AN INTEGRATED USE OF VECTORIAL APPROACH WITH ANALYTICAL AND SYNTHETIC APPROACHES: A TEACHING EXPERIMENT WITH ELEVENTH GRADE STUDENTS ON QUADRILATERALS

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ABSTRACT

AN INTEGRATED USE OF VECTORIAL APPROACH WITH ANALYTICAL AND SYNTHETIC APPROACHES: A TEACHING EXPERIMENT WITH ELEVENTH GRADE STUDENTS ON QUADRILATERALS

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The purposes of this research were to investigate the contributions of the instruction in which vectorial approach was integrated with synthetic and analytic approaches on quadrilaterals to the eleventh grade students' problem solving strategies, to determine how the participants decided the type of approach to be utilized while solving problems on quadrilaterals, to specify major components of the instruction, and to examine the reflections of the participants' on the instruction.

Teaching experiment methodology was utilized to achieve the purposes of the research. Three eleventh grade students from a public high school in one of the cities located in Central Anatolian Region participated in 6-month teaching experiment, which included 37 teaching episodes during 2012-2013 spring semester and summer holiday.

Data was collected by means of pre- and post-interviews, and pre- and post-tests. Moreover, the participants were presented geometry problems containing "proof-based problems" and "classic type-problems" related to quadrilaterals at the end of the chapters. The participants' solutions to these problems were used as another source of the data for this study.

While the interviews were analyzed by means of content analysis method, the students' solutions in the pre- and post-tests, and their solutions for the problems at the end of each chapter were analyzed through descriptive analysis method. Frequency tables were provided for the findings from all of the participants' works.

According to the findings of the study, getting the skills to utilize vector representations, to integrate analytic, synthetic and vectorial approaches within a problem, and to make flexible transitions among these approaches can be stated as the main contributions of the instruction to the participants' problem solving strategies. In addition, the students utilized vectorial approach especially in geometry problems on quadrilaterals that have perpendicular diagonals, that have pair of parallel sides or perpendicular sides, and quadrilaterals that are given on analytic coordinate plane. Furthermore, their solutions indicated that there were some changes in their preferences of approaches depending on the context of the problem. At the end of the instruction, the participants gained skills in comparing the approaches in terms of their advantages and disadvantages, and deciding the approach through which they could solve the problem more conveniently and efficiently.

Keywords: Teaching experiment, mathematics education, quadrilaterals, synthetic approach, vectorial approach, analytic approach, multiple approaches and proof-based problems.

ÖZ

VEKTÖREL YAKLAŞIMIN ANALİTİK VE SENTETİK YAKLAŞIMLARLA ENTEGRE KULLANIMI: 11.SINIF ÖĞRENCİLERİYLE DÖRTGENLERDE BİR ÖĞRETİM DENEYİ

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Bu araştırmanın amaçları; vektörel yaklaşımın analitik ve sentetik yaklaşımlarla entegre edildiği dörtgenler öğretiminin, 11.sınıf öğrencilerinin problem çözme stratejilerine katkılarını incelemek, dörtgenler ile ilgili problemleri çözerken öğrencilerin yararlanılacak olan yaklaşım çeşidine nasıl karar verdiklerini belirlemek, öğretimin ana ögelerini belirlemek ve katılımcı öğrencilerin öğretim ile ilgili düşüncelerini incelemektir.

Araştırmanın amaçlarına ulaşmak için öğretim deneyi metodundan faydalanılmıştır. İç Anadolu bölgesindeki şehirlerin birindeki bir devlet okulunda öğrenim gören üç 11.sınıf öğrencisi, 37 öğretim bölümünden oluşan ve 6 ay süren öğretim deneyine 2012-2013 yılının ilkbahar ve yaz tatili dönemlerinde katılmıştır.

Veriler; ön ve son mülakatlar, ön ve son testler aracılığı ile toplanmıştır. Ayrıca, bölüm sonlarında öğrencilere dörtgenlerle ilgili ispat tabanlı ve klasik tipte geometri problemleri sunulmuştur. Katılımcı öğrencilerin bu problemlere üretmiş oldukları çözümler bu çalışma için başka bir veri kaynağıdır.

Mülakatlar içerik analiz metodu ile analiz edilirken, öğrencilerin ön test ve son testlerdeki ve bölüm sonlarındaki problem çözümleri betimsel analiz metodu ile analiz edilmiştir. Öğrencilerin tüm çalışmalarından elde edilen bulgular için frekans tabloları oluşturulmuştur.

Çalışmanın bulgularına göre, öğrencilerin vektörel gösterimlerden faydalanma, analitik, sentetik ve vektör yaklaşımları bir problem içinde entegre etme ve bu yaklaşımlar arasında esnek geçişler yapabilme becerilerini kazanmaları öğretimin öğrencilerin problem çözme stratejilerine olan başlıca katkıları olarak belirtilebilir. Ayrıca, öğrenciler; köşegenleri birbirine dik olan, kenarlarından en az bir çifti paralel veya dik olan dörtgenler ile analitik koordinat düzleminde verilmiş olan dörtgenleri içeren geometri problemlerinde özellikle vektörel yaklaşımdan faydalanmışlardır. Bununla beraber, öğrencilerin çözümleri problemin içeriğine göre öğrencilerin yaklaşım tercihlerinde bazı değişiklikler yaptıklarını göstermektedir. Öğretimin sonunda, katılımcılar yaklaşımları avantajlar ve dezavantajlar açısından kıyaslama ve bir problemi hangi yaklaşım ile daha rahat çözebileceklerini kararlaştırma becerilerini kazanmışlardır.

Anahtar Kelimeler: Öğretim deneyi, matematik eğitimi, dörtgenler, sentetik yaklaşım, vektörel yaklaşım, analitik yaklaşım, çoklu yaklaşımlar ve ispat tabanlı problemler.

Dedicated to my family

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LIST OF ABBREVIATIONS

ETHE	: Examination for Transition to Higher Education
LDE	: Level Determination Examination
MoNE	: Ministry of National Education
ÖSYM	: Ölçme, Seçme ve Yerleştirme Merkezi (*Measurement, Selection and Placement Center*)
PIF	: Personal Information Form
PKQT	: Prerequisite Knowledge for Quadrilaterals Test
PPGT	: Proof Performance in Geometry Test
QAT	: Quadrilaterals Achievement Test
SESM	: Self-Evaluation Scale in Mathematics
TDS	: The Theory of Didactic Situations
UICSM	: University of Illinois Committee on School Mathematics
UPE	: Undergraduate Placement Examination
VKT	: Vector Knowledge Test

CHAPTER 1

INTRODUCTION

Today, it has been discussed that most of the students in schools have troubles and difficulties with mathematics and especially with geometry, which has a central role in school mathematics (Leikin & Levav, 2008). Specifically, the students have relatively lower performance in mathematics tests including geometry test items with respect to the tests of other academic majors in national high stake examinations such as LDE (level determination examination), ETHE (examination for transition to higher education), and UPE (undergraduate placement examination), which are administered annually in Turkey. To illustrate, the performance of the examinees is the lowest on mathematics test with the mean of 2.35 over 20 items with respect to the other branches in LDE in 2009 (MoNE, 2009c). Similarly, the percentage of mean scores of the tests is the lowest for mathematics (14.88 %) in comparison with tests on Turkish (38.05 %), Foreign Language (42.31 %), Science-Technology (39.94 %) and Social Sciences (39.94 %) for the grade level 6 in LDE in 2009 (MoNE, 2009a). Parallel results were realized in LDE in 2009 for the grade level 7 with the percentage of mean scores of the tests for mathematics (13.33 %), Turkish (39.76 %), Foreign Language (27.07 %), Science-Technology (29.39 %) and Social Sciences (41.61 %) for the grade level 7 in LDE examination in 2009 (MoNE, 2009b).

The figure is nearly the same for university entrance examinations. The average test score is 6.1 over 40 items for the mathematics test in ETHE-2014, as the first stage (ÖSYM, 2014a). The mean score is 5.47 over 30 items for geometry test and 9.72 over 50 items for mathematics test in UPE-2014, as the second stage (ÖSYM, 2014b). In the light of all of these data, the mean score of correctly solved items are not in satisfactory level in terms of examinees' mathematics and geometry achievements in these examinations.

The situation is not different in international examinations "PISA and TIMMS" in terms of Turkish students. Specifically, the mean score of mathematics is the lowest in comparison with mean scores of reading and science tests in PISA 2003, 2006 and 2009 (MoNE, 2010). Similar unsatisfactory results are reported in terms of mean scores of mathematics test in TIMMS 1999 and 2007 (Uzun, Bütüner &Yiğit, 2010).

In order to find possible explanations to the students' low level of success in geometry and mathematics, the researchers have been conducting many studies in mathematics education field. There are studies on students' difficulties related to (a) elementary geometrical concepts such as triangles and quadrilaterals (e.g., Hershkovitz, 1987); (b) defining and recognizing geometric figures (e.g., Hart, 1981, Pyshkalo, 1968); (c) constructing and completing proof of theorems (e.g., Dimakos, Nikoloudakis, Ferentinos & Choustoulakis, 2007; Weber, 2003). Moreover, Regecova (2005) states that the students are not good at geometry and specifically, at spatial skills. According to her, the students; therefore, try to carry out some algebraic or arithmetic methods in solving geometric problems. It can be inferred from these studies that students have difficulties in geometry, which is one of the problematical part of the mathematics in secondary school mathematics.

Majority of the students have troubles with mathematics and geometry. Moreover, there have been many improvements in science and advancements in technology recently. Therefore, there are several modifications realized in teaching programs and curricula of courses. Due to these improvements, many requirements have been emerged accordingly in every field of discipline day to day. Without updating current contents and approaches in curriculum programs, it is impossible to reply these requirements. Hence, there have been many changes in most of countries' curriculum programs in most of the disciplines.

There were some investigations and regulations in various disciplines' curricula realized in Turkey as well. Specifically, mathematics and geometry curriculum programs have been developed and regulated in terms of topics to be taught and the ways how to teach these topics. Alternative approaches in teaching and learning processes have been scrutinized by the researchers in education field. To be more precise, geometry teaching through "*analytic, synthetic and vector approaches*" has been included in geometry teaching for secondary school grade levels in Turkey (MoNE, 2010a & 2010b).

Teaching geometry via vector approach in addition to analytic and synthetic approaches necessitates the utilization of vectors. For the field of mathematics, vector is a facilitator and can be used as a conceptual tool in school mathematics including analytic geometry, algebra, trigonometry and Euclidean geometry (Copeland, 1962; Hausner, 1998; Barbeau, 1988; Bundrick, 1968 & Nissen, 2000). However, vectors have been considered or included as a separate subject from the other topics in mathematics or geometry (Regecova, 2003 & 2005; MoNE, 1991 & 1992; Rumanova, 2006; Foldesiova, 2003; Gagatsis & Demetriadou, 2001). Moreover, when mathematics textbooks are examined, it would be seen that vectors are presented at the end of the chapters (e.g., MoNE, 1991). Nevertheless, vectors can be utilized in solving geometry problems efficiently. In fact, various concepts in plane geometry, plane analytic geometry, solid analytic geometry and space geometry can be taught more sufficiently by means of vectors (Bundrick, 1968). However, to be able to utilize vectors as a tool depends on a treatment by which students can solve problems through vectors or they can learn appropriate subjects via vectors.

Moreover, vectors are "useful and beneficial" tools not only for the other topics in mathematics and geometry but also for the topics in other disciplines. To illustrate, vectors have an important role and a place in various courses at university level such as linear algebra, calculus, physics and engineering etc., as it is known. Specifically, vector is an indispensable part of the units in secondary school and undergraduate physics courses such as Kinematics (*velocity, acceleration*), Dynamics (*mechanical force, torque, impulse and momentum*) and Electromagnetism (*electric force, magnetic force*) (Nguyen & Meltzer, 2003; Knight, 1995 & Küçüközer, 2009).

While this tool is common for mathematics and physics courses, as can be understood from the studies reported in the literature, there are more research conducted in physics education literature (e.g., Kanim, Nguyen & Meltzer, 2003; Knight, 1995; Flores, Kanim, & Kautz, 2004; Van Deventer & Wittmann, 2007;

Küçüközer, 2009) than in mathematics education literature (e.g., Harel, 1990; Fyhn, 2010).

The studies in physics education literature are mostly on *"developing materials for teaching vector concepts and operations on 2D or in 3D with the aid of technology*" in college level physics courses (Nishizawa & Yoshioka, 2008; Tsegaye, Baylie & Dejne, 2010; Nishizawa, Zgraggen & Yoshioka, 2009; Çataloğlu, 2006) or on "*searching for different ways to teach vector concepts effectively*" (Fyhn, 2010; McCusker, Ma & Caserta, 2014; Grant, 1971).

In addition to the wide range usage area of vectors and the requirement of the usage of vectors in mathematics and physics, the researcher gained experiences and had positive attitudes toward "*vectors*", "*teaching geometry vectorially*", "*solving geometric problems with various approaches simultaneously*" and especially toward "*proving geometric propositions and expressions by vectors*" while engaging with vectors during the preparation phase of the materials. Moreover, it was experienced and; hence, inferred in this period that synthetic approach to geometry requires so much information and knowledge about theorems, definitions, formulas etc. This is also reported in several studies (Gagatsis & Demetriadou, 2001; DiFonzo, 2010; Klamkin, 1970). Therefore, the students naturally might prefer memorizing this huge knowledge. It can be predicted intuitively, or can be understood as a result of teaching experiences, and of some studies (Kimball, 1954; Feynman, 1985) in the literature that this memorizing has negative effects on students' mathematics understanding and achievement. On the other hand, vector approach to geometry does not require huge knowledge of theorems, formulas or propositions (Klamkin, 1970; DiFonzo, 2010).

As a teacher-researcher when started to study with vectors and tried to incorporate vectors into "teaching geometry, problem solving and proving", the researcher realized that it is not so easy to work with vectors at the beginning. This is because of the fact that, it is unusual or strange to solve problems and prove geometrical properties by means of vectors till this study. However, as studying with vectors, one might get an alternative way of thinking and has a chance to develop himself in terms of approaching geometric problems from a different perspective. Trying to solve geometry problems via various approaches might have a potential to develop thinking abilities not only for students but also for teachers.

The importance of having an opportunity to make journeys among approaches and to represent mathematical ideas by various ways are stressed in some standards or studies. To illustrate, according to NCTM (1989), students should have many opportunities to compare, contrast and translate among synthetic, coordinate and transformation geometry. Students' ability to understand mathematical concepts depends on their ability **to make translations** among several modes of representations (Sfard & Thompson, 1994). Besides, Kwon (2013) asserts that students can grasp meaning of mathematical conceptions sufficiently if they are able to experience multiple representations of these concepts.

As can be guessed naturally, despite possible contributions and advantages of utilizing vectors in geometry specifically, they cannot be transferred completely to the students without treating geometry teachers accordingly. Because, they are the implementers of teaching programs in the classrooms. Ponte, Matos, Guimarães, Leal and Canavarro (1994) and Sztajn (2003) state that it is not possible to be successful in curriculum reforms if the teachers are not well educated in the targeted direction. According to the pre-interviews conducted with mathematics teachers, it was understood that most of the teachers did not have any idea or knowledge about how to solve geometry tasks by means of vectors. Therefore, most of the teachers naturally objected to teach geometry through vectors. A nice idiom from Turkish language summarizes this fact sententiously. "A *person may directly against to the knowledge that he does not have any idea about it*".

People who have not any education on geometry teaching through vector approach may think that it is unnecessary or useless. In addition, although it is more convenient to work with vectors depending on the problem cases, most of the mathematics teachers and educators consider that teaching geometry by using vector approach is more difficult than teaching geometry via synthetic approach (Ayre, 1964). Probably, this could be as a result of not having sufficient infrastructure, experience or knowledge on the use of vectors in mathematics or geometry. Therefore, this can be accepted as a prejudgment and this prejudgment can be eliminated by using appropriate curriculum designs in which there is an appropriate design of tasks presented. This situation should be taken care of and studied well enough to clarify whether the situation is as thought or not. Accordingly, if geometry teachers could have treatment on teaching geometry through vectors and if they are able to transfer this knowledge and skill to their students, teachers' prejudgment on this issue might be disappeared. Choquet (1969) qualifies the concepts of "*vector space and inner product in teaching geometry*" as a **"royal road".** However, he emphasizes that the pupils should **"not be cannonballed"** in this road without any preparation because the pupils are not in appropriate ages to grasp algebraic operations and concepts adequately. Teaching vectors to the students does not guarantee that they would gain the ability of utilizing vectors in geometry. This is also true for the geometry teachers in that they should not be left alone in this private road.

Training teachers so that they are able to transfer this skill to their students, and supplying or developing appropriate tasks or problems to be solved by means of vectors are vital to make curriculum reforms effective in real classroom environment. How can be the teachers assumed to teach such a knowledge if they do not have any background in this direction? According to the interviews conducted with seven in-service mathematics teachers before this study had started and during the study, it was understood that the teachers had the opinion of previously mentioned prejudgement. In other words, in-service mathematics teachers unfortunately had not been included in any teacher training programs related to the use of vectors in geometry teaching before the implementation of newly implemented curriculum program sufficiently in Turkey. Therefore, it might not be logical or reasonable expect teachers to teach geometry topics by vector approach or by other approaches in schools without educating them in spite of the fact that the newly developed geometry curriculum programs necessitate it. As a result, mathematics teachers do not have enough knowledge on specifying proper problem types to be solved via vectors and proper ways of utilizing vectors in their mathematics teaching. According to most of the teachers, "*vector*" is an isolated or independent subject (Rumanova, 2006; Regecova, 2005; Gagatsis & Demetriadou, 2001). After completing to teach this topic with its properties, vectors are only necessary for solving questions related to vectors merely on university entrance examinations or in class examinations to assign some grades to the students. In other words, vectors have not been utilized in solving geometric problems in other subjects. Therefore, it is very natural that they directly made objections to the new geometry curriculum program.

Furthermore, pre-service teachers do not have any training courses that cover teaching geometry with vector methods in their education faculties. Although vectors are included in some calculus or algebra courses, these courses are not in the scope of "*teaching method courses*" in departments of mathematics education in universities. To illustrate; it has been reported in the study of Bayraklı and Akkoç (2014) that pre-service mathematics teachers' pedagogical knowledge about vector approach is insufficient in each component.

Another important aspect of newly developed geometry curriculum program is related to "*reasoning and proving*" (MoNE, 2010a & 2010b). As a learning skill to be gained by the students, "*reasoning*" and especially "*proving in different approaches*" are emphasized repeatedly in this geometry-teaching program. More generally, various approaches to geometry are desired to be taught to the students. One of these approaches is "*vector approach*" to geometry (MoNE, 2010a & 2010b). However, it is understood that the teachers have not applied these reforms sufficiently in terms of proving. Cansız (2013) states that vector approach to geometry was not utilized. This is probably because of the fact that the implementers of the curriculum in classes: "*the mathematics teachers*" have no idea or knowledge how to achieve this aim. Since the teachers have been educated according to the classical or Euclidean methods in geometry, they naturally have no knowledge about how to teach geometry via vectors to the students. Therefore, some of the mathematics teachers even do not want to teach at 11th grade geometry classes in which geometry is required to be taught by using vectors mostly. On the other hand, the teachers who were obligated to teach at 11th grade geometry classes were teaching geometry by not applying predetermined targets in accordance with recently promulgated curriculum program for 11th grade geometry classes. In other words, the teachers stated that although they referred to the geometry curriculum to determine the topics to be lectured, they did not resort to various approaches (vectors or coordinates) in their teaching. As a result, they continued to teach these topics as they had taught in the past without considering the requirements of the curriculum.

It was expressed that the subject of vectors is introduced and taught as an isolated chapter from the other topics of the geometry. If the math textbooks are examined in this regard, it would be easily seen that there is not any link of vector chapter neither with previous nor with subsequent subjects. It is astonishing that parallel tendencies can be seen in national mathematics or geometry teaching programs that included vectors. To illustrate, vectors is “exactly the last chapter” of the teaching program prepared by MoNE (1991). Actually, this topic is the final unit in some of mathematics textbooks. Moreover, most of the exercises in the vector chapter are presented with vector notations and they are directly related to the properties and basic concepts and operations of vectors. Despite the fact that teaching geometry through vector approach is required in curriculum (MoNE, 2010a & 2010b), this requirement could not be reflected to the official and private geometry textbooks in the complete meaning. As an example, the publishers of the geometry textbooks only add a final test containing problems related to vectors merely. However, it was recognized by the researcher with astonishment that these problems were solvable with vector operations and they were directly related to the topic of teaching basic concepts of vectors. According to the regulations of new curriculum (MoNE, 2010b), these problems should be solved through analytic, synthetic and vector approaches so that vectors and coordinates could be utilized in the solutions. The importance of developing and solving geometry problems through multiple approaches that is analytic, synthetic, vector and transformational approaches is stressed in the studies of Nissen (2000) and Barbeau (1988). Finally, teaching geometry via various approaches especially with vectors could not be explicated by the curriculum developers and; hence, geometry teachers could not understand well enough what is required in the curriculum.

Analytic geometry is accepted as an important subject for students to develop formation of their thinking (Regecova, 2005). In addition, vectors are useful tools for different disciplines in addition to mathematics and physics such as geography, meteorology, electrical and electronics engineering, statistics and mechanical engineering (e.g., Malek, et al., 2014). However, the students do not apply analytic approach and vector approach in solving geometric problems sufficiently (Regecova, 2005; Baki & Akşan, 2014a; Rumanova, 2006; Gagatsis & Demetriadou, 2001). Baki and Akşan (2014a) found that the number of participants who utilized analytic and vector approaches are respectively 2 and 9 among 51 participants when they are asked to prove a geometric statement. However, the frequency is reported as 38 for the utilization of synthetic approach in the same statement. Very similar results are also reported for the other statement asked in the same study. Moreover, students try to solve geometric tasks by means of synthetic approach in spite of the fact that vector methods could be more effective or practical in comparison with synthetic approach especially in solving certain type of tasks. As stated implicitly, it could be because of the fact that neither pre-service teachers (*hence naturally in-service teachers*) and nor the students have experiences with vector methods in geometry. In other words, the students do not have an alternative way of approach in their background or repertoire different from synthetic approach while learning geometry or solving geometry problems. This enrichment cannot be achieved by teaching vector topics independently "as an isolated topic". It is most probably because of the fact that, "teaching vectors" and "teaching geometry by utilizing vectors" are accepted or thought as the same things. However, teaching vectors to the students does not guarantee that they would gain the ability of utilizing vectors in geometry problem solving. In order to clarify this situation, a metaphor in the following paragraph will be utilized.

In one of the most popular films "*Vizontele*" which was written and directed by Yılmaz Erdoğan (2003), the story of bringing television broadcasting service to one of the cities in Turkey is told. The official technicians deliver "*Television, Receiver and necessary equipment*" to the officials in that city. Then, they leave the city without connecting TV with receiver and without explaining the complete installation processes of equipment. Moreover, the technicians do not specify TV broadcasting periods in a day, which is important for the times when 24-hour TV broadcasting was not provided yet. The officials of the city do not have any idea on how to connect the tools and to set the direction of antenna. Most importantly, they do not know the periods of TV broadcasting. With this lack of all pieces of knowledge, it is not reasonable to expect the officers of the city to be able to set correct connections, proper direction of antenna and to switch on the TV on "*proper periods*". As experienced in this story, a parallelism can be established similarly. The expectation that the students are going to utilize vectors in problem solving may not be meaningful and fair if they learn vectors as an isolated unit from the other topics and if they are not specified "how and when" to utilize vectors in problems.

Since quadrilaterals are desired to be taught through multiple approaches in geometry curriculum program for the grade level 11 (MoNE, 2010b) and this unit is determined as the topic to be studied in this research, literature related to quadrilaterals were reviewed. In this review, there are some studies related to *"teaching quadrilaterals with various methods or with specific ways"* (e.g., Erdoğan & Sağan, 2002; de Villiers, 1994; Athanasopoulou, 2009; Monaghan, 2000; Lai & White, 2012; Dağdelen, 2011). Besides some of the researchers studied on "*geometry teaching with the aid of dynamic geometry software or via other technology tools"* (e.g., Gülbağcı, 2009; Güven, 2002; Athanasopoulou, 2009; Healy, 2000; Özçakır, 2013; Boyraz, 2008; Erbaş & Aydoğan, 2011; Aydoğan, 2007; Erez & Yerushalmy, 2006; Obara & Jiang, 2009). It is also possible to find research on *"students' difficulties and misconceptions on quadrilaterals"* (e.g., Başışık, 2010). In addition, there is a little research about *"students' concept image of special quadrilaterals"* (Duatepe, İymen & Gül, 2013) and *"pedagogical and subject matter knowledge of teachers on quadrilaterals"* (Akkaş & Türnüklü, 2014; Baturo & Nason, 1996). However; there are considerable amount of studies about *"classification and hierarchy of the quadrilaterals"* in the literature (e.g., Aktaş & Cansız, 2012; Okazaki & Fujita, 2007; Monaghan, 2000; de Villiers, 1994; Walcott, Mohr & Kastberg, 2009; Karakonstantis & Patronis, 2010; Fujita & Jones, 2007; Çontay & Duatepe, 2012; Okumuş, 2011; Leung, 2008; Fujita, 2012; Richardson, Schwartz & Reynolds, 2010). Besides, there can be found some studies related to *"defining and identifying quadrilaterals"* (e.g., Okumuş, 2011; Ergün, 2010, de Villiers, 1998; Ubuz & Üstün, 2004; Pratt & Davison, 2003). However, it can be said that there is a lack of research on *"quadrilaterals including proving abilities"* (Güven, Çelik & Karataş, 2005).

Research studies related to vectors and vector approaches are also reviewed in accordance with the purposes of this dissertation. There are some investigations about *"students' misconceptions and difficulties on vectors"* (e.g., Appova & Berezovski, 2013; Kanim, Nguyen & Meltzer, 2003; Knight, 1995; Barniol & Zavala, 2009; Kanim, & Kautz, 2004; Poynter & Tall, 2005; Flores et al., 2004; Gagatsis & Demetriadou, 2001; Van Deventer & Wittmann, 2007). While the literature is investigated in terms of vector approach, there could be found papers on "*developing vector approach proofs of some specific theorems and vector approach solutions of problems on specific topics (such as solid geometry, plane geometry)"* for university levels (e.g., Klamkin, 1970; Maynard & Leversha, 2004; Just & Schaumberger, 2004; Bourne, 1952; Amir, 1965). Besides, there exist this kind of studies at high school levels (e.g., White, 1975; Szabo, 1966; Nissen, 2000; Barbeau, 1988; Szabo, 1967; Athen, 1966a; Athen, 1966b; Glicksman, 1965, Vaughan, 1965; Troyer, 1963).

The teaching approach utilized in the research studies on quadrilaterals, which are stated above is synthetic. In other words, there are many studies on geometry teaching and learning through synthetic approach. However, there could not be found studies including vector approach to geometry. Indeed, there is especially lack of research studies on proving theorems through vectors while teaching quadrilaterals. In other words, whereas there are studies on "*quadrilaterals*" and "*vectors*" separately, there could not be reached any research study about teaching quadrilaterals through vectors. As can be seen from the literature stated above, there is a dearth of studies about the use of vectors in teaching geometry, specifically on teaching quadrilaterals. In spite of the fact that there are only few studies (Barbeau, 1988; Nissen, 2000) that present solutions to various geometry problems in high school or university geometry level through multiple approaches (synthetic, analytic, vector and transformational approaches), they are not enough in number and variety.

From different viewpoint, if the students do not have any treatment, naturally they do not think that they are able to utilize vectors in geometry. Specifically, they do not have any knowledge that a geometry problem without vector notations can be solved by means of vector methods. At this point, it can be expressed that there is a lack of problems that can be solved via vector approach in teaching quadrilaterals. To some extent, this gap is also tried to be filled by this study. Moreover, there are some research studies stressing the fact that the teaching of vectors is deficient or non-existent

(Gagatsis & Demetriadou, 2001; Regecova, 2005). In this study, these deficiencies are tried to be resolved as much as possible.

The reasons that the researcher started to conduct this research are expressed below. Although the reformed geometry-teaching program (MoNE, 2010a & 2010b) advised teaching geometry through multiple approaches, it could not supply the necessary background showing how to achieve this target. Moreover, the number and variety of the tasks, which are solvable via multiple approaches was deficient. It was also important to help teachers to teach geometry via multiple approaches specifically by means of analytic and vector approaches. In fact, it would be valuable to help geometry teachers proving geometrical expressions by means of vectors. This could be another way of verifying or stressing that reasoning and proving should be essential part of school geometry courses. Moreover, textbooks and resources do not reflect geometry teaching through vector approach in addition to synthetic approach. When the researcher tried to resolve these deficiencies, he gained positive experiences with studying vectors in geometry problem solving and he could improve himself in this regard. It was important to provide a background especially to the students by which they can solve geometry problems or prove statements via multiple approaches. Therefore, these experiences should have been shared with other people.

The researcher recognized that there is a need for research study related to vectors in geometry teaching specifically in teaching quadrilaterals for secondary school and high school geometry courses because of the gap in the literature. In other words, there is a need to verify scientifically that the students who learned geometry via vector approach in addition to synthetic approach do not have any troubles on their geometry learning or achievement. Besides, contributions of vector approach to students' geometry achievement could be reinforced by means of scientific investigations. Naturally, these contributions to learning geometry and to solving geometry problems can be understood more clearly by this way. It could be recognized that vectors are tools to be utilized in geometry teaching by indicating practical aspects of using vectors. Potentially, it is a way to diminish teachers' or some other related scholars' prejudgments on geometry teaching through vectors. Therefore, it is important to reveal or determine the contributions of vectors for the sake of teachers, curriculum developers and learners as well. To some extent, the scientific verification could be achieved by this research.

In a broad sense, there is a dearth of research studies on the "*teaching of vector*" concept and "*geometry teaching through vector approach*". Specifically, there is a gap in mathematics education literature on "*the teaching of quadrilaterals by means of vector approach in addition to synthetic approach*". Actually, there are some studies (e.g., Pettofrezzo, 1959; Schaumberger, 1962; Bundrick, 1968 & Hershberger, 1970) investigating the effects of vector approach to students' mathematics achievement scores. However, these studies focused on some of basic analytic geometry concepts and space geometry. It is understood that vector approach disappeared in USA geometry teaching programs after 1970s because of the lack of experiential studies on geometry teaching through vectors. Moreover, there could be reached a few studies on vector approach in geometry teaching. However, these studies are problematics in that vector concept is taught as an isolated or disconnected topic. Furthermore, there is almost inexistence of vector approach solutions to the geometry problems and proofs to the geometric statements in secondary school level. The deficiency in number and variety of the geometry problems, which are solvable through multiple approaches, is another gap. Therefore, it was tried to supply collection of tasks solvable by vector approach strategies in addition to synthetic approach strategies. Since these gaps were tried to be filled by this study as much as possible, this research can be accepted as "*valuable, original and significant*". If these are considered, it can be said that this study might be one of the first studies in this area.

1.1 Purpose of the Study

The purpose of this study was to identify contributions of the instruction in which vectorial approach is integrated with synthetic and analytic approaches on quadrilaterals to the eleventh grade students' problem solving strategies. In accordance with this purpose, it was also tried to determine major components of the instruction that includes the integration of vectorial approach with analytical and synthetic approaches. The third purpose was to determine how students decide the type of approach to be utilized while solving problems related to quadrilaterals under this instruction. As the final purpose, it was aimed to investigate eleventh grade students' reflections on this specified type of instruction.

1.2 Research Questions

This section contains the research questions of the present study. Answers to the following research questions were investigated in this teaching experiment as compatible with the purposes of the study.

1. What are the contributions of the instruction in which vectorial approach is integrated with synthetic and analytic approaches on quadrilaterals to eleventh grade students' problem solving strategies?
2. How do students decide the type of approach to be utilized while solving problems related to quadrilaterals during the designed instruction?
3. What are major components of the instruction in which vectorial approach is integrated with analytic and synthetic approaches on quadrilaterals at the 11^{th} grade?
4. What are the eleventh grade students' reflections on the instruction in which vectorial approach is integrated with analytic and synthetic approaches on quadrilaterals?

1.3 Definition of Important Terms

The terms necessary for the present study are defined in the following part of this dissertation.

a) <u>Quadrilateral and Area of a Quadrilateral</u>

According to Argün, Arıkan, Bulut and Halıcıoğlu (2014), a quadrilateral is a four-sided polygon. It is also possible to define a quadrilateral as a closed geometric figure with four vertices and four sides. However, in case of existence or occurrence of some misunderstandings by these two definitions, Öztoprakçı and Çakıroğlu (2013) define quadrilateral as "*a closed shape, which is composed of four line segments combining* *coplanar four points any three of which are non-collinear*. However, since this study is focused on 11th grade geometry course, the definition of a quadrilateral in the geometry program (MoNE, 2010b) will be used. According to this definition, "*a quadrilateral is a closed shape, which is composed of four line segments combining four points any three of which are non-collinear*".

Since there is not a convention about the definition of trapezoid in related the literature, it would be proper to dwell on the definition of trapezoid. It is possible to define trapezoid in two ways. Firstly, "*a trapezoid is a quadrilateral with exactly one pair of parallel sides*". This definition is called as "*exclusive definition*" of trapezoid (Usiskin and Griffin, 2008). Secondly, a trapezoid is defined as "*a quadrilateral with at least one pair of parallel sides*". According to Usiskin and Griffin (2008), the second definition is "*inclusive*" in that it includes the first one. Although, the geometry program (MoNE, 2010b) used the exclusive one, the inclusive definition of the trapezoid is used in this study. In this way, all members of the parallelogram family will be a trapezoid as well. This choice of inclusive definition is not accidental. The reasons for this preference can be explained with the facts that it results in more general concept schema and; hence, makes it easier to infer the properties of more specific quadrilaterals via more general quadrilaterals.

As Athanasopoulou (2008) found, and Usiskin and Griffin (2008) state, it is very natural that different ways of definition lead different hierarchical classifications or different quadrilateral family tree. De Villiers (1994) explains "*hierarchical classification*" as classifying a set of concepts in a way that concepts that are more general include concepts that are more particular.

Baturo and Nason (1996) define the concept of area as "*the amount of surface or region that is enclosed within a boundary*". By saying to calculate the area of a quadrilateral or a rectangle or etc., it is meant that the amount of the region bounded by that quadrilateral. To illustrate, there are problems questioning the area of triangle, rectangle, square etc. in the studies of Ayoub (2006), and Baturo and Nason (1996).

b) Vector

There are several ways of defining vectors. However, in compatible with the grade level of the participants and the focus of the present study "*a collection of all directed line segments having a given magnitude and a given direction*" will be used as the definition of vector in this dissertation.

c) *Synthetic Approach to Geometry*

Synthetic approach to teaching geometry in general meaning and to teaching quadrilaterals in specific meaning includes using some properties of Euclidean geometry and algebra. Specifically, certain types of postulates and theorems are utilized in order to prove mathematical expressions such as propositions or theorems in synthetic approach.

d) *Vector Approach to Geometry*

Vector approach to teaching geometry includes using the concept of vectors and elementary vector algebra in addition to the traditional tools that is synthetic approach. Algebra of vectors and vector concepts such as norm of a vector, addition and subtraction of vectors and inner product are used to prove mathematical statements and to solve geometry problems in this approach.

e) *Analytic Approach to Geometry*

Cartesian coordinate systems are utilized in order to verify the correctness of the mathematical statements, and to solve geometry problems. Proofs through analytic approach include coordinates (ordered pairs or triples) based on algebraic properties and formulae.

f) Multiple Approach Instruction

While teaching quadrilaterals, an instruction with synthetic, vector and analytic approaches were utilized in the study. The properties, definitions and distinctions of the approaches are explained in the literature and methodology chapter. Besides, the elements of this instruction is presented in the methodology chapter in a detailed manner.

g) *Proof-based Problems*

The terms; "*proof-based problems*" (Byrne, 2014; Raman, 2001), "*proof-oriented problems*" (Jiang, Manouchehri & Enderson, 2001; Fabrykowski & Dunbar, 2013), "*proof-focused problems*" (Byrne, 2014), "*proof-type geometry problems*" (Chinnappan, Ekanayake & Brown, 2012), "*geometry proof problems*" (Alvin, Gulwani, Majumdar & Mukhopadhyay, 2014; Golzy, 2008; Leikin & Grossman, 2013) and "*proof problems*" (Rodríguez & Gutiérrez, 2006; Leikin & Grossman, 2013; Chinnappan, Ekanayake & Brown, 2012) are used in some of the studies in mathematics and mathematics education field.

No matter how this type of problems are labeled, as a common feature of these problems, the students are required to provide deductive proofs as the solutions to the given geometry proof problems in a broader sense. To illustrate, Chinnappan et al. (2012) preferred to use the name "*proof problems*" and define them as the type of items, which include proving the given statement. Furthermore, Leikin and Grossman (2013) use "*geometry proof problems*" and classify these problems into three categories, which are namely: verification problems, discovery problems and computational problems. A problem is categorized as "*verification problem*" if it solely requires testing the correctness of a mathematical proposition by means of proving. A problem is regarded as "*discovery problem*" when it necessitates developing some conjectures, analyzing developed conjectures and then proving. Finally, a problem is classified as "*computational problem*", if the problem requires finding the length of a line segment, the measure of an angle, and perimeter and area of a geometric figure. In this dissertation, there are problems of these three types of problems.

Alvin et al. (2014) describe "*geometry proof problem*" as the problem, which contains a figure, some assumptions related to that figure, goals that need to be established about the figure, and the set of axioms that need to be used. However, a problem can be classified as a proof problem, if it does not include a given figure. In that case it could be another objective that the students are required to draw the necessary figure according to the given information or assumptions.

This type of problems are labeled as "*proof-based problems"* in the present study. The participants are expected to investigate and discover the properties of triangles or quadrilaterals. This discovery or investigation can be achieved through proving via multiple approaches as the solution of given proof-based problems.

h) Elegance of Proofs or Solutions

The word "elegance" is defined as "(*of a scientific theory or solution to a problem*) pleasingly ingenious and simple", in the oxford dictionary. Although, Posamentier and Krulik (1998) do not give the explicit definition of elegance, they used "*clever*" and "*efficient*" adjectives accompanying to "*elegant*". Sinclair (2003) simply uses "elegant" to evaluate a solution as "good" and to draw a distinction between "*good*" and "*not-so-good*" mathematical products. Dreyfus and Eisenberg (1986) specified a model consisting of some components to evaluate whether a mathematical entity is elegant or not. However, they do not specify the exact definition of elegance. Instead of defining the term of elegance, they define more general term aesthetic "as the branch of philosophy that provides a theory of beautiful" and use aesthetic synonymously with elegance. Rouse (2005) define elegant solution in terms of various disciplines. While defining elegant solution, she brought optimality into the forefront for engineering, computer sciences and mathematics in that the maximum product or gain is obtained through the least effort or cost.

Sinclair (2003) states that there is no certain consensus among participants in her study to decide that a solution is more aesthetic or elegant. However, the participants could appreciate the existence of different solution methods for the same problem. She accepts this as the main target of dealing with elegant or aesthetic solutions in that the students could have an opportunity to develop a "*value-oriented sense of mathematics*". According to Sinclair (2003), a "*value-oriented sense of mathematics*" makes it possible to include students in classroom as more personal and humanistic. Moreover, students' own experiences would be more lasting and meaningful by this way. Dealing with or pursuing elegant solutions help the process of mathematical inquiry for the mathematicians. As cited from the study of Dreyfus and Eisenberg (1986); "Papert (1980) and Poincare (1956)" put aesthetics or elegance in the central position in the process of mathematical thinking. Therefore, it can be concluded that being able to appreciate beauty of mathematics is a beneficial in terms of mathematical improvement.

As observed in the study of Sinclair (2003), Dreyfus and Eisenberg (1986) stresses the ambiguity in evaluating mathematical values (solution, proof, statement etc.) in terms of elegance. However, when deciding the elegance, they consider the level of prerequisite knowledge, clarity, simplicity, length, conciseness, structure, power, cleverness and whether it contains elements of surprise. According to them, these are key parameters to decide whether a mathematical output is elegant or not. In fact, they conclude that if there is a need to utilize more arguments in terms of prerequisite knowledge, the elegance of the output will be decreased. Dienes (1964) accepted an argument as powerful when the desired conclusions are reached by means of some basic prerequisite knowledge. Since power is one of the indicators of elegance according to the model of Dreyfus and Eisenberg (1986), a single assumption or method through one step leading one or more conclusions can be evaluated as elegant.

In the study of Dreyfus and Eisenberg (1986), five different ways for the proof showing the irrationality of $\sqrt{2}$ were presented to the mathematicians. They were asked to rate the solutions in terms of elegance with underlying reasons. While rating the methods in terms of elegance, these mathematicians stated the "*simplicity*" and "*minimal amount of mathematical background in the solutions*" as the most frequent two reasons to evaluate a solution as more elegant. This is closer to the definition given in the oxford dictionary. However, Posamentier and Krulik (1998) notify that a simple solution is better than an elegant solution if the simple one is in hand whereas the latter one cannot be reached easily. Therefore, elegant solutions are not necessarily to be simple according to them.

Related to the focus of this study, it is seen that the term of elegance was used for the solutions in several studies (e.g., Krech, 1968; Wexler, 1962; Barbeau, 1988; Glicksman, 1965 & Lord, 1985). Barbeau (1988) labeled a solution as "elegant" if it is more aesthetic in comparison with the other approaches. While comparing analytic, synthetic and vector approach solutions, Wexler (1962), Glicksman (1965) and Lord (1985) evaluated the proof of a theorem or a solution of a problem more elegant if it is developed through vector approach. Specifically, Lord (1985) qualified the proofs of theorems including the properties of inner product and cross product as elegant.

i) Auxiliary Elements

Polya (2004) defines auxiliary elements as the tools that present possible opportunities to make progress in a solution of a problem. Adding new lines into given geometric figure or adding unknown terms into given literal expressions are stated as two examples for auxiliary elements in the book of Polya (2004, p 46).

CHAPTER 2

LITERATURE REVIEW

This chapter includes related literature review of the present study under the titles; Vector concept with its history, the use of vectors and students' misconceptions and difficulties with vector concepts and vector operations. These are directly related to vector concept. After that, literature on analytic, synthetic and vectorial approaches with their definitions and elements will be presented. Since the integration of vector approach is one of the main of the main focus for this study, the history of vector approach in geometry teaching with the studies in the literature is included. In the subsequent sections, related literature on multiple approaches, comparison of these approaches, time issue, and vector approach as an alternative to similarity will be given respectively. Theoretical framework and how it leads the present study will be presented. This chapter ends with presenting the literature summary.

2.1 Vector Concept and its History

Vector is a Latin-originated term in mid-19th century and has the meaning of "*carrier*". However, it has special meanings as a term in different sciences such as biology, physics and mathematics. In biology science, "*vector*" is an organism, typically a biting insect, that transmits a disease or parasite from one animal or plant to another (Oxford Dictionary). In physics, a vector is a quantity with direction and magnitude (Kwon, 2011; McCoy, 1968; Schuster, 1962; Adams, 2003 & Stewart, 2003). Displacement, force and velocity can be given as examples to this quantity. Therefore, Sezginman (1974) briefly defines vector as the representative of vectorial quantities.

Since the focus of this dissertation is on geometry teaching, the definition of vector is taken into consideration in the context of mathematics. Argün, Arıkan, Bulut and Halıcıoğlu (2014) and Willmore, Barr and Voils (1971) define each element of a vector space $\left(V, F, \oplus, \square\right)$ as “vector”. However, they qualify this definition as *formal, mathematical or abstract.* In fact, this definition is appropriate for university level algebra courses. Argün et al. (2014) also define vector concept through the help of the concepts “*directed line segment”* and “*equivalence class”.* According to their definition, “*a vector is the equivalence class of all directed line segments (or arrows) having the same magnitude and the same direction*”. However, the concept of “*equivalence class*” can be cumbersome for the students despite it is included in high school mathematics curriculum (MoNE, 2011). In fact, there are some studies defining vectors simply as *directed* (Polya, 2004; Robinson, 2011; Schuster, 1962) or *oriented* (Hausner, 1998) line segment.

White (1975), Grant (1971) and SMSG (1963) define vector as an ordered pairs or array of numbers having direction and distance. However, Rainich and Dowdy (1968) add translation term to this definition. Hence, they state that vector is a *collection of all pairs of points that can be changed into each other by translation*.

In the light of the definitions presented above, as Chiba (1966) states, there are two ways of defining vectors: algebraic and geometric within the boundary of mathematics. In algebraic definition, *vector is an ordered pair of real numbers in two-dimensional plane and an ordered triple of real numbers in three-dimensional space*. In geometric definition, *vector is a directed line segment with a starting point A and a terminal point B and denoted by* $\overrightarrow{AB}$.

These definitions are compatible with Hillel (2002)’s classification of vector representations. There are three modes of representing vectors: abstract mode, algebraic mode and geometric mode. In abstract mode, vector is accepted as any element of vector space that is defined with a set of axioms. This mode is compatible with Argün et al. (2014)’s and Willmore et al. (1971)’s definition. In algebraic mode, a vector is an n-tuple of real numbers. This mode is parallel to the definition of Rainich and Dowdy (1968), White (1975), Grant (1971) and SMSG (1963). In the last mode:

that is geometric mode, a vector is simply a directed line segment. The scholars who define vector in geometric mode is stated in the preceding paragraph.

Finally, since the focus of this dissertation is related to geometry teaching at secondary school level, the vector is defined in geometric mode, which is considered compatible with students' grade level. As a result, the definition of vector will be used as "*a collection of all directed line segments having a given magnitude and a given direction*" (Protter & Morrey, 1985; Bourne, 1952) in this study.

As Nguyen and Meltzer (2003) report, the definition of vector varies from country to country. Specifically, whereas the definition of vector includes "direction" and "magnitude" in U.S., orientation and sense are included instead of direction in the definition of vector outside the U.S. In fact, orientation has the property of line of action and sense has the property of the way where the vector points. This is also the case in Turkey. The geometric definition of vector includes magnitude, sense and orientation.

History of Vectors

Despite of various and wide range of application fields of vectors, they are not based on a long history. Isaac Newton (1643-1727) studied quantities that are called as vectors today; such as, velocity and force. However, he did not use the word "vector" at all in his book "*Principia Mathematica* (1687)". Indeed, vectors emerged as directed line segments in physics and mathematical disciplines in the time of Aristotle and nearly in the middle of 19^{th} century. Therefore, Knott (1978) underlines that the concept of "*vector is older than its name*" which means it was utilized without having a special name. Moreover, Mobius used directed line segments and represented them by letters. He also conducted some operations with these directed line segments such as adding them and multiplying them by constants in his book "The Barycentric Calculus" in 1827. In this book, Mobius studied centers of gravity and projective geometry. The mathematicians: Argand, Wessel, Gauss, Servois and Mobius tried to use vector representations for geometrical treatment of complex numbers during the years 1830s. However, these mathematicians could not be successful in this treatment (Knott, 1978).

William R. Hamilton and Herman G. Grassman were mathematical physicists who firstly utilized vectors in their studies to solve a lot of problems on some topics in physics such as motion and force (Scott & Rude, 1970; Ayre, 1965). Hamilton and Grassman worked on vector concepts in the years 1840s independently. However, both of the mathematicians utilize vectors embedded in quaternions. In the beginning of 1880s, the theory of vectors was developed by considering vectors as independent from quaternions and then vector concept became an independent entity by the studies of J.W. Gibbs and Oliver Heaviside. Hence, we learn that the theory of vectors is mainly the consequences of the studies of Gibbs and Heaviside (Knott, 1978). After the development of this concept since these years, it has been utilized in many branches.

It is important to state that the development of vectors was grounded to the idea of utilizing coordinates of points that is "analytic geometry" of Rene Descartes and Pierre Fermat in the 17^{th} century (Scott & Rude, 1970). According to Robinson (2011, p. 2), Descartes' idea of utilizing analytic geometry is a great contribution to geometry in the year 1637. Hence, it can be said that the vector concept owes its discovery or emergence to the existence of analytic geometry. It is necessary to state that Euclidean geometry was established nearly in the periods BC 300 by Euclid. He set up well-known postulational method of geometry in his well-known book "Elements" containing five basic postulates (Scott & Rude, 1970; Stephenson, 1972).

2.2 The Use of Vectors in Various Fields

As stated in the first chapter, there are several application fields of vectors such as physics, geometry, biology, computer graphical programs, maps, engineering, military, geographic information systems, computer supplemented agriculture and production and so on. The use of vectors in mathematics and out of mathematics will be presented according to the conducted studies in the literature.

Szabo (1966) qualifies vectors as a beneficial and useful link between geometry and algebra. He states that vector space concepts have a potential to relate geometry and algebra. Since school algebra and geometry are mostly being treated independently, vectors can be utilized as a tool to solve this trouble. According to the study of Hershberger (1971), the underlying reason for many mathematicians' recommendations of utilizing vectors is unifying characteristic property of vectors.

Glicksman (1965) expresses the potential of vectors to teach analytic geometry in various dimensions by means of the geometric meaning of algebraic manipulations through vectors. Specifically, Bundrick (1968) states that the relationship between plane (2D) and solid analytic geometry (3D) can be enhanced by vectors. In addition, he also points out that most of the concepts in solid analytic geometry can be developed more than adequately via vectors with respect to traditional analytic geometry teaching. Besides, Athen (1966b) specifies that lines, planes, circles and conic sections in analytic geometry are the application fields of vectors in higher grades of German high schools. In addition, Chiba (1966) emphasizes that vector equations of parabola, hyperbola, ellipse and circle is a way to get the standard equations of these figures.

Commission on Mathematics (1959), Hershberger (1971) and Bundrick (1968) recommend the use of vectors while teaching trigonometry, algebra and analytic geometry in high school mathematics courses. According to Krech (1968) and Hershberger (1971), vectors are useful to understand the connection between a complex number and its conjugate.

In several studies, vectors are used to prove some of the famous theorems such as Ceva, Menelaus, Desargues Theorem and Pappus theorem. Some of the geometry textbooks and curriculum programs, which utilize vector methods as a teaching model, included the proofs of these theorems conducted with vector methods (Chiba, 1966).

As Bundrick's (1968) states, vector algebra plays an important role in several courses at the college level. For example, vectors have an important role in advanced mathematics courses such as algebra, real analysis, geometry and linear topology and physical sciences (Pettofrezzo, 1966). Moreover, since derivative of vector functions has direct applications, vectors have important place in physical sciences, meteorology, aeronautics and engineering (Chiba, 1966; Schuster, 1961). Moreover, Schuster (1961) also underlines the necessity of utilizing vectors as a tool for the social scientists to make analysis in their studies. Besides, Ayre (1965) states the increasing importance of the use of vectors in many areas. To illustrate, he states that vector methods are useful in the solution of the problems on space travel and ordinary air travel on the earth. In addition, utilizing vector approach is stated as a possible way to make transition to linear programing and game theory. This transition can be achieved by means of solving system of linear equations with two or three variables through column vector representations and inner product (Glicksman, 1965).

Bengtsson (2014) suggest vector approach while teaching the theory of Laplace, Fourier and z-transforms to the university students. In the teaching model that he suggests, the signals and transforms are presented as vectors and inner products respectively. Since signals have direction and magnitude, they are treated as vectors. The use of inner product is also necessary in this model. According to the experience of this researcher, the vector approach presents an opportunity to the students "*a whole new level of the understanding of transforms*".

Küçüközer (2009) indicated that vectors are indispensable tools for velocity, acceleration, force (dynamic or mechanics) and electricity and magnetism in the introductory courses of physics. Since these topics include vectorial properties, the students need to understand and comprehend vectors adequately to have ability to conduct vectorial reasoning and manipulations correctly (Knight, 1995; Nguyen & Meltzer, 2003; Flores et al., 2004). Indeed, Knight (1995) qualifies the role of vectors as "*the essential component of the mathematical language of physics*".

McCusker and Caserta (2014) stated that since vector is a basic concept for the courses like statics, control theory and computer graphics in electrical and electronics engineering, mechanical engineering and computer engineering, it is important to enhance students' performances in these courses by providing competent vector knowledge. Moreover, in the report prepared by Aksu (1985), it is pointed out the need for strong infrastructure containing vector and complex numbers knowledge for the university level courses that are specifically necessary for electricity and electro technique. According to Miller (1999), mathematical operations of points and vectors are the base of conducting and utilizing computer graphics and modelling systems.

Biometric recognition such as "fingerprint recognition" (Karar & Kaur, 2015; Sharma, Mishra & Yadav, 2013), "facial authentication system" (Malek, Venetsanopoulos & Androutsos, 2014) is another application field of vectors in the Euclidean domain. Feltens (2009) explains the way of utilizing vectors in face recognition systems and geodesy as another application of vectors. He specifies vector approach calculations as the most practical and the most accurate way of finding the distance of a point to an ellipsoid among many other methods. More interestingly and indirectly, Eghbal-zadeh, Lehner, Schedl and Widmer (2015) utilized vectors in "music similarity estimation tasks" and "artist recognition tasks".

In the light of wide range of usage and area of vectors in mathematics or in other fields as presented above, it is important to include vectors into geometry teaching. Therefore, it is necessary to develop an efficient instruction plan for the teaching of vectors not as a separate topic in geometry.

2.3 Students' Misconceptions and Difficulties with Vector Concepts and Vector Operations

There are some studies reporting that students have difficulties and misconceptions with various vector concepts at various grade levels in middle and high school and even in university grade levels. Gagatsis and Demetriadou (2001) evaluated these errors as non-accidental according to the model developed in the study of Movshovitz, Zaslavsky and Inbar (1987). These common mistakes are interpreted as a result of some quasi-logical process that makes sense to the students in some way. Having foreknowledge of students' common errors on vector concepts and searching for underlying reasons for these errors are important to consider for the sake of realizing an efficient teaching of vectors. Tall (1992) evaluates examining students' approaches and difficulties as necessary.

Overall, students' challenges with vectors in physics are mostly related with vector magnitude, direction, addition, subtraction, dot product, cross product and unit vectors according to studies mainly conducted in physics and physics education field (Barniol & Zavala, 2009; Nguyen & Meltzer, 2003; Knight, 1995; Kanim, 1999 and Flores et al., 2004; Deventer & Wittmann, 2007 and Dimitriadou & Tzanakis, 2011). Therefore, related literature about students' difficulties and misconceptions on vectors will be presented in the following subtitles.

2.3.1 Students' Difficulties with Components of Vectors

Vectors can be represented analytically that is to say that a vector can be utilized by means of having knowledge about its coordinates. Therefore, in some problem cases, a student may need to calculate the components of a vector so that any of the x or y components might be necessary for his solution strategies. However, Gagatsis and Bagni (2000) and Gagatsis and Demetriadou (2001) found that students have difficulties in writing correct components of a vector. In addition, McCusker, Ma and Caserta (2014) state that students have troubles in finding components of vectors correctly. Resolving a vector into its components is a problematic issue that the students have weakness in this area. According to the authors, this weakness may even result in students' not completing their engineering educations.

2.3.2 Students' Difficulties with the Magnitude, Direction and Sense of Vectors

Students have various difficulties and misconceptions related to magnitude and direction of a vector, which are elementary concepts of vectors. As Ortiz (2001) specifies, students have difficulties on basic vector operations. Küçüközer (2009) qualifies these difficulties as "serious" in her study. She states that this difficulty can be explained with incomplete understanding of the concept of position vectors. The problem can be resolved by means of conceptual understanding of position vectors. Gagatsis and Bagni (2000), Gagatsis and Demetriadou (2001) and Flores et al., (2004) are the other researchers reporting the difficulties related to sense, orientation or direction of vectors.

Barniol and Zavala (2009), Küçüközer (2009) and Nguyen and Meltzer (2003) reported almost the same misconceptions in terms of students in their studies. According to their findings, students accepted two vectors as having the same direction if these vectors faced "not exactly" but "nearly" to the same direction. To illustrate, it is enough for two vectors to have the same direction if the vectors are located in the same quadrant with a common tail. Although the vectors have different angles with x-axis, the students were observed that they accepted these vectors as having the same direction. Dimitriadou and Tzanakis (2011) also reported another misconception type of the students on vector direction. They expressed that students do not take care of the directions of the vectors when they need to deal with proportional magnitudes of vectors.

2.3.3 Students' Difficulties with Vector Addition and Subtraction

Vector addition and subtraction are indispensable operations of vectors not only for vector topic merely but also for the ways of utilizing vectors in mathematical problem solving and in physics. However, students' difficulties with the addition and subtraction of vectors are probably the most frequent difficulties related to vectors as understood from research studies in various fields. As Knight (1995) and Flores et al. (2004) report, students do not have sufficient conceptual knowledge on basic concepts such as magnitude, direction and vector addition. Gagatsis and Bagni (2000) and Gagatsis and Demetriadou (2001) encountered students' errors while they were conducting vector addition and subtraction especially during the process of replacing the vectors.

Poynter and Tall (2005) points out that students have problems with adding and subtracting vectors geometrically when their tails are not intersected at the same point. In other words, the students have troubles with these operations if the vectors are given with non-standard positions. Similarly, Watson (2002) reaches the same findings that students have serious problems with the addition of vectors if they are at non-standard position. Besides, Nguyen and Meltzer (2003) find that the students have difficulties with addition of vectors especially when they belong to 2D.

According to the studies of Nguyen and Meltzer (2003) and Knight (1995), the students confused the parallelogram method and triangle method (or tip to tail method) while conducting vector addition. In this type of difficulty, students were observed that they move vectors so that their tails to be intersected and then they combine two tips of the vectors. The students have challenges with adding and subtracting vectors graphically and they could not answer qualitative problems about addition and subtraction of vectors according to the findings of Flores et al. (2004).

Nguyen and Meltzer (2003) states that the students conducted incorrect actions while subtracting two vectors. They explain this error with students' memorization like “*combine the tail of one to the tip of the other*’’. Deciding which of the tips must be the tail of the resultant vector or how to adapt this method to the triangle method are most likely the sources of difficulties in vector subtraction. Moreover, these researchers observed that students made some mistakes with the directions and magnitudes of the vectors while moving them in order to realize vector subtraction.

Küçüközer (2009) found that students have problems with applying Pythagorean Theorem correctly in addition to their difficulties with addition and subtraction of vectors. Moreover, she stated that students have difficulties with solving the resultant vector problems that necessitate the application of geometry knowledge. According to the findings of her study, some of the students did not take care of direction and sense of vectors in vector addition problems. To illustrate; no matter what the direction and sense of two vectors with the same size, their resultant or addition vectors is the same as seen in the Figure 2-1.

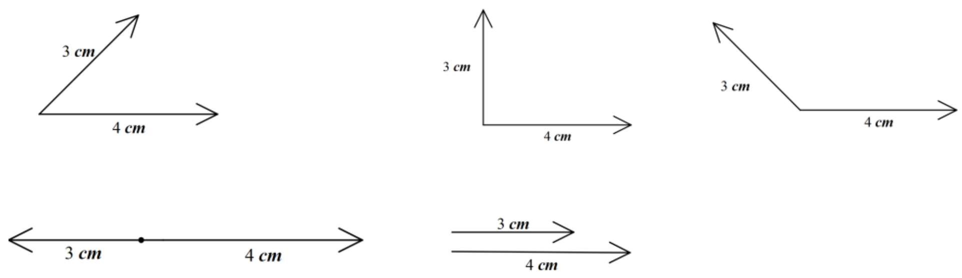


Figure 2-1 Addition of the same pair of vectors in various positions

2.3.4 Students’ Difficulties with Dot Product

Dot product plays an important role during the teaching of vectors. It is also a significant tool while solving geometry problems by means of vectors. However, students have some misconceptions and difficulties while calculating the dot product of two vectors. According to findings of Deventer and Wittmann (2007)’s study, students’ success is not at the satisfactory level to calculate inner product.

Students have difficulty in understanding how the dot product of two perpendicular vectors yield a “zero result” and how the result of the dot product can be “negative” for two vectors with obtuse angle (Christensen, Nguyen & Meltzer, 2004). Gagatsis and Bagni (2000) also observed that students determined the angle between two vectors incorrectly while calculating their dot product. In addition, there were also some misconceptions of students detected like "*dot product of two vectors result in a vector*" (Appova & Berezovski, 2013) and "dot product has a direction" (Van Deventer & Wittmann, 2007). Furthermore, Appova and Berezovski (2013) also encountered with students' misconceptions on vector projections and multiplying a vector with a constant number.

2.3.5 Confusing Scalar Relations with Vector Relations

Until learning vectors, the students have been learning mathematics through real numbers without being in the case of considering any "directional" aspect. In other words, students are used to study with scalar quantities. However, with the learning of vector quantities, the learners need to consider direction concept in their calculations while solving problems. This naturally brings with possibility of some difficulties or confusions in terms of students.

The students might think that they could manipulate with vectors as they get used to manipulate with real numbers or scalars. This situation can be similarly predicted by mathematicians or mathematics educators who study on vectors. Related to this issue, there are several cases in which students have confusions or difficulties in this regard. For example, Grant (1971) reported that students had a confusion about the addition of vectors with the lengths of vectors in this vector addition. Specifically, two vectors with the lengths of 3 and 4 are added but the result of their addition is not equal to 7. This was found as strange for some of the students. Gagatsis and Demetriadou (2001) state that students have this type of difficulties when they convert a relation for vectors to a relation for their lengths. When comparing vectors, Dimitriadou and Tzanakis (2011), and Gagatsis and Demetriadou (2001) emphasize that students' main difficulty was as a consequence of their misconception that equality of magnitudes is sufficient for equality of vectors. In other words, when the lengths of two vectors are the same, so are their directions for these students.

Aguirre and Erickson (1984), Flores et al. (2004), Gagatsis and Demetriadou (2001) and Demetriadou (1994 & 1995) found as a common and one of the most frequent faults that students "*treat a vector as a scalar*" while adding and subtracting vectors without taking into consideration the directional information of vectors. Besides, Flores et al. (2004), and Gagatsis and Demetriadou (2001) reach the conclusion that students subtracted only the lengths of vectors instead of the vectors. Furthermore, Appova and Berezovski (2013) also highlight students' misconception of not being aware of the distinction between vectors and scalars. For example, they encountered that participant students tried to add or subtract a scalar with a vector (or add subtract a vector with a scalar), and as a result of this operation, they got a vector or scalar object.

Gagatsis and Bagni (2000) and Demetriadou and Gagatsis (2001) classified and defined this misconception, as "*vector is equivalent to a line segment*". Among the "sense errors, addition and subtraction errors, errors in dot product and errors in using coordinates", the error of accepting a vector as a line segment and operating accordingly is the most frequent type of error in their studies. The percentage of this error is the highest with the value 49 % among other errors for vector approach students.

2.3.6 Students' Difficulties Related to Angles

Vectors cannot be thought without angle concept. To illustrate, "angle" plays an important place in geometric representation of vectors and geometric definition of dot product. However, students have challenges related to angle concept especially while determining the angle between two vectors correctly. This challenge can be found in several studies. For example; Pavlakos, Spyrou and Gagatsis (2005) states that students have difficulties while determining the correct angle between two vectors as the angle gets larger and larger. Moreover, Gagatsis and Demetriadou (2001) observed that there is a tendency of using incorrect angle among the participant students. Interestingly, the exact values $0°$ and $180°$ between two vectors is a source of challenge for the students according to the study of Nguyen and Meltzer (2003). This finding is very similar to the findings of Pavlakos, Spyrou and Gagatsis (2005). In addition, Deventer and Wittmann (2007) observed students' challenges with relabeling the angles during the process of moving vectors to make the tails of the vectors concurrent.

2.3.7 Students' Difficulties with Vectors in 3D

One of the underlying reason for the use of vectors in geometry is that vectors have a unifying feature or advantage, which presents possibility of time saving because vectors in 2D can be thought as facilitator and prerequisite for vectors in space. However, Nishizawa and Yoshioka (2008) states that students have difficulties with vectors in 3D. Moreover, Hinrichs (2010) found that undergraduate and graduate physics students have difficulties to write position vectors in 3D by utilizing spherical unit vectors.

2.3.8 Students' Difficulties with Vectors in Miscellaneous Contexts

There are several fields for vectors to be utilized, as stated in the earlier parts in this dissertation. Despite the fact that vector solution for some problems in different contexts is the same, students have difficulties with vector solutions dependent on the context of the problem. This conclusion can be accepted as interesting. To illustrate, Deventer and Wittman (2007) found that the participants had difficulties in solving a problem on vectors if the problem is asked in physics context instead of asking the task in the mathematical context. In other words, students' were found as more successful in the times based on mathematical area than physics area. Dependently, Aguirre (1988), Aguirre and Erickson (1984), and Deventer and Wittman (2007) reach the conclusion that the students are less successful in solving a problem when it is presented in physics context in comparison with the same task is presented in mathematical context. Similarly, Ba and Dorier (2010) report students' difficulties with vectors while adding them in the context of velocities and forces at undergraduate grade levels. In addition, Flores et al. (2004) also stated students' troubles on the field of application of vectors in physics. Furthermore, Nguyen and Meltzer (2003) found participant students' strengths and conceptual confusions about elementary vector concepts when they are given in graphical forms. Moreover, Gagatsis and Bagni (2000) encountered with students' errors when the vectors were expressed analytically.

Regecova (2003) points out that students have problems with understanding of concept of vector itself, as well as with its application in solving the different mathematical tasks at secondary schools. As understood from this study, the same findings were also determined in the studies of Rumanova (2004a, 2004b).

2.3.9 Students' Difficulties with Vectors in High School and University Physics Courses

Although students start to learn vectors at various grade levels from 8^{th} grade to 12^{th} grade (*this may vary from country to country*) in geometry, mathematics or physics courses, there are some studies (Barniol & Zavala, 2009; Nguyen & Meltzer, 2003; Knight, 1995; Flores et al., 2004; Kwon 2013 & 2011 and D'Angelo, 2010) reporting that students have difficulties even with elementary vector concepts in high school and university levels. To illustrate, Ba (2007) state that the students' knowledge on vectors is reported as deficient and almost non-existing by the physics teachers in their physics lessons on Motion and Forces.

According to test results in the study of Knight (1995), students have difficulties with vector addition, determining the direction of vectors, dot product and vector product in spite of the fact that 86 % of the students had learnt vectors previously. Moreover, Barniol and Zavala (2010) state that some students continue to have troubles related to basic vector operations although they have attended some introductory physics courses in the universities. Kwon (2011 & 2013) states this fact in the manner that students experience some troubles while utilizing vectors and they have challenges in understanding vectors although they have some experiences with vectors.

2.4 Sources of the Difficulties

In the preceding section, literature review on students' misconceptions and difficulties related with vector concepts and vector operations was presented. In this part of dissertation, possible explanations and underlying reasons for these misconceptions and difficulties, and possible solutions to these challenges given in the literature will be presented.

In the literature, underlying reasons for students' difficulties on vectors are briefly attributed to "*teaching vectors independently among various fields*", "*the use of non-standard vector representations or notations*", "*use of vectors in prototypical positions*" and "*use of different methods for vector operations*". Moreover, these difficulties are also explained with "*insisting of teaching vectors through traditional ways*". In addition, *using ordinary symbols "+" and "="* which have different meanings in real numbers is also a source of difficulties while they are used in vector operations. Besides, "*inadequate use of figures*" and "*teaching algebraic aspects of vectors without presenting their geometric meanings or counterparts*" are asserted as possible reasons for these challenges. In addition, "*inadequate linking among graphical, verbal and symbolic representation of vector concepts*" is presented as another reason. Lastly, "*that the teaching of vectors remains at a procedural or algorithmic level and that not being reached a conceptual quality"* is also indicated as another factor leading difficulties and misconceptions among students. These reasons are tried to be presented comprehensively in the following paragraphs.

Dimitriadou and Tzanakis (2011) reported that vectors have been taught differently in physics and mathematics as if "*vectors in physics*" and "*vectors in mathematics*" are different things. In addition, negative effects of teaching vectors separately in mathematics and physics lessons as if they are different concepts have been reported in the study of Ba and Dorier (2010). A kind of isolated teaching of vectors in different courses results in some conflicts and difficulties in students' minds. Moreover, use of different notations, representations or methods for vectors in different courses might increase probability of emergence of these troubles. Furthermore, students may think that there is not a convention or agreement among different courses in terms of a single concept "vector". As an example, while calculating the resultant vector in vector addition, the use of parallelogram method in physics context and the use of triangle method in geometry is reported as a possible source of difficulty for students according to these researchers. Parallel to this information, Poynter and Tall (2005) reach the conclusion that students relate vectors with various contexts such as triangle method is related with displacement and parallelogram method is related with forces. This cannot be thought independent from vector instruction followed in classes. Hence, this is a potential reason for students to gain an intuition possibly resulting in some challenges. Therefore, this should be overcome as much as possible. Related to this situation, Deventer and Wittmann (2007) found that although the same tasks in the form of mathematical and physical environments were directed to the students, their scores are higher in mathematical form of tasks in comparison with physical form.

Similar results are also indicated in the studies of Aguirre and Erickson (1984) and Aguirre (1988). As the findings of these studies, students performed worse on a vector task with a physical context when compared to the same task in a mathematical context. It is very interesting that a student is able to solve a problem when it is given in mathematical context; however, he cannot solve the same task when it is given in physics context. This can be explained possibly with the facts that

a) vectors are taught differently or independently in physics and mathematics,
b) the teaching of vector remains too procedural that students develop a rote learning.

Nishizawa, Zgraggen and Yoshioka (2009) states that the most difficult topic that students experience is “vectors” in linear algebra at precollege grade levels. These researchers explain students’ difficulties with the fact that the graphical, verbal and symbolic operations of vectors are taught in an isolated or separated manner. In addition, Appova and Berezovski (2013) noticed that they did not encounter any cases in which the participants make use of figures or pictorial depictions in their solutions while utilizing algebraic objects. The authors proposed the use of appropriate figures as a prevention for the sake of decreasing the possibility of having misconceptions related to the vectors. In order to develop a conceptual understanding and; hence, to get rid of these misconceptions, drawing related figures is of utmost importance in the course of teaching vector space topics as stressed in the studies of Gueudet-Chartier

(2002; 2004). Besides, Tabaghi (2010) found that visualization is an efficient way to overcome misconceptions and difficulties of students on some abstract topics of linear algebra such as vector projection. In the light of these experiences, it can be asserted that vectors should be taught through geometric counterparts or geometric meanings and operations should not be conducted only in a procedural or algorithmic manner. However, Poynter and Tall (2005) state that while students adding vectors, they mostly complete their operations procedurally and they follow some routine or predetermined rules or algorithms. They cannot develop any reasoning related to their operations. Hence, their operations do not reflect any conceptual learning.

Tsegaye, Baylie and Dejne (2010) related students' difficulties on understanding and visualization of vectors with the teaching of vectors via traditional ways, which can be defined as "paper and pencil environment". There are also various researchers indicating the difficulty and insufficiency of teaching vectors in paper and pencil environment. Therefore, they suggest the utilization of some technological tools that can be used in the teaching vectors.

According to Dimitriadou and Tzanakis (2011), the use of "+ and =" symbols for vectors as used for arithmetic operations of real numbers is another explanation for the difficulties. Indeed, it is defined as "*epistemological obstacle for the students*". To illustrate, while adding the natural numbers 10 and 10, the result equals to 20 and all of these are symbolized with "10+10=20". However, there is possible range of results between 0 and 20 for the addition of two vectors with the same magnitude "10" although this addition is also represented by $\overrightarrow{F_1} + \overrightarrow{F_2} = \overrightarrow{F}$ containing the same " + and = " symbols. Students possibly have troubles in understanding how the addition of two vectors with the magnitude 10 results in 10 again despite of the same addition "+" symbol. Therefore, it is important to underline and emphasize the difference between the addition of real numbers and addition of vectors in order not to have any difficulties in this direction.

Kwon (2013) specifies another explanation for students' challenges on vectors. He states that students would probably experience some troubles while learning and conceptualizing the concepts free vector and position vector without comprehending equivalence relation completely. According to Kwon (2013), free vector and position vector concepts are of utmost importance during the transition from geometric representation to symbolic representation.

Implicitly, there are some differences among various countries' curricula in terms of notations and definition of terms related to vectors. This might be also another source of difficulties experienced in vector teaching for students. To illustrate, in spite of the fact that vector is defined as "*the set of all line segments having direction, sense and magnitude*" in Turkey (MoNE, 2010a & 2010b), the discrimination between direction and sense is not considered in USA (Nguyen and Meltzer, 2003).

2.5. Definitions and Elements of Synthetic, Analytic and Vector Approaches to Geometry

2.5.1 Definitions of the Approaches

Synthetic Approach

According to Rainich and Dowdy (1968), synthetic method is the well-known way of studying geometry. This method focuses directly on figures and obtains the properties of the geometric figures from other properties by means of logical reasoning. It was recognized in the related literature that synthetic approach (Coxford, 1991), metric approach (Krech, 1968), traditional approach or traditional proof method (Chou, Gao & Zhang, 1994; Wexler, 1962; Bundrick, 1968), Euclidean method (Pitta & Gagatsis, 2001; Rainich & Dowdy, 1968; Scott and Rude, 1970) and classical approach (Gagatsis & Bagni, 2000) are used synonymously. In this study, "*synthetic approach*" will be used.

Analytic approach

Analytic approach to geometry refers to the use of coordinate plane while studying geometry. Chou, Gao and Zhang (1993) labeled analytic approach as coordinate approach in which geometric conditions are represented by the equations of point coordinates. Rainich and Dowdy (1968) explain analytic method as the study of geometry through the use of coordinate system. The underlying reason for studying on coordinate system is to derive the properties of figures by means of coordinates; that is, *numbers*. Coordinate approach and analytic approach are used synonymously in the literature. In this study, analytic approach will be used.

Vector approach

Szabo (1966) defines vector approach as the way of utilizing algebra of points and translations while solving problems and discovering the properties of figures in geometry. According to Krech (1968), vector approach is the employment of vectors as a tool while proving theorems. Furthermore, Krech (1968) also emphasizes the discrimination between "vector approach to geometry" and "vector geometry". While the former one is a way of teaching Euclidean geometry, the latter one is the study of vector spaces. Besides, Bundrick (1968) gives a definition for vector approach specific to teaching of plane analytic geometry. According to his definition, vector approach utilizes the vector concepts and elementary vector algebra with the integration of the use of traditional tools. Chou, Gao and Zhang (1993) define vector approach as a different way of proving theorems in which geometric conditions are handled by vector equations. In addition, Rainich and Dowdy (1968) express vector method as the method of treating geometry by directly studying on figures but not with their representations by numbers like analytic method. This treatment includes derivation of the properties of figures by means of computations with expressions and equations.

In the light of related literature and gained experiences during all phases of this dissertation, the researcher defines vector approach as "*the method of studying geometry and; hence, making explorations in geometry through the vector tool, its properties and operations*".

