne?
Defne: Decreasing… but decreased amount is decreasing… In fourth quadrant, an increased graph is there and increased amount is increased…
Ebru: [In the fourth quadrant, sine] increasingly increase.
Researcher: Where do you look to determine… …unit circle or graph?
Defne: We can say through looking to both. (*Pointing the last endpoints of the quadrants on the x-axis through focusing on the graph on the screen like in Figure 5.24*) first quadrant, second, third and fourth…
Ebru: Yes, I understand, too, from each of both [unit circle, graph]… But unit circle is as if crosschecking…
Researcher: How do you determine, Defne, the quadrants on the graph?
Defne: (*Dragging the point P in the counterclockwise direction through waiting a while on the points in which the quadrants change*) first quadrant, second quadrant, third and fourth (*pointing the projection of the point P' on the x-axis for her waiting points*)…
Ebru: (*Nodding her head up and down*)
Researcher: So, what is the value of here (*pointing the middle intersection point of the traced-sine graph with the x-axis on the screen like in Figure 5.24*)?
Defne: π…
Researcher: What do you see here instead of π (*pointing the middle intersection point of the traced-sine graph with the x-axis on the screen*)?
Ebru: …about 3.1 (*coming closer and looking carefully to the screen*).
Defne: Uh-huh, π is about 3.14…
Researcher: Good. Well, if I ask you that what is sin(2)? How do you think?
Defne: Then, we will find the *y* value… …for *x*=2… (*And then, dragging the point P so that the abscissa of the dynamically linked point P' would be 2; and dragging her index finger from the point P' to the y-axis on the dashed-line*) is it 0.9?
Ebru: Let's we look with calculator (*opening the calculate option of the measure menu of GSP and calculating sin(2) as 0.91*) yes… …that is, about 0.9.
Researcher: Well, what about sin(3)?
Defne: That means… …it is closed to 3.14, but smaller than 3.14, so it [sin(3)] is very close to zero.
Researcher: Why?
Ebru: …because sin(π) is zero.
Researcher: Ok. What about sin(10)?
Defne: We assume 2π as about 6…
Ebru: …4 remain. It [4] is in the third quadrant.
Defne: (*Putting her index finger on 4 on the x-axis on the screen; figuring a parallel segment to the y-axis from here up to the traced-sine graph; and then figuring a*

parallel segment to the x-axis from this intersection point to the y-axis) here… …that is, close to minus 1.

Researcher: But 10 must be above and beyond here (*figuring a ray on the positive x-axis starting from the last intersection point of the traced-sine graph with the x-axis*).

Defne: After 2π, it turns second tour… …and it [graph] would repeat in the same way like that (*figuring with her index finger on the screen a sine-curve through adding it to the end point of the traced-sine graph so as to indicate the* ***right*** *second part of the sine graph corresponding to the second full-round turning in the* ***counterclockwise*** *direction*).

Ebru: Yes.

Researcher: Let's we construct the sine graph through GSP automatically (*constructing the graph of y=sin(x) by the aid of plot new function option of GSP like in Figure 5.26*). This part of the graph (*putting her thumb and index fingers on the endpoints of the* ***right*** *second part of the sine graph corresponding to the second full-round turning in the* ***counterclockwise*** *direction*) means the second tour, you said shortly before.

Defne&Ebru: Yes.

Researcher: Well, what does this part mean (*putting her thumb and index fingers on the endpoints of the* ***left*** *first part of the sine graph corresponding to the first full-round turning in the* ***clockwise*** *direction*)?

Defne: It means turning in the reverse direction… …that is, angles are negative…

Ebru: Yes.

Researcher: Well. In this part (*putting her thumb and index fingers on the endpoints of the* ***right*** *first part of the sine graph corresponding to the first full-round turning in the* ***counterclockwise*** *direction on the screen like in Figure 5.26*), for which angle does sine take the value 1?

Defne: π/2.

Researcher: What about in this part (*putting her thumb and index fingers on the endpoints of the* ***right*** *second part of the sine graph corresponding to the second full-round turning in the* ***counterclockwise*** *direction*)?

Ebru: 2π plus π/2.

Defne: Yes.

Researcher: What about in this part (*putting her thumb and index fingers on the endpoints of the* ***left*** *first part of the sine graph corresponding to the first full-round turning in the* ***clockwise*** *direction*)?

Defne: π/2 minus 2π.

Ebru: (*Nodding her head up and down*)

Researcher: Well. Considering the unit circle, how many angles are there with the 0.5 sine value?

Ebru: Two.

Defne: One when going up (*figuring an arc in the first quadrant on the unit circle starting from the point 1 on the x-axis up to the intersection point with the dashed line in the counterclockwise direction in Figure 5.26*)… …one when going down (*figuring an arc in the second quadrant on the unit circle starting from the point 1 on the y-axis up to the intersection point with the dashed line in the counterclockwise direction in Figure 5.26*)… Yes, two.

Researcher: What can you say about these two angles?

Ebru: That is, here (*pointing an arc in the first quadrant on the unit circle starting from the intersection point with the dashed line up to the point 1 on the x-axis in the clockwise direction in Figure 5.26*) would be same with here (*pointing an arc in the second quadrant on the unit circle starting from the intersection point with the dashed line up to the point -1 on the x-axis in the counterclockwise direction in Figure 5.26*).

Researcher: That is, if this angle (*pointing the arc in the first quadrant on the unit circle starting from the point 1 on the x-axis up to the intersection point of unit circle with the dashed line in the counterclockwise direction*) is alpha, what is the other angle (*pointing the intersection point of the unit circle with dashed-line in the second quadrant*)?

Ebru: π minus alpha.

Defne: Yes.

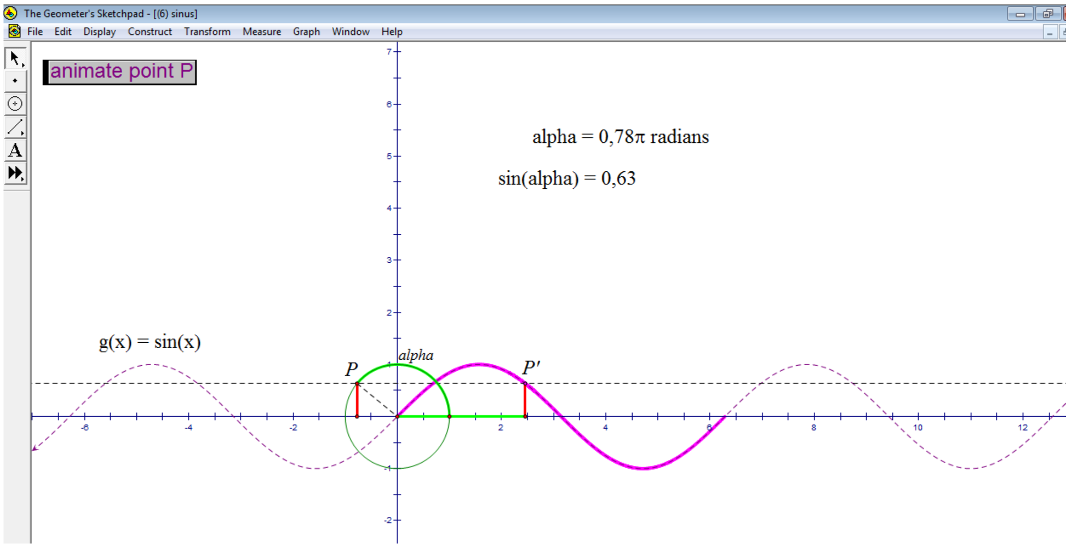


Figure 5.26. Construction of the sine graph with its extended domain from [0,2π) to the *x*-axis

[Cemre & Zafer] Protocol 12

Researcher: Let's we see the graph of sine function (*constructing the graph of y=sin(x) by the aid of plot new function option of GSP like in Figure 5.26 without the traced-part*). Now, please look when rotating the point *P* one full-round… …the variation of the point *P'*.

Zafer: Ah! This (*pointing the point P'*) would go on this (*pointing the sine graph*).

Cemre: (*Smiling*) yes, it [point *P'*] is going on the graph (*dragging the point P in the counterclockwise direction starting from the first quadrant*).

Zafer: Yes… …yes, I understand now.

Cemre: (*Dragging the second-full round of the point P*) but I want this [point *P'*] to go towards here (*pointing the negative x-axis on the screen*).

Researcher: How would it be?

Cemre: How would this [point P'] go here (*dragging her index finger on the **left** sine graph regarding the x-axis*)?

Cemre: (*Looking to the screen without speaking*)

Zafer: How?

Researcher: In the graph, to what does x-axis correspond?

Cemre: …to domain set.

Zafer: Yes.

Researcher: What is in the domain set of sine?

Cemre&Zafer: Angles…

Researcher: For example, here (*pointing -2 on the x-axis*)… …that is, minus 2… …is what?

Zafer: It is a minus [negative] angle…

Cemre: Yes.

Researcher: What does “minus angle” mean?

Zafer: It means turning in the reverse direction.

Cemre: Yes… [Minus indicates] direction…

Zafer: (*Pointing the angle measure on the screen*) if we do this [angle measure] minus, it [point P'] would pass on this side (*pointing the **left** sine graph regarding the x-axis*).

Cemre: Yes, it [point P'] would pass that side.

Researcher: Well, let’s we see the trace of P' when dragging P (*assigning the point P' with the trace property*; *and then animating the point P*).

Cemre: It is from zero up to 2π (*pointing respectively the first and last intersection points of the traced-sine graph with the x-axis on the screen like in Figure 5.26*)

Zafer: That’s exactly following parts repeat.

Cemre: Uh-huh (*nodding her head up and down*).

Researcher: Why does it repeat?

Zafer: After completing the first tour (*figuring a full-round turning with his index finger on the unit circle in the counterclockwise direction*), it would be continue again from 2π (*figuring the second turning with his index finger on the unit circle starting from the intersection point of the unit circle with the positive x-axis in the counterclockwise direction*).

Researcher: What would be in the second tour in the graph?

Cemre: It would start from here (*pointing the last intersection point of the traced-sine graph with the x-axis, i.e. 2π on the x-axis, on the screen like in Figure 5.26*)… …we think as if this (*putting her thumb and index fingers on the endpoints of the **right** first part of the sine graph corresponding to the first full-round turning in the **counterclockwise** direction*) is added to here (*putting her thumb and index fingers on the endpoints of the **right** second part of the sine graph corresponding to the second full-round turning in the **counterclockwise** direction*).

Zafer: Yes.

Researcher: What does “added to here (*pointing the **right** second part of the sine graph*)” mean? For example, now, angle measure is 0.78π radian (*pointing the angle measure on the screen like in Figure 5.26*). How it [0.78π-radian angle] repeats in here (*putting her thumb and index fingers on the endpoints of the **right** second part of the sine graph corresponding to the second full-round turning in the **counterclockwise** direction*).

Zafer: It would be 2π more…

Cemre: It would be here (*pointing the parallel displacement of the point P' on the* ***right*** *second part of the sine graph corresponding to the second full-round turning in the* ***counterclockwise*** *direction based on the dashed line on the screen like in Figure 5.26*).

Zafer: Uh-huh (*nodding his head up and down*).

Researcher: What about here (*pointing the* ***left*** *first part of the sine graph corresponding to the first full-round turning in the* ***clockwise*** *direction*)?

Cemre: Here (*pointing the parallel displacement of the point P' on the* ***left*** *first part of the sine graph corresponding to the first full-round turning in the* ***clockwise*** *direction focusing on the dashed line*)

Zafer: It would be 2π less…

Researcher: So, coordinates of this point (*pointing the parallel displacement of the point P' on the* ***left*** *first part of the sine graph corresponding to the first full-round turning in the* ***clockwise*** *direction*) are what?

Zafer: *x* is 0.78π minus 2π, *y* is…

Cemre: …sine value… that is, 0.63.

Zafer: Yes.

Researcher: Let's we construct a point on the coordinate system with these *x* and *y* values (*after calculating alpha-2π, constructing the ordered pair (alpha-2π,sin(alpha)) taking advantages of the plot as (x,y) option of GSP and labelling with P'' like in Figure 5.27*).

Cemre: Now, this [point *P''*] will go on here (*dragging her index finger on the* ***left*** *first part of the sine graph corresponding to the first full-round turning in the* ***clockwise*** *direction*).

Zafer: Yes.

Researcher: Let's try and see!

(*Students constructed the traced graph of the sine function in the [-2π,0) interval by the researcher's direction like in Figure 5.28*)

Cemre: Yes! I understand better…

Zafer: Me too.

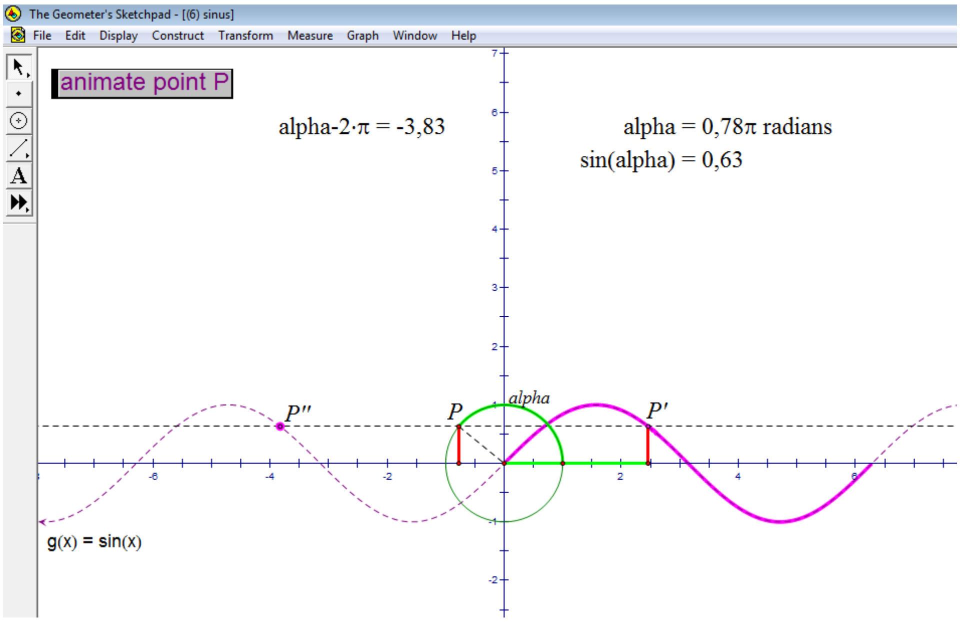


Figure 5.27. Construction of the (*alpha-2π,sin(alpha)*) ordered pair within the *graphical regsiter*

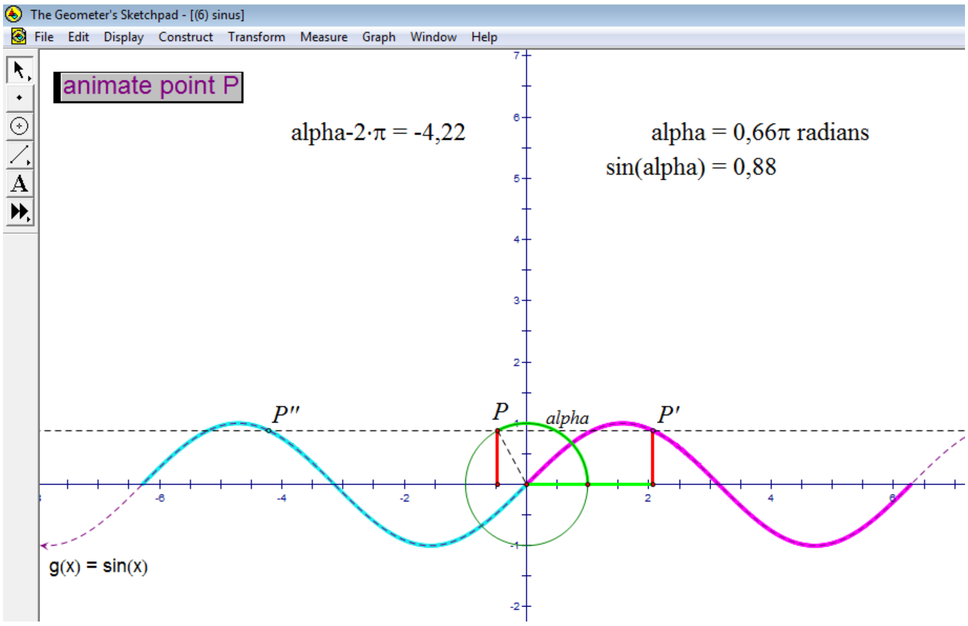


Figure 5.28. Construction of the sine graph's **left** and **right** first parts in the *graphical register* corresponding to the first full-round turnings, respectively, in the **clockwise** and **counterclockwise** directions in the (*unit*) *circle register*

[Defne & Ebru] Protocol 12

Researcher: Where does sine function repeat?
Defne: Here (*pointing the endpoints of the* ***right*** *first part of the sine graph corresponding to the first full-round turning in the* ***counterclockwise*** *direction on the screen like in Figure 5.26*)… …at 2π.
Ebru: Of course, 2π.
Researcher: How did you determine this?
Ebru: Because when this [point P] full-turned around the circle (*dragging her index finger on the unit circle so as to figure a circle starting from the point 1 on the x-axis*), this [point P'] comes to here (*pointing the right-endpoint of the* ***right*** *first part of the sine graph corresponding to the first full-round turning in the* ***counterclockwise*** *direction*).
Defne: Uh-huh (*nodding her head up and down*).
Ebru: It [sine] takes all values in here (*pointing the right-endpoint of the* ***right*** *first part of the sine graph corresponding to the first full-round turning in the* ***counterclockwise*** *direction*)… …that may take anywhere…
Defne: Yes.
Researcher: If we consider this interval (*pointing with her right hands thumb and index fingers respectively the origin and the right-endpoint of the* ***right*** *second part of the sine graph corresponding to the second full-round turning in the* ***counterclockwise*** *direction*), wouldn't sine values repeat?
Defne: It [sine] would repeat… …but period of sine was 2π.
Researcher: Why is the period of sine 2π… …why not 4π?
Defne: I guess it was assumed (*smiling*).
Researcher: Well, what does period mean?
Defne: Period… ….it is values that a function takes. For example, for sine it is part from its highest value up to its lowest value…
Researcher: What do you think Ebru?
Ebru: Frankly speaking, I don't know it [period] as a definition.
Researcher: There is no need to give its mathematical definition. What do you remember about period?
Ebru: (*Waiting without speaking*).
Researcher: Do you agree with Defne about period's meaning?
Ebru: I didn't understand what Defne meant?
Researcher: Can you explain again the meaning of period Defne?
Defne: Period, for example, for sine is part from its highest value up to its lowest value…
Researcher: Is that so Ebru?
Ebru: I don't think so… …but frankly speaking, I don't know what it is [period].
Researcher: Ok. What do you think about this part (*pointing the 2π-length interval from –π to π on the x-axis*)? Would sine take all values in here that may take anywhere?
Defne&Ebru: Yes.
Researcher: Well, can you show me an interval smaller than 2π length… …so that sine could take all values?
Defne&Ebru: (*Shaking their heads right and left*).
Researcher: Then, what was the smallest length of those intervals?
Defne: Is that 2π!
Ebru: Yes, 2π…

Researcher: Ok. Period means that the regular intervals on the graph in which a function repeats its values; for example, like that (*pointing the 2π-length intervals on the screen respectively from –2π to 0, from 0 to 2π and from 2π to 4π on the x-axis*)… …or like that (*pointing the 2π-length intervals on the screen respectively from –π to π and from π to 3π on the x-axis*). What is the magnitude of these intervals?
Defne&Ebru: 2π…
Researcher: Then, 2π is a period of sine function. Is it ok?
Defne&Ebru: Yes.
Researcher: Now that period means the regular intervals in which a function repeats its values… …then, this interval also provides this condition, doesn't it (*pointing the 4π-length interval on the screen from –2π to 2π on the x-axis*)?
Defne&Ebru: Yes.
Researcher: In that case, what is the magnitude of the regular intervals?
Ebru: 4π…
Defne: Yes, 4π.
Researcher: Then, 4π is also a period of sine function. Is it ok?
Defne: Then, 6π would be a period!
Researcher: Good. In that case, how many periods can we find for sine?
Defne: 2π, 4π, 6π, 8π…
Ebru: [Infinitely] many…
Defne: Yes.
Researcher: So, in order to mention the same number about the period of sine, we can assume to consider the smallest one.
Defne: That's 2π.
Researcher: Yes. It is called as the prime period.
Defne: Hmm. I understand.
Ebru: Me too...

5.2. Summary of Students' Developments on Angle and Angle Measure

According to the initial interview results, students' initial concept images on angles included many troubles. There was no clear meaning of angles, angle measures as well as angle measure units on students' concept images (see *Summary of Students' Initial Concept Images* in Chapter 4). Task 2 was the first task of the teaching experiment that *angle* and *angle measure* concepts were brought up for discussion. As a consequence of their degree-dominated angle measure images, discussions started on *degrees* preference, and then, changed to *directed degrees* and *radians* preferences of GSP. When investigating the variation on an angle's measure (in *degrees* preference) through dynamic manipulations of its openness in GSP environment (see *Figure 5.7*), students' two different focuses were observed regarding angle measure. While Defne focused coherently on the same up-part from two ones of the plane separated by the angle, others changed their focuses from one part to the other when their focused-part turned from the obtuse angle to the reflex angle. The researcher inferred that while Defne's seeing of an angle was as a dynamic turning, others' seeing of an angle was as a static interior region separated by its rays.

When the angle measure preference of GSP was changed from degrees to radian, it was observed that initially, none of the students could explain any reason why the angle measure changed from positive to negative. After then, according to their investigations in GSP, they reasoned about the sign of the angle measure as negative [positive] if the position of the angle's interior region was down [up] regarding the horizontal straight angle without mentioning direction. This reasoning prompted Cemre to associate signs of angle measures with the *y*-axis of the coordinate plane, it prompted other students to associate negative measure of an angle with the direction. When direction idea was brought up for discussion, students started to mention the visual variation as rotation.

Although they considered the counterclockwise direction as positive and the clockwise direction as negative, two different points of view having emerged when articulating the direction. When reasoning about angle measures' change from positive to negative, while Zafer and Ebru fixed the initial side of the angle as the starting point of the rotation and changed the direction of the rotation, Defne fixed her focus on the same up-arc but changed the initial side of the angle as the starting point of rotation. Defne's considered the same angle as both negative and positive but only through focusing on the same arc between the angle's sides and changing her point of view between its initial side and terminal side. At that time, Ebru's seeing of an angle was either as negative or as positive through focusing on both the angle's same initial side and its interior region.

And then, discussions on the variation of the angle measure continued considering the role of two arcs of the circle[37] separated by an angle on this variation (see *Figure 5.10*). On this dynamic structure, all students associated the measure of the angle given by GSP with the appropriate arc's angle measure through articulating signs' meaning with the appropriate initial side and direction. Moreover, they interpreted an angle's measure with a positive [negative] measure given by GSP as its negative [positive] equivalence. Unlike their prior concept images on negative angles which was solely based on memorized rules without any reason (see *Summary of Students' Initial Concept Images* in Chapter 4), they had just started to well-define *angle measure* based on its initial side, terminal side and direction of the rotation. This kind of reasoning prompted students' ability to define an angle with two measures (i.e., with the highest negative and the lowest positive measures).

Another theme of Task 2 was on the meaning of the radian measure unit. When discussing the related measures (such as arc lengths, arc angles, radius and angle measure) and the reasons of their mathematical relevance under the variation of an angle's openness, as well as the radius of the circle (centered on the vertex of the angle) in GSP environment, it was observed that initially, none of the students was able to associate an angle's measure in "radians" with the arc length and radius. And then, one student in each group, Cemre and Defne, turned their focus from the variation of measures separately to the relation of these variations. That is, Cemre and Defne, after restricting the variation into the case in which the angle measure was fixed, focused on the relation between variations of two measures [arc length and radius] through specified radii by natural numbers, and then, reasoned their proportional covariations. However, none of the students was able to associate these ratios with the angle measure in radians until the researcher's recommendation for them to consider π in the angle measure. Where, because of students' problematic reasoning about the real value of π in and out the trigonometry context (see *Summary of Students' Initial Concept Images* heading in Chapter 4), the researcher encouraged them to use GSP's calculate option when considering π in the angle measure. GSP's calculation defaults on π as about

[37] This circle centered at the vertex of the angle.

3.14 prompted students' reasoning about π notation in an angle measure in radians with its meaning as a real number. In addition, dynamically-manipulated calculation result of the ratio between (arc length/radius) and (π) prompted a distinct shift on their reasoning about angle measure in radians. That is to say, students had just started to associate arc lengths [when the radius was 1] with the angle measures. Moreover, at the beginning of Task 3, angles were constructed in the unit circle context (see *Figure 5.13*). It was observed that students easily transferred their reasoning mentioned above onto the unit circle.

Students' consideration of angles as dynamic-directed turnings promoted students' advance reasoning about definition of angle measure in the (*unit*) *circle register*. All students started to be able to identify an angle within the (*unit*) *circle register* whose absolute measure was greater than 360 degrees or 2π radians regarding two-step turning in any directions: (1) *principal turning* from the initial side to the terminal side (2) some *full-rounds*. Beside, when identifying angle measures corresponding to the reference points in different quadrants (see *Figure 5.15* and *Figure 5.16*), Cemre and Zafer also reasoned about the *principal turning* from the initial side to the terminal side through considering this turning in two steps regarding the closest coordinate axes: (1) turning from the initial side to the closest coordinate axis in the same direction as the *principal turning* (2) turning from this coordinate axis to the terminal side in the direction so that the way of turning would be the shorter arc. This reasoning prompted a distinct shift on their reasoning about an angle in the (*unit*) *circle register*. That is to say, Cemre and Zafer began to associate an angle within the (*unit*) *circle register* with its complementary and/or supplementary parts in any quadrant. On the other hand, Defne and Ebru reached this level of reasoning when the researcher provoked them to identify an angle in any quadrant within the (*unit*) *circle register* at least in three different ways without considering *full-round turnings*. Students' this level of reasoning on angles was inconsistent with their prior ones revealed throughout the initial interviews.

At the end of Task 3, all students were able to associate a *static angle structure* in the (*unit*) *circle register* with the infinitely many [negative or positive] equivalent measures in the *symbolic register* through considering *dynamic directed turnings*. Inversely, they were able to associate the infinitely many equivalent measures in the *symbolic register* with the **same** *static structure* in the (*unit*) *circle register*.

During Task 5, when angle measures were brought up for discussions together with trigonometric values, students started to reason about a real number as an angle measure with two different ways through considering angle measure unit either in degrees or in radians (see *Summary of Students' Developments on Basic Trigonometric Functions* heading in Chapter 5). When discussing the relation between measures of the angles with the **same** *static structure* but **different** *dynamic structures* in the (*unit*) *circle register* based on trigonometric values, Defne and Ebru initially reasoned about their measures as if equal to each other in the *symbolic register*. After the researcher's provocation of them to discuss this equality as an equation on the paper and pencil environment, they tried to focus on what was mathematically same in this respect; and reasoned about their difference in terms of measures in the *symbolic register*, as well as their sameness in terms of *static angle structures* in the (*unit*) *circle register*. On the other hand, based on the corresponding arc lengths to the angles with the **same** *static structure* but **different** *dynamic structures* in the (*unit*) *circle register*, Zafer and Cemre reasoned that they produced different angle measures in the *symbolic register*. At the end of Task 5, all students had just started to differentiate angles with the **same** *static structure* but **different** *dynamic structures* in the (*unit*) *circle register*, as well as angles with the equivalent but not equal measures in the *symbolic register*.

In Task 6 [Task 7], angles and angle measures were discussed on the dynamic-and-linked simultaneous visual representations of sine [cosine] on the same coordinate plane (i.e., its graph and unit circle representations) (see *Regarding angles* heading in Chapter 5). Dynamic-and-linked manipulations of the point on the unit circle and its correspondence on the sine [cosine] graph fortified students' concept images on the meaning of π, contrary to their initial concept images, through merging its meaning as the angle measure in radians and as a real number (i.e., approximately 3.14) as a consequence of their reasoning about π based on the correspondence of π-radian angle in the (*unit*) *circle register* to about 3.14 on the *x*-axis in the *graphical register*.

Furthermore, on the contrary to the initial interview results, anymore “coordinate plane” was not a *cognitive conflict factor* when the position of π on the *x*-axis is considered simultaneously within the *graphical register* and the (*unit*) *circle register*. Besides, when reasoning about the sine [cosine] value of a real number smaller than 2π in the *symbolic register*, students were able to convert this real number as an angle measure in radians within the *symbolic register* into the same real number on the *x*-axis as the corresponding angle within the *graphical register*, as well as its correspondence arc in the (*unit*) *circle register*. In addition, when reasoning about a sine value of a real number greater than 2π in the *symbolic register,* students were able to differentiate the equivalent but not equal angles’ positions on the *x*-axis with the 2π-length intervals in the *graphical register* based on their ability to convert the **continuously-repeated full-round turnings** in the counterclockwise [clockwise] direction in the (*unit*) *circle register* into the **continuously-repeated regular intervals** on the *x*-axis in the positive [negative] direction in the *graphical register* (see *Regarding angles* heading in Chapter 5). It was the point that students started to reason about the real number set (R) in the *symbolic register* as the domain set of the sine [cosine] function via thinking of the angle concept in the *graphical register* as a continuous and repeated variable on the *x*-axis.

5.3. Summary of Students’ Developments on Basic Trigonometric Functions

According to the initial interview results, students’ reasoning about trigonometric ratios was not based on the angle measure rather than lengths of sides. Although they were able to define sine [cosine] as the ratio of opposite [adjacent] side to hypotenuse in the right triangle context, they did not aware sine [cosine] ratio’s independence from lengths of sides upon entering the teaching experiment. During the first task of the teaching experiment, when discussing this issue under the dynamic-and-simultaneous manipulations on the similar right triangles and their corresponding trigonometric ratios in GSP environment (*Figure 5.1*), all students constructed a new concept definition image on trigonometric ratios as dependent only on angle measure and independent from side lengths.

Moreover, prior to the teaching experiment, students' conceptions of trigonometric values in the unit circle context were unrelated to those in the right triangle context. That is to say, their concept definition images, for example, on sine in the unit circle context did not include any related-part of their concept definition images on sine in the right triangle context. At the end of the first task of the teaching experiment, their conceptions on trigonometric ratios' independence from length brought forth their reasoning about the opposite [adjacent] side of a right triangle with 1-unit hypotenuse as sine [cosine]. It was the first step of their associations of the right triangle context with the unit circle context.

Cognitive analysis of the initial interviews indicated critical troubles on students' conceptions on function concept that were restricted to the polynomial functions in the *symbolic register*, as well as dominated visual representations by the *graphical register*. Therefore, Task 4, started with the definition of the function concept, and continued through discussions of whether the relation mapping the principal arc into the corresponding perpendicular line segment from the reference point of this arc to the *x*-axis was a function or not (*Figure 5.17*). And then, three different measures (i.e., length of this line segment, sine [cosine] of this principal measure, and ordinate [abscissa] of the reference point) were compared and contrasted with each other in four quadrants of the unit circle through taking GSP's dynamically-linked structures. During this process, Ebru's different point of view emerged on the definition of sine based on the directed-opposite length of the reference right triangle regarding the quadrants within the (*unit*) *circle register*. This implies that Ebru's mental image related to sine of an angle just included a reference right triangle in each quadrant with the directed-opposite side rather than a set of geometric procedures (see *Summary of Students' Initial Concept Images* heading in Chapter 4). This mental image prompted a distinct shift on Ebru's reasoning about sine of an angle in any quadrant within the (*unit*) *circle register*. That is to say, Ebru began to reason quickly and accurately about the signs of sine regarding the quadrants within the (*unit*) *circle* *register* without trying to remember the memorized-rules in this regard, as well as the values of sine [cosine] for the angles corresponding to the axes such as 0 and 3π/2 in radians. However, other students did not rich this kind of mental image related to sine [cosine] of an angle within the (*unit*) *circle register* throughout Task 4.

At the beginning of Task 5, it was observed that none of the students was aware of the determination of the sine [cosine] value of an angle in the *symbolic register* by using the coordinate axes as a signed-ruler in the (*unit*) *circle register* despite of their interpretations of sine [cosine] as the ordinate [abscissa] of a point on the unit circle. Therefore, the researcher provoked them to compare two visual objects, namely, the opposite line segment and the radius line segment on the positive *y*-axis in terms of their magnitudes, and then, estimate the exact value referring to the *directed-opposite line segment*. Students were able to quickly and accurately compare these two visual objects in terms of their magnitudes considering their signs, and their estimations were almost same with the sine values of the mentioned angles in any quadrant. It was Task 5 that students had just been able to convert the geometric procedures in their concept definition images on sine within the (*unit*) *circle register* into the values of sine within the *symbolic register*. Besides, in the absence of the reference right triangle regarding the position of the reference point on unit circle, they were able to reason about sine [cosine] of these angles again as the **limit case** of the directed-opposite [adjacent] side; and they were able to convert the existing {non-existing} directed-opposite [adjacent] side within the (*unit*) *circle register* into ±1 {zero} as the value of sine [cosine] corresponding to these angles within the *symbolic register* through considering directions. Therefore, at the end of task 5, all students developed a concept image on sine [cosine] values through determination process of the directed-opposite [adjacent] length of the dynamically-varied reference right triangle in the (*unit*) *circle register*, as well as for its the limit cases, unlike their prior concept images based on the memorized exact values of sine [cosine] at the special angles without any reasons.

However, students these concept images developed in the (*unit*) *circle register*. Therefore, the researcher asked students to reason about a sine value that came first to their mind in order to provoke them to discuss about sine value in the *symbolic register*.

All students expressed a sine value of a real number, such as 30, as an angle in “degrees” that came first to their mind but without stating clearly their “degree” preference as the angle measure unit. Calculation of sin(30) by GSP when the angle measure preference of GSP in radians that produced different output from 1/2 provoked students to reason a real number without π notation as an angle measure. Where, although all students were aware that the difference between two outputs (GSP’s and theirs) arose from the difference between angle measure units, they could not articulate how the sine [cosine] value of this real number corresponded to the appeared output on the screen. When reasoning about an angle’s measure in radians both with and without π notation, students recognized a real number without π notation, for example 30, as an angle measure in radians through transforming it into the symbolic form with π notation by the aid of GSP’s calculate option, and then, locating the reference point on the unit circle through dragging and dropping so as to indicate this angle, and comparing the calculation result of sin(30) with the dynamically-linked sine measure of the angle corresponding to this reference point. Moreover, during this process, they considered π with its approximate real value, i.e., 3.14. Unlike their prior conceptions, students had just started to reason about a real number as an angle measure with two different ways through considering angle measure units. In other words, they were able to convert a real number in the *symbolic register* into two different angles within the (*unit*) *circle register* through considering two angle measure units, i.e., degrees and radians.

Cognitive analysis of students’ initial concept images prior to the teaching experiment indicated “coordinate plane” as a *cognitive conflict factor* when the *graphical register* and the (*unit*) *circle register* were considered simultaneously (see *Summary of Students’ Initial Concept Images* heading in Chapter 4). Visual representations of sine [cosine] on the same coordinate plane both in the (*unit*) *circle register* and the *graphical register*, a theme of Task 6 [Task 7], provided students with the opportunity to compare and contrast the dynamic and simultaneous variations of the reference point on the unit circle and its converted form in the *graphical register*. This opportunity prompted a distinct shift on students’ recognition of the same object (i.e., sine or cosine) represented in different representational registers, as well as their discrimination of what is/is not mathematically relevant in terms of the coordinate plane both in the (*unit*) *circle register* and *graphical register*. That is to say, they were able to differentiate contents of the coordinate planes in the (*unit*) *circle register* and the *graphical register*. For example, they differentiated the meaning of the abscissa [ordinate] of a point in the unit circle from the meaning of the abscissa [ordinate] of a point on the sine [cosine] graph. Moreover, in Task 6 [Task 7], it was observed students were able to convert the meaning of the sine [cosine] function among the registers when interpreting the sine [cosine] value of a real number in the *symbolic register*, such as 2, 3 and 10. For example, when determining sin(2), Defne initially determined 2 as an argument in the domain of the sine function, and its corresponding element in the range of the sine function as the sine value of 2 in the *symbolic register*. And then, considering arguments in the domain set of the sine function as angles, she tried to obtain the angle with the 2-radian measure in the (*unit*) *circle register* through dragging the point *P* on the unit circle (*Figure 5.26*). Next, she focused on the abscissa of the point *P'* in the *graphical register*, which was the correspondence of the point *P* in the (*unit*) *circle register*, to obtain 2 as an angle measure on the *x*-axis in the *graphical register*. Finally, she determined the ordinate of the point *P'* whose abscissa was 2 as the value of sin(2).

Discussions in GSP environment on trigonometric functions within the (*unit*) *circle register* and *graphical register* simultaneously resulted in students' ability to convert the dynamic variation of the sine [cosine] values regarding **continuous turnings** in the counterclockwise {clockwise} direction in the (*unit*) *circle register* into the static representation of the dynamic variation of the sine [cosine] values by associating them with the appropriate right {left} part of the sine [cosine] graph. In other words, they were able to convert the meaning of the sine [cosine] values of the angles with the **same** *static structure* but **different** *dynamic structures* in the (*unit*) *circle register* into the meaning of the *parallel displacement* of the point on the **principal part** of the sine [cosine] graph along the *x*-axis by the 2π-length line segments in the *graphical register* thereby converting the meaning of full-round turnings in the **counterclockwise [clockwise]** direction in the (*unit*) *circle register* into the meaning of the *parallel displacement* along the *x*-axis in the **positive [negative]** direction in the *graphical register* (see *Regarding trigonometric values* heading in Chapter 5). At that point, they started to reason about the sine [cosine] function on the real number set in the *graphical register* based on their ability to transform the principal part of the sine [cosine] graph into the other repeated parts through the parallel displacement of the principal part. This implies that unlike their prior reasoning, anymore students' reasoning on the sine [cosine] values was intimately dependent on the angles instead of considering the sine [cosine] values apart from the angles through restricting them between -1 and 1 as a rule.

5.4. Summary of Students' Developments on Periodicity

According to the initial interview results, the researcher determined that students' concept images on the period concept included crucial troubles within the different representational registers as a consequence of their problematic concept definition images on the periodicity (see *On periodicity* sub-heading for detailed information in Chapter 4).

Task 6 [Task 7] was the first task that the period concept was brought up for discussion based on the variation of the sine [cosine] values regarding the variation of angle measures under their dynamic-simultaneous manipulations of different representations in GSP environment. Dynamic-and-linked manipulations resulted in students' reasoning about the repetition of the sine [cosine] values within the (*unit*) *circle register* as a consequence of full-round turnings, and the repetition of the principal part of the sine [cosine] graph (see *Footnote 34*) into the other repeated parts through the parallel displacement of the principal part along the x-axis by the 2π-length line segments in the *graphical register*. However, when asked students to reason about the meaning of the period concept under the consideration of the sine [cosine] function, none of the students was able to make clear what the period means. At that point, after emphasizing the repetition of the sine values on the regular intervals with different magnitudes, the researcher defined the period concept not going into detail as the regular intervals of the domain set in which a function repeats its values in the *graphical register* and as the lengths of these regular intervals in the *symbolic register*. It was the point that students had just started to reason about the periodicity of the sine function with more than one period. And then, the researcher defined the *prime period* as the smallest period of a periodic function. From this point forward, throughout the teaching experiment, the *period* term was used in the meaning of the *prime period*.

CHAPTER 6

RESULTS FROM TEACHING EXPERIMENT: PART 2

In this chapter, developments of students' understanding on trigonometric functions' general forms is presented from cognitive analyses results of the teaching experiment's second part from each pair [Cemre&Zafer and Defne&Ebru] (see *Instructional Design of This Study* sub-heading in Chapter 3 for detailed description of second part of the designed-instruction). The aim of this chapter is to provide the living models of students' ability to discriminate the *visual features' oppositions* in any representational register in terms of their mathematically relevancies or mathematically differences when dealing with the *representation discrimination tasks* integrated into the conversion tasks.

We defined four main visual features whose oppositions were corresponding to the choice presence/absence of the coefficients (a, b, c and d) in the general form of sine [cosine] function in the *symbolic register* for discrimination tasks of the teaching experiment. These visual feature oppositions were called as A, B, C and D in harmony with a, b, c and d coefficients in the general forms of sine and cosine; i.e., $y=a\sin(bx+c)+d$ and $y=a\cos(bx+c)+d$.

Under the following headings, students' abilities to discriminate these *visual feature oppositions* were presented regarding the (*unit*) *circle register* and *graphical register* in order to answer the research question related to the effect of the dynamically-changed visual components referring to the trigonometric functions on students' *discrimination* ability.

6.1. Development of Students' Concept Images on Trigonometric Functions (General Forms[38]) throughout Teaching Experiment

6.1.1. Visual Feature Opposition A

Visual Feature (A) corresponds to changed-radius in the (*unit*) *circle register* and changed-magnitude in the *graphical register* so that these visual features' opposition corresponds to the choice presence/absence of a coefficient of sine and cosine in the *symbolic register*.

6.1.1.1. Changed-radius in (unit) circle register

Considering students' troubles on the non-unit circle revealed in the initial interviews (see *On Definition of Trigonometric Functions* sub-heading for detailed information in Chapter 4) as well as in the teaching experiment's Task 4 (see lines 20-28 in [Defne & Ebru] Protocol 8), the researcher encouraged students in Task 8 [Task 13] to compare and contrast the principal arcs and the opposite [adjacent] sides of the reference right triangles on the **unit circle** and **non-unit circle** (e.g. lines 1-21 in [Cemre & Zafer] Protocol 13; lines 1-19 in [Defne & Ebru] Protocol 13), as well as the functions mapping the arc angle into the *y*-component [*x*-component] of the point on the **unit circle** and **non-unit circle** in order to see students' abilities to discriminate the role of **the radius as a visual feature opposition (A)** in the (*unit*) *circle register*.

In Task 8, initially, the researcher asked students to compare and contrast their current conceptions on the unit circle with those prior to the teaching experiment (e.g., lines 1-4 in [Cemre & Zafer] Protocol 13). Where, it was observed that all of the students were aware of the role of one-unit radius in the meaning of the opposite side as sine (e.g., lines 5-11 in [Cemre & Zafer] Protocol 13) and in the meaning of the arc length as the angle measure in radians (e.g., lines 19-39 in [Cemre & Zafer] Protocol 13). At that point, the researcher encouraged students to reason about the meaning of the arc length (e.g., lines 19-24 in [Cemre & Zafer] Protocol 13) and the opposite side

[38] See *Definitions of Terms* heading at the end of *Introduction* chapter.

on the non-unit circle (e.g., lines 1-19 in [Defne & Ebru] Protocol 13) through comparing and contrasting with those on the unit circle.

On the one hand, when reasoning about the arc lengths, initially, students interpreted the proportional change of the arc lengths regarding the radius in the *language register* (e.g., lines 19-39 in [Cemre & Zafer] Protocol 13). However, when the researcher asked them to test this proportional relation through taking GSP's "measure" and "calculate" advantages (e.g., lines 40-44 in [Cemre & Zafer] Protocol 13), students encountered a trouble arising from the difference between the distance-measure-unit preference of GSP as centimeter and the visual distance-measure-unit of the coordinate axes (e.g., lines 45-49 in [Cemre & Zafer] Protocol 13). At that point, the researcher provoked students to reason what they called as the "unit" through recommending them to measure it by GSP (e.g., lines 50-57 in [Cemre & Zafer] Protocol 13). It was the point that students had just started to reason about a number different from 1 as a unit (e.g., lines 58-66 in [Cemre & Zafer] Protocol 13). In other words, they had just considered a non-unit circle regarding the centimeter distance-measure-unit as a unit circle regarding the visual distance-measure-unit of the coordinate axes. This reasoning prompted a distinct shift on their association of the arc lengths with the angle measure in radians. That is to say, they began to be able to determine the angle measure in radians corresponding to an arc on a non-unit circle by means of this arc's length through defining their own unit for the distance-measure (e.g., lines 24-39 and 67-102 in [Cemre & Zafer] Protocol 13). In addition, they were able to correctly convert the relation between the arc lengths corresponding to the same angle on the unit circle and non-unit circle in the (*unit*) *circle register* into the proportional relation of their measures with respect to the radii in the *symbolic register* (e.g., lines 24-39 and 100-102 in [Cemre & Zafer] Protocol 13). Furthermore, not only in Task 8 but also in Task 13, when similar discussions were done in each quadrant under the manipulation of the angle's openness as well as under the manipulation of the radius, students' reasoning in the same way was observed that the changing the radius did not cause the changing the angle measure, but caused the proportional change of the arc lengths in the (*unit*) *circle register*.

On the other hand, in Task 8, when reasoning about the opposite sides of the reference right triangles on the unit circle and the non-unit circle in terms of sine, students' initial way of reasoning were based on the equality of sine ratios for these similar reference right triangles (e.g., lines 8-11 in [Cemre & Zafer] Protocol 13; lines 1-12 in [Defne & Ebru] Protocol 13). Where, students' language implies that their visual focus on the structure of the unit circle and the non-unit circle together with their reference right triangles in the (*unit*) *circle register* (*Figure 6.1*) was the similar [reference] right triangles related to sine instead of the ordinates of the points on the circles. So, considering the changed-radius as the similarity ratio, they were able to determine the proportional relation between opposite sides of the reference right triangles on the non-unit circle and the unit circle (e.g., lines 13-38 in [Defne & Ebru] Protocol 13). In order to ease students' generalization of this reasoning for any radius, the researcher encouraged them to control this proportional relation through taking GSP's "measure", "calculate" and "drag-drop" advantages (e.g., lines 39-52 in [Defne & Ebru] Protocol 13). When students confirmed the GSP's outputs to their reasoning on the proportional relation between the opposite sides and the radii (e.g., lines 53-62 in [Defne & Ebru] Protocol 13), the researcher encouraged them to investigate the variation of this relation under the dynamic-simultaneous manipulations of the angle measure (e.g., lines 63-64 in [Defne & Ebru] Protocol 13) and the radius of the non-unit circle (e.g., line 67 in [Defne & Ebru] Protocol 13) in the (*unit*) *circle register*. Where, it was observed that students were able to reason about the co-equally variation of the radius and the ratio of the opposite side on the non-unit circle to the opposite side on the unit circle (e.g., line 65-66 and 68-71 in [Defne & Ebru] Protocol 13). At this point, the researcher encouraged them to reason about the opposite side of the reference right triangle on the non-unit circle with respect to sin(x) as the opposite side of the reference right triangle on the unit circle (e.g., line 72-79 in [Defne & Ebru] Protocol 13). Where, students were able to accurately reason about the opposite side on the non-unit circle as the multiplication of sin(x) with the radius of the non-unit circle in the *symbolic register* (e.g., lines 80-94 in [Defne & Ebru] Protocol 13). In addition, they were able to generalize this reasoning when investigating the variation of the dynamically-linked-measures y_R and (changed radius).sin(x) (*Figure 6.6.4*) through manipulating the angle's openness and the radius of the non-unit circle in GSP environment (e.g., lines 95-106 in [Defne & Ebru] Protocol 13). Furthermore, when similar discussions were done in Task 13 for the adjacent sides of the reference right triangles on the unit circle and the non-unit circle in terms of cosine, it was observed that students reasoned in the same way about the adjacent side of the reference right triangle on the non-unit circle as the multiplication of cos(x) with the radius of the non-unit circle in the *symbolic register*.

From this point forward of Task 8 [Task 13], the researcher preferred to scrutinize students' reasoning about the functions mapping the arc angle to the corresponding opposite [adjacent] side of the reference right triangle on the unit circle and non-unit circle in a more detailed way in the *graphical register* as well. Henceforward, the subsequent progress of students' concept images on the changed-radius in the (*unit*) *circle register* was presented together with its effect on the changed-magnitude in the *graphical register* under the following sub-heading *Changed-magnitude in graphical register*.

[Cemre & Zafer] Protocol 13

Researcher: You had discussed on the unit circle in your class period before. That's our eight meeting… …and we had also discussed on the unit circle up to now. What do you think about the unit circle? Are there changed things in your minds about the unit circle comparing to the old ones?

Zafer: Of course! We found an answer to why the unit circle is used.

Cemre: Yes, unit circle…

Researcher: Why?

Zafer: Because its radius is one. And in case the radius is one, sine… …or division of opposite [side] to hypotenuse is equal to opposite [side].

Cemre: That's, it [opposite side] is directly equal to sine itself…

Zafer: If the radius were 2, then we would divide it [opposite side] by 2…

Researcher: Let's we construct and see what happens with two-unit radius (*giving directions for students to construct the* ***non-unit circle*** *and the* ***unit circle*** *together with their reference right triangles in in the (unit) circle register in GSP environment so as the radius of the non-unit circle to be manipulable like in Figure 6.1*)…

(*Cemre and Zafer cooperatively constructed GSP page like in Figure 6.1 with respect to the researcher's directions.*)

Researcher: This is unit circle (*pointing the unit circle on the screen*). What do you know about this length (*dragging her index finger up and down on the opposite side of the reference right triangle of the **unit circle** on the screen like in Figure 6.1*)?
Zafer: It is sine…
Cemre: …sin(*x*)…
Researcher: What is *x* in here?
Cemre: Angle…
Zafer: Here (*drawing the green arc on the **unit circle** with his index finger in the counterclockwise direction on the screen like in Figure 6.1*).
Cemre: …also here (*drawing the green arc on the **non-unit circle** with her index finger in the counterclockwise direction on the screen*) because two arcs refer the same angle.
Zafer: But, it [angle measure] would be half of its length [arc on the non-unit circle] (*at that time the radius of the **non-unit circle** was about 2-unit length*).
Cemre: Yes, it [angle measure] would be half of this length (*pointing the green arc on the **non-unit circle***).
Zafer: (*Nodding his head up and down*).
Researcher: Why did you think to divide it [arc length on the **non-unit circle**] by 2?
Zafer: …due to radius.
Cemre: …it is 2…
Zafer: Uh-huh.
Researcher: Ok. Let's we calculate it and see.
Cemre: Oops! I'm so excited…
Researcher: We can measure initially the radius of the big circle (*pointing the **non-unit circle** on the screen, at that time radius of the **non-unit circle** was about 2-unit length*).
Cemre: (*Measuring the distance between the origin and the intersection point of the **non-unit circle** with the positive x-axis by the aid of GSP's measuring option. When the result appeared on the screen as 5.58cm like in Figure 6.2*) five point fifty eight… it [radius] must have been 2…
Zafer: (*Looking to the screen without speaking*)
Researcher: What do you see as the measure unit near 5.58?
Zafer: Centimeter…
Cemre: Yes, centimeter.
Researcher: Ok. Where do you mean as 1 unit? Or where is the 1-unit length?
Cemre: Here (*pointing the hypotenuse segment of the reference right triangle on the **unit circle***).
Zafer: Uh-huh (*nodding his head up and down*).
Researcher: Please measure it.
Cemre: (*Measuring the radius of the **unit circle**. When the result appeared on the screen as 2.78 like in Figure 6.2*) hmm… that's, one unit is two point seventy five centimeter.
Zafer: Then, we would transform this from centimeter to unit [length] (*pointing the radius measure on the screen like in Figure 6.2*).
Cemre: Then, we will divide this (*pointing the radius measure on the screen*) by this (*pointing the unit measure on the screen*). (*Calculating the division of "radius" by "unit" as in Figure 6.2*) yes, it is 2-unit (*smiling*).
Zafer: Uh-huh (*smiling*).

Researcher: Good. Now, you can see whether the half of this (*pointing on the screen the length of the principal arc on the **non-unit circle** with respect to the unit length*) is equal to this (*pointing on the screen the length of the principal arc on the **unit circle** with respect to the unit length*).

(*Cemre and Zafer cooperatively measured two arcs length in cm; then, transformed distance unit from centimeter to the unit length; and then, calculated the half length of the principal arc on the **non-unit circle** with respect to the unit length like in Figure 6.3.*)

Cemre: (*When the half length of the principal arc on the **non-unit circle** with respect to the unit length appeared on the screen like in Figure 6.3*) yes… …its half…

Zafer: Uh-huh.

Researcher: Ok. What is the angle measure now?

Cemre: Zero point sixty-nine [radians]…

Zafer: Half of this arc length (*pointing the principal arc on the **non-unit circle** on the screen like in Figure 6.3*)...

Cemre: …or this length (*pointing the principal arc on the **unit-circle***)…

Zafer: (*Nodding his head up and down*)

Researcher: Let’s measure both arcs’ angle.

Cemre: (*Measuring the principal arc angle on the **non-unit circle** like in Figure 6.3*) 0.22π radian…

Researcher: Zafer, please measure the other.

Zafer: (*Measuring the principal arc angle on the **unit circle** like in Figure 6.3*) same output.

Cemre: 0.22π radian…

Researcher: (*Opening “new calculation” window and selecting arc angle like in Figure 6.3*) look at this number (*pointing the number 0.6906667 number in the new calculation window*)!

Zafer: It is the multiplied form with 3.14…

Cemre: They imply same things.

Researcher: It is also equal to the arc length on the unit circle (*pointing on the screen the length of the principal arc on the **unit circle** with respect to the unit length*). But as radius changes, arc length also changes (*pointing on the screen the length of the principal arc on the **non-unit circle** with respect to the unit length*).

Zafer: Uh-huh. When radius is 2, it [arc length] doubles… …when it [radius] is 3, it [arc length] three times, etc.

Cemre: Yes.

(*Similar discussions were done on the invariance measures under the variation through manipulating the angle, as well as the radius.*)

[Defne & Ebru] Protocol 13

Researcher: What do you say about the length of this segment (*pointing the opposite side of the reference right triangle on the **non-unit circle** on the screen like in Figure 6.1*)?

Defne&Ebru: (*Looking to the screen without speaking*).

Researcher: Well, you know, for example, on the unit circle this corresponds the sin(alpha) (*dragging her index finger on the opposite side of the reference right triangle of the **unit circle** up and down on the screen*).
Defne&Ebru: Yes.
Researcher: What about this (*dragging her index finger on the opposite side of the reference right triangle of the **non-unit circle** up and down on the screen*)?
Ebru: Since they [reference right triangles] are proportional, wouldn't sine ratios of them be same?
Researcher: Of course, sine ratios from two right triangles would be same (*figuring with her index finger reference right triangles of the **unit circle** and **non-unit circle** both*).
Ebru: Yes.
Researcher: But I ask you to reason about this segment (*dragging her index finger on the opposite side of the reference right triangle of the **non-unit circle** up and down on the screen*) not about the ratios.
Defne: Now, I think it [opposite side of the reference right triangle in the **unit circle**] will be quadrupled [in the **non-unit circle**] (*radius of the **non-unit** circle about 4 units at that time*).
Researcher: Why?
Defne: …because circle's radius is quadrupled… It [radius] increased from 1 to 4.
Researcher: Does the quadrupled-radius require quadrupling this (*pointing the opposite side of the reference right triangle of the **unit circle** on the screen*)?
Defne: Yes.
Researcher: Why?
Defne: Because the maximum value this (*pointing with the cursor the opposite side of the reference right triangle in the **non-unit circle***) can take is 4 (*dragging the cursor on the intersection point of the **non-unit circle** with the positive y-axis*), and the maximum value this (*pointing with the cursor the opposite side of the reference right triangle in the **unit circle***) can take is 1 (*dragging the cursor on the intersection point of the **unit circle** with the positive y-axis*).
Researcher: Yes, it is true in here (*pointing the positive y-axis*). But does it prove quadrupling in everywhere?
Ebru: Yes… …because similarity ratio is 4…
Defne: Yes, similar…
Researcher: Good. Let's we find signed-measures referring to these segments (*pointing respectively opposite sides of the reference right triangles on both **unit** and **non-unit circles***).
Ebru: …*y*-values…
Researcher: Uh-huh...
Defne: (*Measuring y-values of the points P and R like in Figure 6.6.4.*)
Researcher: If I ask you to discuss about the relation between these two measures (*pointing y-values of the points P and R on the screen like in Figure 6.6.4*), what do you do?
Defne: I would proportion them.
Researcher: How?
Defne: I would divide this (*pointing y_R on the screen*) by this (*pointing y_P on the screen*), and find how many times it changes.
Researcher: Please check.

Defne: (*Dividing y_R by y_P like in Figure 6.6.4*) yes about four times.
Researcher: Ok. What is the radius of this circle (*pointing the **non-unit circle** on the screen*).
Defne: …four.
Ebru: Uh-huh.
Researcher: Can we measure it by abscissa of the point B?
Defne&Ebru: Yes.
Researcher: Ebru, please measure it [radius].
Ebru: (*Measuring the radius*) yes, it [radius] is same [with proportion of y_R to y_P].
Defne: Yes.
Researcher: Are they same in everywhere of the circles? Please see the variation when dragging the point P.
Defne: (*Dragging the point P in the counterclockwise direction one full-round*) yes.
Ebru: Uh-huh (*nodding her head up and down*).
Researcher: Well, what about the variation when manipulating the radius?
Ebru: (*Dragging the point B left and right*) radius changed.
Defne: But these are still equal to each other (*pointing the measures of the radius and the proportion of y_R to y_P on the screen like in Figure 6.6.4*).
Ebru: Yes.
Researcher: Well, (*measuring the principal angle*) let's we call angle measure as *x* (*labeling the measure as x*). Then, you know here is sin(*x*) (*calculating sin(x) like in Figure 6.6.4*)…
Defne&Ebru: Uh-huh.
Researcher: If here is sin(*x*) (*dragging her index finger on the opposite side of the reference right triangle of the **unit circle** up and down on the screen*), what would be here regarding sin(*x*) (*dragging her index finger on the opposite side of the reference right triangle of the **non-unit circle** up and down on the screen*)?
Defne: …similar [right] triangles.
Ebru: …2*x*… …pardon two times sin(*x*) (*at that time the radius of the **non-unit circle** was about 2-unit length*)…
Researcher: How do you think?
Ebru: I think… …if we multiply this with radius (*dragging her index finger on the opposite side of the reference right triangle of the **unit circle** up and down on the screen*), do we find this (*dragging her index finger on the opposite side of the reference right triangle of the **non-unit circle** up and down on the screen*)?
Defne: Yes. y_P times radius equal to y_R.
Researcher: y_P was referring to sin(*x*).
Defne: Then, it [y_R].would be sin(*x*) times radius.
Ebru: Yes.
Researcher: …or, radius times sin(*x*).
Defne: Yes.
Ebru: (*Nodding her head up and down*)
Researcher: Let's we calculate radius times sin(*x*)… …and see it is equal to y_R.
Defne: (*Calculating the multiplication of the radius and sin(x) like in in Figure 6.6.4*) equal.
Researcher: Please see the variation when dragging the point P. Are they always equal to each other in everywhere of the circle?

Ebru: (*Dragging the point P in the counterclockwise direction throughout one full-round*) their values changed but still they are equal.

Defne: Uh-huh (*nodding her head up and down*).

Researcher: Well, what about the variation when manipulating the radius?

Defne: (*Dragging the point B in the counterclockwise direction throughout one full-round*) changing but still equal.

Ebru: Yes.

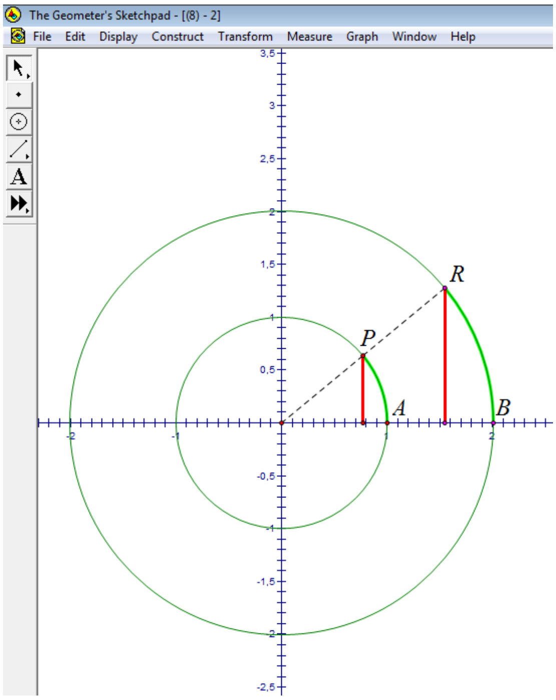


Figure 6.1. Construction of the non-unit circle and the unit circle together with their reference right triangles in in the (*unit*) *circle register*

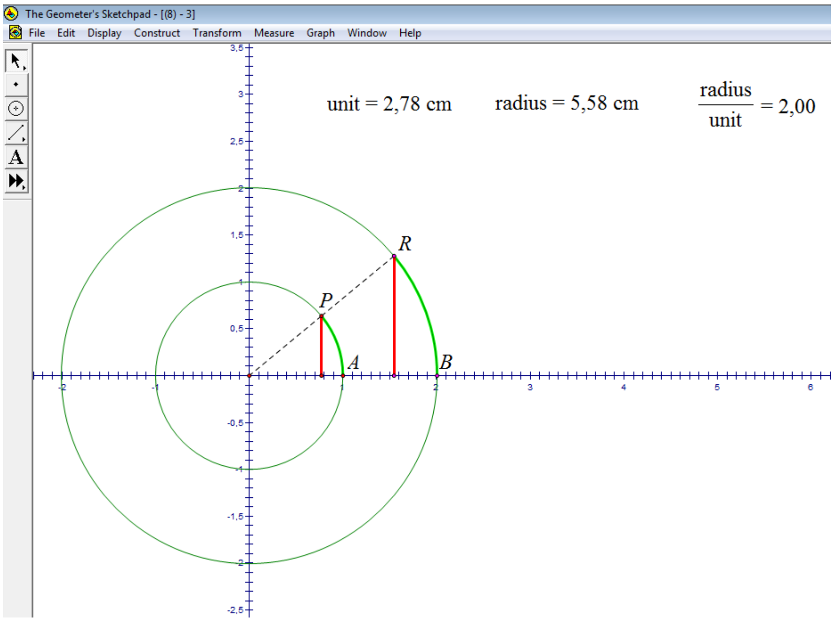


Figure 6.2. Assigning a distance different from 1cm as the unit length in the *symbolic register* by the aid of the coordinate plane in the (*unit*) *circle register*

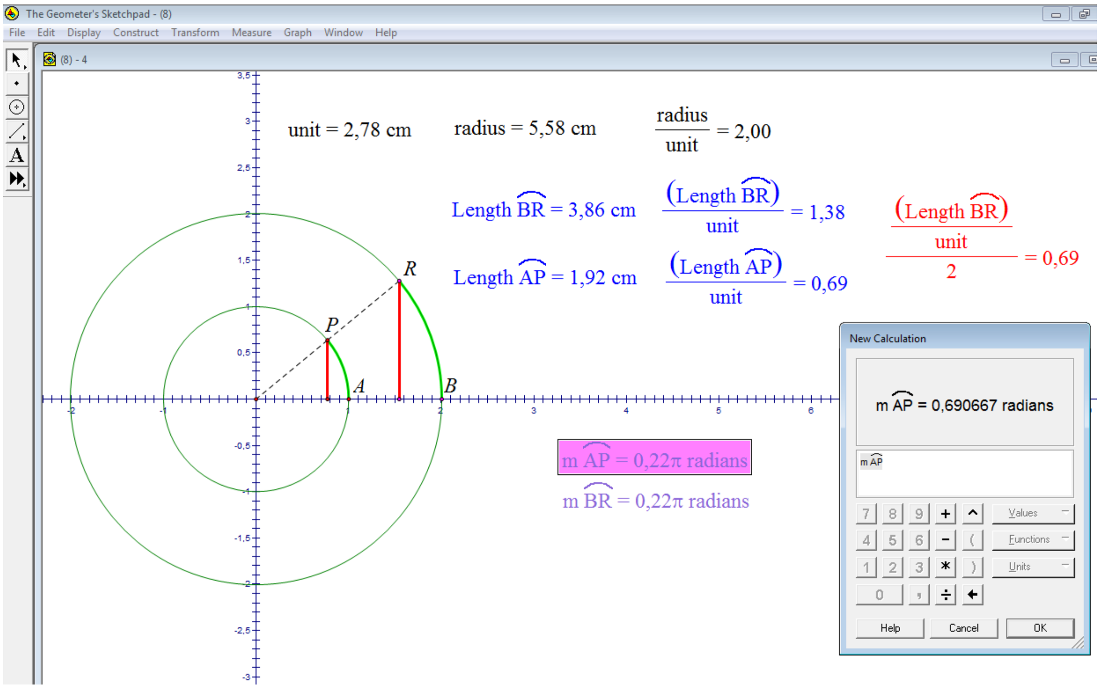


Figure 6.3. Conversion of the arcs remaining in an angle regarding circles with different radii in the (*unit*) *circle register* into this angle's measure in radians in the *symbolic register*

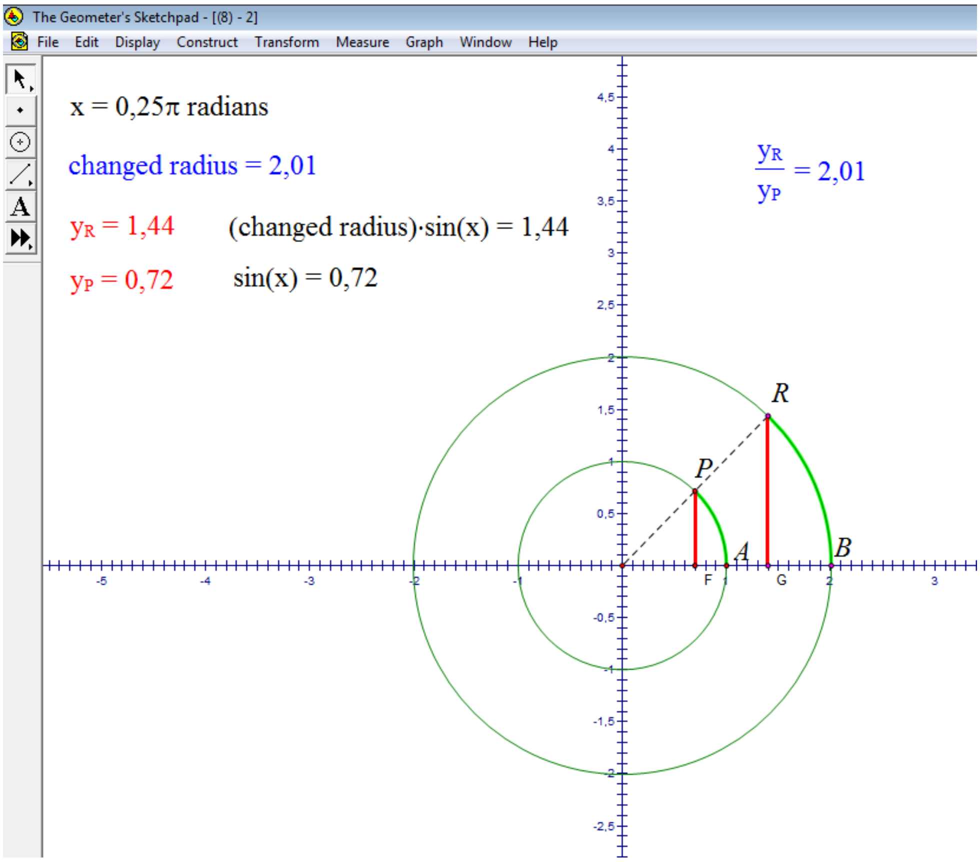


Figure 6.6.4. Generalization of the *y*-values of the points referring to the same angle on circles with different radii with respect to sine of this angle in the (*unit*) *circle register*

6.1.1.2. Changed-magnitude in graphical register

In Task 8 [Task 13], after students' awareness of the variation on the arc angle and the opposite [adjacent] side of the reference right triangle on the non-unit circle with respect to their correspondences on the unit circle (see *Changed-radius in (unit) circle register* heading), the researcher determined to scrutinize students' ability to discriminate the functions mapping the arc angle to the corresponding opposite [adjacent] side of the reference right triangle on the unit circle and non-unit circle when dealing simultaneously with their conversions from the (*unit*) *circle register* into the *graphical register*. For this purpose, she provoked them to convert these two functions [corresponding to sine and *r*.sine functions] in the (*unit*) *circle register* into their dynamically-linked correspondences in the *graphical register* via taking "plot as (*x*,*y*)" and "trace point" advantages of GSP; and then, encouraged them to compare and contrast these functions with each other in a more detailed way in the *graphical register* as well (e.g., lines 1-29 in [Cemre & Zafer] Protocol 14).

In Task 8, when the traced *r*.sine graph in the [0,2π) interval appeared on the screen, without going into details, Cemre associated this traced graph with the right first part of the sine graph –corresponding to the first full-round turning in the counterclockwise direction in the (*unit*) *circle register* (lines 1-11 in [Cemre & Zafer] Protocol 14). But, when reasoning about this traced-graph comparing to that of sine, all students dissociated it from the sine graph in terms of their magnitudes (e.g., lines 11-35 in [Cemre & Zafer] Protocol 14). Where, while Defne's and Ebru's reasoning in the *graphical register* was based on the peak points and intersection points of the graphs, Cemre's and Zafer's reasoning was based on the points *P′* and *R′* which were constructed in the *graphical register* as the dynamically-linked correspondences of the points *P* and *R* in the (*unit*) *circle register* (*Figure 6.6*). Cemre's and Zafer's different focus on reasoning caused Zafer's consideration of "angle" as the unchanging component and "corresponding *y*-values" as the changing components of two functions in the *graphical register* (lines 30-34 in [Cemre & Zafer] Protocol 14), as well as caused Cemre's attribution of the changed-magnitudes regarding points *P′* and *R′* in the *graphical register* to the changed-radii regarding these points' parents in the (*unit*) *circle register* in GSP environment –i.e., points *P* and *R* (e.g., lines 30-35 in [Cemre & Zafer] Protocol 14). While Zafer's this consideration prompted him to establish the proportional –sixfold– relation between the *y*-values of the points *P′* and *R′* in the *symbolic register* (lines 42-50 in [Cemre & Zafer] Protocol 14), Cemre's this attribution prompted her to attribute the coefficient of the 6.sine function in the *symbolic register* to the radius of the six-unit circle in the (*unit*) *circle register* (lines 51-52 in [Cemre & Zafer] Protocol 14). On the other hand, Defne and Ebru did not independently contribute to the discussion with their reasoning about these kinds of relations including conversions of the changed-components in different representational registers until researcher's provocation of them to reason about these aspects. When the researcher asked them to reason about the relations both between and within the registers, Defne and Ebru were able to establish these relations, as well.

However, unlike Cemre's directly association of the coefficient of the 6.sine function in the *symbolic register* with the radius of the six-unit circle in the (*unit*) *circle register* (lines 51-52 in [Cemre & Zafer] Protocol 14), all other students' indirectly association was observed. That is, they attributed the coefficient of *a*.sine function in the *symbolic register* primarily to the magnitude in the *graphical register* (e.g., 51-54 in [Cemre & Zafer] Protocol 14) secondarily to the radius of the non-unit circle in the (*unit*) *circle register* (e.g., 51-56 in [Cemre & Zafer] Protocol 14). At this point, in order to ease students' direct-association between the (*unit*) *circle register* and the *symbolic register*, the researcher encouraged them to reason about the symbolic expression of the visual function on the non-unit circle mapping the principal arc to the corresponding *y*-value (e.g., lines 58-62 in [Cemre & Zafer] Protocol 14). Where, it was observed that none of the students had trouble on expressing this function symbolically (e.g., lines 63-67 in [Cemre & Zafer] Protocol 14). However, in here, except Cemre, none of the other students mentioned the relation between the radius of the non-unit circle and the coefficient of the function, yet. So, the researcher encouraged them to reason about these aspects under the manipulation of the radius of the non-unit circle hoping to provide students with awareness of the role of the radius in the symbolic expression (e.g., lines 1-19 in [Cemre & Zafer] Protocol 15). Where, it was observed that students were able to correctly revise the function's symbolic expression regarding the changed-radius (e.g., lines 18-21 in [Cemre & Zafer] Protocol 15). Moreover, they were aware of this revised-function's correspondence in the *graphical register* (e.g., lines 22-25 in [Cemre & Zafer] Protocol 15). Furthermore, they were able to express this function's symbolic form regarding the radius (e.g., lines 26-28 in [Cemre & Zafer] Protocol 15). And then, when investigating the dynamically-linked components in different registers under the manipulation of the radius and the angle's openness, they were able to generalize their reasoning about the relations among "changed-radius" in the (*unit*) *circle register*, "changed-magnitudes" in the *graphical register* and "changed-coefficient *r* of the *r*.sine function" in the *symbolic register* (e.g., lines 29-36 and 62-71 in [Cemre & Zafer] Protocol 15).

At that point of Task 8, Cemre and Zafer showed more advanced reasoning about these general relations (among *radius*, *magnitude* and *coefficient a*) than the reasoning of Defne and Ebru through extending their reasoning over the patterns of the changes. In the first instance, when investigating the variation of the *r*.sine graph under the manipulation of the radius, Zafer was concerned about the **limit case** of the radius –i.e., *r*=0 condition. That is, through focusing on the pattern of the dynamically-changed graph of the function in the *graphical register*, he considered the graph in this case as the line-shaped (lines 32-39 in [Cemre & Zafer] Protocol 15). However, Cemre objected to Zafer's this reasoning through focusing on the pattern of the dynamically-changed visual representation of the function in the (*unit*) *circle register* (lines 40-45 in [Cemre & Zafer] Protocol 15). At that point, the researcher encouraged them to consider the symbolic expression of the function (lines 46-47 in [Cemre & Zafer] Protocol 15). Where, both Cemre and Zafer were able to interpret the function in the **limit case** of the changed-radius in the (*unit*) *circle register* as the **zero-function** in the *symbolic register* and as the ***x*-axis** in the *graphical register* based on the symbolic expression of the function (lines 46-54 and 72-75 in [Cemre & Zafer] Protocol 15).

In the second instance, when the researcher encouraged them to reason about the negative coefficient of "–sine" function (lines 74-75 in [Cemre & Zafer] Protocol 15), Cemre's different point of view emerged in the *graphical register*. After reasoning about the shape of "–sine" function's graph as the reflection of the sine graph regarding the *x*-axis (lines 76-86 in [Cemre & Zafer] Protocol 15), she considered "–sine" function in the *graphical register* as the *parallel displacement* of the sine graph along the *x*-axis by the π-length in the positive direction (lines 90-103 in [Cemre & Zafer] Protocol 15). On the other hand, Zafer's reasoning was observed again based on the pattern of the dynamically-changed graph of the function in the *graphical register* regarding the coefficient-decrease (lines 87-89 in [Cemre & Zafer] Protocol 15). That is, his actions imply that he reasoned about the negative coefficient of the sine function in the *graphical register* through extending the decreasing-pattern of the *y*-value of the graph's peak point on $\pi/2$ regarding the decreasing-positive-coefficient of sine into the decreasing-negative coefficient of sine. In addition, Zafer also reasoned in the same way with Cemre but through being affected by her reasoning (lines 90-104 in [Cemre & Zafer] Protocol 15). At that point, the researcher constructed a parallel line from the point P' to the x-axis in order to discuss the *parallel displacement* of the graphs based on transformations between the sine and –sine functions in a more detailed way in the *symbolic register* as well (*Figure 6.9*). And then, she encouraged them to establish relations between the coordinates of the point P' and its *parallel displacement* along the x-axis by the π-length in the negative direction (lines 105-114 in [Cemre & Zafer] Protocol 15). Where, it was observed that both Cemre and Zafer were able to express these points' abscissas with respect to each other, as well as the equality of their ordinates in the *symbolic register* (lines 105-133 and 176-186 in [Cemre & Zafer] Protocol 15). Moreover, they were able to correctly reason in the *graphical register* about the positions of the points defined in the *symbolic register* through taking the abscissa of the point P' or its *parallel displacement* point P'' as the reference (*Figure 6.10*); such as the position of $(x\text{-}\pi, \sin(x))$ considering the abscissa of the point P' as x, and the position of $(x+\pi, -\sin(x))$ considering the abscissa of the point P'' as x (lines 140-146 in [Cemre & Zafer] Protocol 15). Where, they were able to easily determine that the position of the ordered pair $(x\text{-}\pi, \sin(x))$ [$(x+\pi, -\sin(x))$] was the point P'' [P'] which was a point on the –sine [sine] graph. At this point, it was observed a distinct shift on their reasoning about the relation between the sine and –sine functions through seeing the point P' [P''] on the sine [–sine] graph as a point on the –sine [sine] function's *parallel displacement* along the x-axis by the π-length in the positive [negative] direction. That is to say, they had just started to reason about the equality between $\sin(x)$ and $-\sin(x\text{-}\pi)$ (lines 140-165 in [Cemre & Zafer] Protocol 15), as well as the equality between $-\sin(x)$ and $\sin(x+\pi)$ (lines 176-191 in [Cemre & Zafer] Protocol 15) in the *symbolic register*.

On the other hand, due to their unwillingness to participate the discussions towards the end of Task 8, the researcher determined to postpone the discussions with Defne and Ebru about the non-positive coefficients of sine to the following episode. At the beginning of Task 9, the researcher encouraged them to interpret *a*.sine function in terms of the effect of the coefficient ***a*** in the *symbolic register* on the shape of the graph in the *graphical register* (lines 1-16 in [Defne & Ebru] Protocol 14). When investigating the dynamic-variation of the *a*.sine graph under the manipulation of the coefficient *a* by a slider (*Figure 6.13*), it was observed that Defne and Ebru were able to reason about the changing pattern of the *a*.sine graph regarding the variation of the positive coefficient *a* through focusing on the peak points (lines 17-29 in [Defne & Ebru] Protocol 14). On the other hand, despite of their interpretations about the changed-direction of the peak points of the graph for the negative coefficient (lines 30-38 in [Defne & Ebru] Protocol 14), none of them was able to reason about the cause of this variation (lines 39-40 in [Defne & Ebru] Protocol 14). So, the researcher encouraged them to consider the negative coefficient on "–sine" function (lines 41-43 in [Defne & Ebru] Protocol 14). Where, both Defne and Ebru interpreted "–sine" function as the opposite of the sine function based on their symmetrical graphs regarding the *x*-axis (lines 44-47 in [Defne & Ebru] Protocol 14). However, unlike Cemre and Zafer, Defne and Ebru did not consider the graphs of –sine and sine as the *parallel displacement* of each other along the *x*-axis by the π-length. But, they were able to interpret, for example, -3.sine function in the *graphical register* based on two steps: (*i*) considering 3 as the magnitude of the graph, and (*ii*) considering the minus sign as the opposite direction of the peak points to those of 3.sine regarding the *x*-axis (lines 48-55 in [Defne & Ebru] Protocol 14). Moreover, by the aid of the slider, they were able to interpret the *a*.sine function in case of the zero-coefficient as the ***x*-axis** in the *graphical register* and as the **zero-function** in the *symbolic register* (lines 56-64 in [Defne & Ebru] Protocol 14).

Where, the researcher preferred not to discuss about the meaning of the non-positive coefficients of sine in the (*unit*) *circle register* until students' awareness of defining a new function related to sine regarding all possible visual-changes in the (*unit*) *circle register*; in addition to the **changed-radius** (theme of *Task 8*); the **changed-center** (theme of *Task 9*), and the **changed-arc** referring to the input of sine [cosine] (themes of *Task 10* and *Task 11*). So, the subsequent progress of students' concept images on the non-positive coefficients of sine in the (*unit*) *circle register* was presented under the sub-headings *Changed-arc with a constant difference in (unit) circle register* and *Changed-arc through folding angle in (unit) circle register*.

[Cemre & Zafer] Protocol 14

(*Students constructed (x,r.sin(x)) ordered-pair taking advantages of the plot as (x,y) option of GSP and labelled as R'; and then, constructed perpendicular lines from this point to the coordinate axes; finally, constructed r.sine graphs in the [0,2π) interval through tracing the point R' like in Figure 6.5 by the researcher's directions.)*

Researcher: Let's we drag the point P.

Zafer: (*Dragging the point P about 15-second without speaking*) yes, these two are equal to each other (*pointing the red segments corresponding to the opposite side of the reference right triangle on the* ***non-unit circle*** *and the perpendicular segment from the point R' on the r.sin(x) graph to the x-axis*).

Cemre: Wasn't this [traced graph] first period of sine?

Zafer: No. (*After 4-second pause*) it's not that of sine… …it's that of six times of sine (*at that time the radius of the* ***non-unit circle*** *was about 6-unit*).

Cemre: Oh yeah, it is first period of sine's six times…

Researcher: If we drew it for sine, how would be the graph?

Zafer: Then, from one… …it would be like that (*figuring with his index finger the* ***right-*** *first part of the sine graph corresponding to the first full-round turning in the* ***counterclockwise*** *direction*).

Cemre: It [a *y*-value on the graph] would be between minus one and plus one, but in here, due to six times of sine, it [GSP] drew regarding that.

Zafer: Uh-huh.

Researcher: Let's we draw it [sine].

(*Students constructed (x,sin(x)) ordered-pair taking advantages of the plot as (x,y) option of GSP and labelled it as P'; and then, constructed perpendicular lines from this point to the coordinate axes; finally, constructed sine graph in the [0,2π) interval through tracing the point P' like in Figure 6.6 by the researcher's directions.*)

Researcher: Let's we drag the point P.

Zafer: It would be at most one…

Cemre: (*Dragging the point P and looking to the screen without speaking*)

Researcher: How was changed the sine graph?

Zafer: Angle didn't change… …only values that sine took (*indicating a magnitude through putting with his right and left hands horizontally parallel to each other*) increased six-times (*indicating a larger magnitude through horizontally moving his hands away from each other*).

Cemre: …as a result of radius…

Researcher: Ok. What else?

Cemre: Period didn't change.

Zafer: Yes.

Researcher: What is the period?

Zafer: Again 2π…

Cemre: 2π.

Zafer: Well, if we plot this function by this program [GSP] (*pointing the blue-traced graph on the screen like in Figure 6.6*)… …that's, we multiple this with six (*pointing the pink-traced graph*), and draw it [graph of 6.sin(*x*)], would it give us this (*pointing the blue-traced graph*)?

Researcher: Let try and see!

(*Zafer constructed the graphs of y=sin(x) and y=6.sin(x) by the aid of plot new function option of GSP like in Figure 6.7.*)

Zafer: Yes, it gave… Good…

Cemre: Very good…

Researcher: Then, what does 6 times sine mean?

Cemre: It means… …a 6-[unit]-radius circle.

Zafer: Between minus six and plus six (*indicating a magnitude through putting with his right and left hands horizontally parallel to each other*)…

Researcher: Good, then, values between minus six and plus six are based on the radius.

Zafer: So is minus one and plus one.

Cemre: Uh-huh.

Researcher: If I consider a function… …so that it is mapping on this circle (*pointing the **non-unit circle** on the screen like in Figure 6.7*) this arc's angle (*pointing the green arc on the **non-unit circle***) to this point's *y*-value (*putting her index finger on the point R, and then dragging downward on the red line segment*), how can you express this function symbolically?

Cemre: …six times sin(*x*) (*at that time the radius of the **non-unit circle** was about 6-unit*)…

Zafer: *h*(*x*) equal to six times sin(*x*) (*pointing the h(x)=6sin(x) equation on the screen like in Figure 6.7*)…

Cemre: Yes.

