ELASTIC ANALYSIS OF ORTHOTROPIC CYLINDERS UNDER
DIFFERENT BOUNDARY CONDITIONS

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ABSTRACT

ELASTIC ANALYSIS OF ORTHOTROPIC CYLINDERS UNDER
DIFFERENT BOUNDARY CONDITIONS

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Analytical solutions are derived to examine the elastic responses of fixed end cylinders made of orthotropic materials. Cylinders are investigated under different boundary conditions which are internal pressure, external pressure, combined pressure and annular rotation respectively. Making use of Maxwell relations, orthotropic cylinders are transformed to isotropic ones. In order to exhibit numerical examples different orthotropic materials are used and compared. It is observed that orthotrophy slightly influences radial and tangential stresses. In other respects, orthotropy largely effects radial displacements and axial stresses.

Keywords: Orthotropic Cylinder, Maxwell Relations, Elastic Analysis

ÖZ

ORTOTROPİK SİLİNDİRLERİN FARKLI SINIR KOŞULLARI ALTINDA ELASTİK ANALİZİ

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Ortotropik malzemeden yapılan, kapalı uçlu silindirlerin verdiği elastik yanıtların incelenmesi için analitik çözümler elde edilmiştir. Silindirler farklı sınır koşulları altından incelenmiştir. Bu sınır koşulları sırasıyla iç basınç, dış basınç, kombine basınç ve dairesel dönüştür. Maxwell ilişkileri kullanılarak, ortotropik silindirler isotropike dönüştürülmüştür. Nümerik örnekleri sunmak için farklı ortotropik malzemeler kullanılmış ve karşılaştırılmıştır. Ortotropinin radyal ve teğetsel stresleri az miktarda etkilediği gözlemlenmiştir. Öte yandan, ortotropi radyal yerdeğiştirme ve eksenel stresleri büyük ölçüde etkilemektedir.

Anahtar Kelimeler: Ortotropik Silindir, Maxwell İlişkileri, Elastik Analiz

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This work is dedicated to my parents

Alev Sezer and Muhittin Farukoğlu

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a	Inner radius of the cylinder
b	Outer radius of the cylinder
c_{ij}	Compliance matrix terms where i, j=1,2,...6
s_{ij}	Stiffness matrix terms where i, j=1,2,...6
u_r, u_θ, u_z	Radial, tangential and axial displacements
$\underline{C}$	Compliance matrix
C_1, C_2	Arbitrary constants
E_r, E_θ, E_z	Elasticity modulus in radial, tangential and axial directions
$G_{r\theta}, G_{rz}, G_{\theta z}$	Directional shear modulus
P_{in}, P_{ex}	Internal and external pressure
R_1, R_2	Orthotropy parameters
$\underline{S}$	Stiffness matrix
$\gamma_{r\theta}, \gamma_{rz}, \gamma_{\theta z}$	Directional shear strains
$\epsilon_r, \epsilon_\theta, \epsilon_z$	Radial, tangential and axial strains
$\nu_{r\theta}, \nu_{rz}, \nu_{\theta z}$	Poisson ratios
$\sigma_r, \sigma_\theta, \sigma_z$	Radial, tangential and axial stresses
$\sigma_r^y, \sigma_\theta^y, \sigma_z^y, \sigma^y$	Radial, tangential, axial and general yield limits
σ_h	Non dimensional Hill's stress
$\tau_{r\theta}, \tau_{rz}, \tau_{\theta z}$	Directional shear stresses
ω	Annular speed

CHAPTER 1

INTRODUCTION

Long hollow cylinders are used frequently in many engineering applications such as liquid and gas transportation. Accordingly, its importance increases and arouses a higher interest in many researches. Cylinders that are made of isotropic materials are previously investigated by many developers. Analytical stress and displacement analysis of isotropic cylinders have been proposed by several books[1, 2]. Unlike the isotropic ones there is less information about orthotropic cylinders.

Abd-Alla, Al-Shehri and Mahmoud[3] have discussed the effect of rotation on orthotropic infinitely long cylinders in their article. They have presented analytical solutions to three different cases which are cylinder with hollow center, cylinder with solid center and cylinder seated on rigid shaft. Tervonen and Pramilla[4] have investigated a similar situation. In orthotropic tubes, they have examined the rotation effect on the structure without applying the stress function model. Zenkour[5] has dealt with elastic solutions of rotating orthotropic cylinders. In his paper, he has performed on orthotropic cylinders that contain solid core. Different solid cores and their effects on the body have been compared by him.

Leu and Hsu[6] have shown plastic analysis of strain hardening orthotropic cylinders under rotation. In the article, they have used Voce hardening principle to search the act of strain hardening. Garmestani and Markiewicz[7] have introduced an elastoplastic analysis on work hardening orthotropic cylinders. They have received solutions under the influence of body forces.

Croccolo and Agostinis[8] have analyzed stresses of the press fitted cylinders of orthotropic materials. They have worked out analytical and numerical solutions on the structure and compared these solutions with diagrams. Jolicoeur and Cardon[9] have prepared general elastic analysis for the orthotropic cylinders. They have examined the stresses on the body under tension and bending. Desai and Kant[10], in their paper, have considered an orthotropic cylinder with layers in which the body was subjected to distributed sinusoidal pressure.

Likely, Lubarda[11] has brought analytical clarifications to orthotropic disks, cylinders and spheres. He has developed models to present elastic responses of these structures under pressure. On the other hand, Eraslan, Kaya and Varlı[12] have designed an analytical model to explore the behavior of variable thickness disk of orthotropic material. They have modeled the general solution with hyper geometric functions.

There are similar works conducted for cylindrical structures made of other than orthotropic materials. Dui, Yang and Zhang[13] have proposed an elastic solution to functionally graded tube which contains two linearly elastic components. They have inspected the body under internal pressure. The influence of volume fraction and Poisson ratio on the displacements and stresses are carried out. Tutuncu[14] has submitted power series solution to cylindrical vessels that are made of functionally graded material. He has assumed that the structure has functionally varying Young modulus and constant Poisson ratio. Later, the author has shown the effects of these non homogenous material properties on the stresses. Eraslan and Akış[15] have recommended an analytical approach to pressurized tube of functionally graded materials. They have postulated that the body has parabolically varying yield limit and modulus of elasticity. Afterwords, writers of the article have solved the problem both elastically and elastoplastically. Finally, non homogenous elastic-plastic approach has been converted to homogenous one by manipulating material parameters. In their paper, You, Ou and Zheng[16] have implemented solutions to functionally graded pressure vessels that have thick walled cylindrical shape. Material properties variate according to Norton's law in their case. After the application of internal pressure to the body, they have examined stresses and creep deformations. Nie and

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Batra[17] have issued a paper that explains functionally graded hollow cylinders under different boundary conditions. They have derived Airy stress function approach on their two dimensional problem.

Ghanda and Nejad[18] have proposed an analytical approach to heterogeneous cylinders. Effect of internal and external pressure on the structure have delved by shear deformation principle. Acquired results from finite element method and plane elasticity are compared by them. Akış and Eraslan[19] have investigated the yielding of tubes that are firmly concentric. They have studied the structure under internal and external pressure. As a result, critical values that cause plastic deformation have been found with Von Misses yield criteria. Yiannopoulos[20] has executed an interpretation on elastic solid cylinders. He has derived a two dimensional stress analysis along the body. His research has been held under non uniform external loads.

The aim of this thesis, in the light of above elucidations, is researching displacements and stresses of the cylinders. For this reason, analytical models are prepared for isotropic and orthotropic cylinders. These analytical models are supported with numerical examples. Later, affect of orthotropy is compared with isotropy.

CHAPTER 2

MANUFACTURING PROCESSES AND THEORY

2.1 Extrusion

Extrusion is a commonly used manufacturing method that creates profiles with fixed cross sections. The advantage of this method is both complex and simple cross profiles can be manufactured. Various materials such as metals, plastics and ceramics are practicable with extrusion process.

Cylinders and pipes that are made of plastic generally produced by plastics extrusion process. Pvc pipes and fiber optic cables can be given as examples.

2.2 Drawing

Drawing is another cylinder production method that is frequently used. In contrary to extrusion, only simple cross profiles can be produced by drawing. Additionally, this method mostly used on metal forming.

Metal cylinders and pipes are commonly manufactured by drawing. According to the working temperature, drawing classifies to cold and hot drawing. Cold drawing is the one that performed in the room temperature. On the other hand, hot drawing is held in high temperatures.

Extrusion and drawing methods require high amount of force to obtain desired cross profiles. Due to these forces, grain structure of materials tend to change. This change cause orthotropy on extruded or drawn materials. For example,

steels illustrate isotropic material behavior. On the other hand after manufacturing processes it becomes orthotropic because of the grain structure.

2.3   Stress-Strain Diagrams

Stress-strain diagrams are the ones that deliver much information about material characteristics. Making use of simple tension tests, those diagrams are obtained. The figure given below represents a stress-strain diagram that we usually attain. Horizontal line of the figure is engineering strain (ε) and longitudinal line of the figure is nominal stress (σ). ε^e and ε^p denotes the elastic and plastic strains respectively.
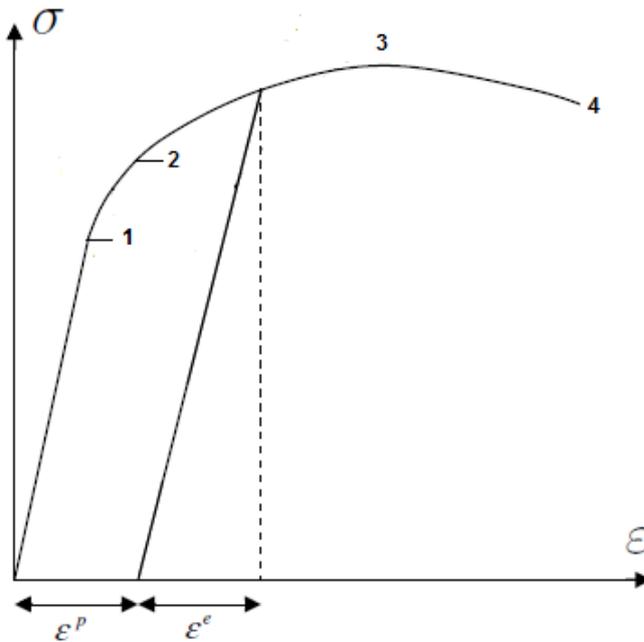
Figure 2.1: Stress strain diagram

Mounted test specimens under incremental force generally behave like the above diagram. Reaction of test specimens are linear until Point 1 or so-called proportional limit. Test specimens increase their length till this point. From Point 1 to 2, materials act elastically but the strains become nonlinear. It means that the Point 2 is the elastic limit. For most of the materials, gap between Point 1 and 2 is very small. Furthermore, response of material agrees with Hook's Law till

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Point 2. Conduction of further load ends up with yielding and plastic deformation occurs. Peak point of the stress-strain diagram, Point 3, is called ultimate strength and beyond this point necking takes a place. Specimens fracture at Point 4.

In the present work, elastic behaviors of the materials are taken into account. Thus, our working subject is up to Point 2.

2.4 Material Characteristics

There are many materials that are used in engineering field such as anisotropic, orthotropic,transversely isotropic and isotropic. All of them have specific features. This study emphasizes orthotropic and isotropic materials. Thus, their characteristic properties are explained in detail.

2.4.1 Orthotropic Materials

When a material has three mutually orthogonal symmetric planes, it is called orthotropic. In other words, orthotropy can be described as direction dependent material property change in perpendicular symmetry planes. Material properties such as elastic modulus, shear modulus and poisson ratio differ along these symmetry directions. Wood is a good example of a natural orthotropic material. On the other hand there are many artificial fiber-reinforced composites that are orthotropic. Formerly, it was stated in stress-strain diagrams that materials adapt to Hook's Law in elastic limits. Hence, strain-stress relation takes the below form.

$$
\begin{bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \varepsilon_z \\ \gamma_{\theta z} \\ \gamma_{zr} \\ \gamma_{r\theta} \end{bmatrix}
=
\begin{bmatrix}
\frac{1}{E_r} & -\frac{v_{\theta r}}{E_\theta} & -\frac{v_{zr}}{E_z} & 0 & 0 & 0 \\
-\frac{v_{r\theta}}{E_r} & \frac{1}{E_\theta} & -\frac{v_{z\theta}}{E_z} & 0 & 0 & 0 \\
-\frac{v_{rz}}{E_r} & -\frac{v_{\theta z}}{E_\theta} & \frac{1}{E_z} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{G_{\theta z}} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{G_{zr}} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{G_{r\theta}}
\end{bmatrix}
\begin{bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{\theta z} \\ \tau_{zr} \\ \tau_{r\theta} \end{bmatrix}
\tag{2.1}
$$

The above relation can be written as $[\underline{\varepsilon}] = [\underline{C}][\underline{\sigma}]$. Due to the symmetry planes compliance matrix $[\underline{C}]$ is symmetric. It must be positive and definite as well. In order to be more compact, terms of the compliance matrix can be represented by index notation. Its terms are given as c_{ij} where i and j is from one to six.

Taking the inverse of the compliance matrix defines the stress-strain relation for orthotropic materials. The matrix that connects stress and strain is called stiffness matrix $[\underline{S}]$. Similarly, it is positive definite and symmetric.

$$
\begin{bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{\theta z} \\ \tau_{zr} \\ \tau_{r\theta} \end{bmatrix}
=
\begin{bmatrix}
\frac{1-v_{\theta z}v_{z\theta}}{E_\theta E_z \Delta} & \frac{v_{\theta r}+v_{\theta z}v_{zr}}{E_\theta E_z \Delta} & \frac{v_{zr}+v_{\theta r}v_{z\theta}}{E_\theta E_z \Delta} & 0 & 0 & 0 \\
\frac{v_{\theta r}+v_{\theta z}v_{zr}}{E_\theta E_z \Delta} & \frac{1-v_{rz}v_{zr}}{E_r E_z \Delta} & \frac{v_{z\theta}+v_{r\theta}v_{zr}}{E_r E_z \Delta} & 0 & 0 & 0 \\
\frac{v_{zr}+v_{\theta r}v_{z\theta}}{E_\theta E_z \Delta} & \frac{v_{z\theta}+v_{r\theta}v_{zr}}{E_r E_z \Delta} & \frac{1-v_{r\theta}v_{\theta r}}{E_r E_\theta \Delta} & 0 & 0 & 0 \\
0 & 0 & 0 & G_{\theta z} & 0 & 0 \\
0 & 0 & 0 & 0 & G_{zr} & 0 \\
0 & 0 & 0 & 0 & 0 & G_{r\theta}
\end{bmatrix}
\begin{bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \varepsilon_z \\ \gamma_{\theta z} \\ \gamma_{zr} \\ \gamma_{r\theta} \end{bmatrix}
\tag{2.2}
$$

where $\Delta = \frac{(1-v_{r\theta}v_{\theta r}-v_{rz}v_{zr}-v_{\theta z}v_{z\theta}-2v_{\theta r}v_{z\theta}v_{rz})}{E_r E_\theta E_z}$

The above stress-strain relation can be described with $[\underline{\sigma}] = [\underline{S}][\underline{\varepsilon}]$. Furthermore, index notation is applicable for stiffness matrix. Its terms are denoted as s_{ij} where i and j is from one to six.

2.4.2 Isotropic Materials

Isotropic material is the one that acts uniformly in all directions. Material properties are identical regardless of symmetry planes for isotropic materials.

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Hence, there is not any preferred directions. Elastic modulus, shear modulus and poisson ratio do not alter through the symmetry directions because all planes are plane of symmetry. Glass and ceramic are some examples of isotropic material. Strain-stress relation for isotropic materials is defined with the below compliance matrix.

$$
\begin{bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \varepsilon_z \\ \gamma_{\theta z} \\ \gamma_{zr} \\ \gamma_{r\theta} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{(1+\nu)}{E} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{(1+\nu)}{E} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{(1+\nu)}{E} \end{bmatrix} \begin{bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{\theta z} \\ \tau_{zr} \\ \tau_{r\theta} \end{bmatrix} \quad (2.3)
$$

Only one elastic modulus, shear modulus and poisson ratio is defined this time according to uniformity. Additionally, this matrix is symmetric, positive and definite. Simply the inverse of it is the stiffness matrix for isotropy.

$$
\begin{bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{\theta z} \\ \tau_{zr} \\ \tau_{r\theta} \end{bmatrix} = D \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-2\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-2\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-2\nu \end{bmatrix} \begin{bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \varepsilon_z \\ \gamma_{\theta z} \\ \gamma_{zr} \\ \gamma_{r\theta} \end{bmatrix} \quad (2.4)
$$

where $D = \frac{E}{(1+\nu)(1-2\nu)}$

2.5 Three Dimensional Elasticity

In solid mechanics, meaning of elasticity is an ability to counter a force that is applied to a body and a tendency to return its initial shape and size. Elastic materials would return to their original shape and size when the applied forces are removed.

To obtain an elastic solution there are three essential conditions to be satisfied:

▼ The elastic body should be in equilibrium.

▼ Compatibility equation should be satisfied.

▼ Boundary conditions of the body should be fulfilled.

Due to the complexity of these conditions and three dimensional elastic equations, generally elasticity problems are reduced to two dimensional simpler forms. Brief of the two dimensional elastic problems are explained below.

2.6   Two Dimensional Elasticity

2.6.1   Plain Strain

Plain strain is a case that one direction of a structure, usually z direction in coordinate system, is very long when it is compared to other two directions. In a more formal form, normal strain and shear strains at the longer side of the body are assumed zero for plain strain state. The shape of the body plays an essential role to determine elastic problems. Pipelines, tunnels and dams are some of the real world examples of the plain strain problems. In the current case, long cylinders of orthotropic and isotropic materials are examined. Length of the structure is longer than the other directions. Thus, this paper deals with plain strain case.

2.6.2   Plane Stress

Plane stress is another type of two dimensional application of elasticity where the one dimension of the body, generally z direction, is thinner than the other two. Plane stress theory assumes that the thinner section of the body is stress free. Therefore, normal stresses and shear stresses are considered zero. Airplane fuselage design or car exterior design can be given as examples.

CHAPTER 3

PROBLEM DEFINITION AND ANALYTICAL SOLUTIONS

Consider a fix ended, long and thick walled cylinder whose center is the origin of the polar coordinate system in which r, θ and z represent radial, tangential (circumferential) and axial(longitudinal) directions respectively. For cylindrical shapes, usage of polar coordinate system would be appropriate to determine stresses. The figure given below visualizes the coordinate system and the cross section of the structure in which a is the inner radius and b is the outer radius.
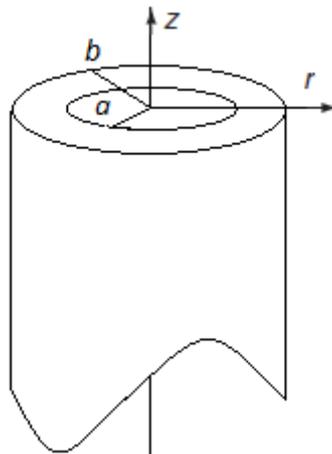
Figure 3.1: Cross section of the structure

The investigated cases examine cylinder subjected to internal pressure, external pressure , combined pressure and annular rotation.

Theses cases are carried out for orthotropic and isotropic materials in principal directions. Effect of orthotropy is compared with that of isotropy. The cases we study cause stress through the structure. These stresses and their directions are illustrated in Figure 3.2.
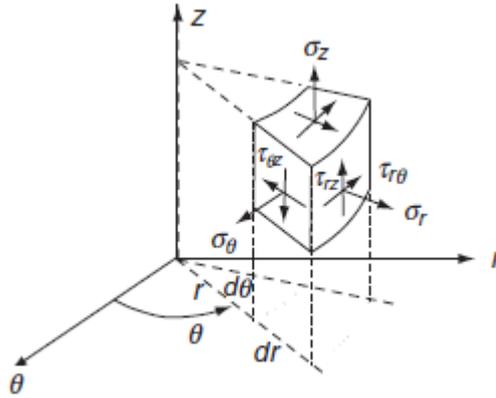
Figure 3.2: Infinitesimal element of the cylinder

The above figure describes the general state of stresses but this work is considered in principal directions. Therefore, shear stresses vanish. Moreover, the stress directions that are given in Figure 3.2 are considered positive. Thus, stresses at these directions cause tension. Similarly, opposite directions are considered negative and cause compression on the structure.

3.1 Basic Formulation to Isotropic Cylinders

In polar coordinate system, displacement field is a function of (r, θ, z) for general three dimensional elasticity problems. For this particular case, it is independent of z direction because of the fix ended structure. The boundary conditions that we taken into account are symmetric through the body. It means that θ dependency disappears. These constrains reduce displacement functions to the following form.

$$u_r = u_r(r), u_\theta = 0 \text{ and } u_z = 0 \tag{3.1}$$

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There is only one displacement function. Hence, it is convenient to use $u_r = u$. Under these conditions strains can easily be obtained from the displacement.

$$\varepsilon_r = \frac{du}{dr}, \varepsilon_\theta = \frac{u}{r} \text{ and } \varepsilon_z = 0 \tag{3.2}$$

It is mentioned that the research is held in principal directions. Thus, shear strains vanish.

$$\gamma_{r\theta} = \gamma_{rz} = \gamma_{\theta z} = 0 \tag{3.3}$$

General strain-stress relation for isotropic cylinder is then

$$\begin{bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \varepsilon_z \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & \frac{-v}{E} & \frac{-v}{E} \\ \frac{-v}{E} & \frac{1}{E} & \frac{-v}{E} \\ \frac{-v}{E} & \frac{-v}{E} & \frac{1}{E} \end{bmatrix} \begin{bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \end{bmatrix} \tag{3.4}$$

Inverse of the above compliance matrix gives the stiffness matrix in principal directions for the isotropic cylinder.

$$\begin{bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \end{bmatrix} = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & v \\ v & 1-v & v \\ v & v & 1-v \end{bmatrix} \begin{bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \varepsilon_z \end{bmatrix} \tag{3.5}$$

All of the displacements, strains, stresses and their relations with each other are stated for the isotropic cylinders. The next step is the application of elastic equations. As it is mentioned before, the body must be in equilibrium in order to have an elastic solution. Three dimensional equilibrium equations in polar coordinates are then

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\tau_{r\theta}}{\partial \theta} + \frac{\tau_{rz}}{\partial z} + \frac{1}{r}(\sigma_r - \sigma_\theta) + f_r = 0 \tag{3.6}$$

$$\frac{\tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\tau_{\theta z}}{\partial z} + \frac{2}{r}\tau_{r\theta} + f_\theta = 0 \tag{3.7}$$

$$\frac{\tau_{rz}}{\partial r} + \frac{1}{r}\frac{\tau_{\theta z}}{\partial \theta} + \frac{\partial \sigma_z}{\partial z} + \frac{1}{r}\tau_{rz} + f_z = 0 \tag{3.8}$$

Eq(3.7) and Eq(3.8) vanish because of the absence of shear stresses and the related constrains. Only the first equilibrium equation is required to be satisfied. Eq(3.6) takes the below form without the shear components.

$$\frac{d\sigma_r}{dr} + \frac{1}{r}(\sigma_r - \sigma_\theta) + f_r = 0 \tag{3.9}$$

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where f_r is the body force term and is $f_r = \rho\omega^2 r$. Radial and tangential stress terms in the above equation can be represented in terms of strains by using stress strain relations. So, Eq(3.9) becomes

$$\frac{d}{dr}(s_{11}\varepsilon_r + s_{12}\varepsilon_\theta + s_{13}\varepsilon_z) + \frac{1}{r}(s_{11}\varepsilon_r + s_{12}\varepsilon_\theta + s_{13}\varepsilon_z - s_{21}\varepsilon_r - s_{22}\varepsilon_\theta - s_{23}\varepsilon_z) + \rho\omega^2 r = 0$$
(3.10)

Strains can be written as a function of displacement. Moreover, due to the material symmetry of isotropy $s_{22} = s_{11}$ and $s_{21} = s_{12}$. Substituting these parameters to the above equation gives the following.

$$\frac{d}{dr}(s_{11}\frac{du}{dr} + s_{12}\frac{u}{r}) + \frac{1}{r}(s_{11}\frac{du}{dr} + s_{12}\frac{u}{r} - s_{12}\frac{du}{dr} - s_{11}\frac{u}{r}) + \rho\omega^2 r = 0 \quad (3.11)$$

Rearranging above equation gives a non homogenous Cauchy-Euler differential equation.

$$r^2\frac{d^2u}{dr^2} + r\frac{du}{dr} - u = -\frac{\rho\omega^2 r}{s_{11}}$$
(3.12)

When the equilibrium equation is solved for our system, we end up with the below equation as a function of r. In the following expression C_1 and C_2 are arbitrary constants to be determined according to the boundary conditions.

$$u(r) = C_1 r^{-1} + C_2 r - \frac{\rho\omega^2 r^3}{8s_{11}}$$
(3.13)

The term that describes the rotation behavior in the displacement function can be taken as $C_3 = -\frac{\rho\omega^2}{8s_{11}}$. So that radial displacement function can be put into a more proper form.

$$u(r) = C_1 r^{-1} + C_2 r + C_3 r^3$$
(3.14)

Strains at r and θ directions are derived from the displacement function.

$$\varepsilon_r(r) = \frac{du}{dr} = -C_1 r^{-2} + C_2 + 3C_3 r^2$$
(3.15)

$$\varepsilon_\theta(r) = \frac{u}{r} = C_1 r^{-2} + C_2 + C_3 r^2$$
(3.16)

Since the strains are found for the isotropic cylinders, stresses can be obtained from the stiffness matrix given in Eq(3.5). In the following stress functions s_{ij} terms are those belonging to the same matrix. Combining radial and tangential strains with Eq(3.5) indicates the directional stresses.

$$\sigma_r(r) = C_1 r^{-2}(-s_{11} + s_{12}) + C_2(s_{11} + s_{12}) + C_3 r^2(3s_{11} + s_{12})$$
(3.17)

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$$\sigma_\theta(r) = C_1 r^{-2}(-s_{21} + s_{22}) + C_2(s_{21} + s_{22}) + C_3 r^2(3s_{21} + s_{22}) \qquad (3.18)$$

$$\sigma_z(r) = C_1 r^{-2}(-s_{31} + s_{32}) + C_2(s_{31} + s_{32}) + C_3 r^2(3s_{31} + s_{32}) \qquad (3.19)$$

Stress at axial direction can be exposed as the combination of radial and tangential stresses. In the following equation c_{ij} terms are compatible with Eq(3.4).

$$\sigma_z(r) = -\frac{c_{31}}{c_{33}}(\sigma_r + \sigma_\theta) \qquad (3.20)$$

The proof of the above expression simply originates from the compliance matrix given in Eq(3.4). It is known that $\varepsilon_z = 0$. Moreover, using the last row of Eq(3.4) strain at z direction can be written as.

$$\varepsilon_z = c_{31}\sigma_r + c_{32}\sigma_\theta + c_{33}\sigma_z \qquad (3.21)$$

Combination of the above identity with $\varepsilon_z = 0$ gives

$$\sigma_z = -\frac{c_{31}}{c_{33}}\sigma_r - \frac{c_{32}}{c_{33}}\sigma_\theta \qquad (3.22)$$

Due to the plane symmetry, $c_{32} = c_{31}$ for isotropic materials. So, we finally arrive at Eq(3.20).

Because of the fix ended structure, stress takes a place at the axial direction. External force at this direction can be calculated from σ_z with the following integration.

$$F_z = \int_a^b 2\pi r \sigma_z dr \qquad (3.23)$$

Equilibrium equations are satisfied and stress functions are determined accordingly. The next step is to apply compatibility equation for the general solution. Corresponding compatibility equation for this purpose is

$$r\frac{d\varepsilon_\theta}{dr} + \varepsilon_\theta - \varepsilon_r = 0 \qquad (3.24)$$

If the radial and circumferential strains that are given in Eq(3.15) and Eq.(3.16) substituted into the above equation, compatibility condition gets satisfied identically.

3.2 Boundary Conditions for Isotropic Cylinders

In this section, application of the boundary conditions to the elastic bodies are displayed separately for each case. General solutions are put into specific forms.

