REASONING ABOUT AND GRAPHING THE RELATIONSHIP BETWEEN COVARYING QUANTITIES: THE CASE OF HIGH SCHOOL STUDENTS AND PROSPECTIVE MATHEMATICS TEACHERS

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I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

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ABSTRACT

REASONING ABOUT AND GRAPHING THE RELATIONSHIP BETWEEN COVARYING QUANTITIES: THE CASE OF HIGH SCHOOL STUDENTS AND PROSPECTIVE MATHEMATICS TEACHERS

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The purpose of this study is to investigate high school students' and prospective mathematics teachers' graphing, their reasoning and the relation between their graphing and reasoning in the context of a modeling task that requires graphing and covariational reasoning. This study is conducted within a larger project designed to develop in-service and prospective mathematics teachers' knowledge and skills about modeling and using modeling in mathematics education.

Qualitative method is used in the study and high school students and prospective mathematics teachers are treated as two cases. 24 prospective mathematics teachers and 107 10^{th} and 11^{th} grade level high school students participated in the study. Data for the study is collected through worksheets, and audio and video recordings. Qualitative data analysis methods are used to analyze the data. Analysis of data revealed that students' graphs can be categorized into four groups: smooth, smooth chunk, uniform chunks, non-uniform chunks; and their covariational reasoning related to rate of change can be categorized into three groups: i) using extensive quantities, ii) creating intensive quantity-comparing intensities, iii) creating intensive quantity-consider variation in intensity. While, the prospective mathematics teachers constructed graphs in

smooth or smooth chunks and considered variance in the intensity, the high school students rarely drew smooth graphs and usually constructed graphs in smooth chunks, uniform chunks and non-uniform chunks. Furthermore, the high school students considered variance of intensities, compared intensities, and extensive quantities in their reasoning.

There exists a consistency between participants' sketches of graphs and their reasoning about rate of change. The students who constructed smooth graphs considered variation in intensities with a global approach. All the students who drew smooth graphs took slope of the graph into consideration. Students who sketched graphs in smooth chunk considered variation in intensity, and compared intensities with a more local approach. Students who drew chunky graphs used extensive quantities with a local approach. However, some students who drew non-uniform chunks changed the slope of the graph to represent variation similar to students who drew smooth graphs. Associating students' sketches of graphs with their reasoning provided us a further insight into how students interpret a covariational situation; and how students' understanding of covariation can be deduced from their graphing.

Keywords: Constructing graph, covariational reasoning, rate of change, students, prospective mathematics teachers

ÖZ

KOVARYASYONEL OLARAK DEĞİŞEN NİCELİKLER ARASINDAKİ İLİŞKİ HAKKINDA AKIL YÜRÜTME VE GRAFİK ÇİZME: LİSE ÖĞRENCİLERİ VE MATEMATİK ÖĞRETMEN ADAYLARI ÖRNEĞİ

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Bu çalışmanın amacı lise öğrencileri ve matematik öğretmen adaylarının kovaryasonel düşünme gerektiren bir modelleme sorusu kapsamında grafik çizimleri, akıl yürütmeleri ve grafik çizimleri ile akıl yürütme yolları arasındaki ilişkiyi incelemektir. Bu çalışma, hizmet-içi matematik öğretmenleri ve matematik öğretmen adaylarının modelleme ve modellemenin matematik eğitiminde kullanımı hakkında bilgi ve becerilerini geliştirmek için tasarlanan kapsamlı bir proje dâhilinde yürütülmüştür.

Bu çalışmada nitel araştırma yöntemleri kullanılmıştır ve lise öğrencileri ile matematik öğretmen adayları iki örnek durum olarak ele alınmıştır. Bu çalışmaya 24 matematik öğretmen adayı ve 107 10. ve 11. sınıf seviyesinde lise öğrencisi katılmıştır. Çalışmada veriler çalışma kâğıtları, ses ve video kayıtları yardımıyla toplanmıştır. Verileri analiz etmek için nitel veri analizi yöntemleri kullanılmıştır. Veri analizlerine göre öğrencilerin grafikleri dört grupta sınıflandırılabilir: düzgün, düzgün parçalı, düzenli parçalı, düzensiz parçalı. Öğrencilerin değişim oranı ile ilgili kovaryasyonel akıl yürütmeleri ise üç grupta sınıflandırılabilir: i) ölçülebilir nicelik kullanma, ii) dolaylı nicelik oluşturma-değişimi karşılaştırma, iii) dolaylı nicelik oluşturma-değişim varyansını dikkate

alma. Matematik öğretmen adayları düzgün ve düzgün parçalı grafikler çizip değişim varyansını dikkate alırken, lise öğrencileri nadiren düzgün grafik çizmişlerdir ve genellikle düzgün parçalı, düzenli parçalı ve düzensiz parçalı grafikler oluşturmuşlardır. Ayrıca lise öğrencileri akıl yürütürken değişim varyansını dikkate almış, değişimleri karşılaştırmış ve ölçülebilir nicelik kullanmışlardır.

Katılımcıların grafik çizimleri ile değişim oranı ile ilgili akıl yürütmeleri arasında bir tutarlılık bulunmuştur. Düzgün grafik çizen öğrenciler daha genel bir yaklaşım sergileyerek değişim varyansını ve grafiğin eğimini dikkate almışlardır. Düzgün parçalı grafik çizen öğrenciler daha yerel bir yaklaşımla değişim varyansını dikkate almışlarıdır ve değişimleri karşılaştırmışlardır. Parçalı grafik çizen öğrenciler yerel yaklaşımla ölçülebilir nicelik kullanmışlardır. Ancak, düzensiz parçalı grafik çizen bazı öğrenciler değişimi açıklamak için, düzgün grafik çizen öğrencilerin yaklaşımına benzer bir şekilde grafiğin eğimini değiştirmişlerdir. Öğrencilerin grafik çizimlerini ve akıl yürütmelerini ilişkilendirmek, onların kovaryosyanel durumları yorumlayabilme ve kovaryasyonel düşünmelerini grafik çizimlerinden çıkartabilme konularında bize geniş bilgi sağlamıştır.

Anahtar kelimeler: Grafik Oluşturma, Kovaryasyonel Düşünme, Değişim Oranı, Öğrenciler, Matematik Öğretmen Adayları

To my daughter

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CHAPTER 1

INTRODUCTION

Education does not serve its purpose unless students can transform what they learn to their lives. To adapt students to their working life following their education, modeling problems and activities gained importance in mathematics and science education (Dijk, Oers, & Terwel, 2003; Lesh & Doerr, 2003; Lesh & Fennewald, 2010; Orfanos 2010)

One of the most common subjects we deal in real life is covariation. Covariation is a major concept in understanding the relationship between two variables and interpreting them depending on the other (Fitzallen, 2012). Covariation helps us explain the relations and patterns in a phenomenon, which mathematics, in real-life, intends. Understanding covariation is one of the most difficult and problematic issues for high school students (Fitzallen, 2012). Modeling real life situations gives students a chance to realize the relationship between covarying variables as they know the relationship between the variables in daily life situations (Carlson, Larsen, & Lesh, 2003). Dealing with real life problems enhances students' understanding of important subjects of mathematics with the familiarity of the topic (Leinhardt, Zaslavsky, & Stein, 1990).

Graphs are one of the most frequent ways to model covariation (Fitzallen, 2012). Thus, its importance in mathematics education and work life is preserved. From statistics to science, medicine to engineering, graphs are used to summarize and manipulate the data or variables and to investigate relations in a compatible and communicable manner. Students' use of graphs in representing the relation of real-life phenomena can give clues about their understanding of covariation.

Students meet graphing in mathematics education in early ages with bar or figure graphs of representing data in elementary school in Turkey. Cartesian graph and statistics graphs expanding to the linear graphs of relations are placed in middle school levels. At the secondary level, students learn to relate sets with functions and graphing of functions, so covariation gains importance at this level. Considering the educational research on secondary school level, there have been many studies conducted to investigate students understanding of graphs of functions and relations. Most of them investigated the students' interpretation of graphs (Araujo, Veit, & Moreira, 2008; Bektaşlı & White, 2012; Fitzallen, 2012; Glazer, 2011; Nathan & Bieda, 2006; Perez-Goytia, Dominuguez, Zavala, Singh, Sabella, & Rebello, 2010; Tebabal & Kahssay, 2011;), while some of them gave space to students' construction of graphs from a table or an algebraic equation (Berry & Nyman, 2003; Even, 1998; Hattikudur et al., 2012; Tairab & Al-Naqbi, 2004). Construction of graphs of real-life related situations, which requires a higher level reasoning (Leinhardt et al., 1990), takes place in modeling or covariational understanding studies (Araujo et al., 2008; Köklü & Jakubowski, 2010; Maverech & Kramarsky, 1997; Şen-Zeytun, Çetinkaya, & Erbaş, 2010). These studies focus on the problems of students in modeling activities and how they use covariational reasoning. One of them (Castillo-Garsow, Johnson, & Moore 2013) just focuses on the shapes of students' graphing by associating it to the students' type of thinking. Castillo-Garsow and others (2013) indicate that students who reason covariation in discrete, measurable intervals graph covariation in chunky style, whereas students who consider covariation in a continuous manner, graph covariation smoothly. In her research, Johnson (2011) reports a student that uses both types and graph in "smooth chunk" approach. Students may think smoothly in intervals/chunks which they determine. Kertil (2014) also reported that students may graph transition points of these intervals inconsistent with the rate of change of the covariational situation.

Moreover, Johnson (2011) provides deep information about how students think and what types of reasoning they use while dealing with covariation with her research studies. Ratio based and not ratio based reasoning focus on covariational reasoning types (Johnson, 2011). Some studies on covariational understanding (Koklu & Jakubowski, 2010; Şen-Zeytun et al., 2010; Yemen-Karpuzcu, Ulusoy, & Işıksal-Bostan, 2015) use Carlson's (Carlson et al., 2002; Carlson et al., 2003) covariational understanding steps which classify students covariational understanding regarding the ratio over change in variables and scaling down the intervals of change to focus on the change on initial point. Carlson et al. (2003) reported that students may use cross-sectional area of the bottle which is not a ratio based reasoning way at the classical bottle problem that relates the change in one variable directly to another variable or situation, which does not fit with the covariational steps she defined. Thus Carlson et al. (2003) called for a revision of these steps. Johnson (2011) also proposed that while dealing with covariation, some students use not ratio-based reasoning such as tracing the change in one of the variables depending on the other, rather than using ratio-based reasoning that is considering the change in the ratio of variables. Johnson (2012) investigated how students may reason about rate of change regarding their reliance on the use of variables. There are three categories she reported; using extensive quantities, creating and comparing intensities of variables, or creating intensities and considering the variance in the intensity. While using extensive quantities and comparing intensities of variables depend on ratio reasoning, considering the variance in the intensity of one variable includes a not ratio-based way of the reasoning about the rate of change.

In interpretation and construction of graphs, we face other facts effecting students' reasoning types such as local-global approaches and quantitative-qualitative types of reasoning. Students may approach functions locally, such as by giving importance to certain points or intervals; or they may be able to see and use more global aspects, such as general trend of the graph, while they interpret or construct graphs. Not necessarily, but generally global aspects are related to qualitative reasoning, whereas local properties are handled by quantitative methods (Leinhardt et al., 1990). In Johnson's (2012) categorization of the extensive quantity measuring, students' use of quantitative operations to explain the change in variables is included. Moreover, Castillo-Garsow and others (2013) mention student' chunky graphing in discrete and countable intervals. In general, global approach or reasoning qualitatively requires a higher level of understanding than local approach and quantitative reasoning. Students have more difficulty in interpreting or constructing graphs with a global point of view and they have problems in tasks that require qualitative reasoning of the graphs (Leinhardt et al., 1990).

1.1.Purpose and research questions

It is stated in the literature that there are students who graph covariation smoothly or in chunks. Also students may use ratio or non-ratio based reasoning in different ways and at different levels for a given covariational situation. However, an extensive review of related literature shows no studies that investigate both students' graphing and their reasoning and connections between them. Furthermore, smooth chunk graphing and inappropriate transition points in graphs are reported in the literature but not studied deeply. There is a need to study the relation between reasoning and graphing of students in a deeper qualitative research to explore reasoning behind students' graphing. In addition, previous studies only focus on students who have not taken calculus or students who have already taken calculus (Carlson et al., 2002; Johnson, 2011; Köklü & Jakubowski, 2010; Şen-Zeytun et al., 2010; Yemen-Karpuzcu et al., 2015). Studying both groups at the same time will provide a wider range of perspectives into students understanding of covariational reasoning.

Thus, the purpose of this study is to investigate high school students' and prospective mathematics teachers' sketches of graphs for covariational situations, their reasoning about covariation and the relation between their graphing and reasoning while they are working on a modeling task, by qualitative analysis methods. Below are the research questions that guide this study.

1. How do high school students and prospective mathematics teachers draw graphs showing the relationship between covarying quantities in the context of a task that involves covariational situation?
2. How do high school students and prospective mathematics teachers reason about covarying quantities in the context of a task that involves covariational situation?
3. How are the students' sketches of the graphs related to their reasoning about covarying quantities?

1.2.Significance

Besides the importance of graphical understanding in education as a necessity in daily life, investigating students' understanding and constructing graphs have essentiality in educational research. Covariational reasoning that is to explain the relation between variables that change dependently is a major object in education due to its reflection in real life. Students' understanding of both covariation and graphs may be reflected in their sketches of graphs and can be explained through them. As seen in literature, some students may graph covariation smoothly whereas others may think in chunks where two variables change simultaneously and continuously. Also, there exists smooth chunk thinking style which may sometimes cause inconsistent graphing (Johnson, 2012; Kertil, 2014). Kertil (2014) suggests warning students about the rate of change before and after transition points to make them draw smoother. A more detailed qualitative study that focuses on students graphing and reasoning behind it will give researchers and teachers a chance to find a more naturalistic/ intuitive way to improve students' graphing and reasoning. Moreover, detecting deficiencies in reasoning behind improper graphing will give clues on how to improve students' graphing.

By investigating these types of graphing together with students' ways of thinking, a deeper explanation can be given for why some students can graph covariation smoothly. Different approaches to reasoning about the rate of change or varieties in the use of concepts may affect their success. Furthermore, we can see why students who draw chunky graphs cannot think smoothly, and which points they miss when compared to smooth thinking students. This study can especially help us to conceive how the reasoning of students who draw smooth chunk graphs differs from that of students who draw smooth graphs. Hence, we can get clues to improve students' thinking from chunky to smooth or smooth chunk to smooth.

Following Carlson et al. (2002), studies that investigate students' understanding level of the covariation mostly use Carlson's steps of covariational understanding levels (e.g., Köklü & Jakubowski, 2010; Şen-Zeytun et al., 2010). Carlson et al. (2003) explained that these steps do not fully explain the covariational understanding of students, thus these steps should be revised. Contrary to other studies, this study does not depend on Carlson's steps of covariational reasoning

(Carlson et al. 2002, Carlson et al. 2003). This study approaches covariational understanding from the graphing point of view and tries to describe students graphing covariation with their rate of change reasoning or other ways of thinking by analyzing students' works descriptively. It is not aimed to create a framework for leveling students' covariational reasoning. However, connecting students graphing with their reasoning will help us to understand their needs in interpreting covariational situations. If there is a consistency or a meaningful pattern among their reasoning, this may help us to classify students reasoning by looking at their graphing.

Some of the studies focused on the importance of investigating early attempts of understanding calculus concepts. Investigating early learning will help understanding students' conceptualizations, intuitions and initiations of the understanding and how it improves. This will enhance the organization of instruction and especially the starting points of the concepts. Studies including high school students show that students can covariate variables and reason with rate of change in tasks that ask the relation between simultaneously and dependently changing variables even before students receive calculus education (Fitzallen, 2012; Johnson, 2011; Köklü & Jakubowski, 2010; Şen-Zeytun et al., 2010). While the previous studies focus on either covariational understanding of secondary students (e.g., Fitzallen, 2012; Johnson, 2011), or university students (e.g., Köklü & Jakubowski, 2010) this study investigates both high school students' and prospective teachers' (university students) reasoning about and graphing of the relationship between covarying quantities.

Mathematical education of students begins with the education of their teachers. The importance of teacher education and studying the conceptualization of prospective teachers to improve their understanding of the topics is unquestionable as teachers transfer their understanding to their students. Investigating how prospective teachers graph and reason in covariational situations can give clues to improve teacher education in covariational topic. However high school students' needs in this topic may differ from prospective teachers. Investigating both high school and prospective teachers graphing and reasoning in covariational situations will provide initial information about improving instruction for both levels. Moreover, investigation on both groups can provide a wider range of perspectives on graphing and reasoning.

As a result, varieties in constructing the graphs depicting the relationship between covarying quantities can be documented better with a broader sample with different levels of mathematics background. High school students at 10^{th} and 11^{th} grade levels do not have formal knowledge about derivative, and how change in the graph is represented by the change in rate. At the 12^{th} grade they meet properties of the change in graphs by maxima points and concavity related to derivative concepts. The other group, prospective mathematics teachers, have calculus education and experience on working with curves more than high school students. Students' intuitions, capabilities, and needs can be different before they take this relevant knowledge on curves, and after sometime experience with calculus concepts. Thus investigating on these two groups will provide a broader sample to observe students' understanding of covariation.

Briefly, this study is unique because it investigates students' sketches of graphs together with their covariational reasoning. Describing students' different ways of thinking related to graphing types may help to develop a way to improve students' covariational reasoning. Moreover, this study has a sample including high school students and prospective mathematics teachers. Prospective mathematics teachers' understanding of covariational situations is important as they have the potential to convey their understanding to their students in the future. Investigating high school students' and their prospective teachers' covariational reasoning is important for the efforts to improve covariational reasoning of at these two levels. Moreover, studying two groups can lead to a broader view related to the students' graphing and reasoning, as high school group represents a sample which has not had any instruction on graphing of 2^{nd} degree curves and prospective mathematics teachers have more experience with these types of graphs. At last, this study also differs from the others by approaching covariational reasoning from the graphing perspective. By using qualitative methods with an inductive perspective, and by focusing on graphing, new knowledge about how students covariate variables and visualize covariation by graphs can be provided.

1.3. Definitions of important terms

Smooth graphing

Continuous and smooth curve with smooth transitions (Kertil, 2014) over the points where intensity of change in dependent variable changes from one type to another (e.g. from decreasingly increasing to constant increase)

Smooth chunks graphing

Smooth curves of variation for chunks that are determined considering the differences in the intensity of change, with sharp transitions where curves are joint. (Kertil, 2014).

Uniform chunky graphing

Graph is consistent of linear several line segments joint together, where each interval/chunk seems to be equal in length.

Non-uniform chunky graphing

Graph consists of linear chunks/line segments joint together, where chunks are not equal in length.

Examples for each type of graphing representing the height of the water depending on the amount of the water filled in a given tank is provided in the Figure 1.