It was recognized in the related literature that vector approach (Just & Schaumberger, 2004; Vaughan & Szabo, 1973; Krech, 1968; Zou et al., 2012; Bengtsson, 2014), vector method (Scott & Rude, 1970; Glicksman, 1965; Vaughan, 1965; Copeland, 1962), and vector geometry (Gagatsis & Demetriadou, 2001; Troyer, 1963; Giles, 1964) are used synonymously. As the name for this approach, the researcher preferred to use vector approach for this study.

The similarities and differences among the approaches

Coxford (1991) states the differences among three of the approaches as follows. In synthetic approach, while figures are studied on planes without any references of points or lines; in analytic approach, there is a need to specify an important point or vertex of given geometric figure as the reference of the system on which the figure is located. In vector geometry; however, the movements of geometric figures on planes with or without any references are of the main principle while studying the geometry.

The differences among approaches in proving theorems are stated in the study of DiFonzo (2010) as while analytic proofs utilizing coordinate plane and relying mostly on algebraic formulas and properties, synthetic proofs are conducted by referring geometric properties, theorems or postulates such as Pythagorean Theorem and similarity-congruence postulates. However, vector addition, subtraction and scalar multiplication are necessary tools in vector approach proofs when proving hypothesis. Besides, Chou, Gao and Zhang (1993) state that vector approach is different from coordinate approach in a way that theorems are proved by eliminating points instead of coordinates. By regarding these difference, Rainich and Dowdy (1968) indicate that vector method takes place in between synthetic and analytic methods. The distance concept is taught by means of coordinatizing a line in synthetic and analytic approaches; however, the dot product is the way of presenting the distance concept. Moreover, since dot product contains the angle concept in its nature, it is also a way to teach angle concept in vector approach. Finally, although congruence and similarity has a considerable role in synthetic approach, they are not resorted in vector approach solutions at all.

2.5.2 Elements of the Approaches

Elements of synthetic approach

Coxford (1991) specifies elementary elements of synthetic approach as "the *intersection, the measure of segments, angle and the measure of angle, parallelism and congruence and similarity*".

Specific to this study, the synthetic approach utilizes (1) Euclidean properties (2) the law of cosines and the law of sines (3) similarity and congruence theorems of triangles and (4) the solution of two equations with two unknowns in this teaching experiment.

The prerequisite knowledge for synthetic approach was composed of concepts and topics that the participants learned in mathematics and geometry courses at grade levels 9 and 10. At least the participants can be accepted, as they are familiar with these topics.

Elements of analytic approach

Coxford (1991) indicates elementary elements of analytic approach as "*distance, midpoint and line equations*". Troyer (1963) adds "*parallelism, orthogonality and length of segments with given ratios*" to prove theorems and solve problems in geometry by means of analytic approach.

Elements of vector approach

According to Vaughan and Szabo (1971) and Rainich and Dowdy (1968), "*translation (or vectors) and points*" are the primitive terms in vector approach. While developing a geometry instruction in this approach, Stephenson (1972) defines lines and planes in terms of points and vectors. Despite the fact that parallelism is lectured through "Parallel Postulate" in synthetic approach, the idea of vectors with the same direction is utilized in vector approach.

While teaching (a) "*Quadrilaterals*" for the first term of the 11^{th} grade level (b) the topics that would be lectured before "*Quadrilaterals*" specifically "*triangles and plane geometry*", vectors are utilized as an approach in addition to synthetic and analytic approaches. In fact, the participants of the study are anticipated to integrate the traditional tools with the concept of vectors and basic vector algebraic operations in this approach. In other words, they are expected to be able to utilize vectors in geometry, in problem solving, in proving geometrical statements and in different kind of representations. Specific to this approach, students utilize (1) the theorem "*for two non-zero vectors* $\vec{a}$ *and* $\vec{b}$ *, their inner product* $\langle \vec{a}\ , \vec{b} \rangle = 0$ *if and only if* $\vec{a}$ *and* $\vec{b}$ *are perpendicular to each other*" (2) the definition and properties of inner product of vectors and (3) addition and subtraction of vectors.

Instead of solving problems on triangles and quadrilaterals by means of synthetic methods, which is usual, it is important and necessary to develop primarily vectorial solutions for geometric problems and to construct proofs of geometric statements by means of vectors, which are unusual situations. Hence, this part of the study was developed earlier from any stage of the research for this study. While solving geometry problems including proof based problems in quadrilaterals and triangles; solutions, proofs and content of instruction to be followed in this research were developed by the researcher originally in compatible with the definition and components of vector approach.

2.6. History of Vector Approach

Although the history of vectors dated back to 1830s, the use of vectors in geometry teaching became popular in the year 1959. As cited from the study of Stephenson (1972), vector approach was firstly presented by Dieudonne and Choquet from France in an Organization for European Economic Co-operation seminar in 1959. However, Ba and Dorier (2010) give information that vectors were used to present Thales' theorem in the 9th grade for the first time in 1947 in France. In addition, homothety, analytic geometry and barycenter are the other examples for the use of vector approach in the same periods in France according to these researchers.

It is obvious that in the years between 1960 and 1970, there can be reached many textbooks, research studies and dissertations (such as, Stephenson, 1973; Bundrick, 1968; Hershberger, 1971 and Johnson, 1967) on vector approach as understood from the related literature. Furthermore, a geometry program with the aim of integrating algebra and geometry was developed by University of Illinois Committee on School Mathematics (UICSM) in 1963 in USA. Vaughn and Szabo were two writers of the book “*A Vector Approach to Euclidean Geometry*” with two-volume text. In this valuable book, geometry is designed to be taught through vectors in addition to the use of Euclidean geometry and analytic geometry tools. In addition, Vaughan (1967) describes the development of materials and the course as:

> "Three dimensional Euclidean geometry is developed as a theory of an inner product space T— the set of translations— acting on a set £ of points— the points of Euclidean space (p:24).

There were some in-service geometry teacher preparation courses and seminars prepared in the summers before the administration of the mentioned geometry program. However, it is understood that this program was not widely utilized in USA for these years (Stephenson, 1973).

Related to this program, Johnson (1967) showed that the twenty-two postulates of SMSG geometry could be proved by assuming the properties of a Euclidean vector space with inner product in his doctoral dissertation. While teaching of analytic geometry was mostly utilizing the algebra of real numbers, a Cartesian plane and some properties of Euclidean geometry during the years 1960s, vectors with the properties of elementary vector algebra were included to the teaching of analytic geometry as a tool in the years between 1960 and 1970 (Bundrick, 1968).

Nissen (2000) express that several approaches such as –coordinate (analytic), vector and transformation approaches were developed and tried to be integrate to geometry teaching during the years 1960s and 1970s. However, the developers of these approaches could not be successful in a satisfactory level. Instead of teaching geometry through multiple approaches as a system, supporters of the approaches tried to prove that these approaches had a potential to supply a perspective to the students. In fact, the contributions of multiple approaches has been appreciated in American National Standards.

Athen (1966a, 1966b) reported the use of vectors in geometry teaching in the middle and high school levels since 1955. As he states, geometry was taught by vector methods in these years. Indeed, vectors were presented as translation of points while vectors were being taught in German Gymnasiums. Although vector teaching started at middle grade levels and continued at high school levels, application of vectors in geometry teaching was not included at each level. As an application, vectors were used in analytic geometry of lines, circles, spheres and conic sections.

2.7 Multiple Approaches in Geometry Teaching

2.7.1 The Need for Multiple Approaches

According to the report prepared by the Cambridge Conference on School Mathematics (1963), it is understood that "there are many different routes to follow in teaching geometry and that each has its advantage”. The value and importance of geometry teaching through multiple approaches with miscellaneous contributions have been stated in several studies (e.g., Barbeau, 1988; Nissen, 2000; Kwon, 2013; Bundrick, 1968 and Gagatsis & Demetriadou, 2001), in various programs (NCTM, 1989; MoNE, 2010a, 2010b & 2010c) and reports (CCSM, 1963; CEEB, 1959). In fact, geometry is an appropriate science to be taught by means of multiple approaches. According to Barbeau (1988), geometry is a particularly fruitful area for synthetic geometry, analytic geometry, vectors, trigonometry, complex numbers, and transformation geometry.

Nissen (2000) states that students should be encouraged to solve geometry problems by means of several approaches as much as possible. In addition, the students should have an experience in this direction so that they have opportunity to conclude that there is no single way of solving geometry qualified as the best way. Actually, geometry and searching for the simplest way of learning geometry has a long history. A king in Egypt in the time period of 2000 BC had asked to the scholars around him whether there existed a simple way of learning geometry or not. He was replied that there was not a royal way of learning geometry. Roughly 4000 years later, three mathematicians expressed nearly the same expressions. According to Chou, Gao and Zhang (1994), there is not such a royal way of learning geometry; however, the students should be presented geometry with several alternatives.

Sfard and Thompson (1994) attribute students' comprehending of mathematical concepts to being able to make transitions among different representations. Similarly, Janvier (1987) and Kwon (2013) assert that students can take in the meaning of mathematical conceptions sufficiently if they are able to experience multiple representations of these concepts. Since geometry teaching through analytic, synthetic and vector approaches necessitate different ways of representations because of their natures, students who learn geometry through multiple approaches also need to learn their ways of representation rituals. Therefore, students might have an opportunity to make journeys among several approaches. This might me an indication of the fact that students comprehend mathematical concepts sufficiently to some extent by means of learning geometry through several approaches.

2.7.2 Curricula and Reports Suggesting Multiple Approaches

It is suggested in the Curriculum and Evaluation Standards (NCTM, 1989) that geometry should be studied from multiple approaches. These approaches are specified namely as synthetic, coordinate, transformation and vector approach. Moreover; vector, analytic and synthetic approaches are strongly and repeatedly recommended in high school geometry curriculum in Turkey (MoNE, 2010a & 2010b) during the teaching and learning process of geometry. Specifically, the use of appropriate approach by deciding according to its convenience was one of the important objectives of these teaching programs. Besides, the Commission in Program for College Preparatory Mathematics (1959) reported that teaching Euclidean geometry without including algebra is a kind of defect that should be remedied. Therefore, Kwon (2013) states the importance of learning geometric objects synthetically, analytically and vectorially for the sake of realizing a complete teaching. According to him, transitions among various approaches, making miscellaneous combinations among thinking systems of these approaches and finally examining and studying geometric figures and objects are all clues for an epistemological shift in terms of students.

Among the history of synthetic, analytic and vector approaches, Euclidean geometry (synthetic) is the oldest and vector geometry is the youngest one. Sfard (1995) establishes a parallelism between the development of a mathematical concept within the history of mathematics and the development of that concept for the learners. When the history of mathematics is examined, it can be observed that there were great and rapid steps or progress in seventeenth century that is the years of expressing the thoughts of ancient mathematicians with the rituals of coordinate geometry systems. In other words, it is the time of for the birth of analytic geometry. This is explained with the fact that the development of new tools gave opportunity to solve problems in variety of ways and to produce new problems to be solved (NCTM, 1989). By combining these facts together, there can be expected great improvements in success and conceptual developments of students after they start to learn geometry via vector and analytic approaches in addition to synthetic approach.

2.7.3 Specified Contributions of and Underlying Reasons for Multiple Approaches

Kwon (2013) states that while solving geometry problems, applying multiple approach strategies is a valuable engagement. According to him, this value is because of the fact that solving a problem with a variety of techniques presents a different and a new consideration to observe the problem and the solution has a potential to enhance students' further understanding. In addition, Stephenson (1972) qualifies each approach as a worthwhile part in students' mathematical education life since these approaches present alternative concepts and techniques for problem solving processes. In addition to these contributions, another contribution of teaching geometry through multiple ways is stated as its potential to enhance students' creativity (Lee, Tay, Toh & Dong, 2003). Students might gain a different insight by each of the approaches and this gives an opportunity to understand better the whole picture in geometry (Barbeau, 1988).

University of Illinois Committee on School Mathematics (UICSM) was the prominent institute in combining synthetic, analytic and vector approaches as multiple approaches in geometry teaching during the periods of 1970s. According to UICSM, teaching geometry through multiple approaches provide possibility of integrating algebra, geometry and trigonometry without an isolated manner or partitioned format. In addition, this way of teaching geometry makes students feel themselves as privileged because of the fact that teaching geometry via various approaches has an original aspect.

Scott and Rude (1970) explain the reasons for including "analytic and vector approach" to geometry teaching as follows. These two approaches:

a) make easier to conduct and understand proofs,
b) to some extent help students be more successful in geometry,
c) are useful tools not only for mathematics but also for other sciences and engineering,
d) have motivating power in geometry teaching, and
e) enhance logical thinking and deductive reasoning because of their natures.

According to the Curriculum and Standards for School Mathematics (NCTM, 1989), developing students' reasoning abilities is one of the emphasized objectives in school geometry. In addition to synthetic approach, utilizing analytic approach is suggested in order to achieve this development.

An approach might be more appropriate or convenient for the solution of a problem or might present more elegant solution way for a problem. This possibly gives discriminating sense to the students. Alternative solutions have potential to open new doors for the sake of users in their mathematics and geometry courses or implicitly in their daily lives. Specifically, algebra can be alternative to arithmetic used for word problems; trigonometry or vectors can be alternative to similarity. Therefore, students might be able to appreciate the power and beauty of mathematics, by experiencing these alternatives and by having journey among these combinations. Besides, Ayre (1965) evaluates different ways of solutions as an experience that facilitates learning of further mathematics and other science courses.

Gagatsis and Demetriadou (2001) found that students who utilized two approaches: vector approach and synthetic approach together in their solutions are more successful than the students who utilize only one of the two approaches in the geometry achievement test. This could be evaluated as an evident indicating the contribution of multiple approach to students' geometry achievement. In addition, considering individual differences among students, Baki and Akşan (2014a) suggest to teachers the necessity of employing various techniques and avoiding single way of teaching geometry in classrooms according to the finding of their study.

Kwon (2013) expresses that a student prefers the most convenient way of solution among available ways according to his comfort if he is capable of solving problems through several approaches. Similarly, Bundrick (1968) found that the students in the vector treatment group in his experimental study could be able to utilize vector approach and traditional approach (*combination of synthetic approach and analytic approach*) together. However, the students in the control group could utilize traditional approach only. The students in vector group were also observed that they preferred the approach by which they felt in confidence and convenient. In addition, Schuster (1961) states that students have a chance to solve problems through several approaches and to decide the most appropriate way of solution according to the type of problem if they are taught how to prove theorems by means of several approaches depending on the appropriateness of time and students' levels and fields.

2.7.4 Approaches Complete Each Other

Robinson (2011) and Hausner (1998) state the need for preliminary analytic geometry and vector courses to learn the relationship between algebra and geometry. They emphasize that the knowledge of algebra and geometry cannot be accepted as complete if one of them is missing or deficient. As algebra and geometry have a completing role on each other, intrinsically analytic, synthetic and vector approaches have complementary role for each other because they are accepted as having a bridge role between algebra and geometry.

The interaction between algebra and geometry is accepted as an efficient way for developing students' problem solving skills (NCTM, 1989). According to these standards specified by NCTM (1989), students should gain the ability of transitions among approaches, comparisons and integration of approaches as much as possible. In this way, they are able to reach a conclusion that a certain type of problems can be solved much better by means of a certain approach. An approach might be a hint, mnemonic, facilitator or complementary of another approach for the students who are able to utilize various approaches while solving problems. As an example, utilizing analytic approach together with vector approach might be easier than synthetic approach to reach the properties of a geometric object.

Klamkin (1970) and Lines (1965) utilize synthetic approach (*trigonometry knowledge*) and vector approach (*inner product and vector product*) as completing each other in their studies. Moreover, Miller (1999) states complementary strengths of coordinate based geometry and vector geometry in addition to their power of supplying requirements for computer graphics. Furthermore, Schuster (1961) underlines the more explicitness of the advantages of analytic approach in geometry problem solving in terms of students if they have knowledge on inner product and vector product.

In a different manner, according to Coxford (1991) the description of geometric concepts by synthetic approach might be more cumbersome in comparison with analytic or vector methods. In fact, algebraic description can be preferred instead of synthetic approach because of easiness and directness without yielding emergence of any conflict. Essentially, he expresses the possibility of different descriptions as the complimentary aspect of approaches for each other. To illustrate, while a student represents required geometric object in an approach and then he can continue to solve the problem through another approach. In brief, being aware of and capable of utilizing multiple approaches is important for the sake of developing students' problem solving capacity.

2.7.5 Studies Recommending Vector Approach

So far, some suggestions and results of studies related to multiple approaches were presented. After this point, since vector approach is one of the main focus for this study, research studies that are specifically on vector approach will be examined in this part. There are some mathematicians and mathematics educators (such as Schuster, 1962; Troyer, 1963; Glicksman, 1965; Robinson, 2011) strongly advise including vector approach to the geometry teaching. Furthermore, it is interesting that Indian Parliament discussed the advantages of utilizing vector approach to geometry teaching rather than one based on transformations as understood from the study of Howson (1980). In addition, Wong (1970) found that 40 percent of the mathematicians and mathematics educators desire including vector approach in their geometry courses.

However, 22 percent of the participants did not want to include vector approach in this research and 38 percent of the participants were unsure about this issue. In fact, the researcher explains nearly the same percentage for unsure participants as the vector approach supporters with the fact that the teachers had not enough knowledge about vector approach. Moreover, he adds that vector approach together with transformation approach is still in experimental stage and there is a few materials readily available for these approaches. Besides, 73 % of the participants wanted to utilize coordinate approach. Hence, this rationale justifies the researcher in that whereas the teachers have necessary knowledge and infrastructure about analytic approach in geometry teaching, the situation is not true for vector approach knowledge.

Robinson (2011) accepts vectors as having a central significance in Euclidean geometry. Indeed, he explains this significance by stating that using geometric illustration of vector properties contributes by preventing rote learning even for middle grade students.

Studying geometry from only synthetic and analytic perspective are insufficient to respond today's requirements that there are many improvements in computer technology and various software programs. According to Coxford (1991), these developments also make necessary to utilize vector knowledge.

Chiba (1966) notes that teaching geometry via vector approach presents an opportunity to learn Euclidean geometry topics from a different perspective, to relate analytic geometry and synthetic geometry and to develop space concepts. Troyer (1963) and Hershberger (1971) specify vector approach as an excellent tool that constitutes a closer link between algebra and geometry. In addition, Stephenson (1972) found in her dissertation that the synthetic approach to geometry yields insufficient integration of algebra and geometry. Related to the connectivity role of vectors between algebra and geometry, Harel (1990) proves the ratio of the line segments in a triangle when the medians of that triangle are intersected as an application of vectors in geometry. At the end, he remarks the contributions of vectors as they provide an opportunity to teach geometry topics, to solve geometry problems and to show how geometry topics are related with each other in middle and high school grade levels. In the light of these facts, there emerges a need to include vectors in geometry teaching so as to supply integration among geometry, algebra and coordinate geometry. Indeed, according to very similar results of the studies conducted by Hershberger (1971), Bundrick (1968), Schaumberger (1962) and Pettofrezzo (1959), they suggest including vector approach in geometry teaching at least for the topics of analytic geometry of the line and plane in two and three dimensions.

Kemeny (1964) reports another aspect of vectors in geometry teaching. In terms of bringing out an analogy between the 2D, 3D and more dimensions of geometry, it is beneficial and practical to utilize vector approach in geometry. Moreover, he adds that many of the geometric proofs essentially become easier if vectors are treated as coordinate-free format. Besides, Bundrick (1968) states the unifying property of vectors as Kemeny (1964) reports. He specifies that many of the concepts taught through vectors in 2D is a facilitator or an analogous for the same concepts in 3D. For example, the distance of a point to a line on 2D and to a plane in 3D are nearly the same in calculation by vector method. However, this calculation present some differences in analytical approach. Therefore, it is possible to mention time saving aspect of vector approach that will be explained subsequent parts in detail.

2.7.6 To What extent will Vector Approach be Included in Geometry Teaching?

Scott and Rude (1970) do not argue that vector approach and / or analytic approach would be replaced with synthetic approach completely. They state that vector and analytic methods have to assist or enhance Euclidean methods, and they should be preferred when they present an easier or more convenient way of solving problems or proving theorems in comparison with Euclidean methods. Actually, this is very similar approach recommended in the national geometry curriculum in Turkey (MoNE, 2010a & 2010b). In addition, Schuster (1961) also states that vector approach can supply more pleasure to the students as far as it is not so much overwhelming in the teaching process. In fact, negative reactions to the overwhelming use of vectors in geometry teaching by the teachers (Aktaş & Cansız, 2012) and by high school students (Baki & Akşan, 2014b) are reported.

As pointed out by Ayre (1965), the researcher also realized that it was not reasonable to argue superiority of any approaches to the others. A certain problem can be solved by an approach more conveniently and easily in comparison with the other approaches. Instead of talking about superiority or priority of any approaches, increasing the variety of available methods and enhancing students' products should be the main points. Besides, solving a problem by means of an unfamiliar or a novel way might increase students' interest to the lectures. Therefore, the teachers' role here is to teach how to handle geometry through various ways i.e. multiple approaches to their students. However, they should not enforce students that they use any of the approaches invariably. Furthermore, the teachers should give opportunity of freedom to their students in their preferences.

2.7.7 When to Start to Include Vector Approach?

As stated above, there are several studies suggesting the integration of vector approach into geometry teaching. However, a debate related to the time of starting to include vector approach is easily distinguished in these studies. In other words, there is not an agreement when to start vector approach in teaching geometry. To illustrate, according to Troyer (1963), analytic, synthetic and vector approaches can be integrated to treating the geometry, to prove geometric theorems and to solve exercises immediately after the students learn coordinate geometry, which means middle or high school grade levels. However, Rosenbloom (1969) states the time of starting vector approach in geometry teaching as high school level if the students have necessary preliminary vector knowledge. In terms of being able to utilize vectors in middle and high school geometry courses, it is important to supply appropriate tasks and to embed vectors to the geometry teaching. If this can be provided, then the students can be expected to learn geometry from vector perspective. However while doing this, it should not be forgotten that the superiority of any approaches is not the main concern. Instead, the focus should be the necessity of the approaches. Another researcher, Athen (1966b) suggests the "earlier" teaching of vectors, which yields a chance to enhance physics teaching in his study. He meant middle and high school grade levels by using "earlier" word.

As understood from Gagatsis and Demetriadou (2001), vector geometry is taught at 12th grade level. However, it is not reasonable to expect students to use vectors as a tool in their problem solving strategies. Since students have been getting used to solve all of their requirements via synthetic approach for 11 years, it would be not easy to change solution ways in terms of students. They naturally indicate some resistance to new approaches whatever it is. Therefore, it seems non-reasonable to postpone vector approach geometry teaching to the last year of high school.

2.7.8 The Challenges of Utilizing Multiple Approaches

Despite the stated potential contributions and importance of multiple approaches, Romanova (2006) and Foldesiova (2003) state that the mathematics textbooks do not reflect the idea of utilizing multiple approaches in geometry teaching which means the difficulty of developing materials for geometry teaching through multiple approaches. Similarly, Dorier, Robert, Robinet and Rogalski (2000) note that the teaching of geometry in high school grade levels is mostly based on synthetic approach in France in spite of the fact that Cartesian and vector geometry are targeted to instruct as well. Therefore, teaching fundamentals of analytic geometry and vector geometry is not enough for the sake of including these approaches to geometry teaching. Furthermore, Athen (1966a) expresses this situation very briefly and beautifully as "*not instruction in vectors but vectorial methods in the instruction*". If vector approach is really desired to be included in geometry teaching, it should be utilized at different parts of the courses in different grade levels.

Similarly, there is a lack in variety and in number of geometry problems, which can be solvable through several approaches. During the preparation phases of this dissertation, the researcher realized that the encountered tasks were nearly the same or very similar in the examined textbooks and academic studies. To illustrate, Nissen (2000) states the difficulty of developing and hence finding appropriate geometry tasks that can be solved via several approaches. Parenthetically, Nissen (2000) defines this type of problems as hybrid in his study. Moreover, Ba and Dorier (2010) underlines the difficulty of finding geometry problems that can be solved more easily and efficiently by means of vector methods in comparison with the other traditional approaches for the students at the grade level 10. This is important to make students appreciate the power of vector approach in problem solving and theorem proving. Students' realizing and appreciating the power and beauty of application of vector approach and analytic approach to the geometric problems are evaluated as significant by Ayre (1965). Besides, he states that a successful start to teach vector approach and analytic approach can be achieved by proving theorems that are familiar to the students. At least, this can be accepted as a hint how to start vector approach teaching.

Naturally, it should not be ignored that teaching geometry via various approaches might be a source of difficulty or extra workload for some of the students and there is possibility of not being able to set up the relations well enough among the approaches. However, it is worth including multiple approaches in geometry teaching because of valuable advantages of each approach in terms of the students.

2.8 Comparison of the Approaches

There are some studies in which the researchers and mathematicians compare vector approach with synthetic and analytic approaches in terms of advantages and disadvantages. Although there are some mathematicians (Randolph, 1961 and Protter & Morrey, 1966 as cited from the study of Bundrick, 1968) found vector approach teaching as more sophisticated to some extent, there are mathematicians or mathematics educators (such as: Copeland, 1962) having opposing view.

Glicksman (1965) states that synthetic approach solutions necessitate using auxiliary and additional lines, verifying similarity or congruence of some triangles and constructing parallelograms in geometry problems. He points these phases as difficulties in terms of students both in understanding and in following. However, according to Glicksman (1965), vector approach solutions are easier to learn and to follow since they are neat. Because of not depending on dimension and easiness of proving via vector approach, Bourne (1952) accepts vector approach superior than conventional slope approach to teach coordinate geometry from mathematical perspective.

Stephenson (1972) states that whereas vector approach proofs necessitates prerequisite vector knowledge including algebra of vectors with properties; synthetic approach geometry teaching requires some knowledge of theorems and postulates. As an example, Klamkin (1970) made this comparison by proving Carnot theorem through three of the approaches. According to his inferences, analytic approach is very direct but require lots of arithmetic operations, vector approach requires less effort but it is less direct, synthetic approach necessitates some theorem repertoire. The directness of a proof is exemplified by Stephenson (1972) as “while proving some of the properties of a parallelogram there is a need to similarity and congruence postulates, and while proving Ceva’s theorem there is a need to similar triangles theorems or postulates in synthetic approach proofs”. Since vector approach proofs do not necessitate these theorems or postulates, they are more direct than synthetic approach proofs.

Krech (1968) compares vector approach and synthetic approach proofs of 78 theorems in her study. She states none existing advantage of one approach over the other approach for the proofs of 57 theorems. In other words, two approaches are similar in elegance or difficulty for 57 theorems. However, five of the proofs were evaluated as better with synthetic approach and 18 were judged as better by means of vector approach.

After presenting some comparisons of approaches, advantages and disadvantages of synthetic and vector approaches will be presented separately in the subsequent sections.

2.8.1 Advantages-Disadvantages of Vector and Synthetic Approaches

A problem can be solved through several ways. The number of ways of solution is dependent on students’ or problem solvers’ repertoire or background. In other words, the more the number of the approaches by which a student learns a course, the more diversity emerges in his solutions. It is very natural that a problem can be solved easily by a specific method. DiFonzo (2010) sates that one of the analytic, synthetic and vector proof strategies is more appropriate for a certain problem in many instances. Furthermore, Miller (1999) notes that this is also valid for the computer programs that each approach can be ideal for different problem cases. Moreover, Zou, Zhang and Rao (2012) specify that a vector approach is a good shortcut for some of the geometry tasks. However, Nissen (2000) states the ultimate target for students as being able to realize that a certain problem can be generally solved more efficiently and conveniently by one of the approaches. The students are expected to gain ability in deciding the most appropriate strategy among the alternatives. To illustrate, in the course with the name “A Vector Approach to Euclidean Geometry” developed by Vaughan and Szabo (1973), they reach a conclusion that use of vectors is more efficient than the algebra of analytic approach.

Barbeau (1988) defines some terms related to the solutions of geometry problems via multiple approaches. Specifically, these terms are “*clarity, security and elegance*”. A solution via an approach will be entitled as having “clarity” if it is the easiest way of presenting orally or in writing. Besides, a solution will be labeled as “elegant” if it is more aesthetic in comparison with the other approaches. Finally, when a method of solving offers the least possibility of making an error, than the method of solving will labelled “secure”.

Krech (1968) evaluates a solution as “<u>better</u>” if:

a) It does not necessitate indirect knowledge such as “drawing auxiliary lines or line segments,
b) The result can be reached immediately after the given data organization
c) If more than one results can be achieved by one-step.

In the light of all of these facts, each of approaches has advantages and disadvantages during the learning and teaching processes in terms of teachers and students. The advantages and disadvantages of the approaches according to the several studies are presented separately in the next sections.

2.8.2 Advantages of Vector Approach

<u>Requirement of less knowledge</u>

DiFonzo (2010) states that vector geometry requires less numbers and less complicated formulae. Similarly, vector proofs necessitates less pre-existing or prerequisite knowledge than synthetic proofs.

Convenience to teach specific topics

According to the study of Athen (1966b), trigonometry becomes simple and easily comprehended study if it is introduced through vectors.

Achieve two aims at once

Hajja and Martini (2013) expressed another advantage of vector approach in proving a theorem besides its being shorter in comparison with other proof strategies. According to their studies, one can prove “concurrence of the altitudes of a triangle” and “Euler theorem” at one-step or one stroke by means of vector approach. Krech (1968) qualifies a proof by an approach as “better” if more than one statement can be verified at one-step. This is accepted as the advantage of the approach in her study. To illustrate, a line segment combining the midpoints of two sides of a triangle is parallel to the third side and its length is half of the third side. This simple theorem can be proved at one stroke by means of vector approach. However, this is not the case for synthetic approach. In fact, parallelism and side relation can be proved by two separate parts in synthetic approach.

Unifying and Generalizing Feature

Grant (1971) reports that vector approach solutions in plane and in space are very similar to each other. The only difference is the number of the components in vector approach; however, the situation is different in analytic approach. Athen (1966a) defines this property of vector approach as having “*unifying and generalizing*” feature. Moreover, unifying and generalizing feature of vectors in geometry teaching as an advantage of vectors is expressed by other researchers (Bundrick, 1968; Hershberger, 1971; CEEB, 1959; Pettofrezzo, 1966; Fehr, 1963). As an example, teaching solids through vector approach yields comprehension not only for 1, 2 or 3-dimension but also for n-dimension. Moreover, geometry teaching by vector approach will give an extension of students’ intuition from familiar cases to unfamiliar cases.

Being easier and more direct

Klamkin (1970) stresses directness and simplicity of vector approach solutions in most of cases in comparison with synthetic approach. However, he does not want to be understood that he eschews from synthetic approach. According to him, the vector proof lies in between analytic proof and synthetic proof in terms of simplicity and directness. Specifically, it can be possible to solve problems by means of only arithmetic of vectors that is algebra of vectors without the need to utilize properties of inner product and vector product (Zou, Zhang, & Rao, 2012) while solving problems via vector approach.

Stephenson (1972) illustrates the directness of a proof with the facts that similarity and congruence postulates are necessary to prove some of the properties of a parallelogram and similar triangles theorems or postulates are necessary to prove Ceva's theorem in synthetic approach. However, there does not exist any necessities to these theorems or postulates in vector approach strategies. As a result, vector approach proofs are interpreted as more direct than synthetic approach proofs.

The simplicity of vector proofs was explained by Stephenson (1972) with the fact that vector approach proofs are similar to proofs by real numbers, which is more familiar to the students. Therefore, he evaluates proofs conducted by vector approach as considerably different from the other type of proofs.

Relating geometry with algebra

Szabo (1967) states that there is a disconnection between geometry and algebra since these branches are taught to the students in two different course. According to Szabo (1967), vectors can be utilized to get rid of this problem. Hence, a student is able to make easy transitions among analytic, synthetic and vector approaches and can "relate geometry with algebra" if he learns geometry by vectors. This fact is evaluated as an advantage of vector approach in the studies of Krech (1968), Stephenson (1972) and Vaughan and Szabo (1971). Furthermore, Szabo (1966) says that "*vector is a beautiful and useful bridge between algebra and geometry*".

In the light of the literature review for this dissertation, the researcher concluded that there is a tendency of relating sub-branches of mathematics and integrating geometry and algebra in the reform endeavors for mathematics curricula. Therefore, vector approach is important to realize to set up the bridge between these sub branches of mathematics.

Appealing and powerful aspect of mathematics

Chatwin (1985) emphasizes that vector approach makes students see the power and beauty of mathematics. In addition, Bundrick (1968) specifies geometry teaching through vector approach as a source of pleasure and feeling the need for more studying because of emergence of a novel way of learning geometry. Moreover, Glicksman (1965) qualifies vector approach solutions as elegant and natural with supplying deep understanding of geometry problems and proofs. Besides, the proof of theorems especially developed by means of properties of inner product and cross product is evaluated as more elegant than synthetic proof by Lord (1985).

More effective with analytic geometry

Schuster (1961) states that it will be easier to solve a problem if the coordinate logic is also utilized in addition to vector approach. In other words, the effect or power of vector approach will be enhanced by the integration with analytic approach. Moreover, Schuster (1961) mentions pedagogical advantage of this combination in addition to mathematical aspects of utilizing vector approach and analytic approach together. While the degree of comprehending the knowledge by which the students can jump from one approach to another one is the indication for the mathematical advantage of utilizing these two approaches, the degree of students' self-confidence can be accepted as pedagogical advantage.

In brief, the advantages of vector approach can be seen in the study of Gagatsis and Demetriadou (2001). The students were asked to state the advantages of vector approach in their study. According to the participants' reflections, these advantages were grouped under the following titles as

a) Vector approach is recent (novel) and up to date.
b) Drawing auxiliary lines are not necessary.
c) Vector approach solutions and steps are easy and comprehensible.
d) Less knowledge requirement and no imagination requirement
e) Vector approach solutions are elegant both mathematically and logically.
f) Vector approach strategies seem standardized ways of solution.
g) No need to localize something in the given figure.

2.8.3 Disadvantages of Vector Approach

Barbeau (1988) points out that geometry presents limited number and variety of problems in which use of vectors is available. However; in these cases, vectors make it easy to reach solutions interestingly. In the study of Gagatsis and Demetriadou (2001), the students were asked to state the disadvantages of vector approach. Students stated that they have lack of experience with vector approach. However, this inexperience is because of the fact that the teaching of geometry via vector approach was taught to the students at the final year of high school level in this study. Therefore, it is specific to this study and, this disadvantage is not due to vector approach. Moreover, the concept of "sense" might be a source of confusion for some of the students. In addition, Wexler (1962) notes that vector representation sometimes can be more cumbersome than the traditional; however, it can be ignored because of simplicity and elegance.

Stephenson (1972) states the artificial nature of vector approach as disadvantage of vector approach. As a result, thinking geometry from vector perspective takes time getting used to. This is also expressed in the report of Cambridge Conference on School Mathematics (1963) that the simplicity of a vectorial approach to the geometry may not appear in a first treatment (p. 79).

2.8.4 Disadvantages of Synthetic Approach

DiFonzo (2010) points out that since synthetic approach is based on theorem knowledge, it necessitates theorem knowledge and the proof of these theorems frequently. The requirements of adding imaginary lines and auxiliary line segments are possible sources of difficulties and hence it is a disadvantage of synthetic approach, as noted by many of the mathematicians and researchers in this dissertation, like Krech (1968). In addition, Glicksman (1965) states that synthetic approach solutions require using auxiliary and additional lines, verifying similarity or congruence of some triangles and constructing parallelograms in geometry problems.

According to the study of Gagatsis and Demetriadou (2001), the disadvantages of synthetic approach are stated by the students as follows.

a) Requires large pieces of knowledge,
b) Complicated thought,
c) Figure difficulties,
d) High possibility of forgetting relative theory.

Lee, Tay, Toh and Dong (2003) stress some requirements of set of tricks in synthetic approach, which might not an easy stuff in terms of most of the students. Hence, they qualify these subtle actions as disadvantage of synthetic approach. Naturally, looking for easier ways of learning geometry has continued constantly.

In spite of these disadvantages, the students or problem solvers mostly resort to synthetic approach, as the first way. Gagatsis and Demetriadou (2001) explain this fact with the long history of synthetic approach in comparison with the history of vector approach. However, it should not be forgotten that synthetic approach is the most utilized approach in school mathematics and geometry courses. The prevalence of synthetic approach is a natural consequence of the most frequent utilization and preference of this approach by mathematics teachers and textbooks. In fact, it is very natural that students' solutions reflect these sources (Baki & Akşan, 2014b). Harel and Sowder (1988) explains this situation by the term external schema by which students use their teachers' strategies or textbooks' strategies. Therefore, it would be unreasonable to expect students prefer vector approach or analytic approach if their teachers and textbook do not apply these approaches.

2.9 Timing Issue

Teaching geometry through vector approaches in addition to synthetic approach yields timing problem according to some of the teachers (Aktaş & Aktaş, 2012) and to the students (Baki & Akşan, 2014b). However, Copeland (1962) mentions that the amount of time saved by utilizing vectors is enough for developing required prerequisite vector knowledge to teach geometry through vectors. This is also verified in the study of Bundrick (1968) and Hershberger (1971). The mean time necessary to study the vector approach and traditional approach treatments was determined equal approximately. Furthermore, the time allocated for the treatment given to the vector approach group was recorded less than traditional approach group. However, it was not reported as significant. In addition, Hershberger (1971) reported that teaching analytic geometry through vectors is 15 % more economical in terms of time exposure with respect to the traditional approach.

2.10 Vector Approach as an Alternative to Similarity

Nissen (2000) states that similarity of triangles, which is one of tools in synthetic approach, is the most known tool by the students while solving problems or proving theorems. As an alternative to similarity and congruence, Lee, Tay, Toh and Dong (2003) utilize algebra of vectors and inner product to show congruence of two triangles. In addition, Stephenson (1972) explain one of the advantages of vector approach as there is no need to postpone the proof of Pythagorean Theorem till the students learn similarity and congruence of triangles. Hence, it is understood that the students learn the proof of Pythagorean Theorem through similarity and congruence of triangles. However, a student can be taught this proof by means of the properties of inner product by vector approach without delaying it to the later grades. Moreover, Vaughan and Szabo (1973) prove most of the theorems without using similarity and congruence theorems for the courses and textbooks on vector approach geometry. Instead of this, they utilize vector algebra and inner product. Similarly, this is also seen in Choquet’s textbook. Actually, Choquet (1969) states that they really did not need to utilize congruence and similarity at any stage of course development because of the fact that they accepted similarity and congruence as an obstacle to develop a vector approach to geometry. Although similarity and congruence has a considerable place in synthetic approach, this is not the case in vector approach geometry. In addition, Krech (1968) states that instead of allocating more time and giving more importance to similarity and congruence in school geometry, geometric topics having more importance can be emphasized.

In the light of all of the facts above, it might be possible to infer that vector approach solution can be an alternative method to solve some sort of geometry problems through similarity and congruence of triangles.

2.11 Theoretical Framework

The theoretical ground on which this teaching experiment study relies will be explained in this part of the study. Theoretical frameworks are vital for research studies in terms of constructing the research study, planning and implementing the research and analyzing and interpreting the results of the study. Therefore, theoretical framework gives an opportunity to accomplish all of these steps in a coherent manner.

2.11.1 The Theory of Didactic Situations

"*The Theory of Didactic Situations*" (TDS) was utilized as a conceptual framework in this teaching experiment study. This theory was founded and developed by Guy Brousseau, Yves Chevallard and Anna Sierpinska. TDS rests on the students' reactions in a given or constituted didactic situation. Compatible with the purposes of the teaching experiment methodology, it makes possible to observe and learn students' mathematical learning and reasoning at firsthand (Steffe & Thompson, 2000).

TDS framework enables us to analyze students' works and students' solutions from various perspectives by considering individual level differences as well. Hence, these analyses constitute the base of a didactic research. Students' responses or reactions to a constituted or arranged "*didactic situation*" are a basis of "*didactic research*". It was utilized in French didactic educational settings under the leading of Guy Brousseau. This theory facilitates analyzing specified problems in different perspectives.

The following table summarizes the integration of milieu and related didactic situations from the study of Margolinas (1994) (*as cited from the study of Rumanova, 2006*). The levels will be presented in detail in the next paragraphs.

Table 2-1 Didactic Situations (Margolinas, 1994)

M_3 *Constructional* milieu		P_3 Teacher – *didactic*	S_3 *Noosferic* situation
M_2 *Project* milieu		P_2 Teacher – *constructor*	S_2 *Constructional* situation
M_1 *Didactic* milieu	E_1 *Reflective* student	P_1 Teacher – *designer*	S_1 *Project* situation
M_0 Milieu of *learning*	E_0 *Student*	P_0 *Teacher*	S_0 *Didactic* situation
M_{-1} *Modeling* milieu	E_{-1} *Cognizant intellect student*	P_{-1} Teacher – *scrutator*	S_{-1} Situation of *learning*
M_{-2} *Objective* milieu	E_{-2} *Active* student		S_{-2} *Modeling* situation
M_{-3} *Material* milieu	E_{-3} *Objective* student		S_{-3} *Objective* situation

2.11.2 Analyses of Teacher (researcher)'s Work

S3 - Noosferic Situation

This is the first stage in which the teacher-researcher examines and analyzes mathematics textbooks in middle and high school grades. He investigates various mathematical materials (such as academic calendar, agenda, or curriculum programs) specifically related to his specific topics and the purpose of the study. The teacher-researcher looks for problems that are non-routine problems. In other words, he seeks for the problems, which are not solvable via simple memorized algorithms. Another feature of these problems is that they require integrating or utilizing students' knowledge from different topics in mathematics or in other disciplines as much as possible. Searching for or developing geometry problems or tasks that can be solved by means of various approaches is an important step in terms of the study. Finally, completing these studies in Noosferic Situation is going to be a milieu for the next situation.

S2 - Constructional Situation

The teacher-researcher continues searching for or developing problems of which properties were described in the previous step. However, the purpose of this stage is to determine problems to be directed to the participants to be able to answer the research questions of the study.

S1 - Project Situation

The teacher presented the problems that he selects ultimately to the students. The teacher projects his solutions as he decides earlier. This phase also includes all kind of participants' activities. Steffe and Thompson (2000) called this as "students' mathematics' which is understood by "what they say and do" during their involvements in a mathematical activity. Therefore, the participants are under the control of their teacher. In other words, the teacher takes care of the students' activities and reactions in this phase. The students solve the problem in any way, method or by any approach that they prefer individually. Specific to this dissertation, the solution can be through either synthetic, vector, analytic or combination of these approaches in this study.

S0 – Didactic Situation

In this phase, new knowledge is analyzed by the researchers. They try to institutionalize the recent knowledge that they obtain finally in the research. Then, the problems of the research are tried to be formulated and expressed clearly. Therefore, the teacher-researcher takes care of the preset purposes of the study and the students' solutions, actions and language come out during all phases of the research. This is the level of situation at which analyses of works of teacher and works of students meet at a common point. Moreover, this is the phase where the didactic situation is realized as a result of teaching experiment processes.

2.11.3 Analyses of Students' Work

S_{-3} – Objective Situation

The students try to get used of the purposes of the research, the problems to be asked in the research and material milieu. This is the phase in which the students encountered with the problems or tasks in the research. There are material milieu, cognitive component of milieu and social component of milieu, which are specific to each problem in the study. Material milieu can be stated as the given information related to the problem.

S_{-2} – Modelling Situation

The students try to solve the problem without any contributions of and intervention with the teacher. They utilize their prerequisite and specific knowledge on necessary topics to solve the problem. He makes some facilitating actions on material milieu. The student needs to be careful about his solutions because of non-existing feedbacks or help from the teacher.

S_{-1} – Situation of Learning

The student acts as the teacher. He collects data from the text of the problem and he makes queries specific to each problem in order to reach final results. In other words, he tries "*think-aloud*" processes in a sense. Rather than focusing merely on the problems, the student focuses on thinking and on expressing his own results clearly. The teacher is a researcher or inspector trying to help the students if they are not in right ways in solving processes.

S_{0} – Didactic Situation

It is evident that the products, works of studies are affected by the teacher-researcher in this situation. While solving problems, the students may resort to their teachers' advices by consulting them in order to institutionalize the new knowledge that they gain as a result of the study they participate in. The teacher-researcher considers solutions developed by the participant students. The teacher's guidance or help may vary according to the nature of the problem situation.

2.11.4 How did the Theory of Didactic Situations Lead this Study?

According to the requirements of a didactic research, participant students' responses to predetermined didactic situation constituted the foundation for this didactic research. It gives opportunity to analyze a specific didactical problem in educational process and in didactical environment.

The learning process in this didactic environment contains; a predetermined goal, realizing sequence of activities, taking care of the effects of this sequence of activities, recording of experiences and reflections of the participants and implementers of the study.

While preparing all kind of teaching materials and conducting lessons in teaching episodes, the teacher-researcher followed the steps explained in the phases of Noosferic Situation, Constructional Situation, Project Situation and Didactic Situation respectively. This order constituted the analysis of teacher's works. Moreover, the stages that participants demonstrated were considered and analyzed in the order of the phases: Objective Situation, Modelling Situation, Situation of Learning and finally Didactic Situation.

2.12 Literature Summary

In the light of literature review related to this study, it can be summarized as follows.

1. Geometry teaching through multiple approaches enhances geometry learning and it is a skill that should be acquired by the students (NCTM, 1989; Barbeau, 1988; Nissen, 2000; Kwon, 2013; Bundrick, 1968, Gagatsis & Demetriadou, 2001; MoNE, 2010a, 2010b & 2010c; CCSM, 1963 & CEEB, 1959).
2. It is expressed that teaching a mathematical concept can be accepted as complete when students can make transitions among various approaches or

representations (Kwon, 2013; Sfard &Thompson, 1994; Schuster, 1961; Dreyfus & Eisenberg, 1986).

3. Rather than asserting priority or superiority of an approach to the other approaches, it is more important to notice that an approach is missing without other approaches and each of these approaches has a complementary feature for the other approaches in geometry teaching and learning (Klamkin, 1970; Lines, 1965; Miller, 1999; Krech, 968 & Robinson, 2011).
4. Searching different ways of solving geometry problems through multiple approaches provide discovering opportunities for students (Glicksman, 1965; Star & Rittle, 2008; Akkoç & Katmer, 2014; Schoenfeld, 1983; Robinson, 2011; NCTM, 1989; Zou, Zhang & Rao, 2012).
5. It is understood that a specific approach might be more appropriate or feasible for certain type of problems (DiFonzo, 2010; Miller, 1999, Coxford, 1993; Regecova, 2005; Appova & Berezovski, 2013; Lee, Tay, Toh & Dong, 2003; Ayre, 1965 & Nissen, 2000). To be able to decide which approach is more appropriate according to problem type necessitates some degree of experience or maturity in learning geometry through multiple approaches (Stephenson, 1972; Cambridge Conference on School Mathematics, 1963).
6. While solving geometric problems by means of analytic approach via placing the given geometric object on imaginary Cartesian plane, it necessitates some period to have some experience and maturity in terms of the learners to decide the most appropriate vertex or point of the given object to be the origin of Cartesian plane. Moreover, assigning different points as the origin of the Cartesian system generates or yields different solution ways (Coxford, 1993 & 1991; Craine, 1985; Ayre, 1965).
7. Vector approach solutions on the plane can be accepted as a preparatory and easiness for the treatment of 3D geometry (Bundrick, 1968; Hershberger, 1971; Athen, 1966a; CEEB, 1959; Pettofrezzo, 1966; Fehr, 1963 & Grant, 1971).
8. Rather than teaching vector as if it is an independent or separate topic, it is emphasized that vector should be embedded in geometry teaching. Vectors should be taught through geometric counterparts or geometric meanings

(Bergman, 2010; Regecova, 2003; Novakovski, 2001; Gueudet-Chartier, 2002 & 2004; Appova & Berezovski, 2013; Tabaghi, 2010; Konyalıoğlu, İpek & Işık, 2003; Nishizawa & Yoshioka, 2008; Stephenson, 1972; Athen, 1966b & Fyhn, 2010)

9. Vector teaching is a problematic field and students have misconceptions and difficulties while learning vectors (Barniol & Zavala, 2009; Nguyen & Meltzer, 2003; Knight, 1995; Kanim, 1999 and Flores et al., 2004; Deventer & Wittmann, 2007 and Dimitriadou & Tzanakis, 2011; Poynter and Tall, 2005; Aguirre & Erickson, 1984; Pavlakos et al., 2005; Nishizawa & Yoshioka, 2008).
10. Solving geometry problems via analytic and vector approaches necessitates some degree of prerequisite knowledge or infrastructure related to vectors and coordinate geometry. Specifically, it is reasonable and recommended repeatedly to teach vectors in earlier ages or times before teaching geometry topics instead of postponing its teaching to the later periods or higher-grade levels (Stephenson, 1972; Krech, 1968; Hershberger, 1971; Choquet, 1969; Athen, 1966b; Troyer, 1963). In addition, for slow learners and the students at earlier grades can be taught vectors in translation context which is stated as simple and efficient way of teaching vectors (Szabo, 1966; Athen, 1966a, Sünker & Zembat, 2012; Coxford, 1993; Poynter & Tall, 2005a & 2005b; Regecova, 2003; Rosenbloom, 1969 & Grant, 1971).
11. It is necessary to teach vector concept in the context of translation so that the students have a chance to embody it (Szabo, 1966; Athen, 1966a, Vaughan & Szabo, 1973; Coxford, 1991 & 1993; Poynter & Tall, 2005a & 2005b; Nguyen & Meltzer, 2003; Rosenbloom, 1969; Stephenson, 1972; Faydacı & Zembat, 2012).
12. It is understood that utilizing various software tools, applets and games during the teaching of vectors and giving importance to visualization of vector concepts make vector teaching more effective and important for the sake of saving time and embodiment of vector concepts (Nishizawa & Yoshioka, 2008;

Tsegaye, Baylie & Dejne, 2010; Nishizawa, Zgraggen & Yoshioka, 2009; Çataloğlu, 2006).

13. To be able to assert non-missing of vector teaching, it is important to teach vectors with various contexts such as mathematics, physics etc. (Dimitriadou & Tzanakis, 2011; Poynter & Tall, 2005a & 2005b; Deventer & Wittmann, 2007; Aguirre & Erickson, 1984 & Aguirre, 1988) and with various positions (non-prototypic positions or non-standard positions) (Poynter & Tall, 2005; Watson, 2002; Barniol & Zavala, 2010; Fujita, 2012; Pavlakos et al., 2005; Gagatsis, 2005 & Gagatsis and Demetriadou, 2001).
14. Teachers are the implementers of the curriculum programs in classrooms. The desired goals cannot be reached if the teachers are not treated according to the specified innovations and regulations. In other words, whichever innovation or approach are included in a curriculum as a reform, the success will not be realistic (Bye, 1968; Ba & Dorier, 2009; Rosenbloom, 1969; Ponte et al., 1994 & Sztajn, 2003).
15. There is a tendency of relating sub disciplines or branches of mathematics with each other and integrating geometry and algebra in the development of new geometry and mathematics curriculum endeavors according to the results of related literature that the researcher reviewed (Stephenson, 1972; Szabo, 1966; Cansız, 2013; Regecova, 2005; Krech, 1968, Troyer, 1963; Harel, 1990; Rumanova, 2006; Dimitriadou & Tzanakis, 2011; Chiba, 1966; Flores et al., 2004; Okolica & Macrina, 1992 & Stephenson, 1972).

CHAPTER 3

METHODOLOGY

In the preceding chapter, the literature focused on the research questions for the current study was presented. In this chapter, methodology of the study will be presented. The methodology chapter includes participants, data sources and data collection, instruments, data analysis, design of the study and teaching experiment methodology. In order to answer the research questions, teaching experiment methodology was conducted by the researcher. This teaching experiment consisted of teaching sessions, pre-tests and post-tests and pre- and post- interviews to realize the investigation. Then, development and components of the instruction, which was prepared for and followed in the study, will be presented in detail. Finally, the issues on procedure, trustworthiness, ethics and assumptions and limitations will be discussed in the methodology chapter.

3.1 Participants

Yin (2011) labels the method of sampling as purposive sampling by which participants are selected deliberately. Patton (1990) expresses the logic and power of the purposive sampling as working on "*information rich cases*". Merriam (1998) also stresses the benefits of studying with information rich cases when gathering data from these cases. By means of this sampling method, a researcher can reach plenty of data related to the main focus of his or her research. Specifically, "*critical case sampling*" is one of the strategies to have a purposive sample (Patton, 1990). In this strategy, the most important point is looking for critical cases. Related to the present study, the participants as critical cases were selected from relatively higher achieving level. The rationale to utilize this sampling strategy is the statement that “while teaching quadrilaterals, if utilizing vector approach does not work in this group, it won’t work in other similar and less successful groups. In other words, if these participants are having troubles with this treatment, then we can infer that all of the groups are having troubles most probably. Therefore, “*critical case sampling strategy*” as purposive sampling method is deliberately preferred for this dissertation. It is vital to underline that it is not intended to make broad generalizations by means of working on one critical case. However, it can be possible to reach logical generalizations in this way.

The participants of the study were selected from one of the public Anatolian High schools in Keçiören, Ankara. The students of this school were selected according to the results of LDE. Approximately 1,200,000 examinees enter this high stake examination, which is conducted annually in Turkey. To be registered to this school, a student’s relative position or percentile value needs to be at least 3.25 %. In other words, to have a right to register this school, a student needs to be located in the first 39000^{th} position overall.

The participant students were primarily selected according to the recommendations of their regular mathematics and geometry classroom teachers in their school. Moreover, among volunteer students, the researcher made each student evaluates the other students as peer assessment as a precaution in order for participant students not dropping out the study and not resulting in any problem during the course of the research. In addition, before the research started, the researcher arranged a meeting with the parents and teachers of the students who were selected as participants. The purposes of the meeting were to give necessary information about the purpose, the process, the place where the sessions to be held, the duration and requirements-principles of the study. Since the majority of the teaching sessions would be conducted in the summer holiday of students, the supports and helps of parents were requested principally and importantly so that not to have any problem or abscission possibly occurred in the course of teaching sessions and during the administration of the tests. The participants were also informed that they would have a chance to get private mathematics and geometry tutorials from the researcher for the next academic years as a gift if they completed all of the teaching episodes entirely.

The participants' being able to express their operations, opinions and thinking ways clearly was very important for this study without feeling any fear or anxiety. This was vital in terms of the study because it was the focus of this research method to probe and to have idea about students' thinking and learning processes transpired throughout the study. Hence, this characteristic property for the participants was taken into consideration and was consulted to their regular classroom teachers before selecting the participants. In the light of these considerations, despite the fact that it was preferred to study with volunteers, the participants were not selected completely at random. In fact, the researcher preferred to study with students who were extrovert, willing and capable of discuss their ways of thinking.

The participants were 10-grade students from previously mentioned high school. These students had completed two years of geometry in which they were supposed to be taught the following topics in the grade levels 9 and 10:

a) *Plane geometry (basic concepts such as; point, line segments, distance, lines and equation of a line)*
b) *Vectors (vector algebra, linear dependence, Euclidean inner product and right projection),*
c) *Triangles (congruence and similarity, metric relations, areas of triangles and some theorems such as Carnot, Ceva and Menelaus theorems),*
d) *Polygons,*
e) *Transformations (translation, rotation, dilation and reflection*
f) *Circles,*
g) *Solids (prisms and pyramids, sphere)*
h) *Euclidean postulates and types of proof*

The number of participants was five at the beginning of the study. In spite of the precautions taken by and efforts of the researcher, one of the students who initially agreed to participate in the research had to drop out the study on July the 11th of 2013 right in the middle of the research because of his parental problems (*they had to move in another city*). The second student had to drop out the study on September 5 of 2013 towards to the end of the research because of his health problems. His health problems started at the date September 23 of 2013 as understood from the conversation with his parent. Therefore, he was excluded from the study necessarily and unwillingly after that date unfortunately. He could not attend the last five teaching sessions and additionally he could not take any of the post-tests and post-interviews. Therefore, these two students were excluded from the data analysis.

The remaining three students regularly attended all of the pre-tests, teaching sessions, interviews and post-tests despite the fact that majority of teaching sessions were conducted in the days of their summer holiday. They did not miss any part of the research fortunately. Therefore, these three students with the pseudonyms Ahmet, Naci and Ömer were finally accepted as the participants of the present study.

All of these students were male in gender. This choice is not accidental. The teaching sessions were necessarily arranged after regular school lessons (04:00 pm) in week days and the sessions lasted till 19:30 or 20:00. Since it could be a problem for female participants' parents to study till the late hours, the researcher preferred to study with male students. Moreover, the participant students' homes were close to their schools.

The number of items solved correctly in mathematics test of LDE by the participants Ahmet, Naci and Ömer are 20, 21 and 20 out of 21 items respectively. The participant students can be considered as above average students according to their math test scores on LDE. However, the situation was different at the high school. Ahmet and Ömer were average students according to their teachers' views, and to their mathematics and geometry grades (Table 3-1) and pre-test scores (Figure 4-28, Figure 4-29 and Figure 4-32). However, Naci was an above-average student with regard to the given criteria.

Table 3-1 Information about the participants

		Course Grades			
		9th-Grade Level		**1st Semester of 10th-Grade Level**	
Participant	**Gender**	**Mathematics**	**Geometry**	**Mathematics**	**Geometry**
Ahmet	Male	3	3	5	5
Naci	Male	5	5	5	5
Ömer	Male	3	4	4	2

Grades are out of 5

3.2 Data Sources and Data Collection

Multiple data sources were utilized in this study. The variety of data sources presents an opportunity for a researcher to constitute an organization of data analysis and interpret the data in an appropriate manner (Fraenkel & Wallen, 2006). In order to figure out the pattern in participants' solutions, and make consistent and reasonable inferences about students' products, the researcher utilized triangulation method. In this method, the researcher utilize two or more data collection methods to engage with the participants' products and to find out common features about participants' behaviors (Cohen, Manion & Morrison, 2000). In the light of these, the researcher tried to make inferences and classifications about multiple approach instruction on quadrilaterals unit by students' solutions and reflections to open-ended questions or problems.

The topics taught in this study and detailed meeting periods will be presented in the next pages (Table 3-7, Table 3-8 and Table 3-9). Although a regular schedule was arranged with participants according to their available times during school-term period, the researcher met with students 2 or 3 times a week during summer holiday period. During the implementation of the lessons throughout entire teaching experiment sessions, students' proposals or suggestions related to the arrangement of teaching periods were considered and hence necessary changes or regulations were made as a deal. Since the ultimate target was to complete all of the sessions successfully, the duration of the courses was not standardized, as will be seen in these tables. Rather, a flexible work schedule was preferred and furthermore, majority of the meeting days and periods were decided by the students. These are all preferred to complete data collection steps in success.

In order to satisfy the requirements of triangulation method, the researcher utilized various data collection tools. These tools are (a) pre-tests, (b) video recordings and (c) audio recordings of teaching episodes, (d) one-to-one interviews, (e) artifacts (all kind of participants' written works emerged during the course of teaching sessions) and (f) home works, (g) the field notes taken by the researcher and the observers during and after teaching sessions and (h) post-tests. In addition, the topics were taught by utilizing "smart board". Therefore, the files including solutions and ideas by the researcher or the participants, and the things emerged during teaching sessions were regularly saved after each teaching session. After that, these files were stored as a folder in a laptop computer, in a desktop computer and in an external hard disk synchronously. This folder is another available source of data for the current study.

Each teaching episode was videotaped by means of two cameras. One of the cameras was utilized so as to record students' participations and actions. The other camera was focused on the researcher and the smart board on which the topics were lectured. However, the periods including four pre-tests and four post-tests were videotaped by only one camera.

The class discussions were audio-recorded by an audio-recording instrument besides having recorded the teaching episodes by digital cameras. This was preferred so that there was not encountered any loss of data or information.

All of the written works of the participants in the study were collected, photocopied, digitized and stored in a computer hard disk and an external hard disk in pdf format as a folder. There are in-class individual assignments and homework assignments given to the students at the end of each teaching episode. Moreover, this folder includes researcher's and the observers' field notes.

3.2.1 Instruments

In this section, the instruments that were used as pre-tests and post-tests will be explained in detail. Prerequisite Knowledge for Quadrilaterals Test (PKQT), Proof Performance in Geometry Test (PPGT), Vector Knowledge Test (VKT), Quadrilaterals Achievement Test (QAT) and semi-structured interviews were utilized to obtain necessary data in order for answering the research questions of the present study. The tests are in constructed-response format by which the participants are expected to provide the response. Haladyna and Rodriguez (2013) state one of the aspects of this type in that it is possible to obtain higher fidelity in the targeted domain with this item format. They also stress the requirement of rubric to score students'' responses subjectively. Therefore, it was necessary utilizing a rubric to evaluate students' products in these tests in order to minimize the bias of the researcher. Since the way of utilizing the rubric is common for all the tests, it will be described in this part of the study.

In order to assess participants' solutions and answers, a rubric was utilized which was developed before administering the tests. Related to the assessment of students' solutions, a simpler version of a rubric from the literature was preferred to utilize. Senk (1985) developed and utilized a rubric for her study in order to assess students' problem solving strategies specifically. This rubric, which is a holistic scoring technique, is as in the Table 3-3.

Table 3-2 Senk's rubric

Score	Criteria
0	Student writes nothing or repeating the given or stating invalid deductions.
1	Student writes at least one valid deduction.
2	Student seems that he makes use of some correct reasoning; however, he stops because of faulty reasoning early in the steps.
3	Despite of some mistakes in notation or stating wrong names, student almost completed a proof.
4	Student completed proof although there is ignorable or simple mistakes.

Since the problem solving strategies are not the main focus for this dissertation and the approaches that were preferred by the participants were rather more important focus, the rubric criteria was limited to 3 scoring scale criteria levels from 5 levels as seen in the Table 3-4.

Table 3-3 Rubric to assess students' PKQT, VKT and QAT scores

Score	Students' works in the solution
0	Student write nothing or writes meaningless relations or deductions.
1	Students are partly successful by writing some relations, or attempt to solve the problem but the solution is not complete
2	Student solves the problem completely with supplying necessary steps.

While this rubric was utilized for the tests PKQT, VKT and QAT, a slightly different version of this rubric was used for PPGT as can be examined in the Table 3-5.

Related to rater reliability of the instruments, two mathematics education graduate students assessed students' tests according to rubrics prepared for these tests. These graduate students (abbreviated as M and Z) also enrolled as observers in this study. As mentioned earlier, they have 15 and 12 years teaching experience in mathematics and geometry courses. The correlation coefficient values between the researcher's and the observers' assessment scores were found as the values between 0,71 and 0,99 as can be seen in the Table 3-5.

Table 3-4 Rater reliability of the instruments

Time of calculating the coefficients	**PKQT**		**VKT**		**PPGT**	
	Z	**M**	**Z**	**M**	**Z**	**M**
Before Reassessment	0,98	0,98	0,98	0,99	0,71	0,75
After Reassessment	0,99	0,99	0,98	0,99	0,86	0,87

Despite the fact that the correlation coefficient values were high enough, the researcher assessed the students' tests more than once. After the second assessment process, the correlation values have become a level that is more satisfying. At this time, the correlations were realized between the values 0,86 and 0,99.

3.2.1.1 Prerequisite Knowledge for Quadrilaterals Test

Prerequisite Knowledge for Quadrilaterals Test (PKQT) included 21 classic test items. This test is administered to find out to what extent the participants have prerequisite knowledge to learn Quadrilaterals Unit. These items are selected from the topics "*Basic concepts in geometry, Lines, Triangles, Transformation geometry, Numbers and algebra, Polynomials and Trigonometry*".

These topics were embedded in Triangles Instructional Module, Analytic Geometry Module and Basic Algebra Instructional Module. They were prepared to reply possible requirements of students when they were to learn quadrilaterals via analytic, synthetic and vector approaches. While specifying these topics, the researcher's experiences with preparing and studying quadrilaterals through analytic, synthetic and vector approaches would be helpful in addition to taking care of previous years' mathematics and geometry programs. According to the results of this test, these modules were revised. After the revision completed, then the lecturing was started. In this test, the works and operations of the students were required to be written down in detail.

Table of content for PKQT and the test itself are included in this dissertation as Appendices E and F respectively.

3.2.1.2 Proof Performance in Geometry Test

There are 15 proof-based items in Proof Performance in Geometry Test (PPGT). The test items were directly selected from 9th and 10th grade geometry curriculum (MoNE, 2010a). These proof-based items are included in the geometry program and hence they would be taught to the students. This test was developed to identify to what extent the participants had proving skills. The mathematical expressions selected for the proof test were the theorems or propositions, which are more frequently utilized statements in high school mathematics courses.

In order to assess participants' solutions and answers in PPGT, the rubric in the Table 3-6 was utilized.

Table 3-5 Rubric to assess students' PPGT scores

Score	Students' works in the solution
0	Student write nothing or writes meaningless relations or deductions.
1	Students are partly successful by writing some relations, or attempt to solve but proof is not complete
2	Student proves completely with supplying necessary steps.

The students were requested to show their works in detail, in the instructions of PPGT. Table of content for PPGT and the test itself are included in this dissertation as Appendices G and H respectively.

3.2.1.3 Vector Knowledge Test

Vector Knowledge Test (VKT) consisted of 18 open-ended items to determine achievement, missing parts and difficulties of the participants related to basic vector concepts and operations. This test was applied twice to the participants. The first administration of the test served two goals. Firstly, it served to determine students' prerequisite knowledge level on vectors. Secondly, according to the results of this test, the elementary vector algebra curriculum part was revised and then it was formed for the final version. The participants were reminded to demonstrate their procedures and operations in a detailed manner.

Table of content for VKT and the test itself are included in this dissertation as Appendices I and J respectively.

3.2.1.4 Quadrilaterals Achievement Test

The Quadrilaterals Achievement Test (QAT) was developed to determine to what extent the participants have knowledge of the subjects matter to be included in the current study. QAT was composed of three parts: (a) Fill in the blanks part, (b) Classic type-items part and (c) Proving items part.

Fill in the blanks part was comprised of two sub-parts. The first sub-part includes 10 fill in the blanks items that are related to the definitions and terms in "Quadrilaterals unit". The second sub-part contains fill in the blanks items that are related to the properties of quadrilaterals and classification of quadrilaterals in a table format.

Classic type items part was composed of 21 items that require not only the results but also the solutions of the items step by step. The last part of QAT includes five proof-based tasks. The students were reminded for the requirement to show their works in a clear and detailed format in the introduction part of this test as instructions.

Table of content for Quadrilaterals Achievement Test and the test itself are included in the dissertation as Appendices K and L respectively.

3.2.1.5 Interviews

As it can be understood from academic calendar of meetings and the lectured lessons (teaching episodes, Table 3-8), the researcher conducted one-to-one interviews with the students in between and after the completion of the teaching episodes. As Clement (2000) states, a researcher can learn students' ways of understanding and thinking in a situation by conducting interviews with them. Therefore, to collect and analyze data about students' reasoning, interviews are utilized in this study. Related to the focus of the study, the aim of these interviews is to bring out each student's perceptions, reflections, attitudes and understanding about learning geometry via vector and analytic approaches in addition to synthetic approach. Another aim of these interviews was to determine the effects, pros and cons of the teaching that was implemented in teaching sessions. Specifically, the researcher tries to learn and probe underlying reasons for students' preferences and operations instead of merely determining what the participants do in their assigned tasks or problems. Therefore, the interview questions were specified by the researcher according to the purpose of the study. However, some of similar studies (e.g., Gagatsis and Demetriadou, 2000) were also taken into consideration while specifying the questions. Interview questions can be found in Appendix B.

3.3 Data Analysis

In this study, there are quantitative and qualitative data to reply the research questions. The quantitative data were collected through the tests: Prerequisite Knowledge for Quadrilaterals Test, Proof Performance in Geometry Test, Vector Knowledge Test and Quadrilaterals Achievement Test. These tests were administered twice to the participants as "*pre-test and post-test*". Some data about participants; such as gender, pre-year geometry and mathematics grades, pre-semester geometry and mathematics grades, the number of correctly solved items on mathematics test in LDE, their self-evaluation of geometry skills and achievements were obtained. Moreover, the frequency of proving mathematical and geometrical statements in their classes or examinations were asked.

In the analysis of pre-tests and post-tests, it was not just focused on participants' relative achievement scores. A descriptive analysis method (Yıldırım & Şimşek, 2006 & Fraenkel & Wallen, 1996) was utilized. In this method of analysis, participants' responses or solutions were assessed according to the classification or themes, which were developed throughout the study and reached the final version at the end of all analysis. To be able to reflect participants' ideas, thoughts or rationale, direct quotations were given in all process. It was tried to classify participants' preferences and to seek for a pattern in students' solutions. Therefore, whole process of data analysis was aimed to constitute a thematic frame or a general schema. After that, the data collected in this study was examined and interpreted according to this frame (Yıldırım & Şimşek, 2006). Necessary sub-categories were also determined according to transpired situations in the data. In addition, the frequencies of common groups were utilized generally in order to enhance the findings in this descriptive analysis according to the recommendations of (Cohen, Manion, & Morrison, 2000). Moreover, the findings from interviews, all kind of artifacts such as participants' written responses to problems asked at the end of each chapter, and to individually assigned homeworks, and fields notes were utilized to enhance the classifications. Therefore, students' solutions were presented with interviews.

It can be said that data analysis was not a simple process for this study. All of the data collected from the participants were iteratively (at least three times) investigated and analyzed from various viewpoints in order to reach the most appropriate and accurate version. Students' solutions and reflections were matched with related video recordings.

Analysis of pre-tests, video recordings and audio recordings of teaching episodes, one-to-one pre- and post-interviews, the participants' written works transpired during the course of teaching sessions and home works given after each of teaching session, the field notes taken by the researcher and by the observers during and after teaching sessions helped the researcher to answer the research questions.

3.4 Design of the Study

This study primarily depended on the teaching experiment methodology of Steffe and Thompson (2000) in order to answer the research questions. The steps, which were followed in this study is roughly given in the Table 3-6.

Table 3-6 Research design of the study

Participants	Pre-tests				Post-tests
Naci Ömer Ahmet	PKQT VKT PPGT QAT Pre-interviews	Teaching Experiment Sessions	Interviews	Teaching Experiment Sessions	PKQT VKT PPGT QAT Post-interviews

In the following part, teaching experiment methodology (Steffe and Thompson, 2000) will be described in essence.

3.5 Teaching Experiment Methodology

Steffe and Thompson (2000) define "*teaching experiment*" as "*a series of teaching episodes containing a teaching agent, one or more students, an observer and a method of recording what goes on in the episodes*". According to them, these are the indispensable elements of a teaching experiment.

The basic purpose of utilizing teaching experiment methodology for this study is to experience firsthand students' mathematical learning and reasoning. It would be deficient to understand mathematical concepts and operations constructed by students without forming a teaching and learning process on the topic specific to this study. During the teaching episodes, the challenges that the researcher experienced are a basis to understand students' mathematical reactions.

Despite of the fact that curriculum developers plan some innovations or teachers plan to realize some regulations, their reflections to the students' minds in the real classroom environment can be different. Because students have their own realities. Steffe and Thompson (2000) use two different but interrelated terms as "*students' mathematics*" and "*mathematics of students*". The students' realities as being different from the teachers or planners is called as "students' mathematics".

Students' mathematics can be understood what they say and do as they study on predetermined tasks or problems. During these engagements, the researchers try to find some explanations or to form models for "students' mathematics". However, "mathematics of the students" is related with the formed models and it also deals with the changes that the students make in their ways of operating.

Looking back to revise and analyze what the participants say and do and to try to understand students' realities are important and indispensable parts of teaching experiment. Von Glasersfelds (1995) called this analysis as "*conceptual analysis*". It is vital for this study to probe students' reactions. With the help of this teaching experiment methodology, the researcher try to determine students' mathematical concepts, operations and students' mathematics in order to reach a conceptual base for school mathematics and hence to constitute a model.

The teaching-experiment sessions were conducted between the dates 16 April and 6 October of the year 2013. During this period, the teaching experiment included 37 classroom teaching episodes. The total time allocated to the instruction and the interviews was 80 hours (80*60=4800 mins). Besides, there were four pre-test and four post- test administrations apart from the instructions and interviews. The aims of conducting this teaching experiment were given in Chapter 1.

In the following sections, major components of this teaching experiment are defined in detail. These are namely "teaching agent and observers", which are stated as two of the major components in a teaching experiment according to Steffe and Thompson (2000). Moreover, the method of recording is also explained in the subsequent paragraphs. The place of the teaching agent, observer and recording tools will be specified by means of physical configuration of the classroom setting.

Teaching Agent

In the present study, the researcher enacted as the teaching agent of the sessions because of several reasons. First of all, since the researcher had an experience of teaching mathematics and geometry for ten years in public and private schools at middle and high school levels and he has been teaching geometry and statistics courses for six years in undergraduate levels, inexperience or clumsiness would not be a problem or in question in classroom environment. Secondly, it would be difficult to find a geometry teacher to work with regularly during the period of summer holiday at which the majority of the teaching sessions conducted. Thirdly, the lesson hours would be changeable or flexible according to the students' availability, which could potentially be a problem to arrange teaching sessions. Finally and most importantly, it would not be easy and feasible to work with a geometry teacher who was not experienced in teaching geometry via vectors and educated accordingly. This unfamiliarity would cause so much loss of time and waste of effort. The researcher had improved himself in learning and teaching geometry via vectors. Further, he had developed materials in accordance with vector approach for at least two years. For all of these reasons, it was decided to teach the preplanned topics to the participants as a teacher-researcher in all phases of the study. In brief, I enrolled as the teacher-researcher throughout the study.

Observers

Two mathematics education graduate students alternately helped the researcher as much as possible in observing the teaching episodes and in taking field notes about lessons, teaching materials and participants' actions (*what participants say and do*). These graduate students had teaching experience for 15 years and 12 years respectively in high school mathematics and geometry courses. One of them has also been conducting a teaching experiment for her dissertation. These were why they were negotiated to follow the teaching sessions of the present study. As the researcher and the observers of this study, we had a frank exchange of views carefully and regularly after the teaching episodes.

During the teaching sessions, majority of the sessions were witnessed by these graduate students in the classroom. While observing the sessions, none of them affected the implementation of teaching and was included in the recordings. As another possible source of data to be analyzed for the research, the main objectives of including the observers were to follow participants' ways and thoughts, to note especially different ways of operations and innovational thinking ways of the participants from a different point of view. Moreover, they followed the teacher-researcher's ways of teaching and intervention with the participants in a critical manner. The teacher-researcher assessed situations for each session after the lesson periods with the observers and supervisor of the study. The process of the sessions, the development of the students' understandings and the difficulties faced in the course of sessions were discussed with the observers.

Physical Configuration of Classroom

The Figure 3-1 depicts the classroom environment in which the teaching sessions were implemented.