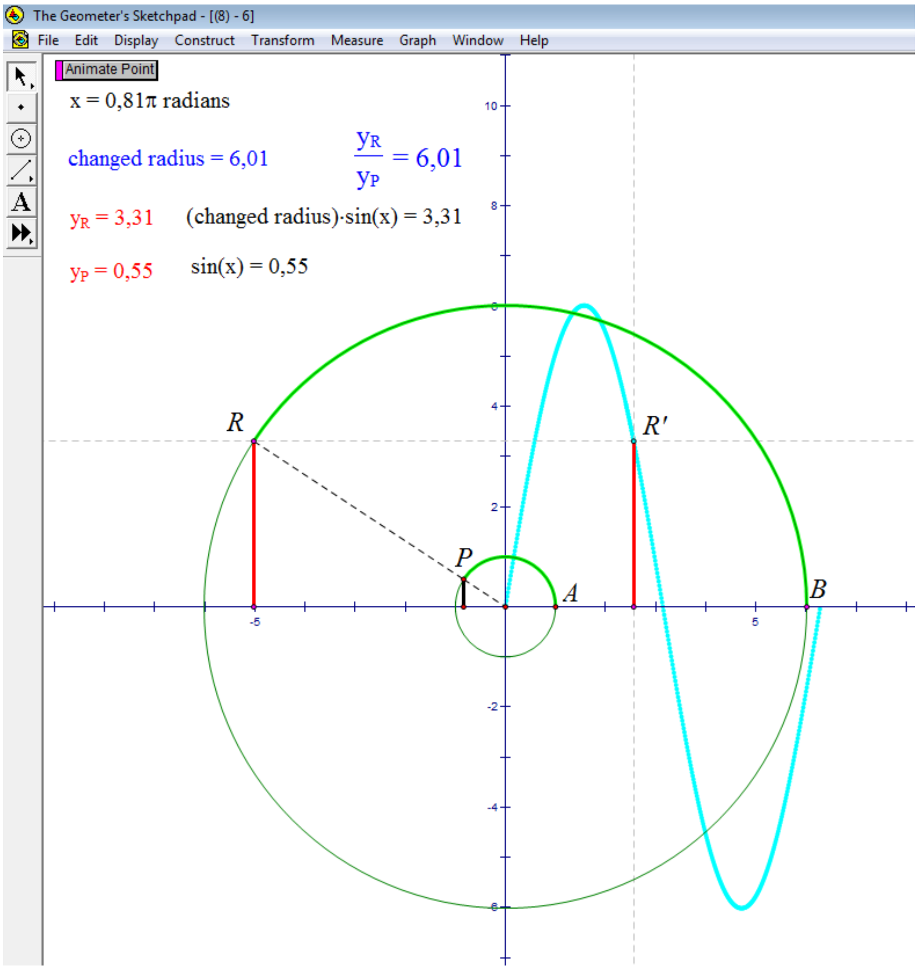


Figure 6.5. Construction of *r*.sine graph in the $[0,2\pi)$ interval via taking the trace point (for the point *R′*) and animate point (for the point *P*) advantages of GSP

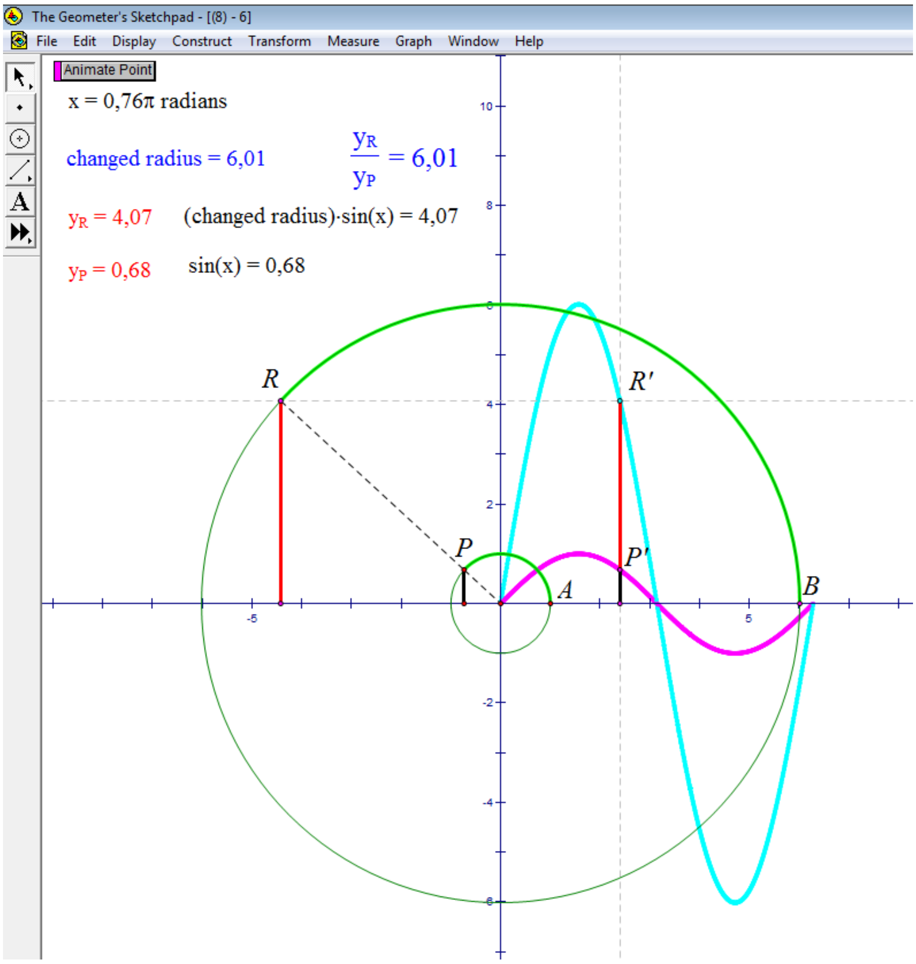


Figure 6.6. Construction of sine and *r*.sine graphs' **right** first parts in the *graphical register* corresponding to the first full-round turning in the **counterclockwise** direction respectively on the unit circle and r-unit circle in the (*unit*) *circle register*

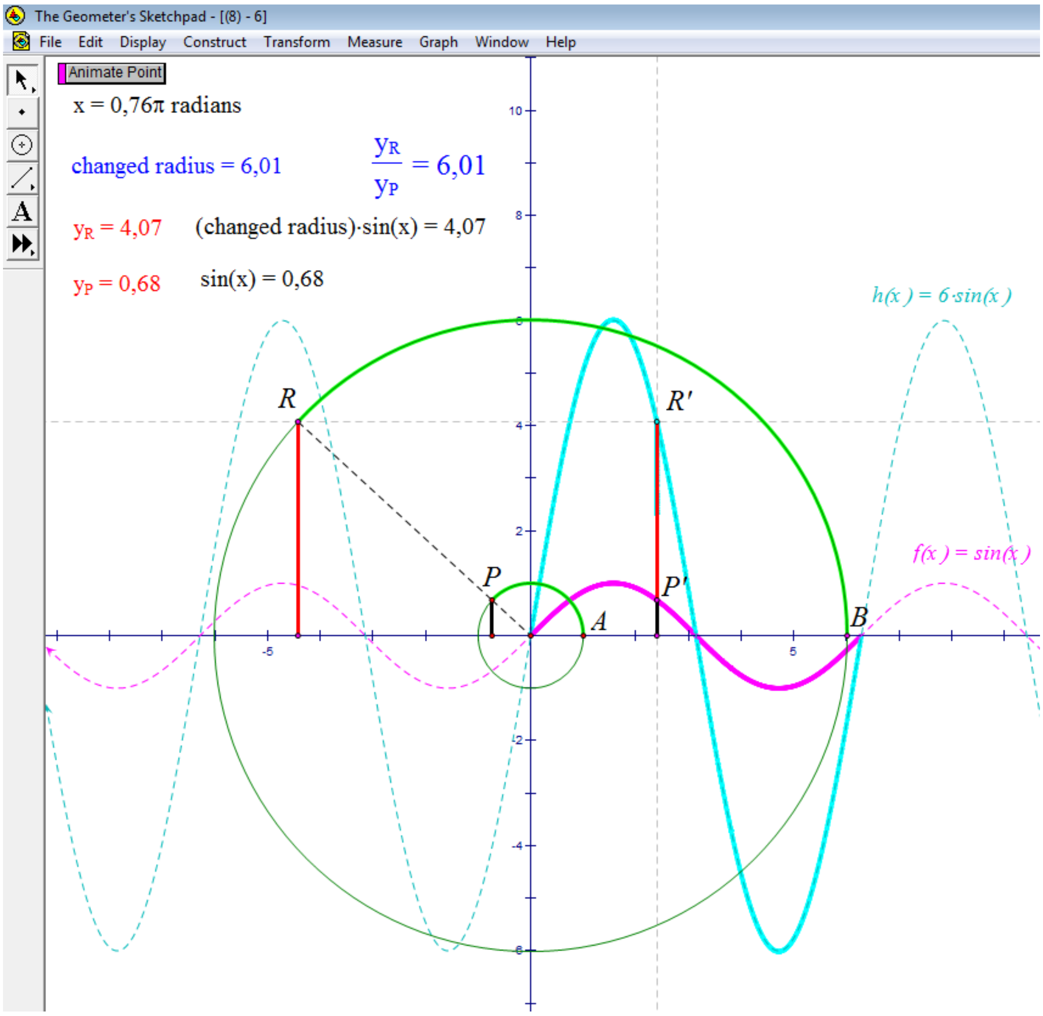


Figure 6.7. Construction of the sine and 6.sine graphs with their extended domains from [0,2π) to the *x*-axis

[Cemre & Zafer] Protocol 15

Researcher: Let's we change the radius of this circle (*pointing the* ***non-unit circle*** *on the screen, and changing the radius as about 3 like in Figure 6.8*). I am erasing all traces (*erasing all traces*).
Cemre: [points *P'* and *R'*] re-trace…
Researcher: Yes (*animating the point P and constructing the new-traces like in Figure 6.8 in case the radius of the non-unit circle was about 3*). Look! How was changed our graph? Its magnitude decreased from 6.
Cemre&Zafer: Yes.
Researcher: What is its maximum value?
Cemre: About 3…
Researcher: What is radius?

Cemre: 2.98…
Zafer: Uh-huh (*nodding his head up and down*).
Researcher: So, this graph was shaped as a result of the circle. How was it? Whose radius was…?
Cemre: 2.98…
Zafer: (*Nodding his head up and down*).
Researcher: Zafer's function in here (*pointing on the screen h(x)=6sin(x)*)… What would it be now (*in that time the radius was about 3*)?
Cemre: Instead of six [times sin(x)], it would be 3 [times sin(x)].
Zafer: …radius (*nodding his head up and down*)…
Researcher: Will 3.sin(x) overlap with our traced graph part?
Cemre&Zafer: Yes.
Researcher: (*Editing h(x)=6sin(x) function as h(x)=3sin(x) like in Figure 6.8*).
Cemre&Zafer: …overlapped.
Researcher: Well, how do you express this function regarding radius?
Zafer: …radius times sin(x).
Cemre: Yes.
Researcher: (*Plotting new function g(x)=(changed radius)sin(x)*).
Cemre: Yes.
Zafer: Uh-huh (*nodding his head up and down*).
Researcher: (*After 4-second pause*) now, I will change the radius? Please look how would change the graph?
Zafer: (*When the researcher changed the radius through dragging the point B, looking to the dynamically-linked-change of the r.sin(x) graph*) wow!
Cemre: (*Smiling*).
Zafer: (*When the radius decreasing up to zero, or the point B coming closer to the origin, looking to the dynamically-linked change of the r.sin(x) graph in terms of the magnitude*) in here, is it [graph of r.sin(x)] linear… …or, very straight…
Researcher: (*Stopping to drag the point B so that the radius of the non-unit circle could very close to zero*) how would it be?
Cemre&Zafer: (*Looking to the screen without speaking*)
Researcher: What is the radius at that time?
Zafer: Zero…
Cemre: …but radius mustn't zero! Then, it isn't a circle… …it would be a point.
Researcher: Yes. How did we define the function g (*pointing the g function's symbolic expression on the screen*)?
Zafer: Radius times sin(x)…
Cemre: Yes.
Researcher: What was the radius at that time?
Cemre: Zero…
Researcher: If I use zero for the radius, then what would be the multiplication of sin(x) with zero?
Cemre&Zafer: Zero.
Researcher: Good. Then, our function would be g(x) equal to zero, doesn't it?
Cemre&Zafer: Yes.
Researcher: In that time, this graph would be the y=0 line (*dragging her index finger on the x-axis left and right*).
Zafer: If radius differs from zero, then it [graph of r.sin(x)] isn't linear.

Researcher: Yes.

Zafer: (*Nodding his head up and down*)

Researcher: Well, let's we look again the effect of the radius. (*Dragging the point B from six up to one*) radius decreasing, decreasing, decreasing… …for which value of the radius, is the function [*r*.sin(*x*) function] referring to sin(*x*)?

Cemre&Zafer: …one…

Researcher: …when the radius is one. Well, what happens when the radius is smaller than one (*decreasing the radius from one to zero through dragging the point B*)?

Zafer: Then, maximum value wouldn't be one.

Cemre: It would be smaller than one…

Zafer: …as much as radius.

Cemre: Yes.

Researcher: What is it [graph of *r*.sin(*x*)] when the radius is being closer to the zero?

Zafer: …zero-line…

Cemre: Here (*dragging her index finger on the x-axis from left to right*).

Zafer: (*Nodding his head up and down*)

Researcher: Ok. (*After 4-second pause*) well, radius would never be negative. What do you think about -sin(*x*)? What does it mean?

Cemre: Is it from opposite direction (*putting her right hand's external part over the x-axis on the part of the sine graph in the (0,π) interval horizontally; and then rotating her hand around the x-axis by 180 degrees so as to indicate the reflection of the sine graph in the (0,π) interval regarding the x-axis*)?

Zafer: (*Looking to the screen without speaking*)

Researcher: What would be its graph for example?

Cemre: This peak would be down (*pointing the peak point of the sin(x) function's graph in the (0,π) interval*), this peak would be up (*pointing the peak point of the r.sin(x) function's graph the (π,2π) interval*).

Zafer: It [*r*.sine graph] seems as if continuing to decrease (*putting his index finger on the peak point of the r.sin(x) function's graph in the (0,π) interval, and dragging his finger vertically downward up to the under of the x-axis*)…

Cemre: (*After 5-second pause*) in fact, it [-sin(*x*)] means… …you know we do the same things with those we did in here (*pointing the **unit circle***)… …actually through transforming them in here (*pointing the sine graph*). So does in -sin(*x*), it is taking the minus side of the graph (*pointing with her index and thumb fingers the part of the sine graph in the interval (-π,0)*), and bringing to here (*dragging her fingers on the x-axis from the interval (-π,0) to the interval (0,π) through keeping the distance between them same*).

Researcher: Let's we plot –sin(*x*) by GSP (*after hiding all objects except the unit circle and the sine graph, plotting the –sine graph like in Figure 6.9*). Now, Cemre, do you mean if we translate sine graph horizontally about π unit distance like that (*pointing with her index and thumb fingers the part of the sine graph in the interval (-π,0); and then dragging her fingers on the x-axis from the interval (-π,0) to the interval (0,π) through keeping the distance between them same*)…

Cemre: Yes (*nodding her head up and down*).

Zafer: …it [translated sine graph] will be minus sine.

Researcher: Ok, let's we construct this parallel line to speak clearly (*constructing parallel line from the point P' to the x-axis like in Figure 6.9*). Now, please only focus on this pink graph (*pointing the sine graph on the screen*), let here be an angle

(*dragging the point P on the* ***unit circle*** *so as the dynamically-linked point P' to be in the (0,π) interval; and then putting her index finger on the projection point of P on the x-axis*)… …then, its sine (*pointing the perpendicular line segment from the point P' to the x-axis through putting her index and thumb fingers on its endpoints*), will be same with what on this blue graph (*translating her fingers simultaneously along the x-axis by the π-length line segment in the negative direction*)?
Cemre: π less than [it]…
Zafer: π less…
Researcher: So, what would this point's coordinates be (*pointing the point on the –sine graph which is the parallel displacement of the point P' on the sine graph along the x-axis by the π-length line segment in the negative direction on the screen like in Figure 6.9*)? What is its abscissa?
Zafer: …its π less (*putting his index finger on the projection point of the point P' on the x-axis; and then, dragging his finger on the x-axis in the negative direction up to the projection of the point on the –sine graph which is the parallel displacement of the point P' on the sine graph along the x-axis by the π-length line segment in the negative direction*).
Cemre: …angle minus π…
Researcher: If we call this angle as *x* (*putting her index finger on the projection point of the point P' on the x-axis*), how do you express this one symbolically (*pointing the projection point of the parallel displacement of the point P' along the x-axis by –π on the screen like in Figure 6.9*)?
Cemre&Zafer: …*x* minus π…
Researcher: (*Calculating x*- π like in *Figure 6.9*) ok. What about the *y*-value of this point (*pointing the parallel displacement of the point P' along the x-axis by –π which is on the –sine graph*)?
Zafer: Same with this (*pointing the perpendicular line segment from the point P' to the x-axis*).
Cemre: Yes.
Researcher: How do you express it as symbolically?
Cemre&Zafer: …sin(*x*)…
Researcher: Then, if I construct a point so as its abscissa to be (*x-π*) and ordinate to be sin(*x*), is it here (*pointing the parallel displacement of the point P' along the x-axis by –π*)?
Cemre&Zafer: Yes.
Researcher: (*Constructing the (x-π,sin(x)) ordered-pair taking advantages of the plot as (x,y) option of GSP and labelling with P'' like in Figure 6.10*).
Cemre&Zafer: Yes (*smiling*).
Researcher: Ok. This pink graph (*pointing the sine graph on the screen like in Figure 6.10*) was the graph of whom?
Cemre&Zafer: …sine…
Researcher: Sine graph. Then, how do you express this value (*pointing the perpendicular line segment from the point P' to the x-axis*)?
Cemre: This angle's sine value (*pointing the projection of the point P' on the x-axis*)…
Zafer: …sin(*x*)…

Researcher: Ok. What about this angle (*pointing the projection point of the point P'' on the x-axis*)… …regarding this angle (*pointing the projection point of the point P' on the x-axis*)?

Cemre&Zafer: x minus π.

Researcher: Considering this blue graph (*pointing the –sine graph*), how do you express this value (*pointing the perpendicular line segment from the point P'' to the x-axis*)?

Cemre: Minus sine of… …x minus π.

Zafer: Uh-huh.

Researcher: Did you say this (*writing on a paper* $-\sin(x-\pi)$)?

Cemre&Zafer: Yes.

Researcher: Then, you said that this was $\sin(x)$ (*pointing the perpendicular line segment from the point P' to the x-axis on the screen like in Figure 6.10*) and this was $-\sin(x-\pi)$ (*pointing the perpendicular line segment from the point P'' to the x-axis*), ok?

Cemre&Zafer: Yes.

Researcher: Then, can we say $\sin(x)$ is equal to $-\sin(x-\pi)$?

Cemre&Zafer: Yes.

Researcher: Let's construct $-\sin(x-\pi)$ graph, and see indeed it is same with sine graph.

Zafer: (*Constructing the* $-\sin(x-\pi)$ *graph*) same.

Cemre: Yes, same (*nodding her head up and down*).

Zafer: (*Nodding his head up and down*)

Researcher: Well, conversely if we consider here as x (*pointing the projection point of the point P'' on the x-axis*)… …and focus on this blue graph or –sine function (*pointing the –sine graph*), then what would be here (*pointing the perpendicular line segment from the point P'' to the x-axis*)?

Cemre&Zafer: …minus $\sin(x)$.

Researcher: What about here (*pointing the perpendicular line segment from the point P' to the x-axis*)?

Zafer: Then, here (*pointing the projection point of the point P' to the x-axis*) would be x plus π.

Cemre: …so, $\sin(x+\pi)$…

Zafer: …minus $\sin(x)$ equal to $\sin(x+\pi)$.

Cemre: Let's we look (*plotting* $\sin(x+\pi)$ *function's graph. When appearing its overlapped-graph with –sine function's graph like in Figure 6.12*) yes. (*After 5-second pause*) I had never known about that before [the reason about the symbolic relation between sine and –sine]…

Zafer: Uh-huh (*nodding his head up and down*).

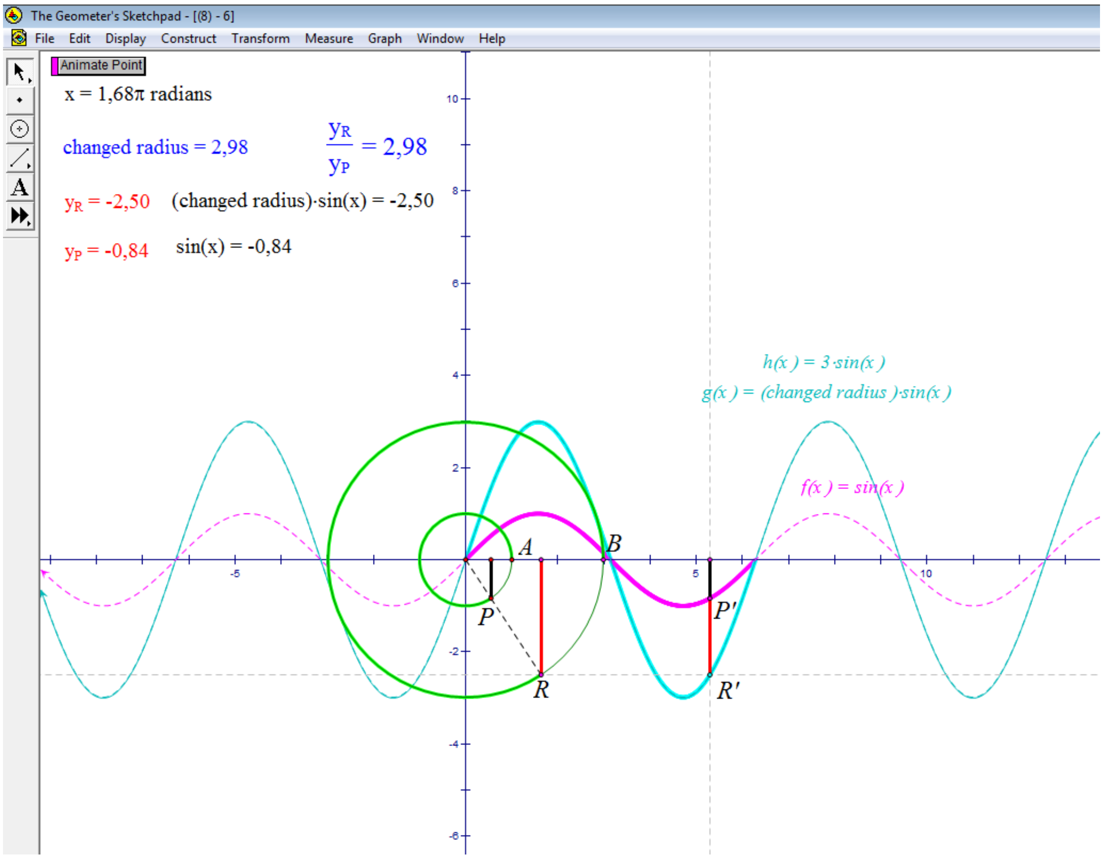


Figure 6.8. Construction of the generalized *r*.sine graph together with its aritmetic form (e.g., h(*x*)=3.sin(*x*) in case the radius was about 3) in order to ease the conversion of the sine and *r*.sine functions between the (*unit*) *circle register* and the *graphical register*

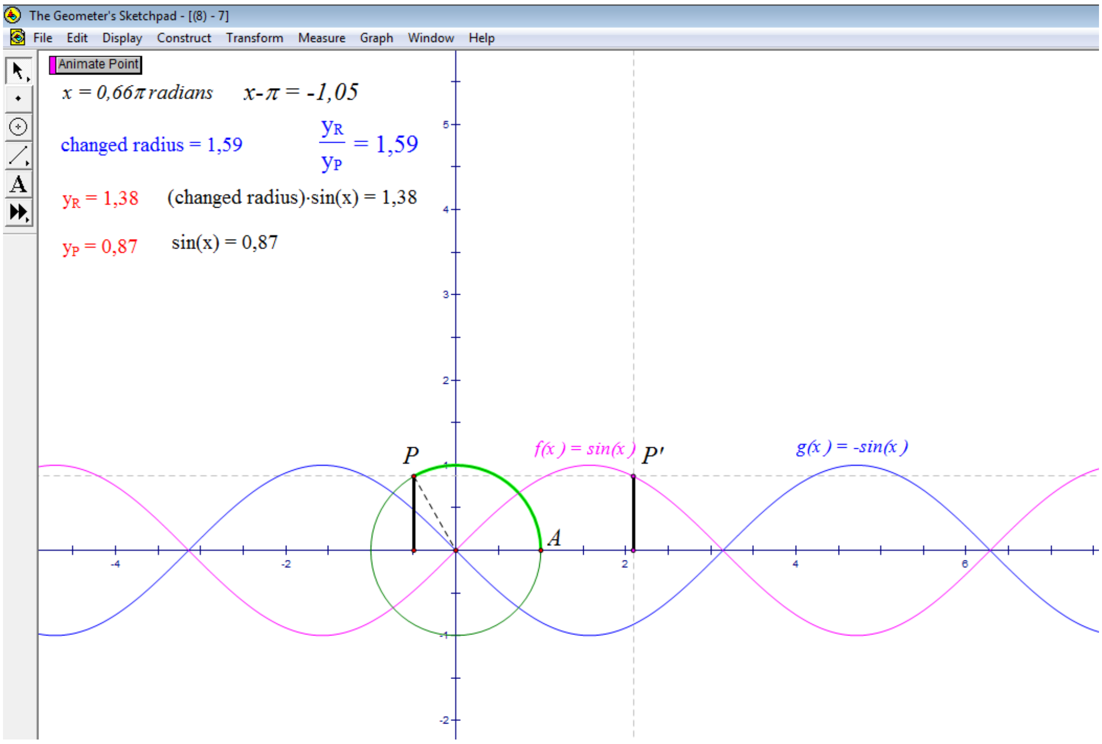


Figure 6.9. Construction of the simultaneous graphs of the sine and –sine functions in the *graphical register*

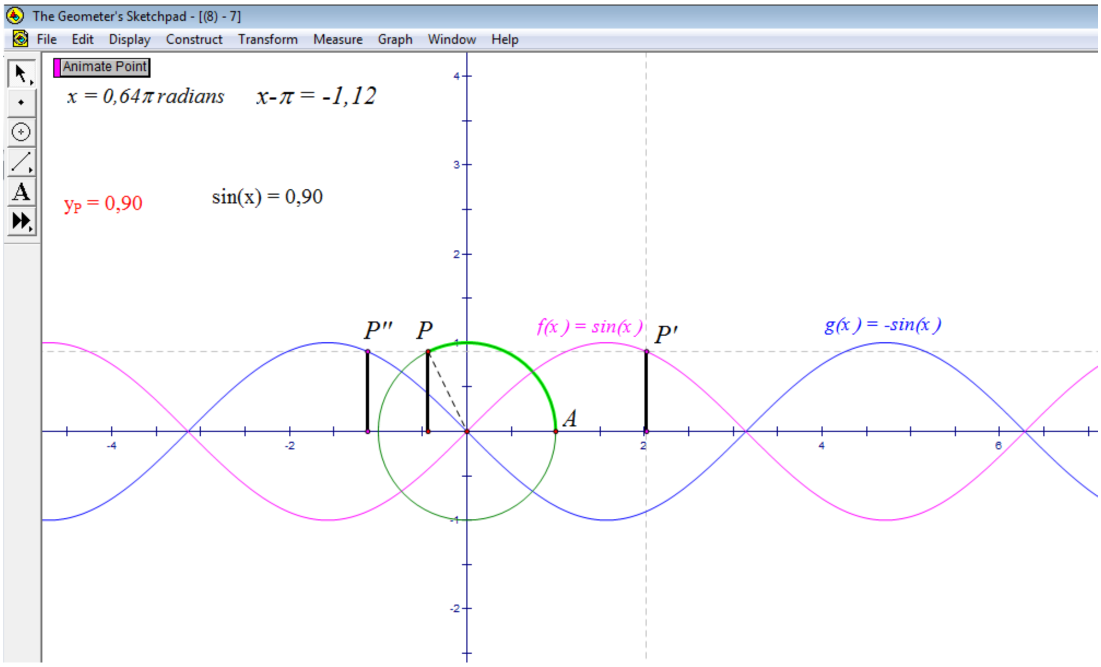


Figure 6.10. Construction of the parallel displacement of the point *P'* in the *graphical register* along the *x*-axis by π in the negative direction

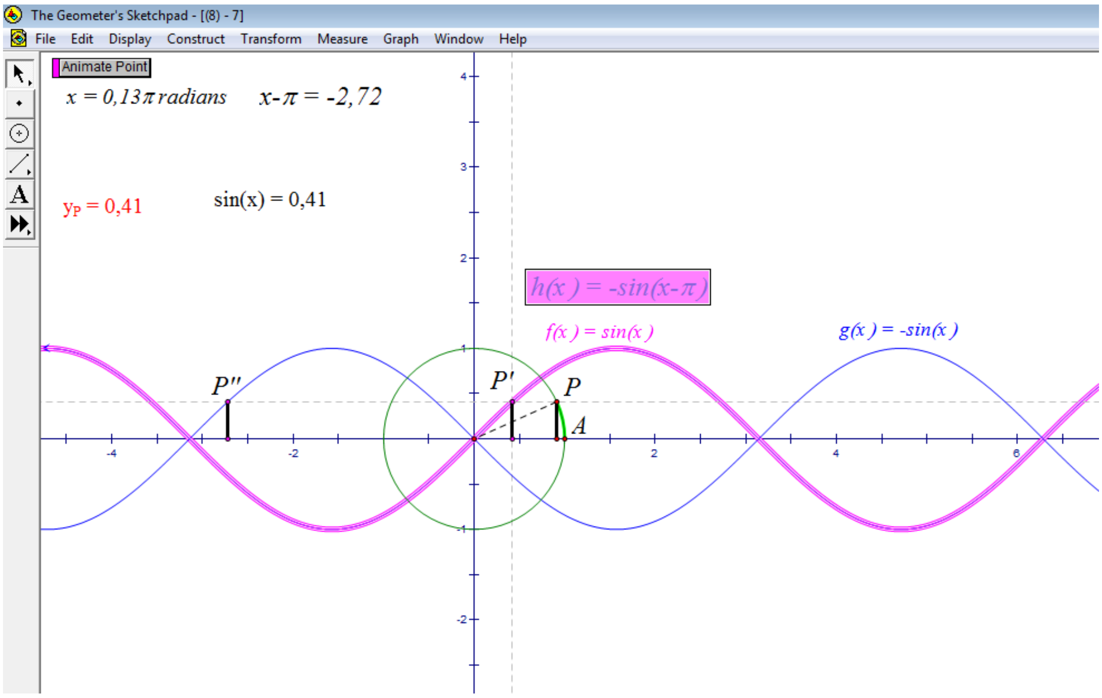


Figure 6.11. Construction of an equavalence of the sine function in the *graphical register* via the parallel displacement of the –sine function along the *x*-axis by π in the positive direction

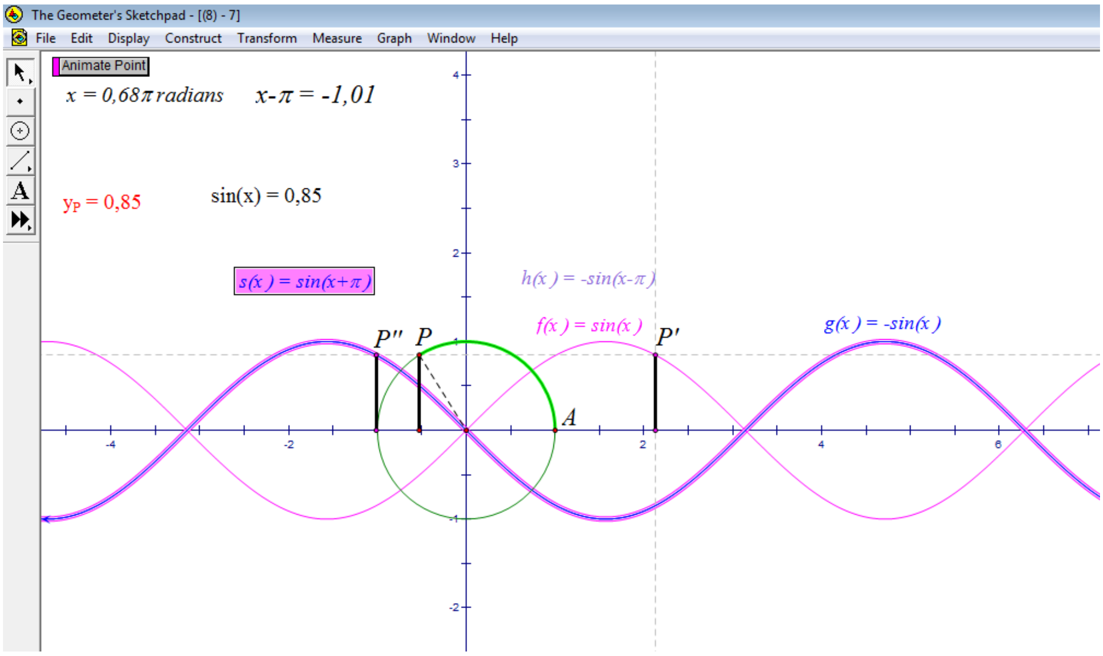


Figure 6.12. Construction of an equavalence of the –sine function in the *graphical register* via the parallel displacement of the sine function along the *x*-axis by π in the negative direction

[Defne & Ebru] Protocol 14

Researcher: Last week, we have discussed about functions, for example, like 3.sine. What do you remember? What does coefficient 3 mean?
Defne: Radius was 3 (*figuring a circle through indicating arcs with her right and left hands' index and thumb fingers, and then a bit more dilating this circle through taking her hands away from each other*).
Ebru: Uh-huh.
Researcher: What else?
Ebru: Magnitude was 3 (*figuring a shape with her index finger as the first peak of the sine in the (0,π) interval*).
Defne: Yes.
Researcher: Well, let's we discuss about coefficient of sine in a bit more detail (*constructing a slider to manipulate the coefficient of a.sine function so as to take values between -5 and 5, and then a.sine function's graph like in Figure 6.13*).
…
Researcher: Please investigate the variation of this blue graph compared to the sine graph (*pointing to the g(x)=a.sin(x) functions graph on the screen like in Figure 6.13*).
Defne: (*Dragging the slider in the positive direction*) it's going up.
Ebru: Can I look (*taking the manipulation control of the slider to her*)! (*Dragging the slider in the negative direction until the 1*) now it's going down.
Defne: Yes.

Researcher: What is going up or going down?
Defne: [*a*.sine] graph (*pointing the positive peak point of the a.sine graph in the* $(0,\pi)$ *interval*)…
Ebru: Maximum value [of *a*.sine]…
Researcher: Ok. What about *a*… …less than 1?
Ebru: (*Dragging the slider in the negative direction and dropping on a position in which a is 0.68*) maximum value is less than 1.
Defne: (*Nodding her head up and down*) 0.68 is maximum value…
Ebru: Yes.
Researcher: Please continue to decrease *a* value?
Defne: (*Dragging the slider firstly in the negative direction until it passing from positive to negative region; secondly turning the direction of dragging from negative to positive; and then, again from positive to negative*) it [peak point in $(0,\pi)$] passed down.
Ebru: (*Looking to the screen without speaking*)
Researcher: When did it pass from up to down?
Defne: When it [coefficient *a*] was minus.
Ebru: Uh-huh.
Researcher: Why?
Defne&Ebru: (*Looking to the screen without speaking*)
Researcher: Ok. Let's we discuss on –sin(*x*). What is the coefficient in –sin(*x*)?
Defne&Ebru: …minus one.
Researcher: How would be its graph?
Ebru: (*Dragging the slider and dropping it at (-1) like in Figure 6.14*) opposite of sine…
Defne: Yes… …here is up (*pointing the positive peak point of the sine graph in the* $(0,\pi)$ *interval*), here is down (*pointing the negative peak point of the a*sine graph in the* $(0,\pi)$ *interval in case the coefficient a was -1*).
Researcher: What do you say… …about the graph of -3.sine function without GSP? How would be graph of -3.sine?
Ebru: It would be between minus 3 and plus 3…
Defne: But opposite manner [with sine].
Ebru: Yes.
Researcher: Let's control by GSP.
Defne: (*Dragging the slider and dropping it at (-1)*) yes, true.
Ebru: Yes (*smiling*)
Researcher: Ok. What about zero as *a* coefficient?
Ebru: (*Dragging the slider and dropping it at zero like in Figure 6.15*) line…
Defne: Yes… …*x*-axis.
Ebru: Uh-huh.
Researcher: In that case, what would be the expression of *g* [function] (*pointing g(x)=a.sin(x) expression on the screen*), *a* was zero?
Defne: …zero times sin(*x*)… …zero…
Ebru: g(*x*) equal to zero.
Defne: Yes.
...
(*At that point, the researcher assigned a.sine function to Defne and the sine function to Ebru*).

Researcher: Well, if we discuss about their periods [periods of sine and *a*.sine], what do you say?
Defne: My function's period [period of *a*.sine] is 2π (*pointing the (0,2π) interval with the cursor on the screen like in Figure 6.13*).
Researcher: Do we have to take that interval (*pointing the (0,2π) interval with her index and thumb fingers*)? Can we take this, for example (*pointing the (-π,π) interval with her index and thumb fingers*)?
Defne: Yes, we can. (*After 3-second pause*) 2π-length is enough.
Ebru: (*Nodding her head up and down*)
Defne: …[this length includes] all values that it [*r*.sine] took… …until turning a tour around circle's circumference (*figuring a circle in the counterclockwise direction with her index finger through starting from a point which was as if the far right point of this circle*).
Ebru: Uh-huh (*nodding her head up and down*).
Researcher: What about your function's period [period of sine] Ebru?
Ebru: 2π. It [sine graph] is repeating in 2π.
Defne: (*Nodding her head up and down*)

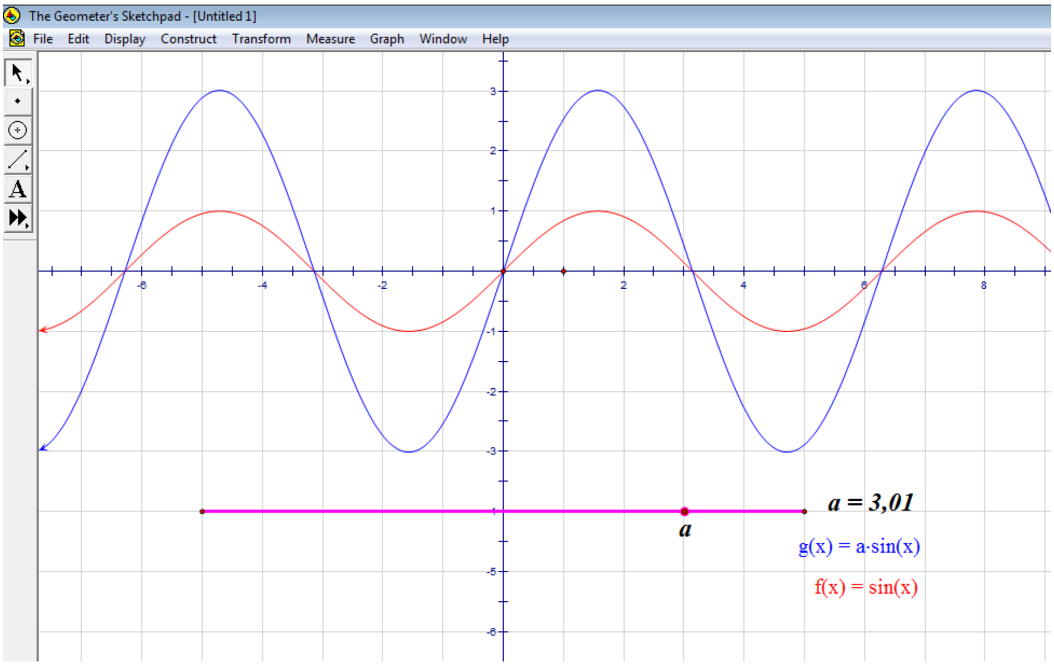


Figure 6.13. Construction of the manipulable *a*.sine function in the *graphical register* by a slider for the coefficient *a* between –5 and 5

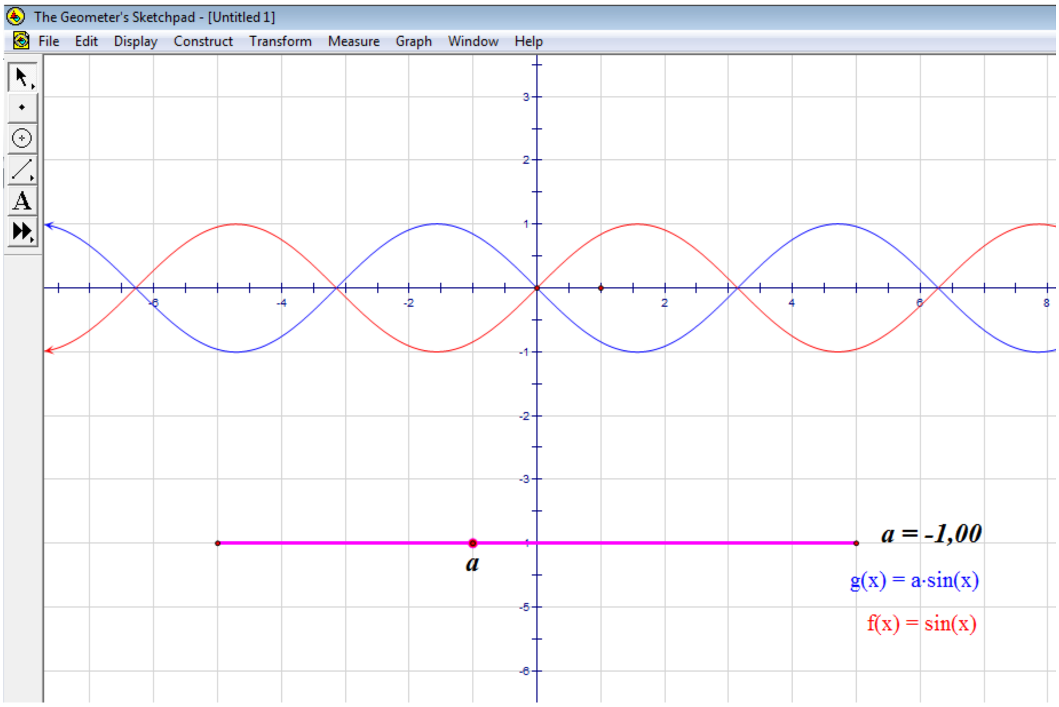


Figure 6.14. Shape of the *a*.sine graph in case of the slider's (–1) position for the coefficient *a*

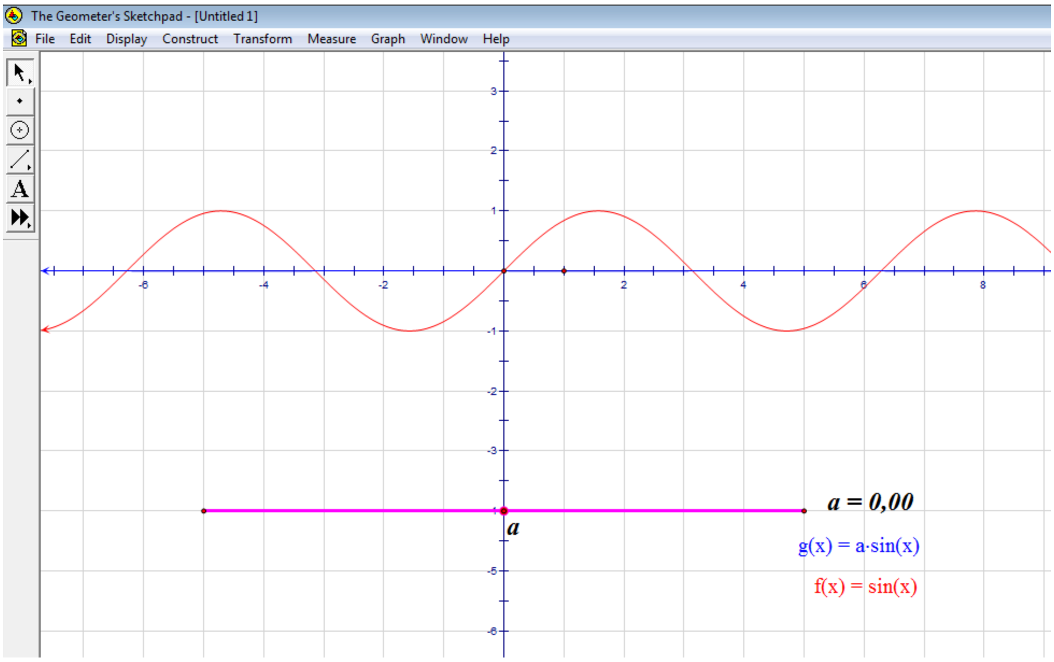


Figure 6.15. Shape of the *a*.sine graph in case of the slider's zero-position for the coefficient *a*

6.1.2. Visual Feature Opposition D

Visual Feature (D) corresponds to changed-center in the (*unit*) *circle register* and parallel-displacement along the *y*-axis in the *graphical register* so that these visual features' opposition corresponds to the choice presence/absence of a constant of sine and cosine in the *symbolic register*.

6.1.2.1. Changed-center in (unit) circle register

In Task 9 [Task 14], students were encouraged to reason about the function mapping the principal arc into the *y*-component [*x*-component] of the point on a unit circle with a manipulable-center on the coordinate system (*Figure 6.16*) through comparing and contrasting with that on the unit circle whose center was located on the origin (e.g., lines 1-8 in [Defne & Ebru] Protocol 15; lines 1-19 in [Cemre & Zafer] Protocol 16).

Initially, the researcher asked students to reason about whether this visual relation was a function or not on the unit circle with a different center from the origin –e.g., a point in the second quadrant of the coordinate system. Where, all students interpreted this relation as a function based on its property that each arc (i.e., input in the domain) must be related to exactly one *y*-component [*x*-component] (i.e., output in the range) (e.g., lines 6-12 in [Defne & Ebru] Protocol 15; lines 17-22 in [Cemre & Zafer] Protocol 16). And then, each group of students reasoned about this function on a unit circle with a manipulable-center on the coordinate system through comparing and contrasting with that on the unit circle with the origin-center –i.e., sine [cosine].

In task 9, on the one hand, the group of Cemre and Zafer started the comparison of this function with the sine function by means of Zafer's concern about its differentiation from sine (lines 17-23 in [Cemre & Zafer] Protocol 16). On the other hand, Defne and Ebru did not concern about this comparison by themselves. So, the researcher dragged the center onto the origin; and then, asked them to reason about the same visual relation in that case (lines 13-15 in [Defne & Ebru] Protocol 15). In case of the unit circle whose center was located on the origin, Defne and Ebru reasoned about this relation as the sine function (lines 16-19 in [Defne & Ebru] Protocol 15). And then, the researcher again dragged the center from the origin onto a point in the second quadrant of the coordinate system. In that case, when asked to reason about the relation between the *y*-value and the sine value corresponding to the point on the unit circle, also Defne and Ebru started to compare this function with the sine function (e.g., lines 13-23 in [Defne & Ebru] Protocol 15).

However, from this point forward of Task 9, discussions with Cemre and Zafer shaped differently from Defne and Ebru in terms of the direction in the conversion of this function between the (*unit*) *circle register* and the *graphical register*. That is to say, as a consequence of Cemre's desire to reason about this function in the *graphical register,* as well (lines 24-25 in [Cemre & Zafer] Protocol 16), the direction in the conversion formed from the *graphical register* to the (*unit*) *circle register* in the group of Cemre and Zafer ([Cemre & Zafer] Protocol 16). Conversely, in the group of Defne and Ebru, the direction in the conversion formed from the (*unit*) *circle register* to the *graphical register* ([Defne & Ebru] Protocol 15). Despite of their different directions in the conversion of this function between the (*unit*) *circle register* and the *graphical register*, each student showed, in general sense, the similar developments within each register. Their developments in the *graphical register* were presented under the following sub-heading *Parallel-displacement along the y-axis in graphical register*. Conversely, their developments in the (*unit*) *circle register* were presented in the following paragraph.

In Task 9 [Task 14], through comparing this manipulable-function defined visually from the principal arc to the corresponding *y*-component [*x*-component] in the (*unit*) *circle register* with the sine [cosine] function, all students were able to express this function in the *symbolic register* as an additive operation between sine [cosine] and the directed-distance of the manipulable-center to the *x*-axis [*y*-axis] (e.g., lines 6-33 and 54-101 in [Defne & Ebru] Protocol 15; lines 78-116 in [Cemre & Zafer] Protocol 16). Moreover, they were able to truly revise this expression in the *symbolic register* regarding the variation of the manipulable-center in the (*unit*) *circle register*;

furthermore, they were able to reason its independence from the horizontal [vertical] variation and dependence only the vertical [horizontal] variation of the manipulable-center (e.g., lines 102-130 in [Defne & Ebru] Protocol 15; lines 78-116 in [Cemre & Zafer] Protocol 16).

Despite of, in general sense, students' similar developments in the (*unit*) *circle register* mentioned above, throughout the Task 9, in special sense, students in each group were different in terms of the concept development stages related to this function defined visually from the principal arc to the corresponding *y*-value on a unit circle with a manipulable-center on the coordinate system. That is to say, on the one hand, despite of their *condensation* stages[39] in case of the unit circle whose center was located on the origin, Defne and Ebru were at the *interioriorization* stage[40] in other cases of the unit circle. In other words, when reasoning about this function defined on a unit circle with a different center from the origin through comparing and contrasting with sine, Defne's and Ebru's actions and language imply that their focuses were predominantly on the processes related to the *y*-components in the (*unit*) *circle register* (lines 20-101 in [Defne & Ebru] Protocol 15). For example, when defining in the *symbolic register* the red line segment referring to the *y*-component in the (*unit*) *circle register* as an additive relation between sine and its residual (i.e., sine+ $y_{O'}$), they focused on the determination process of the ordinates (e.g. lines 20-53 in [Defne & Ebru] Protocol 15). So, their first encounter with the negative sine values caused a trouble for them on extending this additive relation into the cases of the negative sine values or the negative ordinate of the center (e.g. lines 54-80 in [Defne & Ebru] Protocol 15). In other words, for example, when having dragged the point *P* into the third quadrant of the unit circle (*Figure 6.19*), they doubted about whether or not in that case the ordinate of the point *P* could be expressed as an additive relation between sine and the center's ordinate. Where, their efforts on making sense of the operations in their minds by the aid of the calculate option of GSP caused their overcoming this

[39] *Condensation* stage means for students becoming skilled with seeing of the process as a condensed whole without going into details.

[40] *Interiorization* stage means for students becoming familiar with the processes on the mathematical object.

trouble through leading them to concentrate on the negative sign of the sine value in that case (e.g. lines 81-101 and 120-130 in [Defne & Ebru] Protocol 15).

On the other hand, Cemre and Zafer were at least in the *condensation* stage in each case of the unit circle. That is, when reasoning about this function defined on a unit circle with a different center from the origin through comparing and contrasting with sine, Cemre's and Zafer's actions and language imply that their focuses was predominantly on the condensed-whole of the processes instead of their details (lines 6-95 in [Cemre & Zafer] Protocol 16). For instance, Zafer's concern about the differentiation of this manipulated-function from sine in the (*unit*) *circle register* implies his skill on seeing their definition-processes as a condensed whole without going into details (lines 17-23 in [Cemre & Zafer] Protocol 16). Conversely, Cemre's concern about the resemblance of this manipulated-function with sine also implies her skill on seeing their definition-processes as a condensed whole without going into details (lines 22-43 in [Cemre & Zafer] Protocol 16). Beside, their some actions and language indicated the *reification* stage[41] as well. In other words, Cemre and Zafer showed an instantaneous shift on seeing familiar processes of these manipulated-functions as a reified-object in all representational registers (e.g., lines 17-34, 44-58 and 78-95 in [Cemre & Zafer] Protocol 16). For example, when reasoning about this function in the *graphical register*, while Zafer mentioned the horizontal axis from the center as the *x*-axis considering the *x*-axis as if a reified-object without going into details but with awareness of its different location (e.g., lines 36-46 in [Cemre & Zafer] Protocol 16), Cemre mentioned the sine wave as if a reified-object without going into details but with awareness of its different location from the sine graph (e.g. lines 17-58 in [Cemre & Zafer] Protocol 16). Moreover, when reasoning in the (*unit*) *circle register*, Cemre mentioned the unit circle whose center located on the origin as if a reified-object; and was able to change its position up-down and left-right in her mind as a whole on the coordinate system (e.g., lines 78-95 in [Cemre & Zafer] Protocol 16).

[41] *Reification* stage is defined as an instantaneous shift the ability to see familiar processes as a reified-object.

Unfortunately, Defne and Ebru did not rich this level of conception in the (*unit*) *circle register* throughout Task 9. As a cause of this, the researcher hypothesized that students' differentiation on the level of conception arose from their preferences of the direction in the conversion of this function between the (*unit*) *circle register* and the *graphical register*. Therefore, henceforward, she determined to encourage also Defne and Ebru to reason about the considered-functions in the following tasks simultaneously in the (*unit*) *circle register* and the *graphical register* in order to ease for them to develop such a skill in seeing these processes as a condensed whole without going into details, or better a skill in seeing the familiar processes within the registers as a reified-object. Specifically, in Task 14, the researcher encouraged them to discuss about the role of the changed-location of the unit circle based on the function defined in the (*unit*) *circle register* from the principal arc to the corresponding *x*-value simultaneously with its graphical representation (e.g., lines 1-36 in [Defne & Ebru] Protocol 16). Henceforward, the subsequent variation of students' conception-levels on the changed-center in the (*unit*) *circle register* was presented together with its effect on the changed-location in the *graphical register* under the following sub-heading *Parallel-displacement along the y-axis in graphical register*.

[Defne & Ebru] Protocol 15

Researcher: (*Giving directions for students to construct a unit circle in GSP environment whose center can be manipulable and the parallel axes from its center to the coordinate axes in the (unit) circle register like in Figure 6.16*)…
(*Defne and Ebru cooperatively constructed GSP page like in Figure 6.16 with respect to the researcher's directions.*)
Researcher: Well, if we define a relation… …that is mapping this green arc to this *y*-value (*pointing the green arc and red line segment on the screen like in Figure 6.16*)… Is it a function?
Defne: Yes, it is.
Researcher: Why?
Defne: Each one [green arc] is going to only one [*y*-value]…
Ebru: Yes.
Researcher: Ok. (*Dragging the center of the unit circle and dropping on the origin*) now, what do you say about the same function… …that is again mapping this green arc the corresponding *y*-value…
Defne&Ebru: Sine…
Researcher: Please find *y*-value and sine corresponding to the point *P*.

Ebru: (*Measuring y_P and calculating sin(x) by GSP*) they are same.
Defne: Uh-huh.
Researcher: (*Dragging the center of the unit circle from the origin to the position in the second quadrant of the coordinate system like in Figure 6.17*) what about now? Are they [y_P and sin(x)] still same?
Ebru: No. they are different.
Defne: (*After 4-second pause*) because there is an extra (*pointing the red line segment's downward regarding the horizontal axis constructed from the manipulable-center*).
Researcher: Where is the extra?
Defne: Here (*pointing with her index finger the red line segment's downward regarding the horizontal axis constructed from the manipulable-center on the screen like in Figure 6.17*)…
Ebru: …from here to here (*pointing with her pen the intersection points of the red line segment respectively with the manipulable-horizontal axis and the x-axis*).
Defne: Yes.
Ebru: …about 2.20 or like that (*dragging her pen on the manipulable-horizontal axis from the intersection point with the red line segment to the y-axis on the screen like in Figure 6.17*)…
Defne: 2.30 may be…
Researcher: How do you determine this measure?
Ebru: …like that (*dragging her pen on the manipulable-horizontal axis from the intersection point with the red line segment to the y-axis*).
Defne: Yes. So, it is between 2.20 or 2.30.
Ebru: Uh-huh.
Researcher: Do you mean this point's *y*-value (*pointing the intersection point of the red line segment with the manipulable-horizontal axis*)?
Defne&Ebru: Yes.
Researcher: What about center's *y*-value (*pointing the point O'*)?
Defne: Same with this (*pointing the intersection point of the red line segment with the manipulable-horizontal axis*)…
Ebru: Yes, same.
Researcher: Let's measure of center's *y*-value.
Defne: (*Measuring the ordinate of the point O' by GSP, when the result was appeared on the screen like in Figure 6.18*) 2.26…
Ebru: (*Smiling*)
Researcher: You said before, this part was an extra from sine, wasn't it?
Defne&Ebru: Yes.
Researcher: Then, what do you say about this red part (*dragging her index finger from up to down on the red segment, at that time the point P was in the second quadrant like in Figure 6.18*)… …considering it as two parts like that (*dragging her index finger on the red line segment from up to down through dropping for a while on the intersection point of the red line segment and the manipulable-horizontal axis*)?
Defne: We add them.
Ebru: (*Looking to the screen without speaking*).
Researcher: Let's add and see. Measures you need are there (*pointing the measures on the screen*).

Defne: (*Calculating sin(x)+* $y_{O'}$) yes…
Ebru: But in here (*dragging the point P in the counterclockwise direction and dropping it in the third quadrant like in Figure 6.19*), don’t we obtain this (*pointing the red segment*) through subtracting this (*pointing the opposite side of the reference right triangle*) from 2.26 (*indicating the distance between the manipulable-horizontal axis and the x-axis through figuring a perpendicular segment to both so that the point P was on it*)…
Defne: (*Looking to the screen without speaking*)
Ebru: (*After 4-second pause*) actually, you know here (*figuring the distance segment between the manipulable-horizontal axis and the x-axis from the point P*) is same with between O' and *x* [axis] (*figuring the perpendicular segment from the point O' to the x-axis*), that is 2.26… … here (*dragging her index finger on the red segment*) is *y*-value of *P* [point]… …and here (*pointing the opposite side of the reference right triangle*) is sine as well.
Defne: (*Looking to the screen without speaking*)
Researcher: Ebru, do this operation you have expressed.
Ebru: (*Calculating the difference between the ordinates of the point O' and the point P like in Figure 6.20*) it is opposite sign with sine… …of course, sine is negative in third quadrant. I understand.
Defne: (*Nodding her head up and down*)
Researcher: So, which operation is conducted in third quadrant… …subtraction or addition?
Defne&Ebru: Addition.
Researcher: Why?
Ebru: Because sine is negative in third quadrant…
Defne: Uh-huh… …but as if subtraction of this length (*putting her right hand’s index and thumb fingers on the endpoints of the opposite side of the reference right triangle on the screen like in Figure 6.20*).
Researcher: Good, that is, when thinking about sine it means addition… …and thinking about lengths it means subtraction.
Defne&Ebru: Yes.
Researcher: Please control… …are these same (*pointing* y_P *and sin(x)+* $y_{O'}$ *measures on the screen like in Figure 6.20*) in everywhere on the circle?
Ebru: (*Dragging the point P in the counterclockwise direction until completing a full round*) yes, same.
Defne: Uh-huh.
Researcher: Ok. Let’s we drag the circle throughout this way without changing its distance from the *x*-axis (*dragging the point O' by using the right and left arrows of the keyboard like in Figure 6.21*). (*After 6-second pause*) which measures are changing on the screen?
Defne&Ebru: None.
Researcher: Ok. Now, I want to drag the circle throughout this way without changing its distance from the *y*-axis (*dragging the point O' by using the up and down arrows of the keyboard like in Figure 6.22*). (*After 6-second pause*) which measures are changing on the screen?
Defne: Now, they changed…
Ebru: Yes.
Researcher: Which ones changed?

Defne: (*Dragging the point O' toward downward by using the down arrow of the keyboard and dropping in the third quadrant of the coordinate system like in Figure 6.22*) these changed (*pointing* y_P *and sin(x)+* $y_{O'}$ *measures on the screen*)…

Ebru: These also changed (*pointing* $y_{O'}$, $-y_P$ *and* $y_{O'}$ *measures on the screen*).

Defne: Yes.

Researcher: These measures changed (*pointing* y_P *and sin(x)+* $y_{O'}$ *measures on the screen like in Figure 6.22*)… …but still equal to each other, is that right?

Defne&Ebru: Yes.

Researcher: How this *y*-value (*pointing* y_P *measure on the screen at that time the unit circle was in the third quadrant of the coordinate system like in Figure 6.22*) is equal to again this addition of sin(*x*) and $y_{O'}$ (*pointing sin(x)+* $y_{O'}$ *measure on the screen*)? Did we do a subtraction?

Ebru: No… Due to minus [sign of $y_{O'}$] and plus [sign of sin(*x*)] we should add (*at that time the unit circle was in the third quadrant of the coordinate system like in Figure 6.22*)…

Defne: (*Nodding her head up and down*)

(*Similar discussions were done on the equality between* y_P *and sin(x)+* $y_{O'}$ *measures under the manipulation of the angle's openness in the third quadrant, as well as in the fourth quadrant of the coordinate system*).

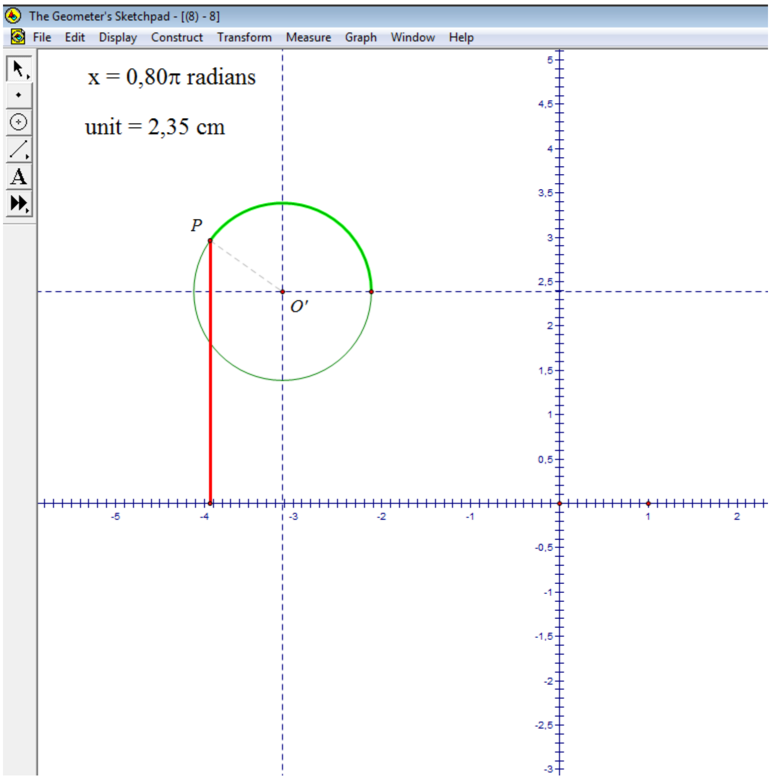


Figure 6.16. Construction of the unit circle with a manipulable-center in terms of sine in the (*unit*) *circle register*

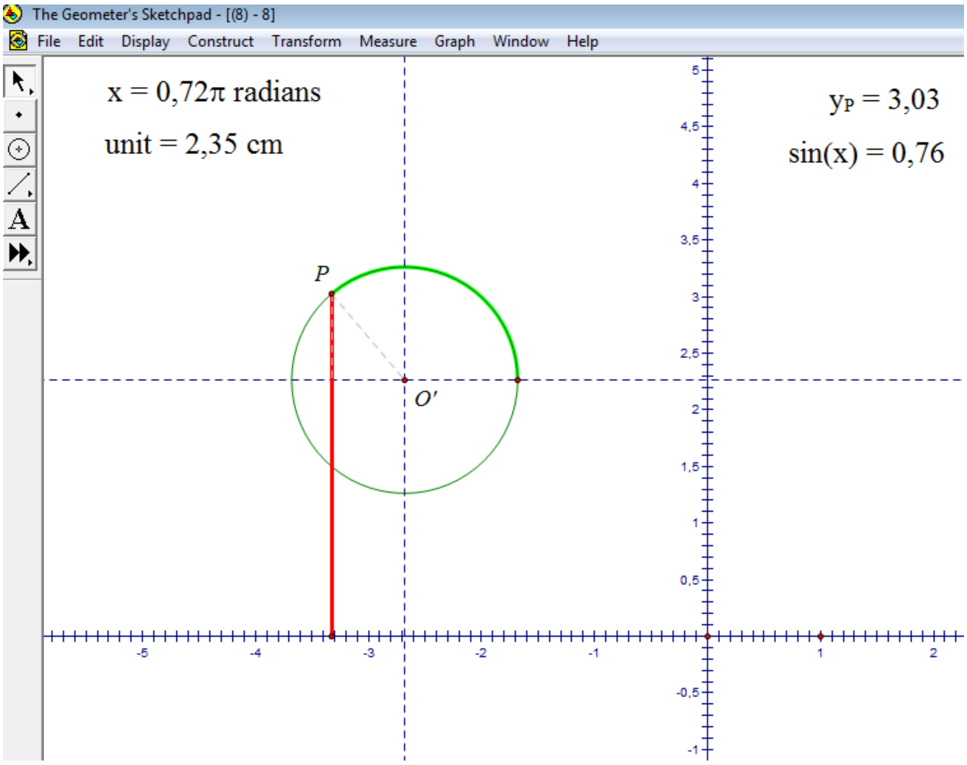


Figure 6.17. The ordinate and sine value corresponding to the point P on the unit circle with the manipulable-center

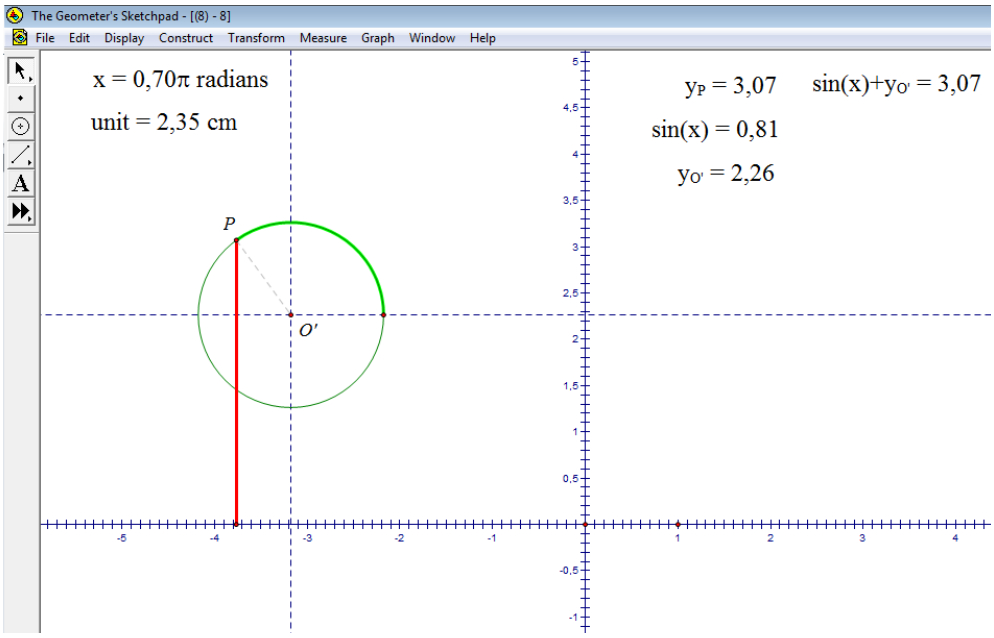


Figure 6.18. Ordinate of the point *P* as an additive relation of the corresponding sine value and the ordinate of the center point

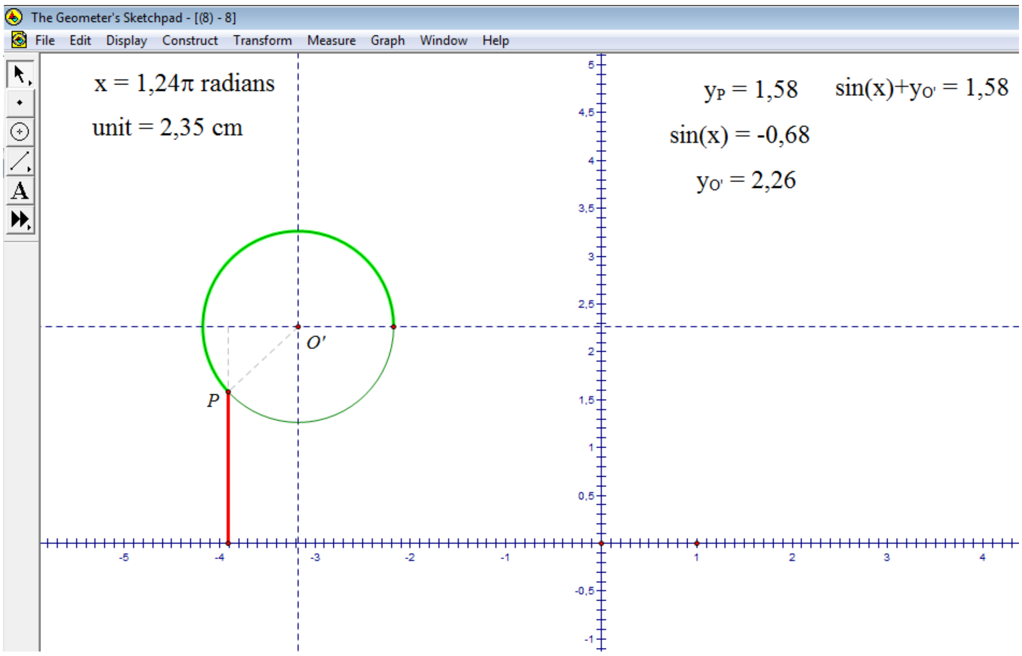


Figure 6.19. Ordinate of the point *P* as an additive relation in case of the negative sine values

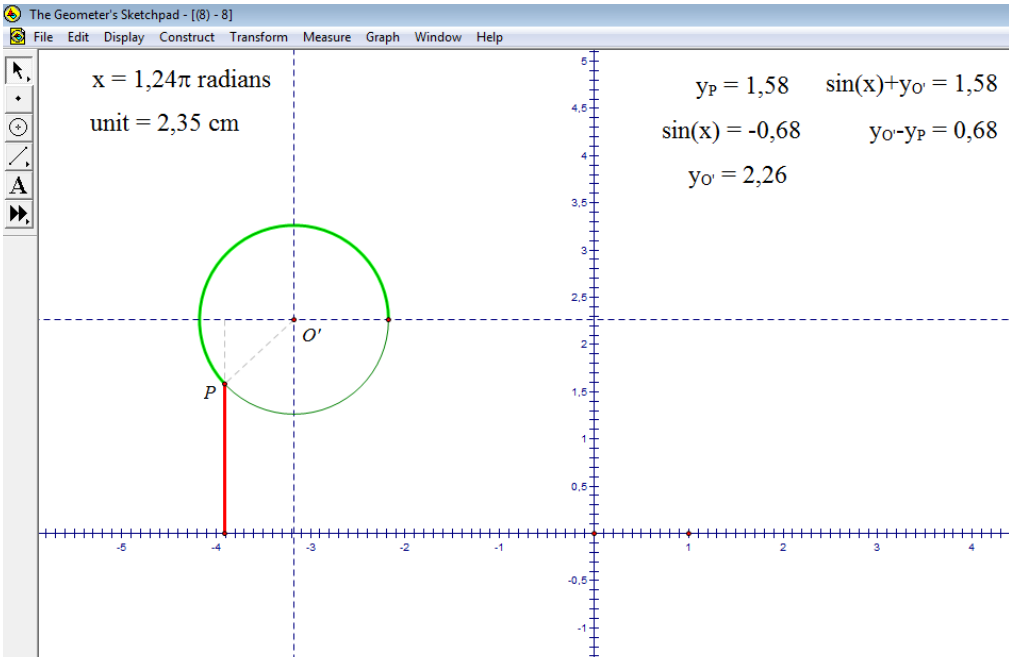


Figure 6.20. Role of the directed-opposite side in the (*unit*) *circle register* on the additive relation indicating the ordinate of the point *P*

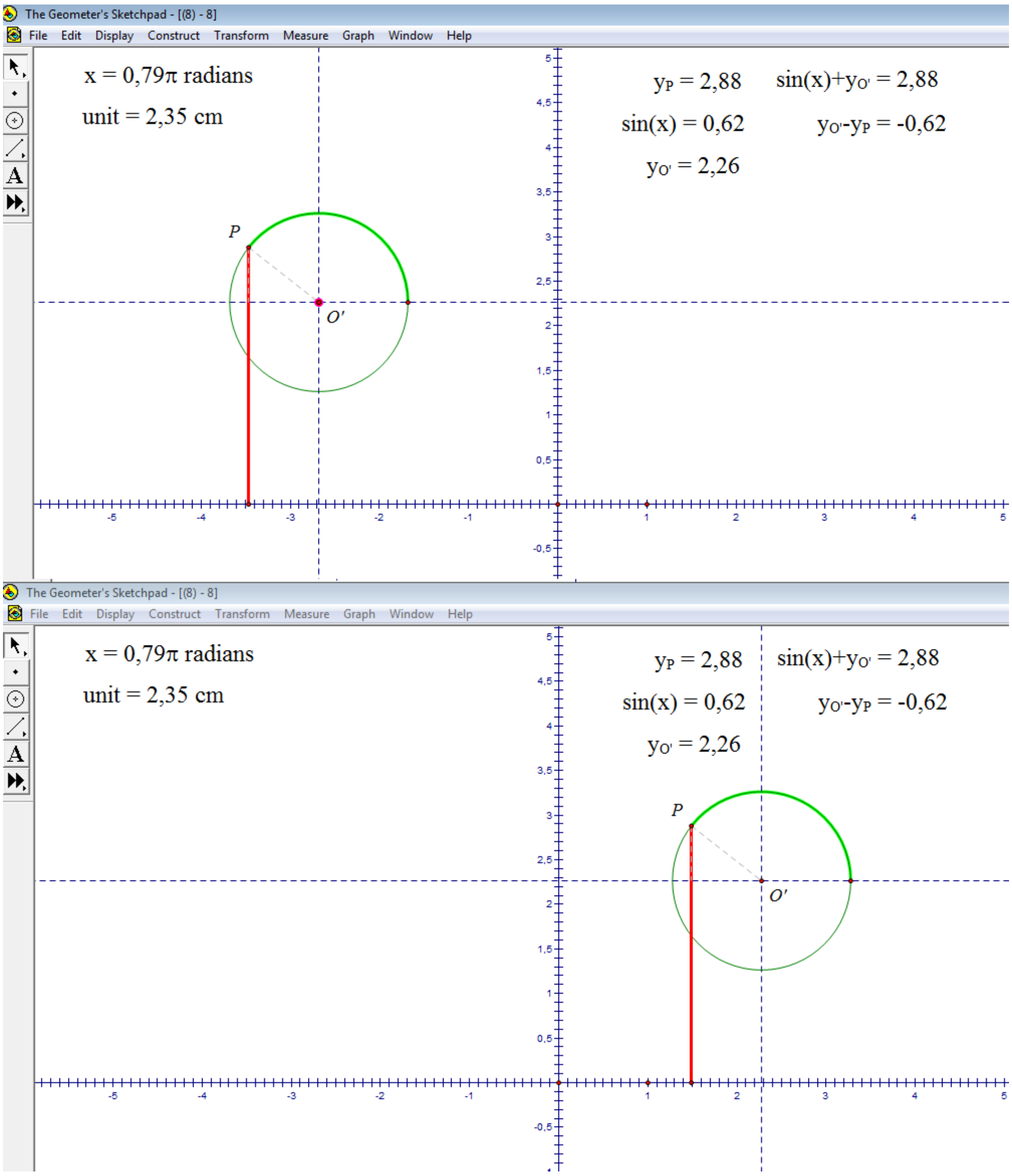


Figure 6.21. Parallel displacement of the unit circle in terms of sine along the *x*-axis in the (*unit*) *circle register*

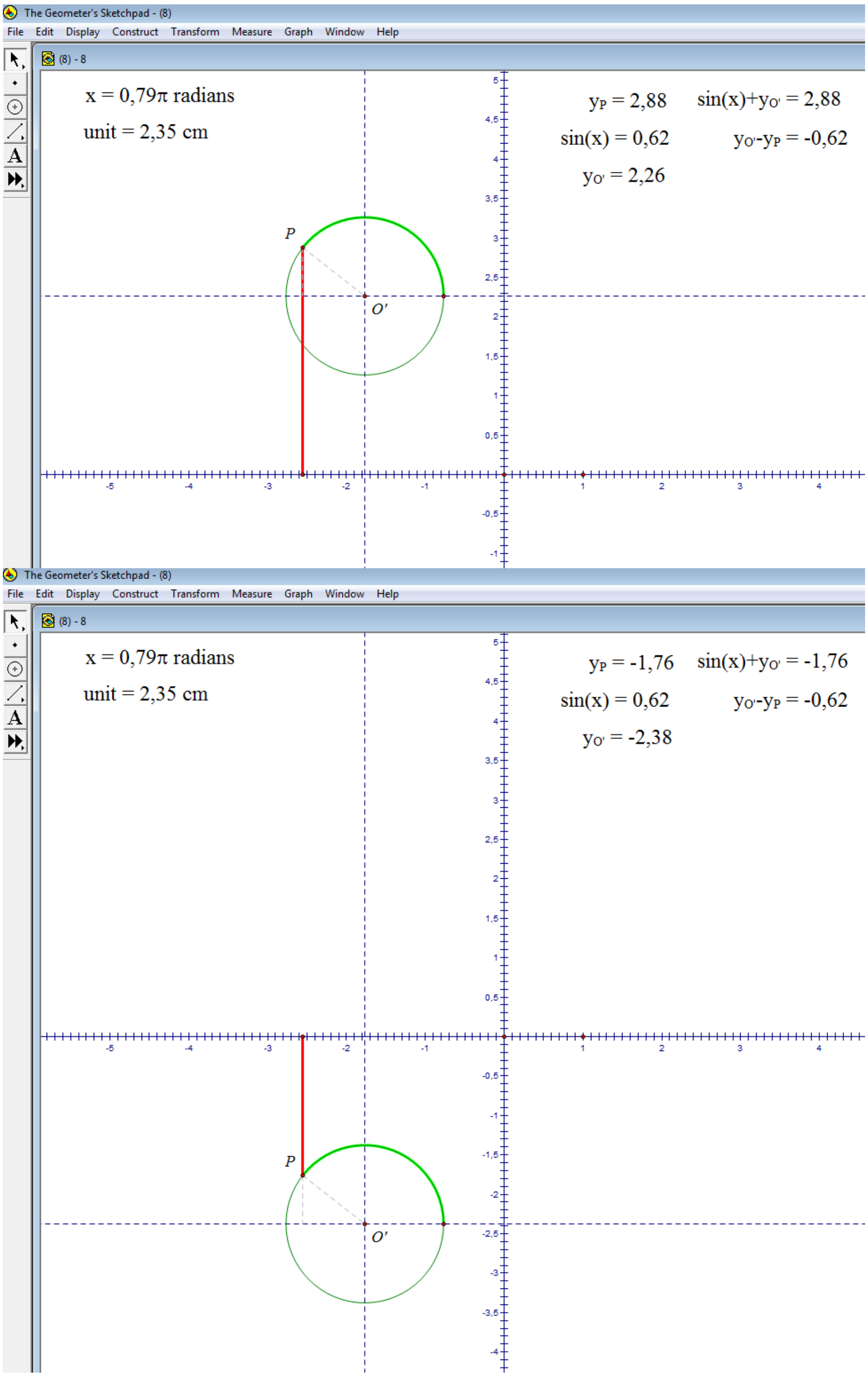


Figure 6.22. Parallel displacement of the unit circle in terms of sine along the *y*-axis in the (*unit*) *circle register*

6.1.2.2. Parallel-displacement along the *y*-axis in graphical register

In addition to the role of the changed-location of the unit circle on the coordinate system, its conversion into the *graphical register* was another important focus in Task 9 [Task 14] by means of the functions defined in the (*unit*) *circle register* from the principal arc to the *y*-component [*x*-component] of the reference point of this arc on a unit circle with a manipulable-center on the coordinate system (*Figure 6.16* and *Figure 6.26*). Where, the researcher encouraged students to interpret the variation of these functions in the *graphical register* under the manipulation of the unit circle's location through comparing and contrasting with that on the unit circle whose center was located on the origin.

When the traced-graph of the mentioned-function was appeared on the screen, all students were able to associate this function with the sine [cosine] function in the *graphical register* based on its visual-shape (e.g., lines 27-58 in [Cemre & Zafer] Protocol 16; lines 33-39 in [Defne & Ebru] Protocol 16). Moreover, they were able to reason about this traced graph as a parallel displacement of the sine [cosine] graph along the *y*-axis in the positive/negative direction but initially without considering the displacement amount (e.g., lines 36-55 in [Cemre & Zafer] Protocol 16; lines 37-45 in [Defne & Ebru] Protocol 16). At that point, the researcher encouraged them to compare and contrast this traced graph with the sine [cosine] graph in a more detailed way (e.g., lines 59-62 in [Cemre & Zafer] Protocol 16; line 46 in [Defne & Ebru] Protocol 16). Where, it was observed that students were able to reason about the parallel displacement amount through focusing on the intersection points with the *y*-axis of the traced graph and the sine [cosine] graph (e.g., lines 59-77 and 96-116 in [Cemre & Zafer] Protocol 16; lines 43-48 in [Defne & Ebru] Protocol 16). Moreover, their actions and language imply that they associated this (directed) amount of the parallel displacement of the sine [cosine] graph with the center's (directed) distance from the *x*-axis [*y*-axis] (e.g., lines 67-74 and 96-116 in [Cemre & Zafer] Protocol 16; lines 46-71 in [Defne & Ebru] Protocol 16), as well as with the (signed) constant of the sine [cosine] function in the *symbolic register* (e.g., lines 96-116 in [Cemre & Zafer] Protocol 16).

Finally, at the end of Task 11 [Task 16] when the researcher provoked students to reason about a general form of sine [cosine] with a constant in the *symbolic register,* students were able to convert the positive {negative} constant term in the *symbolic register* into the parallel displacement amount along the *y*-axis in the positive {negative} direction in the *graphical register*, as well as into the manipulation-distance of the circle from the *x*-axis [*y*-axis] in the positive {negative} direction in the (*unit*) *circle register* (see "*Composed-Coefficients' Visual Oppositions*" heading).

[Cemre & Zafer] Protocol 16

Researcher: (*Giving directions for students to construct a unit circle in GSP environment whose center can be manipulable and the parallel axes from its center to the coordinate axes in the (unit) circle register like in Figure 6.16*)…

(*Cemre and Zafer cooperatively constructed GSP page like in Figure 6.16 with respect to the researcher's directions.*)

Researcher: Let's drag the center point. I want you to see that you can move this circle to anywhere.

Zafer: (*Dragging the center point on the coordinate system's each quadrant*).

Cemre: (*Looking to the screen*)

Researcher: Today, we will see what are changing when moving it [circle].

Zafer: (*Continuing to drag the center point on the coordinate system and then putting it on the origin*) here is the best (*smiling*).

Cemre: (*Smiling*)

Researcher: (*Smiling*) the most familiar position is here, yes. But put it [center] on anywhere from the origin please.

Zafer: (*Putting the center point in the second quadrant*) but it didn't befit (*smiling*).

Researcher: It will befit at the end (*smiling*). (*After 4-second pause*) well, if we define a relation… …that is mapping this green arc to this *y*-value (*pointing the green arc and red line segment on the screen like in Figure 6.16*)… Is it a function?

Cemre&Zafer: Yes.

Cemre: …because for every arc, we find a *y*-value… only one *y*-value…

Zafer: (*Nodding his head up and down*) uh-huh… …but it becomes different from sine anymore…

Cemre: (*After 5-second pause*) then, its graph [how would be]… …oho! Can we construct it?

Researcher: Of, course.

(*Cemre and Zafer cooperatively were measuring* y_P*; and then, constructing* (x, y_P) *ordered-pair taking advantages of the plot as (x,y) option of GSP; finally, constructed its graph in the* $[0,2\pi)$ *interval through tracing the point P′ by the researcher's directions*).

Researcher: Let's we animate the point P.

Cemre: (*Before animating the point P*) I think our doing [sine wave] would appear…

Researcher: That is, what?

Cemre: …period [sine wave in the (0,2π) interval]…
Researcher: Let's we animate the point *P*.
Zafer: (*Constructing the animation button for the point P in the counterclockwise direction with the medium speed with respect to the researcher's directions; and then clicking the button. When the traced graph was appeared on the screen like in Figure 6.23*) this is again like sine function… …that's, it [graph] didn't start from here (*pointing with his index finger the origin on the screen*), started from here (*pointing the intersection point of the y-axis with the manipulable-horizontal axis constructed from the manipulable-center*).
Cemre: Uh-huh (*nodding her head up and down*).
Zafer: If we think here (*dragging his index finger from left to right on the manipulable-horizontal axis on the screen like in Figure 6.23*) as if *x*-axis, then this (*dragging his finger from left to right on the traced graph*) is sine function…
Cemre: That is, we moved sine upward… If we changed [position of] circle (*dragging the center in another position in the second quadrant of the coordinate system*)…
Zafer: …then, it would start from here (*pointing the new position of the intersection point of the y-axis with the manipulable-horizontal axis on the screen like in Figure 6.24(a)*)… …as if here as *x*-axis (*dragging his index finger on the new position of the manipulable-horizontal axis*).
Cemre: Uh-huh. Same thing [sine wave] would be appeared (*animating the point P by using the animation button, when the traced new graph appeared on the screen like in Figure 6.24(b)*) yes…
Researcher: So, it seems… …this function is related to sine…
Zafer: Yes, it is related.
Cemre: Absolutely...
Researcher: Let's we construct sine graph (*plotting sine graph*) and then discuss this relation for that case (*dragging the center point so as the ordinate of the point O' to be about 1 like in Figure 6.25(a)*). Ok, what about now? What do you say in that case?
Zafer: In case we add one to sine…
Cemre: (*Looking to the screen without speaking*)
Researcher: (*Animating the point P, and constructing the traced graph like in Figure 6.25(b)*) Zafer said that we add one to sine… Why do you think so, Zafer?
Zafer: Actually, this is one more (*putting his left hand's index and thumb fingers respectively on the center point O' and its projection on the x-axis on the screen like in Figure 6.25(b)*)… Normally [sine graph], starting from here (*pointing the origin*)… …for example, its highest point is one (*pointing the peak point of the sine graph in the (0,π) interval*). But now (*putting his index finger on the intersection point of the y-axis with the manipulable-horizontal axis*), starting from one… There is one [unit] difference between them. Anymore, the highest point is two (*pointing the peak point of the traced graph in the (0,π) interval*).
Cemre: Yes.
Researcher: Good. What about the lowest value of new function?
Cemre&Zafer: Zero.
Researcher: Well, you reasoned through looking to the graphs (*pointing the traced graph on the screen like in Figure 6.25(b)*). What would you say if you focused only this circle (*pointing green arc and then the red segment on the unit circle*)?

Zafer: Then, If we consider *y*-value (*figuring the perpendicular line segment from the point P to the y-axis*) regarding this (*figuring perpendicular line segment from the center point O' to the y-axis*)… …as well as we add this (*pointing the opposite side of the reference right triangle on the unit circle through putting his right hand's index and thumb fingers on its endpoints*), then it is same… …sine plus one… …same thing…

Cemre: Yes. I would think so that… …if we take this part (*pointing the opposite side of the reference right triangle on the unit circle through putting her right hand's index and thumb fingers on its endpoints, at that time the unit circle was in the second quadrant of the coordinate system like in Figure 6.25(b)*) and move it up one unit (*through keeping the distance between her index and thumb fingers same, moving her fingers initially downward until her thumb finger on the x-axis; and then, upward until her index finger on the point P*), then same thing will continue again here (*moving her index finger horizontally from the point P to the y-axis rightward; and then from the y-axis to the point P leftward*).

Researcher: Good. You were able to reason truly both on graph and unit circle… What about its symbolic form?

Zafer: …from *x* to sine *x* plus one…

Cemre: Uh-huh (*nodding her head up and down*)…

Researcher: With sine *x* plus one, do you mean this (*writing on a paper sin(x+1)*)?

Zafer: No, this's not.

Cemre: No. One should be outside of the parenthesis…

Zafer: This adding one to angle... …we add one to sine *x*.

Cemre: Yes.

Researcher: Please plot this function's graph.

Cemre: (*Plotting the h(x)=sin(x)+1 graph*) yes.

Zafer: Uh-huh (*nodding his head up and down*).

Researcher: Ok. What about now (*dragging the unit circle in the third quadrant of the coordinate axis so as its center's ordinate to be about -2*).

Zafer: Then, it would be from *x* to sine *x* minus two…

Cemre: Now, we moved down two units (*putting her index finger on the intersection point of the manipulable-vertical axis with the x-axis; and then, dragging her finger downward up to the center point*).

Zafer: Yes.