3.2.1   Isotropic Cylinders Under Internal Pressure

In the case of cylinders subjected to internal pressure, boundary conditions take this form

$$\sigma_r(a) = -P_{in}, \sigma_r(b) = 0 \tag{3.25}$$

where a is the inner and b is the outer radius of the cylinder. P_{in} is the limit elastic pressure to be determined. Substituting boundary conditions into σ_r gives C_1 and C_2.

$$C_1 = -\frac{a^2 b^2 P_{in}(1+\nu)}{(a^2 - b^2)E} \tag{3.26.a}$$

$$C_2 = -\frac{a^2 P_{in}(1+\nu)(1-2\nu)}{(a^2 - b^2)E} \tag{3.26.b}$$

If the arbitrary constants C_1 and C_2 are put into the displacement and stress functions, proper forms of the these functions are obtained as below.

$$u(r) = -\frac{P_{in}(1+\nu)a^2 b^2}{(a^2 - b^2)E}\left[\frac{(1-2\nu)r}{b^2} + \frac{1}{r}\right] \tag{3.27}$$

$$\sigma_r(r) = -\frac{P_{in}a^2}{a^2 - b^2}\left[1 - \frac{b^2}{r^2}\right] \tag{3.28}$$

$$\sigma_\theta(r) = -\frac{P_{in}a^2}{a^2 - b^2}\left[1 + \frac{b^2}{r^2}\right] \tag{3.29}$$

$$\sigma_z(r) = -\frac{2\nu P_{in}a^2}{a^2 - b^2} \tag{3.30}$$

3.2.2   Isotropic Cylinders Under External Pressure

This time, cylinders are under external pressure. So, the boundary conditions read

$$\sigma_r(a) = 0, \sigma_r(b) = -P_{ex} \tag{3.31}$$

Similarly, P_{ex} is the limit external pressure. Substitution of the boundary conditions to the radial stress provides C_1 and C_2.

$$C_1 = \frac{a^2 b^2 P_{ex}(1+\nu)}{(a^2 - b^2)E} \tag{3.32.a}$$

$$C_2 = \frac{b^2 P_{ex}(1-2\nu)(1+\nu)}{(a^2 - b^2)E} \tag{3.32.b}$$

Rearranging C_1 and C_2 with displacement and stresses, gives the below functions.

$$u(r) = \frac{P_{ex}(1+\nu)a^2b^2}{(a^2-b^2)E}[\frac{(1-2\nu)r}{a^2} + \frac{1}{r}] \tag{3.33}$$

$$\sigma_r(r) = \frac{P_{ex}b^2}{a^2-b^2}[1-\frac{a^2}{r^2}] \tag{3.34}$$

$$\sigma_\theta(r) = \frac{P_{ex}b^2}{a^2-b^2}[1+\frac{a^2}{r^2}] \tag{3.35}$$

$$\sigma_z(r) = \frac{2\nu P_{ex}\, b^2}{a^2-b^2} \tag{3.36}$$

3.2.3 Isotropic Cylinders Under Combined Pressure

Isotropic cylinders are subjected to both internal and external pressure. Boundary conditions are then

$$\sigma_r(a) = -P_{in}, \sigma_r(b) = -P_{ex} \tag{3.37}$$

Likely,

$$C_1 = \frac{a^2b^2(P_{ex}-P_{in})(1+\nu)}{(\,a^2-b^2)E} \tag{3.38.a}$$

$$C_2 = \frac{(1-2\nu)(1+\nu)(b^2P_{ex}-a^2P_{in})}{(\,a^2-b^2)E} \tag{3.38.b}$$

Displacement function is

$$u(r) = \frac{(1+\nu)a^2b^2}{(a^2-b^2)E}[\frac{P_{ex}-P_{in}}{r} - \frac{(1-2\nu)(P_{ex}b^2-P_{in}a^2)r}{a^2b^2}] \tag{3.39}$$

Directional stresses are

$$\sigma_r(r) = \frac{P_{ex}b^2-P_{in}a^2}{a^2-b^2} - \frac{a^2b^2(P_{ex}-P_{in})}{(a^2-b^2)r^2} \tag{3.40}$$

$$\sigma_\theta(r) = \frac{P_{ex}b^2-P_{in}a^2}{a^2-b^2} + \frac{a^2b^2(P_{ex}-P_{in})}{(a^2-b^2)r^2} \tag{3.41}$$

$$\sigma_z(r) = \frac{2\nu(P_{ex}b^2-P_{in}a^2)}{a^2-b^2} \tag{3.42}$$

It is essential to note that combined pressure case is the superposition of the internal and external pressure cases. For instance, consider the displacement function of the combined pressure case given in Eq(3.39). If $P_{ex} = 0$ in Eq(3.39), we end up with Eq(3.27) which is the displacement function for the internally pressurized cylinder. Likely, If $P_{in} = 0$ in Eq(3.39), we come up with Eq(3.33). This situation is valid for directional stresses too.

3.2.4 Isotropic Cylinders Under Annular Rotation

In this part, the effect of annular rotation on the isotropic cylinders is investigated. For this purpose, boundary conditions become

$$\sigma_r(a) = 0, \ \sigma_r(b) = 0 \tag{3.43}$$

Arbitrary constants of the structure are

$$C_1 = \frac{\rho\omega^2 a^2 b^2 (1 + \nu)(2\nu - 3)}{8E(\nu - 1)} \tag{3.44.a}$$

$$C_2 = \frac{\rho\omega^2 (a^2 + b^2)(1 + \nu)(2\nu - 3)(1 - 2\nu)}{8E(\nu - 1)} \tag{3.44.b}$$

Displacement and directional stresses are then

$$u(r) = \frac{\rho\omega^2 (1 + \nu)(2\nu - 3)}{8E(\nu - 1)} \Big[\frac{a^2 b^2}{r} + (1 - 2\nu)[(a^2 + b^2)r + \frac{r^3}{(2\nu - 3)}]\Big] \tag{3.45}$$

$$\sigma_r(r) = \frac{\rho\omega^2 (2\nu - 3)}{8(\nu - 1)} \Big[a^2 + b^2 - r^2 - \frac{a^2 b^2}{r^2}\Big] \tag{3.46}$$

$$\sigma_\theta(r) = \frac{\rho\omega^2 (2\nu - 3)}{8(\nu - 1)} \Big[a^2 + b^2 + \frac{1 + 2\nu}{2\nu - 3}r^2 + \frac{a^2 b^2}{r^2}\Big] \tag{3.47}$$

$$\sigma_z(r) = \frac{\rho\omega^2 \nu(2\nu - 3)}{8(\nu - 1)} \Big[2a^2 + 2b^2 + \frac{4}{2\nu - 3}r^2\Big] \tag{3.48}$$

As it is seen from the above work general displacements, strains and stresses are adopted to the structures that we are using. Followingly, various boundary conditions are applied and analytical solutions are obtained for the isotropic cylinders. The same procedure is going to be conducted to the orthotropic cylinders.

3.3 Basic Formulation to Orthotropic Cylinders

Basic constrains that we used on the isotropic structures are valid for the orthotropic ones. Thus, the conditions stated in Eq(3.1), Eq(3.2) and Eq(3.3) are the same. On the other hand, the difference between isotropy and orthotropy is defined by the material characteristic matrices. Hence, compliance and stiffness

matrices should be defined once again. The compliance matrix for orthotropy in principal directions is then

$$
\begin{bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \varepsilon_z \end{bmatrix} = \begin{bmatrix} \frac{1}{E_r} & \frac{-v_{\theta r}}{E_\theta} & \frac{-v_{zr}}{E_z} \\ \frac{-v_{r\theta}}{E_r} & \frac{1}{E_\theta} & \frac{-v_{z\theta}}{E_z} \\ \frac{-v_{rz}}{E_r} & \frac{-v_{\theta z}}{E_\theta} & \frac{1}{E_z} \end{bmatrix} \begin{bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \end{bmatrix} \tag{3.49}
$$

It is known that compliance matrix is symmetric. As a result of this, the below equivalence can be written in index notation.

$$
c_{ij} = c_{ji} \tag{3.50}
$$

The above identity can be expressed as

$$
\frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j} \tag{3.51}
$$

Eq.(3.51) is called Maxwell Relations which are applicable to every term of the compliance matrix. The algebraic manipulations give us an opportunity to parametrize the matrix given in Eq(3.49).

$$
\frac{v_{\theta r}}{v_{r\theta}} = \frac{E_\theta}{E_r} = R_1 \quad v_{\theta r} = R_1 v_{r\theta} \quad E_\theta = R_1 E_r \tag{3.52}
$$

$$
\frac{v_{zr}}{v_{rz}} = \frac{E_z}{E_r} = R_2 \quad v_{zr} = R_2 v_{rz} \quad E_z = R_2 E_r \tag{3.53}
$$

$$
\frac{v_{z\theta}}{v_{\theta z}} = \frac{E_z}{E_\theta} = R_3 \quad v_{z\theta} = R_3 v_{\theta z} \quad E_z = R_3 E_\theta \tag{3.54}
$$

R_1, R_2 and R_3 are the proportions among elastic moduli. These proportions gauge the degree of orthotropy. Manipulating Eq(3.49) with the Maxwell relations gives the parametrized form of the compliance matrix. Rearranging compliance matrix reduces the number of poisson ratios and elastic moduli that we need to determine. Instead of six different poisson ratios we only determine three of them which are $v_{r\theta}, v_{rz}$ and $v_{\theta z}$. Similarly, setting the elasticity modulus at radial direction is enough to determine whole strain-stress relation.

$$
\begin{bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \varepsilon_z \end{bmatrix} = \begin{bmatrix} \frac{1}{E_r} & \frac{-v_{r\theta}}{E_r} & \frac{-v_{rz}}{E_r} \\ \frac{-v_{r\theta}}{E_r} & \frac{1}{R_1 E_r} & \frac{-v_{\theta z}}{R_1 E_r} \\ \frac{-v_{rz}}{E_r} & \frac{-v_{\theta z}}{R_1 E_r} & \frac{1}{R_2 E_r} \end{bmatrix} \begin{bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \end{bmatrix} \tag{3.55}
$$

Moreover, after some algebraic manipulations R_3 term disappears in the calculations. It can be written as $R_3 = R_2/R_1$. Thereby, strain-stress relation takes a

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new shape. Simply, taking the inverse of Eq(3.55) supplies the new stress-strain relation.

$$\begin{bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \end{bmatrix} = \begin{bmatrix} \frac{(R_2 v_{\theta z}^2 - R_1)}{\Delta R_1} & \frac{-(R_1 v_{r\theta} + R_2 v_{rz} v_{\theta z})}{\Delta} & \frac{-R_2(v_{rz} + v_{r\theta} v_{\theta z})}{\Delta} \\ \frac{-(R_1 v_{r\theta} + R_2 v_{rz} v_{\theta z})}{\Delta} & \frac{R_1(R_2 v_{rz}^2 - 1)}{\Delta} & \frac{-R_2(R_1 v_{rz} v_{r\theta} + v_{\theta z})}{\Delta} \\ \frac{-R_2(v_{rz} + v_{r\theta} v_{\theta z})}{\Delta} & \frac{-R_2(R_1 v_{rz} v_{r\theta} + v_{\theta z})}{\Delta} & \frac{R_2(R_1 v_{r\theta}^2 - 1)}{\Delta} \end{bmatrix} \begin{bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \varepsilon_z \end{bmatrix} \quad (3.56)$$

where $\Delta = \frac{R_1(R_1 v_{r\theta}^2 - 1) + R_1 R_2 v_{rz}(v_{rz} + 2 v_{r\theta} v_{\theta z}) + R_2 v_{\theta z}^2}{E_r R_1}$

Displacements, strains and stresses are conducted for the orthotropic cylinders. Additionally, strain-stress and stress-strain relations are parametrized with Maxwell relations and transformed to new forms. These new forms facilitate to demonstrate displacements and stresses for the orthotropic structures. Right after these clarifications, equilibrium equations can be applied to the orthotropic bodies. The method used for the isotropic cylinders is applied to the orthotropic structures. Once again, only one equilibrium equation should be satisfied.

$$\frac{d\sigma_r}{dr} + \frac{1}{r}(\sigma_r - \sigma_\theta) + \rho\omega^2 r = 0 \quad (3.57)$$

If the radial and tangential stresses are written as a function of radial displacement, equilibrium equation takes the below form. Corresponding s_{ij} terms are belonging to Eq(3.56).

$$\frac{d}{dr}\left(s_{11}\frac{du}{dr} + s_{12}\frac{u}{r}\right) + \frac{1}{r}\left(s_{11}\frac{du}{dr} + s_{12}\frac{u}{r} - s_{12}\frac{du}{dr} - s_{22}\frac{u}{r}\right) + \rho\omega^2 r = 0 \quad (3.58)$$

Simplifying the above identity gives a non homogeneous Cauchy-Euler differential equation.

$$r^2\frac{d^2 u}{dr^2} + r\frac{du}{dr} - \frac{s_{22}}{s_{11}}u = -\frac{\rho\omega^2 r^3}{s_{11}} \quad (3.59)$$

For the orthotropic cylinders, solution of this differential equation is the displacement function.

$$u(r) = C_1 r^{-\sqrt{\frac{s_{22}}{s_{11}}}} + C_2 r^{\sqrt{\frac{s_{22}}{s_{11}}}} + \frac{\rho\omega^2 r^3}{s_{22} - 9s_{11}} \quad (3.60)$$

Displacement function can be written as

$$u(r) = C_1 r^{-\xi} + C_2 r^{\xi} + C_3 r^3 \quad (3.61)$$

Where $\xi = \sqrt{\frac{s_{22}}{s_{11}}} = \sqrt{\frac{R_1^2(R_2 v_{rz}^2 - 1)}{R_2 v_{\theta z}^2 - R_1}}$ and $C_3 = \frac{\rho\omega^2}{s_{22} - 9s_{11}}$

Radial and tangential elastic strains are acquired from Eq(3.61)

$$\varepsilon_r(r) = \frac{du}{dr} = -C_1\xi r^{-\xi-1} + C_2\xi r^{\xi-1} + 3C_3r^2 \tag{3.62}$$

$$\varepsilon_\theta(r) = \frac{u}{r} = C_1r^{-\xi-1} + C_2r^{\xi-1} + C_3r^2 \tag{3.63}$$

After obtaining strains, stresses can be derived from the stiffness matrix in Eq.(3.56).

$$\sigma_r(r) = C_1r^{-\xi-1}(-s_{11}\xi + s_{12}) + C_2r^{\xi-1}(s_{11}\xi + s_{12}) + C_3r^2(3s_{11} + s_{12}) \tag{3.64}$$

$$\sigma_\theta(r) = C_1r^{-\xi-1}(-s_{21}\xi + s_{22}) + C_2r^{\xi-1}(s_{21}\xi + s_{22}) + C_3r^2(3s_{21} + s_{22}) \tag{3.65}$$

$$\sigma_z(r) = C_1r^{-\xi-1}(-s_{31}\xi + s_{32}) + C_2r^{\xi-1}(s_{31}\xi + s_{32}) + C_3r^2(3s_{31} + s_{32}) \tag{3.66}$$

Once again longitudinal stress can be indicated as a combination of radial and tangential stresses.

$$\sigma_z(r) = -\frac{c_{31}}{c_{33}}\sigma_r - \frac{c_{32}}{c_{33}}\sigma_\theta \tag{3.67}$$

The above assertion is developed from the parametrized compliance matrix of orthotropy. Similar to isotropic structures, force in axial direction is figured out from Eq(3.23).

3.4   Boundary Conditions for Orthotropic Cylinders

In this section, the boundary conditions considered for the isotropic cylinders are applied to the orthotropic structures in a similar way.

3.4.1   Orthotropic Cylinders Under Internal Pressure

For the orthotropic cylinders under internal pressure, boundary conditions are $\sigma_r(a) = -P_{in}$, $\sigma_r(b) = 0$. These are the same boundary conditions we used on the isotropic cylinders. Arbitrary constants are

$$C_1 = -\frac{a^{1+\xi}b^{2\xi}P_{in}}{(a^{2\xi} - b^{2\xi})A^*} \tag{3.68.a}$$

$$C_2 = -\frac{a^{1+\xi}P_{in}}{(a^{2\xi} - b^{2\xi})B^*} \tag{3.68.b}$$

As it is seen, A^* and B^* variables are introduced to arbitrary constants. Due to their complexity, they are given separately.

$$A^* = \frac{E_r(\xi R_2 \nu_{\theta z}^2 - \xi R_1 + R_1^2 \nu_{r\theta} + R_1 R_2 \nu_{rz} \nu_{\theta z})}{R_1^2 \nu_{r\theta}^2 + R_2 \nu_{\theta z}^2 + R_1 R_2 \nu_{rz}(\nu_{rz} + 2\nu_{r\theta}\nu_{\theta z}) - R_1} \qquad (3.69.\text{a})$$

$$B^* = \frac{E_r(\xi R_2 \nu_{\theta z}^2 - \xi R_1 - R_1^2 \nu_{r\theta} - R_1 R_2 \nu_{rz} \nu_{\theta z})}{R_1^2 \nu_{r\theta}^2 + R_2 \nu_{\theta z}^2 + R_1 R_2 \nu_{rz}(\nu_{rz} + 2\nu_{r\theta}\nu_{\theta z}) - R_1} \qquad (3.69.\text{b})$$

Displacement

$$u(r) = -\frac{a^{1+\xi}b^{2\xi}P_{in}}{a^{2\xi} - b^{2\xi}}[\frac{r^{-\xi}}{A^*} + \frac{r^{\xi}}{b^{2\xi}B^*}] \qquad (3.70)$$

Stress functions in radial direction

$$\sigma_r(r) = -\frac{P_{in}a^{1+\xi}}{a^{2\xi} - b^{2\xi}}[1 - \frac{b^{2\xi}}{r^{2\xi}}] \qquad (3.71)$$

$$\sigma_\theta(r) = -\frac{P_{in}a^{1+\xi}\xi}{a^{2\xi} - b^{2\xi}}[1 + \frac{b^{2\xi}}{r^{2\xi}}] \qquad (3.72)$$

$$\sigma_z(r) = -\frac{R_2 P_{in}a^{1+\xi}}{a^{2\xi} - b^{2\xi}}[\nu_{rz}(1 - \frac{b^{2\xi}}{r^{2\xi}}) + \frac{\xi\nu_{\theta z}}{R_1}(1 + \frac{b^{2\xi}}{r^{2\xi}})] \qquad (3.73)$$

3.4.2 Orthotropic Cylinders Under External Pressure

Boundary conditions for externally pressurized orthotropic cylinders are $\sigma_r(a) = 0$, $\sigma_r(b) = -P_{ex}$. Conforming arbitrary constants are

$$C_1 = \frac{a^{2\xi}b^{1+\xi}P_{ex}}{(a^{2\xi} - b^{2\xi})A^*} \qquad (3.74.\text{a})$$

$$C_2 = \frac{b^{1+\xi}P_{ex}}{(a^{2\xi} - b^{2\xi})B^*} \qquad (3.74.\text{b})$$

Similarly, A^* and B^* terms are used here. Rearranging arbitrary constants with displacement and stresses gives the following.

$$u(r) = \frac{a^{2\xi}b^{1+\xi}P_{ex}}{a^{2\xi} - b^{2\xi}}[\frac{r^{-\xi}}{A^*} + \frac{r^{\xi}}{a^{2\xi}B^*}] \qquad (3.75)$$

$$\sigma_r(r) = \frac{P_{ex}b^{1+\xi}}{a^{2\xi} - b^{2\xi}}[1 - \frac{a^{2\xi}}{r^{2\xi}}] \qquad (3.76)$$

$$\sigma_\theta(r) = \frac{P_{ex}b^{1+\xi}\xi}{a^{2\xi} - b^{2\xi}}[1 + \frac{a^{2\xi}}{r^{2\xi}}] \qquad (3.77)$$

$$\sigma_z(r) = \frac{R_2 P_{ex}b^{1+\xi}}{a^{2\xi} - b^{2\xi}}[\nu_{rz}(1 - \frac{a^{2\xi}}{r^{2\xi}}) + \frac{\xi\nu_{\theta z}}{R_1}(1 + \frac{a^{2\xi}}{r^{2\xi}})] \qquad (3.78)$$

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3.4.3 Orthotropic Cylinders Under Combined Pressure

This time orthotropic structures are examined under both internal and external pressure. Boundary conditions are $\sigma_r(a) = -P_{in}$, $\sigma_r(b) = -P_{ex}$. The same procedure is applied.

$$C_1 = \frac{a^\xi b^\xi (a^\xi b P_{ex} - ab^\xi P_{in})}{(a^{2\xi} - b^{2\xi})A^*} \qquad (3.79.\text{a})$$

$$C_2 = \frac{b^{1+\xi}P_{ex} - a^{1+\xi}P_{in}}{(a^{2\xi} - b^{2\xi})B^*} \qquad (3.79.\text{b})$$

Radial displacement function is

$$u(r) = \frac{1}{a^{2\xi} - b^{2\xi}}\Big[\frac{(a^{2\xi}b^{\xi+1}P_{ex} - a^{\xi+1}b^{2\xi}P_{in})r^{-\xi}}{A^*} + \frac{(b^{1+\xi}P_{ex} - a^{1+\xi}P_{in})r^\xi}{B^*}\Big] \quad (3.80)$$

Stresses at r, θ and z directions are

$$\sigma_r(r) = \frac{(b^{1+\xi}P_{ex} - a^{1+\xi}P_{in})r^{\xi-1}}{a^{2\xi} - b^{2\xi}} - \frac{(a^{2\xi}b^{\xi+1}P_{ex} - a^{\xi+1}b^{2\xi}P_{in})r^{-\xi-1}}{a^{2\xi} - b^{2\xi}} \qquad (3.81)$$

$$\sigma_\theta(r) = \xi\Big[\frac{(b^{1+\xi}P_{ex} - a^{1+\xi}P_{in})r^{\xi-1}}{a^{2\xi} - b^{2\xi}} + \frac{(a^{2\xi}b^{\xi+1}P_{ex} - a^{\xi+1}b^{2\xi}P_{in})r^{-\xi-1}}{a^{2\xi} - b^{2\xi}}\Big] \quad (3.82)$$

$$\sigma_z(r) = \Big[\frac{\xi R_2\nu_{\theta z}}{R_1} + R_2\nu_{rz}\Big]\frac{(b^{1+\xi}P_{ex} - a^{1+\xi}P_{in})r^{\xi-1}}{a^{2\xi} - b^{2\xi}}$$
$$+ \Big[\frac{\xi R_2\nu_{\theta z}}{R_1} - R_2\nu_{rz}\Big]\frac{(a^{2\xi}b^{\xi+1}P_{ex} - a^{\xi+1}b^{2\xi}P_{in})r^{-\xi-1}}{a^{2\xi} - b^{2\xi}} \qquad (3.83)$$

It should be clarified that if the internal and external pressure cases are super-posed, displacement and stress functions are carried out as above.

3.4.4 Orthotropic Cylinders Under Annular Rotation

Boundary conditions we used in annular rotation are $\sigma_r(a) = 0$, $\sigma_r(b) = 0$. Arbitrary constants are determined as

$$C_1 = \frac{\rho\omega^2(a^{2\xi}b^{\xi+3} - a^{\xi+3}b^{2\xi})C^*}{(a^{2\xi} - b^{2\xi})(\xi^2 - 9)A^*} \qquad (3.84.\text{a})$$

$$C_2 = -\frac{\rho\omega^2(a^{\xi+3} - b^{\xi+3})C^*}{(a^{2\xi} - b^{2\xi})(\xi^2 - 9)B^*} \qquad (3.84.\text{b})$$

A new variable C^* is introduced to simplify arbitrary constants.

$$C^* = 3 + \frac{R_1(R_1\nu_{r\theta} + R_2\nu_{rz}\nu_{\theta z})}{R_1 - R_2\nu_{\theta z}^2} \qquad (3.85)$$

Radial displacement function is

$$u(r) = \frac{\rho\omega^2 C^*}{(a^{2\xi} - b^{2\xi})(\xi^2 - 9)} \Big[\frac{(a^{2\xi}b^{\xi+3} - a^{\xi+3}b^{2\xi})r^{-\xi}}{A^*}$$
$$- \frac{(a^{\xi+3} - b^{\xi+3})r^\xi}{B^*} + \frac{(a^{2\xi} - b^{2\xi})}{D^*}r^3 \Big]$$

(3.86)

In a similar manner D^* is determined.

$$D^* = -\frac{E_r(3R_1 + R_1^2\nu_{r\theta} + R_1 R_2\nu_{rz}\nu_{\theta z} - 3R_2\nu_{\theta z}^2}{R_1^2\nu_{r\theta}^2 + R_2\nu_{\theta z}^2 - R_1 + R_1 R_2\nu_{rz}(\nu_{rz} + 2\nu_{r\theta}\nu_{\theta z})}$$

(3.87)

Radial stress

$$\sigma_r(r) = -\frac{\rho\omega^2 C^*}{(a^{2\xi} - b^{2\xi})(\xi^2 - 9)} [(a^{2\xi}b^{\xi+3} - a^{\xi+3}b^{2\xi})r^{-\xi-1}$$
$$+ (a^{\xi+3} - b^{\xi+3})r^{\xi-1} - (a^{2\xi} - b^{2\xi})r^2]$$

(3.88)

Tangential stress

$$\sigma_\theta(r) = -\frac{\rho\omega^2 C^*}{(a^{2\xi} - b^{2\xi})(\xi^2 - 9)} [E^*(a^{\xi+3}b^{2\xi} - a^{2\xi}b^{\xi+3})r^{-\xi-1}$$
$$+ F^*(b^{\xi+3} - a^{\xi+3})r^{\xi-1} + G^*(a^{2\xi} - b^{2\xi})r^2]$$

(3.89)

Where

$$E^* = \frac{R_1^2(\xi\nu_{r\theta} + R_2\nu_{rz}^2 - 1) + \xi R_1 R_2\nu_{rz}\nu_{\theta z}}{\xi(R_1 - R_2\nu_{\theta z}) - R_1(R_1\nu_{r\theta} + R_2\nu_{rz}\nu_{\theta z})}$$

(3.90.a)

$$F^* = \frac{R_1(R_1 - R_1 R_2\nu_{rz}^2 + \xi R_1\nu_{r\theta} + \xi R_2\nu_{rz}\nu_{\theta z})}{\xi R_1 + R_1^2\nu_{r\theta} + R_1 R_2\nu_{\theta z}\nu_{rz} - \xi R_2\nu_{\theta z}^2}$$

(3.90.b)

$$G^* = \frac{R_1(R_1 - R_1 R_2\nu_{rz}^2 + 3R_1\nu_{r\theta} + 3R_2\nu_{rz}\nu_{\theta z})}{3R_1 + R_1^2\nu_{r\theta} + R_1 R_2\nu_{\theta z}\nu_{rz} - 3R_2\nu_{\theta z}^2}$$

(3.90.c)

Axial stress

$$\sigma_z(r) = \frac{-\rho\omega^2 C^* R_2}{(a^{2\xi} - b^{2\xi})(\xi^2 - 9)R_1} [(R_1\nu_{rz} + E^*\nu_{\theta z})(a^{2\xi}b^{\xi+3} - a^{\xi+3}b^{2\xi})r^{-\xi-1}$$
$$+ (R_1\nu_{rz} + F^*\nu_{\theta z})(a^{\xi+3} - b^{\xi+3})r^{\xi-1} - (R_1\nu_{rz} + G^*\nu_{\theta z})(a^{2\xi} - b^{2\xi})r^2]$$

(3.91)

Although new variables have been introduced, functions cannot be simplified any further. Since, behavior of annular rotation on orthotropic cylinders is very complicated.

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3.5   Combined Cases

In this section, particular boundary conditions are compounded with each other and analytical solutions are obtained accordingly. For this purpose, annular rotation case is analytically cascaded with internal and external pressure cases respectively.