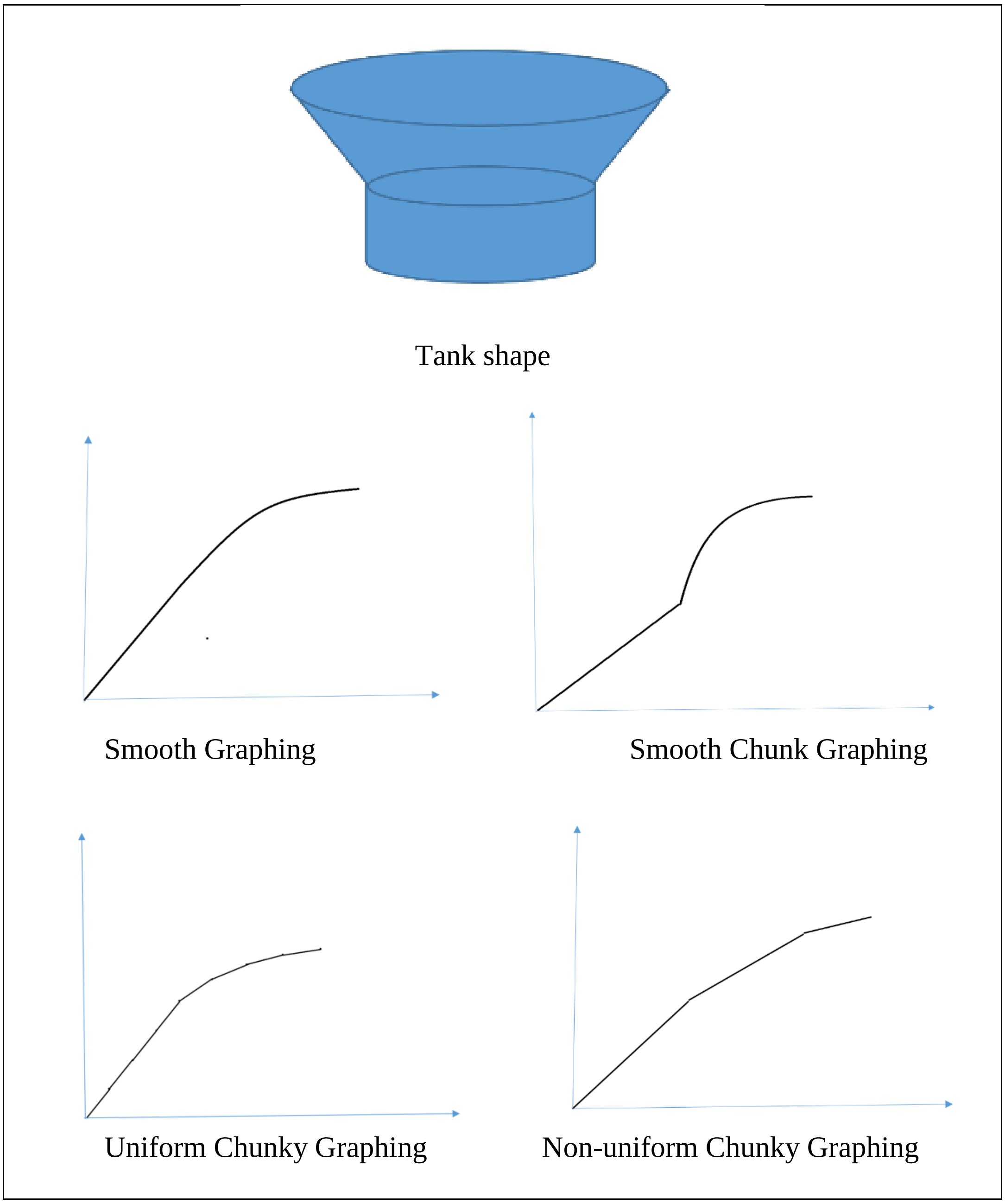


Figure 1. Examples of types of graphing for a given tank: smooth, smooth chunk, uniform chunky and non-uniform chunky

CHAPTER 2

LITERATURE REVIEW

This chapter includes review of literature related to this study. Firstly, literature on graphing in mathematics education is presented. As a subtitle of graphing, construction of graphs is summarized. Secondly, covariational reasoning studies are mentioned followed by a subheading of graphing covariation. At last, reasoning in graphing covariation/ functions is handled and under this title rate of change reasoning in covariation is also examined as a subtitle.

2.1. Graphing

Due to its unquestionable importance in daily life and mathematical development, graphical understanding is highly valued in education field. However, dealing with graphics is not extremely easy for students since it is a new form of interpreting mathematical information that they face with coordinate systems after grades 6 or 7 in Turkey. It is a way of picturing concrete information as abstract models. Moreover, interrelating graphs to graphs or other abstract mathematical objects like equations requires reasoning from abstracts to abstracts which makes it a difficult process for students (Leinhardt et al., 1990).

In its nature dealing with graphs includes reading, interpreting, transforming (Leinhardt et. al., 1990), reasoning, and constructing. Constructing seems most complicated and difficult one compared to others (Leinhardt et al., 1990; Sezgin, 2013; Tairab & Al-Naqbi 2004); however, students may still have difficulties in interpreting and more problematically in transforming and reasoning (Baştürk, 2010; Bayazıt, 2011; Leinhardt et al., 1990; Tekin, Konyalıoğlu, & Işık, 2009).

In educational studies, interpreting graphs takes place more (Araujo et al., 2008; Bektaşlı & White, 2012; Deniz & Dulger, 2012; Fitzallen, 2012; Glazer, 2011; Leinhardt et al., 1990; Mitnik, 2009; Tebabal & Kahssay, 2011). Then, transforming among graphs or transforming between algebraic equations and graphs follows that (Baştürk, 2010; Bayazıt, 2011; Tekin et. al., 2009).

Even though reading and interpreting a graph seem easier for students (Bayazıt, 2011), the problems dealing with qualitative and global aspects of graphs such as commenting on how graph would change if there exist a change on independent variable is difficult to students (Bayazıt, 2011; Tairab & Al-Naqbi, 2004). In addition to this problem, scaling is another issue that students have difficulties in interpretation (Bayazıt, 2011). Studies are conducted to improve students' abilities to interpret graphs by technology enhanced education which shows important success (Berry & Nyman, 2003; Deniz & Dulger, 2012; Mitnik, 2009; Wu & Wong, 2007).

2.1.1. Constructing graphs

Construction seems more difficult compared to interpretation and transformation whereas depending on the nature of the question asked to the students, construction is not necessarily difficult compared to interpretation or others all the time (Leinhardt et al., 1990). However, construction includes a more complex process than interpretation. Leinhardt et al. (1990) explain the complexity of construction as construction includes interpretation in itself but interpretation does not include any construction, thus no difficulty of construction.

Studies that directly focus on constructing graphs is much less, compared to others such as interpretation and transformation (Leinhardt et al., 1990; Şen-Zeytun et al., 2010; Tekin et al., 2009). Presence of constructing graphs in Turkish mathematics curricula or practice is questionable, too (Tekin et al., 2009).

While graphing from a data table can be done successfully by students (Sezgin-Memnun, 2013; Tekin et al., 2009) transforming algebraic or verbal functions to graphs may bring difficulties within (Baştürk, 2010; Bayazıt, 2011; Thomas, 2010). Most well-known problem among them is the tendency to graph even parabolic, logarithmic, trigonometric, or other curved functions linearly (Hadjidemetriou & Williams, 2002; Hadjidemetriou & Williams, 2010; Lovell, 1971; Leinhardt et al., 1990; Maverech 1997; Tekin et al., 2009). There are other problems that students have while graphing from algebraic or verbal expressions of functions such as scaling (Friel, Curcio, & Bright, 2001; Bayazıt 2011), mixing axes (Carlson et al., 2002, Şen-Zeytun et al., 2010), placing values on x and y axes oppositely (Tekin 2009), mixing up different types of kinematics graphs (Demirci & Uyanık, 2009), graphing entire function as one point, conserving increase in graph (Maverech, 1997).

While graphing real-life situations, students may mix the roles of dependent and independent variables (Yemen-Karpuzcu et al., 2015 Another typical difficulty observed in graphing daily life situations is having to do with picture like graphing (Leinhardt et al., 1990; Maverech 1997). Students may think the shape of the distance-time graphs as the road the vehicle moves on (Demirci & Uyanık, 2009).

Graphing a derivative graph of a function graph or converting between derivative and original graphs is another very complicated and difficult task to improve students' capability on (Asiala, Cottrill, & Dubinsky, 1997; Orhun, 2012; Ubuz, 2007).

2.2. Covariational understanding

Not only graphing data, or turning tables or equations to graphs is in use, but also simulation of daily life, constructing graphs from real life situations is one of the most important issues to meet the demands of modern life. New studies are interested in students' covariational understanding. Covariational reasoning is "cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other." (Carlson et.al. 2002, p.466). According to Johnson (2011, p. 2141) "If a student were reasoning about rate of change as a relationship between varying quantities, then the student would also be reasoning covariation ally.". Graphing is considered as one of the most essential ways to interpret covariation in mathematics (Leinhardt et al., 1990). Graphing daily life situations is very difficult for students as they need to move from very concrete life situations to very abstract mathematical interpretations. It is not like graphing done by moving from abstract tables, formulas, or equations to abstract lines.

Covariational studies mostly focus on students' level of how they can covariate variables that include awareness of rate of change in relation with these variables (Köklü & Jakubowski, 2010, Şen-Zeytun, et al., 2010) by using Carlson and others' steps listed below:

Categories of Mental Actions (MA)

MAI) An image of two variables changing simultaneously;

MA2) A loosely coordinated image of how the variables are changing with respect to each other (e.g., increasing, decreasing);

MA3) An image of an amount of change of the output variable while considering changes in fixed amounts of the function's domain;

MA4) An image of rate/slope for contiguous intervals of the function's domain;

MAS) An image of continuously changing rate over the entire domain;

MA6) An image of increasing and decreasing rate over the entire domain.

(Carlson et al., 2003, p. 467)

That is not what happened when these steps were tested with the bottle problem; students preferred to look at the change in height depending on the cross-sectional area of the bottle rather depending on equal amounts of water or considering rate over / ratio of variables. (Carlson et al., 2003). Students do not consider the rate over variables or rate of change at a specific time. They relate the change in one variable to the width, to the area of the disk which is horizontal cross-section of the bottle. Also Johnson's (2011; 2012) covariational studies, which investigate how students use rate of change to interpret covariation, prove the fact that students prefer to trace the change in height with a not ratio-based reasoning.

2.2.1. Graphing covariation

Among covariational studies one of them mainly focuses on the types of students graphing (Castillo-Garsow et al., 2013) and associates them with their thinking. According to Castillo-Garsow and others (2013) there are two types of graphing when students interpret covariation of two variables with a graph; smooth graphing and chunky graphing. Smooth graphing is described as the interpretation of relationship between covarying variables with smooth and non-segmented (and if necessary curved) graphs. In smooth thinking student graphs the covariation is in a continuous process. Chunky graphing is composed of discrete parts of change in covarying variables (Castillo-Garsow et al., 2013; Kertil, 2014). Castillo-Garsow and others (2013) explain chunky and smooth images as chunky images of change are based in countable and completed amounts whereas smooth images of change are based in imagining a continually changing experience.

Castillo-Garsow and others (2013) exemplify chunky and smooth thinking by the classical bottle problem of Swan & Shell Centre Team, 1999 which asks students to draw the height of the water that is dispensed in a bottle, depending on the amount/volume of the water in the bottle. If student sections the bottle into heights and assign volumes for each height, this chunky way of thinking results in graphing which is composed of heights and corresponding volumes. Vice versa is also possible. Student can think in each time an amount of water is added to the bottle (like it was poured a cup in each time) and calculate or guess height of water for each time. As a result, graph is formed by considering change in "discrete chunks". In smooth thinking student can think volume and height change at the same time and interpret it with a continuous increase like the water is coming from a hose.

Castillo-Garsow and others (2013) argue that students' graphing (chunky or smooth) can tell about how they conceptualize variation and how they reason covariation. Castillo-Garsow and others (2013) also claim that thinking independent variable in chunks and calculating dependent variable from independent variable "is not necessarily covariational". Students just determine some points and calculate or estimate variables and then draw the graph by joining these points. In this way student thinks function as correspondence instead of covariation/relation (Şen-Zeytun et al., 2010). Another thing Castillo-Garsow and others (2013) point is no matter how small the student takes the chunks, student still thinks in chunks and cannot move to smooth thinking. However, Castillo-Garsow et al. (2013) give examples of how a student that uses smooth thinking can also advance chunks. For the example of bottle problem Castillo-Garsow and others (2013) point Hannah from Johnson's (2011) research during which Hannah segments graph in chunks/parts considering the points where graph shows shift in different types of rate of change such as from decreasingly increasing to increasingly increasing. However, in these chunks Hannah thinks smoothly and her graphing is smooth.

Another example of Castillo-Garsow and others (2013) is that drawing sine function considering the points $\frac{\pi}{2}, \pi, \frac{3\pi}{2} \ldots$ etc. and filling out these intervals smoothly. In this example it is clear that choosing the number of chunks is important since if the intervals are too big, drawing the smooth parts appropriately cannot be manageable.

Castillo-Garsow and others (2013) conclude that smooth thinking is more powerful than chunky thinking as students that use smooth thinking do not have difficulties that students using chunky thinking face (when dealing with covariation) and smooth thinking also includes the ability to think chunky. He suggests beginning analysis with smooth images of change to ease students' understanding limit, differentiation and integration by pointing the development of calculus from smooth Oresmo's and Leibniz's instantaneous rate of change to chunky (epsilon-delta) approach by modern analysis

As mentioned above, in Johnson's (2012) article about reasoning variation, Hannah uses "smooth chunk" type of thinking to graph covariation. In her study, Johnson uses the classical bottle problem in a reverse way to observe students' graphing of covariation. She gives the graph of height versus volume and asks the shape of the bottle to eliminate considering time as independent variable in covariation activities. The term "smooth chunk" is used in the article to describe Hannah's way of thinking of covariation because Hannah distributes the graph of $h\ vs\ v$ in parts/chunks where the graph shifts from one type of change to another such as from increasingly increasing to decreasingly increasing. Besides, in these chunks, Hannah thinks smoothly and draws the shape of the bottle smoothly connected on the points of change.

Castillo-Garsow and others (2013) imply that smooth reasoning can also include chunky reasoning, "at least one chunk". If non-ratio reasoning is related to smooth thinking and ratio reasoning to chunky thinking, it can be concluded that non-ratio reasoning abilities complement ratio reasoning abilities. Thus, students that use non-ratio reasoning could alter ratio approach of calculus with epsilon-delta.

In his doctoral dissertation Kertil (2014) classifies students' graphs in terms of transition at the points where change is changing as transition with sharp corners and transition with "nearly" smooth corners. By transition with sharp corners he means the points where different parts of graphs such as linear and parabolic or different parabolas join together seem like a corner which is non-differentiable.

Sharp transitions are not mentioned in Carlson's (2002, 2003) and Johnson's (2012) studies. In the examples they provide students are drawing just smooth graphs. In testing of Carlson's covariational steps all of the six students graphed smoothly (Carlson et al., 2003). One of the reasons of this seems that students use smooth reasoning, also it is likely that they use chunks just to determine turning points. When Carlson and others. (2002) asked the student why she drew a smooth curve she replied "I imagined the slope changing as the water was pouring at a steady rate" (p. 365). In interviews, when researchers questioned students, none of them could provide a direct explanation for their smooth drawing; even some of them just say they don't know. However, it is clear that all of these students use slope or steepness of the curve through their reasoning. In addition to the excerpt given above stating that the student "imagined changing the slopes in her drawing" the following quotations also indicate that students consider slopes and steepness concepts in the graphs.

> "It is going to be filling rapidly, so you are going to have greater slope" (Carlson et al. 2002 p.366)

> "The greater the height the steeper the graph will be." (Carlson et al. 2003, p. 475)

In Johnson's (2012) example of Hannah it is documented that she draws the shape of the bottle from the height vs volume graph by using "smooth chunk" thinking. At the end the bottle needs not to be smooth at all. On the other hand, Kertil's (2014) way of asking the bottle problem gives the chance to observe "smooth chunk" graphing with sharp transitions. In the problem which is provided by Kertil (2014) there are several types of water tanks and students (pre-service teachers) are asked to draw height vs volume graph when the tanks are filled with water. For the graphing of some tanks students' graphs included transitions from parabolic parts to linear parts. In Kertil's (2014) way of asking the problem, drawing transition points sharply seems frequent, which is not observed in other studies (Carlson et al., 2003; Johnson 2011).

Kertil (2014) asks students about rate of change before and after these points to make them recognize that graph should be smoother at these points. However, what about the students who are already drawing the graphs smoothly? What is the difference between thinking of the students who draw smooth graphs and the students who draw “smooth chunk” graphs? The difference between the thinking of these two groups of students lies on their reasoning. If what type of reasoning students began, what type of concepts they use, and with what type of approaches they come up to these types of reasoning are determined, it can be clarified what a prerequisite is to graph smoothly and what is needed to improve smooth chunk graphing to smooth graphing. Hence the following section is devoted to the literature on approaches and reasoning types that might affect students’ graphing.

2.3. Reasoning and approaches in constructing graphs of functions

Students may approach construction and as well as interpretation of functions globally or locally. While approaching locally, students focus on certain points or intervals. For the construction of graphs, sketching some points for calculated amounts of an algebraic function can be an example for focusing locally which is the way students tend to approach functions. Nevertheless, global approach is more complicated and difficult for students. Sketching the graph by considering more general aspects of the function, such as increase or growth of a phenomenon, is an example for global approach. Yet, explaining or using general/global properties of the function or its graph requires a much wider perspective (Leinhardt et al., 1990).

While interpreting or graphing a function or a covariation between variables students may reason quantitatively or qualitatively (Leinhardt et al., 1990; Johnson, 2012). In construction students may use quantitative variables to complete a graph, and depend on calculations and numerical results to defend their graphing. Conversely, they can reason with qualitative aspects of the graph; such as its shape, slope, or trend.

Not necessarily but usually, quantitative reasoning is related to focusing on local aspects. Similarly, qualitative reasoning seems mostly in the situations where students approach more globally (Leinhardt et al., 1990). In quantitative reasoning, focus is on mostly where specific points are placed and where they correspond on the axes which reflect a local focus. On quantitative reasoning, however, the focus is not on the axes or labeling the points but the focus is on the shape of the graph which usually does not need to focus on limited (specifies of) points, and requires a more global perspective (Leinhardt et al., 1990).

2.3.1. Reasoning about rate of change in covariation

In graphing covariation or particularly in classical bottle problem students may employ quantitative reasoning by applying certain numerals to graph the relation between variables, or they may approach more globally and qualitatively by determining change in the relation to reflect it on the graph. To define how variables change with respect to each other, students may look at the ratio between them, which is called ratio reasoning. On the other hand, Johnson (2012) explains that students may also use not ratio based reasoning while they work in covariation situations. This is a fact that exceeds the boundaries of Carlson’s covariational steps (Carlson et al. 2002, Carlson et.al. 2003) that highly depends on reasoning about rate of change in covariation.

In a study, Johnson (2011) explains how secondary school students use variation in rate of change in a reversed version of classical bottle problem. She defines covariation as “reasoning about rate of change as a relationship between varying quantities” (p. 2141). Also covariational reasoning involves continuously varying one quantity and examining the change in the related quantity. She exemplifies three different types of quantification of variation in rate of change.

The first and the least complex (and least improved) type of reasoning is associating extensive quantities. In this type to reason variation student just uses extensive quantities that are quantities that can be directly measured (Schwartz 1988 as cited in Johnson, p. 2141), say numerically. In the bottle problem, student just focuses on how much height and how much volume increases and graphs just basing on these numerical amounts. This type of reasoning depending on numerical calculation does not seem to be supporting covariational reasoning (Castillo-Garsow et al., 2013; Johnson, 2011).

The second type of reasoning is “constructing an intensive quantity” with comparing intensities. Intensive quantities can be measured indirectly. In bottle problem situation, student focuses on the variation of change of intensive quantities that are volume and height and compares the intensities of increase of these variables. In the example that Johnson provides, student thinks if the increase of the volume and the height will occur with equivalent intense, then the graph will pass the points on the line $y = x$. However, while graphing, student do not focus on the intensity of change of one variable as it is dependent on the other. There is no thinking of covariation or it is poor. Student just compares the intensities of the change of variables at the same time or according to time.