Cemre: So, sine *x* minus two…

Zafer: (*Nodding his head up and down*)

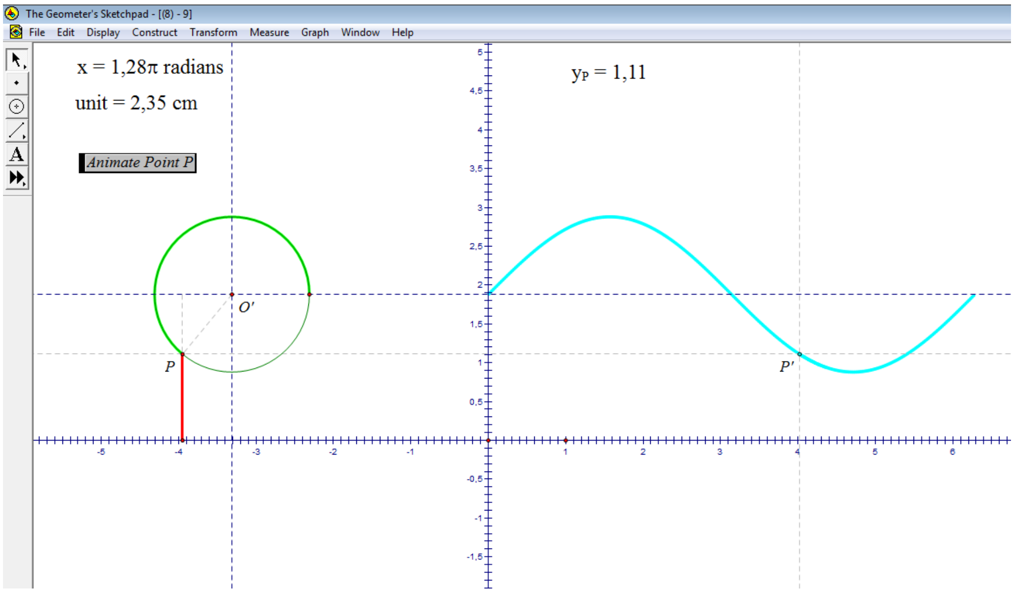


Figure 6.23. Construction of the traced graph of the ordered-pair (x,y_P) in the *graphical register* as dynamically-linked to the point P on the unit circle whose center is located on the different point from the origin in the (*unit*) *circle register*

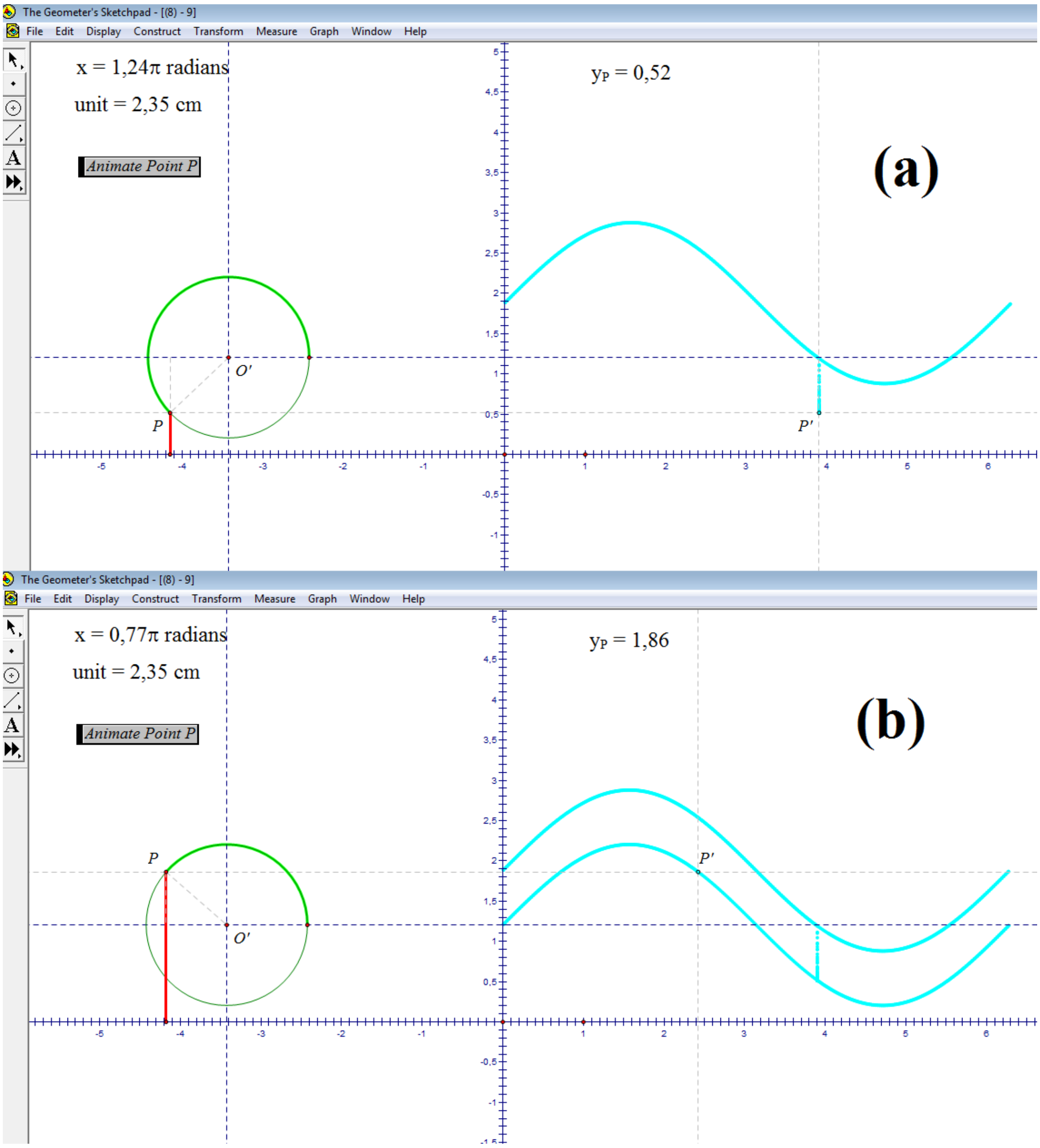


Figure 6.24. Construction of the manipulated-traced graph of the ordered-pair (x,y_P) in the *graphical register* by the manipulation of the center in the (*unit*) *circle register*

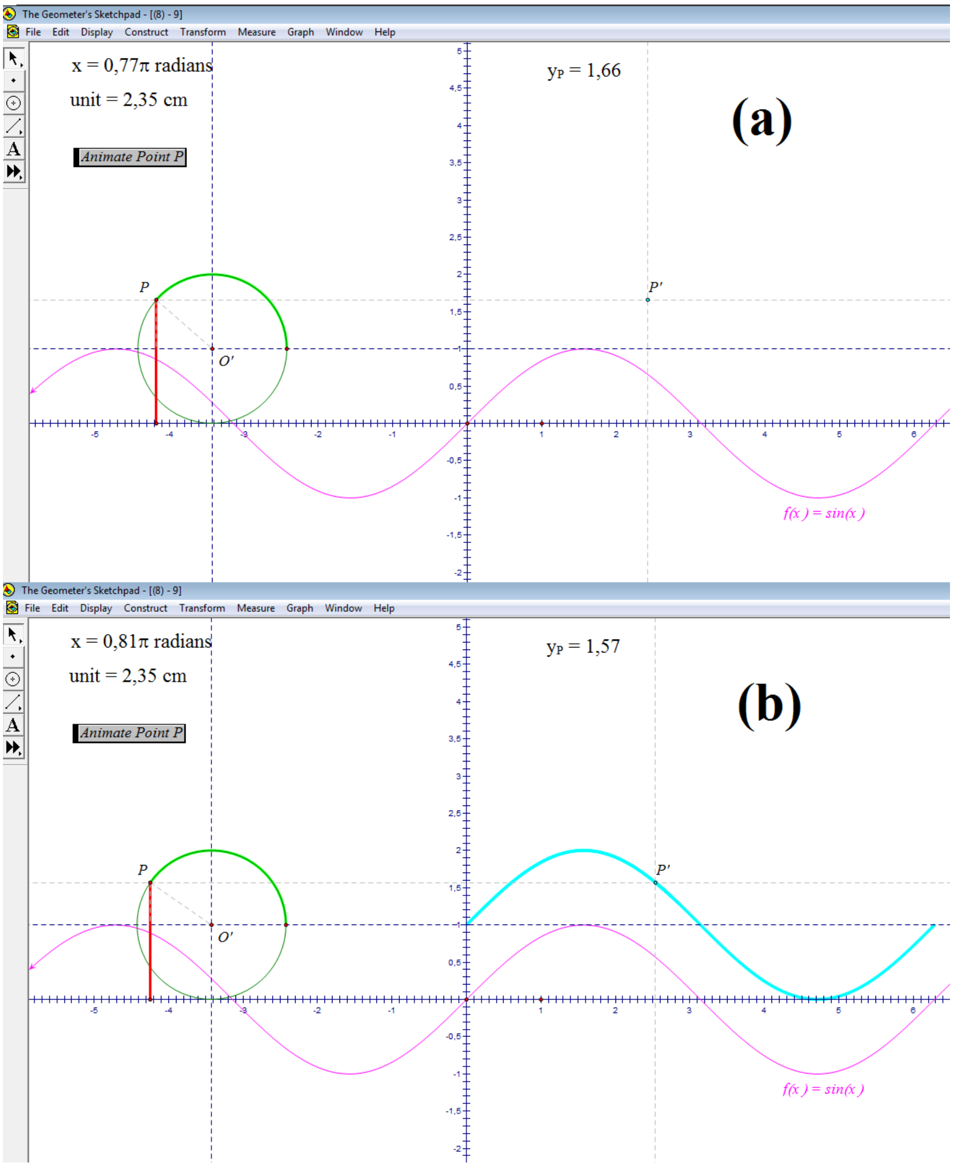


Figure 6.25. Construction of the traced graph of the ordered-pair (x,y_P) in the *graphical register* together with the sine graph

[Defne & Ebru] Protocol 16

Researcher: In the previous week, we discussed about unit circle's different positions on the coordinate system in terms of sine…

Defne&Ebru: Uh-huh.

(*Defne and Ebru constructed a unit circle and the principal arc on it in GSP so as its position on the coordinate system to be changeable by the researcher's directions.*)

…

Researcher: (*Dragging the center of the constructed manipulable-unit circle onto the origin, and constructing the line segment between the origin and the projection point of the point P to the x-axis like in Figure 6.26(a)*) you know here what (*pointing the constructed blue line segment on the screen*)?
Defne&Ebru: Cosine…
Researcher: (*Dragging the center point along the x-axis in the positive direction like in Figure 6.26(b)*) what about now? This blue segment (*pointing the blue line segment on the screen like in Figure 6.26(b)*) refers to what?
Defne&Ebru: (*Looking to the screen without speaking*)
Researcher: Well, how do you correlate it with cosine?
Defne: Cosine (*putting her index finger on the adjacent side of the reference right triangle on the screen like in Figure 6.26(b)*) plus here (*dragging her index finger from the origin to the point O'*)…
Ebru: Yes.
Researcher: Now that we mention about the changed location of the unit circle, it should be better to express this part (*dragging her index finger from the origin to the point O' on the screen like in Figure 6.26(b)*) in terms of coordinates. What do you think? How can we express it regarding its coordinates?
Defne: 2 (*putting her index finger on the point O'*)…
Ebru: x…
Defne: Yes, its *x*-value…
(*Where the researcher encouraged them to measure x-value and calculate the cosine value of the angle corresponding to the point P; and then, calculate their sum like in Figure 6.26(b).*)
…
Researcher: Let's we construct the traced graph of the function… …mapping this angle (*dragging her index finger on the green arc in the counterclockwise direction on the screen like in Figure 6.26(b)*) to this blue line segment on the coordinate system (*dragging her index finger on the blue line segment*).
(Defne and Ebru constructed the traced graph of the ordered pair (x,cos(x)+ $x_{O'}$*) like in Figure 6.27(a).)*
Defne: It resembles to cosine…
Researcher: Let's draw cosine graph.
Defne: (*Drawing the h(x)=cos(x) function's graph like in Figure 6.27(b)*)
Ebru: …its [cosine's] upward translated version.
Defne: Yes. Instead of one (*pointing the intersection point of the cosine graph with the y-axis on the screen like in Figure 6.27(b)*), it starts from three (*pointing the intersection point of the traced graph with the y-axis*)…
Researcher: So, it is translated how many units?
Defne: …two units…
Ebru: Uh-huh.
Researcher: How is it, two units, related to the position of the unit circle (*pointing the center of the unit circle on the screen*)?
Ebru: Its *x*-value…
Defne: Uh-huh (*nodding her head up and down*).
Researcher: Well, if we put the circle in here (*dragging the unit circle from the positive x-axis to the negative x-axis like in Figure 6.26(c)*), what will happen (*animating the point P*)?

Defne: (*When the traced graph appeared on the screen like in Figure 6.27(c)*) it went downward.
Ebru: (*Nodding her head up and down*)
Researcher: How many units?
Ebru: (*Coming closer to the screen like in Figure 6.27(c), putting her index finger respectively on the intersection points with y-axis of the cosine graph and the traced graph*) two units.
Researcher: Cosine graph went downward two units, is it ok?
Defne&Ebru: Uh-huh.
Researcher: What about center's *x*-value?
Defne: Minus two… …then, in case of negative *x*-value [of center], it [traced graph] goes down, in case of positive *x*-value [of center], it [traced graph] goes up…
Ebru: Yes.
(*Similar discussions were done under the horizontal and vertical manipulations of the center; they were able to reason in this way about the relation between the variations of center and the corresponded traced graph.*)

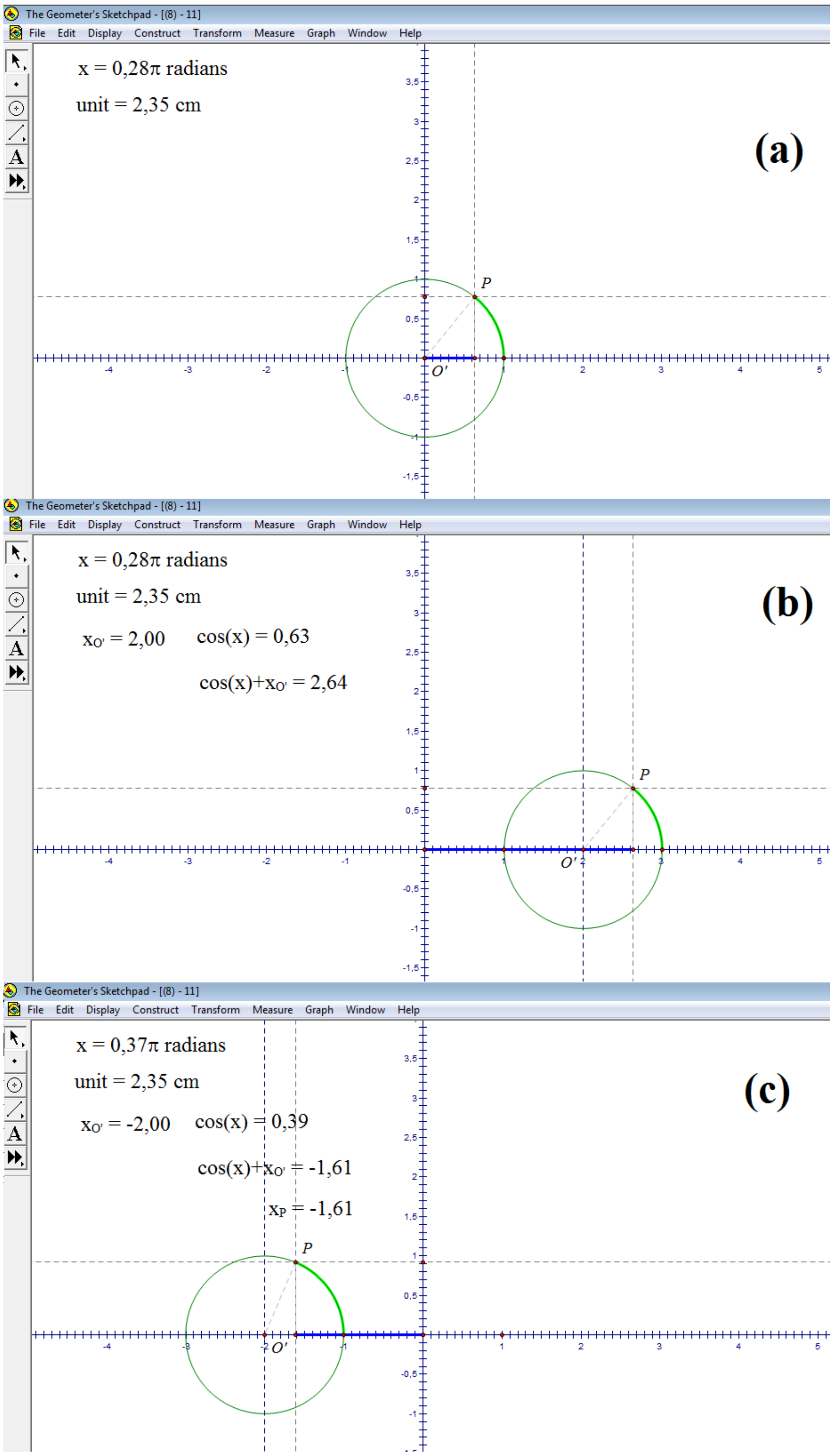


Figure 6.26. Parallel displacement of the unit circle in terms of cosine along the *x*-axis in the (*unit*) *circle register*

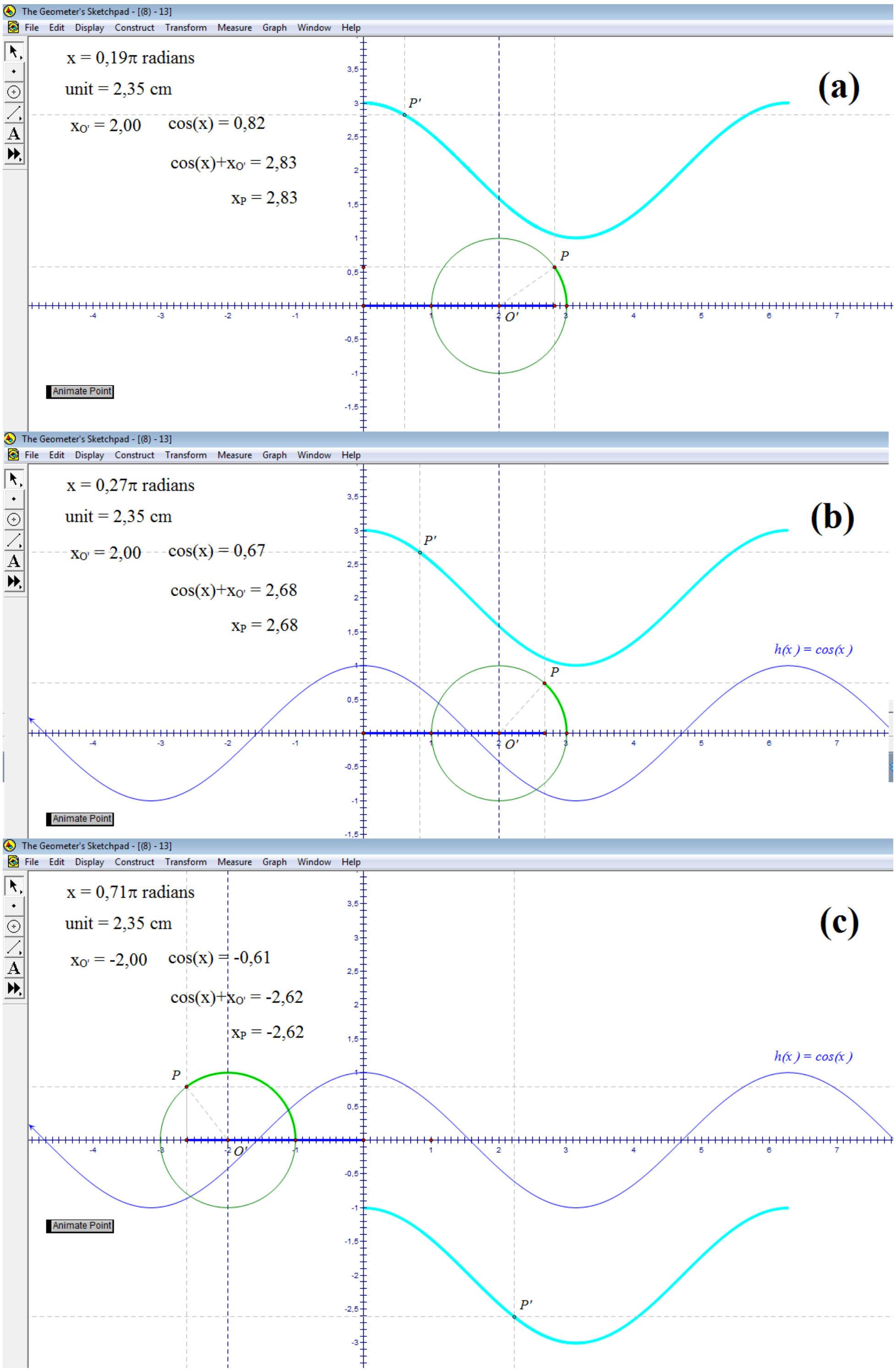


Figure 6.27. Construction of the traced graph of the $(x,\cos(x)+x_{O'})$ ordered pair in the *graphical register* together with the cosine graph

6.1.3. Visual Feature Opposition C

Visual Feature (C) corresponds to changed-arc with a constant (angular) difference in the (*unit*) *circle register* and parallel-displacement along the *x*-axis in the *graphical register* so that these visual features' opposition corresponds to the choice presence/absence of a constant of the input of sine and cosine in the *symbolic register*.

6.1.3.1. Changed-arc with a constant difference in (unit) circle register

At the beginning of Task 10, the researcher asked students in each group to construct a transformed-position of the point *P* on the unit circle through rotating about the origin by a fixed-measure based on their choice (e.g., lines 1-22 in [Cemre & Zafer] Protocol 17). As a fixed rotation measure, while Cemre and Zafer preferred 4 (radian), Defne and Ebru preferred π+4 (radian). Where, it was observed that none of the students in each group encountered a trouble about the rotated-position of the point *P* on the unit circle (e.g., lines 23-29 in [Cemre & Zafer] Protocol 17; lines 69-83 in [Defne & Ebru] Protocol 17). However, as a consequence of groups' different choices as the rotation amount, their different reasoning emerged based on their same focuses on the *principal-rotation path* referring to the principal-arc in the counterclockwise direction from the point *P* to its rotated position.

On the one hand, the choice of Group 1 [Cemre&Zafer] was a principal measure as the rotation amount. Therefore, in their construction, the rotation-path was the principal arc from the point *P* to its rotated position (*Figure 6.28*). This structure caused for Cemre and Zafer to directly reason about the rotation amount through focusing on the principal-rotation path. Moreover, this structure caused Cemre's association of this principal-rotation path by 4-radian with the rotation by 2.28-radian in the reverse direction through considering these two rotation paths as complements of each other to a full round (e.g., lines 25-29 in [Cemre & Zafer] Protocol 17). By this way, when the new function mapping the angle of the point *P* to the ordinate of its rotated position in the (*unit*) *circle register* was converted into the *graphical register*, they reasoned about the differentiation of the new graph from the sine graph based on these two rotation amounts (4 and about 2.3) (e.g., [Cemre & Zafer] Protocol 17). That is to say, they considered either 4 or 2.3 as the parallel displacement amount in order to coincide the new graph with the sine graph (see following sub-heading *Parallel-displacement along the x-axis in graphical register*). Thus, they were able to attribute the variation of the new graph from the sine graph to directly the rotation amount.

On the other hand, the choice of Group 2 [Defne&Ebru] as the rotation amount was not a principal measure, i.e., (π+4)-radian. Therefore, in their construction (*Figure 6.32*), the principal arc from the point *P* to its rotated-position did not refer to the rotation path; instead, referred to the principal-rotation path with a measure approximately 0.86-radian. This structure obstructed Defne and Ebru to directly reason about the rotation amount; instead, they reasoned about the rotation based on the principal-rotation path but without considering its measure as about 0.86 in radians (e.g., lines 75-83 in [Defne & Ebru] Protocol 17). Therefore, when the new function mapping the angle of the point *P* to the ordinate of its rotated position in the (*unit*) *circle register* was converted into the *graphical register*, they reasoned about the variation of the new graph from the sine graph based only on the graphs' visual differences without associating this variation with the rotation amount in the (*unit*) *circle register* (e.g., [Defne & Ebru] Protocol 17). In other words, they were unable to associate the differentiation of the new function from sine in the *graphical register* with the rotation in the (*unit*) *circle register* (see following sub-heading *Parallel-displacement along the x-axis in graphical register*). So, the researcher determined to specify in Task 15 the rotation amount herself so as to refer a principal-rotation path in the (*unit*) *circle register* (lines 1-10 in [Defne & Ebru] Protocol 19).

When investigating the variation of these two points on the unit circle under the manipulation of the point *P*, all students recognized that the arcs between the point *P* and its rotated-position remained invariant in the (*unit*) *circle register* (e.g., lines 30-37 in [Cemre & Zafer] Protocol 17). At that point, the researcher introduced the new function to the students so that it could map the angle of the point *P* to the ordinate of its rotated-position (e.g., lines 38-41 in [Cemre & Zafer] Protocol 17). Except Zafer, none of the other students associated this function with sine until the construction of its graphical representation (e.g., lines 42-49 in [Cemre & Zafer] Protocol 17; lines 7-

12 in [Defne & Ebru] Protocol 17). Although Zafer associated this function with sine based on its visual definition on the unit circle (line 42 in [Cemre & Zafer] Protocol 17), he did not reason about its symbolic representation in the (*unit*) *circle register*; instead, he tried to reason in the symbolic register based only on the construction of its graphical representation (lines 72-122 in [Cemre & Zafer] Protocol 17). In other words, throughout Task 10, none of the students reasoned in the (*unit*) *circle register* about the ordinate of the rotated-position of the point *P* as the sine value of the angle corresponding to the rotated-position through combining the angle of the point *P* and the angle of the rotation. So, in the following tasks (Tasks 11, 12, 15 and 16), the researcher determined to provoke students to identify the abscissa/ordinate of the rotated-position of the point *P* in the (*unit*) *circle register* in terms of cosine/sine in accordance with the tasks' themes (e.g., lines 1-29 in [Defne & Ebru] Protocol 18; lines 1-15 in [Defne & Ebru] Protocol 19).

For example, in Task 11, the researcher provoked students to identify the ordinate of the rotated position[42] of the point *P* in terms of sine of the new angle (e.g., lines 22-31 in [Cemre & Zafer] Protocol 18; lines 1-22 in [Defne & Ebru] Protocol 20). Where, all students were able to identify (*i*) the principal arc corresponding to the rotated position of the point *P* as *2x*, and then, (*ii*) its ordinate as sine of 2*x* through considering the definition of sine (from the arc corresponding to a point on the unit circle into its ordinate) (e.g., lines 23-33 in [Cemre & Zafer] Protocol 18; lines 1-33 in [Defne & Ebru] Protocol 20). It was the point that students had just started to reason about a general form of the sine function from an arc, i.e., *x*, to the ordinate of another arc defined dependently on *x* in the (*unit*) *circle register*. It was observed that in Task 12, 15 and 16, they were able to extend this reasoning into a general form of the cosine function from an arc, i.e., *x*, to the abscissa of another arc defined dependently on *x* in the (*unit*) *circle register* (e.g., lines 1-50 in [Defne & Ebru] Protocol 18; lines 1-19 in [Defne & Ebru] Protocol 19).

[42] It was a point on the unit circle that was constructed through rotating the point *P* by a marked angle, e.g., by marking the *x* angle corresponding to the point *P*.

In Task 12[43], this reasoning prompted a distinct shift on their reasoning about the relation between sine and cosine in the (*unit*) *circle register*. That is to say, they were able to convert the relation between the directed measures of the adjacent [opposite] side of the reference right triangle and the opposite [adjacent] side of its rotated-counterpart by π/2 radian about the center in the (*unit*) *circle register* into the equality between cos(x) [sin(x)] and sin(x+π/2) [-cos(x+π/2)] in the *symbolic register* (e.g. lines 52-87 in [Defne & Ebru] Protocol 18). Moreover, they were able to extend this reasoning based on the structure obtained through rotation by π/2 radian in the counterclockwise direction into the reasoning on the structures obtained through rotations by the integer multiples of π/2 radian in any directions (e.g., lines 89-117 in [Defne & Ebru] Protocol 18), which emerged as a result of the teaching experiment (e.g., lines 119-122 in [Defne & Ebru] Protocol 18). In special sense, Task 12 was the first task that students made sense of the negative coefficient of –sine [–cosine] function in the (*unit*) *circle register* through considering it as a function from the x angle to the perpendicular line segment from the point corresponding to the (x±π) angle to the x-axis [y-axis]. Furthermore, Task 12 was the first task that students had an alternative visual focus referring to sine [cosine] in the (*unit*) *circle register* instead of the opposite [adjacent] side of the reference right triangle. That is to say, students' actions imply that they started to reason about sine [cosine] in the (*unit*) *circle register* through exchanging their focuses between the opposite [adjacent] side of the reference right triangle and its facing-side of the *reference-rectangle*[44] (e.g., lines 104-115 in [Defne & Ebru] Protocol 18). This point of view prompted in Modeling Task of the teaching experiment, Task 17, students' easily and truly modeling of the distance to the point gotten in the car on the Ferris wheel throughout the turning (see *Modeling Task with Ferris Wheel: Modeling Process* heading in Chapter 7).

[43] In this task, after constructing the rotated-version of the reference right triangle about the origin by π/2 radian (*Figure 6.33-Figure 6.35*), discussions were done on their related legs in terms of sine and cosine.

[44] We call a rectangle in the (*unit*) *circle register* as the "reference rectangle" that is restricted by the coordinate axes and the perpendicular lines from a point on the (unit) circle to the axes (*Figure 6.34*).

[Cemre & Zafer] Protocol 17

(*Cemre and Zafer cooperatively constructed a unit circle located on the origin with the principal arc with respect to the researcher's directions.*)

Researcher: Today, we will discuss about a function… …mapping the angle (*dragging her index finger on the principal arc in the counterclockwise direction*) to… …the ordinate of another point on the circle… …you will construct after a while regarding my directions…

Zafer: (*Smiling*)

Researcher: Ok. We will find a new point on this circle… …but it would not be an arbitrary point… …it would be a point so that as you could animate the representation point of the angle (*pointing the point P on the screen*), it would be act equidistantly to this point (*putting her right hand's index finger on the point P and thumb finger another point on the unit circle, and then, dragging her fingers on the circle in the counterclockwise direction in the same speed*)…

Cemre: Hmm…

Zafer: (*After 4-second pause*) how do we construct it [new point]?

(*Cemre and Zafer cooperatively constructed a point on the unit circle, the point P+4, through rotating the point P in the positive direction by a 4-radian fixed angle and its corresponding y-component like in Figure 6.28 with respect to the researcher's directions. Where, 4-radian was preferred by Cemre and Zafer as a fixed rotation measure on the unit circle.*)

Researcher: How it is located here (*pointing the point P+4 on the screen like in Figure 6.28*)?

Zafer: Because it rotated by 4 radian (*dragging his index finger on the unit circle in the counterclockwise direction from the point P to the point P+4*).

Cemre: Then, whole [circumference] is 6.28… …in the reverse direction (*putting her index finger on the point P and dragging in the clockwise direction*), here is 2.28 (*dragging her index finger from the point P to the point P' in the clockwise direction*).

Zafer: Uh-huh (*nodding his head up and down*).

Researcher: Ok. Now, please drag the point *P*.

Cemre: This part exists invariably between them [points *P* and *P+4*] (*pointing the arc from point P to the point P+4 in the counterclockwise direction on the screen like in Figure 6.28*).

Zafer: That's to say, between them there's a fixed distance…

Researcher: Yes, this distance is our rotation amount (*pointing the arc from point P to the point P' in the counterclockwise direction*)… …that's 4.

Cemre&Zafer: Uh-huh (*nodding their heads up and down*).

Researcher: Well… …now we will discuss about the function… …so that it is mapping this green arc (*dragging her index finger on the green arc in the counterclockwise direction on the screen like in Figure 6.28*) into this red line segment (*dragging her index finger on the red line segment up and down*)…

Zafer: Its [new point's] sine would be…

Cemre: Let's we construct its traced graph.

(*Cemre and Zafer cooperatively constructed the* (x,y_P) *ordered-pair and its trace under the animation of the point P like in Figure 6.29(a) with respect to the researcher's directions.*)

Zafer: (*Dragging the point P*)

Cemre: Hmm… Here it is alike sine wave!

Zafer: But its start is a bit more different from sine…

Researcher: Yeah, it [traced-graph] resembles to sine…

Zafer: But it isn't sine…

Cemre: Wait a minute! Yes, it isn't [sine].

Zafer: It is different.

Researcher: It is not $y=\sin(x)$ function. How do you determine this?

Zafer: From their starts for my part (*putting his index finger on the intersection point of the traced graph with the y-axis*)…

Cemre: Hmm… …ok, why did it [traced graph] start from here (*pointing the intersection point of the traced graph with the y-axis on the screen like in Figure 6.29(a)*)?

Cemre&Zafer: (*Looking to the screen without speaking throughout 11-second*)

Researcher: Well, you mentioned the difference of this [traced] graph from sine (*pointing the traced graph on the screen*). Now, I want you to plot sine graph, as well.

Cemre: (*Plotting* $f(x)=\sin(x)$ *function's graph. When the sine graph appeared on the screen like in Figure 6.29(b)*) aha! We move this [sine graph] there (*putting her index finger on the peak point of the sine graph in the* $(0,\pi)$ *interval, and then dragging in the positive direction horizontally rightward up to the positive peak point of the traced graph*).

Zafer: We move the black graph (*putting his index finger on the positive peak point of the traced graph*) toward here (*dragging his index finger horizontally leftward up to the peak point of the sine graph in the* $(0,\pi)$ *interval*) or [sine graph] toward here (*dragging his index finger horizontally rightward from the peak point of the sine graph in the* $(0,\pi)$ *interval up to the positive peak point of the traced graph*).

Researcher: So, how do you express this function (*pointing the traced graph*)?

Cemre: Just a moment! Let's we think about it (*coming closer to the screen like in Figure 6.29(b)*)! (*After 6-second pause*) here (*pointing the first intersection of the traced graph with the x-axis*) is 2.3, is that so?

Zafer: Yeah, something like that... We should do something about its angle… That is, for example, the angle of this (*pointing the first intersection point of the traced graph with the x-axis*) in here (*putting his index finger on the first intersection of the traced graph with the x-axis*)… …in order to bring there (*dragging his index finger leftward up to the origin*), we subtract this amount of distance (*dragging his index finger left and right between the origin and the first intersection of the traced graph with the x-axis*) from this (*putting his index finger on the first intersection of the traced graph with the x-axis, and then dragging leftward on the x-axis up to the origin*)…

Cemre: In my part, you know here is 2.3 (*putting her index finger on the first intersection of the traced graph with the x-axis*) if it was π, or about 3.14… …that is, here was 3.14 (*putting her index finger on the intersection point of the sine graph with the x-axis on π*)… …that's, if to $\sin(x)$, we added π (*holding vertically her right hand and then moving it rightward without changing its vertical stance*), black [traced graph] would start from here (*putting her index finger on the intersection point of the sine graph with the x-axis on π*)…

Zafer: This function (*pointing the first intersection point of the traced graph with the x-axis on the screen like in Figure 6.29(b)*)… …we subtract this distance (*dragging his index finger left and right between the origin and the first intersection point of the traced graph with the x-axis*) from its angle (*putting his index finger on the first intersection point of the traced graph with the x-axis, and then dragging leftward up to the origin*)…

Researcher: Do you mean we translate this graph (*pointing the sine graph*) into this graph (*pointing the traced graph*) through adding this difference (*dragging her index finger left and right between the origin and the first intersection point of the traced graph with the x-axis*)… …or translate this graph (*pointing the traced graph*) into this graph (*pointing the sine graph*) through subtracting this difference (*dragging his index finger left and right between the origin and the first intersection point of the traced graph with the x-axis*)?

Cemre&Zafer: Uh-huh (*nodding their heads up and down*).

Researcher: To what you add… …or from what you subtract this difference (*pointing the segment on the x-axis between the origin and the first intersection point of the traced graph with the x-axis*)?

Zafer: From angle…

Cemre: Yes.

Researcher: How do you express it?

Zafer: We need to start this graph (*pointing the origin*) from here (*the first intersection point of the traced graph with the x-axis*)… …for this purpose, what should we do (*after 8-second pause*)… …do we see through constructing the graph from here (*pointing the graph menu of the GSP with the cursor*)?

Researcher: Of course.

Zafer: (*Plotting y=sin(x+2.3) function's graph. When the graph appeared on the screen like in Figure 6.30(a)*) ah! It went toward left!

Cemre: Then, let's we add 4…

Zafer: Wait a minute! It went toward left… …I had hoped it would go toward right however.

Cemre: (*Looking to the screen without speaking*).

Researcher: Why did it so?

Zafer: I think why it did so, too… …(*after 6-second pause*) we added to angle… …when adding to angle, it [sine graph] went toward left… …we added what, 2.3 (*waiting without speaking about 5 seconds*)…

Researcher: What should we do in order to obtain this graph (*pointing the traced graph*)?

Cemre: We should add a greater value [than 2.3]…

Zafer: We should subtract [2.3]…

Researcher: Let's try and see!

Cemre: (*Taking the mouse-control herself*) firstly, which one do we try… Do we add or subtract?

Zafer: (*Looking to the researcher*) when subtracting, would it go rightward?

Researcher: Let's try and see (*smiling*).

Cemre: Then, let's we subtract 2.3 (*constructing sin(x-2.3) graph in GSP*)

Zafer: (*When the sin(x-2.3) graph appeared on the screen like in Figure 6.30(b)*) it was ok when subtracting… But I did not understand why it did so? That is, why the graph act in the reverse direction?

Cemre: (*Laughing*) yes, when adding [a constant to the input of sine], it should have gone toward that side (*indicating the right side with her hands*), but it went toward this side (*indicating the left side with her hands*).

Researcher: Up to understand, we continue to discussion (*smiling*).

Zafer: (*After 5-second pause*) in here sine is 1 (*pointing with his index finger the peak point of the sine graph in the (0,π) interval*). When we add to this (*figuring a perpendicular segment to the x-axis from this peak point*)… …2.3 (*dragging his finger rightward on the x-axis up to about projection point of the positive peak point of the traced graph*), it is again 1 (*pointing the positive peak point of the traced graph*)! Is that so?

Cemre: Please say that again.

Zafer: Well, two [*y*-values]… …one of whom is at the *x* angle (*figuring a perpendicular line segment to the x-axis from the peak point of the sine graph in the (0,π) interval through emphasizing its down-end point*) and the other is at the *x+2.3* angle (*dragging his finger rightward on the x-axis up to about the projection point of the positive peak point of the traced graph*), two [angles] would have same *y*-value (*figuring a perpendicular line segment to the x-axis upward until the positive peak point of the traced graph through emphasizing its up-end point*) or same sine value, is that so?

Cemre: (*Looking to the screen without speaking*)

Researcher: Ok. Let's we construct an arbitrary point on the sine graph (*constructing an arbitrary point A on the sine graph and measured its abscissa and ordinate like in Figure 6.31*). Of this (*pointing the point A on the sine graph*) at the 2.3-unit beyond (*dragging the cursor horizontally rightward from the point A up to intersect with the pink graph on the screen like in Figure 6.31*), which value does this function take (*pointing the pink graph*)?

Zafer: …same with sine.

Cemre: Yes.

Researcher: So, I need to find 2.3 more of this (*pointing abscissa measure of the point A on the screen*).

Cemre&Zafer: Uh-huh (*nodding their heads up and down*).

Researcher: (*Calculating* $x_A + 2.3$*; and then, constructing the* $(x_A + 2.3, y_A)$ *ordered pair*)

Cemre: (*When the* $(x_A + 2.3, y_A)$ *ordered pair appeared on the pink graph like in Figure 6.31*) yes.

Zafer: (*Nodding his head up and down*)

Researcher: (*Labelling this ordered pair as B like in Figure 6.31*). (*After 5-second pause*) well, what about this act from here to here (*dragging the cursor horizontally rightward from the point B up to intersect with the sine graph on the screen like in Figure 6.31*)?

Cemre: Four more of this (*dragging her index finger horizontally rightward from the point B up to intersect with the sine graph on the screen like in Figure 6.31*)…

Researcher: Why?

Cemre: Because here is 2π (*dragging her index finger horizontally right ward between the points A and C*) and here is about 2.3 (*dragging her index finger horizontally right ward between the points A and B*)… …4 complete 2.3 to 2π.

Zafer: (*Looking to the screen without speaking*)

Researcher: Let's we see (c*alculating* $x_B + 4$*; and then, constructing the* $(x_B + 4, y_A)$ *ordered pair as a point labelled as C*).
Cemre: (*When the* $(x_B + 4, y_A)$ *ordered pair appeared on the screen like in Figure 6.31*) yes.
Zafer: Uh-huh (*nodding his head up and down*).
Researcher: If I drag the point A (*dragging the point A on the sine graph*), what do points B and C do?
Zafer: Same thing [with point A]…
Cemre: They also act in the same way.
Zafer: Uh-huh (*nodding his head up and down*).
Researcher: Ok. Now, I want you not to lose your focus. There are three different points on two different graphs (*pointing the points A, B and C on the screen like in Figure 6.31*). But I want you to concentrate on your focus. I am looking for this function's rule (*pointing the pink graph*) in terms of sine (*pointing the blue graph*).
Cemre&Zafer: Uh-huh (*nodding their heads up and down*).
Researcher: Let abscissa of this point be *x* (*pointing the projection point of the point B on the x-axis*). Then, what do you say about here (*pointing the projection point of the point A on the x-axis*)?
Cemre&Zafer: *x* minus 2.3…
Researcher: Ok. What about here (*dragging her index finger on the perpendicular line segment from the point A to the x-axis*)?
Cemre: Sine of…
Zafer: …*x* minus 2.3.
Cemre: Yes, sin(*x*-2.3).
Zafer: (*Nodding his head up and down*)
Researcher: Is here (*dragging her index finger on the perpendicular line segment from the point A to the x-axis*) same as here (*dragging her index finger on the perpendicular line segment from the point B to the x-axis*)?
Zafer: Same.
Cemre: Yes.
Researcher: Then, the pink graph is mapping *x* angle (*pointing the projection point of the point B on the x-axis*) into sin(*x*-2.3), isn't it?
Cemre: Yes.
Zafer: (*Nodding his head up and down*)
Researcher: (*After 4-second pause*) well, here is *x* (*pointing the projection point of the point B on the x-axis*). What about here (*pointing the projection point of the point C on the x-axis*)?
Cemre: *x* plus four.
Zafer: (*Looking to the screen without speaking*)
Researcher: Is it ok, Zafer?
Zafer: Uh-huh (*nodding his head up and down*).
Researcher: When here is *x* plus four (*pointing the projection point of the point C on the x-axis*), what do you say about here (*dragging her index finger on the perpendicular line segment from the point C to the x-axis*)?
Zafer: sin(*x*+4)
Cemre: Yes.

Researcher: Is here (*dragging her index finger on the perpendicular line segment from the point C to the x-axis*) same as here (*dragging her index finger on the perpendicular line segment from the point B to the x-axis*)?

Cemre: Yes, same.

Zafer: (*Nodding his head up and down*)

Researcher: Then, the pink graph is mapping *x* angle (*pointing the projection point of the point B on the x-axis*) into sin(*x*+4), isn't it?

Cemre: Yes.

Zafer: (*Nodding his head up and down*)

Researcher: Is it ok?

Zafer: It is needed to think about it… Since we haven't seen it in this respect before… …that is, we have seen for the first time, first becomes a little complicated…

Cemre: Yes… Especially, when operating with angle, it complicated a little for my part.

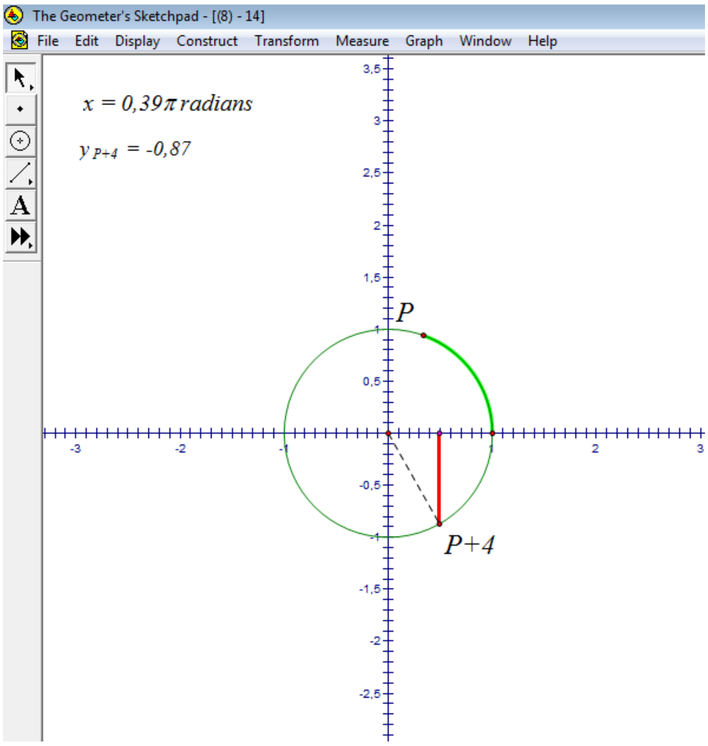


Figure 6.28. Construction of two points *P* and *P+4* as an example of the construction of two points on the unit circle with a constant difference in terms of sine

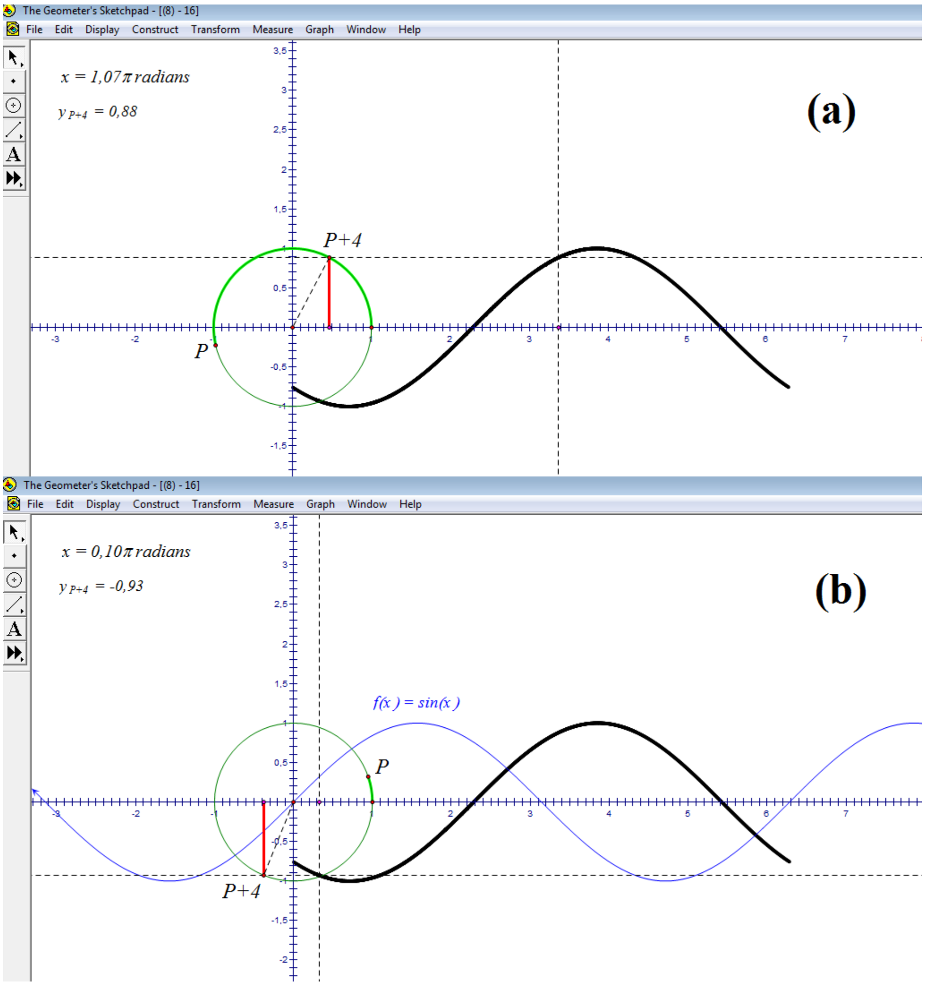


Figure 6.29. An example of the construction of the traced graph of the ordered-pair (x, y_{P+4}) in the *graphical register* so that y_{P+4} would refer to sin(x+4)

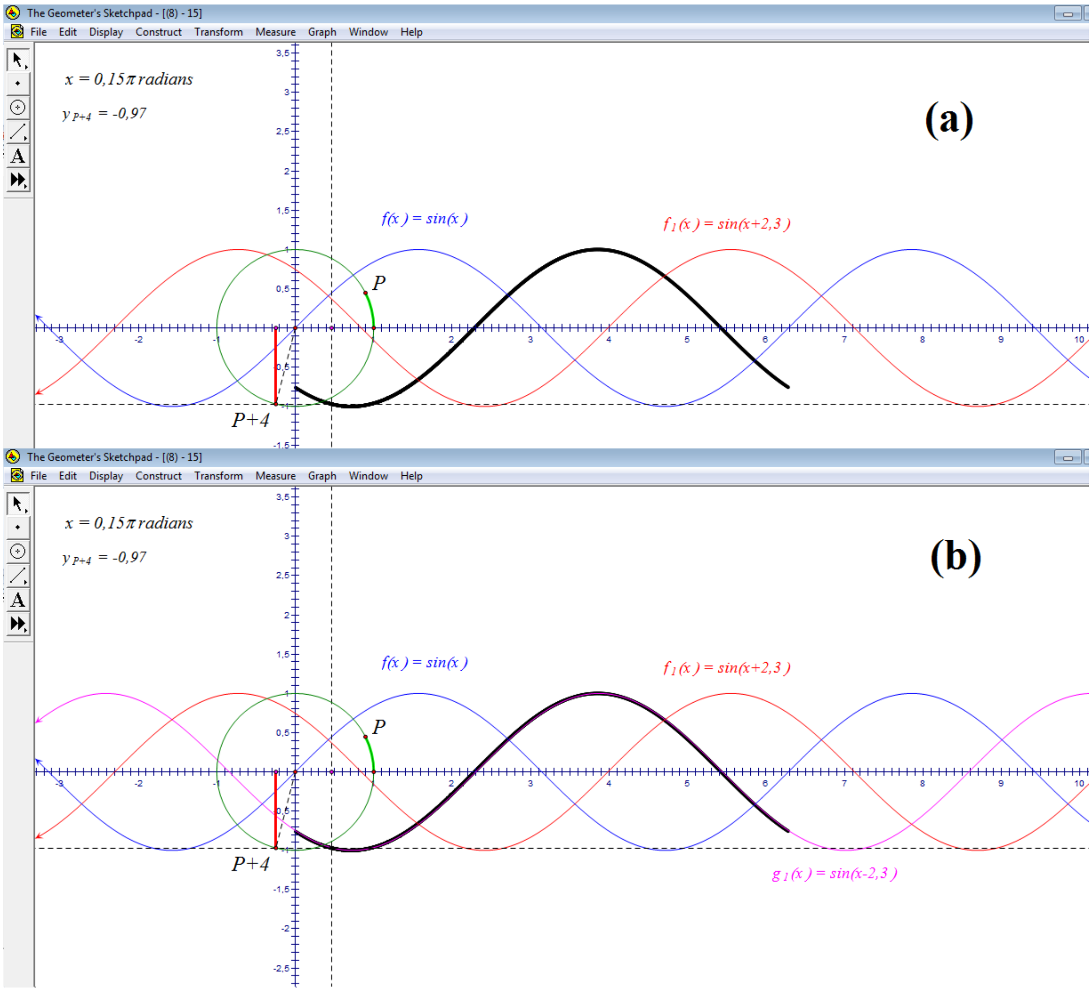


Figure 6.30. Zafer's conversion trouble on transformation of the *y*=sin(*x*±2.3) functions in the *symbolic register* into their representations in the *graphical register* when considering them as the parallel displacements of the sine graph along the *x*-axis by the 2.3-unit length

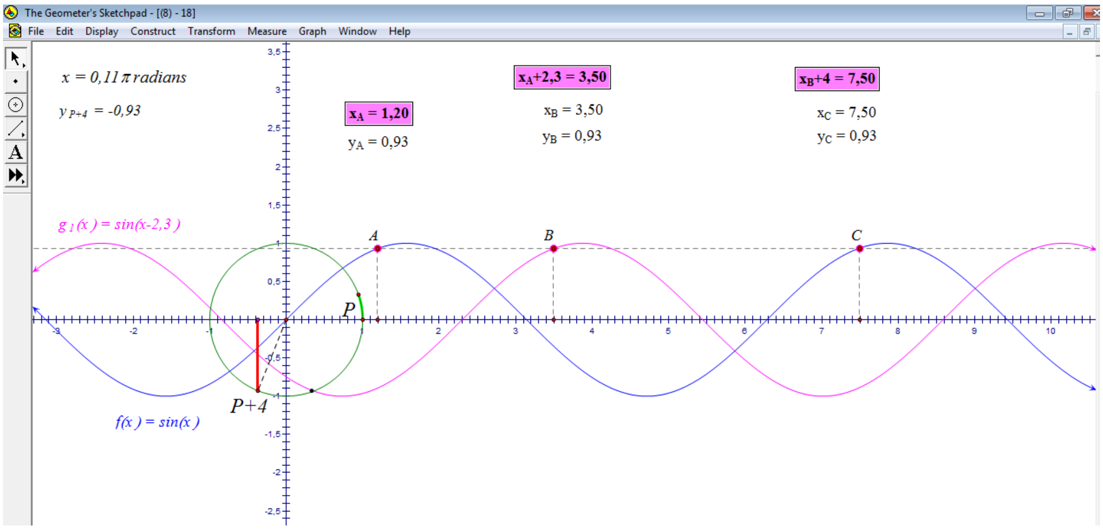


Figure 6.31. Construction of the parallel displacement of an arbitrary point A on the sine graph along the *x*-axis by a constant value –e.g., 2.3– in the positive direction; and then the new point's parallel displacement along the *x*-axis by the complement of this constant value into 2π –e.g., about 4.

[Defne & Ebru] Protocol 17

(*Defne and Ebru cooperatively constructed a unit circle located on the origin with the principal arc represented by the point P with respect to the researcher's directions; and then, rotated position of the point P in the positive direction by a (π+4)-radian fixed angle, labelled as P'. Where, (π+4)-radian was preferred by Defne and Ebru as a fixed rotation measure. Next, they were constructed traced graph of the (principal angle,*$y_{P'}$*,) ordered-pair.*)

Ebru: It [traced graph] resemble to sine.

Defne: Uh-huh… Differently, it [traced graph] started on the *y*-axis from a bit more above (*pointing with her pen the intersection point of the traced graph with the y-axis on the screen like in Figure 6.32(a)*) instead of zero (*pointing the origin*). That is, it starts from above of this amount (*dragging her pen up and down between the origin and the intersection point of the traced graph with the y-axis*).

Researcher: From graph menu, can you plot the sine graph.

Defne: (*Plotting the f(x)= sin(x) graph like in Figure 6.32(b)*)

Researcher: Is there a relation between this graph (*pointing the sine graph on the screen like in Figure 6.32(b)*) and this graph (*pointing the traced graph*)?

Ebru: It is translated toward a bit more this side (*indicating an interval through holding her hands vertically parallel to each other; and then, moving her hands horizontally rightward through keeping the interval between them same*).

Researcher: Which one is translated?

Ebru: This one (*pointing the traced graph on the screen like in like in Figure 6.32(b)*).

Researcher: How do you think?

Ebru: Because if I translate this (*putting her left hand's index finger on the positive peak point of the traced graph*)… …their peak points are same (*dragging her index finger horizontally rightward up to the peak point of the sine graph in the $(0,\pi)$ interval*)… …it [positive peak point of the traced graph] will come right on this (*pointing the peak point of the sine graph in the $(0,\pi)$ interval*).

Defne: Slope in here (*dragging her index finger on the traced graph from its intersection point with the y-axis up to its first intersection point with the sine graph*) decreased comparing to [slope in] here (*dragging her index finger on the sine graph from the origin up to its first intersection point with the traced graph*).

Researcher: Yes. Thus, if we translate them upward or downward, they couldn't coincide with each other.

Defne: Uh-huh.

Researcher: In which direction and which amount should it be translated?

Ebru: Is here about 3.8 (*putting the cursor on the negative peak point of the traced graph; and then indicating its projection point on the x-axis*)?

Researcher: Determine that.

Ebru: Defne, is here about 3.8 (*putting the cursor on the negative peak point of the traced graph; and then indicating its projection point on the x-axis*)?

Defne: I don't see it (*coming closer to the screen*).

Ebru: 3.8 (*putting the cursor on the negative peak point of the traced graph; and then indicating its projection point on the x-axis*)… …this is 4.8 (*putting the cursor on the peak point of the sine graph in the $(\pi,2\pi)$ interval; and then indicating its projection point on the x-axis*). So, does about 1 unit exist between them (*dragging the cursor left and right between 3.8 and 4.8 on the x-axis*)?

Researcher: How do you plot it?

Ebru: (*Plotting $g(x)=\sin(x+1)$ function's graph like in Figure 6.32(c)*) it isn't right. Maybe we add smaller one… …such as 0.8. (*Opening "plot new function" option of GSP, and entering $\sin(x+0.8)$*)

Researcher: (*Before Ebru's click "ok" to plot $\sin(x+0.8)$ graph*) stop! Before seeing this function's graph, why do you determine 0.8? Why it is smaller than 1 instead of greater than 1?

Ebru: It is because I think there is maximum 1-unit between them [sine graph and traced graph].

Researcher: Ok.

Ebru: (*Click ok, and $\sin(x+0.8)$ graph appeared on the screen closer to the traced graph*) it is better [than $\sin(x+1)$].

Researcher: Ok. We obtain this point (*pointing the point P'*) via rotating this point (*pointing the point P*).

Defne&Ebru: Uh-huh.

Researcher: By how much did we rotate?

Defne&Ebru: π plus 4…

Researcher: Defne, can you construct the graph through adding $\pi+4$.

Defne: Hmm! (*Plotting the $y=\sin(x+\pi+4)$ function's graph*)

Ebru: (*When the $y=\sin(x+\pi+4)$ function's graph appeared on the screen as coinciding with the traced graph*) aha! That's, it [graph of $y=\sin(x+\pi+4)$] was that [sine graph] starting from point *P'*, wasn't it?

Defne: That is, with the angle of *P* (*dragging the cursor on the principal arc in the counterclockwise direction*)… …[y-value] of the rotated *P'* (*putting the cursor on* *the point P'*)… …by rotation amount (*dragging the cursor from the point P to P' in the counterclockwise direction*)… …that is, $\pi+4$, when we add it [rotation amount] with the angle of *P*… …then it [$\sin(x+\pi+4)$] give us this graph (*pointing the traced graph*).

Researcher: Is here $\pi+4$ (*dragging her pen on the circle from the point P to P' in the counterclockwise direction*)?

Defne: No… But it [arc from the point *P* to the point *P'* in the counterclockwise direction] implies same where. That is, it [point *P*] went in this way (*dragging the cursor on the arc from the point P to P' in the counterclockwise direction*), or in this way (*figuring with the cursor initially a full round, and then the principal arc from the point P to the point P' in the counterclockwise direction*)… …these are same things.

Ebru: Uh-huh.

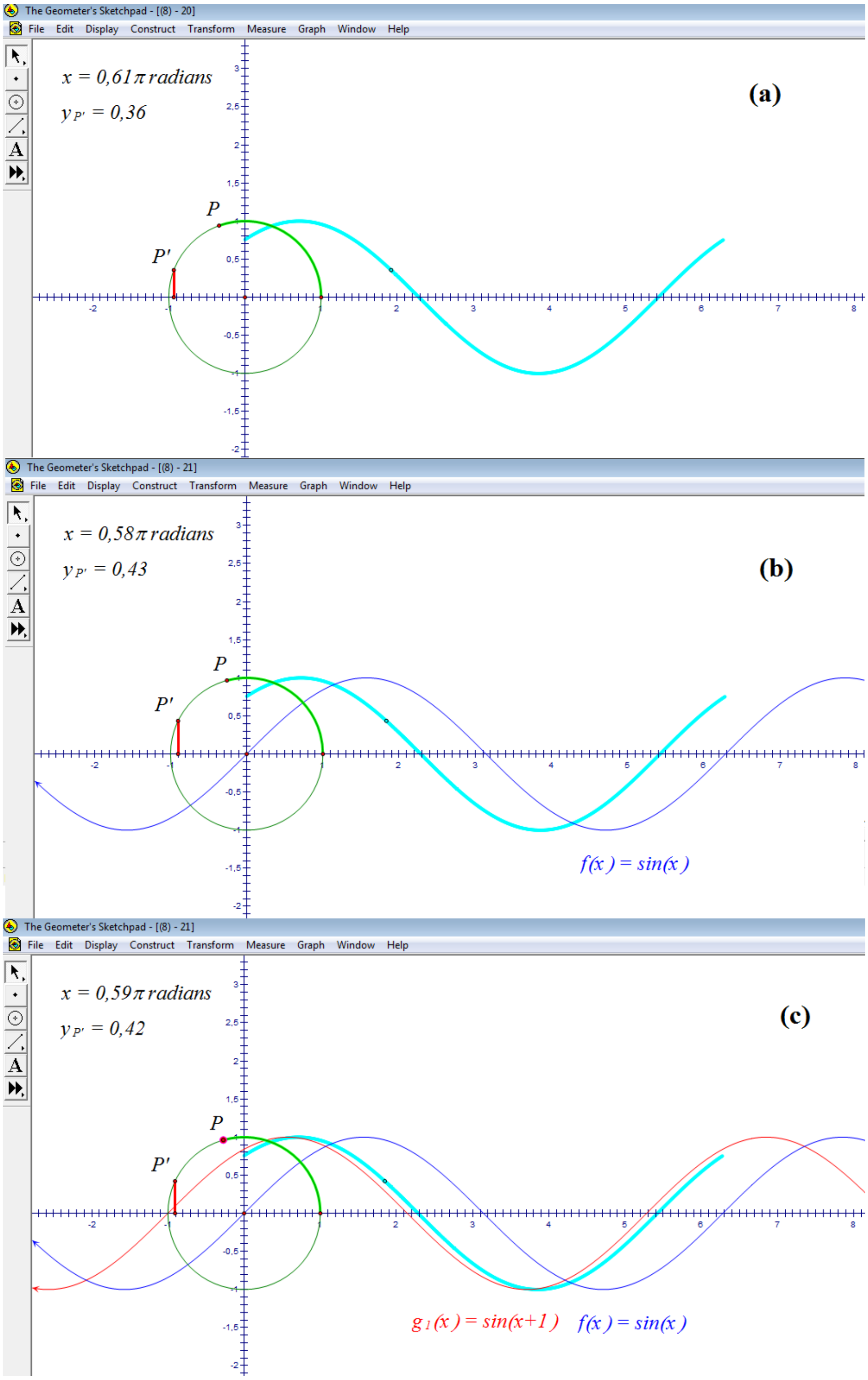


Figure 6.32. An example of the construction of the traced graph of the ordered-pair (x, $y_{P'}$) in the *graphical register* so that $y_{P'}$ would refer to sin(x+π+4)

[Defne & Ebru] Protocol 18

(*Defne and Ebru cooperatively constructed a unit circle located on the origin with the principal arc represented by the point P together with the reference right triangle corresponding to the principal angle with respect to the researcher's directions. And then, they rotated this reference right triangle by a π/2-radian fixed angle about the origin like in Figure 6.33.*)

Researcher: (*Dragging the point P*) look, I am dragging the point *P*. These reds (*pointing respectively, the opposite side of the reference right triangle and its rotated version through putting her index and thumb fingers on their end points*) are same in size, aren't they?

Ebru: Yes, same.

Researcher: …this (*putting her index and thumb fingers on the end points of the opposite side of the reference right triangle*) is only the π/2-rotated version of this (*rotating her fingers about the center in the counterclockwise direction until her fingers coincide with the endpoints of the rotated-red line segment through keeping the distance between them same*).

Defne: Uh-huh (*nodding her head up and down*).

Researcher: Well, I want to ask you… …if this angle (*dragging her index finger on the green arc in the counterclockwise direction*) is *x*, then what is this angle (*putting her index finger on the rotated position of the point P*)? We came from here (*putting her index finger on the point P*) to here (*dragging her finger in the counterclockwise direction until the rotated position of the point P*) as much as π/2…

Ebru: …*x* plus π/2.

Defne: Uh-huh (*nodding her heads up and down*).

Researcher: What of *x*+ π/2 (*dragging her index finger on the principal arc referring to x+ π/2*) is this (*dragging her index finger left and right on the rotated-red line segment*)?

Ebru: Its sine... No… Its cosine…

Defne: Ah! Yes.

(*The researcher constructed the reference right triangle in the second quadrant as the complement of the rotated-construction by π/2-radian of the reference right triangle in the first quadrant like in Figure 6.34.*)

…

Researcher: Then, this red part was sine of *x* (*dragging her index finger on, respectively, the opposite side of the reference right triangle in the first quadrant and the green arc in the counterclockwise direction on the screen like in Figure 6.34*)…

Defne: As we turn additionally π/2 more (*pointing with the cursor the π/2-rotated position of the point P on the screen like in Figure 6.34*), it [sin(*x*)] is equal to cosine (*dragging the cursor on the adjacent side of the reference right triangle in the second quadrant left and right*).

Researcher: Then, how do you express here in terms of cosine (*pointing the adjacent side of the reference right triangle in the second quadrant on the screen like in Figure 6.34*)?

Defne: Cosine of $\pi/2$ plus x.

Researcher: Please label it as you said.

Defne: (*Labelling the adjacent side of the reference right triangle in the second quadrant as* $\cos(x+\pi/2)$ *like in Figure 6.35*)

(*In the similar way, the researcher asked Ebru to determine the opposite side of the reference right triangle in the second quadrant in terms of sine; and then, label it in this way. Ebru determined and labelled it as* $\sin(x+\pi/2)$ *like in Figure 6.35.*)

…

Researcher: Is the length of this blue part (*dragging her index finger left and right on the adjacent side of the reference right triangle in the first quadrant on the screen like in Figure 6.35*) equal to the length of this blue part (*dragging her index finger up and down on the opposite side of the reference right triangle in the second quadrant*)?

Ebru: Uh-huh (*nodding her head up and down*).

Defne: Yes, equal.

Researcher: What about their signs?

Ebru: Both are positive.

Researcher: (*Dragging the point P in the counterclockwise direction so as to be in the second quadrant*) what about now?

Ebru: Do you ask about blues (*pointing the blue segment labelled as* $\sin(x+\pi/2)$)?

Researcher: Yes. That is, I mean… …these blue parts' lengths are always equal to each other (*pointing, respectively, the blue segments labelled as* $\cos(x)$ *and* $\sin(x+\pi/2)$ *in that case the point P in the second quadrant*). I want you to think about their signs, in other words, their equalities considering directed measures.

Ebru: Both are negative… …so, equal.

Researcher: (*Dragging the point P in the counterclockwise direction so as to be in the third quadrant*) what about when the point *P* is in the third quadrant?

Defne&Ebru: [Both] negative…

Researcher: (*Dragging the point P in the counterclockwise direction so as to be in the fourth quadrant*) now, we are in the fourth quadrant. What about now?

Defne: Positive (*pointing the blue segment labelled as* $\cos(x)$), positive (*pointing the blue segment labelled as* $\sin(x+\pi/2)$).

Ebru: Positive.

Researcher: Therefore, now that these blue parts (*pointing the adjacent side of the reference right triangle in the first quadrant and the opposite side of the reference right triangle in the second quadrant on the screen like in Figure 6.35*) are equal to each other everywhere on the unit circle, then, we can say that $\cos(x)$ is equal to $\sin(x+\pi/2)$ (*pointing the labels of the blue segments*) for every x (*writing on a paper* $\cos(x)=\sin(x+\pi/2)$), isn't it?

Defne&Ebru: Yes.

(*At that point, the researcher asked them to plot these two functions' graphs; i.e.,* $y=\cos(x)$ *and* $y=\sin(x+\pi/2)$ *in order to check their reasoning based on the equalities of* $\cos(x)$ *and* $\sin(x+\pi/2)$*. And then, discussions on the red segments were done in the similar way.*)

…

Researcher: If I turned from here (*putting her right hand's index and thumb fingers on the end points of the adjacent side of the reference right triangle from the point P; at that time, it was in the first quadrant like in Figure 6.35*) to here (*rotating her thumb finger by $\pi/2$ in the clockwise direction about her index finger on the origin*), what would you express this (*putting her fingers on the negative y-axis so as to indicate the rotated-position of the adjacent side of the reference right triangle from the point P*)?

Defne: $\pi/2$… …minus [$\pi/2$]… …umm, minus sine of… …x minus $\pi/2$.

Researcher: Then, $\cos(x)$ is equal to what?

Defne: $\sin(x-\pi/2)$.

Ebru: (*Nodding her head up and down*)

(*At that point, the researcher asked Defne to plot these two functions' graphs; i.e., $y=\cos(x)$ and $y=-\sin(x-\pi/2)$ in order to check their reasoning based on the equalities of $\cos(x)$ and $-\sin(x-\pi/2)$.*)

…

Researcher: Ebru, what about when I turned from here to here (*putting her right hand's index and thumb fingers on the end points of the adjacent side of the reference right triangle from the point P; and then, rotating them so as to indicate $3\pi/2$-rotation in the counterclockwise direction by the center*)?

Ebru: $3\pi/2$… …plus $3\pi/2$… …minus sine of… …x plus $3\pi/2$.

Defne: $\cos(x)$ (*lying her head on her right shoulder*) is equal to minus sine… x plus $3\pi/2$ (*changing her head's position from her right shoulder to her left shoulder as if following the $3\pi/2$-radian rotation path*).

Ebru: Yes.

Researcher: Please check that by graphing.

Ebru: (*Plotting $y=-\sin(x+3\pi/2)$. When its graph was appeared on the screen as coinciding with $y=\cos(x)$ graph*) uh-huh, true.

(*Similar discussions on transformations of sine and cosine of $(x\pm k\pi/2)$ in to sine and cosine of x symbolically were done*.)

…

Defne: How easy it was! I had been always confused about them [transformations of sine and cosine of $(x\pm k\pi/2)$ in terms of x symbolically].

Ebru: Me too... (*After 3-second pause*) today was very useful.

Defne: Exactly!

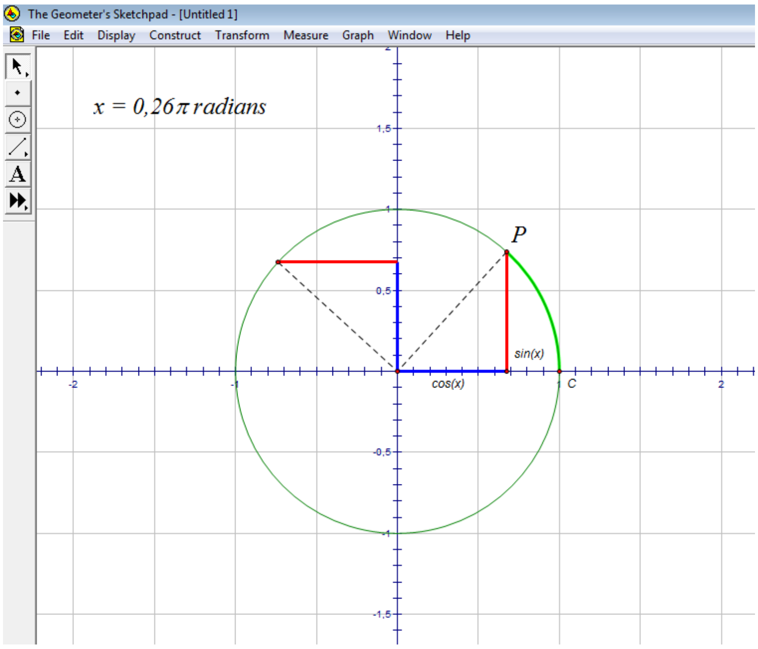


Figure 6.33. Transformed-construction of the reference right triangle in the (*unit*) *circle register* through the rotation about the origin by the π/2-radian fixed angle

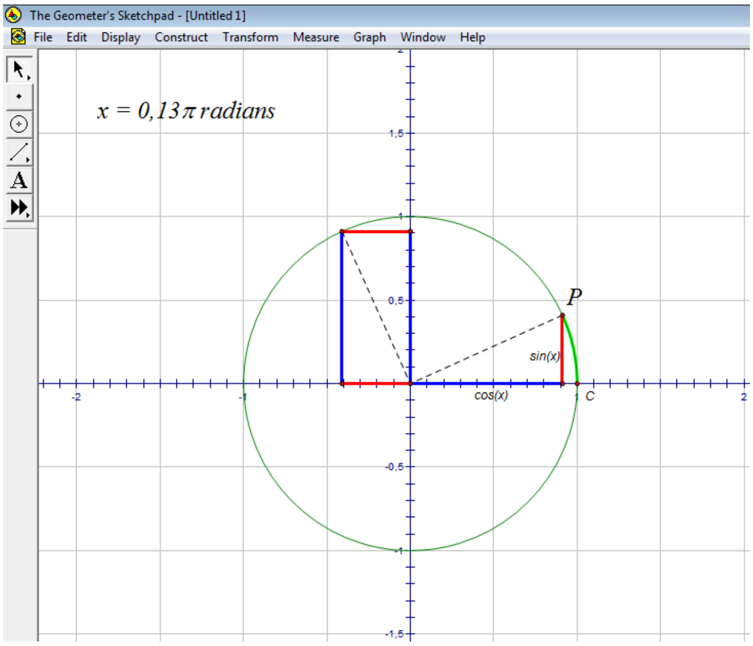


Figure 6.34. Construction of the reference right triangle in the second quadrant as the complemet of the rotated-construction by π/2-radian of the reference right triangle in the first quadrant

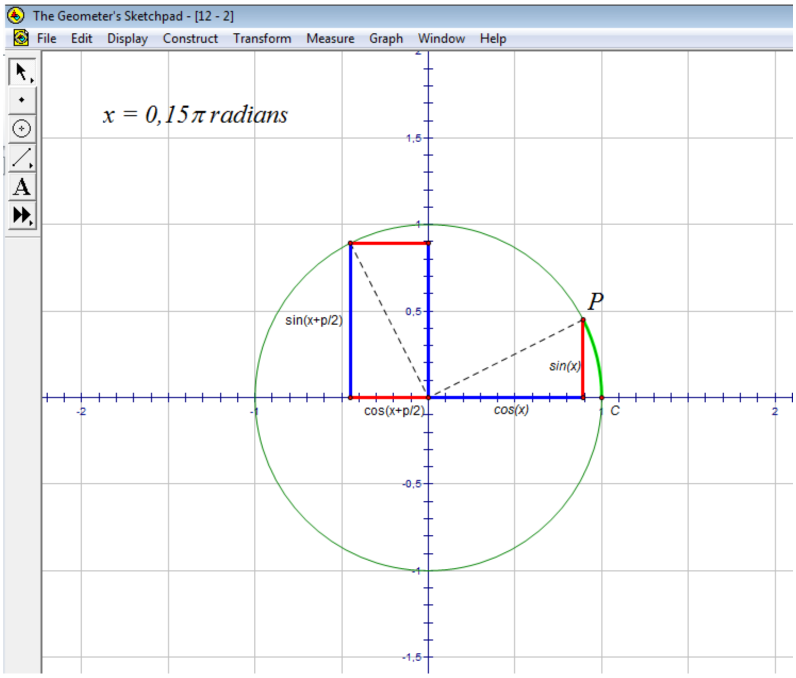


Figure 6.35. Labelling the legs of the reference right triangle in the second quadrant which is the complemet of the rotated-construction by π/2-radian of the reference right triangle in the first quadrant

[Defne & Ebru] Protocol 19

(*Defne and Ebru cooperatively constructed a unit circle located on the origin with the principal arc represented by the point P with respect to the researcher's directions; and then, rotated position of the point P in the positive direction by a 1-radian fixed angle, labelled as P+1. Where, 1-radian was preferred by the researcher as a fixed rotation measure in the light of the analysis result from Task 10. And finally, they constructed the adjacent side of the reference right triangle corresponding to the point P+1 like in Figure 6.36.*)

Researcher: We are dealing with this blue part (*pointing the blue line segment with her pen*). What does this (*dragging her pen on the blue line segment left and right*) call to your mind?

Defne&Ebru: Cosine…

Researcher: Cosine of whom?

Defne: x plus 1 (*dragging her index finger in the counterclockwise direction; initially, on the green arc; and then; on the arc from the point P to the point P+1*).

Ebru: Uh-huh.

(*At that point, the researcher asked students to measure abscissa of the point P+1; and then, to calculate cos(x+1); and finally, to control of these two measures under the manipulation of the point P in order to foster their understanding about these two measures' equalities.*)

…

Researcher: Cosine of x (*pointing with the cursor the pink line segment on the screen like in Figure 6.36*) takes the values between whom?

Ebru: ...minus one and plus one.

Defne: Yes.

Researcher: What about cosine of x+1 (*pointing with the cursor the blue line segment*)?

Defne: Same… …minus one and plus one…

Ebru: Uh-huh (*nodding her head up and down*).

Researcher: Ok. (*When dragging the point P in the first quadrant of the unit circle in the counterclockwise direction*) at which point, *P* or *P+1*… …cosine reaches the minus 1 value earlier than the other?

Defne&Ebru: P+1.

Researcher: Ok (*dropping the point P so as the point P+1 to be on the negative x-axis*). After how much turning, will the point *P* reach here (*dragging the cursor on the arc from the point P to the point P+1 in the counterclockwise direction*)?

Defne: After 1 unit [radian] turning (*dragging her index finger on the arc from the point P to the point P+1 in the counterclockwise direction*).

Ebru: (*Nodding her head up and down*)

Researcher: Why?

Ebru: Because we constructed it [point *P+1*] via rotating [point *P*] by 1 [radian]…

(*At that point, the researcher asked students to plot both y=cos(x) and y=cos(x+1) functions' graphs like in Figure 6.37.*)

…

Researcher: Which graph … …reaches the minus 1 value earlier than the other? This one or this one (*pointing respectively cos(x) and cos(x+1) functions' graphs on the screen like in Figure 6.37*)?

Ebru: This one (*pointing the negative peak point of the blue graph in the (1,3) interval*).

Defne: Yes, this reaches one unit later (*pointing the negative peak point of the pink graph in the (π/2,3π/2) interval; and then, the negative peak point of the blue graph in the (1,3) interval on the screen like in Figure 6.37*).

Ebru: Uh-huh (*nodding her head up and down*).

Researcher: What do you say if you compare cos(x+1) graph with cos(x) graph (*pointing respectively blue and pink graphs on the screen like in Figure 6.37*)?

Defne: It [graph of cos(x+1)] went one-unit leftward.

Ebru: (*Nodding her head up and down*)

Researcher: Why?

Ebru: Because earlier blue reaches to -1 than the other (*pointing -1 on the x-axis on the screen like in Figure 6.37*). Here also blue one reaches -1 earlier (*pointing the negative peak point of the blue graph in the (1,3) interval*).

Defne: Uh-huh (*nodding her head up and down*).

...

Researcher: Well, cos(x) graph horizontally translated one-unit left (*indicating an interval through holding her hands vertically parallel to each other; and then,*

moving her hands horizontally leftward through keeping the interval between them same).
Defne&Ebru: (*Nodding their heads up and down*)
Researcher: Did its period change?
Defne: No.
Ebru: (*Shaking her head right and left*)
Defne: Again 2π.
Ebru: Uh-huh.

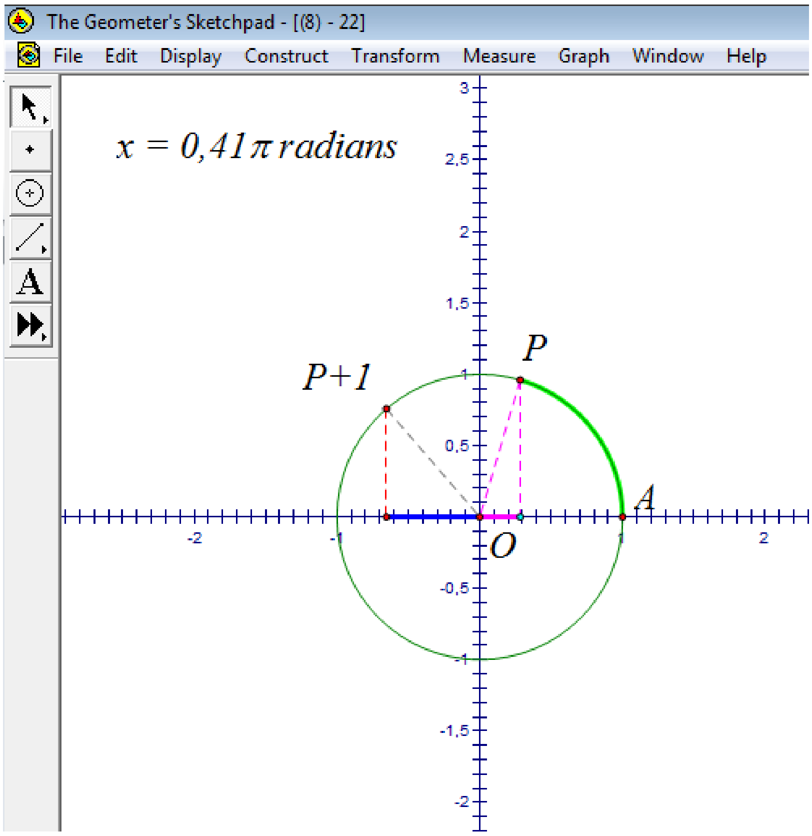


Figure 6.36. Construction of two points *P* and *P+1* as an example of the construction of two points on the unit circle with a constant difference in terms of cosine

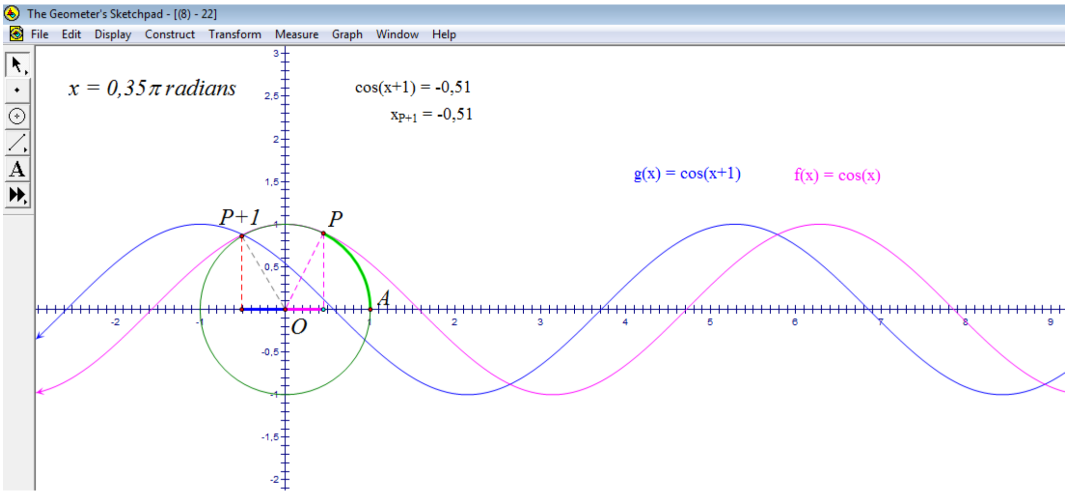


Figure 6.37. Construction of *y*=cos(*x*) and *y*=cos(*x*+*1*) functions in each register

6.1.3.2. Parallel-displacement along the *x*-axis in graphical register

In addition to the role of the changed-arc with a constant difference in the (*unit*) *circle register*, its conversion into the *graphical register* was another important focus of Task 10 by means of a function defined in the (*unit*) *circle register* from the (green) principal arc to the *y*-component of the reference point of the changed-arc with a constant difference from the reference point of the principal arc on the unit circle (*Figure 6.28*).

When the traced graph of the new function was constructed in Task 10, students brought its correlation with sine as well as differentiation from sine up for discussion (e.g., lines 44-53 in [Cemre & Zafer] Protocol 17; lines 1-12 in [Defne & Ebru] Protocol 17). Where, on the one hand, Cemre and Ebru associated the traced graph with sine based on its shape without going into details in terms of its position on the coordinate system (line 48 in [Cemre & Zafer] Protocol 17; line 7 in [Defne & Ebru] Protocol 17). On the other hand, Zafer and Defne differentiated the traced graph from sine based on its *y*-intercept (lines 49, 53-56 in [Cemre & Zafer] Protocol 17; lines 8-12 in [Defne & Ebru] Protocol 17). At that point, the researcher asked them to plot the sine graph in order to provoke them to compare and contrast these two graphs in a more detailed way (lines 60-61 in [Cemre & Zafer] Protocol 17; lines 13 in [Defne & Ebru] Protocol 17).

When the sine graph appeared on the screen together with the traced graph (*Figure 6.29*(*b*) and *Figure 6.32*(*b*)), except Defne, all other students reasoned about these two graphs as the parallel displacement of each other along the *x*-axis (e.g., lines 62-71 in [Cemre & Zafer] Protocol 17; lines 14-31 in [Defne & Ebru] Protocol 17). Where, their actions' and language imply that Ebru reasoned about the sine graph (*Figure 6.32*(*b*)) as the parallel displacement of the traced graph along the *x*-axis in the positive direction (lines 23-27, 36-46 in [Defne & Ebru] Protocol 17); Cemre reasoned about the traced graph (*Figure 6.29*(*b*)) as the parallel displacement of the sine graph along the *x*-axis in the positive direction (e.g., lines 62-66 in [Cemre & Zafer] Protocol 17); and Zafer reasoned in two ways: (*i*) the traced graph (*Figure 6.29*(*b*)) as the parallel displacement of the sine graph along the *x*-axis in the positive direction, as well as (*ii*) the sine graph (*Figure 6.29*(*b*)) as the parallel displacement of the traced graph along the *x*-axis in the negative direction (e.g. lines 67-71 in [Cemre & Zafer] Protocol 17).

At that point, the researcher encouraged them to reason about the parallel displacement idea in a more detailed way (e.g., line 72 in [Cemre & Zafer] Protocol 17; line 35 in [Defne & Ebru] Protocol 17). Their initial reasoning step was to determine the parallel displacement amount through focusing on the distance between the corresponding points of two graphs that were selected according to their preferences. For example, while Cemre and Zafer determined the distance as about 2.3 unit through focusing on the distance between the origin[45] and the first intersection point of the traced graph with the *x*-axis (e.g. lines 73-105 in [Cemre & Zafer] Protocol 17), Ebru determined the distance as about 1 unit through focusing on the distance between the negative peak points of the traced graph and the sine graph in the $(0,2\pi)$ interval (e.g. lines 36-46 in [Defne & Ebru] Protocol 17).

[45] The origin is one of the intersection points of the sine graph with the *x*-axis.