3.5.1   Isotropic Cylinders Under Annular Rotation and Internal Pressure

Combined effect of annular rotation and internal pressure is investigated analytically. For this particular case, boundary conditions are given as the following.

$$\sigma_r(a) = -P_{in}, \ \sigma_r(b) = 0 \text{ where } \omega \neq 0 \tag{3.92}$$

Under these boundary conditions, arbitrary constants read

$$C_1 = \frac{a^2 b^2 (1+\nu)}{8E}\left[\frac{\rho\omega^2(2\nu-3)}{\nu-1} - \frac{8P_{in}}{a^2-b^2}\right] \tag{3.93.a}$$

$$C_2 = \frac{(1-2\nu)(1+\nu)}{8E}\left[\frac{\rho\omega^2(2\nu-3)(a^2+b^2)}{\nu-1} - \frac{8P_{in}a^2}{a^2-b^2}\right] \tag{3.93.a}$$

Displacement function is

$$u(r) = \frac{\rho\omega^2(2\nu-3)(1+\nu)}{8E(\nu-1)}\left[\frac{a^2 b^2}{r} + (1-2\nu)(a^2+b^2)r\right] - \frac{(1+\nu)P_{in}a^2}{E(a^2-b^2)}\left[\frac{b^2}{r} + (1-2\nu)r\right] \tag{3.94}$$

Stresses in radial directions are

$$\sigma_r(r) = \frac{(r^2-b^2)}{8r^2}\left[\frac{\rho\omega^2(2\nu-3)(a^2-r^2)}{\nu-1} - \frac{8P_{in}a^2}{a^2-b^2}\right] \tag{3.95}$$

$$\sigma_\theta(r) = \frac{\rho\omega^2(2\nu-3)}{8(\nu-1)}\left[a^2+b^2+\frac{a^2 b^2}{r^2}+\frac{(1+2\nu)r^2}{2\nu-3}\right] - \frac{P_{in}a^2}{a^2-b^2}\left[1+\frac{b^2}{r^2}\right] \tag{3.96}$$

$$\sigma_z(r) = \nu\left[\frac{\rho\omega^2[2r^2+(2\nu-3)(a^2+b^2)]}{4(\nu-1)} - \frac{2P_{in}a^2}{a^2-b^2}\right] \tag{3.97}$$

Combined influence of annular rotation and internal pressure can be investigated with the above equations. Moreover, if the equations given in Section 3.2.1 and 3.2.4 are superposed we carry out the above equations.

3.5.2 Isotropic Cylinders Under Annular Rotation and External Pressure

Combination of annular rotation and external pressure is described with the below boundary conditions.

$$\sigma_r(a) = 0, \ \sigma_r(b) = -P_{ex} \text{ where } \omega \neq 0 \tag{3.98}$$

Application of the boundary conditions give the below arbitrary constants.

$$C_1 = \frac{a^2 b^2 (1 + \nu)}{8E} \left[\frac{\rho \omega^2 (2\nu - 3)}{\nu - 1} + \frac{8 P_{ex}}{a^2 - b^2} \right] \tag{3.99.a}$$

$$C_2 = \frac{(1 - 2\nu)(1 + \nu)}{8E} \left[\frac{\rho \omega^2 (2\nu - 3)(a^2 + b^2)}{\nu - 1} + \frac{8 P_{ex} b^2}{a^2 - b^2} \right] \tag{3.99.b}$$

Radial displacement takes the below form.

$$u(r) = \frac{\rho \omega^2 (2\nu - 3)(1 + \nu)}{8E(\nu - 1)} \left[\frac{a^2 b^2}{r} + (1 - 2\nu)(a^2 + b^2)r \right] - \frac{(1 + \nu) P_{ex} b^2}{E(a^2 - b^2)} \left[\frac{a^2}{r} + (1 - 2\nu)r \right] \tag{3.100}$$

Radial, tangential and axial stress functions are submitted below.

$$\sigma_r(r) = \frac{(r^2 - a^2)}{8r^2} \left[\frac{\rho \omega^2 (2\nu - 3)(b^2 - r^2)}{\nu - 1} + \frac{8 P_{ex} b^2}{a^2 - b^2} \right] \tag{3.101}$$

$$\sigma_\theta(r) = \frac{\rho \omega^2 (2\nu - 3)}{8(\nu - 1)} \left[a^2 + b^2 + \frac{a^2 b^2}{r^2} + \frac{(1 + 2\nu)r^2}{2\nu - 3} \right] + \frac{P_{ex} b^2}{a^2 - b^2} \left[1 + \frac{a^2}{r^2} \right] \tag{3.102}$$

$$\sigma_z(r) = \nu \left[\frac{\rho \omega^2 [2r^2 + (2\nu - 3)(a^2 + b^2)]}{4(\nu - 1)} + \frac{2 P_{ex} b^2}{a^2 - b^2} \right] \tag{3.103}$$

Similarly, the above equations can be obtained by implementing principal of superposition to the equations given in Section 3.2.2 and 3.2.4.

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3.5.3 Orthotropic Cylinders Under Annular Rotation and Internal Pressure

In the present instance, associated effect of annular rotation and internal pressure is examined for orthotropic cylinders. Boundary conditions are then $\sigma_r(a) = -P_{in}$, $\sigma_r(b) = 0$ where $\omega \neq 0$.

C_1 and C_2 takes the below form.

$$C_1 = \frac{a^\xi b^\xi}{(a^{2\xi} - b^{2\xi})A^*}[\frac{(a^\xi b^3 - a^3 b^\xi)C^* \rho \omega^2}{(\xi^2 - 9)} - ab^\xi P_{in}] \qquad (3.104.a)$$

$$C_2 = -\frac{1}{(a^{2\xi} - b^{2\xi})B^*}[\frac{(a^{\xi+3} - b^{\xi+3})C^* \rho \omega^2}{(\xi^2 - 9)} + a^{\xi+1} P_{in}] \qquad (3.104.b)$$

Displacement in radial direction is found with the below function.

$$u(r) = \frac{1}{(a^{2\xi} - b^{2\xi})}[\frac{\rho \omega^2 C^*}{(\xi^2 - 9)}[\frac{(a^{2\xi} b^{\xi+3} - a^{\xi+3} b^{2\xi})r^{-\xi}}{A^*}$$
$$-\frac{(a^{\xi+3} - b^{\xi+3})r^\xi}{B^*} + \frac{(a^{2\xi} - b^{2\xi})r^3}{D^*}] - a^{1+\xi} b^{2\xi} P_{in}[\frac{r^{-\xi}}{A^*} + \frac{r^\xi}{b^{2\xi} B^*}]] \qquad (3.105)$$

Stresses in polar directions are the following.

$$\sigma_r(r) = \frac{-1}{(a^{2\xi} - b^{2\xi})}[\frac{\rho \omega^2 C^*}{(\xi^2 - 9)}[(a^{2\xi} b^{\xi+3} - a^{\xi+3} b^{2\xi})r^{-\xi-1}$$
$$+(a^{\xi+3} - b^{\xi+3})r^{\xi-1} - (a^{2\xi} - b^{2\xi})r^2] + P_{in} a^{1+\xi}[1 - \frac{b^{2\xi}}{r^{2\xi}}]] \qquad (3.106)$$

$$\sigma_\theta(r) = \frac{-\xi}{(a^{2\xi} - b^{2\xi})}[\frac{\rho \omega^2 C^*}{\xi(\xi^2 - 9)}[E^*(a^{\xi+3} b^{2\xi} - a^{2\xi} b^{\xi+3})r^{-\xi-1}$$
$$+F^*(b^{\xi+3} - a^{\xi+3})r^{\xi-1} + G^*(a^{2\xi} - b^{2\xi})r^2] + P_{in} a^{1+\xi}[1 + \frac{b^{2\xi}}{r^{2\xi}}]] \qquad (3.107)$$

$$\sigma_z(r) = \frac{-1}{(a^{2\xi} - b^{2\xi})}[\frac{\rho \omega^2 C^* R_2}{(\xi^2 - 9)R_1}[(R_1 \nu_{rz} + E^* \nu_{\theta z})(a^{2\xi} b^{\xi+3} - a^{\xi+3} b^{2\xi})r^{-\xi-1}$$
$$+(R_1 \nu_{rz} + F^* \nu_{\theta z})(a^{\xi+3} - b^{\xi+3})r^{\xi-1} + (R_1 \nu_{rz} + G^* \nu_{\theta z})(a^{2\xi} - b^{2\xi})r^2]$$
$$-R_2 P_{in} a^{1+\xi}[\nu_{rz}(1 - \frac{b^{2\xi}}{r^{2\xi}}) + \frac{\xi \nu_{\theta z}}{R_1}(1 + \frac{b^{2\xi}}{r^{2\xi}})]] \qquad (3.108)$$

As it is seen from the above equations, analytical representation of radial displacement and directional stresses are very heterotaxic. Therefore, influence of orthotropy on the structures should be treated attentively.

3.5.4 Orthotropic Cylinder Under Annular Rotation and External Pressure

On orthoropic cylinders, influence of annular rotation and external pressure are investigated. In this case, boundary conditions are the ones that we used in Eq(??) which are $\sigma_r(a) = 0$, $\sigma_r(b) = -P_{ex}$ where $\omega \neq 0$. Under these conditions, arbitrary constants become the following ones.

$$C_1 = \frac{a^\xi b^\xi}{(a^{2\xi} - b^{2\xi})A^*}\left[\frac{(a^\xi b^3 - a^3 b^\xi)C^* \rho \omega^2}{(\xi^2 - 9)} + a^\xi b P_{ex}\right] \qquad (3.109.\text{a})$$

$$C_2 = -\frac{1}{(a^{2\xi} - b^{2\xi})B^*}\left[\frac{(a^{\xi+3} - b^{\xi+3})C^* \rho \omega^2}{(\xi^2 - 9)} - b^{\xi+1} P_{ex}\right] \qquad (3.109.\text{b})$$

Analytical representation of the radial displacement is

$$u(r) = \frac{1}{(a^{2\xi} - b^{2\xi})}\left[\frac{\rho \omega^2 C^*}{(\xi^2 - 9)}\left[\frac{(a^{2\xi}b^{\xi+3} - a^{\xi+3}b^{2\xi})r^{-\xi}}{A^*} - \frac{(a^{\xi+3} - b^{\xi+3})r^\xi}{B^*}\right.\right.$$
$$\left.\left. + \frac{(a^{2\xi} - b^{2\xi})}{D^*}r^3\right] + a^{2\xi}b^{1+\xi}P_{ex}\left[\frac{r^{-\xi}}{A^*} + \frac{r^\xi}{b^{1+\xi}B^*}\right]\right] \qquad (3.110)$$

Directional stresses:

$$\sigma_r(r) = \frac{-1}{(a^{2\xi} - b^{2\xi})}\left[\frac{\rho \omega^2 C^*}{(\xi^2 - 9)}[(a^{2\xi}b^{\xi+3} - a^{\xi+3}b^{2\xi})r^{-\xi-1}\right.$$
$$\left. + (a^{\xi+3} - b^{\xi+3})r^{\xi-1} - (a^{2\xi} - b^{2\xi})r^2] - P_{ex}b^{1+\xi}[1 - \frac{a^{2\xi}}{r^{2\xi}}]\right] \qquad (3.111)$$

$$\sigma_\theta(r) = \frac{-\xi}{(a^{2\xi} - b^{2\xi})}\left[\frac{\rho \omega^2 C^*}{\xi(\xi^2 - 9)}[E^*(a^{\xi+3}b^{2\xi} - a^{2\xi}b^{\xi+3})r^{-\xi-1}\right.$$
$$\left. + F^*(b^{\xi+3} - a^{\xi+3})r^{\xi-1} + G^*(a^{2\xi} - b^{2\xi})r^2] - P_{ex}b^{1+\xi}[1 + \frac{a^{2\xi}}{r^{2\xi}}]\right] \qquad (3.112)$$

$$\sigma_z(r) = \frac{-1}{(a^{2\xi} - b^{2\xi})} \Big[\frac{\rho \omega^2 C^* R_2}{(\xi^2 - 9) R_1} [(R_1 \nu_{rz} + E^* \nu_{\theta z})(a^{2\xi} b^{\xi+3} - a^{\xi+3} b^{2\xi}) r^{-\xi-1}$$
$$+ (R_1 \nu_{rz} + F^* \nu_{\theta z})(a^{\xi+3} - b^{\xi+3}) r^{\xi-1} - (R_1 \nu_{rz} + G^* \nu_{\theta z})(a^{2\xi} - b^{2\xi}) r^2]$$
$$- R_2 P_{ex} b^{1+\xi} [\nu_{rz}(1 - \frac{a^{2\xi}}{r^{2\xi}}) + \frac{\xi \nu_{\theta z}}{R_1}(1 + \frac{a^{2\xi}}{r^{2\xi}})]] \Big]$$
$$(3.113)$$

As can be seen, combined cases are treated and represented analytically. For every particular case, the equations we obtain can be provided with application of superposition as well.

3.6   Transformation from Orthotropy to Isotropy

In the previous sections, analytical models were developed to find displacement, strain and stress functions for the orthotropic and isotropic cylinders. In this section, transformation from orthotropy to isotropy is analyzed by using Maxwell relations. Beginning from the displacement function, strains and stresses are going to be converted. Consider the general displacement function for the orthotropic cylinders.

$$u(r) = C_1 r^{-\sqrt{\frac{s_{22}}{s_{11}}}} + C_2 r^{\sqrt{\frac{s_{22}}{s_{11}}}} + \frac{\rho \omega^2 r^3}{s_{22} - 9 s_{11}} \qquad (3.114)$$

The variable ξ converts orthotropic bodies to isotropic ones, if the following procedure is applied.

$$\xi = \sqrt{\frac{s_{22}}{s_{11}}} = \sqrt{\frac{R_1^2 (R_2 \upsilon_{rz}^2 - 1)}{R_2 \upsilon_{\theta z}^2 - R_1}} \qquad (3.115)$$

Unlike the ortotropic materials, isotropic material properties do not alter through polar directions. Thus, below equivalences can be written for the isotropic materials.

$$\nu_{r\theta} = \nu_{rz} = \nu_{\theta z} = \nu \qquad (3.116)$$

$$E_r = E_\theta = E_z = E \qquad (3.117)$$

In order to convert the material from orthotropic to isotropic, R_1 and R_2 should be taken as the following. When the corresponding elastic moduli are taken

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equally, below equivalence is obtained identically.

$$R_1 = R_2 = 1 \tag{3.118}$$

Under these constrains, ξ is equal to one. If $\xi = 1$ then $s_{22} = s_{11}$. So that, Eq(3.114) takes the below form which is the general displacement function of the isotropic structures.

$$u(r) = C_1 r^{-1} + C_2 r - \frac{\rho \omega^2 r^3}{8 s_{11}} \tag{3.119}$$

Once the orthotropic displacement function is converted to isotropic one, isotropic directional strains and stresses can be derived. These derivations have shown in Section 3.1 with details.

Transformation from orthotropy to isotropy is convenient through the structures. This convenience continues when the boundary conditions are applied too. For instance, consider C_1 and C_2 of the internally pressurized orthotropic cylinder given in Eq(3.68.a) and Eq(3.68.b). In order to simplify these constants, we have introduced A^* and B^*. If Poisson ratios and elastic moduli are taken equally, values of A^* and B^* are simplified to below form.

$$A^* = \frac{E}{1 + \nu} \tag{3.120}$$

3.7 Elastic Limits of the Cylinders

Elastic limit is the maximum value that does not cause plastic flow. In order to have an elastic solution, elastic limits should be found. For this purpose, Hill's yield criteria with the rotational symmetry is used. The below equation is the criteria in principal directions.

$$(\sigma_\theta - \sigma_z)^2 + (\sigma_z - \sigma_r)^2 + R_h(\sigma_r - \sigma_\theta)^2 = (R_h + 1)(\sigma_r^y)^2 \tag{3.122}$$

Yielding criteria can be put into non dimensional form, if it is written as follows. Non dimensional form can be denoted with σ_h.

$$\sigma_h(r) = \sqrt{\frac{(\sigma_\theta - \sigma_z)^2 + (\sigma_z - \sigma_r)^2 + R_h(\sigma_r - \sigma_\theta)^2}{(R_h + 1)(\sigma_r^y)^2}} = 1 \tag{3.123}$$

In the above criteria, σ_r^y is the radial yield limit and R_h is an orthotropy parameter to be determined. This criteria is compatible with general orthotropic materials which have rotational symmetry about z direction. On the other hand, it can be applied to isotropic materials as well. Material properties change in different directions for orthotropy. Therefore, yield limits, σ_r^y, σ_θ^y and σ_z^y, differ in polar directions. These yield limits are equal to each other for isotropic materials. Thus, below equivalence can be written as

$$\sigma_r^y = \sigma_\theta^y = \sigma_z^y = \sigma^y \tag{3.124}$$

Additionally, for isotropic materials

$$R_h = 1 \tag{3.125}$$

Hence, non dimensional yielding criteria takes the below form for isotropic materials.

$$\sigma_h(r) = \sqrt{\frac{(\sigma_\theta - \sigma_z)^2 + (\sigma_z - \sigma_r)^2 + (\sigma_r - \sigma_\theta)^2}{2(\sigma^y)^2}} = 1 \tag{3.126}$$

As can be seen, above equation is the non dimensional form of Von Misses yield criteria. It is necessary to state that regardless of what the boundary condition is, yielding occurs at the inner surface of the cylinder because maximum principal stresses take place at $r = a$. Thus, elastic limit pressure and elastic limit annular speed are calculated at $\sigma_h(a)$.

$$\sigma_h(a) = 1 \tag{3.127}$$

Above expression should be satisfied at $r = a$. If $\sigma_h(a) > 1$ then plastic deformation transpires.

CHAPTER 4

NUMERICAL RESULTS AND DISCUSSIONS

4.1 Dimensionless Forms of Quantities

Previously, analytical derivations are held and shown in details. In the present chapter, these analytical derivations are supported with numerical examples. The numerical examples of radial displacement, directional stresses and elastic limits are shown graphically. These graphics are put into dimensionless forms for convenience. For this purpose, the following form of the displacement is used $\overline{u} = Eu/b\sigma_r^y$ in diagrams. Similarly radial, tangential and axial stresses are transformed to their dimensionless forms by applying the following process $\overline{\sigma}_r = \sigma_r/\sigma_r^y$, $\overline{\sigma}_\theta = \sigma_\theta/\sigma_r^y$ and $\overline{\sigma}_z = \sigma_z/\sigma_r^y$. The axial force F_z is normalized by $\overline{F}_z = F_z/\sigma_r^y b^2$. Additionally, radial coordinate of the structure is converted to its nondimensional form by $\overline{r} = r/b$.

Followingly, pressure values are considered and transformed to their dimensionless forms. Normalized internal pressure can be calculated with the following $\overline{P}_{in} = P_{in}/\sigma_r^y$. Likely, external pressure is $\overline{P}_{ex} = P_{ex}/\sigma_r^y$. Annular speed is slightly different than other ones. It is converted to non dimensional form by $\overline{\omega} = \omega b\sqrt{\rho/\sigma_r^y}$. Finally, arbitrary constants are written as $\overline{C}_1 = C_1/b^2$ and $\overline{C}_2 = C_2$.

As it is seen, all of the non dimensional forms are presented with over lines for convenience. These dimensionless forms are used in all diagrams and tables.

4.2 Results for Theoretical Materials

To exemplify numerical examples, A 36 standard structural steel is used for the isotropic material. Later, its mechanical properties are changed theoretically to illustrate orthotropic material behaviors. These modified materials are named Modified A36-1 and Modified A36-2 for convinience.

Table4.1: Material and structural properties

	Standard A36	Modified A36-1	Modified A36-2
$E_r\ (GPa)$	200	200	200
$\upsilon_{r\theta}$	0.26	0.26	0.26
υ_{rz}	0.26	0.26	0.26
$\upsilon_{\theta z}$	0.26	0.26	0.26
R_1	1	1.5	2
R_2	1	1.5	2
R_h	1	1.5	2
$\rho\ (kg/m^3)$	7.8	7.8	7.8
$\sigma^y\ (MPa)$	250	250	250
$a(m)$	0.05	0.05	0.05
$b(m)$	0.1	0.1	0.1

In the following tables, data that we acquire from the application of boundary conditions are exhibited. The table given below shows the dimensionless elastic limit pressure, force and arbitrary constants for the cylinders subjected to internal pressure.

Table4.2: Calculated values for the internally pressurized cylinder

	$Standard\ A36$	$Modified\ A36-1$	$Modified\ A36-2$
$\overline{P}_{in}$	0.431977	0.40219	0.380651
$\overline{F}_z$	0.176423	0.130051	0.091769
$\overline{C}_1$	$2.26788\ 10^{-4}$	$9.08505\ 10^{-5}$	$4.56171\ 10^{-5}$
$\overline{C}_2$	$1.08858\ 10^{-4}$	$9.00799\ 10^{-5}$	$7.71453\ 10^{-5}$

Similarly, these crucial values are presented for the externally pressurized structures.

Table4.3: Calculated values for the externally pressurized cylinder

	$Standard\ A36$	$Modified\ A36-1$	$Modified\ A36-2$
$\overline{P}_{ex}$	0.417285	0.421938	0.428844
$\overline{F}_z$	-0.681689	-0.780586	-0.883645
$\overline{C}_1$	$-2.19074\ 10^{-4}$	$-8.28387\ 10^{-5}$	$-3.99882\ 10^{-5}$
$\overline{C}_2$	$-4.20623\ 10^{-4}$	$-4.34926\ 10^{-5}$	$-4.46796\ 10^{-4}$

For the case cylinder under both internal and external pressure, important values are given at Table 4.4. These numerical results are calculated under a condition in which the external pressure is two times greater than the internal pressure. In other words, the below values are obtained when $P_{ex} = 2P_{in}$.

Table4.4: Calculated values for both internally and externally pressurized cylinder

	$Standard\ A36$	$Modified\ A36-1$	$Modified\ A36-2$
$\overline{P}_{in}$	0.389611	0.436176	0.487796
$\overline{P}_{ex}$	0.779223	0.872352	0.975592
$\overline{F}_z$	-1.11384	-1.47281	-1.89264
$\overline{C}_1$	$-2.04546\ 10^{-4}$	$-7.27407\ 10^{-5}$	$-3.25133\ 10^{-5}$
$\overline{C}_2$	$-6.87275\ 10^{-4}$	$-8.01511\ 10^{-5}$	$-9.17573\ 10^{-4}$

Likely, the same procedure is implemented to the cylinders under annular rotation.

Table4.5: Calculated values for annularly rotating cylinder

	$Standard\ A36$	$Modified\ A36-1$	$Modified\ A36-2$
$\overline{\omega}$	1.125537	1.10083	1.08120
$\overline{F}_z$	0.485048	0.490126	0.49959
$\overline{C}_1$	$2.08963\ 10^{-4}$	$7.9129\ 10^{-5}$	$3.8339\ 10^{-5}$
$\overline{C}_2$	$5.01512\ 10^{-4}$	$5.5152\ 10^{-4}$	$6.0047\ 10^{-4}$

For the combined cases, following data are established when the boundary conditions are applied. Dimensionless values are tabulated below for the cylinder

subjected to both annular rotation and internal pressure. These values are calculated when $\overline{P}_{in} = 0.2$.

Table4.6: Calculated values for both annularly rotating and internally pressurized cylinder

	Standard A36	Modified A36 − 1	Modified A36 − 2
$\overline{P}_{in}$	0.2	0.2	0.2
$\overline{\omega}$	0.828675	0.781984	0.745278
$\overline{F}_z$	0.344607	0.311993	0.285596
$\overline{C}_1$	$2.18271 \ 10^{-4}$	$8.5107 \ 10^{-5}$	$4.2184 \ 10^{-5}$
$\overline{C}_2$	$3.22251 \ 10^{-3}$	$3.2309 \ 10^{-4}$	$3.2584 \ 10^{-4}$

In a similar manner, combination of annular rotation and external pressure gives the below values which are calculated when $\overline{P}_{ex} = 0.2$.

Table4.7: Calculated values for both annularly rotating and externally pressurized cylinder

	Standard A36	Modified A36 − 1	Modified A36 − 2
$\overline{P}_{ex}$	0.2	0.2	0.2
$\overline{\omega}$	1.36895	1.33650	1.30927
$\overline{F}_z$	0.39081	0.35245	0.32048
$\overline{C}_1$	$2.04117 \ 10^{-3}$	$7.7370 \ 10^{-5}$	$3.7569 \ 10^{-5}$
$\overline{C}_2$	$5.40282 \ 10^{-3}$	$6.0678 \ 10^{-4}$	$6.7213 \ 10^{-4}$

The following procedure explains the graphical representation of displacements, stresses and elastic limits. In the diagrams dashed lines represent the isotropic cylinders. In other words, when $R_1 = R_2 = 1$ dashed lines are used. On the other hand, continuous lines describe the orthotropic behavior.