On the third type of reasoning of variation in rate of change Johnson (2012) gives the example of Hannah who constructed an intensive quantity with also considering variation in intensity. Hannah focuses on how the change in one variable changes; for example, increase is decreasing.

Johnson (2012) states that Hannah’s focusing on the rate of change does not include any ratio reasoning such as looking for the ratio between height and volume or comparing these two quantities. By quantity Johnson means “attributes of objects that can be measured” (Thompson, 1993, 1994b as cited in Johnson 2011, p.314). Johnson (2011) states that students may use rate of change numerically or non-numerically. Moreover, they use non-ratio reasoning while dealing with covariation. Johnson (2011) also points out non-ratio reasoning is more useful to understand derivative, which is a central topic of calculus but ratio reasoning is essential for the topics ratio, limit and function that are pre-requisite for derivative of modern analysis which is an epsilon-delta approach to calculus.

Johnson (2012) reports Hannah is looking for the difference of difference from data tables of covarying quantities. To determine how area of a square changes by increasing the length side, she looks for the difference in area for each length in the table. Then she recognizes a pattern that difference of the areas increases by 0.5 at each time. She interprets this as “difference of difference”. She tries to explain that the area does not increase at a constant rate when length increases, in each time the difference increases by 0.5. Hannah did not consider the change only by creating intensive quantity. She associated extensive quantities to look for how change is changing. However, her attention stayed at quantitative meaning of the change. She couldn't interpret what this change represents in terms of the variables.

As mentioned before Hannah considered variance in the intensity with a non-ratio based reasoning but resulting in a smooth chunk type of thinking. It was not the case in Hannah, but smooth chunk thinking may cause graphing with sharp transitions as seen in Kertil's (2014) study. What is the difference in students' reasoning for graphing smoothly or in smooth chunks? Is there a relation between their reasoning about rate of change and their type of sketches? What else effects their graphing? Investigating how high school students and prospective teachers think in specific types of graphing may give clues about how their graphing covariation and understanding of covariation can be improved.

CHAPTER 3

METHOD

This chapter explains how the data is gathered and analyzed to investigate the types of graphs that high school and prospective mathematics teachers draw for representing covariational situations, and to explore the students' reasoning related to rate of change.

Qualitative methods are employed in this study to observe and analyze students understanding in a natural context (Creswell, 2003). With a qualitative research methodology, it is aimed to conclude on students' reasoning by analyzing their work inductively (Creswell, 2012). Among the qualitative methods, case study is used for this research. In the case studies, participants, phenomena, or certain behaviors are investigated in depth, in their natural context (Gall, Gall, &Borg, 2007). Cases are distinguished from others and studied in depth to generate some patterns specialized to that groups.

For the comparison of prospective teachers and high school students two case groups are formed. Analyzing these two cases of high school students and prospective teachers, a pattern about their graphs and reasoning can be generated (Cohen, Manion, & Morrison, 2007), and these two cases can be compared to each other. The cases are restricted to the groups of participants from particular high school groups and the prospective teachers defined in the latter section.

3.1. Participants

The participants of this study are students from two different high schools and a university in Ankara. One of the high schools is an Anatolian High School and the other one is an Anatolian Teacher High School, both of which admit students who get high points on the national standardized tests. Two classes of students from each of these two high schools (n=107) participated in this study. The classes and grade levels are selected by the teachers based on the teachers' schedule and their availability. One of the classes is 11^{th} grade and the other 3 classes are 10^{th} grade. Students in these classes work in groups of 4 or 5 which make a total of 25 groups. All of the high school students completed at least one year of algebra class, but none of them took any calculus course before.

The other group who participated in this study includes prospective secondary school mathematics teachers attending an elective course "Mathematical Modeling for Prospective Mathematics Teachers" in a state university. Among 24 students who participated in the study, 16 of them are 3^{rd} year students, 6 of them are 4^{th} year students and 2 of them are 5^{th} year students. The students' ages ranges from 19 to 22. The students work in groups of 3-4 which make a total of 7 groups. All of the prospective mathematics teachers completed at least one year of calculus course in addition to some other mathematics courses, such as, analytic geometry, differential equations, Euclidean geometry, introduction to algebra, and set theory.

3.2. The context of the study

Data for this study was collected in a project supported by The Scientific and Technological Research Council of Turkey (TÜBİTAK) under the grand number 110K250. Three major purpose of the project were

> (*i*) to develop mathematical modeling tasks and activities that can be used with both secondary school students and pre-service and in-service teacher education programs;
>
> (*ii*) to develop an in-service mathematics teacher professional development program about mathematical modeling and to investigate how the program would affect teachers' beliefs, knowledge and practices;
>
> (*iii*) to develop an academic course for preservice mathematics teachers and investigate how the course would affect pre-service teachers' knowledge, competencies and attitudes in terms of mathematics, mathematical modeling and using mathematical modeling in mathematics education. (Erbaş, Çetinkaya, & Çakıroğlu, 2013, p. 2)

To support the first purpose of the project, as part of the research, several modeling tasks were developed and tested. After testing these modeling tasks, a professional development program for improving in-service and prospective teachers' knowledge and skills about modelling activities were constructed. Out of 32 modeling tasks developed in the project 7 tasks were revised considering the pedagogical issues and applied in two high schools in Ankara. These modeling tasks mainly covered rate, ratio, trigonometry, exponential functions, tangents, and derivative concepts. 10 mathematics teachers in these high schools gained experience on modelling tasks through meetings twice a month to discuss modelling tasks and implementing them in their classes. Two teachers from each school implemented one of the activities in each month during the academic year 2011-2012. Before and after the implementations, the teachers discussed on implementation plan of the tasks and students' ways of thinking related to the concepts involved in the tasks.

The modeling tasks were also used in an elective course aimed at developing prospective mathematics teachers' knowledge about modeling tasks and their knowledge of and skills about using these tasks in teaching mathematics Out of the 7 activities used in high schools, 6 modeling tasks were implemented to 25 prospective teachers in a course titled "Mathematical Modeling for Prospective Teachers" during the spring semester of the academic year 2011-2012. In addition, the prospective teachers were asked to reflect on students' ways of thinking on four of these modeling tasks. The prospective teachers also developed a modeling task and a lesson plan to implement the task, and at the end of the semester they implemented the task to their classmates.

In this research study, among six modeling tasks that were implemented to improve instruction about covariational reasoning in high school levels and for prospective teachers, "Water Tank" task (see Figure 2) was chosen to investigate students' graphing and covariational reasoning. "Water Tank" task was one of the last two activities that both high school and prospective teachers had worked on.

3.2.1. "Water Tank" Task

The Water Tank modeling task (Erbaş et al., 2016) which is a revised form of classical bottle problem (Carlson et al., 2003) is chosen for this study as it requires covariational reasoning by asking students to draw graphs for different situations. In this version of the problem students are presented four tanks which have parts with different shapes. Students are asked to graph the height of the water on each tank depending on the volume of water. In addition, they are asked to develop a manual that describe how to draw the graph for any tank while it is filled with water. By developing a manual, students are expected to think about and compare different types of graphs, reveal their understanding of graphing and covariational reasoning.

The Water Tank

A software company produces variety of programs to educational institutions. The company had just been contracted to make a short animation that shows a variety of water tanks filled with water and some graphical representations of this process. A team of professionals who work on this project needs a graph that shows the height of the water as a function of the amount of water in the tank.

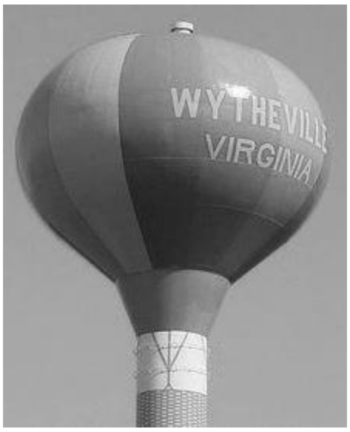
Assume that you are a member of this team. The team members need your help to make sure this animation and accompanied graphs appear realistic. You are expected to provide a graph for each of the water tanks and a manual that tells them how to make their own graph for any tank that is shown in the animation.

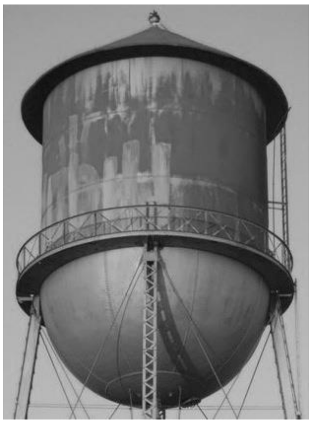
Figure 2. The Water Tank task used in the study [Erbaş et al. (2016*). Lise matematik konuları için günlük hayattan modelleme soruları*. Ankara, Türkiye: Türkiye Bilimler Akademisi.]

Although the shapes of the tanks in the tasks provided to both high school and prospective mathematics teachers are the same for the three of the water tanks. The first tank provided to high school students include only a cylindrical tank to start with on the other hand the first tank provided to the prospective mathematics teachers includes a logarithmic shaped tank (see Figure 3). As tanks have different parts of shape, students can think each part separately for graphing and join these graphs, namely think in smooth chunks. While graphing, joining especially linear and parabolic parts, can cause sharp transitions. Thus, Tank#2 and Tank#3 may help in observing sharp transitions and smooth chunk reasoning implicitly.

Tank #1 (high school students)

Tank #1 (prospective teachers)





Tank #2

Tank #3

Tank #4

Figure 3. Shape of tanks provided to high school students and prospective teachers for “Water Tank” task.

3.3. Task implementation and data collection procedures

Since this study is a part of a larger project, in this study the modeling task was implemented by the participating teachers and an instructor of the project. In fact, for the prospective mathematics teachers the Water Tank task was implemented by an instructor who also implemented several other tasks; and for the two different high schools, the task was implemented by four different mathematics teachers who also implemented at least one more modeling task in their classrooms. The task was implemented by following an implementation plan that includes an outline of the lesson, possible student difficulties, and strategies and methods to overcome these difficulties. During the implementation of the task, the role of the teacher/instructor was to be a facilitator of students' learning.

During the implementation both in the high school classrooms and in the university, the students worked on the task in small groups. The students first worked individually on the task for a few minutes, and after developing some ideas for the task they started to work as a group. At the end, each group presented and discussed their solutions for the modeling task in about 5 minutes. After the presentations, instructors concluded the class with a brief summary and evaluation on the implementation of the activity. Working on the task and presenting the works took about 100 minutes for high school. Since the presentations and discussions were taking place more in university, the implementation of the task took about 140-150 minutes for prospective mathematics teachers.

Data for this study encompass groups' solution papers which include students' graphs for the four tanks, and their explanations related to each graph. The groups' other written work regarding their solutions were also collected for analysis. In the study, each classroom was videotaped, and one group from each high school classroom and three groups from prospective mathematics teachers were videotaped while they were working on the modeling task. While the video cameras that were positioned to capture whole class focused on classroom discussions and groups' presentations, the other video cameras focused on the groups' written and verbal work. Additionally, each group was audiotaped during the course so that the discussions among group members could be examined.

3.4. Data analysis

Students' solution papers and other written works, video recordings, and audio recordings were analyzed by qualitative research methods. At first students' solution papers were analyzed. Then audio and video recordings added to the analysis by considering the groups they belong to.

The analysis of video and audio recordings helped in exploring the details of students' thinking in graphing. Before analyzing the data, the classroom video recordings are transcribed at first. The audio recordings served to fully complete the transcriptions of classroom videos in times when the video recordings were not clear or complete. The transcription of audio recordings was handled by watching the video recordings at the same time to understand how students acted on while graphing and what they referred to while discussing their work. The audio recordings of other groups that do not have video recordings were not analyzed, because students did not talk in proper terminology to help us understand how they reasoned. Moreover, just audio recordings did not provide clues how they acted on graphing.

In analyzing the data, all of the data were read and openly coded at first. As coding went on, returning to the previous groups' work and recordings occurred often. First coding appeared to be the way that students drew the graphs, later similarities and differences showed up to be as codes. After first coding the following codes emerged: smooth, chunky, smooth chunk, linear big chunks, sectional area, radius, volume, unit volume, unit based reasoning, non-unit based reasoning, unit height, tracing trend of increase in height, compare amount of height to volume, time dependence, slope, limit, sectioning tanks into parts, sectioning tanks into unit *h* or depending on shape, quantitative and non-quantitative reasoning, ratio and non-ratio reasoning, global and local approach.

Next, these codes and a list of themes derived from related literature (Castillo-Garsow et al., 2013; Johnson, 2011; 2012) were used to reanalyze the data and finalize the codes. In finalizing the codes, the codes that were derived from the research studies but were not seen in the data were removed from the code list. Hence the following list of codes was used in analyzing the graphs and students' covariational reasoning. The descriptions of final codes and samples quotations are presented in Appendix A.

Table 1. *List of codes and categories used in the analysis*

Graphing	**Reasoning**
Types of students' sketches (Castillo-Garsow et al., 2013): a. smooth b. smooth chunk (Johnson, 2012) c. uniform chunk d. non-uniform chunk	Reasoning about rate of change (Johnson, 2011): a. constructing intensive quantity; considering variation in intensity b. constructing intensive quantity; comparing intensities c. associating extensive quantities
Reflecting change in graph by a. considering slope b. focusing on the trend	Unitization: i. unit volume based ii. unit height based iii. both unit height and unit volume based iv. non-unit based v. continuous volume or continuous height based
Approaches a. Global approach on graphing b. Local approach on graphing.	

Frequencies of slope consideration and global-local approach are questioned in graphing types. Unitization differentiated among types of reasoning about rate of change. Hence unitization helped in determining types of reasoning about rate of change. Relation between students' sketches and their reasoning about rate of change is investigated.

There are 4 types of tanks in "Water Tank" task. Thus students' sketches and reasoning may differ through each of the tanks. For their sketches if there exist at least one smooth chunk graphing among smooth graphs, the group is coded as smooth chunk. The reason is that for some tanks smooth graphing may be easy or trivial. However, some tanks required students to use smooth thinking to be able to sketch accurately. For example, although connecting a decreasingly increasing and an increasingly increasing curve smoothly was easy for Tank #4 to represent the covariational situation, joining linear and parabolic parts smoothly, or differentiating among different parabolic parts were not trivial for the Tank #2 (see Figure 4). If the group of students is aware of the necessity of the smoothness, they would reflect it in all of the graphs (see Appendix A). As seen in Figure 4, a group of students constructed graphs for Tank #1 and Tank #4 smoothly, while they drew graphs for Tank #2 and Tank #3 in smooth chunks. Hence, this group's sketches were coded as "smooth chunk".

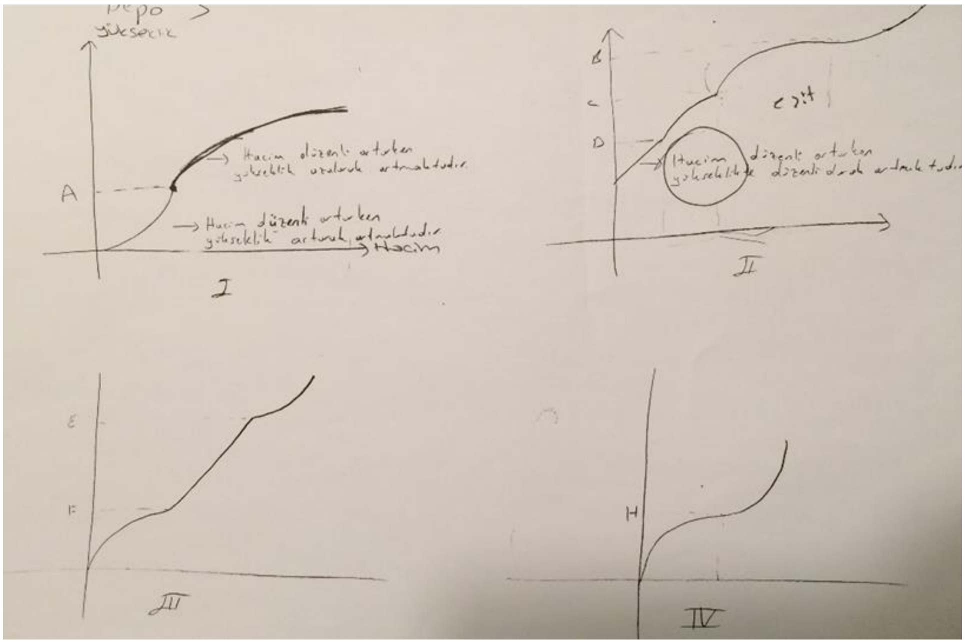


Figure 4. Sample sketches of smooth and smooth chunk graphs belonging to a group

Participants' reasoning about rate of change was more complicated. What kind of reasoning yields to sketches was not clear for all groups. All the data about participants' reasoning was taken into consideration so that participants' different types of reasoning about rate of change throughout the task was revealed.

3.5. Reliability

In order to ensure the reliability, the data was independently coded by the researcher and an expert in mathematics education. Coding by another expert met 80% agreement. Some disagreements occurred in differentiating smooth and smooth chunk graphs, and thus definition of codes revised for eliminating ambiguity. Smooth graphing was previously considered as differentiable graphs, and its description was revised almost differentiable so that rate of change is comparable just before and just after transition points (Kertil, 2014). Furthermore, among the four graphs, some may be graphed smoothly while others may be graphed in chunks. It was made clear that if one of the graphs is drawn as smooth chunk and others. are drawn as smooth then the group's graphing will be considered as smooth chunk instead of smooth assuming that if the groups were aware of the essence or meaning of smooth graphing they had to apply it in all graphs. Drawing some of the graphs smoothly may have to do with just ease of drawing or non-supported good looking reasons, or just be by chance. Moreover, unitization code was understood to be as just assigning points or quantities on the axes. We agreed that any sign of unitization in students' reports had to be considered under unitization category. For example, when students mention "for equal amounts of water" it will be considered that students mean to take unit volumes. Examples that illustrate "unitization" codes provided in the appendix. Eliminating the ambiguities by clarifying the definitions and discussing on the non-agreed codes, 100% agreement was reached with the second coder.

CHAPTER 4

RESULTS

In this chapter, the results of the study are presented under three main sections: i) students' (prospective teachers and high school students) graphical representations of a covariational situation, ii) students' reasoning about rate of change, and iii) relation between students' sketches and their reasoning. The difference between high school and prospective mathematics teachers is noted in relevant sections.

4.1. Students' graphical representations of a covariational situation

Based on the analysis of data the students' sketches of the graphs for a covariational situation are presented in Table 2. The table shows that while the prospective mathematics teachers constructed graphs in smooth or smooth chunks, the vast majority of the groups of high school students (64%) produced graphs in smooth chunk. According to the table, a remarkable number of the high school groups (28%) also sketched graphs in uniform or non-uniform chunks. Although 2 groups (8%) of high school students out of 25 groups sketched smooth graphs, it was 3 out of 7 groups (43%) for prospective teachers groups.

Table 2. *Frequency of prospective mathematics teachers and high school students' sketching types*

	smooth	Smooth chunk	Non-uniform chunk	Uniform chunk	total
Prospective teachers	3 (43%)	4 (57%)			7
High school	2 (8%)	16 (64%)	5 (20%)	2 (8%)	25

In the following sub-sections, excerpts from students' work will be presented to portray the students' different representations of varying quantities.

4.1.1. Smooth graphing

A group's sketches were categorized as smooth if all the graphs were drawn smoothly. The analysis of data indicated that 3 groups of prospective mathematics teachers and 2 groups of high school students draw all of their graphs smoothly. Below, excerpts from a group of prospective mathematics teachers (Prospective Teachers Group #2, [PG-2]) and two groups of high school students are presented to show students' graphical representations of a covariational situation.

The analysis of data indicated that PG-2 drew the graphs without placing off their hands as much as possible. Their drawing showed no traces of joint parts or chunks (see Figure 5 for an example).