Their next reasoning step was to define the function represented by the traced graph regarding sine in the *symbolic register*. During that phase, they were aware that these distances would affect the input of sine as a constant (e.g., lines 73-110 in [Cemre & Zafer] Protocol 17; lines 35-50 in [Defne & Ebru] Protocol 17). When thinking on the sign of this constant in the *symbolic register*, from students' language and actions, the researcher inferred that their concept images on the graphs' parallel displacements along the *x*-axis in the *graphical register* were including a conjecture on the conversion of the positive [negative] direction of the parallel displacement along the *x*-axis in the *graphical register* into the positive [negative] sign of the constant affecting the input variable in the *symbolic register* (e.g., lines 76-122, 124-132, 136-166 in [Cemre & Zafer] Protocol 17; lines 15-27, 35-50 in [Defne & Ebru] Protocol 17).

This wrong-conjecture, on the one hand, caused Cemre's and Zafer's confusion on the conversion of the traced graph in the *graphical register* into its symbolic representation in terms of sine (lines 124-129, 136-166 in [Cemre & Zafer] Protocol 17). In other words, they confused about the location of the graph of $y=\sin(x+2.3)$ on the left with respect to the sine graph on the coordinate system. However, on the other hand, this conjecture did not cause any confusion for Ebru as a consequence of her incoherent-assumptions on the *source*-function of the transformation in the *graphical register* and the *symbolic register*. That is to say, she assumed in the *graphical register* the traced graph (*Figure 6.32*(*b*)) as the *source*-function by means of the parallel displacement along the *x*-axis in the positive direction (lines 23-27, 35-46 in [Defne & Ebru] Protocol 17); conversely, in the *symbolic register,* she manipulated the sine function (i.e., *target*-function of transformation) through adding the parallel displacement amount between two graphs to the input of sine as a constant so as to express the traced-function (lines 15-27, 35-50 in [Defne & Ebru] Protocol 17). So, her conjecture on the conversion of the **positive** direction of the parallel displacement along the *x*-axis in the *graphical register* into the **positive** sign of the constant affecting the input variable in the *symbolic register* did not bring forth any confusion as a consequence of her failure to preserve her assumption on the *source*-function and *target*-function of transformation coherently in the *graphical register* and the *symbolic register*. Although the researcher was aware of Ebru's this problematic reasoning, she did not provoke Ebru to reason about this aspect in a more detailed way in the discussion process of Task 10 as a consequence of her group friend's, Defne's, inability to reason about these two graphs as the parallel displacement of each other along the x-axis.

Throughout Task 10, differently from others, Defne reasoned about these two graphs through comparing their slopes with each other but focusing only on their restricted parts from their y-intercepts up to their first intersection points (lines 28-31 in [Defne & Ebru] Protocol 17). Despite of the researcher's provocation her into the reasoning based on translation (e.g. lines 32-35 in [Defne & Ebru] Protocol 17), she made no interpretation about these two graphs as the parallel displacement of each other along the x-axis throughout Task 10 even when Ebru was reasoning in that way (lines 35-58 in [Defne & Ebru] Protocol 17). Based on her keeping silent, the researcher inferred that she did not recognize that the traced graph had the same shape with the sine graph. In other words, she did not aware that the traced graph and the sine graph would coincide with each other if their positions on the coordinate plane were changed horizontally in the appropriate amount and appropriate direction. Therefore, the researcher asked Defne to explain on a paper her understandings from the discussions of Task 10.

Initially, Defne wrote $f(x)=\sin(x+5)$ function symbolically (*Figure 6.38*(*1*)); and then, drew $y=\sin(x)$ function's graph in the $(0,2\pi)$ interval (*Figure 6.38*(*2*)); finally, constructed another graph so as to refer the graph of $f(x)=\sin(x+5)$ in the same interval (*Figure 6.38*(*3*)) through uttering "it [graph of $\sin(x+5)$] would start a bit more above (*putting her pen on the y-axis a bit more above the origin but below 1*)… …its slope would decrease (*drawing a curve starting from this point through decreasing its slope comparing with sine*) and it [graph of $\sin(x+5)$] would go toward 1 like that (*continuing her drawing so as to coincide with the peak point of the sine graph in the (0,π) interval*)." Her constructions (*Figure 6.38*) and articulations imply that her reasoning in the *graphical register* about a function defined symbolically as $y=\sin(x+c)$ was including an over-generalization of her comparison between the traced graph and the sine graph in GSP environment (*Figure 6.32*(*b*)). As it was mentioned before, her comparison in GSP environment was a restricted-comparison of these two graphs instead of comparing them as a whole. That is to say, she compared the traced graph with the sine graph in terms of their slopes and *y*-intercepts but based only on their restricted-parts from their *y*-intercepts up to their first intersection points with each other; instead of comparing their graphs as a whole. At that point, the researcher inferred that her restricted focus on these graphs prevented her from seeing these two graphs as a parallel displacement of each other. So, considering Ebru's and Defne's quite different reasoning focusses on comparison of two graphs (parallel displacement *versus* their slopes and *y*-intercepts in a restricted interval), as well as their problematic reasoning parts (regarding *source* and *target* functions of the parallel displacement *versus* over generalization of slopes and *y*-intercepts) in the *graphical register*, the researcher preferred to postpone discussions in a more detailed way on these aspects to following episodes, namely, Task 12 and 15.

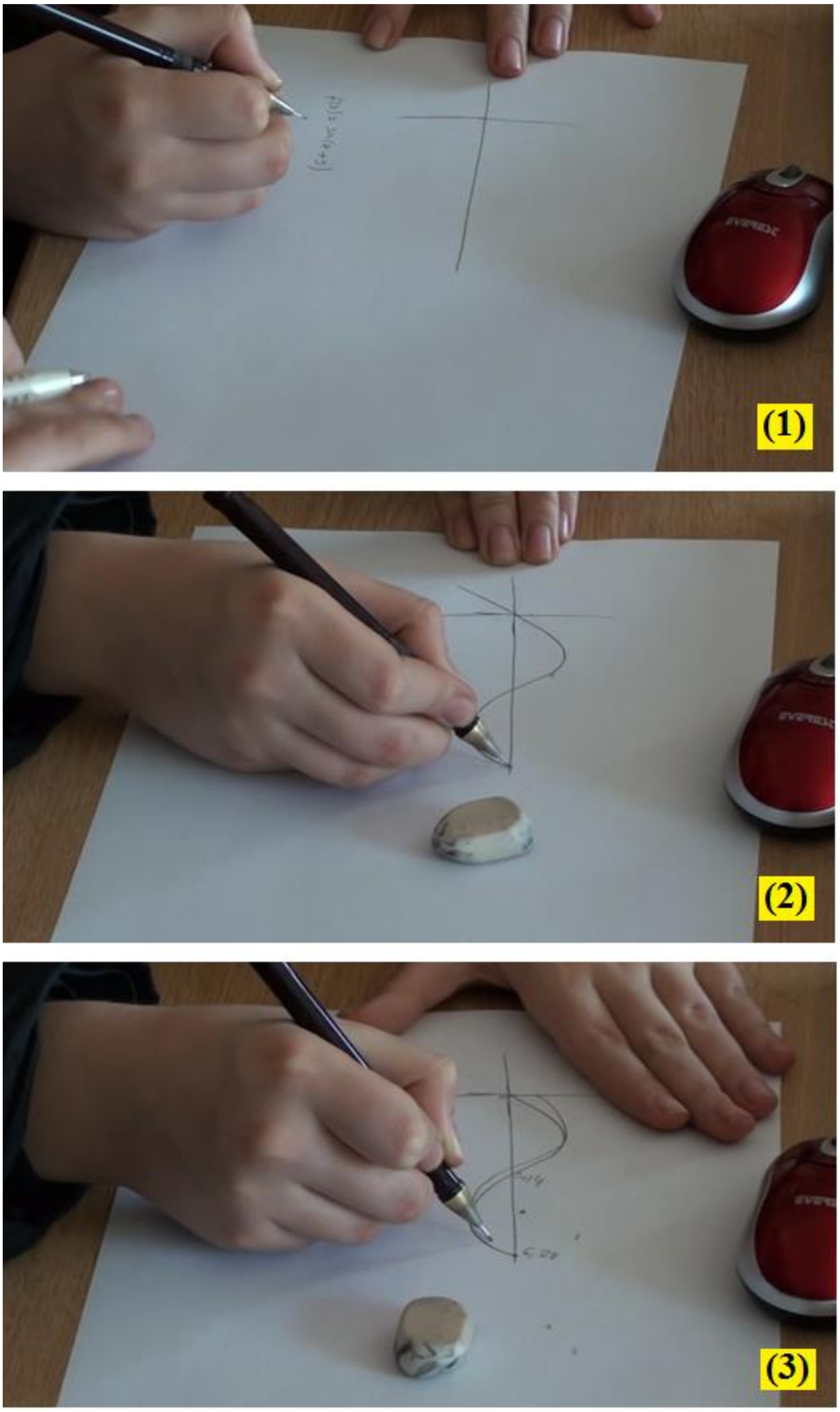


Figure 6.38. At the end of Task 10, Defne's constructions on a paper when reasoning about $f(x)$=sin(x+5) function's graph

On the other hand, as a consequence of Cemre's and Zafer's confusion about the location of the graph of y=sin(x+2.3) [y=sin(x-2.3)] on the left [right] with respect to sine, the researcher provoked them to reason about corresponding points of two graphs in a more detailed way. For this purpose, the researcher constructed an arbitrary point on the sine graph, and then, its two purposively-determined parallel displacements along the x-axis taking "measure", "calculate" and "plot as (x,y)" advantages of GSP so as one of them to be on the graph of y=sin(x-2.3) and the other to be on the next periodic wave of the sine graph (see points *A*, *B* and *C* in *Figure 6.31*). Throughout this construction process, it was observed that neither Cemre nor Zafer encountered a trouble with their conjecture that associated the positive [negative] direction of the parallel displacement along the x-axis with the positive [negative] sign of the constant term added to x in the *symbolic register* (lines 159-204 in [Cemre & Zafer] Protocol 17). This was because this construction process was including the parallel displacement of the points along the x-axis, which required only expressing their abscissas as an additive structure with respect to the abscissa of the point *A* rather than expressing their ordinates dependently on their abscissas in the *symbolic register*. For example, when constructing the point *B* as the parallel displacement of the point *A* along the x-axis by 2.3 unit in the positive direction (*Figure 6.31*), the abscissa of the point *B* was expressed as x_A+2.3, and its ordinate was considered as equal to y_A without reasoning about it dependently on x_A or x_B (lines 163-180 in [Cemre & Zafer] Protocol 17).

However, conversion of the parallel displacement of these two graphs along the x-axis in the *graphical register* into the *symbolic register* required identifying the ordinate of the point *B* dependently on its abscissa with respect to sine (e.g. lines 205-240 in [Cemre & Zafer] Protocol 17). In order to ease their identification process, the researcher motivated Cemre and Zafer to consider the abscissa of the point *B* as x (lines 205-210 and 227-236 in [Cemre & Zafer] Protocol 17), and then, reason about the ordinate of the point *B* with respect to sine dependent on x. In this reasoning process, the researcher provoked them to reason through changing their attention-focus hierarchically in a rectangular path among four points in the *graphical register*: (*i*) the point *B* (*Figure 6.31*) (*ii*) its projection point on the x-axis, (*iii*) the projection point of the point *A* [point *C*] on the *x*-axis (*iv*) the point *A* [point *C*] (*v*) the point *B* (lines 211-208 and 215-228 in [Cemre & Zafer] Protocol 17). This hierarchical-rectangular path of their attention-focuses caused their identification of the pink graph (*Figure 6.31*) with respect to sine as y=sin(x-2.3) [y=sin(x+4)] in the *symbolic register* (lines 194-226 and 232-240 in [Cemre & Zafer] Protocol 17). Although they were satisfied with this identification process in the *symbolic register*, it seemed that they had not been yet satisfied with the location of y=sin(x-2.3) [y=sin(x+4)] on the right [left] with respect to sine in the *graphical register* (e.g., lines 241-244 in [Cemre & Zafer] Protocol 17). Thus, in Task 12 [Task 15], the researcher determined to provoke students to reason about the parallel displacements of the sine [cosine] graph along the *x*-axis in the positive or negative direction in the *graphical register* together with its meaning in the (*unit*) *circle register*.

From Task 11 forward in all tasks, it was observed that all students were able to express the ordinate [abscissa] of an arc as sine [cosine] of this arc but considering it dependently on another arc in the (*unit*) *circle register* (see the previous sub-heading *Changed-arc with a constant difference in (unit) circle register*). This reasoning based on making sense of the symbolic expressions in the form of sin(x+c) and cos(x+c) in the (*unit*) *circle register* prompted a distinct shift on students' reasoning about these expressions in the *graphical register*. That is to say, for example in Task 15, Defne was able to reason about the graph of cos(x+1) as the parallel displacement of the graph of cos(x) by 1-unit along the *x*-axis in the negative direction (e.g. lines 51-59 in [Defne & Ebru] Protocol 19) based on the patterns on their actions in the (*unit*) *circle register* (e.g. lines 21-50 in [Defne & Ebru] Protocol 19). Moreover, both Defne and Ebru interpreted the positive [negative] constant "*c*" as the *c*-unit length before arrival [after arrival] in a value in the *graphical register* as a consequence of the *c*-radian rotation in the counterclockwise [clockwise] direction in the (*unit*) *circle register* (e.g., 28-41, 43-59 in [Defne & Ebru] Protocol 19). In Task 12 and Task 15, Cemre and Zafer were able to interpret the positive [negative] constant "*c*" in the same way with Defne and Ebru. Giving "after arrival" ["before arrival"] meaning in the (*unit*) *circle register* to the parallel displacement along the *x*-axis in the positive [negative] direction in the *graphical register* led to Cemre's and Zafer's satisfaction with the negative [positive] constant "*c*" in the *symbolic register*. That is to say, the role of the coefficient *c* in the *graphical register* made sense for students only when they were able to associate its absolute value with the fixed-rotation amount and its sign with the direction of the rotation in the (*unit*) *circle register* simultaneously.

6.1.4. Visual Feature Opposition B

Visual Feature (B) corresponds to changed-arc through folding the angle variable in the (*unit*) *circle register* and compressed/stretched-wavelength in the *graphical register* so that these visual features' opposition corresponds to the choice presence/absence of a coefficient of the input of sine and cosine in the *symbolic register*.

6.1.4.1. Changed-arc through folding angle in (unit) circle register

Task 11 began with the constructions of two points on the unit circle, namely, *P* and *P′*, by the aid of "rotate" option of GSP so as the point *P′* to have two-fold principal measure of that of the point *P* (*Figure 6.39(a)*), and then, the constructions of the perpendicular line segments from these points to the *x*-axis (*Figure 6.39(b)*). At that point, the researcher provoked students to identify the perpendicular line segment from the point *P′* in terms of sine (e.g., lines 22-31 in [Cemre & Zafer] Protocol 18; lines 1-24 in [Defne & Ebru] Protocol 20). Where, it was observed that considering the principal measure of the point *P* as *x*, all students were able to identify (*i*) the principal arc corresponding to the point *P′* as *2x*, and then, (*ii*) its ordinate as sine of 2*x* through considering the definition of sine (from the arc corresponding to a point on the unit circle into its ordinate) (e.g., lines 22-33 in [Cemre & Zafer] Protocol 18; lines 1-36 in [Defne & Ebru] Protocol 20). In other words, they were able to reason about "twofold" relation between the principal arcs of the points *P* and *P′*, and interpret their *y*-values by means of their twofold-related principal angles.

In addition, they were able to compare their full-round turnings on the unit circle with each other (e.g., lines 72-96, 105-112 in [Cemre & Zafer] Protocol 18; lines

68-95 in [Defne & Ebru] Protocol 20). That is, they interpreted one full-round turning of the point P [point P'] through associating with two full-round [a half-round] turning of the point P' [point P]. Moreover, they were able to extend this comparison beyond for the points with "twofold"-related principal arcs defined in the (*unit*) *circle register* into the comparison of the full-rounds of the points referring to sine of x and sine of a natural number multiple of x defined in the *symbolic register*. For example, when reasoning about the meaning of $\sin(3x)$ [$\sin(5x)$], students were able to convert $\sin(3x)$ [$\sin(5x)$] in the *symbolic register* into the reference point referring to $3x$ [$5x$] in the (*unit*) *circle register* comparing with the point P referring to x (e.g., lines 95-102 in [Defne & Ebru] Protocol 20). And then, they were able to compare its full-rounds with the full-rounds of the reference point referring to x (e.g., lines 127-160, 182-185 in [Defne & Ebru] Protocol 20).

In the same way, they were able to reason about $\sin(x/2)$. That is, they reasoned that while the reference point referring to x completed a full-round, the reference point referring to $x/2$ completed a half-round (e.g., lines 162-181 in [Cemre & Zafer] Protocol 18), or reversely that while the reference point referring to $x/2$ completed a full-round turning, the reference point referring to x completed two full-rounds (e.g. lines 215-225 in [Defne & Ebru] Protocol 20). Moreover, in Task 11, Cemre and Zafer were able to extend this reasoning onto the negative multiples of x through attributing the meaning of the negative sign in the *symbolic register* to the clockwise direction of the rotation in the (*unit*) *circle register* (e.g., lines 183-204 in [Cemre & Zafer] Protocol 18). In other words, they had just defined $y=\sin(-x)$ in the (*unit*) *circle register* as a function mapping the angle of a point on the unit circle to the ordinate of its reflection point regarding the x-axis (lines 183-184, 193-197 in [Cemre & Zafer] Protocol 18). Also, when trying to articulate this function in the *graphical register*, they indicated that its graph would be with the same shape as the graph of $y= -\sin(x)$ through considering it as the reflection of the graph of $y=\sin(x)$ regarding the x-axis (lines 183-202 in [Cemre & Zafer] Protocol 18) in the same way as they had reasoned in Task 8 about the graph of $y= -\sin(x)$ (e.g., lines 76-81 in [Cemre & Zafer] Protocol 15). Therefore, the researcher inferred that Cemre and Zafer combined $y=\sin(-x)$ with $y= -\sin(x)$ in the *graphical register*. On the other hand, in Task 12, they had just defined *y*= –sin(*x*) in the (*unit*) *circle register* as a function mapping the angle of a point on the unit circle to the ordinate of its (±π)-rotated position around the origin (see *Changed-arc with a constant difference in (unit) circle register* heading). Therefore, while they combined *y*=sin(–*x*) with *y*= –sin(*x*) in the *graphical register*, they were also be able to dissociate them in the (*unit*) *circle register*.

With the other group of students, Defne and Ebru, as a consequence of their troubles on the period issue at the beginning of Task 11 (see *Periodicity as pattern based on behaviors of trigonometric functions* heading), the researcher preferred to postpone the discussions about the meaning of the negative coefficient of the input variable in the (*unit*) *circle register* to Task 16. In task 16, when reasoning about *y*=cos(–*x*) in the (*unit*) *circle register,* it was observed that all students were able to attribute the meaning of the negative multiples of *x* in the *symbolic register* to the clockwise direction of the rotation in the (*unit*) *circle register* in the similar manner that Cemre and Zafer reasoned about *y*=sin(–*x*) in Task 11 (e.g., lines 183-184, 193-197 in [Cemre & Zafer] Protocol 18).

Moreover, as a consequence of their developments throughout Task 12 in terms of making sense of sine [cosine] as cosine [sine] (see *Changed-arc with a constant difference in (unit) circle register*), all students were able to transfer their interpretations made on sine in Tasks 8, 9, 10 and 11 onto cosine, respectively, in Tasks 13, 14, 15 and 16. Thus, all these reasoning mentioned above in Task 11 in the (*unit*) *circle register* was observed in Task 16 in terms of cosine, as well. Henceforward, the subsequent progress of students' concept images on the visual feature opposition (B) in the (*unit*) *circle register* was presented under "*Composed-Coefficients' Visual Oppositions*" heading.

6.1.4.2. Compressed/Stretched-wavelength in graphical register

In addition to the role of the changed-arc through folding the angle variable in the (*unit*) *circle register*, its conversion into the *graphical register* was another important focus of Task 11 [Task 16] by means of the function defined from the angle of the initial-arc to the y-component [x-component] of the reference point of the changed-arc (e.g., lines 34-37 in [Cemre & Zafer] Protocol 18).

For example, in Task 11, a new function from the angle of the point P to the y-component of the point P' was considered as an initial example to model the changed-arc as “twofold” (*Figure 6.39*(*b*)). Where, it was observed that students were able to reason correctly about “twofold” situation in the (*unit*) *circle register* (see *Changed-arc through folding angle in (unit) circle register* heading). At that point, the researcher encouraged them to reason about the conversion of the “twofold” idea into the *graphical register*. For this purpose, (x,sin(x)) and (x,sin(2x)) ordered pairs were constructed in the *graphical register* as dynamically-linked to their correspondences in the (*unit*) *circle register* and the *symbolic register* (lines 34-41 in [Cemre & Zafer] Protocol 18; lines 37-38 in [Defne & Ebru] Protocol 20).

When these ordered pairs were appeared on the screen, unlike the group of Defne and Ebru, Cemre and Zafer tried to associate their coordinates with their correspondences on the unit circle before the construction of these ordered pairs’ traced graphs (lines 41-49 in [Cemre & Zafer] Protocol 18, lines 37-41 in [Defne & Ebru] Protocol 20), as well as to estimate the functions indicated by these ordered pairs in the *graphical register* (lines 48-71 in [Cemre & Zafer] Protocol 18). Moreover, Cemre and Zafer tried to make sense of the appeared-position of (x,sin(2x)) ordered-pair considering the position of (2x,sin(2x)) with respect to x and the graph of y=sin(x) (lines 52-67 in [Cemre & Zafer] Protocol 18). That is to say, they reasoned about the position of (x,sin(2x)) through changing their attention-focus hierarchically in a rectangular path among four points in the *graphical register*: (*i*) the point (x,sin(2x)) (*Figure 6.43*) (*ii*) the point (x,0) (*iii*) the point (2x,0) (*iv*) the point (2x,sin(2x)), (*v*) the point (x,sin(2x)) (lines 52-67 in [Cemre & Zafer] Protocol 18). This rectangular reasoning path was indicating a compression of the act of y=sin(x) function in 2-unit-length interval into one-unit-length interval. At that point, the researcher encouraged them to construct their traced graphs in order to provoke them to reason about these two functions in a more detailed way hoping to emerge the compression idea explicitly (lines 68-73 in [Cemre & Zafer] Protocol 18).

When the traced graphs of the ordered pairs $(x,\sin(x))$ and $(x,\sin(2x))$ appeared on the screen, all students started to compare and contrast $y=\sin(2x)$ function with $y=\sin(x)$ in a more detailed way in the *graphical register*. Initially, they interpreted the compression of the graph of $y=\sin(2x)$ as much as half comparing with the graph of $y=\sin(x)$ (e.g., lines 74-80 in [Cemre & Zafer] Protocol 18; lines 37-41, 53-71 in [Defne & Ebru] Protocol 20). When the researcher provoked them to articulate the meaning of compression (e.g. line 76 in [Cemre & Zafer] Protocol 18; lines 39-42 in [Defne & Ebru] Protocol 20), they brought the interval –on the x-axis corresponding to one-full-action– and the period aspects up for the discussion (e.g., lines 77-80 in [Cemre & Zafer] Protocol 18; lines 43-45, 53-59 in [Defne & Ebru] Protocol 20). That is, they articulated the compression of the graph of $y=\sin(2x)$ as much as half comparing with the graph of $y=\sin(x)$ based on the intervals $(0,\pi)$ and $(0,2\pi)$[46] in which, respectively, $\sin(2x)$ and $\sin(x)$ completed their one-full-actions in the *graphical register* as a consequence of one-full-round turnings of their reference points in the (*unit*) *circle register* (e.g., lines 77-122 in [Cemre & Zafer] Protocol 18; lines 43-45, 53-90 in [Defne & Ebru] Protocol 20). In other words, they attributed this compression in half in the *graphical register* to the changed-arc as "twofold" in the (*unit*) *circle register*.

In addition, all students were able to extend this compression idea beyond for the graph of $y=\sin(2x)$ into, for example, the graphs of $y=\sin(3x)$ and $y=\sin(5x)$ functions defined in the *symbolic register* (e.g., lines 126-130 in [Cemre & Zafer] Protocol 18; lines 91-115, 182-213 in [Defne & Ebru] Protocol 20). That is, their actions and language imply that they interpreted the compression of the graph of $y=\sin(3x)$ [$y=\sin(5x)$] as much as one third [one fifth] comparing with $y=\sin(x)$ based on their peak parts in the $(0,2\pi)$ interval (e.g., lines lines 127-130 in [Cemre & Zafer] Protocol 18; lines 102-115 in [Defne & Ebru] Protocol 20). Moreover, they were able to associate one-full-acts of these graphs with their parts including two consecutive peaks (one positive and one negative) (e.g., lines 127-144 in [Cemre & Zafer] Protocol 18; lines 188-213 in [Defne & Ebru] Protocol 20). In other words, reversely, they were

[46] Where, in order not to change the discussion focus from the main idea of the graphs' compression, end points of the intervals were not treated as a point at issue. Therefore, these intervals were specified as open intervals rather than closed intervals.

able to dissociate the peaks of the graph from its repeated-full-acts (e.g., lines 188-203 in [Defne & Ebru] Protocol 20).

In the same way, they were able to reason about the stretched-graphs comparing with y=sin(x). For example, when reasoning about the graph of y=sin(x/2), they expressed its stretched-form as much as twofold comparing with the graph of y=sin(x) through comparing their one-full-actions with each other in the *graphical register* (e.g., lines 162-181 in [Cemre & Zafer] Protocol 18; lines 214-225 in [Defne & Ebru] Protocol 20), as well as through associating their full actions in the *graphical register* with one-full-round turnings of their reference points in the (*unit*) *circle register* (e.g., lines 173-179 in [Cemre & Zafer] Protocol 18).

Moreover, Cemre and Zafer were able to extend this reasoning in Task 11 onto the graph of sine for the negative multiples of x through attributing the meaning of the negative sign of the coefficient of x in the *symbolic register* into the *graphical register* as the reflection of the graph of y=sin(x) regarding the x-axis (lines 183-192 in [Cemre & Zafer] Protocol 18), as well as regarding the y-axis (lines 193-204 in [Cemre & Zafer] Protocol 18). The researcher inferred from their language and actions that their reasoning about y=sin(–x) as the reflection of y=sin(x) regarding the y-axis arose from their consideration of y=sin(–x) as a function from x to sin(–x) in the *graphical register* (lines 193-204 in [Cemre & Zafer] Protocol 18); as for their reasoning as the reflection regarding the x-axis, this arose from their consideration of y=sin(–x) as a function from x to –sin(x) as the reflection of sin(x) regarding the x-axis in the (*unit*) *circle register* (lines 183-196 in [Cemre & Zafer] Protocol 18).

On the other hand, with the other group of students, Defne and Ebru, as a consequence of their troubles on the period issue at the beginning of Task 11 (see *Periodicity as pattern based on behaviors of trigonometric functions* heading), the researcher preferred to postpone the discussions about the meaning of the negative coefficient of the input variable in the *graphical register* to Task 16. In task 16, when reasoning about y=cos(–x) in the *graphical register*, it was observed that all students were able to attribute the meaning of the negative multiples of x in the *symbolic register* to the negative angles represented on the negative x-axis in the *graphical* *register*. By this way, they interpreted the graph of y=cos(–x) as the reflection of y=cos(x) regarding the y-axis through considering y=cos(–x) as a function from x to cos(–x) in the *graphical register* in the similar manner that Cemre and Zafer reasoned about y=sin(–x) in Task 11 (e.g., lines 193-204 in [Cemre & Zafer] Protocol 18).

Moreover, as a consequence of their developments throughout Task 12 in terms of making sense of sine [cosine] as cosine [sine] (see the heading *Changed-arc with a constant difference in (unit) circle register*), all students were able to transfer their interpretations made on sine in Tasks 8, 9, 10 and 11 onto cosine, respectively, in Tasks 13, 14, 15 and 16. Thus, all these reasoning mentioned above in Task 11 in the (*unit*) *circle register* was observed in Task 16 in terms of cosine, as well. Henceforward, the subsequent progress of students' concept images on the visual feature opposition (B) in the *graphical register* was presented under "*Composed-Coefficients' Visual Oppositions"* heading.

6.2. Periodicity as pattern based on behaviors of trigonometric functions

Task 6 [Task 7] was the first task that the period concept was brought up for discussion on the basic form of the sine [cosine] function (see *Regarding periodicity as pattern based on behaviors of trigonometric functions* heading in Chapter 5). Where, it was observed that all students were able to reason about the repetition of the sine [cosine] values as a consequence of the *full-round turning*s of the point P (*Figure 5.21* and *Figure 5.22*) that was referring to both the input and output of the basic form of the sine [cosine] function within the (*unit*) *circle register*; moreover, they were able to reason about the repetition of the sine [cosine] values in the *graphical register* and the *symbolic register*. At that point, the researcher had introduced the definition of the period concept without going into details as the regular intervals of the domain set in which a function repeats its values in the *graphical register* and as the lengths of these regular intervals in the *symbolic register*. From this Task forward, in all tasks, the period concept was brought up for discussion with the *prime period* meaning on a general form of sine or cosine in accordance with the tasks' themes comparing with the period of their basic forms.

Firstly, in Task 8, the discussion focus changed from the basic form of the sine function (i.e., y=sin(x)) into a general form of sine (i.e., y=a.sin(x)). For this purpose, both functions were represented in the (*unit*) *circle register* by their reference points, *P* and *R*, on the origin-centered **unit circle** and **non-unit circle** (*Figure 6.1*). Where, the points *P* and *R* were referring to the same angle in the (*unit*) *circle register*, and moving simultaneously at the same angular speed in the GSP environment. In this task, it was observed that all students reasoned about the new function defined on the **non-unit circle** with the same period as the basic form of sine –i.e., 2π– based on their reference points' same angular speeds in the (*unit*) *circle register*, as well as their *one full-actions* in the same interval in the *graphical register* (e.g., lines 30-41 in [Cemre & Zafer] Protocol 14; lines 66-84 in [Defne & Ebru] Protocol 14).

Next, in Task 9, the discussion focus was another general form of sine (i.e., y=sin(x)+d) comparing with the basic form of the sine function (i.e., y=sin(x)). For this purpose, a unit circle with a manipulable-center was constructed on the coordinate system in order to define a new function by the reference point *P* as a mapping from its angle to its ordinate (*Figure 6.16*). Where, there was only one point, the point *P*, referring to the *full-round turning* in the (*unit*) *circle register*, as well as the *unique shape*[47] of their graphs indicating their *one full-actions* in the same interval in the *graphical register* (*Figure 6.25*(*b*)). Thus, students were able to reason truly about the new function defined on the unit circle **with different-center from the origin** with the same period as the basic form of sine –i.e., 2π (e.g., lines 27-95 in [Cemre & Zafer] Protocol 16).

Afterwards, in Task 10, the discussion was focused on another general form of sine (i.e., y=sin(x+c)) comparing with the basic form of the sine function (i.e., y=sin(x)). For this purpose, in order to represent these two functions, two points were constructed on the unit circle so that one of them was an arbitrary point and the other

[47] Where, the "*unique shape*" of two graphs means that they coincide with each other by the vertical and/or horizontal parallel-displacement.

was its rotated-position about the origin by a fixed-measure (*Figure 6.28*). Where, the new function was defined from the angle of the point *P* to the ordinate of its rotated-position. It was the first task that a function was defined based on two different points on the unit circle so that one of them was referring to the input and the other was referring to the output. However, these two points were moving in the same (angular) speed on the unit circle. Thus, students were able to reason about the new function defined by **two points with the same (angular) speed** in the (*unit*) *circle register* with the same period as the basic form of sine –i.e., 2π– based on their *full-round turning*s as they reasoned in Task 8 (e.g., lines 77-84 in [Defne & Ebru] Protocol 14); as well as based on the *unique shape* of their graphs (*Footnote 47*) indicating their *one full-actions* in the horizontally-translated-intervals in the *graphical register* (e.g., lines 62-71 in [Cemre & Zafer] Protocol 17; lines 17-27 in [Defne & Ebru] Protocol 17).

Lastly, in Task 11, this systematic variation of the general forms of sine was pursued with the functions in the form *y*=sin(*bx*) as the discussion-focus. As an initial example, *y*=sin(2*x*) was handled without mentioning its symbolic representation. For this purpose, two points, the point *P* and *P'* (*Figure 6.39(a)*), and then, the perpendicular segments from these points to the *x*-axis (*Figure 6.39(b)*) were constructed in the (*unit*) *circle register*. Where, the point *P* was an arbitrary point on the unit circle, and the point *P'* was its rotated-position about the origin by the principal measure of the point *P*. Thus, the point *P'* was moving on the unit circle at the double (angular) speed of the point *P* in the GSP environment. It was the first task that a new function was defined by **two points with the different (angular) speeds** from the angle of the point *P* to the ordinate of the point *P'* in the (*unit*) *circle register*. Dynamically-linked conversion of this function from the (*unit*) *circle register* into the *graphical register* together with the sine graph caused for students to bring the period aspect up for discussion (e.g., lines 72-80 in [Cemre & Zafer] Protocol 18; lines 39-45 in [Defne & Ebru] Protocol 20).

On the one hand, Cemre and Zafer interpreted the period of *y*=sin(2*x*) as half of the period of *y*=sin(*x*) through attributing its meaning to (*i*) the half-turning of the point *P* (referring to the input of *y*=sin(2*x*)) so as to bring forth one full-round turning of the point P' (referring to the output of y=sin(2x)) in the (*unit*) *circle register*, (*ii*) (0,π) interval (*Footnote 46*) as the half of (0,2π) interval on the x-axis in which, respectively, sin(2x) and sin(x) completed their one-full-actions in the *graphical register* as a consequence of one-full-round turnings of the point P' and P (referring to the outputs of y=sin(2x) and y=sin(x)) in the (*unit*) *circle register* (lines 79-122 in [Cemre & Zafer] Protocol 18).

On the other hand, Defne and Ebru encountered a trouble on the period of y=sin(2x) (lines 39-52 in [Defne & Ebru] Protocol 20). That is to say, while Defne expressed the period of y=sin(2x) as 4π based on its two full-actions in the (0,2π) interval in the *graphical register* (lines 39-65 in [Defne & Ebru] Protocol 20), Ebru expressed its period as 2π through reasoning about the period as always 2π (lines 49-52 in [Defne & Ebru] Protocol 20). When articulating their thoughts behind the reasoning about the period of y=sin(2x) in this way, their actions and language imply that both assumed its one-full-action in the *graphical register* as 2π through associating one-full-action with the one-full-round turning of the point P' (referring to the output of y=sin(2x)) in the (*unit*) *circle register* (lines 53-82 in [Defne & Ebru] Protocol 20). At that point, the researcher inferred that their concept images on the period concept included a deficient part in the (*unit*) *circle register* in terms of the interpretation of the full-round turnings of two points referring to (*i*) the input variable and (*ii*) the output variable. While Defne interpreted the period as 4π considering two full-rounds of the point P' that was produced by one full-round of the point P, Ebru interpreted the period as 2π considering one full-round turning of only the point P'. In other words, they interpreted the period in the (*unit*) *circle register* based only on the full-round turning of the reference point of the output variable rather than the turning of the reference point of the input variable producing the full-round of the reference point of the output variable.

Therefore, she encouraged them to reason about the full-round-turnings of the point P' dependently on the turning of the point P (lines 83-85 in [Defne & Ebru] Protocol 20). Where, they were able to attribute one-full-round turning of the point P' to the π-radian turning of the point P in the (*unit*) *circle register* (lines 86-90 in [Defne

& Ebru] Protocol 20). Before mentioning its meaning as the period, the researcher provoked them to reason about another function, i.e., $y=\sin(3x)$ (line 91 in [Defne & Ebru] Protocol 20). Without construction of its representations on the GSP page, they were able to determine that three repetitions of its one-full-action would exist in the $(0,2\pi)$ interval (*Footnote 46*) in the *graphical register* (lines 92-110 in [Defne & Ebru] Protocol 20). When its graph plotted in GSP, the researcher provoked them to reason about how to determine the interval length referring to one of three repetitions in the $(0,2\pi)$ interval (lines 111-126 in [Defne & Ebru] Protocol 20). While this reasoning process caused Ebru's inference of its period as a division of 2π by 3 (line 123 in [Defne & Ebru] Protocol 20), it caused Defne's conversion trouble on association of this interval length in the *graphical register* with the full-round turning in the (*unit*) *circle register* (lines 124-126 in [Defne & Ebru] Protocol 20) as a consequence of her deficient concept image mentioned the above paragraph.

At that point, the researcher determined to represent $y=\sin(3x)$ function in the (*unit*) *circle register* (*Figure 6.41*) in order to provoke them, especially Defne, to reason about the full-round turning of the reference point of the output variable dependently on the turning of the reference point of the input variable (lines 127-131 in [Defne & Ebru] Protocol 20). Throughout this reasoning process, they were able to reason about the accurate turning amount of the point P as the cause of the full-round turning of the point P'', as well as for the point P' (lines 133-160 in [Defne & Ebru] Protocol 20). After then, the researcher constructed the traced graphs of three ordered pairs, namely, $(x,\sin(x))$, $(x,\sin(2x))$ and $(x,\sin(3x))$, through dragging and dropping the point P in the counterclockwise direction in the (*unit*) *circle register* so as to emphasize each graph's first full-action in the *graphical register* (*Figure 6.42*). And then, she asked them to reason about their periods through emphasizing its meaning as the smallest-repeated-interval in the *graphical register* and the length of this interval in the *symbolic register* (e.g., lines 162-168 in [Defne & Ebru] Protocol 20). They were able to reason about their periods correctly through focusing on the right endpoint of these intervals (lines 171-181 in [Defne & Ebru] Protocol 20).

At the end of Task 11, not only Defne and Ebru, but also Cemre and Zafer were able to determine the period of some other functions in the form y=sin(bx) accurately in two different reasoning ways in the *symbolic register* (e.g., lines 126-204 in [Cemre & Zafer] Protocol 18; lines 101-120, 167-225 in [Defne & Ebru] Protocol 20). For example, firstly, by the proportional reasoning, they determined the period of y=sin(3x) as 2π/3 through considering it as one third of the basic sine function's period (lines 126-157 in [Cemre & Zafer] Protocol 18; lines 101-120 in [Defne & Ebru] Protocol 20). Secondly, they determined the period of y=sin(3x) as about 2.1 by the reasoning way based on the measurement of the abscissa of the right endpoint of the interval referring to its first one-full action (lines 158-161 in [Cemre & Zafer] Protocol 18; lines 177-181 in [Defne & Ebru] Protocol 20).

Moreover, all students were able to transfer their interpretations on the period of the general forms of sine in Tasks 8, 9, 10 and 11 mentioned above to cosine (e.g., lines 61-70 in [Defne & Ebru] Protocol 19), respectively, in Tasks 13, 14, 15 and 16 as a consequence of their conceptual developments on association of cosine [sine] with sine [cosine] throughout Task 12 (see *Changed-arc with a constant difference in (unit) circle register* heading in Chapter 6).

[Defne & Ebru] Protocol 20

(*Defne and Ebru cooperatively constructed a unit circle located on the origin with the principal arc represented by the point P with respect to the researcher's directions; and then, its rotated position in the positive direction by a marked angle x that was the angle corresponding to the point P, and labelled as P' like in Figure 6.39(a).*)
Researcher: Please look, when x is decreasing (*dragging the point P in the clockwise direction in the first quadrant*), this arc (*dragging the cursor on the arc from the point P and the point P' in the counterclockwise direction*)…
Ebru: …is decreasing.
Researcher: This arc in here (*dragging the cursor on the arc from the point P and the point P' in the counterclockwise direction*) is as much as x.
Ebru: Yes.
Defne: There [up to point *P'*] is 2x in total.
Ebru: (*Nodding her head up and down*)
Researcher: Good. Let's we construct perpendicular segments from these points to the x-axis.

(*Defne and Ebru cooperatively constructed perpendicular segments from the point P and P' on to the x-axis like in Figure 6.39(b) with respect to the researcher's directions.*)

Researcher: Defne just said to us that if here is *x* (*pointing the principal arc corresponding to the point P*), here is 2*x* (*pointing the principal arc corresponding to the point P'*).

Defne&Ebru: (*Nodding their heads up and down*)

Researcher: Well, how do you express this red segment in terms of sine?

Defne&Ebru: (*Looking to the screen without speaking*)

Researcher: Consider this arc (*dragging her index finger on the principal arc corresponding to the point P'*) and this segment?

Defne: Sine of 2*x*…

Ebru: Yes.

Researcher: Please calculate sin(2*x*) and the *y*-value of the point *P'*?

Ebru: They are same, aren't they?

Defne: (*Calculating sin(2x) and the ordinate of the point P' by GSP's calculate option*) uh-huh, same.

Researcher: Please drag the point *P* and control their equalities in anywhere of the circle.

Ebru: (*Dragging the point P*)

Defne&Ebru: Yes, same.

Researcher: Let's we construct their traced graphs (*constructing their traced graphs like in Figure 6.40*)

Defne: (*When the traced graphs were appeared on the screen like in Figure 6.40(a)*) it [wave length of the sine graph] becomes smaller… …that is, twofold cause division into two…

Researcher: What do you mean by "twofold"?

Defne: You see exactly from here (*pointing the position of π on the x-axis*) it is twofold (*pointing respectively the origin and the position of 2π on the x-axis*). I think it as period.

Researcher: When you think in terms of their periods… …which graph is twofold of the other?

Defne: This one (*pointing the blue traced graph on the screen like in Figure 6.40(a)*).

Researcher: What do you say Ebru?

Ebru: I don't think twofold… …the period didn't change…

Defne: Period would be 4π.

Ebru: No, it wouldn't be 4π. It is again 2π or about 6.28…

Defne: For sine, one repetition is there… …but for other, [repeated] interval became shorter (*keeping her hands vertically-parallel so as to indicate an interval, and then, bringing them closer to each other*). That is, from here forward (*separating the blue traced graph into two equal parts through putting her left hand vertically on the position of π on the x-axis*), this (*indicating the left part*) is same of this (*indicating the right part*).

Ebru: Yes.

Defne: Then, twofold means this interval doubled (*indicating the left part of the blue traced graph in the (0,π) interval on the screen like in Figure 6.40(a)*). That is, if this interval is *x* (*putting her right hand's index and thumb fingers on the end points of the (0,π) interval on the x-axis*), this interval is 2*x* (*putting her right* *hand's index and thumb fingers on the end points of the* $(0,2\pi)$ *interval on the x-axis*).

Ebru: You know here is 2π (*putting her left hand's index finger on the origin and the right hand's index finger on the* 2π *on the* x*-axis*) pink one [$\sin(x)$ graph] as well as blue one [$\sin(2x)$ graph] is 2π. This is coming toward half (*pointing the midpoint of the* $(0,2\pi)$ *interval on the* x*-axis*), that is, this (*pointing the point P on the unit circle*) is turning about the half (*figuring the up-half circle in the counterclockwise direction*)… …while this comes to here (*dragging her index finger on the pink traced graph in the* $(0,\pi)$ *interval*), this will come to here (*dragging her index finger on the blue traced graph in the* $(0,\pi)$ *interval*). I mean x comes to half (*figuring the up-half circle in the counterclockwise direction*)… …since this [point P'] is turning more (*figuring the down-half circle in the counterclockwise direction*), it [$\sin(2x)$ graph] repeats many more [than $\sin(x)$ graph], doesn't it?

Defne: Yes. Number of full-rounds of it [point P'] increased [comparing with that of point P].

Ebru: That is, when P completed circle (*figuring a circle in the counterclockwise direction through starting from its far right point*), that [point P'] completed many more.

Defne: Uh-huh.

Researcher: Please concentrate on P'… …after how much turning of P… …does P' complete a full-round (*dragging the point P slowly in the counterclockwise direction from first quadrant to the second quadrant*)?

Ebru: (*When the point P was coming closer to the negative* x*-axis*) at π (*smiles with satisfaction*).

Defne: Yes. At π, $\sin(2x)$ has been completed.

Ebru: At π, it [$\sin(2x)$] completed one tour (*figuring the sine wave*), and then one more (*figuring the following sine wave*).

Researcher: What do you say about $\sin(3x)$?

Defne: Wait a minute! This was x (*dragging her index finger on the principal arc referring to the point P*) to x (*dragging her index finger on the arc from the point P and P' in the counterclockwise direction*), so $2x$ (*dragging her index finger on the principal arc referring to the point P'*). Then it [$3x$] would be x (*dragging her index finger on the principal arc referring to the point P*) to $2x$ (*indicating an arc from the point P to another point beyond the point P' in the counterclockwise direction*), so $3x$ (*dragging her index finger on the principal arc referring to that point*).

Ebru: (*Looking to the screen without speaking*)

Researcher: Well, what do you say about the graph of $\sin(3x)$?

Ebru: It would repeat many more [than $\sin(2x)$]. That is, this point (*pointing the position of* π *on the* x*-axis on the screen like in Figure 6.40(a)*) would be somewhere on this side (*indicating the left side regarding the position of* π *on the* x*-axis*).

Defne: There would be three and three, that is, six peaks.

Ebru: Yes. Three would be at the top… …and three at the bottom.

Defne: Top bottom… …top bottom… …top bottom.

Researcher: So, in the $(0,2\pi)$ interval (*indicating* $(0,2\pi)$ *interval on the* x*-axis*), how many times does it repeat?

Defne: Three.

Researcher: Please plot $y=\sin(3x)$ function's graph in order to see indeed it is like you imagine.

Defne: (*Plotting $y=\sin(3x)$ function's graph like in Figure 6.40(b)*) yes, one (*pointing with the cursor the first smallest repeated part of the $y=\sin(3x)$ in the $(0,2\pi)$ interval*) two (*pointing the second one*) and three (*pointing the second one*).

Researcher: If I know that here is 2π (*putting her index fingers on the endpoints of the $(0,2\pi)$ interval*), I know… …one… two… three (*pointing respectively the first, second and third sub-intervals of the $(0,2\pi)$ interval on the x-axis, in which $y=\sin(3x)$ repeats*)… its three times is 2π.

Defne&Ebru: (*Nodding their heads up and down*)

Researcher: Then, how do you find this interval length (*pointing the first smallest repeated part of the $y=\sin(3x)$ in the $(0,2\pi)$ interval*)?

Ebru: We divide 2π by 3.

Defne: But period was 2π, wasn't it! I mean here corresponds to one full-round (*dragging her index finger on the first smallest repeated part of $y=\sin(3x)$ function's graph*). It doesn't so.

Researcher: Ok. Let's we construct its correspondence on the unit circle (*constructing rotated position of the point P' on the unit circle by a marked angle x in the counterclockwise direction, and labelling it as P'' like in Figure 6.41(a)*). Please focus on the point *P''* (*pointing the point P'' on the unit circle*). I will drag the point *P*. You will say "stop" to me when *P''* complete one full round. Is it ok?

Defne&Ebru: Ok.

Researcher: (*Dragging the point P slowly in the counterclockwise direction starting from the intersection point of the unit circle with the positive x-axis*)

Ebru: (*When the point P'' was nearly completing a full round like in Figure 6.41(b)*) stop.

Defne: Wait a minute! I didn't see.

Researcher: Ok. Let's we look again. Look at *P''* (*pointing the point P'' on the screen like in Figure 6.41(a), and then, dragging slowly in the counterclockwise direction the point P starting from the intersection point of the unit circle with the positive x-axis*).

Defne: (*When the point P'' was nearly completing a full round like in Figure 6.41(b)*) [one full round for *P''*] almost completed.

Ebru: Uh-huh.

Researcher: I stopped (*stopping dragging the point P when the point P'' on the positive x-axis*). How much did I turn (*dragging her index finger on the principal arc corresponding to the point P*)?

Ebru: 0.66π radian.

Researcher: Its 3 fold is 2π (*dragging her index finger on the whole circle so as to indicate the principal arc of the point P''*).

Defne&Ebru: Yes.

Defne: …because it [point *P''*] turns 3-fold faster [than point *P*].

Researcher: Look again! Up to *P* point's completion of a full-round, how many full-rounds are completed for *P''* (*dragging slowly in the counterclockwise direction the point P starting from the intersection point of the unit circle with the positive x-axis*)?

Defne: (*When the point P'' completed the first full-round*) first tour, (*when the point P'' completed the second full-round*) second tour, (*when the point P'' completed the third full-round*) third tour is completed together.

Ebru: Yes.

(*Similar discussions were done for the point P' (see Figure 6.41(c)).*)

…

Researcher: Ok. I want you to say periods of functions through looking to their graphs. This pink one is sin(*x*) (*dragging her index finger on the pink graph on the screen like in Figure 6.42*), and you know its period is 2π (*indicate the (0,2π) interval on the x-axis*).

Defne&Ebru: Yes.

Researcher: What is the period of this blue one (*dragging her index finger on the blue graph on the screen like in Figure 6.42*)? Where is its smallest repeated part?

Defne: Smallest-repeated part is here (*dragging her index finger on the first smallest repeated part of the y=sin(2x) in the (0,2π) interval*).

Researcher: What is the magnitude of this interval (*pointing the (0,π) interval on the x-axis*)?

Defne: 3.14 or something like that.

Ebru: (*Nodding her head up and down*)

Researcher: What about this red one (*dragging her index finger on y=sin(3x) graph on the screen like in Figure 6.42*)?

Defne: For red one… …here (*dragging the cursor on the first smallest-repeated part of the y=sin(3x) in the (0,2π) interval*).

Researcher: So, what about its period?

Defne: …about 2 point one.

Ebru: Uh-huh.

Researcher: Well, if I mention *y*=sin(5*x*) function… What does it evoke for you?

Defne: Until it [point *P* referring to the input] turns a full-round, the other [point *P'* referring to the output] will turn five full-rounds.

Ebru: Yes (*nodding her head up and down*).

Researcher: (*Asking them to plot y=sin(5x) function's graph. When the graph appeared on the screen*) how many repeated parts exist between zero and 2π?

Ebru: (*Counting repeated parts of y=sin(5x) graph through dragging her index finger on each repeated part of y=sin(5x) graph in the (0,2π) interval*) one, two, three, four, five.

Defne: (*Coming closer to the screen and counting the positive peak points of the y=sin(5x) graph in the (0,2π) interval*) one, two, three, four, five, ten.

Defne&Ebru: (*Laughing looking at each other*)

Researcher: Please decide on five or ten (*smiling*).

Defne&Ebru: Five (*laughing*).

Defne: I mean there are ten peaks in total, but I know… …because this up-peaks (*pointing a positive peak point of y=sin(5x) graph*) and down-peaks (*pointing a negative peak point of y=sin(5x) graph*) are not same, five repetitions exist.

Ebru: Five in the sin(5*x*) says to me that it [*y*=sin(5*x*) graph] repeats five times in 2π [length-interval] (*indicating an interval through holding her hands vertically parallel to each other*).

Defne: A job that a worker does in a day… …is done in 5 days by the other worker (*smiling with pleasure*).

Researcher: (*Smiling*) a job, that is, completion of a sine wave (*figuring a sine wave with one positive and one negative peaks*)… …is done by sin(5*x*) in a day (*dragging her index finger on the* ***right*** *first part of the y=sin(5x) graph corresponding to the* *first full-round turning in the* ***counterclockwise*** *direction of the reference point referring to the output, and then, indicating its interval on the x-axis*)… …and is done by sin(*x*) in 5 days (*dragging her index finger on the* ***right*** *first part of the y=sin(x) graph corresponding to the first full-round turning in the* ***counterclockwise*** *direction of the reference point referring to the output, and then, indicating its interval on the x-axis*)

Ebru&Defne: (*Nodding their heads up and down when smiling with satisfaction*).

Researcher: Ok. What about *y*=sin(*x*/2)?

Defne: This (*putting her thumb and index fingers on the end points of the interval on the x-axis so as to refer the first repeated part of the y=sin(5x) function*) would expand even more (*pointing a larger interval on the x-axis that expands beyond 2π*).

Ebru: (*Nodding her head up and down*)

Researcher: So, what would be its period?

Defne: Its period would be… …2π and 2π… …that is, 4π.

Ebru: Yes. 4π…

Defne: Because one full round of sin(*x*/2) requires two full-rounds of *x*…

Ebru: Uh-huh (*nodding her head up and down*).

(*Similar discussions were done on y=sin(x/2) function in the GSP environment, and they reasoned in the same way.*)

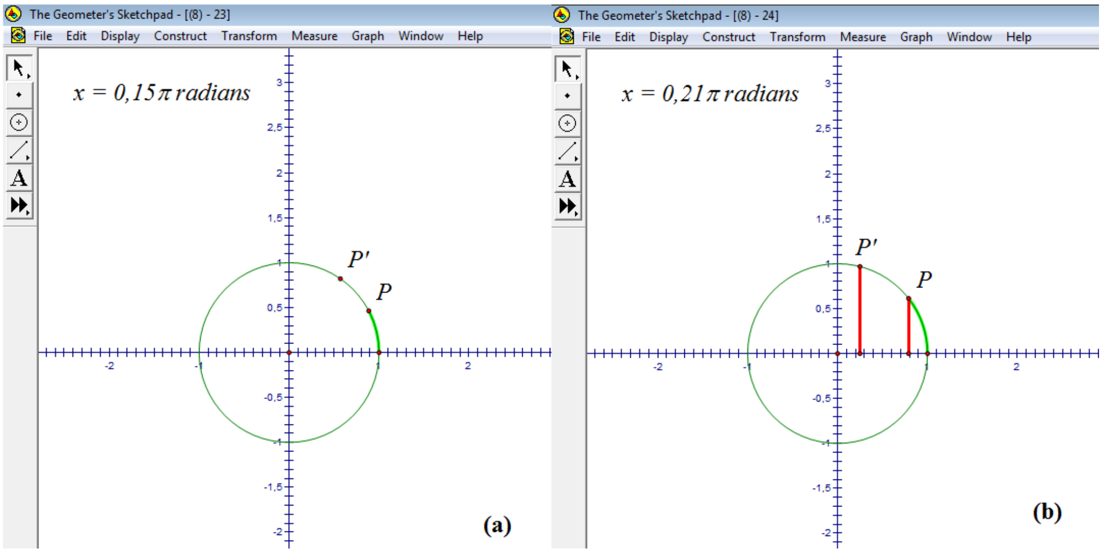


Figure 6.39. Through folding the principal angle corresponding to a point, the point *P*, on the unit circle, construction of another point, the point *P'*

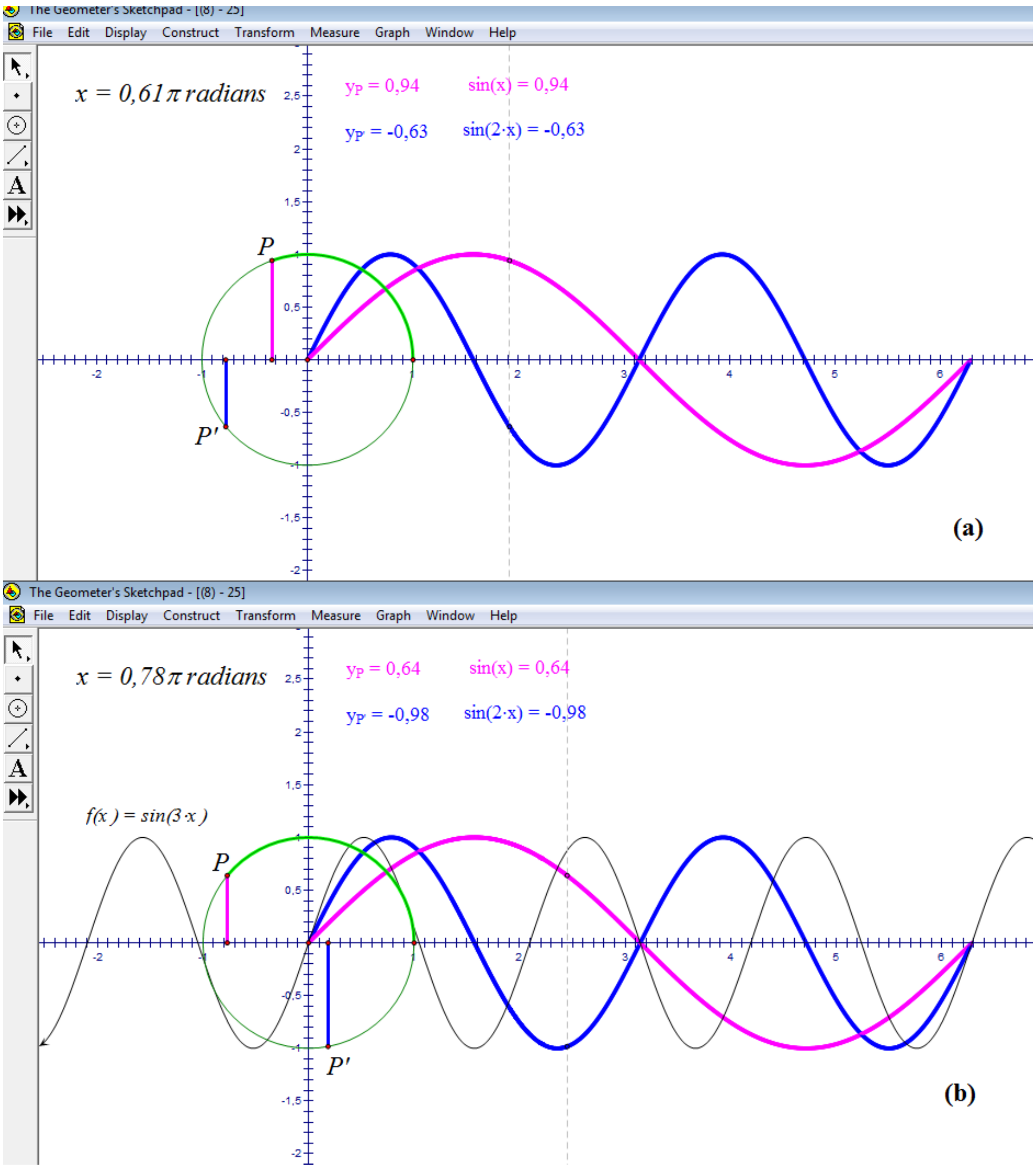


Figure 6.40. Simultaneous constructions of *y*=sin(2*x*) function in the (*unit*) *circle register* and the *graphical register* together with *y*=sin(*x*) function

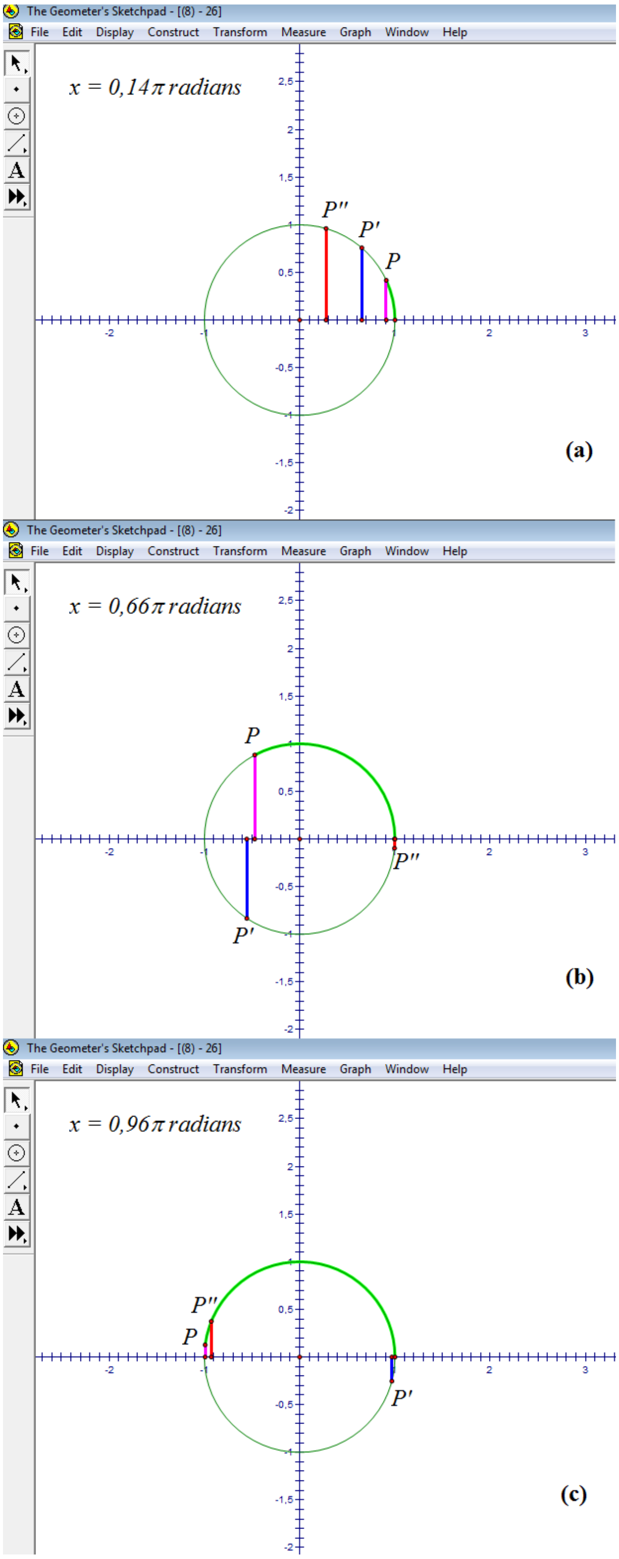


Figure 6.41. Construction of dynamically-linked three points, P, P' and P'', on the unit circle with the principal measures, respectively, x, $2x$ and $3x$

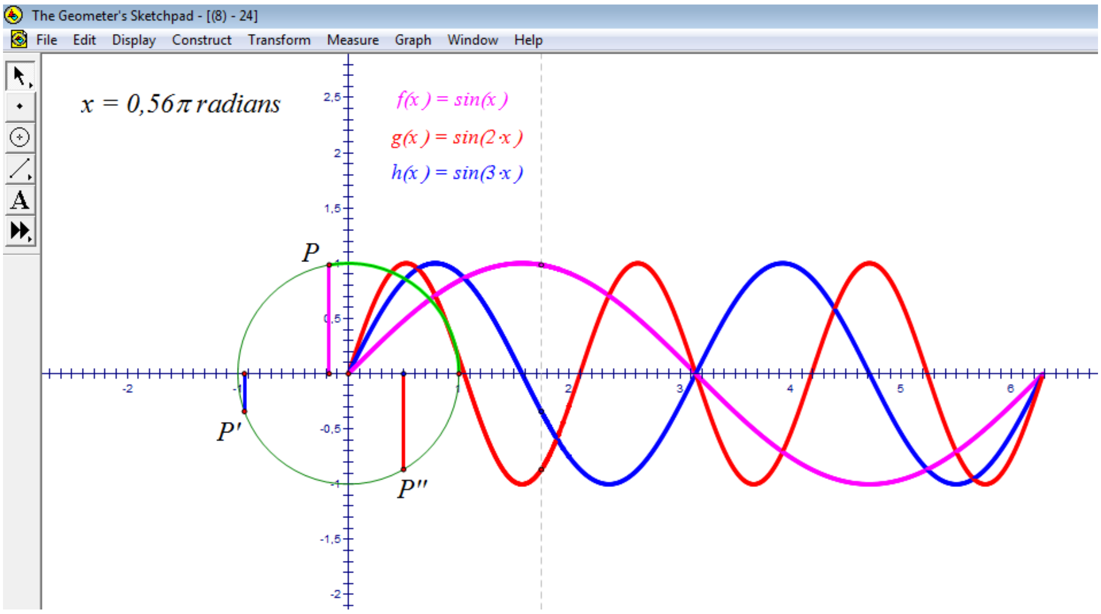


Figure 6.42. Construction of *y*=sin(3*x*) function's graph together with *y*=sin(*x*) and *y*=sin(2*x*)

[Cemre & Zafer] Protocol 18

Researcher: We discussed in the previous tasks about the periods.
Cemre&Zafer: (*Nodding their heads up and down*)
Researcher: You know that *y* is equal to sine *x* function's period is 2π… …the period of *y* is equal to sine *x* out of parenthesis plus something is…
Cemre: …again 2π. All of them [periods of the functions that were discussed in the teaching experiment up to Task 11] were 2π.
Zafer: (*Nodding his head up and down*) period of 2 times sine *x* is 2π, too.
Researcher: …for the functions in the form like that… … *y* is equal to sine in the parenthesis *x* plus or minus something… …period…
Cemre: In each case it [period] is 2π.
Researcher: In each case we mentioned up to this time, the period was 2π.
Cemre: It wouldn't change (*arching her eyebrows and looking to the researcher*), huh?
Researcher: (*Laughing*)
Cemre&Zafer: (*Laughing but looking worried*)
Researcher: Let's we discuss this issue at the end of this lesson.
(*Cemre and Zafer cooperatively constructed a unit circle located on the origin with the principal arc represented by the point P with respect to the researcher's directions; and then, its rotated position in the positive direction by a marked*

angle x that was the angle corresponding to the point P, and labelled as P'; finally, perpendicular segments from the points P and P' like in Figure 6.39(b).)

…

Researcher: If here is x (*dragging her index finger on the principal arc referring to the point P in the counterclockwise direction on the screen like in Figure 6.39(b)*) and here is x (*dragging her index finger on the arc from the point P and the point P' in the counterclockwise direction*), then here is what (*dragging her index finger on the principal arc referring to the point P' in the counterclockwise direction*)?

Cemre: $2x$.

Zafer: Uh-huh.

Researcher: Then, If I ask the y-value of the point P', what is of $2x$ (*dragging her index finger on the principal arc referring to the point P' in the counterclockwise direction and then on the perpendicular segment from the point P' to the x-axis*)?

Cemre: …$\sin(2x)$.

Zafer: Yes.

Researcher: Today, we will discuss about the function mapping x (*dragging her index finger on the principal arc referring to the point P in the counterclockwise direction*) to $\sin(2x)$ (*dragging her index finger on the perpendicular segment from the point P' to the x-axis*).

Zafer: Come on then (*smiling*)!

Cemre: (*Smiling*)

Researcher: Let's we construct its traced graph together with sine.

(*Cemre and Zafer plotted two points as (x,sin(x)) and (x,sin(2x)) ordered pairs*.)

Cemre: (*When these ordered pairs appeared on the screen*) here (*pointing the line segment on the x-axis from the origin to the projection point of these ordered pairs*) is x, isn't it?

(*At that point, the researcher encouraged them to construct related parts of these ordered pairs with the same color like in their correspondences on the unit circle like in Figure 6.43*).

Cemre: This one is on $\sin(x)$ graph (*pointing (x,sin(x)) ordered pair on the screen like in Figure 6.43*).

Researcher: Please plot it and see.

Cemre: (*Plotting the y=sin(x) graph*) uh-huh. (*After 4-second pause*) then, this (*pointing (x,sin(2x) ordered pair on the screen like in Figure 6.43*)… Actually, you know this amount is x (*dragging her index finger on the green segment from left to right on the x-axis*). It should be one x more (*continuing to drag her index finger on the x-axis rightward starting from the right end point of the green segment so as to indicate its iteration*). It (*dragging her index finger vertically upward from the right end point of the iterated segment up to intersect with the sine graph*) gives us P''s doing (*dragging her index finger horizontally leftward from this intersection point up to the ordered pair (x,sin(2x)*)).

Zafer: It [$2x$] affects the y-value… …its angle is again x (*dragging his index finger on the green segment from left to right on the x-axis*), but it [x] gives $\sin(2x)$ (*continuing to drag his index finger; respectively, (i) on the x-axis rightward starting from the right end point of the green segment so as to indicate its iteration, (ii) vertically upward from the right end point of the iterated segment up to intersect with the sine graph, (iii) horizontally leftward from this intersection point up to the ordered pair (x,sin(2x)*)).

Cemre: Uh-huh. I said so.

Researcher: Let's we see… …on how a graph this point (*pointing the ordered pair (x,sin(2x))*) would be?

Zafer: It would be on sin(2*x*).

Cemre: Yes.

(*At that point, the researcher encouraged them to construct traced graphs of both ordered pairs (x,sin(x)) and (x,sin(2x)) like in Figure 6.44.*)

Cemre&Zafer: (*Looking to the screen without speaking until finishing the traced graphs*)

Cemre: (*After 8-second pause*) it became smaller.

Researcher: What became smaller?

Cemre: This distance became smaller (*putting her right hand's thumb and index fingers on the end point of the (0,π) interval on the x-axis*).

Zafer: Period was reduced by half.

Cemre: Yes, because this was twofold (*pointing the point P' on the unit circle*)

Zafer: When it is twofold…

Cemre: Actually you know… …*P'* (*pointing the point P' on the unit circle*) was equal to 2*x* (*dragging her index finger on the principal arc corresponding to the point P' on the screen like in Figure 6.44*).

Zafer: Since it was equal to 2*x*, the way it [point *P*] takes in π (*figuring an up-half circle in the counterclockwise direction*)… …other [point *P'*] takes twice (*figuring a whole circle in the counterclockwise direction starting from its far right point*). That is, while this takes this path with one peak (*dragging his index finger on y=sin(x) graph's part in the (0,π) interval*), this takes with two peaks (*dragging his index finger on y=sin(2x) graph's part in the (0,π) interval*).

Cemre: Yes. The path that sin(*x*) takes in *x* (*dragging her index finger on y=sin(x) graph's part in the (0,2π) interval*)… …is taken twice by sin(2*x*) (*dragging her index finger on y=sin(2x) graph's part in the (0,2π) interval*).

Zafer: You see… when *x* completed a full-round (*figuring a circle in the counterclockwise direction starting from its far right point*), 2*x* completed 2π and further (*figuring the same circle in the same way; and then, an arc*).

Cemre: Actually, it happens so that… …in a full round [of *x*], it [*y*=sin(2*x*) graph] takes this path (*figuring a sine wave with one positive and one negative peaks*) twice. That is, it does this act (*putting her index fingers on the endpoints of the (0,2π) interval on the x-axis so as to indicate the sine wave in this interval*) purely between this hill (*putting her index fingers on the endpoints of the (0,π) interval on the x-axis so as to indicate the positive peak point of the sine wave in this interval*).

Zafer: Yes.

Researcher: Look! I am turning *P* (*dragging the point P in the counterclockwise direction from first quadrant to the second quadrant*).

Zafer: (*When the point P was a bit more beyond π-radian*) *P'* passed to second tour.

Cemre: P'… …yes.

Zafer: That is, it completed 2π [radian turning], and passed second tour.

Cemre: Uh-huh (*nodding her head up and down*). Now, it will turn one 2π [radian] more.

Zafer: Actually, from here to here (*pointing respectively the origin and 2π on the x-axis*), it [*point P'*] comes through rotating 4π. But its period is here (*putting his thumb and index fingers on the end points of the (0,π) interval so as to indicate the smallest repeated part of y=sin(2x) graph*), that is, π.

Cemre: While that [period] of $\sin(x)$ is as much as here (*indicating the smallest repeated part of* $y=\sin(x)$ *graph in the* $(0,2\pi)$ *interval*), that is, it [$y=\sin(x)$] does its whole act in here (*indicating the smallest repeated part of* $y=\sin(x)$ *graph in the* $(0,2\pi)$ *interval*), other [$y=\sin(2x)$] does the same act in here (*indicating the smallest repeated part of* $y=\sin(2x)$ *graph in the* $(0,\pi)$ *interval*), that is, as much as π (*indicating the smallest repeated part of* $y=\sin(2x)$ *graph in the* $(0,\pi)$ *interval*). So, its period became π.

Zafer: Uh-huh (*nodding his head up and down*).

Researcher: Good.

Zafer: (*After 3-second pause*) period hasn't been always 2π.

Cemre: (*Smiling with satisfaction*) yes.

Researcher: What about $y=\sin(3x)$ function?

Zafer: Then, in this interval three times (*indicating* $(0,\pi)$ *interval on the x-axis*)… like that and that (*figuring in the* $(0,\pi)$ *interval the sine wave with three peaks, respectively, up, down and up*).

Cemre: Yes, in here (*indicating* $(0,2\pi)$ *interval on the x-axis*), six hills will be.

Researcher: Let's we plot its graph.

Cemre: (*Plotting* $y=\sin(3x)$ *function's graph*)

Researcher: The most stretched wave is this one (*pointing the graph of* $y=\sin(x)$ *function on the screen like in Figure 6.40*)… …with one up and one down hills (*dragging her index finger on the* $y=\sin(x)$ *graph in the* $(0,2\pi)$ *interval*). So, what is its period?

Cemre&Zafer: 2π.

Researcher: Right! Now, look at this one (*pointing the graph of* $y=\sin(2x)$ *function on the screen like in Figure 6.40*). It was going up, going down, going up, going down (*dragging her index finger on the* $y=\sin(2x)$ *graph in the* $(0,2\pi)$ *interval through emphasizing its consecutive peaks*), and will go on in this way. Where is the smallest repeated part?

Cemre: In here (*indicating the part of the* $y=\sin(2x)$ *graph in the* $(0,\pi)$ *interval*).

Zafer: Yes.

Researcher: Then, what is its period?

Cemre&Zafer: π.

Researcher: Now, we are looking at this one (*pointing the graph of* $y=\sin(3x)$ *function on the screen like in Figure 6.40*). It was going up, going down, going up, going down, going up, going down (*dragging her index finger on the* $y=\sin(2x)$ *graph in the* $(0,2\pi)$ *interval through emphasizing its consecutive peaks*)). Where is the smallest repeated part?

Zafer: Then, it [its period] is two thirds.

Cemre: …two π thirds.

Zafer: Two thirds times π.

Researcher: Both are same.

Cemre: Yes. I understand it.

Zafer: (*Nodding his head up and down*)

Researcher: Well, what is its approximate value?

Zafer: Here it is (*putting her index finger around 2 which was the intersection point of* $y=\sin(3x)$ *graph with x-axis like in Figure 6.40*).

Cemre&Zafer: …2.1

Zafer: So, then… …if we consider sin(x/2), it will come from here to here like that (*figuring a positive peak of the sine wave through extending its interval from (0,π) to (0,2π) on the screen like in Figure 6.40*), and continue like that (*continuing to figure its consecutive part under the x-axis so as to indicate its negative peak in the (2π,4π) interval*).

Cemre: (*Laughing with pleasure*) yes.

Researcher: What about its period?

Zafer: Its period would be 4π.

Cemre: Yes (*smiling with satisfaction*). I think so.

Zafer: That is, the other [point P] completed 2π [a full-round], that [point P'] got behind, that is, at π.

Cemre: Yes. (*Smiling*) heigh-ho! Too much! That is, when the other [point P] comes whole [a full-round] (*dragging her index finger on the unit circle in the counterclockwise direction starting from the intersection point with the positive x-axis so as to indicate a full-round turning*), x/2 [point P'] comes up to here (*dragging her index finger on the up-part of the unit circle in the counterclockwise direction so as to indicate a half turning*).

Zafer: Uh-huh.

(*Similar discussions were done on y=sin(x/2) function in the GSP environment, and they reasoned in the same way.*)

…

Researcher: Well, what do you think about y=sin(-2x) (*opening "plot new function" option of GSP, and entering sin(-2x), but not clicking "ok"*)? How would it be?

Cemre: It would come to this side (*pointing a place under the x-axis so as to indicate the reflection of the positive peak point of y=sin(2x) graph regarding the x-axis*). That is, it would be shaped… …as the opposite of this grey one (*dragging her index finger on the first smallest repeated part of y=sin(2x) graph on the screen like in Figure 6.44*)… …on this side (*putting her right hand's external part over the x-axis on the part of y=sin(2x) graph in the (0,π/2) interval horizontally; and then, rotating her hand around the x-axis by 180 degrees so as to indicate its reflection regarding the x-axis*).

Zafer: You know minus means… …the angle's start of the rotation in this way (*putting his index finger on the intersection point of the unit circle with the positive x-axis; and then, dragging in the clockwise direction*).

Cemre: Yes. It would go from here (*figuring y=sin(2x) graph's left part regarding to the x-axis through starting from the origin*). Minus indicates only direction. Because the minus side is here (*dragging her index finger leftward on the negative x-axis starting from the origin*), this will come to here (*putting her right hand's internal part horizontally on the negative x-axis; and then, rotating her hand around the y-axis by 180 degrees so as to indicate its reflection regarding the y-axis*).

Zafer: Yes. As if a reflection [regarding the y-axis]…

(*At that point, the researcher plotted y=sin(-2x) function's graph, and students reasoned in the similar way.*)

…

Researcher: So, you recognize that period does not have to 2π. You see that… …when the coefficient of x in the sine function changed, its period also changed.

Cemre&Zafer: Yes (*nodding their heads up and down*).

Researcher: At the beginning of this course, you were worried about it. How do you feel now?

Zafer: Good.

Cemre: I feel wonderful. I understand (*smiling with satisfaction*).

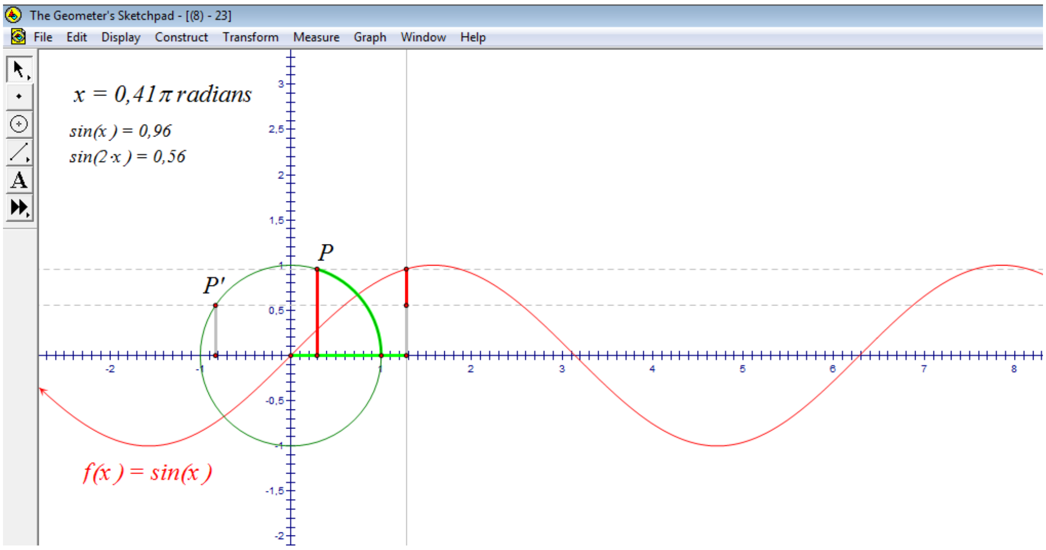


Figure 6.43. Construction of the ordered pairs (*x*,sin(*x*)) and (*x*,sin(2*x*)) as dynamically-linked with their correspondences in the (*unit*) *circle register*

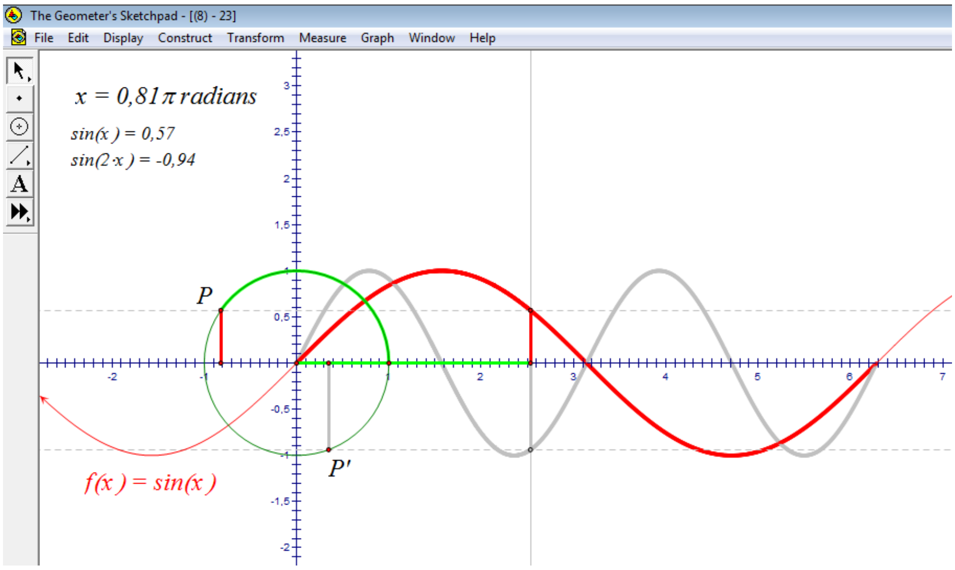


Figure 6.44. Simultaneous constructions of *y*=sin(*x*) and *y*=sin(2*x*) functions in the (*unit*) *circle register* and the *graphical register* through emphasizing the principal arc's conversion into the *graphical register*

6.3. Composed-Coefficients' Visual Oppositions

The main themes of the episodes of the teaching experiment were the conversion tasks based on the discrimination of the *visual features' oppositions* in any representational register. Therefore, while the (*unit*) *circle register* and the *graphical register* were considered as both the *source register* and the *target register*, the *symbolic register* was considered primarily as the *target register* of the conversion tasks. Moreover, the visual features of each task between Task 8 and 11 [Task 13 and 16] were referring in the *symbolic register* to only one coefficient of the general form of the sine [cosine] function. Therefore, at the end of Task 12 [Task 16], the researcher provoked students to reason about the coefficients' visual oppositions on a general form of sine [cosine] stated by all coefficients in the *symbolic register* with two main aims: (*i*) to reverse the role of the *symbolic register* in the conversion as the "*source register*", (*ii*) to bring the composed-coefficients' composed-visual oppositions up for discussion.

For this purpose, at the end of Task 12, a general form of sine function in the form y=3sin(2x+4)-1 was considered as the first example. Initially, the researcher asked students to reason about the visual opposition of the coefficient of sine; i.e., 3 (e.g., lines 1-2 in [Cemre & Zafer] Protocol 19; lines 1-3 in [Defne & Ebru] Protocol 21). Where, students expressions in the *language register* imply that all students were able to associate this coefficient in the *symbolic register* with the tripled-radius in the (*unit*) *circle register*, as well as the tripled-magnitude in the *graphical register* (e.g., lines 1-6 in [Cemre & Zafer] Protocol 19; lines 1-11 in [Defne & Ebru] Protocol 21). Moreover, they expressed the composed-visual oppositions of the coefficient and constant of sine; i.e., respectively 3 and -1, through articulating the upper and lower bounds of the range set in the *graphical register*, as well as the 1-unit down location of the circle in the (*unit*) *circle register* (e.g., lines 7-12 in [Cemre & Zafer] Protocol 19; lines 12-22 in [Defne & Ebru] Protocol 21).

When reasoning in the *language register* about the coefficient "2" and constant "4" of the input of sine in y=3sin(2x+4)-1 function, it was observed that students preferred to reason about "2" and "4" coefficients' visual oppositions mainly in the *graphical register* (e.g., lines 15-26 in [Cemre & Zafer] Protocol 19; lines 25-33 in [Defne & Ebru] Protocol 21). They reasoned about the visual opposition of "4" as the parallel displacement of the graph along the x-axis by 4 units in the negative direction (e.g., lines 21-26 in [Cemre & Zafer] Protocol 19; lines 25-28 in [Defne & Ebru] Protocol 21) and the visual opposition of "2" as the compression of the graph into half (e.g., lines 15-20 in [Cemre & Zafer] Protocol 19; lines 29-33 in [Defne & Ebru] Protocol 21). Where, the researcher inferred that they reasoned about the visual oppositions of these two coefficients separately rather than their composed-visual opposition. Therefore, in order to provoke them to think more deeply about these two coefficients' composed-visual oppositions in the *graphical register*, she asked them to plot y=3sin(2x+4)-1 function's graph in GSP environment (line 27 in [Cemre & Zafer] Protocol 19; lines 34-35 in [Defne & Ebru] Protocol 21).

When the graph of y=3sin(2x+4)-1 appeared on the screen (*Figure 6.45*(*a*)), students started to compare its visual features with their expectations. Their first focus was the visual opposition of the coefficient “3”. They reasoned based on the upper and lower bounds of the range set in the *graphical register* (lines 28-30 in [Cemre & Zafer] Protocol 19; lines 36-38 in [Defne & Ebru] Protocol 21). However, next focusses of the groups were different from each other. While the visual opposition of the coefficient “-1” was brought up for discussion in the group of Defne and Ebru (lines 36-40 in [Defne & Ebru] Protocol 21), the visual opposition of the coefficient “4” was mentioned in the group of Cemre and Zafer (lines 31-36 in [Cemre & Zafer] Protocol 19). Despite of their different second-step-reasoning-focuses, all students encountered a major trouble on reasoning about the composed-visual-opposition of the coefficients “2” and “4” in the *graphical register*.

On the one hand, in their reasoning process that resulted in this major trouble, Ebru –and then, Defne– encountered an initial trouble on the *y*-intercept of the graph as a consequence of their reasoning based only on the composed-effect of two coefficients, i.e., 3 and -1, rather than reasoning based on all four coefficients of y=3sin(2x+4)-1 function (lines 38-40 in [Defne & Ebru] Protocol 21). At that point, the researcher provoked them to reason about the variation of y=sin(x) function’s graph up to y=3sin(2x+4)-1 function’s graph through incorporating a new function into the discussion; respectively, y=3sin(x), y=3sin(x)-1, y=3sin(2x)-1 (lines 41-42, 45-47, 51-52, 57-60 in [Defne & Ebru] Protocol 21). Until the last step of this reasoning process, they had no conflict between the variation of the graphs and their expectations about them in the *graphical register* (lines 41-73 in [Defne & Ebru] Protocol 21). That is to say, their actions and language imply that their concept images on the visual variation in the *graphical register* between (*i*) sin(x) and 3sin(x), (*ii*) 3sin(x) and 3sin(x)-1, (*iii*) 3sin(x)-1 and 3sin(2x)-1 were coherent with the visual variation between these pairs of graphs produced by GSP. However, they encountered the major trouble on the visual variation between the graphs of y=3sin(2x)-1 and y=3sin(2x+4)-1 in terms of the displacement amount between their graphs (lines 74-91 in [Defne & Ebru] Protocol 21) as a consequence of their reasoning about the visual oppositions of the coefficients “2” and “4” separately rather than their composed-visual opposition.

On the other hand, in their reasoning process that resulted in this major trouble, Cemre and Zafer encountered an initial trouble on the reference function when reasoning about the visual opposition of the coefficient “4” as the parallel displacement along the *x*-axis (lines 31-43 in [Cemre & Zafer] Protocol 19). When reasoning by the aid of the researcher’s hint about the reference function of this displacement as *y*=3sin(2*x*)-1 (lines 39-41, 44-45 in [Cemre & Zafer] Protocol 19), like Defne and Ebru, they encountered the major trouble on the displacement amount between the graphs of *y*=3sin(2*x*)-1 and *y*=3sin(2*x*+4)-1 as a consequence of their reasoning about the visual opposition of the coefficient “4” separately from the visual opposition of the coefficient “2” in the *graphical register* (lines 46-56 in [Cemre & Zafer] Protocol 19).