Seven different distributions are presented for each boundary condition. Distribution of radial displacement, radial stress, tangential stress and axial stress are illustrated through radial coordinate. Later, directional stresses are band together in one graphic to compare stresses in polar directions. While comparing directional stresses, distribution of the dimensionless yield criteria is also shown with the denotation σ_h. Ultimately, elastic limit alteration according to orthotropy parameters is illustrated. The effect of R_1 and R_2 are shown separately in plottings.
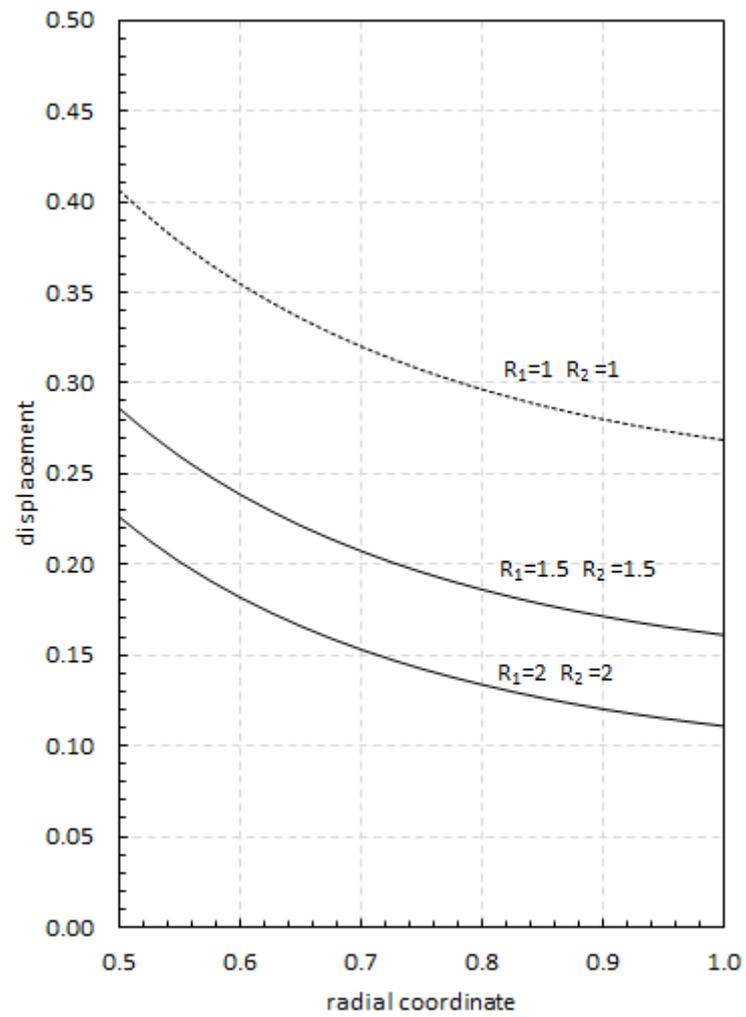
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Figure 4.1: Distribution of non dimensional displacement under internal pressure along radial coordinate
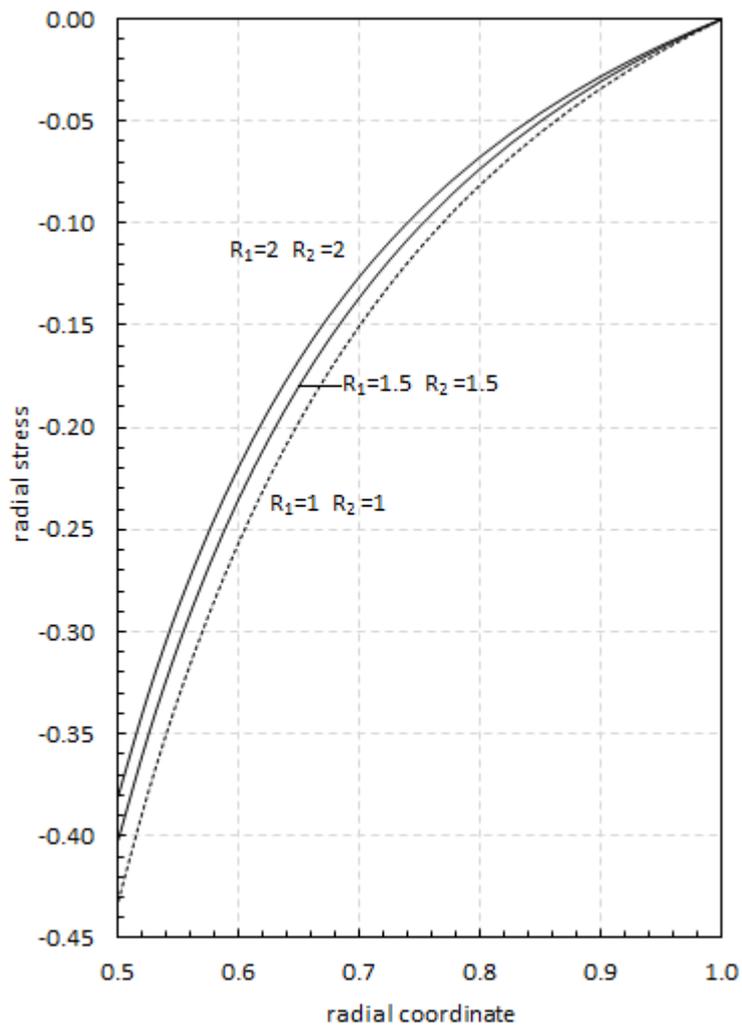
Figure 4.2: Distribution of non dimensional radial stress under internal pressure along radial coordinate
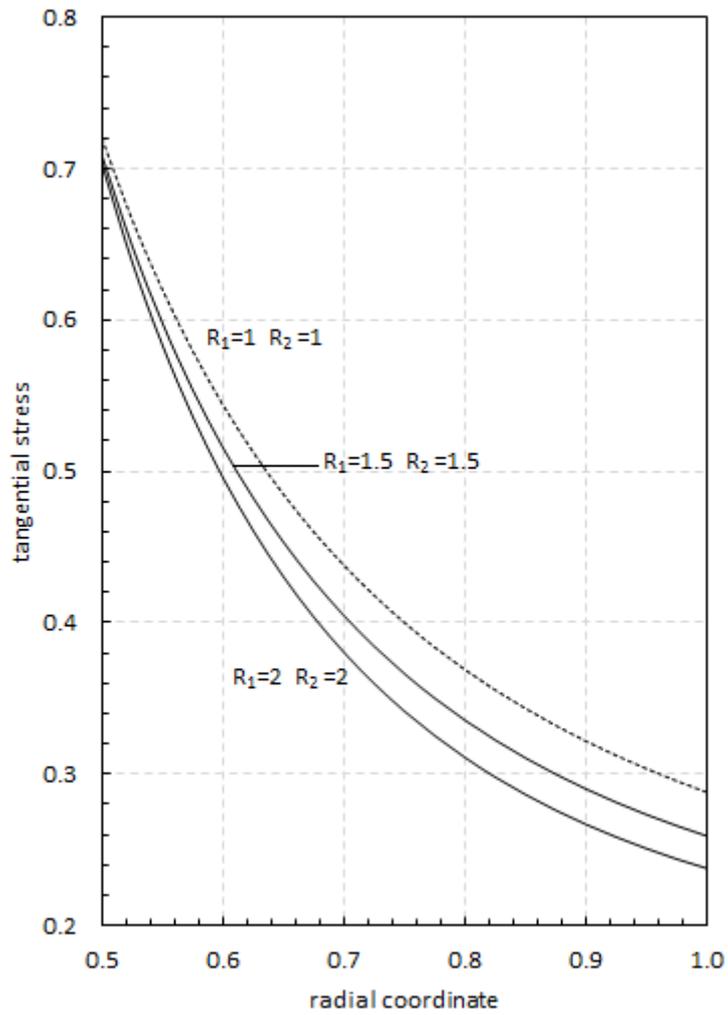
Figure 4.3: Distribution of non dimensional tangential stress under internal pressure along radial coordinate
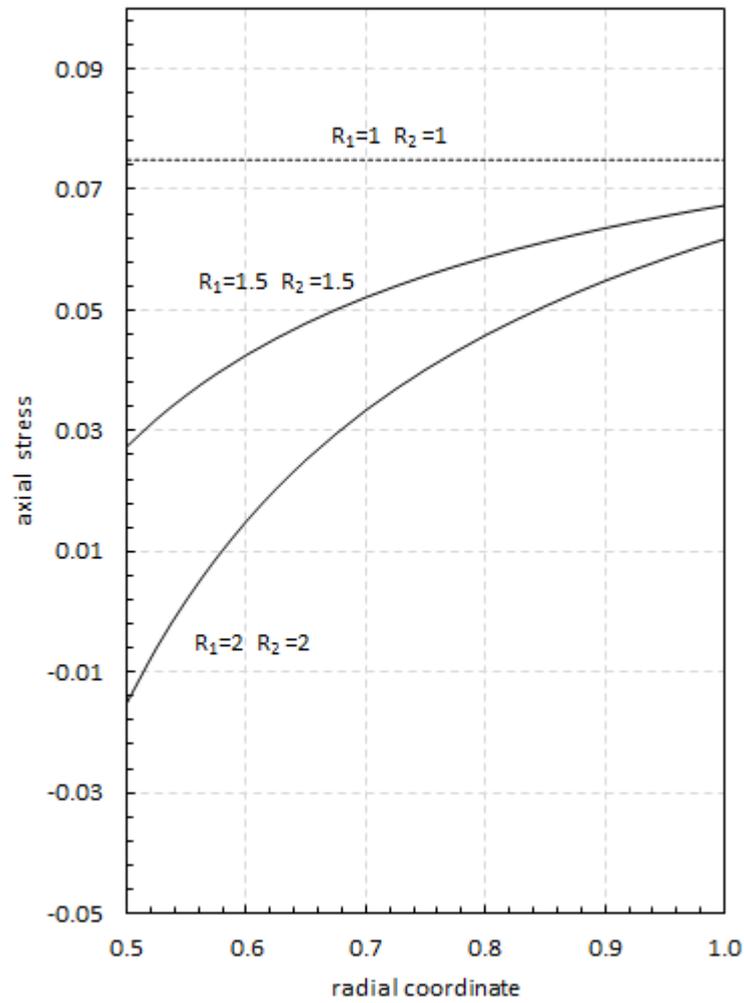
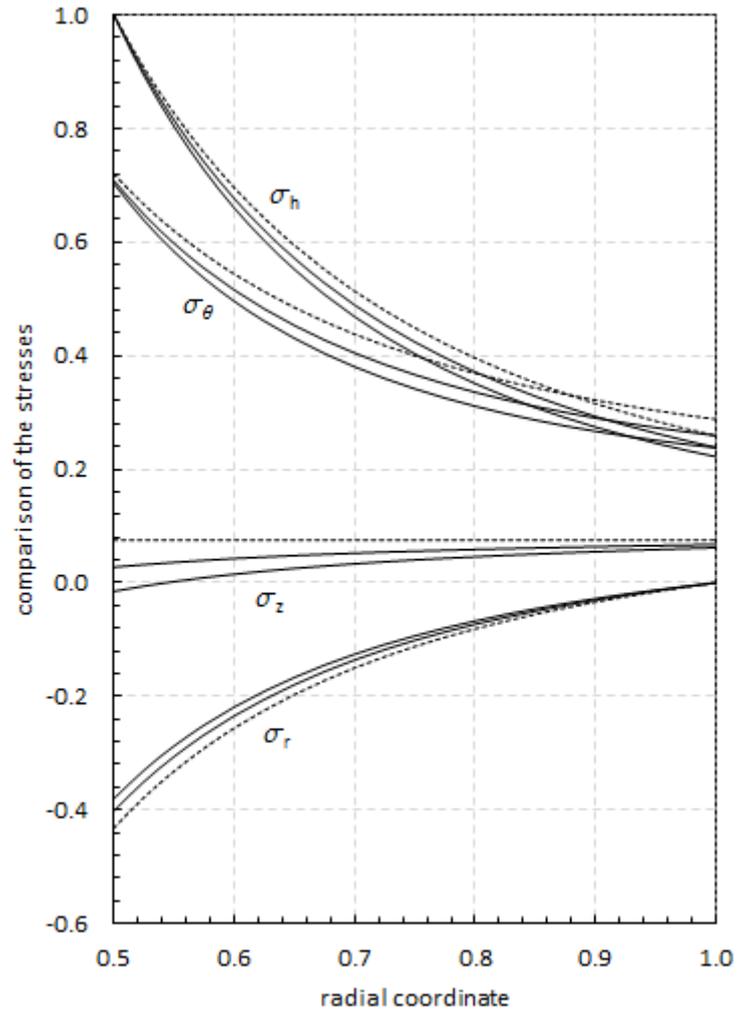Figure 4.4: Distribution of non dimensional axial stress under internal pressure along radial coordinate

Figure 4.5: The states of non dimensional stress under internal pressure along radial coordinate
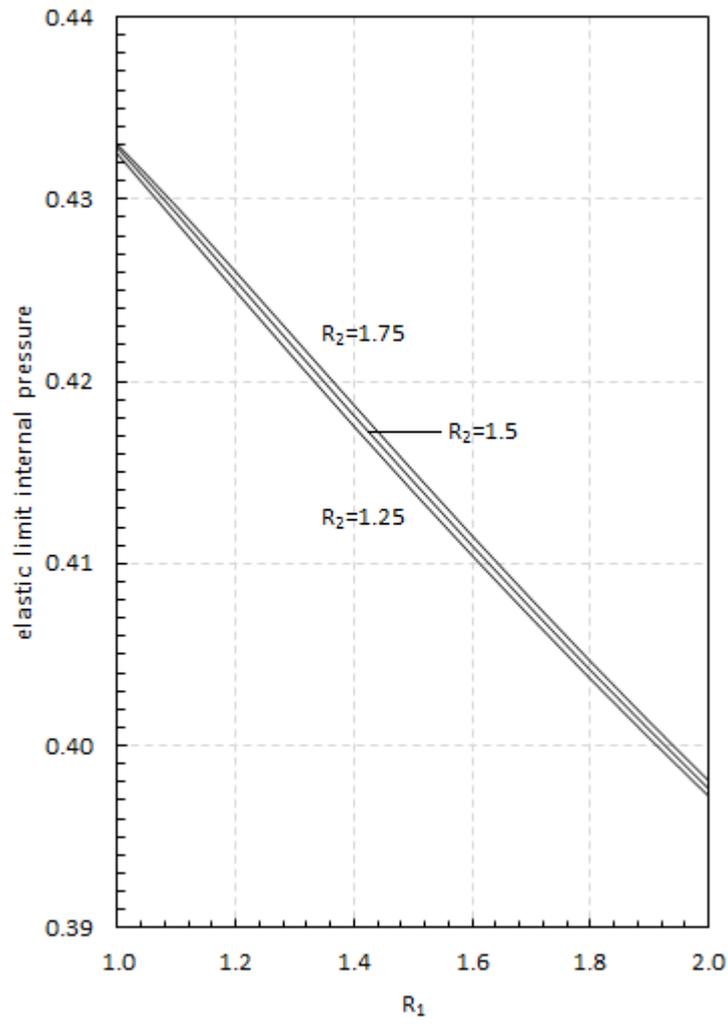
Figure 4.6: Distribution of elastic limit internal pressure in conjunction with orthotropy parameter R_1
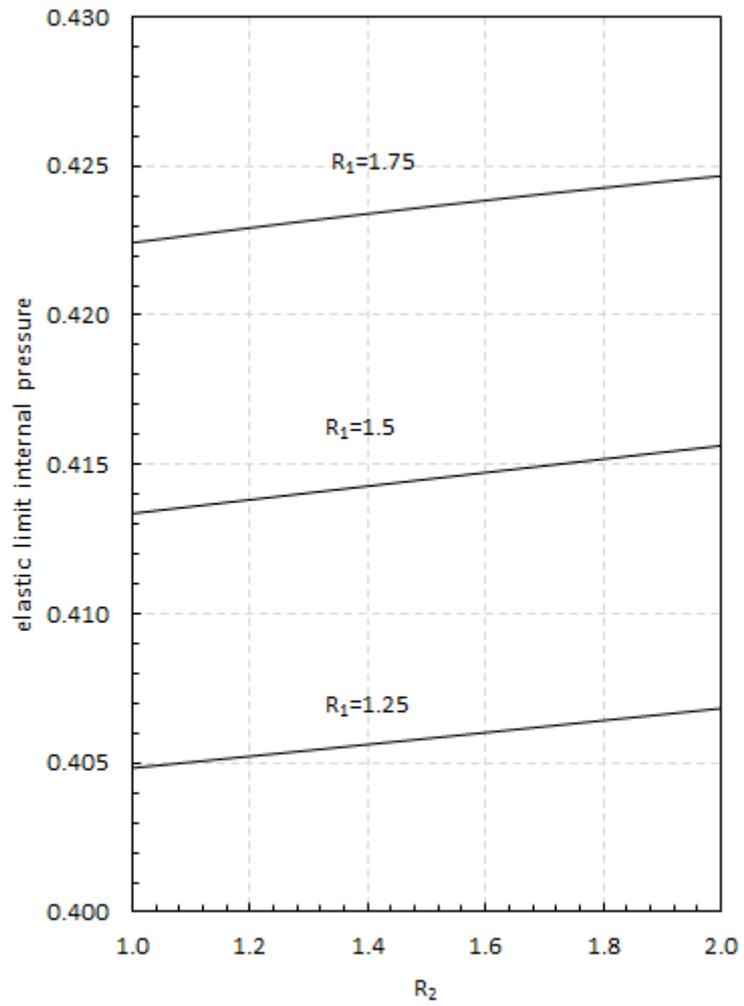
Figure 4.7: Distribution of elastic limit internal pressure in conjunction with orthotropy parameter R_2
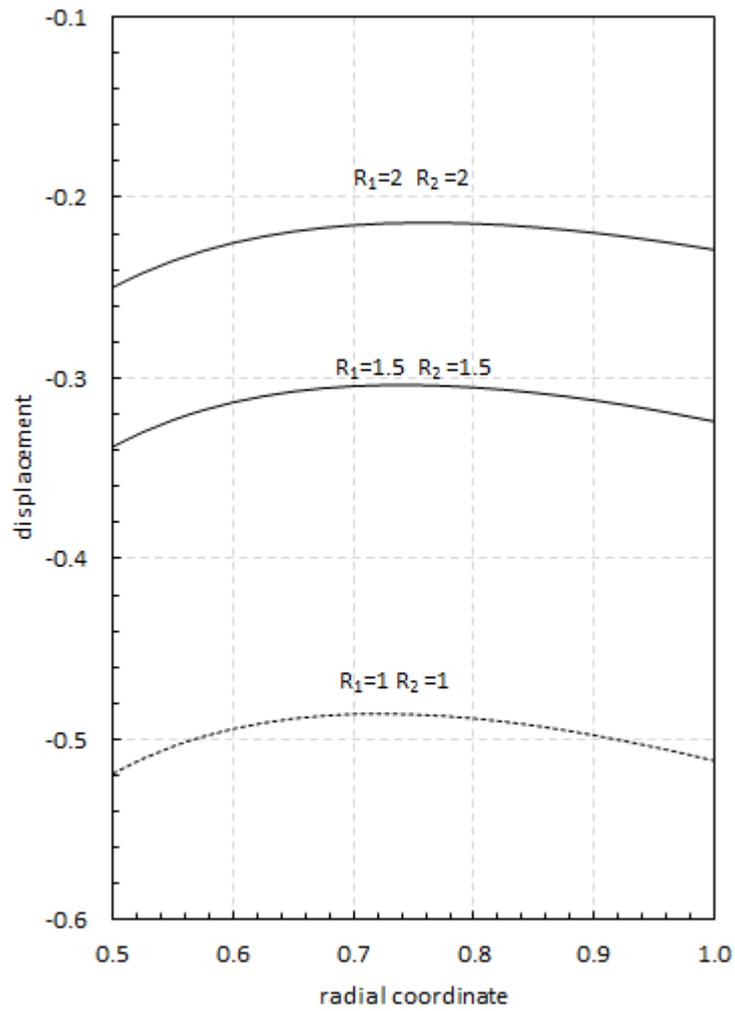
Figure 4.8: Distribution of non dimensional displacement under external pressure along radial coordinate

Figure 4.9: Distribution of non dimensional radial stress under external pressure along radial coordinate

Figure 4.10: Distribution of non dimensional tangential stress under external pressure along radial coordinate

Figure 4.11: Distribution of non dimensional axial stress under external pressure along radial coordinate

Figure 4.12: The states of non dimensional stress under external pressure along radial coordinate

Figure 4.13: Distribution of elastic limit external pressure in conjunction with orthotropy parameter R_1

Figure 4.14: Distribution of elastic limit external pressure in conjunction with orthotropy parameter R_2