Figure 5. PG-2's graph showing the relationship between the growth of height and the volume for water tank #2

PG-2's graphs that showed the relation between covarying quantities included only a few assigned points on the axes. For the x-axis no points were assigned while on the y-axis some points were assigned as a, b, c to show turning points in the graph (see Figure 5), while no points were assigned on any axes in the 3rd graph (see Figure 6). In addition, their sketching seems to occur without lifting the pen in a single motion. Students' not focusing on local points on the axes and graphing in a single motion show that they have a global approach while graphing.

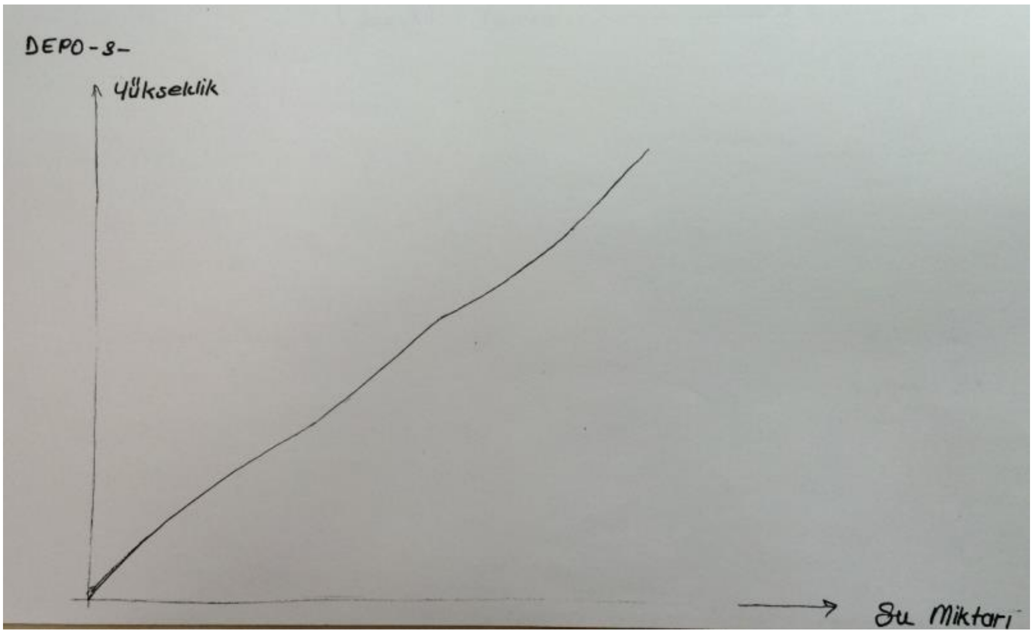


Figure 6. PG-2, graph for tank #3; no points assigned on axes

The other prospective teachers groups (PG-5 and PG-6) that drew smooth graphs, sketched the graphs not in one hand action, but they also assigned only x-axis as a, b, c ; V_1, V_2, V_3 (see Figure 7). They seem to have a tendency to make the linear part of the graph more apparent.

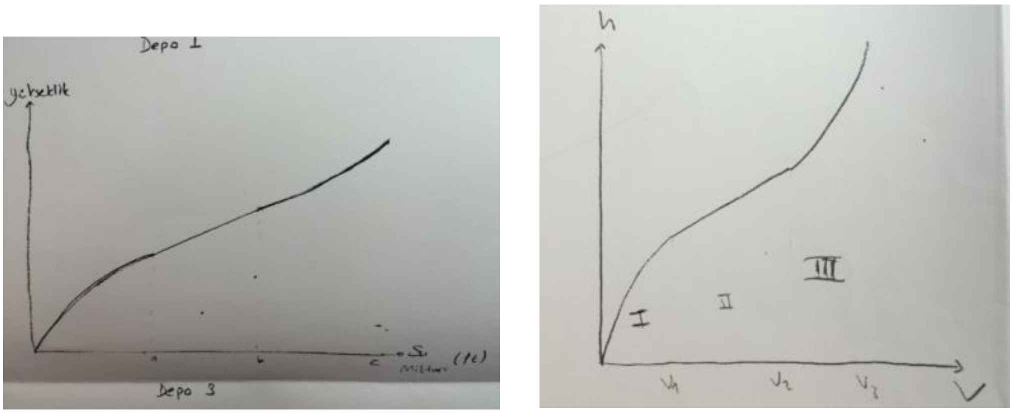


Figure 7. PG-5 and PG-6’s sketches of graphs for tank #3.

The analysis of PG-2’s explanations regarding their sketches revealed that in the beginning the students just confused and automatically produced a graph that represents the increase in the shape of the tank, and then they changed it. In other words, they just focused on the increase of volume at first and observed that the tank is widening in the lower part and it is narrowing than in the second part of the tank (see Figure 8). The students might just find it easy to interpret this covariation or automatically join the shape of tank with the parabolic graphs. Also High School Group #19 (HG-19) have the same difficulty. They explained graphing tank#4 as;

"Because the volume increases, it goes outward. In the second part because volume decreases it goes inward."

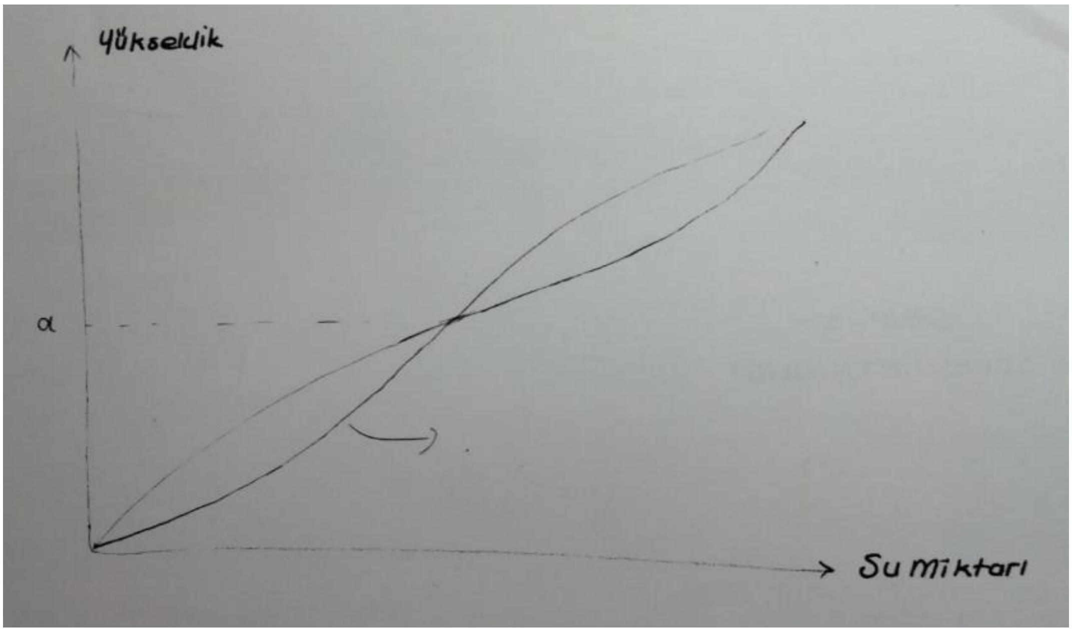


Figure 8. PG-2's graph showing the relationship between the growth of height and the volume for water tank #4

There was no explanation for PG-2's smooth drawing in their work sheet and no clear information in their videos that they argue why it is to be smooth. They tried to explain it to be smooth for transitions while they were presenting their work to other groups. However, they had no enough explanations even to convince their group members. While another group was presenting, they argued and could explain why it should be smooth at transition points. Possibly, it is the "smooth chunk" example what is presented which made them awake and recognize why a smooth chunk graph will not represent the situation. They argued that at the transition point

V1, slope should start to decrease, not an increase is possible as width gets larger at this point (see Figure 9).

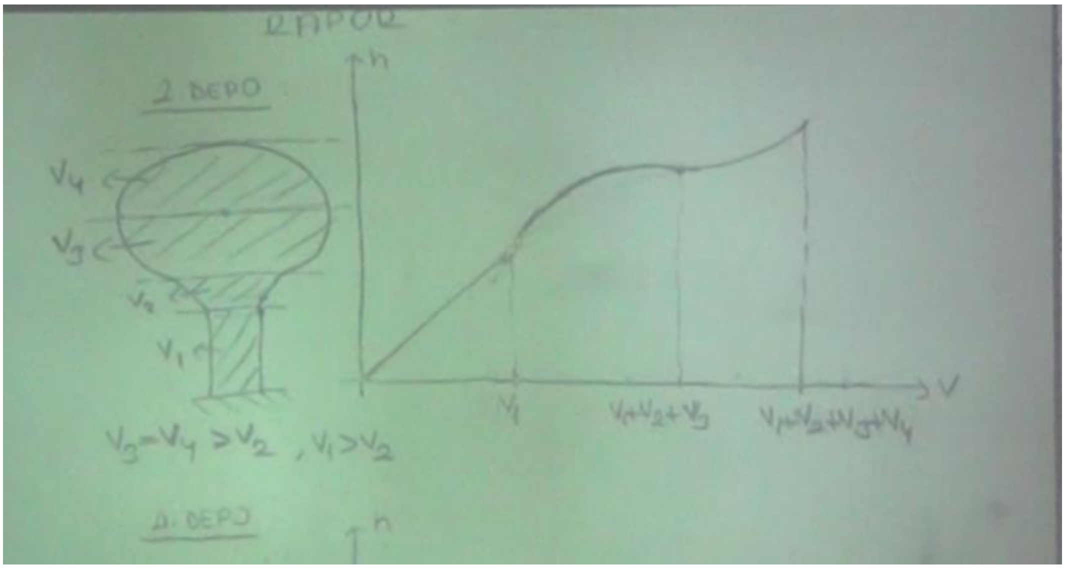


Figure 9. PG-2’s discussion about sharp transitions on the graph of tank #2 of a group presentation.

Also PG-5 and PG-6 explained why the transition points should be smooth by slope and slope like properties of graph.

PG-5: “We draw horizontal lines on profile picture of the tanks. If the lengths of these lines are decreasing, the water accumulated (height) increases and the slope in graph increases…” “If the change of length between consecutive lines decreases the curve in graph is convex; slope is positive.”

PG-6: “We establish the curves as more horizontal or vertical according to change of speed in their heights.” (The words “more horizontal”, “more vertical” prove that they care slope.)

There were two groups of students who drew smooth graphs from high school; HG-12 (High school Group 12) and HG-25 (High school Group 25). Similar to prospective mathematics teachers that draw smooth graphs, they had a global approach while graphing. One of the groups from high school which drew smooth graphs assigned only on the x-axis while the other preferred assigning only on the y-axis, focusing or unitizing/dependent on height maybe. However, both of the high school groups that graph smoothly seem to graph in one action of drawing (see Figure 10).

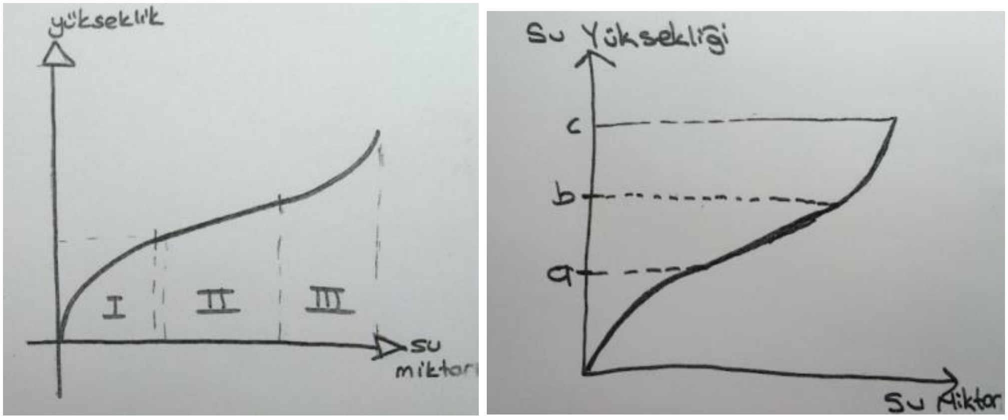


Figure 10. HG-12 and HG-25's graphs for tank #3; just assigning points on one axis

HG-12 depends on changing the slope while sketching their graphs.

HG12: "In every area the volume is the same thus the height and water amount increases without the slope is changing."

For HG-25 we did not see any focus on slope directly. They said "the height is slowing down" defining the action of the graph. They coordinated what slowing down means and how to interpret in graph. There were no memorizing parabolas as increasingly increasing etc. Moreover, they said "the rate of the water amount to height increases and the graph increases with an increasing speed". That means they know the meaning of ratio of covarying variables and how it is interpreted in graph. They relate the ratio to the speed of graph.

One of the students from HG-25 was drawing the graph of covariation between height and volume just near the tank #2. He continued all of the process, drawing without taking his hands off. He moved his hands up or down by continuing an increase of height less or more, at the same time he pointed the corresponding height in the tank #2 with his other hand. He was not thinking locally about rate of change for this point or the points before and after. He passed through the points of inflection, and continued the drawing by just sketching more horizontally or vertically throughout the graph (see Figure 11).

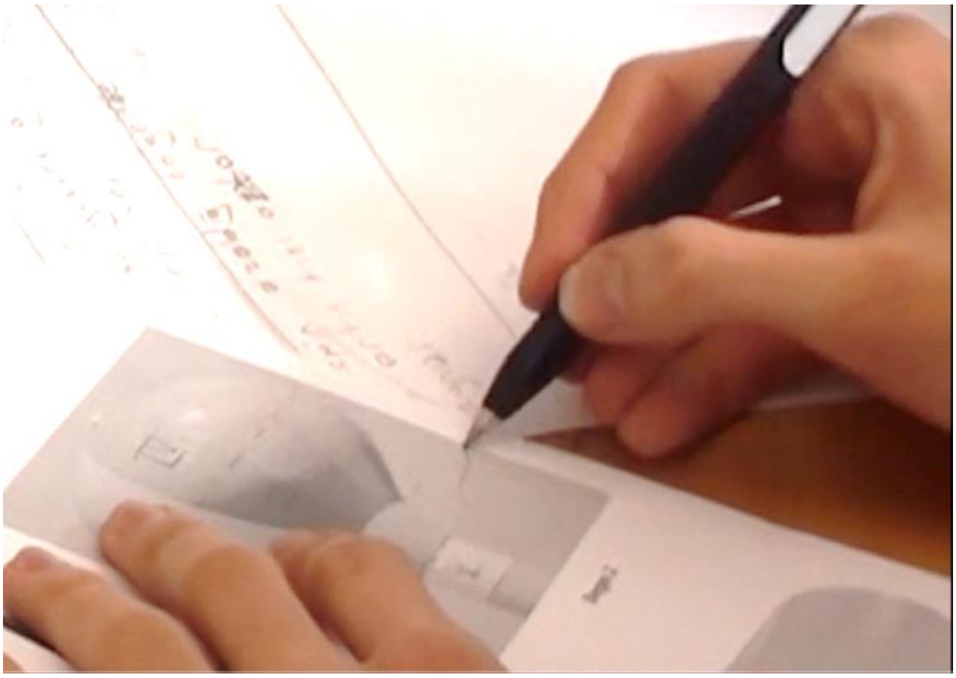


Figure 11. A student’s graphs for tank #2; continuous and smooth

In general, we see that groups who graph smoothly both from prospective teachers and high school students have global approach in graphing. More interestingly all of these groups use slope concept, steepness or slope like properties of curved graphs to reflect the change in covariational situation in their graphs. They give clues that they are aware of what rate of change means in terms of graphing.

4.1.2. Smooth chunk graphing

Four groups of prospective mathematics teachers and 16 groups of high school students’ sketches of graphs were categorized as “smooth chunk”.

In typical smooth chunk graphing we saw “sharp transitions” between chunks that cause inconsistency with the rate of changes of the height that depend on the volume just before and just after the points where chunks are connected (Kertil, 2014). The following sketch of a group of prospective mathematics teachers portrays the smooth chunk graphing.

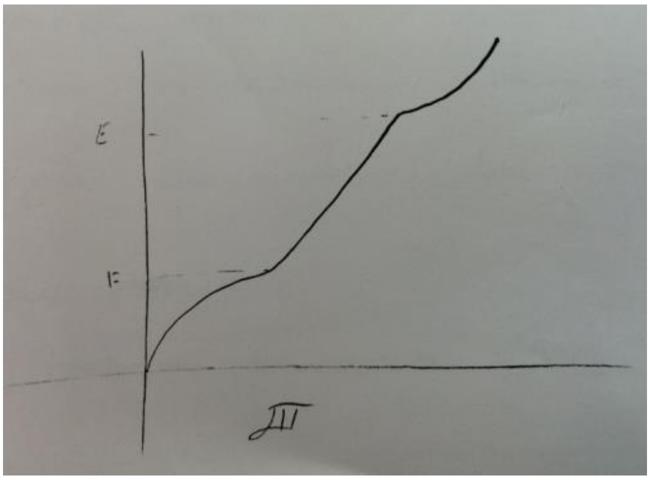
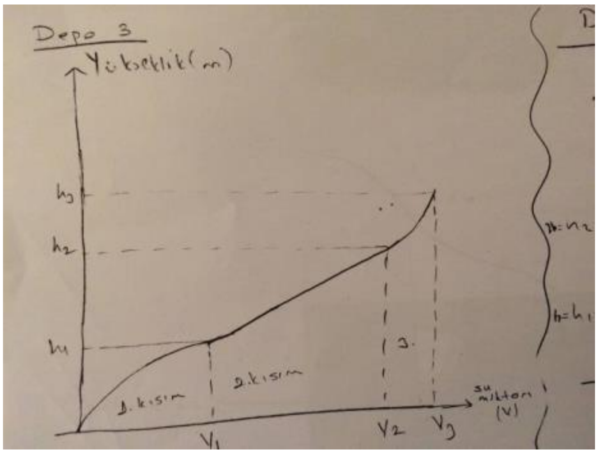
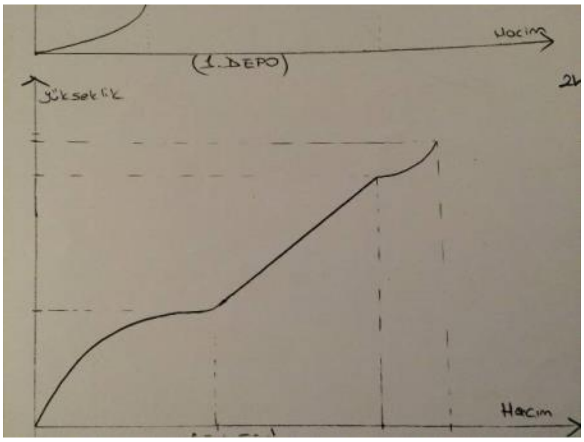


Figure 12. PG-1’s graph showing the relationship between the growth of height and the volume for water tank #3

Prospective teacher groups that drew smooth chunk graphs mostly assigned points on the both axes (see Figure 13 a-b)



A

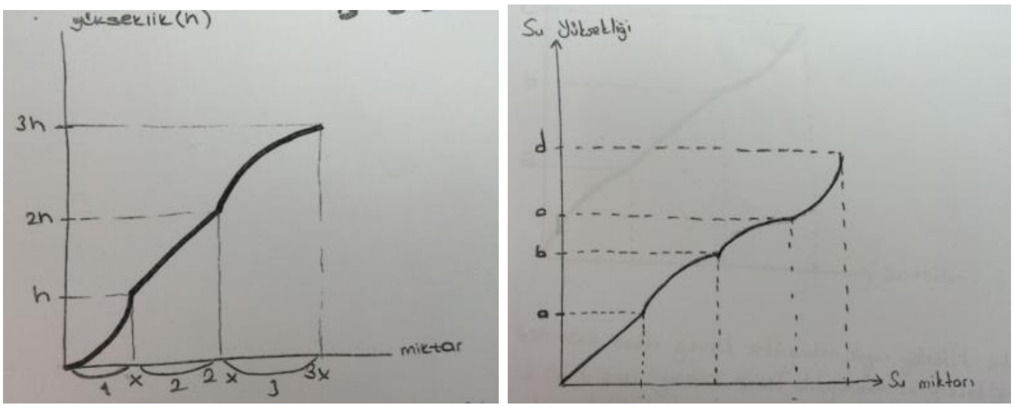


B

Figure 13. PG-3 and PG-7’s sketches for tank #3

Group discussions of PG1 also showed that students never questioned sketches of the graphs at the transition points or just before and after these points. They just focused on how might the graph look like within these chunks and joined them. The analysis of the worksheets and video recordings of all prospective teacher groups who draw smooth chunk graphs showed that they had paid no attention on the appearance of the sketch while joining parabolic parts and linear parts or they did not consider rate of changes of the height while graphing. Moreover, they did not attend to the slope of the graphs.