When this major trouble emerged, the researcher asked students to determine the displacement amount between these two graphs (line 57 in [Cemre & Zafer] Protocol 19; line 92 in [Defne & Ebru] Protocol 21). All students determined the displacement amount as 2 –instead of 4– by the aid of GSP’s “zoom in” and “zoom out” options for the scaled *x*-axis (lines 57-73 in [Cemre & Zafer] Protocol 19; lines 92-107 in [Defne & Ebru] Protocol 21). Next, except Zafer, all other students associated this measure with the half of the constant of the input of sine in *y*=3sin(2*x*+4)-1 function (line 73 in [Cemre & Zafer] Protocol 19; lines 103-110 in [Defne & Ebru] Protocol 21). Moreover, Defne and Ebru attributed the cause of this half-reduced-displacement-amount to the half-reduced-period (lines 103-110 in [Defne & Ebru] Protocol 21).

However, Cemre and Zafer did not make any interpretation about the cause of this half-reduced displacement amount (lines 69-74 in [Cemre & Zafer] Protocol 19). Where, the researcher provoked them to reason about the displacement amount when the constant of the input of sine in *y*=3sin(2*x*+4)-1 function was changed from 4 to 2 (lines 75-76 in [Cemre & Zafer] Protocol 19). When Cemre edited *y*=3sin(2*x*+4)-1 as *y*=3sin(2*x*+2)-1, Zafer expressed the displacement amount as “almost 1 [unit]” but Cemre expressed it as “exactly 1 [unit]” and associated it with again the half of the constant of the input of sine in *y*=3sin(2*x*+2)-1 (lines 77-84 in [Cemre & Zafer] Protocol 19). At that point, the researcher inferred that Zafer needed to know the exact displacement amount (lines 80-84 in [Cemre & Zafer] Protocol 19). For this purpose, she constructed an arbitrary point, $A(x_A, y_A)$, on the graph of y=3sin(2x)-1; and then, another point (x_A-1,y_A) by the aid of "plot as (x,y)" option of GSP (lines 85-87 in [Cemre & Zafer] Protocol 19). When this plotted-point appeared on the graph of y=3sin(2x+2)-1 as the correspondence of the point A on the graph of y=3sin(2x)-1, Zafer ensured about the exact-displacement amount as 1 unit (lines 88-90 in [Cemre & Zafer] Protocol 19). It was the point that Zafer had just started to reason that the displacement amount was exactly the half of the constant of the input of sine in case that the coefficient of the input of sine was 2 (lines 88-100 in [Cemre & Zafer] Protocol 19). When reasoning about the cause of this half-reduced displacement amount, like Defne and Ebru, Zafer and Cemre attributed the cause of this half-reduced-displacement-amount to the half-reduced-period (lines 101-113 in [Cemre & Zafer] Protocol 19).

Students' this attribution of the ratio between the changed-displacement amounts[48] to the ratio between the changed-periods prompted them to adapt the operational-process of the determination of the period[49] to the displacement amount. That is to say, they started to reason about the displacement amount by an operational-process as the division by the coefficient of x as in the case of the determination process of the period (e.g., lines 114-117 in [Cemre & Zafer] Protocol 19; lines 121-133 in [Defne & Ebru] Protocol 21). For example, they determined the displacement amount between the graphs of y=3sin(3x+6)-1 and y=3sin(3x)-1 as 2 through dividing the constant of x (i.e., 6) by the coefficient of x (i.e., 3) (e.g., lines 114-115 in [Cemre & Zafer] Protocol 19; lines 121-133 in [Defne & Ebru] Protocol 21). Moreover, they verified this determination in the *graphical register* based on the distance between the correspondence points on the x-axis of these two functions through using the scaled x-

[48] The same constant of the input of sine [cosine] causes the different displacement amounts along the x-axis in consequence of the different coefficients of the input of sine [cosine]. Accordingly, the displacement amounts change in cases (bx+c) and (x+c) as the input of sine [cosine], where $b \neq \{0,1\}$. We mean by "changed-displacement amounts" the displacement amounts in consequence of the same coefficient "c" on two sine [cosine] functions with the inputs in the form (bx+c) and (x+c).

[49] The operational-process of the period of a general form of sine and cosine is the division of 2π by the coefficient of x.

axis of the GSP as a measuring-tool (e.g., lines 118-123 in [Cemre & Zafer] Protocol 19; lines 125-130 in [Defne & Ebru] Protocol 21). Furthermore, they reasoned in the same way for the some other functions in GSP environment through manipulating the coefficients in the *symbolic register* and observing their simultaneous effects in the *graphical register* in terms of the changed-displacement amount –between two graphs with and without the constant term of the input of sine (e.g., lines 116-117, 132-136 in [Cemre & Zafer] Protocol 19).

Besides, in addition to the *graphical register*, students interpreted the cause of the changed-displacement amount (*Footnote 48*) in the (*unit*) *circle register* as well. Their actions and language imply that they converted the constant "*c*" of the input of sine in the *symbolic register* into a *c*-radian fixed-arc in the (*unit*) *circle register*; and then, interpreted this fixed-arc by means of two different, but dependent, (angular) speeds referring to x and bx. For example, when reasoning about the cause of the half-reduced displacement amount on the case of y=3sin(2x+6)-1, Cemre and Zafer considered the constant "6" of the input of sine as a fixed-arc; and then, interpreted that "6" [radian turning] was completed by the point referring to 2x in double speed of the point referring to x; in other words, "6" [radian turning] was completed dependently on 3-radian turning of the point referring to x (lines 101-110 in [Cemre & Zafer] Protocol 19). In the same way, Defne and Ebru reasoned about the cause of the changed-displacement amount on the case of y=3sin(3x+6)-1 in the (*unit*) *circle register* (lines 134-148 in [Defne & Ebru] Protocol 21). That is to say, they attributed the displacement amount in the *graphical register* to the (c/b)-radian turning of the reference point of (x) that produced the *c*-radian arc as the path of the reference point of (bx) so as to indicate (bx+c) in the (*unit*) *circle register*. It means that students reasoned about the *c*-radian arc between the reference-points of (bx) and (bx+c) in a dynamic-turning-structure through considering the turning-amount of (bx) dependently on the turning amount of (x). The researcher inferred that students' this reasoning arose from their effort to determine how much turning of (x) caused the *c*-radian "before arrival/after arrival" on a specific point between the reference points of (bx) and (bx+c) in the (*unit*) *circle register* (for more detailed articulation about "before arrival/after arrival" aspect, see the last paragraph of the heading *Parallel-*

displacement along the x-axis in graphical register). In the scope of teaching experiment, the researcher preferred not to discuss this issue in the GSP environment because the manipulation of the coefficients "*b*" and "*c*" for the functions in the form $y=a\sin(bx+c)+d$ in the (*unit*) *circle register* required time-consuming-constructions in contrast to the "easy-manipulation" of these coefficients in the *symbolic register* and "fast-observation" of their oppositions in the *graphical register*.

Finally, all these issues of Task 11 mentioned above on the general form of sine were discussed at the end of Task 16 on the general form of cosine. It was observed that students were able to transfer their final interpretations in Task 11 on the composed-visual oppositions of the composed-coefficients for sine mentioned above to those for cosine at the end of Task 16 as a consequence of their conceptual developments on association of cosine [sine] with sine [cosine] throughout Task 12 (see *Changed-arc with a constant difference in (unit) circle register* heading in Chapter 6).

[Defne & Ebru] Protocol 21

Researcher: (*Writing f(x)=3.sin(2x+4)-1 function by the aid of "new function" option under the graph menu of GSP; and then, pointing the coefficient 3 on the screen*) this coefficient, three, is related to what?
Defne: To the radius…
Ebru: Radius tripled.
Defne: Yes.
Researcher: Radius tripled. Then, how a graph do you expect?
Ebru: …higher.
Defne: (*Figuring a sine wave, and then, indicating its positive peak point*) this will go up more… …up to 3…
Ebru: (*Nodding her head up and down*)
Researcher: (*Pointing f(x)=3.sin(2x+4)-1 function's "-1" coefficient on the screen*) what about minus one?
Ebru: It will start from minus one (*pointing the position of -1 on the y-axis*).
Defne: Circle will go down one unit.
Researcher: Well, you say it would start from minus one (*indicating the position of -1 on the y-axis*),
Ebru: …up to two.
Defne: Yes, because it would go up 3 units [from -1].
Researcher: Well, what about down?

Defne: …minus one and minus three, it would be up to minus 4.

Ebru: Minus 4.

Researcher: Don't forget your statements! In just a moment, we plot its graph.

Defne&Ebru: (*Smiling*)

Researcher: Well, what about four (*pointing $f(x)=3.sin(2x+4)-1$ function's "4" coefficient on the screen*)?

Ebru: It will slide four units leftward (*holding her right hand vertically and then dragging horizontally leftward*).

Defne: (*After 4-second pause*) when multiplying by two (*pointing $f(x)=3.sin(2x+4)-1$ function's "2" coefficient on the screen*), period will become smaller (*putting her hands vertically parallel; and then, bringing them closer to each other*).

Ebru: There will be two repetitions [in $(0,2\pi)$ interval].

Defne: Yes. That is, the period will decrease in half [of 2π].

Researcher: Now, I plot it (*plotting the graph of $f(x)=3.sin(2x+4)-1$ function*).Control whether your expectations are true or not.

Defne: (*When the graph appeared on the screen like in Figure 6.45(a)*) it is from minus four to two.

Ebru: Yes. (*After 3-second pause*) but it doesn't start from minus one (*pointing the graph's y-intercept on the screen like in Figure 6.45(a)*). Why?

Defne: Yes. It would have started from minus one?

Researcher: Ok. Let's we start with $y=\sin(x)$ (*plotting $y=sin(x)$ function's graph like in Figure 6.45(b)*).

Defne&Ebru: (*When $y=sin(x)$ graph appeared screen, looking to the screen with satisfaction*)

Researcher: Now, let's we consider the coefficient 3 (*pointing $f(x)=3.sin(2x+4)-1$ function's "3" coefficient on the screen; and then, entering "3sin(x)" in the box of "plot new function" window under the graph menu of GSP*).

Ebru: (*Before its graph's appearance on the screen like in Figure 6.46(a)*) it will go up 3 and down -3.

Defne&Ebru: (*When the graph of $y=3sin(x)$ appeared on the screen*) yes.

Researcher: Now, we mention the coefficient -1 as well (*entering "3sin(x)-1" in the box of "plot new function" window under the graph menu of GSP*).

Defne: (*Before its graph's appearance on the screen like in Figure 6.46(a)*) it [graph of $y=3\sin(x)$] will go down one unit, (*after the graph appeared on the screen like in Figure 6.46(b)*) yes.

Ebru: (*Nodding her head up and down*)

Researcher: Ok, we consider what about addition of the coefficient 2 (*entering 3sin(2x)-1 in the box of "plot new function" window under the graph menu of GSP*). (*Before clicking the "ok" button*) how will its graph be… …comparing this blue graph (*pointing the graph of $y=3sin(x)-1$ on the screen like in like in Figure 6.46(b)*)?

Defne: Its period will become smaller.

Ebru: (*Nodding her head up and down*) there will be two repetitions [in $(0,2\pi)$ interval].

Defne: Yes. Period will decrease in half [of 2π].

Defne&Ebru: (*When the graph of $y=3sin(2x)-1$ appeared on the screen like in Figure 6.47(a), looking to the screen without speaking*)

Researcher: Let's we delete some of them (*deleting $y=3sin(2x+4)-1$, $y=sin(x)$ and $y=3sin(x)$ functions' graphs*). Indeed, is this pink graph (*pointing the graph of* *y=3sin(2x)-1 on the screen like in Figure 6.47(b)*) compressed into half comparing to this blue graph (*pointing the graph of y=3sin(x)-1*)?

Defne: Yes, while blue does one repetition in here (*dragging her index finger on the blue traced graph in the (0,π) interval*), this does two repetitions (*dragging her index finger on the blue traced graph in the (0,π) interval*).

Ebru: Uh-huh (*nodding her head up and down*).

Researcher: Please look at this function (*pointing y=3sin(2x+4)-1 function on the screen like in Figure 6.47(b)*). Only the coefficient 4 is there differently from this function (*pointing y=3sin(2x)-1 function*), is it ok?

Defne&Ebru: Yes.

Researcher: So, how would their graphs differ (*pointing y=3sin(2x+4)-1 and y=3sin(2x)-1 functions' symbolic expressions simultaneously on the screen like in Figure 6.47(b)*)? How will this pink graph change (*pointing the graph of y=3sin(2x)-1*)?

Ebru: It will be translated four units leftward (*moving her right hand horizontally leftward*).

Defne: (*Looking to the screen without speaking*)

Researcher: (*Plotting the graph of y=3sin(2x+4)-1*)

Ebru: (*When the graph of y=3sin(2x+4)-1 appeared on the screen like in Figure 6.48(a)*) it [GSP] drew wrong, didn't it?

Defne: Now, if we assume here as the reference (*pointing the pink graph's first intersection point with the positive x-axis on the screen like in Figure 6.48(a)*), four is here (*pointing -4 on the x-axis*), but this is further back (*pointing the dashed graph's intersection point with the negative x-axis nearest to -2*). Yes, it [dashed-graph plotted by GSP] is wrong.

Researcher: What is the translation amount?

Defne&Ebru: (*Coming closer to the screen so as to determine the horizontal displacement amount between two graphs*)

Researcher: (*Zooming in the unit length through dragging the point 1 on the x-axis rightward so as to provide with a more detailed scale for measuring like in Figure 6.48(b)*)

Defne: (*Pointing the line segment on the positive x-axis from the origin to the first intersection point of the pink graph on the screen like in Figure 6.48(b)*) here is about zero point one and… …a half [0.15]… …and here is also 0.15 (*pointing the line segment on the negative x-axis from -2 to the nearest-intersection point of the dashed-graph with the x-axis*).

Defne&Ebru: It [translation amount of y=3sin(2x)-1 graph onto y=3sin(2x+4)-1] is two.

Defne: (*After 4-second pause*) it may be due to the changed period.

Ebru: Yes, for 2x [input variable of sine], period changed into half… …so, translation amount also changed into half.

Defne: Yes.

Researcher: Do you mean if this coefficient is 6 rather then 4, then the translation amount will be half of six?

Defne&Ebru: Yes.

Researcher: Let's control.

Defne: (*Editing y=3sin(2x+4)-1 function as y=3sin(2x+6)-1*) here is again 0.15 (*pointing the first intersection point of the pink graph with the positive x-axis on the screen like in Figure 6.49(a)*)… …3 (*pointing -3 point on the x-axis, and then, dragging* *her index finger rightward on the x-axis until intersecting with the traced graph which is about 0.15 unit*).

Ebru: Yes. 3.

Defne: Yes. Again is half.

Researcher: Ok. You conjectured that this coefficient, 2, caused half (*pointing the coefficient 2 of* $y=3\sin(2x+6)-1$). If it is 3, what will do?

Ebru: Then, we will divide by 3. So it will be translated by 2 units.

Defne: Yes.

Researcher: Let's control.

Defne: Ebru, now, you edit.

Ebru: (*Editing* $y=3\sin(2x+4)-1$ *and* $y=3\sin(2x)-1$ *functions, respectively, as* $y=3\sin(3x+6)-1$ *and* $y=3\sin(3x)-1$) yes, two (*putting her index finger, respectively, on -2, the nearest intersection point of dashed graph with the x-axis, the origin and the first intersection point of the pink graph with the positive x-axis on the screen like in Figure 6.49(b)*).

Defne: Yes.

Ebru: So, we divide 6 by 3… …that's, it is translated by 2 units (*dragging her right hand's index finger horizontally leftward*).

Defne: Uh-huh (*nodding her head up and down*).

Researcher: What is the correspondence of this situation on the unit circle? I mean what does $\sin(3x+6)$ mean on the unit circle?

Defne: There would be [reference points with the angle measures] x, $3x$… …and 6 more (*figuring an arc as if a principal arc referring to x, and then, a greater arc so as to indicate 3x, finally, a further arc following the last arc*). When x is completing the circle, $3x$… …so, $3x+6$ completes three [full-round].

Ebru: Uh-huh (*nodding her head up and down*).

Researcher: What about two-unit translation of the graph (*pointing respectively the first intersection point of the pink graph with the positive x-axis and its correspondence on the dashed graph around -2 on the screen like in Figure 6.49(b)*)?

Ebru: x turns with one third speed of $3x$. So, 6 [-radian turning] for $3x$ means… …2 for x.

Defne: Yes. Because here is x (*dragging her index finger on the x-axis*), the translation amount is considered as 2.

Ebru: Uh-huh (*nodding her head up and down*)

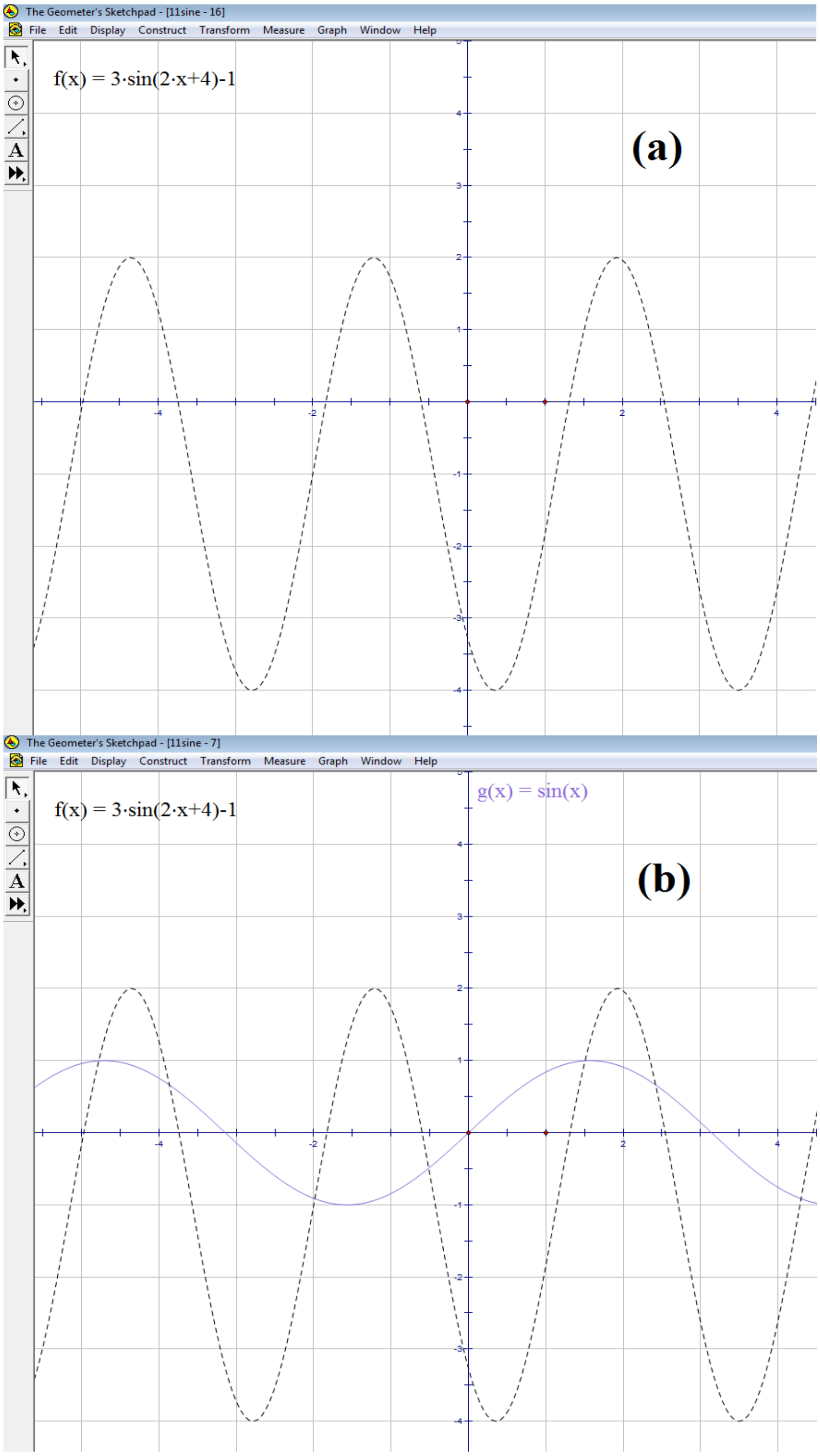


Figure 6.45. Conversion of a general form of sine function composed by all coefficients in the *symbolic register* into the *graphical register*

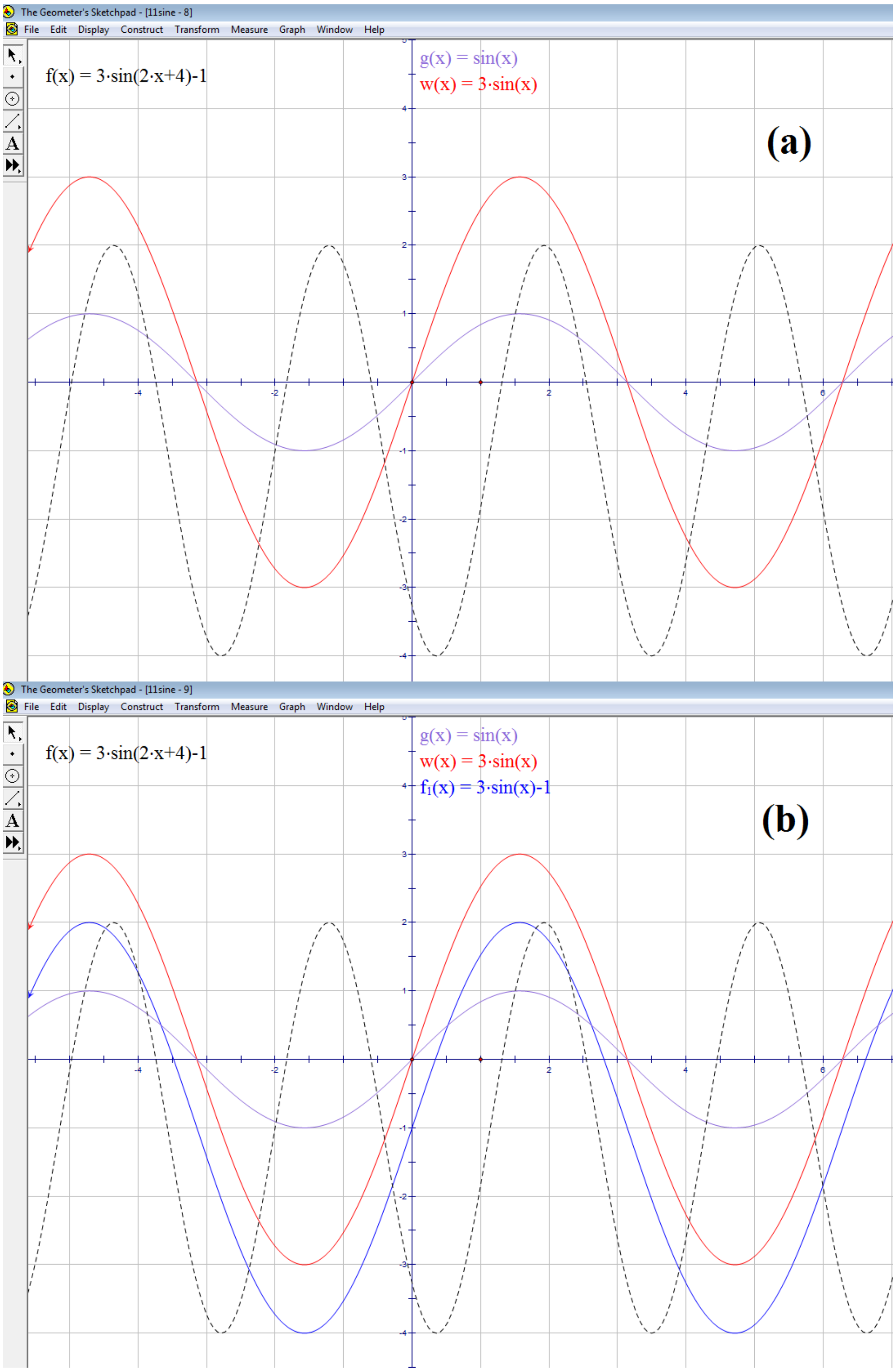


Figure 6.46. An example of the conversion of *a* and *d* coefficients in the *symbolic register* regarding the general form of sine function, i.e., $y=a.\sin(bx+c)+d$, into the *graphical register*

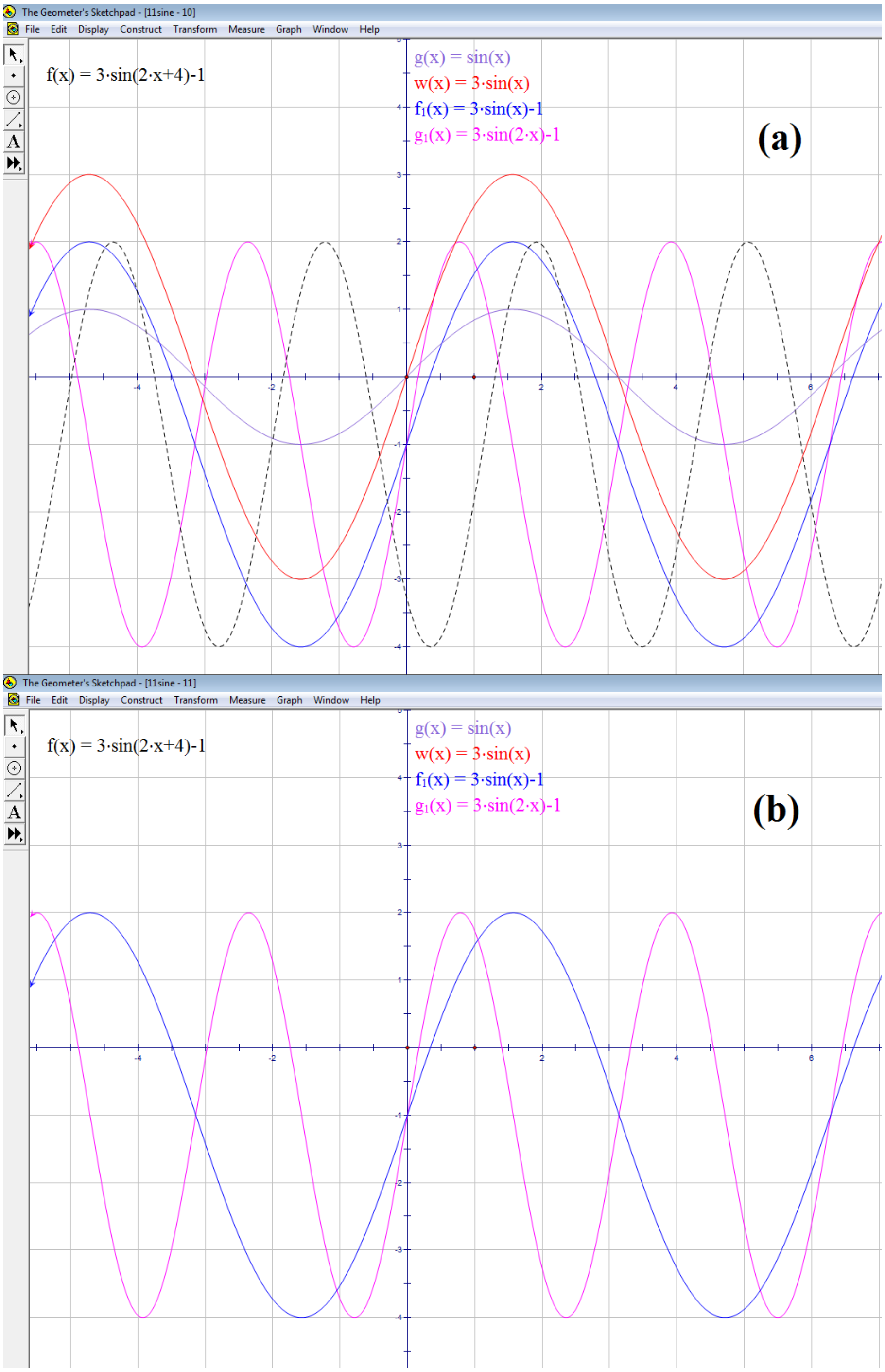


Figure 6.47. An example of the conversion of *a*, *b* and *d* coefficients in the *symbolic register* regarding the general form of sine function, i.e., *y*=*a*.sin(*bx*+*c*)+*d*, into the *graphical register*

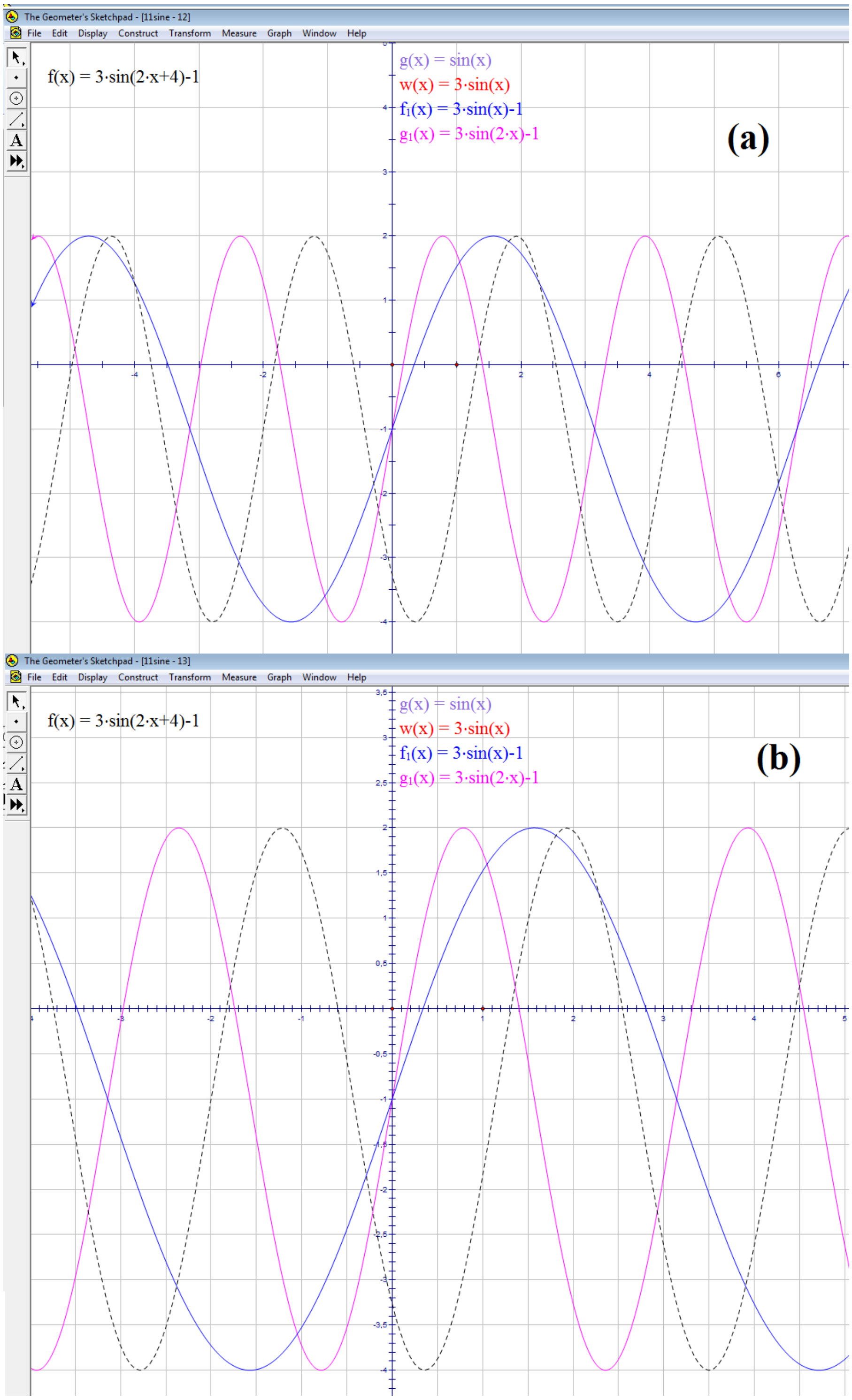


Figure 6.48. An example of the conversion of some combinations of all coefficients in the *symbolic register* regarding the general form of sine function, i.e., $y=a.\sin(bx+c)+d$, into the *graphical register*

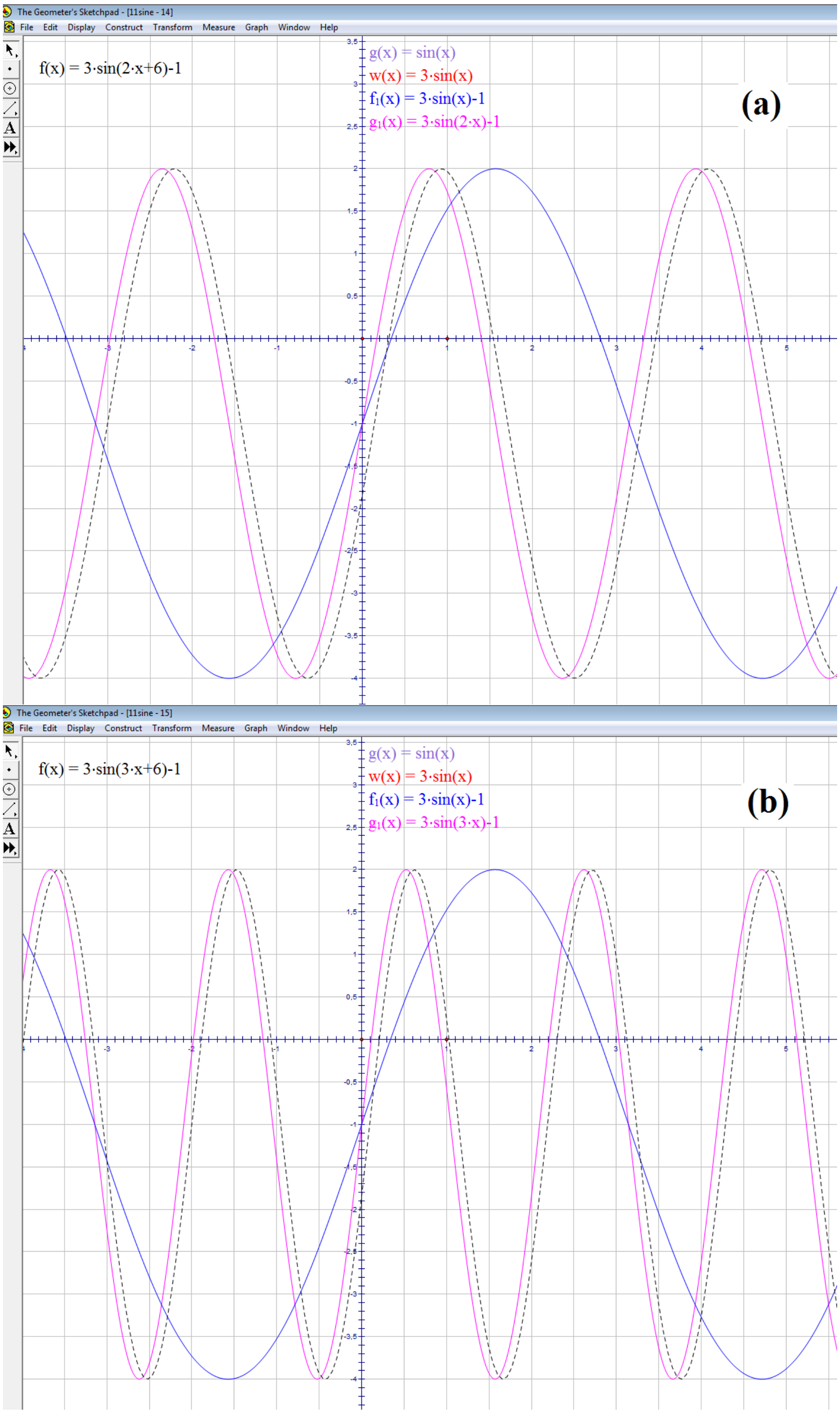


Figure 6.49. Two other examples of the conversion of the general form of sine function in the *symbolic register*, i.e., $y=a.\sin(bx+c)+d$, into the *graphical register* so as to discuss the composed effect of b and c coefficients

[Cemre & Zafer] Protocol 19

Researcher: (*Writing on a paper* $y=3\sin(2x+4)-1$ *function through uttering*) what does this three remind you (*pointing the coefficient of sine*)?

Zafer: The radius.

Cemre: (*Nodding her head up and down*) radius is three… …graph also between -3 and 3

Zafer: Uh-huh (*Nodding his head up and down*).

Researcher: What about this minus one (*pointing the coefficient "-1" of* $y=3\sin(2x+4)-1$)?

Zafer: Graph is one-unit going down (*dragging her hand vertically downward*).

Cemre: (*Nodding her head up and down*) graph is between -4 and 2. (*After 3-second pause*) …or, unit circle is one-unit going down.

Zafer: (*Nodding his head up and down*)

Researcher: What about here… …$2x+4$ (*pointing the input of sine on* $y=3\sin(2x+4)-1$ *function*)?

Cemre: $2x$ indicates… … its full round is completed in half of time [in which x takes one full round]. That is, its graph is compressed into half [regarding $y=\sin(x)$ graph] (*holding her hands vertically parallel to each other; and then, bringing them closer to each other*).

Zafer: (*Nodding his head up and down*) period reduced by half.

Cemre: Yes.

Researcher: Ok. What do you say about this four (*pointing the coefficient "4" of* $y=3\sin(2x+4)-1$)?

Cemre&Zafer: Leftward sliding.

Researcher: In which amount?

Zafer: 4 radians.

Cemre: Uh-huh.

Researcher: Can you plot it.

Cemre: (*Plotting the graph of* $y=3\sin(2x+4)-1$ *function by GSP. When the graph appeared on the screen like in Figure 6.45(a)*) yes, it is like our expectation. We said it would be between 2 and -4 (*pointing respectively 2 and -4 on the y-axis*)… …also we said it would slide leftward about four (*pointing the first positive peak point on the left of the y-axis*)… …and we said period would reduce (*indicating an interval referring to the dashed-graph's one-full-action*). It is like our expectation.

Zafer: (*Listening Cemre's explanations through looking carefully to the screen and nodding his head up and down*) yes.

Researcher: What do you mean by 4-unit sliding?

Cemre&Zafer: (*Holding without speaking about 6 seconds*)

Researcher That is, do you mean that $y=3\sin(2x)-1$ function's graph will slide leftward by 4-unit (*writing* $y=3\sin(2x)-1$ *on the paper under the* $y=3\sin(2x+6)-1$ *expression through uttering*)?

Cemre: …plus four…

Zafer: That's to say… (*Not being sure*) yes.

Researcher: Then, please plot the graph of this (*pointing y=3sin(2x)-1 expression on the paper*).

Cemre: (*Plotting y=3sin(2x)-1 function's graph*)

Zafer: (*When its graph appeared like the pink graph in Figure 6.48(b), coming closer to the screen, and after 2-second going back without speaking*) It is translated leftward, it is ok... …but [translation amount is not] 4…

Cemre: 4 is impossible for this distance (*pointing on the screen like in Figure 6.48(b, respectively, the first positive peak point of the dashed-graph, and then, its first correspondence on the left on the pink graph*).

Zafer: (*Getting a load of the graphs on the screen*)

Cemre: This four was how? What did we do with it (*turning her glance from the screen, and staring*)? I don't understand this four. Our expectation did not come true about this coefficient 4.

Researcher: What is the translation amount between them?

Cemre: Very less than four.

Zafer: If we compare these points (*pointing the first intersection point of the pink graph with the positive x-axis; and then, its correspondence on the dashed graph around -2 on the screen like in Figure 6.48(b)*)…

Cemre: Just a minute! (*Coming closer to the screen*) our concern is this distance (*pointing with her index finger the line segment on the x-axis between two points that Zafer indicated*), isn't it? That is, the distance between two graphs…

Zafer: Yes.

Cemre: There is less than 0.2 (*indicating the line segment on the x-axis from the origin and the first intersection point of the pink graph with the positive x-axis on the screen like in Figure 6.48(b)*).

Zafer: Is it 2?

Cemre: (*Pointing with her index finger "-1" on the x-axis; and then, counting increasing tenth parts from the point "-1" towards the intersection point of the dashed graph with the negative x-axis around "-2"*) one, two, three, four, five six, eight… …it [distance between graphs] is 2… …half of four.

Zafer: (*Looking to the screen without speaking*)

Researcher: What about when this (*pointing the coefficient 4 on the symbolic expression of y=3sin(2x+4)-1*) is 2?

Cemre: I edit it (*editing y=3sin(2x+4)-1 function as y=3sin(2x+2)-1*).

Zafer: (*Coming closer to the screen*) now, it [distance between graphs] was almost 1.

Cemre: It was exactly 1. Aha! Again it is half. That is, half of 2… …1.

Zafer: I don't know it was exactly one.

Cemre: (*Counting the distance between the correspondence points of two graphs on the x-axis by the aid of tenth-scale as she did before*) one, two, three, four, five, six, nine, ten [tenths], so, it is 1…

Zafer: (*Looking to the screen without speaking*)

Researcher: (*Constructing an arbitrary point (A) on the y=3sin(2x)-1 graph; next, measuring its abscissa* (x_A) *and ordinate* (y_A)*; and then, plotting* $(x_A - 1, y_A)$ *ordered pair.)*

Zafer: (*When this plotted-point appeared on the graph of y=3sin(2x+2)-1 as the correspondence of the point A on the graph of y=3sin(2x+2)-1*) it was exactly 1 (*smiling*).

Cemre: I said already it was exactly 1 as much as half of it [constant of the input variable of sine]. (*After 5-second pause*) but I don't understand why it was half? When we considered four, it was again half, that is, 2.

Researcher: If the coefficient is 6 instead of 2 (*pointing the constant coefficient of the input of the y=3sin(2x+2)-1 function on the screen*), will the translation amount 3?

Cemre&Zafer: Yes.

(*In the same way, similar discussion in GSP environment were done on y=3sin(2x+6)-1 function. They determined the translation amount of the graph of y=3sin(2x+6)-1 as 3 units of y=3sin(2x)-1.*)

Researcher: So, why is the translation amount reduced by half?

Zafer: Because, the period was reduced by half.

Cemre: Uh-huh… When the period is π, 6 [fixed-amount] is completed [by $2x$] two times faster [than x]. That is, while $2x$ is going about 6 (*figuring a circle starting from its far right point in the counterclockwise direction*), x is going about 3 (*figuring the up-half circle in the counterclockwise direction*). Actually, if that [coefficient of x] is 3 [instead of 2], that is, if it is $3x+6$, then it [translation amount] will be 2.

Zafer: Yes, yes… …all right (*nodding his head up and down*)…

Cemre: Then, 6 is completed [by $3x$] three times faster [than x on the circle].

Zafer: Uh-huh (*nodding his head up and down*).

Researcher: Indeed is it so?

Cemre: (*Smiling*) yes, it is. Let's we immediately look (*smiling*)!

Zafer: (*Nodding his head up and down when smiling*)

Cemre: In case $3x+6$, it [displacement-amount] will be 2 (*editing y=3sin(2x+6)-1 and y=3sin(2x)-1functions, respectively, as y=3sin(3x+6)-1 and y=3sin(3x)-1*).

Zafer: (*When Cemre was editing these functions, looking to the researcher*) then, if we take $x/2$, when we write $x/2+6$, it [translation amount] will be 12.

Cemre&Zafer: (*When the graphs appeared on the screen like in Figure 6.49(b), they are coming closer to the screen*) yes, it is two.

Researcher: How do you determine?

Zafer: I looked at here (*pointing the first intersection point of the pink graph with the positive x-axis on the screen like in Figure 6.49(b); and then, its first correspondence on the dashed graph on the negative x-axis*). It [distance] is two.

Researcher: But is it exactly two?

Zafer: Exactly yes (*smiling*).

Cemre: Certain sure (*laughing*)!

Zafer: (*Constructing the correspondence points on these two graph with two-unit horizontal distance so as one of them to be an arbitrary point (x,y), and the other to be its horizontal displacement point by 2-unit, (x-2,y) through taking GSP's measure, calculate and plot as (x,y) advantages*) yes.

Cemre: (*Looking to the screen with satisfaction*)

Researcher: If we think about the example as Zafer said before… …that is, what will be the displacement amount if the input is $x/2+6$?

Zafer: It [$x/2$] would be left behind [x on the circle].

Cemre: It will be 12.

Zafer: (*Nodding his head up and down*).

6.4. Summary of Students' Developments on discrimination of Trigonometric Functions from (Unit) Circle Register

During the first seven episodes of the teaching experiment, as a consequence of their dynamically-linked *conversions* between representational registers, students showed significant proggess on recognition of trigonometric functions' basic forms in each registers. They were able to interpret sine [cosine] as a function mapping from arc angle to the corresponding opposite [adjacent] side of the reference right triangle in the (*unit*) *circle register* as well as a function representing the coordinated-variations of angle measures and corresponding sine [cosine] values in the *graphical register* (see *Summary of Students' Developments on Basic Trigonometric Functions* heading in Chapter 5). Moreover, unlike their problematic initial concept images on angle and angle measure, they were able to reason a *static angle structure* in the (*unit*) *circle register* with the infinitely many [negative or positive] equivalent but not equal measures in the *symbolic register* through considering *dynamic directed turnings* (see *Summary of Students' Developments on Angle and Angle Measure* in Chapter 5). All these progresses emerged when students were reasoning about a mathematical object (such as angle measure and sine [cosine] value) under the dynamic-and-linked variations of a point on the unit circle referring to this object and its converted-correspondence in the *graphical register*.

From the 8th episode forward, visual features of the different representations of trigonometric functions (such as radius of the circle, position of the center, reference point referring to trigonometric value in the (*unit*) *circle register*) were systematically varied to provoke students to discuss on a new function what is mathematically relevant or what is mathematically different when compared with trigonometric functions' basic forms in the (*unit*) *circle register*'s content.

In Task 8, the discussion focus changed from the basic form of the sine function (i.e., y=sin(x)) into a general form of sine (i.e., y=a.sin(x)). For this purpose, both functions were represented in the (*unit*) *circle register* by their reference points, *P* and *R*, on the origin-centered **unit circle** and **non-unit circle** (see *Figure 6.1*). Throughout this task, students developed significant understandings referring to the trigonometric functions. On the one hand, when measuring the changed-radius by GSP, students encountered a trouble arising from the difference between the distance-measure-unit preference of GSP as centimeter and the visual distance-measure-unit of the coordinate axes. At that point, the researcher provoked students to reason what they called as the "unit" through recommending them to measure it by GSP. It was the point that students had just started to reason about a number different from 1 as a unit. In other words, they had just considered a non-unit circle regarding the centimeter distance-measure-unit as a unit circle regarding the visual distance-measure-unit of the coordinate axes. This reasoning prompted a distinct shift on their association of the arc lengths with the angle measure in radians by means of this arc's length through defining their own unit (as radius) for the distance-measure. On the other hand, through taking GSP's "measure", "calculate" and "drag-drop" advantages, students were able to discriminate (*i*) the opposite side length of the reference right triangle on the non-unit circle from that on the unit circle, and (*ii*) the principal arc length on the non-unit circle from that on the unit circle, as well as recognize (*i*) the opposite side length of the reference right triangle on the non-unit circle as the multiplication of sin(x) with the radius of the non-unit circle in the *symbolic register*, and (*ii*) the equality between the principal arcs' angle on the unit circle and non-unit circle (see *Changed-radius in (unit) circle register* heading in Chapter 6). It was the point that students had just started to reason about a general sine function in the form $y=a.\sin(x)$ in the (*unit*) *circle register* for positive coefficients through associating the coefficient a with the changed-radius.

In Task 9, the discussion focus was another general form of sine (i.e., $y=\sin(x)+d$) comparing with the basic form of the sine function (i.e., $y=\sin(x)$). For this purpose, a unit circle with a manipulable-center was constructed on the coordinate system in order to define a new function by the reference point P as a mapping from its angle to its ordinate (see *Figure 6.16*). Throughout this task, students developed significant understandings referring to the trigonometric functions. When reasoning about the new function through comparing and contrasting with sine, each student showed, in general sense, the similar developments. That is, all students were able to express this function in the *symbolic register* as an additive operation between sine and the directed-distance of the manipulable-center to the *x*-axis. Moreover, they were able to truly revise this expression in the *symbolic register* regarding the variation of the manipulable-center in the (*unit*) *circle register*; furthermore, they were able to reason its independence from the horizontal variation and dependence only the vertical variation of the manipulable-center. However, in special sense, students in each group were different in terms of the concept development stages related to the new function defined on the unit circle **with different-center from the origin**. That is to say, on the one hand, despite of their *condensation* stages in case of the unit circle whose center was located on the origin, Defne and Ebru were at the *interioriorization* stage in other cases of the unit circle. Defne's and Ebru's focuses were predominantly on the processes related to the *y*-components in the (*unit*) *circle register* based on the determination of the ordinates. On the other hand, Cemre and Zafer were at least in the *condensation* stage in each case of the unit circle. Cemre's and Zafer's focuses were predominantly on the condensed-whole of the processes instead of their details. They focused direcly on the dissociation and association of this manipulated-function from sine in the (*unit*) *circle register* instead of on operational processes. Beside, their some actions and language indicated the *reification* stage as well. For example, Zafer mentioned the horizontal axis from the center as the *x*-axis considering the *x*-axis as if a reified-object without going into details but with awareness of its different location. Cemre mentioned the unit circle whose center located on the origin as if a reified-object; and was able to change its position up-down and left-right in her mind as a whole on the coordinate system (see *Changed-center in (unit) circle register* heading in Chapter 6).

In Task 10, the discussion was focused on another general form of sine (i.e., y=sin(x+c)) comparing with the basic form of the sine function (i.e., y=sin(x)). For this purpose, in order to represent these two functions, two points were constructed on the unit circle so that one of them was an arbitrary point and the other was its rotated-position about the origin by a fixed-measure (see *Figure 6.28*). Where, the new function was defined from the angle of the point *P* to the ordinate of its rotated-position. It was the first task that a function was defined based on two different points on the unit circle so that one of them was referring to the input and the other was referring to the output. When investigating the variation of these two points on the unit circle under the manipulation of the point *P*, all students recognized that the arcs between the point *P* and its rotated-position remained invariant in the (*unit*) *circle register*. Except Zafer, none of the other students associated this function with sine until the construction of its graphical representation. Although Zafer associated this function with sine based on its visual definition on the unit circle, he did not reason about its symbolic representation based on its representation in the (*unit*) *circle register*. In other words, throughout Task 10, none of the students reasoned in the (*unit*) *circle register* about the ordinate of the rotated-position of the point *P* as the sine value of the angle corresponding to the rotated-position through combining the angle of the point *P* and the angle of the rotation. So, in the following tasks (Tasks 11, 12, 15 and 16), the researcher determined to provoke students to identify the abscissa/ordinate of the rotated-position of the point *P* in the (*unit*) *circle register* in terms of sine/cosine in accordance with the tasks' themes.

In Task 11, this systematic variation of the general forms of sine was pursued with the functions in the form y=sin(bx) as the discussion-focus. As an initial example, y=sin(2x) was handled without mentioning its symbolic representation. For this purpose, two points, the point *P* and *P′* (see *Figure 6.39(a)*), and then, the perpendicular segments from these points to the x-axis (see *Figure 6.39(b)*) were constructed in the (*unit*) *circle register*. Where, the point *P* was an arbitrary point on the unit circle, and the point *P′* was its rotated-position about the origin by the principal measure of the point *P*. When the researcher provoked students to identify the ordinate of the rotated position of the point *P* in terms of sine of the new angle, all students were able to identify (*i*) the principal arc corresponding to the rotated position of the point *P* as *2x*, and then, (*ii*) its ordinate as sine of 2x through considering the definition of sine (from the arc corresponding to a point on the unit circle into its ordinate). It was the point that students had just started to reason about a general form of the sine function from an arc, i.e., x, to the ordinate of another arc defined dependently on x in the (*unit*) *circle register*. In Task 12, this reasoning prompted a distinct shift on their reasoning about the relation between sine and cosine in the (*unit*) *circle register*. That is to say, they were able to convert the relation between the directed measures of the adjacent [opposite] side of the reference right triangle and the opposite [adjacent] side of its rotated-counterpart by $\pi/2$ radian about the center in the (*unit*) *circle register* into the equality between cos(x) [sin(x)] and sin($x+\pi/2$) [-cos($x+\pi/2$)] in the *symbolic register*. Moreover, they were able to extend this reasoning based on the structure obtained through rotation by $\pi/2$ radian in the counterclockwise direction into the reasoning on the structures obtained through rotations by the integer multiples of $\pi/2$ radian in any directions, which emerged as a result of the teaching experiment. In special sense, Task 12 was the first task that students made sense of the negative coefficient of –sine [–cosine] function in the (*unit*) *circle register* through considering it as a function from the x angle to the perpendicular line segment from the point corresponding to the ($x\pm\pi$) angle to the x-axis [y-axis]. Furthermore, Task 12 was the first task that students had an alternative visual focus referring to sine [cosine] in the (*unit*) *circle register* instead of the opposite [adjacent] side of the reference right triangle. That is to say, students' actions imply that they started to reason about sine [cosine] in the (*unit*) *circle register* through exchanging their focuses between the opposite [adjacent] side of the reference right triangle and its facing-side of the *reference-rectangle* (see *Footnote 44*).

In Task 11, all students were able to easily and truly reason about functions, for instance, sin($2x$), sin($3x$), sin($x/2$), in the (*unit*) *circle register* through comparing full-round turnings of their respective reference points on the unit circle with each other by the aid of drag and drop option of GSP. Moreover, Cemre and Zafer were able to extend this reasoning onto the negative multiples of x through attributing the meaning of the negative sign in the *symbolic register* to the clockwise direction of the rotation in the (*unit*) *circle register*. In other words, they had just defined y=sin($-x$) in the (*unit*) *circle register* as a function mapping the angle of a point on the unit circle to the ordinate of its reflection point regarding the x-axis. With the other group of students, Defne and Ebru, as a consequence of their troubles on the period issue at the beginning of Task 11 (see *Periodicity as pattern based on behaviors of trigonometric functions* heading in Chapter 6), the researcher preferred to postpone the discussions about the meaning of the negative coefficient of the input variable in the (*unit*) *circle register* to Task 16. In task 16, when reasoning about y=cos($-x$) in the (*unit*) *circle* *register*, it was observed that all students were able to attribute the meaning of the negative multiples of x in the *symbolic register* to the clockwise direction of the rotation in the (*unit*) *circle register* in the similar manner that Cemre and Zafer reasoned about y=sin(–x) in Task 11.

Moreover, as a consequence of their developments throughout Task 12 in terms of making sense of sine [cosine] as cosine [sine] (see *Changed-arc with a constant difference in (unit) circle register* heading in Chapter 6), all students were able to transfer their interpretations made on sine in Tasks 8, 9, 10 and 11 onto cosine, respectively, in Tasks 13, 14, 15 and 16.

6.5. Summary of Students' Developments on discrimination of Trigonometric Functions from Graphical Register

Integration of the graphical representations in last two episodes of the teaching experiment's first part fortified students' recognition of trigonometric functions in each representational registers. Visual representations of sine [cosine] on the same coordinate plane both in the (*unit*) *circle register* and the *graphical register* provided students with the opportunity to compare and contrast the dynamic and simultaneous variations of the reference point on the unit circle and its converted form in the *graphical register*. This opportunity prompted a distinct shift on students' recognition of the same object (i.e., sine or cosine) represented in different representational registers, as well as their discrimination of what is/is not mathematically relevant in terms of the coordinate plane both in the (*unit*) *circle register* and *graphical register*. For example, dynamic-and-linked manipulations of the point on the unit circle and its correspondence on the sine [cosine] graph fortified students' concept images on the meaning of π. Anymore "coordinate plane" was not a *cognitive conflict factor* when the position of π on the x-axis is considered simultaneously within the *graphical register* and the (*unit*) *circle register* (see *Summary of Students' Developments on Angle and Angle Measure* heading in Chapter 5). In addition, they were able to differentiate contents of the coordinate planes in the (*unit*) *circle register* and the *graphical register*. For example, they differentiated the meaning of the abscissa [ordinate] of a point on the unit circle from the meaning of the abscissa [ordinate] of a point on the sine [cosine] graph (see *Summary of Students' Developments on Basic Trigonometric Functions* heading in Chapter 5).

In the second part of the teaching experiment, visual features of the different representations of trigonometric functions (such as magnitude, parallel-displacement along the y-axis, parallel-displacement along the x-axis and compressed/stretched wavelength in the *graphical register*) were systematically varied to provoke students to discuss on a new function what is mathematically relevant or what is mathematically different when compared with trigonometric functions' basic forms in the *graphical register*'s content.

In Task 8, the discussion focus changed from the basic form of the sine function (i.e., y=sin(x)) into a general form of sine (i.e., y=a.sin(x)). For this purpose, both functions were represented initially in the (*unit*) *circle register* by their reference points, P and R, on the origin-centered **unit circle** and **non-unit circle** (see *Figure 6.1*), and then, in the *graphical register* by their reference points P' and R' constructed as dynamically linked to the point P and R (see *Figure 6.6*). When the traced graphs by their reference points P' and R' appeared on the screen, students determined a proportional relation between their magnitudes, i.e., the changed-visual features, and expressed the new function in terms of sine (for example, 6.sin(x)). Initially, only Cemre associated the changed-magnitude in the *graphical register* directly with the changed-radius in the (*unit*) *circle register*. So, the researcher encouraged them to reason about these functions under the manipulation of the radius of the non-unit circle. All students were able to correctly revise the function's symbolic expression regarding the changed-radius. Furthermore, they were able to express this function's symbolic form regarding the radius. When investigating the dynamically-linked components in different registers under the manipulation of the radius and the angle's openness, they were able to generalize their reasoning about the relations among "changed-radius" in the (*unit*) *circle register*, "changed-magnitudes" in the *graphical register* and “changed-coefficient *r* of the *r*.sine function” in the *symbolic register* (see *Changed-magnitude in graphical register* heading in Chapter 6).

In Task 9, the discussion focus was based on another general form of sine (i.e., $y= \sin(x)+d$) comparing with the basic form of the sine function (i.e., $y=\sin(x)$). For this purpose, a unit circle with a manipulable-center was constructed on the coordinate system in order to define a new function by the reference point *P* as a mapping from its angle to its ordinate (see *Figure 6.16*). When this function was constructed in the *graphical register* as dynamically-linked to its representation in the (*unit*) *circle register*, all students were able to associate this function with the sine [cosine] function in the *graphical register* based on its visual-shape. Moreover, they were able to reason about its graph as a parallel displacement of the sine [cosine] graph along the *y*-axis in the positive/negative direction initially without considering the displacement amount. Through reasoning about the variation of the graph under the manipulation of the unit circle’s location, students started to reason considering displacement amount. In other words, they associated this (directed) amount of the parallel displacement of the sine graph with the center’s (directed) distance from the *x*-axis, as well as with the (signed) constant of the sine function in the *symbolic register* (see *Parallel-displacement along the y-axis in graphical register* heading in Chapter 6).

In Task 10, the discussion was focused on another general form of sine (i.e., $y=\sin(x+c)$) comparing with the basic form of the sine function (i.e., $y=\sin(x)$). For this purpose, in order to represent these two functions, two points were constructed on the unit circle so that one of them was an arbitrary point and the other was its rotated-position about the origin by a fixed-measure (see *Figure 6.28*). Where, the new function was defined from the angle of the point *P* to the ordinate of its rotated-position. When this function was constructed in the *graphical register* as dynamically-linked to its representation in the (*unit*) *circle register*, whereas Cemre and Ebru associated this graph with sine based on its shape without going into details, Zafer and Defne differentiated it from sine based on its *y*-intercept. When reasoning about this graph in a more detailed way through comparing and contrasting with the sine graph, except Defne, all other students reasoned about these two graphs as the parallel displacement of each other along the *x*-axis. Their initial reasoning about the parallel displacement idea was to determine the parallel displacement amount through focusing on the distance between the corresponding points of two graphs that were selected according to their preferences. Their next reasoning step was to define the function represented in the *graphical register* regarding sine in the *symbolic register*. During that phase, they were aware that the horizontal distance between corresponding points of two graphs would affect the input of sine as a constant. However, their concept images on the graphs' parallel displacements along the *x*-axis in the *graphical register* were including a conjecture on the conversion of the positive [negative] direction of the parallel displacement along the *x*-axis in the *graphical register* into the positive [negative] sign of the constant affecting the input variable in the *symbolic register*. While this conjecture caused Cemre and Zafer's confusion about the location of the graph of y=sin(x+2.3) [y=sin(x-2.3)] on the left [right] with respect to sine, Ebru did not encounter any confusion as a consequence of her failure to preserve her assumption on the *source*-function and *target*-function of transformation coherently in the *graphical register* and the *symbolic register* (see *Parallel-displacement along the x-axis in graphical register* heading in Chapter 6). On the other hand, throughout Task 10, differently from others, Defne reasoned about these two graphs based on their slopes with each other but focusing only on their restricted parts from their *y*-intercepts up to their first intersection points. With the other group, Cemre and Zafer, discussions were done based on three purposively constructed points, i.e., A, B and C (see *Figure 6.31*) in order to provoke them to reason through changing their attention-focus hierarchically in a rectangular path among four points in the *graphical register*: (*i*) the point *B* (*Figure 6.31*) (*ii*) its projection point on the *x*-axis, (*iii*) the projection point of the point *A* on the *x*-axis (*iv*) the point *A* and (*v*) the point *B*. This hierarchical-rectangular path of their attention-focuses caused their identification of the graph with respect to sine. However, even though they were satisfied with this identification process in the *symbolic register*, it seemed that they had not been yet satisfied with the location of y=sin(x-2.3) on the right with respect to sine in the *graphical register*. Considering Cemre's and Zafer's unsatisfuction and Ebru's and Defne's quite different reasoning focusses on comparison of two graphs as well as their problematic reasoning parts, the researcher preferred to postpone discussions on the parallel displacement of the sine [cosine] graph along the *x*-axis in the *graphical register* in Task 12 [Task 15] considering its meaning simultaneously in the (*unit*) *circle register*. In Task 11, students' making sense about a general form of the sine function from an arc, i.e., *x*, to the ordinate of another arc defined dependently on *x* in the (*unit*) *circle register* (see *Summary of Students' Developments on discrimination of Trigonometric Functions from (Unit) Circle Register* heading in Chapter 6) prompted a distinct shift on their making sense of the symbolic expressions in the form of sin(x+c) and cos(x+c) in the (*unit*) *circle register*. As a consequence, for example, in Task 15, Defne was able to reason about the graph of cos(x+1) as the parallel displacement of the graph of cos(x) by 1-unit along the *x*-axis in the negative direction based on the patterns on their actions in the (*unit*) *circle register*. Moreover, all students interpreted the positive [negative] constant "*c*" as the *c*-unit length before arrival [after arrival] in a value in the *graphical register* as a consequence of the *c*-radian rotation in the counterclockwise [clockwise] direction in the (*unit*) *circle register* in Task 12 and Task 15 (see *Parallel-displacement along the x-axis in graphical register* heading in Chapter 6).

In Task 11, this systematic variation of the general forms of sine was pursued with the functions in the form y=sin(bx) as the discussion-focus. As an initial example, y=sin(2x) was handled without mentioning its symbolic representation. For this purpose, two points, the point P and P' (see *Figure 6.39(a)*), and then, the perpendicular segments from these points to the *x*-axis (*Figure 6.39(b)*) were constructed in the (*unit*) *circle register*. Where, the point P was an arbitrary point on the unit circle, and the point P' was its rotated-position about the origin by the principal measure of the point P. When the function mapping from the angle of the point P to the *y*-component of the point P' was constructed in the *graphical register* as dynamically-linked to its representation in the (*unit*) *circle register*, all students reasoned about the compression of the graph of y=sin(2x) as much as half comparing with the graph of y=sin(x) based on the intervals (0,π) and (0,2π) in which, respectively, sin(2x) and sin(x) completed their one-full-actions in the *graphical register* as a consequence of one-full-round turnings of their reference points in the (*unit*) *circle register*. In the same way, they were able to reason about the stretched-graphs comparing with y=sin(x). Moreover, Cemre and Zafer were able to extend this reasoning in Task 11 onto the graph of sine for the negative multiples of x through attributing the meaning of the negative sign of the coefficient of x in the *symbolic register* into the *graphical register* as the reflection of the graph of y=sin(x) regarding the x-axis. On the other hand, with the other group of students, Defne and Ebru, as a consequence of their troubles on the period issue at the beginning of Task 11 (see *Periodicity as pattern based on behaviors of trigonometric functions* heading), the researcher preferred to postpone the discussions about the meaning of the negative coefficient of the input variable in the *graphical register* to Task 16. In task 16, when reasoning about y=cos(–x) in the *graphical register*, it was observed that all students were able to attribute the meaning of the negative multiples of x in the *symbolic register* to the negative angles represented on the negative x-axis in the *graphical register*. By this way, they interpreted the graph of y=cos(–x) as the reflection of y=cos(x) regarding the y-axis through considering y=cos(–x) as a function from x to cos(–x) in the *graphical register* in the similar manner that Cemre and Zafer reasoned about y=sin(–x) in Task 11.

Moreover, as a consequence of their developments throughout Task 12 in terms of making sense of sine [cosine] as cosine [sine] (see the heading *Changed-arc with a constant difference in (unit) circle register* heading in Chapter 6), all students were able to transfer their interpretations made on sine in Tasks 8, 9, 10 and 11 onto cosine, respectively, in Tasks 13, 14, 15 and 16.

6.6. Summary of Students Developments on Periodicity

In last two episodes of the teaching experiment's first part, dynamic-and-linked manipulations of sine [cosine] between different registers promoted students' reasoning about the repetition of the sine [cosine] values within the (*unit*) *circle register* as a consequence of full-round turnings of the point P (see *Figure 5.21*), and the repetition of the sine [cosine] values in the *graphical register* (see *Summary of*

Students' Developments on Periodicity heading in Chapter 5). In these tasks, there was only one point on the unit circle referring to the full-round turnings.

In the second part of the teaching experiment, visual features of the different representations of trigonometric functions were systematically varied to provoke students to discuss on a new function what is/is not mathematically relevant when compared with trigonometric functions' basic forms. Period was an important discussion focus of each episode in this part of the teaching experiment.

Initially, in Task 8, definition of sine on the **origin-centered unit circle** was transformed into a new definition on the **origin-centered non-unit circle** with the same visual objects (a reference point on the circle referring to an arc and a perpendicular line-segment to the *x*-axis). Reference points for both function (points *P* and *R*) were constructed in GSP environment in a dynamically-linked way (see *Figure 6.1*). As a consequence of their simultaneous movements at the same angular speed in the GSP environment in the (*unit*) *circle register* as well as their *one full-actions* in the same interval in the *graphical register*, all students reasoned about the new function defined on the **non-unit circle** with the same period as the basic form of sine –i.e., 2π.

Next, in Task 9, definition of sine on the **origin-centered unit circle** was transformed into a new definition on the unit circle **with different-center from the origin** with the same visual objects (a reference point on the circle referring to an arc and a perpendicular line-segment to the *x*-axis) (see *Figure 6.16*). Where, there was only one point, the point *P*, referring to the *full-round turning* in the (*unit*) *circle register*, as well as the *unique shape* (see *Footnote 47*) of their graphs indicating their *one full-actions* in the same interval in the *graphical register* (see *Figure 6.25*(*b*)). Thus, students were able to reason truly about the new function defined on the unit circle **with different-center from the origin** with the same period as the basic form of sine –i.e., 2π.

Afterwards, in Task 10, definition of sine by only one reference point on the unit circle was transformed into a new definition based on two different reference points on the unit circle. These two points were constructed on the unit circle so that one of them was an arbitrary point and the other was its rotated-position about the origin by a fixed-measure (see *Figure 6.28*). These two points were moving in the same (angular) speed on the unit circle under the manipulation of them in GSP environment. Thus, students were able to reason about the new function defined by **two points with the same (angular) speed** in the (*unit*) *circle register* with the same period as the basic form of sine –i.e., 2π– based on their *full-round turning*s, as well as based on the *unique shape* of their graphs (see *Footnote 47*) indicating their *one full-actions* in the horizontally-translated-intervals in the *graphical register*.

Lastly, in Task 11, definition of sine by only one reference point on the unit circle was transformed into a new definition based on two different reference points on the unit circle **with the different (angular) speeds** in GSP environment. One of these points was constructed on the unit circle in GSP environment as an arbitrary point and the other as its rotated-position by a measure dependent on the principal measure of the first point so as to be its integer multiples. As an initial examples, the point *P* as an arbitrary point and the point *P′* as its rotated-position about the origin by the principal measure of the point *P* were constructed in GSP environment. Thus, the point *P′* was moving on the unit circle at the double (angular) speed of the point *P* in the GSP environment (see *Figure 6.39(a)* and *Figure 6.39 (b)*). Dynamically-linked conversion of the new function (mapping from the angle of the point *P* to the ordinate of the point *P′*) from the (*unit*) *circle register* into the *graphical register* together with the sine graph caused for students to bring the period aspect up for discussion.

On the one hand, Cemre and Zafer interpreted the period of y=sin(2x) as half of the period of y=sin(x) through attributing its meaning to (*i*) the half-turning of the point *P* (referring to the input of y=sin(2x)) so as to bring forth one full-round turning of the point *P′* (referring to the output of y=sin(2x)) in the (*unit*) *circle register*, (*ii*) (0,π) interval (*Footnote 46*) as the half of (0,2π) interval on the x-axis in which, respectively, sin(2x) and sin(x) completed their one-full-actions in the *graphical register* as a consequence of one-full-round turnings of the point *P′* and *P* (referring to the outputs of y=sin(2x) and y=sin(x)) in the (*unit*) *circle register*).

On the other hand, Defne and Ebru encountered a trouble based on the full-round turnings of two points referring to (*i*) the input variable and (*ii*) the output variable. While Defne interpreted the period of y=sin(2x) as 4π considering two full-rounds of the point P' that was produced by one full-round of the point P, Ebru interpreted the period as 2π considering one full-round turning of only the point P'. In other words, they interpreted the period in the (*unit*) *circle register* based only on the full-round turning of the reference point of the output variable rather than the turning of the reference point of the input variable producing the full-round of the reference point of the output variable. Therefore, the researcher encouraged them to reason about the full-round-turnings of the point P' dependently on the turning of the point P. Where, they were able to attribute one-full-round turning of the point P' to the π-radian turning of the point P in the (*unit*) *circle register*. When similar discussion were done on another function, i.e., y=sin(3x), based on its dynamic-and-linked representation within and between the (*unit*) *circle register* and the *graphical register*, they were able to reason about the full-round turning of the reference point of the output variable dependently on the turning of the reference point of the input variable. When the researcher asked them to reason about their periods through emphasizing its meaning as the smallest-repeated-interval in the *graphical register* and the length of this interval in the *symbolic register*, they were able to reason about these functions' periods correctly through using the x-axis as a measuring tool to determine the length of these intervals.

At the end of Task 11, all students were able to determine the period of some other functions in the form y=sin(bx) accurately in two different reasoning ways in the *symbolic register*. For example, firstly, by the proportional reasoning, they determined the period of y=sin(3x) as 2π/3 through considering it as one third of the basic sine function's period. Secondly, they determined the period of y=sin(3x) as about 2.1 by the reasoning way based on the measurement of the abscissa of the right endpoint of the interval referring to its first one-full action.

Moreover, all students were able to transfer their final interpretations on the period of the general forms of sine in Tasks 8, 9, 10 and 11 mentioned above to cosine, respectively, in Tasks 13, 14, 15 and 16 as a consequence of their conceptual developments on association of cosine [sine] with sine [cosine] throughout Task 12 (see *Changed-arc with a constant difference in (unit) circle register* heading in Chapter 6).

6.7. Summary of Students' Reasoning on Composed-Coefficients' Visual Oppositions

The main themes of the episodes of the teaching experiment were the conversion tasks based on the discrimination of the *visual features' oppositions* in any representational register. Therefore, while the (*unit*) *circle register* and the *graphical register* were considered as both the *source register* and the *target register*, the *symbolic register* was considered primarily as the *target register* of the conversion tasks. Moreover, the visual features of each task between Task 8 and 11 [Task 13 and 16] were referring in the *symbolic register* to only one coefficient of the general form of the sine [cosine] function. Therefore, at the end of Task 11 [Task 16], the researcher provoked students to reason about the coefficients' visual oppositions on a general form of sine [cosine] stated by all coefficients in the *symbolic register* with two main aims: (*i*) to reverse the role of the *symbolic register* in the conversion as the "*source register*", (*ii*) to bring the composed-coefficients' composed-visual oppositions up for discussion.

For this purpose, at the end of Task 12, a general form of sine function in the form y=3sin(2x+4)-1 was considered as the first example. When reasoning in the *language register* about the visual opposition of the coefficient of sine; i.e., 3, all students associated this coefficient in the *symbolic register* with the tripled-radius in the (*unit*) *circle register*, as well as the tripled-magnitude in the *graphical register*. Moreover, they were able to expressed the composed-visual oppositions of the coefficient and constant of sine; i.e., respectively 3 and -1, through articulating the upper and lower bounds of the range set in the *graphical register*, as well as the 1-unit down location of the circle in the (*unit*) *circle register*.

When reasoning in the *language register* about the coefficient “2” and constant “4” of the input of sine, students preferred to reason about “2” and “4” coefficients’ visual oppositions mainly in the *graphical register*. They reasoned about the visual opposition of “4” as the parallel displacement of the graph along the *x*-axis by 4 units in the negative direction and and the visual opposition of “2” as the compression of the graph into half. That is, they reasoned about the visual oppositions of these two coefficients separately rather than their composed-visual opposition.

When the graph of this function was plotted in GSP environment, all students encountered a major trouble on reasoning about the composed-visual-opposition of the coefficients “2” and “4” in the *graphical register*. At that point, the researcher provoked them to reason about the variation of y=sin(x) function’s graph up to y=3sin(2x+4)-1 function’s graph through incorporating a new function into the discussion; respectively, y=3sin(x), y=3sin(x)-1, y=3sin(2x)-1. Until the last step of this reasoning process, they had no conflict between the variation of the graphs and their expectations about them in the *graphical register*. That is to say, their concept images on the visual variation in the *graphical register* between (*i*) sin(x) and 3sin(x), (*ii*) 3sin(x) and 3sin(x)-1, (*iii*) 3sin(x)-1 and 3sin(2x)-1 were coherent with the visual variation between these pairs of graphs produced by GSP. However, they encountered the major trouble on the visual variation between the graphs of y=3sin(2x)-1 and y=3sin(2x+4)-1 in terms of the displacement amount between their graphs.

When this major trouble emerged, the researcher asked students to determine the displacement amount between these two graphs. All students determined the displacement amount as 2 –instead of 4– by the aid of GSP’s “zoom in” and “zoom out” options for the scaled *x*-axis. Where, the researcher encouraged them to reason about the displacement amount when the constant of the input of sine was changed from 4 to 2. When students edited y=3sin(2x+4)-1 as y=3sin(2x+2)-1, they determined displacement amount as 1 –instead of 4– based on dynamically-changed graphs, as well as associated the displacement amount with the half of the constant of the input of sine. When reasoning about the cause of this half-reduced displacement amount, all students attributed the cause of this half-reduced-displacement-amount to the half-reduced-period. At that point, the researcher encouraged them to reason about the displacement amount when the coefficient of the input of sine was changed. Again, students attributed the ratio between the changed-displacement amounts (see *Footnote 48*) to the ratio between the changed-periods. That is to say, they started to reason about the displacement amount by an operational-process as the division (of the constant) by the coefficient of *x* as in the case of the determination process of the period. Moreover, they verified this determination in the *graphical register* based on the distance between the correspondence points on the *x*-axis of these two functions through using the scaled *x*-axis of the GSP as a measuring-tool. Furthermore, they reasoned in the same way for the some other functions in GSP environment through manipulating the coefficients in the *symbolic register* and observing their simultaneous effects in the *graphical register* in terms of the changed-displacement amount – between two graphs with and without the constant term of the input of sine.

Besides, in addition to the *graphical register*, students interpreted the cause of the changed-displacement amount (*Footnote 48*) in the (*unit*) *circle register* as well. They converted the constant "*c*" of the input of sine in the *symbolic register* into a *c*-radian fixed-arc in the (*unit*) *circle register*; and then, interpreted this fixed-arc by means of two different, but dependent, (angular) speeds referring to *x* and *bx*. That is to say, they attributed the displacement amount in the *graphical register* to the (*c*/*b*)-radian turning of the reference point of (*x*) that produced the *c*-radian arc as the path of the reference point of (*bx*) so as to indicate (*bx*+*c*) in the (*unit*) *circle register*. It means that students reasoned about the *c*-radian arc between the reference-points of (*bx*) and (*bx*+*c*) in a dynamic-turning-structure through considering the turning-amount of (*bx*) dependently on the turning amount of (*x*) (see *Composed-Coefficients' Visual Oppositions* heading in Chapter 6). The researcher inferred that students' this reasoning arose from their effort to determine how much turning of (*x*) caused the *c*-radian "before arrival/after arrival" on a specific point between the reference points of (*bx*) and (*bx*+*c*) in the (*unit*) *circle register* (for more detailed articulation about "before arrival/after arrival" aspect, see the last paragraph of the heading *Parallel-displacement along the x-axis in graphical register*). In the scope of teaching experiment, the researcher preferred not to discuss this issue in the GSP environment because the manipulation of the coefficients "*b*" and "*c*" for the functions in the form $y=a\sin(bx+c)+d$ in the (*unit*) *circle register* required time-consuming-constructions in contrast to the "easy-manipulation" of these coefficients in the *symbolic register* and "fast-observation" of their oppositions in the *graphical register*.

Finally, all these issues of Task 11 mentioned above on the general form of sine were discussed at the end of Task 16 on the general form of cosine. It was observed that students were able to transfer their final interpretations in Task 11 on the composed-visual oppositions of the composed-coefficients for sine mentioned above to those for cosine at the end of Task 16 as a consequence of their conceptual developments on association of cosine [sine] with sine [cosine] throughout Task 12 (see *Changed-arc with a constant difference in (unit) circle register* heading in Chapter 6).

CHAPTER 7

RESULTS FROM TEACHING EXPERIMENT: PART 3

In this chapter, students' abilities to transform their understanding having developed throughout the teaching experiment on trigonometric functions (see *summary* parts in Chapter 5 and 6) as well as their conversion troubles in paper-and-pencil environment when dealing with a mathematical modeling task on trigonometry context are presentated.

The aim of this chapter is to provide the model of *students' trigonometry* in paper-and-pencil environment at the end of the teaching experiment conducted in GSP environment through articulating their reasoning steps in the modeling process of the ground clearance of a car on the Ferris wheel as well as the distance to the position gotton in the car.

7.1. Modeling Task with Ferris Wheel: Modeling Process

Although students were studied previous tasks separately as two pairs, throughout this task, all of four students were worked as a group on a mathematical modeling task with Ferris wheel (Appendix C).

The first step of students' reasoning on the modeling task was the visualization of the task context. Initially, students focused on the positions of 36 cars on the Ferris wheel with a 10-degree distance between two successive cars as if a static structure instead of the dynamic structure of one car. Initially, Cemre and Defne started to draw a circle and a base line so as to model this Ferris wheel (e.g., *Figure 7.1*). They put the first car on the bottom and the following three cars on the right, top, and left of the circle; and then, Defne reasoned that other cars located at a 10-degree interval (*Figure 7.2*); Cemre placed the other 32 cars on the circle so as to be 8 cars in each quadrant (*Figure 7.3*). At that point, Ebru constructed her own drawing in that way; and then, reasoned about the ground clearance on four positions, namely, bottom, right, top and left as 4 *meter*, 74 *meter*, 144 *meter* and 74 *meter* respectively (*Figure 7.5*). After observing others' drawings, Zafer said that "we can consider it as an upward-translated version of the unit circle (*putting his right hand horizontally on the horizontal line referring to the ground on the Ebru's drawing like in Figure 7.5; and then, dragging his hand vertically-upward until the center of the circle*)"; and then, started to construct his model for the Ferris wheel (*Figure 7.6*). Where, Defne objected to the unit meaning through saying "but not unit, whose radius is 70". Cemre responded her "we can assume 70 *meter* as the unit length". Zafer confirmed Cemre's articulation; and wrote -1 and 1 near the bottom and top intersection points of the circle with the vertical line in his model (*Figure 7.6*). The researcher inferred that at the end of the teaching experiment, Cemre and Zafer were able to reason about any non-unit circle as a unit circle through considering a new measure unit as the units of another unit as they did in Task 8 (see *Changed-radius in (unit) circle register* heading in Chapter 6).

When the unit circle idea was brought up for discussion, Cemre proposed to discuss the ground clearance in the first quadrant in the (*unit*) *circle register*; which was the second step of their reasoning on the modeling task. Where, Zafer asked Ebru to calculate the value of sine of 10 degrees by GSP. After GSP gave 0.17 as the value of sin(10) under "degrees" preference as the angle measure unit, Zafer wrote $f(x)$=70sin(x)+74 function on his paper (*Figure 7.6*) through uttering "radius was 70" when writing the coefficient of sine. And then, he articulated his reasoning way on this function (lines 1-25 in [Modeling Task] Protocol 1). Where, his actions and language (e.g., lines 1-10 in [Modeling Task] Protocol 1) imply that he was able to reason about the length of the opposite side of the reference right triangle through considering the same circle either as a unit circle or as a non-unit circle; respectively, as the sine value and the multiplication of the sine value by the radius as he did in Task 8 (see *Changed-radius in (unit) circle register* heading). Moreover, he was able to reason about a non-unit circle's changed-location through extending his reasoning about the unit circle's changed-location mentioned in Task 9 (see *Changed-center in (unit) circle register* heading). Zafer's reasoning in this way was confirmed by Cemre and Defne (lines 11-21 in [Modeling Task] Protocol 1). And then, without going into details, Zafer asserted that $f(x)$=70sin(x)+74 function gave the ground clearance of a car on the Ferris wheel (lines 22-25 in [Modeling Task] Protocol 1).

At that point, Defne tried to reason about the ground clearance of the top position arithmetically based on the nine 10-degree arcs between 10 cars in the first quadrant, as well as the value of sin(10^{o}). That is to say, she reasoned that each 10-degree arc would cause the increase of the height as much as 11.9 [value of 70sin(10^{o})] (lines 28-38 in [Modeling Task] Protocol 1). All other students objected to this idea as a consequence of the circular shape (lines 35-45 in [Modeling Task] Protocol 1). Zafer articulated for the sine function that the proportional increase of inputs did not give proportional increase of outputs through giving the relation between sin(10^{o}) and sin(20^{o}) as an example based on the curvilinear shape in the *graphical register*, as well as the circular shape in the (*unit*) *circle register* (lines 36-43 in [Modeling Task] Protocol 1). The researcher inferred that Defne's concept image on the rate of change of the sine values was including a trouble arising from her reasoning about "change" idea predominantly in the *symbolic register* as if a constant rate of change exists between angles and their sine values. In her construction in the (*unit*) *circle register* (*Figure 7.2*), there were six perpendicular segments with the equal magnitude that were located at equivalent-angular intervals in the first quadrant. These perpendicular segments did not represent the increasing amount of the height regarding to the previous one. Despite of this visualization by herself, she considered the sum of their lengths as the same length with the up-vertical radius segment (lines 28-38 in [Modeling Task] Protocol 1). This implies that she did not reason about the relation between these perpendicular line segments in the first quadrant and the up-vertical radius segment visually in the (*unit*) *circle register*. Instead, she reasoned algebraically about the multiplication of the number of these perpendicular segments by their magnitude as the radius predominantly in the *symbolic register*.

From this point forward, a 3-minute pause was observed. Then, the researcher provoked them to think loudly (lines 1-2 in [Modeling Task] Protocol 2). Where, Zafer expressed the ground clearance was found, and the distance to the position gotten in the car was needed to be found (lines 3-7 in [Modeling Task] Protocol 2). Where the researcher provoked them to think more deeply about $f(x)=70\sin(x)+74$ function's suitability (lines 8-10 in [Modeling Task] Protocol 2). At that point, Ebru expressed her dissatisfaction on this function's modeling the ground clearance anywhere of the circle (lines 11-14 in [Modeling Task] Protocol 2). Zafer claimed that this function gave the ground clearance anywhere of the circle through attributing any position to an angle measure (lines 15-21 in [Modeling Task] Protocol 2). And then, Defne agreed with Zafer through articulating on two points referring to 10-degree and 20-degree angles in the first quadrant (lines 23-35 in [Modeling Task] Protocol 2). Also, Cemre approved this functions' suitability (lines 36-38 in [Modeling Task] Protocol 2). Next, Zafer confirmed the function's suitability in case of the negative sine values, which was approved by Cemre and Defne, through considering an arbitrary position in the third quadrant, as well as the bottom position on the circle (lines 39-68 in [Modeling Task] Protocol 2). However, Ebru again expressed her dissatisfaction about this function in terms of the meaning of x in $f(x)$; and brought "turning" up for the discussion (lines 69-70 in [Modeling Task] Protocol 2). Zafer explained the meaning of x in $f(x)$ as the angle referring to the principal arc (lines 71-74 in [Modeling Task] Protocol 2). At this point, the researcher provoked them to reason about the angle through associating with the turning of the Ferris wheel (line 75 in [Modeling Task] Protocol 2).