Figure 4.15: Distribution of non dimensional displacement under combined pressure along radial coordinate
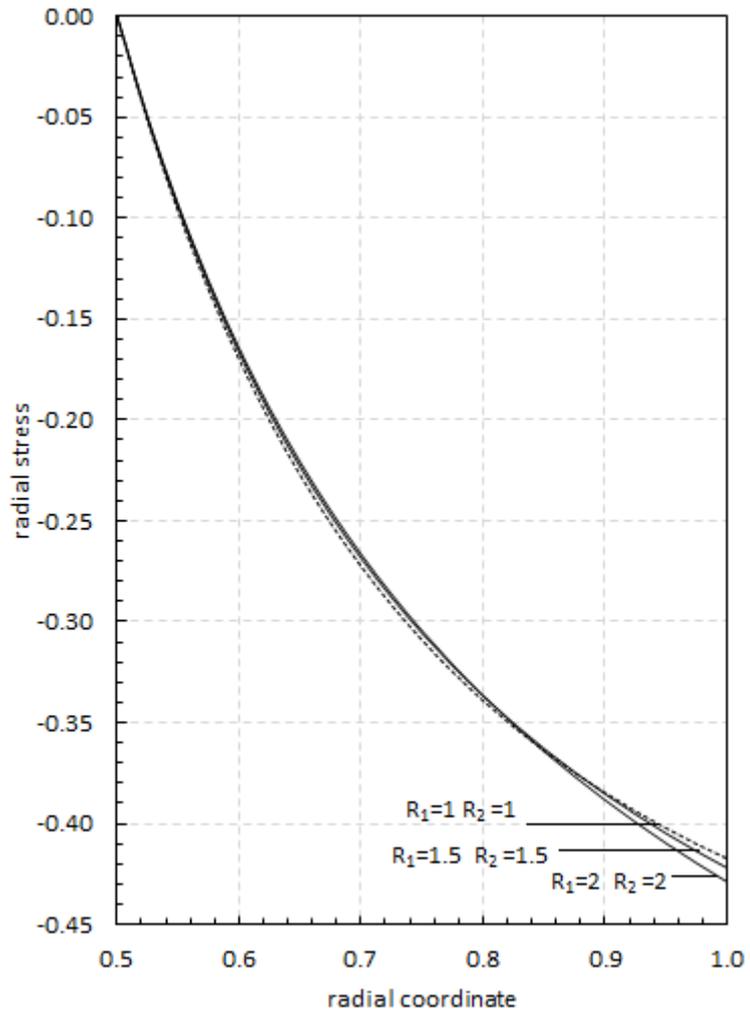
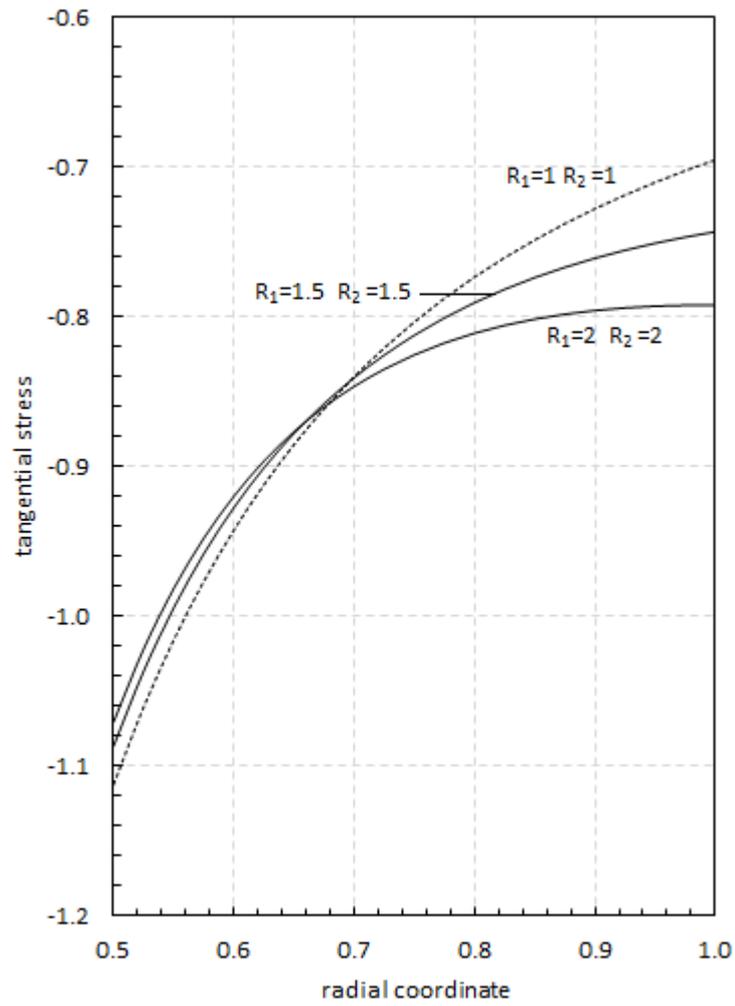Figure 4.16: Distribution of non dimensional radial stress under combined pressure along radial coordinate
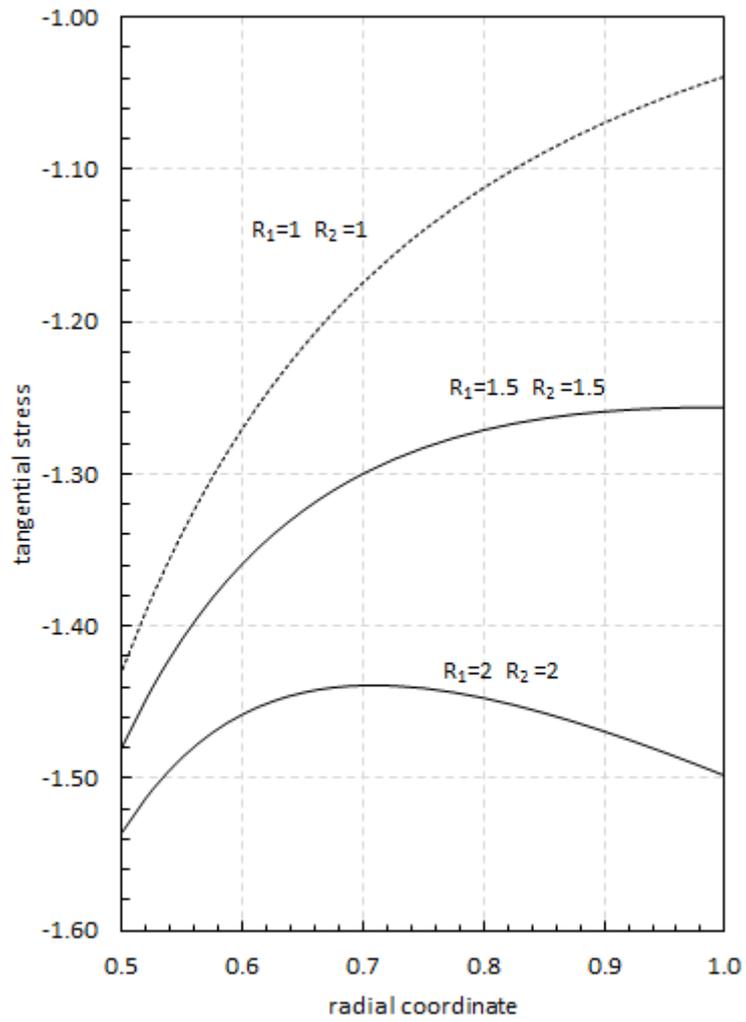
Figure 4.17: Distribution of non dimensional tangential stress under combined pressure along radial coordinate
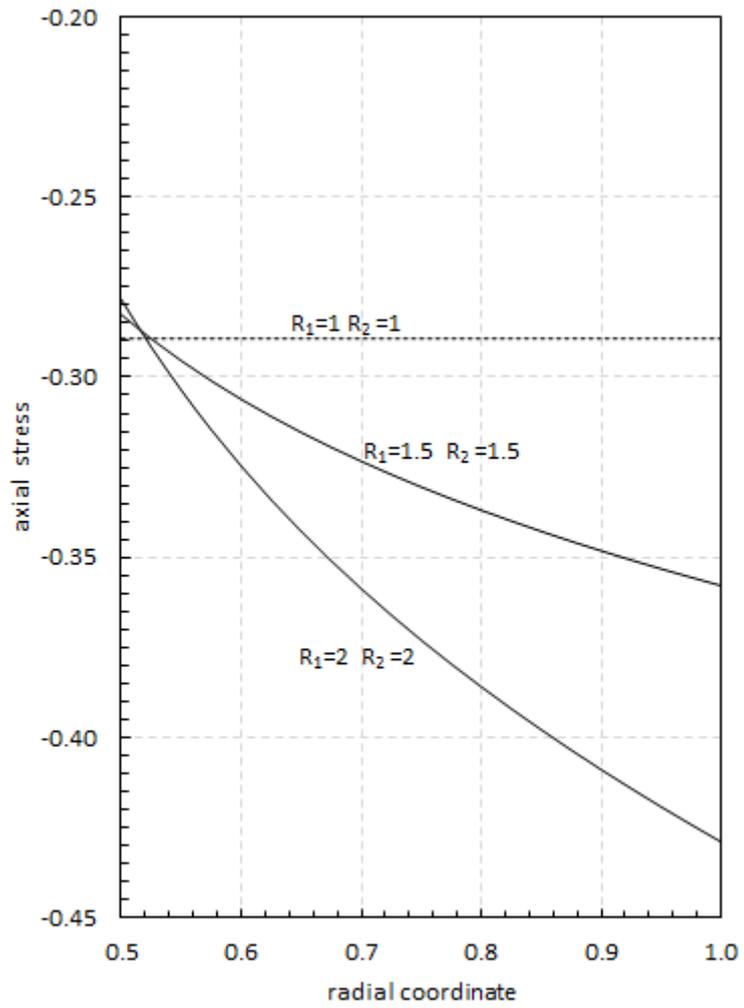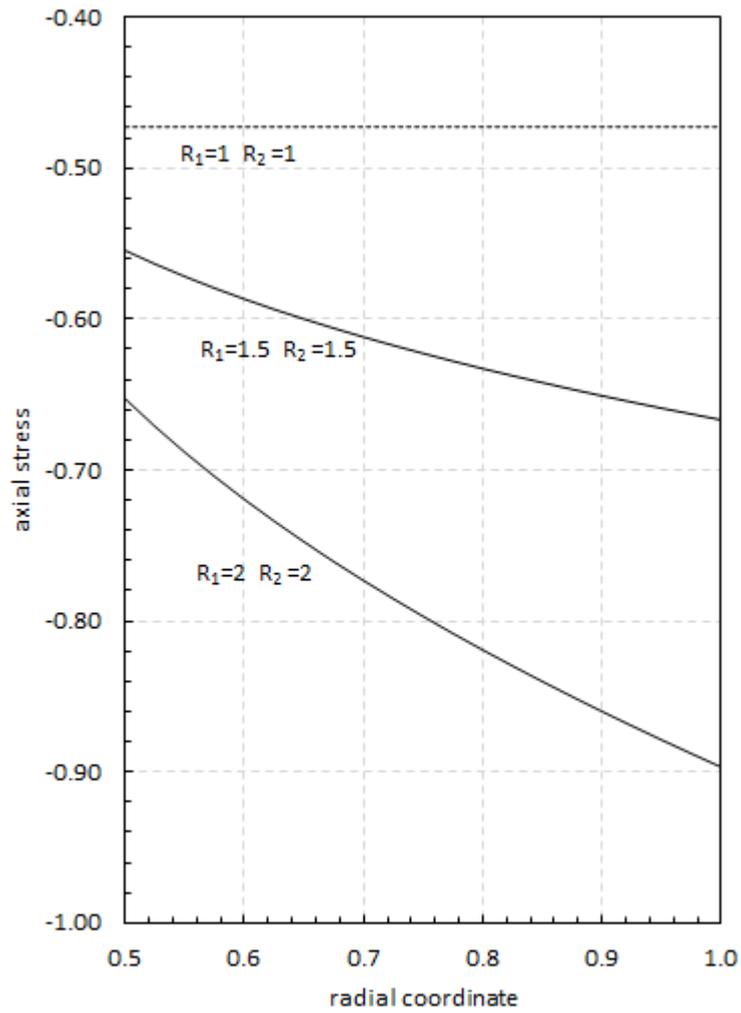
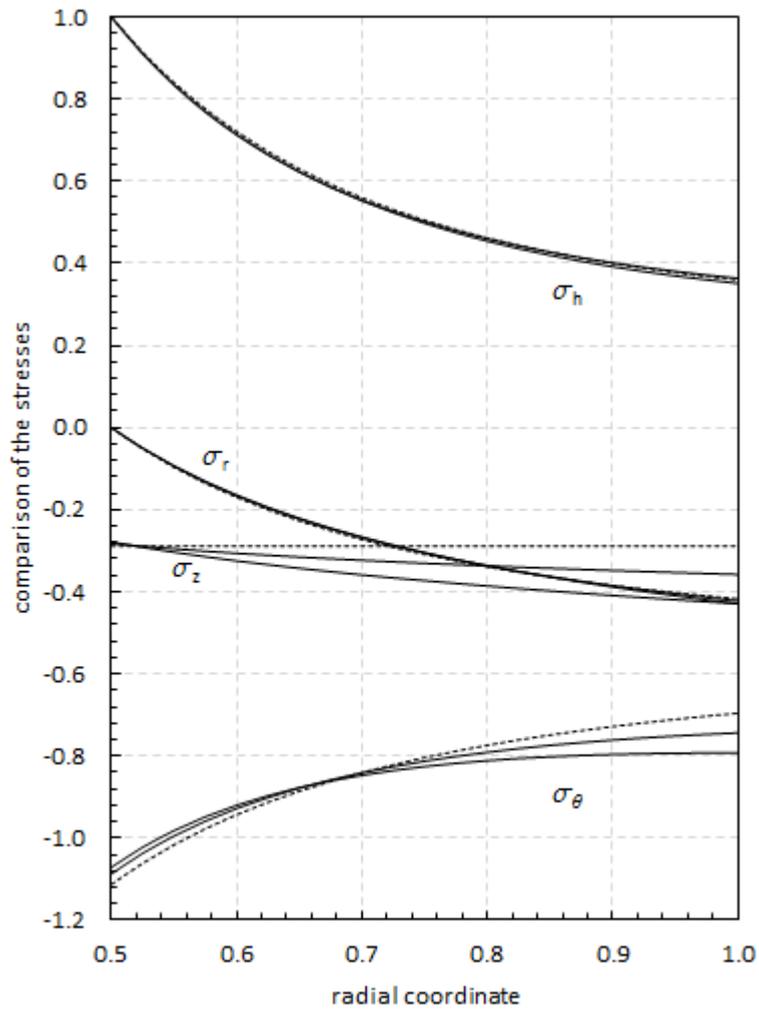Figure 4.18: Distribution of non dimensional axial stress under combined pressure along radial coordinate
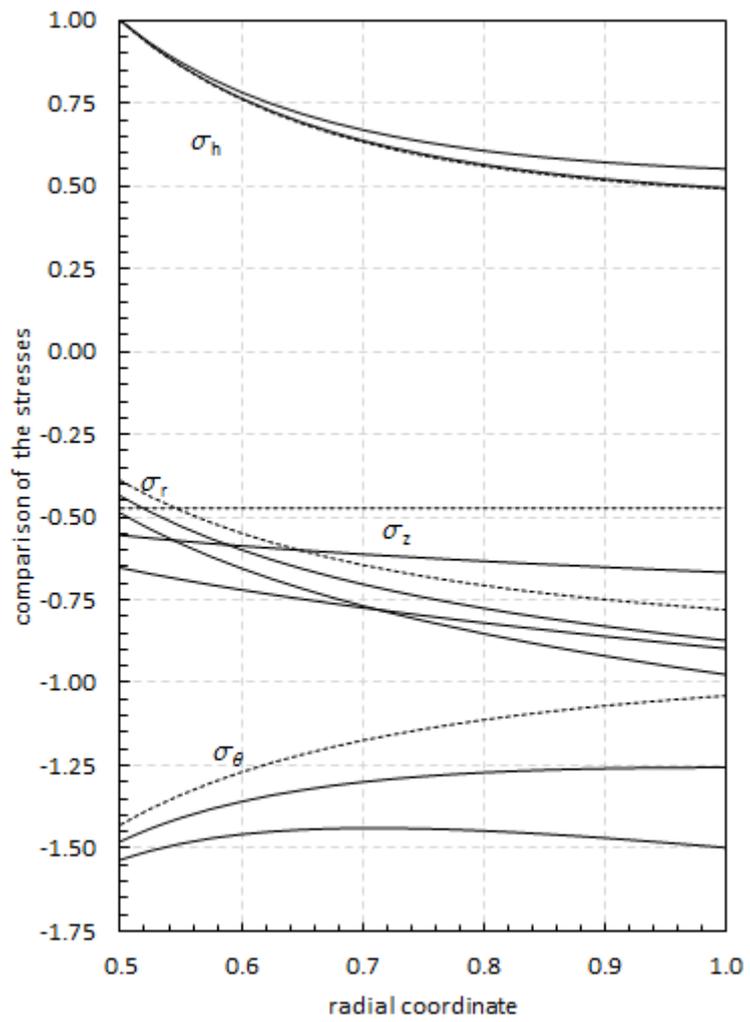
Figure 4.19: The states of non dimensional stress under combined pressure along radial coordinate
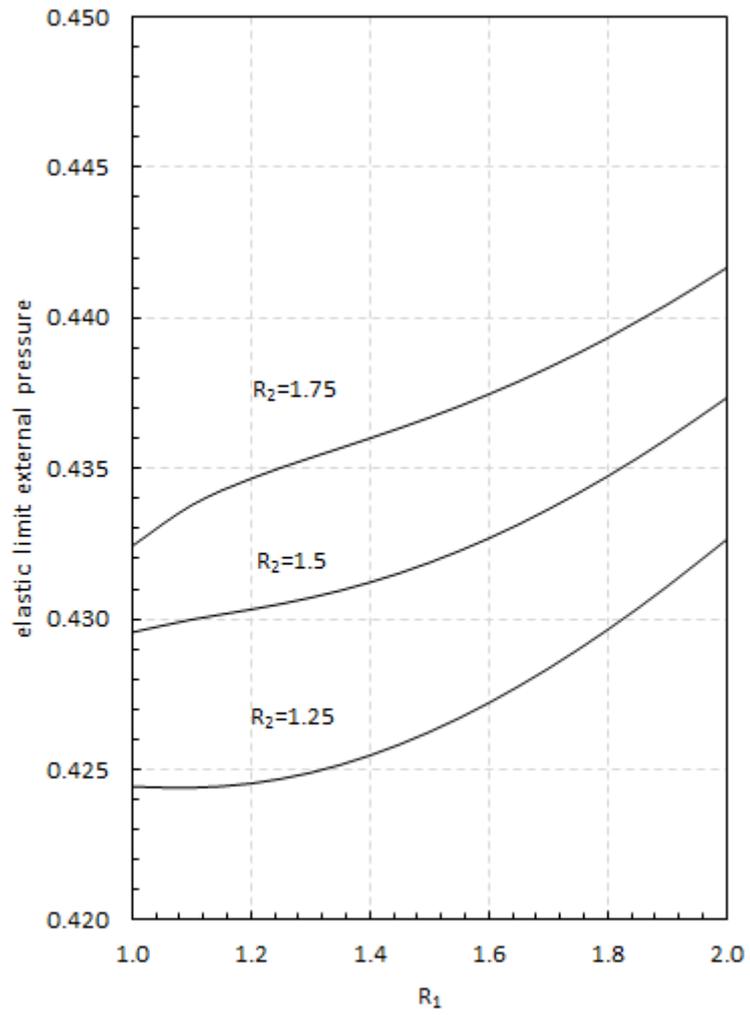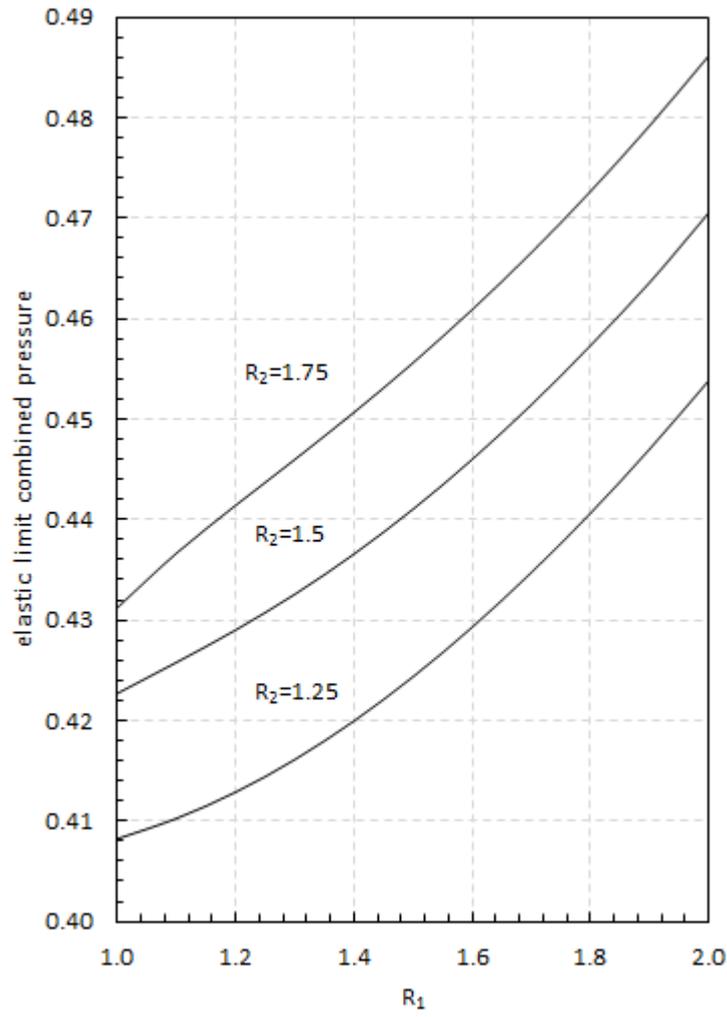
Figure 4.20: Distribution of elastic limit combined pressure in conjunction with orthotropy parameter R_1
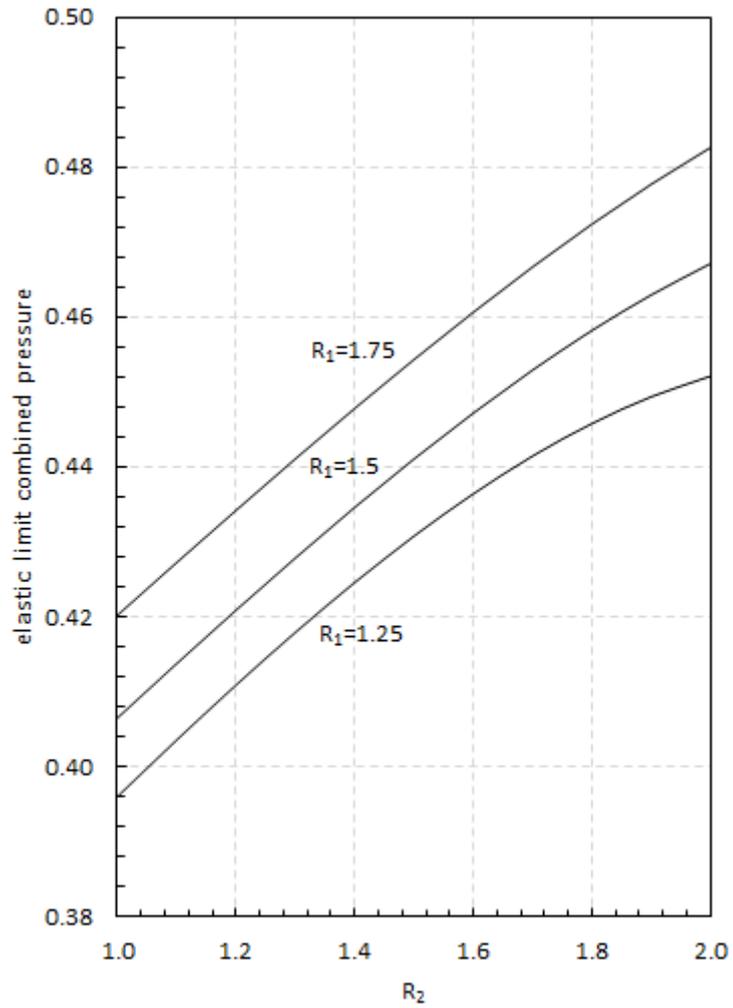
Figure 4.21: Distribution of elastic limit combined pressure in conjunction with orthotropy parameter R_2
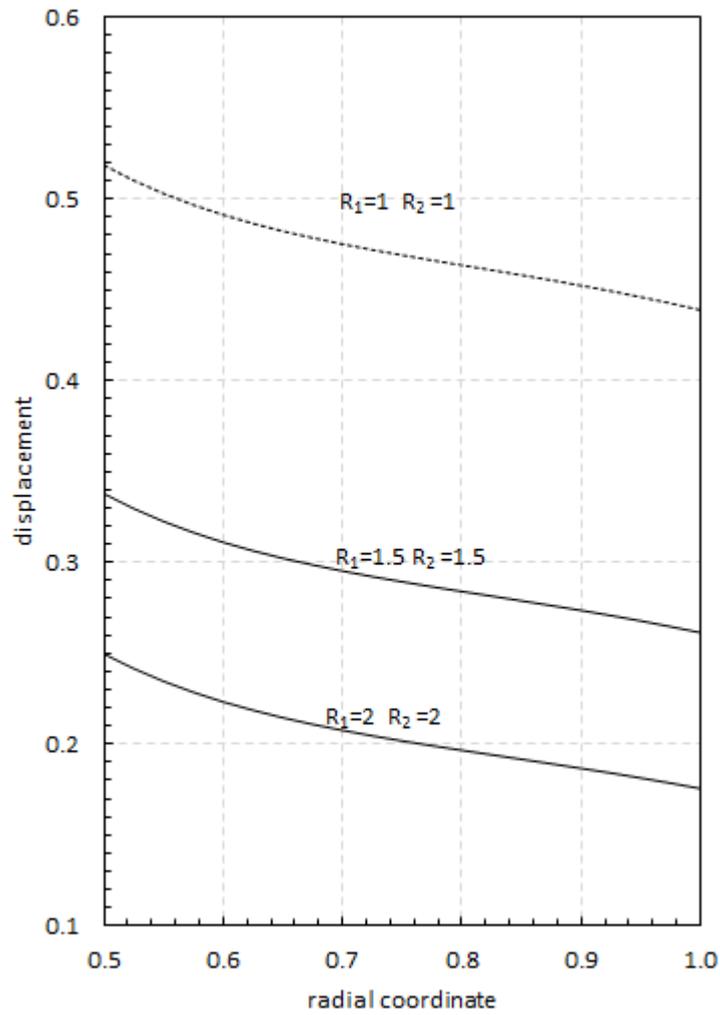
Figure 4.22: Distribution of non dimensional displacement under annular rotation along radial coordinate
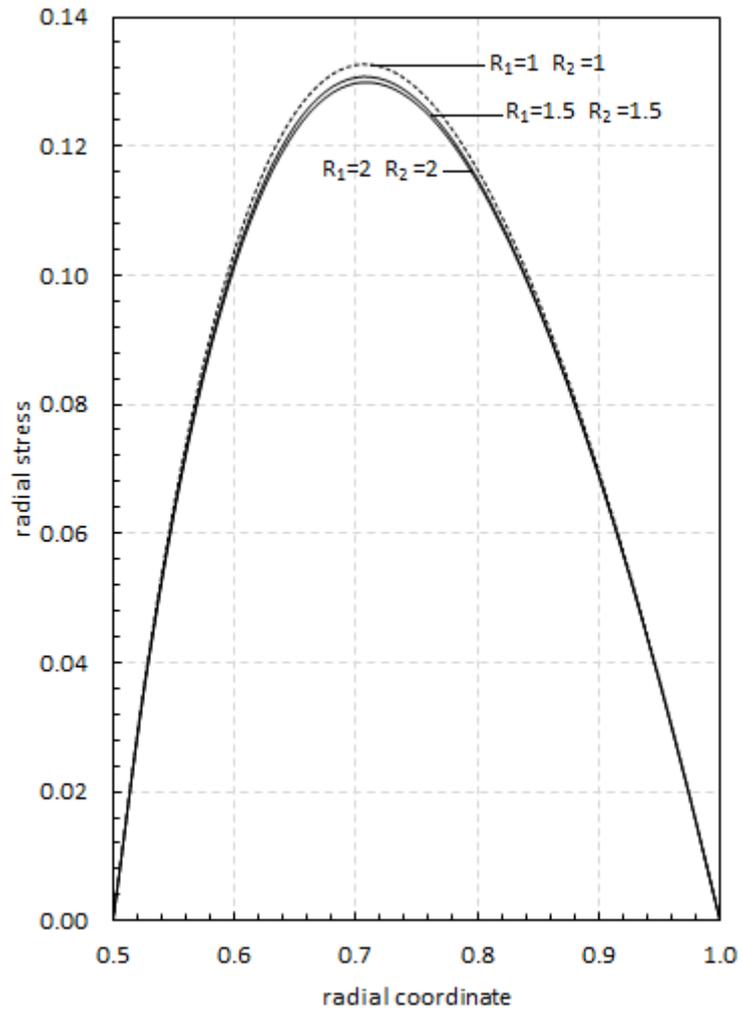
Figure 4.23: Distribution of non dimensional radial stress under annular rotation along radial coordinate
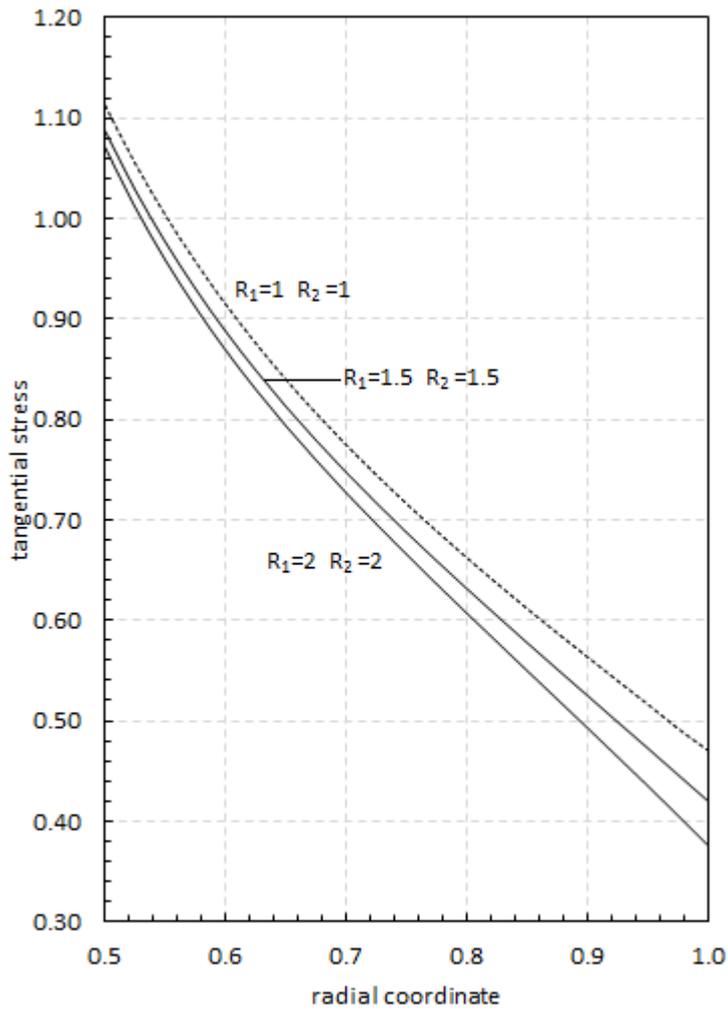
Figure 4.24: Distribution of non dimensional tangential stress under annular rotation along radial coordinate
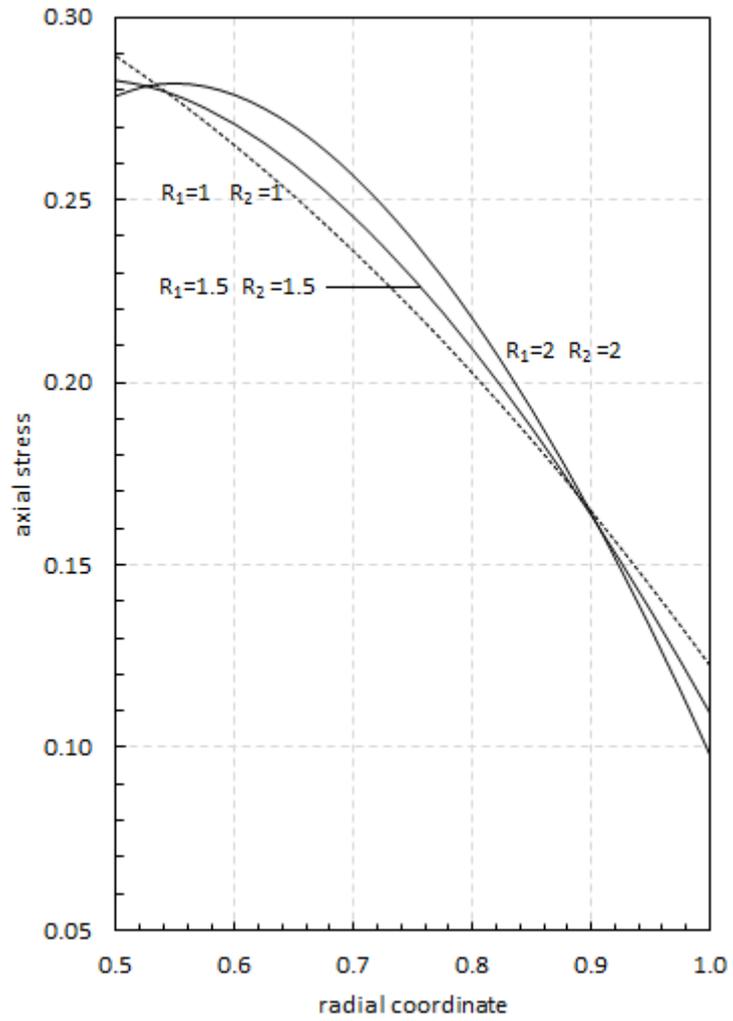
Figure 4.25: Distribution of non dimensional axial stress under annular rotation along radial coordinate
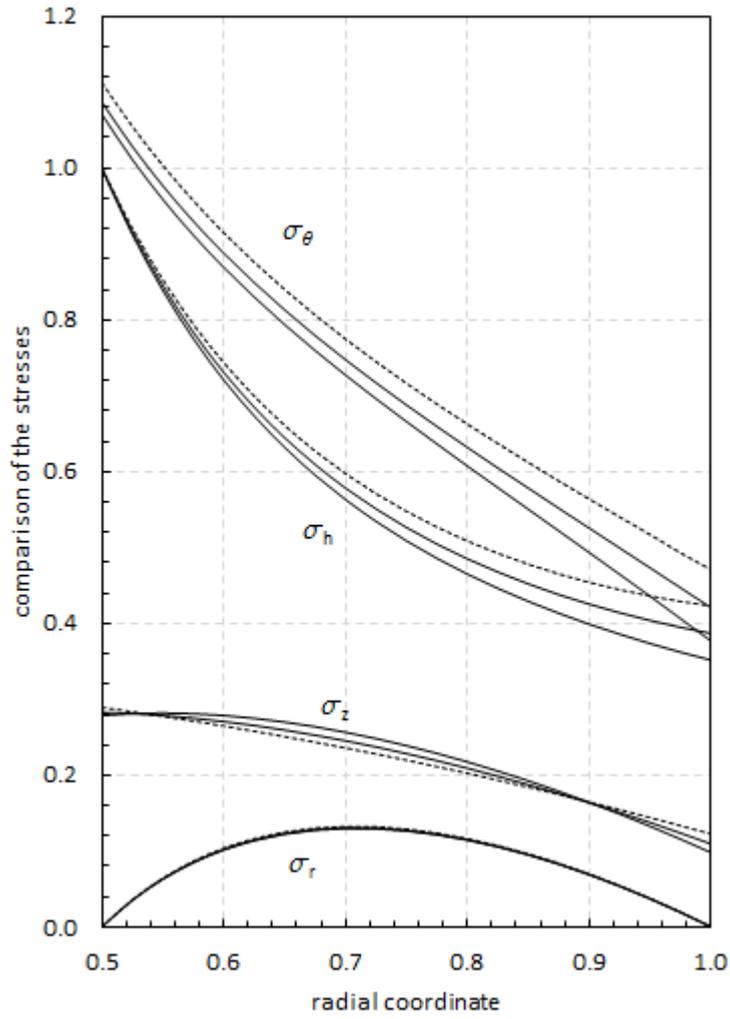
Figure 4.26: The states of non dimensional stress under annular rotation along radial coordinate
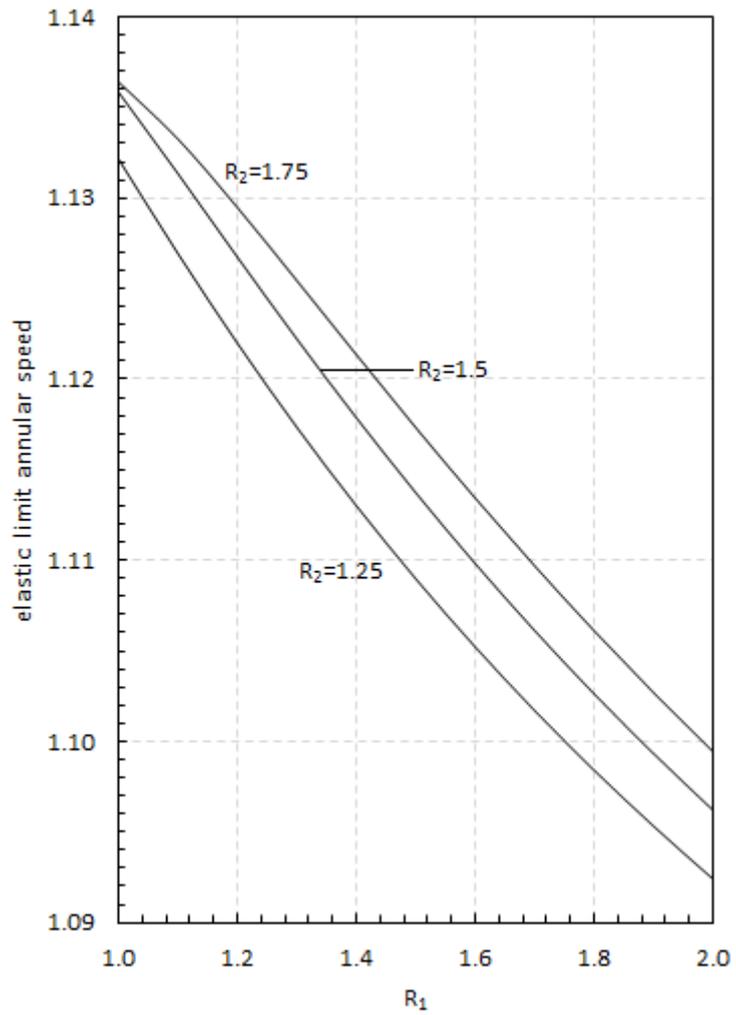
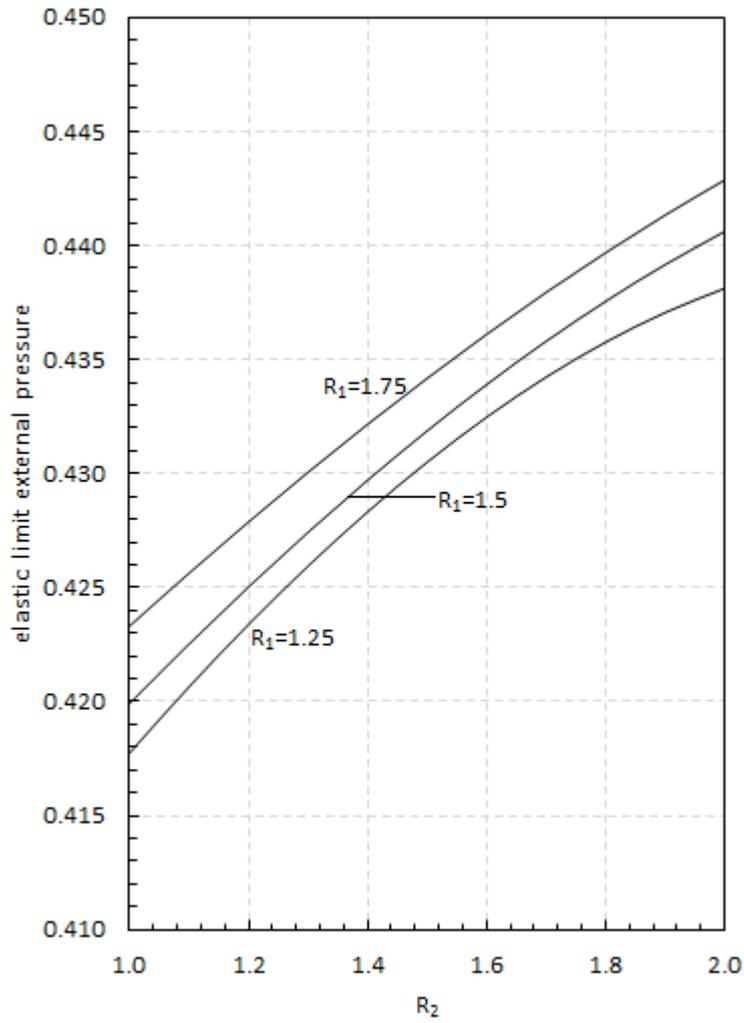Figure 4.27: Distribution of elastic limit annular speed in conjunction with orthotropy parameter R_1
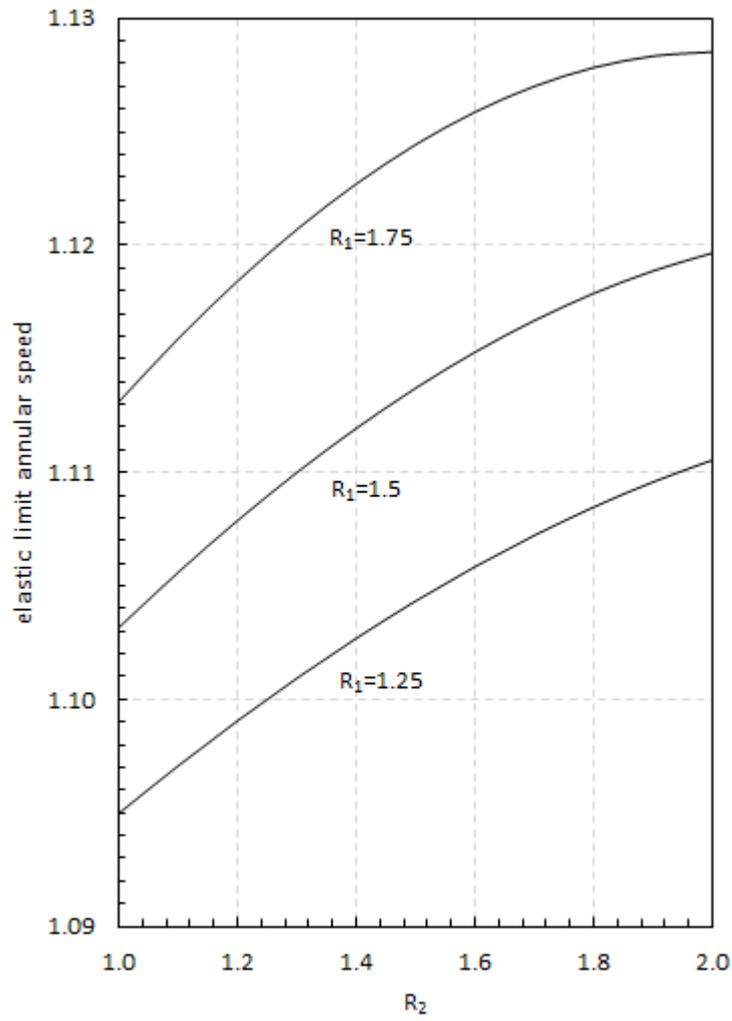
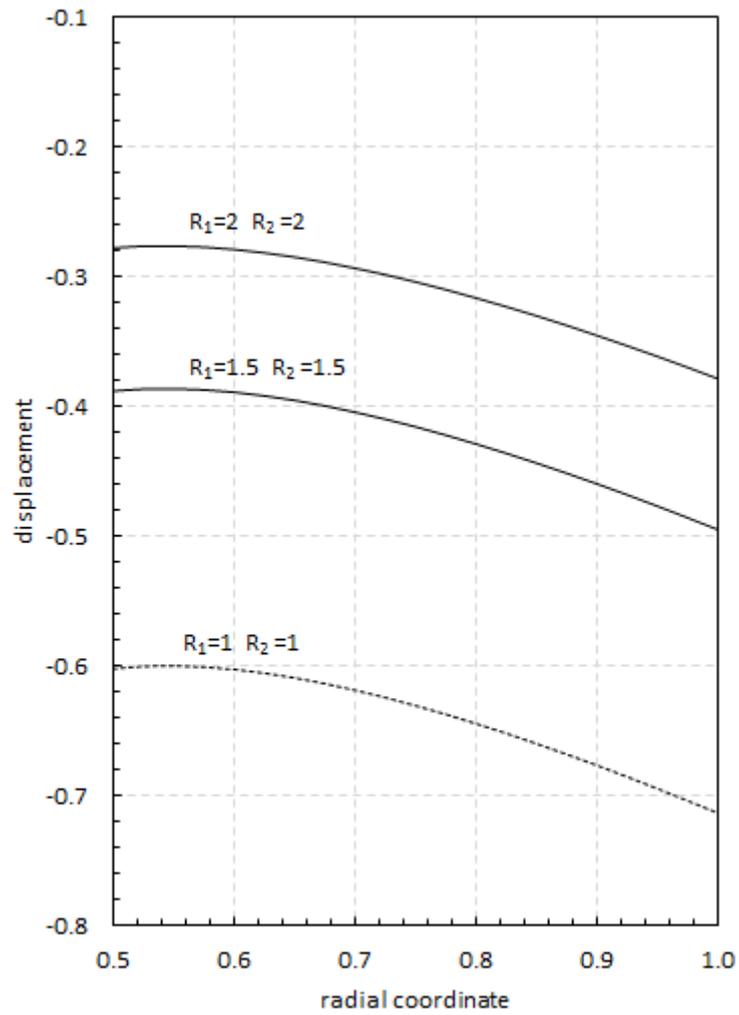Figure 4.28: Distribution of elastic limit annular speed in conjunction with orthotropy parameter R_2