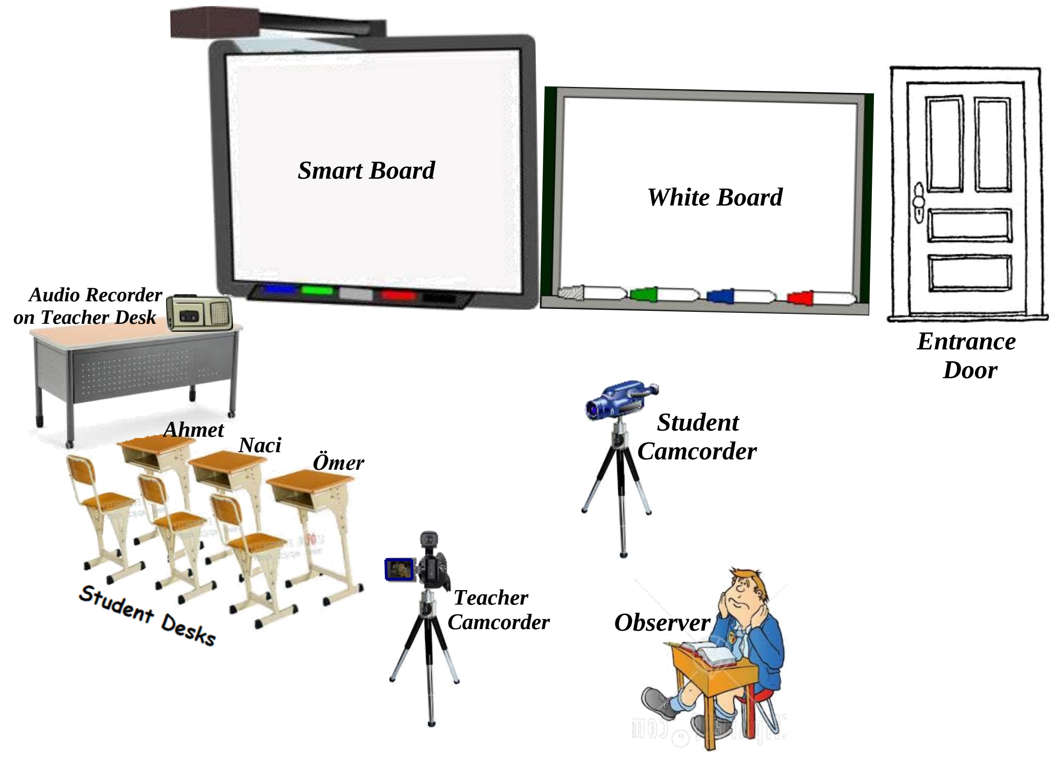


Figure 3-1 An illustration of classroom environment during teaching sessions

3.6 Instruction

3.6.1 Design of the Instruction

As stated before, the researcher enacted as the classroom teacher. The reasons for this were explained in the earlier parts. The instruction implemented during the teaching episodes supplemented with proof-based problems. Moreover, the topics were taught continuously in cause-and-effect relationship throughout the study. In other words, reasoning and proving were the constant feature of the instruction throughout the entire teaching experiment. In fact, each of the expressions included in this curriculum parts was discussed and taught with underlying reasons. Actually, this is one of the objectives of teaching geometry program for the grade levels 9-12 (MoNE, 2010a, b & c), as well.

Moreover, it is desired to teach geometry by means of multiple approaches that is synthetic, analytic and vector approaches in geometry teaching programs (MoNE, 2010a, b & c). However, geometry has been teaching through synthetic approach, which has been followed for many years in school mathematics and geometry (e.g., MoNE, 1992; Dorier et al., 2000). The teacher-researcher taught geometry in compatible with the requirements of geometry teaching program in this teaching experiment.

Information about the features and elements of the instruction and the common features of the instructional modules will be explicated in detail in the following parts. It is important to emphasize two matters at this point. Firstly, all of the instructional modules, lesson plans and teaching materials had been planned and prepared before the administration of the pre-tests. The contents of the subject matters were prepared and developed in a programmed manner by the researcher. The contents of the topics, teaching materials such as problems and homework assignments were developed by the researcher himself in accordance with the boundary of Turkish national high school mathematics (MoNE, 2011) and geometry curricula for the grade levels 9, 10 and 11 (MoNE, 2010a and 2010b). While developing the tasks for this study, the following criteria were taken into consideration as much as possible.

1) The tasks should be solved by both vector approach and synthetic approach as much as possible.
2) The properties of vectors, which were necessarily used for the solutions in vector approach, should be appropriate for 10-grade students, regarding their geometry curriculum.
3) The duration allocated for the solutions of the problems and for teaching subjects matters should be reasonable and applicable in real classroom environment.
4) The mathematical concepts to be utilized in the solution of geometric problems or in the teaching episodes should be appropriate for the 11^{th} grade students.
5) The tasks should not necessitate merely routine procedural algorithms as much as possible.

Secondly, as can be seen in the teaching experiment schedule in Table 3-7, before the teaching episodes started, the participants were administered the pre-tests devoted to measure prerequisite knowledge level and to determine the students' deficient knowledge and difficulties on Prerequisite Knowledge for Quadrilaterals Test, Proof Performance in Geometry Test, Vector Knowledge Test and Quadrilaterals Achievement Test. The pre-planned and previously prepared instructional modules were revised according to the pre-test scores and works of the students on these tests. Primarily, the most problematic topics and issues were stressed and the number of examples about these topics was increased.

The following plane geometry subject matters were developed and prepared by the researcher in two approaches: synthetic approach and vector approach separately.

1) Some topics in plane analytic geometry (*the details will be presented under the title "Analytic Geometry* Instructional Module*").*
2) Triangles (*the details will be presented under the title "Revision of Triangles Instructional Module*").
3) Quadrilaterals Unit (*the details will be presented under the title "Quadrilaterals Instructional Module"*)

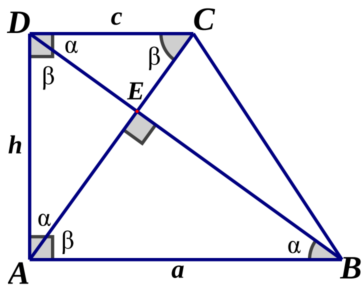
It should be stated that while developing the contents of the teaching experiment, the contents were prepared as if the research design would be an experimental research. Therefore, problems and topics were developed in accordance with both of the approaches, as much as possible. To illustrate; among all of the problems, proof-based tasks were developed so that they could be solved by means of vectors and Euclidean elements. As a result, the preparation period of the teaching materials took a long time. Preparing necessary infrastructure for this study and developing required materials in order to teach the unit "quadrilaterals" through vectors and coordinates in addition to synthetic methods necessitated a considerable amount of time. This is very normal situation since this might be one of the first studies in this area. It could be understood better if the historical developments of concepts are considered in terms of allocated time. Despite the fact that the historical backgrounds of concepts dated back for many years to be developed, these concepts are lectured in couple of lesson hours. To illustrate, the time allocated to teach taking square root of numbers is 12 lesson hours (MoNE, 2009); however, the history of taking square roots of numbers dated back to 1650 BC in The Rhind Mathematical Papyrus (Anglin, 1994).

Mathematical Aspects of Various Approaches

The definitions, components and distinctions among the approaches are presented in previous chapters. However, an illustration for solving a problem through analytic, synthetic and vector approaches separately is presented in this part so that the readers can understand the difference among the approaches on a solution process. In the problem, the participants are required to "*prove that the length of height is geometric mean of the length of the bases in right trapezoids, which have perpendicularly intersecting diagonals*". This problem will be solved via synthetic, vector and analytic approaches respectively to better express what is meant by solving a problem through these approaches.

First way: *Solving the problem by Synthetic Approach*

As one of the synthetic approach strategies, similarity of triangles is utilized to prove this relation. The synthetic approach solution is as follows (Figure 3-2).



By AAA triangle similarity theorem $\triangle ACD \sim \triangle BDA$. Therefore,

$$\frac{|AC|}{|BD|} = \boxed{\frac{|AD|}{|BA|} = \frac{|CD|}{|DA|}}.$$

In this relation, take

$$\frac{|AD|}{|BA|} = \frac{|CD|}{|DA|}.$$

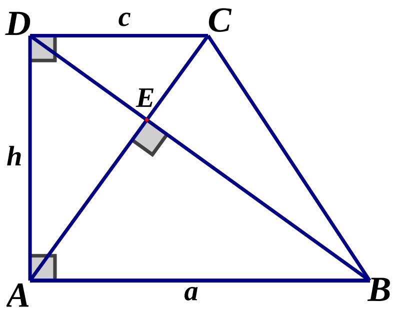
Hence, $\boxed{\frac{h}{a} = \frac{c}{h}}$.

Finally, we can get $\boxed{h^2 = a.c}$

Figure 3-2 Solving a problem through synthetic approach

Second way: *Solving the problem by Vector Approach*

Vector algebra and inner product are utilized in order to prove that that the length of height is geometric mean of the length of the bases in right trapezoids having perpendicularly intersecting diagonals. The vector approach solution is as follows (Figure 3-3).



Since $\overrightarrow{AC}$ *and* $\overrightarrow{BD}$ are perpendicular vectors, the result of their inner product equals to 0.

$$\overrightarrow{AC} \cdot \overrightarrow{BD} = 0$$

Figure 3-3 Solving a problem through vector approach

$\overrightarrow{AC}$ *and* $\overrightarrow{BD}$ are re-written in terms of $\{\overrightarrow{AD}, \overrightarrow{DC}\}$ and $\{\overrightarrow{BA}, \overrightarrow{AD}\}$ respectively.

Then,

$$\left(\overrightarrow{AD}+\overrightarrow{DC}\right)\cdot\left(\overrightarrow{BA}+\overrightarrow{AD}\right)=0$$

$$\underbrace{\overrightarrow{AD}\cdot\overrightarrow{BA}}_{=0}+\underbrace{\overrightarrow{AD}\cdot\overrightarrow{AD}}_{\downarrow}+\underbrace{\overrightarrow{DC}\cdot\overrightarrow{BA}}_{\downarrow}+\underbrace{\overrightarrow{DC}\cdot\overrightarrow{AD}}_{=0}=0$$

$$0+\left|\overrightarrow{AD}\right|.\left|\overrightarrow{AD}\right|.\cos 0+\left|\overrightarrow{DC}\right|.\left|\overrightarrow{BA}\right|.\cos 180+0=0$$

$$\left|\overrightarrow{AD}\right|^2-\left|\overrightarrow{DC}\right|.\left|\overrightarrow{BA}\right|=0$$

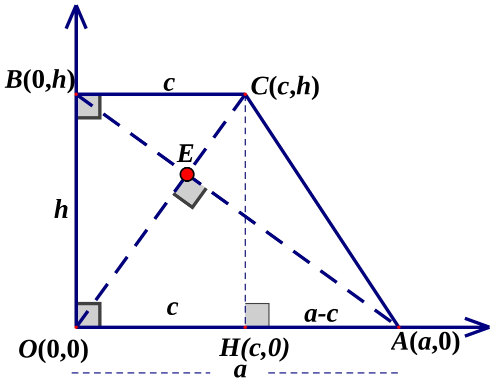
$$\left|\overrightarrow{AD}\right|^2=\left|\overrightarrow{DC}\right|.\left|\overrightarrow{BA}\right|$$

$$\boxed{h^2=c.a}$$

As a result, the relation can be proved via vector approach.

Third way: *Solving the problem by Analytic Approach*

To verify this relation through analytic approach, the right trapezoid is located on coordinate plane with the origin as O vertex. The properties of perpendicularly intersecting lines can be utilized in analytic approach. Specifically, the product of the slope of perpendicularly lines can be used. Since $[OC]$ *and* $[AB]$ are perpendicular to each other, the product of their slopes equal to -1. Then, the given relation can be verified via analytic approach as follows (Figure 3-4).



$$m_{OC}\ .\ m_{AB}=-1$$

$$\frac{h-0}{c-0}.\frac{h-0}{0-a}=-1$$

$$\frac{h}{c}\ .\ \frac{h}{-a}=-1$$

$$h^2=(-1)(c)(-a)$$

$$\boxed{h^2=a.c}$$

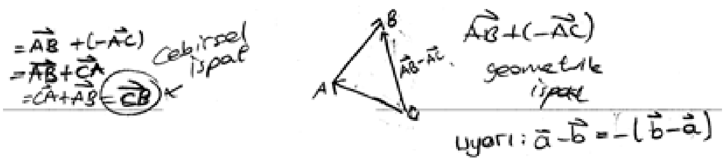
Figure 3-4 Solving a problem through analytic approach

Illustrations for synthetic, vector and analytic approaches to geometry were also given in geometry curriculum (MoNE, 2010a). In these illustrations, Pythagorean Theorem is proved by means of three approaches. The authors of the programs define one of the ways as "*vector approach*". They use a law of cosine in this solution. However, they only put arrows over the line segments. This cannot make the solution as vector approach solution. Maybe, this solution can be accepted as the combination of synthetic approach and vector approach. However, Pythagorean Theorem was proved by utilizing algebra of vectors and some properties of inner product by the participants of this dissertation. The proof does not include any use of law or formula. Besides, as another solution way, which was referred as analytic approach solution, the right triangle is transferred to analytic coordinate plane. The vertex that has the right angle is set as the origin of the plane. Despite the fact that the solution is described as analytic method in the program (MoNE, 2010a), it also contains various vector concepts such as inner product, position vector, magnitude of a vector etc. Therefore, this way cannot be accepted as analytic method merely. However, combining or integrating various approaches are not criticized here. On the contrary, utilizing various approaches is one of the aims and profits of this dissertation. The aim of presenting a solution via various approaches is to be clear and aware of the type of approach by which the solution is completed.

Proving and Reasoning

Reasoning and proving were stated as the indispensable component of the instruction in this study. There are several types of proving. Among these types, proofs by giving counter examples, was taught to the participants. Moreover, the inductive reasoning and the deductive reasoning were included in the study with the differences between these two reasoning method. Whereas the first one utilizes some number of specific examples to reach a reasonable conclusion, the latter one utilizes some of the ruler, definitions or properties to arrive at a logical conclusion. The difference was emphasized repeatedly in the study. In other words, it was repeatedly stated and emphasized that giving a numerical value or trying lots of numerical values satisfying the correctness of a statement (*experimental verification*) cannot be a proof for these statements.

Participants were required to prove some geometrical statements through vectors. However, there emerges the need for emphasizing the difference between algebraic proof and geometric proof in vector approach proofs. Therefore, some of the theorems or mathematical expressions were proved by means of algebraic proof and geometric proof, in order to distinguish the difference between the two types. As an example, the resultant vector of $\overrightarrow{AB} - \overrightarrow{AC}$ was respectively modelled algebraically and geometrically as in the Figure 3-5.



Algebraic proof Geometric Proof

Figure 3-5 Algebraic and geometric proofs

3.6.2 The Reasons for Implementing a Long-term Instruction

The reasons for allocating longer time to teaching sessions are stated as follows. To start with, it was important to provide a teaching-learning medium with an effective interaction and a healthy communication among students and the teacher-researcher. For example, in order to provide familiarization and adaptation among participants and the teacher-researcher, the researcher preferred to start to the sessions by solving some problems on geometry, algebra and analytic geometry right at the beginning of the teaching episodes.

Preparing an environment in which all of the participants can communicate effectively was also necessary because there would be several new situations to be faced in terms of students. These were namely; a new teacher would teach not only topics they had learnt before but also the new teacher would teach topics they had not learnt before. Moreover, the new topics would be taught in multiple approaches that they had not even any idea about them. Specifically, the students had not received any treatment by their teachers related to vector approach and analytic approach, which are two of the approaches in geometry teaching in terms of them. Moreover, the topics would be taught continuously with proof-based problems, which would be indispensable or constant component of the instruction followed in the classroom.

In addition to novelties related to teaching issues, there would be unfamiliar cases for the participants in terms of physical features of the classroom environment. For example, the courses would be lectured through smart board, the lessons would be audio recorded, and video recorded. In fact, one of the video cameras would be focused on the participants. Actually, although the students found being video recorded in the sessions very strange at the beginning, they got used to this situation. In fact, they forgot about being recorded at further parts of lessons. Furthermore, the sessions would be witnessed by an observer. In spite of the fact that the observer would not have any intervention with participants or the teacher, this would not be normal case for the participants. All of these were the reasons for preparing and arranging relatively long-term teaching episodes in this research.

In addition to the reasons presented above, another reason for arranging a long-term study is coming from the literature. Firstly, Steffe and Thompson (2000) states that a teaching experiment is conducted to understand students' progresses over the "*extended periods*". Furthermore, they underline insufficiency of short periods of teaching students while trying to figure out students' thinking comprehensively. Moreover, the duration of the research studies (e.g., Gagatsis & Demetriadou, 2001) about the use of vectors in geometry or mathematics teaching implemented so far was not long enough. It was found in the study that the students who preferred synthetic approach were found more successful than those who preferred vector approach in solving geometry problems (Gagatsis & Demetriadou, 2001). Moreover, the students were reported, as they were more apt to work with synthetic approach than vector approach. These conclusions cannot be fair, reasonable and scientific because of the fact that students who were inferred as successful had been educated in synthetic approach for long years. However, the students who were inferred as less successful had been only treated accordingly since participating in the mentioned study. That is, the students have 12-year-experience with synthetic approach and only 1 year-experience with vector approach. Therefore, the inferences drawn from the comparison of these two groups of students is not honest. Moreover, the students were not taught how to integrate vectors in geometry and how to utilize vectors in problem solving. Instead, they learned vectors with properties only as an isolated topic from the other geometric topics. Learning vector as a separate topic and learning geometry by vectors are different things. In other words, they were not instructed how they would utilize vectors to construct a bridge between algebra and geometry. Therefore, teaching geometry through vectors, utilizing vectors in problem solving by allocating sufficient time to work with vector approach in addition to synthetic approach were aimed for this study. The students were supposed to use vectors as a facilitator and a conceptual tool in geometry problem solving by the instruction given in this study.

In brief, the duration of the teaching experiment was planned long enough in terms of the time allocated for the instruction and the preparation phases of all kind of materials. Therefore, the researcher had a chance to make necessary regulations, corrections and revisions by means of these opportunities.

Handouts

In order to use the time effectively and economically during the instruction or teaching periods, the participants were supplied worksheets of the all-teaching episodes. They are also another data source for the study. Moreover, these were student-version materials in which there are “fill in the blank type exercises”, tasks, homework assignments and definitions. These handouts were also given to the observers. The student-versions of handouts were transferred into pdf-format and then they were projected on to the smart board and followed during teaching episodes. The teacher-researcher utilized teacher version of the handouts, which were prepared and filled completely before the related teaching session and topic.

The written works of the students were regularly collected and after each lesson, they were scanned by a scanner. After scanning processes, the original documents were given back to the students so that they could study what they learned.

In this way, all of the documents were digitized and then stored in various hard disks in order for not encountering any loss of data. The students were assigned homework regularly after each teaching episode throughout the study.

3.6.3 Development of the Instructional Modules

The following instructional modules constituted preliminary courses for the main instructional module "*Quadrilaterals*" in this teaching experiment research.

a) Elementary Vector Algebra Instructional Module,
b) Revision of Triangles Instructional Module,
c) Basic Algebra Instructional Module,
d) Analytic Geometry Instructional Module

These modules will be explained in detail in the following subtitles.

3.6.3.1 Elementary Vector Algebra Instructional Module

The purpose of preparing and teaching elementary vector algebra module was primarily to provide necessary prerequisite knowledge and abilities related to the vectors for the participants. This module consists of elementary vector algebra, which is essential for the students to study geometry via vectors.

The module was prepared in detail so that none of the participants faced with any problem in this domain. In addition, this basic vector algebra module was prepared by the researcher especially to provide prerequisite knowledge, which would be necessary for the students who would study several geometry topics such as triangles, some plane analytic geometry topics and quadrilaterals by means of vectors. In this way, the use of vectors was tried to be integrated or embedded into geometry teaching and to constitute the idea that vectors should not be thought as a separate topic from other geometrical subjects.

The following topics are included in the elementary vector algebra instructional module:

directed line segments, congruent directed line segments, definition of vector, unit vectors, zero vector, equality of vectors, opposite vectors, orthogonal vectors, addition and subtraction with vectors, vector addition by polygon law, triangle law and parallelogram law, displacement analogy in vector addition, resultant vector, analytic representation of vectors and operations on coordinate plane, definition and properties of scalar multiplication, linear dependence and independence of vectors, use of vectors in polygons, definition of position vector of a point, Euclidean inner product, magnitude (norm) and direction of vectors, unit vector of a vector with the same direction and opposite direction, the angle between two vectors, properties of inner product, right projection of a vector over a line and over another vector, area of polygons on the Cartesian coordinate plane, parametric and standard equations of line, normal and direction vector of a line, distance from a point to a line, and distance between two parallel lines.

The researcher prepared the elementary vector curriculum part in accordance with formal teaching mathematics program (MoNE, 2011) and geometry programs for the grade levels 9-10 and 11 (MoNE, 2010a & 2010b). Moreover, officially approved textbooks and some other reliable sources were examined and utilized as valuable references. However, great majority of the examples, problems and teaching materials are the work of the researcher originally. Moreover, the participants were required to solve vector algebra chapter of another predetermined two textbooks as homework.

In spite of the fact that participants had learned majority of aforementioned topics related to vectors before this teaching experiment started, it was observed that there was inadequate knowledge of the participants on most of the vector topics. In addition, they did not have any information about some topics (*these will be presented under the title "4.1.8 Statistical Analysis of Pre-tests and Post-tests Scores"*). Therefore pre-planned and pre-prepared elementary vector algebra curriculum part was revised according to the pre-test scores and works of participants on the VKT.

After these revisions, the final version of vector algebra module was constituted and contents of the revised module were taught to the students at the end.

It is important to emphasize more than once that while teaching this curriculum part, vectors were linked and integrated with other geometrical topics. In doing this, it was aimed that students could realize that vectors subject was not separate from other geometrical entities or it was not a useless unit. On the contrary, vectors can be used in solving several geometry problems. As will be stated in results section, the students have no idea or information about this situation by the time they participated in this teaching experiment.

Specific to this module, the researcher developed two analogies in order to teach related concepts better for this teaching experiment.

Utilizing Analogies

The researcher developed two analogies and then utilized them while teaching vectors. One of the though topic among contents of vector is to grasp visualization of right projection of a vector onto another vector or onto a line (Appova & Berezovski, 2013). To overcome this difficulty to some extent, light of a cell phone was used to enlighten the stylus pen for smart board from the top of it. As a result, there emerged stylus pen's shadow over the desk. It was used to illustrate right projection of a vector. This contributed for "the though subject" being clear or concrete on their minds. The analogy used here was called as "*shadow analogy*". An illustration for this analogy is given in the Figure 3-6.

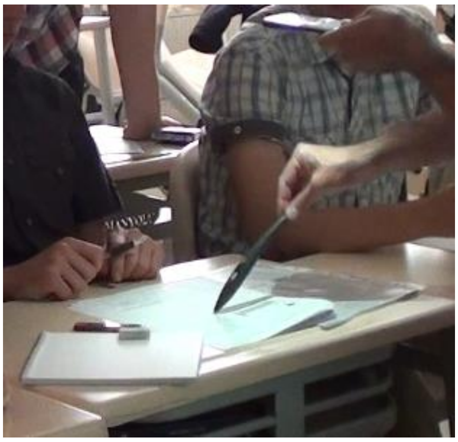


Figure 3-6 Illustration of shadow analogy to depict right projection of a vector

The second analogy used for this study is "*displacement vector*" analogy. In this analogy, a side of a polygon (*triangle or quadrilateral*) is accepted as a displacement vector. One of the endpoints of the current side is set as initial point and the remaining one is set as the terminal point. Then the students were asked to determine all possible alternative paths by which we could have a trip in order to reach from initial point to the terminal point. This rationale was utilized frequently in solving geometric problems especially containing geometrical statements or propositions to be proved.

An illustration of utilizing this analogy is given in the Figure 3-7. The participants were explained that in order to reach L point from K, either "K-A-B-L" or "K-D-C-L" paths should be followed. After determining these paths, the pupils transferred these possibilities into vectorial representations.

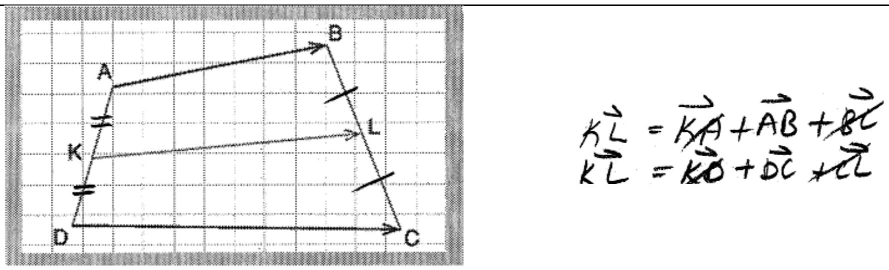


Figure 3-7 An illustration for displacement vector analogy

The idea of displacement vector analogy for algebra of the vectors was developed by the researcher after studying vectors and trying to determine the ways to integrate vectors into geometry teaching.

In this stage of the teaching experiment, all of the students were taught elementary vector algebra module and upon completion of the module, the students took a test on vectors as homework. According to the written works of the students on vectors, the missing points and the difficulties related to vectors that they had were tried to be resolved.

Some considerations taken care of as precaution

Another difficulty in the literature (Nguyen & Meltzer, 2003; Van Deventer & Wittmann, 2007) is that students have difficulties with vector subtraction. Specifically assigning correct endpoints in subtracting two vectors is one of the problematic points. Specifically, the students have difficulties in writing the resultant vector as $\overrightarrow{CB}$ for the subtraction operation $\overrightarrow{AB}-\overrightarrow{AC}$. Instead of being able to write correct resultant vector $\overrightarrow{CB}$, the reverse order "$\overrightarrow{BC}$" can be frequently encountered in students' answers. This was as well, difficulty of the researcher in studying vectors during the preparation phase for the dissertation and in his high school days as a student. Therefore, the researcher tried to find alternative and effective ways to teach vector subtraction. Because of these reasons, the researcher preferred to switch the subtraction operation to addition of two vectors. While performing this style, the minus sign in the subtraction operation is changed with plus sign and the order of the letters in the subtrahend vector is reversed simultaneously. In other words, instead of writing $\overrightarrow{CB}$ directly for the subtraction of $\overrightarrow{AB}-\overrightarrow{AC}$, the following operations were preferred in teaching subtraction of two vectors.

$$\begin{aligned}\overrightarrow{AB}-\overrightarrow{AC} &= \overrightarrow{AB}+\left(-\overrightarrow{AC}\right)\\ &= \overrightarrow{AB}+\overrightarrow{CA}\\ &= \overrightarrow{CA}+\overrightarrow{AB}\\ &= \overrightarrow{CB}\end{aligned}$$

The researcher recommended the participant students subtract vectors as shown above instead of memorizing place of letters for the resultant vector. This is not only for preventing them from memorizing operations but also make them familiarizing with conducting operations algebraically. Moreover, changing subtraction operation with addition of vectors can be accepted as more conceptual and beneficial than writing the resultant vector directly. Particularly, this preference contains application of vector addition, multiplication of a vector with a scalar and commutative property of vector addition; hence, this is more mathematical than the way of memorizing the order of letters. In the latter case, the students' were to memorize the order of the endpoints that possibly yields writing incorrect vectors. After reviewing related literature, it was found that some of authors preferred to use this method (e.g., Ayre, 1965 p: 86)

Pavlakos, Spyrou and Gagatsis (2005) indicate that students have difficulties in determining the angle between two vectors especially for straight and obtuse angles. Moreover, they state that students have difficulties in recognizing for zero angle. The researcher determined earlier that some of the solutions for geometric problems specifically for proof-based problems related to quadrilaterals that have parallel sides necessitate the use of parallel vectors and inner product of these side vectors. Therefore, correctly determining the angle (either $0°$ or $180°$) between two parallel vectors is important. Moreover, this is vital in the problem cases when the vectors are not on the same side or line. Consequently, correctly determining angles between two vectors was stressed when preparing Elementary Vector Algebra Module.

Another concept that students have difficulties with vectors is the inner product of vectors in introductory physics courses as reported in the study of Ortiz (2001). Since inner product is an important part of vectors in setting a bridge between algebra and geometry, it was preferred to teach the properties of inner product with their proofs. However, firstly the students were required to prove these properties on their own.

Lastly, the participant students were emphasized with the fact that the vectorial relations are valid also for length of vectors for the parallel vectors. However, they are reminded that the reverse is not true by giving counter examples.

3.6.3.2 Revision of Triangles Instructional Module

Triangles module with its subtitles can be stated as a prerequisite unit for the main subject matter "*Quadrilaterals*". From his teaching experiences, the researcher has the idea that when one of the diagonals of a quadrilateral is drawn, then there emerge two triangles. In other words, a quadrilateral can be thought as "two triangles with a common side". Therefore, the stronger the infrastructure on triangles can be structured, the less problems and difficulties related to quadrilaterals would be encountered according to the researcher. In other words, stronger knowledge on triangles unit yields stronger knowledge on quadrilaterals. When this logic was shared with the participants before teaching triangles part, the participants appreciated it. Therefore, the researcher developed this revision of triangles module more carefully, importantly and in detail. The students as well paid more importance to triangles in this respect. Besides, it is necessary to note that the entire triangle unit is one of the most important subjects for school geometry and mathematics courses and especially for university entrance examinations for the sake of students.

Triangles module also constitutes the first opportunity for the integration of vectors with geometry teaching after learning elementary vector algebra unit and before learning quadrilaterals unit. In one sense, teaching triangles with synthetic, analytic and vector approaches can be considered as the first pilot study of teaching quadrilaterals with analytic, synthetic and vector approaches. Therefore, the researcher made the triangles unit last longer in order to see the problematic issues and to monitor progressing of the treatment through analytic, synthetic and vector approaches.

Specifically, the proof of the following statements are constructed with students in triangles instructional module during the teaching experiment.

> *synthetic and vector proofs of Pythagorean's theorem, AA triangle similarity theorem (butterfly similarity), triangle proportionality theorem, Euclidean metric relations, Thales' theorem, The Law of Cosines, The Law of Sines, Vector intersecting theorem (Zou, Zhang & Rao, 2012) in triangles, the ratio of length of emerging parts of medians when two medians intersect.*

While teaching all of these topics, vector proof for these theorems were assigned as a homework task and the participants were required to complete them. Most of the time, they proved the theorems with vectors in success. Frankly, the construction of these studies took longer in the research. However, it was important to see how the training through analytic, synthetic and vector approaches progressed and at which points there arose difficulties and problems in the classroom during the instruction.

In the light of these, the purpose of this module was not merely to teach triangles as a prerequisite knowledge for quadrilaterals. In addition, it was also aimed to observe how vectors and coordinates were utilized in geometry teaching specifically in classroom environment with students.

The revision of triangles instructional module included the following topics that were supposed to be taught to the students when they were at grade levels 9 and 10.

These topics are namely,

> *Trigonometric formula for the area of a triangle, side-area relationship in a triangle, Thales' theorem, congruence and similarity of triangles, Pythagorean's theorems, sign of trigonometric functions, Euclidean metric relations, triangle proportionality theorem, Menelaus's theorem, Ceva's theorem, Stewart's theorem, Carnot's theorem, the angle bisector theorem, median length: the "Apollonius' theorem", use of these theorems in comparing area of triangles and quadrilaterals constructed in a triangle.*

In addition to these subject matters, centroid of triangle and area of triangular or quadrilateral regions formed by centroid of a triangle, vector intersecting theorem (Zou et al., 2012) and center of mass or balance model (Hausner, 1998) were included in this module. The last part of this module was taught to the students to present a chance to recognize various uses of vectors in different subjects as an application of vector approach. In order to rate the area of different regions that are constituted in a triangle, various methods were applied. Solving this kind of problems by several approaches was a novelty in terms of the participants.

Among the aforementioned topics on triangles module, center of mass model (Hausner, 1998) for comparing area of triangular or quadrilateral regions constituted in a triangle and application of vector intersecting theorem (Zou et al., 2012) were out of boundary of formal geometry curriculum program (MoNE, 2010a & 2010b). However, they were utilized to present several illustrations of analytic, synthetic and vector approaches to geometric problems for the students.

3.6.3.3 Basic Algebra Instructional Module

This elementary algebra instructional module contains teaching some parts of literal and algebraic expressions that would be necessary in teaching geometry especially by means of vectors. This requirement emerges in two aspects according to the experiences of the researcher during the preparation of the materials. Firstly, in developing this module, students would especially be required to learn how to manipulate with algebraic and hence numerical expressions in order to be able to calculate the area of quadrilaterals that are given in coordinate plane. Area of these quadrilaterals can be computed through synthetic approach; however, vector approach solution was desired in the geometry curriculum (MoNE, 2010b). Therefore, the students were to compute area of quadrilaterals in coordinate plane by means of predetermined vectorial formula: $\frac{\sqrt{\|\vec{p}\|^2 . \|\vec{q}\|^2 - \langle \vec{p}, \vec{q} \rangle^2}}{2}$ where $\vec{p}$ *and* $\vec{q}$ are the diagonal vectors of the quadrilateral (MoNE, 2010b). As seen, this formula includes multiplication of two squared numbers. In order not to engage with large numbers because of this multiplication, the researcher recognized that an algebraic manipulation to the numerical quantities could be utilized to overcome this difficulty. The underlying reason for this manipulation can be explained by algebra of literal expressions. As a result, this part was incorporated into this module.

Secondly, as stated earlier, inner product of the vectors takes an important place in teaching quadrilaterals by means of vector approach according to the experiences of the researcher when he studied utilization of vector approach in geometry. This fact is also stated in the related literature (e.g., Vaughan & Szabo, 1973; Johnson, 1967 and Choquet, 1969)

The researcher realized that inner product is useful especially for proof-based problems in quadrilaterals units. The following topics were included in the form of algebra of literal expressions because they necessitate “addition, subtraction and distributive property (multiplication over addition and subtraction)”.

(a) Algebra of vectors (vector addition and subtraction, multiplication of a vector with a scalar),

(b) Properties of inner product (inner product of a vector with the addition of another two vectors or inner product of addition of two vectors with subtraction of another two vectors)

(c) Transition from a vectorial quantity to scalar quantity

In this way, an infrastructure of this respect would be constructed.

3.6.3.4 Analytic Geometry Instructional Module

While lecturing quadrilaterals, starting to teach all of the quadrilaterals with specified coordinates of their vertices on coordinate plane is recommended in the Turkish national high school geometry-teaching program (MoNE, 2010b) and in the standards of NCTM (1989). This is continuously desired for each special quadrilateral such as trapezoid, kite, parallelogram, square etc. The aim of this recommendation is explained with possible increase in students’ motivations toward geometry in the classrooms. Moreover, students are expected to deduce the properties of geometric figures and the relations among these properties by means of these trials and explorations (NCTM, 1989). Hence, all of the quadrilaterals were presented firstly with a specific example on coordinate plane as a task called as *entering assignment*, in this teaching experiment (*a sample for entering assignment is presented in Appendix C*).

Since one of the approaches included in this teaching experiment is analytic approach, it is necessary for students having knowledge on some of the elementary analytic plane concepts so that the participants are able to utilize these concepts in geometry problem solving. In other words, studying quadrilaterals analytically requires some prerequisite knowledge on analytic geometry. The purpose of this instructional module was to provide this knowledge.

Related to the purpose expressed above, analytic geometry module included the following topics that were supposed to be taught to the students when they were at grade levels 9 and 10 in geometry courses.

These plane topics are namely,

the distance between two points, mid-point of given two points, equation of a line, finding intersection of two intersecting lines, slope of a line, slope-shape relation, direction and normal vector of a line, distance from a point to a line and distance between two parallel lines.

This prerequisite part would be also necessary especially when vectors were represented in coordinate plane.

3.6.3.5 Quadrilaterals Instructional Module

This section explicates major components and features of the main subject matter "*Quadrilaterals*". Detailed information on content, reference, requirements of the geometry curriculum are presented in the following subsections.

Content and Boundary

This module constitutes the main subject matters of the current teaching experiment study. The contents of this module are composed of the following topics: "*Quadrilaterals, Trapezoid, Parallelogram, Rectangle, Rhombus, Square, Deltoid and Classification of Quadrilaterals*" respectively. These subjects are going to be taught to the students in geometry lessons when they attend grade level 11. Hence, the participant students would learn this unit before they attend to their regular geometry course.

As similarly in the other curriculum parts, "Quadrilaterals" unit was also completely prepared and planned by teacher-researcher of the study. While preparing the module for quadrilaterals, Turkish national teaching geometry program for the grade level 11(MoNE, 2010b) and mathematics program (MoNE, 2011) for grade levels 9, 10 and 11 were continuously and primarily taken into consideration to define a boundary for the study.

Officially approved textbooks and some other reliable sources of different private publishing firms were examined and utilized as valuable references. However, great majority of the examples, tasks, problems and teaching materials are the product of the researcher's works after studying with vectors and of his teaching experiences.

Targets on Curriculum Standards

1. Cause-and-effect relation

While instructing quadrilaterals unit, a learning medium was designed in which participants were made discover and infer definitions, properties, theorems and results related to quadrilaterals. One of the reasons for this rationale is because of the fact that the importance of learning in a cause-and-effect learning environment is repeatedly stressed in the curriculum program for all grades in high schools. Consequently, none of the properties was presented directly to the participant students. The results and features were attained after some endeavors in classroom, discussions and dialogs among the students and the teacher. Reasons underlying mathematical or geometrical statements were continuously questioned and discussed in this study.

2. Utilizing "Analytic, Synthetic and Vector Approaches" and "Reasoning and Proving"

The importance of the developing proving and reasoning skills for the students is emphasized during the learning process in geometry teaching for high school geometry courses. In addition, multiple approaches those are namely synthetic, analytic and vector approaches to proving are specifically desired to be developed in this period with regard to the geometry teaching program (MoNE, 2010a & 2010b). Related to this issue, the number of proofs desired to be constructed is 12 out of 15 main learning objectives in the first semester of the 11th grade geometry course. However, there are specifically emphasized 30 properties, propositions or theorems to be proved or justified in this formal geometry curriculum (MoNE, 2010b). All of the proofs of mentioned geometrical statements are included in this teaching experiment without any exceptions.

It is stated in geometry curriculum for grade level 11 (MoNE, 2010b) that the option of approach among synthetic, analytic and vector approaches is preferred according to convenience and easiness of the approaches while proving geometrical statements via various approaches. This was followed throughout this teaching. Moreover, after some steps and progress in teaching experiment, this choice is released to students' preferences.

In short; "*proving*", "*teaching in cause-and-effect relation" and "utilizing analytic, synthetic and vector approaches"* are clearly indispensable components of the teaching in this study.

3. Discussion

Instead of forming a teaching-learning environment in which the teacher is active transmitter or instructor and the participants are passive collectors, a teaching-learning medium in which students actively enact is preferred and tried to be realized. By means of this characteristic of learning environment, changes and improvements in students interactions are anticipated from students' being inactive learner to active learner toward to the end of this teaching experiment.

4. Increasing Students' Motivation

The importance of increasing students' motivation is stressed repeatedly in geometry curriculum for 11th grade level (MoNE, 2010b). The researcher was as well aware of the importance of increasing students' motivation in terms of students' success in geometry or in other courses from his teaching experiences and from the literature. To illustrate; in the study of Middleton and Spanias (1999), they state that supplying chance for students to improve motivation in mathematics is accepted as important in terms of success in mathematics. In order to increase students' motivation in geometry course, some suggestions were specified in the curriculum program (MoNE, 2010b). In addition to these suggestions, the teacher-researcher made use of additional endeavors to provide an increase in students' motivation.

As stated in the analytic geometry curriculum part, in order for increasing the students' motivation, the subject matters: i.e. quadrilaterals are recommended to be started firstly with solving specific numerical examples on coordinate plane i.e. by using analytic approach. That is to say, quadrilaterals' vertices are given with their coordinates. In order to response this requirement, the students were supplied student-version of handout teaching materials. They were additionally provided "*Coordinate Plane Worksheets*" as graphing papers so that they could use coordinate plane. The students utilized these graphing papers, as they desired. Parallel to this advice, starting with numerical analytic examples was implemented throughout the teaching episodes.

Quadrilaterals were instructed and introduced firstly with discovering their properties on coordinate plane by solving numerical examples via analytic approach in the format of an assignment sheet. After that, they were asked to report general characteristics of the quadrilateral on which they engage. The initial properties of each quadrilateral discovered or inferred by the students were compared with the characteristics that they reached at the end of each quadrilateral section conducted with the teacher-researcher. Furthermore, in order for not giving prototypical examples (Fujita, 2012), the quadrilaterals were placed in different positions (*non-standard positions*) as much as possible while presenting these quadrilaterals on coordinate plane.

The following activities were preferred to be included in the study by the researcher to increase participants' motivation. Firstly, the geometrical and mathematical terms were presented in English to the participants while teaching topics in the scope of this study. Secondly, appealing historical background about some terms was shared with the participants. Giving some historical information about mathematical concepts is thought to increase students' motivation and excitements towards mathematics courses (e.g., Farmaki & Paschos, 2007; Tattersall & McMurran, 2004). As an example, origin of the name "Algebra" from the word "al-jabr" in the book on calculation of al-Jabr-vel Mukabalah of the scholar Al-Khwarizmi (Katz, 1997), the trapezoid proof of Pythagorean theorem by James A.Garfield the 20th president of U.S.A (Nelsen, 1993) were utilized in teaching episodes.

In addition, the researcher put a square to the right and bottom of the proof as a sign indicating the completeness of proof. This is a habit or ritual that he learned during his undergraduate education. This is also a way of making geometry as fruitful study. In students' proofs, the reflection of putting square was observed so many times. Since it is not the scope of the study, it was not included in results chapter.

At some points of the teaching experiment during the instruction, different types of activities such as "*proofs without words*" activities (Nelsen, 1993) were studied with the participants. Specifically, in order to prove Pythagorean Theorem by means of drawings or diagrams, students were supplied necessary materials and then they were required to prove this theorem without using any mathematical statements. Besides, expressions related to calculating the area of quadrilaterals were verified or justified by means of proofs without words activities as well. To illustrate, the area of a trapezoid was verified by means of cut and paste method (Özdural, 2000).

When developing items to be solved in this instructional module or in other modules as well, high-stakes university entrance examinations (ETHE and UPE) were considered. In other words, problems similar to items of those aforementioned examinations were frequently asked and solved in order to increase students' motivation and; hence, students' attendance to the teaching episodes. In terms of students and their parents, it is important to be prepared for university entrance examinations besides participating in a research study like this teaching experiment.

During teaching sessions in the study, besides dealing with proving geometrical statements or theorems and dealing with solving proof-based geometrical problems, some practical solution ways and tricks for geometric problems were shared with the students. This is also another source of motivation for the participants during the teaching episodes.

5. Enrichment of Learning Environment

In their problem solving processes in geometry, students' developing alternative approaches depends on the training that they receive on geometry or other disciplines in their classrooms. The richer the learning environment provided to the participants, the more diverse learning outcomes to be observed and could be accordingly anticipated. Participant students could develop alternative solution strategies and make flexible transitions among different approaches by means of such a rich teaching-learning medium. Moreover, combining aforementioned approaches interchangeably in necessary problem situations and being able to use these approaches together to complete solving geometric problems could possibly have a potential to increase students' achievement in geometry and mathematics. A step in one of the approaches can be a hint or trigger for the other approach to carry on solutions.

According to the requirements of geometry teaching program (MoNE, 2010b) teaching materials were enriched and supported with real life examples. It was observed that students had difficulties in solving this kind of examples. Moreover, the researcher made use of transformations (*translation, rotation, reflection, dilation*) and homothetic-translation as an application of vector approach in geometry. As known, angle and symmetry are parameters for rotation and reflection respectively. Similarly, vectors can be used as a parameter for translations (Faydacı & Zembat, 2012). Therefore, transformations especially transition was included in this module as an application of vector approach.

6. Duration

The time allocated for teaching "*quadrilaterals unit*" is 48 lesson-hours in 11th grade geometry curriculum. This unit is taught throughout the first semester of 11th grade level. Except for periods allocated for interviews, general reviews and make up lessons, 49 lesson hours were totally allocated for teaching quadrilaterals unit in this teaching experiment. This resembles an appropriate situation for requirements of formal geometry program in terms of time allocated for this unit.

Use of Analogies

As stated before, some analogies, which were developed earlier in elementary vector algebra curriculum, were started to be integrated into teaching of quadrilaterals unit. To illustrate, the displacement analogy was developed and used in teaching quadrilaterals. In this analogy, despite the fact that there are not any vectors given in the problems, one of the diagonals (or side) of the quadrilateral is accepted or determined as "displacement vector". One of the vertices of the selected diagonal is set as initial point and the remaining vertex is set as the terminal point. After that stage, students were asked to determine possible paths to link these points by using sides or elements of the quadrilateral. Then addition, subtraction and inner product of vectors that constitutes the path were utilized to solve geometric problems in vector approach to geometry.

Pilot study

There are some issues can be considered as pilot study for the main application of this study. Firstly, the triangles unit was taught via vector approach in addition to the other approaches. Therefore, revision of triangles unit can be accepted as the first pilot study for the main subject matter of the present research. That is to say, triangles can be thought as the first application field for vector approach. This was presented under the title Revision of Triangles Instructional Module.

The iterative aspect of quadrilaterals can be thought as the second opportunity to pilot the teaching of subsequent topics. In other words, as the teaching experiment progressed, it was understood that each of the subject in quadrilaterals unit functioned as a pilot study and an iteration for the next chapter within this module. The researcher had already this perception before the administration of the teaching experiment while preparing the teaching materials. In fact, this perception was verified once more again in the classroom environment while teaching topics.

While defining quadrilateral, inclusive definition of quadrilaterals was preferred. This preference also feeds the iterative feature of the quadrilaterals. This was explained in the section "Definition of Important Terms". In spite of the fact that the number of special cases or properties increase in number for the subsequent quadrilaterals as the study progresses, the students started to develop a rationale and to adapt and apply this rationale to the next quadrilaterals. Naturally, teaching started to last shorter periods in comparison with the earlier subjects (Figure 4-78).

Handouts

Students were supplied handouts right at the beginning of each section for quadrilaterals module. In these handouts, there are intentionally left blank parts in the pages so that the students could write down the necessary information and could follow the teacher easily by this way. Moreover, since the students were required to solve geometric problems by three approaches (*synthetic, analytic and vector*) as much as possible, there are provided spaces on which the participants were desired to solve the problems in various approaches. In addition, they were asked to write the name of the approach that they preferred on these handouts. Besides, the students were required to state the difficulties, conveniences, advantages and disadvantages of the that they preferred in solving problems at some points of teaching episodes approaches (*these are exemplified in the results chapter in detail*). They were additionally asked to exhibit the underlying reasons for their preferences. This was in the form of interviewing with the participants or requesting them to document on the supplied handouts. Participants' being aware of the method, which they preferred, was aimed by filling those blank spaces throughout the quadrilaterals unit. An example of a handout containing lesson plan can be found in the Appendix D.

Each quadrilateral is composed of two sections. Therefore, the participants were provided two booklets for each quadrilateral in this module. In the first section, general definitions, terms and properties of the related quadrilateral were presented to the students. Some theorems specified in the curriculum (MoNE, 2010b), the formulas that express perimeter and area of the quadrilaterals are presented in the second section. The students were given homework, which was predominantly composed of conjecture and proof-based problems at the end of the second section.

Priority or Superiority of Approaches

It is important to emphasize that the researcher does not claim priority or superiority of any certain approach to the other approaches. Specifically, eschewing synthetic methods was not asserted or advised to the pupils in any point of the teaching experiment. Instead of asserting and hence, trying to prove superiority or priority of approaches, enhancing students' current problem solving strategies and students' gaining the ability to make flexible transitions among approaches were aimed in this study which is compatible with the related literature.

In the study during the problem solving phases, solving each geometric problem or justifying the correctness of all mathematical statements by three approaches was not required or entailed. Utilizing from three approaches was achieved as much as possible and as curriculum program necessitates (MoNE, 2010b). Therefore, some of the problems were not solved or properties were not justified by means of vector approach.

Preliminary Preparation for the Next Grade Levels

Finally, it should be reminded that multiple approaches to geometry is also desired for geometry courses at grade level 12 (MoNE, 2010c). The most important discrimination is "*working in 3D space*" instead of working on 2D plane. Because of the unifying character of vectors, if students can grasp the logic and principles behind 11^{th} grade geometry course that are presented in detail above, they probably will not have so much difficulties in subsequent geometry courses through in high school and university. In other words, probably it can be said that achievement on 12^{th} grade geometry course depends on the success of 11^{th} geometry course. This is explained under the title “Advantages of Vector Approach” in literature review chapter.

3.6.3.6 Revising the Instructional Materials

After the researcher completed the first draft of the instructional materials, three experienced mathematics-geometry teachers reviewed these materials. As well as the teachers examined instructional materials in terms of mathematical compatibility, they also took care of appropriateness of the contents of the materials with regarding to high school geometry and mathematics curriculum programs. According to their comments and corrections, the researcher revised the necessary parts of the teaching materials. Besides, since each special quadrilateral contains common properties with preceding quadrilaterals that were taught previously, the researcher had an opportunity to revise the materials continuously as the study progressed.

Each of the final versions of instructional materials was distributed to the subjects of the study as a separate handout. Each booklet has two versions: student version and teacher version. An example for student version is presented in Appendix D.

The following tables present the date, time, duration and order of tests, interviews and instructions in the teaching episodes.

Pre-Tests

Table 3-7 Meeting schedule and application of pre-tests

Test	Date	Duration
Prerequisite Knowledge for Quadrilaterals Test	April 16, 2013	90 mins
Proof Performance in Geometry Test	April 18, 2013	90 mins
Vector Knowledge Test	April 19, 2013	90 mins
Personal Information Form	April 20, 2013	30 mins
Quadrilaterals Achievement Test	June 26, 2013	120 mins
	Total	**7 hours**

Teaching Episodes

Table 3-8 Meeting schedule and teaching experiment sessions

Episode	Lesson Topics	Date	Duration
1	Equation of a Line, Slope of a Line, Slope-Shape Relation, Types of Angles, Area of a Triangle, Area of a Region Bounded by Two Intersecting Lines and Axes, Sign of Trigonometric Functions	May 01, 2013	120 mins
2	Vectors (*basic definitions and key concepts*), Vector Algebra (*vector addition, vector subtraction, multiply vector by a scalar, linear dependence and independence of vectors)*	May 07, 2013	180 mins
3	Euclidean Inner Product and Properties of Inner Product, Unit Vector	May 10, 2013	180 mins
4	Euclidean Inner Product and Properties of Inner Product, Unit Vector	May 14, 2013	100 mins
5	Right Projection of a Vector Computing Area of Quadrilaterals on Coordinate Plane via Vectors	May 17,2013	180 mins

Table 3.8 continued

6	Synthetic and Vector Proof of Theorems in Learned Subjects so far, Trigonometric Formula for Area of a Triangle Congruence and Similarity of Triangles (*SSS Congruence, SSS Similarity, AAA Congruence, AAA Similarity*) Triangle Proportionality Theorem	May 20, 2013	180 mins
7	Ceva' s Theorem, Menelaus's Theorem Stewart's Theorem, Carnot's Theorem The Angle Bisector Theorem Median length "*Apollonius' Theorem*" Use of Theorems Above in Comparing Area of Triangles and Quadrilaterals Constructed in a Triangle.	June 02, 2013	150 mins
8	Congruence and Similarity of Triangles Euclidean Metric Relations Triangle Proportionality Theorem	June 07, 2013	150 mins
9	Congruence and Similarity of Triangles Euclidean Metric Relations Triangle Proportionality Theorem Thales' Theorem An Application of AA Similarity "*Butterfly Similarity*"	June 10,2013	150 mins
10	General Revision	June 11, 2013	90 mins
11	Literal Algebraic Expressions	June 12, 2013	75 mins
12	Analytic Geometry The Distance Between Two Points Equation of Lines Distance from a Point to a Line Direction and Normal Vectors	June 13, 2013	150 mins
13	General Revision	June 18, 2013	50 mins
14	Distance from a Point to a Line and its Proof Distance Between Two Parallel Lines and its Proof The Proof of Pythagorean Theorem in Synthetic and Vector Approaches	June 19, 2013	60 mins

Table 3.8 continued

15	Awesome Triple Distance Between Two Parallel Lines and its Proof Area of Polygons in Coordinate Plane Proof of Vector Intersecting Theorem and its Application The Center of Mass Model	June 21, 2013	180 mins
16	Awesome Triple Area of Polygons on Coordinate Plane Proof of Vector Intersecting Theorem and its Application The Center of Mass Model Centroid of Triangle and Area of Regions formed by Centroid of a Triangle	June 25, 2013	180 mins
	Quadrilaterals Achievement Test	June 26, 2013	120 mins
17	Quadrilaterals 1st Part	July 01, 2013	75 mins
18	Quadrilaterals 1st Part	July 03, 2013	150 mins
19	Quadrilaterals 2nd Part	July 05, 2013	150 mins
	Interview	July 05, 2013	60 mins
20	Quadrilaterals 2nd Part	July 08, 2013	60 mins
	Interview	July 08, 2013	60 mins
21	Trapezoid 1st Part	July 11, 2013	135 mins
22	Trapezoid 1st Part	July 14, 2013	60 mins
	Trapezoid 2nd Part	July 14, 2013	60 mins
23	Trapezoid 2nd Part	July 17, 2013	135 mins
24	Parallelogram 1st Part	July 20, 2013	100 mins
25	Parallelogram 1st Part	July 24, 2013	60 mins
	Interview	July 24, 2013	60 mins

Table 3.8 continued

26	Parallelograms 2nd Part	July 27, 2013	180 mins
27	General Revision	July 30, 2013	150 mins
	Interlude and homework assignment		
28	Rectangle 1st Part (*power cut*)	August 29, 2013	60 mins
29	Rectangle 1st Part	September 01, 2013	120 mins
30	Rectangle 1st Part (*make up lesson*)	September 04, 2013	90 mins
31	Rectangle 1st Part	September 05, 2013	60 mins
31	Rectangle 2nd Part	September 05, 2013	75 mins
32	Rectangle 2nd Part Ömer and Naci	September 10, 2013	45 mins
32	Rhombus 1st Part Ömer and Naci	September 10, 2013	90 mins
33	Rectangle 1st and 2nd Part Ahmet	September 13, 2013	45 mins
33	Rhombus 1st Part Ahmet	September 13, 2013	45 mins
34	Rhombus 2nd Part Ömer and Naci	September 14, 2013	75 mins
35	Rhombus 2nd Part Ahmet	September 18, 2013	60 mins
36	Square	September 26, 2013	60 mins
37	Kite	September 28, 2013	40 mins
37	Classification of Quadrilaterals	September 28, 2013	30 mins
		Total	**80 hours**

Post-tests

Table 3-9 Meeting schedule and application of post-tests

Test	**Date**	**Duration**
Prerequisite Knowledge Test for Quadrilaterals	September 30, 2013	90 mins
Proof Performance in Geometry Test	October 01, 2013	90 mins
Vector Knowledge Test	October 02, 2013	90 mins
Quadrilaterals Achievement Test	October 06, 2013	120 mins
	Total	**6,5 hours**

3.7 Procedure

The procedure to conduct this teaching experiment included several steps. These steps were listed below:

a) According to the researcher's interest on improvement of students' conceptual understandings through multiple approaches in geometry, students' problem solving strategies and proving skills, and since the researcher has teaching experiences, a teaching experiment was determined a research method for this study.
b) Several key terms such as; "teaching experiment", "synthetic approach", "analytic approach", "vector approach", "transformational approach", "multiple approach instruction", "proof", "vector proof", "synthetic proof", "analytic proof", "Euclidean geometry", "vector geometry", "coordinate or analytic geometry", "misconceptions and difficulties", "quadrilaterals" and "activities, tasks and problems on quadrilaterals" were utilized to make literature review. Reviewing literature was a long process and this process was carried out in every steps of the study.
c) The research problem were specified after initial literature review performed with predetermined key terms through databases (Educational Resources Information Center (ERIC), EBSCOhost, ProQuest Dissertations and Theses (PQDT), and Education Research Complete), Science Direct, Google Scholar, METU Library Theses and Dissertations, and Turkish Higher Education Council National Dissertation Center.
d) After reading process, theoretical framework of the present study was constructed with the help of related studies.
e) Approximately 25-30 geometry textbooks and additional sources were examined and studied in detail to see to what extent the requirements of curriculum standards and requirements were reflected to these resources.
f) The researcher studied on the theorems specified in the curriculum and tried to develop vector approach and coordinate approach proofs of these theorems. He tried to generate problems to be solved through several approaches as much as possible during the course of teaching experiment.

g) Lesson plans and instructional materials were developed according to reviews of a professor majoring mathematics and three mathematics and geometry teachers.
h) Permissions from the students' families were obtained to conduct the teaching experiment sessions for this study.
i) Available places were determined and necessary permissions were obtained from the owners or principles of these settings.
j) As much as possible a closer and continuous contact with the parents were provided before, during and after the study.
k) Pre-tests, pre-interviews, teaching experiment sessions, in-term interviews, post-tests and post-interviews were completed successfully.
l) Written and oral data from participant students were analyzed continuously. The interviews were transcribed by the researcher himself. Necessary tables, graphics and figures were formed to have an idea about the frequency of emerging situations.
m) Dissertation was completed.

3.8 Trustworthiness

It is important to enhance the quality and credibility of either a quantitative or a qualitative research by providing some criteria through some of the strategies. Although these strategies are similar in a broader meaning in terms of a quantitative and qualitative research designs, there are some differences among the ways of establishing reliability and validity issues for quantitative and qualitative studies. "*Internal validity, external validity, reliability and objectivity*" are the ways of implementing and obtaining a credible quantitative research (Yıldırım and Şimşek, 2006). However, Lincoln and Guba (1985) utilize the criteria "*credibility, transferability, dependability and confirmability*" to satisfy the requirements of the trustworthiness of a qualitative research.

3.8.1 Credibility

Credibility is one of the criteria to establish trustworthiness of a research. According to Lincoln and Guba (1985), while conducting a scientific investigation, credibility has two role. In the first one, it serves to the aim of promoting the probability of the credibility of prospective findings. The second role is for the aim of revealing the credibility of the results through getting them approved. Furthermore, the process and the results of an investigation should be accessible for any researcher and be open to any verification process by any interested person. Therefore, a researcher should present necessary proofs showing that the conducted research is credible. In the light of these requirements, Lincoln and Guba (1985) state seven techniques to establish credibility of a qualitative study. However, the researcher utilized five of these techniques in this study to have a credible research. These are "*prolonged engagement, persistent observation, triangulation, peer debriefing and member checking*". They are presented in detail as follows.

3.8.1.1 Prolonged Engagement

The researcher should interact with the source of the data (*the participants, the products obtained from the participants*) with a considerable period. Having enough time with the data is called as "*prolonged engagement*". Lincoln and Guba (1985) state that there are two purposes of prolonged engagement. The first purpose is to understand the research environment as better as possible in order not to distort the data. The second goal is to establish trust with the participants. In order to achieve these goals, the researcher also acted as the teacher in the study. Therefore, he could interact effectively and sufficiently with the participants because the study lasted 7 months totally. In this period, the researcher met with the participants 45 times. Furthermore, the researcher preferred to transcribe audio and video recordings of the teaching session by himself instead of hiring anyone to transcribe the data in order to have a comprehensive knowledge about the source of the data. Moreover, the researcher had teaching experiences in geometry and mathematics for nine years in middle and secondary school levels and six years in teaching geometry courses as undergraduate level. Therefore, he did not have any difficulties during the teaching sessions not only in teaching the subject matters but also in establishing a healthy, friendly and cozy relationship with the students. As an example, before the main teaching sessions implemented, the researcher arranged problem solving and revision sessions for their mathematics and geometry courses. Furthermore, a picnic was arranged with the participants. Preparing a teaching-learning medium in which the participants feel themselves comfortable and confident during their participations, and they act as they are is important in terms of qualitative research (Bogdan and Biklen, 2007). As a result, the researcher could have many opportunities to know the participants and to establish the trust of the participants.

Yıldırım and Şimşek (2006) emphasized that the interviewees feel themselves in confident if the duration of the interviews are arranged rather longer in time. Moreover, conducting the interviews at different times will result in more credible research analyses. In addition to allocating rather longer time for the implementation of the current study, the researcher conducted the interviews at several points of the study.

3.8.1.2 Persistent Observation

Persistent observation is the second technique to set up the credibility of a research study. Lincoln and Guba (1985) identify two goals for "*persistent observation*". In the first goal, the researcher tries to determine components and properties, which are most related to the current problem in the study. The second aim is to probe these component and properties in detail. Therefore, while the prolonged engagement technique presents an opportunity to specify the frame, the persistent observation supplies deepness.

Especially after completing the first three subjects: quadrilaterals, trapezoid and parallelogram in the current study, the researcher continuously compared and interpreted students' written products so that he could obtain a possible pattern in their solutions. In these comparisons, a student was compared within himself on other topics and with respect to the other participants. At the end of these comparisons and interpretations, students' works could be classified and a pattern could be figured.

3.8.1.3 Triangulation

Researchers can utilize the triangulation method to enhance the credibility of a qualitative study. In the triangulation strategy, a researcher make use of several data collection methods and utilize different data sources. Therefore, a researcher need to utilize multiple source of data while making an inference related to the study.

In this teaching experiment, the data sources are pretests, video and audio recordings of teaching episodes, one-to-one (pre, in-term and post) interviews, artifacts: the participants' written works emerged during the course of teaching sessions, home works, the field notes taken by the researcher and the observers during and after teaching sessions and post-tests. The researchers triangulate the finding from these rich data sources to make reasonable and plausible inferences. In compatible with the nature of a teaching experiment (Steffe and Thompson, 2000), data from various instants of the study were taken into consideration.

3.8.1.4 Peer Debriefing

The purposes of "*peer debriefing*" or "*peer examination*" are to learn whether emergent hypotheses or inferences are reasonable or not, to become aware of researcher's situation toward data and analysis and to overcome his or her bias (Lincoln & Guba, 1985, p. 308). This is a way of considering reached inferences from an external perspective. Peer examination provides an opportunity to the researcher for catharsis. Therefore, the observers who were doctoral students in the same institute with the researcher, helped to the researcher to discuss findings, classifications and inferences until reaching a convention. Since they observed the teaching sessions, they had knowledge about the purpose of this study. They examined some of the products of participants and the researcher's inferences. After that, they gave their feedbacks on the issues where the researcher had troubles or incorrect. By means of this, the researcher had a chance to review and defend himself by his colleagues. In addition, the thesis advisory committee was regularly informed about the methodology, data collection tools and data analysis procedures regularly within every 6 months. They gave crucial feedbacks to the researcher.

3.8.1.5 Member Checking

Another strategy to establish a credible research is "*member checking*". In this strategy, the researcher discussed the findings and inferences with the participants who are the main source of the data (Lincoln & Guba, 1985, p. 308). Member checking is also recommended as beneficial to verify the results that the researcher obtains in accordance with the data on participants' works (Yıldırım & Şimşek, 2006). It is important to diminish researcher's misinterpretations due to his subjective conjectures. Furthermore, Steffe and Thompson, (2000) underline the fact that a researcher's imputation might be constrained by participants' speech and actions (what they say and do) while interpreting their mathematical understandings. Therefore, the researcher met with the participants to share his inferences and interpretations about their written products. It was important to learn students' intentions clearly. This activity was implemented with the participants individually so that non-existence of any interactions. Member checking was conducted for students' written and oral products. As an example, the researcher made the participants re-label the name of the approach that they utilized and re-express the underlying reasons of their preferences. At the end of these reviews, there were beneficial feedbacks from the participants and the researcher had a chance to better understand, revise and re-interpret students' reasoning and ways of thinking

3.8.2 Transferability

Despite the fact that generalizability of the results or external validity is possible with some error in quantitative research designs, it is not possible for qualitative research studies (Yıldırım & Şimşek, 2006). In qualitative research, in terms of generalizability, the concept of transferability is used. Lincoln and Guba (1985) define “*transferability*” as the way of showing that the findings of a study can be also applicable in other but similar contexts. The authors specified only one technique “*thick description*” to establish transferability. In this technique, the researcher gives necessary and detailed information such as time, place, situations, and the participants in his study. By this way, an interested researcher can compare these results with his own. The researcher gave detailed information about the duration, the process, the place and the participants of the study. Moreover, the researcher also presented direct quotations from the interviews and students’ solutions on open-ended questions.

In addition to thick description, Erlandson, Harris, Skipper and Allen (1993) specify, “*utilizing purposive sampling*” in qualitative studies as another way of establishing transferability of the results. In this study, “*critical case sampling*”; one of the purposive sampling methods was preferred. It was aimed to work on critical cases. Specific to this study, the participants were selected from relatively higher achieving level in order to argue that “while teaching quadrilaterals, if utilizing vector approach does not work for this group, it won’t work in other similar and less successful groups. Alternatively, if participants of this study are having troubles with this treatment, then we can infer that all of the groups are going to have similar troubles most probably. Therefore, it is not intended to make generalizations in this study. However, it can be possible to establish transferability for this study. In other words, the results and inferences of this study can be transferred to other contexts by considering the settings of this study. To sum up, an interested researcher can learn about the information on whole settings of this study and sampling strategy in that he can determine to what extent the findings of the current study can be transferred to his or her investigation.

3.8.3 Dependability and Confirmability

“*Reliability*” in quantitative research design corresponds “*dependability*” in qualitative research studies. Cohen, Manion and Morrison (2000) accept reliability as the consistency and replicability over time. Lincoln and Guba (1985) count dependability as one of the criteria of establishing trustworthiness of a study by demonstrating that the findings of that study are consistent and can be repeated. In order to provide dependability for this research, the researcher, had observations and students' written and oral products at various times throughout the entire teaching experiment. He looked for if the same or similar observations and conclusions were the case or not. In other words, the researcher tried to be sure about whether his inferences were accidental or not. Emergence of a conclusion is not accepted as accidental if it is observed at the different times (at least twice) of a teaching experiment (Steffe and Thompson, 2000). In order to enhance the degree of dependability, the researcher recorded all type of data sources, procedures and details. The ways of reaching a conclusion were shared with the thesis advisory committee regularly and their feedbacks and interpretations were taken into consideration to have coherent and consistent conclusions.

The last criteria in establishing trustworthiness of a research study is confirmability, which corresponds objectivity in quantitative designs. Lincoln and Guba (1985) express that the findings and hence the interpretations and inferences of the study are based on the data sources and not based on researcher's bias, motivation or interest. In order to get rid of these threats, the researcher drew conclusions by utilizing triangulation of the data. This is explained in detail under triangulation title. Moreover, the inferences were verified through students' solutions and interviews frequently. Finally, the students' pre-test and post-test scores were evaluated according to a rubric which is prepared before the administrations of the tests. Students' solutions were scored without looking at their names. Two observers of the teaching sessions also evaluated students' works on these tests. Their evaluations were correlated with the researchers' evaluation. In spite of higher correlations among these assessments, the researcher evaluated students' solutions once again and then a little bit higher correlations were obtained.

3.9 Ethical Issues

It is an indispensable requirement for an educational research that certain ethical rules have to be to take into considerations by the implementers of the researcher. These rules are not to cause any harm to the participants, to guarantee the confidentiality of the data, and not to deceive the participants at all (Fraenkel & Wallen, 2006). In this dissertation, the researcher tried to consider about these ethical issues from beginning to the end of the study.

After determining the students who would participate in the study voluntarily, the subjects of the study, their families and the teachers were informed on entire procedures to be followed for the study before the research study started. Moreover, they were informed that data collected from or about the participants to be saved in confidence. They also were stated that their real names never be used in any publications. They were guaranteed that instead of using real names, fictitious names were preferred under necessary situations. At any point of the study, withdrawing right from the study was expressed to the students. In fact, two of the participant students withdrew from the research study despite the fact that they took all of the pre-tests and they attended most of the teaching episodes. However, at the beginning of the study and during the course of the study, the teacher-researcher of the study continuously stressed the aim of the study, the importance of results and possible negative effects of absenteeism for the data collection.

Besides, a meeting with the students' parents, mathematics and geometry teachers, students' were arranged in order to give necessary information about the purpose, the process, the place where the sessions to be held, the duration and requirements-principles of the study. The students and parents were also informed about the aim of the interviews to be conducted throughout the study. They informed that there would not be any annoying questions and there would not be any questions on their private lives. Finally, the parents approved their children to participate in the current study by filling and signing "*Parent Consent Forms*" before the study started. These forms were saved by the researcher. Further, the parents expressed their appreciates and gratitude to the researcher repeatedly not only in the course of the study but also after the completion of the study. Lastly, the researcher tried to be fair both in data collection and in data analysis phases.

3.10 Assumptions and Limitations

Assumptions

The researcher as a teacher of the sessions applied and followed the lesson plans and materials as much as possible. Moreover, the teacher-researcher administered the instruments of the study under standard conditions for each of teaching episodes. Further, the participants answered the items of the instruments honestly and they stated their opinions cordially. As much as possible, the researcher tried to conduct interviews under standard conditions with all of the participant students. Participant students replied interview questions cordially and honestly.

Limitations

The number of the participant students was three for this study. This can be considered as one of the limitations for that study. However, the data collected in this study is large enough. There are both qualitative and quantitative data in the study. As known, the qualitative data makes it possible to probe and understand students' ways of operations and logical thinking. However, quantitative data is a possible way to have an idea about the participants' performances. This set of data help and complete each other to reach some conclusions. Therefore, the number of the participants for this study should not be evaluated as a limitation.

In the course of the interviews, the participants had conversations with the researcher who was a new identity in terms of them. Hence, their replies to the questions could be possibly affected. They could hinder their exact ideas about the instruction and the process. As a precaution to this situation, the researcher preferred to arrange a longitudinal study and to conduct the interviews at various points of the study. In addition, what the participants said also was triangulated with their written products.

There were 5 months and 15 days between the administration dates of pre-tests and post-tests. The variation is 3 months and 10 days for the administration of Quadrilaterals Achievement Test. These variations are accepted as enough for qualitative research so that the participants do not remember the items administered more than once (Fraenkel & Wallen, 2006).

The researcher's bias can be thought as another limitation for this study, as it is the case for the other quantitative and qualitative studies. The data was collected and analyzed by the researcher. This is actually appropriate with the nature of the teaching experiment methodology. In fact, the main aim of conducting a teaching experiment methodology is to experience, "at firsthand", the participants' mathematical reflections. Therefore, while trying to figure out what is going on in the classroom as a researcher, he also needs to be a teacher to reach the data at the firsthand. In spite of this fact, the researcher tried to minimize researcher's bias by means of several precautions. Firstly, it was stated that the lessons were witnessed by two of graduate and experienced mathematics teachers alternately, as the observers. In fact, presence of the observers was the indispensable component in a teaching experiment methodology. Secondly, there were several types of the data such as teaching episodes, students' written works, interviews, the products of the participants on the instruments, which were administered twice. Moreover, retrospective analysis of the data is another precaution to overcome the researcher's bias. Finally, it was stated by the researcher repeatedly throughout the instructional periods that, the researcher has not any idea asserting superiority or priority of any approaches in geometry teaching, which was the main focus of the study.

Two other limitations can be related to the gender and success level of the participants. It was stated that the participants were needed to be chosen among male students. The reasons for this selection was presented in chapter 3 under the title "Participants". However, it might be more appropriate including female students to the research as well. In addition, students were selected from relatively successful students. However, the selection of these students were not completely at random. The reasons for this preference was also explained at the "*Participants*" part in detail.

CHAPTER 4

FINDINGS AND DISCUSSION

Findings from the analysis of the data collected from various data gathering tools will be presented in this chapter. The data gathering tools are presented in the preceding chapter. The findings of this study were reported under four titles. The titles are related to the research questions respectively. These titles are; contributions of analytic, synthetic and vector approach instruction, the way of participants that they decide the approach to be utilized, major component of the designed instruction and participants' reflections on the current instruction. In addition, discussion of the findings will be presented together with each of the findings of the study.

4.1 What are the contributions of the instruction in which vectorial approach is integrated with synthetic and analytic approaches on quadrilaterals to eleventh grade students' problem solving strategies?

The contributions to problem solving strategies as a result of the instruction followed in this study are presented under the next sub-titles. Students' representations of geometric objects via vectors, being able to integrate multiple approaches and making journeys among approaches will be given in detail. Students' ways of constructing bridge between algebra and geometry, utilizing analytic approach instead of algebra of vector are presented with students solutions in this section. Participants' endeavors to develop new proofs are other contributions of the present study. After presenting these contributions, statistical analysis of students' scores on PKQT, VKT, PPGT and QAT will be given at the end of the section.

4.1.1 Using Vector Representations

After a certain stage in the study, the participants of the study were observed that they used vector notations in their solutions. Specifically, they started to represent sides and diagonals of the quadrilaterals by means of vectors when coordinates of the vertices of quadrilaterals are given. This is also the case when the quadrilaterals are directly given on Cartesian plane.

The students solved certain kind of problems by sketching the given the quadrilateral on coordinate plane or without placing it on the plane when the coordinates of the vertices are provided. In both of the cases, they were observed that they represented sides and/or diagonals of the quadrilaterals by means of vectors. Moreover, the participants also represented the required line segments on quadrilaterals or triangles by vectors.

Vector representations were utilized in geometry problems including the length of sides, the slope of lines on which the sides of polygons lie, translation of geometric objects and determining relative positions of sides according to the other sides. Vector representations were also utilized to find out unknown coordinates of a vertex of the given quadrilaterals especially for the family of parallelogram. Determining whether the diagonals are perpendicularly intersected is another topic for the use of these representations. In essence, the participants were observed that they used vector representations in problems for which they preferred vector approach to solve them. However, there are some exceptions. Specifically, the students used vector notations despite the fact that they did not continue solving the problem through vectors. In the following subtitles, vector representations encountered in the students' solutions are given in detail.

Utilizing Vector Representations for Geometric Objects Given Analytically

There are entering assignments at the beginning of each quadrilaterals. These assignments are defined in the methodology chapter in detail. In the tasks included in these assignments, there are quadrilaterals given with coordinates of vertices or they are directly given on Cartesian plane. For example in the following problem, the students were required to "*compute the length of the sides and the length of the diagonals of a quadrilateral whose coordinates of vertices are given as in specified*" in the problem.

It is seen in the Figure 4-1 that Ahmet firstly started to represent the sides and diagonals of the quadrilateral vectorially. After that, he calculated the magnitude of the vectors that he wrote.