When reasoning about the changed-angle as a result of the Ferris wheel's turning (lines 69-78 in [Modeling Task] Protocol 2), "time" idea was brought up for the discussion by Zafer (line 1 in [Modeling Task] Protocol 3). He reasoned that the angle was changing at a constant speed so that one full-round (or 360 degrees) was completed in 30 minutes (lines 1-5 in [Modeling Task] Protocol 3). It was the third step of students' reasoning on the modeling task that provoked students to determine the angular speed of the Ferris wheel. They tried to make sense of the relation between "time" and "angle" by the direct proportions considering the full-round of the Ferris wheel at 30 minutes. Firstly, Cemre calculated the elapsed time to take a 10-degree angular path as 50 seconds so as a car to reach the position of the next car (lines 6-15 in [Modeling Task] Protocol 3). Zafer calculated the elapsed time to take a 1-degree angular path as 5 seconds (lines 16-17 in [Modeling Task] Protocol 3). And then, he changed his reasoning focus on "time" based on "angle" into the reasoning on "angle" based on "time". That is to say, he calculated the angular path taken at 60 seconds as 12 degrees (lines 17-19 in [Modeling Task] Protocol 3). At that point, Ebru proposed to determine the angular path taken at 1 second –which was referring to the angular speed as degree/second (line 24 in [Modeling Task] Protocol 3). Thereupon, Zafer and Cemre calculated the angular speed as 0.2 degree/second without mentioning the angular speed term (lines 25-29 in [Modeling Task] Protocol 3). At that point, Defne proposed to determine the angular path taken at an arbitrary time, i.e., "*t*" (line 30 in [Modeling Task] Protocol 3). However, they tried to determine the angular path arithmetically for a specific time, i.e., 15 seconds, as 3 degrees, rather than for an arbitrary time, i.e., "*t*" (lines 31-35 in [Modeling Task] Protocol 3). And then, they mentioned whether or not they could find the ground clearance at 15th second for 3 degrees (lines 36-38 in [Modeling Task] Protocol 3). This reasoning process prompted a distinct shift on students reasoning about x in $f(x)=70\sin(x)+74$, which was their first model for the ground clearance in the *symbolic register*. That is to say, they started to dissociate x in the $f(x)$ from x in the $\sin(x)$ through attributing the meaning of the former x as "time" and the latter as "angle" based on "time" in the *language register* (e.g., lines 40, 73-74, 81-88 in [Modeling Task] Protocol 3). However, they could not express the input of sine dependently on x in $f(x)$ in the *symbolic register* yet.

Their expression process of the input of sine dependently on "time" variable in the *symbolic register* started by Defne's reasoning about the angular path based on the speeds of the points as they did in Task 11 and 16 on two points with different speeds on the unit circle (lines 73-79 in [Modeling Task] Protocol 3). She propounded the pattern of the relation between "time" and "angle" up for discussion through articulating the first three angular paths respectively at the ends of the first, second and third minutes of the turning (lines 81-87 in [Modeling Task] Protocol 3). Moreover, she generalized this pattern at an arbitrary time in minutes as the multiplication of "time" by 12 (which was the angular speed in degree/minute) but in the *language register* without symbolizing it (line 88 in [Modeling Task] Protocol 3). Zafer converted this generalization into the *symbolic register* as 12*x* through uttering (line 89-92 in [Modeling Task] Protocol 3). However, they did not symbolize it explicitly on their model function (lines 105-120 in [Modeling Task] Protocol 3) until they considered the importance of the boarding point on the angular position with respect to the turning (lines 41-71, 93-108 in [Modeling Task] Protocol 3).

Their consideration of the boarding position started to Zafer's mention about the position of the car at the last-1-minute as 258 in degrees in the third quadrant in the (*unit*) *circle register* (lines 41-46 in [Modeling Task] Protocol 3). He reasoned about the position of a car at the last 1 minute as 258 in degrees through (*i*) assuming the turning of the Ferris wheel in the counterclockwise direction, (*ii*) considering the 12-degree angular path at each 1-minute, (*iii*) assuming the boarding –bottom– position as 270 degrees, (*iv*) considering the 12-degree turning from the boarding position in the clockwise direction (lines 48-58 in [Modeling Task] Protocol 3). Cemre accepted Zafer's articulation (lines 59-62 in [Modeling Task] Protocol 3). Defne and Ebru did not make any objection or confirmation to this articulation. Where, the researcher asked students to reason about 258 and 12 degrees at the same time considering the turning (lines 61-65 in [Modeling Task] Protocol 3). Their sum as 270 in degrees caused a trouble for Cemre as a consequence of her expectation on one-full-round as 360 in degrees (line 67 in [Modeling Task] Protocol 3). However, Zafer did not encounter any trouble about the difference between this addition (270 degrees) and one-full-round turning (360 degrees) as a consequence of his ability to dissociate the turning amount of a car from his definition of the angle corresponding to the position of the car (lines 67-72 in [Modeling Task] Protocol 3). Zafer's this ability prompted another viewpoint to express the input of sine dependently on "time" as from the boarding –bottom– position.

This viewpoint was the next step of students' reasoning on the modeling task that provoked students to consider the boarding position as an important idea for the input of sine for modeling the ground clearance. Initially, Zafer tried to modify his model function through considering the elapsed-time between the boarding position and the far-right position as 7.5 minutes (lines 93-94 in [Modeling Task] Protocol 3). Where, Cemre mentioned the angular path taken at the first 7.5 minutes as a constant of the input of sine of the model function without specifying its sign (line 95 in [Modeling Task] Protocol 3). At that point, Zafer tried to specify its sign. Initially, he expressed that their model function gave the results after the first 7.5 minutes rather than from the beginning (lines 96-99 in [Modeling Task] Protocol 3); and then, determined that the first 7.5-minute turning affected the input of sine as a positive constant (lines 98-99 in [Modeling Task] Protocol 3). Even though he was aware of the far-right position as the position of a car at the first 7.5 minutes regarding the turning from the boarding position in the counterclockwise direction, when manipulating their model function so as to transform into the expected function, he missed out the reference position between the boarding position and the far-right position in terms of which position was transformed onto the other one. At that point, Defne calculated the angular path taken at 7.5 minutes arithmetically as 90 degrees (lines 100-104 in [Modeling Task] Protocol 3); and then, revised their model function as $f(x)$=70sin(x+90)+74 through emphasizing x in $f(x)$ as "time" and x+90 in sine as "angle" (*Figure 7.19*). Next, Zafer revised their model function as $f(t)$=70sin(x+90)+74 through representing "time" variable by the letter "*t*" and correspondence "angle" variable by the letter "*x*" (*Figure 7.6*); and then, reasoned in order to express x based on "time" (lines 105-108 in [Modeling Task] Protocol 3). Defne again propounded x in sin(x+90) as the multiplication of "time" (in minutes) by 12 (line 109 in [Modeling Task] Protocol 3). Cemre and Zafer confirmed this; and then, Zafer revised verbally the input of sine as 12x+90 through articulating the meaning of x in here as "time" in minutes by an example for 1 minute [as a "time" value] and 12 degrees [as an "angle" value taken at 1-minute] (lines 110-118 in [Modeling Task] Protocol 3). Then, Defne revised in written the model function as $f(x)$=70sin(12x+90)+74 (*Figure 7.19*). At that point, Zafer wanted to control this function's suitability through calculating its value for 1 minute (line 119-121 in [Modeling Task] Protocol 3). When Ebru calculated 70sin(12+90)+74 as 142.47 by GSP, they determined that this function did not model the ground clearance from the beginning of the turning (lines 122-128 in [Modeling Task] Protocol 3). The first cause of it that comes to their mind was the addition of 90 in the input of sine (lines 124-125, 129 in [Modeling Task] Protocol 3). They wanted to consider 90 as a subtracted-term of the input of sine in $f(x)$=70sin(12x+90)+74 instead of the added-term (line 129 in [Modeling Task] Protocol 3). However, they reasoned again based only on the calculation result of 70sin(12-90)+74 by GSP, they could not ensure whether $f(x)$=70sin(12x-90)+74 models the ground clearance or not (lines 129-133 in [Modeling Task] Protocol 3).

At that point, the researcher modeled in GSP environment a similar but not same situation with the modeling Task (lines 1-5 in [Modeling Task] Protocol 4) by a circle with the 7-unit radius whose center was located on the origin (*Figure 7.20*). Where, it was observed that they were able to revise their last function modeling the ground clearance (i.e., $f(x)$=70sin(12x-90)+74) into the function modeling the distance of a point on the circle to y=-7 line in GSP environment (i.e., y=7sin(x-90)+7) (lines 6-26 in [Modeling Task] Protocol 4). They controlled this function's suitability on two angle measures (12 and 24 degrees) which corresponded to two positions at the first and second minutes of the turning of the Ferris wheel. Initially, they calculated the output for 12 degrees as 0.15 (lines 6-10 in [Modeling Task] Protocol 4); next, arranged the point *A* on the circle so as to obtain approximately the 12-degree angle (*Figure 7.20*(*a*)) from the point *C* to the point *A* in the counterclockwise direction (lines 11-14 in [Modeling Task] Protocol 4); and finally, measured the distance between the points *D* and *A*, as well as compared it with the output (lines 14-17 in [Modeling Task] Protocol 4.) In the similar way, after controlling its suitability for 24 degrees in GSP environment, they determined that their model function (i.e., $f(x)$=70sin(12x-90)+74) was suitable for the ground clearance of a car during the turning of the Ferris wheel (lines 17-26 in [Modeling Task] Protocol 4). When they ensured its suitability, Zafer reasoned about the cause of subtracting 90 [degrees] rather than adding it based on the *zero-points*[50] of the sine function and the Ferris

[50] "Zero-point" was used in the meaning of the "zero-degree" angles but from different initial sides in the (*unit*) *circle register* (i.e., as from the positive *x*-axis) for sine and (i.e., as from the negative *y*-axis) for the Ferris wheel.

wheel. That is to say, he expressed the position gotten in the car as the 90-degree before of the far right position of the circle through assuming the counterclockwise direction as the turning direction (lines 25-32 in [Modeling Task] Protocol 4). All other students agreed on this reasoning (lines 33-38 in [Modeling Task] Protocol 4). They put the function modeling the ground clearance with respect to time in minutes into the final form as $f(x)$=70sin(12x-90)+74 (lines 39-44 in [Modeling Task] Protocol 4).

When reasoning in order to model the distance to the point gotten in the car with respect to time, students' initial constructions were in the fourth quadrant (*Figure 7.4* and *Figure 7.6*). And then, they determined the distance from the bottom position to the far right position as $70\sqrt{2}$ by Pythagorean Theorem (lines 3-7 in [Modeling Task] Protocol 2; lines 5-10 in [Modeling Task] Protocol 5). This reasoning provoked students to reason on a right triangle by Pythagorean Theorem. When Ebru dragged the point *A* in the first quadrant on GSP file (*Figure 7.22*), students started to reason about the distance to the point gotten in the car in the first quadrant by Pythagorean Theorem on a right triangle whose hypotenuse was referring to this distance (lines 11-35 in [Modeling Task] Protocol 5). Initially, they constructed some rectangular structures in the first quadrant on their drawings in order to specify the legs of this right triangle in terms of sine and cosine (*Figure 7.4*, *Figure 7.23*, *Figure 7.24*). Students' drawings (*Figure 7.23* and *Figure 7.24*), actions and language (e.g., lines 12-14, 39-43 in [Modeling Task] Protocol 5) imply that on this rectangular structures, they were able to reason through exchanging their focuses between the opposite [adjacent] side of the reference right triangle and its facing-side in the *reference-rectangle* (see *Footnote 44*) in terms of sine [cosine].

Again, the symbolic expression of the model function for the distance to the point gotten in the car was proposed by Zafer. He applied the Pythagorean Theorem on a right triangle whose vertexes were the boarding –bottom– position, another position in the first quadrant of the (*unit*) *circle register,* and its projection on the vertical axis (*Figure 7.23*). His initial symbolization was $f(x)^2=(70+sin\alpha)^2+(cos\alpha)^2$, where α referring to the principal angle in the (*unit*) *circle register* (*Figure 7.23*). This expression implies that he dissociated the independent (time) variable of the function from the input of sine and cosine. His second step reasoning on the symbolization was to express α with respect to *x*. That is, he wrote 12*x* under α notations; and then, revised this rule as $(70+sin(12x\text{-}90))^2+(cos(12x\text{-}90))^2$ (*Figure 7.23*). Finally, when he propounded it to discuss (line 36 in [Modeling Task] Protocol 5), he recognized that he missed the coefficients (i.e., 70) of sine and cosine and revised the function rule as $(70+70sin(12x\text{-}90))^2+(70cos(12x\text{-}90))^2$ through adding these coefficients (lines 44-46 in [Modeling Task] Protocol 5).

All students confirmed this function's suitability to model the distance to the bottom position (lines 37-48 in [Modeling Task] Protocol 5). But yet, Cemre wanted to control it in GSP environment (line 49 in [Modeling Task] Protocol 5). As they did for the function modeling the ground clearance (see lines 1-26 in [Modeling Task] Protocol 4), students controlled their model function for the distance to the position gotten in the car through adopting it in case of the circle in GSP environment with the 7-cm radius whose center was located on the origin, as well as comparing with the measure of the red line segment that was referring to the distance to the bottom position on the circle (see *Figure 7.25*).

At the end of the modeling task, they modeled the ground clearance and the distance to the position gotten in the car with respect to "time" in minutes by two functions like in *Figure 7.26*.

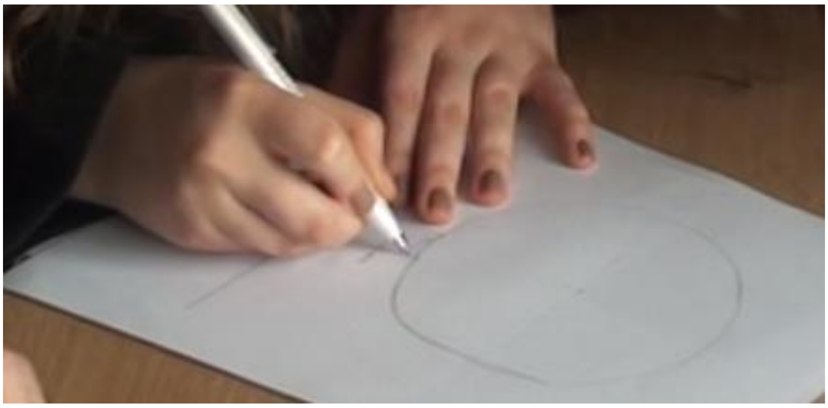


Figure 7.1. Defne's drawing of a circle and a base line to model the Ferris wheel and the floor

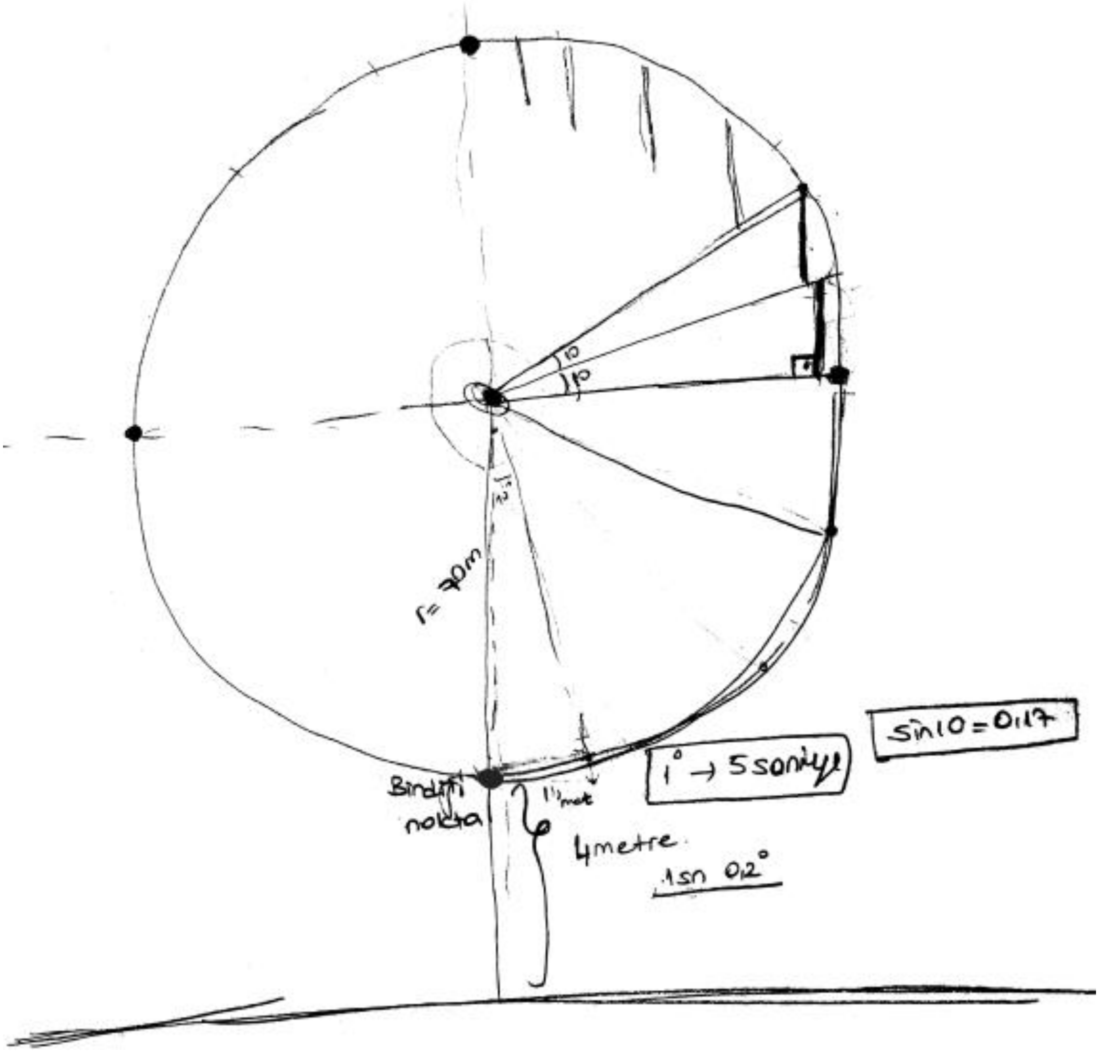


Figure 7.2. Defne's drawing to model Ferris wheel

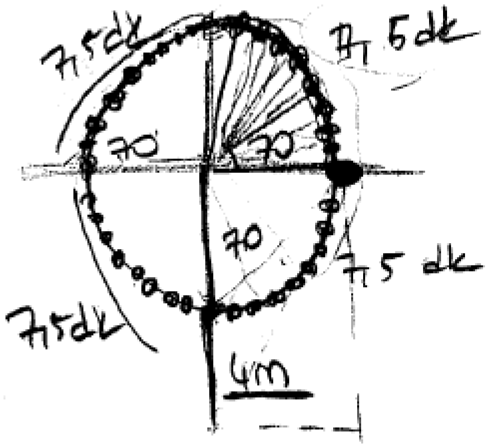


Figure 7.3. Cemre’s first drawing to model Ferris wheel

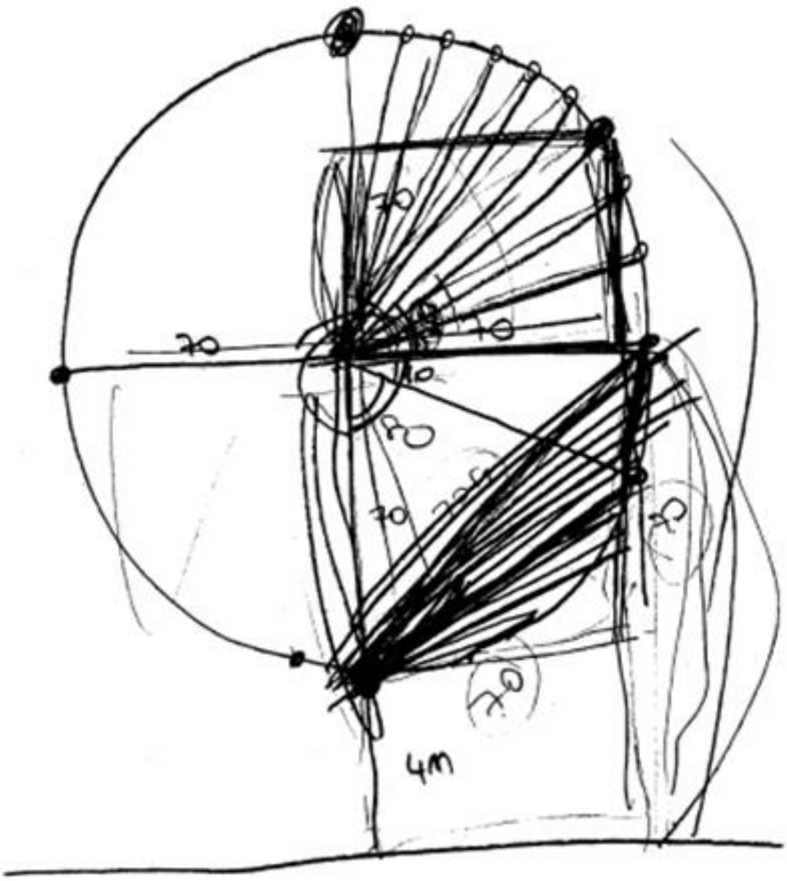


Figure 7.4. Cemre’s second drawing to model Ferris wheel

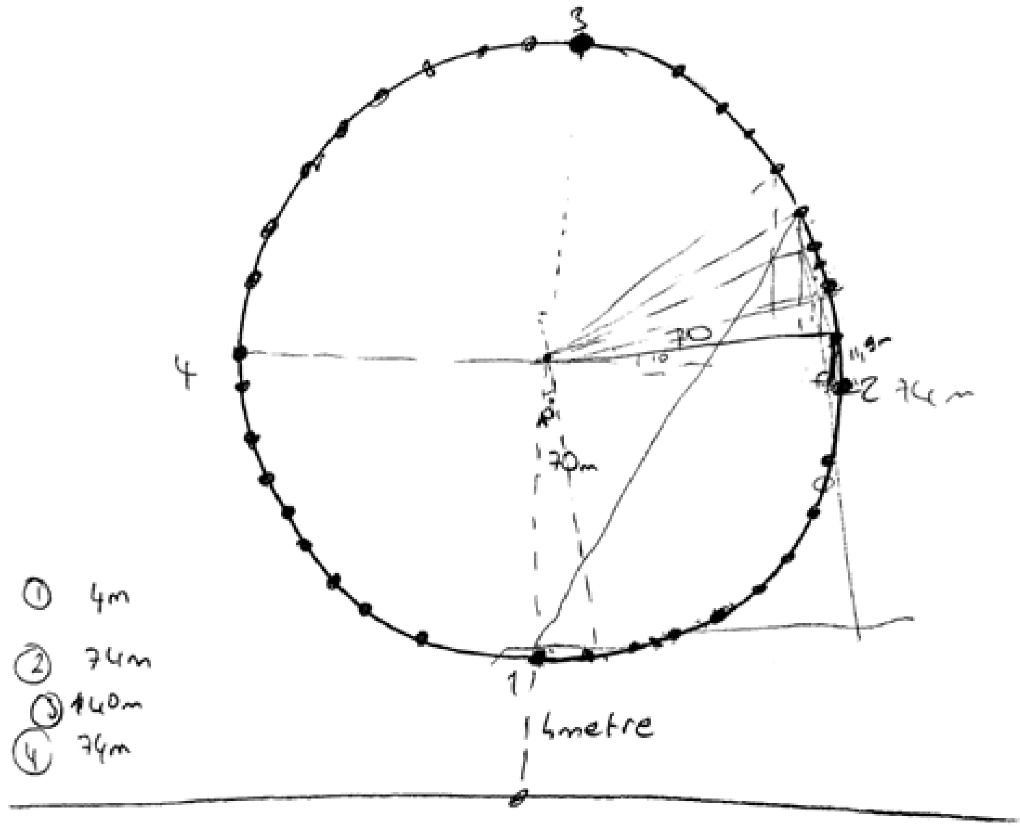


Figure 7.5. Ebru’s drawing to model Ferris wheel

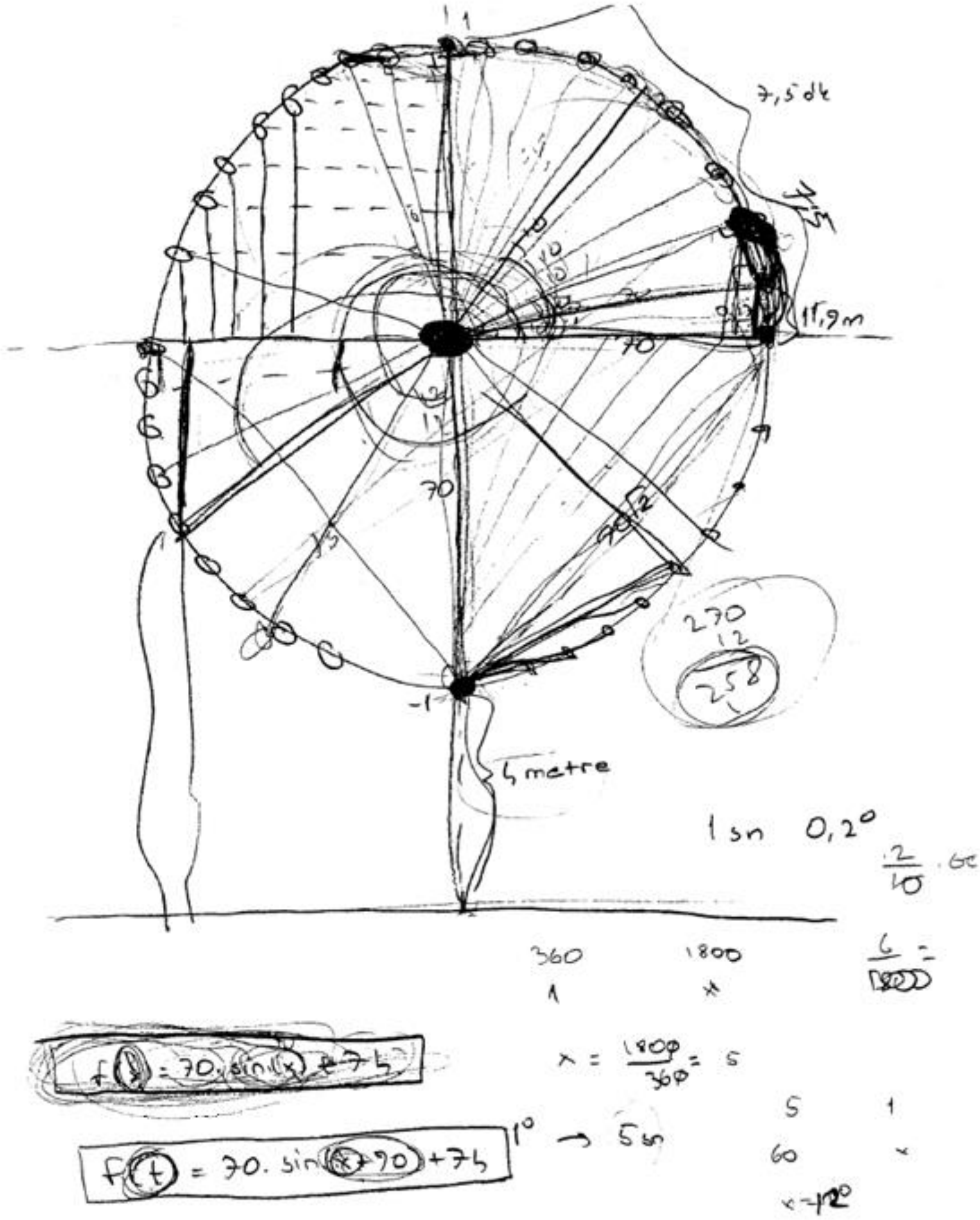


Figure 7.6. Zafer's drawing to model Ferris wheel

[Modeling Task] Protocol 1

Zafer: Now look! If the radius were 1, then here would be sine of 10 degrees (*dragging his pen on the opposite side of the reference right triangle referring to the 10-degree principal arc on his drawing like in Figure 7.6*). Similarly, here would be sine of 20 degrees (*dragging his pen on the opposite side of the reference right triangle referring to the 20-degree principal arc*). We multiple it [sin(20^{o})] by 70 in order to find this height (*dragging his pen on the opposite side of the reference right triangle referring to the 20-degree principal arc*). And from here to here is 74 meters (*pointing the distance between the horizontal line from the center and the line referring to the ground*). So, we add 74 [to 70sin(x)] (*pointing 74 in f(x)=70sin(x)+74 function on the paper like in Figure 7.6*).

Cemre: Yes. We are going to do it… …and do it [for every positions] in that way (*pointing respectively third and fourth points in the first quadrant on Zafer's construction like in Figure 7.6*).

Zafer: Let's multiply 0.17 [the value of sin(10^{o})] by 70.

Ebru: (*Calculated this multiplication by GSP*)

Zafer: (*When the result appeared on the screen as 11.9*) 11.9… If here is 11.9 (*dragging his pen on the opposite side of the reference right triangle referring to the 10-degree principal arc; writing 11.9 on its right side like in Figure 7.6*)... …this (*drawing on the opposite side of the reference right triangle to the 10-degree principal arc*) is the distance to here (*drawing the horizontal line from the center*).

Defne: Ok. I agree with you.

Zafer: Actually, it [car] is moving toward like that (*dragging his pen on the first quarter arc in the counterclockwise direction on the paper like in Figure 7.6*). I think the ground clearance can be found by this function (*pointing f(x)=70sin(x)+74 function on the paper like in Figure 7.6*).

Cemre: Zafer, make a trial (*pointing f(x)=70sin(x)+74 function on the paper like in Figure 7.6*)!

Defne: Wait a minute! There are nine parts in here, because ten cars are there (*dragging her pen on the first quarter arc in the counterclockwise direction on the paper like in Figure 7.2*). So, 11.9, 11,9, like that and like that (*drawing some vertical line segments in the first quadrant from the points at approximately equal intervals so as to refer each car of the Ferris wheel on her drawing like in Figure 7.2*)… if multiplying 11.9 by nine… ...it should be 70 (*indicating the radius line segment from the top point*). Ebru, please multiply 11.9 by 9.

Cemre: No, it shouldn't.

Defne: I mean… …in each car, it [height] increases 11.9 (*drawing again same perpendicular line segments in the first quadrant on her drawing like in Figure 7.2*).

Zafer: No. In each car, it [height] doesn't increase 11.9. Actually, sine is… …umm… …you know remember its graph (*figuring a sine wave*)… For example, sin(20^{o}) isn't double of sine (10^{o}). When going upward (*figuring the first-quarter-arc in the counterclockwise direction*), these values [sin(10^{o}), sin(20^{o}), sin(30^{o}), etc.] is increasing but decreasing the increased amount.

Cemre: Yes (*nodding her head up and down*), because of the circle shape.

Ebru: Yes. (*Calculating (9)(11.9) as 107.1*)

Defne: (*Looking to her paper without speaking*)

(*Each of the students was looking their own paper without speaking during 3 minutes.*).

[Modeling Task] Protocol 2

(*Each of the students was looking their own paper without speaking during 3 minutes.*)

Researcher: What do you think? Where did you encounter a trouble?

Zafer: We found the ground clearance (*$f(x)=70sin(x)+74$ function on his paper like in Figure 7.6*) but we have to find the distance to here (*pointing the bottom of the circle on his paper*). (*Drawing some line segments to represent the distance to the gotten point in the car; and then, drawing a specific line segment from the bottom position to the far right position and determining its distance as $70\sqrt{2}$*).

Researcher: The thing that was asked from you is not to determine each car's ground clearance. Instead, it is to determine the ground clearance of a car that you are in during the turning.

Ebru: Yes. When we are anywhere (*pointing another point on the first quartile of the circle from the points represent to the location of 36 cars on her paper like in Figure 7.5*), it should show us at how much height we are. We didn't write this function.

Cemre: (*Reading the asked items loudly*)

Zafer: We need to find so thing [function rule] that what we write in degrees [referring to a position]… …it should give us the ground clearance.

Cemre: But in the question it [ground clearance] is asked as from the entrance (*pointing the bottom position on Zafer's drawing like in Figure 7.6*).

Zafer: I think ground clearance is found from here (*pointing $f(x)=70sin(x)+74$ function on his drawing like in Figure 7.6*). What do you think?

Cemre: Let's we try it.

Defne: On this function (*circling $f(x)=70sin(x)+74$ function on Zafer's drawing like in Figure 7.6*), for each position (*pointing a position on the circle in the first quadrant*), when we put its angle, we find its sine (*drawing the perpendicular segment from this point to the horizontal line from the center*). This point's ground clearance is obvious (*indicating the distance between the horizontal line from the center and the line referring to the ground clearance*). Then, we add this height (*again indicating the line segment referring to the sine*), and find the ground clearance.

Zafer: (*Nodding his head up and down*)

Defne: If we put 10 [degrees] for *x* (*pointing x in $70sin(x)+74$*), we find here (*figuring the ground clearance of the position in the first quadrant referring to 10 degrees in her drawing like in Figure 7.2*). For 20 [degrees], we find here (*figuring the ground clearance of the position in the first quadrant referring to 20 degrees in her drawing*).

Cemre: Yes. This function is true I think (*writing on a paper this function through labelling it as "yerden yükseklik" like in Figure 7.7 which means the ground clearance*).

Zafer: Additionally, let's think about the case that sine is negative in the third and fourth quadrants… For example, when we look it [ground clearance] in here (*drawing a radius segment in the third quadrant like in Figure 7.8 so as to refer to the terminal side of the angle*), this angle is greater than 180 degrees (*drawing an arc inside of the circle from the initial side to the terminal side in the third quadrant like in Figure 7.9*).

Cemre: Yes.

Zafer: When we find its sine value, that is, here (*drawing the opposite side of the reference right triangle corresponding to the angle in the third quadrant like in Figure 7.10*)

Cemre: …negative.

Zafer: Negative, uh-huh. That is here (again *drawing the opposite side of the reference right triangle corresponding to the angle in the third quadrant like in Figure 7.10*).

Cemre: Yes.

Zafer: When we distracted it [length of the opposite side] from 74, we find this length (*drawing the perpendicular line segment to the base line from the reference point of the angle in the third quadrant like in Figure 7.11*).

Defne: Yes.

Cemre: Ok.

Zafer: We can check it in here (*indicating the bottom position on the circle like in Figure 7.12*). If we look regarding this point (*indicating the bottom position on the circle*), [angle is] 270 [degrees], sin(270) is minus one…

Defne: Yes.

Zafer: I wrote 270 (*Pointing "x" in 70sin(x)+74 on Cemre's paper like in Figure 7.7*), it gave minus one (*underlining sin(x)*), then plus seventy-four and minus seventy (*pointing respectively 74 and 70 on f(x)=70sin(x)+74 function*), that is, 4. Regarding here, the ground clearance is four (*pointing the bottom position on the circle like in Figure 7.12*).

Cemre: …four. Then, it is true.

Zafer: Uh-huh. Consistent in itself.

Ebru: But where, *x* is what (*pointing "x" in f(x) on Cemre's writing in Figure 7.7*)? It should be related to our turning speed.

Zafer: *x* is angle… …for example, if it is in here (*pointing a position on the circle in the fourth quadrant; and then, drawing the radius segment from this point like in Figure 7.13*), *x* is its angle from here to here (*drawing an arc inside of the circle so as to indicate its principal arc like in Figure 7.14*).

Researcher: Who calculates it?

Ebru: (*Smiling*) yes, that's what I'm saying.

Zafer: (*Laughing*) let someone else calculate it.

Cemre: (*Laughing*) let computer calculate it.

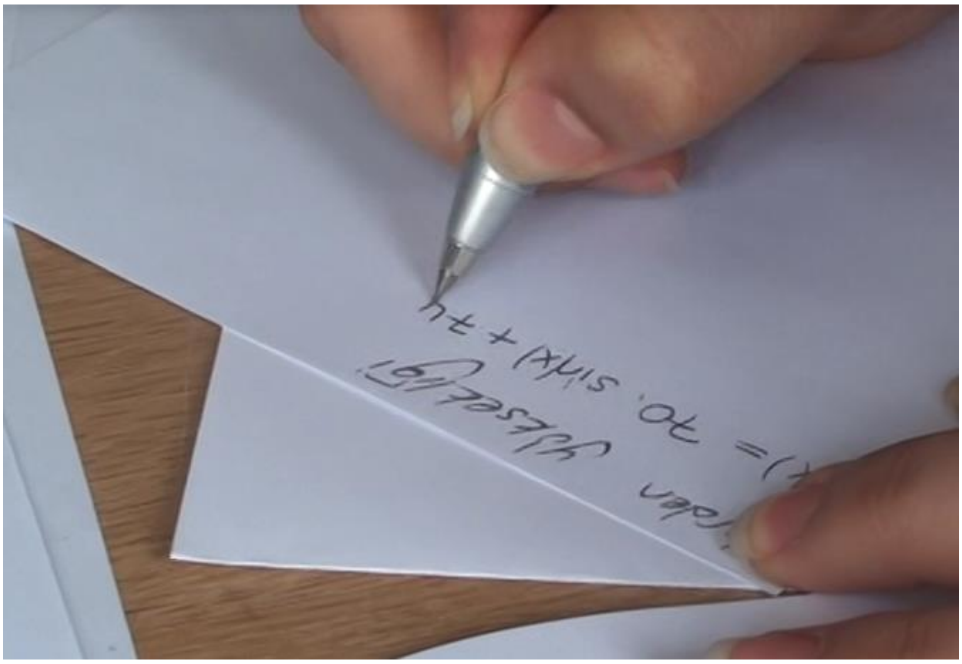


Figure 7.7. Cemre's writing the function to model the ground clearance

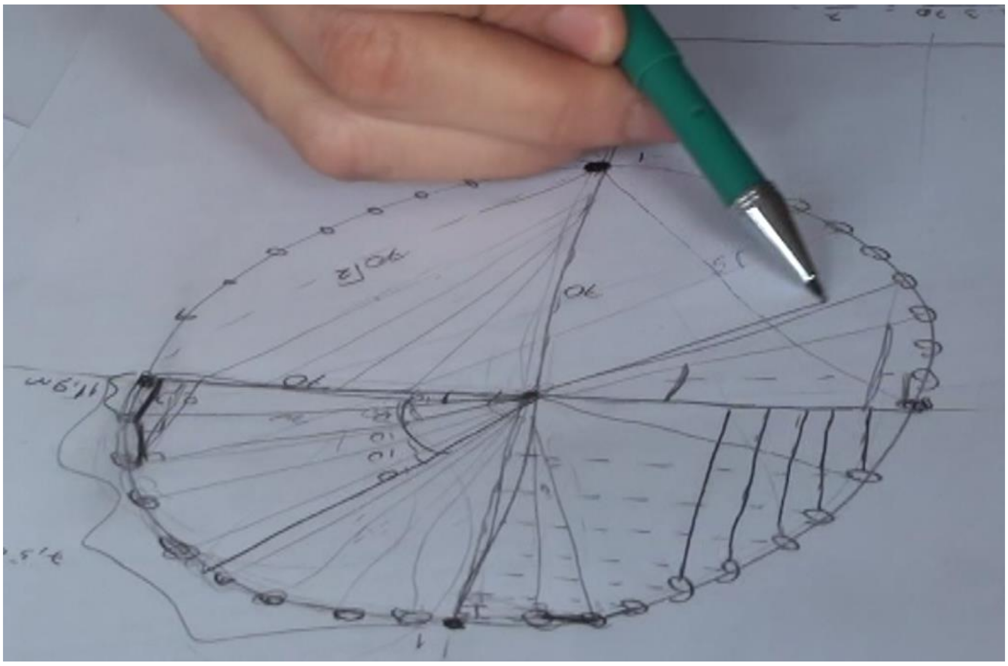


Figure 7.8. Zafer's consideration of the ground clearance by $f(x)=70\sin(x)+74$ function in case the position of the car was inside of the third quadrant (*Step1. modeling the terminal side of the angle*)

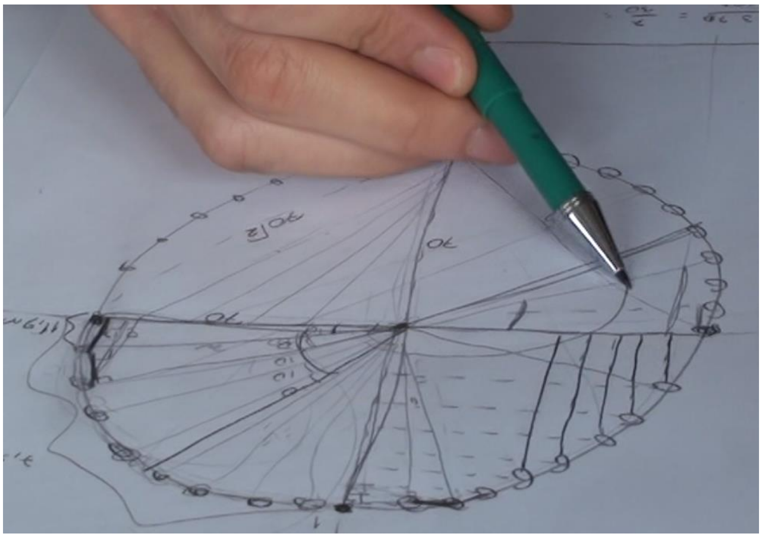


Figure 7.9. Zafer’s consideration of the ground clearance by $f(x)$=70sin(x)+74 function in case the position of the car was inside of the third quadrant (*Step2. modeling the angle as a dynamic turning by its principal arc in the (unit) circle register*)

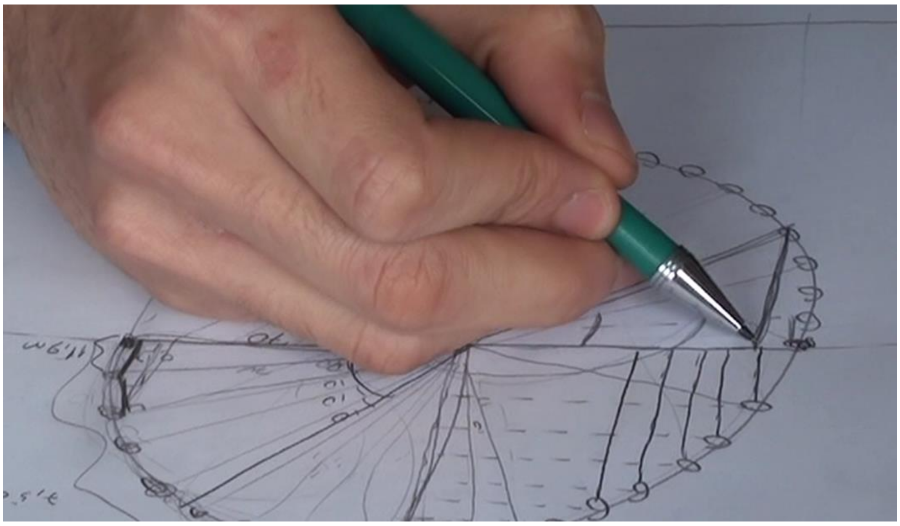


Figure 7.10. Zafer’s consideration of the ground clearance by $f(x)$=70sin(x)+74 function in case the position of the car was inside of the third quadrant (*Step3. modeling the opposite side of the reference right triangle*)

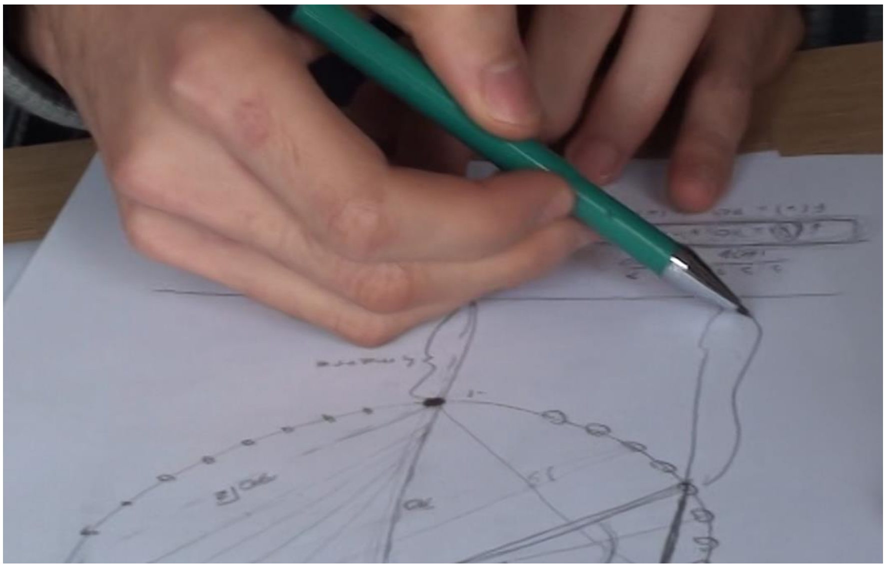


Figure 7.11. Zafer's consideration of the ground clearance by $f(x)$=70sin(x)+74 function in case the position of the car was inside of the third quadrant (*Step4. modeling the ground clearance*)

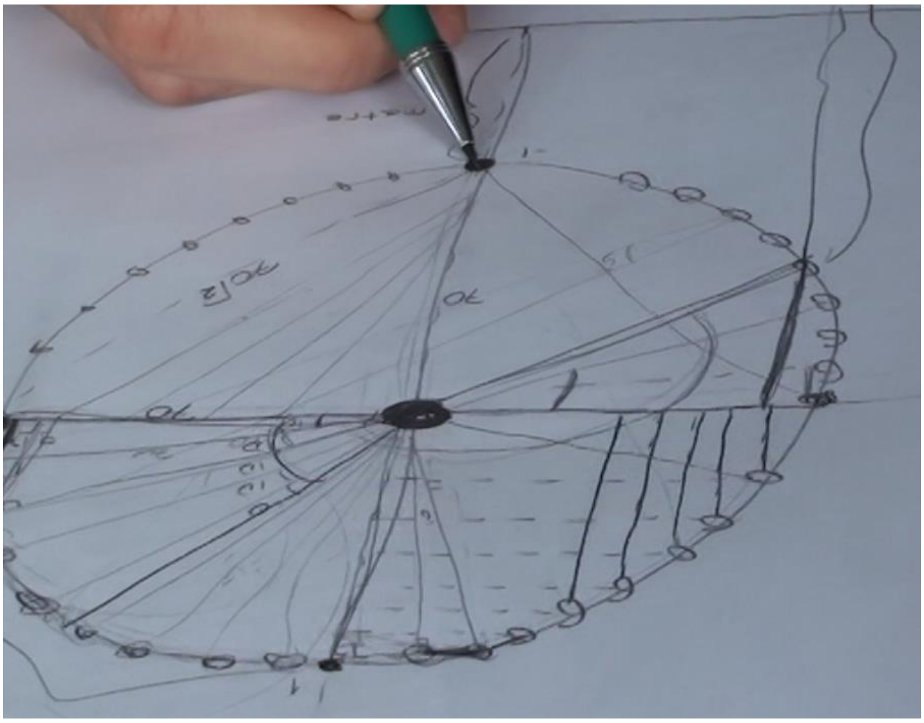


Figure 7.12. Zafer's checking of $f(x)$=70sin(x)+74 in case of sin(x)= –1

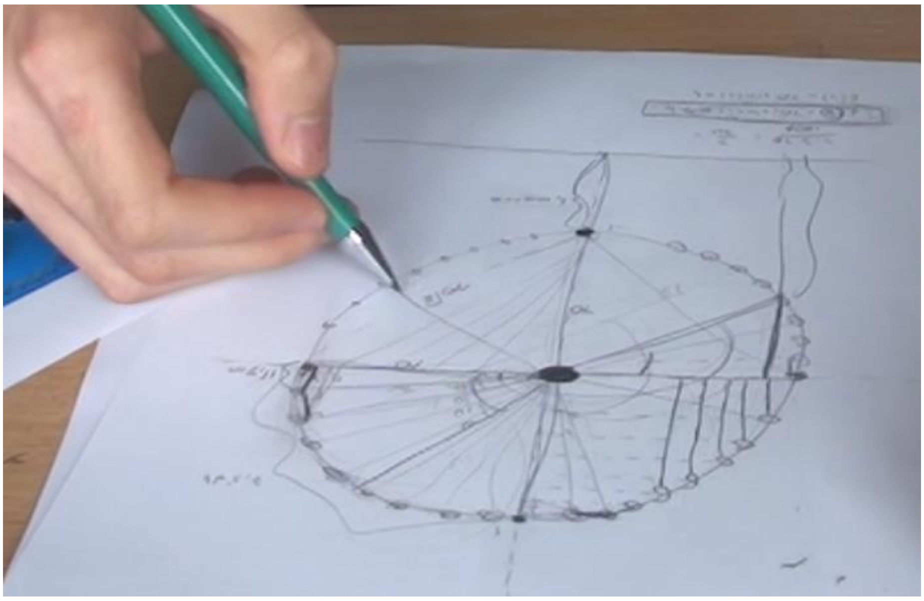


Figure 7.13. Zafer's drawing to articulate the meaning of *x* in *f*(*x*)=70sin(*x*)+74 (*Step1. modeling the terminal side of the angle*)

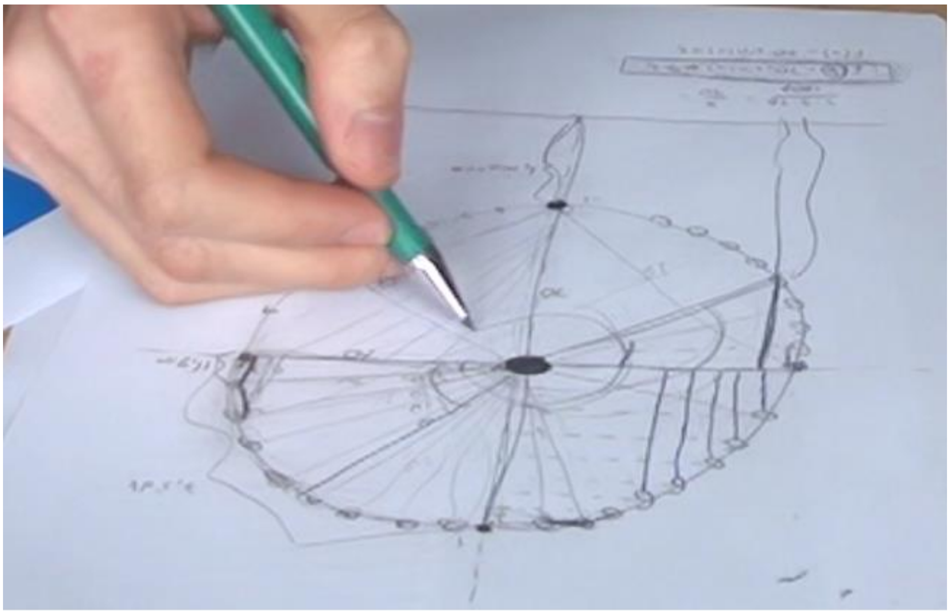


Figure 7.14. Zafer's drawing to articulate the meaning of *x* in *f*(*x*)=70sin(*x*)+74 (*Step2. modeling the angle as a dynamic turning by its principal arc in the (unit) circle register*)

[Modeling Task] Protocol 3

Zafer: Computer calculates it with respect to the time, I think. I mean… …actually, you know it is turning in 30 minutes at a constant speed (*drawing a principal arc in the first quadrant in the counterclockwise direction*). The angle increases in a constant rate (*drawing a principal arc in the first quadrant in the counterclockwise direction*).

Cemre: 30 minutes, 36 cars… …when changing the position of a car onto the position of the next car…

Ebru: It does not change onto the position of the next car. Their positions aren't fixed.

Zafer: That's, the time until one car comes to the next car.

Cemre: Huh. Zafer understood me (*smiling*).

Zafer: 30 minutes are 1800 seconds.

Cemre: The [angular] distance between two cars was 10 degrees. If 360 degrees are taken at 1800 seconds, 10 degrees are taken at how much time (*calculating this time by a direct proportion like in Figure 7.15*)… …50 seconds. That is, one car takes the next car's position at 50 seconds.

Zafer: (*Without speaking, initially, calculating a direct proportion to determine 1 degree is taken at how much time; and finding 1 degree at 5 seconds. And then, calculating another proportion to find how much degree are taken at 60 seconds; and finding it as 12 degrees like in Figure 7.6.*)

Cemre: Zafer, what are you doing now?

Zafer: 360 degrees are taken at 1800 seconds, I wondered 1degree was taken how much seconds. I found it at 5 seconds.

Defne: Yes, Cemre found 10 degrees at 50 seconds.

Ebru: I think we should find how much degrees are taken at 1 second.

Zafer: At 1 second, very small degrees is taken… …umm, 1 over 5, that is, 0.2 degrees is taken at 1 second (*writing 2/10 on his paper as a transformed form of 1/5; and then, its result as 0.2 degrees like in Figure 7.6*).

Cemre: (*Calculating a direct proportion to determine how much degree is taken at one second like in Figure 7.16.*)

Defne: If it takes 0.2 degree at one second, at *t* second how much does it take?

Cemre: Huh, it should give me, for example, at fifteenth seconds what is my ground clearance? We associate seconds [time] with degrees [angle]… …also degrees [angle] with the heights [ground clearance].

Defne: Lets we calculate. (*Calculating a direct proportion to determine how much degree is taken at 15 second like in Figure 7.17*) at 15 degrees it takes 3 degrees.

Cemre: Ok. Can we find the ground clearance for 3 degrees?

Zafer: Yes. We find it whatever we put in degrees.

Cemre: Then, it is ok.

Zafer: And, there is also 258 degrees…

Cemre: We should embed *t* [time] in it [function].

Zafer: It should give a warning at 258th degrees.

Cemre: Are you sure?

Zafer: I think so. Actually, I said before… …that 1 degree is taken at 5 seconds (*pointing his calculation by the direct proportion on the paper like in Figure 7.6*). Thus, at

60 seconds, 12 degrees are taken (*pointing his calculation by the direct proportion*). That is, at 1 minute, it takes 12 degrees.

Cemre: Ok.

Zafer: You know the boarding point is here (*pointing the bottom position on the circle on his drawing like in Figure 7.6*). So, its 12-degree before is here (*drawing a radius segment in the third quadrant; and then, indicating the arc inside of the circle from this segment to the down-vertical radius segment in the counterclockwise direction; and labelling it as 12*). That is, the time for coming from here to here is 1-minute (*indicating the endpoint of the radius segment on the circle referring to 12-degree before of the bottom position; and then, the bottom position on the circle*). When subtracting 12 from 270 (*dragging his pen on the down-vertical radius segment*), we find here (*drawing a principal arc inside of the circle from the right-horizontal radius segment to the radius segment in the third quadrant*). That is, at 258th degree, 1-minute remains to landing.

Cemre: Ok. You are right. That is, when it is at 258 degrees, it give a warming.

Zafer: Uh-huh (*nodding his head up and down*).

Researcher: What is 258 degrees?

Cemre: It refers the position of the car at the last one minute before landing.

Researcher: How much degrees does it take in the last one minute?

Cemre&Zafer: 12 degrees.

Researcher: What is the addition of 258 and 12?

Zafer&Cemre: 270.

Cemre: (*Looking to the Zafer*) it should have been 360?

Zafer: No. The boarding point is here (*marking the bottom point on the circle on his paper like in Figure 7.6*). But we calculated angle with respect to here (*drawing an arc inside of the circle from the right-horizontal radius segment to the down-vertical radius segment in the counterclockwise direction*).

Cemre: Hmm.

Ebru: (*After 10-second pause*) we need to write *x* based on the time.

Cemre: Yes, but how do we do it?

Defne: Well, when thinking about *x* based on time… …umm, you know we did for *x* and 2*x* [in Task 11 and 16]… …actually, it [reference point of 2*x*] was turning in double speed of other [reference point of *x*]. Remember we did on the unit circle… …at the same time, while that [reference point of 2*x*] was turning that much path, the other [reference point of *x*] was turning its half.

Zafer: At one second, it [a car on the Ferris wheel] takes 0.2 degree.

Defne: I mean now, we think that (*writing f(x)=70sin(x)+74 function like in Figure 7.18*) … …at 1 minute (*writing "1dk" referring to 1 minute near x in the f(x)*), it would take 12 degrees (*pointing x in the sin(x)*), at 2 minutes (*writing "2dk" referring to 1 minute near x in the f(x)*), it would take 24 degrees (*pointing x in the sin(x)*).

Zafer: Uh-huh (*nodding his head up and down*).

Defne: …at 3 minutes, it would take three times of 12. It is going on this way (*putting the points under "2dk" to indicate the pattern between time and angle measure like in Figure 7.18*). So, we multiply time by 12.

Zafer: Then, if writing 12*x* instead of *x* in here (*pointing the x in 70sin(x)+74 expression on Defne's writing like in Figure 7.18*), we have associated minutes [time] with degrees [angle].

Defne: (*Writing 12x below of x in sin(x) like in Figure 7.18*)

Zafer: Then, it should take in 1 minute… …umm… It will start from here (*pointing the bottom position on the circle*). We should consider the first 7.5 minutes also.

Cemre: We add or subtract the angle of 7.5 minutes, in my opinion.

Zafer: Our function starts from here (*pointing the far right position on the circle*) as if this part is passed over (*dragging the quarter arc from the bottom position to the far right position in the counterclockwise direction*). Then, we should add 7.5 minutes.

Defne: If 1 minute refers to 12 degrees, 7.5 minutes refer to 90 degrees (*writing a direct proportion to determine the angular path taken at 7.5 minutes like in Figure 7.18*). Then we should add 90 to here (*pointing the input of sine, and writing* $f(x)=70\sin(x+90)+74$ *through emphasizing x in f(x) as time and input of sine as angle like in Figure 7.19*).

Zafer: Yes of course (*indicating the quarter arc from the bottom position to the far right position in the counterclockwise direction*)! (*Writing* $f(t)=70\sin(x+90)+74$ *on his paper like in Figure 7.6*). Now, if we put here 1 minute (*circling t in f(t)*), here is ok. Where, *x* in degrees must be 12 (*circling x in sin(x+90)*).

Defne: It [*x*] should be 12 times of minutes [*t*].

Cemre: Uh-huh.

Zafer: Then, it [input of sine] should be $12x + 90$. You know we want to have a thing in degrees in here (*circling the input of sine on* $f(t)=70\sin(x+90)$). I mean for example for 1 minute, there must be in here 12 plus 90. Why? Because we cannot write 12 to here (*circling t in f(t)*), it [t] is the time.

Cemre: Ok. We write there [for *t*] 1.

Zafer: For being able to write 1 to here (*circling t in f(t)*), in order to have 12 in here (*circling x in sin(x+90)*), we have to write 12*x* to here (*circling x in sin(x+90)*. You know for 1 it [12*x*] gives 12.

Defne: (*Revising her writing as* $f(x)=70\sin(12x+90)+74$ *like in Figure 7.19*)

Cemre: Yes, it is ok.

Zafer: Let's we control it for 1 [minute].

Ebru: (*Calculating* $70\sin(12+90)+74$ *as 142.47*) 142.47.

Defne: It should have been smaller than 74.

Zafer: Why did it so?

Defne: Because we add 90.

Cemre: It didn't give us our expected value.

Defne: (*Crossing out of writing* $f(x)=70\sin(12x+90)+74$ *like in Figure 7.19*)

(*All of the students were looking to their own papers without speaking during 5 minutes.*)

Zafer: What about we subtract 90 instead of adding 90.

Cemre: Let's we try through subtracting. (*Calculating* $70\sin(12-90)+74$ *as 5.53*) 5.53. Is it possible for the ground clearance at 1 minute?

Zafer: I am not sure.

(*Another pause was observed in that time.*)

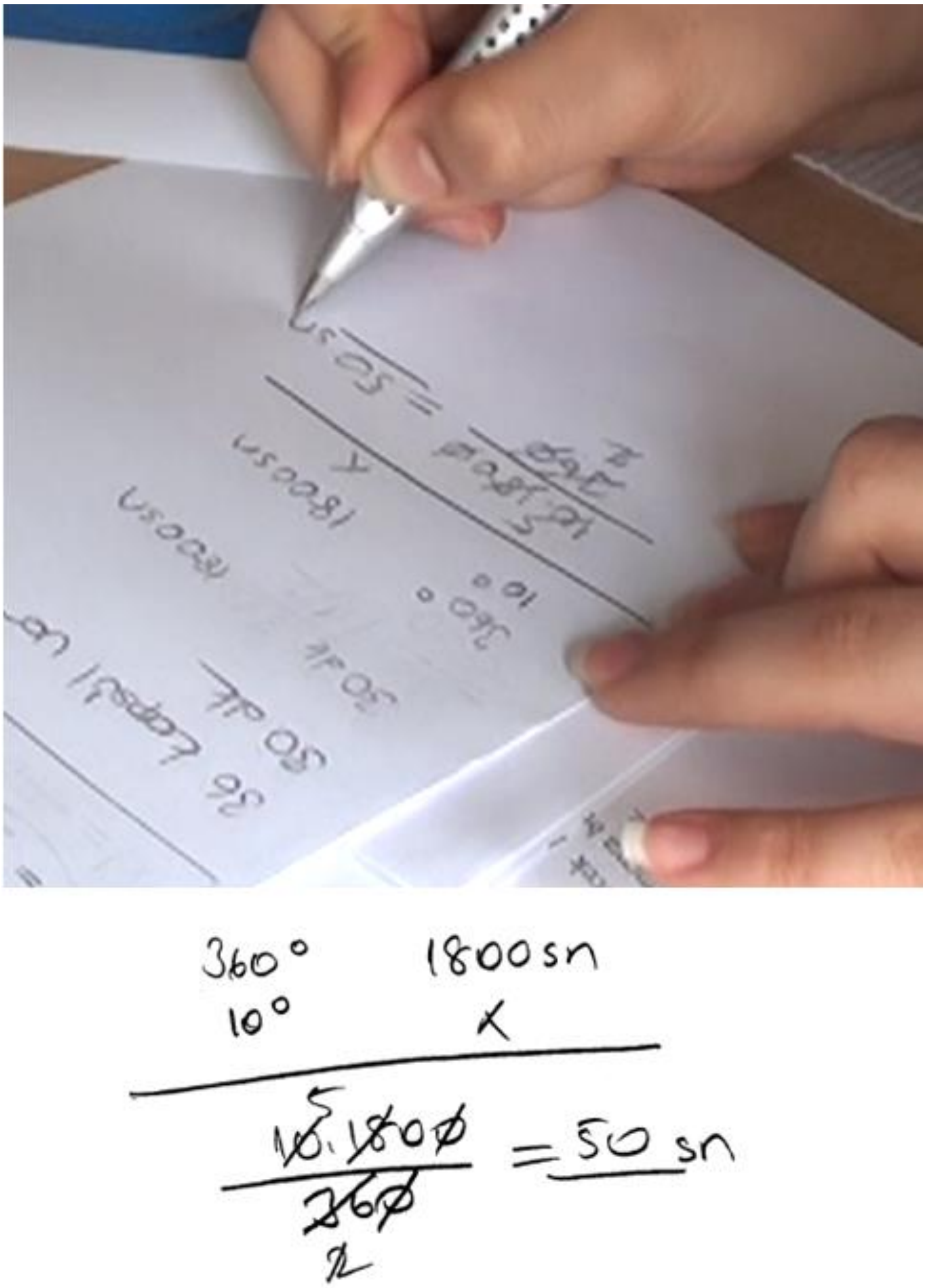


Figure 7.15. Cemre's direct proportion to determine the time at which one car replaced on the next car's position

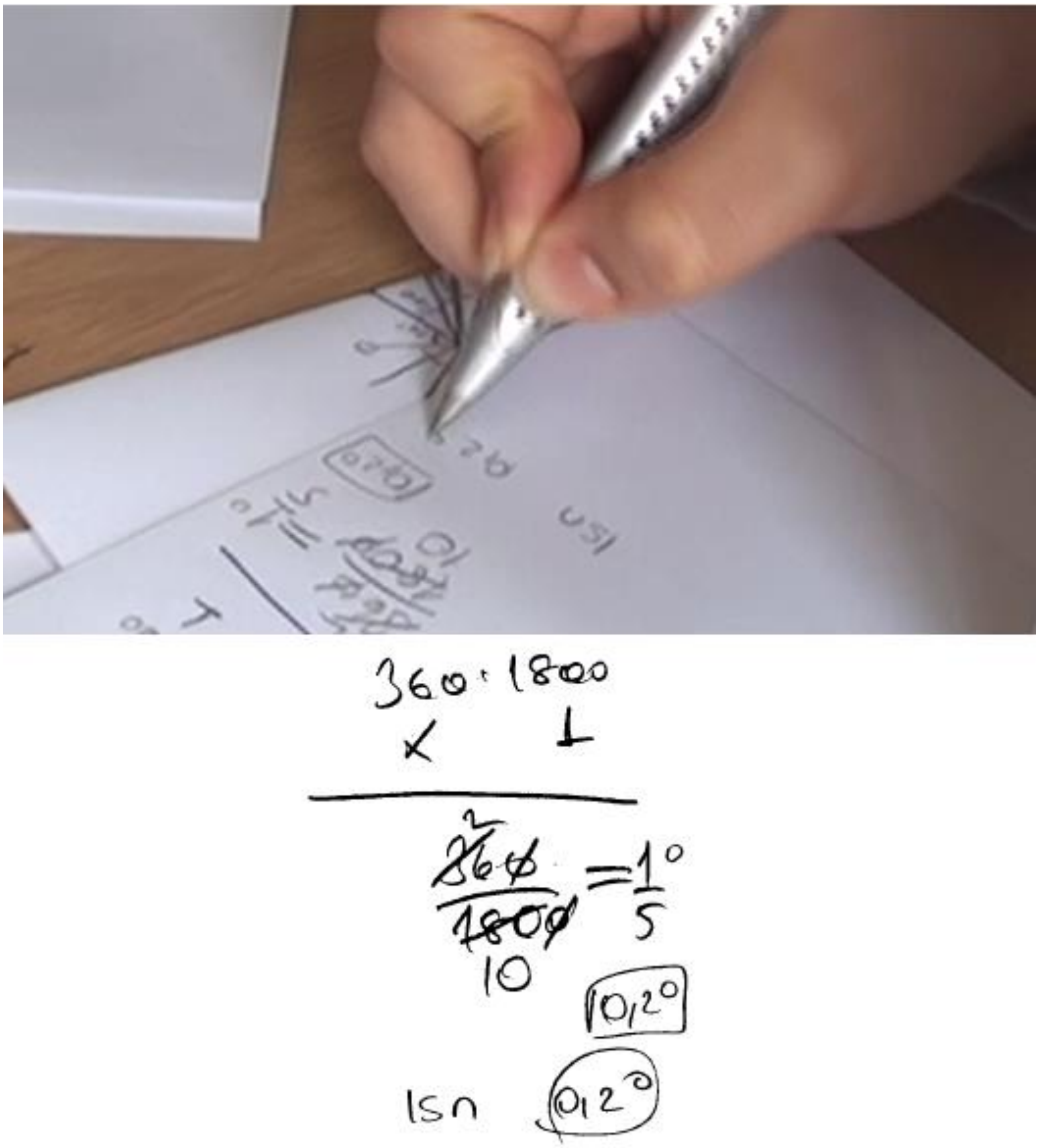


Figure 7.16. Cemre’s direct proportion to determine the turning amount in degrees at 1 second

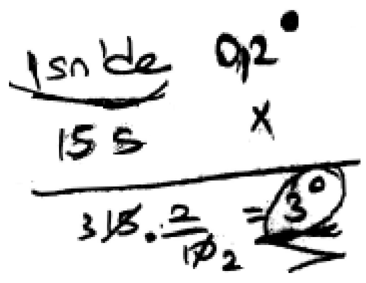


Figure 7.17. Defne's direct proportion to determine turning amount at 15 seconds

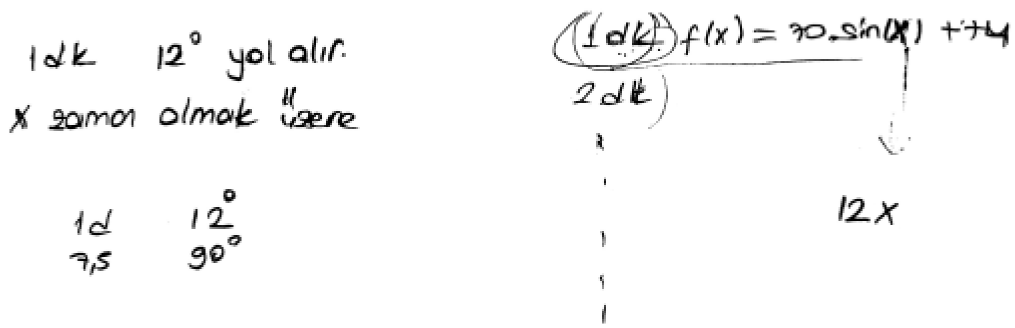


Figure 7.18. Defne's reasoning about "angle" based on "time" in minutes

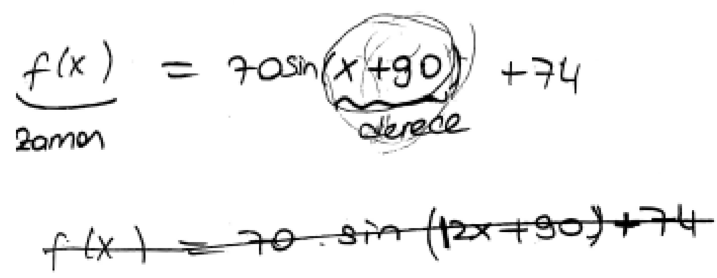


Figure 7.19. Defne's revision of the function so as to model the ground clearance

[Modeling Task] Protocol 4

Researcher: (*Constructing an origin-centered circle with 7-unit radius; as well as the line segments referring to the ground clearance and the distance to the gotten in the car through considering y= -7 line as the ground like in Figure 7.20. Moreover, she measured the distance of DA line segment by GSP*) let's control your model in case of this circle.

Cemre: Well, (*opening the calculate menu of GSP and typing 7sin(12-90)+7 through uttering*) the radius is 7. It must be seven times sine 12 minus 90… …where I took *x* as 1 minute… and out of parenthesis plus 7.

Defne: …zero point fifteen (*uttering the result of 7sin(12-90)+7 by GSP under "degrees" angle measure unit*).

Zafer: Do this angle 12 degrees (*pointing the point A on the circle*).

Cemre: Firstly, I measure it (*measuring <CBA angle*)

Zafer: Now, bring it to 12 degrees.

Cemre: (*Dragging the point A so as to obtain 12 degree as the measure of <CBA*) now I measure here (*pointing the distance from the point A to D Figure 7.20(a)*). (*When the distance measure of DA segment were appeared as 0.15cm like in Figure 7.20(a)*) yes. But, let's we control also for 24.

Zafer: Then, it for 2 minutes.

Cemre: (*Editing 7sin(12-90)+7 as 7sin(24-90)+7; and then, dragging the point A so as to obtain 24 degrees as the measure of <CBA. When the result of her calculation and the distance measure of DA segment were appeared as the same output like in Figure 7.20(b)*) yes (*laughing*).

Zafer: True.

Defne: Then, our [model] function is true.

Cemre: (*Laughing*) I am very pleased. We do it. That is to say, we should have subtract 90 degrees.

Researcher: Why?

Zafer: I understand it. We start from here for the sine function (*pointing the intersection point of the circle with the positive x-axis on the screen like in Figure 7.20(b); and then, dragging his index finger on the circle in the counterclockwise direction*). However, we start turning its 90 degrees before (*dragging his index finger in the clockwise direction on the quarter arc of the circle in the fourth quadrant*).

Cemre: Yes. That is, we are going 90 degrees in the negative direction (*figuring an arc in the clockwise direction*).

Ebru: Uh-huh (*nodding her head up and down*).

Defne: …as we mentioned earlier courses… …there were "before arrival", "after arrival" issues.

Zafer: Yes.

Cemre: Then, our function is $f(x)$=70sin(12x-90)+74 (*writing f(x)=70sin(12x-90)+74 like in Figure 7.21 and labelling it as "yerden yükseklik" which means the ground clearance*).

Defne&Ebru: Yes.

Zafer: Finally, we did (*laughing*).

Cemre: I am very happy (*laughing*).

45 *Researcher:* Unfortunately, you did not finish it!
46 *Zafer:* Why?
47 *Researcher:* Still, you did not find the distance to the point gotten in the car (*smiling*).

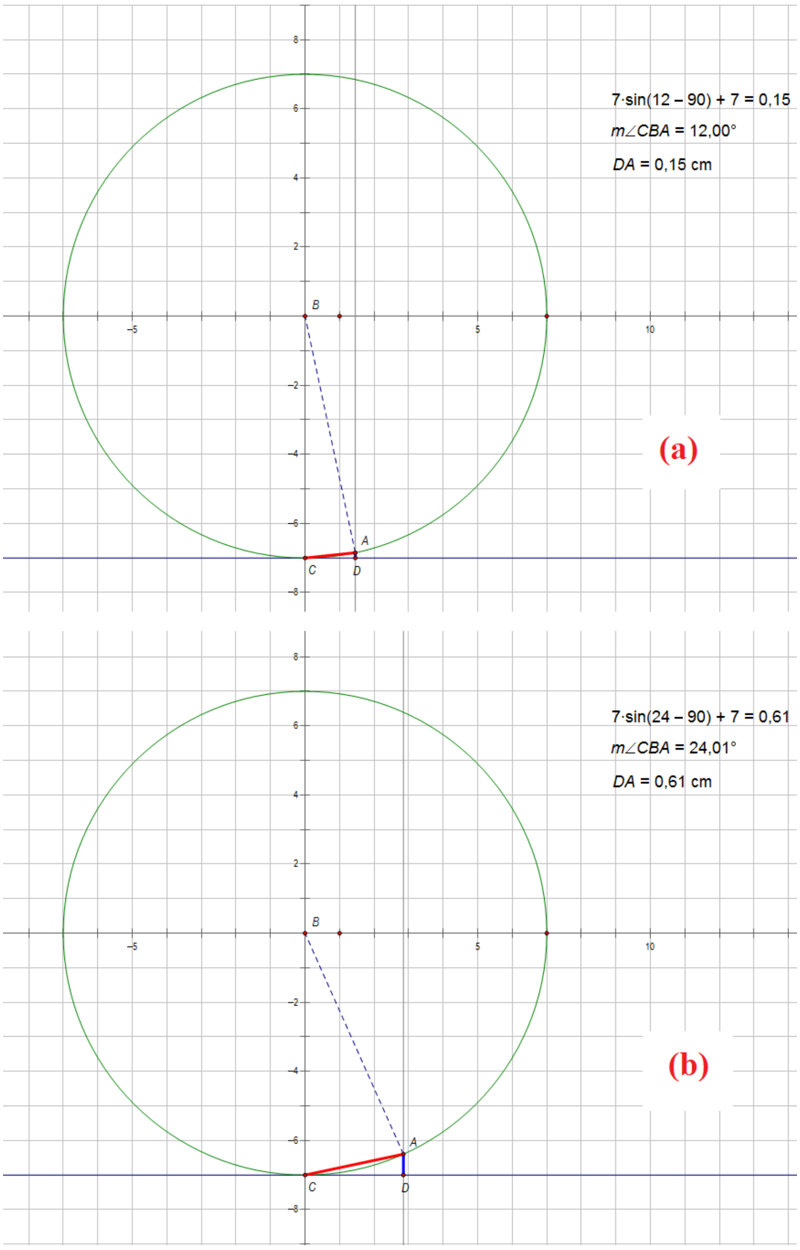


Figure 7.20. The researcher's construction in GSP environment to model for another Ferris wheel with a different radius and a different ground line from those in the modeling task

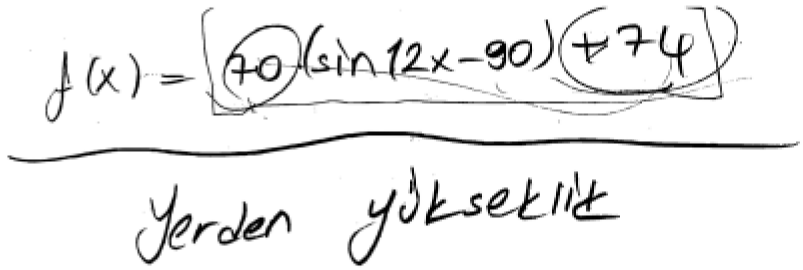


Figure 7.21. Cemre’s revision of the function modeling the ground clearance

[Modeling Task] Protocol 5

Researcher: Still, you did not find the distance to the point gotten in the car (*smiling*). You should identify another function that gives this red segment’s distance with respect to time (*dragging her index finger on the red line segment from the point C to the point A*).

Cemre: (*Drawing some line segments to represent the distance to the gotten point in the car on her drawing like in Figure 7.4*)

Zafer: It is obvious that here is 70√2 (*pointing the line segment from the bottom position to the far right position on his drawing like in Figure 7.6*)

Cemre: Yes (*drawing the down-right triangle with the hypotenuse from the bottom position to the far right position on her drawing like in Figure 7.4*).

Ebru: (*Dragging the point A in the first quadrant on GSP page like in Figure 7.22*)

Defne: Here is sin(alpha) (*dragging her index finger between the projection point of the point A on the y-axis and the origin on the screen like in Figure 7.22*)…

Ebru: Uh-huh (*nodding her head up and down*).

Cemre: (*Without speaking, constructing a reference rectangle (see Footnote 44) in the first quadrant of the circle on her drawing like in Figure 7.4*)

Zafer: (*Without speaking, drawing on a paper a new circle with the perpendicular axes and a right triangle so as its hypotenuse to be the line segment referring to the distance to the position gotten in the car like in Figure 7.23*)

Defne: Here is sine of alpha plus 70 (*dragging her index finger on the y-axis from the projection point of the point A to the bottom position of the circle*). Thus, its square… …I mean we can consider by Pythagorean.

Zafer: (*Labelling the angle in the first quadrant as α, the opposite side of the reference right triangle as “sin(α)”, the adjacent side as “cos(α)” and the leg on the y-axis of the big right triangle as “70+sin(α*)” *like in Figure 7.23. And then, writing symbolic expression of the relation between the legs and the hypotenuse of the big right triangle by Pythagorean like in Figure 7.23.*)

Defne: (*Drawing a unit circle through separating its quadrants, as well as a reference right triangle in the first quadrant; and then, labelling the below vertical radius segment as “70”, and the projection segment of the opposite side of the reference* *right triangle on the y-axis as "sin(α)" on a paper like in Figure 7.24*). Here is (*pointing the opposite side of the reference right triangle*)…

Ebru: It is cosine, isn't it?

Defne: Yes, it is cosine of alpha (*labelling the adjacent side of the reference right triangle as "cos(α)" like in Figure 7.24*).

Zafer: That is not (*pointing his symbolic expressions on the paper like in Figure 7.23*)?

Defne: (*Writing Zafer's symbolic expression like in Figure 7.23 on her own drawing like in Figure 7.24*) yes, it's by Pythagorean.

Cemre: Can I look (*looking to Zafer's paper like in Figure 7.22*). Yes, here is cosine (*dragging her index finger on the horizontal side of the big right triangle whose hypotenuse was referring to the distance to the bottom point on Zafer's drawing like in Figure 7.22*), here is 70 plus sine (*dragging her index finger on its vertical side*), you did it by Pythagorean. Yes.

Zafer: Oops! There must be in here 70s (*adding coefficient70 for sine and cosine on his symbolic expression; and then, writing it on the next line in a more expanded spacing like in Figure 7.23*).

Defne: Yes.

Ebru: Uh-huh (*nodding her head up and down*).

Cemre: Let's we control it.

(As they did for the function modeling the ground clearance (see lines 1-25 in [Modeling Task] Protocol 4), students controlled their model function for the distance to the position gotten in the car through adopting it in case of the circle in GSP environment with the 7-cm radius whose center was located on the origin, as well as comparing with the measure of the red line segment that was referring to the distance to the bottom point on the circle (*see Figure 7.25*).)

Cemre: (*Writing their final symbolizations to model respectively the ground clearance and the distance to the position gotten in the car during the turning by two time-dependent functions like in Figure 7.26*).

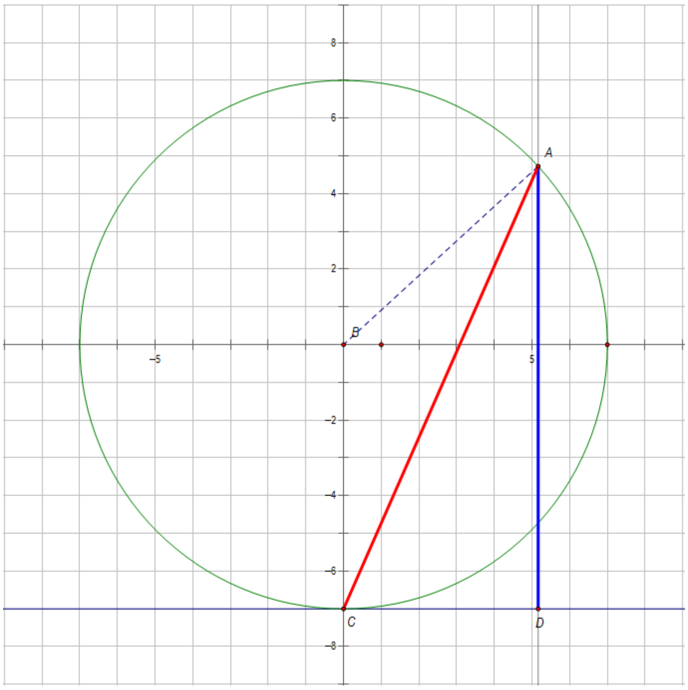


Figure 7.22. Students’ reasoning efforts on the distance to the position gotten in the car by a simpler analogy in the first quadrant without loss of generality

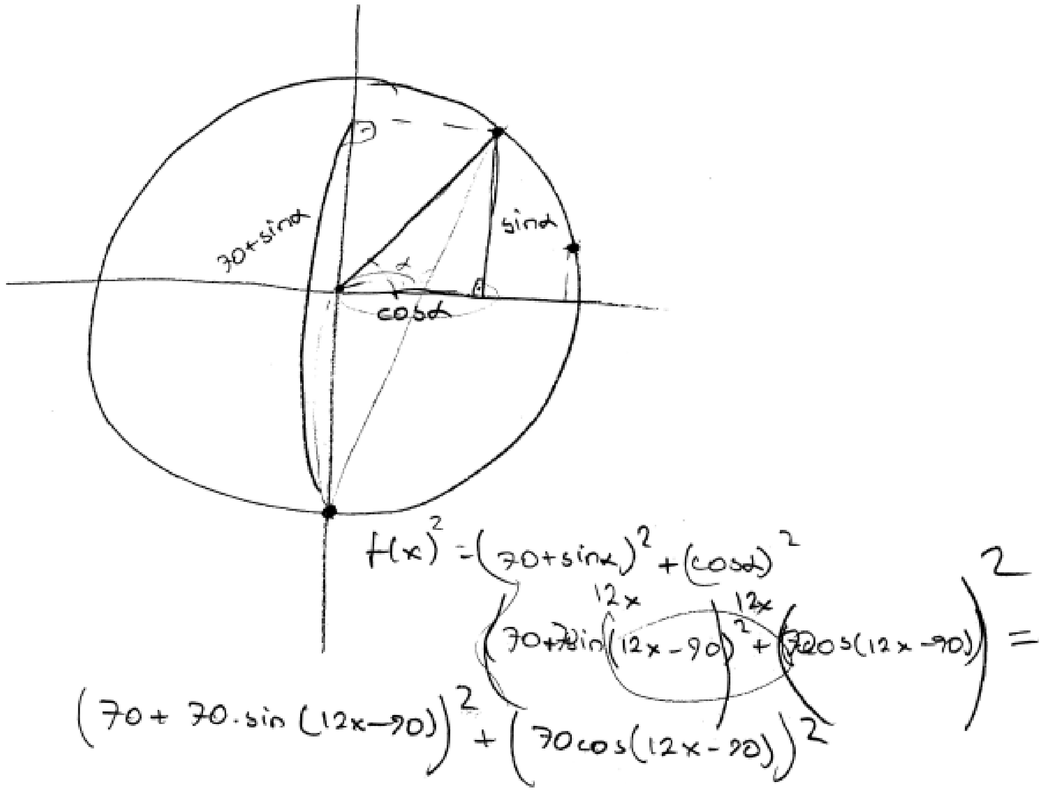


Figure 7.23. Zafer's drawing to model the distance to the point gotten in the car

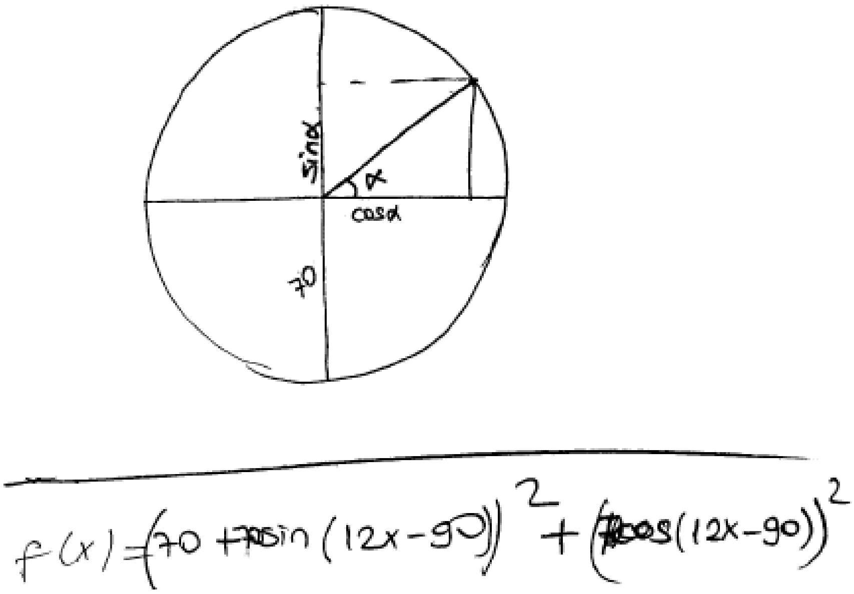


Figure 7.24. Defne's drawing to model the distance to the point gotten in the car

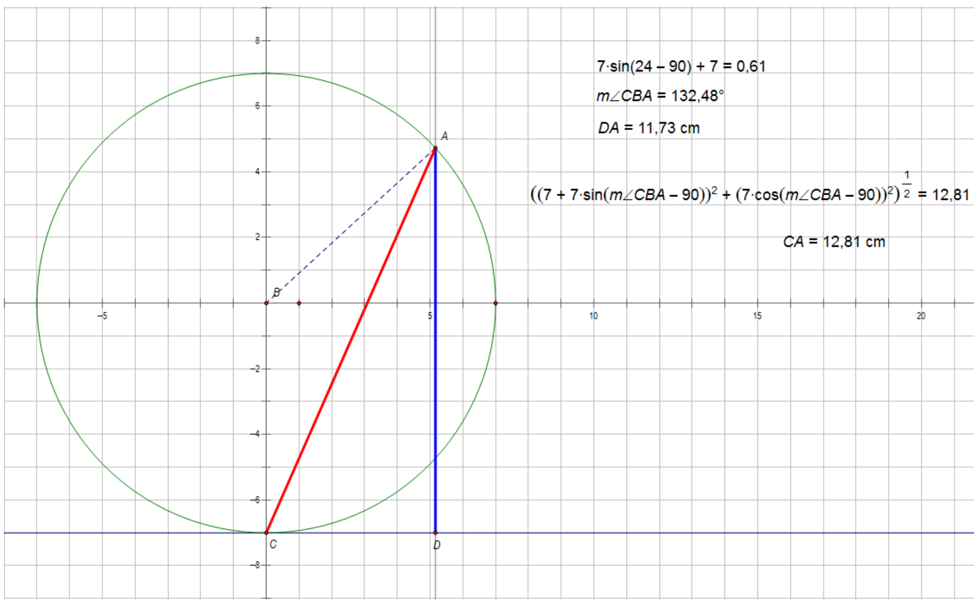


Figure 7.25. Students' controlling their model function for the distance to the position gotton in the car through adopting it in case of the circle in GSP environment with the 7-*cm* radius whose center was located on the origin

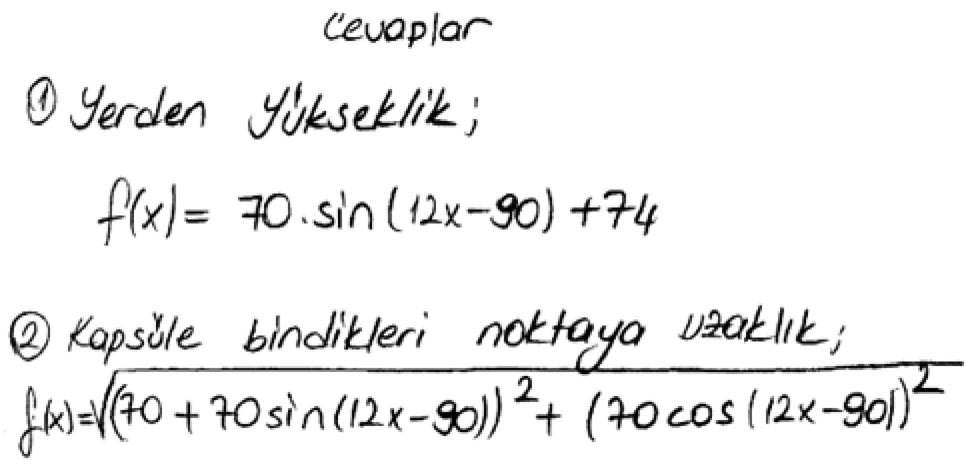


Figure 7.26. Final forms of two functions modelling respectively the ground clearance and the distance to the position gotten in the car with respect to "time" in minutes

7.2. Summary of Students' Way of Reasoning in Modeling Task

Throughout the teaching experiment important developments on students' concept images related to trigonometric functions were observed in GSP environment (see *summary* parts in Chapter 4, 5 and 6). As the last episode of the teaching experiment, a modeling task, Ferris wheel, was used to understand students' abilities to transform their progressed-concept images in GSP environment into reasoning in paper-and-pencil environment. Although students were studied previous tasks separately as two pairs, throughout this task, all of four students were worked as a group on a mathematical modeling task with Ferris wheel (Appendix C).