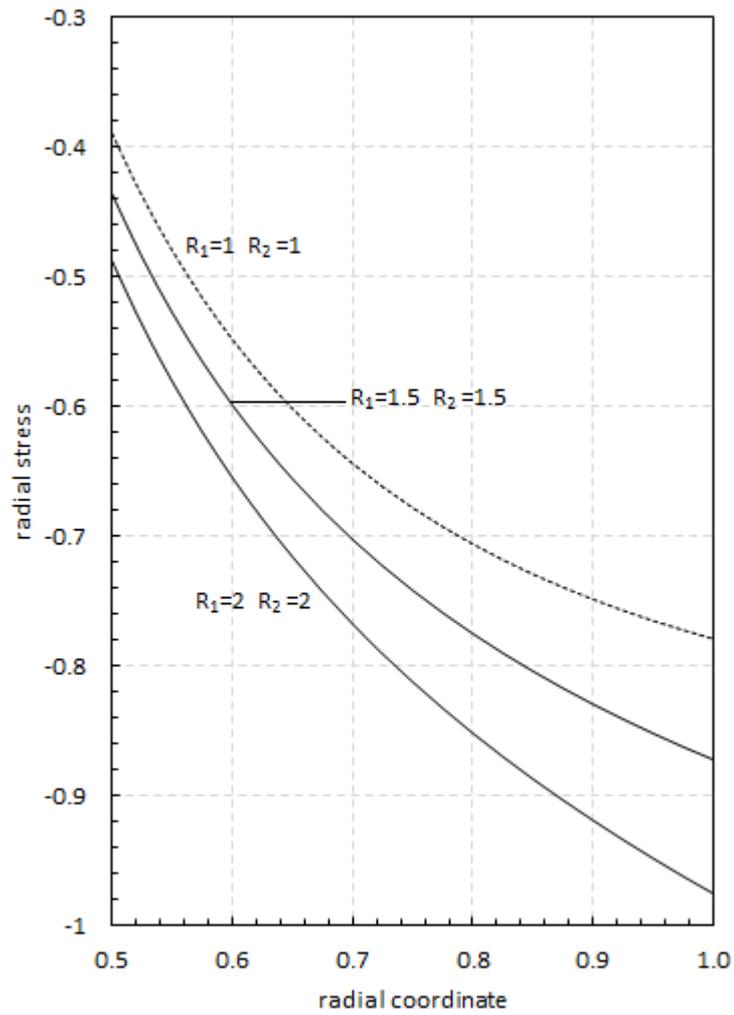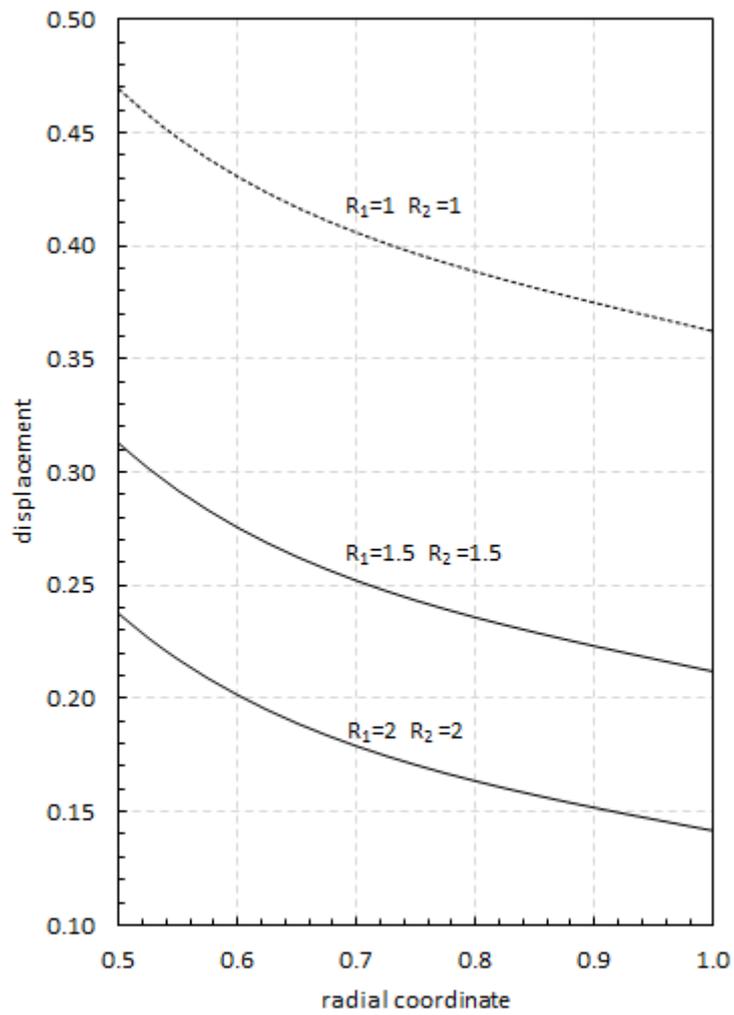Figure 4.29: Distribution of non dimensional displacement under annular rotation and internal pressure along radius
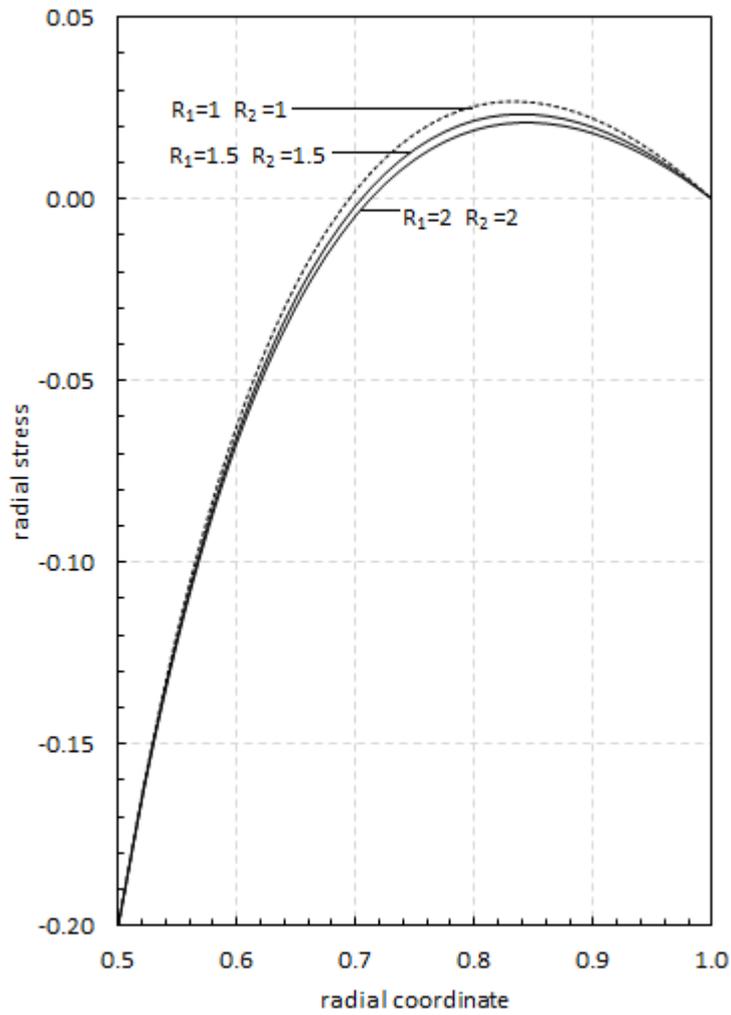
Figure 4.30: Distribution of non dimensional radial stress under annular rotation and internal pressure along radius
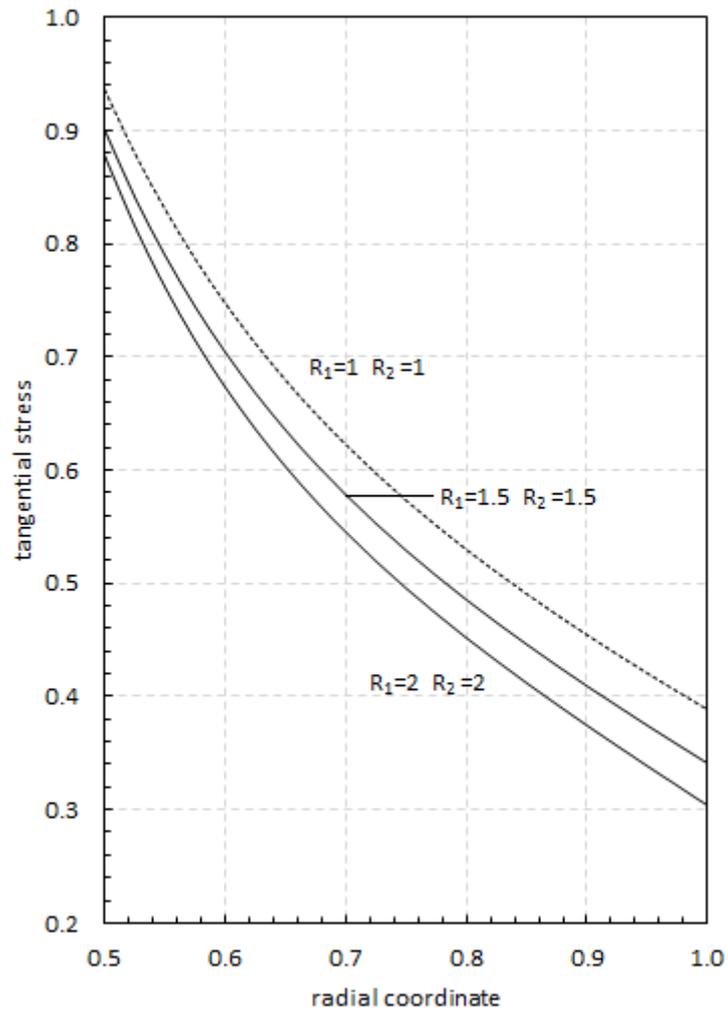
Figure 4.31: Distribution of non dimensional tangential stress under annular rotation and internal pressure along radius
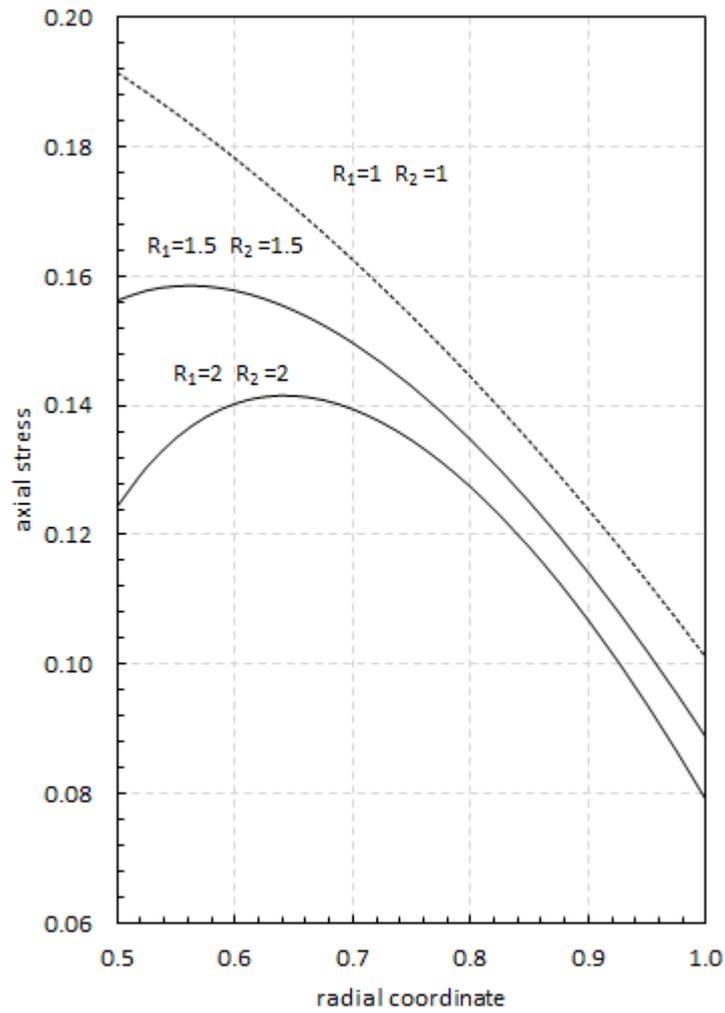
Figure 4.32: Distribution of non dimensional axial stress under annular rotation and internal pressure along radius
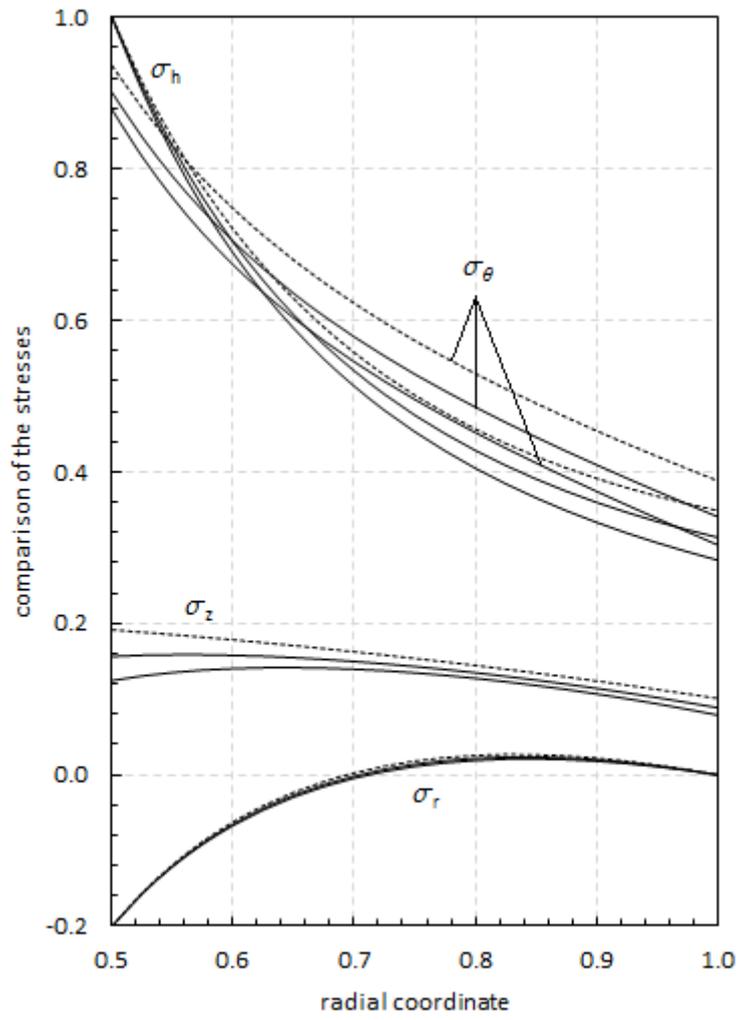
Figure 4.33: The states of non dimensional stress under annular rotation and internal pressure along radial coordinate
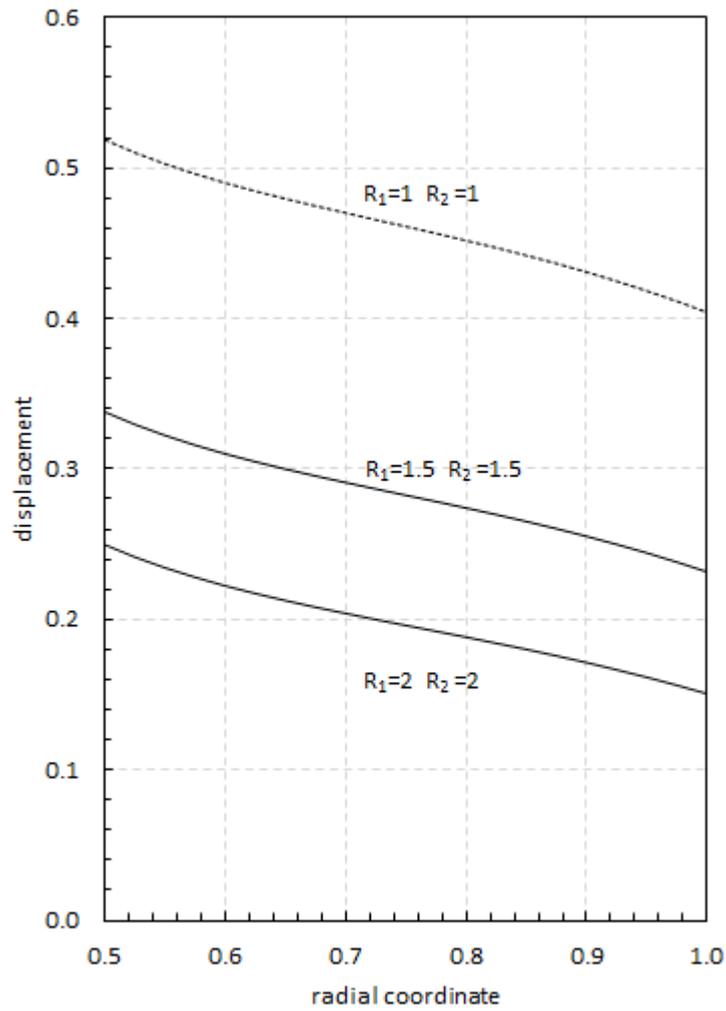
Figure 4.34: Distribution of non dimensional displacement under annular rotation and external pressure along radius
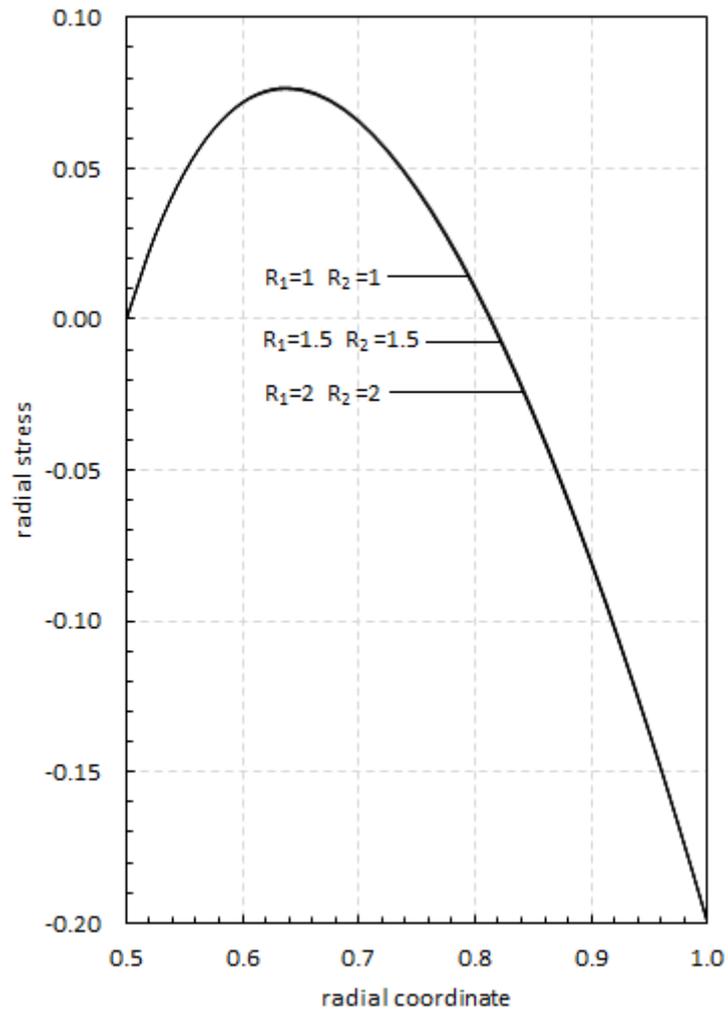
Figure 4.35: Distribution of non dimensional radial stress under annular rotation and external pressure along radius
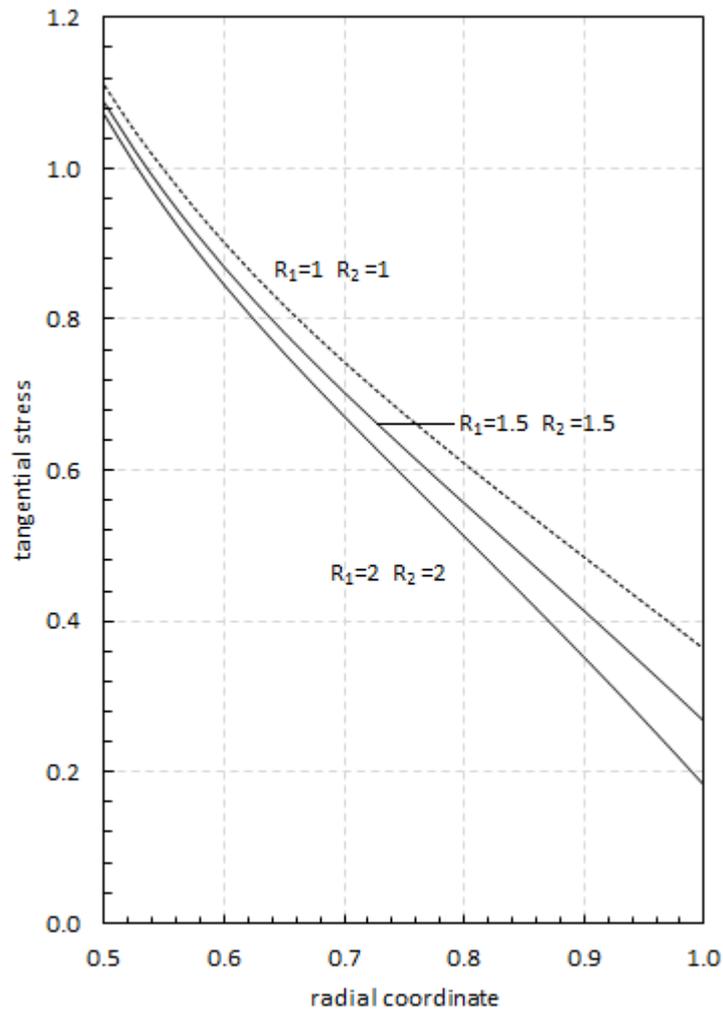
Figure 4.36: Distribution of non dimensional tangential stress under annular rotation and external pressure along radius
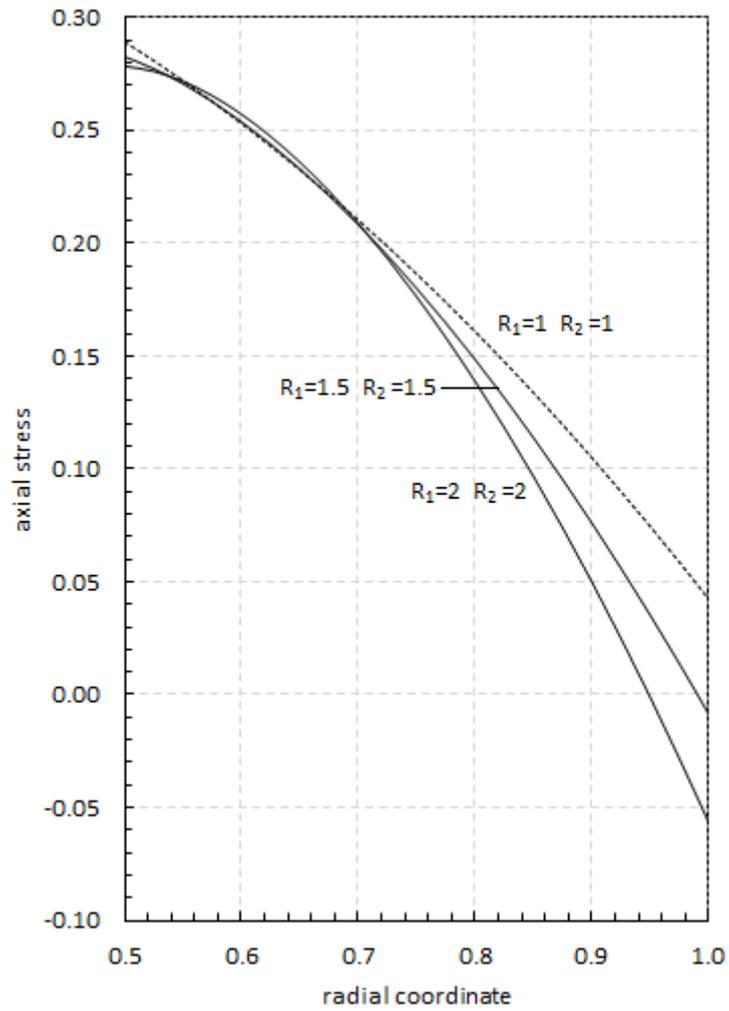
Figure 4.37: Distribution of non dimensional axial stress under annular rotation and external pressure along radius
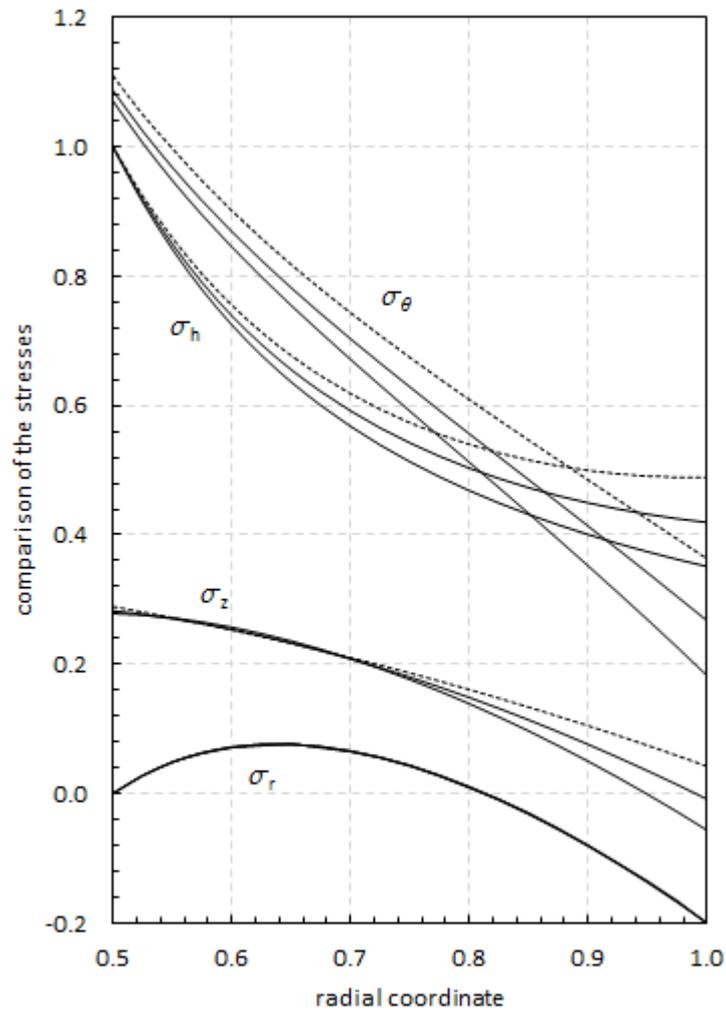
Figure 4.38: The states of non dimensional stress under annular rotation and external pressure along radial coordinate