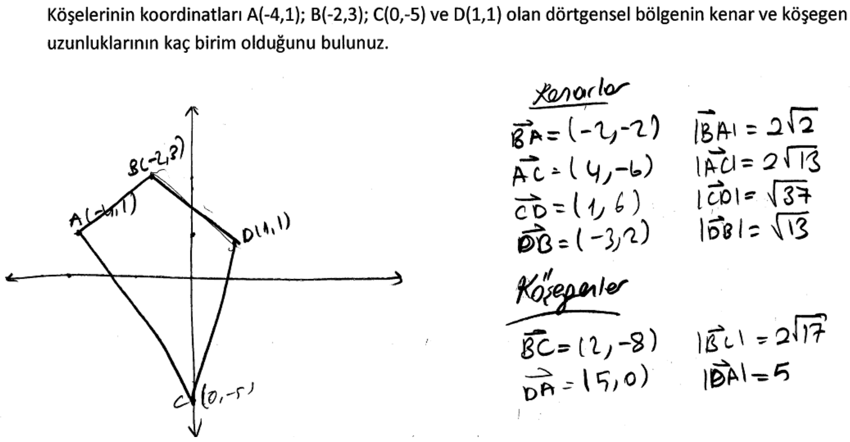


Figure 4-1 Ahmet's solution to the problem 5 on Quadrilaterals 2nd section

In addition to this, it was frequently observed that the participants utilized vector representations on their solutions:

a) to find the slope of a line passing through two points,
b) to find the distance between two points on plane,
c) to determine relative position (*parallel, perpendicular, intersecting or coincident*) of two lines carrying the sides and the diagonals of the quadrilaterals on plane and
d) to calculate the area of polygons on plane.

As an illustration, the students were required to "*determine relative positions of sides and diagonals where coordinates of the vertices are provided",* in the following problem. Firstly, Naci represented sides and diagonals through vectors in the Figure 4-2. Then, he determined parallelism and perpendicularity of required line segments.

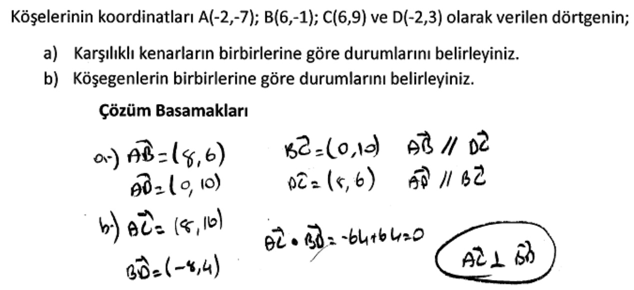


Figure 4-2 Naci’s solution to the entrance assignment for Rhombus 1st section

To compute the area of quadrilaterals whose coordinates of vertices are given, the following solution illustrates how participants symbolized the sides via vectors. This is work of Ömer (Figure 4-3) in QAT pre-test for the problem C3. The problem necessitates “*calculating the area of a quadrilateral region whose coordinates are given*”. As seen in his solution, he symbolized two of the sides of given quadrilateral by vectors.

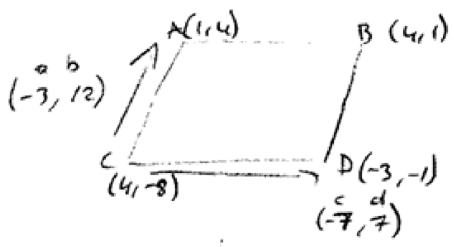


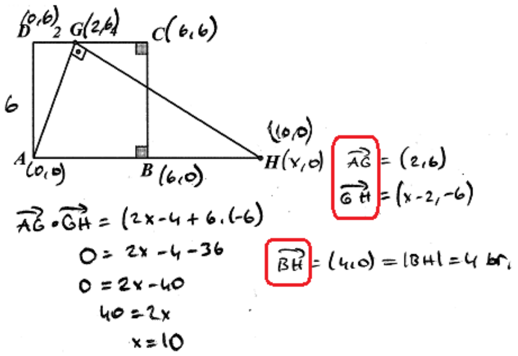
Figure 4-3 Ömer’s solution to the problem B3 on QAT pre-test

Utilizing Vector Representations for Geometric Objects not given Analytically

Besides representing geometrical concepts via vectors for the shapes given analytically, the students made use of vector representations, which were not given analytically. In geometric problems related to quadrilaterals that have parallel sides or perpendicular elements but not given on the coordinate plane, students themselves developed a solution way.

In this newly developed solution method, the students set one of the vertices as the origin O (0,0). After decision process of assigning one of the vertices as the origin of Cartesian plane, the students determined coordinates of the other vertices according to the length of the sides or diagonals. Then using these dependent coordinates, the sides of the given quadrilateral were represented by vectors.

In the following problem, "*ABCD square is given with the lengths of the line segments* $|DG| = 2\,\text{cm}, |CG| = 4\,\text{cm}$. *The length of* $[BH]$ *is to be computed*". The Figure 4-4 exemplifies the use of vector representation of the sides $[AG]$, $[BH]$ and $[GH]$ in the solution of the problem with framed objects.



ABCD bir kare olmak üzere şekilde verilen bilgilere göre BH uzunluğunu bulunuz.

Figure 4-4 Ömer's solution to an exercise on Square 1st section

Robinson (2011) states that any point on the plane or in the space can be accepted as the origin of coordinate system. Ayre (1965) named the way of "*assigning any point in the space as an origin*" as "*Origin Principle*" in his book (p.84). He states that usefulness of this principle will be obvious if the vectors are utilized in the solutions. Coxford (1991) specifies that after the selection of an important point on the figures or one of the vertices of the figures as the origin of given object; the other vertices or points would be represented by ordered pairs in coordinate approach solutions. Lastly, Craine (1985) notes that while solving geometry problems via analytic approach, one of the vertices of the quadrilateral is specified as the origin and one of the sides of the quadrilateral is placed on positive x-axis. These informations are very similar to the cases observed so many times in this study. This is a spontaneous improvement for the participants. Besides, utilizing origin or analytic approach in problem solving can be thought as a result of geometry teaching through vector approach.

Utilizing Vector Representation for "Translation"

It was seen that Ahmet represented translations of the vertices of a quadrilateral that were given on coordinate plane by vectorial symbols as seen in the Figure 4-5. In the problem, "*A, B and C vertices are translated 6 units to the right and 4 units to the down. At the end, the area of* $A'B'C'D'$" is asked. Ahmet expressed the combination of the movements of the object in the direction of x and y-axes as the translation vector $\vec{u}$ as shown in the framed expression.

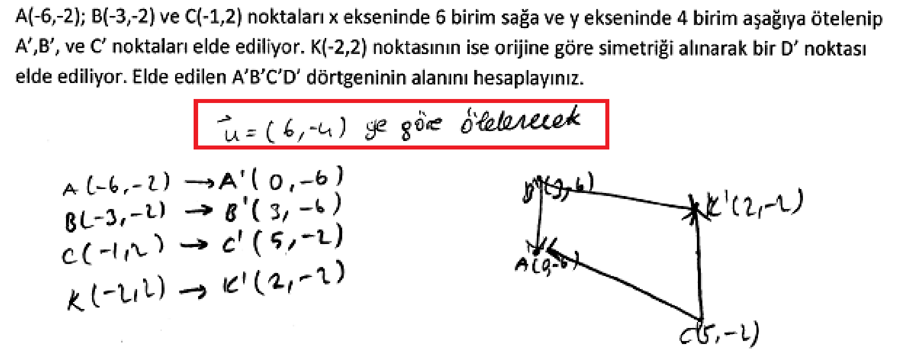


Figure 4-5 Ahmet's solution to an exercise on Parallelogram 2nd section

Ahmet also represented the movement of given quadrilateral as the translation vector for another two problems in A156 and A166. However, the translation of the quadrilaterals were not represented by vectors in some cases. The students made these translations without demonstrating or conducting any operations. They were seen that they could directly write the final position of the points without conducting any operations. N147 and N157 are of this kind. This could be a modus operandi, which can change from person to person. Ömer preferred to demonstrate these translations or movements on the Cartesian plane. Since this subtitle is related to the use of vectors for the translation, only Ahmet's solutions are presented here.

Faydacı and Zembat (2012) found that while constructing the meaning of translation, the understanding of vector is necessary. Moreover, translation is reported as beneficial and effective to teach vectors so that the students can embody or conceptualize vectors in several studies (e.g., Szabo, 1966; Athen, 1966a, Vaughan & Szabo, 1973; Coxford, 1991 & 1993; Poynter & Tall, 2005a & 2005b; Nguyen & Meltzer, 2003; Rosenbloom, 1969; Stephenson, 1972; Faydacı & Zembat, 2012). Hence, students' use vector representations for the movements of figures can be accepted as a sign for conceptual understanding of vectors for this study.

So far, it was understood that the participants made use of vector notations for several geometrical objects while solving geometry problems by means of vector approach. This was stated with illustrations before. However, they were observed that they made use of vector representations for the solutions that they did not prefer to continue with vector approach. The following solution illustrates this situation (in Figure 4-6). "*A quadrilateral with coordinates of vertices as* $\mathrm{A}(-4,-6)$*;* $\mathrm{B}(4,-2)$*;* $\mathrm{C}(1,4)$ *and* $\mathrm{D}(-7,0)$ *is given and the properties of the quadrilateral is asked to be reported*". For this introductory problem, in spite of the fact that Ahmet represented the sides of the polygon as vectors, he did not continue utilizing vector approach in his solution. Instead, he calculated the slope of the lines from analytic representation of vectors. This is also the case for Ömer's solutions.

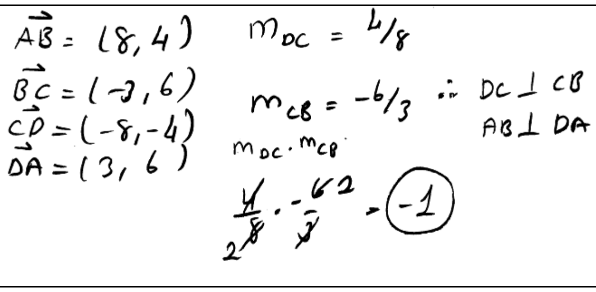


Figure 4-6 Ahmet's solution to the entrance assignment for Rectangle 1st section

4.1.2 Integration of the Approaches

While solving geometry problems, integration of the approaches i.e. the use of analytic, synthetic and vector approaches with variety of combinations can be accepted as one of the most important and valuable contributions for the sake of students in this teaching experiment. These combinations are presented in the following subtitles with students' solutions.

Case 1: Analytic+Synthetic+Vectorial

In the following illustrations, the students made use of analytic, synthetic and vector approaches together in a single task's solution.

Example 1

It was observed that students simultaneously utilized both synthetic approach and vector approach in the same geometry problem. Additionally, the students used analytical representation of vectors (i.e. position vector of a point) while utilizing vector approach in solving problems. Thus, three of the approaches were integrated only for single geometry problem as understood from the students' solutions.

This situation can be found in Naci's solution to the following problem. "*The area of quadrilateral OABC is required to be computed with the given information on the figure"* for this problem (Figure 4-7). The quadrilateral is given on Cartesian plane with some unknown coordinates of vertices. It is seen that synthetic, analytic and vector approaches were utilized to support and complete each other in the solution.

In the first step, he drew an "*auxiliary line segment*" in order to have two right triangles. After that, he applied Pythagorean Theorem on these triangles. These two steps are the strategies, which are generally attributed to synthetic approach. While applying this theorem as a synthetic approach, he represented sides of the triangles by vectors in the equation that he obtained. In these representations, despite the fact that the algebra of vectors is also available, the student preferred to show the vectors by analytical coordinates. He made use of magnitude of a vector concept to express the length of the sides. Finally, he obtained an equation with an unknown and he could solve it without having any mistakes.

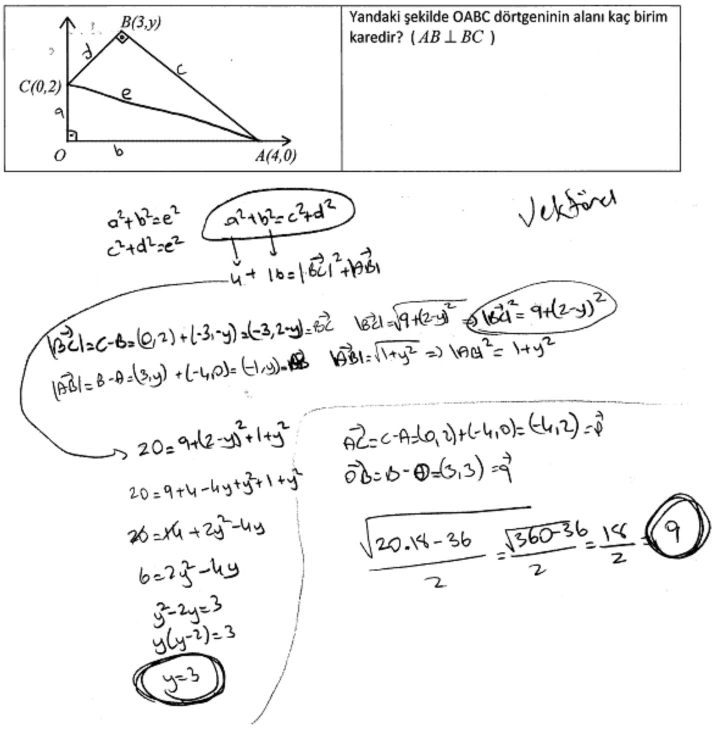


Figure 4-7 Naci's solution to the problem 6 on Quadrilaterals 2nd section

Naci attributed a "complementary role" to the multiple approaches while solving geometry problems in Excerpt 4-1. Moreover, Naci stated that he could make the transitions among approaches easily and successfully in case of not being able to passing further steps in his solution as understood from Excerpt 4-2.

Excerpt 4-1 Excerpt from an interview with Naci on 05.07.2013

Researcher: What do you think about solving geometry problems through multiple approaches? Is it unnecessary?

Naci: No, these are pretty nice. We are going to need each of the approaches, because they are going to "complete each other". Some of them will come to our minds or some of them will not; however, I think that these approaches will complete each other.

Example 2

Ömer solved the following problem in the Figure 4-8 by integrating three of the approaches in a single problem. An ABCD isosceles trapezoid with perpendicularly intersecting diagonals is given in the problem. The area of the trapezoid is asked for which the length of the bases are specified as 3 and 7 units. In his solution, the participant envisioned the trapezoid as if it was given on a coordinate plane by assigning "vertex A" as the origin of that system. Then the diagonals were represented by vectors. Inner product of the diagonal vectors gave the value of "h" which is specified as the length of height of the trapezoid. Finally, the area of the quadrilateral was computed by synthetic area formula, which was proved during the teaching experiment. Although, he could use vectorial approach formula giving the area of quadrilaterals, he preferred to utilize synthetic approach formula.

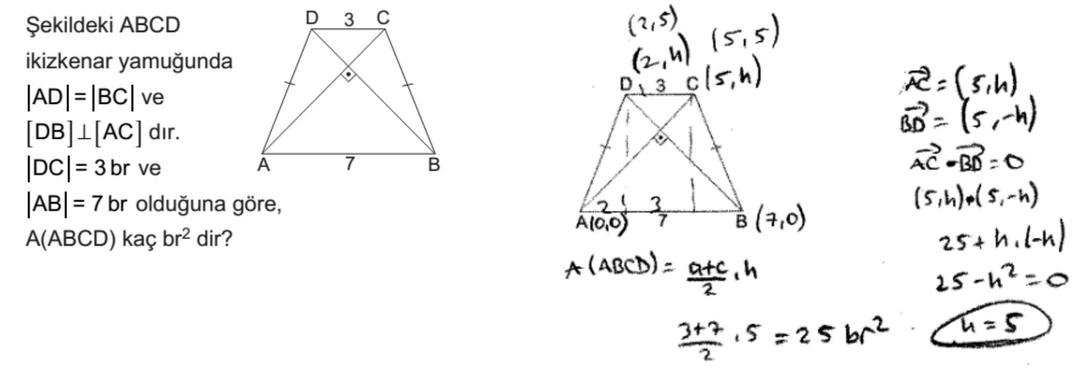


Figure 4-8 Ömer's solution to the problem B11 on QAT post-test

Case 2: Analytic+Vectorial

In this case, vector approach is integrated with analytic approach. The way how the participants integrate analytic and vector approaches are exemplified in the following solutions.

Example 1

In the problem (Figure 4-9) "*A quadrilateral with the coordinates of vertices as* $A(-4,-6)$; $B(4,-2)$; $C(1,4)$ *and* $D(-7,0)$ *is given and the properties of the quadrilateral is asked to be reported*". After determining sides of the quadrilateral vectorially, Ahmet preferred to utilize the definition of slope of a line passing through origin and a certain point. Then, the slopes of the sides were compared. The sides with equal slopes were classified as "*parallel*" and the sides were classified as "*perpendicular*" when the product of their slopes equals to "-1".

The students described the sides carried by equal vectors as "*parallel*". After denoting sides of the quadrilateral by vectors in the form of $\vec{u}=(x,y)$, the relative position of the sides that are possibly perpendicular or parallel to each other were determined by the use of the relation "$\text{slope}=\frac{y}{x}$". For this problem, while the student used vectorial symbols to represent sides and diagonals of the given quadrilateral, he conducted the operations analytically, as seen in the Figure 4-9. Giles (1964) calls $\frac{y}{x}$ as "*the gradient of the vector* $\vec{u}=(x,y)$" and denoted it with $\text{grad}\,\vec{u}$. The same way of solution is seen in the other participant students' solutions (e.g., *Ö 161*) as well.

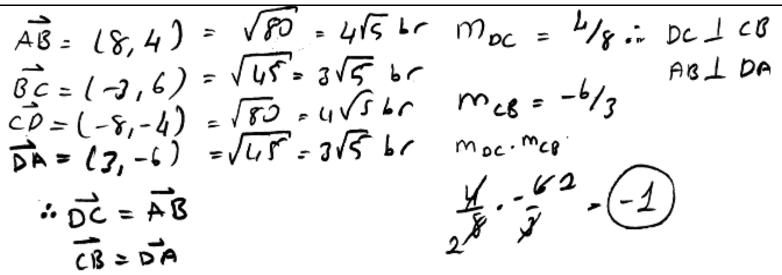


Figure 4-9 Ahmet's solution to the entrance assignment for Rectangle 1st section

Example 2

The following problem in the Figure 4-10 was also solved by integration of analytic and vector approaches. In the problem, "*An ABCD rectangle with* $|AB| = 2|AD|$ *and* $|DP| = \frac{3}{4}|DC|$ *is given where P is a point on* $[DC]$ *. It is asked to show whether* $[\mathrm{AC}]$ *is perpendicular to* $[\mathrm{BP}]$ *or not*". Ömer located the given rectangle in coordinate plane although it was not specified with coordinates of vertices. In doing this, an appropriate point (D) in the shape was assigned as the origin of the analytic system. The coordinates of the other points (A, B, C and P) were determined according to the ratio of other line segments respectively. After that, line segments were symbolized as vectors. Then, analytical representation of vectors were used to solve this problem. Ömer utilized inner product of vectors, which are carried by possibly perpendicularly intersected line segments in this rectangle. He applied properties of inner product and finally he could have the correct result in the light of operations conducted for this problem.

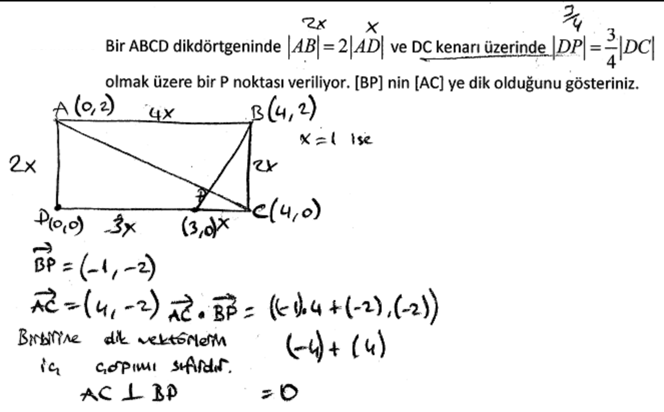
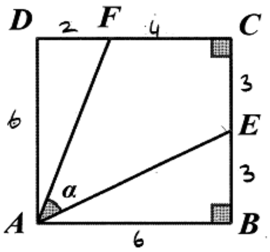


Figure 4-10 Ömer's solution to the problem 4 on Rectangle 1st section

Example 3-4

This trend; that is, the integration of analytic and vector approaches was also observed in the students' solutions for the problems that necessitated "*trigonometry knowledge in rectangles*" (Figure 4-11) and for the problems that are specifically related to "*rectangles and squares*" (Figure 4-12). "*An ABCD square with* $|FC| = 2|DF|$ *and E as the midpoint is given*" and the value of the angle α *and* $\cos\alpha$ are asked in the problem (Figure 4-11). "*An AOCD rectangle with perpendicularly intersecting line segments* $[DO]$ *and* $[AE]$ *is given. The coordinates of the vertex D are specified as* $D(6,8)$ *and the length of* $[DE]$ *is asked"* in the problem (Figure 4-12).



Yandaki şekilde ABCD bir kare, E orta nokta ve $|FC| = 2|DF|$ olmak üzere "α *ve* $\cos\alpha$" değerlerini bulunuz.

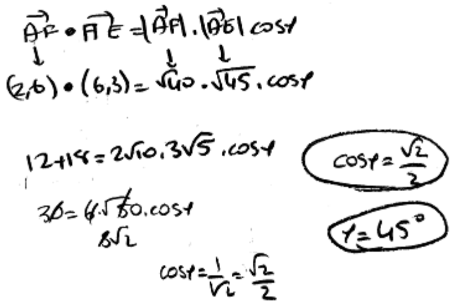


Figure 4-11 Naci's solution to an exercise on Square 1st section

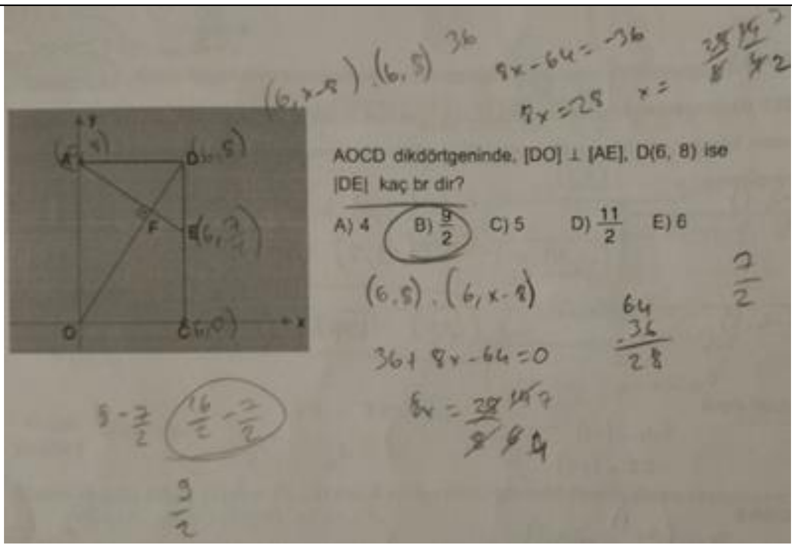


Figure 4-12 Ömer's solution to the problem 7 on Rectangle 2nd section

Example 5

In the problem, "*the geometric relation* $|PA|^2 + |PC|^2 = |PB|^2 + |PD|^2$ *is to be proved for a rectangle where P is interior or exterior point of the rectangle*". In order to justify this relation, it is seen more than once that "analytic and vector approaches" were integrated in the students' solutions for the problem. Firstly, to be able to locate the given geometric figure into analytical coordinate plane, relatively the most appropriate point (*P*) in the shape is determined and then it is set as the origin of that system. Later, line segments whose one of the end points is located at the origin (*P*) were considered as position vectors. In other words, the students made use of line segments by representing them as position vectors with the use of analytic system. After that, the sum of squares of norm of position vectors was compared. Then, Ahmet simply obtained an equation as seen in the Figure 4-13. Consequently, he could prove the geometric relation for rectangles.

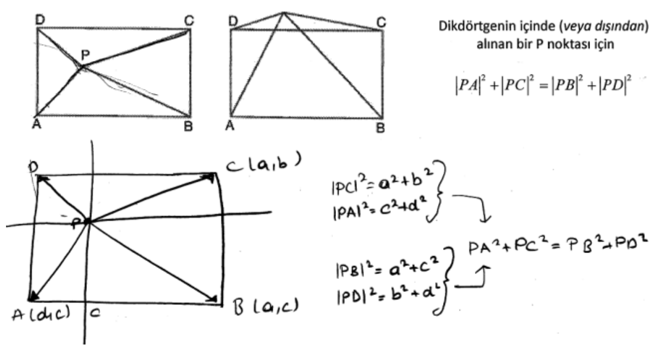


Figure 4-13 Ahmet's proof to the 6th property for Rectangles

Example 6

In the following problem, "*the area of the quadrilateral whose coordinates of the vertices are specified as* $A(1,4); B(4,1); C(4,-8)$ and $D(-3,-1)$ " is asked. The area was calculated through analytic and vector approaches. Therefore, Naci's solution for this problem is another example for the integration of "*analytic and vector approaches*". Besides utilizing vectorial notations and vector concepts, Naci utilized slope of lines on which there are side vectors. Moreover, he wrote equation of lines passing through a certain point and having certain slope. After determining the parallelism of the sides AB and DC, the student found the height of the quadrilateral by calculating the distance of a point to a line. While trying to find this distance, he made vectorial application of the formula, which yields this distance. As can be seen in the solution in the Figure 4-14, he could successfully integrate analytic and vector approaches.

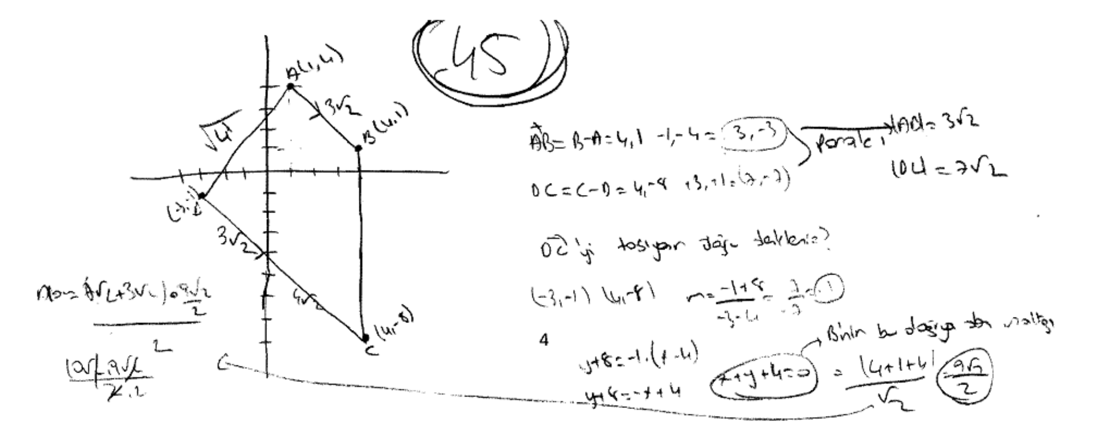


Figure 4-14 Naci's solution to the problem B3 on QAT pre-test

Allendoerfer (1969) determines "an integration of geometric ideas with other parts of mathematics" as one of the goals in high school geometry course. Moreover, Klamkin (1970) and Lines (1965) made use of synthetic approach (*trigonometry knowledge*) and vector approach (*inner product and vector product*) to complete each approach in their teaching. Furthermore, Miller (1999) reminds the complementary strength of analytic geometry and vector geometry in his study. Besides, the complementary role of approaches is was one of the objectives specified in the geometry program of UICSM prepared by Vaughan and Szabo (1973). In their study, the possibility of integrating knowledge from different parts of mathematics was predicted and expected if the students would be treated by multiple approach instruction. This expectation was realized in this study that the participants had some developments in this direction according to the students' solutions and interviews. In other words, as understood from students' solutions, participants were able to integrate their knowledge on algebra, geometry, analytic geometry and trigonometry by utilizing analytic, synthetic and vector approaches.

In the standards of NCTM (1989), this integration is also evaluated as an important way of developing and enhancing students' problem solving abilities. Regecova (2005) states that students have difficulties in integrating vector approach and analytic approach in their problem solving steps because of the fact that the teaching of vector and analytic geometry concepts were realized separately or isolated in the schools. Since these topics were not taught as an isolated manner in this study, the integration of approaches were realized in a considerable degree. This is accepted as important development because Stephenson (1972) finds utilizing various algebraic concepts in geometry as important to show that geometry is not an isolated subject. The use of several approaches in solving geometry problems is explained with the fact that an approach can be a facilitator for the other approach or has a complementary role on the other approach in terms of students (NCTM, 1989).

Moreover, Stephenson (1972) found that geometry instruction by synthetic approach is not enough to relate algebra and geometry. However, the students seems to have realized this relation considerably in the present study. Therefore, in order to realize this integration, it is important to include vector approach into geometry teaching.

4.1.3 Flexible Transitions among Approaches

It is stated in the "*Methodology*" chapter under the title "*Enrichment of Learning Environment*" for this teaching experiment that, a learning environment was prepared and accordingly designed for the students in which they could make flexible transitions among various approaches. It is aimed by this learning environment that the students would produce alternative ways of solutions to a single problem. Students' solutions and ideas realized as findings of the study reflect the results of such a rich teaching-learning medium. In other words, it is understood that participants started to make journeys among the approaches.

The students stated that it was important to reach the correct answer in a way for solving problems till participating in this study. This was accepted as enough by them. However, they started to develop and search for alternative ways of solutions and to solve problems from different perspectives with the help of this teaching experiment. To illustrate; despite the fact that different type of approaches and ways of solutions were not required in the following problem, Ömer preferred to solve this problem with three ways (*i. the law of cosines, ii. sum of interior angles of a triangles and iii. inner product*). He utilized synthetic approach twice and vector approach once.

In this problem, the students were asked to show that "*the diagonals of a parallelogram intersect perpendicularly when the sides of the parallelogram are of the same length*" (Figure 4-15). While proving this statement the student firstly, utilized properties of an isosceles triangle. He used three or four isosceles triangles formed by the sides of the parallelogram and utilized the concept of alternate interior angles. In the second way, he used the law of cosines and properties of trigonometric value of supplementary angles. These are steps in synthetic approach generally. Lastly, he solved the problem by means of vector approach in which he used algebra of vectors (*displacement analogy*) and inner product. In the light of these solutions, the student can be said that he could make transitions between approaches effectively to solve this problem.

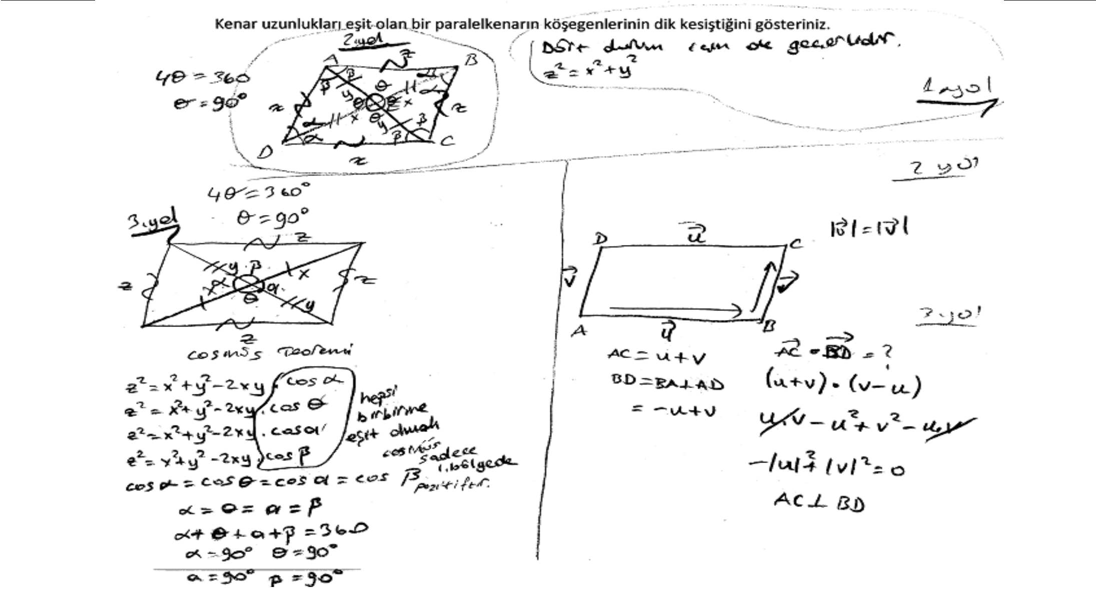


Figure 4-15 Ömer's solution to the problem 7 on Parallelogram 1st section

As the second example for the flexible transitions among the approaches, the next solution can be examined. The participants are asked to find "*the intersection point of the diagonals of a rectangle whose coordinates of the vertices are given*". In order to find out coordinates of the intersection point, Naci determined the equation of lines passing through two points (Figure 4-16). This way is an analytical method. After that, he solved a system of linear equation and finally he could obtain the coordinates of the intersection point.

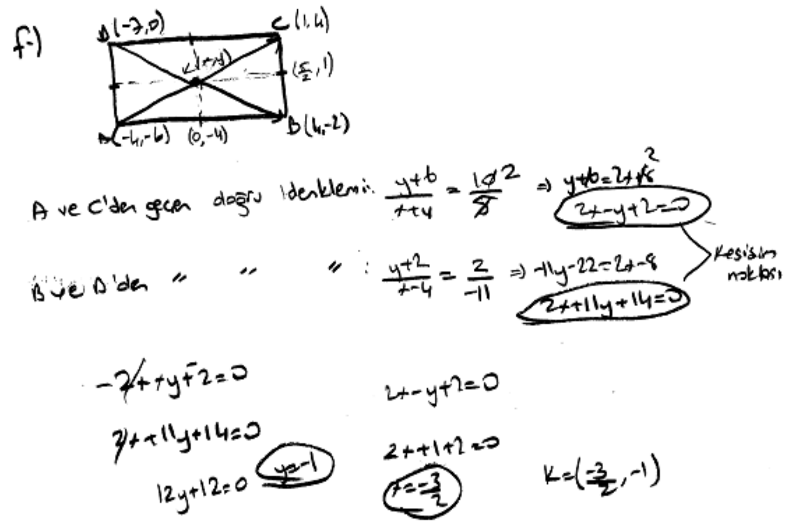


Figure 4-16 Naci’s solution to the entrance assignment for Rectangles 1st section

Secondly, he solved this problem by using vector approach in which he used algebra of vectors and linear dependence of vectors (Figure 4-17). While solving this problem by means of vector algebra, he could write parallel vectors as a scalar multiple of each other. Despite the fact that the second way was novel and more complex way to follow for the students, he could solve it successfully and completely. It should be expressed also that the student did not sketch the given rectangle on the analytic coordinate plane; instead, he drew an imaginary rectangle with its coordinates of vertices. Whereas the first way is very common among students, it can be said obviously that the second way of the solution cannot be frequently encountered among students at high school level.

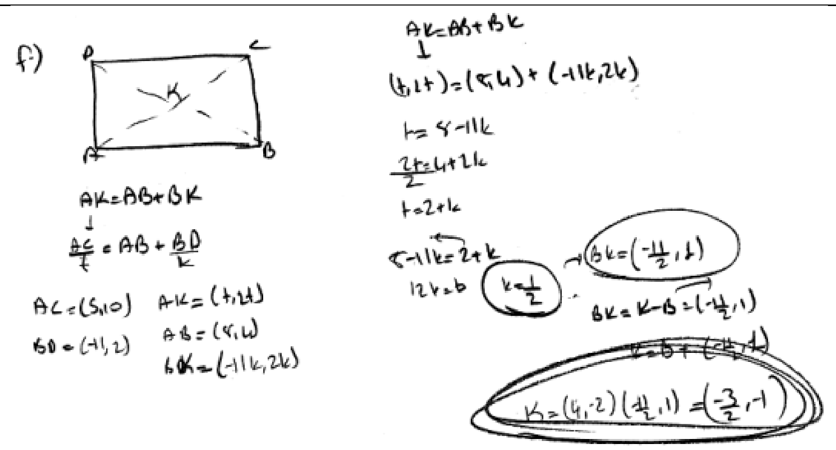


Figure 4-17 Naci's solution to the entrance assignment for Rectangles 1st section

The third example illustrating the flexible transitions among the approaches is as follows. As seen in the Figure 4-18, "$|DF| = x$ *in ABCD rectangle is asked with the given information on the figure*". The student solved the problem in two approaches. He firstly sketched two auxilary line segments and then utilized Pythagorean theorem to solve this problem. This way is attributed to synthetic approach. Among students, this solving method is common and the teachers mostly preferred this method in their classes. However, the student was able to solve the problem in an unusual manner that he used combination of analytic and vector approaches simultaneously in his second solution way. As explained previously; deciding an appropriate point (*F*) to set origin in the rectangle, then assigning this point as origin (*O*) of an imaginary analytic plane and using length of sides to determine coordinates of the points are the steps of analytic approach. Using the determined coordinates to symbolize some of the line segments as vectors and utilizing inner product of these vectors are the steps in vector approach. The student used position vectors of two points (E and G) in coordinate plane and he computed inner product of two vectors that were perpendicular to each other. Finally, he could obtain so simple relation that he could solve the relation in his mind. Students were observed that they were happy and satisfied with this solution. The second way of solution is frequently observed for the other two participants for this type of problems.

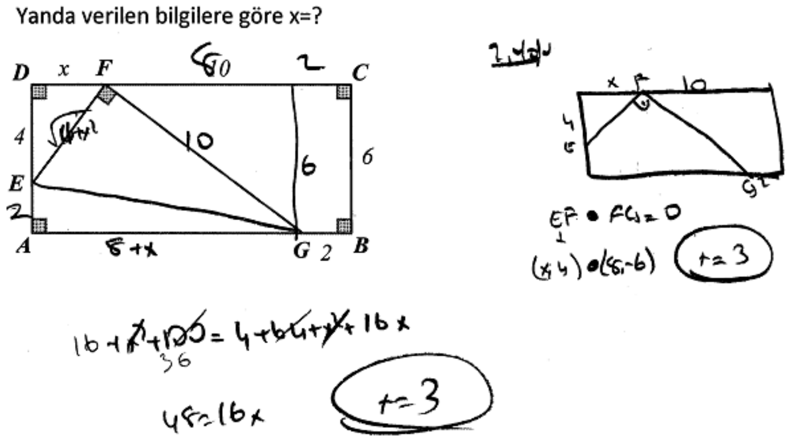


Figure 4-18 Naci's solution to the problem 5 on Rectangles 1st section

Naci stated that he could make the transitions among approaches easily and successfully in case of not being able to passing further steps in his solution as understood from Excerpt 4-2.

Excerpt 4-2 Excerpt from an interview with Naci on 24.07.2015

> *Researcher: I see that you solved this problem (*Task 5- parallelogram 1st chapter*) by means of vector approach as well. Can you explain your rationale?*
>
> *Naci: I can make transitions to the other approaches conveniently if I recognize that I cannot make any progress in my solution. I say myself that this problem can be solved in this way and I am mostly successful. I think that the actions that conduct in these operations support each other. For example, I start with vector approach. The problem is not progressing and hence it is not solvable. Then I try to solve through synthetic approach. My actions in synthetic approach might support the missing solution parts of vector approach.*

Star and Rittle-Johnson (2008) define "*strategy flexibility*" as "*the knowledge of multiple strategies and their relative efficiency*". From their point of view, as a consequence of being exposured multiple strategies in learning geometry, a student improves himself in the direction of having the ability of flexibility in problem solving.

In students' solutions presented above, the students were able to make transitions among various approaches. In fact, each transition means solving a problem at least two methods from two different approaches broadly. In the standards of NCTM (1989), transitions among synthetic, coordinate and transformation geometry are stated as the opportunities that the students need to have. In fact, transitions among approaches, comparisons and integration of approaches are stated as the abilities that the students should gain in these standards. As cited from Levav and Leikin (2012), while solving a problem, the utilization of different methods, strategies or approaches is a frequently recommended issue in the mathematics education literature (e.g., Polya, 1963, 1973, 1981; Vinner, 1989; Schoenfeld, 1983, 1988; Dhombres, 1993; House & Coxford, 1995 and NCTM, 2000). This advice is attributed to the reason that solving a problem through different strategies deepen mathematical knowledge and understanding and develop mathematical creativity. Similarly, according to Dreyfus and Eisenberg (1986), there are two practical profits of seeking two or more solution ways in problem solving processes for the learners. Having a chance of becoming familiar with several methods is the first one. These researchers accept the second profit as more important which yields having a deeper conceptual understanding in terms of the students. Furthermore, Leikin and Levav (2008) found that there was a change in teachers' mathematical knowledge on the topics in which the teachers preferred the utilization of multiple solution tasks in their classes. Therefore, if there exists an improvement in teachers' mathematical knowledge through the utilization of multiple approaches, why is not the case for the students?

Nissen (2000) named the problems as hybrid since they can be solved through several approaches. Furthermore, Nissen (2000) accepts the experience of solving hybrid problems via several approaches as beneficial since it is an opportunity to observe the creativity aspect of mathematics in terms of the students.

Since students make frequent transitions among approaches while studying on geometrical problems, students should have flexibility skill according to Richardson, Reynolds and Schwartz (2012). While learning geometric objects and figures, and making journeys among the properties of geometric objects by means of vectors, the transitions among synthetic, analytic and vector approaches are related to an epistemological shift and also related to process-object encapsulation as an onthological shift (Kwon, 2013 & 2011).

Moreover, these transitions are important because Sfard and Thompson (1994) argue that students' comprehending of mathematical concepts depend on how well and effective they make transitions among different representations. Since each approach has a different representation way, the transitions among various approaches observed in this study can be evaluated as in Sfard and Thompson (1994)' s category. Similarly, Kwon (2013) also states that a mathematical concept can be understood well if it is experienced through multiple representations. Besides, Schuster (1961) expresses that problem solving can be easier if coordinate logic is utilized in addition to vector approach. He underlines pedagogical aspect of this combination in addition to mathematical consideration of using vector and analytic approaches together. Schuster (1961) accepts the degree of comprehending the knowledge by which the students can jump from one approach to another as an indication for mathematical advantage of utilizing the two approaches. Furthermore, he accepted the degree of students' self-confidence as pedagogical advantage during the journeys among multiple approaches.

4.1.4 A Way to Construct a Bridge between "Algebra and Geometry"

Another contribution of utilizing analytic, synthetic and vector approach instruction on quadrilaterals for eleventh grade students' problem solving strategies is that students had a chance to set up a bridge between algebra and geometry. The participant students solved some geometry problems related to quadrilaterals by including vector approach despite these problems do not contain any vector representations or vectorial symbols. Moreover, they mostly used to solve these problems by synthetic approach till participating this study. In other words, they started to solve some of the geometry problems via an algebraic tool.

While solving the problems via vector approach, for example, the students accepted one of the points or vertices in the quadrilaterals or triangles as origin of a Cartesian plane. Therefore, it can be thought that a bridge was constructed between geometry and algebra in that geometrical entities (*line segments or sides*) were represented by algebraic entities (*vectors*). It is important in terms of students to understand that algebra and geometry are not disconnected mathematical fields. Moreover, this bridge provide an opportunity for the participants to be able to integrate analytic, synthetic and vector approaches in a task, to make flexible transitions among these approaches. Moreover, they represented line segments with vectors while assigning a point as origin. The process how students solved set a point as origin and then solved the problem with vector approach with various examples were presented at the following subtitles: "*Using Vector Representations*", "*Integration of Approaches*", "*Flexible Transitions among Approaches*" and "*Utilizing Analytic Approach as an Alternative to Algebra of Vectors*".

According to the report of NCTM (1989), the interaction between algebra and geometry is evaluated as an important way of developing problem solving skills in terms of students. Furthermore, Szabo (1966) asserts that vector is a beautiful and useful bridge between algebra and geometry. Krech (1968), Stephenson (1972) and Vaughan and Szabo (1971) state that teaching or / and learning geometry via a medium enhanced with vector approach make it possible to relate geometry with algebra. Moreover, Robinson (2011) and Hausner (1998) accepted geometry and algebra teaching as deficient if one of them is missing. Specifically, Athen (1966) states that expressing an algebraic object in terms of a geometric object or the reverse is a nice idea. That is, while solving an equation system, the resultant vector is thought as the diagonal of a parallelogram which is generated by two vectors whose coefficients are the unknown in this system. In the light of these facts, participants' use of vectors in their problem solving processes in geometry is of utmost importance to be aware of the relation between algebra and geometry.

4.1.5 Utilizing Analytic Approach as an Alternative to Algebra of Vectors

The students were observed that they developed some alternative ways in solving geometry problems by vector approach that necessitates the use of vector algebra. This might be thought as a natural result of lack of students' experiences with vectors in learning geometry. For instance, instead of expressing a vector in terms of linear combination of the other vectors that is to say processing algebra of vectors or linear dependence, the students preferred to use the way that they developed on their own in the course of this study.

This method includes setting an appropriate point as the origin and then utilizing coordinate plane. Accordingly, the students utilized two approaches: analytic and vector approaches, in a problem. Whole processes explained here is given as an advice in a guideline listed in the study of Ayre (1965) while deciding the approach to be utilized according to the problem case. Ayre (1965) recommended selecting coordinates or vectors for the aim of simplifying the algebra. In order to reflect this situation, the following students' solutions are presented.

In the first of these problems, the area of a parallelogram is asked in the Figure 4-19. A diagonal and a side of the parallelogram are given as vectors in the problem. In spite of the fact that the parallelogram is not given on the coordinate plane and the coordinates of the vertices are not specified, Naci selected an arbitrary vertex as the origin (vertex A) and; hence, he determined the other vertices' coordinates (B, C and D) with respect to given "side and diagonal vectors" accordingly. Instead of utilizing algebra of side - diagonal vectors (*vector addition*), Naci utilized position vectors to solve this problem. In other words, Naci did not prefer to add or subtract the given vectors $\overrightarrow{AC}$ *and* $\overrightarrow{DC}$. He preferred to operate with respect to the subsequently assigned origin. Finally, he was able to find the area of parallelogram by using vector formula that gives the area of parallelogram.

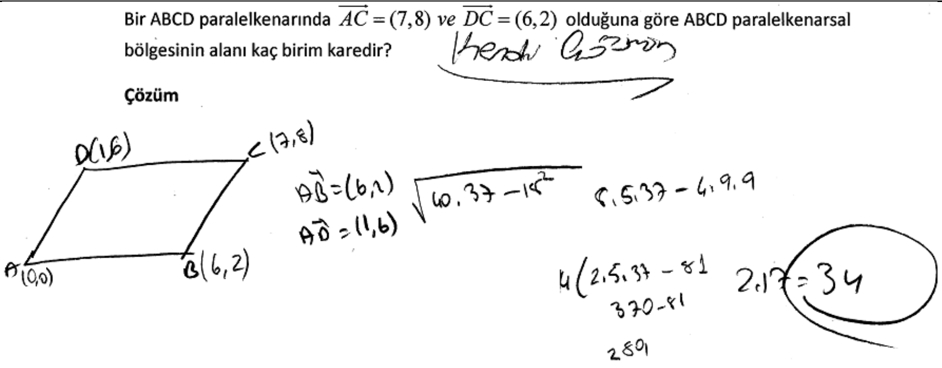


Figure 4-19 Naci's solution to an exercise on Parallelogram 2nd section

The second illustration is Ahmet's solution, which reflects the use of analytic operations of vectors instead of algebra of vector. In the problem (Figure 4-20), "*an ABCD rectangle with* $|AB| = 2|AD|$ *and* $|DP| = \frac{3}{4}|DC|$ *are given where P is a point on* $[DC]$". It is asked to show whether $[AC]$ is perpendicular to $[BP]$ or not in this problem (Figure 4-20). Although, Ahmet did not clearly demonstrate the point that he set as the origin, he benefited from a coordinate plane as seen in the following solution. The student used analytical representation of vectors. Parenthetically, there are also many solutions for which the participants utilized analytic approach by setting a vertex as the origin. However, they did not show or indicate the origin directly.

Ahmet could sketch the rectangle according to given information in the problem and could construct the necessary relation correctly. However, he could not write components of the vectors correctly. Therefore, he could not reach what he wanted to desire. However, the idea for solving this problem is more important here. Most probably, he wanted to get the result of "0" from the inner product of the vectors $\overrightarrow{AC}$ and $\overrightarrow{BP}$ and; hence, he would reach the perpendicularity of these vectors. However, he could not. Therefore, he gave up solving the problem. As in Naci's solution, Ahmet did not write the vectors $\overrightarrow{AC}$ and $\overrightarrow{BP}$ as the combination of the pair of vectors ($\overrightarrow{AB}$ and $\overrightarrow{BC}$) and ($\overrightarrow{BC}$ and $\overrightarrow{CP}$) respectively. Instead of utilizing algebra of vectors, he utilized analytic representations of vectors.

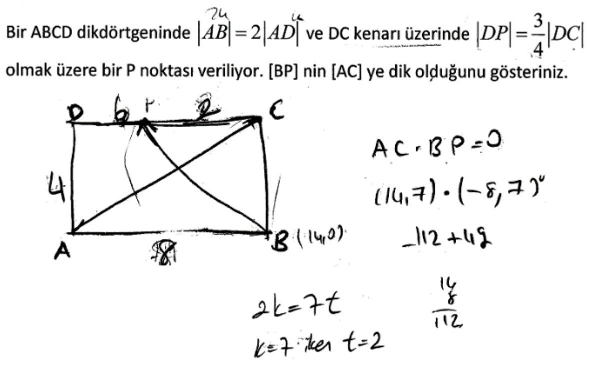


Figure 4-20 Ahmet's solution to the problem 4 on Rectangle 1st section

The same logic was also observed in several solutions like problem 12 in pre-test of QAT by Naci and problem 15 in post-test of QAT by Ömer. Moreover, Example 2 of the Case 1, Example 2, 3, 4 and 5 of the Case 2 under the subtitle "*Integration of Approaches*" exemplify the idea of setting a point as origin and not using vector algebra.

In brief, assigning a proper point as the origin is a way to jump from algebra of vectors to the use of analytical representations and operations of the vectors. According to Schuster (1961), a problem can be solved more easily via vectors if analytic approach is also included in the solution. As Ayre (1965) and Schuster (1961) state, the participants used analytic representation of vectors and analytic operations of vectors instead of vector algebra for the simplicity or because of the fact that the participants are not experienced enough in conducting vector algebra operations while solving geometry problems.

4.1.6 Students' Endeavors of Exploring and Developing New Proofs

After being treated in accordance with analytic, synthetic and vector approach instruction, which was specifically enhanced with proof-based tasks, students were observed that they started to search for geometrical expressions to be proved. Vector approach strategies emerged as frequent tools in completing these proofs. To illustrate, especially Ahmet and Naci repeatedly stated their ambitions to be the first person in proving at least one of the theorems or expressions in geometry by means of vectors. They asked to learn how to achieve this aim to the researcher. They brought their proof endeavors to the class and desired to share with friends and the researcher. Some of participants' proofs were saved by the researcher and presented under the following title "*Elegant Proofs and Solutions Developed by the Participants*". One of the examples illustrating these endeavors is presented in the following paragraph.

Naci stated that he started to question underlying reasons for the formula distance $\left(d_1, d_2\right) = \frac{\left|c_1 - c_2\right|}{\sqrt{a^2 + b^2}}$ giving the distance between two parallel lines $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$. He could develop the proof of this formula in the light of the proof of formula $\frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$, which yields the perpendicular distance of the point $A(x_{0,}y_0)$ to the line $ax + by + c = 0$.

Naci was able to prove this formula through vector concepts. His proof is given in the Figure 4-21. Although this theorem was not included for this teaching experiment since it is not included in formal curriculum of MoNE (2011), it was included as a task to be proved because of the efforts of Naci. After its inclusion, the other participants were given a period to prove this geometric statement. Ahmet also proved this theorem by means of vector approach (A69) on his own successfully.

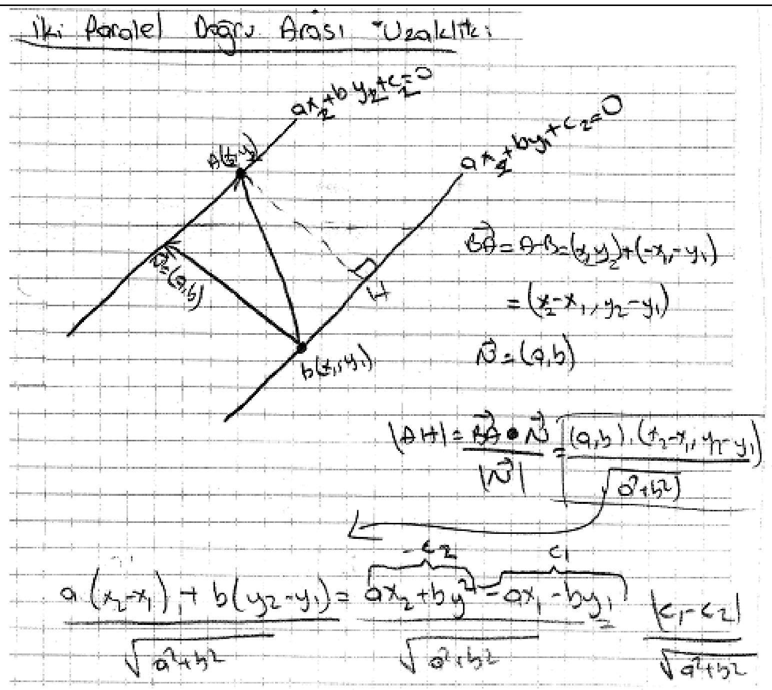


Figure 4-21 Naci’s proof of the formula giving the distance between two parallel lines

It was stated that proving and reasoning was ongoing component of the instruction followed in this study. Participants stated at different periods of the study, in the interviews and during the teaching episodes that they had not learnt geometry through reasoning and proving. In other words, the students repeatedly stated at different instants of the study that they had never learned geometry by utilizing proving and reasoning. However, they were observed that they could prove geometrical statements in the study. Furthermore, they wanted to prove some geometrical statements, which were not included in the study. Although the formula that gives the distance between two parallel lines is not included in the geometry curriculum program for grade level 11 (MoNE, 2010b), Naci’s efforts to prove this formula by himself and completion of the proof successfully are very pleasing developments in terms of the success of the research and the students. At the same time this situation can be accepted as a nice example to the “*discovery function*”, that is one of the functions of proof according to de Villiers (1990). Moreover, Naci’s thinking to prove of this formula as a problem solving assignment and sharing it with us in the classroom were a source of motivation for all of us. This situation resembles an example to the communication function of the proof by which transferring of mathematical knowledge realized according to the same study of de Villiers (1990). According to de Villiers (1990), proof is accepted as a unique way to interchange mathematical results among teachers and students. It could be stated that including multiple approaches and reasoning and proving to geometry instruction may enhance the emergence of these two functions of proof “discovery and communication” in terms of student.

In this study, a multiple approach strategies were utilized to teach quadrilaterals unit. During the course of literature review, it was concluded that teaching learning processes via multiple approaches and especially via vector approach has a potential of providing discovery opportunity to the students (Glicksman, 1965; Star & Rittle, 2008; Akkoç & Katmer, 2014; Schoenfeld, 1983; Robinson, 2011; NCTM, 1989; Zou, Zhang & Rao, 2012). Specifically, Schoenfeld (1983) claims that having a choice to be able to engage with a problem by means of various methods makes it possible to discover different routes in students’ mathematical knowledge. Therefore, students’ endeavors to explore properties of or other relations for geometric objects is compatible with the claims of these researchers, actually.

4.1.7 Elegant Proofs and Solutions Developed by the Participants

Some of the solutions and proofs are evaluated as elegant by the researcher according to the studies in the related literature and his teaching experiences. The reasons why these proofs and solutions are classified as elegant are presented in the definition of the terms part. These proofs and solutions of the participants are presented in the following paragraphs.

A) It is seen that the students started to prefer completely applying vector approach to find out unknown coordinate of a vertex of quadrilaterals that have two pairs of parallel sides (*parallelogram, rhombus, rectangle and square*). This is very similar to the idea that the participants represented the sides of quadrilaterals by vectors in geometry problems in which the vertices are specified analytically. There remains to the students only to equate the opposite side vectors. In the problem, "*the coordinates of three vertices (K, L and M) of the parallelogram KLMN are given and the coordinates of the vertex N is asked*". The solution is as can be seen in the Figure 4-22.

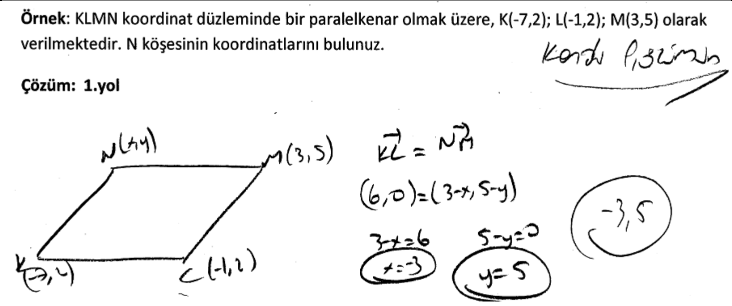


Figure 4-22 Naci's solution to an exercise on Parallelogram 1st section

Before having participated in this study, the students were determining these unknown coordinates by equalizing the sum of the opposite pairs of vertices. That is, they were utilizing the relations: $x_1 + x_3 = x_2 + x_4$ and $y_1 + y_3 = y_2 + y_4$ when the vertices of a parallelogram are given as A(x_1, y_1); B(x_2, y_2); C(x_3, y_3) and D(x_4, y_4). However, they used this way of solution without knowing or questioning the underlying reason. Actually, it only means that they were memorizing a procedural formula then applying it on problems. However, the situation is different in students' vector approach solutions in that they utilized definition of equality of vectors. They could conceptualize an operation in their solution. The students solved the problem having knowledge about the underlying reason in vector approach in spite of the almost the same energy and time expended for synthetic approach. While finding the coordinates of the fourth vertex of a parallelogram, a student in the study of Giles (1964) also used the same strategy with Naci. The researcher describes student's solution as intuitive and argues that the students' choices of vector concept is implicit.

B) This is a nice example that reflects an elegant idea to determine the type of a quadrilateral. The quadrilateral is not given on the coordinate plane or which is given without being specified coordinates of vertices. In the task, "*the type of quadrilateral EBZD is to be determined where E and Z are the points on the diagonal AC of the parallelogram ABCD. The relation among A, E, C and Z are given as* $|AE| = |ZC| = \frac{1}{4}|AC|$". It is evident that there is not any representation related to vector concepts in the problem. Despite of these facts, the students solved the tasks through vector approach. The solutions of Naci and Ömer are as follows in the Figure 4-23 and Figure 4-24 respectively.

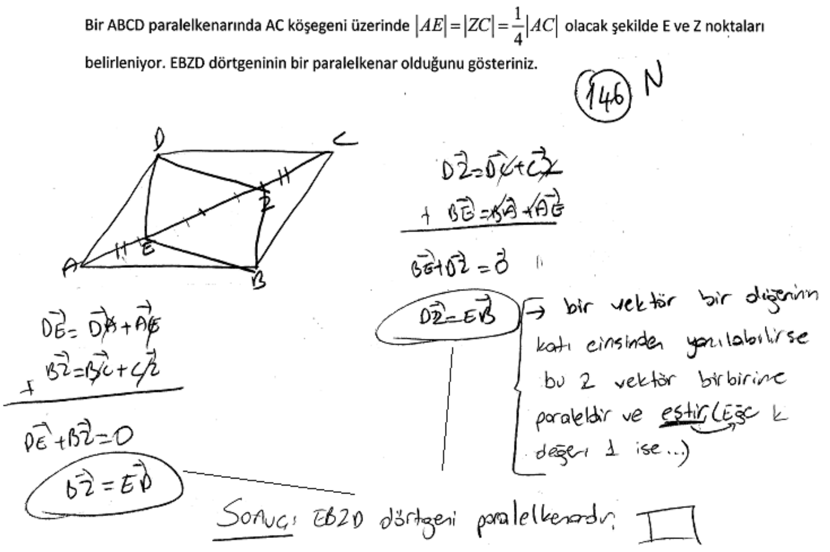


Figure 4-23 Naci's solution to the problem 5 on Parallelogram 1st section

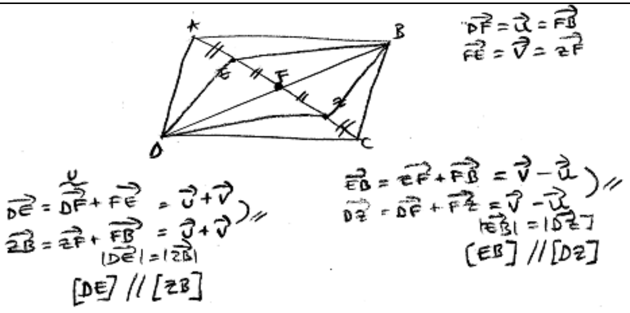


Figure 4-24 Ömer's solution to the problem 5 on Parallelogram 1st section

C) An elegant solution by using vector-intersecting theorem (Zou et al., 2012) and algebra of vectors is given in the Figure 4-25. In the task, it was asked to verify, "*The diagonals in a parallelogram bisect each other*". Ömer proved this statement by vector approach. He represented parallel vector as a scalar multiple of each other. In addition, the student made use of algebra of vectors and displacement analogy for this problem.

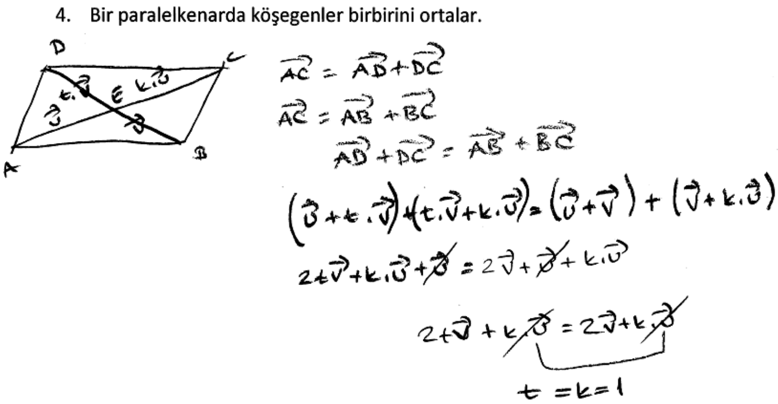


Figure 4-25 Ömer's proof to the 4th property for Parallelograms

D) In the following task in the Figure 4-26, the students were required to show that "*a parallelogram is a rectangle if its diagonals are equal in length*". Naci utilized equal length of vectors to make transition to inner product. Then, the properties of inner product was considered to prove this theorem. Finally, he could get the perpendicularity of the adjacent sides of the parallelogram. That is to say that the parallelogram is a rectangle at the final step.

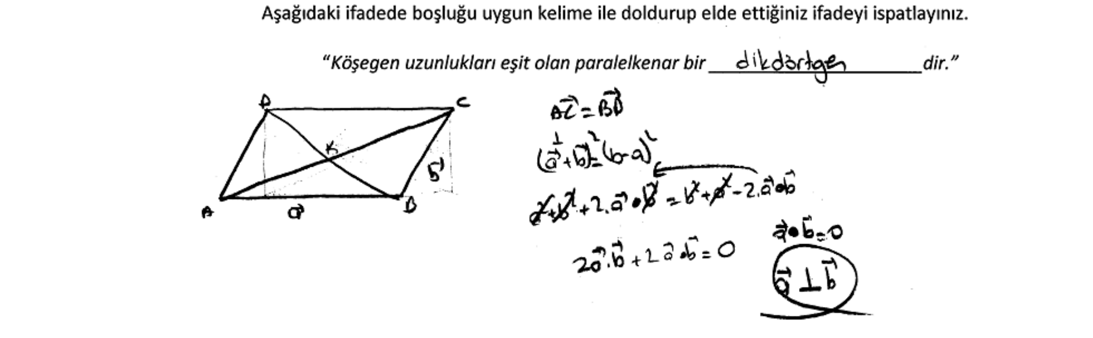


Figure 4-26 Naci's solution to the problem 1 on Rectangles 1st section

E) Ahmet himself developed the following proof for the geometric formula $A(ABCD)=\sqrt{\left|\overrightarrow{AB}\right|^2\left|\overrightarrow{AD}\right|^2-\left\langle\overrightarrow{AB},\overrightarrow{AD}\right\rangle^2}$, which gives the area of ABCD parallelogram generated by side vectors $\overrightarrow{AB}$ *and* $\overrightarrow{AD}$. Ahmet could select appropriate area relation and apply necessary manipulations for this relation as seen in the Figure 4-27. Moreover, he could utilize inner product to enhance the notion of area, as Athen (1966b) and Chiba (1966) state.

Area of a Parallelogram

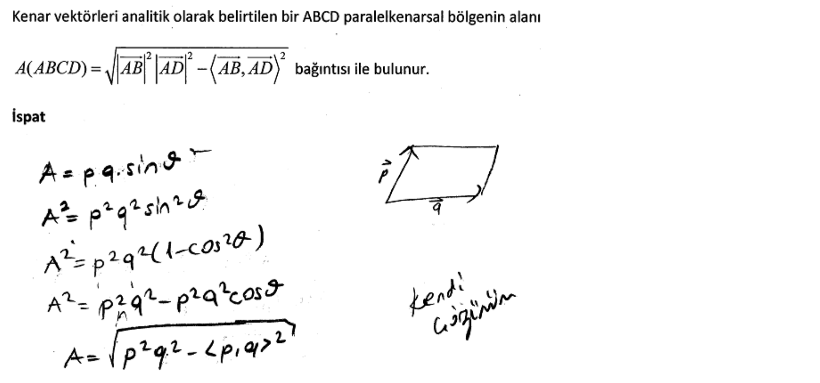


Figure 4-27 Ahmet's proof of vectorial area formula for a parallelogram

Related to the elegance of students' products, there are five illustrations of the examples presented. As a common aspect of these solutions, it is understood that participants utilized vector approach in the solutions. This is compatible with the statements of Chatwin (1985) that vector approach solutions gave opportunity to aware of the beauty of and power of mathematics for the sake of learners. In addition, Glicksman (1965) expresses that vector approach solutions can supply natural and elegant proofs for theorems. Wexler (1962), Glicksman (1965) and Lord (1985) evaluated the proof of a theorem or a solution of a problem more elegant if it is developed through vector approach in comparison with the other approaches. Moreover, participants resorted to inner product frequently in their solutions. Lord (1985) qualified the proofs of theorems including the properties of inner product as elegant as observed in these solutions.

4.1.8 Statistical Analysis of Pre-tests and Post-tests Scores

The contribution of utilizing analytic, synthetic and vector approach instruction on eleventh grade students' PKQT, VKT, PPGT and QAT scores will be analyzed in this part of the study. In order to achieve this aim, pre-test and post-test scores for the tests PKQT, VKT, PPGT and QAT are given in the following tables and figures. The scores for all instruments are converted to corresponding values out of 100.

According to the graphics, the following summary can be useful.

a) According to the pre-test and post-test scores, and students' solutions on these tests for corresponding problems, it can be concluded that necessary prerequisite knowledge to teach "quadrilaterals unit" could be acquired by the participants. Besides students' considerable increase in scores on post-test in comparison with pre-test scores of PKQT (Figure 4-28), students' deficient knowledge on the following topics could be completed. These topics are namely; i) solving first degree equations in two variables ii) operations with literal expressions iii) the sign and value of trigonometric functions of angles on various quadrants iv) finding area of triangular regions v) similarity and congruence of triangles vi) the concepts of analytic coordinate geometry and vii) setting the relation between the sides and area of triangles.

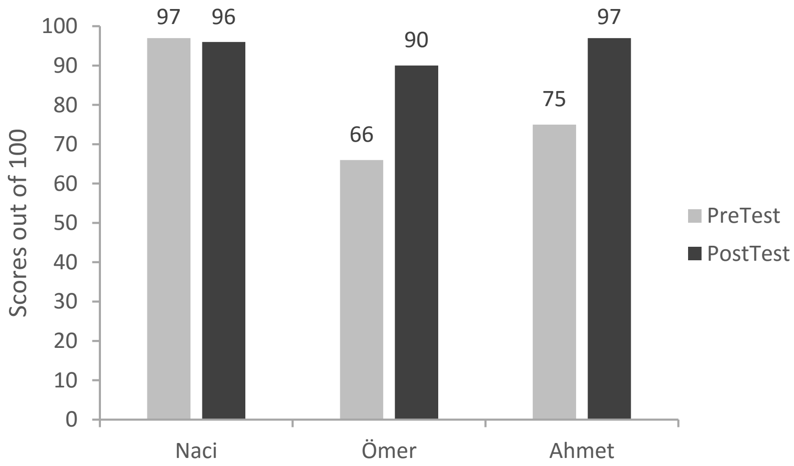


Figure 4-28 Students' scores on Prerequisite Knowledge for Quadrilaterals

b) All of the participant students had almost no sufficient knowledge on linear independence and dependence, projection of vectors and some applications of inner product as understood from their works on VKT pre-test. Besides, Ömer and Ahmet had deficient knowledge on unit vector concept. Lastly, Ömer had troubles with some basic concepts on vectors. According to the pre-test and post-test scores of VKT (Figure 4-29) and students' solutions, it can be said that the students gained necessary prerequisite knowledge of vectors to solve problems on vectors and more importantly to solve geometry problems via vectors.

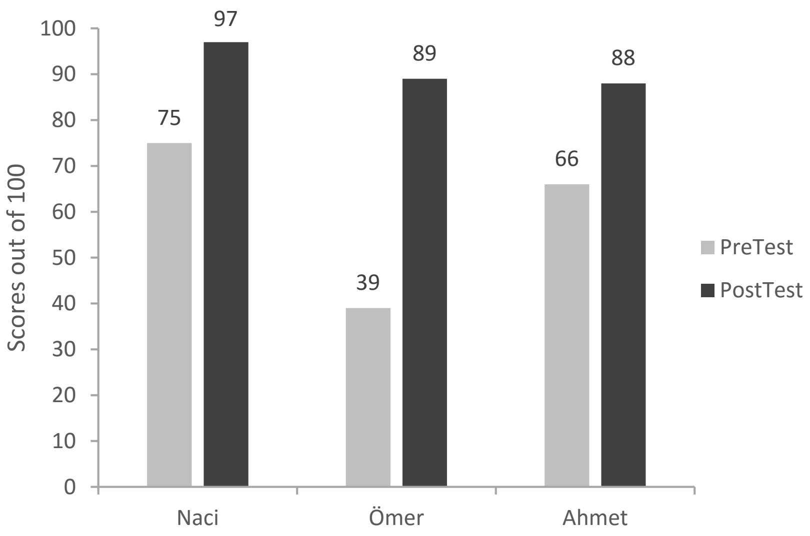


Figure 4-29 Students' scores on Vector Knowledge Test

It is observed that participants' solutions become more practical and compact in comparison with the solutions given on pre-test of VKT according to students' solutions. To illustrate; Naci's and Ömer's solutions (Figure 4-30 & Figure 4-31) to the same problem on VKT pre-test and post-test demonstrated the change in their solutions in terms of practicality. Furthermore, the participants seemed to gain necessary skills related to conducting algebraic operations with vectors in the further steps of the teaching episodes, as it can be observed from students' solutions and works on their written documents throughout the entire teaching sessions,

In Cartesian plane, the points A(-2,3) and B(0,5)

are given. What is the value of $\left\|\overrightarrow{AB}\right\|$?

$a)\ \sqrt{5} \quad b)\ \sqrt{6} \quad c)\ 2\sqrt{2} \quad d)\ 3 \quad e)\ 2\sqrt{3}$

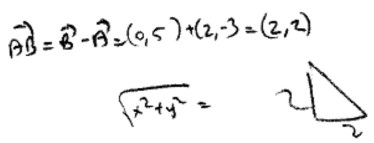
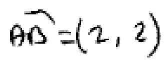
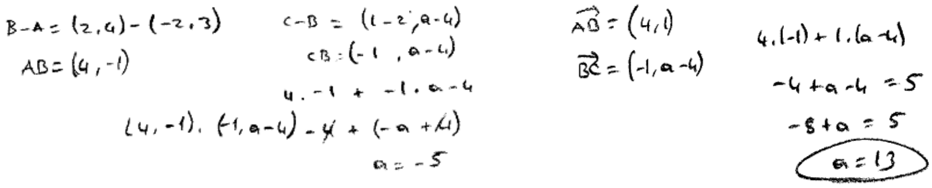
 (pre-test)	 (post-test)

Figure 4-30 Naci's solution to the problem 7 on VKT pre-test and post-test

The points $A(-2,3)$; $B(2,4)$ and $C(1,a)$ are given.

If $\langle \overrightarrow{AB}, \overrightarrow{BC} \rangle = 5$ then what is the value of a?



Pre-test Post-test

Figure 4-31 Ömer’s solution to the problem 10 on VKT pre-test and post-test

c) According to the pre-test and post-test scores of Proof Performance in Geometry Test, participants’ performances increased by 29%, 44% and 39% on this test for Naci, Ömer and Ahmet respectively (Figure 4-32). It is useful to keep in mind that the problems on PPGT was from geometry courses for the grade level 9 and 10. That is, they would be expected to have learnt the topics covered in this test.

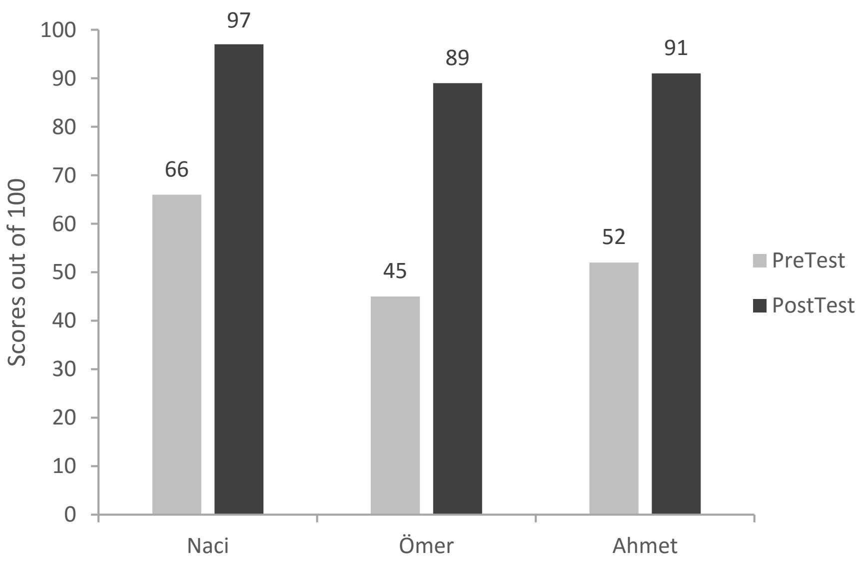


Figure 4-32 Students’ scores on Proof Performance in Geometry pre-test and post-test

In spite of the fact that geometry teaching curriculum program for the grade levels 10 and 11 require reasoning and proving to be utilized and to be improved during geometry teaching, it was understood that proving rarely (almost never) had been utilized in students' geometry classes. In addition, proof-based problems were never asked to the students in mathematics and geometry examinations administered in their schools. Although the subjects of the study are above average students and they are good at solving mathematical and geometric problems that require procedural knowledge, the students were not successful in proof-based problems adequately according to the Figure 4-32. Moreover, despite the fact that they had knowledge about facts, most of the rules, theorems and postulates in geometry, they did not know how to use them in formal proofs of theorems or even in proof-based tasks. This situation actually contradicts with the findings in the study of Senk (1985). The researcher reports that there is a high Pearson correlation coefficient as 0,67 between geometry problem solving and content knowledge of students in geometry. However, the participants could not solve proof-based problems in "*Proof Performance in Geometry Test*". Besides, the students were not be able to solve proof-based problems not only by synthetic approach but also by vector approach, by the time they participated in this teaching experiment according to their solutions on their pre-tests and according to their teachers and to pre-interviews that the researcher conducted. Moreover, at the beginning of the study, they could not solve proof-based problems. They also indicated that they had never learned geometry via vectors at all.