Students started to visualize task context in the (*unit*) *circle register*. In this visualization process, Zafer associated this visualization with the upward translated-version of the unit circle. Defne dissociated it from unit circle based on the radius of the Ferris wheel, i.e., 70 *meter*. Cemre mentioned a circle with 70-meter radius as a unit circle through considering 70-*meter* as the unit length. Zafer confirmed this reasoning. Cemre and Zafer were able to reason about any non-unit circle as a unit circle through considering a new measure unit as the units of another unit as they did in Task 8 and 13 (see *Changed-radius in (unit) circle register* heading in Chapter 6).

When the unit circle idea was brought up for discussion, Cemre proposed to discuss the ground clearance in the first quadrant in the (*unit*) *circle register*. When reasoning in the (*unit*) *circle register*, even though *radian* measure unit had been used during the teaching experiment, during the modeling task students preferred to use *degree* measure unit. When reasoning about the ground clearance of a car on the Ferris wheel, Zafer was able to reason about the length of the opposite side of the reference right triangle through considering the same circle either as a unit circle or as a non-unit circle; respectively, as the sine value and the multiplication of the sine value by the radius as he did in Task 8 (see *Changed-radius in (unit) circle register* heading). Moreover, he was able to reason about a non-unit circle's changed-location through extending his reasoning about the unit circle's changed-location mentioned in Task 9 (see *Changed-center in (unit) circle register* heading).

When reasoning about the ground clearance of the top position arithmetically based on the nine 10-degree arcs between 10 cars in the first quadrant and the value of $\sin(10^{o})$, Defne considered that the proportional increase of angle measure produced the proportional increase of vertical distance. Zafer articulated for the sine function that the proportional increase of inputs did not give proportional increase of outputs through giving the relation between $\sin(10^{o})$ and $\sin(20^{o})$ as an example based on the curvilinear shape in the *graphical register*, as well as the circular shape in the (*unit*) *circle register*. The researcher inferred that Defne's concept image on the rate of change of the sine values was including a trouble arising from her reasoning about "change" idea predominantly in the *symbolic register* as if a constant rate of change exists between angles and their sine values. In her construction in the (*unit*) *circle register* (*Figure 7.2*), there were six perpendicular segments with the equal magnitude that were located at equivalent-angular intervals in the first quadrant. These perpendicular segments did not represent the increasing amount of the height regarding to the previous one. Despite of this visualization by herself, she considered the sum of their lengths as the same length with the up-vertical radius segment. This implies that she did not reason about the relation between these perpendicular line segments in the first quadrant and the up-vertical radius segment visually in the (*unit*) *circle register*. Instead, she reasoned algebraically about the multiplication of the number of these perpendicular segments by their magnitude as the radius predominantly in the *symbolic register*.

Zafer introduced his initial model of ground clearance as $f(x)$=70sin(x)+74, and confirmed the function's suitability for some case including the negative sine values. Cemre and Defne approved this function's suitability. However, Ebru expressed her dissatisfaction about this function in terms of the meaning of x in $f(x)$; and brought "turning" up for the discussion. Zafer explained the meaning of x in $f(x)$ as the angle referring to the principal arc. When reasoning about the changed-angle as a result of the Ferris wheel's turning, "time" idea was brought up for the discussion by Zafer. This provoked students to determine the angular speed of the Ferris wheel. Although "angular speed" idea was not mentioned explicitly anywhere of the teaching experiment, as a consequence of Defne's reasoning about the angular path based on the speeds of the points as they did in Task 11 and 16 on two points turning dynamically with different speeds on the unit circle in GSP environment, they were able to determine the angular speed of the Ferish wheel as 12x (where, x refers to time in minutes).

When the boarding –bottom– position was mentioned, Zafer expressed that their model function gave the results after far-right position of the Ferris wheel. They were able to determine this situation affected their model function's input variable as a constant additive term. However, when revising their model function so as to transform it into the expected function, they missed out the reference position between the boarding position and the far-right position in terms of which position was transformed onto the other one. That is, they revised it through adding 90 to the input of sine. When they tested this function's suitability by using GSP's calculate option, they were aware the error, and revised it through subtracting 90.

At that point, the researcher modeled in GSP environment a similar but not same situation with the modeling Task by a circle with the 7-unit radius whose center was located on the origin. When they ensured its suitability, Zafer reasoned about the cause of subtracting 90 [degrees] rather than adding it based on the *zero-points* (see *Footnote 50*) of the sine function and the Ferris wheel. That is to say, he expressed the position gotten in the car as the 90-degree before of the far right position of the circle through assuming the counterclockwise direction as the turning direction. All other students agreed on this reasoning.

When reasoning the distance to the point gotten in the car with respect to time, they constructed some rectangular structures in the first quadrant on their drawings in order to specify the legs of this right triangle in terms of sine and cosine. On this rectangular structures, they were able to reason through exchanging their focuses between the opposite [adjacent] side of the reference right triangle and its facing-side in the *reference-rectangle* (see *Footnote 44*) in terms of sine [cosine], which emerged as a result of the teaching experiment in Task 12.

CHAPTER 8

DISCUSSIONS, IMPLICATIONS AND SUGGESTIONS

This chapter summarizes the four secondary students' development in understanding fundamental concepts of trigonometry during the teaching experiment. Also, this chapter presents some conceptual frameworks that were grounded as a result of the on-going design process of the instruction of the teaching experiment based on the prospective and retrospective cognitive analysis of the data. Firstly, *cognitive concept maps* on angles, sine [cosine] function and periodicity that were revealed as foundational for students' *recognition* of trigonometric functions are presented to provide a lens for the reasoning ways of the students' understanding fundamental concepts of trigonometry when integrating a new concept and the related current concepts into students' cognitive knowledge structure. Secondly, *cognitive networks* that were revealed as foundational for visual discrimination of the sine function both in the (*unit*) *circle register* and the *graphical register* are presented. This chapter also presents implications and suggestions for curriculum and instruction. Eventually, limitations of the study and recommendations for future research are presented.

8.1. Students' Initial Concept Images on Trigonometric Functions

Prior to the teaching experiment, the four students of the study (Cemre, Defne, Ebru and Zafer) had just completed trigonometry course successfully. Under this heading, their *recognition* and *discrimination* problems on foundational trigonometric concepts (such as angle, angle measure, trigonometric value, trigonometric functions, and periodicity) as well as *potential conflict factors* in their *concept images* are summarized.

8.1.1. Students' recognition problems

Cognitive analysis of the initial interviews revealed that the students' concept images included a lot of critical troubles on foundational concepts related to trigonometric functions. Initially, as a consequence of their restricted concept definition images on functions into the polynomial functions in the *symbolic register*, they were unable to recognize trigonometric functions as functions (Weber, 2005). An additional problem of students' recognition of trigonometric functions as functions was their dominated-mental images on functions' visual representations by the *graphical register*. It may be the result of their inability to associate the right triangle definitions and unit circle definitions of trigonometric functions with functions.

Except Defne, other students' mental images on angles were *static* instead of *dynamic turning*. However, many of the studies in the literature emphasize the importance of thinking about angles in terms of rotations (Brown, 2005; Fi, 2003; Mitchelmore & White, 1996; Moore, 2010) which is critical in comprehension of trigonometric functions in different representations, as well as using in the different contexts.

Another trouble of the students was based on the angle concept. Although students' concept definition images on angles included an intuitive relation between openness and measure of an angle, they also included a trouble consistent with the literature (Thompson, 2008; Moore, 2010) that what was the measured part to determine an angle's measure. In spite of the students' circular constructions inside of an angle [such as arcs and arc sectors], they did not recognize the meaning of these circular constructions in terms of angle measure. Nevertheless, they reasoned about the angle measure intuitively not to vary with respect to variations of radii of these circular constructions. Thompson (2008; 2011) mentioned the inner arc of an acute angle of a right triangle only as a pointer in a diagram without associating with angle measure. Thus, the students' circular constructions were playing a part on their concept images on angles only as a "pointer".

Akkoç (2008) and Fi (2006) propose the presence of π in the radian measures as a source of the students' difficulty with the radian measure unit due to their consideration of π as the unit for the radian measure instead of its real value. During the initial interviews, when reasoning about transformations of π between different representational registers, the students encountered many troubles based on π as a consequence of the change of the source register from *symbolic* to *graphical* and as a consequence of the variation on the scaling of the *x*-axis from numbers regarding π to the real values. All students' reasoning on π in the *graphical register* was constrained by the trigonometry context. When reasoning about the location of π as the abscissa of an ordered-pair within the *graphical register*, Cemre and Defne preferred to transfer their reasoning on π within the *symbolic register* in the trigonometry context (i.e., π as equal to 180) to that out of the trigonometry context. That is, they located π in the *graphical register* on 180 on the *x*-axis. Conversely, Ebru and Zafer preferred to transfer their reasoning on π within the (*unit*) *circle register* and considered respectively $\pi=-1$ and $2\pi=1$ in the *graphical register*, respectively. That is to say, they were unable to dissociate two meanings of the intersection points of the unit circle with the *x*-axis [i.e., the points (-1,0) and (1,0)] as an **abscissa** corresponding to these points within the *graphical register* and as an **angle** in radians corresponding to these points within the *unit circle register*. This arose from their mental images on angles as points on the unit circle instead of corresponding arcs. Besides, when trying to convert her reasoning on π within the (*unit*) *circle register* into the *graphical register*, Defne considered origin (i.e., zero point on the *x*-axis) as 2π on the *x*-axis based on their same positions when referring to angles in radians within the (*unit*) *circle register*. It means that she was unable to dissociate the meaning of the equivalence between "zero" and 2π in radians from the equality between them within the *graphical register*, which arose from her dominated concept image on principal angles despite of her mental image on angles as dynamic turning. Therefore, the students did not recognize the same mathematical object, i.e., π, within different representational registers.

Consequently, students' reasoning on π in different representational registers indicates that "**coordinate plane**" became a *cognitive conflict factor* when the position of π on the *x*-axis is considered simultaneously within the *graphical register* and the (*unit*) *circle register*.

In comprehension of trigonometric functions, there is a need to associate them properly with the appropriate geometric models (Brown, 2005; Weber, 2005). Two common geometric representations exist to model trigonometric functions visually: *right triangle representation* and *unit circle representation*. When asked the students to define sine [cosine], their initial attempts were based on the right triangle context. They defined sine [cosine] as ratio of opposite [adjacent] side to hypotenuse. However, none of the students was aware that this ratio for an angle in a right triangle were the same as that in all similar right triangles. This unawareness arose from their reasoning about sine and cosine within the right triangle context as calculations instead of ratios obtained from proportions in the similar right triangles. Thus, the students' concept definition images in right triangle trigonometry did not include similarity. Thompson (2008) stresses similarity –that the similar right triangles have same trigonometric ratios– as a fundamental starting point to reason about triangle trigonometry. On the other hand, when trying to define sine [cosine] in the (*unit*) *circle register*, although their constructions included a right triangle in the first quadrant, none of the students associated this right triangle with the definition of sine[cosine] on the unit circle. That is to say, they dissociated the definition of sine [cosine] on the unit circle from that on the right triangle. They defined sine[cosine] on the unit circle by a set of geometric procedures including drawing an angle in the first quadrant, concretizing its reference point on the unit circle, drawing a dashed-perpendicular line segment from this point to the *y*-axis [*x*-axis], and concretizing the intersection point of this segment with the *y*-axis [*x*-axis]. Thus, none of the students recognized the relation between right triangle trigonometry and unit circle trigonometry, which is mentioned in the literature as a root of students' difficulties in trigonometry (Thompson, 2008).

All recognition problems of the students mentioned above arise from their problematic concept definition images. Tall and Vinner (1981) assert the weak understanding of the *concept definition* as a source of students' problems in mathematics. First requirement of the ability to recognize same object represented in different registers is to know this mathematical object's definition.

8.1.2. Students' discrimination problems

Consistent with the prior literature that indicates students' troubles on making sense of the radian measure unit apart from the degree measure unit (Akkoç, 2008; Fi, 2003; Topçu, Kertil, Akkoç, Yılmaz & Önder, 2006), none of the students was able to reason about the meaning of radian apart from degree. That is to say, the meaning of radian measure was dominated by degree meaning (Akkoç, 2008; Fi, 2003) and restricted only into transformations between degree and radian measures (Topçu, et al., 2006) within the *symbolic register* without any other meaning of the radian measure. Topçu et al. (2006) characterize the equation $\frac{D}{180} = \frac{R}{\pi}$ as a possible source of this understanding. During the initial interviews, the students' transformations between degree and radian implied their *discrimination* problems in the *symbolic register* on the meaning of the equal sign. In other words, they did not aware of the existence a proportional equality with degree and radian notations [such as πR=180^{o} and $\frac{\pi}{6}$ R=30^{o}] rather than a computational equality without degree and radian notations [such as π=180 and $\frac{\pi}{6} = 30$].

Another *discrimination* problem was observed on π notation in the *symbolic regsiter*. In spite of their reasoning that the same π notation must be refer to the same object within the *symbolic register*, all students dissociated π in and out of the trigonometry context in terms of its real value. In other words, they treated π in the *symbolic register* as if it had two different real values; i.e., “180” in the trigonometry context and about 3.14 out of the trigonometry context (Akkoç, 2008).

> Therefore, within the *symbolic register*, “π” had a potential to become a *cognitive conflict factor* in and out of the trigonometry context when considered simultaneously.

During the initial interviews, it was observed that systematic variations of situations in a representational register caused the students’ *discrimination* problems. For example, despite of her definition on sine of an angle in the first quadrant by a set of geometric procedures, Cemre was unable to reason about sine of angles corresponding to the axes such as π/2 and 3π/2 in radians. Furthermore, Ebru converted her definition of sine [cosine] within the (*unit*) *circle register* into the *symbolic register* as ordinate [abscissa] of a point on the unit circle. Unfortunately, she reasoned an arbitrary ordered-pair’s ordinate [abscissa] as sine [cosine] when reasoning about the position on the coordinate plane of the ordered-pair whose ordinate was defined in terms of cosine. She made an overgeneralization of the ordered-pair definitions of sine and cosine. It may be due to the students’ consideration of the whole *x*-axis [*y*-axis] as the cosine [sine] axis without awareness of the correlation between the *x*-axis [*y*-axis] and cosine [sine].

Another *discrimination* problem of the students was based on reasoning about angles with the same principal measure. Consistent with the literature reporting students’ inadequate knowledge of coterminal angles (Fi, 2003), the students were unable to discriminate coterminal angles (i.e., angles with the same principal measure) in the (*unit*) *circle register*. That is to say, they were unable to generate a negative equivalent meaure from a principal measure. It was due to their reasoning about negative angles based solely on the memorized-rules without any reasons within the *symbolic register* as well as their restricted concept images on angle measures only to principal measures. However, coterminal angles are the related and necessary knowledge to model periodic phenomena as well as generate angle measures other than the principal ones (Fi, 2003).

In addition, concept images of the students indicated another *discrimination* problem regarding trigonometric functions' general forms in terms of their ranges and values especially within the (*unit*) *circle register* and the *graphical register*. That is to say, they were unable to dissociate, for examples, a sinusoidal graph from the graph of $y=\sin(x)$ within the *graphical register*, as well as the ordinate [abscissa] of a point on the *non-unit circle* from the ordinate [abscissa] of a point on the *unit circle* in terms of sine [cosine] within the (*unit*) *circle register*. In the light of historical development of trigonometry (Katz, 2009), *discrimination* of "what is mathematically relevant, and what is not mathematically relevant" (Duval, 2006, p. 115) on the unit-circle and a non-unit circle is an important cognitive ability for strong understanding of the periodic functions' trigonometry. Despite of difficulties in reasoning about trigonometric functions on the non-unit circles (Moore, LaForest, & Kim, 2012), the ability to use trigonometric functions in any circular context is crucial.

Finally, students' concept images on the period concept included serious troubles within the different representational registers as a consequence of their problematic concept definition images on the periodicity. None of the students was able to appropriately associate the meaning of the repetition in the *graphical register* with the meaning of the period in the *symbolic register*. That is, when reasoning about a function based on its sinusoidal graph, despite of their awarenss of its repetition within the *graphical register*, none of the students was able to determine its period truly as a consequence of their confusion about the meaning of the period. For example, Defne got confused about the period of the function concerning whether the period of a function with respect to sine (or cosine) could be different from 2π as a consequence of her transformation of her reasoning about the period of the basic forms of the sine (or cosine) functions as 2π onto her reasoning about that of their general forms within the *graphical register*.

Once more, the students' weak understanding of the *concept definition* emerged as a source of the students' problems (Tall & Vinner, 1981) in trigonometry that prevented them from *discrimination* of a mathematical object from the content of the representation (Duval, 2006).

8.2. Development of Students' Concept Images on Trigonometric Functions

Under this heading, the four students' (Cemre, Defne, Ebru and Zafer) developments during the teaching experiment are presented in accordance with the research questions of the study. Initially, developments of students' *recognition* that emerged as a result of the dynamically-linked *conversions* between representational registers are presented under three sub-heading based on foundational trigonometric concepts (i.e., *angle and angle measure*, *trigonometric functions*, and *periodicity*). And then, developments of students' *discrimination* of trigonometric functions that emerged as a result of the dynamically-changed visual components referring to the trigonometric functions are presented in the (*unit*) *circle register* and the *graphical register*.

8.2.1. Students' recognition of angle and angle measure

When investigating the variation on an angle's measure (in *degrees, radians* and *directed degrees* preferences) through dynamic manipulations of its openness in GSP environment, the students conceptualized angles as dynamic turnings [rather than static structures] and started to well-define *angle measure* based on its initial side, terminal side and direction of the rotation [rather than focusing on its interior openness]. They constructed this conception when dynamically manipulating the angle's openness –that was constructed together with a circle centered at its vertex and its two arcs separated by the angle (see *Figure 5.10*)– and observing dynamic-and-linked changes of measures of angle and two arcs (in *directed degrees* preference) in GSP environment. Moreover, they interpreted an angle measure given with a positive [negative] measure by GSP as its negative [positive] equivalence. Unlike their prior concept images on negative angles which was solely based on memorized rules without any reason, the students had just started to well-define *angle measure* based on its initial side, terminal side and direction of the rotation with two measures (i.e., with the highest negative and the lowest positive measures) (see *Figure 8.1*).

When investigating the relations among arc lengths, arc angles, radius and angle measure (in *radians* preference) under the manipulation of the openness as well as the radius of the circle (centered on the vertex of the angle) in GSP environment, initially none of the students was able to establish any relation between them. When reasoning about the relation between the changing-measures in a way through restricting the variation into the cases in which the angle measure was fixed and the radius was changed by natural numbers, they started to recognize the proportional covariations of arc length and radius. At that point the researcher encouraged them to calculate this proportion by GSP, and then, investigate variations and invariations when manipulating the angle's openness. Although they determined the simultaneous variation of the angle measure and the ratio of the arc length to the radius, none of the students was able to associate this ratio with the angle measure in radians. It was due to their problematic prior concept definition images on radian measure based only on transformations in the *symbolic register* and π notation in the radian measure, as revealed in the literature (Akkoç, 2008; Fi, 2003; Topçu, et al., 2006). At that point, the researcher provoked them to consider π in the angle measure by the aid of GSP's calculate option. GSP's calculation defaults on π as about 3.14 prompted students' reasoning about π notation in an angle measure in radians with its meaning as a real number. In addition, dynamically-manipulated calculation result of the ratio between (arc length/radius) and (π) prompted a distinct shift on their reasoning about angle measure in radians. That is to say, students had just started to associate arc lengths [when the radius was 1] with the angle measures in radians.

When the angles were constructed in the (*unit*) *circle register*, more advanced-developments were observed in the students' concept images on angle and angle measure. As a consequence of their ability to consider angles as dynamic-directed turnings, the students started to identify an angle within the (*unit*) *circle register* whose absolute measure greater than 360 degrees or 2π radians regarding two-step turning in any directions: (1) *principal turning* from the initial side to the terminal side and (2) some *full-rounds*. Beside, when identifying angle measures corresponding to the reference points in different quadrants (see *Figure 5.15* and *Figure 5.16*), they reasoned about the *principal turning* from the initial side to the terminal side through considering this turning in two steps regarding the closest coordinate axes: (1) turning from the initial side to the closest coordinate axis in the same direction as the *principal turning* (2) turning from this coordinate axis to the terminal side in the direction so that the way of turning would be the shorter arc. This reasoning prompted a distinct shift on their reasoning about an angle in the (*unit*) *circle register*. That is to say, they began to associate an angle within the (*unit*) *circle register* with its complementary and/or supplementary parts in any quadrant, as well as its reference angle. Moreover, they started to associate a *static angle structure* in the (*unit*) *circle register* with the infinitely many [negative or positive] equivalent measures in the *symbolic register* through considering *dynamic directed turnings*. Inversely, they were able to associate the infinitely many equivalent angle measures in the *symbolic register* with the **same** *static structure* in the (*unit*) *circle register* (see *Figure 8.2*).

Consideration of angle measures together with trigonometric functions prompted the students to reason about angles in a more detailed way. On the one hand, they recognized real numbers as angle measures with two different ways through considering angle measure unit either in degrees or in radians. They constructed this recognition when reasoning about a sine value that came first to their mind. Initially, they expressed a sine value of a real number, for example 30, as an angle in "degrees" that came first to their mind but without stating clearly their "degree" preference as the angle measure unit. Calculation result of sin(30) by GSP in *radians* preference that produced a different output from 1/2 provoked students to reason about a real number without π notation as an angle measure. When reasoning about an angle's measure in radians both with and without π notation, the students recognized a real number without π notation, for example 30, as an angle measure in radians through transforming it into the symbolic form with π notation by the aid of GSP's calculate option, and then, locating the reference point on the unit circle through dragging and dropping so as to indicate this angle, and comparing the calculation result of sin(30) with the dynamically-linked sine measure of the angle corresponding to this reference point. Moreover, during this process, they considered π with its real value, i.e., about 3.14. Unlike their prior conceptions, the students had just started to reason about a real number as an angle measure with two different ways through considering two angle measure units, i.e., degrees and radians.

On the other hand, consideration of angle measures together with trigonometric functions within the dynamic-and-linked visual representations on the same coordinate plane (i.e., their graphs and (unit) circle representations) promoted the students' reasoning about angles. They recognized angle measures as real numbers based on their simultaneous reasoning about the principal arc's angle in the *unit circle register* and its conversion on the *x*-axis scaled with real numbers in the *graphical register*. In special sense, dynamic-and-linked manipulations of the point on the (unit) circle and its correspondence on the sine [cosine] graph fortified students' concept images on the meaning of π, contrary to their initial concept images, through merging its meaning as an angle measure in radians and as a real number (i.e., approximately 3.14) due to their association of π-radian angle in the (*unit*) *circle register* with its conversion on the *x*-axis in the *graphical register*. Furthermore, on the contrary to the initial interview results, anymore "coordinate plane" was not a *cognitive conflict factor* when the position of π on the *x*-axis is considered simultaneously within the *graphical register* and the (*unit*) *circle register*. In addition, the students recognized angles in the *graphical register* as a continuous and repeated variable on the *x*-axis. Their reasoning based on dynamic-and-linked covariations of the point on the (unit) circle and its correspondence on the sine [cosine] graph provided them with the ability to convert the **continuously-repeated full-round turnings** in the counterclockwise [clockwise] direction within the (*unit*) *circle register* into the **continuously-repeated regular intervals** on the *x*-axis in the positive [negative] direction in the *graphical register*.

8.2.2. Students' recognition of trigonometric functions

When reasoning on dynamic-and-simultaneous manipulations of the similar right triangles and their corresponding trigonometric ratios in GSP environment (see *Figure 5.1*), contrary to their prior conceptions, all students constructed a new concept definition image on trigonometric ratios as dependent only on angle measure and independent from side lengths, which is a fundamental starting point to reason about triangle trigonometry (Thompson, 2008). In special sense, they constructed a right triangle conception with 1-unit hypotenuse that included association of its opposite [adjacent] side regarding an acute angle with sine [cosine] of this angle. This conception led them to associate the meaning of the opposite [adjacent] side of the reference right triangle in the unit circle with the corresponding sine [cosine] value. In other words, they had just merged their unrelated concept images on sine [cosine] in the right triangle context and the unit circle context (see *Figure 8.3*).

Consideration of the unit circle together with the reference right triangle that was constructed as a dynamic structure in GSP environment (see *Figure 5.17*[51]) promoted the students' reasoning about the measures of the visual structures (such as principal arc and legs of the reference right triangle) referring to sine [cosine] in the (*unit*) *circle register*. In special sense, the students found a chance to observe three different measures' covariation (i.e., length of opposite [adjacent] side of the reference right triangle, corresponding sine value to the principal angle, and ordinate of the reference point of the principal arc on the unit circle) under the manipulation of the reference point of the principal arc. This observation process prompted a distinct shift on Ebru's recognition of sine [cosine] in the (*unit*) *circle register*. She had just started to reason about sine [cosine] as the directed-opposite [adjacent] side length of the reference right triangle in any quadrant of the unit circle. However, the other students did not rich this kind of reasoning during this task. Therefore, the researcher preferred in the following task to study on the unit circle integrated with a reference right triangle

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[51] Until the 8th episode of the teaching experiment, the students studied on pre-constructed GSP pages by the researcher. When measuring lengths of the reference right triangle's legs (see *Figure 5.17*), the researcher used the unit of the coordinate axes as the measurement unit.

whose opposite [adjacent] leg was labeled as sine [cosine] (*Figure 5.19*), and asked the students to interpret the value of sine of the principal angle without seeing the dynamically-linked sine measure on the screen. Up to that time even though all students were able to interpret sine of an angle as the ordinate of the reference point both in the (*unit*) *circle register* and in the *language register*, it was interestingly observed that they were unable to convert their interpretations arithmetically into the *symbolic register*. That is, none of the students was aware of the determination of the sine [cosine] value of an angle in the *symbolic register* by using the *y*-axis [*x*-axis] as a signed-ruler in the (*unit*) *circle register*. Their concept images on the values of sine in the *symbolic register* were restricted to the memorized exact values of sine at the special angles without any reasons. At that point, the researcher provoked students to estimate the exact value referring to the directed-opposite [adjacent] line segment by using the coordinate axes as a signed-ruler in the (*unit*) *circle register*. Their estimations were almost same with the sine [cosine] values of the mentioned angles in any quadrant. When the researcher unveiled sine [cosine] measure on the screen which was given by GSP as the dynamically-linked measure to the principal angle measure, all students were surprised and excited. This implies that each student had just been able to convert the geometric procedures in their concept definition images on sine [cosine] within the (*unit*) *circle register* into the values of sine within the *symbolic register*. Unlike their prior concept images, the students recognized sine [cosine] in the (*unit*) *circle register* as same with sine [cosine] in the *symbolic register*. Furthermore, as a consequence of dynamic-manipulation of the reference right triangle, they were able to reason about sine [cosine] of the angles in cases of the absence of the reference right triangle regarding the position of the reference point on the axes. That is to say, they constructed a dynamic-conception of the reference right triangle that enabled their *recognition* of sine [cosine] in the **limit case** of the directed-opposite [adjacent] side (see *Figure 8.4*).

Dynamic-and-linked visual representations of trigonometric functions on the same coordinate plane within the (*unit*) *circle register* and the *graphical register* promoted the students' ability to convert the dynamic variation of the sine [cosine] values regarding **continuous turnings** in the counterclockwise {clockwise} direction in the (*unit*) *circle register* into the static representation of the dynamic variation of the sine [cosine] values by associating them with the appropriate right {left} part of the sine [cosine] graph. In other words, they were able to convert the meaning of the sine [cosine] values of the angles with the **same** *static structure* but **different** *dynamic structures* in the (*unit*) *circle register* into the meaning of the *parallel displacement* of the point on the **principal part** of the sine [cosine] graph along the *x*-axis by the 2π-length in the *graphical register* thereby converting the meaning of full-round turnings in the **counterclockwise [clockwise]** direction in the (*unit*) *circle register* into the meaning of the *parallel displacement* along the *x*-axis in the **positive [negative]** direction in the *graphical register*. At that point, they started to reason about the sine [cosine] function on the real number set in the *graphical register* based on their ability to transform the principal part of the sine [cosine] graph into the other repeated parts through the parallel displacement of the principal part. That is to say, all students reconized the systematic variation of sine [cosine] both in the (*unit*) *circle register* and the *graphical register*. The students' reasoning about systematic covariation of trigonometric functions based on systematic variation of the angle measure in GSP environment is consistent with Kaput's (1992) argument that making variation is needed for understanding of invariance which is the very important aspect of mathematical thinking.

8.2.3. Students' recognition of periodicity

In last two episodes of the teaching experiment's first part, observations of the systematic covariations of angle measures and corresponding sine [cosine] values between dynamic-and-linked different representational registers promoted students' reasoning about (*i*) the repetition of the sine [cosine] values within the (*unit*) *circle register* as a consequence of full-round turnings of the reference point on the unit circle and (*ii*) the repetition of the sine [cosine] values in 2π-length intervals in the *graphical register*.

In the second part of the teaching experiment, visual features of the different representations of trigonometric functions were systematically varied to provoke the students to discuss on a new function "what is mathematically relevant, and what is not mathematically relevant" (Duval, 2006, p. 115) when compared with trigonometric functions' basic forms. Period was an important discussion focus of each episode in this part of the teaching experiment.

Initially, when definition of sine on the **origin-centered unit circle** was transformed into a new definition on the **origin-centered non-unit circle** with the same visual objects (a reference point on the circle referring to an arc and a perpendicular line-segment to the *x*-axis), two reference points' simultaneous movements at the same (angular) speed on the unit-circle and the non-unit circle in the GSP environment supported the students' recognition of the new function's period[52] –defined on the **non-unit circle**– as the same period with the basic form of sine –i.e., 2π.

Next, definition of sine on the **origin-centered unit circle** was transformed into a new definition on the unit circle **with different-center from the origin** with the same visual objects (i.e., a reference point on the circle referring to an arc and a perpendicular line-segment to the *x*-axis). As a consequence of existence of only one point referring to the *full-round turning* in the (*unit*) *circle register*, as well as the *unique shape* (see *Footnote 47*) of their graphs indicating their *one full-actions* in the same interval in the *graphical register,* the students were able to reason truly about the new function defined on the unit circle **with different-center from the origin** with the same period as the basic form of sine –i.e., 2π.

Systematic variations of visual features of the representations continued the transformation of the definition of sine by only one reference point on the unit circle into a new definition based on two different reference points on the unit circle. These two points were constructed on the unit circle so that one of them was an arbitrary

[52] From 5th episode forward throughout the teaching experiment, the *period* term was used in the meaning of the *prime period* (see *Regarding periodicity as pattern based on behaviors of trigonometric functions* heading in Chapter 5).

point and the other was its rotated-position about the origin by a fixed-measure. These two points were moving in the same (angular) speed on the unit circle under the manipulation of them in GSP environment. Thus, students were able to reason about the new function defined by **two points with the same (angular) speed** in the (*unit*) *circle register* with the same period as the basic form of sine –i.e., 2π– based on their *full-round turning*s, as well as based on the *unique shape* of their graphs (see *Footnote 47*) indicating their *one full-actions* in the horizontally-translated-intervals in the *graphical register*.

Finally, definition of sine by only one reference point on the unit circle was transformed into a new definition based on two different reference points on the unit circle **with the different (angular) speeds** in GSP environment. One of these points was constructed on the unit circle in GSP environment as an arbitrary point and the other as its rotated-position by a measure dependent on the principal measure of the first point so as to be its integer multiples. These two points were moving at different (angular) speeds in GSP environment. Dynamically-linked conversion of the new function (mapping from the angle of the arbitrary point to the ordinate of its rotated position) from the (*unit*) *circle register* into the *graphical register* together with the sine graph caused for students to bring the period aspect up for discussion.

On the one hand, Cemre and Zafer interpreted the new function's period through attributing its meaning to (*i*) the turning amount of the arbitrary point (*referring to the input of the new function*) so as to bring forth one full-round turning of the other (*referring to the output of the new function*) in the (*unit*) *circle register* and (*ii*) the interval in which the new function completed one-full-action in the *graphical register* as a consequence of one-full-round turning of the reference point referring to the output.

On the other hand, Defne and Ebru encountered a trouble based on the full-round turnings of two points referring to (*i*) the input variable and (*ii*) the output variable. Emergence of Defne's and Ebru's this trouble on periodicity is consistent with Duval's (2006) argument that conversion troubles (or cognitive distances between registers) are observed only when tasks –in which a representation within a *source register* is systematically varied into its converted representation in the *target register*– are given to students. They did not recognize the meaning of full-round turnings in terms of periodicity. While Defne interpreted, for example, the period of $y=\sin(2x)$ as 4π considering two full-rounds of the point P' that was produced by one full-round of the point P, Ebru interpreted the period as 2π considering one full-round turning of only the point P' (see *Figure 6.40(a)*). In other words, they interpreted the period in the (*unit*) *circle register* based on the full-round turning of the reference point of the output variable rather than the turning of the reference point of the input variable producing the full-round of the reference point of the output variable. Therefore, the researcher encouraged them to reason about the full-round-turnings of the point P' dependently on the turning of the point P. Where, they were able to attribute one-full-round turning of the point P' to the π-radian turning of the point P in the (*unit*) *circle register*. When similar discussion were done on another function, i.e., $y=\sin(3x)$, based on its dynamic-and-linked representation within and between the (*unit*) *circle register* and the *graphical register*, they were able to reason about the full-round turning of the reference point of the output variable dependently on the turning of the reference point of the input variable. This finding supports Kaput's (1992) argument that making variation is needed for understanding of invariance which is the very important aspect of mathematical thinking.When the researcher asked them to reason about their periods through emphasizing its meaning as the smallest-repeated-interval in the *graphical register* and the length of this interval in the *symbolic register*, they were able to reason about these functions' periods correctly through using the scaled x-axis of GSP as a measuring tool to determine the length of these intervals (see *Figure 8.5*).

When all discussions on sine mentioned above were done on cosine in the following episodes of the teaching experiment, they were able to reason in the same way about the period of the new function –that was formed through systematically varied-visual features of the different representations of the cosine function.

8.2.4. Students' discrimination of trigonometric functions in (unit) circle register

As mentioned in the previous heading (i.e., *Students' recognition of periodicity*), visual features of the different representations of sine [cosine] were systematically varied during the second part of the teaching experiment to provoke the students to discuss on a new function "what is mathematically relevant, and what is not mathematically relevant" (Duval, 2006, p. 115) when compared with its basic form.

From the 8th episode forward, in each task transformations of the circular representation into the graphical representation were investigated in GSP environment in order to support as well as understand the students' ability to distinguish the represented mathematical object (i.e., sine and cosine functions) from the representational register's content (Duval, 2006).

Initial visual feature of the systematic variation was the radius of the circle. When the unit circle was changed into a non-unit circle, the students developed significant understandings referring to trigonometric functions. To begin with, they constructed "unit" meaning that enabled them to consider a unit circle as a non-unit circle. They constructed this meaning when measuring the changed-radius by GSP as a consequence of their troubles arising from the difference between the distance-measure-unit preference of GSP as centimeter and the visual distance-measure-unit of the coordinate axes. They had just considered the same circular structure in GSP environment as both a unit circle and non-unit circle through reasoning about it (*i*) as a unit circle according to the unit of the scaled-coordinate axes and (*ii*) as a non-unit circle according to the centimeter distance measure unit. Reversely, in the modeling task, Cemre and Zafer considered a non-unit circle as a unit circle through identifying its radius as *unit*. This consideration promoted their reasoning about general forms of trigonometric functions in the (*unit*) *circle register*. For example, in the modeling task, they considered initially the Ferris wheel with 70-meter radius as a unit circle and associated the vertical (directed) distance to the horizontal axis from its center with the sine value of the corresponding angle. And then, they dissociated this vertical (directed) distance from sine (or, associated this vertical distance with 70.sin(x)) through considering the Ferris wheel as a non-unit circle with 70-meter radius. Thus, they were able to discriminate sine represented in the (*unit*) *circle register* from the (*unit*) *circle register*'s content. This finding supports Moore, LaForest, and Kim's (2012) conclusion that provoking students to consider units with different measures promoted their unit circle understandings from their restricted understanding of the unit circle notion with "one" radius to any given circle through considering its radius as "one" unit. On the other hand, through taking GSP's "measure", "calculate" and "drag-drop" advantages, Defne and Ebru also discriminated the opposite side of the reference right triangle on the non-unit circle from that on the unit circle, as well as associated the (directed) length of the opposite side of the reference right triangle on the non-unit circle with the multiplication of sin(x) with the radius of the non-unit circle in the *symbolic register*. This reasoning generated a new conception that enabled the students to reason about a general sine function in the form y=a.sin(x) in the (*unit*) *circle register* but for only positive coefficients through associating the coefficient a with the changed-radius.

Next visual feature of the systematic variation was the center of the unit circle. When the location of the center of the unit circle was changed from the origin to any other position on the coordinate plane, Cemre and Zafer showed more advanced developments on the level of ability to reason about the basic properties of the unit circle representation than Defne and Ebru. In general sense, each of the students determined that the new function –from the angle measure to the corresponding ordinate on the unit circle with different center from the origin– was expressed as an additive operation in the *symbolic register* between sine and directed-distance of the manipulable center to the x-axis. Moreover, based on the dynamic manipulations of the center, they determined this additive expression's independence from the horizontal variation and dependence only the vertical variation. However, in special sense, their reasoning processes indicated their different stages in terms of these conceptions. On the one hand, Defne's and Ebru's focuses were predominantly on the processes related to the y-components in the (*unit*) *circle register* based on the determination of the ordinates. On the other hand, Cemre's and Zafer's focuses were predominantly on the condensed-whole of the processes instead of their details. That is to say, they focused direcly on the dissociation and association of this manipulated-function from sine in the (*unit*) *circle register* instead of operational-processes. For example, Zafer mentioned the horizontal axis from the center as the *x*-axis considering the *x*-axis as if a reified-object without going into details but with awareness of its different location. Cemre mentioned the unit circle whose center located on the origin as if a reified-object; and was able to change its position up-down and left-right in her mind as a whole on the coordinate system. Even though Cemre and Zafer reasoned based on the visual objects' basic properties and their significance for sine function, Defne and Ebru reasoned based on the details of the processes. During this mathematical activity, Cemre and Zafer performed at higher reasoning stage (i.e., *condensation*/*reification*) than Defne and Ebru (i.e., *interiorization*) considering Sfard's (1991) hierarchical stages of the concept development. This level of reasoning promoted in the modeling task that Cemre and Zafer's transformation abilities of the basic properties within and between the (*unit*) *circle register* and the *symbolic register*, which is consistent with Duval's (2006) argumentation that transforming one semiotic representation to another one is to be only at the *level* of grasping the basic properties of semiotic representations and their significance for mathematics.

Final visual feature of the systematic variation was the reference point referring to trigonometric value. That is to say, the only one reference point on the unit circle referring to both angle and corresponding trigonometric value was varied from only one point to two points so as one of them to refer the input and the other to refer the output of the function. As a consequence of this variation, the students' *discrimination* problems were observed regarding the meaning of the reference points in the (*unit*) *circle register*. Except Zafer, none of the other students associated the new function – from the angle measure indicated by one point into the ordinate of the other point on the unit circle– with the sine function until the construction of its graphical representation. Although Zafer associated this function with sine based on its visual definition on the unit circle, he was also unable to reason about this function in terms of sine in the *symbolic register*. Only when the researcher provoked them to identify this function in terms of sine in the *symbolic register* through (*i*) considering the reference point referring to the sine value as well as (*ii*) defining its angle dependent on the other reference point's angle, the students started to be able to express this new function symbolically in terms of the sine function through revising the input variable of the basic form of the sine function. This conception prompted a distinct shift on their reasoning about the relation between sine and cosine in the (*unit*) *circle register*. For example, unlike their prior concept images on transformations between sine and cosine that were based only on the memorized-rules in the *symbolic register*, they started to be able to convert the relation between the directed-measures of the adjacent [opposite] side of the reference right triangle and the opposite [adjacent] side of its rotated-counterpart by $\pi/2$ radian about the center in the (*unit*) *circle register* into the equality between $\cos(x)$ [$\sin(x)$] and $\sin(x+\pi/2)$ [$-\cos(x+\pi/2)$] in the *symbolic register*. Moreover, they were able to extend this ability onto the structures obtained through rotations by the integer multiples of $\pi/2$ radian in any directions. Therefore, in special sense, they made sense of the negative coefficient of –sine [–cosine] function in the (*unit*) *circle register* through considering it as a function from the x angle to the perpendicular line segment from the point corresponding to the ($x\pm\pi$) angle to the x-axis [y-axis]. Their ability to transform sine [cosine] into cosine [sine] in the (*unit*) *circle register* made easier their reasoning about cosine in the following episodes of the teaching experiment due to their conceptions on sine mentioned above.

8.2.5. Students' discrimination of trigonometric functions in graphical register

Integration of the graphical representations in episodes of the teaching experiment fortified students' understanding of trigonometric functions in each representational registers. Initially, visual representations of sine [cosine] on the same coordinate plane both in the (*unit*) *circle register* and the *graphical register* provided the students with the opportunity to compare and contrast the dynamic-and-linked variations of the reference point on the (unit) circle and its converted form in the *graphical register*. This opportunity supported the students' *discrimination* of the coordinates of a point in the (*unit*) *circle register* and the *graphical register*. In other words, unlike many trigonometry students' troubles on the role of axes in a sine graph (Brown, 2005), they constructed an ordered-pair conception so that it enabled them to distinguish the meaning of the abscissa [ordinate] of a point on the (unit) circle from the meaning of the abscissa [ordinate] of a point on the corresponding graph. Moreover, this opportunity supported the students' *discrimination* of the meaning of the positive [negative] direction in the (*unit*) *circle register* and the *graphical register* in terms of both angle measures and trigonometric measures. Furthermore, this opportunity supported the students' *discrimination* of the coterminal angles represented by the **same** *static structure* but **different** *dynamic structures* in the (*unit*) *circle register* and by the **equivalent** but **not equal** measures in the *symbolic register*. That is to say, visual representations' of the coterminal angles in the *graphical register* provided the students with the ability to differentiate visually the equivalent but not equal angles' positions on the *x*-axis in the *graphical register*. This kind of reasoning about the coterminal angles is the related and necessary knowledge to understand periodicity (Fi, 2003).

As mentioned in the previous heading (i.e., *Students' discrimination of trigonometric functions in (unit) circle register*), visual features of the different representations of sine [cosine] in the (*unit*) *circle register* were systematically varied, we obtained the opportunities to discuss these visual features dynamically-linked oppositions in the *graphical register* as well.

Initially, when the radius of the circle was changed, this visual feature's dynamic-and-linked opposition was constructed in the *graphical register* based on the new function (defined in the same way with sine but on the non-unit circle) via taking *plot as (x,y)*, *trace point* and *animate point* advantages of GSP. When the changed-magnitude of the graph appeared on the screen as a result of dynamic manipulation of the point on the non-unit circle, all of the students were able to distinguish the traced-graph from the sine graph regarding their magnitudes in the *graphical register*. Moreover, through focusing on the proportional relation between their magnitudes, they were able to convert the new function represented by its traced-graph into its symbolic expression, e.g., as 6sin(x). Duval (2006) argues this ability as a deep cognitive condition to be “able to discern *how two graphs that seem visually alike are mathematically different*” (p. 124). However, initially, only Cemre attributed the changed-magnitude in the *graphical register* directly to the changed-radius in the (*unit*) *circle register*. As a consequence of *seeing* the systematic covariations between the radius and magnitude, all of the students were able to reason about relations among “changed-radius” in the (*unit*) *circle register*, “changed-magnitude” in the *graphical register* and “changed-coefficient r of the r.sine function” in the *symbolic register*. Coherent understanding and flexible use of trigonometric functions in different representations are difficult for most students (Brown, 2005). This important ability were constructed by the students as a result of their *seeing* for understanding of invariance in dynamic-and-linked different representations when making variation, which is consistent with Kaput’s (1992) idea that emphasizes this kind of activity as a very important aspect of mathematical thinking.

Next, when the location of the center of the unit circle was changed on the coordinate plane, this visual feature’s dynamic-and-linked opposition was constructed in the *graphical register* based on the new function –from the angle measure to the corresponding ordinate on the unit circle with different center from the origin. When the traced-graph appeared on the screen as a result of dynamic manipulation of the point on the unit circle, the students were able to associate this function with the sine [cosine] function in the *graphical register* based on its visual-shape, as well as reason about it as the parallel-displacement of the sine graph along the y-axis in the positive/negative direction. Moreover, they converted this represented function in the *graphical register* into the *symbolic register* considering the (directed) displacement amount as the constant of the sine function.

And then, when the reference point referring to the sine value was changed from the reference point referring to the input, this visual feature’s dynamic-and-linked opposition was constructed in the *graphical register* based on a new function that was defined on the unit circle but based on two points so as one of them to refer to the input and the other to the output.

On the one hand, in 10th episode, these two points were constructed on the unit circle so that one of them was an arbitrary point on the unit circle and the other was its rotated version about the origin by a fixed-measure. When comparing and contrasting the sine graph with the new function's graph, except Defne, all other students considered these two graphs as visually alike. However, they had difficulty in recognition "how two graphs that seem visually alike are mathematically different" (Duval, 2006, p. 124). In other words, they encountered a trouble when trying to reason about the new function in terms of sine as a consequence of their conjecture about conversion of the parallel-displacement along the *x*-axis in the *graphical register* into the transformation of sine to the new function in the *symbolic register*. This conjecture was the conversion of the positive [negative] direction of the parallel displacement along the *x*-axis in the *graphical register* into the positive [negative] sign of the constant affecting the input variable in the *symbolic register*. This wrong-conjecture caused Cemre's and Zafer's confusion about the location of the graph, for example, $y=\sin(x+2.3)$ on the left, with respect to the sine graph on the coordinate system. However, this conjecture did not cause any confusion for Ebru as a consequence of her incoherent-assumptions on the *source*-function of the transformation in the *graphical register* and the *symbolic register*. Differently from others, Defne reasoned about these two graphs through comparing their slopes with each other but focusing only on their restricted parts from their *y*-intercepts up to their first intersection points. That is to say, none of them were able to make sense of the parallel-displacement along the *x*-axis idea during this episode. Even though this task was also including the unit circle representation, as a consequence of the students' difficulties in reasoning about the new function (from the angle measure indicated by one point into the ordinate of the other point on the unit circle) in terms of sine in the (*unit*) *circle register* (see the previous heading), the researcher preferred to postpone detailed-discussions on the meaning of the parallel-displacement of the graphs along the *x*-axis in the (*unit*) *circle register*. Therefore, this problem was discussed in the episode on the relation between sine and cosine (i.e., 12th episode) as well as in the corresponding episode on the cosine function with same visual features (i.e., 15th episode) to make sense of the graphs' parallel-displacement along the *x*-axis in the (*unit*) *circle register*. From 12th episode forward, their ability to make sense of the symbolic expressions in the form of sin(x+c) and cos(x+c) in the (*unit*) *circle register* caused their interpretations about the positive [negative] constant "c" as the c-unit length before arrival [after arrival] in a value in the *graphical register* as a consequence of the c-radian rotation in the counterclockwise [clockwise] direction in the (*unit*) *circle register*. It was the point that the students had just constructed an ability to make sense of the parallel-displacement of the graphs along the x-axis only when they considered its meaning together with their conversions in the (*unit*) *circle register*. This finding supports Duval's (2006) recommendation that "in order to make students notice the basic visual features *oppositions that are mathematically relevant and cognitively significant*, any representation discrimination task has to be integrated into a conversion task" (p. 125).

On the other hand, in 11th episode, two points were constructed on the unit circle so that one of them was an arbitrary point on the unit circle and the other was its rotated version about the origin by a marked-angle (i.e., an integer multiple of the principal measure of the angle indicated by the initial point). It was the first episode that the students had to reason about a new function defined in the same way with sine but based on two reference points in the (*unit*) *circle register* that were moving in the different , but dependent, (angular) speeds on the unit circle under the manipulation of them in GSP environment. When the traced-graph of the new function (from the angle of the arbitrary point to the ordinate of its rotated position) appeared together with that of sine, the students compared them based on their *one-full-action*s in the *graphical register* that were formed as a consequence of *one-full-round turning*s of their reference points referring to their outputs in the (*unit*) *circle register*. They interpreted the new function's graph as the compressed [stretched] form of the sine graph through attributing the meaning of the compressed [stretched] wave in the *graphical register* to the faster [slower] turning of its reference output point than the reference output point of sine in the (*unit*) *circle register*. Moreover, they were able to express this function in the *symbolic register* in the form y=sin(bx) considering the compression [stretch] ratio in the *graphical register* as well as ratio between the reference output points' turnings at the same time in the (*unit*) *circle register*.

To sum up, until the end of 12th episode of the teaching experiment, the students constructed some cognitive networks that enabled them to discriminate the visual feature oppositions of (*i*) “changed-magnitude”, (*ii*) “parallel-displacement” along the *y*-axis, (*iii*) “parallel-displacement” along the *x*-axis, and (*iv*) “compressed/stretched” of graphs. Duval’s (2006) cognitive analysis emphasizes the visual discrimination of graphs (*i*) as nothing obvious especially when their forms and contents are seemed very similar, and (*ii*) that requires the construction of *cognitive networks* so that visual feature oppositions are merged as if only one. Therefore, at the end of 12th episode, composed-coefficient’s visual oppositions were handled so as to observe as well as support the students’ mergense of these cognitive networks. For this purpose, a general form of sine function, i.e., y=3sin(2x+4)-1, was considered as an initial example. When reasoning about the variation of y=sin(x) function’s graph up to y=3sin(2x+4)-1 function’s graph through incorporating a new function into the discussion; respectively, y=3sin(x), y=3sin(x)-1, y=3sin(2x)-1 in GSP environment, it was observed that all of the students encountered a major trouble on reasoning about the composed-visual-opposition of the coefficients “2” and “4” in the *graphical register*. Although until the last step of this reasoning process, they had no conflict between the variation of the graphs produced by GSP and their expectations about them in the *graphical register*, they encountered the major trouble on the visual variation between the graphs of y=3sin(2x)-1 and y=3sin(2x+4)-1 in terms of the displacement amount between their graphs. When the constant and coefficient of the input of sine was systematically changed, they started to reason about the displacement amount by an operational-process as the division of the constant by the coefficient of x based on their determinations of the displacement amount between two graphs by the aid of GSP’s “zoom in” and “zoom out” options for the scaled x-axis. In addition to the *graphical register*, they interpreted the cause of the changed-displacement amount (*Footnote 48*) in the (*unit*) *circle register* as well. They converted the constant “c” of the input of sine in the *symbolic register* into a c-radian fixed-arc in the (*unit*) *circle register*; and then, interpreted this fixed-arc by means of two different, but dependent, (angular) speeds referring to x and bx. That is to say, they attributed the displacement amount along the x-axis in the *graphical register* to the (c/b)-radian turning of the reference point of (*x*) that produced the *c*-radian arc as the path of the reference point of (*bx*) so as to indicate (*bx+c*) in the (*unit*) *circle register*. It means that students reasoned about the *c*-radian arc between the reference-points of (*bx*) and (*bx+c*) in a dynamic-turning-structure through considering the turning-amount of (*bx*) dependently on the turning amount of (*x*). The researcher inferred that students' this reasoning arose from their effort to determine how much turning of (*x*) caused the *c*-radian "before arrival/after arrival" on a specific point between the reference points of (*bx*) and (*bx+c*) in the (*unit*) *circle register*.

8.3. Conceptual Frameworks of Trigonometry

Under this heading, some conceptual frameworks are presented that were grounded as a result of the on-going design process of the instruction of the teaching experiment based on the prospective and retrospective cognitive analysis of the data. Firstly, *cognitive concept maps* on angles, sine [cosine] function and periodicity that were revealed as foundational for students' *recognition* of trigonometric functions are presented to provide a lens for the reasoning ways of the students' understanding fundamental concepts of trigonometry when integrating a new concept and the related current concepts into students' cognitive knowledge structure. Secondly, *cognitive networks* that were revealed as foundational for visual discrimination of the sine function both in the (*unit*) *circle register* and the *graphical register* are presented.

8.3.1. Cognitive concept maps

<u>*Cognitive concept map on angles*</u> (*Figure 8.1*).

Development in the students' concept definition images on angles and angle measures revealed that angle and angle measure are two different concepts that is needed to be dissociated from each other (Argün, Arıkan, Bulut, & Halıcıoğlu, 2014). An angle is constructed by two rays with common initial point. An angle is defined by its openness which means angle measure. However, for an angle, two different openness; i.e., interior openness and exterior openness, can be considered. Without specifying its *initial side*, *terminal side* and *direction*, a static angle structure produces four different angle measures. Therefore, the well-defined angle measure requires to specify its *intial side*, *terminal side* and *direction*. When angle measure units are considered, for example, *degree* and *radian*, a static angle structure produces eight different angle measures. Therefore, conception of angle measure is cognitively complex process unless students construct well-defined concept definition images on angle as summarized in *Figure 8.1*.

<u>*Cognitive concept map on angles in the (unit) circle register*</u> (*Figure 8.2*).

When the circular representations of trigonometric functions was brought up for discussion in the teaching experiment, development in the students' concept definition images on angles and angle measures revealed a more complex process of angle and angle measure conception. The positive horizontal axis and a ray from the center constitude an angle in the (*unit*) *circle register*. However, its **unique** *static* structure indicates **infinite** equivalent *dynamic* structures through considering *dynamic-directed-turnings* from the positive horizontal axis to the terminal side in the (*unit*) *circle register*. These *dynamic-directed-turnings* can be considered a combination of (1) *principal turning* from the initial side (i.e., positive horizontal axis) to the terminal side and (2) some *full-rounds* in any direction. *Full-round turnings* are critical in comprehension of the meaning of coterminal angles that are the related and necessary knowledge to understand periodicity, as well as to generate angle measures other than the principal ones (Fi, 2003). Furthermore, a *principal turning* can be considered as a *piecewise principal turning* in two steps regarding the closest coordinate axes: (1) turning from the initial side to the closest (coordinate) axis in the same direction as the *principal turning*, and (2) turning from this (coordinate) axis to the terminal side in the direction so that the way of turning would be the shorter arc. *Piecewise principal turnings* are critical in comprehension of the connection between a principal angle and its reference angle. Therefore, conception of angle and angle measure in the (*unit*) *circle register* is cognitively complex process unless students construct well-defined concept definition images on angles as summarized in *Figure 8.2*.

Cognitive concept map on sine [*cosine*] (*Figure 8.3*).

Tall and Vinner (1981) assert the weak understanding of the *concept definition* as a source of the students' problems in mathematics. Thompson (2008) mentiones the right triangle trigonometry and the periodic functions' trigonometry as two unrelated trigonometries of elementary and secondary school mathematics. Therefore, well-structured concept definition images (Tall & Vinner, 1981) are critical in coherent understanding and flexible use of trigonometric functions in different representations (Brown, 2005; Thompson, 2008). The findings that emerged from this study support Thompson's (2008) suggestion that stresses similarity –that the similar right triangles have same trigonometric ratios– as a fundamental starting point to reason about triangle trigonometry. Indeed, it is important to start discussions on sine [cosine] on the similar right triangles. Similar right triangles have proportional sides regarding their corresponding interior angles. These proportional sides produce the same sine [cosine] ratio for an acute angle in all similar right triangles. A sine [cosine] ratio is dependent only on an acute angle and independent from side lengths. Specially, in case the hypotenuse measure is equal to 1, opposite [adjacent] side length regarding the acute angle corresponds to sine [cosine] value. Therefore, opposite [adjacent] side of the reference right triangle corresponds to sine [cosine] in the first quadrant of the unit circle. In other words, ordinate [abscissa] of the reference point on the unit circle corresponds to sine [cosine] value of the reference angle in the first quadrant. This conception is the first step in coherent understanding of the right triangle trigonometry and the unit circle trigonometry.

Cognitive concept map on sine [*cosine*] as a function (*Figure 8.4*).

As revealed in the cognitive analysis of the initial interviews of this study, because trigonometric functions cannot be expressed as algebraic formulas involving arithmetical procedures, students have trouble on reasoning about them as functions (Weber, 2005). Therefore, functionality idea must be a fundamental starting point to define sine [cosine] in any representational register. Sine [cosine] must be defined as a function from a domain set to the range set so that each argument in the domain set corresponds to exactly one element in the range set. If we define sine [cosine] in the *symbolic register*, then arguments are real numbers. Development in the students' concept images on sine [cosine] of a real number revealed that students need to encounter the reality that real numbers one-to-one correspond to angles in the (*unit*) *circle register* as long as angle measure unit is described clearly. Otherwise, a real number corresponds to (generally) two different angles as a consequence of *degree* or *radian* preference as the angle measure unit. It means that a unique real number generates two different sine [cosine] value. For example, below representation of the sine function in the *symbolic register* is not a function because it is not *well-defined*[53] without making clear about the angle measure unit.

$$\textit{sine}: \quad \mathrm{R} \to [-1,1]$$

$$x \to \sin(x)$$

The students in this study recognized this idea as a consequence of their trouble on different outputs of GSP for sin(*30*). While sin(*30*)=*0.50* (in *degree* preference), sin(*30*)=*-0.99* (in *radians* preference). Therefore, definition of trigonometric functions in the *symbolic register* requires to "make clear which unit we are using when we work with" them (Cooney, Beckmann, Lloyd, Wilson, & Zbiek, 2010, p. 61).

Making clear about angle measure is not enough to define sine [cosine] in coherent understanding of the right triangle trigonometry and the unit circle trigonometry. Findings of this study revealed that defining sine [cosine] in the (*unit*) *circle register* as the ordinate [abscissa] of the reference point on the unit circle does not provide students with the ability to discriminate the ordinate of a point on a non-unit circle from sine. It is important for students to semantically merge the right triangle trigonometry and the unit circle trigonometry that are students' fragmented

[53] $f: A \to B$ is well-defined if for each x in A, there is a unique y in B so that $f(x)=y$.

and unrelated comprehensions (Thompson, 2008). Thus, definition of sine [cosine] should draw from the dynamic view of the reference right triangle integrated into the circular representation in the (*unit*) *circle register*. In special sense, on the unit circle after students' *recognition* of the opposite [adjacent] side of the reference right triangle as sine [cosine] from the similarity perspective (see <u>*Cognitive concept map on sine* [*cosine*]</u> (*Figure 8.3*)), the meaning of the opposite [adjacent] side within each quadrant as well as the *limit cases*[54] of the reference right triangle must be discussed through comparing with the corresponding sine [cosine] value. This activity is critical in *recognition* of sine [cosine] as directed opposite [adjacent] side lengths of the reference right triangle. The meaning of sine [cosine] as ordinate [abscissa] of the reference point on the unit circle is the next step of the *recognition* tasks.

Final step of the *recognition* tasks of sine [cosine] function is to investigate its systematic covariation with respect to the variation of angle measure in order to distinguish the role of coterminal angles, principal angle and reference angle (*Figure 8.4*). This ability is constructed only at the level of grasping the angles as dynamic turnings in the (*unit*) *circle register* (see <u>*Cognitive concept map on angles in the (unit) circle register*</u> (*Figure 8.2*)).

<u>*Cognitive concept map on periodicity of core trigonometric functions*</u> (*Figure 8.5*).

Development in the students' concept definition images on periodicity revealed that *recognition* of periodicity requires students' reasoning beyond the full-round turning in the (*unit*) *circle register*. In fact, it requires reasoning about a turning amount of the reference point referring of the input so as to generate full-rounds of the reference point referring to the output.

On the one hand, for sine and cosine, periodicity is caused by full-round turnings of the reference point referring to the output in the (*unit*) *circle register*. Full-round turnings produce period which is generally represented by the *prime period*. Prime

[54] The *limit case* of the reference right triangle means the absence of the reference right triangle regarding the position of the reference point on unit circle; for example, its positions on the coordinate axes.

period is defined based on the least principal turning of the reference point referring to the input that generates a full-round turning of the reference point referring to the output. On the other hand, periodicity is caused by regular intervals of the domain set in which values of sine [cosine] repeat in the *symbolic register* and the *graphical register*. Prime period is defined based on the least interval of the domain set in which values of sine [cosine] repeat in the *symbolic register* and the *graphical register*. Prime period in the *symbolic*/*graphical* register corresponds to that in the (*unit*) *circle register* if and only if angle measure preference is same.

Recognition of the period in this way is a crucial cognitive condition to reason about periodicity outside of the basic forms of sine [cosine]. For example, when the visual representation of sine on the unit circle was changed into a representation of another function on the unit circle so that it was defined based on two different points moving in the different angular speeds on the unit circle, Defne and Ebru encountered a trouble in reasoning about period of this function based on the full-round turnings of two points referring to (*i*) the input variable and (*ii*) the output variable. They interpreted the period in the (*unit*) *circle register* based on the full-round turning of the reference point of the output variable rather than the turning of the reference point of the input variable producing the full-round of the reference point of the output variable. Therefore, conception of periodicity for sine and cosine requires *recognition* of the meaning of full-round turnings and regular intervals for period as summarized in *Figure 8.5*.

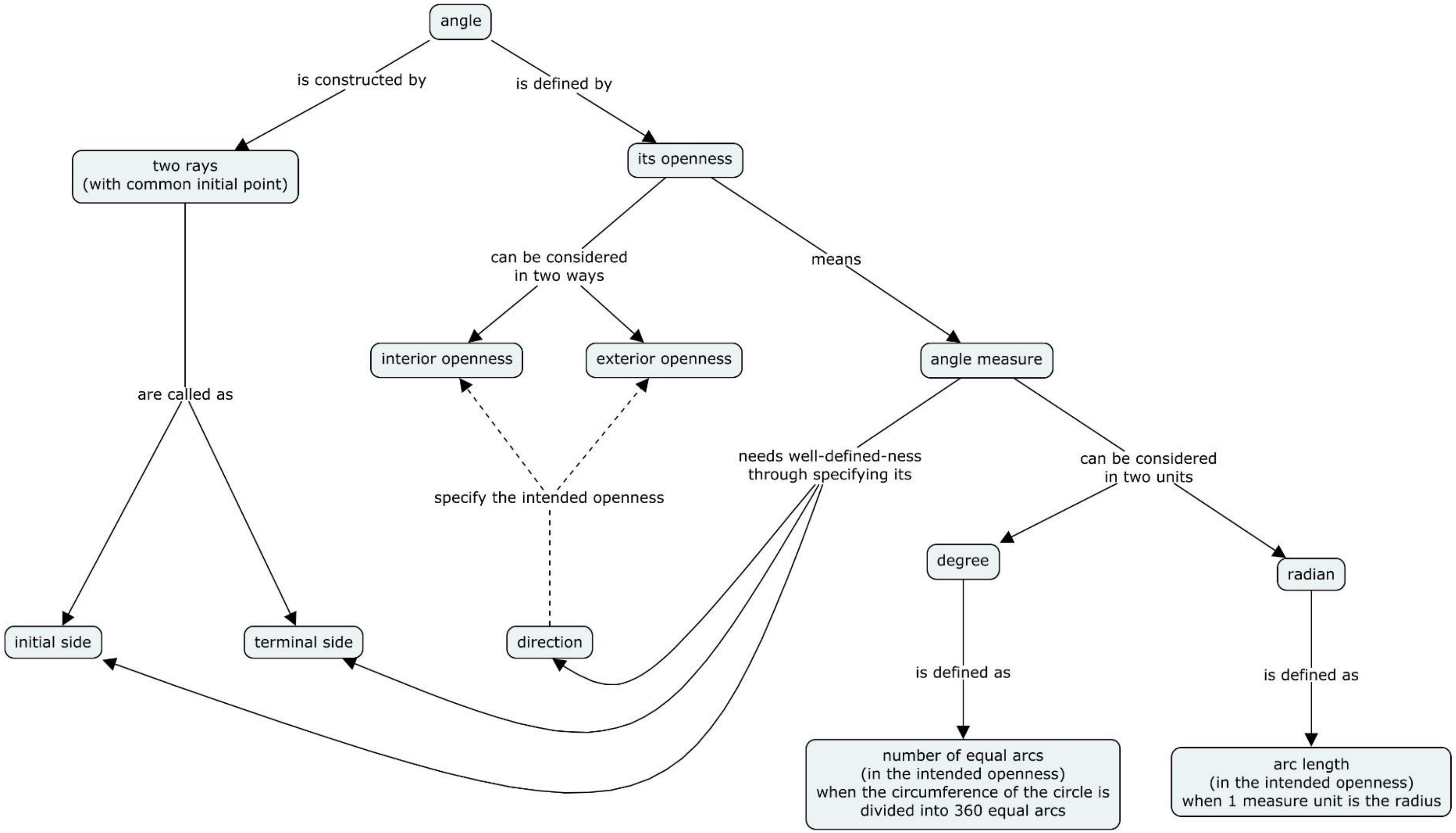

Figure 8.1. Cognitive concept map on angles that models the integration of a new concept and the related current concepts into students' cognitive knowledge structure

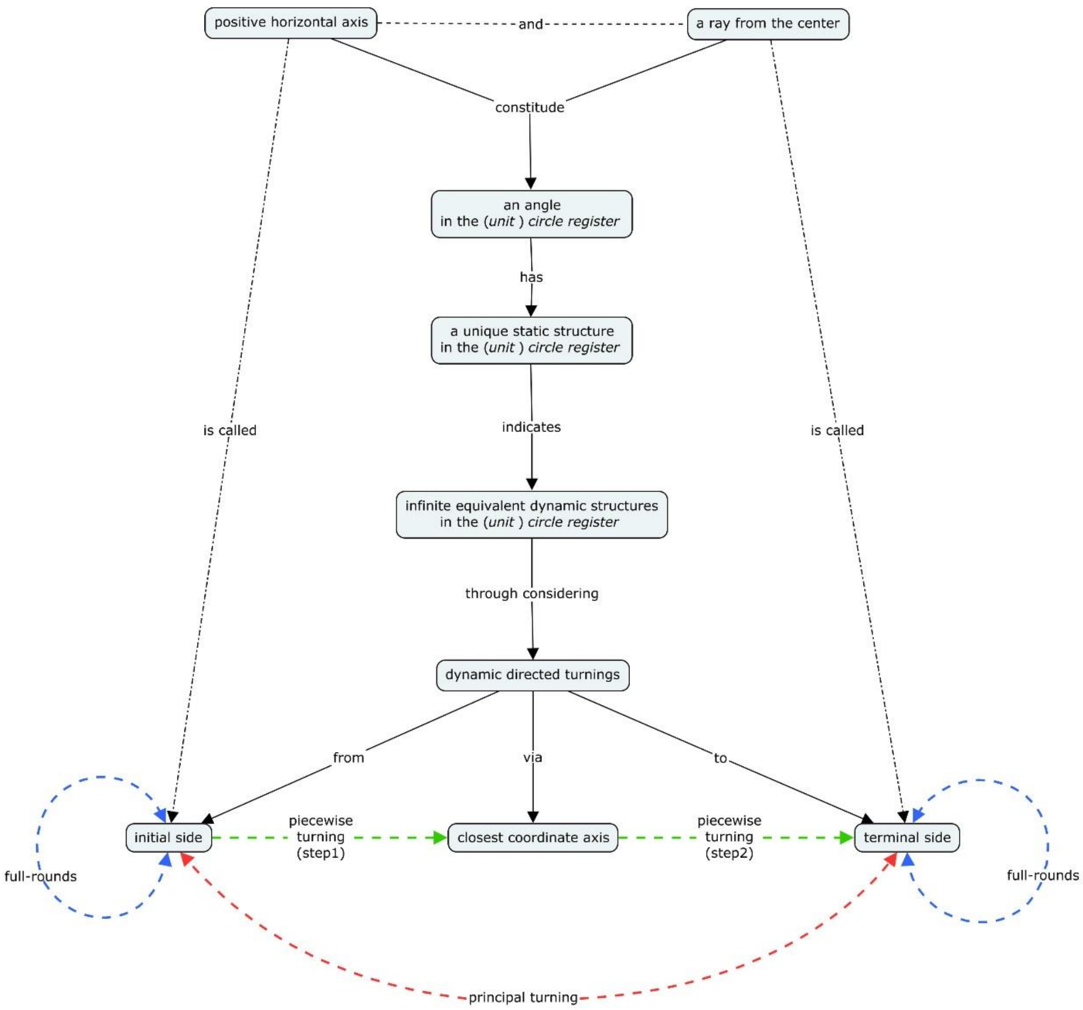

Figure 8.2. Cognitive concept map on angles in the (*unit*) *circle register* that models integration of a new concept and the related current concepts into students' cognitive knowledge structure

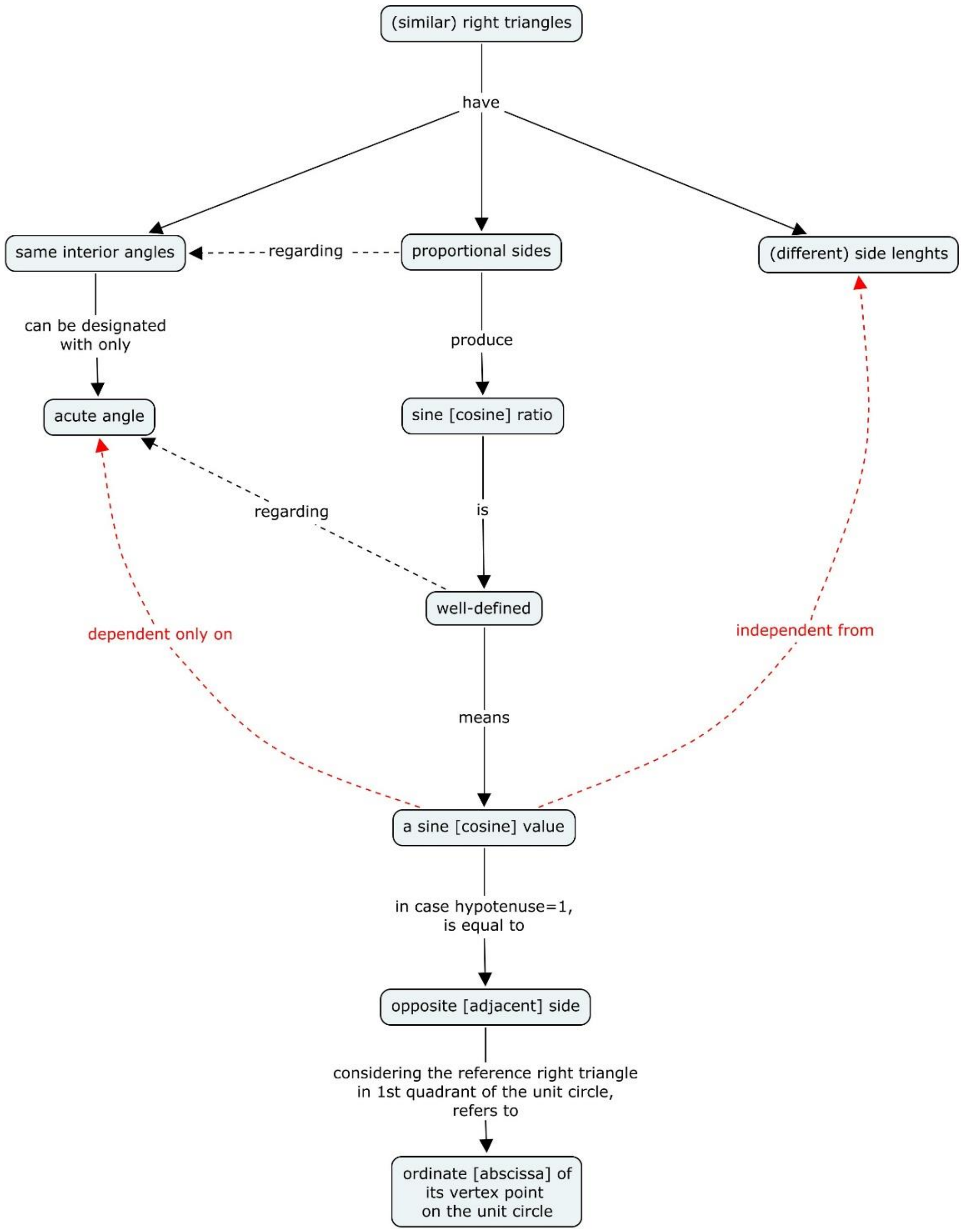

Figure 8.3. Cognitive concept map on sine [*cosine*] that models integration of a new concept and the related current concepts into students' cognitive knowledge structure

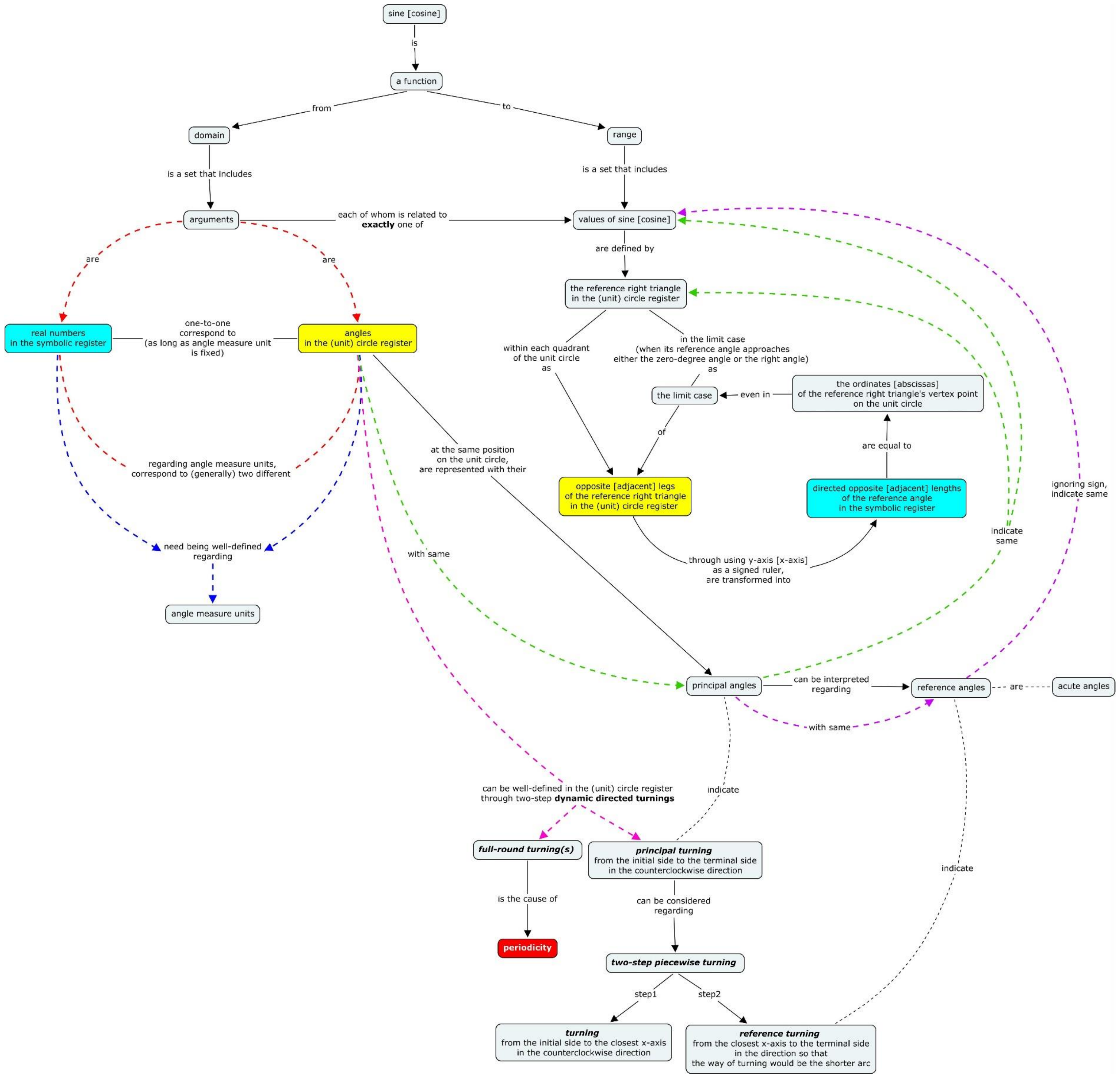

Figure 8.4. Cognitive concept map on sine [*cosine*] *as a function* that models integration of a new concept and the related current concepts into students' cognitive knowledge structure

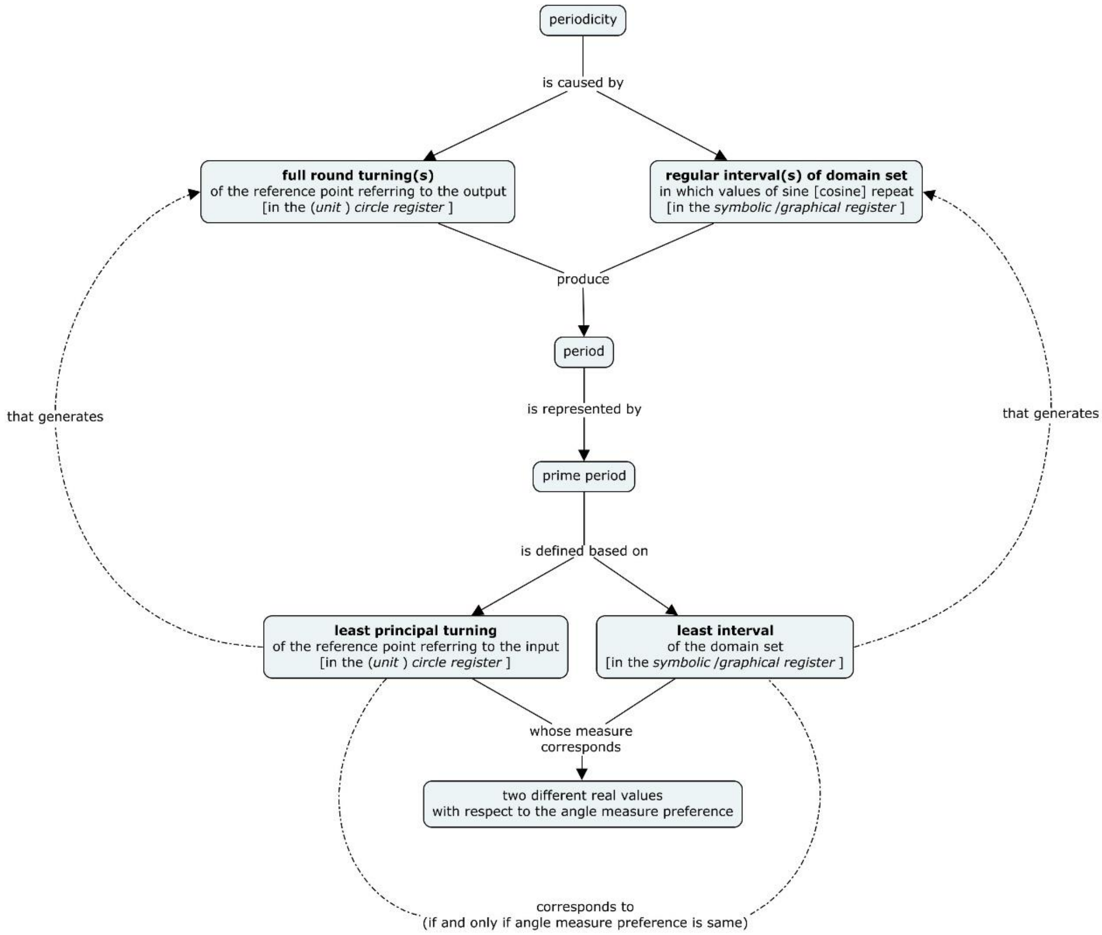

Figure 8.5. Cognitive concept map on periodicity of core trigonometric functions that models integration of a new concept and the related current concepts into students' cognitive knowledge structure

8.3.2. Cognitive networks

Duval (2006) articulates a cognitive network through exemplifying on linear functions' graphs (see *Figure 2.4* in Chapter 2) as a cognitive condition referring to the ability "to discern *how two graphs that seem visually alike are mathematically different*" (p. 124). In realizing such a cognitive network, he stresses the importance of making "students notice the basic visual features *oppositions that are mathematically relevant and cognitively significant*" (p. 125) through investigating representation variations in a conversion task both in the source register and in the target register. Our instructional design was based on investigations of systematically varied representations and their oppositions in the other representational registers. Therefore, through inspiring from Duval's (2006) cognitive network (see *Figure 2.4*), we formed some cognitive networks based on *mathematically relevant and cognitively significant visual features* in the (*unit*) *circle register* and the *graphical register*.

Visual feature opposition (A)

Visual Feature (A) corresponds to **changed-radius** in the (*unit*) *circle register* (e.g., *Figure 8.6-Figure 8.8, Figure 8.12*) and **changed-magnitude** in the *graphical register* (e.g., *Figure 8.13-Figure 8.15, Figure 8.19*). These visual features' opposition in the *symbolic register* corresponds to the choice presence/absence of the coefficient (a) in a general form of sine [cosine] function, e.g., $y=a\sin(x)$ [$y=a\cos(x)$]. These visual features indicate only for $a>0$ the *discrimination* of $a=1$ or $a\neq1$ in the *symbolic register*. Discrimination of the coefficient a (for $a<0$) in the *symbolic register* needs consideration of another visual feature, i.e., *Visual Feature (C),* together with *Visual Feature (A)* in the (*unit*) *circle register* (see *Figure 8.7*) and the *graphical register* (see *Figure 8.14*).

Visual feature opposition (B)

Visual Feature (B) corresponds to **changed-arc through folding the angle variable** in the (*unit*) *circle register* (e.g., *Figure 8.8*, *Figure 8.10*, *Figure 8.11*) and **compressed/stretched-wavelength** in the *graphical register* (e.g., *Figure 8.15, Figure 8.17*, *Figure 8.18*). These visual features' opposition in the *symbolic register* corresponds to the choice presence/absence of a coefficient (b) of the input of a general form of sine [cosine] function, e.g., $y=\sin(bx)$ [$y=\cos(bx)$], in the *symbolic register* for discrimination (the choice of $b=1$ or $b\neq1$). Where, the negative coefficient b refers to turning in the clockwise direction in the (*unit*) *circle register* and the reflection of the graph regarding the y-axis in the *graphical register*.

Visual feature opposition (C)

Visual Feature (C) corresponds to **changed-arc with a constant (angular) difference** in the (*unit*) *circle register* (e.g., *Figure 8.7*, *Figure 8.9*, *Figure 8.11*) and **parallel-displacement along the *x*-axis** in the *graphical register* (e.g., *Figure 8.14*, *Figure 8.16*, *Figure 8.18*). If we change our reasoning focus from one reference point to two reference points with a constant angular difference in the (*unit*) *circle register*, this visual feature's opposition is parallel-displacement of graphs along the *x*-axis in the *graphical register*. These visual features' opposition in the *symbolic register* corresponds to the choice presence/absence of a constant (c) of the input of a general form of sine [cosine] function, e.g., $y=\sin(x+c)$ [$y=\cos(x+c)$].

Visual feature opposition (D)

Visual Feature (D) corresponds to **changed-center** in the (*unit*) *circle register* (e.g., *Figure 8.6*, *Figure 8.9*, *Figure 8.10*, *Figure 8.12*) and **parallel-displacement along the *y*-axis** in the *graphical register* (e.g., *Figure 8.13, Figure 8.16, Figure 8.17, Figure 8.19*). These visual features' opposition in the *symbolic register* corresponds to the choice presence/absence of a constant (d) in a general form of sine [cosine] function, e.g., $y=\sin(x)+d$ [$y=\cos(x)+d$].

Visual feature opposition (R)

Visual Feature (R) corresponds to the rotation of the reference right triangle about the origin by an integer multiple of $\pi/2$-radian in the (*unit*) *circle register*.

Visual feature opposition (SRRT)

Visual Feature (SRRT) corresponds in the (*unit*) *circle register* to the selection of the reference right triangle in the reference rectangle (see *Footnote 44* in Chapter 6). As a consequence of the visual features (R) and (SRRT), we can transform sine [cosine] into cosine [sine] in the *symbolic register*.

Visual feature opposition (S)

Visual Feature (S) corresponds in the *graphical register* to the separation of the sine [cosine] graph into the parts referring the variation in the quadrants of the unit circle.