Similar to prospective teachers groups, high school students groups that drew smooth chunk graphs assigned both axes at transition points (see Figure 14 a-b). Boundaries of chunks are determined on both of the axes.



A B

Figure 14. High school students' assigning on both axes for tank #3 and #2

No slope or rate of change focus is observed among high school student groups who draw smooth chunk graphs, except one group: the HG-11.

HG-11 used slope in their explanation:

> …Height depending on the water amount will increase with a constant slope, linearly. … in the same time interval less water fills. Thus the slope of the graph of height depending on water amount will decrease. … The width of the shape decreases when you move to the upper part. Thus the growth rate of height will increase. A parabola that goes vertical to graph is drawn.

We see from their explanations that this group also had a good covariation understanding as they say height is dependent on the water amount and they focus on the change in the increase of height. However, their graphs were not smooth in their final report. When we look at their other worksheets we see some smoother drawings of one student (see Figure 15) and also in explanations of the same student we see focus on slope.

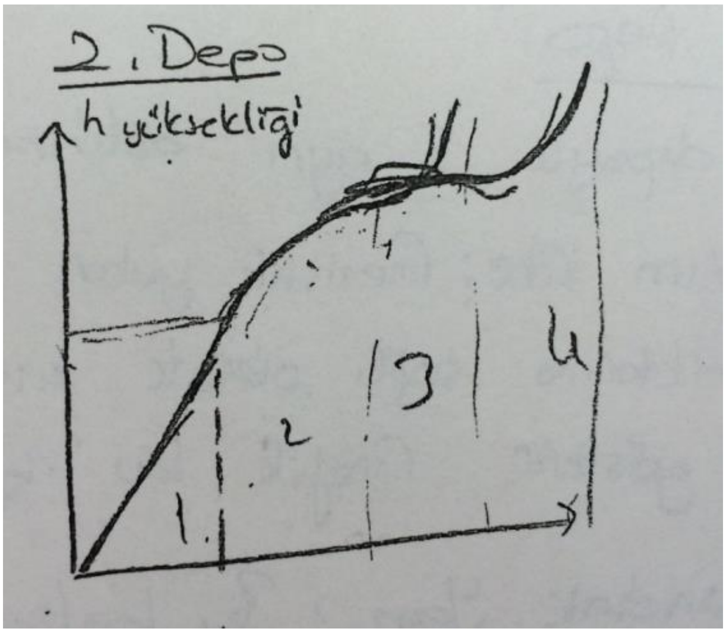


Figure 15. HG-11's sketches for tank #2

The analysis of classroom videos showed that all the students in this group, HG-11, except one who does not graph anything, were graphing the second tank smoother at first (see Figure 16). After some time, they recognized other groups graph in smooth chunk. They thought that other groups' graphing is wrong. However, sometime later researcher came to the group and asked for the difference between 2nd and 3rd part. They tried to explain the difference as; for the 3rd part decrease will be more (see Figure 17). In their final reporting the student who did not graph anything when working by herself graphed all the graphs, deciding to graph in smooth chunks to show the difference between 2nd and 3rd parts. It is observed that interference changed their minds and they gave up insisting on smooth graphs. Explanations and graphing come from different students which also caused this inconsistency.

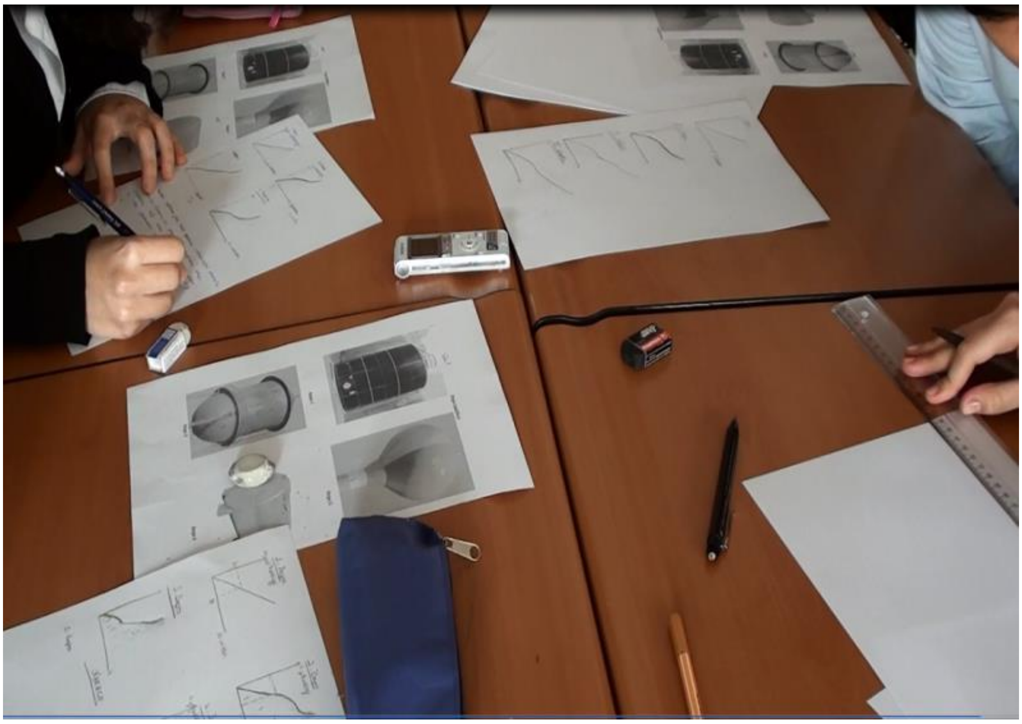


Figure 16. HG-11's attempt to draw smooth graphs for tank #2.

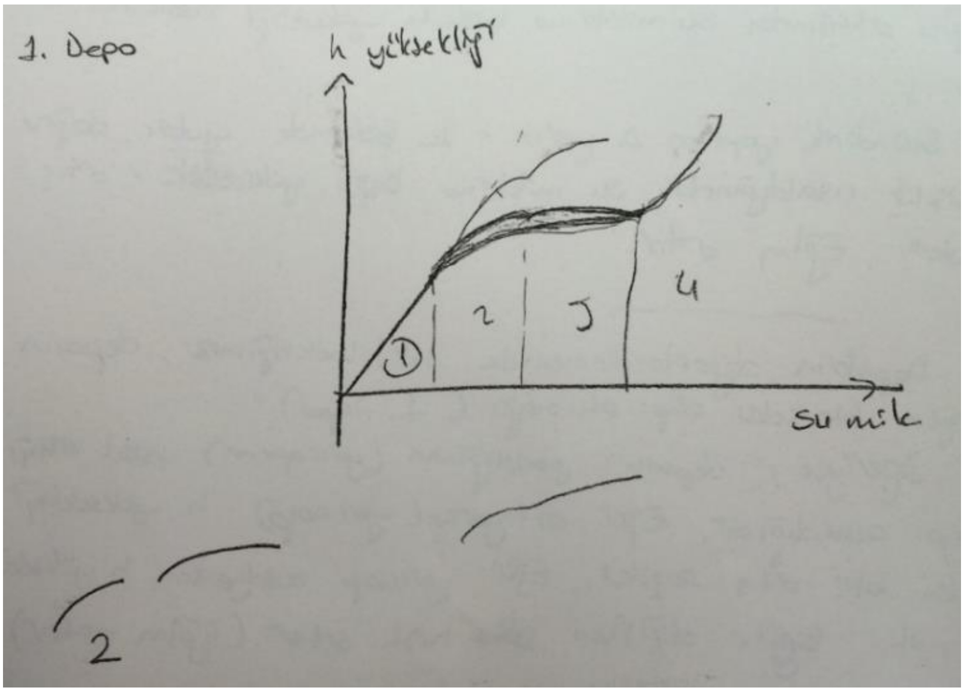


Figure 17. HG-11's attempt to explain the differences between 2nd and 3rd part of the graph.

This group is considered to be as graphing in smooth chunks regarding their last graphing on their report. However, before the interference of the researcher, this group acted actually as a group that draws graph smoothly both on their graphing and reasoning.

Briefly, it can be said that both prospective teachers and high school students groups who drew smooth chunk graphs preferred to graph the change of height depending on the volume in chunks which makes them seem more locally oriented when compared to student groups who draw smooth graphs. Their pointing on axis and assigning values proposed this fact (In smooth graphing we observed that students were just assigning only one axis or none of them.). Moreover, both prospective teacher and high school student groups who draw smooth chunk graphs did not focus on the slope like groups who draw smooth graphs. Their focusing on the change in chunks and not throughout the chunks, and also their not paying attention to slope change throughout the chunks resulted in sharp transitions between chunks.

4.1.3. Uniform chunky graphing

There were no groups from prospective teachers who drew uniform chunky graphs. There are two groups of high school students who drew their graphs in uniform chunks; one graph is from HG-20, the other is from HG-4 group.

The group HG-20 graphed in typical chunky style as Castillo (2010) defines that they prefer to segment the graph into little chunks and quantitatively sign values on the axes. They separated the tanks into equal heights and then estimate volumes for each height. Chunks were taken in little parts, all the tanks were separated into $6h$'s on the axes values are labeled as $h, 2h, 3h\ldots$ for heights and $x, 2x, 3x, \ldots$ for volumes (see Figure 18). Assigning volumes as x can make us think that they take volume as dependent variable but it was not. They took equal h's and estimate volumes for each and they said, "for the same time there will be more height but fewer amounts". (We see there is a comparison.). Unitization, estimation and comparing the variable amounts in the chunks dominated all their work.

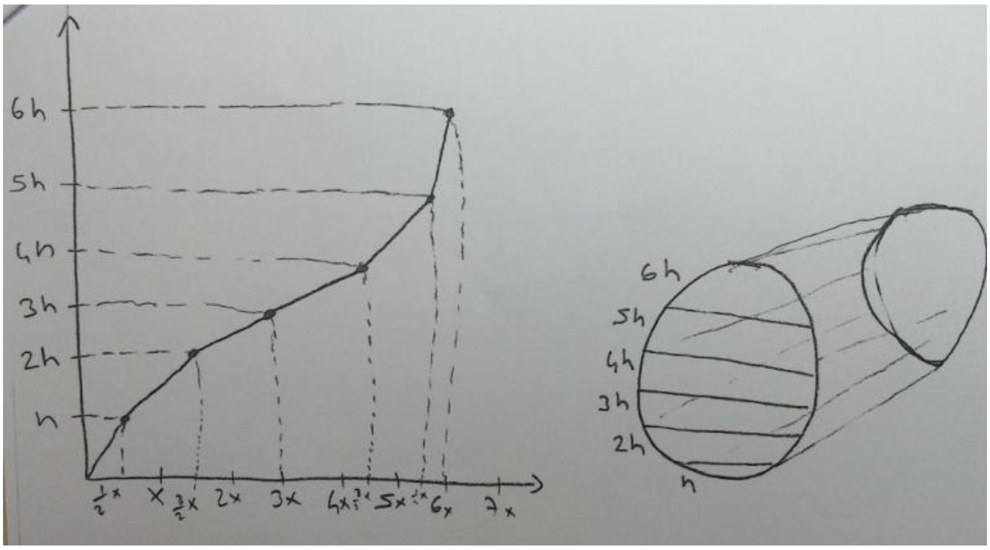


Figure 18. HG-20’s sketch of graph indicating unitization, approximation, chunky graphing

The other group HG-4 who draw uniform chunky graphs shows similar properties. This time, they segment the graph into equal volumes. They take volume as the independent variable and try to estimate heights for equal amounts of water that is added instead of estimating volume for unit h’s as the other group hg-20. They prefer directly assigning values rather than h’s or x’s (see Figure 19 a-b).

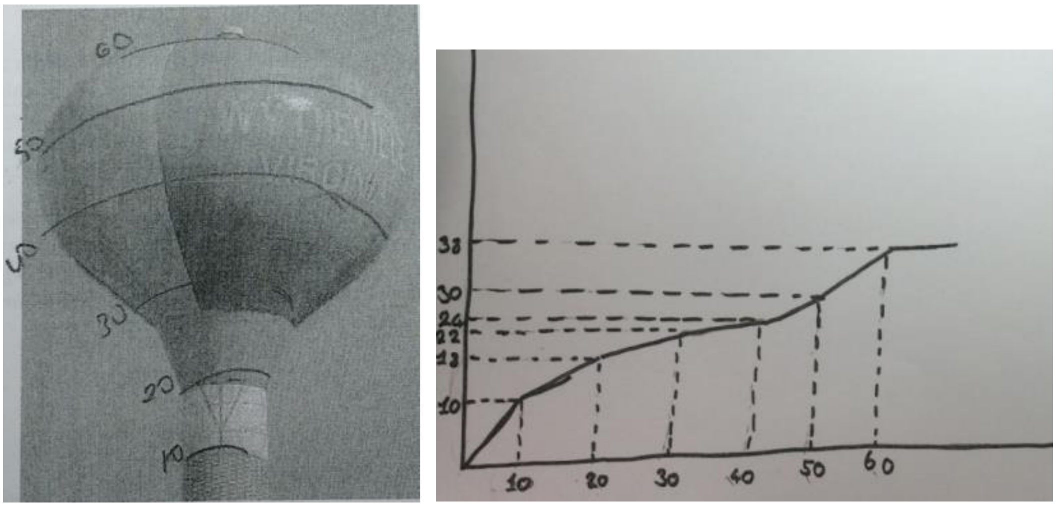


Figure 19. HG-4’s unitization and tank segmentation (a), sketch and approximation of height (b) for the tank #2

In addition to HG-20 group that graph in uniform chunks, they have another strategy. Not only do they use calculations or estimate how much height or volume will increase, they say they also consider the slope of the graph in each chunk. They graphed the 4th graph by changing the slope of the line through three chunks. In their graph of 4th tank also it is observed no quantitative values are assigned on the axes (see Figure 20). However, for their graphing, it cannot be stated that they aim to determine the chunks uniformly by taking equal lengths of intervals, nor these intervals are as small as in the other graphs. It is seen that chunks are determined regarding the wide and narrow parts of the tank #4.

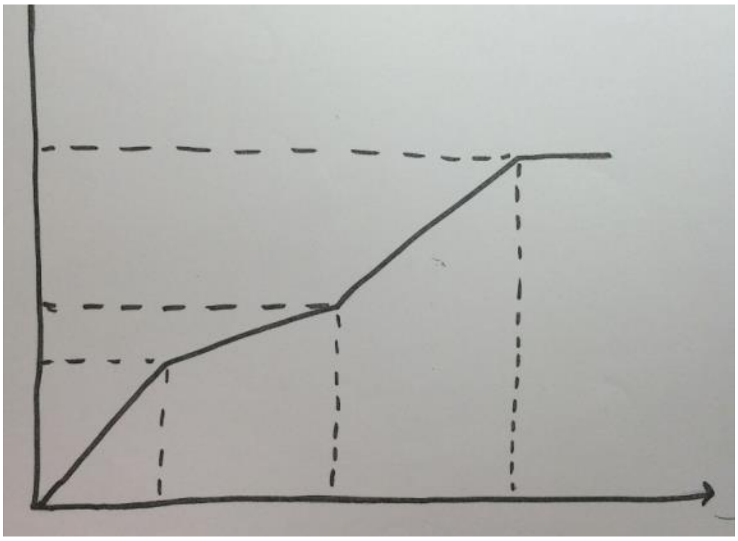


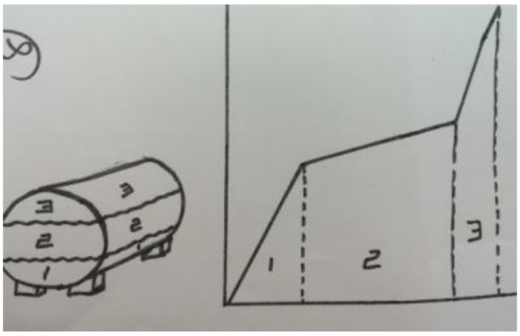
Figure 20. HG-4's sketch for tank #4

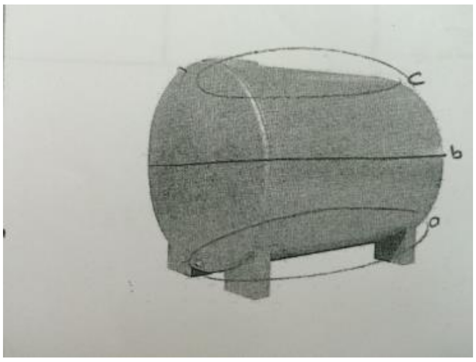
4.1.4. Non-uniform chunky graphing

There are no prospective teacher groups but 5 high school student groups that draw non-uniform graphs.

At first sight, big chunks and big linear line segments were seen in all of the non-uniform chunky graphs of students' works. Graphs were constructed mostly in 3 parts, 3 chunks, and the graph of tank#2 can be graphed in more than 3 parts. In 6 groups, we observed this type of graphing.

Chunks seemed to be determined considering sometimes shapes of the tanks, sometimes turning points from widening to narrowing, sometimes just wide or narrow parts of the tanks. Most of them (and the uniform chunky graphing group from this class) preferred to section 4th tank into 3 parts (see Figure 21 a. b.).

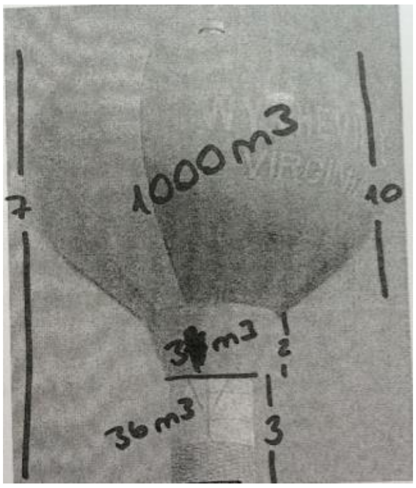


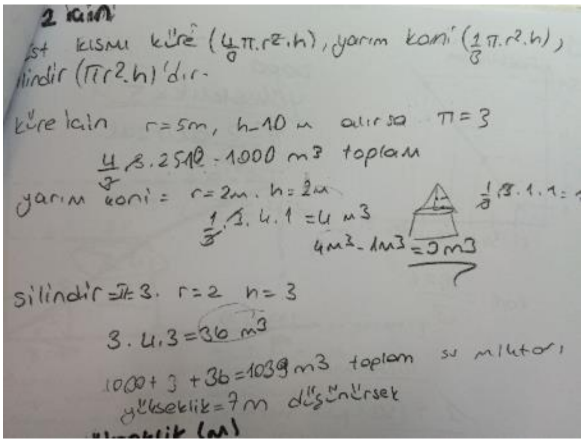


A B

Figure 21. HG-6’s and HG-2’s sectioning tank #4 into 3 parts

There appear two strategies in graphing non-uniform chunks; calculating amounts, and changing slope. We see calculating amounts in little chunks in the first tank in works of most of the groups that graphs in chunks. HG-3 typically calculated amounts of volumes for chunks which are determined by heights (see Figure 22 a. b. c.).