At the beginning of this teaching experiment, the participants had been informed that they would have an opportunity to learn geometry in detail and more conceptually than they had experienced in their regular geometry classes. In fact, they were said that the researcher would frequently resort to proof activities. In addition, the participants were also informed that vectors would be utilized during the course of whole teaching experiment. The students did not demonstrate any indication of dissatisfaction or inconvenience at any moment of the study despite spending much time on conceptual teaching, solving various types of problems such as proof-based tasks. On the contrary, they reacted positively to the teaching style followed in this study. The increase in PPGT and the solutions reflect the students' improvements in terms of proving skills.

d) According to the pre-test and post-test scores of Quadrilaterals Achievement Test, participants' performances increased by 38%, 53% and 50% on this test for Naci, Ömer and Ahmet respectively (Figure 4-33). In other words, all the participants gained scores over 85 on QAT.

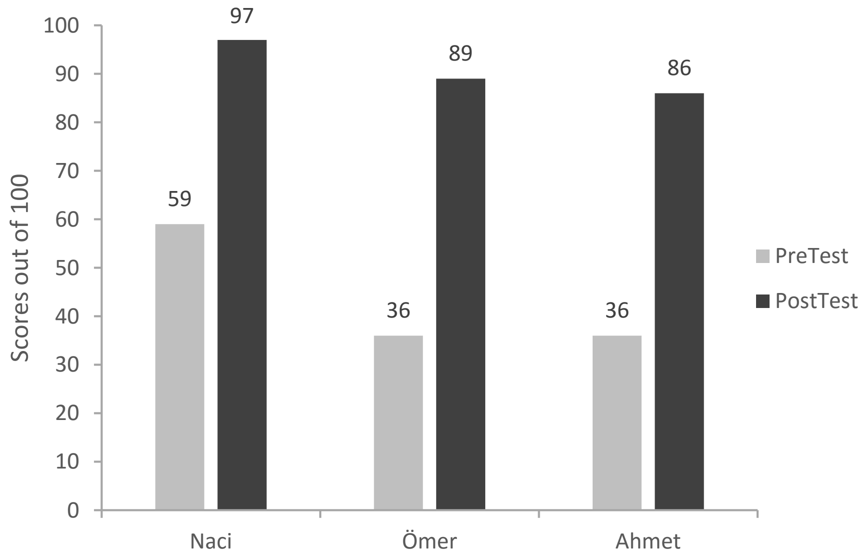


Figure 4-33 Students' scores on Quadrilaterals Achievement pre-test and post-test

4.2 How do students decide the type of approach to be utilized while solving problems related to quadrilaterals during the designed instruction?

The second research question will be answered in this part of the study. The ways how participants solved the problems and how they decided the approach to solve the problems will be examined. In the light of their ways of solutions, a possible classification of approaches with respect to subjects will be presented as a summary of participants' preferences.

4.2.1 Specifying the Type and Properties of Quadrilaterals Given on Coordinate Plane

At the beginning of the study, the students preferred two methods in order for solving problems related to specifying the type and properties of quadrilaterals, which are given with coordinates of vertices. The first group students preferred sketching the given quadrilateral on coordinate plane according to given specific coordinates. After completing to draw the figure of the quadrilateral, they tried to determine the type and then to deduce the characteristic properties of the given quadrilateral. To illustrate; Ahmet drew the quadrilateral whose coordinates of vertices were specified as A(1,2); B(7,2); C(10,6) and D(4,6) on Cartesian plane in the Figure 4-34. After that, he specified the type and properties of the quadrilateral.

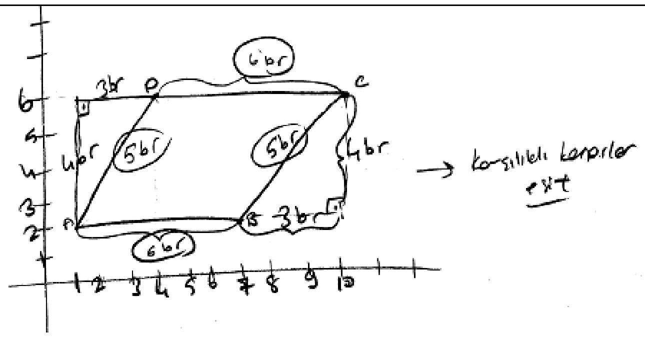


Figure 4-34 Ahmet's solution to the entrance assignment for Parallelogram 1st section

The second group of students preferred to calculate the slope of line segments passing through two points. Later, they made inferences like that "*the segments having the same slopes are parallel, or the segments are perpendicular if the product of their slopes equals to -1*". They tried to solve this type of problems by the way as seen in the Figure 4-35. In the problem, the type of the quadrilateral with the coordinates of $A(-7,2)$; $B(2,-4)$; $C(1,2)$ and $D(-2,4)$ is to be determined. Ömer stated the reasons for the preference of analytic approach as (a) he found this method easier, (b) he felt himself in confidence with this method and (c) the education that he received in schools (Excerpt 4-16 & Excerpt 4-17).

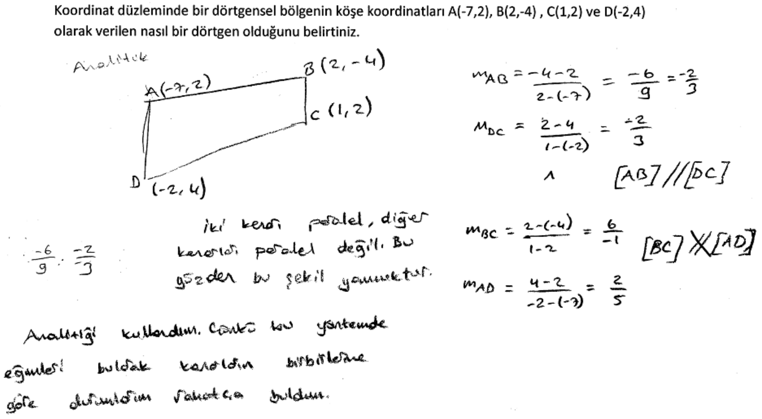
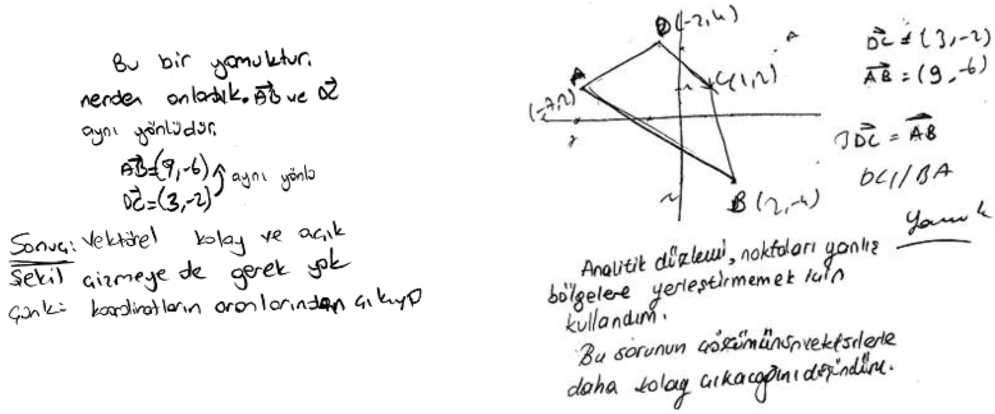


Figure 4-35 Ömer's solution to the problem 1 on Trapezoid 1st section

However, the same problem was solved through vector approach by the other two participants "Naci and Ahmet" although they sketched the figures on Cartesian plane. Naci explained his preference of vector approach in the Figure 4-36-A as the easiness and clearness of the vector approach and no need to draw any figure. Ahmet also stated the reason for utilizing vector approach Figure 4-36-B as the easiness of the vector approach in comparison with other approaches. Both of the students made use of the fact that if a vector is a scalar multiple of the other vector then they are parallel vectors.



A B

Figure 4-36 Naci's & Ahmet's solutions to the problem 1 on Trapezoid 1st section

As the study progressed, especially after completing the first two topics in quadrilaterals unit, the students changed their previous strategies and started to utilize vector approach in determining the type and deducing properties of quadrilaterals. An example for this change, A140 part 2 (Figure 4-34) and A205 (Figure 4-37) can be examined. Although these tasks are very similar to each other, Ahmet solved the problems via different ways. In the first one (Figure 4-34), he sketched the quadrilateral on analytic plane and utilized analytic and synthetic approaches. However; in the latter one (Figure 4-37), Ahmet was observed that he did not sketch the quadrilateral on Cartesian plan and he made use of vector approach (A205) to determine the type and properties of the given quadrilateral whose coordinates of vertices are given. The same change is also observed in terms of Ömer and Naci as well.

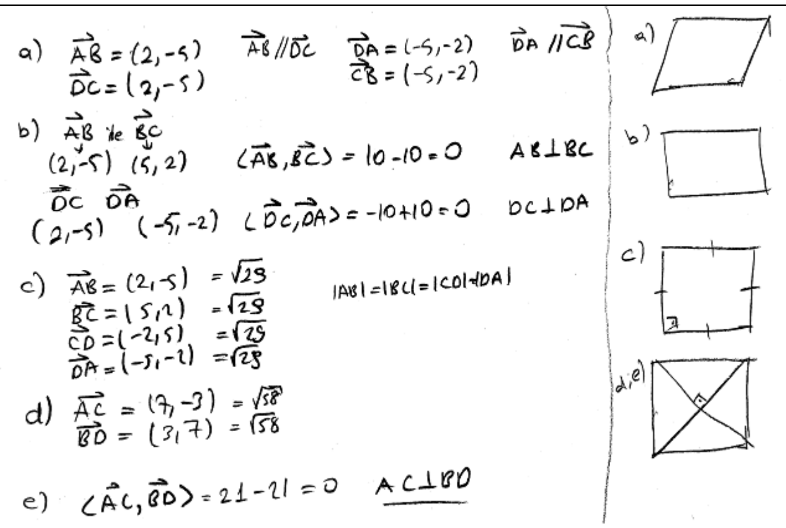


Figure 4-37 Ahmet's solution to the entrance assignment for Square 1st section

So far, the given solutions are directly related to the specifying the type of quadrilaterals given with coordinates of vertices. Parallel approach was also utilized to determine the type of quadrilateral that are not given with coordinates of vertices. In the following problem, the participants were asked "*to determine the type of quadrilateral when midpoints of the sides of a rhombus are respectively combined*". Naci utilized vector addition (*displacement analogy*), and properties of inner product.

He was seen that he did not utilize synthetic approach at all in his solution in the Figure 4-38.

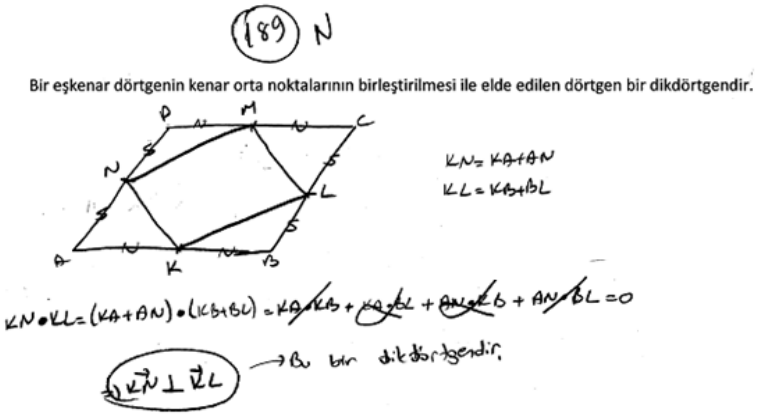


Figure 4-38 Naci’s solution to the problem 4 on Rhombus 1st section

In this part, the quadrilaterals were almost completely given with coordinates. In the eighth standard of NCTM (1989), it is strongly advised that the participants should able to deduce the properties of geometrical figures and to solve problems by means of “*coordinates and transformations*”. In their solutions, the participants utilized coordinates frequently. Moreover, the participants are understood that they utilized one of the transformations implicitly. Indeed, translation is one of the Euclidean transformations. Furthermore, translation includes vectors in itself because translating a point to a new position in the plane means the addition of a vector to the components of the given point. In the light of these, it was understood that the aim of the eighth standard of NCTM (1989) as “*Geometry from an Algebraic Perspective*” could be realized in this study. This is also compatible with the information given by Ayre (1965) and Schuster (1961). According to them, it is very natural that the participants prefer utilizing vector approach in addition to analytic approach because of the easiness and effectiveness of this integration.

4.2.2 Finding the Length of Line Segments through Vectors

During the preparation phase of the teaching materials while looking for the ways to integrate vectors in geometry teaching and during the course of the teaching experiment sessions in the classroom, it was deduced that the way to transfer a vector quantity into a scalar quantity is "scalar product (*inner product*) of a vector with itself" i.e. taking square of a vector. As the study progressed, it was understood that this deduction was adopted and conceptualized by the students spontaneously. The participants: Naci, Ömer and Ahmet utilized taking square of vectors to compute and compare the lengths of the vectors in their solutions. It was observed that this modus operandi was frequently utilized at various points of the study. The frequencies are 17, 6 and 7 for the participants respectively (Table 4-1).

Table 4-1 Frequency of taking square of vectors

Participant	**Frequency**
Naci	17
Ömer	6
Ahmet	7

As an example, while proving the law of cosines or solving a problem related to the law of cosines, students utilized inner product of a vector with itself in order to switch from a vector quantity to a scalar quantity (Figure 4-39)

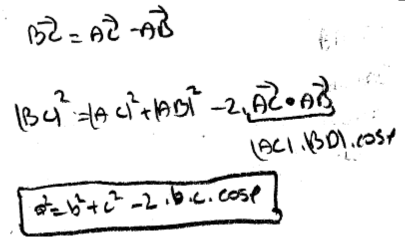


Figure 4-39 Naci's solution to a problem on Vectors 2nd section

The following solution (Figure 4-40) also illustrates how Ömer utilized taking square of a vector to make a transition from a vector quantity to a scalar quantity. The participants are asked to show that $a^2 + b^2 = c^2 + d^2$ for the given ABCD quadrilateral in the problem.

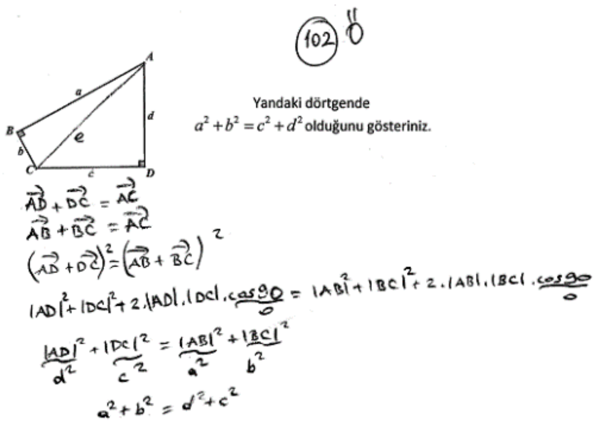


Figure 4-40 Ömer's solution to the problem 2 on Quadrilaterals 2nd section

Vaughan and Szabo (1973) developed the distance concept based on the norm of a vector in their course "A Vector Approach to Euclidean Geometry" and in their textbooks. Norm of a vector is handled through inner product. Moreover, Troyer (1968) called inner product as "*Measuring Rod*" in Euclidean Plane. In addition, although distance concept is presented by coordinatizing a line in synthetic approach, Stephenson (1972) developed the distance concept by inner product in his dissertation showing the differences between the postulational structures of synthetic and vector approaches to plane geometry. These are all compatible with students' conceptualizing of inner product to reach length of a line segment.

4.2.3 Area of Polygons on Coordinate Plane

At the beginning of the study, the students were observed that they were calculating the area of triangles or quadrilaterals whose coordinates of vertices are given by means of placing the given polygon on a coordinate plane. As seen in the following solution in the Figure 4-41, Ömer firstly sketch the given quadrilateral on Cartesian plane. After that, he calculated the area of parallelogram by applying the synthetic formula $\text{"Area} = \text{base} \times \text{height"}$, which gives the area of the required parallelogram. To find the height of the parallelogram, he considered 3 – 4 – 5 special triangle.

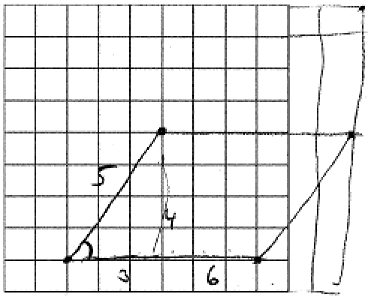
What is the area of PRST parallelogram with the coordinates of the vertices P(2,1); T(5,5) and S(11,5)?	

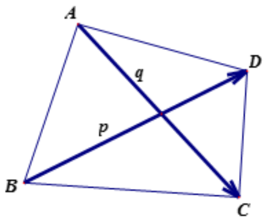
Figure 4-41 Ömer's solution to an exercise on Vectors 2nd section

In addition, at the very beginning of the study, the students were observed that they were failing to calculate the area of some of the polygons on coordinate plane because of their positions or shapes. At this point, the participants seemed to grasp that the use of vector approach met their needs to find the area of polygons, which are given analytically, no matter what the shape, or position of the polygon is.

Here, two kinds of change were observed according to participants' solutions. In the first of these, while finding the area of aforesaid quadrilaterals whose diagonal vectors are either directly given or can be determined, students began to prefer using the following vectorial area formula (Formula 1) that results in the area of the quadrilaterals.

Area of any ABCD quadrilateral with diagonal vectors $\vec{p}$ *and* $\vec{q}$ is computed by the following formula.

$$A(ABCD)=\frac{\sqrt{\|\vec{p}\|^2\cdot\|\vec{q}\|^2-\langle\vec{p},\vec{q}\rangle^2}}{2}$$

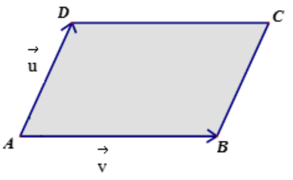


Formula 1 Area of a quadrilateral

Secondly, they started to divide the questioned quadrilateral region into two triangular regions and then to sum the area of these triangular regions. They thought the area of subsequently obtained triangles as the half of the area of parallelograms.

While calculating the area of parallelogram, they utilized the following vectorial formula (Formula 2) in which ABCD is a parallelogram generated by two non-zero vectors $\vec{u}$ and $\vec{v}$.

$$A(ABCD) = \sqrt{\left|\vec{u}\right|^2 \left|\vec{v}\right|^2 - \left\langle \vec{u}, \vec{v} \right\rangle^2}$$



Formula 2 Area of a parallelogram

The participants were observed that they represented the sides of parallelogram through the vectors by using coordinates of the vertices. Then, they applied this formula.

The participants utilized vectors and vectorial representations while they were roughly drawing the parallelograms or other quadrilaterals with the given coordinates of all vertices on analytical plane. In order to find the area of parallelogram, students might be expected that they would locate parallelogram into a rectangle or divide the parallelogram into two triangular regions in analytical coordinate plane. After this stage, they would compute and add the total area as a combination of analytical and synthetic approaches. However, it is easily observed that they tended to solve this kind of geometry problems via vector approach.

In the following example Figure 4-42, Naci calculated the area of parallelogram by means of vector approach as depicted above. However, he did not draw the quadrilateral on coordinate plane. In addition, despite the fact that he was also required to solve the problem analytically as homework, he only drew the parallelogram on coordinate plane roughly and then he continued to solve the problem vectorially again. He did not give up vector approach for this problem. Although he was expected to solve this problem by means of combination of analytic and synthetic approaches in this homework, this was not realized.

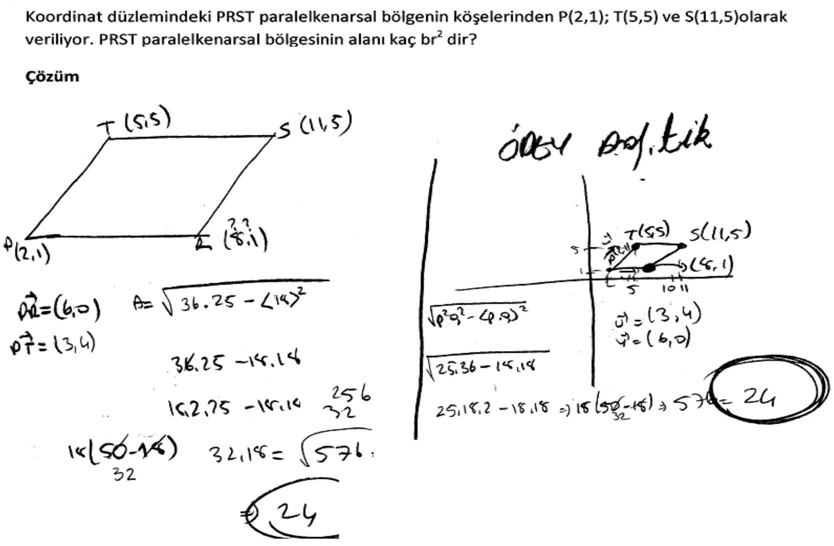


Figure 4-42 Naci's solution to an exercise on Parallelogram 2nd section

Naci preferred utilizing Cartesian plane at the beginning as seen in the Figure 4-43. The side vectors of the parallelogram are given and the area of the parallelogram is asked in this task. Although the problem contains vectors, he did not make use of vectors. Instead, he tried to locate the parallelogram on Cartesian plane. However, he could not solve the problem correctly and left the problem incomplete.

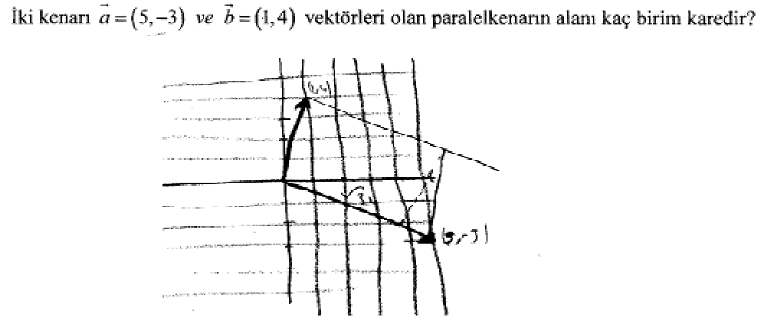


Figure 4-43 Naci's solution to the problem 15 on VKT pre-test

Besides, the students were observed that they acquired a tendency while calculating area of parallelograms generated by two vectors by means of the "practical method", which was developed by the researcher. Especially, it is seen that Ahmet preferred this method in his solutions very frequently, almost completely. After

Ahmet's persistency on solving this kind of problems correctly and practically, the other students started to apply this method on their solutions.

Ahmet states the way of his solution and the reason of preferences of his own strategy as in Excerpt 4-3.

Excerpt 4-3 Excerpt from an interview with Ahmet on 08.07.2013

Researcher: Why did you choose this method to solve area problems and how did you construct your own strategy?
Ahmet: While finding area of a quadrilateral whose coordinates of vertices are given, I firstly divide the quadrilateral into two triangular regions by drawing one of the diagonals of the quadrilateral. Here, I thought the area of each of the triangles as the half of area of two different parallelograms. I represented two sides of any post-constructed triangles as vectors and then I used the practical method in order to calculate the area of parallelogram generated by two side vectors. I repeat the same procedure for the second triangle. After that, I add the area of triangles to reach the area of initial quadrilateral. The reason for my preference is easiness and convenience of the application of this method. Moreover, since I could develop the proof of this method myself, I feel myself happy in applying this method. I applied my own strategy to all of the similar problems repeatedly. Then, I could solve all of the problems correctly throughout this study.

As Ahmet stated, he frequently resorted to this method throughout the study. The following illustration (Figure 4-44) shows briefly the system that the student constructed and followed to calculate the area of a quadrilateral that is given on coordinate plane.

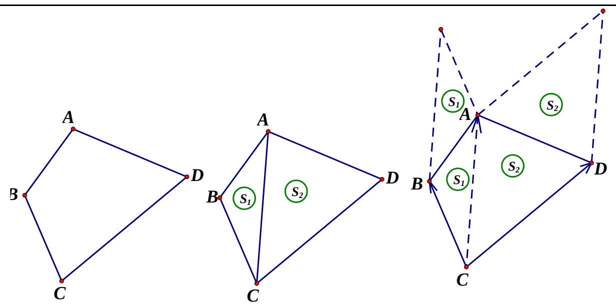


Figure 4-44 Ahmet's strategy to calculate the area of a quadrilateral via vectors

The solution below (Figure 4-45) illustrates how Ahmet solved this type of problems. In the problem, "*coordinates of the vertices are given and the area of the quadrilateral is asked*". He drew the Cartesian plane so as to be sure about the position of the vertices. Actually, this was not necessary to solve the problem. In the solution, he also stated the steps that he followed to calculate the area.

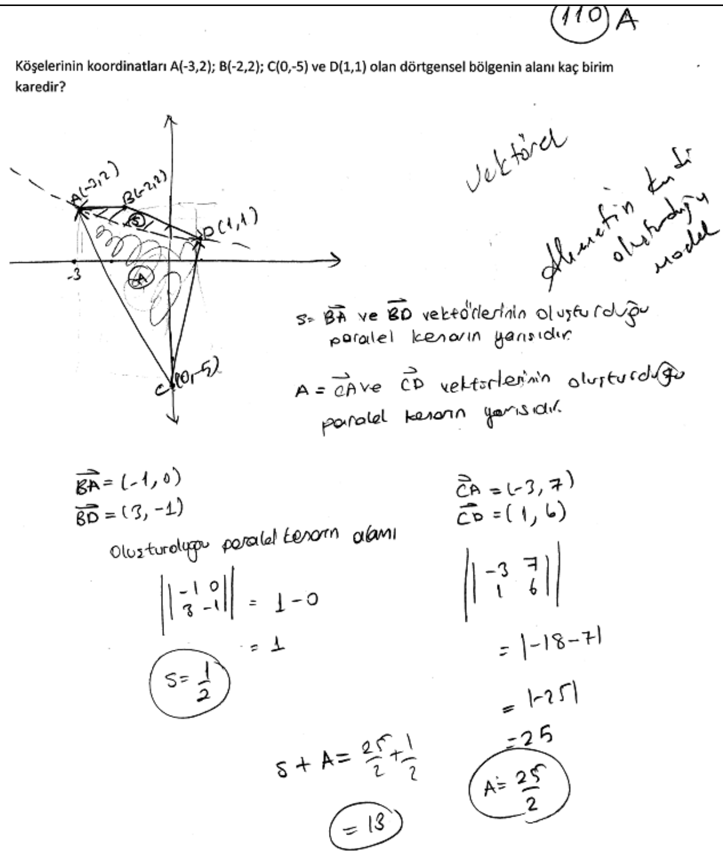


Figure 4-45 Ahmet's solution to the problem 3 on Quadrilaterals 2nd section

Ahmet's own process of finding area of a quadrilateral whose vertices are given is known as "*The Surveyor's Formula*" as understood from the study of Braden (1986). Krech (1968) also states this method as the preliminary stage to calculate the area of polygons on plane.

It was observed that Ahmet did not make any mistakes while solving this type of problems. Actually it can be concluded that the students possibly have lower error rates or situations if they could develop their own strategies or systems but under the supervision of their teachers.

Moreover, specifically related to this type of problem, Ahmet used to draw analytic plane necessarily and then to locate the given polygon on this plane in the past. However, he stated that he left this way of solution with the help of vector-based experiences. In other words, he expressed that he started to choose vector approach strategies instead of combination of synthetic and analytic approaches in order to solve these problems. He explained this change with stressing the fact that placing the given quadrilateral on analytic coordinate plane was waste of the time. Instead, he roughly specified the quadrants on which the coordinates of the vertices to be placed when it is necessary. After this stage, he continued with vector methods. Further, he talked about the pleasure or enthusiasm of utilizing recently learned tools such as vector methods or practical area method. The Figure 4-46 illustrates that he could find the area of the quadrilateral region without preparing any coordinate plane or drawing any figure in spite of the fact that the coordinates of the vertices were given in the problem.

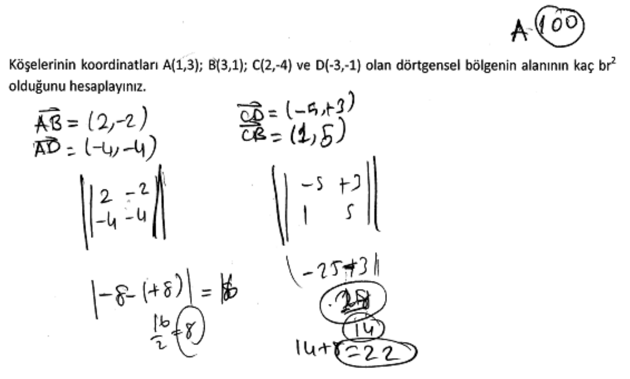


Figure 4-46 Ahmet's solution to an exercise on Quadrilaterals 2nd section

The following two solutions illustrate the approaches preferred by Naci at the beginnings of the study (17.05.2013) and towards to the end of the study (14.09.2013). In the former case, the area of PRST parallelogram with the coordinates of the vertices P(2,1); T(5,5) and S(11,5) is questioned in the problem. He located the given parallelogram on analytic plane by utilizing provided coordinate plane worksheets by the researcher. After that, he solved the problem via synthetic and analytic approaches (Figure 4-47) as expressed in the earlier passages. In the second problem, the area of the rhombus is asked. However, he preferred to use vector approach in solving the task in the latter case (Figure 4-48) although the problems are very close to each other essentially.

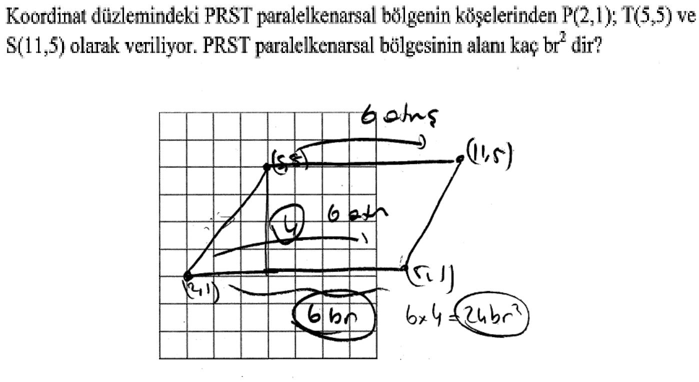


Figure 4-47 Naci’s solution an exercise on Vectors 2nd section

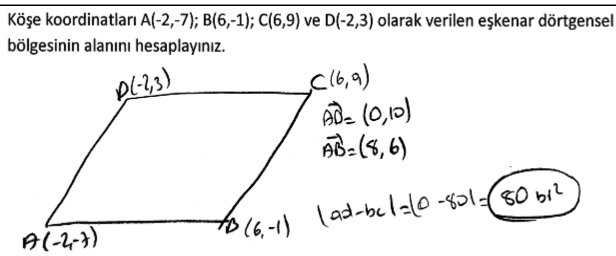


Figure 4-48 Naci’s solution to the problem 4 on Rhombus 2nd section

Further, the following solution illustrates (Figure 4-49) the use of practical method. The proof of the method was developed through vector methods with the participants actively. In other words, they could learn underlying principles for this method by actively participated in the process of proving. Despite the fact that the triangle is already given on the analytic plane and it is easy to embed the triangle into a rectangle, the students preferred to use the practical method instead of using previous solving strategies i.e. combination of analytical and synthetic approaches. In this preference, two of the sides in the triangle were represented as vectors and after that, the problem was solved by vector approach as seen in the following figure.

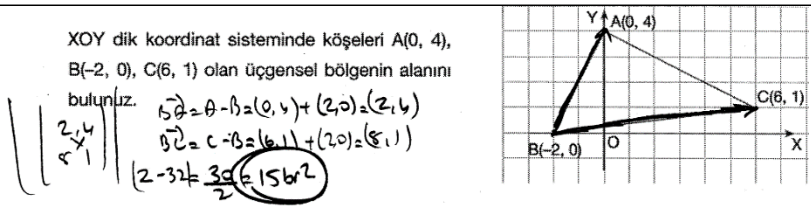


Figure 4-49 Naci's solution to the problem 23 Vectors 2[nd] section

For the solution to this kind of problems, there exist two alternative ways to "*vector approach*". To begin with, it could be solved through calculating the area of a triangle with the given vertices that include calculating the determinant of 3×3 matrix. However, matrices and determinant are not appropriate concepts since they are not included currently in curriculum of the geometry for the grade levels 9-10 and 11 (MoNE, 2010a & 2010b). Secondly, the given quadrilaterals can be sketched on coordinate plane and then the steps in synthetic method need to be completed. However, as stated by the students, this method takes too much time. As a result, Naci calculated the area of ABCD quadrilateral through the use of vectors in the Figure 4-50. In the problem, the area of ABCD quadrilateral is asked where E, F, G and H are the midpoints with E(-2,6); H(4,-2) and G(0,-9).

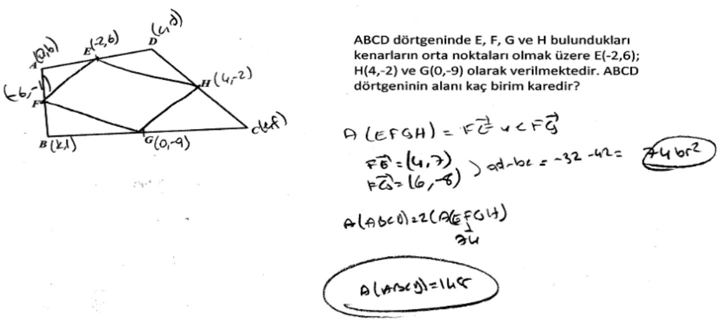


Figure 4-50 Naci's solution to the problem 3 on Parallelogram 2[nd] section

The participants used this method for the test items “3, 12 and 15” in both pre-test and post-test on QAT as seen in the Table 4-2. This method was preferred 11 times in total to solve three different problems on QAT. It is also important note that the participants had learnt vector approach strategies to compute to the area of polygons on Cartesian planes in the context of triangles before the administration of pre-test of quadrilaterals. They could convert vector approach strategies in the pre-test of QAT before lecturing the main unit “quadrilaterals”.

Table 4-2 The frequency of the use of practical method for the problem in QAT

Participant	**Pre-test**	**Post-test**
Naci	12,15	15
Ömer	3	3,15
Ahmet	3,12,15	3,15

The strength of the utilization of vector approach with coordinates transpires once more for the solutions of the problems on the area of polygons, which are given with coordinates. In addition to the power of this integration (Ayre, 1965 & Schuster, 1961), Bundrick (1968) states that a student prefers the easiest way of solving a problem among several approaches by means of which they learn geometry. Moreover, Ahmet’s way of calculating the area of a quadrilateral whose vertices are given is reasonable and known as “*The Surveyor’s Formula*” (Braden, 1986). As a result, students’ preference of vector approach is meaningful while calculating the area of a quadrilateral on coordinate plane.

4.2.4 Quadrilaterals with Perpendicular Elements

As a common feature of students’ solutions, it is observed from students’ products that the students made use of inner product of vectors in the problems on quadrilaterals that have perpendicular elements (diagonals, sides or line segments).

Since the result of inner product of two non-zero vectors is 0 when two vectors are perpendicular, the students might have preferred to utilize this property. While participants were utilizing inner product, they expressed the vectors in the inner product as the sum or subtraction of elements of the quadrilaterals. Although, there are not any vectorial representations on quadrilaterals, they made use of sides, diagonals or/and line segments as if they were given vectorially. As a result of this operation, they could find magnitude of required quantities as metric quantity by means of vector approach. The frequency of cases that the students utilized vector approach in solving problems about quadrilaterals, which include perpendicular elements, is given in the Table 4-3.

Table 4-3 The frequency of utilizing vector approach in problems containing perpendicular elements

Participant	**Frequency**
Naci	18
Ömer	12
Ahmet	17

To illustrate; in the following problem (Figure 4-51), the participants were required to verify the relation "$(a+c)^2 = e^2 + f^2$" where “a and c” are the bases and “e and f ” are the diagonals of a right trapezoid with perpendicularly intersecting diagonals. Algebra of vectors and properties of inner product were utilized by explaining underlying reasons, as clearly seen in the following solution. Moreover, Naci stated the reason why he took the square of vectors in his solution. It is important to remind that Naci proved the relation by vector approach although there is not any vectorial notation in the problem.

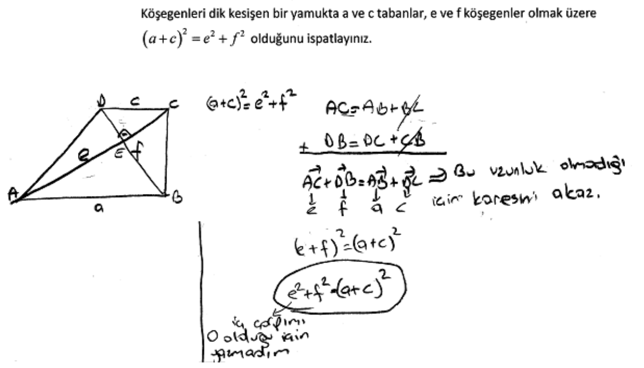


Figure 4-51 Naci's solution to the problem 2 on Trapezoid 1st section

One of the Ahmet's solutions to the mentioned problems is given in the Figure 4-52. The value of x is to be found out in the rectangle according to the given information in the figure. He used inner product of the vectors that he obtained from the line segments $[FA]$ and $[FE]$ by means of origin principle.

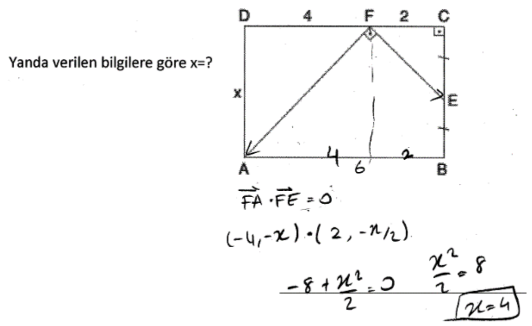


Figure 4-52 Ahmet's solution to the problem 6 on Rectangle 1st section

The following geometry problem (Figure 4-53) contains a right trapezoid with perpendicular diagonals with the lengths of the bases 2 and 8 units respectively. Computing the area of the trapezoid is the ultimate target for the participants. In spite of the facts that solution through the use of vectors is available and the pupils had applied vector approach to similar problems earlier, they directly applied a formula for this special case.

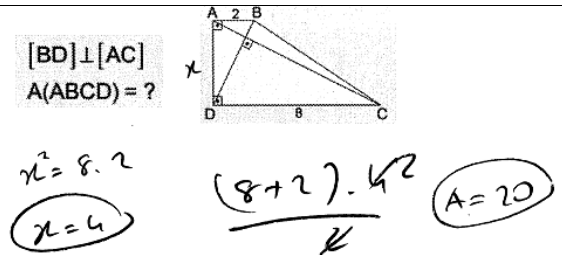


Figure 4-53 Ahmet's solution to an exercise on Trapezoid 2nd section

Probably, this could be thought as a result of long-term experience with synthetic approach and treatment under this approach in schools. The students cannot give up applying this approach in the solutions of tasks generally. Moreover, it should be noted that the proof of special formula "*the length of the height equals to geometric mean of the bases in a right trapezoid with perpendicularly intersecting diagonals*" was asked to the students. It had been proven by means of synthetic and vector approaches by the students as an assignment before solving this problem. Naturally, the students might have applied the formula directly to achieve an answer by calculating the height of the trapezoid with the operation of taking square root of multiplication of base lengths. These are all possible explanations for the student's choice.

There are totally 57 problems asked to the participants at the end of the units. 23 of these problems are proof based problems. In other words, these problems contain show that, or verify that or prove that statements. Table 4-4 shows the frequency of preferred approach while solving proof-based problems. In this table, Naci utilized vector approach 18 times out of 23 proof-based problems. The frequencies are 10 and 13 for Ömer and Ahmet respectively. This can be explained with the fact that when the students were presented proof-based problems, they preferred to solve these problems via vector approach because they were to write the reasons in a formal way. It seems that the students preferred vector approach rather than synthetic approach in proving geometrical facts. In other words, the students seemed that they solved the tasks by means of traditional ways if the tasks required only an answer, such as a multiple-choice item. Glicksman (1965) and Klamkin (1970) explain this choice with the fact that vector proofs can be constructed and organized more easily than constructing synthetic proofs. Moreover, they add that synthetic approach solutions necessitate using auxiliary and additional lines, verifying similarity or congruence of some triangles and constructing parallelograms in geometry problems. Furthermore, Scott and Rude (1970) assert that analytic and vector approaches make easier to conduct and understand the proofs. Therefore, this preference is very natural. Bundrick (1968) also stated that the participants in his study prefer the easiest way of solution if they learn geometry through both synthetic and vector approaches. Moreover, Kwon (2013) found that while solving problems students show a tendency of preferring the most convenient approach for the sake of themselves when they have knowledge on problem solving through multiple approaches. Parallel to these claims, a similar pattern transpired in this study as can be seen in the Table 4-4.

Table 4-4 The frequency of preferred approach while solving proof-based problems

	Participant		
Approach	Naci	Ömer	Ahmet
Synthetic	2	11	6
Vector	18	10	13
Synthetic + Vector	3	1	1
Unsolved	0	1	3
Total	**23**	**23**	**23**

Szabo (1967) states that orthogonality and length of vectors are useful while studying properties of quadrilaterals. Hence, orthogonality can be utilized to specify the type of quadrilaterals and to classify the quadrilaterals. Specifically, he accepted vector solutions as complete in classifying quadrilaterals with perpendicularly intersecting diagonals. Besides, Maynard and Leversha (2004) called quadrilaterals with orthogonal diagonals, as "*Pythagorean Quadrilaterals*" since the sum of the squares of the length of opposite sides are equal to each other in a quadrilateral. However, whereas Josefsson (2012) called these quadrilaterals as "*orthodiagonal quadrilaterals*", De Villiers (1994) labeled these quadrilaterals as "*perpendicular quadrilaterals*".

4.2.5 Quadrilaterals having Pair (s) of Parallel Sides

As another common feature of students' solutions, it is observed that the students utilized vector approach in problems on quadrilaterals that have one pair or two pairs of parallel sides. During the course of teaching "*Elementary Vector Algebra Instructional Module*", the participants had been emphasized that parallel vectors were scalar multiple of each other and vector relations were also valid for the metric relation of these vectors when they were parallel to each other. Specifically, when parallel vectors are added or subtracted, their magnitudes can be added and subtracted, as well. However, this is not the case for non-parallel vectors. Possibly because of this knowledge, the participants were observed that they utilized vector approach to solve problems on quadrilaterals containing pair(s) of parallel sides. Furthermore, this prediction seems reasonable since Vaughan and Szabo (1973) include quadrilaterals in their geometry course, which is mostly based on vector approach. They explain the inclusion of quadrilaterals with the fact that parallelism of lines, planes and ratios are the major focus of their course.

The number of cases that the students utilized vector approach while solving problems about quadrilaterals including parallel sides is given in the Table 4-5.

Table 4-5 The frequency of utilizing vector approach to solve problems containing parallel sides

Participant	Frequency
Naci	17
Ömer	10
Ahmet	12

As the first example; in the following problem, the relation between the lengths of $|AB|, |CD|$ and $|EF|$ is asked for ABCD trapezoid in which E and F are the midpoints in the Figure 4-54. It is understood that the participant students preferred vector approach to solve this problem. Ömer utilized displacement logic in his solution as given in the Figure 4-54. However, synthetic approach (*similarity*) could be utilized for this problem. Furthermore, it is again worth pointing out that although the following problem does not include any vectorial representations originally, three of the students preferred to solve it through vectors.

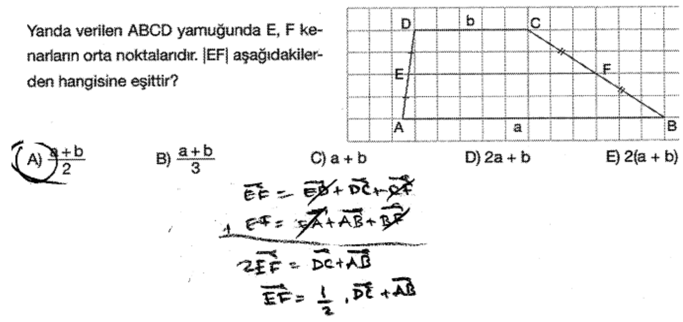


Figure 4-54 Ömer's solution to the problem 17 Vectors 1st section

As the second illustration, in the following task the students are asked to find unknown vertex (N) of KLMN parallelogram whose coordinates of three vertices (K, L, M) are given. Ahmet found the coordinates of the unknown vertex by equating opposite side vectors of the parallelogram as seen in the Figure 4-55. While Ahmet and Naci solved this problem via vector approach, Ömer preferred analytic approach to solve this problem.

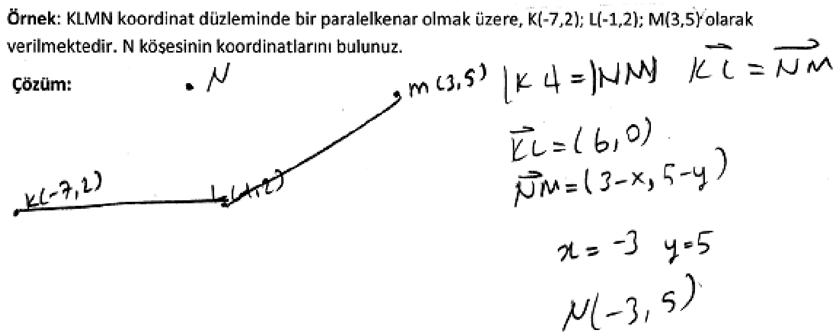


Figure 4-55 Ahmet's solution to an exercise on Parallelogram 1st section

The third illustration for the use of vector approach for quadrilaterals having parallel sides is Naci's solution. In the task, the type of quadrilateral EBZD is to be determined where E and Z are the points on the diagonal AC of the parallelogram ABCD. The relation among A, E, C and Z are given as $|AE| = |ZC| = \frac{1}{4}|AC|$. In his work in the Figure 4-56, Naci stated that *"two vectors are parallel to each other if one can be written as a scalar multiple of the other one (two vectors are equal if k=1)".*

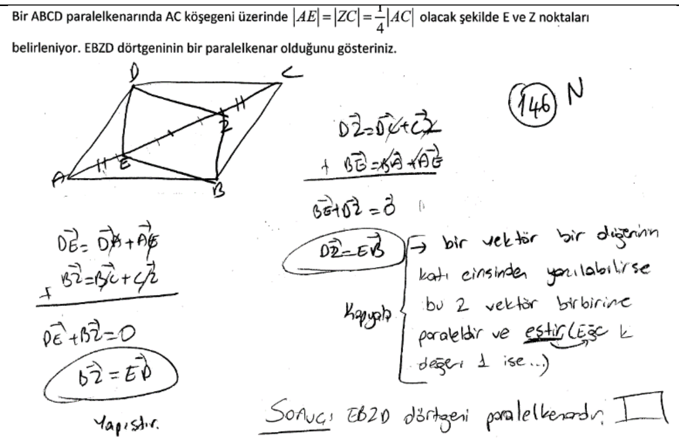


Figure 4-56 Naci's solution to the problem 5 on Parallelogram 1st section

As seen in the three of the solutions, the participants utilized vector approach for the quadrilaterals having parallel sides.

4.2.6 Participants' Preferences in the Entrance Assignments

Information about entrance assignments was given under the title "*Quadrilaterals Instructional Module*" in detail. Briefly, length of the sides and diagonals, relative position of the sides and diagonals, measure of interior angles, coordinates of intersection point of diagonals and coordinates of midpoint of sides or diagonals are studied in these assignments to deduce the properties of special quadrilaterals. According to the study of Richardson, Reynolds and Schwartz (2012), studying rich mathematical tasks specifically related to finding and determining the type of quadrilaterals on grid paper may enhance adaptibility in any classroom.

The students solved this entrance assignment sheets independently and individually. In spite of the fact that there were some interventions, helps or guidance of the researcher at the first topic "*Quadrilaterals*" for these entrance assignments, the students started to solve them without any contributions of anyone for the subsequent special quadrilaterals. Throughout the solution processes and ways of participants in these assignments, there were different developmental stages and different solution strategies emerged in this manner in terms of the participants of the study. These improvements are presented in the following paragraph. An example for this part of lesson plans for quadrilaterals can be found at the Appendix C.

The participants' preferences for the entrance assignments are given in the Table 4-6. According to this table, all of three participants utilized analytic and vector approaches together in order to calculate the lengths of sides for the first two quadrilaterals (*trapezoid and parallelogram*). However, they used vector approach for the rest of four quadrilaterals (*rectangle, rhombus, square and deltoid*). Parallel to this pattern, the participants preferred vector approach at least four times for the five quadrilaterals to find the length of the diagonals for these quadrilaterals. This case is compatible with the situations explained under the title *"4.2.2Finding the Length of Line Segments through* Vectors*"*.

All of the students were observed that they made use of vectors as a tool to determine relative position of the sides and diagonals for all of the quadrilaterals with different frequencies. Vector approach was merely preferred for 5 quadrilaterals by Naci, 1 quadrilateral by Ömer and 3 quadrilaterals by Ahmet. However, analytic and vector approaches were utilized together 5-times by Ömer, 3-times by Ahmet and once by Naci. Therefore, the use of vectors were very frequent in participants' works to determine the relative position of the sides and diagonals in the given quadrilaterals.

All of the participant students applied only analytic methods for all of cases requiring finding the coordinates of midpoints of sides or diagonals. None of them utilized vector approach to find the coordinates of midpoint of line segments on quadrilaterals. The participants made use of three of the approaches with various combinations in order to calculate the measure of interior angles of quadrilaterals.

They utilized both analytic and vector approach for the problem related to relative position and length of median in a trapezoid. They could complete entire steps correctly to make inferences about the median. Related to coordinates of intersection point of diagonals in two problems; Naci utilized 2-times analytic and vector approach simultaneously, Ömer used only analytic approach twice and Ahmet preferred analytic approach for the first problem and vector approach for the second problem separately.

In the following interview with Ahmet (Excerpt 4-4), the reason why he prefers vector approach in the solution of entrance assignments and whether he prefers this approach deliberately or not can be examined.

Excerpt 4-4 Excerpt from an interview with Ahmet

> *Researcher: In this entrance assignment (parallelogram 1st section), the prevalence of vector approach is an outstanding aspect of your solution in determining the type of quadrilateral given with coordinates of vertices. Have you made this choice consciously?*
> *Ahmet: I know that vector solution is more convenient and practical than analytic and synthetic approaches while calculating the length of sides and specifying the relative positions of sides and diagonals to determine the type of the quadrilateral that is given with coordinates of vertices. This is why I prefer vector approach in my solutions. Instead of synthetic approach or sketching the given quadrilateral on Cartesian plane, I prefer vector approach. I made it consciously.*

Participants' preference of vectors in the entrance assignment is not accidental with the fact that Vaughan and Szabo (1973) reach a conclusion that use of vectors is more efficient than the algebra of analytic approach. Hence, the simplicity and efficiency of the use of vectors when the coordinates available are possible explanations for the frequent preference of vector approach in entrance assignments. Furthermore, According to Ayre (1965), the use of vectors are quite versatile and they are appropriate for a wide range of conditions. Specifically, concurrence, parallelism, and perpendicularity of lines are cases that might lead someone to prefer a vector approach.

Table 4-6 Participants' preferences in the entrance assignments

Participant	**Length of side**	**Relative position of sides and diagonals**	**Length of diagonal**	**Angle**	**Intersection point**	**Midpoint coordinate**	**Median of the quadrilateral**
Naci	2AV **4V**	1AV 5V	**5V**	2S3V	2VA	2A	1A**V**
Ömer	1A 1AV **4V**	5AV 1V	1AV **4V**	2S 1A 1V 1AVS	2A	3A	1A**V**
Ahmet	1AV 1ASV **4V**	3AV 3V	**5V**	2S 1A 1V	1A 1V	3A	1A**V**

A: Analytic approach S: Synthetic approach V: Vector approach

4.2.7 Vector Approach: As an Alternative to "Similarity and Congruence of Triangles"

According to teaching experiences of the researcher and related literature (Senk, 1985; Gagatsis & Demetriadou, 2001), "*Similarity and Congruence of Triangles*" is one of the most problematic topics in geometry teaching. Specifically, determining which of the triangles are similar or congruent and; hence, matching corresponding congruent angles and determining three pairs of corresponding proportional (or equivalent) sides are difficult steps for students. In other words, the students have difficulties in writing corresponding congruent or similar triangles. Although, the order of vertices is very important in writing proportional relation for the equivalence or similarity of two triangles, the students frequently write the order incorrectly. Moreover, setting the equivalence or similarity of the triangles is mostly ignored. Table 4-7 shows briefly the situation whether the participant students set the similarity and whether they could solve the problems related to similarity and congruence. The problems, the instruments (PKQT, PPGT and QAT) and the participants are included in this table. According to the table, similarity and equivalence was utilized 13 times without setting the similarity and equivalence relation in the pre-tests. However, the participants could use similarity and equivalence 24 times with setting the relation only 5 times in the post-tests.

Table 4-7 Participants' setting correspondence and success in similarity and congruence problems on the instruments

Instrument	Participant	Problem	Pre-test		Post-test	
			Setting the Correspondence	Problem Solved Correctly	Setting the Correspondence	Problem Solved Correctly
Prerequisite Knowledge for Quadrilaterals Test	Naci	15	0	1	0	1
		17	0	1	0	1
		18	0	1	0	1
	Ömer	15	0	0	0	0,5
		17	-	-	0	0,75
		18	0	1	0	1
	Ahmet	15	0	1	0	1
		17	0	0,5	0	1
		18	0	1	-	-
Proof Performance in Geometry Test	Naci	12	-	-	0	1
		13	0	1	-	-
	Ömer	12	-	-	1	1
		13	-	-	1	1
	Ahmet	13	0	0,5	-	-
Quadrilaterals Achievement Test	Naci	C4	0	1	0	1
		C7	0	0	0	1
		C8	-	-	0	1
		C9	0	1	-	-
		C11	0	0	-	-
	Ömer	C4	-	-	0	1
		C7	-	-	0	1
		C8	-	-	0	0
		D1	-	-	1	1
		D3	-	-	1	0,75
		D4	-	-	1	1
	Ahmet	C4	0	1	0	1
		C7	-	-	0	1
		C8	0	1	0	1
		C9	0	0	-	-
		C11	0	0	-	-
		C16	0	0	-	-
		D1	-	-	0	1
		D3	-	-	0	0,75
			A	**B**	**A**	**B**

A	**0**	Correspondence was not set
	1	Correspondence was set
	-	Left Empty

B	**0**	incorrect or irrelevant solution
	0,5- 0,75	solved partially
	1	solved completely
	-	Left Empty

Teaching geometry through the use of vector concepts can be an alternative way to solve certain type of problems in geometry. Specifically, vector approach might be concluded as an alternative to the use of theorems of similarity and congruence of triangles especially for certain types of problems. This is because of the fact that participants could solve problems that were solvable via similarity and congruence of triangles by means of vector approach. At least, the students may have a chance to verify or check the similarity that they set by means of vector approach. This could be benefits of teaching geometry with vectors in addition to synthetic approach.

Towards the end of the study, the students were observed that they tried vector approach strategies in solving geometry problems that the researchers had not predicted earlier. There are totally 20 problems, which can be solved via the theorems of similarity and congruence of triangles. None of the problems includes any vectorial representation or clue. The distribution of these problems with respect to the subjects is given in Table 4-8. Despite the facts that these problems can be solved via similarity and congruence as a synthetic approach and the participants might be anticipated to solve these problems through synthetic approach, they frequently resorted to vector approach to solve these problems. The frequency of students' use of vector approach is presented in Table 4-9.

Table 4-8 The distribution of the problems that can be solved via similarity and congruence by the subjects.

Subject	**Number of Problems**
Quadrilaterals	2
Trapezoid	3
Parallelogram	4
Rectangle	6
Rhombus	3
Square	2
Total	20

Table 4-9 The frequency of utilizing vector approach as an alternative to the use of similarity and congruence

Participant	Frequency
Naci	17
Ömer	11
Ahmet	15

Out of 20 problems

To illustrate, Naci solved the following geometry problem by utilizing vector approach in spite of the fact that the problem does not contain any vector notations or symbols or any clue to guide the students use of vectors. In the problem, *an ABCD rectangle with* $|AB| = 2|AD|$ *and* $|DP| = \frac{3}{4}|DC|$ *are given where P is a point on* $[DC]$ in the Figure 4-57. It is asked to show whether or not $[AC]$ is perpendicular to $[BP]$. Generally, the students were used to solve this kind of problems through synthetic approach in schools. Particularly, the teachers generally set similarity proportional relation in order to solve this type of problems. The situation was not different for the researcher in his school life as a student and in his professional teaching experiences as a teacher. Even, the similarity type that can be used in solving this type of problems has a special name "*butterfly similarity*" as a mnemonic device so that the students can apply it easily and practically. Actually, it is a SAS similarity. However, Naci solved this problem by vector approach in which he utilized algebra of vectors and inner product instead of setting a similarity. This solution is elegant and not common among neither students nor teachers.

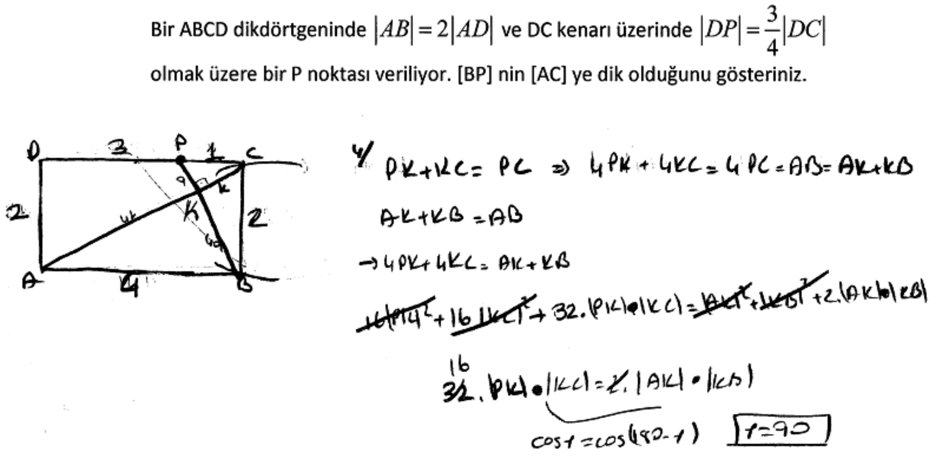


Figure 4-57 Naci's solution to the problem 4 on Rectangle 1st section

The following problem solution is another illustration for the utilization of vector approach as an alternative to congruence of triangles. In the task (Figure 4-58), the students are asked to verify that *"a quadrilateral is a parallelogram if its diagonals bisect each other"*. Despite the fact that SAS equivalence relations for triangles are appropriate to show that the opposite sides of the quadrilaterals are equivalent, Ömer chose vector approach to solve this problem. He questioned the equivalence of the opposite side vectors by means of vector algebra of equal vectors.

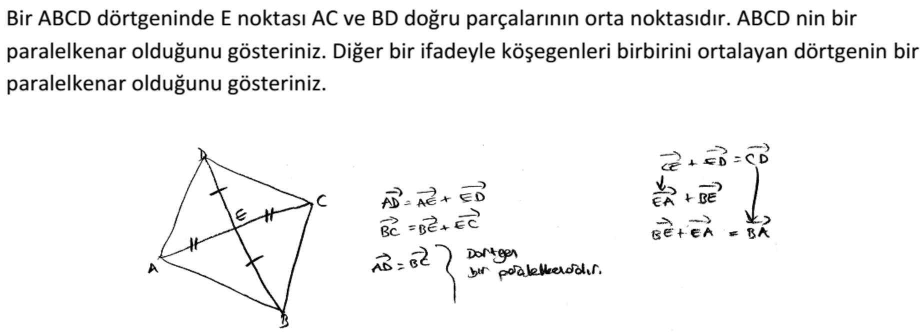


Figure 4-58 Ömer's solution to the problem 2 on Parallelogram 1st section

Finally, Ahmet preferred vector approach instead of applying AAA congruence theorem for the triangles $\triangle ICD$ and $\triangle JCB$ for the problem in the Figure 4-59. In the problem, "*I is the midpoint, the area of square ABCD is given as 100 cm^2 and the* *length of* $[\mathrm{CJ}]$ *is asked*". Ahmet utilized analytic representation of vectors and inner product to find $|BC| = y$. However, he calculated $\boldsymbol{x}$ by means of Pythagorean Theorem.

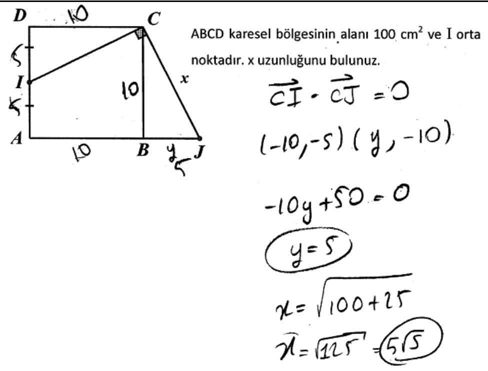


Figure 4-59 Ahmet's solution to problem 4 on Square 2nd section

Related to the solution of the problem in the Figure 4-58, Ahmet expressed that vector approach can be alternative way of solving similarity and congruence problems. It is understood that he was aware of solving the problems in two ways and he is aware of pros and cons of both of the approaches as understood from Excerpt 4-5.

Excerpt 4-5 Excerpt from an interview with Ahmet on 24.07.2013

Researcher: Which approach did you utilize?
Ahmet: Vector approach!
Researcher: Why?
Ahmet: It can be solvable and understandable easily via vector approach.
Researcher: Ok! What would you use if you were to solve it through synthetic approach?
Ahmet: Similarity.
Researcher: Which similarity theorem would be necessary?
Ahmet: SAS!
Researcher: How many times would you need to apply similarity?
Ahmet: Twice.
Researcher: What are you trying to get?

Ahmet: I want to determine corresponding congruent angles and; hence, to show that the given quadrilateral is a parallelogram.

Researcher: You solved the problem through vector approach as a homework yesterday and now you have just explained how to solve the same problem through synthetic approach. Can you compare the approaches in the light of these?

Ahmet: It seems that vector approach is more advantageous to solve this kind of problems. I can solve more practically and in a compact manner through vector approach. You do not need to SAS and proportional relations in vector approach as in synthetic approach. Besides, I can write down whatever I think in vector approach. If I were to solve the problem via similarity and congruence, I had to think about and write down two different pairs of triangles. It would be more challenging. However, it is not like this in vector approach. I can understand what I do with vector approach.

As understood from students' solutions, interviews and the frequencies of their resorting to vector approach, it can be inferred that vector approach can be an alternative way of solution for the problems on similarity and congruence. This inference can also be encountered in some of the studies in the literature. First of all, Lee et al. (2003) utilized vectors and inner product instead of resorting the theorem of congruence to verify that two given triangles are identical. Furthermore, Stephenson (1972) asserts that vector approach is beneficial to teach the proof of Pythagorean Theorem to the students without needing similarity and congruence of triangles. Vaughan and Szabo (1973) proved most of the theorems by means of vectors instead of similarity and congruence theorems in their courses and textbooks based on vector approach while teaching geometry. Specifically, these scholars utilized vector algebra and inner product. Finally, Choquet's (1969) textbook is a nice example to support the idea that vector approach can be utilized as an alternative to similarity and congruence. Choquet (1969) notes that they did not need to apply congruence and similarity at any phase of the development of their course. The underlying reason for this rationale is that they think similarity and congruence as an obstacle while developing geometry via vector approach. In the light of all of the facts above, it could be possible to deduce that vector approach can be an alternative to solve some sort of geometry problems related to similarity and congruence.

4.2.8 Participants' Preferences of Approaches in Solving Problems

The students were asked to solve geometry problems related to quadrilaterals at the end of each section of quadrilaterals. The most important feature of these tasks is that most of them can be solved by means of any of the approaches. The number of problems assigned as an individual task for each participant is presented in Table 4-10.

Table 4-11 shows the frequency of analytic approach, synthetic approach and vector approach as the first preference of the participants while solving these problems. It is important to stress that besides the problems represented in the Table 4-11 were solved by the students correctly, they were also solved by means of the other approaches. In order to understand better, the preferences of approaches as the first choice regarding the participants is also given as graphically in the Figure 4-60, Figure 4-61 and Figure 4-62. Since the number of problems is not the same across the topics, the frequencies are converted to the percentages.

Table 4-10 The number of problems at the end of each subject

Subject	Number of Problems
Quadrilaterals	6
Trapezoid	8
Parallelogram	8
Rectangle	13
Rhombus	11
Square	7
Deltoid	4
Total	**57**

Table 4-11 Participants' preferences of approaches while solving end of chapter problems

	Quadrilaterals (6)			Trapezoid (8)			Parallelogram (8)			Rectangle (13)			Rhombus (11)			Square (7)			Deltoid (4)		
	A	V	S	A	V	S	A	V	S	A	V	S	A	V	S	A	V	S	A	V	S
Naci	0	5	1	1	6	1	0	6	2	0	9	4	0	8	3	0	6	1	0	4	0
Ömer	0	4	2	2	4	2	2	2	4	2	6	5	1	5	5	0	5	2	0	3	1
Ahmet	0	5	1	1	5	2	1	5	2	0	11	2	1	5	5	0	7	0	0	3	1

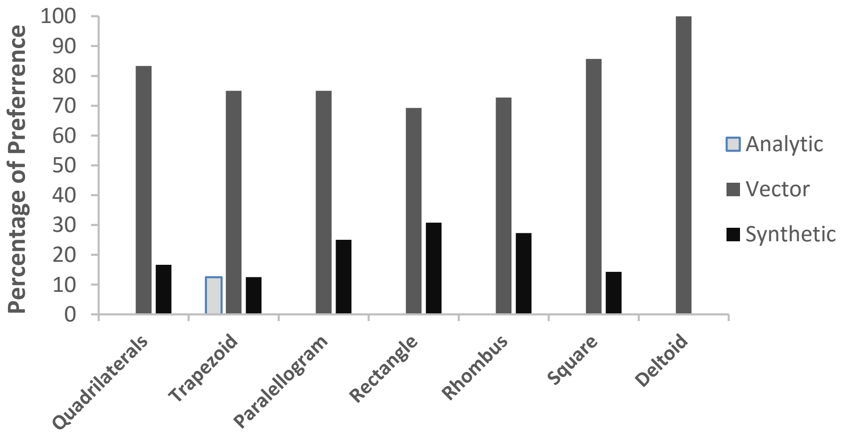


Figure 4-60 Preference of approaches as the first choice by Naci

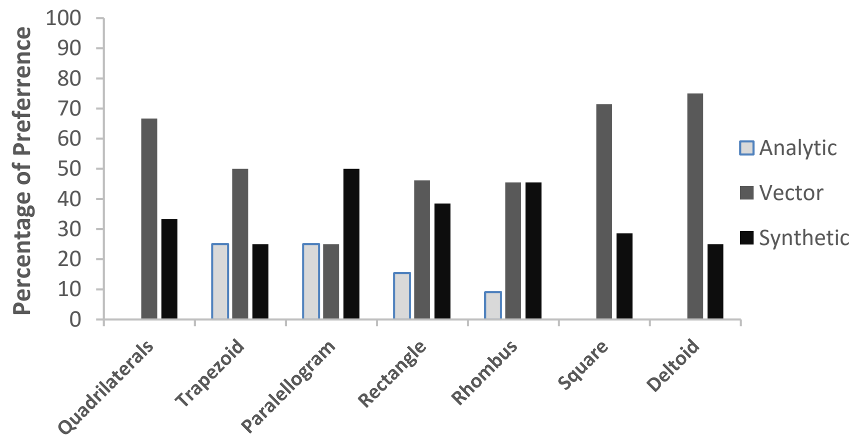


Figure 4-61 Preferences of approaches as the first choice by Ömer

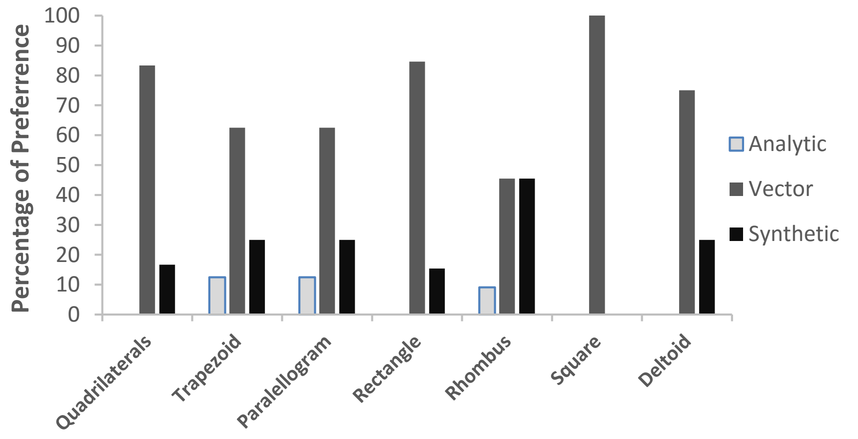


Figure 4-62 Preferences of approaches as the first choice by Ahmet

The different preferences among the approaches for the participants can be explained with the following facts:

a) The participants might be in tendency to prefer the easiest or newest way of solution when they have knowledge or infrastructure to solve a problem through multiple approaches. Indeed, it can be said that newly gained knowledge of conducting operations or solving problems might be a source of enthusiasm.

b) According to the students' products, the participants can be accepted as they are knowledgeable on several approaches because of the instruction followed for this study. Therefore, the participants might deduce that an approach is more suitable or optimal than the other approaches with respect to the problem cases. Similarly, Star and Rittle-Johnson (2008) state that flexible problem solvers know more than one way to complete tasks and they have knowledge on which strategies are more efficient than others under particular circumstances.

c) The participants might think that newly learned methods should be improved through solving more problems. Previously learned methods already have been experienced sufficiently so far.

4.2.9 The Change of Approaches in Solving Problems

Variety of approaches, techniques or methods in students' products depends on their variety of repertoire or knowledge of various approaches. In order to provide this enrichment, the quadrilaterals unit was taught through multiple approaches: analytic, synthetic and vector approaches. Whether the participants could gain or to what extent they could gain this variety and how students' solutions reflect the treatment are important results of the present study. Therefore, participants' solutions were analyzed in this perspective.

To illustrate; in the following two problems, it was clearly distinguished that the problems are similar in terms of their contents. Cosine of the angles $"x \text{ and } \alpha"$ are required to be found respectively in both of the problems related to squares with the given information in the figures. However, two different approaches were observed in the student's solutions. While Naci solved the item by means of the law of cosine as synthetic approach right at the beginnings of the study (14.05.2013), he solved the similar question through the concept of vectors towards the end of the study (21.09.2013). It is also interesting for these two items that the student solved the first problem (Figure 4-63) by synthetic approach although it contains vectorial notations. In contrast, he solved the second problem (Figure 4-64) by vector approach despite the fact that the problem does not have any clue or symbol about vectors. This change can be explained by the result or the effect of the teaching experiment.

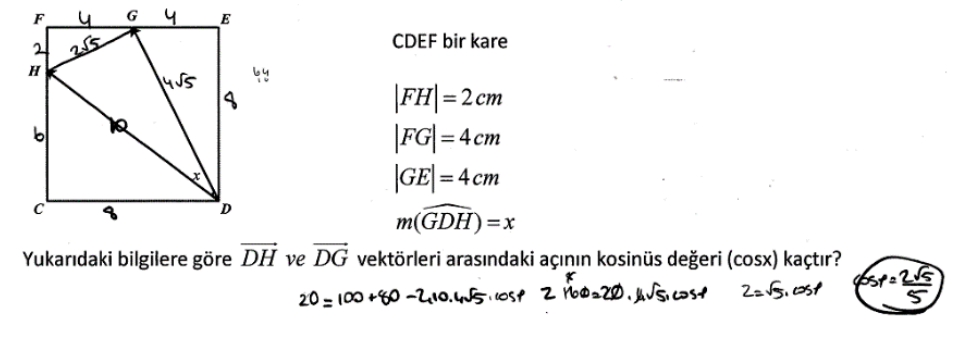


Figure 4-63 Naci's solution to the problem 7 on Vectors 2nd section

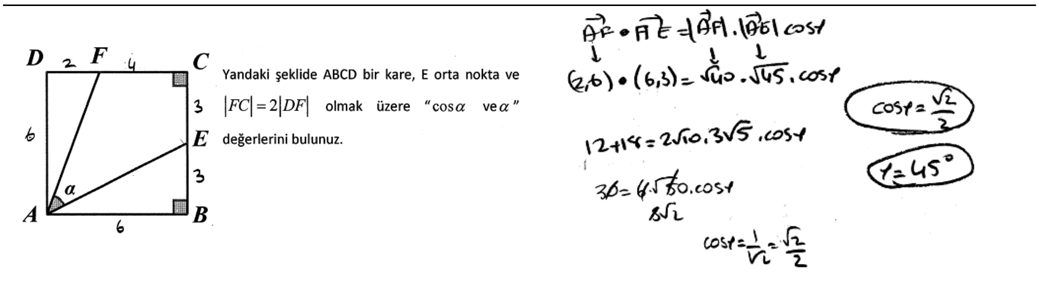


Figure 4-64 Naci's solution to the problem 1 on Square 1st section

In the light of this change, the researcher examined the students' preferences on the instruments administered to the participants at the beginning and at the end of the study. When students' solutions were examined in terms of preferred approaches, the researcher realized that there were some changes in their preferences. The following four tables (Table 4-12, Table 4-13, Table 4-14 and Table 4-15) summarize the shifts in preferred approaches to solve the items in the instruments, which are administered as pre-test and post-test.