4.3 Results for Real Materials

In this section, 3-D glass epoxy and drawn 7050 aluminum alloy is used to illustrate numerical examples. Mechanical properties of glass epoxy is found from literature[21]. On the other hand, material properties of 7050 aluminum are obtained experimentally in the Metallurgy Laboratory of METU. To measure mechanical properties of drawn 7050 aluminum Instron 5582 tension test machine is used, which is pictured in Fig 4.39.
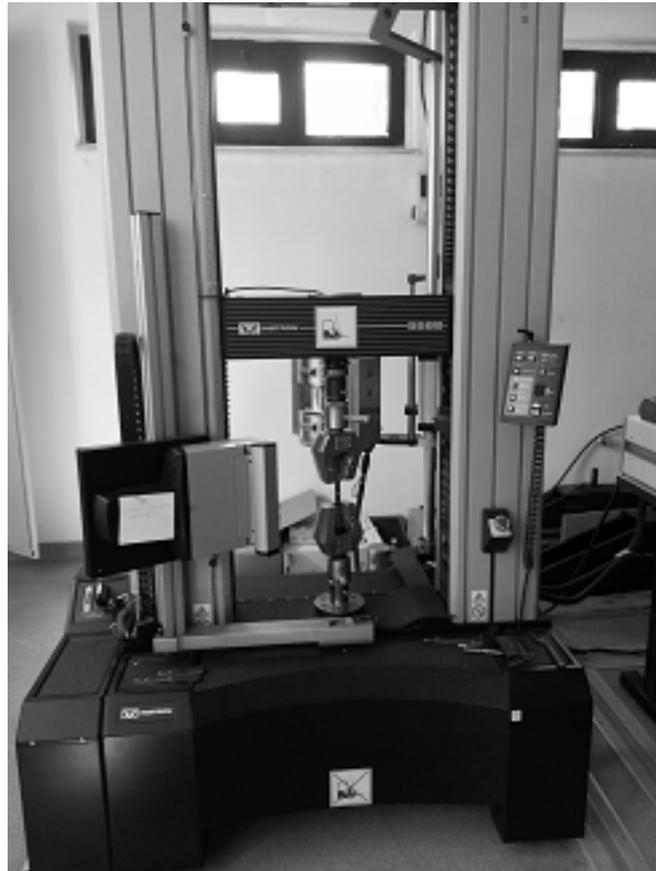
Figure 4.39: Instron 5582 tension test machine

Furthermore, to illustrate an example, experimentally obtained stress-strain diagram in radial direction is given.
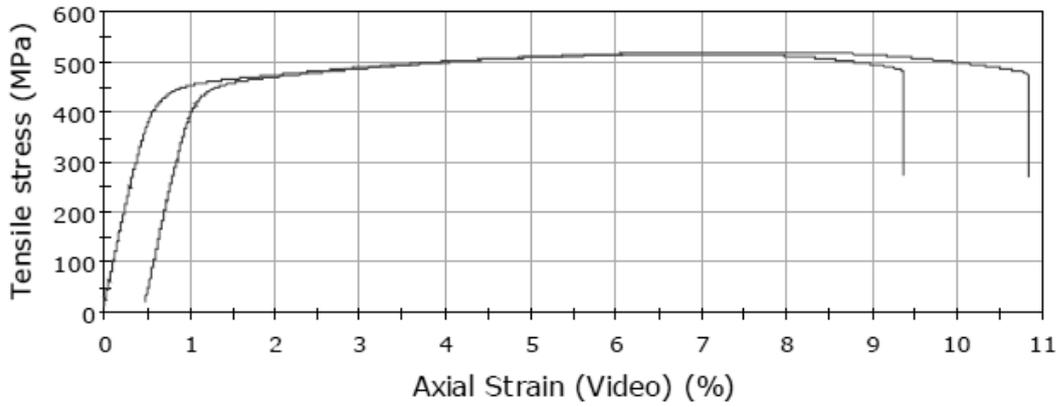
Figure 4.40: Stress-strain diagram of 7050 aluminum in radial direction

It is previously mentioned that manufacturing processes such as drawing and extrusion cause orthotropy on materials. In the experiment, 3-D properties of drawn 7050 aluminum are obtained. For each direction, two different test specimens are used. In total, six different tension test are held. The tables given below express the experimental properties of the aluminum.

Table4.8: Test data in radial direction

	Specimen 1	Specimen 2
E_r (GPa)	74.571	75.395
σ_r^y (MPa)	437.430	443.378
$v_{r\theta}$	0.31	0.31

Table4.9: Test data in tangential direction

	Specimen 1	Specimen 2
E_θ (GPa)	76.874	72.228
σ_θ^y (MPa)	437.128	412.453
$v_{\theta z}$	0.33	0.33

Table4.10: Test data in axial direction

	Specimen 1	Specimen 2
E_z (GPa)	66.152	68.124
σ_z^y (MPa)	403.561	423.631
v_{rz}	0.33	0.33

Followingly, average values of the experiments are taken and used to plot the graphics.

Table4.11: Material and structural properties for glass epoxy and 7050 aluminum

	Glass Epoxy	7050 Aluminum
E_r (GPa)	44.6	75
E_θ (GPa)	17	74.5
E_z (GPa)	16.7	67
σ_r^y (MPa)	1240	440
σ_θ^y (MPa)	43.9	425
σ_z^y (MPa)	31.3	413
ρ (kg/m^3)	1.9	2.8
$v_{r\theta}$	0.262	0.31
v_{rz}	0.264	0.33
$v_{\theta z}$	0.350	0.33
$a(m)$	0.06	0.06
$b(m)$	0.1	0.1

For the glass epoxy, orthotropy parameters are assumed as $R_1 = 0.381$, $R_2 = 0.374$ and $R_h = 1.33$. Similarly, these parameters are the following for the drawn 7050 aluminum $R_1 = 0.993$, $R_2 = 0.893$ and $R_h = 0.75$. The R_h parameter of the aluminum could not find in the experiment but it is between 0.6-0.8 for aluminum alloys[22].

After material properties are determined, boundary conditions are applied and the following data are tabulated. In the graphical illustrations, dashed lines represent glass epoxy and continuous lines represent 7050 aluminum.

Table4.12: Calculated values for internally pressurized cylinder made of glass epoxy and 7050 aluminum

	Glass Epoxy	7050 Aluminum
$\overline{P}_{in}$	0.372388	0.377173
$\overline{F}_z$	0.351692	0.253714
$\overline{C}_1$	$3.8116 \ 10^{-2}$	$1.6523 \ 10^{-3}$
$\overline{C}_2$	$4.7034 \ 10^{-3}$	$6.1706 \ 10^{-4}$

Table4.13: Calculated values for externally pressurized cylinder made of glass epoxy and 7050 aluminum

	Glass Epoxy	7050 Aluminum
$\overline{P}_{ex}$	0.347603	0.366608
$\overline{F}_z$	-0.635344	-0.682576
$\overline{C}_1$	$-4.2615 \ 10^{-2}$	$-1.6086 \ 10^{-3}$
$\overline{C}_2$	$-1.0196 \ 10^{-2}$	$-1.6633 \ 10^{-3}$

Table4.14: Calculated values for both internally and externally pressurized cylinder made of glass epoxy and 7050 aluminum

	Glass Epoxy	7050 Aluminum
$\overline{P}_{in}$	0.316602	0.348502
$\overline{P}_{ex}$	0.633203	0.697004
$\overline{F}_z$	-0.858349	-1.0633
$\overline{C}_1$	$-4.5178 \ 10^{-2}$	$-1.5316 \ 10^{-3}$
$\overline{C}_2$	$-1.4575 \ 10^{-2}$	$-2.5923 \ 10^{-3}$

Table4.15: Calculated values for annularly rotating cylinder made of glass epoxy and 7050 aluminum

	Glass Epoxy	7050 Aluminum
$\overline{\omega}$	1.13004	1.12099
$\overline{F}_z$	0.57337	0.50963
$\overline{C}_1$	$4.0221 \ 10^{-2}$	$1.5215 \ 10^{-3}$
$\overline{C}_2$	$1.1633 \ 10^{-2}$	$2.1373 \ 10^{-3}$
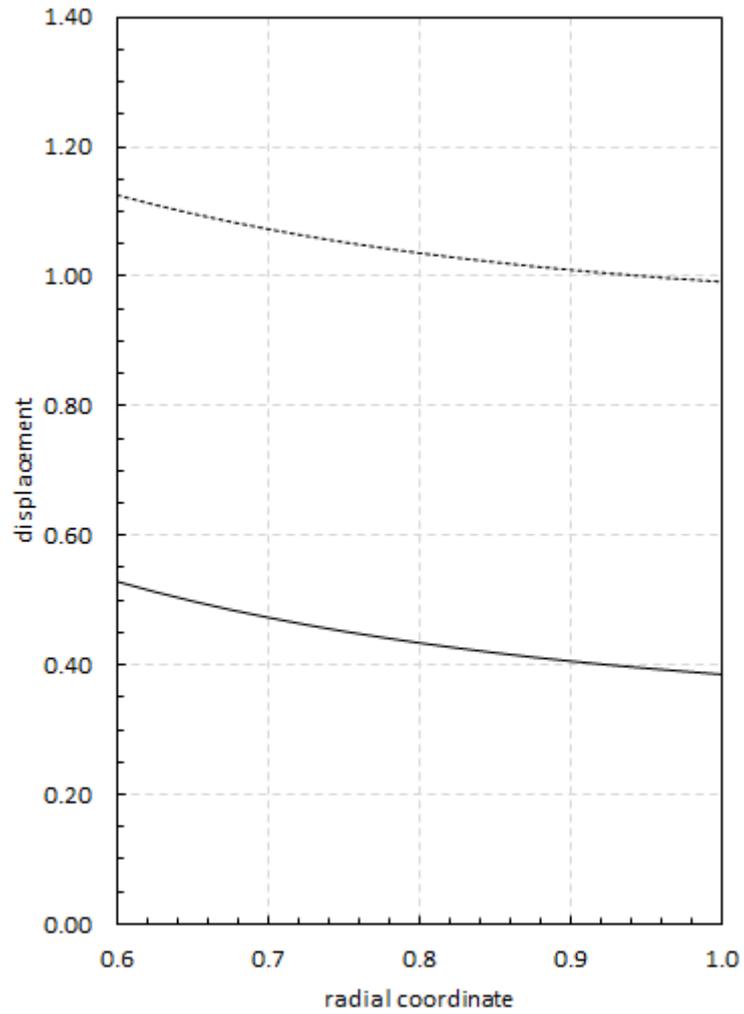
Figure 4.41: Distribution of non dimensional displacement under internal pressure along radial coordinate for glass epoxy and 7050 aluminum
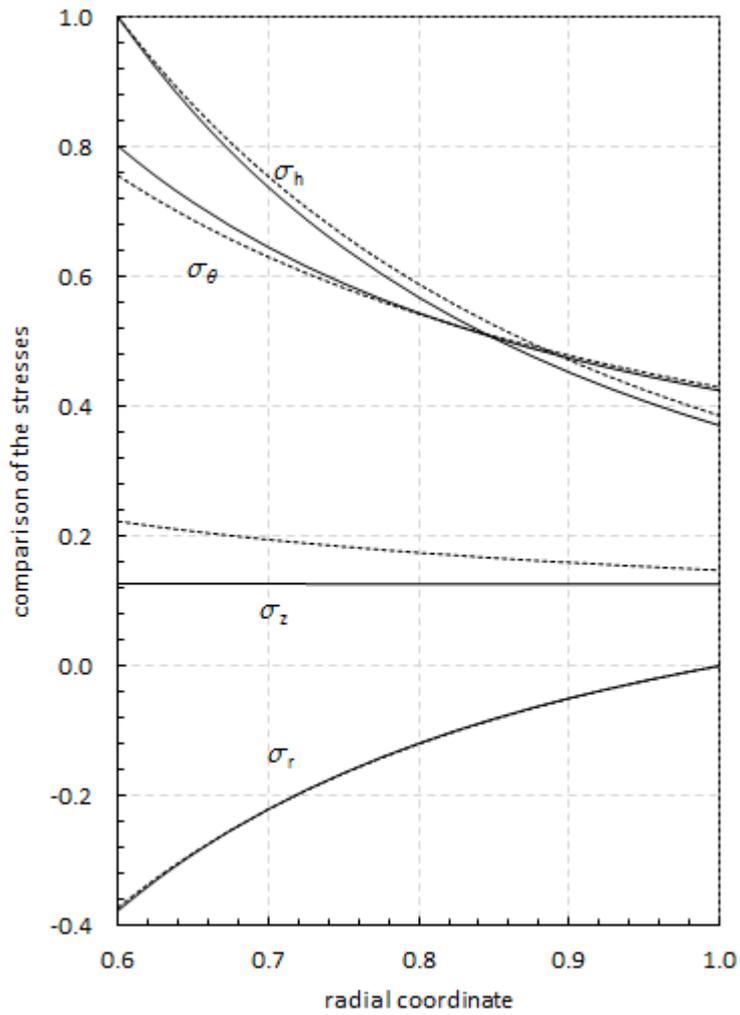
Figure 4.42: The states of non dimensional stress under internal pressure for glass epoxy and 7050 aluminum
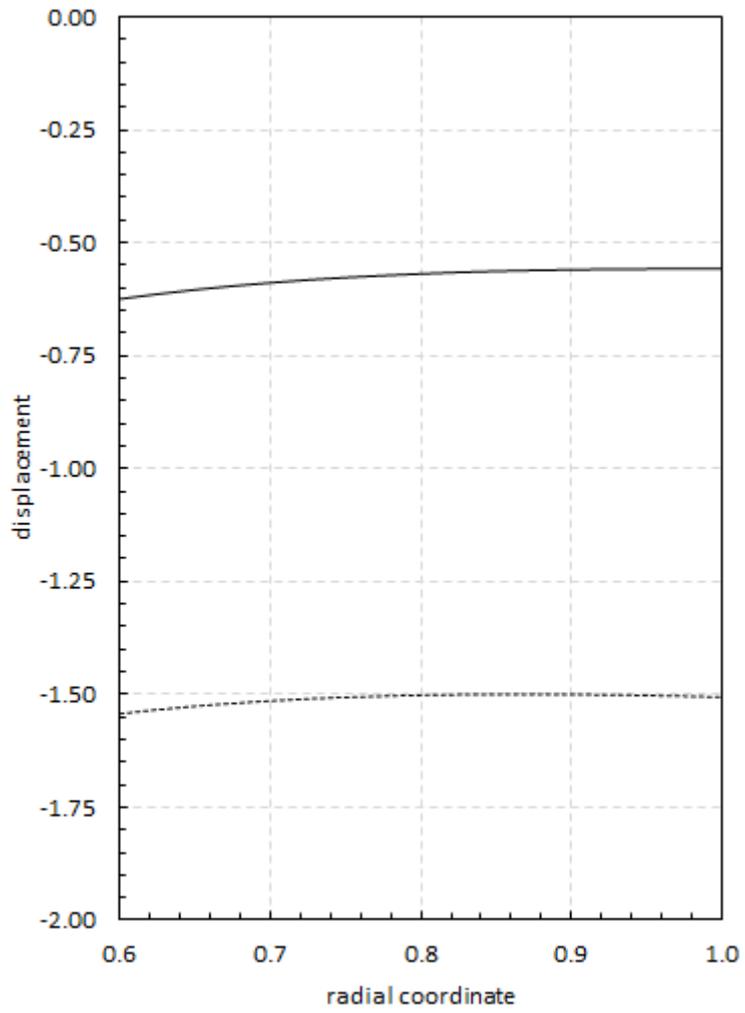
Figure 4.43: Distribution of non dimensional displacement under external pressure along radial coordinate for glass epoxy and 7050 aluminum
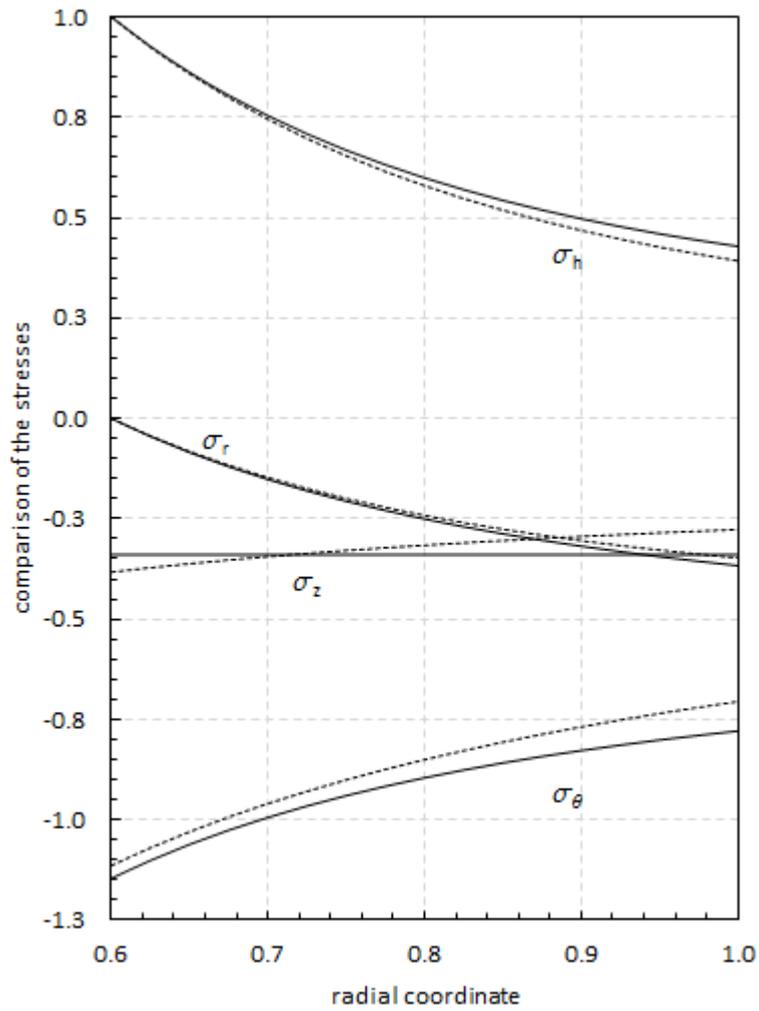
Figure 4.44: The states of non dimensional stress under external pressure for glass epoxy and 7050 aluminum
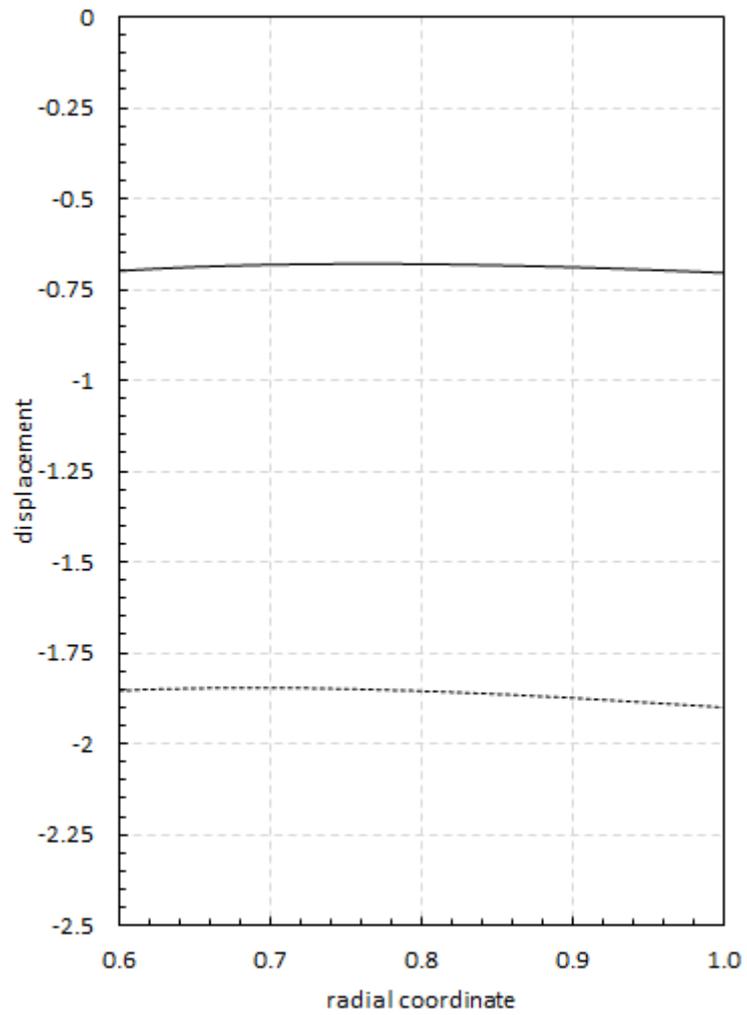
Figure 4.45: Distribution of non dimensional displacement under combined pressure along radial coordinate for glass epoxy and 7050 aluminum
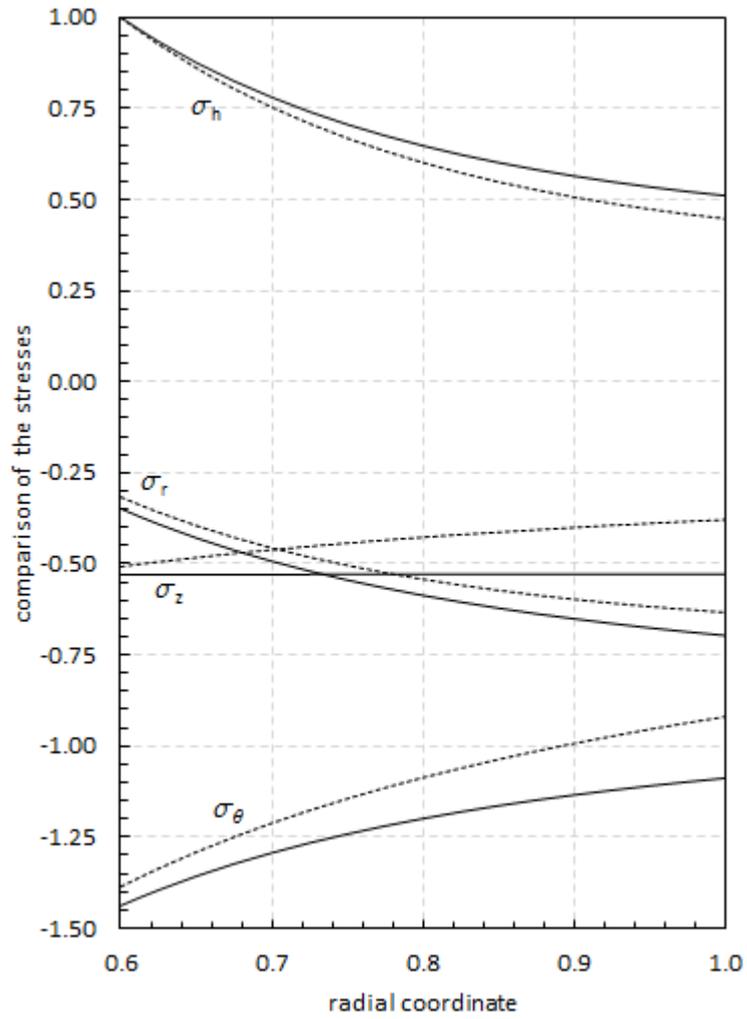
Figure 4.46: The states of non dimensional stress under combined pressure for glass epoxy and 7050 aluminum
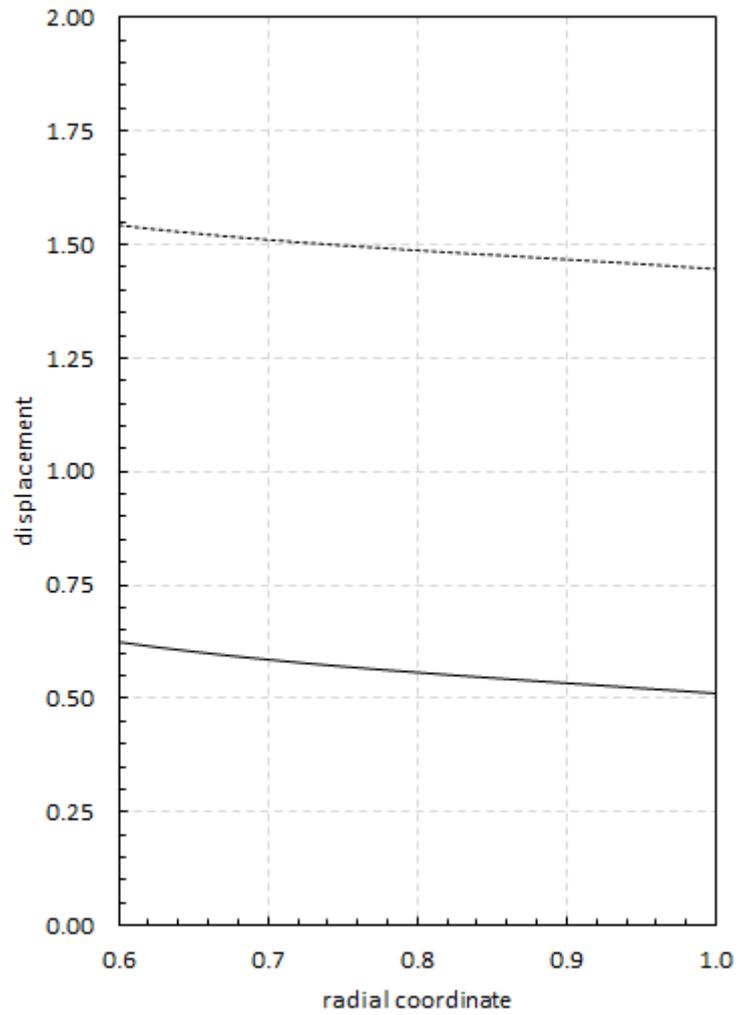
Figure 4.47: Distribution of non dimensional displacement under annular rotation along radial coordinate for glass epoxy and 7050 aluminum
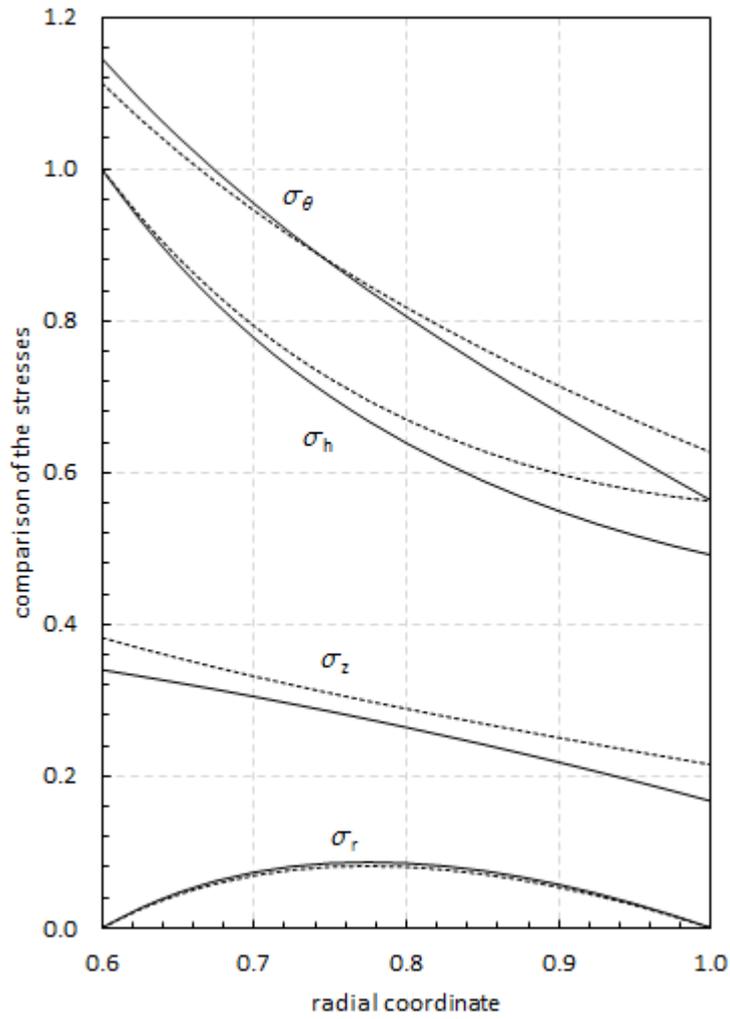
Figure 4.48: The states of non dimensional stress under annular rotation for glass epoxy and 7050 aluminum