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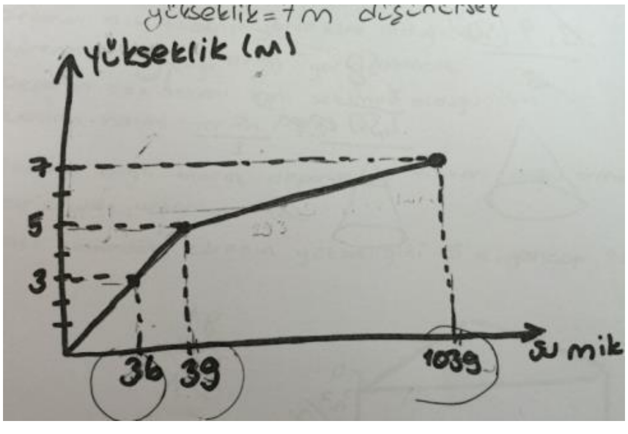


C

Figure 22. HG-3’s calculation strategies

For the calculations to be easy they prefer choosing big intervals/ chunks even segmenting the tank#2 as the upper balloon part to be as a whole spherical chunk. In tank#1 they were typically using uniform chunky approach. In others they also prefer unitizing height. However, calculations get more difficult and they move on to non-uniform chunk. Rather by using direct calculations, if they used estimation, probably they would work on with little uniform chunks.

Similarities can be observed depending on calculations in works of HG-1. When calculations get harder they prefer not to calculate anymore and estimate how lines would be in the graph; possibly changed slopes, because in the 4th graph we see they changed the slope of the lines comparing it to line/constant increase (see Figure 23).

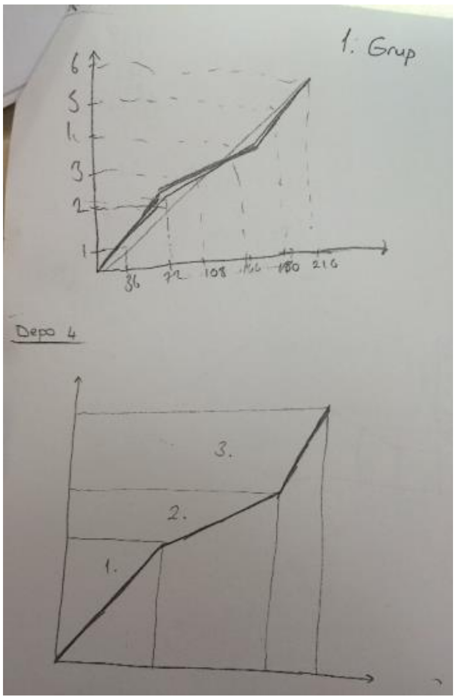


Figure 23. HG-1's sketch indicating the change in slopes comparing to the line

This group and the rest (4 groups) used this type of changing slope strategy while only one group was reasoning with direct calculations. In changing slope strategy, the process starts with determining wide and narrow parts of the tanks and then students change the slope of the lines accordingly. HG-2 also tried to change the slope but couldn't handle it in the 4th graph. In the second chunk, they draw a decreasing line with negative slope. They might think in the wide part increase will decrease but do not know how to interpret it with graph and just draw a decrease. We also see in y axis c, a, b order; b is smaller than a. This fact and the decrease in height is not considered by the group (see Figure 24).

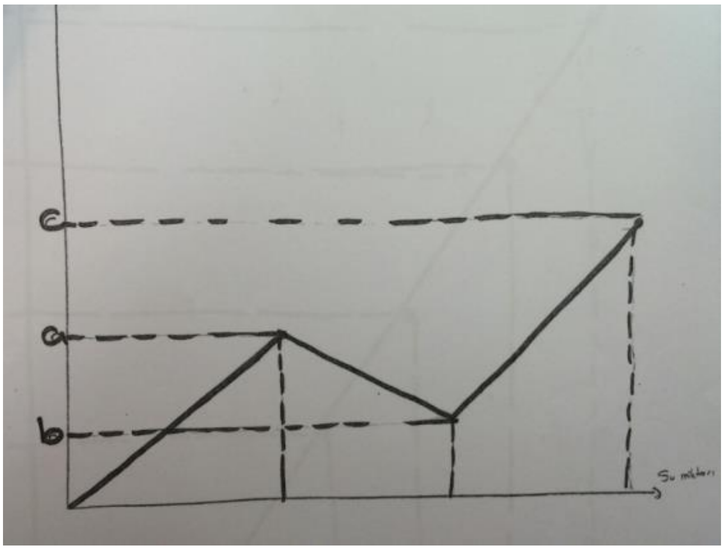


Figure 24. HG-2, changing slope in 4th graph

HG-6 explains their graphing procedure as they showed the narrow parts of the tanks with more slope, wide parts with less slope on the graph providing a very typical example of changing slope strategy (see Figure 25).

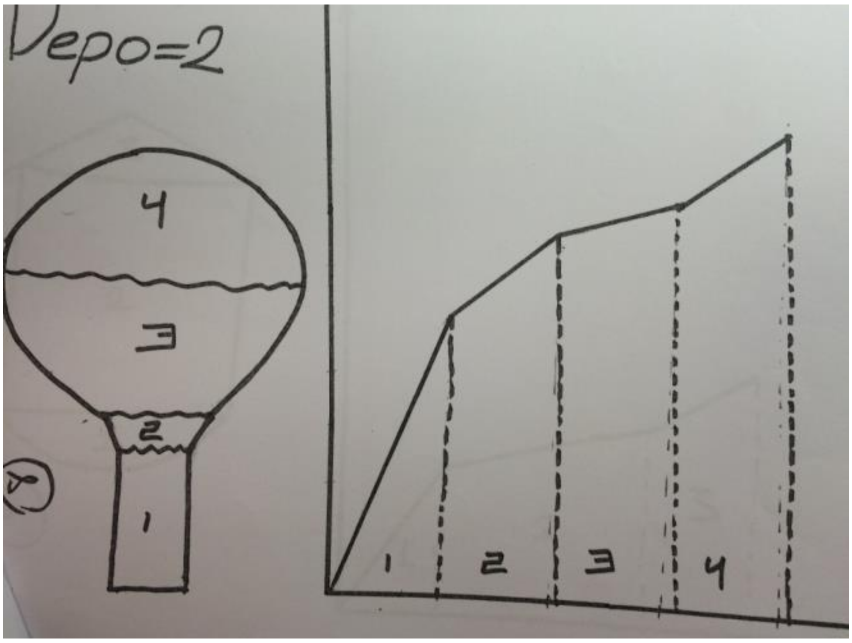


Figure 25. HG-6, changing slope for tank #2

Similarly, HG-5 sectioned tank as widening and narrowing parts and they say that “we show the increase of water (height) fast in narrow parts, slow in wide parts”. They seem to graph the fast water rise (height) with lines with greater slope and the slow rise with lines with fewer slopes. (It is like the speed concept from physics kinematics graphs). Different from group 8 they focus on the speed of increase, and change in increase. They are aware of variance of the intensity. Moreover, they do not segment tank#4 into 3 parts as narrow or wide as all the other groups drawing non-uniformly chunky, but they segment it into 2 as considering narrowing and widening trend. Other groups were sectioning tank #4 into 3 parts to show narrow and wide parts. They have another focus. “We determine the increase in the water amount according to tanks widening or narrowing trends in width.” They do not segment tanks into wide or narrow part, but they think about where the parts are narrowing or widening (see Figure 26). It is a more global point of view with also considering change in change. It results in their graphs with some curved parts (see Figure 27 a. b.). As it is seen, in both part parabolic parts are interpreted with the same curve consistently. Unfortunately, their curves are wrong, but it was a nice try. They are aware of the change in change but maybe they do not know anything about parabolic curves or properties of them.

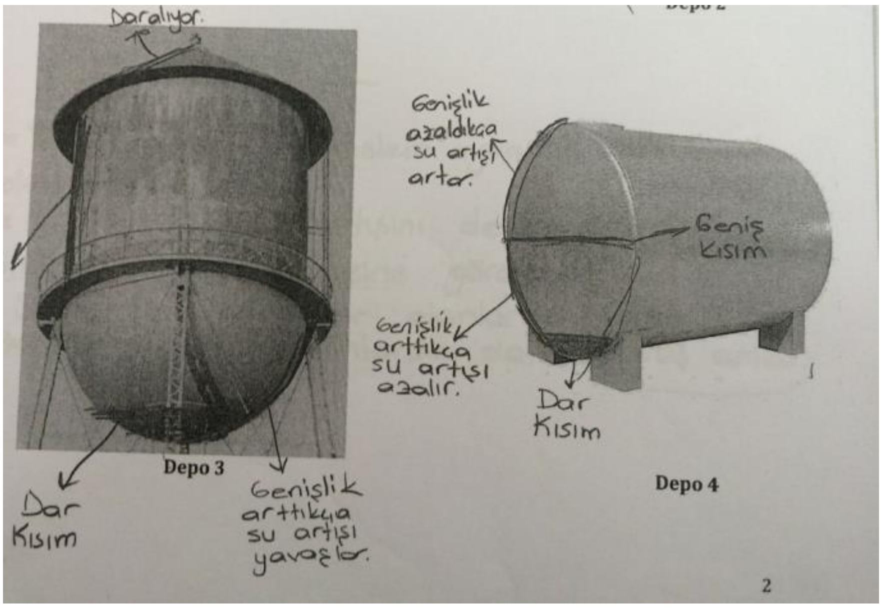
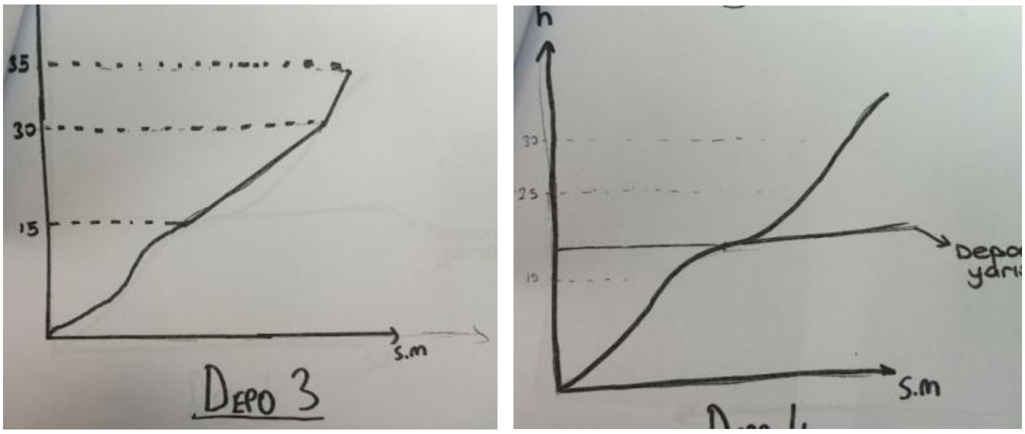


Figure 26. HG-5, sectioning tanks into narrowing widening parts



A B

Figure 27. HG-5’s curved graphs for tanks #3 and #4

Briefly, if students use estimation with unitization they prefer uniform chunky graphing, but if they use direct calculation they need non-uniform chunky graphing in respectively difficult graphs. Also students, who use slope changing strategy according to the width of the tank, graph in non-uniform chunks due to their choice of wide and narrow parts on the tank.

4.2 Students’ reasoning about rate of change

Students’ reasoning about rate of change was categorized by utilizing Johnson’s (2011) framework. The analysis of data related to students’ reasoning about rate of change revealed that the students considered variance of intensity, comparing intensities, extensive quantities and some combinations of these (see Table 3).

Table 3. Frequency of prospective mathematics teachers and high school students’ reasoning types about rate of change.

	Variance	**Variance & Comparison**	**Comparison**	**Comparison & Extensive**	**Extensive & Variance**	**Extensive**	**Unidentified**	**Total**
Prospective teachers	6	1						7
High school	14	3	2	2	2	1	1	25

The table shows that all of the prospective mathematics teachers reasoned considering the variance of intensity, and one group of prospective mathematics teachers used both variance of intensity and comparison of intensities in reasoning about rate of change. The majority of the high school students (n=12) used variance of intensity, while some of the groups used comparison of intensities (n=7) or associating extensive quantities (n=6) in reasoning about rate of change. Moreover, one high school group's written work and explanations did not reveal their reasoning about rate of change.

4.2.1. Creating intensive quantity; Variance of intensity

While interpreting their reasoning about covariation, all of the prospective mathematics teachers groups (7) and the majority of high school groups (17 out of 25) considered variance of the intensity. Totally there were 24 groups that have this approach in their reasoning about rate of change.

The data analysis revealed that most of these groups (13 out of 24) preferred to consider the horizontal cross sections of the tanks, and used the changes in radius, width, area of cross-sections, or lengths of lines drawn from profile to interpret the change in height of the water in the tank. Most of them; 10 groups out of 13, related the change in radius or similar quantities directly to the change in height, while the other 3 groups related the change in radius or alike quantities to the speed of the graph. We observe this strategy among both prospective teachers and high school students:

> $r_3 › r_2 › r_1$ Thus, because radius goes increasingly, height increases decreasingly." (See Figure 28) (radius) (PG-4)
>
> On the profile, horizontally parallel lines are drawn. The change in the length of these lines is considered. If the length is decreasing, the water filled/accumulated (height) increases and the slope of the graph increase. (lines) (PG-5)
>
> Since the width increases between *a* and *b*, increase of height decreases. (width) (HG-13)
>
> If there is widening in shape, the increase in the height will decrease. The reason for this is that the same amount of water will share more area" (area) (HG-21)
>
> "Because area increases up to the middle of the tank, the increase of speed decreases. Then again speed of increase increases because area decreases" (area) (HG-5)

> "In the second part in cut off cone, because the horizontal section increases, the height of water will increase slowing." (horizontal section) (HG-12)

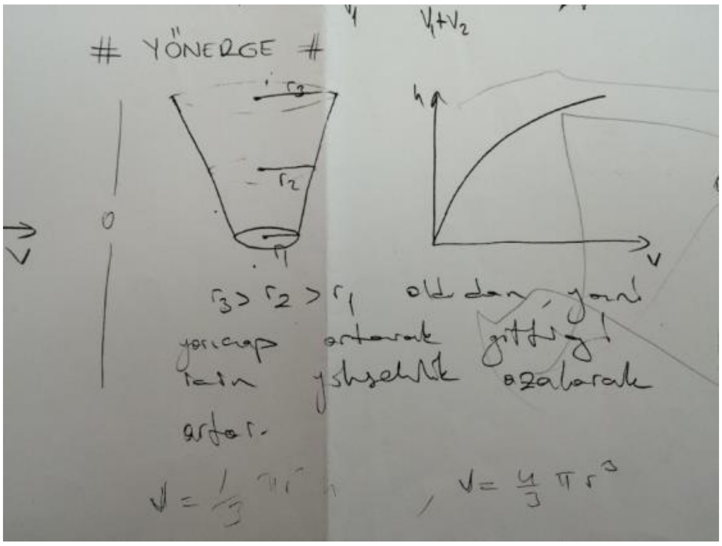


Figure 28. PG-4's sketches showing the change in radius

Some of the groups related width to the speed of height or speed of rise of the height (that might remind a consideration of speed and time). HG-25 is an example explaining their reasoning as "Because the shape widens, the increase in the height slows down." Some of the groups such as HG-12 was able to directly relate the change in the volume to the change in height.

> "In the part between the points *a* and *b*, because the volume does not increase with a constant ratio but increases increasingly, decreasing parabola is drawn." (volume) (HG-12)

None of the groups who used variance of intensity considered the amount of change in volume per unit height or the amount of change in height per unit volume except for the groups that both compared the intensities of covarying quantities and considered variation in the intensity of increase in quantity with respect to the increase in other quantity. Rather the results showed that some of the groups had a continuous image of increase in height/volume or understanding that constant increase in one of these variables in order to examine the difference/increase in the other variable (see Figure 29).

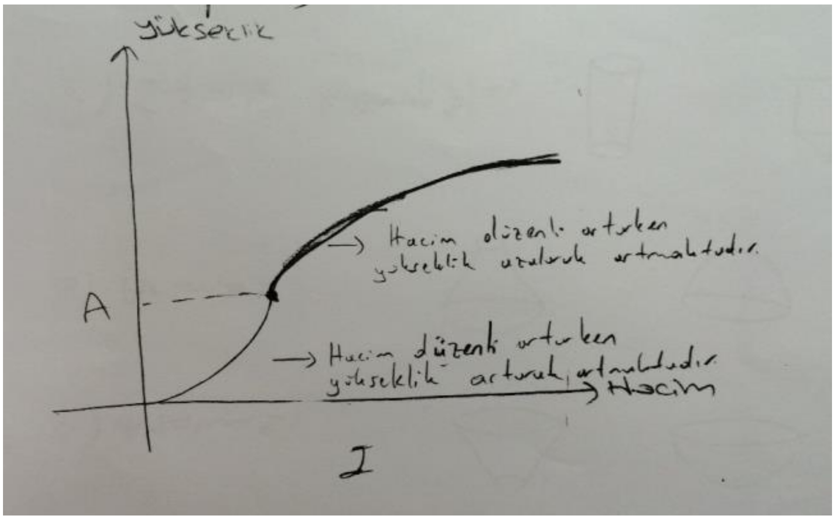


Figure 29. PG-1’s sketches and explanations for Tank #4

Prospective mathematics teachers explained their graphs as, “While volume increases constantly, height increases decreasingly” (UG-1), “If volume increases increasingly [describing the tank #1], while height is increasing constantly, the graph will increase decreasingly…” (UG-2). With the expression “while height is increasing constantly”, UG-2 seemed to refer to the increase in height, without unitizing, or segmenting the height into equal amounts. Imagining a continuous increase in height but not volume they seemed to consider the height as an independent variable.

The groups that drew uniform and non-uniform chunky graphs attempted to explain the covariational situation using the variance of intensity reasoning although their sketches had flaws in representing the situation very well. Two of the explanations were: “Because these parts are narrower, the speed of increase increases.” (HG-4), and “When width increases, the increase in water slows down.” (HG-5).

Examining the expressions that the students used to interpret variance of intensity, differences are observed between prospective teachers and high school students. While the prospective mathematics teachers mostly used expressions such as, “Increasingly increasing graph/parabola” or “Height increases decreasingly,” the high school students mostly focused on the speed (rate) of the increase or interpreted the change in intensity with a less formal terminology such as,

…the increase is slowing down

…the increase of water level (height) decreases

…increasing parabola, but the increase is slowing down

…speed of rise increases

The tank gets narrower, thus the increase in the height of the water level increases.

The increase in the height of the water level will decrease, slow down.

The high school groups' use of these expressions seemed to come from their experience with speed concepts and graphs in Physics courses. Since they have not learned the properties of parabolas and how to interpret them in math classes, the teacher of one the classes asked students where they had learned to sketch curved graphs. Students replied that they had learned from Physics lessons.

A discussion on the expression "decreasingly increasing", during a group of high school students' presentation, also indicated the students' difficulty in interpreting curved graphs. In the discussion, one group used "decreasingly increasing" interpretation and a student in the classroom opposed them telling that they should use "the increase will slow down" interpretation claiming that there is no interpretation like "decreasingly increasing" or "increasingly increasing".