Table 4-12 Shifts in preferred approaches on PKQT

Problem	Participant	Change	
		Pre-test	Post-test
2a 2c 21	Naci	Analytic	Vectorial
2a 2c	Ömer	Analytic	Vectorial
21		Analytic	Analytic+Vectorial
2a	Ahmet	Vectorial*	Vectorial
2c		Analytic	Vectorial
21		Analytic	Analytic

* This is the single case in which a student solved a geometry problem via vectors in pre-tests.

Table 4-13 Shifts in preferred approaches on VKT

Problem	Participant	Change	
		Pre-test	Post-test
5	Naci	Synthetic	Vectorial
10		Analytic+Synthetic	Vectorial
11 14 16 17		Analytic	Vectorial
6 11 14 17	Ömer	Analytic	Vectorial

The problem 14 on VKT is an application of vector concept in geometry. It can play an important role for the sake of linking algebra with geometry. Despite the fact that Naci and Ömer resort to analytic approach to solve this problem and Ahmet could not solve the problem at all in the pre-test, all of them could solve the problem via vectors on the post-test. This could be accepted as an evidence showing the deficient and isolated teaching of vectors in schools. In other words, this situation can be stated as an example for inexistence of application of vectors in geometry in schools as found in the study of Rumanova (2006).

Table 4-14 Shifts in preferred approaches on QAT

Problem	**Participant**	**Change**	
		Pre-test	**Post-test**
C3	Naci	Analytic+Vectorial	Vectorial
D1		-	
D2		Synthetic	
D3		Synthetic	
D4		Synthetic	
C11	Ömer	Analytic+Synthetic	Analytic+Synthetic+Vectorial
C15		Synthetic+Vectorial	Analytic+Synthetic+Vectorial
D2		Synthetic	Vectorial
D2	Ahmet	Synthetic	Vectorial
D4			

Table 4-15 Shifts in preferred approaches on PPGT

Problem	Participant	Change	
		Pre-test	Post-test
6	Naci	-	Vectorial
7		-	Synthetic
8		-	Synthetic
11		Synthetic	Vectorial
13		Synthetic	Vectorial
6	Ömer	-	Vectorial
11		-	Vectorial
12		-	Synthetic
13		-	Synthetic
6	Ahmet	Synthetic	Vectorial
7		-	Synthetic
11		-	Vectorial
12		-	Synthetic
13		Synthetic	Vectorial

The problems D1, D2, D3 and D4 on QAT and all of the problems on PPGT are proof-based problems. According to the Table 4-14 and Table 4-15, there has been observed that vector approach was preferred frequently by the students especially for these problems on the post-tests. For the problems asking to verify the correctness of some mathematical statements in the post-test of QAT, the students resorted to vector approach operations according to Table 4-14. This tendency is parallel to the preferences of students on proof based problems, which were asked to participants at the end of each section of quadrilaterals. It can be examined in Table 4-4.

4.2.10 The Topics of the Problems for Which Students' Preferences of Approaches Changed in Solving Problems.

During the course of the study, as students attended the instruction, there were some changes observed in students' problem solving strategies in terms of their preferences of the approaches. The researcher wanted to probe whether these changes could be classified or not. In order to construct a classification, the problems and topics of problems for which there were changes observed in the solutions of participants were determined. While this classification was being constructed, the participants' solutions for the same problem were compared individually on the pre-tests and post-tests. At the end of this analysis, the following Table 4-16 and Figure 4-65 were obtained.

In the Table 4-16, one can see the changes from one approach to another approach or combination of approaches in terms of subjects of the problems. The problem, and which test it belongs, the preferred approaches in pre-test and post-test and the subject of the problems were shown in this table.

The changes from one approach to another approach within the contexts of subject and sub-topics are also presented in the Figure 4-65. In this figure, the ellipse represent the changes among the approaches. In addition, while the rectangles are representing the subjects, the rounded rectangles are representing the sub-topics under these subjects, in the Figure 4-65. Besides, to better understand what is aimed with the figure, two explanations will be presented. Firstly, a change from synthetic approach to vectorial approach was observed for the sub-topics: "*Medians of polygons and Quadrilaterals with perpendicular diagonals*" in the "*Quadrilaterals*" subject. Another result to be inferred from the Figure 4-65 is that while participants were proving Pythagorean Theorem in Triangles subject through synthetic approach, they started to prefer vectorial approach to prove this theorem. Secondly, it can be understood by this figure that while the students were calculating the distance and slope values in plane analytic geometry subject through analytic approach, they started to calculate these values through vectorial approach. Similar conclusions can be inferred with the help of this figure.

Table 4-16 The change of approaches with respect to the topics

Change				
Pre-test	**Post-test**	**Instrument**	**Problem**	**Topics**
Analytic	Vectorial	PKQT	2a	The distance between two points.
Analytic	Vectorial	PKQT	2c	Finding the slope of a line passing through two points.
Analytic	Vectorial	PKQT	21	The distance from the point to the straight line
Analytic	Vectorial	VKT	6	The distance between two points.
Analytic	Vectorial	VKT	11	The angle between two vectors whose end-point coordinates are given.
Analytic	Vectorial	VKT	14	Area of polygons on coordinate plane.
Analytic	Vectorial	VKT	16	Inner product of two vectors, which are given on a triangle or quadrilateral.
Analytic	Vectorial	VKT	17	Orthogonal vectors which are given analytically.
Synthetic	Vectorial	VKT	5	Finding midsegments (medians) of quadrilaterals.
Analytic+ Synthetic	Vectorial	VKT	10	Inner Product.
Synthetic	Vectorial	PPGT	6	The Law of Cosines.
Synthetic	Vectorial	PPGT	11	Triangle proportionality theorem.
Synthetic	Vectorial	PPGT	13	Pythagorean Theorem.
Synthetic	Vectorial	QAT	D1	Quadrilaterals whose midpoints of consecutive sides are joined.
Synthetic	Vectorial	QAT	D2	Properties of quadrilaterals that have perpendicular diagonals.
Synthetic	Vectorial	QAT	D3	Quadrilaterals that have pair(s) of parallel sides
Synthetic	Vectorial	QAT	D4	
Analytic+ Vectorial	Vectorial	QAT	C3	The area of polygons whose vertices are given by their Cartesian coordinates.
Analytic+ Synthetic	Analytic + Synthetic+ Vectorial	QAT	C11	Area of quadrilaterals polygons with perpendicularly intersecting diagonals.
Synthetic+ Vectorial		QAT	C15	Finding the area of quadrilaterals whose side or/and diagonal vectors are given.

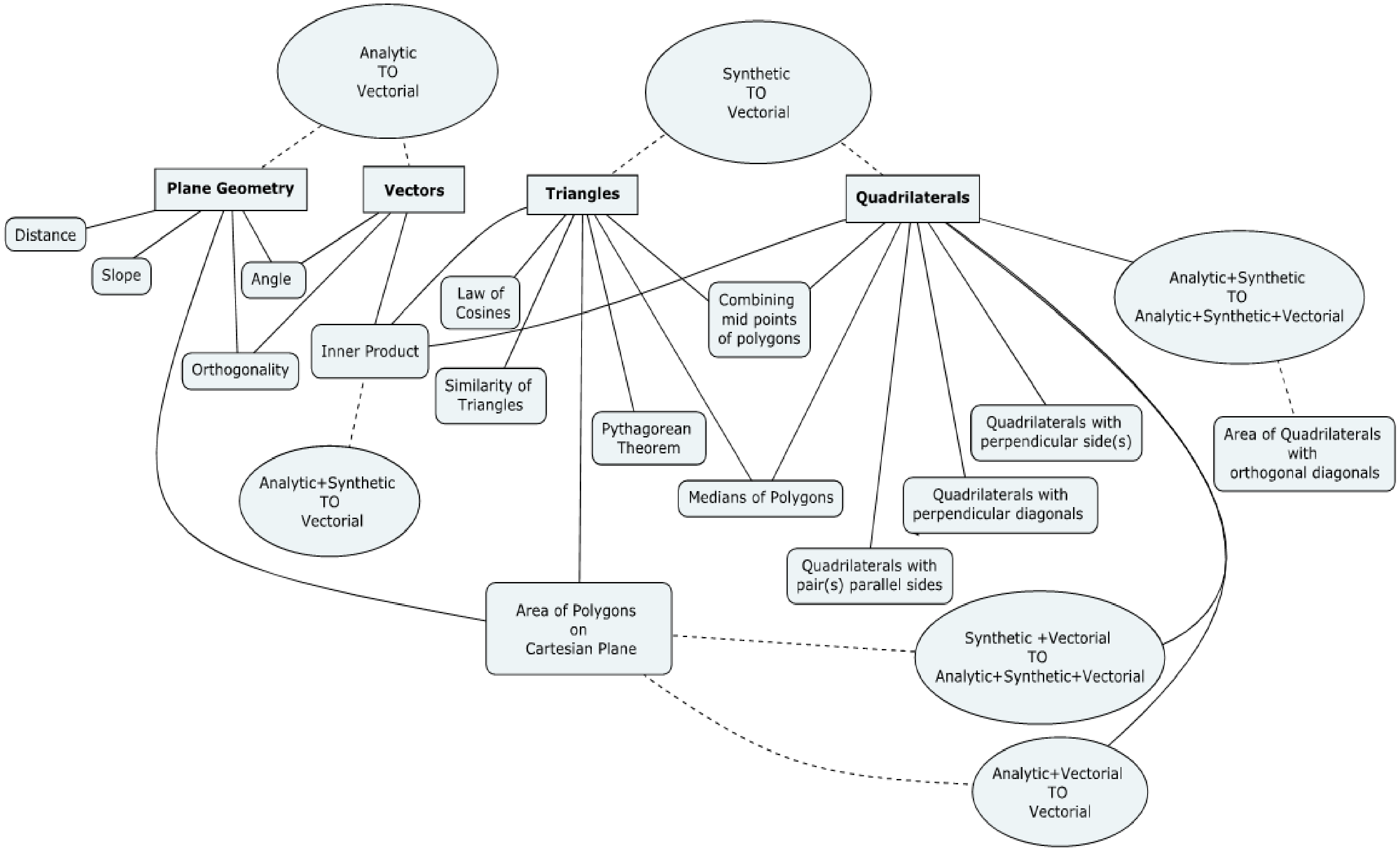


Figure 4-65 The change of approaches with respect to subjects with subtitles

4.3 What are major components of the instruction in which vectorial approach is integrated with synthetic and analytic approaches on quadrilaterals for the grade level 11?

The components of the instruction followed in this study were presented in detail in the Methodology chapter. While preparing this instruction, the boundary was geometry curriculum program for 11th grade level (MoNE, 2010b). Furthermore, the researcher experienced that some special tools such as displacement analogy, shadow analogy and literal manipulations were beneficial to conduct operations for this study. The students' products reflected the effects of these tools. In this part of the dissertation, these effects will be presented through students' products.

4.3.1 Use of Analogies

Shadow Analogy

As stated in the "Elementary Vector Algebra Instructional Module", *shadow analogy* was utilized while teaching the right projection of a vector onto another vector. It was observed that students utilized this analogy in their solutions for different geometry problems. To illustrate; Ömer drew the following picture to depict right projection of $\vec{A}=(4,3)$ onto $\vec{B}=(7,0)$ while solving a problem in the Figure 4-66.

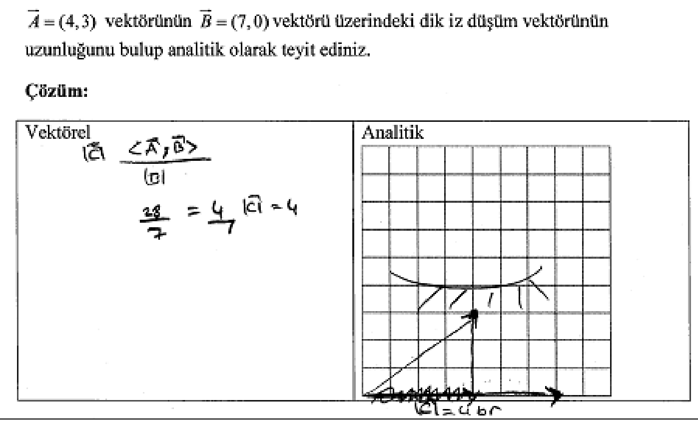


Figure 4-66 Ömer's solution to an exercise on Vectors 2nd section

This is important in order to conceptualize vector projections for the sake of the participants. He did not apply the formula directly. Instead, he prepared the configuration with the help of coordinate plane worksheet and shadow analogy. Appova and Berezovski (2013) found that there were students' misconceptions on vector projections. Moreover, the researcher experienced that applying the formula of vector projection is a source of difficulty and error. Since the participants could be able to overcome these troubles by the help of shadow analogy, it can be taught to the students in geometry instruction, which includes vectorial approach. Moreover, Tabaghi (2010) found that visualization is an efficient way to overcome misconceptions and difficulties of students on some abstract topics of linear algebra such as vector projection.

Displacement Analogy

It is stated earlier that "*displacement vector*" analogy was developed by the researcher for this study. This analogy was utilized while teaching algebra of vectors (*especially for vector addition*) and vector proof of geometric statements.

It was frequently observed in students' solutions that the participant students made use of this analogy for different purposes spontaneously such as in solving problems related to the vector algebra, in proving some geometrical statements or to determine the order of letters correctly in some of the geometry formulas. These will be presented in detail with students' solutions.

Firstly, students made use of displacement idea while solving the following problem in the Figure 4-67. In this problem, it is required to show that "*the mid-segment combining two sides of a triangle is parallel to the third side of the triangle and the length of the mid-segment is half of the length of this third side*".

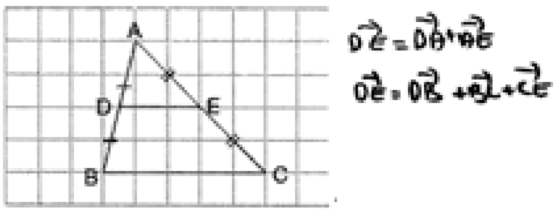
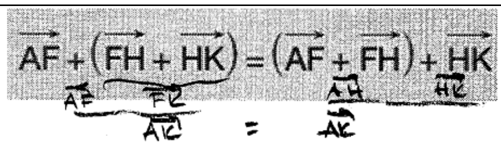


Figure 4-67 Naci’s solution to the problem 15 on Vectors 2nd section

Secondly, the use of *displacement vector analogy* was encountered in proving some geometric statements. As an example, all of the participants used this analogy in proving “*associative property of vector addition*” as seen in the following work in Figure 4-68. It should be noted that this task was assigned as a homework for the students. The second solution is another example of utilizing displacement analogy. Therefore, the participants might be inferred that they could conceptualize this analogy not only for vectors, which are given geometrically but also for vectors, which are given algebraically as seen in the following solution.



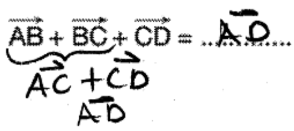


Figure 4-68 Ömer’s and Ahmet’s solutions to the exercises 3 and 5 on Vectors 1st section respectively

The use of displacement analogy was also observed in the pre-tests and post-tests. The frequency of the use of displacement analogy in the pre-tests and post-tests was presented in the Table 4-17. According to this table, the participants utilized this analogy 25 times in total for the items in pre-tests and post-test. Moreover, the participants had not used this analogy before the instruction. This fact can be said clearly because of the fact that the pre-test for QAT was administered after instructions for vectors, triangles and plane geometry topic. Indeed, they learned this analogy before the administration of QAT during the preliminary courses for the main units. Furthermore, the use of this analogy was not observed in the pre-tests of the instruments VKT, PPGT and QAT.

Table 4-17 The frequency of displacement analogy in pre-tests and post-tests

Participant	VKT		PPGT		QAT	
	Pre	**Post**	**Pre**	**Post**	**Pre**	**Post**
Naci	0	2	0	2	1	5
Ömer	0	2	0	2	0	1
Ahmet	0	2	0	3	2	3

Two of 25 use of this analogy in the "instruments" are presented below (Figure 4-69 & Figure 4-70).

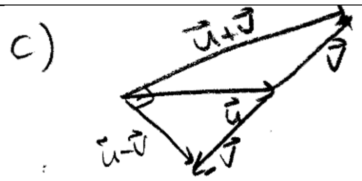


Figure 4-69 Ahmet's solution to the problem 14 on VKT post-test

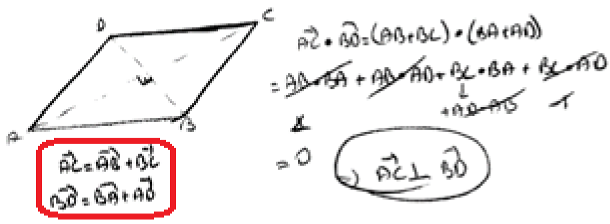


Figure 4-70 Naci's solution to the problem C2 on QAT post-test

Participants very often resorted to displacement analogy in their vector approach solutions to the problems asked at the end of each sections. The frequency of utilizing displacement analogy by topics is presented in Table 4-18. According to this table, it can be inferred that, displacement analogy was an important part of vector approach solutions hence vectorial approach instruction. While assessing this table, the frequency of students' non-vector solution should be taken into consideration. According to Table 4-20, in the solutions of 57 problems, the frequencies of resorting vector approach as the first choice were 44, 29 and 41 for Naci, Ömer and Ahmet respectively. Hence, the ratio of the frequency of use of displacement analogy to the frequency of vector approach solution is $\frac{24}{44}$ for Naci, $\frac{13}{29}$ for Ömer and $\frac{12}{41}$ for Ahmet according to Table 4-18 and Table 4-20. These frequencies cannot be ignored.

Table 4-18 The frequency of utilizing displacement analogy in problems at the end of the subjects

Subject	**Naci**	**Ömer**	**Ahmet**
Quadrilaterals	3	3	3
Trapezoid	3	1	0
Parallelogram	5	4	4
Rectangle	4	0	1
Rhombus	5	1	1
Square	3	3	3
Deltoid	1	1	0
Total	**24**	**13**	**12**

The reason why this analogy was used very often can be explained with Naci's interpretation in Excerpt 4-6.

Excerpt 4-6 Underlying reason for the preference of displacement analogy by Naci

> *...Especially the logic "a student moved from A to B and then from B to C" is an easy and understandable logic in terms of student. It is the logic similar to "going from school to home, then from home to grocer"....*

Besides practical aspects of displacement analogy, it is also reasonable in terms of having a meaning in students' daily life. In other words, displacement analogy has a counterpart in daily life, which is important in geometry teaching (MoNE, 2010b).

Thirdly and more indirectly, although the researcher used this analogy for the mentioned purposes, it was observed that the participants themselves adapted vector displacement analogy for remembering the direction and positions of elements written in the formulas for Ceva’s, Menelaus’ and Carnot’s theorems at different points of this teaching experiment. This adaptation or similarity might be evaluated as a good and satisfactory development in terms of students.

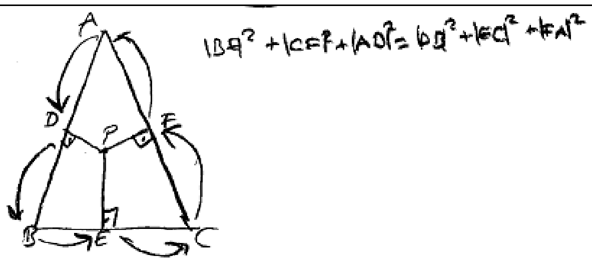


Figure 4-71 Ömer’s drawing related to Carnot’s Theorem

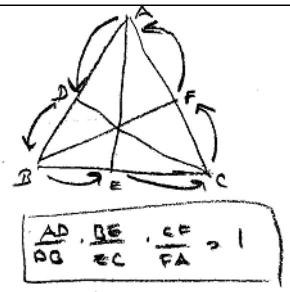


Figure 4-72 Ömer’s drawing related to Ceva’s theorem

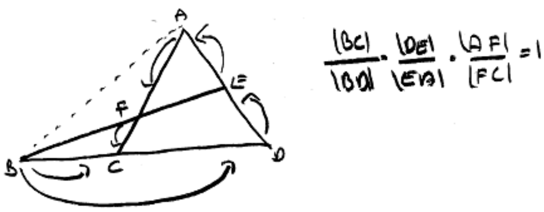


Figure 4-73 Naci’s drawing related to Menelaus’ theorem

Furthermore, the participants expressed that these theorems were taught them in the format of remembering the formulas and then applying the formulas on numerical examples directly at grade level 10. However, they said that they forgot these formulas even after short time periods. Specifically, they expressed that they have difficulties in remembering the formulas correctly or completely forget them at the end of the semester. Ahmet stated that

"*when the semester ends our works with formulas end*".

The researcher subsequently found in the literature that the displacement analogy was utilized as "*relation de Chasles'* " which can be translated into English as "*Chasles' Relation*", as Ritzenthaler (2004) reported. Athen (1966a) used two alternative names for this operation as "*Detour Rule*" and "*Vector chain*".

Dimitriadou et al., (2011) found that students are more successful with triangle method in comparison with parallelogram method while adding vectors. Since triangle method is very close to displacement analogy, the preference of the analogy seems appropriate.

Poynter and Tall (2005a) recommend the use of journey idea for vector addition in order to enhance the conceptualization of this operation in terms of students. This idea is similar to the displacement logic. Furthermore, Poynter and Tall (2005a) assert that vector addition can be encapsulated by the students when it is taught through displacement idea according to the theories of embodiment and APOS. In addition, Poynter and Tall (2005a) observed a significant improvement for their students in their experimental study in which they utilized displacement analogy. Barniol and Zavala (2010) found that the students are more successful in solving the problems on vector addition based on displacement context in comparison with the problems based on force context or non-contextual problem. They explained the difference in success with the fact that the students are more familiar to displacement context. The familiarity of displacement analogy is also reported in the study of Nguyen and Meltzer (2003) so that the students would not have trouble in vector addition. Moreover, Hawkins, Thompson, Wittmann, Sayre and Frank (2010) state that the students are more likely prefer head-to-tail method than tail-to-tail method while seeking for resultant vector in a vector addition. Head-to-tail method is similar to displacement idea. Watson, Spyrou and Tall (2002) assert that associating vector with journey in vector addition results in the use of triangle method. This is very similar to the students' products for vector addition that participants frequently utilized displacement in the addition of vectors. However, parallelogram method is also taught to the students for the addition of vectors. While teaching this method, the teachers should be careful that this is not a source of conflict. Specifically the teachers should be aware that the students might related triangle method with mathematics and parallelogram method with physics as if vector addition is different for these two majors.

In the light of these findings, while designing a geometry teaching through multiple approaches one of which is vector approach, displacement analogy is beneficial for the sake of the students for many aspects such as familiarity to the students, conceptualization or embodiment of vector addition and convenience to the students.

Vector Subtraction

It is presented in the previous section that while teaching vector addition, the researcher utilized displacement analogy. This analogy was used so many times by the participants. However, vector subtraction is another issue that the participants have troubles. In order to utilize displacement analogy in subtraction of vectors and to resolve students' difficulties related to subtraction of vectors, this operation was changed to vector addition by reversing the order of initial and terminal points of the second vector. The participants' solutions reflected the traces of this way of operation. In this way, they could conceptualize this rationale to some extent. To illustrate; Ahmet could solve the following tasks as they were taught in this teaching experiment (Figure 4-74) and he also utilized "*the change of vector addition to vector subtraction*" and "*displacement analogy*" together in the Figure 4-69.

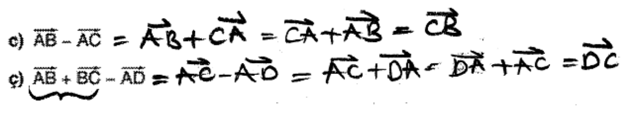


Figure 4-74 Ahmet's solution to exercise 3 on Vectors

For the following task, the participants preferred shifting the vector subtraction to the vector addition to obtain $2\vec{u} - \vec{v}$ from the given vectors " $\vec{u}$ *and* $\vec{v}$ " in the problem. The work of Ahmet is presented in the Figure 4-75.

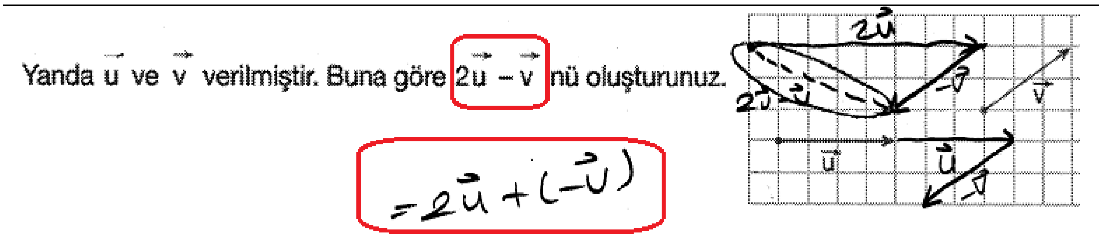


Figure 4-75 Ahmet’s solution to the problem 12 on Vectors 1st section

It can be concluded that use of displacement analogy possibly made vector subtraction almost unnecessary in solving problems. Indeed, the participants used displacement analogy nearly for all of the problems for which it was possible to utilize vector subtraction. This was realized by means of selecting appropriate direction of vectors. To illustrate, participants conducted vector addition to reach a final destination from a starting point via two different routes. This was observed in their solutions to prove Pythagorean Theorem and the law of cosine. Although most of the geometry textbooks utilized vector subtraction to prove these theorems, the participants made use of displacement analogy instead. This situation can be examined in the following solution of the problem asking to write and prove Pythagorean Theorem in the Figure 4-76. Three of the participants proved Pythagorean Theorem and the law of cosine by utilizing displacement analogy in their solutions instead of using vector subtraction.

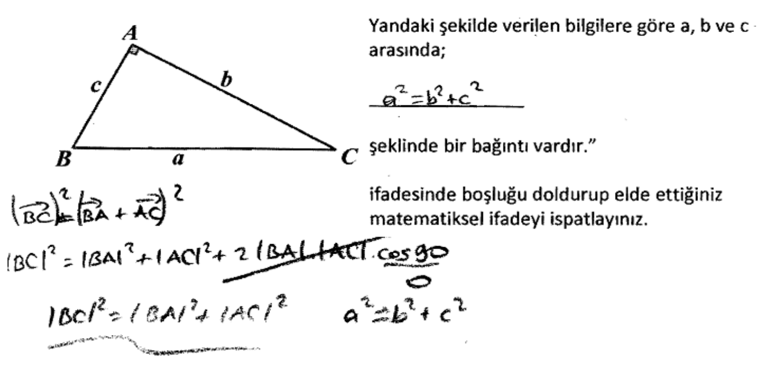


Figure 4-76 Ömer’s solution to the problem 11 on PPGT post-test

As can be encountered many geometry textbooks and studies (Chiba, 1966; Rosenberg, 1967; DiFonzo, 2010 and Schuster, 1961), vector subtraction is one of the essential components of vector algebra or geometry teaching through vector approach.

In addition, Athen (1966a) specifies vector addition and subtraction as a way to integrate algebra and geometry if these operations for real numbers are taught in vector context on number line. However, as stated in the literature chapter, there are some studies (e.g., Küçüközer, 2009; Van Deventer & Wittmann, 2007; Appova & Berezovski, 2013; Nguyen & Meltzer, 2003; Flores et al., 2004) reporting students' difficulties or misconceptions with vector subtraction. Nguyen and Meltzer (2003) explain the reason for this difficulty with students' memorization of the place of the tail of one vector to the tip of the other vector. Some of the other researchers (e.g., Kustusch, 2011; Aguirre & Erickson, 1984; Flores et al., 2004; Gagatsis & Demetriadou, 2001; Van Deventer & Wittmann, 2007; Nguyen & Meltzer, 2003 and Gagatsis & Bagni, 2000) connect students' troubles with vector subtraction as not taking the direction of vectors into consideration or treating a vector as a scalar. Because of these difficulties, it is advised in the curriculum of secondary school mathematics (MoNE, 1992) that the teaching of vector subtraction should be presented with geometric counterparts or interpretations as an application. Ayre (1965) recommends the teaching of vector subtraction to be shifted to the vector addition. He states underlying reasons as

> "An emphasis on subtraction of vectors defined in terms of addition should be made. This should be done not only for purely algebraic reasons, but also to simplify finding the difference of two vectors in a vector diagram" (p. 86).

In brief, changing vector subtraction operation to the vector addition might be a precaution to overcome mentioned problems. Therefore, it would be beneficial to teach vector subtraction by this was especially for geometry teaching through vector approach.

4.3.2 The Development of Use of Literal Manipulations in Numerical Expressions

As the researcher predicted the necessity of the use of literal manipulations in numerical expressions, the students utilized these manipulations in computing the area of quadrilaterals that are given on analytic coordinate plane. This necessity was expressed thoroughly in the "Methodology" chapter under the title "Basic Algebra Instructional Module".

An illustration can be seen in Naci's solution in the Figure 4-77. The area of ABCD parallelogram with $\overrightarrow{AC} = (7,8)$ and $\overrightarrow{DC} = (6,2)$ is asked in this problem. One of the reasons for these manipulations was not to struggle with great numbers. A solution to this challenge can be thought as factoring out the greatest common terms or factors as implemented in the framed part in his solution.

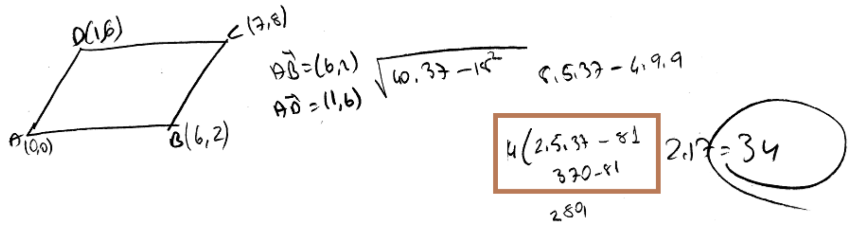


Figure 4-77 Naci's solution to an exercise on Parallelogram 2nd section

This requirement can be evaluated as very specific to computing the area of polygons, which are given on Cartesian plane. However, it takes a considerable place in geometry curriculum for 11th grade (MoNE, 2010b). Therefore, it is a way to overcome waste of time due to engaging with large numbers. Therefore, the students should learn how to manipulate with numbers in the context of factoring literal expressions in a geometry-teaching course including vector approach strategies and the use of vectorial formulas according to the experiences of the researcher.

4.3.3 Time Allocated for the Subject Matters

It is stated in the literature chapter that some of the teachers (Aktaş & Aktaş, 2012) and students (Baki & Akşan, 2014b) think that teaching or learning geometry through vector approach in addition to synthetic approach is waste of time. Even if these researchers are right in their assertions, allocated time for teaching geometry through vector approach might be disregarded when the advantages and gains of the students through this approach are considered. Moreover, the situation was different in this study. Specifically, it was noticed that teaching the initial subjects; specifically the first three subjects (i.e. *quadrilaterals, trapezoid and parallelogram*), took longer time in comparison with the teaching of subsequent subjects. As the participants conceptualized the approaches and became familiar to the content and the context followed throughout the study, it was observed and experienced that allocated time and expended effort for teaching the remaining subjects decreased in spite of the fact that the subjects changed and the number and the type of problems and properties increased. The time allocated for teaching each of quadrilateral in the study are given in the Figure 4-78. In fact, teaching the last two special quadrilaterals (*square and deltoid*) was completed in a shorter time considerably in comparison with teaching the earlier subjects. The students themselves could complete fill in the blanks type lecturing parts. Besides, they could pass directly to solving problems for these topics independently from the teacher-researcher.

The finding is actually compatible with the findings of Copeland (1962), Bundrick (1968) and Hershberger (1971). Specifically, Hershberger (1971) states that teaching analytic geometry through vectors is 15 % more economical in time in comparison with teaching analytic geometry via traditional approach. Bundrick (1968) and Hershberger (1971) state that the average time necessary to study the vector approach and traditional approach treatments was approximately equal. Indeed, the time allocated for the treatment given to the vector approach group was recorded less than traditional approach group although it was not reported as significant. Besides, Copeland (1962) asserts that the amount of time saved by utilizing vectors is sufficient for developing the necessary vector algebra. On the other hand, Krech (1968) expresses that synthetic approach has a timing advantage because of the fact that there exists the necessity of providing some prerequisite knowledge to teach geometry through vector approach, which means extra time allocation. The researcher is not completely wrong in their assertion; however, Krech (1968) forgets considering allocated time to teach necessary prerequisite knowledge to teach geometry via synthetic approach. Moreover, it is not reasonable to exclude vector approach in geometry teaching because of considering timing issue merely. Instead, some of the practical methods or technological tools can be utilized to provide necessary background related to vector approach. To illustrate, Çataloğlu (2006) found that one of the software applets “FOSS simulations” could be helpful for the sake of students to understand vector topics and to shorten the time necessary for learning these concepts. In addition, McCusker, Ma and Caserta (2014) utilized MATLAB application to develop students’ performances on trigonometry of vector problem solving. Besides, there can be many tools in several studies (e.g., Nishizawa & Yoshioka, 2008; Tsegaye et al., 2010; Nishizawa et al., 2009) to realize these requirements.

From different viewpoint, vector is an indispensable tool for physics course. Therefore, providing vector-based background to the students is also beneficial for physics courses. In fact, Aksu (1985) underlines the importance of synchrony of time and consistency of the contents while teaching vectors in physics and mathematics. However, Szabo (1966) expresses that school algebra and geometry are mostly being treated independently.

Moreover, Bundrick (1968), Grant (1971) and Hershberger (1971) state that vectors has a unifying character. That is, the students might have a chance to transfer their knowledge on 2D to 3D. Specifically, the distance of a point to a line on 2D and to a plane in 3D are very similar when the operations are conducted through vector approach. However, this was not the case in traditional approach. Bundrick (1968) found that the students in vector approach group performed significantly better (.05) than the students in traditional approach group on the “transfer test”. The transfer test was administered to examine whether there would exist any difference between control group and experimental group for the solid analytic geometry topics, which were not lectured for both of the groups. Hence, Bundrick (1968) specifies that lots of concepts in solid analytic geometry or in space geometry can be taught more sufficiently by means of vectors. These are all possible ways to save of the time and to gain the advantages of vector approach while teaching geometry via vectors.

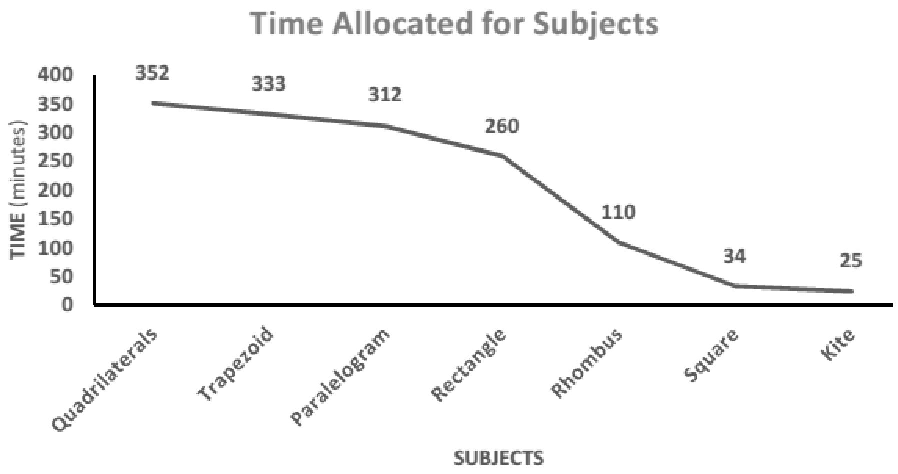


Figure 4-78 Time allocated for the subjects

4.3.4 Feeling in Need to Work with Proof-based Problems

It was observed that students find proof-based problems more interesting and valuable in comparison with routine problems whose solutions require routine or certain algorithms. They also expressed their desires to the researcher to allocate more time to work on this kind of problems. They stated that they could be more concentrated on working geometry with these tasks. According to them, solutions for these problems are unusual and necessitate some tricks or subtle actions. As an example, after learning Stewart's theorem, utilizing this theorem to prove the angle bisector theorem and conducting necessary algebraic manipulations such as adding or subtracting terms, expressing some terms as a function of other quantities, factorization by means of common factors etc. made them happy and provide a satisfaction for them. These manipulations were not familiar to them.

After a certain stage in this teaching experiment while dealing with this kind of tasks, the participant students accepted and expressed themselves as not having learnt or done mathematics in their mathematics lessons so far. They stated that they had memorized certain algorithms to solve certain types of problems and they were not involved in mathematical thinking processes sufficiently. These are inferences from the observation of students in teaching episodes and interviews conducted with them, such as Excerpt 4-7.

Excerpt 4-7 Ömer's opinions about regular geometry courses in his school

> *11. 09.2013*
> *I think that the way of our geometry teachers have been following in teaching geometry is unreasonable. Because, they only give formulas and keep going on solving similar type of numerical examples. Now that, I want to learn underlying reasons for geometrical statements. In addition, I think that studying with proof-based problems and solving this kind of problems are more important.*

Specifically, despite the fact that they applied and used Euclidean relations so many times in problem solving, they were not taught the proof of Euclidean relations at all, as they indicated. Moreover, they stated that they had not utilized these relations to prove Pythagorean Theorem as an application of these relations. Even, they did not have any idea in this direction. Therefore, the situation was accepted as very natural. They stated that if they could have instruction including proof-based problems, they would utilize theorems or geometrical propositions to solve these problems or to prove some mathematical statements and theorems, at least for the most familiar ones. Naci stated his opinions in this direction in the Excerpt 4-8.

Excerpt 4-8 Naci's desires about the need for learning proofs in geometry

> *11.09.2013*
> *While the teacher gave a formula related to the current subject of the day, I questioned the underlying reason for this formula and requested to learn proof of the formula. The teacher tried to explain the reasons for the formula. After my learning desire of proofs of formulae or theorems, my mathematics teacher in private tutorial institution started to teach formulae with underlying reasons from then. This is because of my curiosity and attitude. He was looking into my eyes while he was presenting subjects in this manner.*

Thus, it can be concluded that the questions and demands of the students have the potential to change their teachers' teaching styles and methods accordingly. It could be explained with the expression, "*Liquids take the shape of the container they are in*".

Parallel to the way of instruction followed throughout this study, it was observed that the students could utilize and retrieve earlier proofs in order to solve proof-based problems or to justify the statements. In the Figure 4-79, the students were required to show that "*two line segments connecting midpoints of two non-adjacent sides of a quadrilateral bisect each other*". In order to verify this statement, the student used formerly proved statement. In his solution, Naci stated that a quadrilateral, which is constructed through combining four midpoints of the four sides of another quadrilateral, is a parallelogram. After that, he accepted given two line segments as the diagonals of the parallelogram. Finally, he concluded that these two line segments bisect each other because of the fact that the diagonals bisect each other in a parallelogram. This development can be observed in the other students' solutions as well (such as N142)

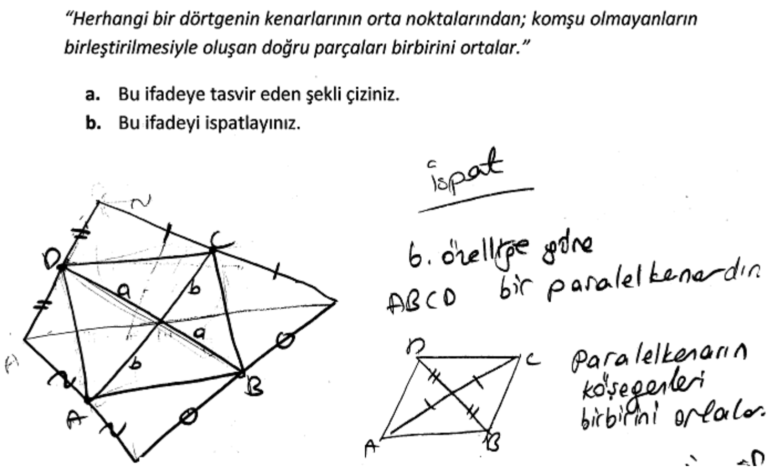


Figure 4-79 Ahmet's solution to problem 1 on Parallelogram 1st section

Another illustration that the participants could be able to model previous proofs in order to solve proof-based geometry problems and to prove, justify or falsify geometric expressions is presented in the following solution (Figure 4-80). Naci utilized the proof of a different theorem in order to show that *"the area of shaded* *region is the half of the area of ABCD trapezoid where E is the midpoint”* in the problem in the Figure 4-80. In fact, while solving this problem, Naci utilized the proof of AA triangle similarity theorem, which is known as butterfly similarity among the students.

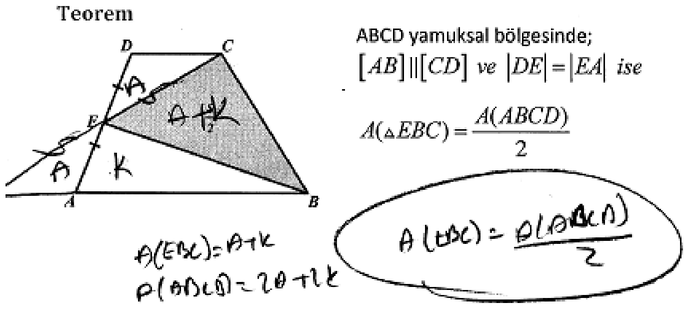


Figure 4-80 Naci’s proof of a theorem on Trapezoid 2nd section

Weber (2003) states that students have difficulties when rewriting the proof of theorems in their textbooks and proving simple statements in Euclidean geometry. Furthermore, Healy and Hoyles (1998) and Senk (1985) report students’ difficulties with producing the proofs. Despite of these difficulties, the participants in this study demonstrated improvements in understanding and developing proofs in geometry. Most importantly, the developed a positive attitude toward to engaging with proofs. As a result, it can be concluded that the students can understand the proofs from their teachers or from books as long as the teachers prove mathematical expressions without any prejudgment like “*the students don’t understand proofs*”. Ultimately, each of students can utilize from teachers or textbooks according to their capacities. They can also develop their own proofs or justifications when they are treated in accordance with instruction that include effective ways of proving. Therefore, it would be meaningless to think that the students cannot understand proofs or proving is difficult for all of the students. Hence, excluding proofs from geometry or mathematics teaching would not be appropriate or reasonable choice. In addition, the participants stated their desire to learn geometry, which is enhanced with proof-based problems and multiple approaches. Therefore; when it is planned to design a geometry teaching through analytic, synthetic and vector approaches, it is important to enhance the teaching with reasoning and proving. Naturally, proof-based examples should necessarily take a considerable place in addition to routine problems or questions in this planned curriculum.

4.4 What are the eleventh grade students' reflections on analytic, synthetic and vector approach instruction on quadrilaterals for the grade level 11?

In this study, the students were taught geometry via an instruction that they were not familiar with. In fact, utilizing multiple approaches in teaching process was a novelty in terms of the students. However, the strangest aspect of the instruction was the inclusion of vector approach. Therefore, as the fourth research question for the study, it was important to learn; at firsthand, students' reflections or reactions to the geometry learning through vector approach in addition to synthetic and analytic approaches. According to participants' reflections, the advantages and disadvantages of vector approach and synthetic approach strategies were determined from students' views.

4.4.1 Participants' Reflections on Approaches

The researcher asked students to compare approaches at various times in the study in terms of difficulties, easiness, advantages and disadvantages of the approaches especially when the problems required to be solved by two or more approaches. In these comparisons, the participants frequently reported that solution through the use of vectors is more elegant, easier to understand and easier to explain to someone else, and more mathematical with respect to synthetic approach solutions.

In the following excerpts and solutions, students' reflections and comparisons are presented.

1) It is observed that the students wrote down some positive expressions such as "*wonderful, very important"* or added *"five stars"* etc. (*framed part in the* Figure 4-81) especially to the solutions conducted with vector approach at various instants of teaching sessions. In the problem, students are asked to

"*calculate the inner product of the diagonal vectors of a rhombus*". Naci marked five stars for the solution of the problem.

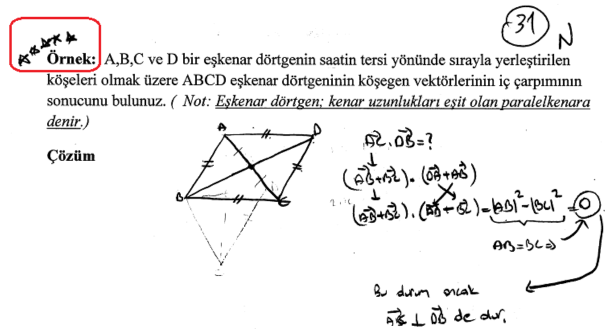


Figure 4-81 Naci's solution to an exercise on Vectors 2nd section

2) The participants stated that they would rather like solving some sort of problems via vector approach. Moreover, in terms of them, solving geometry problems by means vector methods makes them happy in comparison with solving the problems with traditional ways merely. While solving the problem in the Figure 4-82, Naci expressed that

   "*I solved through vector approach. I like it more!*"

   The problem requires calculating "*the area of a quadrilateral whose coordinates of vertices are given*". Naci stated his getting more pleasure with vector approach solutions in different solutions (N110) and at two different interviews (08.07.2013). Ahmet also stated that he likes vector approach for problem solving on two different instants of the study (08.07.2013 and 10.05.2013)

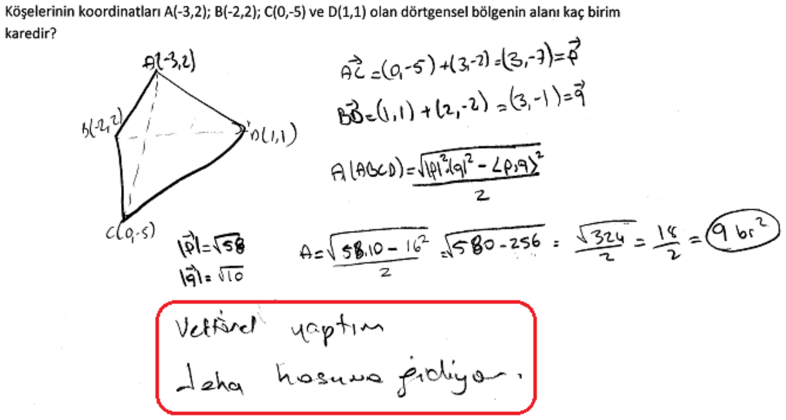


Figure 4-82 Naci's solution to the problem 3 on Quadrilaterals 2nd section

3) The participants are asked "*to show* $a^2+b^2=c^2+d^2$ *for ABCD quadrilateral*" in the problem in the Figure 4-83. Related to the participant's solution given to the following problem, an interview was conducted with Ahmet (Excerpt 4-9):

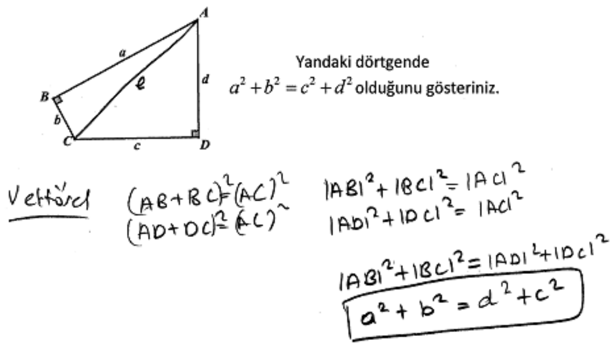


Figure 4-83 Ahmet's solution to problem 2 on Quadrilaterals 2nd section

Excerpt 4-9 Excerpt from an interview with Ahmet on 08.07.2013

08.07.2013

Researcher: Which approach did you utilize to solve the problem 2?

Ahmet: I have solved the problem by adapting vector proof of Pythagorean Theorem.

Researcher: Why did not you utilize synthetic approach?

Ahmet: Actually, there is not much difference between them.

Researcher: Then why did not you prefer synthetic as the first way?
Ahmet: Solving through synthetic approach is <u>*routine*</u> *in my opinion. It seems ordinary. However, when I solve through vectors I feel happy. Since I proved Pythagorean Theorem via vectors by myself, solving this problem in this manner makes me happy. I have already solved this problem via synthetic approach; however, I wrote down it by vector approach.*

4) Ahmet states that geometry teaching through vector approach is more enjoyable in the Excerpt 4-10.

Excerpt 4-10 Ahmet' thoughts about the vector approach solutions on 26.08.2013

I noticed that following the lessons and solving problems through classical way i.e. with synthetic approach got boring me. Now that, teaching only via synthetic approach became ordinary in my opinion. On the contrary, teaching subjects via vector approach seems more appealing, understandable, elegant and innovative from my point of view. Sometimes I lose my attention to the lessons; however, I can follow the teacher in vector approach instruction without getting bored in spite of the fact that proving was frequently included in the classroom.

5) Ömer stated the reasons why he needs to learn vector approach, the contributions of vector approach, how he decides the type of approach to solve the problems and the comparison of approaches in Excerpt 4-11.

Excerpt 4-11 Ömer's opinions about vector approach solutions on 24.07.2013

Researcher: While solving problems, mostly you are utilizing analytic and synthetic approaches in your solutions. However, you solved this task via vector approach. What is the reason for this? Ömer: I can apply synthetic and analytic approaches mostly. However, I also want to develop myself on vector approach solutions.
Researcher: Why?

Ömer: I think that I know the other two approaches. I want to gain problem solving ability via an extra approach.

Researcher: What kind of contributions does it provide to you?

Ömer: I think that it is going to provide some easiness and convenience to me for some kind of problems.

Researcher: What kind of problems do you talk about? Alternatively, How do you determine your preference?

Ömer: I can make my choice when I see the problem.

Researcher: As an example, how do you solve this problem if you were to solve it through synthetic approach?

Ömer: Similarity. I would utilize SAS similarity.

Researcher: In this situation, If I want you to compare the approaches, what are you going to say?

Ömer: I can understand clearly what I conduct if I solve the problem through vector approach. However, I need to think about and try to understand what I did with my synthetic approach solutions.

Briefly, in the light of the students' reflections, it can be said that the participants evaluated vector approach solutions as important, enjoyable, recent, innovative, appealing and elegant, and convenience. In spite of novelties of the instruction followed in this study, the students had never showed any indication of dissatisfaction or displeasure. They tried to learn every point of teaching experiment sessions. Moreover, the participants expresses the necessity of learning one more approach as vector approach. On the contrary, they express that solution through synthetic approach as routine or ordinary and the lessons are boring.

4.4.2 Advantages of Vector Approach in Geometry

In this part of the dissertation, advantages of vector approach solutions will be presented. These advantages were written down or stated by the participant students after solving geometry problems or when they were interviewed.

1. Vector approach require less knowledge of mathematical statements

While solving geometry problems via vectors, the students stated that they need less knowledge of theory, definitions, relations, facts or formulae. In fact, they stated that there is less need to memorization actually. On the other hand, the solutions with synthetic approach necessitated more knowledge about theorems, facts, rules or etc. Therefore, synthetic approach requires much more memorization than vector approach solution.

An example to "*the need for more mathematical theorems or knowledge to solve problems by synthetic approach*", the following solution and the solver's ideas can be considered. In the problem, the students were asked to verify the relation "*the sum of the squares of the length of the diagonals in a parallelogram is twice the sum of squares of length of two different sides for the same parallelogram i.e.* $e^2 + f^2 = 2\left(a^2 + b^2\right)$ *where "a and b" are the sides, e and f are the diagonals of the parallelogram*". If the students want to prove this statement via Euclidean methods, they need to utilize "*Apollonius' Theorem*"; that is, "*in any triangle, the sum of the squares on any two sides is equal to twice the square of half the third side together with twice the square on the median that bisects the third side*". In fact, this formula is also known as "*median theorem*" among students in Turkey. Therefore, it is important to memorize this relation resulting in calculating median length of a triangle. However, two difficulties emerge in this situation. First, the solvers need to specify or decide the appropriate mathematical relation, which is necessary for their solutions. Secondly, this relation should be recalled and written correctly by the students. Naci expressed the following considerations related to synthetic approach solution in Excerpt 4-12.

Excerpt 4-12 Naci's opinions about the difficulty of recalling formulas in geometry

> "Th*e median length theorem never came to my mind. I even forgot what the median length theorem was and that it is impossible to recall this formula. These are all results of rote learning. I could recall some theorems but this theorem… It did not come to my mind. This is actually a difficult stuff and a heavy duty. Moreover, it is unlikely to remember*

this relation correctly. In other words, only a few number of students can recall this formula correctly because the relation is not a simple relation to be recalled easily."

Naci's prediction; that is, remembering the formulae incorrectly already occurred in the present study. It is presented under the title "*Disadvantages of Synthetic Approach*" in one of the subsequent sections. On the other hand, he stated that there is no need to the use of any theorem or formula to solve this problem in vector approach. Naci solved this problem through vector approach. His solution reflects (Figure 4-84) what is meant by the student. Moreover, his solution is elegant and unusual.

In the problem (Figure 4-84), it is asked to verify the relation "$2\left(a^2+b^2\right)=e^2+f^2$ *where a and b are the sides and e and f are the diagonals of ABCD parallelogram"*. Naci verified this relation by utilizing algebra of vectors, properties of inner product and some algebraic manipulations of the literal expressions. It should be emphasized that the student started to prove the relation firstly by expressing the sides of the parallelogram by vectors.

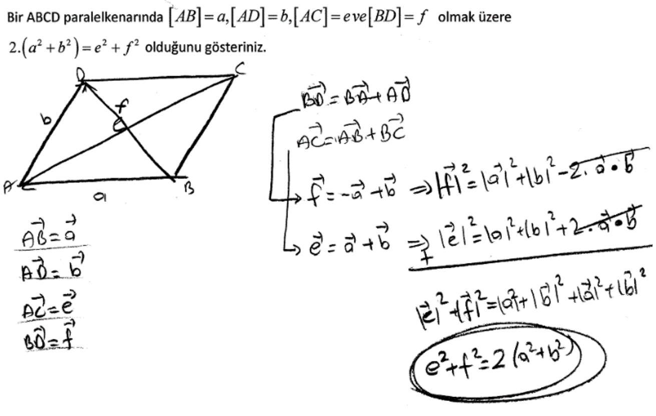


Figure 4-84 Naci's solution to the problem 4 on Parallelogram 1st section

Naci also states the <u>need for more knowledge</u> while solving another two problems. One of them is presented in the Figure 4-85. In this problem, it was asked to show that "*a rectangle with perpendicularly intersecting diagonals is a square"*. He solved the problem via vector and synthetic approaches. He stated under his solution in the Figure 4-85 that (*framed expression*)

> "*As it can be seen, this problem can be solved in two ways. It can be done via two of them. However, this cannot be verified by synthetic approach without knowing "awesome triple (he refers to the converse of Thales' Theorem)". However, there is no need know this theorem in vector approach".*

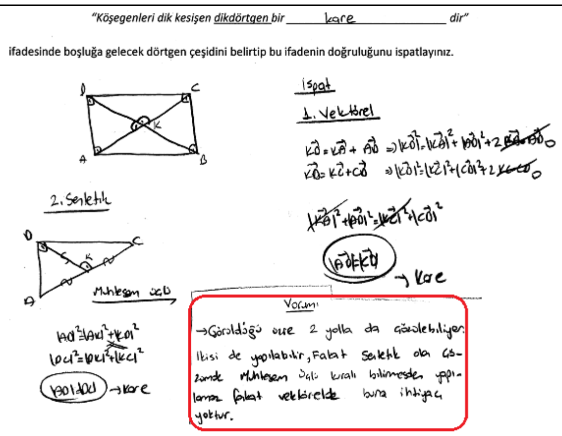


Figure 4-85 Naci's solution to the problem 3 on Square 1st section

In an interview with Ahmet, he stated in Excerpt 4-13 that a powerful background is necessary to conduct operations in synthetic approach. However, this is not the case for vector approach according to the student.

Excerpt 4-13 Excerpt from an interview with Ahmet on the comparison of approaches in terms of background knowledge for geometry

> *Till participating this study, I thought that synthetic geometry was sufficient to learn geometry because geometry was turning into mathematics by means of synthetic geometry. However, there is a need to have a certain level of knowledge or infrastructure. If there is this knowledge level then it is easy to proceed and continue. On the other hand, there is no need to have a great knowledge level in vector approach even for the problems that necessitate longer procedures. There are a few things; you need to know in vector approach. Those are namely: "scalar product, vector addition and a little*

bit thinking". The experience with vector approach is a facilitator for your processes because so far we have been educated via synthetic approach. We need to have so much knowledge of theorems, formulae and etc. in synthetic approach. Your knowledge must be adequate.

As DiFonzo (2010) states, the participants realized that operations and proofs in vector approach necessitate less pre-existing or prerequisite knowledge than synthetic proofs. Moreover, the students in the study of Gagatsis and Demetriadou (2001) and Gagatsis and Bagni (2000) state their positive opinions about vector geometry as the solutions through vector approach necessitate less knowledge of types and theorems. Besides, they interpret this less knowledge as a chance of making fewer errors in the operations. Furthermore, Athen (1966b) states that vector approach can be a prevention to students' memorizing because of less knowledge requirement in comparison with synthetic approach.

2. Expressing Works and Ideas Easily

Parallel to the fact given above, it is easier in vector approach than synthetic approach to document or write down what the participants think related to the solution of the problems according to them. However, sometimes it could not be possible to express in writing what they think in synthetic approach. They added that they needed to upgrade continuously the figure of the given crude or original geometric object as they add some additional or auxiliary lines or drawings at each step. This fact reveals the conclusion that it is difficult to understand what is meant by the solution with only final and complex drawing that contains all of sequential solution steps. These are expressed by Ömer in the Excerpt 4-14 and by Ahmet in Excerpt 4-15.

Excerpt 4-14 Excert from an interview with Ömer on 24.07.2013

While solving this problem via synthetic approach, I conducted many operations on the figure of given quadrilateral. Moreover, I could not write down all of my operations on the figure. I could not reflect my thoughts to *the figure. However, I saw that I could reflect all of my thoughts to the paper in vector approach. Therefore, despite solution via synthetic approach is simpler than vector approach, it started to become confusing and difficult when writing down your thoughts in synthetic approach.*

Ahmet stated the difficulty of reflecting or writing down all of the solution steps to the paper at "five different instants" of the project. Two of them are presented in Excerpt 4-5 and Excerpt 4-15.

Excerpt 4-15 Excert from an interview with Ahmet on 05.08.2013

Researcher: What do you think about vector approach solutions?

Ahmet: I like tricky points and subtle manipulations in vector approach solutions. While proving in synthetic approach or solving via synthetic approach, I know many subject matters. However, there are many things, which we cannot do or we cannot write down. While proving statements or solving geometry problems, I am much more satisfied with my solutions and I am getting more pleasure. The solutions are short and compact.

3. Easier way of solving especially certain type of problems

While solving geometry problems related to determining the type of quadrilaterals, which are given on analytical coordinate plane, representing sides via vectorial notations seems more practical. Moreover, calculating the lengths of the sides, determining the relative position of the sides; hence, specifying the type of the given quadrilateral is easier by means of vector approach in comparison with analytic and synthetic approaches. To illustrate, it is necessary to apply the distance formula that gives the distance of two end-points of sides on coordinate plane in order to compare the length of the sides. Further, to determine the relative position of the sides with respect to each other, there is a need to compare the slopes of sides. To make this comparison, finding the slopes of sides that passing through the end-points is inevitable. These two steps are essential analytical methods so as to determine the type of a quadrilateral. However, it seems longer in time and process and a little bit difficult with respect to vector approach. Despite of these difficulties, Ömer preferred analytic approach to solve this kind of problem. In the problem, "*the type of quadrilateral with given coordinates of vertices is asked*" in the Figure 4-86. He states under his solution that (*expressions in the frame in the Figure 4-86 as* Excerpt 4-16).

Excerpt 4-16 The reason for the preference of analytic approach by Ömer

> "*I used analytic approach because I could conveniently find the relative positions of the sides by calculating the slopes*."

It is beneficial to note that Ömer utilized vector approach for four times to find the length of sides and diagonals; however, he utilized analytic approach for the five times to determine the relative position of sides or diagonals according to Table 4-6.

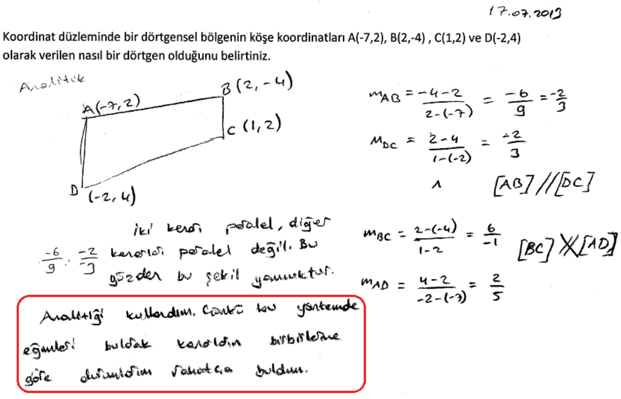


Figure 4-86 Ömer's solution to the problem 1 on Trapezoid 1st section

To be able to solve this problem by utilizing synthetic approach, it is also necessary to locate the given quadrilateral to an analytic coordinate plane. After that, the lengths of the sides of the given quadrilateral can be calculated by constituting right triangles and by applying Pythagorean Theorem on these triangles respectively. After calculating the length of the sides, there emerges two alternatives to compare the slope of the sides. In the first choice, the students need to utilize analytic approach to specify relative positions of sides as explained in the paragraph above. Therefore, only synthetic approach is not enough to reach the solution in this choice. In other words, combination of synthetic and analytic approaches is inevitable for this case. The second alternative emerges as calculating the slope of the sides by utilizing trigonometry knowledge. That is, the slopes of sides can be calculated by forming right triangles and then utilizing the definition of tangent of an angle. After that, the slopes of the sides can be compared. As a result of this workload, it can be observed that the students naturally might have preferred to solve this kind of problems by means of vector approach after the second special quadrilateral topic "*parallelogram*". This result can be explained with the easiness and convenience of vector approach in comparison with mere analytic approach, mere synthetic or the combination of analytic and synthetic approaches.

It is also important to state that these students did solve this kind of problems by analytic approach or synthetic approach in the past as understood from Excerpt 4-17 and Excerpt 4-18.

Excerpt 4-17 Excerpt from an interview with Ömer on 24.07.2013

Researcher: I see that you solved entrance tasks by means of analytic and vector approaches. Were you be able to solve this type of tasks through multiple approaches or specifically through vectors before this project?

Ömer: I could be able to solve through analytic methods; however, I could not be able to solve via vectors.

Researcher: Why?

Ömer: Because our teachers did not teach it to us.

Researcher: I understood that you have given priority to analytic approach in your solutions. How can you explain this situation?

Ömer: I could explain it with the education on analytic approach that I had in my school.

Excerpt 4-18 Excerpt from an interview with Ahmet on 08.07.2013

> *Researcher: In the light of the task: "find the area of quadrilateral region whose coordinates of the vertices are given as A(-3,2); B (-2,2); C(0,-5) and D(1,1)"Have you ever encountered with this type of problems before?*
>
> *Ahmet: Yes, but mostly I could not solve these problems.*
>
> *Researcher: If you wanted to solve, how would you solve it?*
>
> *Ahmet: By placing on coordinate plane as analytic approach, by completing the given quadrilaterals to rectangles or other known quadrilaterals and then excluding area of unwanted regions as synthetic method.*

The students started to prefer studying through vectors as the study progressed. It can be interpreted as a change and this fact exemplifies the advantage of utilizing vectors in terms of solving this kind of problems specifically. The following elegant and compact solution in the Figure 4-87 illustrates how students' utilize vectors in solving the mentioned geometry problems. In the task, coordinates of all vertices of a quadrilateral are given. The lengths of sides and diagonals, the intersection point of diagonals, relative positions of sides, sum of interior angles of the quadrilateral and; finally, the properties of the quadrilaterals are required to be reported in this task in the Figure 4-87.

As can be seen in Ahmet's work, he calculated the length of sides and diagonals, and specified the relative position of opposite sides in a compact and understandable manner via vector approach.

Students' preferences of particular approach among the available alternatives is compatible with some of the researchers' claims or findings (e.g., DiFonzo, 2010; Miller, 1999, Coxford, 1993; Regecova, 2005; Appova & Berezovski, 2013; Lee, Tay, Toh & Dong, 2003; Ayre, 1965 & Nissen, 2000). These researchers state that an approach is more appropriate to solve a problem according to the context or scope of the problem. Moreover, students preferred the most convenient way to solve a problem if they are knowledgeable on problem solving through multiple approaches according to the findings of the study conducted by Kwon (2013). Similarly, the participants gain an experience that a strategy is more convenient in comparison with the other approaches under particular circumstances (Star & Rittle-Johnson, 2008). However, making a proper decision necessitates some degree of experience while learning geometry via various approaches (Stephenson, 1972; Cambridge Conference on School Mathematics, 1963).

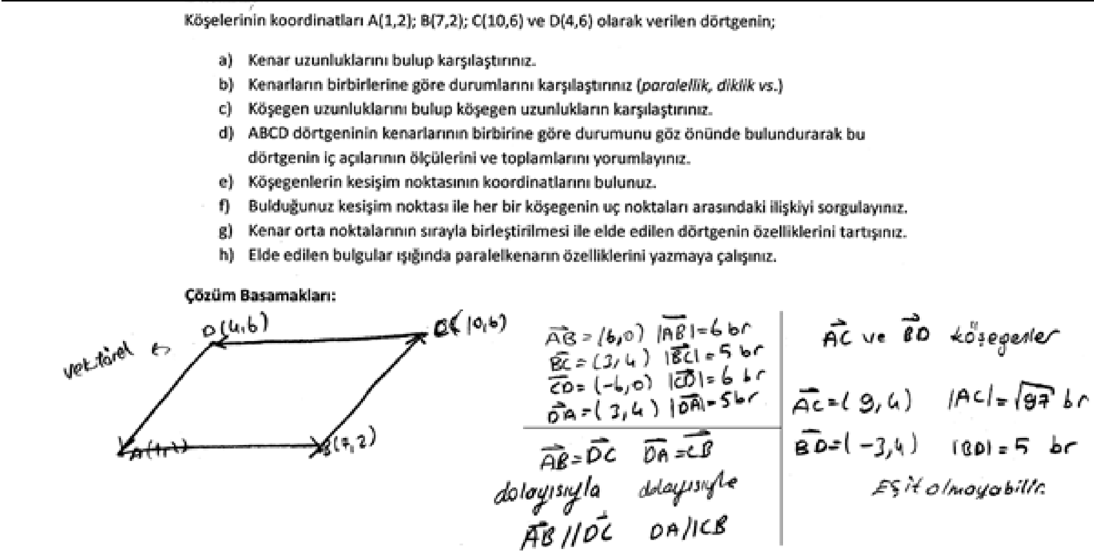


Figure 4-87 Ahmet’s solution to the entrance assignment for Parallelogram 1st section

4. *Easiness of vector approach in solving proof-based problems*

The students recognized that solving geometry problems containing nested figures and proof-based problems via analytic or synthetic approaches requires more effort and time in comparison with solving these tasks via vector approach. As a result, it was observed that the students began solving mentioned problems via vector approach after a while in the study. In the Table 4-4, the frequency of solving proof-based problems through vector approach is 18, 10 and 13 out of 23 proof-based problems for Naci, Ömer and Ahmet respectively. Moreover, vectors were also utilized in combination of synthetic and vector approach solutions five times in total. According to the following three problems (Figure 4-88, Figure 4-89 & Figure 4-91), it can be easily seen that students started to solve the problems by representing sides of quadrilaterals with vector notations.

In the first of them (Figure 4-88), the students are required to show that "*midpoints of consecutive sides of a quadrilateral form a parallelogram*". To verify whether the constituted quadrilateral is a parallelogram or not, Naci preferred to show the equivalence of opposite sides and; hence, the parallelism of these pairs of sides by utilizing vector approach. Analytical representation of vectors were utilized since the coordinates were specified.

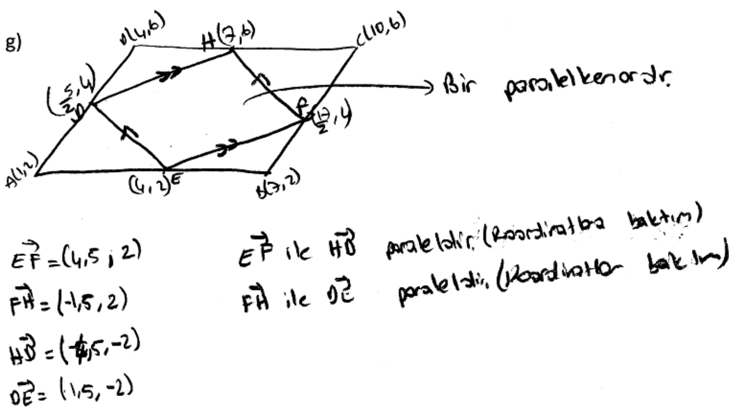


Figure 4-88 Naci's solution to the entrance assignment for Parallelogram 1st section

The second example (Figure 4-89) is another proof-based task in which the students were required to verify that "*a quadrilateral is a parallelogram if its diagonals bisect each other*". This task was assigned as an individual task for the participants. In his solution, Naci stated that "*his first preference was vector approach*" (*expression in frame*). According to his solution, algebra of vectors and definition of equivalent vectors were utilized to prove this theorem.

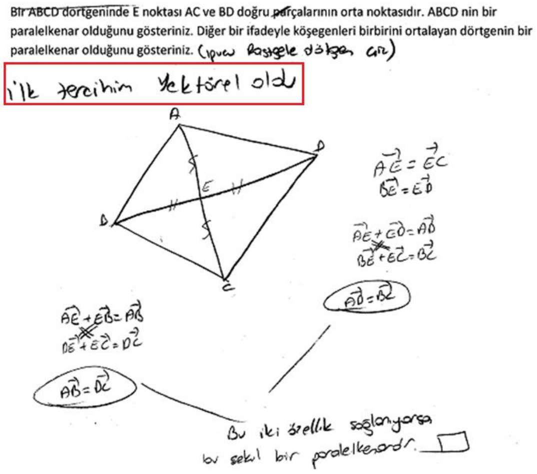


Figure 4-89 Naci’s solution to the problem 2 on Parallelogram 1st section

The researcher requested him to solve this problem by means of another approach as a homework to make a logical comparison among approaches. At this time, he preferred to solve this task via synthetic approach (Figure 4-90). In his solution, it is evident that E is the common midpoint of the line segments $[AC]$ and $[BD]$ in the quadrilateral ABCD. The student utilized similarity and congruence theorems as synthetic approach. He set up SAS congruence relation for two pairs of corresponding triangles. The student correctly constructed entire solution steps for both of the ways. Consequently, he could successfully determine the type of the quadrilateral as “*parallelogram*”.

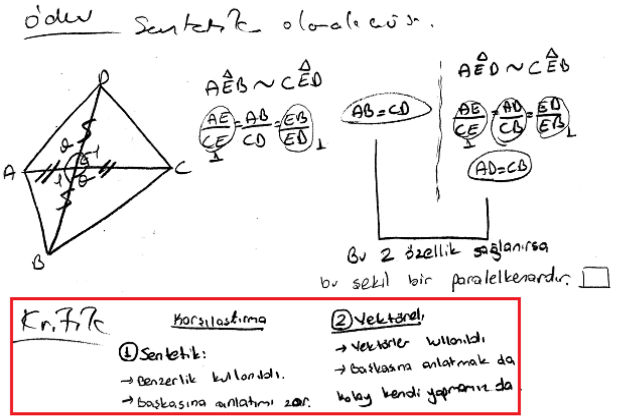


Figure 4-90 Naci’s solution to the problem 2 on Parallelogram 1st section

Finally, he was asked to compare two of the approaches. Naci stated that proving this item via vector approach is easier than synthetic approach.

The same comparison was also asked to Ömer after he solved the problem in two approaches. Ömer states under his solution in the Figure 4-91 that

> *"In my opinion, both of the approaches for the solution of the problem are beautiful and understandable. However, vector approach is slightly more superior and elegant than the synthetic approach."*

Although solutions in two approaches seem nearly identical in terms of workload, Ömer stated the superiority of vector approach. This could be as a result of gained ability to apply analytic, synthetic and vector approaches in geometry problems.

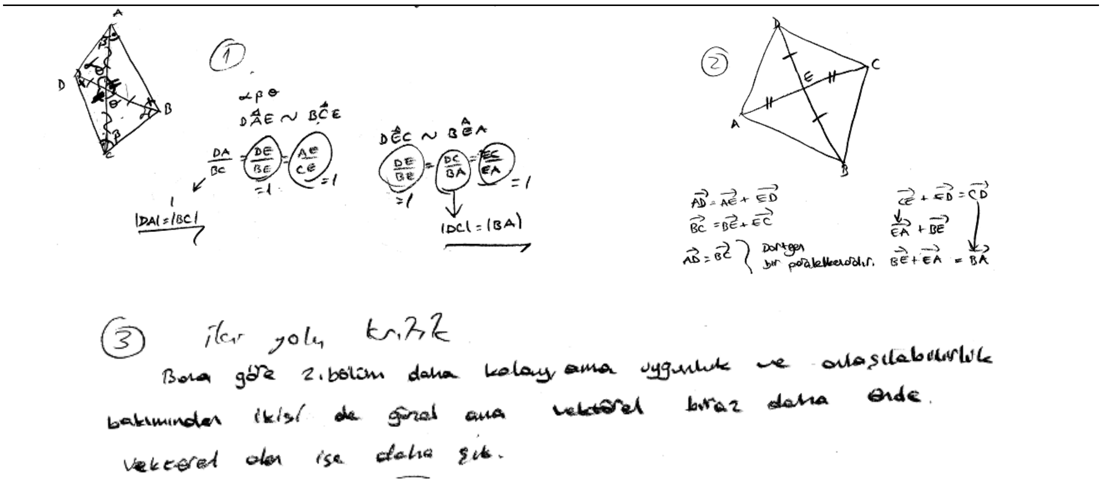


Figure 4-91 Ömer's solution to the problem 2 on Parallelogram 1st section

According to Table 4-4, the participants resorted to vector approach 46-times totally in order to solve 69 proof-based problems. Only three of the solution are presented in this part. Therefore, it can be inferred that participants resorted to vector approach possibly because of easiness of vector approach in solving proof-based problems and easiness of being able to reflect their thoughts. Easiness of vector approach in solving proof-based problems was discussed by considering the literature under the title "*0 4.2.4 Quadrilaterals with Perpendicular* Elements*"*.

5. Convenience of vector approach for low achiever students

Besides convenience of solving these problems by vector approach in terms of the students, Naci and Ahmet also stated that it is more convenient to explain the solutions via vectors for certain geometry problems to their friends especially for unsuccessful or slow-learner students. In other words, it could be easy to learn from their friends or to explain another person among students. These were stated under his solution in the Figure 4-90 (*expressions in frame*) and in Excerpt 4-19 and Excerpt 4-20.