Visual feature opposition (C^*) [(C^{**})]

Visual Feature (C^*) [(C^{**})] corresponds in the *graphical register* to the parallel-displacement of sine or cosine graph defined in radians along the *x*-axis by π/2-length in the positive [negative] direction. These visual features are special cases of *Visual Feature (C)*. As a consequence of the visual features (S) and (C^*) [(C^{**})], we can transform sine [cosine] into cosine [sine] in the *symbolic register*.

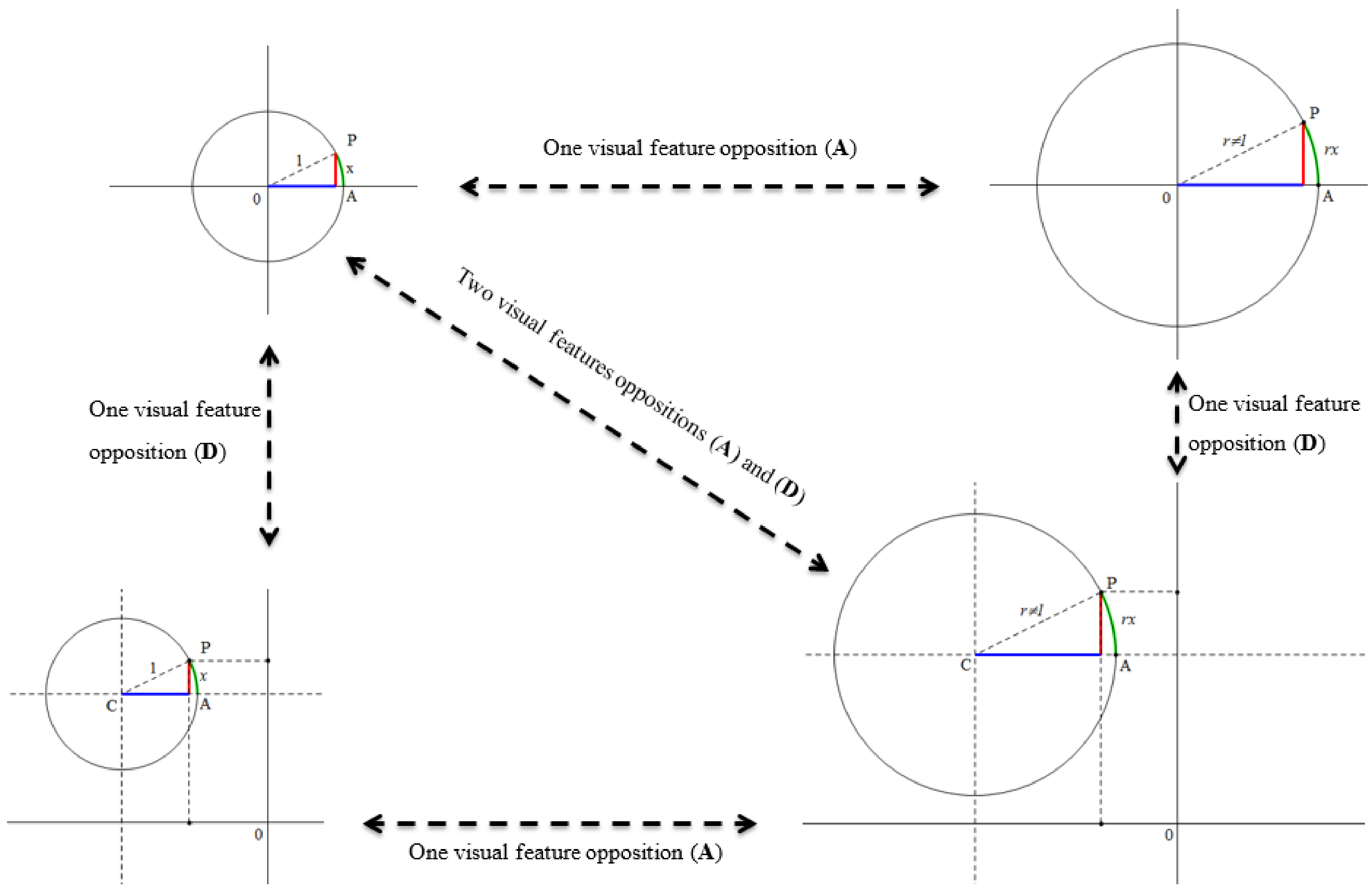

Figure 8.6. Cognitive network of the connections between *visual feature oppositions* (*A*) and (*D*) for the representation discrimination in the (*unit*) *circle register*

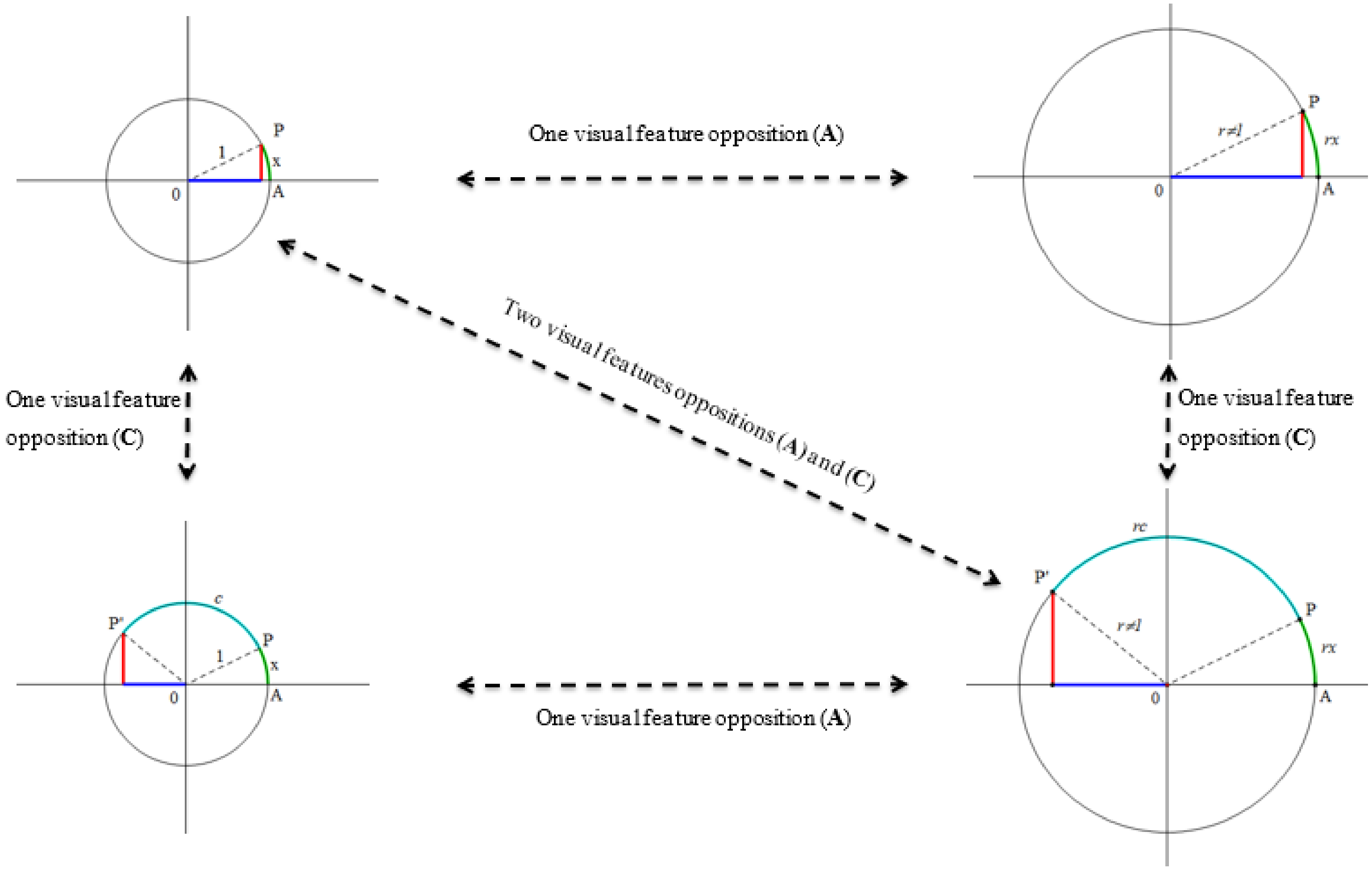

Figure 8.7. Cognitive network of the connections between *visual feature oppositions* (*A*) and (*C*) for the representation discrimination in the (*unit*) *circle register*

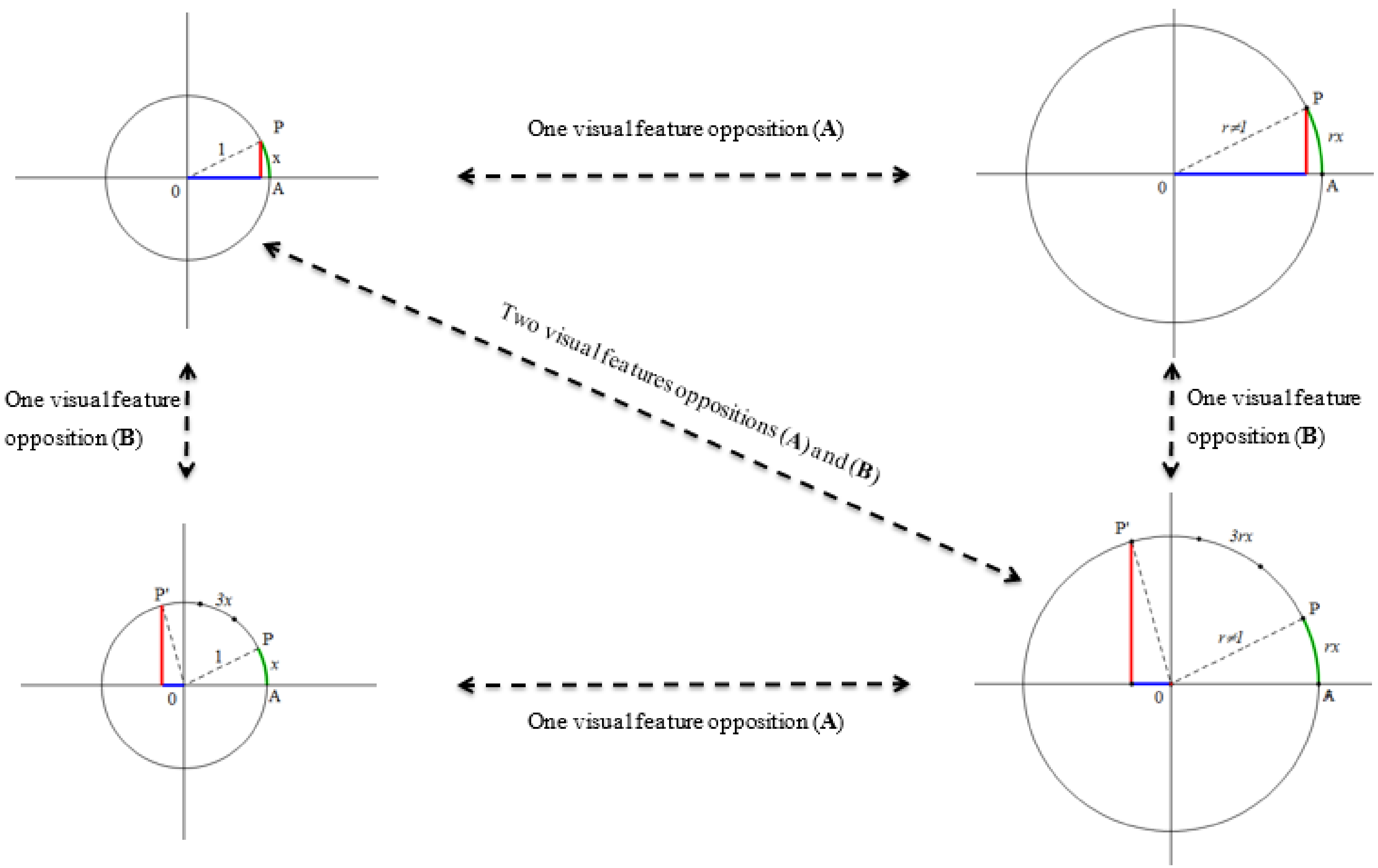

Figure 8.8. Cognitive network of the connections between *visual feature oppositions* (*A*) and (*B*) for the representation discrimination in the (*unit*) *circle register*

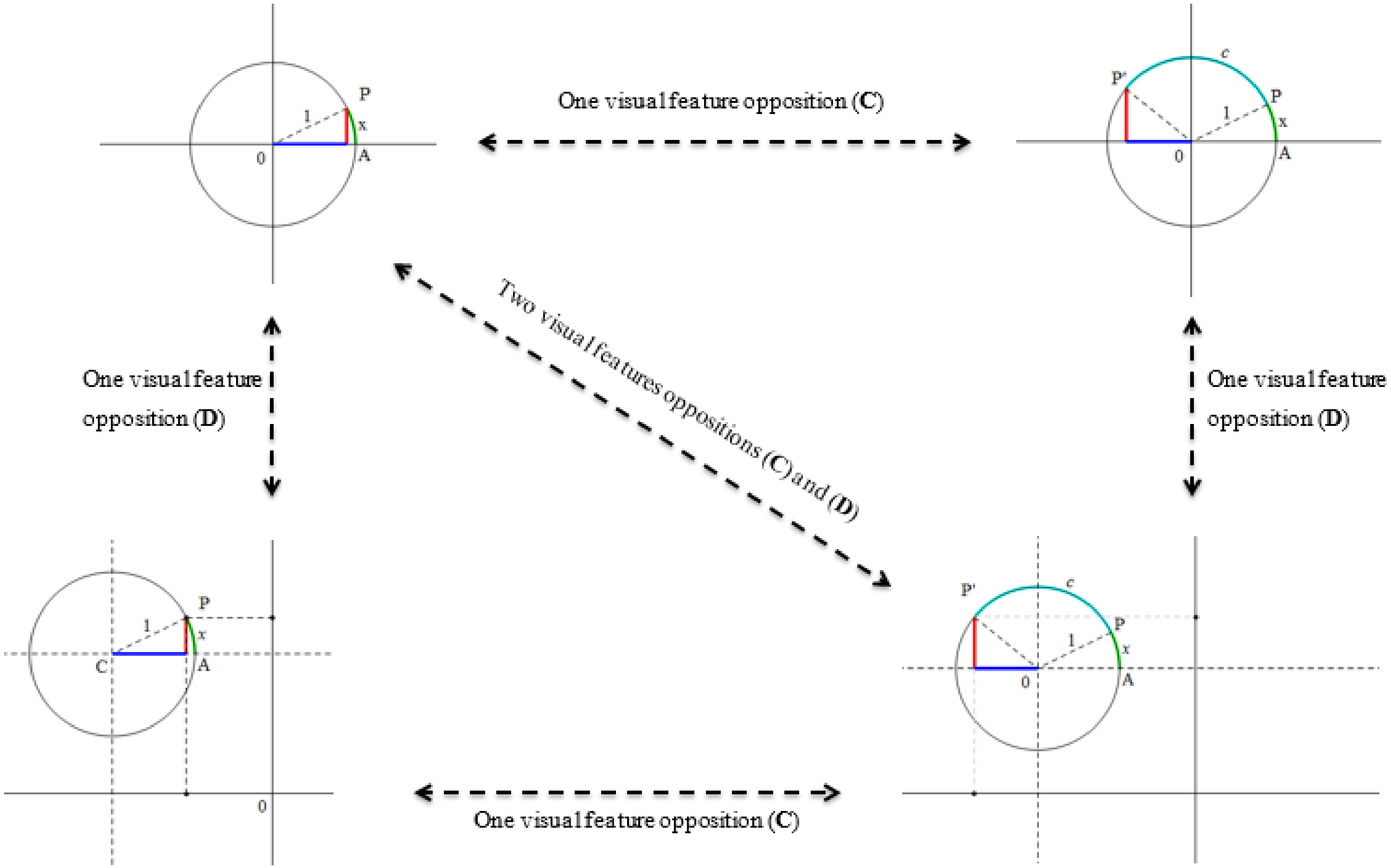

Figure 8.9. Cognitive network of the connections between *visual feature oppositions* (*C*) and (*D*) for the representation discrimination in the (*unit*) *circle register*

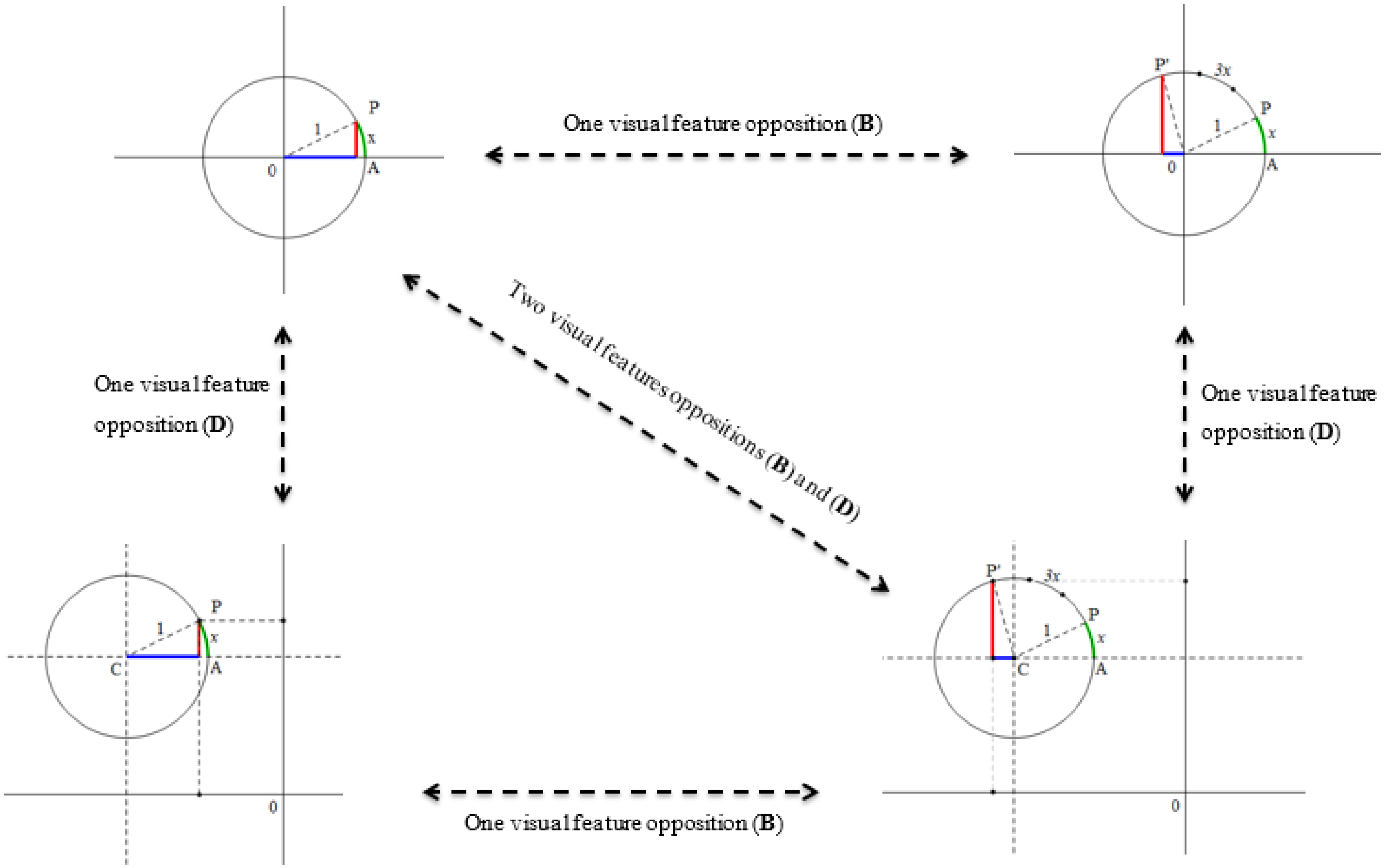

Figure 8.10. Cognitive network of the connections between *visual feature oppositions* (*B*) and (*D*) for the representation discrimination in the (*unit*) *circle register*

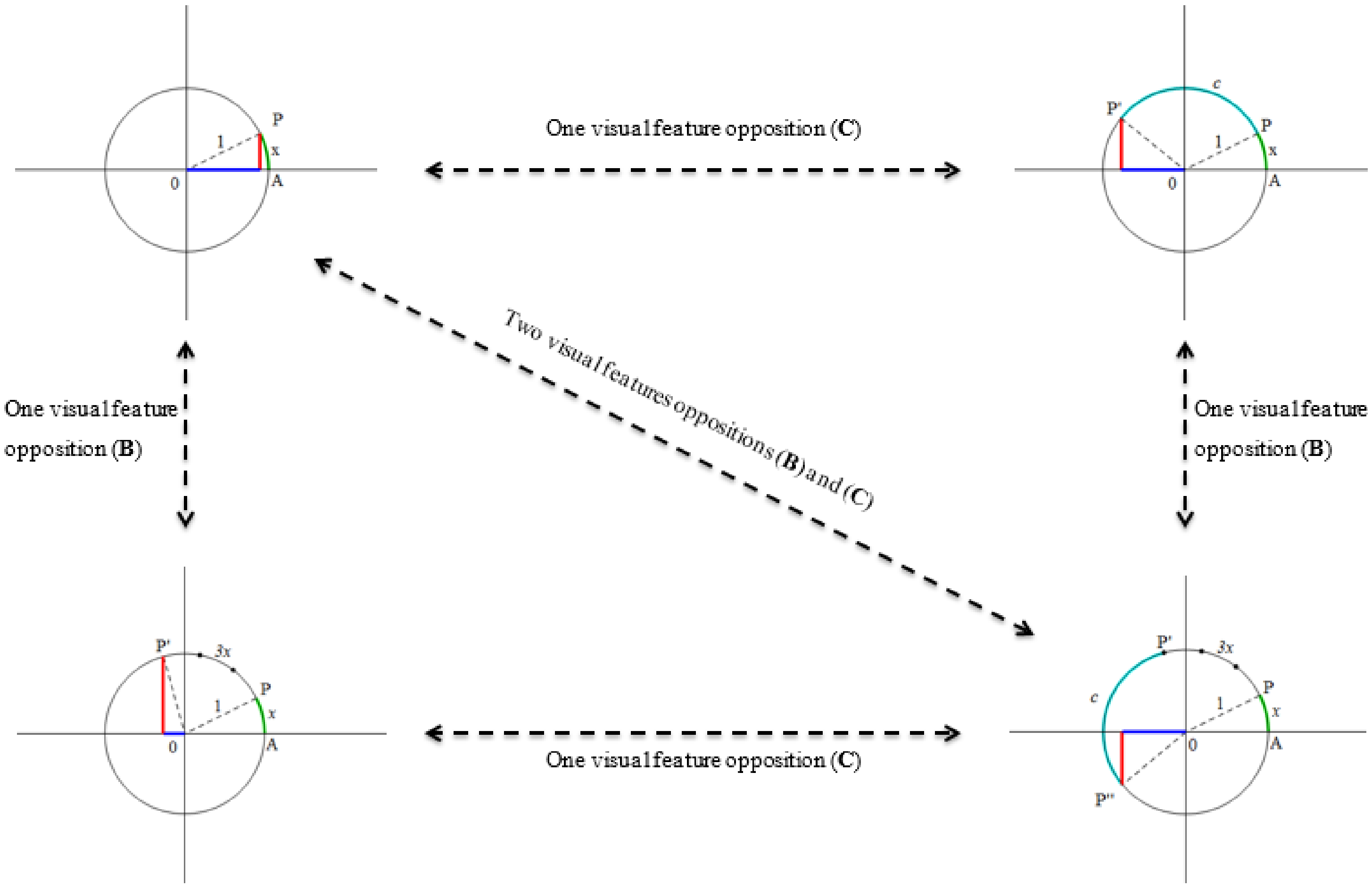

Figure 8.11. Cognitive network of the connections between *visual feature oppositions* (*B*) and (*C*) for the representation discrimination in the (*unit*) *circle register*

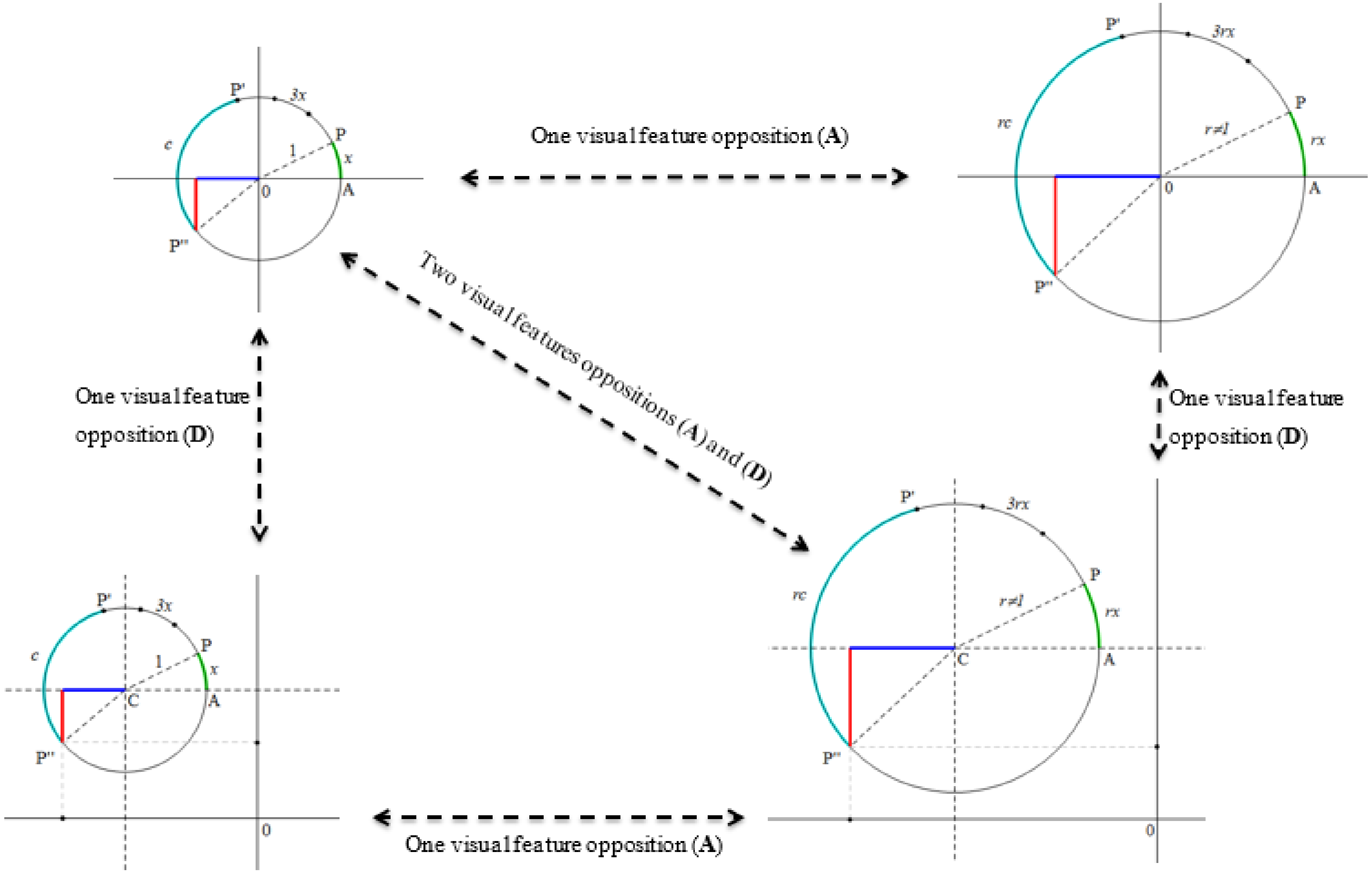

Figure 8.12. Cognitive network of the connections between *visual feature oppositions* (*A*), (*B*), (*C*) and (*D*) for the representation discrimination in the (*unit*) *circle register*

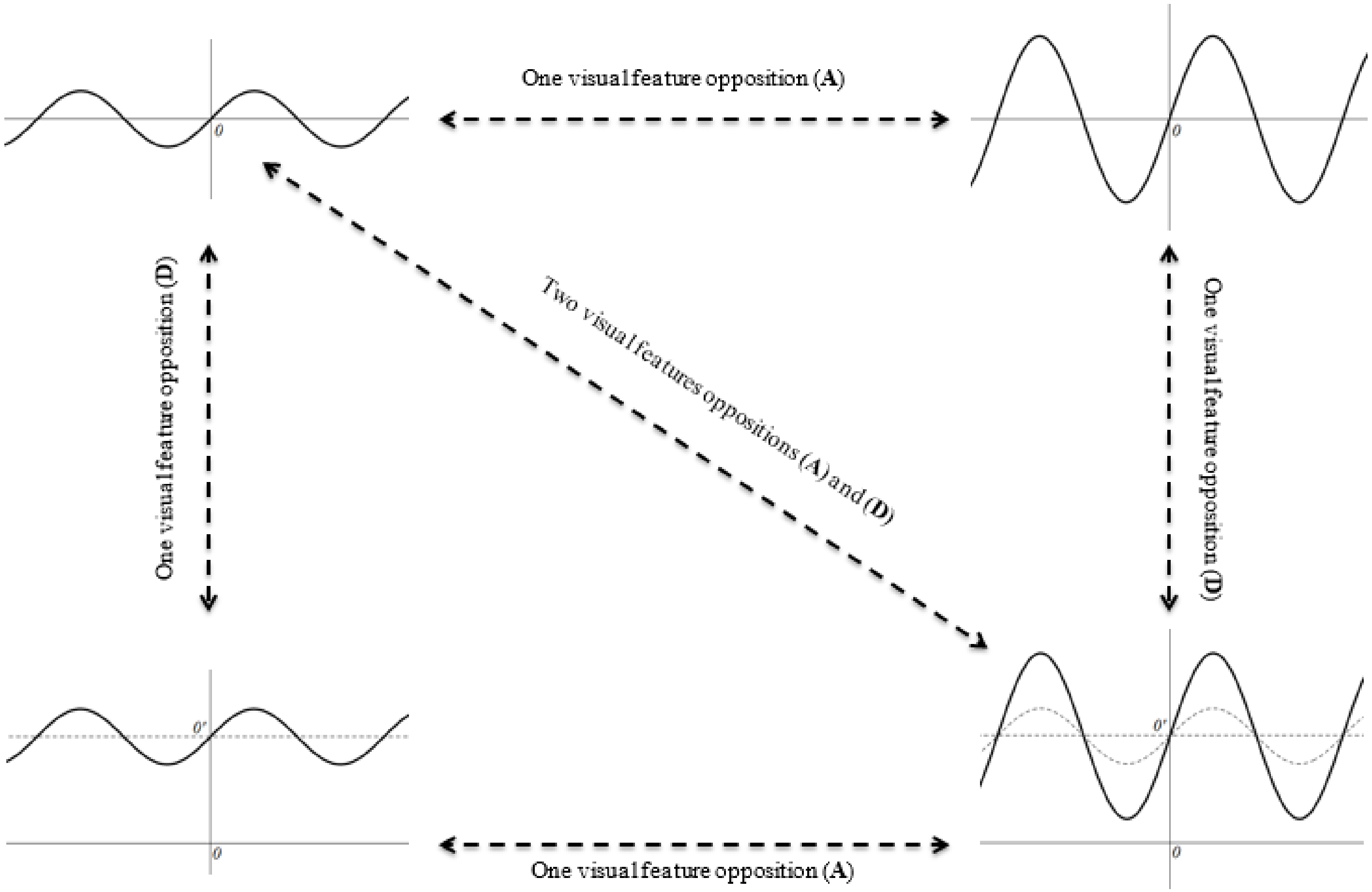

Figure 8.13. Cognitive network of the connections between *visual feature oppositions* (*A*) and (*D*) for the representation discrimination in the *graphical register*

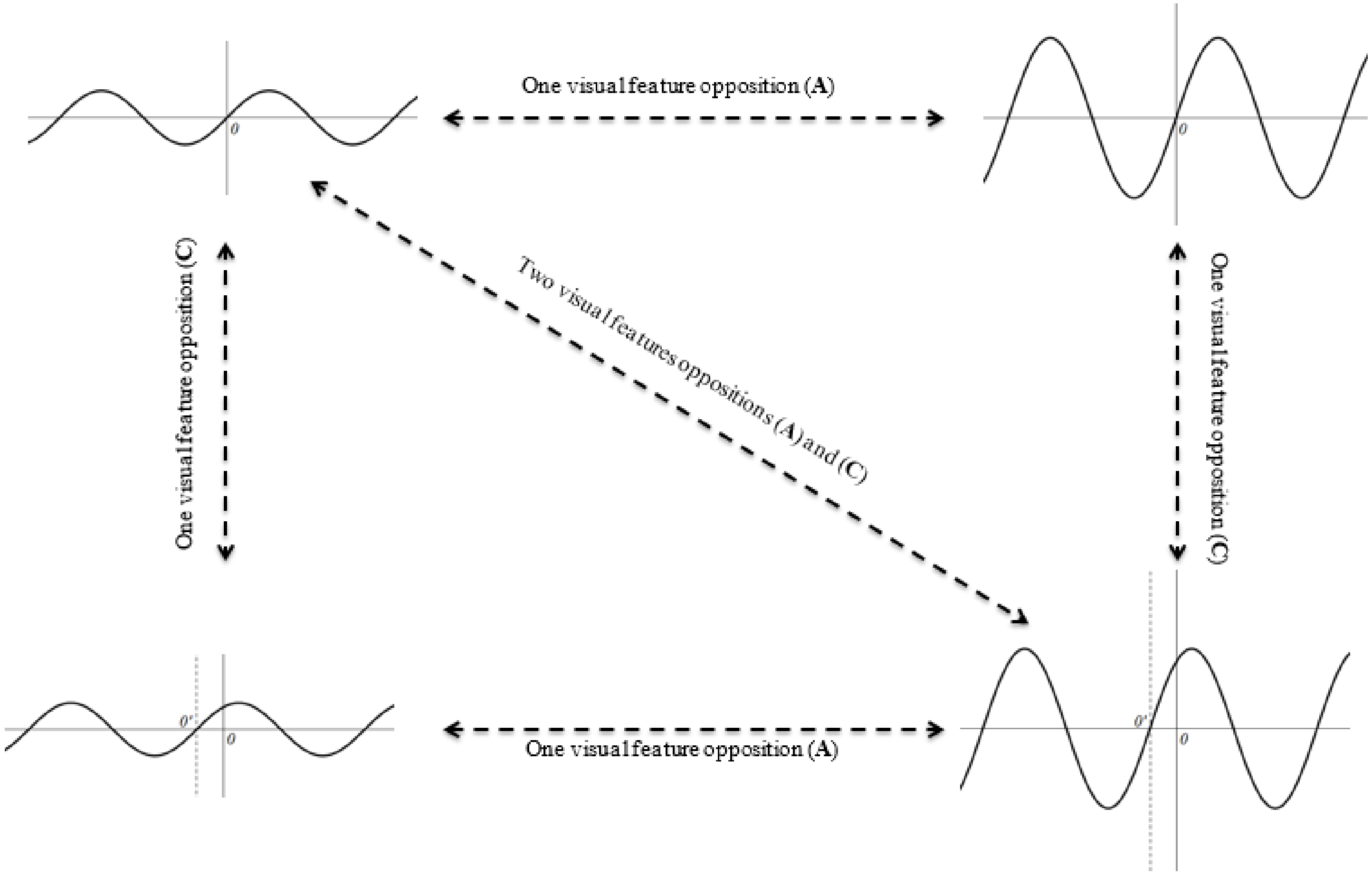

Figure 8.14. Cognitive network of the connections between *visual feature oppositions* (*A*) and (*C*) for the representation discrimination in the *graphical register*

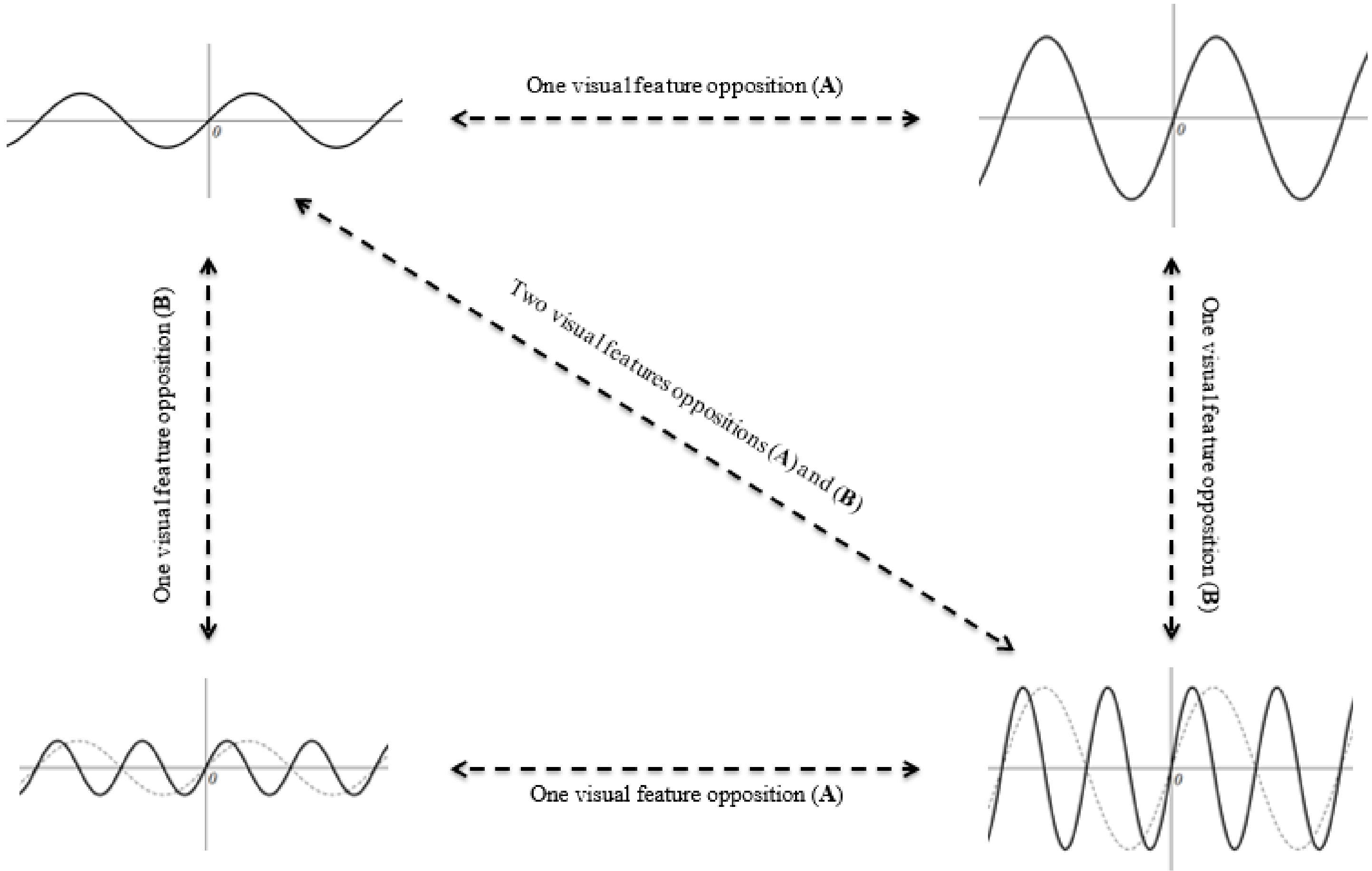

Figure 8.15. Cognitive network of the connections between *visual feature oppositions* (*A*) and (*B*) for the representation discrimination in the *graphical register*

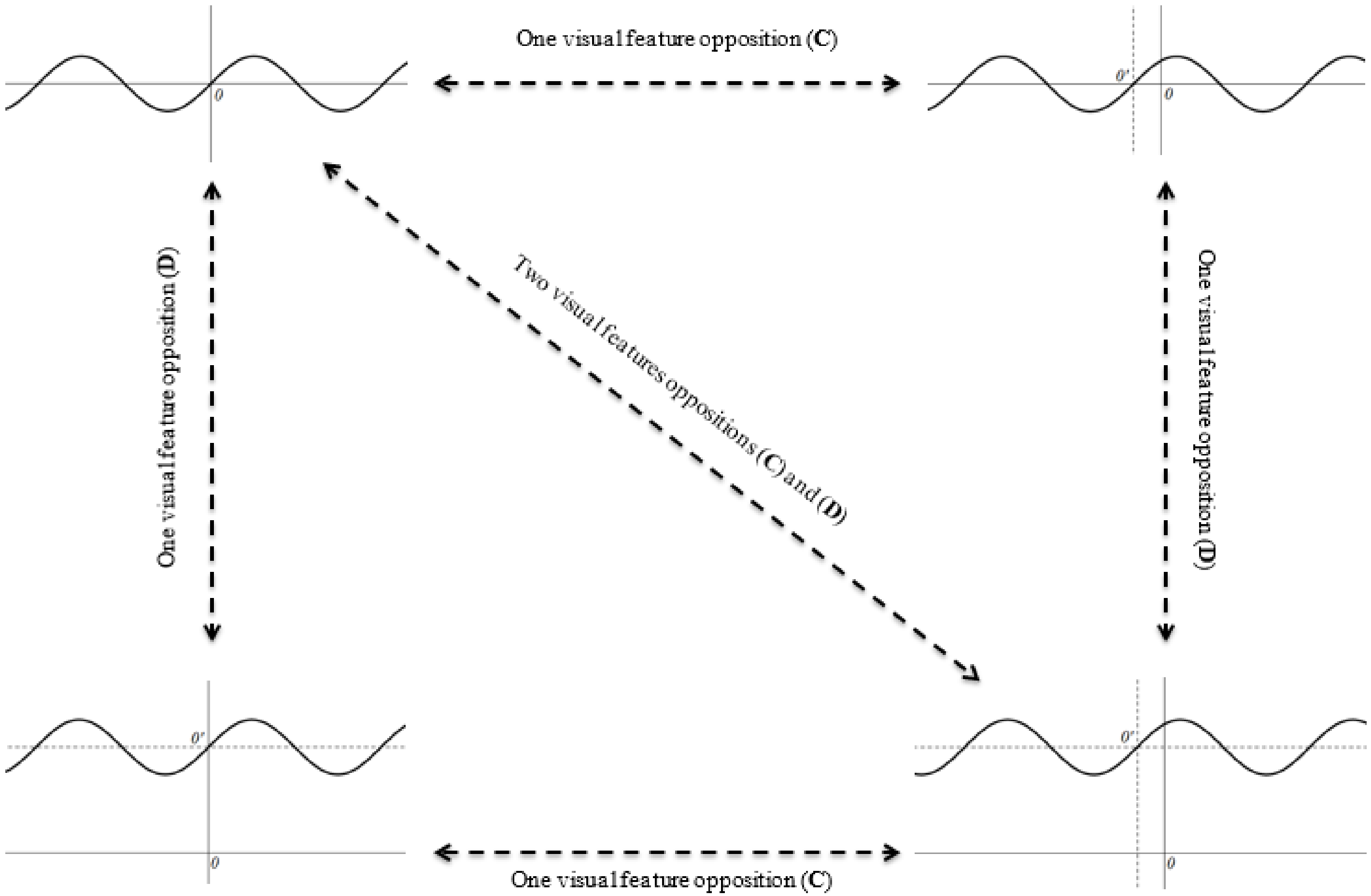

Figure 8.16. Cognitive network of the connections between *visual feature oppositions* (*C*) and (*D*) for the representation discrimination in the *graphical register*

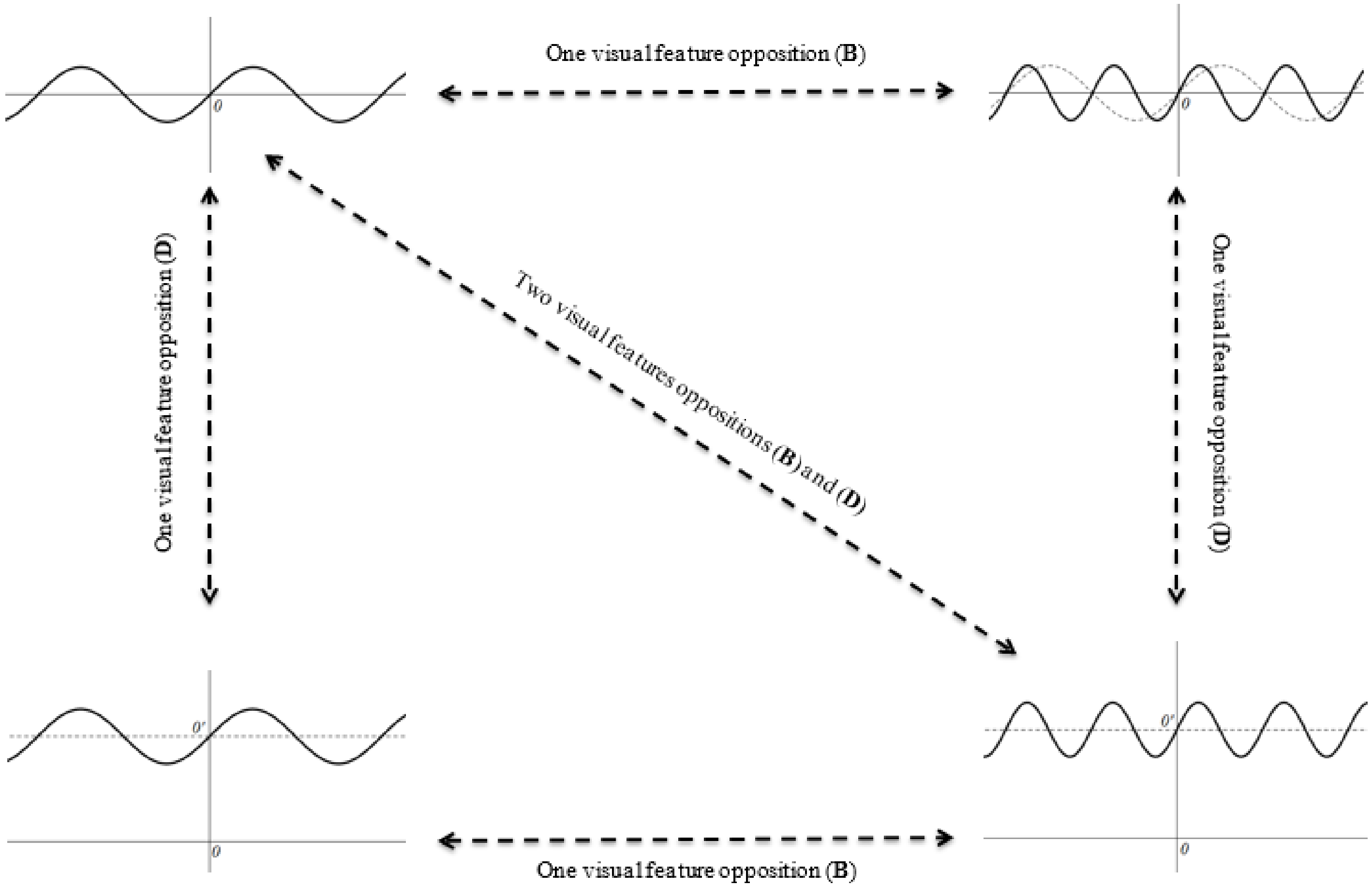

Figure 8.17. Cognitive network of the connections between *visual feature oppositions* (*B*) and (D) for the representation discrimination in the *graphical register*

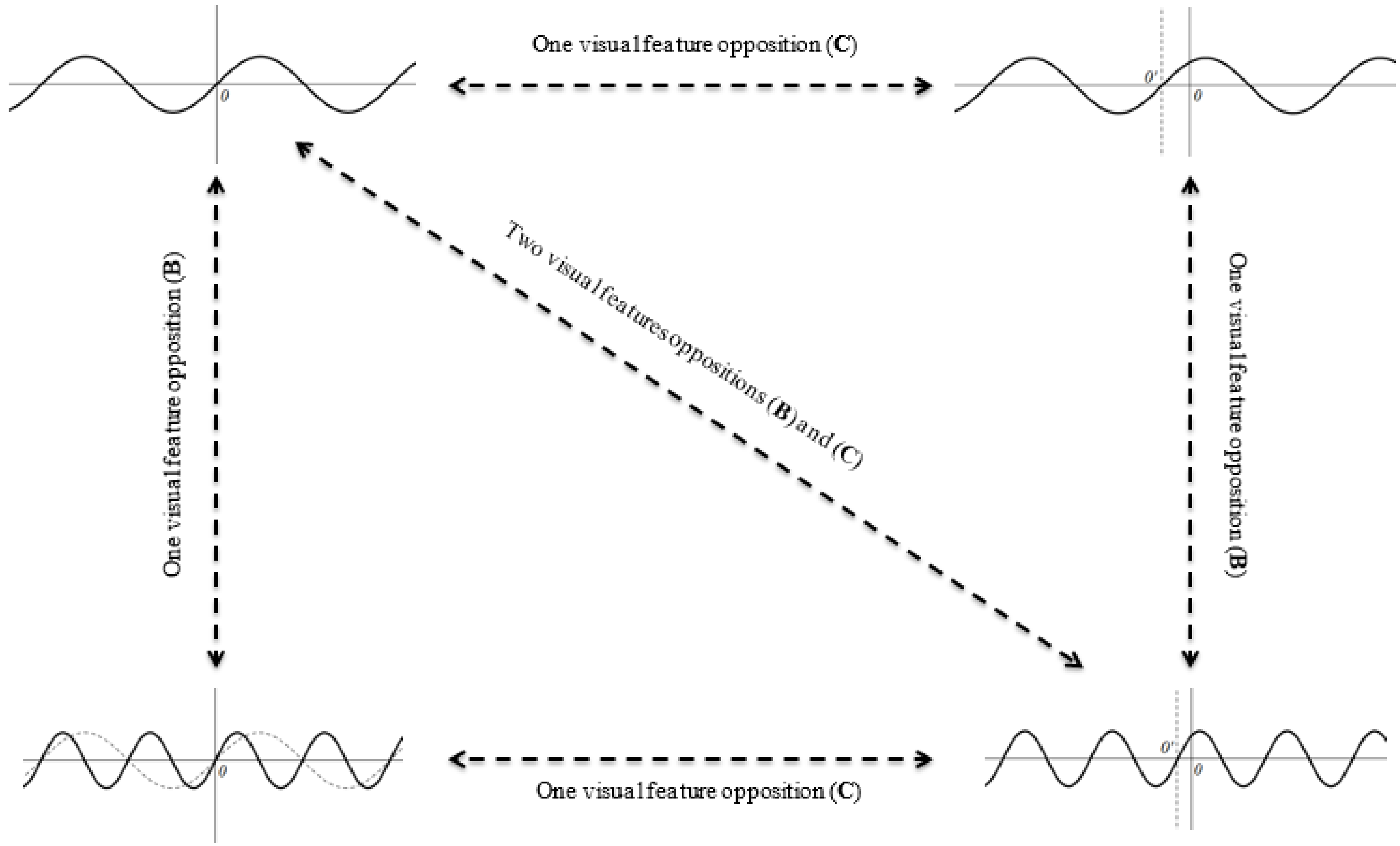

Figure 8.18. Cognitive network of the connections between *visual feature oppositions* (*B*) and (*C*) for the representation discrimination in the *graphical register*

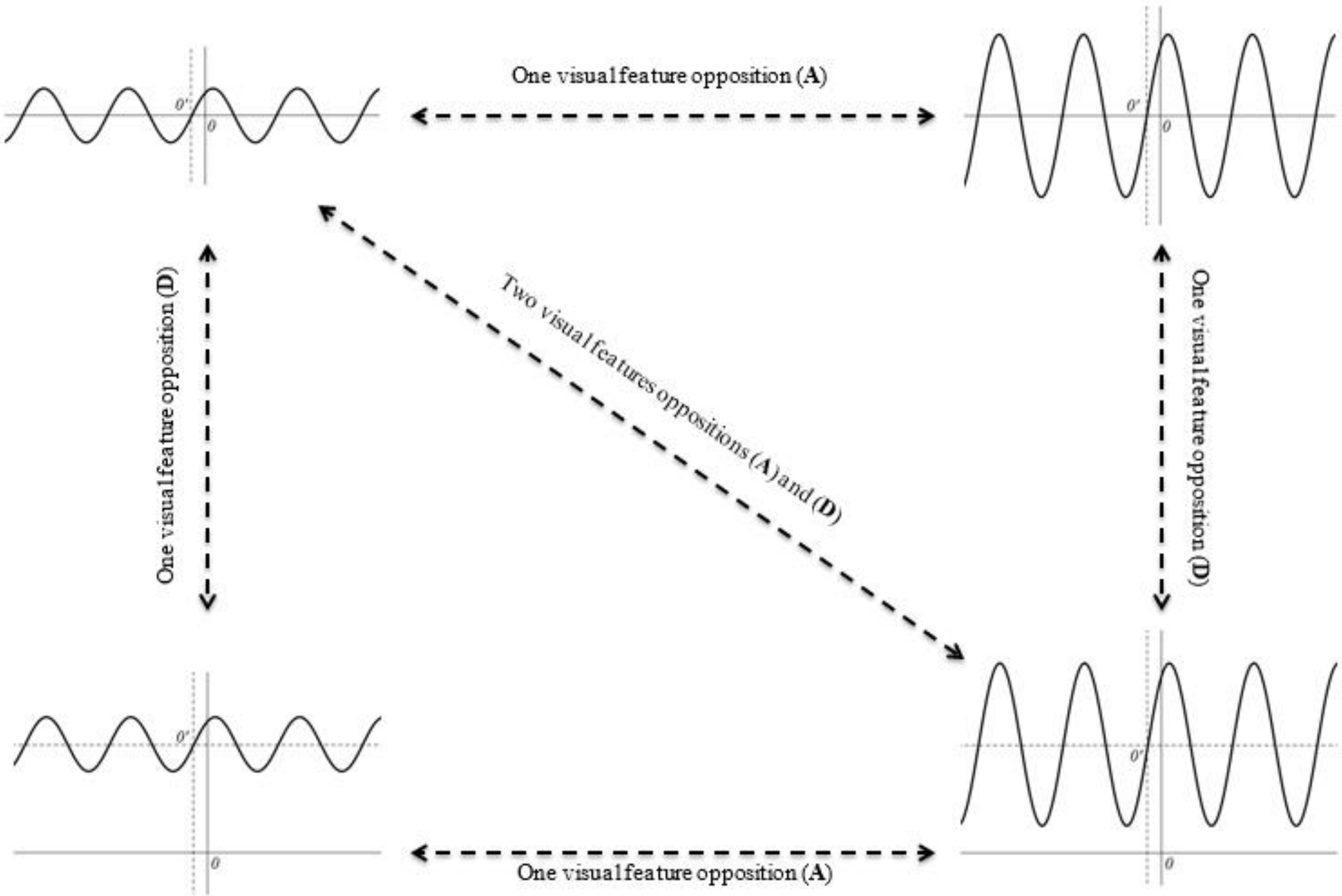

Figure 8.19. Cognitive network of the connections between *visual feature oppositions* (*A*), (*B*), (*C*) and (*D*) for the representation discrimination in the *graphical register*

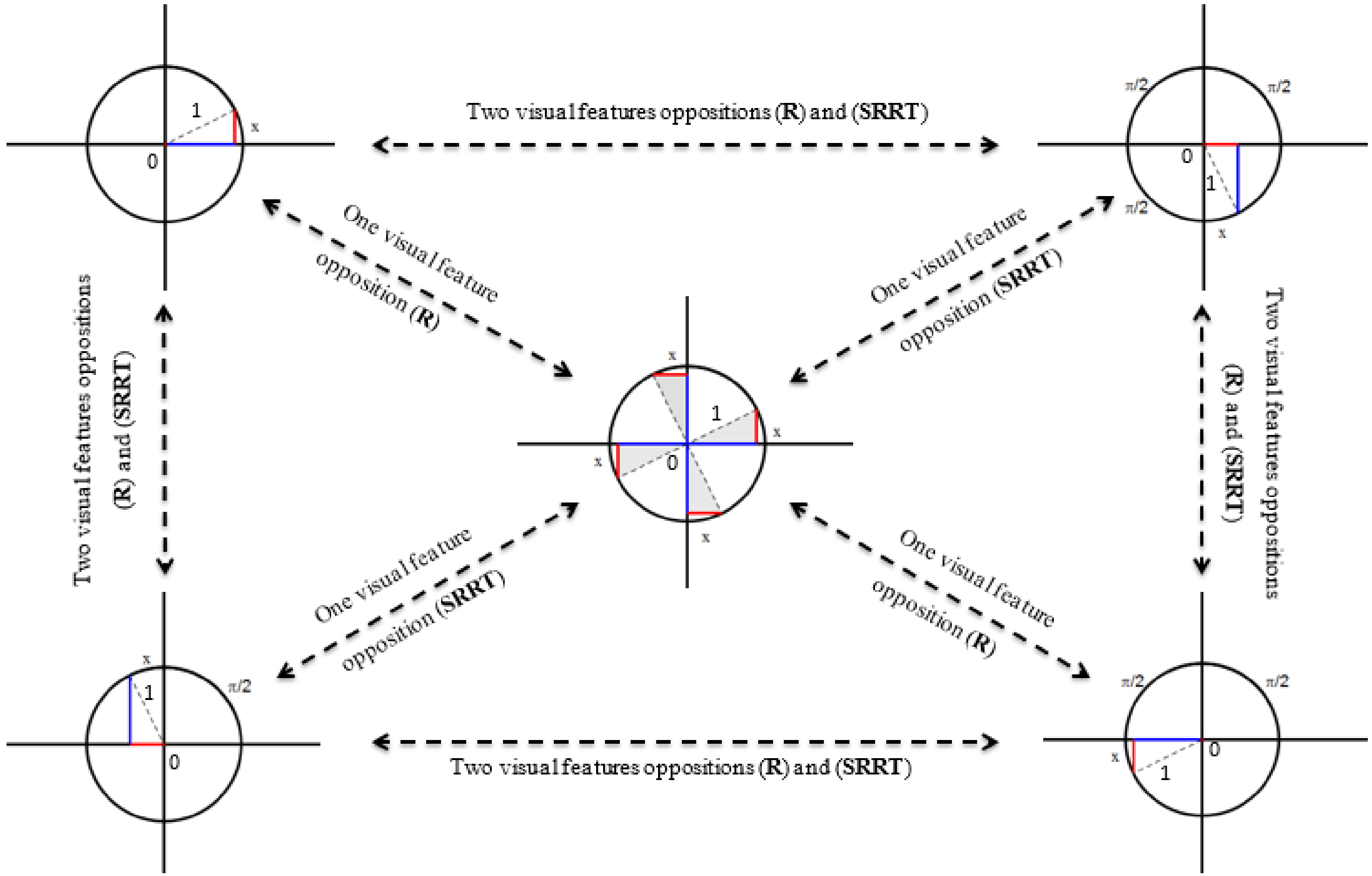

Figure 8.20. Cognitive network of the connections between sine and cosine in the (*unit*) *circle register* as a consequence of the *visual feature oppositions* (*R*) and (*SRRT*)

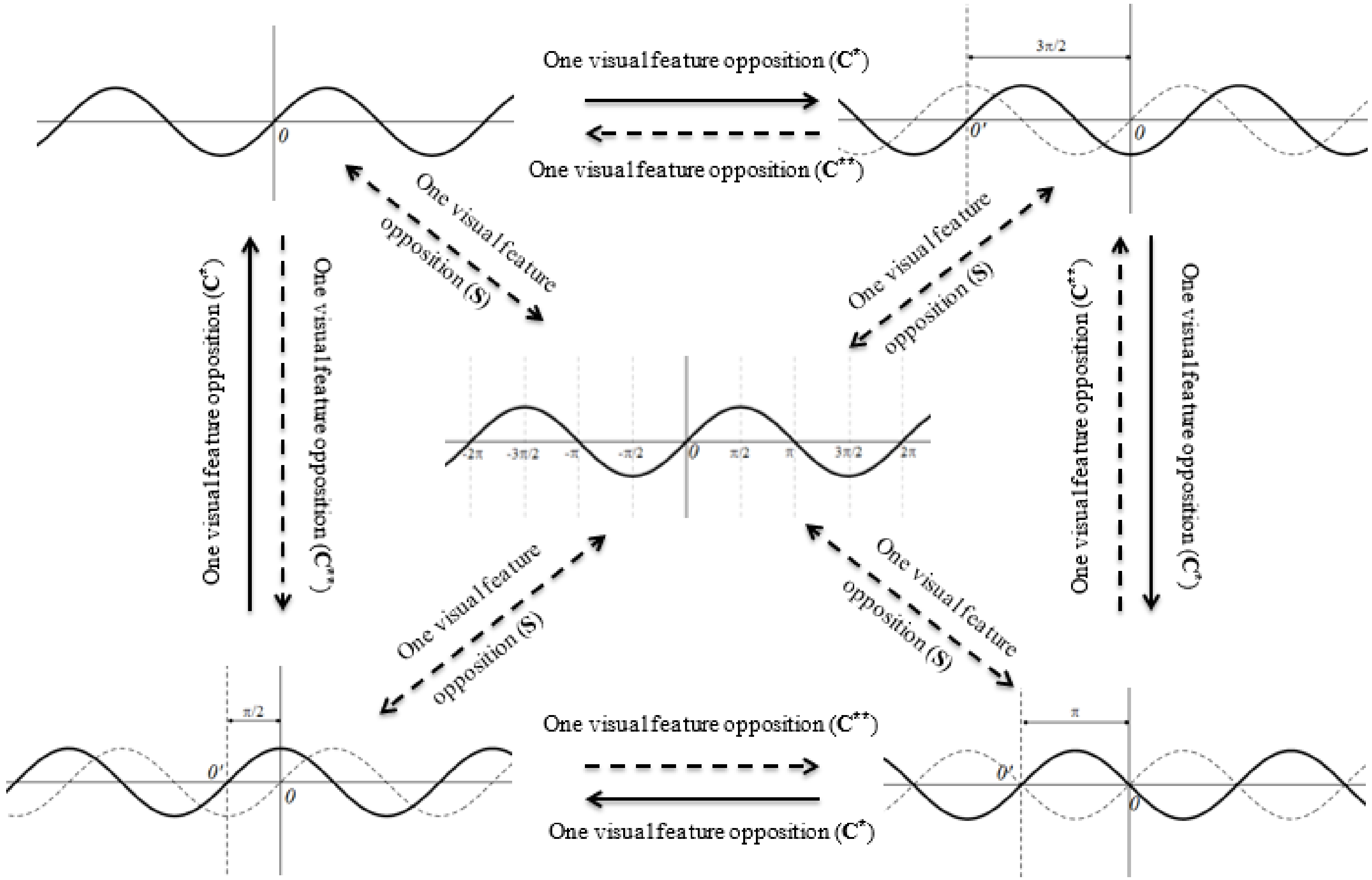

Figure 8.21. Cognitive network of the connections between sine and cosine in the *graphical register* as a consequence of the *visual feature oppositions* (*S*) and (C^* and C^{**})

8.4. Implications and Suggestions for Curriculum and Instruction

This study included a trigonometry instruction that was designed to support students' concept images on trigonometric functions in different representations (i.e., symbolic, circular, and graphic) through initially inspiring from research literature on trigonometry, historical development of trigonometry, our exploratory teaching experience, and initial interview results, and then, revising as a result of the on-going prospective and retrospective cognitive analysis during the 17-week experimentation process. We categorized semiotic representations of trigonometric functions into four: (*unit*) *circle register*, *graphical register*, *symbolic register* and *language register*. Our design was including a sequential *recognition* tasks (based on dynamically-linked *conversions* of trigonometric functions between representational registers) and *discrimination* tasks (based on dynamically-changed visual components referring to trigonometric functions) in the dynamic geometry environment with GSP. In conversions of trigonometric functions, the way followed was from the (*unit*) *circle register* to the *graphical register* together with the *symbolic register*.

The cognitive analysis of the data revealed the students' serious *recognition* problems on *foundational trigonometric concepts* (i.e., *angle*, *angle measure*, *trigonometric value*, *trigonometric functions*, and *periodicity*) based on their weak concept definition images. As the study progressed, when reasoning about these concepts in GSP environment, they constructed *well-defined* concept definition images as summarized in *cognitive concept maps* that were grounded based on the result of this study.

To begin with, when making variations of an angle constructed in GSP, reasoning about its dynamically-changed measure in *directed angles* preference produced the students' *recognition* of the invariance components referring to the angle measure (*initial side*, *terminal side* and *direction* of the rotation). Moreover, they constructed dynamic-and-directed turning view on angles in the (*unit*) *circle register* that enabled them to associate a unique *static* angle structure with infinitely many equivalent *dynamic* structures. Thus, it is important that curriculum and instruction promote students' *well-defined* concept definition images on angles (see *cognitive concept maps on angle* in *Figure 8.1-Figure 8.2*) including dynamic-turning view through specifying the rotation's *initial side, terminal side* and *direction*.

Secondly, the findings that emerged from this study emphasize similarity as a fundamental starting point to reason about triangle trigonometry. When dynamically manipulating similar right triangles, and observing corresponding changes of trigonometric ratios between similar right triangles, the students recognized that a sine [cosine] ratio is dependent only on an acute angle and independent from side lengths, specially, in case the hypotenuse measure is equal to 1, opposite [adjacent] side length regarding the acute angle corresponds to sine [cosine] value. This was the first step of semantically mergence of their fragmented concept images on trigonometric functions in the right triangle context and the unit circle context. Therefore, students' *well-defined* concept definition images on trigonometric functions in the right triangle context (see *cognitive concept maps on sine* [*cosine*] in *Figure 8.3*) should arise from similarity before defining them on the unit circle context.

On the other hand, as revealed in the cognitive analysis of the initial interviews of this study, because trigonometric functions cannot be expressed as algebraic formulas involving arithmetical procedures, students have trouble on reasoning about them as functions. Therefore, functionality idea must be a fundamental starting point to define sine [cosine] as a function in any representational register. Development in the students' concept images on sine [cosine] of a real number revealed that students need to encounter the reality that real numbers one-to-one correspond to angles as long as angle measure unit is described clearly. Otherwise, a real number corresponds to (generally) two different angles as a consequence of *degree* or *radian* preference as the angle measure unit. It means that a unique real number generates two different sine [cosine] value. For example, below representation of the sine function in the *symbolic register* is not a function because it is not *well-defined* (see *Footnot 53*) without making clear about the angle measure unit.

$$\mathit{sine}: \quad \mathrm{R} \rightarrow [-1,1]$$

$$x \rightarrow \sin(x)$$

The students in this study recognized this idea as a consequence of their trouble on different outputs of GSP for sin(*30*). While sin(*30*)=*0.50* (in *degree* preference), sin(*30*)=*-0.99* (in *radians* preference). Therefore, definition of trigonometric functions in the *symbolic register* requires to make clear about the angle measure unit that is used. Making clear about angle measure is not enough to define sine [cosine] in coherent understanding of the right triangle trigonometry and the unit circle trigonometry. Findings of this study revealed that defining sine [cosine] in the (*unit*) *circle register* as the ordinate [abscissa] of the reference point on the unit circle does not provide students with the ability to discriminate the ordinate of a point on a non-unit circle from sine. Thus, definition of sine [cosine] should draw from the dynamic view of the reference right triangle integrated into the circular representation in the (*unit*) *circle register*. In special sense, on the unit circle after students' *recognition* of the opposite [adjacent] side of the reference right triangle as sine [cosine] from the similarity perspective, the meaning of the opposite [adjacent] side within each quadrant as well as the *limit cases* (see *Footnote 54*) of the reference right triangle must be discussed through comparing with the corresponding sine value. The meaning of sine [cosine] as ordinate [abscissa] of the reference point on the unit circle sould be the next step of the *recognition* tasks. Final step of the *recognition* tasks of sine [cosine] function sould be to investigate its systematic covariation with respect to the variation of angle measure in order to distinguish the role of coterminal angles, principal angle and reference angle (see *cognitive concept maps on sine* [*cosine*] *as a function* in *Figure 8.4*). This ability is constructed only at the level of grasping the angles as dynamic turnings in the (*unit*) *circle register*. Thus, it is important that curriculum and instruction promote students' *well-defined* concept definition images on trigonometric functions in the (*unit*) *circle register* arising from the meaning of the reference right triangle.

The importance of the basic visual features' *discrimination* is also revealed in this study. When the basic visual features referring to trigonometric functions (i.e., radius of the circle, position of the center, position of the reference point on the circle referring to trigonometric value) were systematically varied in the (*unit*) *circle register*, and their dynamic-and-linked oppositions in the *graphical register* were constructed, the students developed significant understandings referring to trigonometric functions.

Initially, when the unit circle was changed into a non-unit circle, they constructed "unit" meaning that enabled them to consider a unit circle as a non-unit circle. They constructed this meaning when measuring the changed-radius by GSP as a consequence of their troubles arising from the difference between the distance-measure-unit preference of GSP as centimeter and the visual distance-measure-unit of the coordinate axes. This ability to reason about the same circular structure as both a unit circle and non-unit circle is necessary to recognize general forms of trigonometric functions in the (*unit*) *circle register*. When the radius changed from 1 to $r \neq 1$, *discrimination* of trigonometrically relevant and cognitively significant visual features (such as the reference point, arc length, arc angle, reference right triangle –see *Visual Feature Opposition (A)* in *Figure 8.6-Figure 8.8*) is an important cognitive ability to use trigonometric functions in any circular context. This discrimination requires to be able to not only associate the opposite side of the reference right triangle in the r-unit circle with $\sin(x)$ through considering r is the unit, but also associate it with $r.\sin(x)$ through considering $r \neq 1$. This visual feature's opposition is changed-magnitude in the *graphical register* (see *Visual Feature Opposition (A)* in *Figure 8.13-Figure 8.15*). Cognitive analysis of the data revealed that, *seeing* the systematic covariations between the radius and magnitude is necessary condition of the ability to reason about relations among "changed-radius" in the (*unit*) *circle register*, "changed-magnitude" in the *graphical register* and "changed-coefficient r of the r.sine function" in the *symbolic register*. However, these visual features by oneself indicate (for $r>0$) the *discrimination* of $r=1$ or $r \neq 1$ in the *symbolic register*. *Discrimination* of the coefficient a (for $a<0$) in the *symbolic register* needs consideration of another visual feature, i.e., *Visual Feature (C),* together with *Visual Feature (A)* in the (*unit*) *circle register* (see *Figure 8.7*) and the *graphical register* (see *Figure 8.14*).

Next visual feature of the systematic variation was the center of the unit circle. Changed-location of the center from the origin to any other position on the coordinate plane revealed that the *discrimination* ability requires to focus on mathematically relevant objects (e.g., reference right triangle, displacement amount and direction) in reasoning about the new situation instead of processes (e.g., ordinate of a point, procedural definition of sine or cosine). For example Zafer reasoned about the new representation focusing on the horizontal axis from the center as the *x*-axis considering the *x*-axis as if a reified-object without going into details but with awareness of its different location. Cemre mentioned the unit circle whose center located on the origin as if a reified-object; and was able to change its position up-down and left-right in her mind as a whole on the coordinate system. On the other hand, Defne and Ebru focused predominantly on the processes related to the *y*-components in the (*unit*) *circle register* based on the determination of the ordinates. For this reason, Cemre and Zafer performed at higher reasoning stage (i.e., *condensation/reification*) than Defne and Ebru (i.e., *interiorization*) considering Sfard's (1991) hierarchical stages of the concept development. Thus, *discrimination* of trigonometric functions represented in the (*unit*) *circle register* from this register's content needs to reason about them based on trigonometrically relevant and cognitively significant objects (such reference point(s), as arcs, reference right triangle, horizontal axis from the center and radius) rather than the detailed processes.

Final visual feature of the systematic variation was the reference point referring to trigonometric value. That is to say, the only one reference point on the unit circle referring to both angle and corresponding trigonometric value was varied from only one point to two points so as one of them to refer the input and the other to refer the output of the function (see *Visual Feature Oppositions (B) and (C)* in *Figure 8.11*). Findings from these tasks of the study reveal the importance of students' *recognition* of sine [cosine] considering its angle dependent on another angle for *discrimination* of, for example, $y=\sin(x)$, $y=\sin(x+1)$ and $y=\sin(2x)$ represented in the (*unit*) *circle register* from each other. These visual features' oppositions in the *graphical register* (see *Visual Feature Oppositions (B) and (C)* in *Figure 8.18*) also reveal that (*i*) students' making sense of the parallel-displacement of the sinusoidal graphs along the *x*-axis is possible only when they are interpreted together with their dynamic-and-linked conversions in the (*unit*) *circle register*, (*ii*) for students' construction of *well-defined* concept definition images on periodicity (see *cognitive concept maps on* *periodicity of core trigonometric functions* in *Figure 8.5*), it is crucial to reason about trigonometric functions that are defined based on two points turning at different (angular) speeds in the (*unit*) *circle register*.

Finally, this study reveals the importance of dynamic-and-linked conversions of trigonometric structures between representational register in students' *discrimination* of trigonometrically relevant and cognitively significant structures. For example, visual representations of sine [cosine] on the same coordinate plane both in the (*unit*) *circle register* and the *graphical register* are vital in promoting students' *discrimination* of the coordinates of a point in the (*unit*) *circle register* and the *graphical register* only when the opportunity to compare and contrast dynamic-and-linked variations of the reference point on the (unit) circle and its converted form in the *graphical register* is given to students. This opportunity is also important in promoting students' *discrimination* of the meaning of the positive [negative] direction in the (*unit*) *circle register* and the *graphical register* in terms of both angle measures and trigonometric measures. Furthermore, representing the coterminal angles in the *graphical register* is the best way to differentiate visually the equivalent but not equal angles' positions on the *x*-axis unlike their unique static structures in the (*unit*) *circle register*. In addition, simultaneous reasoning about the principal arc's angle in the *unit circle register* and its conversion on the *x*-axis scaled with real numbers in the *graphical register* is an important opportunity for students to recognize angle measures as real numbers. In special sense, if the angle measure unit is considered in radians, this opportunity fortifies students' concept images on the meaning of π through merging its meaning as an angle measure in radians and as a real number (i.e., approximately 3.14) via simultaneous representation of π-radian angle in the (*unit*) *circle register* and its conversion on the *x*-axis in the *graphical register*.

All in all, it is important that curriculum and instruction promote students' *recognition* and *discrimination* of foundational trigonometric concepts within and between different representational register via taking into the consideration important ideas that are mentioned above.

8.5. Limitation of the Study and Suggestions for Future Research

In this study, secondary students' concept images on trigonometric functions were investigated both after their school trigonometry course and during the experimentation of the designed-instruction in GSP environment including sequential tasks that were revised in an on-going way to influence *students' recognition* and *discrimination* of foundational trigonometric concept within and between different representational registers.

Participants of the study were four successful secondary students who had just completed the trigonometry course in their schooling time with high grades on the written exams conducted at their schools. Thus, the experimentation did not include students with different ability levels in general sense, as well as those who did not experience trigonometric functions before. Also, interactions of the teaching experiment were not social interactions as a part of classroom mathematical practices; rather, interactions were social interactions with two students at a time to gain detailed insights into their reasoning ways throughout the 17-week experimentation. Moreover, except in the modeling task, representations on which discussions were done constructed in the dynamic geometry environment rather than paper-and-pencil environment. Lastly, trigonometric functions was delimited in the scope of this study into sine and cosine functions.

Due to the limitations mentioned above, the result of this study may not be generalized to all secondary students with different ability levels. However, it is reasonable to presume that the participants' cognitive troubles after their trigonometry course and prior to the teaching experiment would be valid for others including average and lower achievers. Moreover, the results of this study provide a lens for mathematics instructors at tertiary education to recognize their students' background knowledge on trigonometry that have been acquired during their secondary education. Also, conceptual frameworks that were created in this study based on the cognitive analysis of the experimentation process can serve a valuable guide for trigonometry teachers, other researchers and educators in designing more effective trigonometry tasks, as well as in analyzing *students' trigonometry*.

To build a broader picture of students' concept images on trigonometry, it is needed to investigate the effect of dynamically-linked conversions of trigonometric functions between representational registers on students' *recognition* and *discrimination* abilities in terms of (*i*) students with different ability levels, (*ii*) students at different grades and experience levels, (*iii*) many other topics related to trigonometry, as well as (*iv*) social interactions as a part of classroom mathematical practices. Also, it is needed to investigate (*i*) the critical visual components referring to other trigonometric functions and many other topics related to trigonometry in students' *recognition* and *discrimination* abilities.

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APPENDIX A

INITIAL INTERVIEW QUESTIONS

(Q1) Matematikteki *fonksiyon*u kendi cümlelerinizle tanımlayınız.

(Q2) Küçük bir salyangoz bir parça kâğıt üzerinde aşağıda görülen yolu takip ederek yürüyor.

- Salyangozun kâğıt üzerindeki konumunu, zamana bağlı bir fonksiyon olarak ifade edebilir miyiz? Cevabınızın nedenini açıklayınız.
- Zamanı, salyangozun kâğıt üzerindeki konumuna bağlı bir fonksiyon ifade edebilir miyiz? Cevabınızın nedenini açıklayınız.

(Q3) Aylin hafta sonu gittiği lunaparkta dönme dolabın kırmızı renkli salıncağına biniyor; ve dönme dolabın hareketi süresince zamana göre yerden yüksekliğinin nasıl değiştiğini düşünüyor. Aylin'in bindiği salıncağın yerden yüksekliği zamana bağlı olarak nasıl değişir? Ne dersiniz, bu değişim bir fonksiyon olur mu?

Olursa neden olur?
Olmazsa neden olmaz?
Açıklayınız.

(Q4) Trigonometrideki *birim çember*i kendi cümlelerinizle tanımlayınız.

(Q5) "sinüs" ve "kosinüs" deyince aklınıza neler geliyor? Açıklayınız.

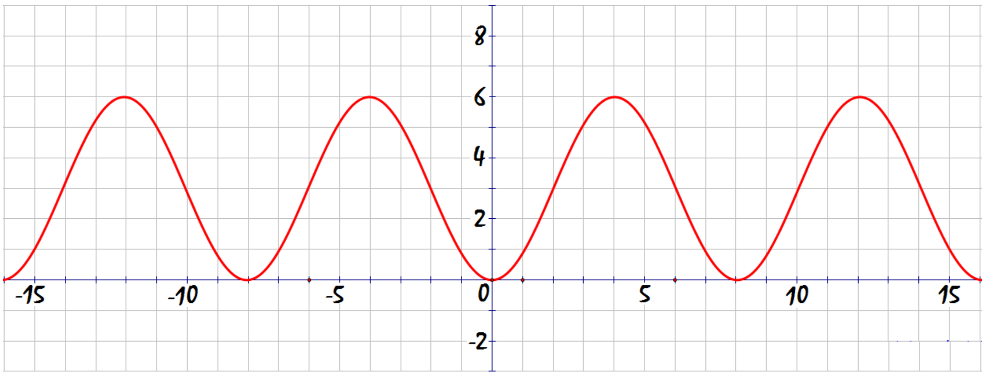
(Q6) $f: R \to R$ ve $f(x) = 2cosx$ fonksiyonu veriliyor. Aşağıdaki noktaları dik koordinat düzleminde gösteriniz.

- $(30, f(30))$
- $(\pi, f(\frac{\pi}{6}))$
- $(1, f(\frac{\pi}{3}))$

(Q7) $f: R \to R$ ve $f(x) = sinx$ fonksiyonu veriliyor. $f(x) = -\frac{\sqrt{3}}{2}$ eşitliğini sağlayan

- en büyük negatif sayı kaçtır?
- en küçük pozitif sayı kaçtır?

(Q8) Aşağıdaki grafiği inceleyiniz. Bu grafik hangi fonksiyonun grafiği olabilir? Açıklayınız.



APPENDIX B

PARENTAL PERMISSION FORM

ORTA DOĞU TEKNİK ÜNİVERSİTESİ
MIDDLE EAST TECHNICAL UNIVERSITY
06531 ANKARA-TURKEY

Ortaöğretim Fen ve Matematik Alanları Eğitimi Bölümü
Department of Secondary Science and Mathematics Education

Tel: 90 (312) 210 4049
Faks:90 (312) 210 79 71

Veli Onay Mektubu (Ornek)

Sayın Veliler, Sevgili Anne-Babalar,

Orta Doğu Teknik Üniversitesi Ortaöğretim Fen ve Matematik Alanları Eğitimi Bölümü olarak "Öğrencilerin Trigonometrik Fonksiyonlar Üzerine Kavram Görüntülerini Geometer's Sketchpad ile Zenginleştirmek: Bir Oğretim Deneyi" başlıklı araştırma projesini yürütmekteyiz. Araştırmamızın amacı öğrencilerin trigonometrik fonksiyonlar üzerine kavram görüntülerini zenginleştirmelerine yardımcı olacak Geometer's Sketchpad dinamik geometri ortamında bir eğitim dizayn etmek, ve daha sonra bu eğitimin öğrencilerin trigonometrik fonksiyonlara dair kavram görüntüleri üzerine etkisini araştırmaktır. Bu amacı gerçekleştirebilmek için çocuklarınızın katılımıyla bir öğretim uygulaması yapmaya ihtiyaç duymaktayız.

Katılmasına izin verdiğiniz takdirde çocuğunuzla birlikte, bir öğrenci, çocuğunuzun matematik öğretmeni ve araştırma görevlisinin bulunacağı bir ortamda haftada yaklaşık bir saatlik bir öğretim uygulaması yapılacaktır. Uygulama okulda ders saatleri dışında yürütülecektir. Çocuğunuzun çalışmaya katılmasının onun psikolojik gelişimine olumsuz etkisi olmayacağından emin olabilirsiniz. Öğretim deneyi uygulama sürecinden elde edilen veriler tamamıyla gizli tutulacak ve bu veriler sadece bilimsel araştırma amacıyla kullanılacaktır. Bu formu imzaladıktan sonra çocuğunuz katılımcılıktan ayrılma hakkına sahiptir. Araştırma sonuçlarının özeti tarafımızdan okula ulaştırılacaktır.

Araştırmayla ilgili sorularınızı aşağıdaki e-posta adresini veya telefon numarasını kullanarak bize yöneltebilirsiniz.

Saygılarımızla,

Araştırma Görevlisi Zülâl ŞAHİN
Ortaöğretim Fen ve Matematik Alanları Eğitimi Bölümü
Orta Doğu Teknik Üniversitesi, Ankara
Tel: (0312) 210 7501
GSM: (0545) 639 0016
e-posta: zulal@metu.edu.tr

Doç. Dr. A. Kürşat ERBAŞ
Ortaöğretim Fen ve Matematik Alanları Eğitimi Bölümü
Orta Doğu Teknik Üniversitesi, Ankara
Tel: (0312) 210 3652
e-posta: erbas@metu.edu.tr

Lütfen çocuğunuzun bu araştırmaya katılması konusundaki tercihinizi aşağıdaki seçeneklerden size en uygun gelenin altına imzanızı atarak belirtiniz ve bu formu çocuğunuzla okula geri gönderiniz.

A) Bu araştırmaya çocuğum ....'nın katılımcı olmasına izin veriyorum.

Baba Adı-Soyadı.................................. Anne Adı-Soyadı......................................

İmza .... İmza ....

B) Bu çalışmaya çocuğum ....'nın katılımcı olmasına izin vermiyorum.

Baba Adı-Soyadı.................................. Anne Adı-Soyadı......................................

İmza .... İmza ....

APPENDIX C

MODELING TASK

Dönme Dolap

İngiltere'nin başkenti Londra'daki "London Eye" ismiyle bilinen dönme dolap Londra'yı kuşbakışı izlemek isteyenler için tavsiye edilmektedir. 1999 yılında inşa edilen ve dünyanın en büyük dönme dolaplarından birisi olan yapı, yıllık 4 milyon civarında ziyaretçisiyle Londra'nın önemli turizm kaynaklarından biri haline gelmiştir. 135 metre yüksekliğindeki bu dönme dolap her biri 25 kişi kapasiteli, içinde insanların rahatça dolaşabileceği genişlikte 32 kapsülden oluşmaktadır. Dönme dolabın bir diğer özelliği de hiç durmadan hareketine devam etmesidir. Yani yolcu indirmek ya da bindirmek için durmayan dolap, insanların yer seviyesinde kapsüllere rahatlıkla inip binebileceği kadar yavaş hareket etmektedir.

Londra'daki bu yapıyı inceleyen ve müşteri potansiyelinden etkilenen bir yatırımcı, benzer bir dönme dolabı İstanbul'da Çamlıca tepesine yapmaya karar veriyor. Çapı 140 metre olması planlanan dönme dolap, yerden yüksekliği 4 metre olan bir platform üzerine kurulacaktır. Dönme dolap üzerine eşit aralıklarla her biri 25 kişi kapasiteli 36 kapsülün yerleştirilmesi düşünülmektedir. Dönme dolabın bir tam turunu tamamlama süresi 30 dakika olarak planlanmaktadır. Kapsüllerin içerisine yerleştirilecek olan elektronik göstergelerde müşteriye anlık olarak aktarılması planlanan bilgiler şunlardır:

- *Yerden yükseklik*,
- *Kapsüle bindikleri noktaya olan uzaklık*,
- *Hız*,
- *Bir tam turun tamamlanmasına ne kadar zaman kaldığı* (bir tam turun bitmesine 1 dakika kala yolcuların iniş hazırlığı için erken uyarı devreye girecektir).

Bu bilgileri anlık hesaplayabilecek yazılımı geliştirecek bilgisayar programcısına yardımcı olmanız istenmektedir. Bu çerçevede, programcıya bu bilgilerin matematiksel olarak nasıl hesaplanabileceği konusunda bir yöntem öneriniz.

CURRICULUM VITAE

PERSONAL INFORMATION

Surname, Name	:	Şahin (Kabalak), Zülal
Nationality	:	Turkish (TC)
Date and Place of Birth	:	25 April 1979, Ankara
Marital Status	:	Married
Phone	:	+90 312 210 75 03
Fax	:	+90 312 210 79 71
Email	:	zulaluz@gmail.com, zulaluz@yahoo.com

EDUCATION

Doctor of Philosophy, Mathematics Education, Middle East Technical University, Ankara, October 2015.

Master of Science, Mathematics (Algebra), Hacettepe University, Ankara, June 2005.

Bachelor of Science, Mathematics Education, Gazi University, Ankara. September 2000.

WORKING EXPERIENCE

Years	Place	Enrollment
2001-2004	Elementary School, Ankara	Mathematics Teacher
2004-2008	Secondary School, Ankara	Mathematics Teacher
2008-2015	METU, Ankara	Research Assistant

PUBLICATIONS

Kabalak, Z. (2005). *Bir radikal halkanın adjoint grubu ve lie halkası üzerine çalışmalar* (Unpublished master thesis). Hacettepe University, Graduate school of Natural and Applied Sciences, Ankara, Turkey.

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