4.2.2. Creating intensive quantity; Comparing intensities

The analysis of data indicated that one prospective teachers group and seven high school groups compared intensities in interpreting the covariational situation. In this type of reasoning, the students mostly but not necessarily compare the change in one quantity to unit amount in the other quantity.

5 of the 8 groups that compared intensities used unit volume based or both unit volume and unit height based approach. Of these groups, 2 of them (one of which is a prospective teacher group) used only unit volume based approach. The other three groups used both unit volume and unit height based reasoning. The following quotations are examples that portray students' unit volume based and both unit volume and unit height based approaches.

… for this part, for equal amounts of water increase height will decrease by time. (PG-3)

… for equal amounts of height, the increase in volume will decrease; for equal amounts of volume, the height needed will increase. (HG-15)

3 of the 8 groups preferred non-unit way of comparing intensities. One of them, HG-25, compared the water level to the increase in height, and the other group, HG-22, compared the amount of increase in height and volume: “Although the amount of water increases, water level (height) will increase less compared to the amount of increase in amount of water (volume). The graph will be decreasing parabolic (parabola).” HG-20 who drew uniformly chunky graphs directly compared the amount of height and volume with respect to time: “In less time, in more height, less amount of water will be filled up (for the spherical part of the tank #3). Then it will increase constantly (for the cylindrical part). Then in the conic part, in less time, the height will be more, but amount of water will not be much.”

Sketching the graph for the first tank and interpreting the graph by stating that “the amount of water and the height of the water level have a constant ratio/direct proportion” was also common even high school groups did not compare this graph with other graphs.

Another interesting example of comparison of intensity came from HG-1 who drew non-uniform chunky graphs for graphing tank #4 by comparing the change with a constant situation (see Figure 30).

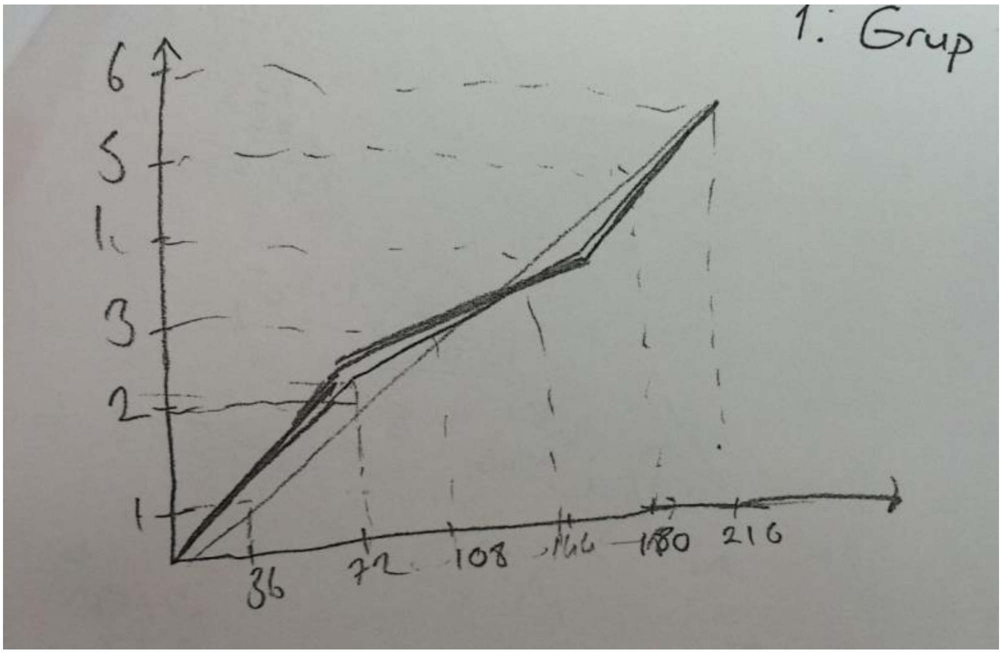


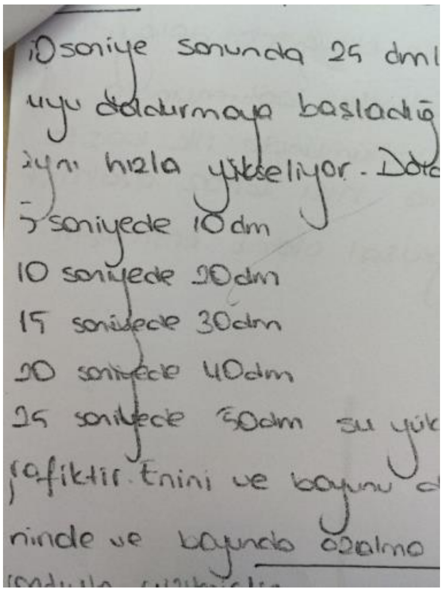
Figure 30. HG-1’s sketch for the tank #4 indicating the comparison to a line.

The group seemed to plot points or a line that connects origin with the points in the form of (36x, x) plotted on the Cartesian coordinate plane to try to show constant increase, or to compare the slopes of the piecewise linear graphs with the slope of a line that they drew. The group drew lines with same slopes for the first and third part of the tank and constructed a line with a smaller slope for the middle part of the tank where the tank's shape gets larger.

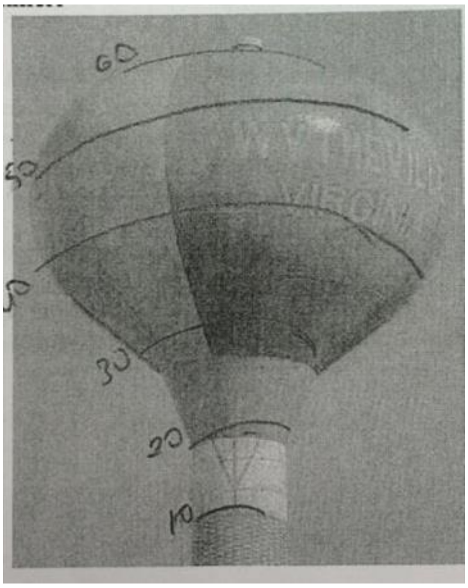
In general, the results of the study showed that the comparison of intensities with a unitization approach was the most prevalent reasoning for the students. Furthermore, the students that directly compared the increase in the amounts of quantities or the students that compared the ratio of the two quantities were also observed in the analysis. The results also indicated that the students compared the given covariational situation to a constant situation.

4.2.3. Creating an extensive quantity

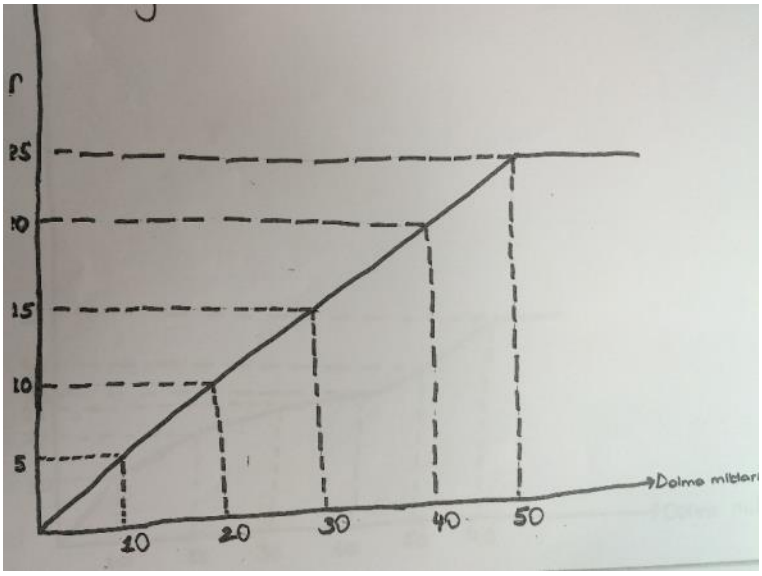
The analysis of data revealed that four groups, all of which from high schools, used extensive quantity measuring reasoning in interpreting the covariational situation. Especially in the two groups, HG-4 and HG-20, who drew uniform chunky graphs, sketches by placing values such as, x, 2x, ... on the axis (unitization) and approximating the values in the other variable. In the following figure, a group of students' unitization in their graphs, calculations, and segmentations can be seen (see Figure 31 a-c).



A



B

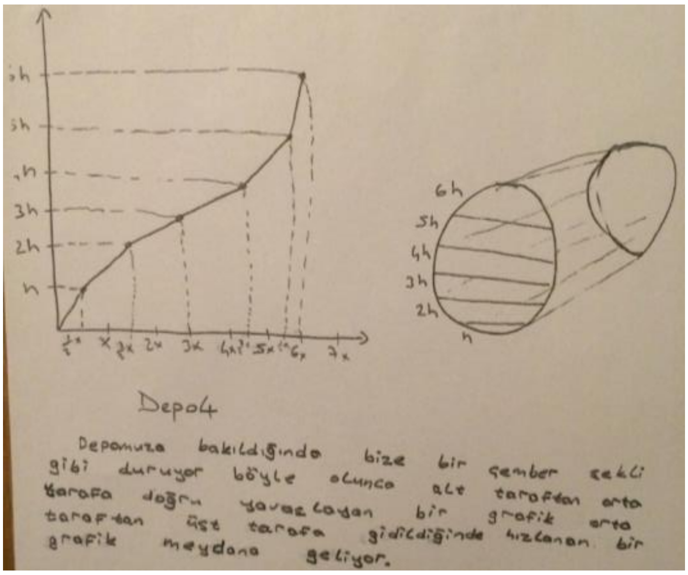


C

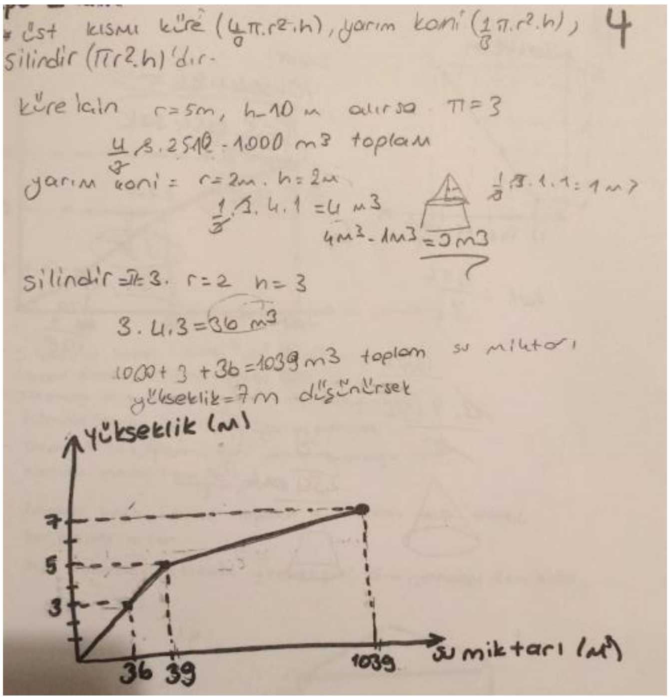
Figure 31. A group's unitization and tank segmentation (a and b) sketch (c) for the bottom part of the tank #2

In their drawings, both HG-4 and HG-20 considered the amount of gain in each chunk, and how much more or less it is compared to the previous chunk when the unitized variable increases equally at each time. HG-4 chose the volume to unitize and approximated the values for the heights while HG-20 preferred to segment the tanks and graphs into equal heights and approximated the volumes with respect to these values. These two groups used this way of thinking for all the tanks.

It is observed that students' approximations resulted in uniform chunky graphs, however when they used calculations they had to take some big parts of the tanks, and thus non-uniform graphing became inevitable (see Figure 29 a-b).



A



B

Figure 32 a-b. HG-3's calculation through non-uniform chunks.

Briefly, high school students chose extensive quantity measuring approach, and mostly as a strategy for the first tank as volume can be easily calculated by changing the height for cylindrical tank. Unitization and approximation were observed to be together, while calculation required use of non-uniform chunks.

Moreover, extensive quantity measuring method was observed to be used with comparison of intensities method as seen in HG-1: “At 6h (height) because shape is conic, in less time there will be more h but less amount.”

The data analysis also showed that one prospective teachers group used extensive quantity measuring to support or check for their solutions, but not to relate the covariational quantities and interpret the relationships (see Figure 33).

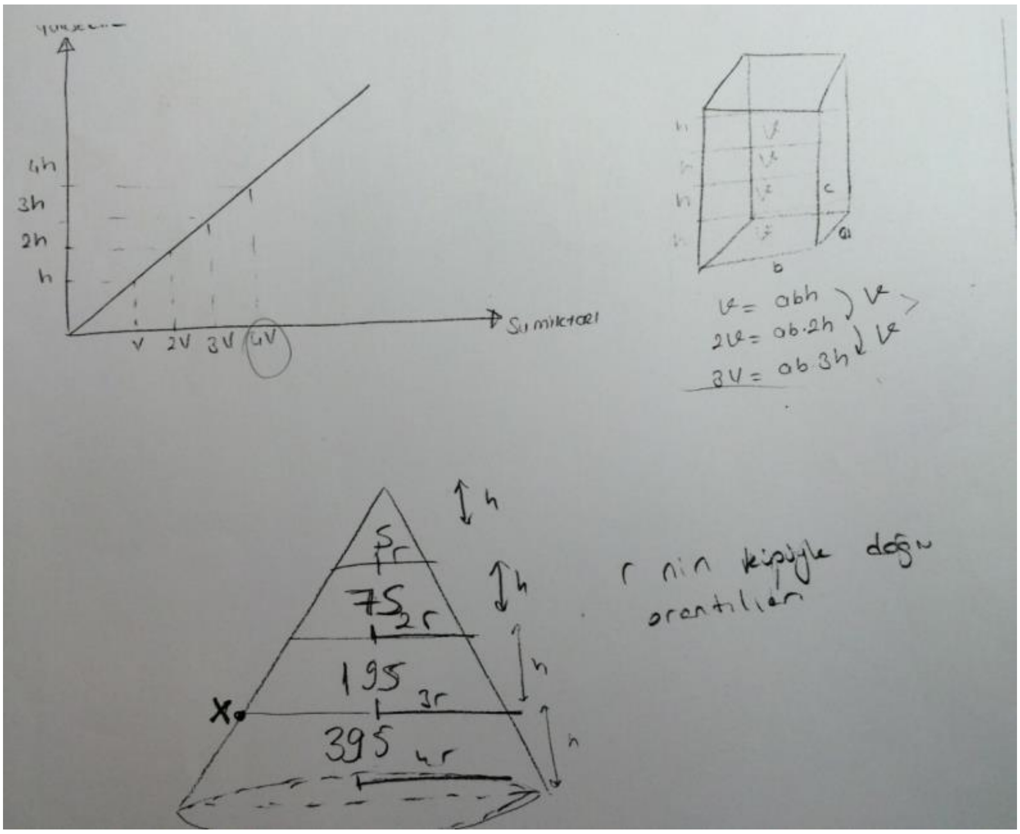


Figure 33. PG-2’s use of extensive quantity measuring.

As seen in the figure, PG-2 drew conic and rectangular prism, and sectioned them into equal heights and calculated volumes for them. By doing this, they tried to determine the differences between successive increases in volumes and attempted to see if the differences were increasing or decreasing. For example, for the conic part of the tank, they observed that the amount of change between successive parts was getting smaller (see Figure 33).

Prospective mathematics teachers mostly used quantitative reasoning for justification of their drawings or conclusions. High school students usually used it for the first step or for the whole reasoning of their work. If they use it as reasoning for all of their work it ends up with extensive quantities reasoning of chunky graphing. Briefly extensive quantity measuring appears when estimation, calculation, or convincing exists.

To sum up, the students' reasoning exists in the following forms: i) variance, ii) variance and comparison, iii) comparison and extensive, iv) extensive and variance, and v) extensive. According to the results of the study, it can be concluded that students do not use just one way of reasoning about rate of change, but they consider different ways in examining and interpreting the covariational situation. Sometimes they use two types of reasoning in interpreting the covariational situation for different tanks or different parts of the same tanks.

4.3. Relation between students' graphing types and their reasoning about rate of change

In the study, it is examined if the groups that had a particular type of graphing had a common reasoning about rate of change. According to the results of the data analysis, all of the groups that drew smooth graphs considered variance of the intensities (see Table 4). There were groups that considered variance of intensity or compared intensities or both thinking among the groups that drew graphs in smooth chunks. The students who drew uniform chunky graphs used only extensive quantity measuring reasoning or comparison of intensities reasoning beside extensive quantity measuring. The students who drew non-uniform chunky graphs used extensive measuring quantity reasoning or variance of intensity reasoning.

Table 4. *Relation between students' sketching types and students' reasoning about rate of change*

	Smooth	Smooth chunk	Non-uniform chunk	Uniform chunk
Variance	5 (3 PG)*	14 (3 PG)	1	
Variance and comparison		4 (1 PG)		
Comparison		2		
Comparison and Extensive			1	1
Extensive and Variance			1	1
Extensive			1	

* The number in the parenthesis shows the number of groups of prospective mathematics teachers

Recognizing that the smooth graphing is more complicated and requires higher level thinking than smooth chunk graphing, and smooth chunk graphing requires higher level thinking than chunky graphing; we get a consistent relationship between graphing types with the forms of reasoning about rate of change since the variance of intensity reasoning is a more advanced conception of rate of change than the comparison of intensities reasoning, which is a more advanced conception than the extensive quantity measuring reasoning.

Based on the analysis of data, for the students to be able to draw smooth graphs they should consider variance of the intensities first. Then, using the global approach and the meaning of slope in curves, they managed to draw extremely smooth graphs of covariation. Variance of intensity reasoning was also essential in smooth chunk graphing. However, focusing on local points or having improper knowledge about the slope of a curve led them to produce graphs with inappropriate transitions between chunks. The uniform chunky graphing requires extensive quantity measuring reasoning with approximating values for each chunk consistent with the level of steps that Carlson et al. (2003) proposed. However, expecting students to imagine smaller and smaller chunks and reach smoother graphs was questionable. On the other hand, these students' sketches represented the covariational situation better than the sketches of students who drew non-uniform chunky graphs or used extensive quantity measuring. Changing the slopes of the lines in chunks to interpret the change was another strategy that students who drew non-uniform chunky graphs used. They knew what slope meant in terms of change, but were not able to apply that knowledge in interpreting the curves or slope of curves. Attending on the changes in the quantities and how to interpret in the graph, they potentially used some variance of intensity approach.

Based on the results of the study, the relation between students' sketches of graphs for a covariational situation and the forms of reasoning about rate of change can be represented with the following model. This model displays the process of students' representation of a covariational situation which encompasses their reasoning and the types of sketches.