Excerpt 4-19 Naci's thoughts about the suitability of vector approach for low achieving students

Researcher: According to your solution to the 3rd task, you were observed that you preferred vector approach. How can you interpret your solution?
Naci: I think that the solution by vector approach is more appropriate for middle and low achieving students. Especially the logic "a student moved from A to B and then from B to C" is an easy and understandable in terms of these students. It is the logic similar to "going from school to home, then from home to grocer". The alternative way of solution to this problem is similarity and congruence of triangles. This solution is mostly appropriate for high achieving students. It is not a simple task to set similarity and congruence.

Excerpt 4-20 Ahmet's thoughts about the suitability of vector approach for low achieving students

Researcher: As a successful student, can you interpret vector approach solution and synthetic approach solution to task 4 in terms of low achieving and high achieving students?
Ahmet: Definitely, vector approach is more appropriate for these students because recalling the formula and applying necessary manipulations are not simple tasks in my opinion. However, thinking a vector as the summation of other vectors is rather easier. After that, a

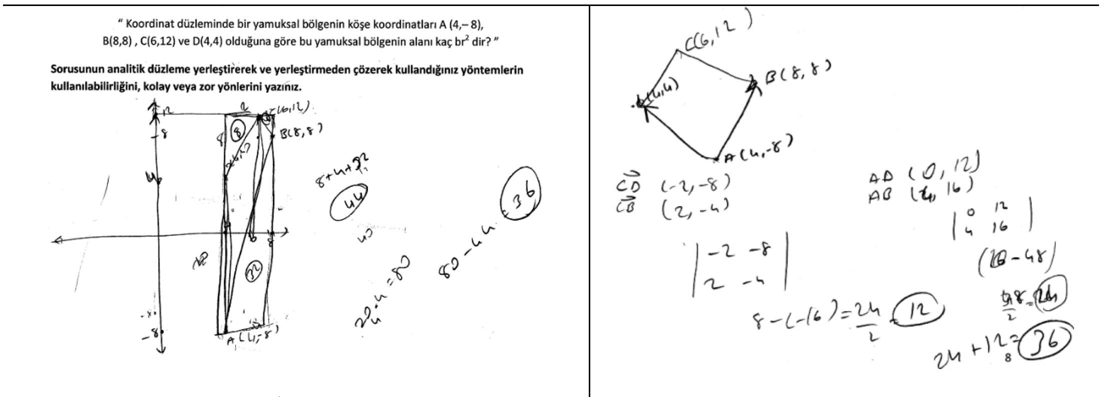
student needs to know taking square of a vector as the way of passing from a vectorial quantity to a scalar quantity.

Bundrick (1968) found that the mean scores of the students from low-level group under vector approach treatment were significantly higher than the mean scores of the students from low-level group under traditional approach treatment on criterion test for plane analytic geometry topics. Furthermore, the mean scores of the students from low-level group under vector approach treatment were nearly equal to the mean scores of the students from high-level group under traditional approach treatment on criterion test for the same topics. A similar pattern was also found in terms of transfer test for low-level of students. Furthermore, the mean scores of the students from low-level group under vector approach treatment were slightly higher than the mean scores of the students from high-level group under traditional approach treatment on transfer test for solid analytic geometry topics. However, the difference was not reported as significant. However, it is an outstanding finding. These findings are compatible with participants' reflections for the appropriateness of the vector approach in geometry teaching for low achieving students.

6. Representing geometric shapes easily through vectors as an alternative to drawing on Cartesian plane.

While calculating the area of quadrilaterals or specifying the type of quadrilaterals whose coordinates of vertices are given, in case of being given relatively larger numerical values of coordinates for vertices (*such as (8, 8) or (6, 12) or like this)* or in case of having relatively distant vertices (*such as (4,-8) and (6, 12) or like this*), the students stated the difficulty of displaying these points; hence, sketching the required polygons on coordinate plane.

Ahmet pointed the complexity of the solution when firstly placing the given polygon on coordinate plane and then continuing with synthetic approach as understood from his solution in the Figure 4-92 and Excerpt 4-21. Moreover, he attributed this complexity to larger distance among the vertices. On the other hand, he expressed the convenience and the simplicity of vector approach solutions in comparison with the other approaches for this type of problems in terms of practically representing the given geometrical objects. In the problem (Figure 4-92), “*the area of a trapezoid whose coordinates of vertices are given is asked to find through placing and without placing on Cartesian plane and the comparison of these methods is required to be reported*”. He solved the problem in two approaches and then he compared his solutions.



1st Part | 2nd Part

Figure 4-92 Ahmet’s solution to the problem 3 on Trapezoid 2nd section

Excerpt 4-21 Ahmet’s opinion about the comparison of approaches

> *When firstly placing the given quadrilateral into coordinate plane, operations have become more complicated because of the larger distance among the vertices. Therefore, the progress has become more complicated. Instead of placing the quadrilateral to the coordinate plane, if we continue via vectors, our work is getting easier and clearer as can be seen in my solution.*

After solving the same problem via vector approach and combination of analytic and synthetic approaches, Ömer states in the Excerpt 4-22 that

Excerpt 4-22 Ömer's opinions about the comparison of approaches

> *In analytic approach, drawing the shape of the given quadrilateral on coordinate plane is waste of time. However, the rest of the solution becomes enjoyable after drawing the picture. After determining diagonal vectors, the vector formula is applied in vector approach. However, great numbers is disadvantage of vector approach here. After all, vector approach should be used primarily, in my opinion."*

The students recognized that it takes too long to locate the given quadrilateral on coordinate plane to solve the problem by analytic approach or synthetic approach. Therefore, the students might have preferred to solve the problem by vector approach, which was determined as more practical in their perspectives. In order to reach the correct answer in vector approach, Naci did not draw the quadrilateral on Cartesian plane despite it is given with coordinates of vertices. After that, he represented the related sides with vectors as seen in the Figure 4-93. In the problem, it is asked to "*determine the type of the quadrilateral whose coordinates of vertices are given*". He states in Excerpt 4-23 that:

Excerpt 4-23 Naci's opinions" about vector approach solutions

> *"Vectorial is easy and clear. There is no need to draw any figure because it can be solved through the ratio of the coordinates."*

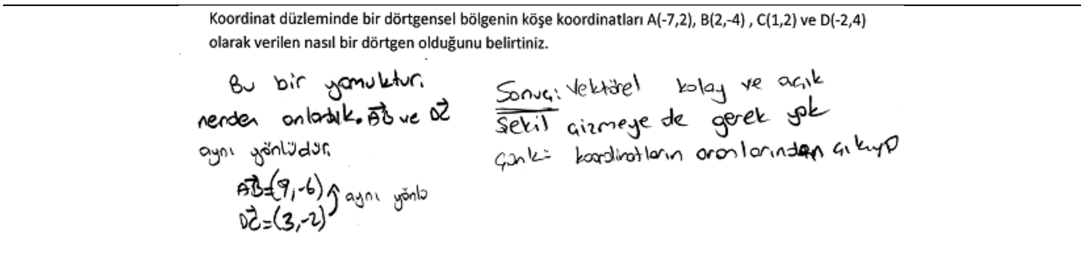


Figure 4-93 Naci's solution to the problem 1 on Trapezoid 1st section

Ahmet also stated the easiness of vector approach for the same problem in his solution (Excerpt 4-24). He states that:

Excerpt 4-24 Ahmet's opinions about vector approach solutions

> *I <u>drew analytic plane roughly</u> so as not to place the points (vertices) to the quadrants incorrectly. I thought it would be easier to solve this problem via vectors.*

As understood from Excerpt 4-25, Excerpt 4-26 and Excerpt 4-27, participants stated drawing geometric figures on analytic plane to determine the type and to calculate the area of quadrilaterals as <u>waste of time</u>.

Excerpt 4-25 Ahmet's opinions about vector approach solutions

> *Researcher: Why didn't you solve this problem by drawing the quadrilateral on analytic plane?*
> *Ahmet: Because, I think it is waste of time. In my opinion it is more practical solve this problem by means of vectors. Moreover, solution with vectors is more compact.*

Excerpt 4-26 Ömer's opinions about vector approach solutions

Researcher: You solved this problem through vectors. Why didn't you draw the figure on coordinate plane?
Ömer: I did not want to experience waste of time by drawing the picture on coordinate plane. It could be possible if coordinate plane graph worksheets was been provided. However, I would still switch to vector approach because vector approach is less tiring for this type of problems in my opinion.

Excerpt 4-27 Naci's opinions about vector approach solutions

Researcher: Why did you solve the problem without placing the given quadrilateral on Cartesian plane while calculating the area of the quadrilateral with coordinates of vertices A(-3,2); B(-2,2); C(0,-5) and D(1,1)?
Naci: I just like to solve with vector approach more. Moreover, vector approach is easier than the other method. Because placing the points on the plane is waste of time. I am not good at drawing. I cannot draw pictures well enough. Besides, think about greater coordinates like 12 or 13, how would it be then? Probably, it would be difficult for me. However, I do not draw figures in vector approach. It is getting simple for me.

To overcome the waste of time issue with drawing the figures on coordinate plane for these type of problems, Ömer proposed an alternative idea in Excerpt 4-26. Besides, the students who are not good at sketching will probably have additional difficulties with drawing the figures on coordinate plane according to Excerpt 4-27. In addition, Meserve and Meserve (1986) express the aims of utilizing vectors. One of the aims is that vector can be used to represent figures and the teachers should develop themselves by considering this aspect of vectors. In summary, because of stated difficulties and reasons, the students preferred to represent geometric shapes roughly through vectors and then they continued solving the problem via vector approach.

7. *Vector Approach to Reduce the Possibility of Making Operational Errors*

Naci (11.07.2013) describes himself as a student who makes operational mistakes frequently in his solutions, in the Excerpt 4-28.

Excerpt 4-28 Naci's spontaneous ideas about vector approach solutions

> *While determining the type of quadrilaterals with the given coordinates of all vertices, it is necessary to be sure about relative positions of sides of the quadrilateral. In order for achieving this purpose, it is necessary to infer that the sides are parallel or perpendicular. Therefore, calculating the slopes of the sides are required. Moreover, the formula giving the distance between two points is necessary so as to compare the length of the sides. As you see, making these analytical and synthetic operations <u>requires lots of arithmetic operations and calculations</u>. <u>I make arithmetical mistakes frequently in my operations and calculations</u>. Therefore, there is rather high probability of making errors in my solutions in these two approaches. On the contrary, it is not necessary to make use of distance formula for two points to compare the length of the sides in vector approach solutions. In addition, there is no need to calculate the slope of the sides one by one in order to determine the positions of sides relative to each other in vector approach. It is enough to represent side of the quadrilaterals as vectors, which gives information about whether the sides are equivalent and whether the sides are parallel to each other. Related to the case of sides' being perpendicular to each other, it is easy to decide whether the sides are perpendicular or parallel after representing them by means of vectorial ways without conducting any additional operations. In this manner, besides its being practical aspects, <u>vector approach strategy would decrease probability of making operational errors</u> in my opinion.*
>
> (*Naci stated these interpretations spontaneously and explained his solution steps without being directed any questions to him by the researcher*.)

Naci's interpretation about relatively low probability of making operational errors in vector approach solutions is compatible with the findings of the study conducted by Gagatsis and Demetriadou (2001). In their study, there are students who preferred Euclidean method, vector method and both Euclidean and vector methods in their solutions. Whereas the number of error cases was 104 for Euclidean solvers, it was 67 for vector solver students. Furthermore, within Euclidean-vector solver type group, whereas the errors related to general errors was 153 in classical approach, it was determined as 81 for the errors on vectors. In brief, the students who preferred vector approach in their solutions had fewer errors in comparison with the students who preferred synthetic approach. Furthermore, Gagatsis and Bagni (2000) found that the frequency of errors were lower for the students who utilized vectors in their solutions than the students who used traditional approaches for the problems requiring the use of theorem and definition. In addition, Barbeau (1988) labelled a method of solution as secure when this method offers the least possibility of making an error. Therefore, according to the Naci's interpretations and Barbeau's (1988) definition, it can be concluded that the more frequency of including vectors is realized in problem solving processes, the less probability of making operational errors might be experienced in the solutions.

8. Vector approach as a source of satisfaction and pleasure

If the protocols from participants' interviews are examined in the preceding sections, it will be easily inferrred that they had pleasures and satisfactions with their vector approach solutions in the study. Specifically, after solving the problem in the Figure 4-94, the dialogue between the researcher and Naci was realized as in Excerpt 4-29. The participants are asked to show $a^2 + b^2 = c^2 + d^2$ for ABCD quadrilateral in the problem in the Figure 4-94. In this dialogue, he stated his satisfaction with vector approach in his solution.

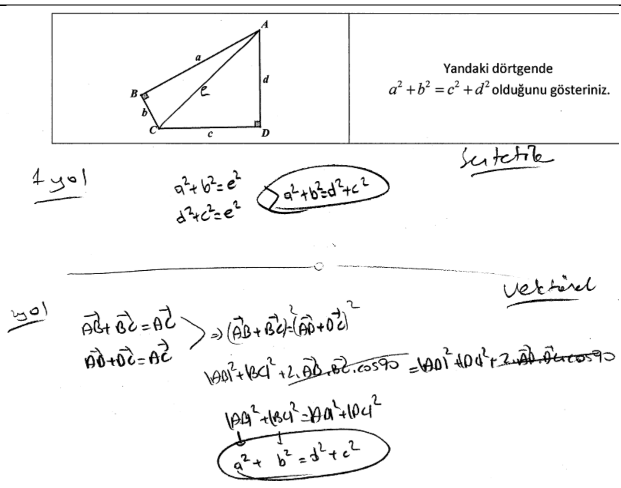


Figure 4-94 Naci's solution to the problem 2 on Quadrilaterals 2nd section

Excerpt 4-29 Naci's ideas about satisfactory feature of vector approach solutions

> *Researcher: Why did you solve this problem via vectors in addition to synthetic approach?*
>
> *Naci: I firstly chose synthetic approach. I was sure about my solution however since it is so much simple I could not be satisfied my answer. Therefore, I looked for the second way. I solved the problem through vector approach. My second solution also gave the same answer and it would be more elegant. As a result, I felt satisfied with my second solution.*

Moreover, Ahmet stated his satisfaction with vector approach solutions in the Excerpt 4-30, after he solved the problem in the Figure 4-95. He also stated his satisfaction with vector approach proofs and solutions in Excerpt 4-15.

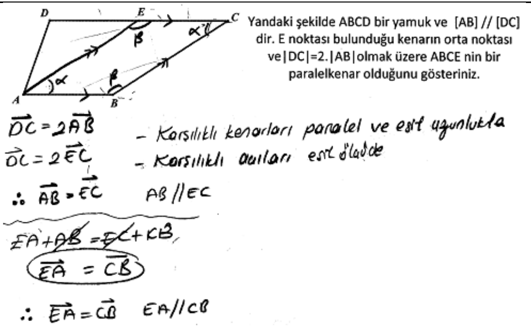


Figure 4-95 Ahmet's solution to the problem 3 on Parallelogram 1st section

Excerpt 4-30 Ahmet's ideas about satisfactory feature of vector approach solutions

Researcher: You solved this problem through vectors again. Why didn't you try a second method?

Ahmet: Because the way that I solved though <u>vectors made me satisfied</u>. If I had not felt satisfied with my solution even I could solve the problem correctly, I would definitely try synthetic approach as well. To illustrate, I would utilize similarity and congruence theorems for triangles. I would do this if there were something that I did not know the underlying reason. However, the solution that I did made unnecessary to resort to synthetic approach.

Researcher: What do you think about the reflection of vector approach to your success?

Ahmet: I certainly think that it will reflect to my success. Because, having been able to use vector in my solution makes me feeling <u>so happy</u>. In addition, I think that I am going to be able to prove other theorems while proving some other statements.

While proving a statement or solving a problem, students' feeling necessity to the second way of proving to be satisfied with is a considerable improvement for this study. This is important from two aspects in terms of the students. Firstly, looking for alternative proofs of the same geometrical argument is a way to enhance their logical and deductive reasoning (Hansen, 1998). Secondly, an argument becomes more convincing by means of searching for alternative ways of proving (Neubrand, 1998).

However, this not specific to vector approach. Any other way of solution or proving is also a source of convincing. In addition, Chatwin (1985) states that the students have an opportunity to appreciate the power and beauty of mathematics by means of vector approach in problem solution. In the light of students' reflections and some of the researcher's assertions, it is possible to conclude that v*ector approach might be evaluated as a source of satisfaction and pleasure.*

4.4.3 The Challenges in Utilizing Vector Approach

In this section, students' common difficulties and errors in their solutions will be presented. These challenges were written down after solving geometry problems or stated by the participant students when they were interviewed. These common challenges, difficulties and mistakes can be interpreted as a result of short-term experience with studying vector approach in geometry problem solving. These challenges are presented separately as follows.

1. *Difficulty in determining the angle between two vectors*

The students had difficulties in determining the angle between two vectors especially when their initial points are not common or when the vectors are situated on different lines. Students' difficulties of this type; that is, they have difficulty with vector operations when the vector are not in standard position is stated in the study of Poynter and Tall (2005). The researcher was also aware of this challenge before the study started because of his experiences and related literature (Pavlakos, Spyrou, & Gagatsis, 2005; Gagatsis and Demetriadou, 2001). This difficulty is explained mostly with the vector teaching that includes acute angle between two vectors as a prototype angle (Pavlakos, Spyrou, & Gagatsis, 2005). Moreover, Barniol and Zavala (2010) found that students are more successful in finding addition of two vectors when the vectors are given in standard position in comparison with separated vectors. Therefore; to overcome this difficulty, the following problem (Figure 4-96) was solved as a precaution and the participants were emphasized in this regard. In the exercise, they were asked "*to compute the inner product of two pairs of vectors which are on different lines"*.

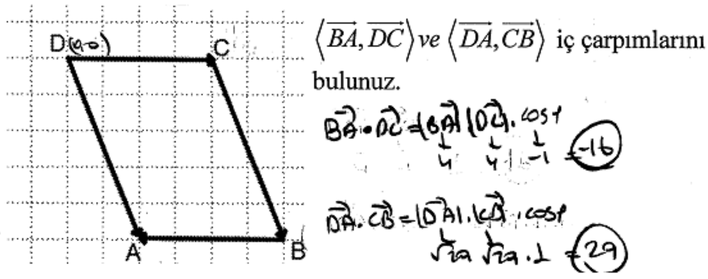


Figure 4-96 A precaution example to determine the angle between vectors of non-standard position

This difficulty was observed generally, when the students were utilizing inner product. For example, Ömer and Naci incorrectly marked the angle between the vectors $\overrightarrow{AB}$ *and* $\overrightarrow{BD}$ as "x" in the following problem (Figure 4-97). Actually, the value of the correct angle between $\overrightarrow{AB}$ *and* $\overrightarrow{BD}$ is supplementary of the angle ABD, that is: $180 - x$.

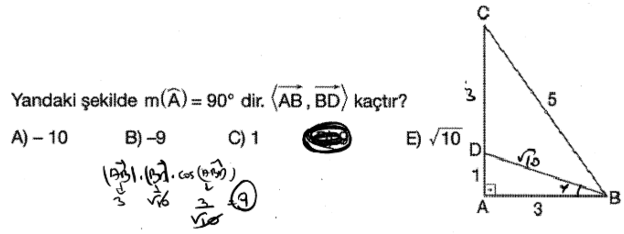


Figure 4-97 Naci's solution to the problem 24 on Vectors 2nd section

Similarly, it was understood that Ömer also had difficulty in correctly determining the angle between two vectors when they are not in standard position as in the following solution in the Figure 4-98. In other words, he had difficulty with two vectors, which are not in standard position i.e. the tails of the vectors are not intersected at the same point. In the problem, the participants are asked to show that *"a rhombus whose diagonals are equal in length is a square"*. Ömer considered the supplementary angle of the actual angle (*denoted by arrows in the* Figure 4-98). Instead, he should have considered supplementary of the angles B *and* C in the inner products.

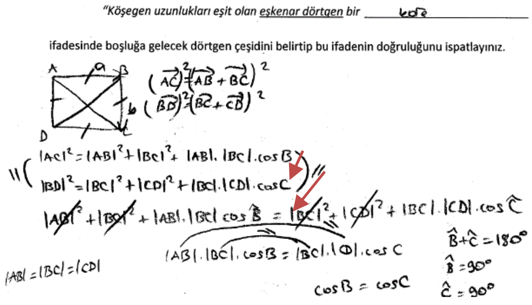


Figure 4-98 Ömer's solution to the problem 4 on Square 1st section

This difficulty was also encountered when the given vectors were parallel, especially when they are on parallel lines. While proving the relation "*if* $\vec{u} \parallel \vec{v}$ *then* $\langle \vec{u}, \vec{v} \rangle = |\vec{u}||\vec{v}|$", Ahmet and Ömer (Figure 4-99 & Figure 4-100) forgot to consider $180°$ as a possible angle between parallel vectors in addition to the angle with the measure of $0°$, as understood from their statements and solutions to this problem. Since both of the participants considered the measure of the angle between two parallel vectors as $0°$, they accepted the statement as if it was always true.

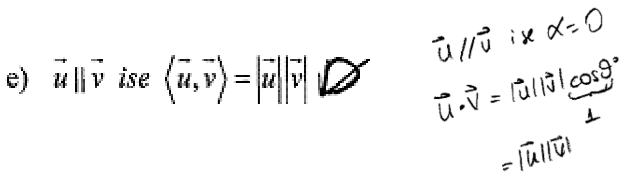


Figure 4-99 Ahmet's solution to the problem 14 on VKT post-test

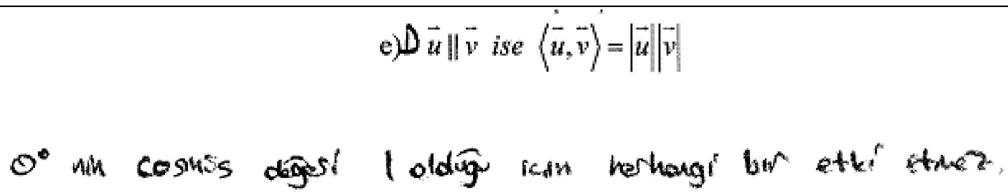


Figure 4-100 Ömer's solution to the problem 14 on VKT post-test

However, it seems that this difficulty was resolved for some of the problems by the students towards to the end of the study, as seen in the following solution with framed parts in the Figure 4-101. “*The value of x is asked to the students in ABCD rectangle where* $[DB] \perp [CE]$, $|AD| = 6$ cm and $|DC| = 8$ cm ” in the problem.

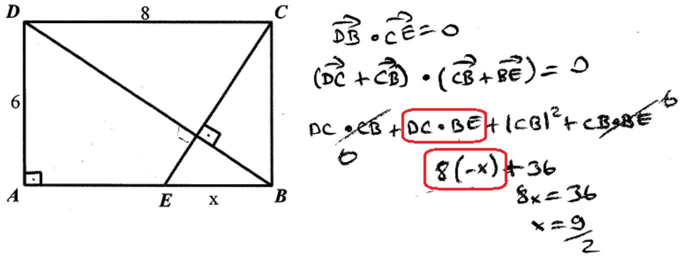


Figure 4-101 Ömer’s solution to an exercise on Rectangle 1st section

It can be appropriate to state that setting one of the vertices of the quadrilaterals as origin and; hence, utilizing analytic representation of the vectors in problem solutions can be an alternative way of solving geometry problems without determining the angle between two vectors. This difficulty could be eliminated by this way instead of utilizing vector algebra that necessitate determining and then the use of the angle between two vectors. In the following problem, “*an ABCD rectangle with* $|AB| = 2|AD|$ *and* $|DP| = \frac{3}{4}|DC|$ *are given where P is a point on* $[DC]$”. It is asked “*to show whether or not* $[AC]$ *is perpendicular to* $[BP]$”. As seen in the Figure 4-102, Ömer assigned “D vertex” as the origin of ABCD rectangle. He did not need to consider the angle between two vectors to be necessary for inner product. Instead, he utilized analytical representation of the vectors and analytic definition of inner product.

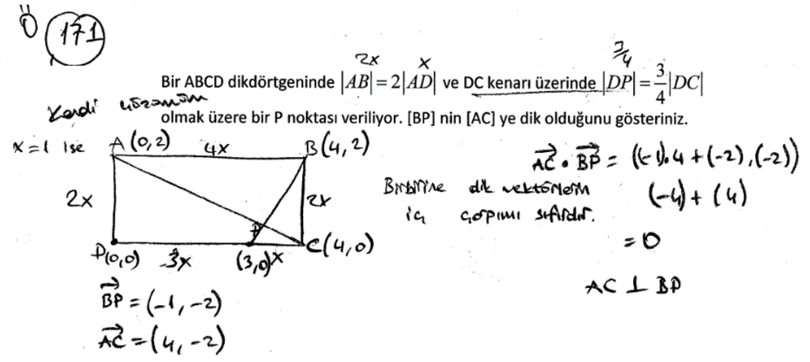


Figure 4-102 Ömer's solution to the problem 4 on Rectangle 1st section

2. Difficulty in expressing a vector in terms of other vectors

While utilizing algebra of vectors, the students had difficulties in expressing a vector as a combination of correctly chosen vectors. In other words, although it is possible to express a line segment in the given geometry problem as a combination of other vectors more easily, the student might choose more difficult or indirect way to represent the required segment vector in terms of other segment vectors. Possibly, it is based on lack of experience with studying vector approach in geometry problem solving. The participants explained underlying reason for this difficulty with the fact that they were inexperienced with vector approach. Therefore, this trouble can be resolved as the students have experiences on studying with vectors in geometry solving. This is a similar situation for the researcher as well. That is, he had similar problems when he started to work with vectors in geometry.

To illustrate the difficulty explained above, the following problem solution can be considered in the Figure 4-103. In the problem, the students are asked to show that "*the line segments which are constituted by combining midpoints of non-adjacent sides of a quadrilateral bisect each other*". In his solution, Ömer expressed $\overrightarrow{HK}$ as a combination of rather indirect vectors in spite of the fact that there is more practical and direct way. Specifically, $\overrightarrow{HK}$ could be expressed as a combination of $\overrightarrow{HF}$ and $\overrightarrow{FK}$; however, the student preferred to write $\overrightarrow{HK} = \overrightarrow{HG} + \overrightarrow{GF} + \overrightarrow{FK}$. This preference is prevalent in all of the steps. It is seen that he had problem in expressing vector as a combination of other vectors at this stage.

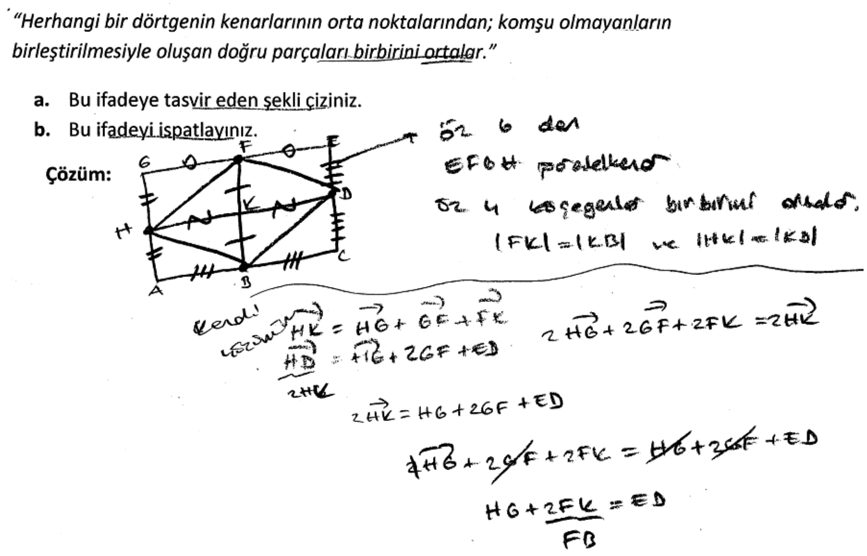


Figure 4-103 Ömer's solution to the problem 1 on Parallelogram 1st section

It is important to state that, the participants had this difficulty when they utilized vectors in geometry problem solving as a tool. However, when they are directly asked to write a vector as a linear combination of other vectors, they could reach the correct solution. For example, the students are required to "*express* $\overrightarrow{EG}$ *in terms of* $\vec{a}$ and $\vec{b}$" in the Figure 4-104. Three of the students could solve the problem correctly. Therefore; as stated repeatedly, solving problems related to vectors and using vector as a tool in geometry problem are different things. The former one is prerequisite for the latter one but it does not guarantee that a person who is good at solving vector problems is also good at solving geometry problems through the use of vectors.

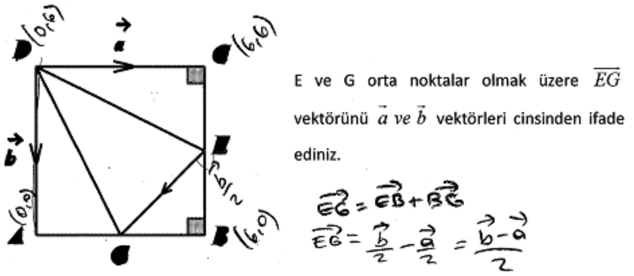


Figure 4-104 Ömer’s solution to an exercise on Square 1st section

This difficulty was also stated by the participants. They can be examined in the Excerpt 4-31 and Excerpt 4-32.

Excerpt 4-31 Excerpt from an interview with Ömer and Ahmet on 05.07.2013

Researcher: What are the disadvantages or difficulties of vector approach?
Ahmet: You select appropriate vectors according to your rationale and these vectors seem as if they were reasonable for your targets. You think that you are going to be able to solve the problem by the vector that you select. However, it is possible not to have any progresses.
Ömer: Yes, teacher. That was also the case for me. For example, in proofs! There are many alternatives possibly to be used in vector approach. To illustrate; $\overrightarrow{AD}$ *and* $\overrightarrow{DA}$ *are different vectors. However, they refer to the same thing as the length in synthetic approach. This may lead to confusion.*
Researcher: In this situation what is the reason for this confusion that you experienced?
Ömer: It might be because of short-term experience with studying geometry via vectors.
Ahmet: While proving Pythagorean Theorem, I selected vectors randomly without being sure about which are necessary for me. However, I selected the vectors without having idea where these

vectors will take me at the end. After that, to make transition to the length concept, taking square of vectors that I decide to use and inner product spontaneously came to my mind.

Researcher: This brings to mind the discovery function of proof.

Ahmet: Yes teacher! I also experienced this in different proofs. To illustrate, I could be able to verify the ratio of 2:1 for intersecting medians of triangles via vectors that I selected without being sure of them. Therefore, I want to be able to utilize vectors effectively for the next mathematics and geometric topics.

Excerpt 4-32 Ahmet's ideas about the difficulty of vector approach on 24.07.2013

Researcher: What are the difficulties of vector approach that you encountered?

Ahmet: We need to decide the vectors that we are going to find and write here. That might be difficult at the beginning. However, the remaining steps are much easier after deciding stage.

Researcher: What do you attribute the reason for these challenges?

Ahmet: That we have not long experience with vector approach! We learnt vector approach just in this study. However, we have been learning synthetic approach for many years.

Researcher: So why didn't you prefer synthetic approach to solve this problem if you have been learning synthetic approach for many years?

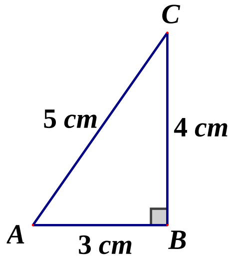
Ahmet: Despite the fact that I am more experienced with synthetic approach than vector approach, why don't I prefer a more practical method? Ultimately, I am open to innovations!

At the beginning of the study, the participant students had deficient knowledge on linear independence-dependence, as understood from their works on VKT pre-test although they had learnt linear dependence in their geometry lessons before taking this test. In addition, it was concluded under the title "*4.1.5 Utilizing Analytic Approach as an Alternative to Algebra of Vectors*" that the participants utilized analytic representation of vectors instead of expressing a vector in terms of linear combination of the other vectors. This preference was explained with the fact that vector approach becomes more powerful and easier with the help of coordinates (Ayre, 1965; Schuster, 1961 & Schuster, 1961). Moreover, the researcher added that it could be accepted as natural because students had no sufficient experience with vectors to solve problems. Therefore, it is meaningful for students' searching for alternative ways to utilize vector approach in their solutions. Furthermore, Maracci (2005) reported graduate and undergraduate students' difficulties with the notion of linear combination because of having difficulties in perceiving linear combination as object and process. In conclusion, participants' difficulties with expressing a vector in terms of vectors is not specific to these students only.

3. Difficulty in Discriminating Vector Relations and Scalar Relations

One of type of misconceptions that can be seen in students' solutions was writing scalar relation as if it was also valid for vector quantities. Specifically, the students had the opinion or knowledge that the equality valid for vectors is also valid for their magnitudes. In other words, they had difficulties in discriminating the relations valid for vectors and their magnitudes.

This challenge was predicted by the researcher to be possibly encountered before the sessions started. Therefore, the following counter example in the Figure 4-105 was taught and emphasized to the students as a precaution to overcome this difficulty.



In spite of the fact that the relation for the vectors

$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ is correct,

it cannot be mentioned for the length of the vectors as depicted in the following:

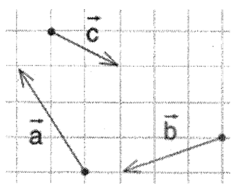
$|\overrightarrow{AB}| + |\overrightarrow{BC}| = |\overrightarrow{AC}|$

$3 \;+\; 4 \underset{?}{=} 7$

Figure 4-105 A precaution example to distinguish vector and scalar quantities

Similarly, the following expression at the top-right of the Figure 4-106 was also shared with the participants in order to make them pay attention to the mentioned misconception type. That is to say, when a vector is the resultant of the addition of two vectors, the length of the resultant vector may not equal to the addition of the lengths of these two vectors. In other words, while addition of two vectors is a vectorial operation, the addition for the lengths of the vectors is a scalar operation. Therefore, the equivalence of a scalar quantity and vectorial quantity is meaningless.

Find $\vec{a} + \vec{b} + \vec{c} =$?



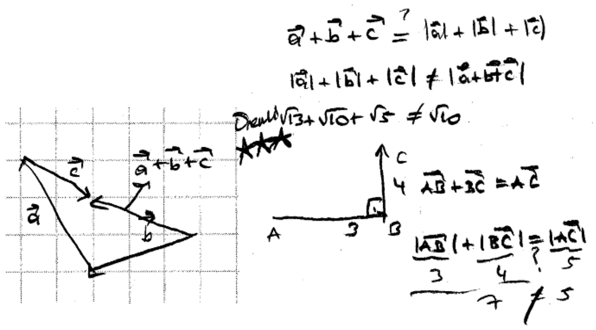


Figure 4-106 A precaution example to distinguish vector and scalar quantities

However, it was seen that these precautions did not aid completely to overcome this trouble. The following two solutions in the Figure 4-107 and Figure 4-108 clearly illustrate that the participants had this type of difficulty. In the first problem in the Figure 4-107, the students were asked to show that *"the length of the diagonals are equal in an isosceles trapezoid" and "the length of* $|AH| = |EB| = \dfrac{a-c}{2}$ *is required to* *be verified"* in the second problem in the Figure 4-108. In both of the solutions, Naci correctly wrote addition of vectors. However, he made operations for scalar quantities as he did with vectorial equations as if it was also valid for the length of vectors without considering the direction of the vectors in his solution. Indeed, this could be correct if the vectors were parallel to each other actually. However, this was not the case for these problems.

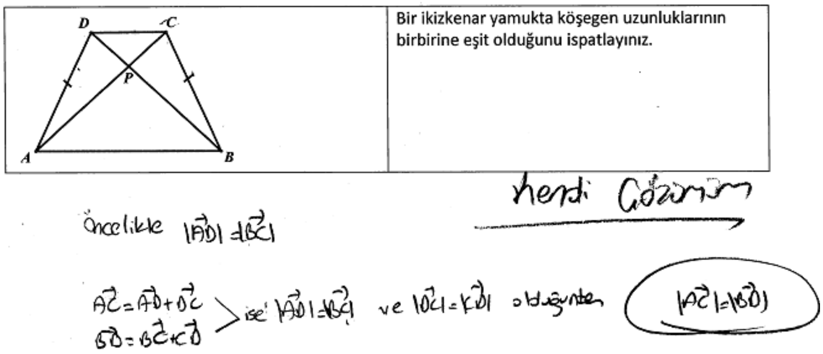


Figure 4-107 Naci's solution to the problem 4 on Trapezoid 1st section

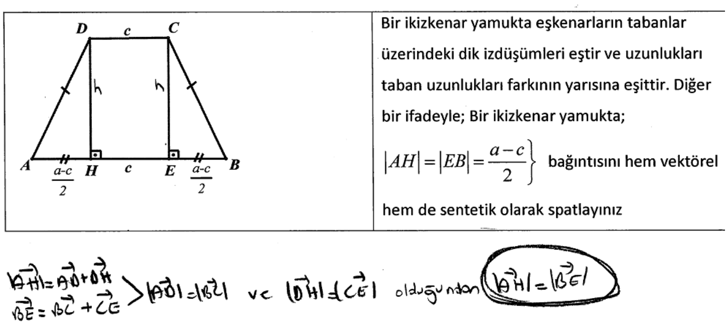


Figure 4-108 Naci's solution to the problem 5 on Trapezoid 1st section

The misconception of this type was also encountered in Ömer's solutions for the problems in the Figure 4-109 and Figure 4-110. "*The equivalence of the length of the diagonals in a rectangle and in an isosceles trapezoid are desired to be verified*" for the problems in the Figure 4-109 and Figure 4-110 respectively.

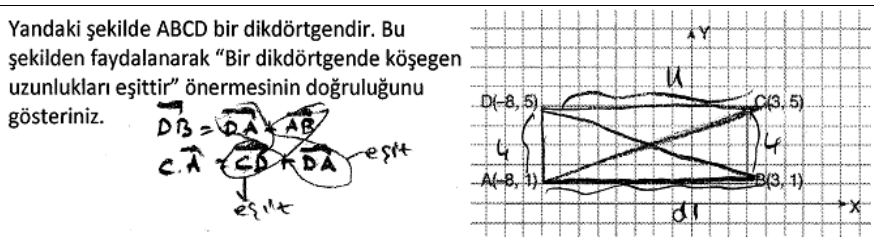


Figure 4-109 Ömer’s solution to the problem 18 on Vectors 2nd section

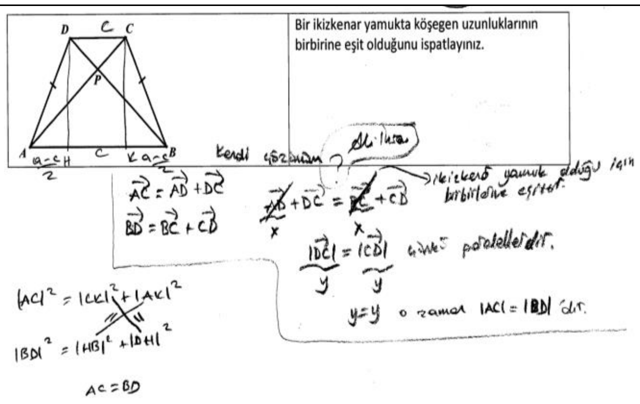


Figure 4-110 Ömer’s solution to the problem 4 on Trapezoid 1st section

Although Naci repeated the same misconception symbolically (*the misconception is framed part in the* Figure 4-111), it is understood that he could eliminate this misconception and solve the problem correctly.

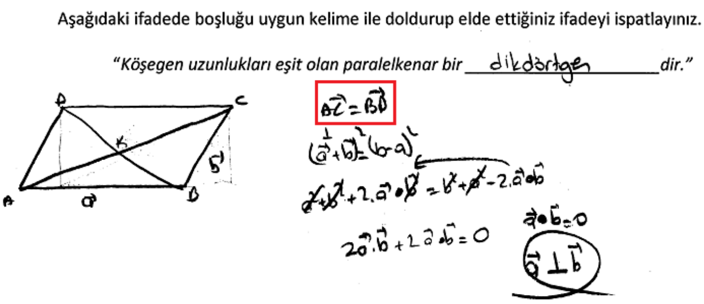


Figure 4-111 Naci’s solution to the problem 1 on Rectangle 1st section

For the following solution in the Figure 4-112, Naci was seen that he could aware of his mistake and he could solve the problem correctly. In the solution, he understood that the relations valid for vectors were also correct for their magnitudes <u>if</u>

the vectors are parallel. "*Determining the type of the quadrilateral is the main focus of the problem in the Figure 4-112 when the midpoints of the sides of a rectangle is combined respectively*".

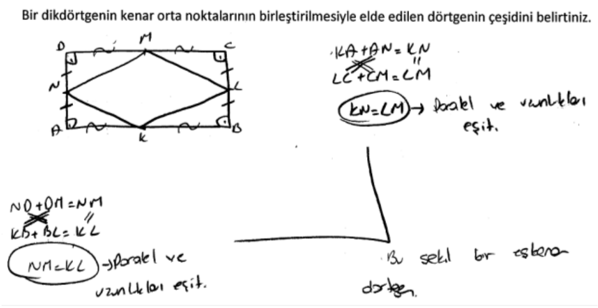


Figure 4-112 Naci's solution to the problem 3 on Rhombus 1st section

In vector addition, the necessary condition for vectors to add their lengths is that the vectors must be parallel. In other words, the lengths of vectors can be added only for the vectors having the same direction. The participants could conceptualize this knowledge as understood from their solutions in the Figure 4-112 and Figure 4-113, and Excerpt 4-33 and Excerpt 4-34. "*Maximum integer value of* $|EF|$ *is asked in ABCD quadrilateral in which E and F are midpoints of the sides* in the problem in the Figure 4-113". Ahmet and Ömer preferred vector approach to solve the problem in the Figure 4-113. After solving the problem, the questions in Excerpt 4-33 and Excerpt 4-34 were directed to Ahmet and Ömer.

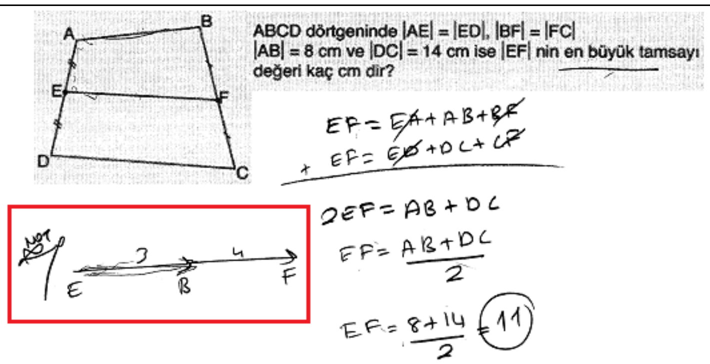


Figure 4-113 Ahmet's solution to the problem 5 on Quadrilaterals 2nd section

Excerpt 4-33 Ahmet's knowledge on addition of vectors

08.07.2015

Researcher: While solving the problem in the Figure 4-113 , you wrote down a note in the figure (framed parts). What do you mean by this figure or note?

Ahmet: While adding vectors, to be able to add their lengths as well, the vectors should be placed end to end.

Researcher: That is?

Ahmet: Vectors must <u>be parallel</u> to each other. Under other conditions, there is nothing like that, you cannot add lengths of vectors while adding these vectors.

Researcher: Nice! You could utilize triangle inequality to solve this problem. Why didn't you use it?

Ahmet: I accepted it as rote learning; therefore, I did not want to use it. Moreover, it seemed difficult to me.

Excerpt 4-34 Ömer's knowledge on addition of vectors

Researcher: Why did you need to add vectors?

Ömer: Because I was able to <u>add the lengths of vectors</u> in case of having <u>parallel vectors</u>.

Having a difficulty in discriminating scalar and vectorial quantities in terms of the participant students in this study can also be seen in the related literature. In the studies of Gagatsis and Demetriadou (2001), Demetriadou (1994) and Demetriadou (1995), the most frequently encountered misconception is "*thinking a vector being equivalent to a line segment*". In this misconception, students treat a vector as if it is a line segment. Gagatsis and Demetriadou (2001) found that 49 percent of the students who preferred vector approach in their solutions and 57 percent of the students who preferred Euclidean and vector approach in their solution had an error of confusing scalar quantity with vectorial quantity. Similarly, students' treating vectors as a scalar without considering direction of vectors while operating with vectors is reported as one of the most common students' faults in some of the studies (e.g., Flores et al., 2004; Gagatsis & Demetriadou, 2001; Aguirre & Erickson, 1984 ; Appova & Berezovski, 2013). More specifically, Grant (1971) reminds that students might confused the vector addition with the lengths of vectors in this vector addition.

4. Difficulty in putting arrows continuously in vector approach solutions

Naci stated the difficulty of continuously putting arrow sign at the top of the vectors in vector approach solutions in the Excerpt 4-36. Therefore, the researcher checked students' solutions from start to end of their written products in this regard. After that, the frequency of not putting arrow on the top of letters for vectors for each of the participants was determined. This case was observed especially in the Naci's and Ahmet's solutions after they started to utilize vectors in problem solutions i.e. especially after special quadrilaterals unit. Moreover, it was an outstanding situation that they started not to put arrows for vectors especially in problems for which they utilized algebra of vectors. In these solutions, they were to write down many vectors actually. Ahmet stated that the constant use of arrows was waste of time and effort in the Excerpt 4-35. Naci and Ömer were observed that they did not add vector arrows for their solutions to 26 problems and the frequency is 5 for Ahmet as seen in the Table 4-19. Ömer and Naci did not need to add the vector sign for their vectorial approach solutions. However, Ahmet's insisting on putting arrow sign in his vectorial approach solutions might be explained with the fact that he is more rigorous in his writing or notes. This is obvious in his written products throughout the study. A reader may distinguish tidiness of Ahmet's handwritings in the solutions presented in the findings chapter. Despite the fact that he almost consistently added arrow sign to his vectorial approach solutions, he evaluated it as a waste of time. An illustration of not putting vector arrow to the vectorial approach solutions for each participant is given in the Figure 4-114, Figure 4-115 and Figure 4-116.

Table 4-19 The frequency of not putting arrow signs in the solutions

Participant	Frequency
Naci	26
Ömer	26
Ahmet	5

An ABCD rectangle with $|AB| = 2|AD|$ and $4|DP| = 3|DC|$ are given where P is a point on $[DC]$. Show that $[AC]$ is perpendicular to $[BP]$.

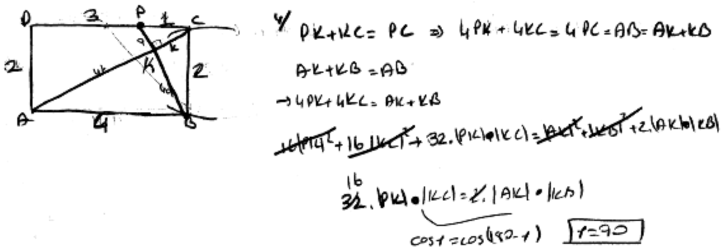


Figure 4-114 Naci's solution to the problem 4 on Rectangle 1st section

Prove that the diagonals of a parallelogram are perpendicularly intersecting if the sides of that parallelogram are equal in length.

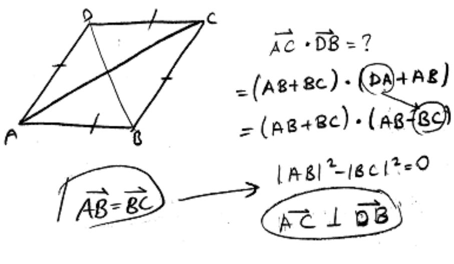


Figure 4-115 Ahmet's solution to the problem 7 on Parallelogram 1st section

Prove that the sum of the squares of two opposite sides equals that of the other two opposite sides for any deltoid.

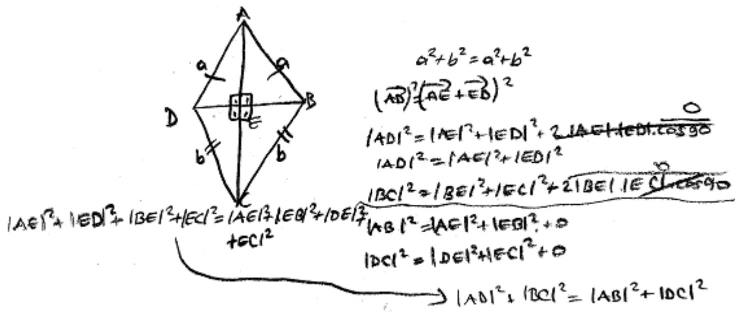


Figure 4-116 Ömer’s solution to the problem 3 on Deltoid 1st section

Actually, they can be accepted as right in their opinions to some extent because there can be found some studies not using arrow sign for vectors in mathematics and mathematics education literature. Instead of using arrow sign, Chiba (1966) states that there are studies representing vectors with capital letters, boldfaced lower case letters. In addition, she states that position vector is represented by means of brackets in some sources. To illustrate; Vaughan (1965) used lower case and bold-faced letters, Klamkin (1970) used single capital and bold-faced letters. However, the manner how they represent vectors are stated in the beginning of their studies. In other words, they represent vectors according to their predefined manner in their studies. Moreover, Engel (1998) states that for the sake of being practical, the solvers may drop out the arrows from vectors “after a while” in their vector approach solutions as long as they care of the difference between “a point” and “a vector” (p.289). Therefore, participants’ natural preference of not putting arrows on vectors can be understood or tolerated. However, it is important to state that there were cases in which participants ignored this distinction. To illustrate; Ömer incorrectly wrote the following position vectors of points (Figure 4-117) and he repeated this wrong representations in different parts of the study (Ö26, Ö103, Ö134 and Ö192 etc.).

However, as stated earlier passages, some of the mathematicians use their own representation style at the beginning of their studies. Chou et al. (1993) state that they represent a vector by AB . They use this representation to denote the vector from a point A to point B. They also used the relation $AB = B - A$. By this way of representation, Ömer's representation shown in the Figure 4-117 can be meaningful.

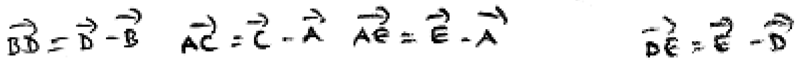


Figure 4-117 Ömer's works for entrance assignment for Parallelogram 1st section

Excerpt 4-35 The reason for not adding arrow by Ahmet on 08.07.2013

Researcher: Which approach did you prefer to solve this problem number 1?

Ahmet: Vector approach.

Researcher: However, you did not put arrow sign *at the top of vectors.*

Ahmet: For the sake of being practical *in my solutions, I started not to put arrow signs.*

Excerpt 4-36 The reason for not adding arrow by Naci

Researcher: Which approach did you prefer to solve this problem number 6?

Naci: I utilized vectors.

Researcher: How can I understand that you have utilized vectors? Because you did not write arrow signs.

Naci: I did not use in order for being practical.

4.4.4 Disadvantages of Synthetic Approach

In this section, students' difficulties and errors in their synthetic approach solutions will be presented. These are reported according to their written products and oral data sources. The need for more knowledge and dependently possibility of making errors in writing formulas and making operational errors are concluded as disadvantages of synthetic approach strategies. Although students have long-term experience with geometry via synthetic approach, they had common challenges, difficulties and errors in their operations. These challenges are presented in this section.

As expressed under the title "*Advantages of Vector Approach in Geometry*", the students stated that solutions in synthetic approach necessitate more knowledge of theorems, formulas or auxiliary statements in mathematics and geometry in comparison with solutions in vector approach. In the following problem (Figure 4-118), the students are asked to show that *"a rectangle is a square if the diagonals intersect perpendicularly"*. Naci was able to verify this statement by means of two approaches: vector approach and synthetic approach. He utilized length of vectors, vector addition and inner product in vector approach solution. However, he used two theorems: Pythagorean Theorem and the theorem "*the length of the median on the hypotenuse of a right triangle equals half of the length of the hypotenuse*" in his synthetic approach solution. The second theorem can be considered as the converse of Thales' theorem. In his solution, Naci stated his thoughts under his solution (*expressions in frame*) in the Excerpt 4-37.

Excerpt 4-37 The comparison of synthetic and vector approaches by Naci

> *As can be seen, the problem can be solved via two approaches. It can be done by both of the approaches. However, the problem cannot be solved via synthetic approach if you do not know converse of Thales' theorem ("muhteşem üçlü"). However, <u>knowledge of these theorems are not necessary in vector approach solution</u>.*

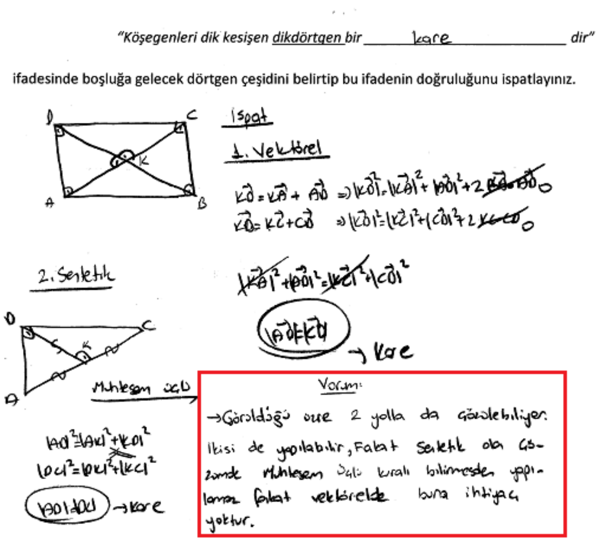


Figure 4-118 Naci's solution to the problem 3 on Square 1st section

The students were observed that they needed to utilize median theorem and the law of cosines in their solutions when they solved the problems through synthetic approach. This requirement of more knowledge brings with the possibility of making some degree of making mistakes in their solutions at the same time. To illustrate, there is possibility of remembering and writing mathematical formulas incorrectly. Ömer tried to solve the problem through the use of the law of cosines in the Figure 4-119. However, he wrote "*sine*" instead of "*cosine*" of the angle in solving the problem in his synthetic approach solution. This is also understood in Excerpt 4-38.

Show that "$2\left(a^2+b^2\right)=e^2+f^2$, where a and b are the sides; and e and f are the diagonals of ABCD parallelogram.

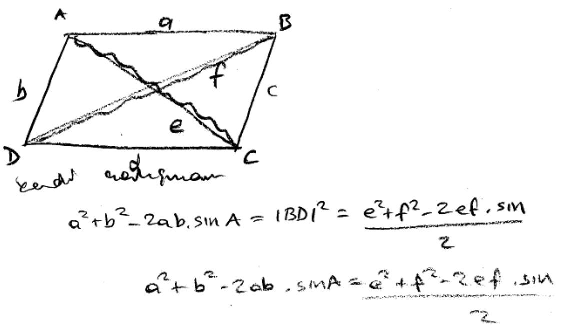


Figure 4-119 Ömer's solution to the problem 4 on Parallelogram 1st section

Excerpt 4-38 Incorrectly remembered formula by Ömer

Researcher: While solving problem 4, you tried to utilize law of cosine. However, you wrote the relation incorrectly while solving the problem. You wrote sine instead of cosine!

Ömer: Actually, I knew this formula but I remembered it incorrectly.

Researcher: Therefore, you could not solve the problem correctly.

Ömer: Unfortunately. I see my fault.

Confusing *sine* and *cosine* in the formulas was also encountered in students' solutions while utilizing the trigonometric formula for the area of a triangle. Ahmet wrote cosines instead of sines in the formula for the problems 7 and 9 in the pre-test of PKQT in the Figure 4-120. The same mistake was also repeated by Ömer in the Figure 4-121.

Figure 4-120 Ahmet's solutions to the problems 7 & 9 on PKQT pre-test

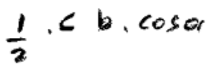


Figure 4-121 Ömer's solution to the problem 10 on PPGT pre-test

Another specific example related to the requirement of more knowledge and dependently making possible mistakes was observed in students' utilizing the formula, which gives the distance of a point to a certain line in the plane. As can be examined in the following solution (Figure 4-122), Ömer wrote the formula incorrectly that is one of the most frequent mistakes made by the participants. Despite the fact that the square root sign is only necessary for the expression in the denominator, Ömer took the square root of the numerator as well. However, the rest of his solution is correct. Since his starting point is incorrect, he could not reach the right answer. Related to this formula, Ahmet stated that he could not recall this formula while solving problem 21 in the pre-test of PKQT despite the fact that he was aware of the necessity to this formula. As a result, Ahmet left the problem empty.

The side AB of ABCD parallelogram is on the line $3x+4y-12=0$. *The coordinates of the vertices C(13,2) and D(5,8) are given. Calculate the area of ABCD parallelogram.*

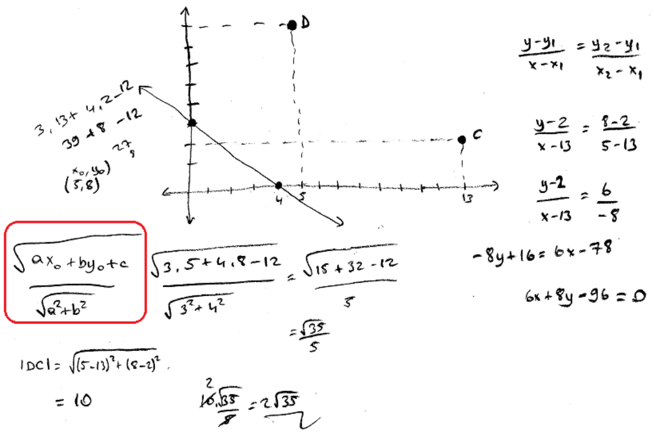


Figure 4-122 Ömer's solution to the problem 1 on Parallelogram 2nd section

Parallel to this finding, Nishizawa and Yoshioka (2008) report the existence of some evidence that students recalled some of the formulas incorrectly. The formula that gives the distance of a point to a plane in the space is an example. They state that the students do not know graphical meaning of the formula and they do not have any idea about the derivation of the formula. This is very similar findings observed in the present study. Therefore, it is better to teach how to derive the algebraic expressions and to interpret their graphical and geometrical meanings of these relations. A nice Chinese idiom expresses this situation sententiously as "it is better to teach a man to fish than to give him a fish". While providing this gain to the students, vectorial approach can be a tool because of the convenience of conducting and developing proofs by this approach.

While solving problems via synthetic approach, the participants utilized "*previously studied theorems*" in this study. However, while utilizing these theorems, they did not consider all of necessary conditions. They ignored some of the conditions while applying these formulae. For example, Ahmet computed the height of isosceles trapezoid with perpendicularly intersecting diagonals by calculating geometric mean of length of the bases in the Figure 4-123. However, this was valid only for right trapezoids. Since he did not consider the trapezoid to be right trapezoid, he calculated the length of the height and dependently calculated the area of the trapezoid incorrectly. Moreover, it should be reminded that this theorem was proved in the classroom via both of the approaches. Despite of these endeavors and facts, it can be said that there is more possibility of incorrectly remembering the formulas and theorems for synthetic approach solutions in comparison with vector approach solutions.

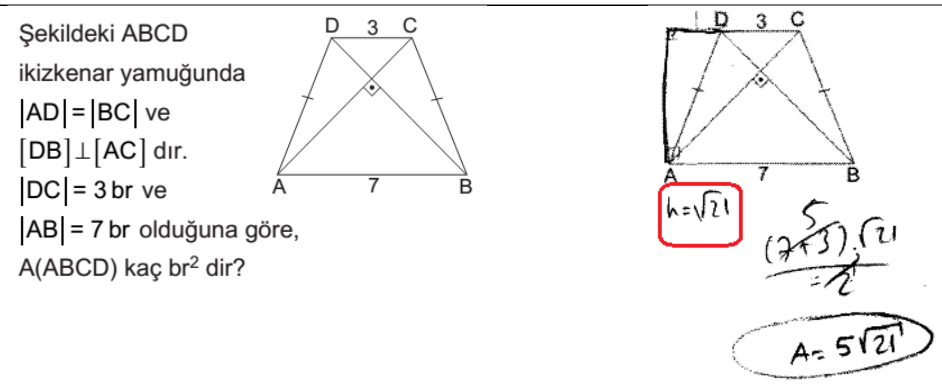


Figure 4-123 Ahmet's solutions to the problem B11 on QAT post-test

The following solution is another illustration for the use of specific geometrical relations without satisfying all of the necessary conditions. In the problem "$[AB] \perp [CD]$ *is given and the area of OABC quadrilateral is to be computed where O is the origin of Cartesian plane*". Ahmet drew $[\mathrm{AC}]$ and then accepted it as the angle bisector in the Figure 4-124. He ignored the necessary condition of the equality of

$|BC|$ and $|OC|$ for $[AC]$ to be the angle bisector. According to theorem "*when a point on angle bisector of an angle then this point is equidistant from the sides of that angle*". Since Ahmet did not consider this equidistant requirement, he thought $[AC]$ as if it was an angle bisector. As a result, he solved the problem incorrectly. Underlying the reason for his fault can be understood in the Excerpt 4-39.

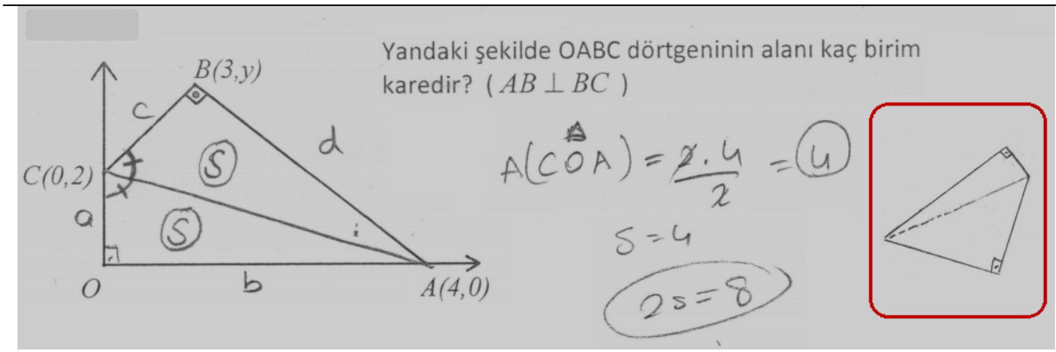


Figure 4-124 Ahmet's solution to the problem 6 on Quadrilaterals 2nd section

Excerpt 4-39 Ahmet's error on angle bisector theorem

Researcher: How do you know that $[AC]$ *is angle bisector and that the diagonal you drew bisects the quadrilateral region in two equal triangular regions?*

Ahmet: Since both of $[AO]$ and $[AB]$ *are perpendicular to* $[OC$ *and* $[BC]$ *respectively, I thought as* $[AC]$ *was angle bisector. Because, when we draw two perpendicular line segments to the sides of angle from a point on angle bisector then the line segments are equal in length.*

Researcher: However, you do not know whether $[AC]$ *is angle bisector or not. Think about this case:(the researcher is drawing a figure of a quadrilateral (framed part) in the Figure 4-124 in order to give a counter example showing that it does not guarantee that a point lies on the angle* *bisector of an angle it if is the intersection point of two perpendicular lines drawn from the sides of that angle). For this quadrilateral, Is subsequently drawn diagonal an angle bisector?*

Ahmet: That is OK. I understood my error.

To sum up, the participants expressed that they need to have more knowledge of theorems and formulae. Moreover, they started to be aware of the fact that they needed frequently to draw additional auxiliary lines, line segments or imaginary tricks that were not given directly in the problems. All of these emerge as challenges for students in synthetic approach.

Similar thoughts or findings for synthetic approach are reported in the literature. Firstly, DiFonzo (2010) reports the need for more knowledge in synthetic approach because synthetic approach is based on theorem knowledge. Krech (1968) accepted drawing auxiliary line segments and Lee et al. (2003) reported the requirement of various set of tricks as the disadvantage of synthetic approach. In the study of Gagatsis and Demetriadou (2001), the participant students' negative opinions as the need for large pieces of knowledge, figure difficulties, difficulties in recalling some theorems and complicate thought emerged as the negative aspects of Euclidean geometry. These are similar findings or statements with the findings in experienced in the current study.

4.4.5 The Effects of Long-Term Training by Synthetic Approach

In spite of inferences or reflections about disadvantages of synthetic approach, it was observed that the students could not give up synthetic approach strategies in their solutions after being educated for many years in their school life. This is already what was targeted in this study. In other words, superiority or priority of any approaches was not argued or eschewing the synthetic approach was not asserted at any step of the study by the researcher. It was seen that the idea of not asserting the priority or obligation of any approaches was understood by the participants according to Ahmet's expressions in the Excerpt 4-40.

Excerpt 4-40 Ahmet's considerations about the priority of approaches

I understood at the end of the study that there is not a strict rule entailing the use of vectors for all problems. It is more important to learn when, how and why to prefer a certain approach in problem solving. Giving more easier or practical proofs of the properties for the subjects might have probably made the topics more understandable in my opinion. This should be considered by curriculum developers. In this way, we would not be afraid of proofs. I would like the other students to receive a treatment prepared for this study. Rather than preferring or searching practical ways to prove statements in mathematics and geometry lessons, selecting harder proofs makes proofs inaccessible, non-achievable and tedious in terms of us, as learners.

It can be said that the participants seemed to conceptualize and grasp when to and how to use vectors in geometry problems according to problem types in the context of quadrilaterals unit. In spite of this fact, the students were observed that they indispensably and firstly tried to solve some of problems by synthetic approach. The Table 4-20 shows the frequency of students' first preferences that they resorted to solve problems at the end of units.

Table 4-20 Participants' first preferences in solving the problems

Participant	**Analytic Approach**	**Synthetic Approach**	**Vector Approach**	**Total**
Naci	1	12	44	57
Ömer	7	21	29	57
Ahmet	3	13	41	57

The participants were understood that they were free to select the approach while solving the problems according to the Table 4-20 because of the different pattern of preferences among participants. This was also asked to Ömer who utilized vector approach less than Ahmet and Naci. His answer is given in the Excerpt 4-41.

Excerpt 4-41 Underlying reason for the preference of vector approach by Ömer

Researcher: What is the reason for utilizing various approaches in your solutions? Is it because of imposition of the researcher?
Ömer: Any of the imposition would not make any effect on me. I learnt how to utilize vectors in the lessons and I am impressed with this.

According to the Table 4-20, Naci, Ömer and Ahmet preferred synthetic approach for 12, 21 and 13 times as the first method to solve problems. These frequencies cannot be ignored and need to be taken into consideration. Rumanova (2006) and Baki and Akşan (2014a) found that students frequently resorted to synthetic approach in their studies. Gagatsis and Demetriadou (2001) explain students' high frequency of resorting synthetic approach with the fact that it had a long and prevalent place in the history of mathematics. Moreover, the most frequent idea of teaching geometry is Euclidean in the schools. For example, Dorier et al. (2000) state that the teaching of geometry is mostly based on synthetic approach in France in spite of the fact that Cartesian and vector geometry are targeted to instruct. Furthermore, Stephenson (1972) indicates that "the preparation of secondary teachers in the area of geometry is primarily concentrated on Euclidean geometry from the synthetic or the metric approach". This is the case for our country as well. If the geometry textbooks are examined in this direction, it can be easily inferred that the prevalent in the textbooks is also synthetic approach. By considering all of these situations, Harel and Sowder (1988)'s explanation with the fact that students use their teachers' strategies or textbooks' strategies that they call it external schema. Therefore, participants' application of synthetic approach strategies as the first preference can be understood more clearly.

4.4.6 Students’ Habits of Listening and Studying Lessons after this Teaching Experiment

The students stated that there has been a change in their listening and studying habits after participating in this study as understood from their reflections in the Excerpt 4-42 and Excerpt 4-43.

Excerpt 4-42 Ahmet’s reflection about the effects of the implementation of the teaching experiment 10.09.2013

> *By the help and in the light of this teaching experiment, I have learnt how to listen teachers in the lessons and understood how to establish links among the topics. As a result, I think that I began to get better performances on courses from then.*

Excerpt 4-43 The change in Naci’s studying habit on 26.08.2013

> *While studying by myself I do not directly accept a theorem or rule in geometry or mathematics without learning its proof. Moreover, I necessarily question underlying reasons for the statement that I encounter. I try to prove the mathematical sentences on my own without looking for proofs on different textbooks while studying. I am able to prove these sentences mostly and this makes me happy. In addition, I study other courses in the same manner.*

These shifts can be interpreted as the pedagogical effects of teaching geometry through multiple approaches. Schuster (1961) asserts that being able to solve geometrical problems by means of analytic and vectorial approaches provides pedagogical advantages to the students in addition to mathematical advantages of utilization of these two approaches. Moreover, according to UICSM, students feel themselves as privileged when they learn geometry via various approaches because of the original aspect of it. Similarly, Bundrick (1968) notes that students need to study more and they feel themselves happier because of the novelty effect of learning geometry through a new way: vectorial approach. Besides, Barbeau (1988) advise that students should have an opportunity to defense their solution ways through different methods and then there should be provided a classroom environment in which the students can talk about various methods. According to him, students' excitements might be stimulated in this way. Lastly, Scott and Rude (1970) note that including analytic and vector approaches into geometry teaching has a motivating power. In the light of these, it might be inferred that including various approach in teaching processes might have some effects on students' habits or teachers' rituals in classrooms.

As understood form participants' reflections, they think that they have become better follower or listener of the courses or teachers. This might be beneficial for their success in mathematics because Dursun and Dede (2004) report that following the courses in a better manner is one of the most important factors in terms of students' achievements.

4.4.7 Initial and Final Situations of Participants Related to Geometry Teaching through Vector Approach according to Pre and Post-Interviews

It was expressed earlier that although the teachers are required to teach geometry in a medium enhanced by analytic, synthetic and vector approaches and reasoning-proving, this has not been the case in a complete meaning in the classes according to interviews conducted with students and geometry teachers as well. Particularly, it is understood form the participants that vector approach had never been utilized in their geometry classrooms. The students mostly thought that their teachers did not have any knowledge about geometry solutions through the use of vector concepts. Moreover, Naci stated the same situation also for private tutorial lessons in different settings called as "*Dershane*". Moreover, Ahmet thought that if their teachers had knowledge related to use of vectors in this manner, they would have utilized this method in classroom for geometry teaching. In the light of the interviews conducted with participants at various instants of the study, the researcher has almost become sure that participant students did not have any idea or knowledge that the geometrical problems could be solved through the use of vectors by the time they participated in this teaching experiment. According to the students pre-test solutions and their expressions, the students were not taught vectors in the manner utilized in this study. On the contrary, their teacher never mention about the use of vectors in geometry problem solving. They stressed that they learned vectors as an isolated or disconnected topic at different phases of the study. These inferences are made according to the Excerpt 4-44, Excerpt 4-45 and Excerpt 4-46.

Excerpt 4-44 Ömer's thoughts about vector approach in miscellaneous context by considering his knowledge before participating the study

> *I had never thought that vectors could be used in geometry problem solving. Despite the fact that we learnt vectors in grade levels 9 and 10, I do not think that neither our teachers nor my friends had such a knowledge about the use of vectors in problem solving in geometry. We had not learned vectors as a method or tool for problem solving. Therefore, I really astonished when I learned vector solutions of geometry problems in this study. I did not have any knowledge on this field. At the beginning, I did not think that I would utilize vectors in problem solving and I would continue to solve geometry in classical ways that I familiar with. However, as the study progressed and I started to grasp vector approach, I began to think that I needed to learn this approach necessarily. Although I firstly utilized synthetic approach solutions in certain type of problems, I enforced myself to solve these problems by means of vectors as a second way.*

Excerpt 4-45 Excerpt from an interview with the participants on 05.07.2013

> *Researcher: My friends: How is it going on the studies in this study?*
> *Ömer: My opinions have changed a lot. At the beginning, I was thinking that vector is something whose length is found and it is an independent topic. However, I learnt that vector could be utilized in teaching topics and in solving geometry problems.*

Naci: Yes, there is an improvement in terms of us. Before, I was confusing the formulae related to vectors. Now that, I also learnt how to derive the formulae. Proof based geometry teaching was very important factor for me to gain this skill. There was a great difference between proof based geometry teaching and geometry teaching dependent on remembering.
Ahmet: I was thinking that vector was just a topic. I never thought vectors as a tool. I had no such information. I was mostly thinking that there was a topic, which was namely "vector", and we are asked to some questions on this topic on the examinations. After the examinations, I will not have anything to do with vectors. That is the end of my job with vectors. It was very interesting for me to learn that vectors can be used as a tool in geometry teaching and problem solving. I concluded that vector solutions are brief, compact and reasonable.

Excerpt 4-46 Ahmet's and Naci's thoughts about vector approach by considering their situations before participating the study

24.07.2013
Ahmet: We have been learning vectors in geometry course because it is included in our geometry curriculum. However, we were not taught any geometry topic through vectors. We learnt this idea in this project. However, the most important thing is that we should use vectors in problem solving and continue using them. It should not be something like we learnt and we are finished. Because vector approach is ultimately a useful method.
Researcher: I saw that you utilized vectors for all items in entrance assignment for parallelogram. All of your answers are correct! Why did you prefer using vectors in a problem related to parallelogram? I expected you to prefer analytic methods since it is given with coordinates of vertices.
Naci: But, vectors are also useful in analytic geometry.

Researcher: Did you know this before this project? Were you be able to do this earlier?

Naci: Absolutely no! I have learnt the use of vectors in geometry with this project. We learnt vectors in 9^{th} and 10^{th} grade levels; however, just a subject in itself such as what is a vector? What are equivalent directed line segments? We learnt vectors like this.

Researcher: I know that you learnt triangles and vectors when you were students at grade level 9 and 10. I want to know whether you have learnt triangles via vectors either.

Naci: No. We did not learn triangles through vectors.

In order to determine the knowledge level or initial situations of students about teaching geometry through vector approach, the participants were interviewed at the beginning (20.04.2013) and at the end (04.10.2013) of the study. In order not to repeat the questions, only students' responses are presented in the following excerpts. Interview questions are presented at the Appendix B. The students' answers to these questions in the pre-interview and post-interview are given individually as follows.

Excerpt 4-47 Naci's answers to pre-interview questions

I do not have any idea about vector approach solutions. I have not heard anything about vector approach from my friends so far. I have not any experience on studying geometry through vectors. My geometry teacher have never used vectors neither in geometry teaching nor in solving geometry problems and proving. I have no any knowledge whether my teacher has positive or negative opinion about vector approach. I do not have any idea if vector approach solutions are more understandable, reasonable or elegant.

Excerpt 4-48 Naci’s answers to post-interview questions

I had experiences on studying geometry through vector at the end of this study. However, I do not think that my friends have knowledge or experience in learning geometry via vectors. Our teacher never utilized vectors in geometry courses and he did not solve geometry problems with the help of vectors. I think that our geometry teacher has negative opinion about vector approach. Moreover, he does not have any idea about how to teach geometry through vectors. In my opinion, vector approach solutions are more elegant, evident and reasonable than synthetic approach solutions.

Excerpt 4-49 Ömer’s answers to pre-interview questions

I am not sure whether I am knowledgeable about vector approach or not. However, I think that my classmates do not have any experience on vector approach. Besides, I think that our geometry teacher might use vectors in problem solving. However, he does not use vectors frequently. Moreover, I am not sure what our geometry teacher think about vector approach. Since I am not sure about vector approach, I have not any idea about vector approach solutions and proofs.

Excerpt 4-50 Ömer’s answers to post-interview questions

At the end of this study, I can say that I have enough experience in learning geometry and solving geometry problems through vectors. However, I think that my friends do not have information or experience related to vector approach. I do not think that our geometry teachers utilize vectors in geometry problem solving. Since my geometry teacher has never utilized vectors in classes, I guess that he does not know how to teach geometry via vectors. Moreover, I do not think that my geometry *teacher has positive attitude toward vector approach. The geometry solutions, which are constructed by vector methods, are more reasonable, elegant and easier to understand in comparison with synthetic approach solutions.*

Excerpt 4-51 Ahmet's answers to pre-interview questions

I do not have any knowledge about learning geometry through vector approach. I have not heard anything about vector approach from my friends or geometry teachers so far. I have never witnessed my geometry teacher's utilizing vectors neither in lecturing geometry topics nor in solving geometry problems. However, I think that our teacher has positive thinking about vector approach. Since I have no any knowledge about what the vector approach is, I cannot say anything about vector approach solutions whether they are reasonable, understandable or not.

Excerpt 4-52 Ahmet's answers to post-interview questions

I have knowledge about teaching and learning geometry via vector approach. I have also experience in solving geometry problems and proving mathematical statements through vectors. My friends do not have any knowledge on this issue, in my opinion. Although I am sure that our teacher has not used vectors in our geometry courses, I do not have idea about whether he is positive or negative towards vector approach. In my opinion, vector approach proofs and solutions are more reasonable and more elegant than synthetic approach proofs and solutions. I also think that vector approach solutions are more obvious than synthetic approach solutions.

CHAPTER 5

IMPLICATIONS AND RECOMMENDATIONS

As the final chapter of the study; chapter five includes two sections. Implications and recommendations to the researchers, teachers and curriculum developers and to the further studies will be presented.

5.1 Implications

In this study, the participants had an opportunity to learn quadrilaterals through an instruction including integrated use of vector approach with analytic and synthetic approaches. In other words, a multiple approach instruction was utilized to teach quadrilaterals. It was aimed to identify contributions of the instruction in which vectorial approach is integrated with synthetic and analytic approaches on quadrilaterals to eleventh grade students' problem solving strategies. Specifying students' firsthand reflections and experiences related to designed instruction was important to seek for answers to the research questions of the study.

Analysis of all kind of data from participants showed that while the students did not have any idea about the use of vectors in problem solving and proving at the beginning of the study, they started to utilize vectors frequently to solve problems and prove geometrical statements towards end of the study. The changes in students' solutions and outstanding inferences are presented in the following paragraphs.

Participants were observed that they started to utilize vectors to solve the problems of which they had solved through similarity and congruence of triangles until having participated in this study. Therefore, it is understood that vector approach can be an alternative to similarity and congruence (SAS and AAA) of the triangles. Moreover, the students began assigning a vertex of or a point on the given quadrilateral as the origin of a coordinate plane while solving some of the problems. However, the quadrilateral was not given on Cartesian plane for these problems actually. They could be able to utilize vector approach efficiently by this way. This utilization; that is, "*analytic representation of vectors*" can be an alternative to algebra of vectors.

The students could conceptualize that taking square of a vector is a measuring rod (Troyer, 1968) for a vector. In other words, transition from a vectorial quantity to a scalar quantity can be achieved through inner product of a vector with itself. Furthermore, the students preferred vector approach strategies for the problems on quadrilaterals containing parallel or / and perpendicular components.

Participants frequently resorted to vector approach to calculate the area of a quadrilateral, which is given with coordinates of vertices. They were observed that they understood sufficiently the steps to be followed. They started to partition the given polygon into triangles and to compute the area of each triangle through vector approach. Finally, they added all of the areas of subsequently formed triangles. Actually, this method is known Surveyor's area formula (Braden, 1986). In addition, while the participants had difficulties in calculating the area of a polygon given on coordinate plane depending on position of it in analytic and synthetic approaches, it was understood that the position is immaterial in vector approach calculations. It should be noted that the participants were trying to apply analytic or synthetic strategies to solve this kind of problems before participating in this teaching experiment. Moreover, representing sides of a quadrilateral through vectors and roughly drawing the picture was found as practical when it is difficult to draw the given picture on coordinate plane. It is also worth mentioning that students' being able to manipulate with literal expressions makes it easier to calculate area of polygons on coordinate plane though vectorial area formula, which might yield large numbers.

Students were understood that they frequently resorted to vector approach strategies to solve proof based problems in this study. The underlying reasons for this preference was attributed to several factors. Firstly, they found vector approach as making easier to organize or to develop a proof or a solution. Secondly, they concluded that a solution through vector approach did not necessitate huge knowledge of theorems and formulae. In addition, the students expressed that vector approach solutions were more elegant and secure in comparison with the other approaches, in most of cases in the study. Although these were positive opinions of students related to vector approach, they were observed that they had some sort of difficulties with vector strategies. To illustrate, they were understood that they had difficulties in determining the angle between two vectors when they were not in a standard position. Moreover, expressing a vector in terms of other vectors was another difficulty for them. While implementing operations on vectors, they could possibly forget that the operations for scalar might not correct for the vectors. Finally, they complained frequently for the necessity of putting arrow sign at the top of vectors in vector approach solutions. These difficulties can be reported as the disadvantages of vector approach solutions in terms of the participants.

The time for providing necessary prerequisite knowledge on vectors to teach a geometric topic through vector approach was not found as an issue to discuss on. As the students get familiar with studying quadrilaterals through vectors, the practical aspect of vector approach is a way to compensate this allocated time. Therefore, it is not reasonable to argue that teaching geometry through vector approach is a waste of time.

It was experienced as beneficial that geometric figures should firstly be presented on coordinate plane so that the students can explore and deduce their properties.

Although the students were taught quadrilaterals through multiple approaches under the same conditions, the students' preferences of approaches indicated a different pattern when they learnt geometry through multiple approaches. In fact, a student wants to select the easiest one according to his or her convenience if he or she learns geometry through multiple approaches (Kwon, 2013). Finally, a student feels himself exclusive or privileged if he learns geometry via a novel way. Particularly, vector approach could provide this feeling to the participant students in this study.

The participants had a chance to learn geometry through vectorial approach in addition to analytic and synthetic approaches in this study and; hence, they could utilize these approaches within a problem. As stated in the literature review, an approach might have a complementary role on the other approaches. In addition, the participant students could be able to solve lots of problems by means of several ways in this study. Each way of the solution is as a result of applying different approaches, which the participants learnt in the present study. Therefore, the students' solutions ways can be enhanced or diversified if they have an opportunity to learn geometry through multiple approaches.

5.2 Recommendations

5.2.1 Recommendations for the Practice

In this part of the study, recommendations for mathematics and geometry teachers, teacher educators and curriculum developers in mathematics education area will be presented according to the findings and implications of the study, and the researcher's experiences transpired throughout the current study. The suggestions are as follow.

In the new geometry curriculum development studies, sub-branches of mathematics such as geometry, analytic geometry and algebra should be related to each other. While setting a connection among these sub-branches, vector is a beautiful tool to integrate analytic geometry, algebra and geometry. Furthermore, mathematics should be related with other sciences such as physics. Actually, vector is a nice tool to realize this aim, too. Therefore, while teaching materials are being prepared, they should be prepared by considering different contexts. That is to say, an example on vectors from physics content should be studied in geometry courses so that the students understand that the vectors in physics and in geometry are not different things. This was accepted one of the sources of some misconceptions about vectors in some of the studies (Dimitriadou & Tzanakis, 2011; Ba & Dorier, 2010).

If vectors are desired to be used in problem solving and proving in geometry as vector approach, they should be one of the continuous parts of geometry teaching. By teaching vectors separately as a disconnected topic, teaching geometry through vectors cannot be provided. Choquet (1969) summarize this fact very nice in the following quotation.

> We have a "royal" road based on the concepts of "vector space and inner product" but pupils cannot be cannonballed along this road without preparation, especially at an age when they are not very familiar with algebraic operations (p.14).

Therefore, the use of vectors in problem solving cannot be expected from the students unless they learn how to achieve this aim. In addition, vectors should not be postponed to the later years in academic calendar. Similarly, vectors should not be the final chapter of a geometry courses.

While teaching vectors, the consideration of the following points will be beneficial for the teachers and curriculum developers for the sake of preparing an efficient lesson plan to teach vectors and utilize vectors in geometry. Based on the findings, not only the students, but also the teachers had no idea about how to integrate vectors with other approaches to teach geometry. As a result, curriculum designers should prepare in-service teacher training courses to make curriculum innovations effective in the classes.

Vectors should not be presented frequently in standard positions so that students can apply their knowledge any of the object in various positions. In other words, vectors should be given in different positions in the teaching materials on vectors. This is also true for geometric figures. The teachers should not use prototypical shapes or positions for the geometrical figures while teaching them.

While teaching vector addition and subtraction, displacement analogy and changing subtraction to addition were found as effective and conceptual methods in terms of students. In addition, shadow analogy used in this study was found effective and made vector projection more understandable for the sake of the students. Besides, since vectorial formula to calculate the area of polygons necessitates dealing with large numbers, it would be helpful to teach how to manipulate with numbers in the context of literal expressions. Therefore, these methods should be utilized while teaching vectors in the classrooms.

For the vector addition of two vectors: $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$, this relation may not true for the length of the vectors. That is to say $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ is not enough to conclude that $\left|\overrightarrow{AB}\right| + \left|\overrightarrow{BC}\right| = \left|\overrightarrow{AC}\right|$. Despite the necessary precautions were considered and applied in the preliminary courses in this study, the participants were observed that they continued to make this type of error in some of their solutions. Therefore, the probability of encountering with this misconception that is confusing scalar relations with vector relations should not be neglected while developing geometry programs and courses.

The importance of the order of the quadrilaterals' vertices whose coordinates are specified should be highlighted so that the students do not draw any other figure, which is not asked in the problem. In fact, a convex polygon may be turned into a concave polygon if the order is not taken into consideration. To emphasize the importance of the order, clockwise rotation or counter clockwise rotation can be utilized.

Students had difficulties with expressing a vector as a linear combination of other vectors while solving a problem through vector approach. In other words, they had difficulty to decide which vectors were the most appropriate to choose in order to express or reach a resultant vector. As a result, this is another point to be taken into consideration. Displacement analogy can be a solution for this trouble. However, it is also interesting that different selection of vectors to express resultant vector might result in the proofs or solutions.

The use of vectors with coordinates was found as efficient and convenient way while solving problems. Therefore, it is recommended that the students should be taught how to utilize coordinates if any information about them is not given in the problem. In fact, assigning any of the vertex or any point on the figure as an origin of a coordinate system is labelled as origin principle by Ayre (1965).

At the beginning and during the course of the study, the students were observed that they preferred to utilize synthetic approach or analytic approach to solve problems, which contain vectorial components. As the study progresses, there were encountered with the cases that a problem without any vectorial clue was solved via vector approach by the participants. Therefore, it is important that the teachers should have knowledge and experience on this issue. The students should learn such cases from their teachers in their geometry lessons. In other words, the teachers should teach that a problem can be solved through vectors when there are not any vectorial representations in the problem. Actually, the students might be expected to solve this kind of problems via synthetic approach.

There is a lack of problems to be solved through vector approach in number and variety. In this study, this gap was tried to be filled to some extent in quadrilaterals context. However, there is still need to develop this kind of problems not only in quadrilaterals unit but also for other geometric figures to show the beauty and power of vectors in geometry problem solving. Moreover, it should be expressed that vector proof of some geometric properties and vector solution to some geometric problems could not be developed by the researcher. This presents a gap to be filled for interested persons. The teachers can develop these proofs and solutions with their students' in their courses.

It is important to share a student's solution with other students if multiple approaches are utilized together for the solution of the problem in terms of other students. The teachers should appreciate solution methods via an unfamiliar way, which are constituted by the students. However, a solution through an approach can be the best, easier or more convenient for some of the students. In fact, efficiency, simplicity and elegance of a solution through an approach can be evaluated as subjective. Individual differences should not be neglected or forgotten. Therefore, superiority or priority of any approaches should not be asserted by considering individual differences or preferences of the students. In other words, the teachers should not enforce their students to use any of the approaches. Instead of this, providing a variety of alternatives to the students should be the focus.

In this study, proof and reasoning was the indispensable component of the instruction. The participants did not learn any of the geometric property, theorem or statement without studying on their proofs. These statements were proved through multiple approaches as much as possible. The participants did not express any kind of unsatisfaction while studying on proofs. Therefore, geometric theorems or statements ought to be taught with proofs and underlying reasons by considering students' understanding levels. In other words, each of the students can acquire proving abilities after the application of well-planned and proof-based teaching programs.

In the light of all of the experiences as a result of this study, it is necessary to re-examine high school geometry programs in terms of multiple approaches and textbooks should be revised and prepared in accordance with the given suggestions.

5.2.2 Recommendations to the Further Studies

As a result of the conclusions drawn from this study and the implications of the findings, the writer presents the following suggestions for the further studies.

This study will possibly contribute to mathematics education literature in terms of geometry teaching via analytic, synthetic and vector approach by means of its qualitative results and some additional quantitative results. Therefore, it seems important to implement this study through experimental research design, one of the quantitative studies in order to search for the effects of analytic, synthetic and vector approach instruction on high school students' geometry achievement. In this way, the effects of a geometry instruction through vector approach integrated with analytic and synthetic approaches on students' geometry achievement and students' solutions can be investigated experimentally.

In this study, vector approach with the integration of analytic and synthetic approaches were utilized to teach quadrilaterals unit. In order to grasp the complete picture, other subjects such as triangles, plane analytic geometry, solid analytic geometry, complex numbers, trigonometry and conics should be taught through multiple approaches. After that, the effects of the designed instructions and hence students' reflections can be investigated through qualitative and quantitative research designs. In addition, this study was conducted with relatively higher achieving students at eleventh grade because of predetermined reasons, which are presented under the title "P*articipants*". Hence, this study can also be repeated with the participants from various achievement levels and different grade levels. In this way, the findings observed in this study and to be observed in future studies can be compared, which is important to reach a broader or more general interpretations or inferences.

After determining the topics in mathematics and geometry that can be taught by an instruction including multiple approaches in middle school levels, the effects of the instruction on students' success and solutions in terms of variety can be investigated through experimental and qualitative studies.

In this study, it is concluded that this concept can be utilized as a tool in problem solving in geometry. Since vector is a common concept for mathematics and physics, the utilization of vector in problem solving is evident for the nature of physics. There are some studies (Dimitriadou & Tzanakis, 2011; Ba & Dorier, 2010) reporting the reasons for students' experiencing difficulties and misconceptions on vectors because of the inconsistency of teaching vectors in physics and geometry. By considering these facts, an experimental research can be implemented to reveal the difference between fundamental physics achievement test mean scores of the group of students who learn geometry through vector approach integrated with traditional approaches and the group of students who learn geometry through traditional approach.

The topics in plane analytic geometry and in space are nearly the same and the operations are conducted in a similar way. One of the difference between them is the number of the component (Bundrick, 1968; Hershberger, 1971; CEEB, 1959; Pettofrezzo, 1966; Fehr, 1963). Therefore, an experimental study can be conducted to seek for the difference between transfer test mean scores of the group of students who learn plane analytic geometry through vector approach integrated with traditional approaches and the group of students who learn plane analytic geometry through traditional approach. Meanwhile, the transfer test includes the items from the topics, which are common for plane and space analytic geometry in accordance with the curriculum.

According the findings of the study, it is concluded that vector approach can be alternative to similarity and congruence in synthetic approach. There were some researchers or mathematicians (Lee et al., 2003; Vaughan & Szabo, 1973 & Choquet, 1969) in mathematics education field utilizing vectors instead of improving geometry courses or requirements via similarity and congruence theorems. Firstly, the scope of vectors being alternative to similarity and congruence needs to be specified explicitly. After that, an experimental study can be implemented to examine the effects of geometry instruction through vector approach integrated with traditional approaches to teach similarity and congruence of triangles on students' geometry achievement.

It would be beneficial to examine students' errors when they solve geometry problems through vector approach. In this study, students were observed that they might have confused the logic for scalar quantities with the logic valid for vectorial quantities. Therefore, a qualitative study focusing on the points where the students have confusion between scalar and vectorial properties can be conducted. Furthermore, a vector teaching considering these errors and solutions for these errors can be developed. In fact, the effects of such an instruction on students' products can be investigated qualitatively and quantitatively.

So far, the recommendations for the further studies presented above are mostly based on the students. However, considering teachers and preservice teachers is also important while presenting recommendations for further studies in mathematics education field. The following recommendations for future research are focused on teachers and preservice teachers.

One of the deficient field in the literature is on candidate and in-service teachers' knowledge on geometry teaching via multiple approaches in addition to synthetic approach. Therefore, these teachers' subject matter knowledge and pedagogical knowledge on geometry teaching through vectors and on geometry teaching through multiple approaches can be investigated. It is important to reveal existing knowledge level of teachers on and probe their reflections about teaching through multiple approaches in order to realize an effective teaching in classes. Moreover, Bayraklı and Akkoç (2014) found that pre-service mathematics teachers' pedagogical knowledge about vector approach is insufficient in each component. Therefore, it is necessary to heal this problem. After developing teaching geometry through multiple approaches course for preservice geometry teachers in mathematics education departments and after developing in-service teacher training courses, a comprehensive qualitative study can be implemented with these teachers to determine in-service and pre-service teachers' degree of readiness level to teach geometry through multiple approaches and their actual reflections on this instruction.

Specific to Turkey, since geometry teaching through multiple approaches for high school levels were included for the years 2011-2015, a qualitative study can be implemented with in-service geometry teachers who taught geometry in these years in order to determine to what extent they could realize the requirements of geometry curriculum, the difficulties that they experienced with teaching geometry through multiple approaches and especially with vector approach, and their explanations for these difficulties. In line with these situations, their beliefs on this issue can also be studied.

If the contributions or effects of vector approach in geometry teaching are desired to be examined fairly and comprehensively, the treatment of any topic through vectors should be long enough in time for future studies because of the two reasons. Firstly, it is not easy to shift from a familiar and habitual way to a novel way. Secondly, getting familiar with vectors and understanding how to utilize vectors in geometry problem solving necessitate sufficient time. Studies on well-designed lessons to teach geometry through multiple approaches can be useful and helpful for the teachers and researchers in mathematics education area.

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APPENDIX A

OBJECTIVES OF THE UNIT

1. Related to Quadrilaterals, the students will;

    1.1 explain quadrilateral and its basic elements and make applications.
    1.2 prove theorems related to quadrilaterals and make applications.
    1.3 calculate perimeter of a quadrilateral, prove theorems related to the area of quadrilaterals and make applications.

2. Related to Known (*Special*) Quadrilaterals, the students will;

    2.1 explain trapezoid and prove theorems related to properties of trapezoid.
    2.2 derive the formula giving area of trapezoid and make applications.
    2.3 explain parallelogram and prove theorems related to the properties of parallelogram and make applications.
    2.4 derive the formula giving area of a parallelogram and make applications.
    2.5 explain rectangle and its properties.
    2.6 derive the formula giving area of a rectangle and make applications.
    2.7 explain rhombus and prove theorems related to properties of rhombus and make applications.
    2.8 derive the formula giving area of a rhombus and make applications.
    2.9 explain square and prove theorems related to properties of square and make applications.
    2.10 derive the formula giving area of a square and make applications.
    2.11 explain deltoid and its properties and make applications.
    2.12 derive the formula giving area of a deltoid and make applications.

2.13 classify quadrilaterals and explain the relations among them.

APPENDIX B

INTERVIEW QUESTIONS

Interview with participants on vectors:

The following questions were asked to the students to understand their initial positions related to vector approach. The questions are:

1) Do you have any knowledge or experience with learning geometry or through vector approach?
2) Do you have any knowledge or experience with proving through vectors?
3) Does any of your friends mention about teaching geometry through vector approach in their classes?
4) Does your geometry teacher utilize vector approach while teaching geometry?
5) Does your geometry teacher utilize vectors while solving geometry problems?
6) What is the opinion of your geometry teacher about vector approach?
7) Are vector approach solutions more understandable than synthetic approach solutions?
8) Are vector approach solutions more elegant than synthetic approach solutions?
9) Are vector approach solutions more reasonable than synthetic approach solutions?

APPENDIX C

SAMPLE ENTERING ASSIGNMENT FOR RECTANGLES

Konu: DİKDÖRTGEN

I. BÖLÜM

Kazanım 1

Dikdörtgeni ve özelliklerini açıklar.

Problem

Köşelerinin koordinatları A(-4,-6); B(4,-2); C(1,4) ve D(-7,0) olarak verilen ABCD dörtgeninin;

a) Karşı kenarların birbirlerine göre durumlarını karşılaştırınız (*paralellik, diklik vs*)
b) Kenar uzunluklarını bulup karşılaştırınız.
c) Komşu kenarların birbirine göre durumlarını karşılaştırınız (*paralellik, diklik vs*).
d) Bu dörtgeninin hangi çeşit dörtgen olduğunu sebepleri ile birlikte belirtiniz.
e) Köşegen uzunluklarını bulup karşılaştırınız.
f) Köşegenlerin kesişim noktasının koordinatlarını bulunuz.
g) Köşegen parçalarının uzunluklarını karşılaştırınız.
h) Elde edilen bulgular ışığında bu dörtgenin özelliklerini yazmaya çalışınız.

APPENDIX D

SAMPLE HANDOUT THAT CONTAINS LESSON PLAN
(Student Version)

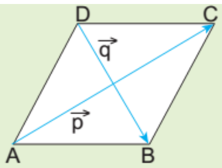
Konu: Eşkenar Dörtgen

Kazanım: Eşkenar dörtgensel bölgenin alan bağıntısını elde eder ve uygulamalar yapar.

EŞKENAR DÖRTGEN II.BÖLÜM

Eşkenar Dörtgensel Bölgenin Alanı

$\vec{p}$ ve $\vec{q}$ ABCD eşkenar dörtgensel bölgenin köşegen vektörleri olmak üzere, eşkenar dörtgensel bölgenin alanı $A(ABCD) = \frac{\|\vec{p}\|.\|\vec{q}\|}{2}$ bağıntısı ile bulunabilir.



veya

Köşegenleri e ve f olan eşkenar dörtgeninin alanı……………….............bağıntısı ile bulunur.

İspat _________________________Yaklaşım

İspat __________________________Yaklaşım

Örnek
Köşegenleri 12 birim ve 16 birim olan eşkenar dörtgenin alanı kaç birim karedir?
Çözüm

Not: ABCD eşkenar dörtgeni aynı zamanda bir paralel kenar olmasından dolayı alanı

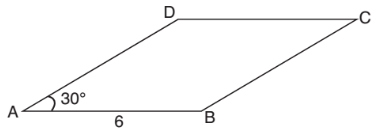
………………..............bağıntısı ile de hesaplanabilir.

Örnek

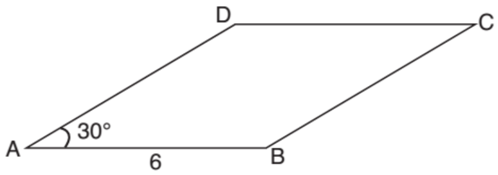
Yandaki ABCD eşkenar dörtgeninde,

$m(\widehat{BAD}) = 30°$ ve $|AB| = 6$ cm

olduğuna göre A(ABCD) kaç cm² dir?



Çözüm

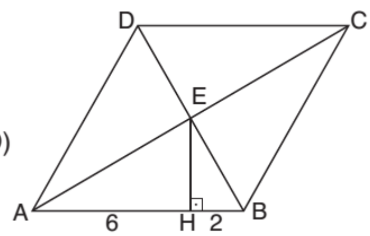


Örnek

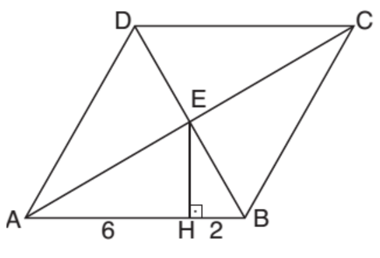
Yandaki ABCD eşkenar dörtgeninde,

[AC] ve [BD] köşegen,

$[EH] \perp [AB]$, $|BH| = 2$ cm, $|AH| = 6$ cm olduğuna göre A(ABCD) kaç cm² dir?



Çözüm



Örnek Köşe koordinatları A(2,1); B(7,4); C(10,9) ve D(5,6) olan eşkenar dörtgensel bölgesinin alanını, vektörel, analitik ve sentetik yaklaşımlar ile hesaplayınız.

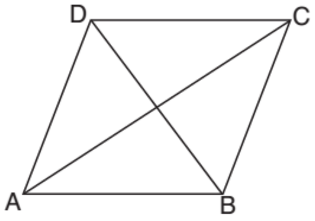
Problem 1 Bir ABCD eşkenar dörtgeninde $\overrightarrow{AD} = (3,4)$ *ve* $\overrightarrow{CA} = (-8,-4)$ olarak verilmektedir. ABCD eşkenar bölgesinin alanını hesaplayınız

Problem 2 Köşe koordinatları A(2,2); B(7,1) ve D(1,7) olarak verilen eşkenar dörtgensel bölgesinin alanını hesaplayınız.

Problem 3 Çevre uzunluğu 40 cm ve köşegenlerinden birinin uzunluğu 16 cm olan bir eşkenar dörtgensel bölgesinin tabana ait yüksekliğini hesaplayınız.

Problem 4 Köşe koordinatları A(-2,-7); B(6,-1); C(6,9) ve D(-2,3) olarak verilen eşkenar dörtgensel bölgesinin alanını hesaplayınız.

Problem 5



Yandaki ABCD eşkenar dörtgeninde $\|\overrightarrow{AB}\| = 6$ ve $\langle \overrightarrow{AB}, \overrightarrow{AC} \rangle = 54$ ise eşkenar dörtgenin alanını bulalım.

Problem 6 Bir ABCD eşkenar dörtgeninde e ve f köşegenler

$e + f = 15$ *ve* $e^2 + f^2 = 117$ olarak verilmektedir. ABCD eşkenar bölgesinin alanını hesaplayınız.

APPENDIX E

TABLE OF CONTENT FOR PREREQUISITE KNOWLEDGE FOR QUADRILATERALS TEST

Table E-1

Item (s)	Topics / Objectives	Basic Geometric Concepts	Lines	Triangles	Transformation Geometry	Numbers and Algebra	Polynomials	Trigonometry
1, 5	To be able to explain angle, measure of an angle and to make applications.	10%						
2, 3	To be able to construct coordinate axes and to make applications. To be able to find equation of a line and make applications.	10%						
4	To be able to find solution sets for the equations and inequalities of first degree with two unknowns.					5%		
6	To be able to find the slope of a line with respect to a perpendicular coordinate system.		5%					
7, 8	To be able to prove relations which results in area of triangular regions and to make applications.			10%				
9, 14	To be able to prove metric relations in a right triangle and to make applications.				10%			
10	To able to find trigonometric ratios of acute angles in a right triangle. To be able to express trigonometric functions in terms of each other.							5%
11, 12, 15, 18	To be able to prove the relations among sides and angles of a triangle and to make applications.			20%				
13, 19	To be able to prove properties of similar triangles and to make applications.				10%			
16	To be able to apply methods of factoring out the greatest common factor and factoring by grouping.						5%	
17	To be able to express Thales', Menalaus' and Ceva's theorems and to make applications.				5%			
20	To be able to prove theorems for two congruent triangles and to make applications.				5%			
21	To be able to find the distance of a point to a line and to make applications.		5%					

APPENDIX F

PREREQUISITE KNOWLEDGE FOR QUADRILATERALS TEST
(DÖRTGENLER İÇİN ÖNBİLGİ TESTİ)

Soru 1

Ölçüleri $135°, 89°, 90°, 68°, 360°, 92°, 45°, 180°\ ve\ 168°$ olarak verilen açıları aşağıdaki listede uygun olan açı çeşitlerinin altına yazınız.

	Dar Açı	**Dik Açı**	**Geniş Açı**	**Doğru Açı**	**Tam Açı**
1					
2					
3					
4					

Soru 2

A(-2,3) ve B(4,-5) noktaları için;

a) A ve B noktaları arasındaki uzaklığı bulunuz.
b) [AB] nın orta noktasının koordinatlarını bulunuz.
c) A ve B noktalarından geçen doğrunun eğimini bulunuz.
d) A ve B noktalarından geçen doğrunun denklemini bulunuz.

Soru 3

$k: y + x - 5 = 0\ ve\ m: y - \frac{1}{2}x - 2 = 0$ düzlemde iki doğrudur. Buna göre;

a) k ve m doğrularının kesişim noktasını bulunuz.
b) k, m doğruları ile eksenlerin sınırlandırdığı bölgenin alanının kaç br^2 dir?

Soru 4

$$\left.\begin{aligned} 3x-2y &= 6 \\ -5x+3y &= -11 \end{aligned}\right\}$$ denkleminin çözüm kümesini bulunuz.

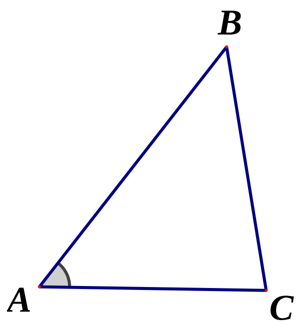
Soru 5

Bütünler iki açıdan büyük olanın ölçüsü küçük olanın ölçüsünün 5 katından 24 eksiktir. Buna göre ölçüsü küçük olan açının tümlerinin ölçüsü kaç derecedir?

Soru 6

k, d ve m doğrularını eğimleri sırasıyla $\frac{1}{3}, -3 \ ve \ \frac{1}{3}$ tür. k, d ve m doğrularının birbirlerine göre durumlarını şekil olarak gösteriniz.

Soru 7



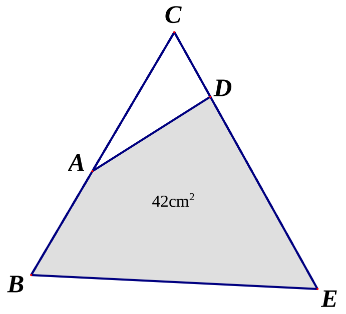
Yandaki şekilde

$$\left.\begin{aligned} |AB| &= 8 \ cm \\ |AC| &= 6 \ cm \\ m\left(BAC \right) &= 53^\circ \end{aligned}\right\}$$ olarak verilmiştir.

Buna göre ABC üçgenin alanını hesaplayınız.

(*Not:* $\sin 53 \cong 0{,}8$ *ve* $\cos 53 \cong 0{,}6$ *olarak alabilirsiniz*)

Soru 8

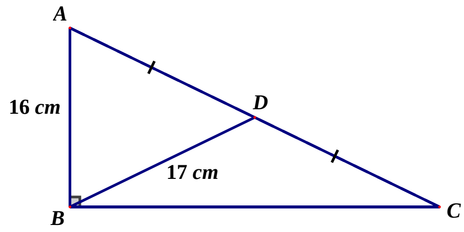


Yandaki şekilde

$\frac{|AB|}{|AC|} = 0{,}75 \ ve \ \frac{|DE|}{|DC|} = 3$ ve

A(ABED)= 42 cm^2 dir. Bu bilgilere göre CBE üçgeninin alanı kaç cm^2 dir?

Soru 9



ABC üçgeninde D bulunduğu kenarın orta noktasıdır.

$|AB| = 16\ cm,\ |BD| = 17\ cm\ ve\ [AB] \perp [BC]$

olduğuna göre $A(\triangle ABC) = ?$

Soru 10

a) Yazılacak açıların ölçüleri $180°$ den küçük olmak üzere aşağıdaki boşlukları doldurunuz.

b) Açılardan trigonometrik değerlerini bildiklerinizin değerlerini yazınız.

$\sin 30° =\ \sin ....° = ...\qquad \cos 30° =\ \cos ......°\ = ...$

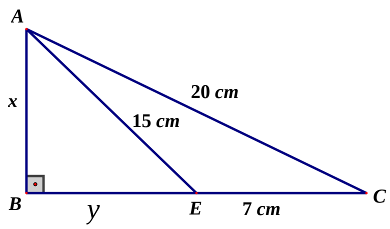
$\sin 45° =\ \sin ....° = ...\qquad \cos 45° =\ \cos ....° = ...$

$\sin 60° =\ \sin ....° = ...\qquad \cos 60° =\ \cos ....° = ...$

$\sin 72° =\ \sin ....° = ...\qquad \cos 75° =\ \cos ......°\ = ...$

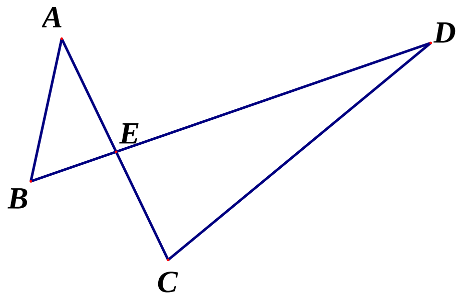
$\sin 145° = \sin ....° = ...\qquad \cos 156° =\ \cos ......°\ = ...$

Soru 11



Yandaki şekilde verilen bilgilere göre *x ve y* uzunluklarını bulunuz.

Soru 12

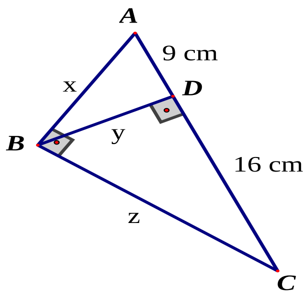


Yandaki şekilde

$2|CE| = 3|AE|$ ve $|BD| = \frac{9}{7}|ED|$ ise

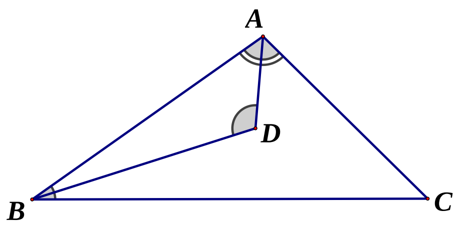
$\frac{A(\triangle ABE)}{A(\triangle DEC)} = ?$

Soru 13



Yandaki şekilde verilenlere göre x, y ve z uzunluklarını bulunuz.

Soru 14

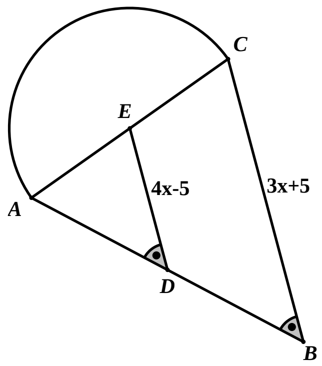


$[AD]$ ve $[BD]$ sırasıyla A ve B açılarının açıortaylarıdır.

$m(\widehat{ADB}) = 112^\circ$ olduğuna göre

$m(\widehat{C}) = ?$

Soru 15

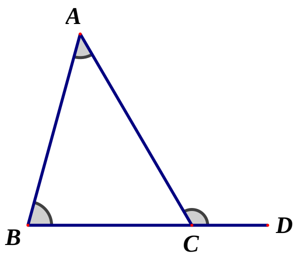


Yandaki şekilde E merkezli yarım çember verilmiştir.

$m(\widehat{ADE}) = m(\widehat{ABC})$

$|DE| = 4x - 5,$ $|BC| = 3x + 5$ olduğuna göre $|BC|$ uzunluğunun kaç birim olduğunu bulunuz.

Soru 16



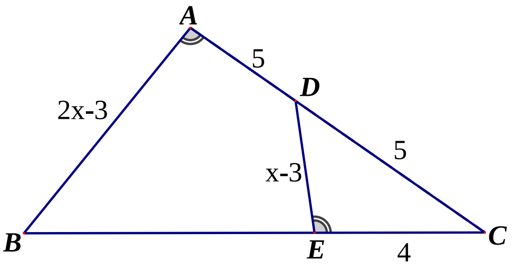
Yandaki şekilde

$m(A) = 5x + 13°$
$m(B) = 6x - 19°$
$m(ACD) = 9x + 22°$

olarak verildiğine göre

$m(ACB) = ?$

Soru 17



Yandaki şekilde

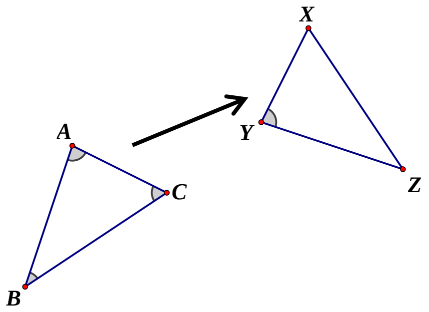
$m(BAC) = m(DEC)$ dir.

Şekilde verilen bilgilere göre

$|AB|, |BE|$ ve $|DE|$ uzunluklarını

bulunuz.

Soru 18



XYZ üçgeni ABC üçgenine uygulanan çeşitli dönüşüm hareketleri ile elde edilmiştir.

Yanda verilen bilgilere göre

$m(YXZ)$, $m(YZX)$, $|YZ|$ ve $|XZ|$

değerlerini bulunuz.

$m(BAC) = 82°$

$m(ACB) = 60°$

$|AB| = 6\,cm$

$|AC| = 5\,cm$

$|BC| = 7\,cm$

$m(ZYX) = 82°$

$|XY| = 5\,cm$

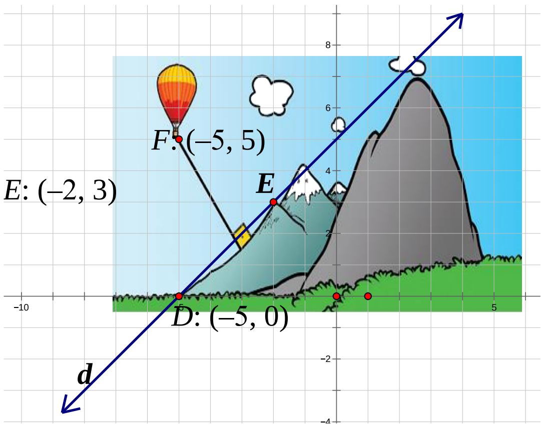
Soru 19

Çevresi 13 birim ve kenar uzunlukları tam sayı olan ikizkenar üçgenleri çiziniz.

Soru 20

$b = a + 7 \;\; ve \;\; c = 11 + d \;\; ise \;\; ac + bd - cb - ad = ?$

Soru 21



Bir balonun dağın yüzeyine olan uzaklığını bulmak isteyen Ali resmi yandaki şekilde olduğu gibi bir koordinat düzlemine yerleştirmiştir. Ali D,E ve F noktalarının koordinatlarını belirlemiştir. Koordinat düzleminde bir birim 10 metreye karşılık gelmektedir. Buna göre balonun dağın yüzeyine olan uzaklığı kaç metredir?

Not:

1.Dağın yüzeyi ile d doğrusu arasındaki uzaklık ihmal edilecektir.

2. $\sqrt{2} \cong 1,4 \quad \sqrt{3} \cong 1,73 \quad \sqrt{5} \cong 2,24$

APPENDIX G

TABLE OF CONTENT FOR PROOF PERFORMANCE IN GEOMETRY TEST

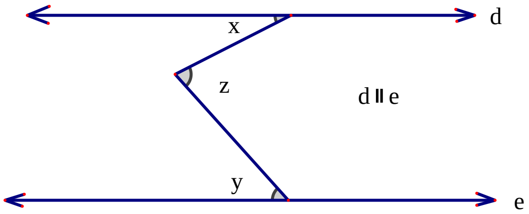
Table G-1

Item(s)	Topics / Objectives	Angles	Polygons	Triangles	Quadrilaterals	Circles
1, 4	To be able to explain angle, measure of an angle and to make applications.	13%				
2	To be able to specify the relations among basic elements of a convex polygon.		7%			
3, 5	To be able to prove the relations among sides and angles of a triangle.			13%		
6	To be able to specify remaining basic elements of a triangle which is given with necessary basic elements and to make applications.			7%		
7	To be able to prove law of sines and to make applications.			7%		
8	To be able to find the point that divides a side of triangle in a given ratio depending on sides of the triangle and on this ratio.			7%		
9, 10	To be able to prove relations which results in area of triangular regions and to make applications.			13%		
11, 12	To be able to prove metric relations in a right triangle and to make applications.			13%		
13	To be able to prove properties of similar triangles and to make applications.			7%		
14	To be able to derive relations which results in area of polygons.				7%	
15	To be able to explain angles in circles.					7%

APPENDIX H

PROOF PERFORMANCE IN GEOMETRY TEST
(GEOMETRİDE İSPAT-ÖN BİLGİ BAŞARI TESTİ)

Soru 1



"Yandaki şekilde verilen bilgilere göre x, y ve z arasında;

………………..............

şeklinde bir bağıntı vardır."

ifadesinde boşluğu doldurup elde ettiğiniz matematiksel ifadeyi ispatlayınız.

Soru 2

"Düzlemde kenar sayısı n olan bir dış bükey çokgenin köşegen sayısı

________________ dır."

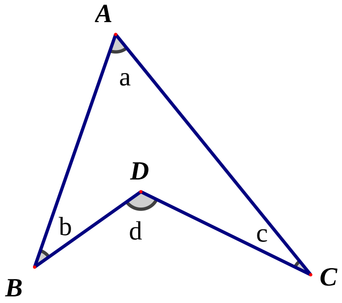
ifadesinde boşluğu doldurup elde ettiğiniz matematiksel ifadeyi ispatlayınız.

Soru 3

"Üçgenin iç açılarının ölçüleri toplamı ………………dir."

ifadesinde boşluğu doldurup elde ettiğiniz matematiksel ifadeyi ispatlayınız.

Soru 4



Yandaki şekilde verilen bilgilere göre a, b, c ve d arasında;

____

şeklinde bir bağıntı vardır."

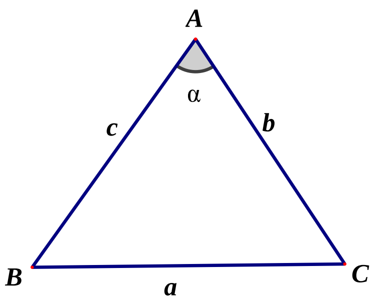
ifadesinde boşluğu doldurup elde ettiğiniz matematiksel ifadeyi ispatlayınız.

Soru 5

"Bir üçgenin iki iç açısından ölçüsü daha büyük olan açı karşısındaki kenar uzunluğu daha ……………….................."

ifadesinde boşluğu doldurup elde ettiğiniz matematiksel ifadeyi ispatlayınız.

Soru 6



Yandaki şekilde verilen ABC üçgeninde $|AB| = c, |BC| = a \text{ and } |AC| = b$ uzunlukları ile $m(A) = \alpha$ ölçüsü verilmiştir.

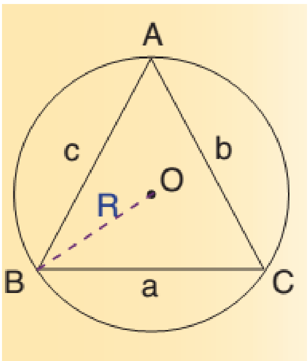
"a, b, c ve α arasında;

____

şeklinde bir bağıntı vardır. "

ifadesinde boşluğu doldurup elde ettiğiniz matematiksel ifadeyi ispatlayınız.

Soru 7



"Kenar uzunlukları a, b ve c; iç açılarının ölçüleri A, B ve C ve çevrel çemberinin yarıçapı R olan ABC üçgeninde a, b, c, A, B, C ve R arasında;

____

şeklinde bir bağıntı vardır."

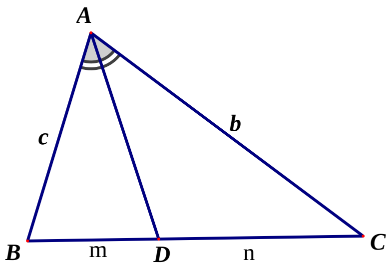
Matematiksel ifadesinde;

a) boşluğu doldurup

b) elde ettiğiniz ifadeyi ispatlayınız**.**

c) Bu bağıntı _________________olarak adlandırılır.

Soru 8



Yandaki ABC üçgeninde [AD]; A açısının açıortayıdır. Verilen bilgilere göre " b, c, m ve n arasında;

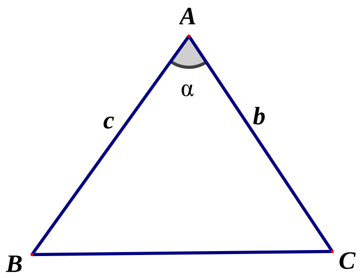
____

şeklinde bir bağıntı vardır."

ifadesinde boşluğu doldurup elde ettiğiniz matematiksel ifadeyi ispatlayınız.

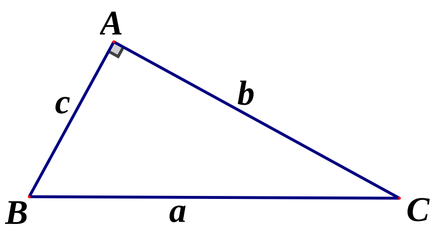
Soru 9
Bir üçgenin alanını taban ve yüksekliklerinin uzunlukları cinsinden ifade edip elde ettiğiniz matematiksel ifadeyi ispatlayınız.

Soru 10



Yanda verilen ABC üçgeninde |AB|=c ve |AC|=b uzunlukları ile $m\left(A\right)=\alpha$ ölçüsü verilmiştir. ABC üçgeninin alanını bu verilenler cinsinden ifade edip elde ettiğiniz matematiksel ifadeyi ispatlayınız.

Soru 11

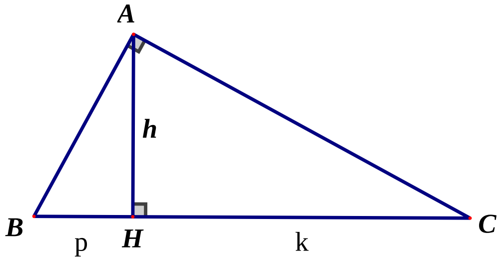


Yandaki şekilde verilen bilgilere göre a, b ve c arasında;

………………...............
şeklinde bir bağıntı vardır."

ifadesinde boşluğu doldurup elde ettiğiniz matematiksel ifadeyi ispatlayınız.

Soru 12



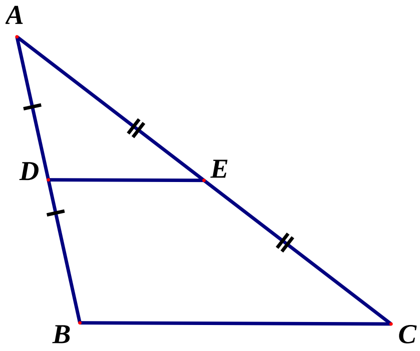
Yandaki şekilde verilen bilgilere göre h, p ve k arasında;

………………..............

şeklinde bir bağıntı vardır."

ifadesinde boşluğu doldurup elde ettiğiniz matematiksel ifadeyi ispatlayınız.

Soru 13



ABC üçgeninde $[BC] \parallel [DE]$ dir. D ve E noktaları bulundukları kenarların orta noktaları olmak üzere;

$\frac{|DE|}{|BC|} =$ ………………..............

eşitliğinde boşluğu doldurup elde ettiğiniz matematiksel ifadeyi ispatlayınız.

Soru 14

"Taban uzunlukları a ve c, yüksekliği h olan bir yamuğun alanı;

………………..............   formülü ile bulunur."

ifadesinde boşluğu doldurup elde ettiğiniz matematiksel ifadeyi ispatlayınız.

Soru 15

"Bir çemberde çevre açısının ölçüsü aynı yayı gören merkez açının ölçüsünün

……………….............. eşittir."

İfadesinde boşluğu doldurup elde ettiğiniz ifadeyi ispatlayınız.

APPENDIX I

TABLE OF CONTENT FOR VECTOR KNOWLEDGE TEST

Table I-1

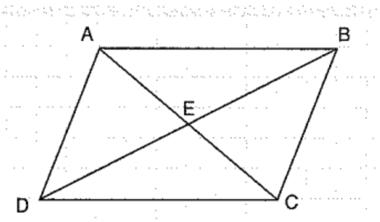
Item(s)	Topics / Objectives	Basic Vector Concepts	Algebra of Vectors	Norm of a Vector	Inner Product	Projection of Vectors
1, 2	To be able to explain vector and to explain point-vector correspondings.	11%				
3, 4	To be able to add vectors and to apply properties of addition.		11%			
5, 6	To be able to explain linear independence and dependence of vectors.		11%			
7, 8	To be able to find norm (length) of a vector.			11%		
9, 10, 14, 17, 18	To be able to explain Euclidean inner product and to make applications with this product.				28%	
11, 12	To be able to find the angle between two vectors.				11%	
13, 15, 16	To be able to find orthogonal projection of a vector on another vector and to make applications.					17%

APPENDIX J

VECTOR KNOWLEDGE TEST
(VEKTÖR BİLGİ TESTİ)

AÇIKLAMA: Merhaba Arkadaşlar. Aşağıda verilen soruların çözümlerini açıklamalarıyla beraber yapınız. Çoktan seçmeli sorularda doğru cevabı işaretlemekle beraber çözümünüzü de yapınız. Bu test 10 sayfa ve 18 sorudan oluşmaktadır. Başarılar dilerim.

Soru 1



Yanda ABCD paralelkenarı veriliyor. Buna göre aşağıdaki ifadelerden hangileri doğrudur?

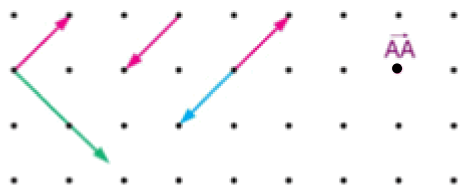
a) [AB] ve [DC] nın doğrultuları aynıdır.

b) [AC] ve [AB] nın doğrultuları aynıdır.

c) [AD] ve [CB] nın doğrultuları farklıdır.

ç) [AE] ve [CE] nın doğrultuları aynıdır.

Soru 2



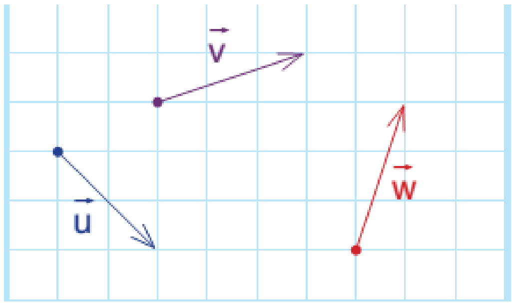
Şıklarda verilen terimlerin tanımlarını yazıp noktalı kâğıt üzerinde örnek gösterimi olmayan terimi belirtiniz.

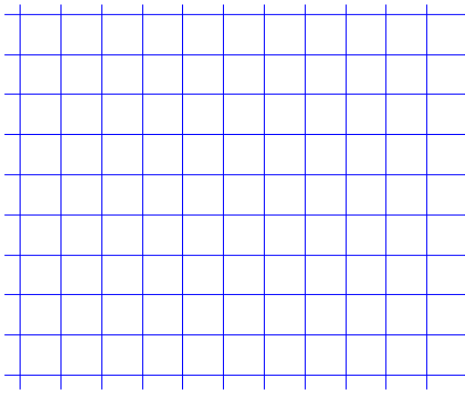
a) Birim vektör b)Sıfır vektörü

c) Eş vektörler d)Zıt vektörler

e)Dik vektörler

Soru 3





Koordinat düzleminde verilen üç vektörün toplamını gösteren vektörü bulunuz.

Soru 4

$\vec{A}=(-2,4)$ *ve* $\vec{B}=(1,3)$ vektörleri veriliyor. $\vec{A}+\vec{B}$, $\vec{A}-\vec{B}$ ve $2\vec{A}+3\vec{B}$ vektörlerini bulunuz.

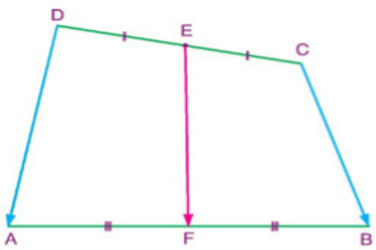
Soru 5

$\vec{A}$ = (7, –1) vektörünün, $\vec{B}$ = (1, –2) ve $\vec{C}$ = (5, 3) vektörlerinin lineer bileşimi şeklinde yazılımı aşağıdakilerden hangisidir?

A) $\vec{A} = 2\vec{B} + \vec{C}$ B) $\vec{A} = \vec{B} - 2\vec{C}$ C) $\vec{A} = 3\vec{B} - \vec{C}$

D) $\vec{A} = 2\vec{C} - \vec{B}$ E) $\vec{A} = 4\vec{B} + 3\vec{C}$

Soru 6



Şekildeki ABCD dörtgeninde; |AF| = |FB| ve |DE| = |EC| olduğuna göre, **$\vec{EF}$ vektörünün $\vec{DA}$ ve $\vec{CB}$ türünden eşitini bulalım.**

Soru 7

Düzlemde A(–2, 3) ve B(0, 5) noktaları veriliyor.

Buna göre, $\|\vec{AB}\|$ kaç birimdir?

A) $\sqrt{5}$ B) $\sqrt{6}$ C) $2\sqrt{2}$ D) 3 E) $2\sqrt{3}$

Soru 8

$\vec{u}$ = (−3, 4) vektörü ile zıt yönlü ve doğrultusu aynı olan birim vektör aşağıdakilerden hangisidir?

A) $\left(-\frac{3}{5}, \frac{4}{5}\right)$ B) $\left(-\frac{3}{5}, -\frac{4}{5}\right)$ C) $\left(\frac{3}{5}, \frac{4}{5}\right)$

D) $\left(\frac{3}{5}, -\frac{4}{5}\right)$ E) $\left(\frac{4}{5}, -\frac{3}{5}\right)$

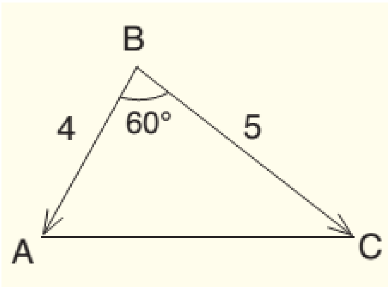
Soru 9

$\vec{u} = (3, -1)$ *ve* $\vec{v} = (-2, 4)$ vektörleri veriliyor. $\langle \vec{u}, \vec{v} \rangle$ işleminin sonucunu bulunuz.

Soru 10

$A(-2,3); B(2,4)$ *ve* $C(1, a)$ noktaları veriliyor. $\langle \overrightarrow{AB}, \overrightarrow{BC} \rangle = 5$ ise a kaçtır?

Soru 11



Yandaki şekilde;

$\|\overrightarrow{BA}\| = 4$ birim, $\|\overrightarrow{BC}\| = 5$ birim ve $m(\widehat{ABC}) = 60°$ olarak verilmiştir.

$\langle \overrightarrow{BA}, \overrightarrow{BC} \rangle$ değerini hesaplayınız.

Soru 12

Analitik düzlemde A(1 , 2); B(−1 , 3); C(3 , 0) ve D(4 , 2) noktaları veriliyor.

$\overrightarrow{AB}$ *ve* $\overrightarrow{CD}$ vektörleri arasındaki açıyı hesaplayınız.

Soru 13

$\vec{u} = (3, -1)$ vektörünün $\vec{v} = (2, 3)$ vektörü üzerindeki dik izdüşümünün uzunluğunu hesaplayınız.

Soru 14

$\langle \vec{u}, \vec{v} \rangle$; u ve v vektörlerinin iç çarpımını göstermektedir. Yandaki özelliklerden doğru olan için **D** ve yanlış olanlar için **Y** harfini kullanıp sebeplerini açıklayınız.

a) $\langle \vec{u}, \vec{v} \rangle = -\langle \vec{v}, \vec{u} \rangle$

b) $\langle \vec{u}, \vec{u} \rangle = |\vec{u}|^2$

c) $\langle \vec{u} + \vec{v}, \vec{u} - \vec{v} \rangle = |\vec{u}|^2 - |\vec{v}|^2$

d) $\vec{u} \perp \vec{v}$ *ise* $\langle \vec{u}, \vec{v} \rangle = 0$

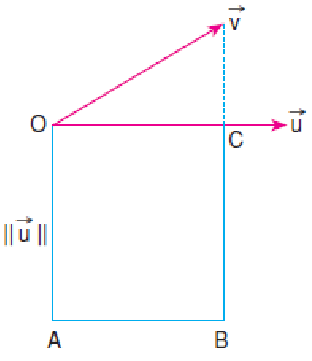
e) $\vec{u} \parallel \vec{v}$ *ise* $\langle \vec{u}, \vec{v} \rangle = |\vec{u}||\vec{v}|$

Soru 15

İki kenarı $\vec{a}=(5,-3)$ *ve* $\vec{b}=(1,4)$ vektörleri olan paralelkenarın alanı kaç birim karedir?

a) 14 b) 16 c) 20 d) 23 e) 29

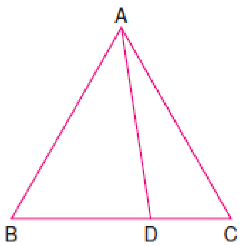
Soru 16



Şekilde OABC dikdörtgen, $|OA| = ||\vec{u}||$, $\vec{u} = (4, 2)$ ve $\vec{v} = (5, 1)$ ise A(OABC) kaç br^2 dir?

Soru 17

ABC eşkenar üçgeninin kenar uzunluğu 6 br ise $\langle \overrightarrow{AB}, \overrightarrow{AD} + \overrightarrow{DC} \rangle$ kaçtır?



A) 6 B) 12 C) 18 D) 24 E) 36

Soru 18

$\vec{a}=(m+1,\ 2)$ *ve* $\vec{b}=(1,4)$ vektörleri birbirine dik ise, m kaçtır?

a) -12 b) -9 c) 3 d) 7 e) 11

Test Bitti.

APPENDIX K

TABLE OF CONTENT FOR QUADRILATERALS ACHIEVEMENT TEST

Table K-1

Item(s)	Topics / Objectives	Quadrilaterals			Trapezoid			Parallelogram			Rectangle			Rhombus			Square			Deltoid			Classifiaction of Quadrilaterals		
		K	C	A	K	C	A	K	C	A	K	C	A	K	C	A	K	C	A	K	C	A	K	C	A
A1, A2, A3 A6, A7	To be able to explain quadrilaterals and their basic elements, to make applications.	5																							
A4, A5	To be able to explain trapezoid and to prove theorems related to properties of trapezoid.				2																				
A8	To be able to derive the formula giving area of square and to make applications.																		1						
A9	To be able to explain rhombus and to prove theorems about properties of rhombus and to make applications.															1									
A10	To be able to explain deltoid and its properties, to make applications.																					1			
A11	To be able to classify quadrilaterals, to explain the relations among them.																								1
B1, B2, B4	To be able to prove theorems related to quadrilaterals and to make applications.			3																					
B3, B5	To be able to calculate perimeter of a quadrilateral, to be able to prove theorems related to the area of quadrilaterals, and to make applications.			2																					
B6, B7, B8	To be able to explain trapezoid and to prove theorems related to properties of trapezoid.					**1**	2																		
B9, **B10**, B11	To be able to derive the formula giving area of a trapezoid and to make applications.					**1**	2																		
B12, B14	To be able to derive the formula giving area of a parallelogram and to make applications.									2															

Table K-1 continued

Item(s)	Topics / Objectives	Quadrilaterals			Trapezoid			Parallelogram			Rectangle			Rhombus			Square			Deltoid			Classifiaction of Quadrilaterals		
		K	C	A	K	C	A	K	C	A	K	C	A	K	C	A	K	C	A	K	C	A	K	C	A
B13	To be able to explain parallelogram and to prove theorems related to properties of parallelogram and to make applications.									1															
B15, B16	To be able to derive the formula giving area of a rectangle and to make applications.												2												
B17	To be able to explain rhombus and to prove theorems related to properties of rhombus and to make applications.															1									
B18	To be able to derive the formula giving area of a rhombus and to make applications.															1									
B19	To be able to explain square and to prove theorems related to properties of square and to make applications.																		1						
B20	To be able to derive the formula giving area of a square and to make applications.																		1						
B21	To be able to derive the formula giving area of a deltoid and to make applications.																					1			
C1, C4, C5	To be able to prove theorems related to quadrilaterals and to make applications.			3																					
C2	To be able to explain rhombus and to prove theorems related to properties of rhombus and to make applications.															1									
C3	To be able to explain parallelogram and to prove theorems related to properties of parallelogram and to make applications.									1															

K: Knowledge **C**: Comprehension **A**: Application

APPENDIX L

QUADRILATERALS ACHIEVEMENT TEST
(DÖRTGENLER BAŞARI TESTİ)

A) Boşluk Doldurma

Aşağıda nokta koyularak bırakılan boşlukları doldurunuz. Boşlukların uzunlukları eşit olarak ayarlanmıştır. Bu boşlukların uzunluğu ile boşluklara yazılacak kelime veya kelimelerin uzunlukları arasında bir ilişki yoktur.

1. Bir dörtgenin komşu olmayan iki kenarının orta noktalarını birleştiren doğru parçasına ....denir.

2. Bir dörtgenin komşu olmayan iki köşesini birleştiren doğru parçasına ....denir.

3. Herhangi bir iç açısının ölçüsü 180° den büyük olan dörtgene .... denir.

4. Paralel olmayan kenarlarının uzunlukları birbirine eşit olan yamuğa .... denir.

5. Yamuğun paralel olan kenarlarına ....denir.

6. Dörtgenlere uygulanan öteleme hareketleriyle iç açılarının ölçülerinin değerleri ....

7. Dörtgenlere uygulanan dönme hareketleriyle kenarlarının uzunlukları ....

8. Kenar uzunlukları eşit olan bir kare ve eşkenar dörtgenden .... nin alanı daha büyüktür.

9. Bir eşkenar dörtgenin kenarlarının orta noktalarının birleştirilmesiyle elde edilen dörtgen ....dir/dır.

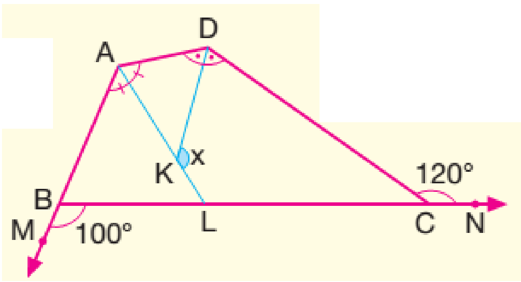
10. Deltoidin köşegenleri eşit uzunlukta olduğunda dörtgen .... adını alır.

11. Aşağıdaki tabloda verilen özellikleri inceleyerek tablodaki dörtgenlerin bu özellikleri sağlayıp sağlamama durumuna göre + veya – işaretlerinden uygun olanını yazınız.

DÖRTGENLERİN ÖZELLİKLERİ						
ÖZELLİKLER	Yamuk	Paralelkenar	Dikdörtgen	Eşkenar Dörtgen	Kare	Deltoid
Karşılıklı kenar uzunlukları eşittir.						
Bütün kenar uzunlukları eşittir.						
Karşılıklı kenarları paraleldir.						
Her bir açısı diktir.						
Köşegenleri birbirini ortalar.						
Köşegenleri eştir.						
Köşegenleri dik kesişir.						
Ardışık açıları bütünlerdir.						
Köşegenleri açıortaydır.						
İç açılarının ölçüleri toplamı 360° dir.						

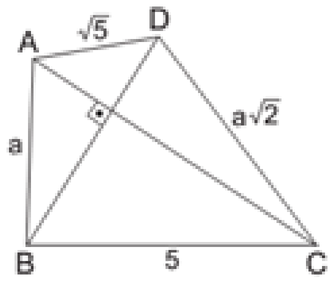
B) Klasik Tipte Dörtgenler Testi
Soru 1

Yandaki verilen şekle göre x açısı kaç derecedir?



Soru 2

Şekildeki ABCD dörtgeninde,
$[BD] \perp [AC]$,
$|AD| = \sqrt{5}$ br,
$|AB| = a$ br,
$|BC| = 5$ br ve
$|DC| = a\sqrt{2}$ br olduğuna göre,
$|AB|$ kaç br dir?

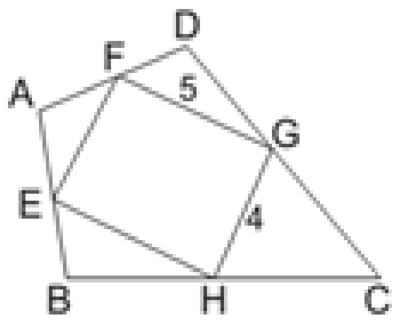


Soru 3

Köşelerinin koordinatları $A(1,4);\ B(4,1);\ C(4,-8)\ ve\ D(-3,-1)$ olan ABCD dörtgensel bölgesinin alanını hesaplayınız.

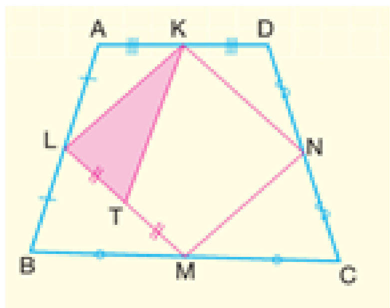
Soru 4

Şekildeki ABCD dörtgeninde E, F, G ve H bulundukları kenarların orta noktalarıdır.
$|GH| = 4$ br ve $|FG| = 5$ br olduğuna göre, ABCD dörtgeninin köşegenlerinin uzunluklarının toplamı kaç br dir?



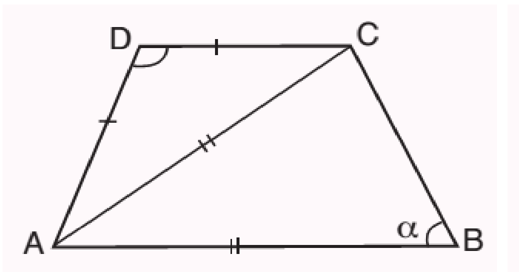
Soru 5

ABCD dörtgensel bölgesinin alanı 120 br^2 dir.
K, L, M, N ve T bulundukları kenarların orta noktaları olduğuna göre A($\widehat{KLT}$) kaç br^2 dir?



Soru 6

Yandaki ABCD yamuğunda, [AB] // [DC], |AD| = |DC|, |AB| = |AC|, m($\widehat{DAB}$) = 80° olduğuna göre m($\widehat{B}$) = α kaç derecedir?

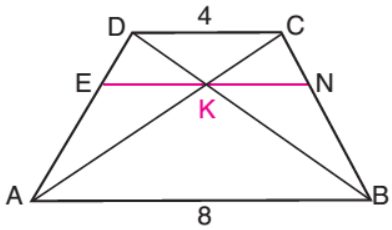


Soru 7

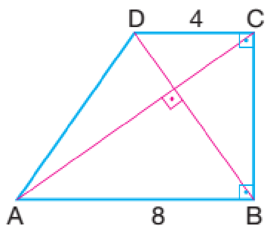
Bir yamuğun alt tabanının uzunluğu 7 br ve orta tabanın köşegenler arasında kalan parçasının uzunluğu 2 br olduğuna göre yamuğun orta taban uzunluğu kaç birimdir?

Soru 8

Yandaki ABCD yamuğunda,
[DC] // [EN] // [AB],
[AC] ∩ [BD] = {K}
|AB| = 8 cm, |DC| = 4 cm olduğuna göre
[EN] nın uzunluğunu bulunuz.

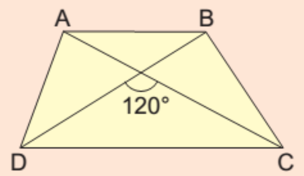


Soru 9



A(ABCD) kaç br^2 dir?

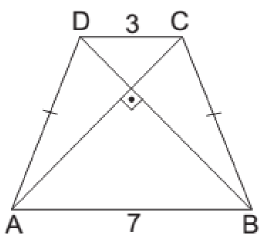
Soru 10



|AC| = 12 br, |BD| = 10 br
A(ABCD) = ?

Soru 11

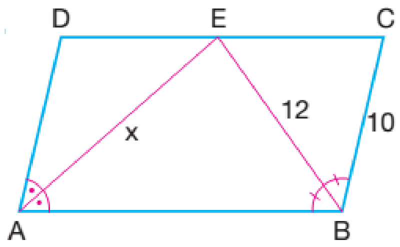
Şekildeki ABCD ikizkenar yamuğunda
|AD| = |BC| ve
[DB] ⊥ [AC] dır.
|DC| = 3 br ve
|AB| = 7 br olduğuna göre,
A(ABCD) kaç br^2 dir?



Soru 12

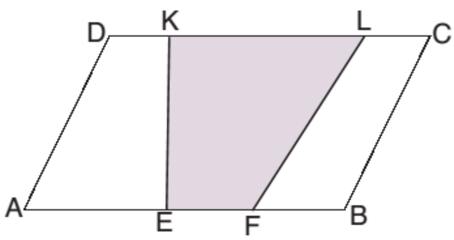
A, B, C ve D sırasıyla bir paralelkenarsal bölgenin köşeleri olmak üzere köşegen vektörleri $\overrightarrow{AC}=(10,3)$ *ve* $\overrightarrow{BD}=(-4,\,3)$ olarak verilmektedir. Buna göre ABCD paralelkenarsal bölgesinin alanı kaç birim karedir?

Soru 13



Soru 14

Yandaki ABCD paralelkenarında,
$3|KL| = 2|DC|$
$|AB| = 4|EF|$
$A(EFLK) = 66\ cm^2$ olduğuna göre,
$A(ABCD)$ kaç cm^2 dir?

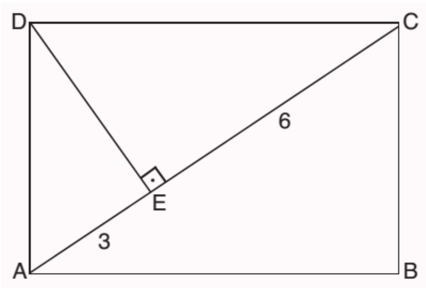


Soru 15

Kenar vektörü ve köşegen vektörü $\overrightarrow{DB}=(8,\,1)$ olan ABCD dikdörtgensel bölgesinin alanını bulunuz.

Soru 16

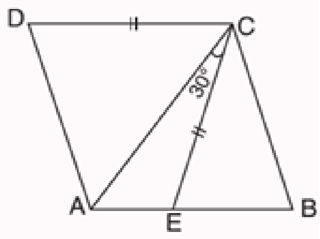
Yandaki ABCD dikdörtgeninde,
$[DE] \perp [AC]$, $|AE| = 3$ cm,
$|EC| = 6$ cm olduğuna
göre A(ABCD) kaç cm^2 dir?



Soru 17

Yandaki ABCD eşkenar dörtgeninde,

$|CE| = |CD|$, $m(\widehat{ACE}) = 30°$ olduğuna göre CAB açısı kaç derecedir?



Soru 18

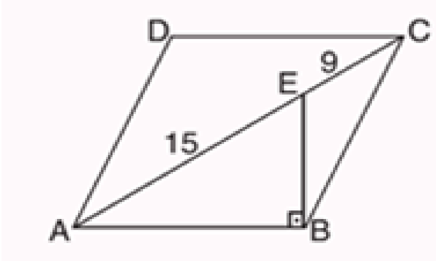
ABCD eşkenar dörtgen,

$[AB] \perp [BE]$

$|AE| = 15$ cm

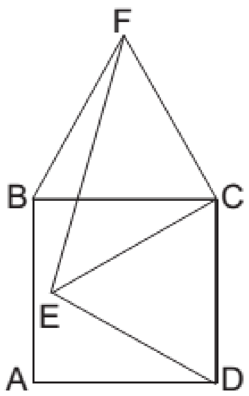
$|EC| = 9$ cm

Yukarıdaki verilere göre A(ABCD) kaç cm² dir?



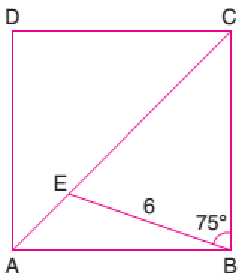
Soru 19

Şekildeki ABCD karesinde DEC ve BFC eşkenar üçgenler olduğuna göre CEF açısının ölçüsünü bulunuz.

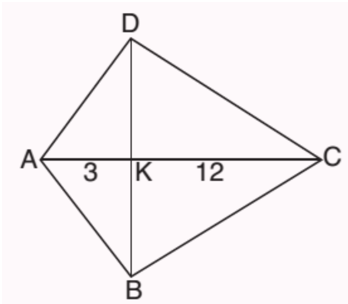


Soru 20

ABCD karesinin alanını bulunuz.



Soru 21



Yandaki ABCD deltoidinde,

|AD| = |AB|

|AD| ⊥ |DC|

|AK| = 3 cm

|KC| = 12 cm dir.

Yukarıdaki verilere göre
A(ABCD) kaç cm^2 dir?

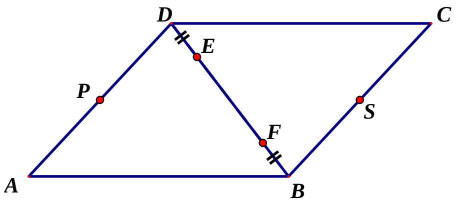
C) İspat Testi

Problem 1
Herhangi bir dörtgenin kenar orta noktalarının sırayla birleştirilmesiyle hangi geometrik şekil elde edilir? İspatlayınız.

Problem 2

Bir eşkenar dörtgende köşegenler arasındaki açının kaç derece olduğunu bulunuz.

Problem 3



ABCD paralelkenarında P ve S orta noktalar ve $[DE]=[FB]$ olmak üzere PFSE nin bir paralel kenar olduğunu ispatlayınız.

Problem 4
Köşegenleri birbirini ortalayan bir dörtgenin hangi çeşit bir dörtgen olduğunu ispatlayınız.

Problem 5
"Tüm dörtgenlerde köşegenlerin kesişim noktası dörtgenin ağırlık merkezinin yeridir" ifadesinin doğruluğunu ya da yanlışlığını belirtip ispatlayınız.

CURRICULUM VITAE

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Year	Place and Enrollment
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1998-1999	Middle East Technical University, Student Assistant, SSME, Ankara.

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PUBLICATIONS

Mut, A.İ. (2013). The Effects of Dynamic Geometry Software to the Frequency of Proving in In-service Mathematics Teachers' Practices: Teachers' Views. In T. Bastiaens & G. Marks (Eds.), *Proceedings of World Conference on E-Learning in Corporate, Government, Healthcare, and Higher Education 2013* (pp. 904-908). Chesapeake, VA: AACE, Las Vegas, U.S.A.

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Mut, Ali İhsan (2003). Investigation of Students' Probabilistic Misconceptions. Unpublished Master Thesis, Middle East Technical University, Ankara.

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