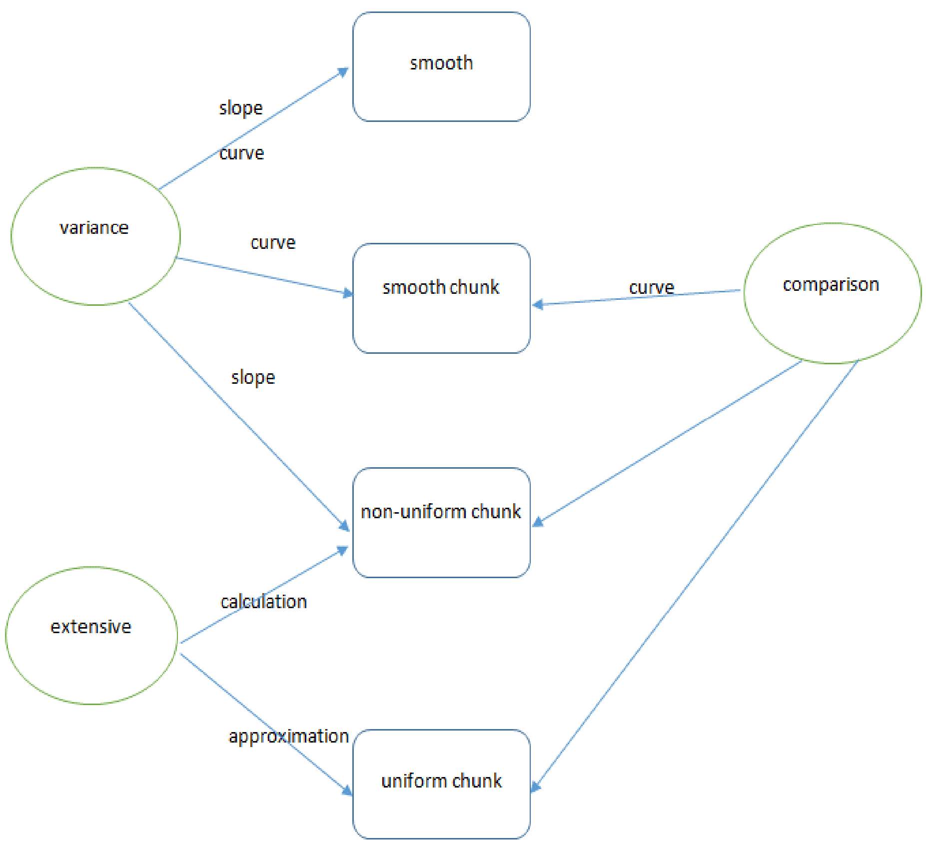


Figure 34. A model encapsulating the relations between students' reasoning and their sketches of graphs.

When students used variance of intensity forms of reasoning, it could end in three different types of graphing; smooth, smooth chunk, or non-uniform chunk. If the students made use of their knowledge about curves and their properties, they could construct graphs in smooth or smooth chunk. Smooth graphing required students to consider the slope of the curves together with using a more global approach. Variance approach without the knowledge on curves resulted in non-uniform chunky graphing when the students used the changing slope strategy.

On the other hand, reasoning about rate of change through comparison of intensities resulted in smooth chunk or chunky sketches; however, this type of reasoning seems not enough to sketch smooth graphs. Reasoning about rate of change by associating extensive quantities resulted in chunky graphing as the construction of the graphs fully focused on quantitative and local aspects rather than qualitative and global aspects such as trends of the change on the graphs. If the students used calculations in reasoning about quantities, they preferred non-uniform chunks to create easily calculable parts. If they used approximations, they produced uniform chunks.

CHAPTER 5

CONCLUSION

This chapter includes a summary and a discussion of the results. In the chapter, the implications of this research study are presented and studies for further research are offered.

5.1. Summary of the findings

In this study, prospective mathematics teachers and high school students' sketches while graphing covariation and their reasoning about rate of change while working on a covariation activity "Water Tank" were investigated.

We observed four different types of graphing in this sample of high school students and prospective teachers; smooth, smooth chunk, uniform chunks, non-uniform chunks. Their reasoning while working on the activity centered on Johnson's (2012) categorization of rate of change reasoning; creating intensive quantities; variance of intensity, creating intensive quantity; comparing intensities, and associating extensive quantities.

The analysis of data revealed that prospective mathematics teachers graphed in smooth and smooth chunk by considering variance in the intensities. High school students' sketches varied in all types of graphing, and they also reasoned about rate of change by comparing intensities and associating extensive quantities in addition to considering the variance in the intensity.

There existed a relation between students' sketches and their reasoning about rate of change. The groups that drew smooth and smooth chunk graphs seemed to be considering variance of intensity. Also, the comparison of intensities reasoning sometimes led to smooth chunk sketches. Comparing intensities together with associating extensive quantities, students ended up with chunky graphing. The difference between the reasoning of groups who drew smooth chunk and smooth graphs was related to the use of variance of intensity. The groups who sketched graphs smoothly represented the variance of intensity by imagining the slope of the curve to be changing according to the covariation situation. Moreover, the groups who drew smooth graphs had a more global approach. Chunky graphing occurred in uniform and non-uniform chunks depending on whether the students used approximation or calculation when they used extensive quantities. Some of the groups that used the variance of intensity reasoning could produce chunky sketches. However, the groups that drew graphs in chunks seemed to have a lack of knowledge on curves or knowledge about how a change could be represented by curves.

5.2. Discussions on findings

Carlson's (2002, 2003) steps remind chunky approach in calculus that defines differentiation as rate of change in little chunks and in limiting chunks to zero length. On the other hand, in this study the students do not focus on chunks or scaling down the chunks and their reasoning is rather not ratio-based (Johnson, 2012). Students prefer to use horizontal sections, radius, or width to relate the change in height and volume (Carlson 2003; Johnson, 2012). They covariate the height directly to the cross-sectional area, which represents the change in volume at that time. This fact is also supported by this study. Students just trace the change in height, without looking at the ratio of some unit or compare to the change in one quantity to the other quantity. How differentiation developed in mathematics history is also similar to this orientation. Firstly, smooth and differential calculus was founded and used, and then chunky definition which was a delta- epsilon approach of differentiation came (Castillo-Garsow et al., 2013).

It is observed that some students who considered variance in the intensity, covariate one variable depending on a continuous change in the other. They do not think in a unit-based approach. Their consideration of the constant increase in the independent variable, not an increase in chunks shows that they have a smooth thinking (Castillo-Garsow et al., 2013). However, this type of reasoning does not always result in smooth graphs. As observed in our study, sometimes smooth thinking may be bounded in chunks, resulting in a smooth chunk type of thinking (Johnson, 2011).

According to the results, if the students covariational understanding levels are ordered according to their sketches of graphs as smooth, smooth chunk and chunky; and according to their reasoning about rate of change are ordered as variance of intensity, comparison of intensities, and extensive quantity measuring, it is clear that a more advanced reasoning is linked with more advanced sketches and vice versa. For the students, who draw smooth chunk graphs, use of variance of intensity reasoning stays at local consideration of parts of the graphs. If the students do not focus on the change in chunks throughout the chunks, and do not pay attention to change in slopes throughout, then the sketches resulted in sharp transitions between chunks. This finding is in accordance with Yemen-Karpuzcu and others (2015) that report the students may just focus on the properties of the structure of the graph such as "decreasingly increasing" rather than associating the change in variables with respect to each other.

The comparison of intensities reasoning is detected in some works of the students who draw smooth chunk graphs which makes them move to a ratio-based reasoning (Johnson, 2011). Comparison of intensities is not sufficient to reason and graph covariation. Our study shows that students need to consider variance in intensities to construct smooth graphs for a covariation situation. Moreover, comparison of intensities which is related to thinking about ratios over changes in variables and how one variable change is compared to the others change, is a correspondence approach on functions. Students think about the change in one quantity and the change in the corresponding quantity, and then compare the changes. In this way, the students may miss the relation between variables. Thinking function as correspondence, instead of relation is a poor understanding of covariation (Şen-Zeytun et al., 2010).

The results indicate that the students who draw smooth graphs have an understanding of covariation with considering variance of intensity approach. They are also globally thinking about the change throughout graphing. Moreover, they are able to consider the covariational situation through slopes and they can interpret the changes for the curved graphs. Students who draw smooth chunk graphs may also compare intensities which depends on the ratio between increments of the variables. Consistent with the fact that non-ratio reasoning is more powerful (Johnson, 2012), it is again showed that smooth graphing is more powerful than smooth chunk graphing. However, there are students who draw graphs in smooth chunks and consider variance in the intensity with a non-ratio based approach by relating height directly to the change in cross-sectional area. The students' sketches of smooth and smooth chunk graphs may be related to their covariational reasoning. In Carlson's (2002, 2003) and Johnson's (2011) studies we see students who graph smoothly but cannot explain why they graph in that way. The results of this study revealed that the key difference between students who draw smoothly and students who draw in smooth chunks is considering the change in slope to represent the change. This is consistent with the interviews addressed in Carlson's studies (2002, 2003). In these studies, although the students could not explain why they graphed smoothly, they used word "slope" in their explanations of how they graphed the covariation.

If we look at the reasoning of the students who draw chunky graphs, it cannot be stated that uniform chunky thinking is powerful than non-uniform chunky thinking in understanding covariation or vice versa. Some of the groups who draw non-uniform chunky use interpretations as if they consider the variance of intensities and also use change in slopes as a base to produce graphs like smooth thinking groups. However, other groups that draw graphs in non-uniform chunks just rely on calculations, like the students who draw graphs in uniform chunks that use approximations. Castillo-Garsow and others (2013) defines chunky graphing as it is a result of using quantitative methods. However, we see that some groups that graphs in non-uniform chunks used a strategy of changing slope of lines while graphing covariation. From their interpretations, it can be concluded that they also consider about change in changes, variance in the intensities. In this point of view, they have a more powerful reasoning about rate of change than the groups who use comparison of intensities. They care about the increase in the increase of one quantity and try to transfer it to graphing. They differentiate two different increases by examining the change in slopes. They seem just don't know how to work with curves in reasoning about rate of change. The difference of this type of reasoning from the reasoning of the students who draw smooth graphs seems to be a lack of knowledge on curves and what slope or change means in curves.

In addition, it is observed that a prospective teachers group used extensive quantities to investigate the change in the volumes, which was similar to the case Hannah (Johnson, 2012) who was looking for the difference of the quantities on data table to quantify the change in change of the variables. Her reasoning couldn't make her interpret this change in change in terms of the variance in the covarying variables. In our case prospective teachers group did not take the calculations of differences of the volumes forward to quantify the variance. Their purpose was to check their arguments on the variance by quantitatively on certain shapes of tanks.

In the study, an important difference in the sketches and reasoning about rate of change between prospective mathematics teachers and high school students is noticed. The prospective mathematics teachers do not depend on quantitative reasoning except for checking their conclusions. They focus on the variation in change and clearly represent the covariational situation. However, a limited number of groups of high school students can covariate height and volume and transfer this relation to the graph. Slope consideration and knowledge on parabolas serve the basis for the reason. Their deficiency shows up in just interpreting covariation. They do not know the formal terms to interpret rate of change.

High school students learn to interpret the increasing functions, maxima, and concavity of the second-degree functions in 12th grade in mathematics class. However, before that in the 10th grade they are provided with acceleration and instantaneous speed concepts in physics classes. Tenth and eleventh grade high school students who can draw the graphs in smooth or smooth chunks may use the knowledge and skills about these topics from Physics courses. Prospective mathematics teachers' graphing smooth graphs is plausible since they have learned to use the derivative of function to interpret the increase in covarying quantities and concavity of functions. As it is seen in Table 4, prospective mathematics teachers only graphed smooth and smooth chunks, and also they considered variance in the intensity. However, high school students had various forms of graphing and reasoning. Prospective mathematics teachers' reluctance to curved graphs might help them in graphing in smooth and smooth chunk forms and their experience in derivative topic might lead them consider the variance in the intensity.

5.3. Implications of the study

According to the study, even though the prospective mathematics teachers coordinated the changes in one quantity (height) with respect to changes in other quantity (volume), almost half of them graphed covariation in smooth chunks. The reason may be that they are using the properties of parabolas that they memorized and join the graphs of parabolas without questioning the connection points and behavior of graphs just before and just after these points. They can even say, “Which was increasingly increasing, this parabola or this one?” and they may not consider what the change in slopes or rate of change is. To improve students understanding of covariation from smooth chunk thinking to smooth thinking, one way is to have students consider about the rate of change just before and after the inflection points where chunks are joint (Kertil, 2014). Moreover, in the study it is revealed that students were not able to consider the slope in reasoning about rate of change and examine the situation in a more global approach. To prevent smooth chunk graphing of covariation, we may need to familiarize students with a more global and continuous approach. Students can be guided to discover what the slope means in curves (and in lines) and what change and rate of change mean in terms of graphing curves by activities related to real life situations. This may also help students understand the definition of derivative with the limit of ratios. Considering the reality that high school students can also covariate variables (Johnson, 2011), high school students might also be led to focus on the meaning of the rate of change in terms of graphing.

According to this study, it can be concluded that the students’ reasoning about rate of change can be identified by examining their graphs for covariational situations. However, it should be kept in mind that even students that graph in non-uniform chunks may have a variance of intensity reasoning. It should not be stated that students who draw non-uniform chunky graphs just use quantitative methods. Moreover, by using “Water Tank” task, students who have a good understanding of rate of change from the ones who just focus on visual properties of curves can be recognized, among the students who draw smooth and smooth chunk graphs.

It is clear that physics instruction of kinematics may have a bearing on students’ understanding of graphs of a covariational situation. In addition to the students' reports on the role of physics course, the students' interpretation of change in graphs in terms of increase or decrease in speed or their dependence on time supports this view. This result may indicate a need to synchronize mathematics curriculum with the physics curriculum to improve students' understanding of graphs for covariational situations. At the time that our participants were taking physics course, 2011 physics curriculum were in effect and this curriculum at the 10^{th} grade suggests calculating instantaneous speed for the situations where constant acceleration exists. In the same concept, it is recommended to comment on the movements of objects based on the distance-time graphs for second degree functions. In 2013 Physics curriculum, the suggestion of "calculating instantaneous speed" is removed and it is recommended not to do mathematical calculations for instantaneous speed but recognize average speed and instantaneous speed. In 2011 mathematics curriculum, we see the properties of second-degree graphs in 12^{th} grade under the derivative concept. Students use the derivative to determine increase, maxima, and concavity of the graphs. However, the concept of functions is begun to be served in 9^{th} grade with a correspondence definition. This definition can be supported by a relation that indicates how one variable depends on the other. Students know linear graphs and relations. Starting from linear relations and carrying it to varying change situation can be helpful to understand functions as relations. In mathematics education, we should focus on properties of lines and curves, and what change and rate of change mean in lines and curves. Moreover, discussing the change in graphs will support physics concepts, such as average and instantaneous speed in the situation of acceleration. Supporting student discussions on rate of change with modeling real life situation such as in speed concept can give the possibility to integrate physics and mathematics topics in one activity.

5.4. Further research

Relations between types of graphs students draw for covaritional situations and their reasoning about rate of change can be further investigated and enlarged with other studies as this study is limited to group work. Case studies with in-depth interviews and quantitative studies testing the relations can provide meaningful information in this topic. The proposed model of the relation between students' sketches of graphs and their reasoning about rate of change presented in Figure 34 can be tested with large sample of students.

The non-ratio reasoning is more powerful than ratio reasoning. However, the ratio reasoning and chunky thinking are essential for the topics of ratio, limit, and function (Johnson, 2012). The success of the students who draws graphs smoothly on these topics comes to be suspicious. But at least their ability to think about the slope of curves, meaning it to the rate of change may help them to handle these topics easily. Moreover, Castillo-Garsow et al. (2013) report that students who hold smooth thinking have the ability to think in chunks. Smooth thinking students' capability to think in chunks can be researched in depth.

This study revealed that students who draw non-uniform chunky graphs may also be considering variance in intensity. Are there also students who draw graphs in uniform chunks and also consider variance in intensities? Or can we guide students who draw non-uniform chunky graphs and consider the variance in intensities to develop their graphing into uniform chunks and make uniform chunks get smaller? What can be the result of this orientation? Whether this can help them to improve a way to get closer to the instantaneous rate of change like in Carlson's covariational steps (Carlson et al., 2002; Carlson et al., 2003) can be investigated.

Focusing on slope and rate of change in curves is important. What can be the reason for students not attending slope and rate of change in curves? Their conception of slope and rate of change needs to be investigated in depth. Students' conception of slope on lines and curves could be a good starting on an investigation. Further, difference on the conception of the slope between students that draw smooth and smooth chunk graphs can be investigated. Moreover, studying the effect of their conceptions on rate of change and slope to their understanding of other calculus subjects such as derivative or limit definition can be meaningful.

During analysis, some difficulties students have while graphing covariation are observed. Firstly, graphing covariation for tank #4 was difficult for some groups of the students even they were reasoning correctly in the other tanks. Secondly, students may have an attempt to graph linear graphs representing constant increase with an angle of 45º. This fact caused them to draw transition points with sharp corners. Thirdly, students consider time as the independent variable. However, it does not affect their final graphs. They might use time issue to imagine a continuous decrease in their minds or just discover the time variable would act like volume variable in the case volume increases constantly. These difficulties while students graph covariation can be further investigated in depth.

Conducting further research on and referring the results of this study will enhance us to gain deep information about how students reason covariation and how their reasoning is reflected in their graphs.

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APPENDIX A

DESCRIPTION OF FINAL CODES AND SAMPLES

Codes		Definitions and examples
Graphing Types	Smooth	Smooth graphing is the graphing of the covarying variables with a completely continuous and smooth curve with smooth transitions (Kertil, 2014) over the points where intensity of change in dependent variable changes from one type to another (e.g. from decreasingly increasing to constant increase). The graph is consistent in terms of rate of change, with the situation in the task, before and after the transition points. When there is at least one smooth chunk graphing, while others are smooth, group is to be considered as smooth chunk graphing group.
	Smooth chunk	Graphing in smooth chunks is graphing the covarying variables with smooth curves of variation for chunks that are determined considering the differences in the intensity of change. However, the curve shows sharp transitions (Kertil, 2014) which cause inconsistency with the conditions in terms of rate of change before and after these points.
	Uniform chunks	Graph is consistent of linear several line segments joint together, where intervals are determined by unitizing dependent or independent variable. Each interval/chunk seems to be equal in amount.
	Non-uniform chunks	Graph consists of linear chunks/line segments joint together. Intervals of chunks are determined not equally depending on the shape of the tanks; such as narrow or wide parts or narrowing or widening parts.

Types of reasoning about rate of change	Associating extensive quantities	"Student focuses on numeric amounts of change in volume and height on interval that he chose." (Johnson 2011, p.2143)
	Constructing an intensive quantity: comparing intensities	Student compares amounts of change in variables (height and volume) with respect to each other, or compares the variation of change with a constant situation. Students' considering rate of change among the variables or increase in variables is also considered in this type of reasoning. "Volume increases more than height." "Increase in volume is more than increase in height" "Rate of change between volume and height is constant, thus linear…"
	Constructing an intensive quantity: considering variation in the intensity	Student considers the variation in the change of a variable considering its dependence on the other variable. "Height is increasingly increasing when volume increases." "Volume dependent on height increases increasingly" "Increase of height decreases" "Increase of height slows down"
Unitization	Unit height based	Student segments tank or graph into equal heights or students tells "for the same h, there will be less volume…" "For equal amounts of heights…." Independent variable seems to be height, but height could also be used for just separation in ease and no dependency can be mentioned.
	Unit volume based	Student thinks at each time equal amounts of water is poured into the tank and think about the increase in height Or student tells "for the same amount of water, h will increase more…."
	Both h and v based	Student can think two sided and interpret covariation by taking each variable as independent variable.

	Non-unit based	There is no focus on unitization or dependency. Student just focuses on the intensity of change on one variable.
	Continuous height or volume based	Student not unitizes one of the variables but assign it to be increasing uniformly, or constantly. “while volume increases, h increase increasingly” “When height is constantly increasing, volume increases more” “when (increase) volume is constant, height will increase increasingly”
	Slope consideration	Students mention about slope of the graph or changing slope, or change in slope of the graph. Steepness and more horizontal/ more vertical type of words also considered to be related to slope. Comparing slope of the parabolas in the second graph such as “second parabola has less slope” type of interpretation is not determined to be in this categorization as they are talking about the tilt of whole of the parabola for just comparing the two of them as if they were lines. Slope of the curve is not the matter of consideration there.
	Global approach on graphing	Students focus on the change of the whole graph. Mostly graph is drawn at one hand action. Local points are not considered or assigned in drawing
	Local approach on graphing	Graphing appears in parts and more than one action of drawing. Points are considered basis of graphing. Assigning on both axes strengthens the position of the points.