4.4   Thick Walled to Thin Walled Transition

In this section, wall thickness of the cylinders are considered and effect of the wall thickness is briefly explained. Cylinders are assumed to be thin walled when $t/b < 0.1$ in which t is the wall thickness and equals to $t = b - a$. In the sequal, a and b are the inner and outer radius of the cylinders. This assumption allows us to consider the wall thickness as a surface when it is thin enough. Under the light of this consideration, radial stress ,σ_r, converges to zero. In other words, for thin walled cylinders $\sigma_r = 0$.

In contrary to thick walled cylinders, directional stresses of thin walled ones do not notably change with respect to radial distance. Thus, tangential and axial stress components are considered to be constant through the wall thickness. In realty tangential and axial stresses vary through t but variations are small enough to be considered as constant.

Hereby, the analytical solutions for the thick walled cylinders are applied for different t/b ratios. The graphics given below illustrate the cylinders subjected to annular rotation for different t/b ratios. Annular rotation case is chosen purposefully because radial stress alteration with respect to t/b can be seen obviously in the graphics. In order to plot the graphics, the data given in the first column of Table 4.1 is used. For different t/b ratios, following data are found at the elastic limit.

Table4.16: Calculated values for annularly rotating cylinder with respect to different t/b ratios

	$t/b = 0.4$	$t/b = 0.3$	$t/b = 0.2$	$t/b = 0.1$
$\overline{\omega}$	1.11428	1.1014	1.08708	1.07151
$\overline{F}_z$	0.441368	0.37648	0.284948	0.161257
$\overline{C}_1$	$2.9491 \ 10^{-4}$	$3.9219 \ 10^{-4}$	$4.9901 \ 10^{-4}$	$6.1361 \ 10^{-4}$
$\overline{C}_2$	$5.3478 \ 10^{-4}$	$5.7244 \ 10^{-4}$	$61379 \ 10^{-4}$	$6.5814 \ 10^{-4}$
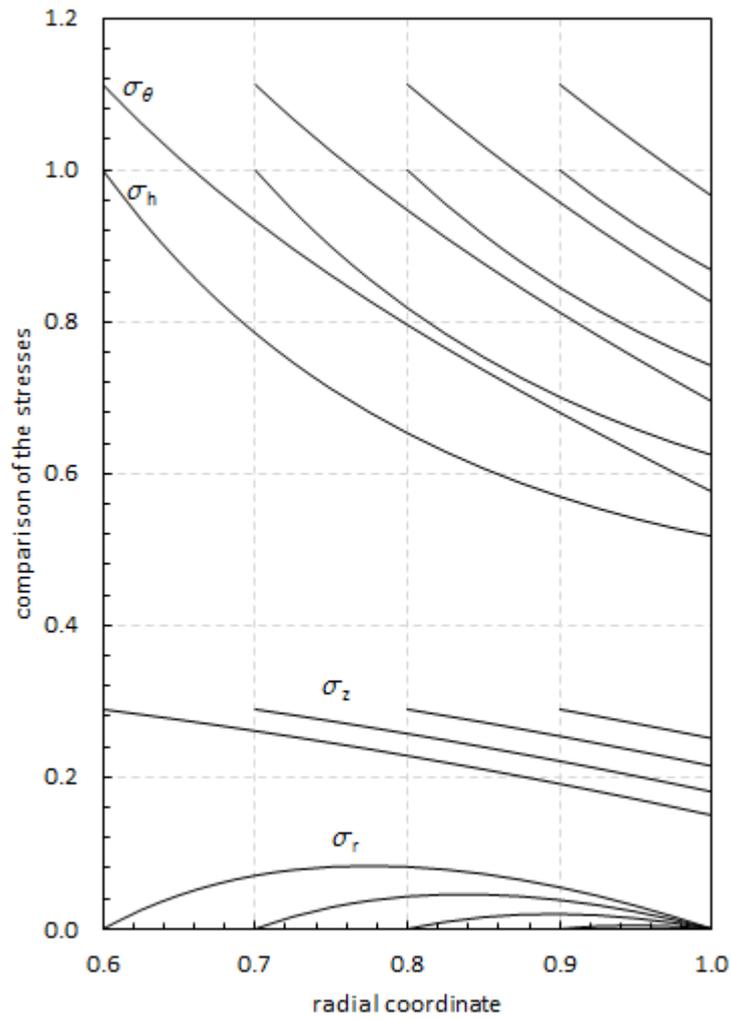
Figure 4.49: The states of non dimensional stress under annular rotation for different t/b ratios

CHAPTER 5

CONCLUSION

In the past, researchers have investigated isotropic cylinders considerably. First investigations had begun in 19th century with the famous french engineer Lame. Following Lame's path, the number of new researches were expanded in the past decades.

In contrary to isotropy, orthotropy has not gained attention greatly. Thus, this research has focused on orthotropic structures. Hereby, orthotropic cylinders have examined spaciously. Analytically treated orthotropic cylinders are prospected in the elastic state. In the sequel, Maxwell relations are introduced to analytical solutions and ratios between elastic moduli are taken as orthotropy parameters. Effect of these parameters on the variation of radial displacement, directional stresses and elastic limits are determined. Afterwords, these variations are plotted.

In order to illustrate numerical examples graphically, theoretical and real materials are used. It is observed that results resemble with each other even though different materials are used. Radial displacement and directional stress curves in the graphics are coherent.

When the directional stress diagrams are taken into consideration, it is seen that radial and tangential stress components slightly influenced by R_1 and R_2. In other words, orthotropy parameters do not greatly change the variation of radial and tangential stresses. On the other hand, reactions of the axial stress components are distinct to alteration of R_1 and R_2.

Displacement diagrams signify that orthotropy highly influences radial displacement like axial stresses. Even the small changes on orthotropy parameters differ the behavior of radial displacement. It is seen from the radial displacement diagrams that when the degree of orthotropy increases, radial displacement decreases.

For further work, to extend this research, orthotropic cylinders can be investigated at the elastic-plastic or plastic state. Additionally, analytical solutions can be compared with numerical approaches.

REFERENCES

[1] A. C. Ugural and S. K. Fenster. *Advanced strength and applied elasticity.* Prentice Hall, New Jersey, 2004.

[2] S. Timoshenko and J.N. Goodier. *Theory of elasticity.* McGraw-Hill, New York, 1951.

[3] A.M. Abd-Alla, S. R. Mahmoud, and N.A. Al-Shehri. Effect of the rotation on a non-homogeneous infinite cylinder of orthotropic material. *Applied Mechanics and Computation*, 217:8914–8922, 2011.

[4] M. Tervonen and A. Pramila. Stresses in a hollow rotating cylindrically orthotropic tube. *Mechanics of Composite Materials*, 6:577–581, 1996.

[5] A.M. Zenkour. Rotating variable thickness orthotropic cylinder containing a solid core of uniform thickness. *Archive of Applied Mechanics*, 76:89–102, 2006.

[6] S. Y. Leu and H. C. Hsu. Exact solutions for plastic responses of orthotropic strain-hardening rotating hollow cylinders. *International Jornal of Mechanical Sciences*, 52:1579–1587, 2010.

[7] H. Garmestani, M.R. Vaghar, D. Markiewicz, and N. Chandra. Stress analysis of an orthotropic work-hardening cylinder with body force. *Mechanics of Structures and Machines*, 23:521–548, 1995.

[8] D. Croccolo and M. DeAgostinis. Analytical solution of stress and strain distributions in press fitted orthotropic cylinders. *International Journal of Mechanical Sciences*, 71:21–29, 2013.

[9] C. Jolicoeur and A. Cardou. Analytical solution for bending of coaxial orthotropic cylinders. *Journal of Engineering Mechanics*, 120:2556–2574, 1994.

[10] P. Desai and T. Kant. Stress analysis of finite length cylinders of layered media. *Applied and Computational Mechanics*, 5:129–142, 2011.

[11] V.A. Lubarda. On pressurized curvilinearly orthotropic circular disk, cylinder and sphere made of radially non uniform material. *Journal of Elasticity*, 2:103–133, 2012.

[12] A.N. Eraslan, Y. Kaya, and E. Varlı. Analytical solutions to orthotropic variable thickness disk problems. *Pamukkale University Journal of Engineering Sciences*, 22:24–30, 2016.

[13] L. Xin, G. Dui, S. Yang, and J. Zhang. An elasticity solution for functionally graded thick walled tube subjected to internal pressure. *International Journal of Mechanical Sciences*, 89:344–349, 2014.

[14] N. Tutuncu. Stress in thick walled fgm cylinders with exponentially varying properties. *Engineering Structures*, 29:2032–2035, 2007.

[15] A.N. Eraslan and T. Akis. Plane strain analytical solutions for a functionally graded elastic-plastic pressurized tube. *International Journal of Pressure Vessels and Piping*, 83:635–644, 2006.

[16] L.H. You, H. Ou, and Z.Y. Zheng. Creep deformations and stresses in thick walled cylindrical vessels of functionally graded materials subjected to internal pressure. *Composite Structures*, 78:285–291, 2007.

[17] G.J. Nie and R.C. Batra. Material tailoring and analysis of functionally graded isotropic and incompressible linear elastic hollow cylinders. *Composite Structures*, 92:265–274, 2010.

[18] M. Ghannad and M.Z. Nejad. Elastic analysis of heterogeneous thick cylinders subjected to internal or external pressure using shear deformation theory. *Acta Polytechnica Hungarica*, 9:117–136, 2012.

[19] A.N. Eraslan and T. Akis. Yielding long concentric tubes under radial pressure based on von mises criterion. *Journal of the Faculty of Engineering and Architecture of Gazi University*, 20:365–372, 2005.

[20] A. C. Yiannopoulos. A general formulation of stress distribution in cylinders subjected to non-uniform external pressure. *Journal of Elasticity*, 56:181–198, 1999.

[21] D.D. Samborsky, J.F. Mandell, and P. Agastra. 3-d static elastic constants and strength properties of a glass/epoxy unidirectional laminate. url: http://www.coe.montana.edu/composites/.

[22] J.R. Davis. *Tensile Testing*. Asm International, Materials Park, Ohio, 2004.

APPENDICES A

FORTRAN CODES

A.1   Internally Pressurized Cylinders

This program solves the internally pressurized cylinders made of orthotropic and isotropic materials at the elastic limits. If R1 and R2 are taken as R1=R2=Rh=1 then program solves isotropic structures. Other R1,R2 and Rh values give solutions for orthotropic structures.

INPUTS:

Er: Elasticity Modulus in radial direction. It should be inserted in MPa unit.

Vrt, Vtz, Vrz: Poisson ratios

R1, R2, Rh: Orthotropy parameters

A, B: Inner and outer radius of the cylinders. These values should be inserted in meters.

SYR: Yield limit in radial direction(MPa).

OUTPUTS:

cij: Compliance matrix terms

sij: Stiffness matrix terms

C1 and C2: Arbitrary constants

Pin: Dimensionless elastic limit internal pressure

Fz: Dimensionless axial stress

U: Dimensionless radial displacement distribution

SR, ST, SZ, Hill: Radial, tangential, axial and Hill's stress distributions

```
PROGRAM ORT

IMPLICIT NONE

DOUBLE PRECISION::E1,V12,V13,V23,R1,R2,R3,SYR,SRD,STD,SZD,PH2,FZ

DOUBLE PRECISION::S11,S12,S13,S22,S23,S33,DEN,C11,C12,C13,C22,C23,C33

DOUBLE PRECISION::F,G,H,P,CR,CT,CZ,PIN,K1,K2,U,SR,ST,SZ,SM,PH,PIN2

DOUBLE PRECISION::R,A,B

WRITE(6,*)' Please enter Er(MPa),Vrt,Vrz,Vtz,R1,R2'

READ(5,*) E1, V12,V13,V23,R1,R2

WRITE(6,*)'Please insert inner(meters) and outer radius(meters) of the cylin-
der'

READ(5,*) A,B

WRITE(6,*)'Please enter radial yielding strength(MPa) and Rh'

READ(5,*) SYR, R3

DEN= R1**2 * V12**2 + R2 * V23**2 + R1 * (-1 + R2* V13 * (V13 + 2*V12
* V23))

S11=E1*(-R1+R2*V23 **2)/DEN

S12=-E1*R1*(R1*V12+R2*V13*V23)/DEN

S13=-E1*R1*R2*(V13+V12*V23)/DEN

S22=E1*R1**2 * (-1+R2*V13**2) / DEN

S23=-E1*R1*R2*(R1*V13*V12+V23)/DEN
```

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S33=E1*R1*R2*(-1+R1*V12**2)/DEN

C11=1/E1

C12=-V12/E1

C13=-V13/E1

C22=1/(R1*E1)

C23=-V23/(R1*E1)

C33=1/(R2*E1)

P=SQRT(S22/S11)

CR=-1.0

CT=-((A**(2*P)+B**(2*P))*P) / (A**(2*P)-B**(2*P))

CZ=-(C13/C33)*CR - (C23/C33)*CT

PIN=(Sqrt(1 + R3)*SYR)/Sqrt(CR**2 + CT**2 - 2*CR*CZ - 2*CT*CZ + 2*CZ**2 + CR**2*R3 - 2*CR*CT*R3 + CT**2*R3)

PIN2=PIN/SYR

K1=-(A**(1+P) * B**(2*P)*(S12+S11*P)*PIN) / ((A**(2*P)-B**(2*P))*(-S12**2 + S11*S22))

K2=-(A**(1+P) * PIN)/ ((A**(2*P)- B**(2*P))*(S12 + S11*P))

IF (R1 .EQ. 1.0) THEN

FZ=((-2*A**2*C13*3.14159265*PIN)/C33)/(SYR*B**2)

ELSE

FZ=(-2*3.14159265*PIN*A*(-2*A**P *B**(1+P)*(C13-C23)*P + A*B**(2*P) *(1+P)*(C13-C23*P) +A**(1+2*P)

(-1+P)(C13+C23*P))/((A**(2*P)-B**(2*P))*C33*(-1+P**2)))/(SYR* B**2)

END IF

DO 10 R=A,B,0.0001

U=(K1*R**(-P) + K2*R**(P)) *(E1/(SYR*B))

SR=(K1 * R**(-1-P) * (-P*S11+S12) + K2 * R**(-1+P) * (P*S11+S12))

SRD=SR/SYR

ST=(K1 * R**(-1-P) * (-P*S12+S22) + K2 * R**(-1+P) * (P*S12+S22))

STD=ST/SYR

SZ=(K1 * R**(-1-P) * (-P*S13+S23) + K2 * R**(-1+P) * (P*S13+S23))

SZD=SZ/SYR

PH2=((ST-SZ)**2 + (SR-SZ)**2 + R3*(SR-ST)**2)/((R3 +1)*SYR**2)

WRITE(6,*)' U SR ST SZ Hill '

WRITE(6,*)'————————————————————————————————'

WRITE(6,100) U,SRD,STD,SZD,PH2

10 CONTINUE

WRITE(6,*)'————————————————————————————————'

WRITE(6,*)

WRITE(6,*)' CONSTANTS '

WRITE(6,*)

WRITE(6,*)'s11:',S11, 's12:',S12

WRITE(6,*)'s13:',S13, 's22:',S22

WRITE(6,*)'s23:',S23, 's33:',S33

WRITE(6,*)'c11:',C11, 'c12:',C12

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WRITE(6,*)'c13:',C13, 'c22:',C22

WRITE(6,*)'c23:',C23, 'c33:',C33

WRITE(6,*)'C1: ',K1, 'C2: ',K2

WRITE(6,*)'Internal Pressure (MPa):',PIN

WRITE(6,*)'Dimensionless Pin:',PIN2

WRITE(6,*)'Dimensionless Fz:',FZ

PAUSE

100 FORMAT (3X,5(F11.6,3X))

STOP

END PROGRAM

A.2   Externally Pressurized Cylinders

The below program solves the externally pressurized cylinders made of orthotropic and isotropic materials at the elastic limits. If R1, R2 and Rh are taken as 1, program solves isotropic structures. In order to use the below program inputs should be declared.

INPUTS:

Er: Elasticity Modulus in radial direction. It should be inserted in MPa unit.

Vrt, Vtz, Vrz: Poisson ratios

R1, R2, Rh: Orthotropy parameters

A, B: Inner and outer radius os the cylinders. These values should be inserted in meters.

SYR, SYT, SYZ: Radial ield limit (MPa)

OUTPUTS:

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cij: Compliance matrix terms

sij: Stiffness matrix terms

C1 and C2: Arbitrary constants

Pex: Dimensionless elastic limit external pressure

Fz: Dimensionless axial stress

U: Dimensionless radial displacement

SR, ST, SZ, Hill: Radial, tangential, axial and Hill's stress distributions

PROGRAM EXT

IMPLICIT NONE

DOUBLE PRECISION::E1,V12,V13,V23,R1,R2,R3,SYR,SRD,STD,SZD

DOUBLE PRECISION::S11,S12,S13,S22,S23,S33,DEN,C11,C12,C13,C22,C23,C33

DOUBLE PRECISION::F,G,H,P,CR,CT,CZ,PEX,PEX2,K1,K2,U,SR,ST,SZ,PH,FZ

REAL::R,A,B

WRITE(6,*)' Please enter E1(MPa),V12,V13,V23,R1,R2'

READ(5,*) E1, V12,V13,V23,R1,R2

WRITE(6,*)'Please insert inner(meters) and outer radius(meters) of the cylinder'

READ(5,*) A,B

WRITE(6,*)'Please enter radial yielding strength(MPa) and Rh'

READ(5,*) SYR,R3

DEN= R1**2 * V12**2 + R2 * V23**2 + R1 * (-1 + R2* V13 * (V13 + 2*V12 * V23))

S11=E1*(-R1+R2*V23 **2)/DEN

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S12=-E1*R1*(R1*V12+R2*V13*V23)/DEN

S13=-E1*R1*R2*(V13+V12*V23)/DEN

S22=E1*R1**2*(-1+R2*V13**2)/DEN

S23=-E1*R1*R2*(R1*V13*V12+V23)/DEN

S33=E1*R1*R2*(-1+R1*V12**2)/DEN

C11=1/E1

C12=-V12/E1

C13=-V13/E1

C22=1/(R1*E1)

C23=-V23/(R1*E1)

C33=1/(R2*E1)

P=SQRT(S22/S11)

CR=0

CT=(2*A**(-1 + P) * B**(1 + P) *P) / (A**(2*P) - B**(2*P))

CZ=(-C13/C33)*CR + (-C23/C33)*CT

PEX=(Sqrt(1 + R3)*SYR)/Sqrt(CR**2 + CT**2 - 2*CR*CZ - 2*CT*CZ + 2*CZ**2 + CR**2*R3 - 2*CR*CT*R3 + CT**2*R3)

PEX2=PEX/SYR

IF (R1 .EQ. 1.0) THEN

FZ=((2*B**2*C13*PEX*3.14159265)/C33)/(SYR*B**2)

ELSE

FZ=2*PEX*3.14159265*B*(2*A**(1+P)*B**P *(C13-C23)*P

+A**(2*P)*B*(1+P)*(-C13+C23*P) -B**(1+2*P) *(-1+P)*(C13+C23*P))

/((A**(2*P)-B**(2*P))*C33 *(-1+P**2))/(SYR*B**2)

END IF

K1=(A**(2*P) * B**(1+P) * (S12 + S11*P) * PEX) / ((A**(2*P) - B**(2*P)) * (-S12**2 + S11*S22))

K2=(B**(1+P) * (-S12 + S11*P) * PEX) / ((A**(2*P) - B**(2*P)) * (-S12**2 + S11*S22))

DO 10 R= A,B,0.0025

U=(K1*R**(-P) + K2*R**(P))*E1 / (SYR*B)

SR=(K1 * R**(-1-P) * (-P*S11+S12) + K2 * R**(-1+P) * (P*S11+S12))

SRD=SR/SYR

ST=(K1 * R**(-1-P) * (-P*S12+S22) + K2 * R**(-1+P) * (P*S12+S22))

STD=ST/SYR

SZ=(K1 * R**(-1-P) * (-P*S13+S23) + K2 * R**(-1+P) * (P*S13+S23))

SZD=SZ/SYR

PH=((ST-SZ)**2 + (SR-SZ)**2 + R3*(SR-ST)**2)/((R3 +1)*SYR**2)

WRITE(6,*)' U SR ST SZ Hill'

WRITE(6,*)'————————————————————————————'

WRITE(6,100) U,SRD,STD,SZD,PH

10 CONTINUE

WRITE(6,*)'————————————————————————————'

WRITE(6,*)

WRITE(6,*)' CONSTANTS '

WRITE(6,*)

WRITE(6,*)'s11:',S11,'s12:',S12

WRITE(6,*)'s13:',S13,'s22:',S22

WRITE(6,*)'s23:',S23,'s33:',S33

WRITE(6,*)'c11:',C11,'c12:',C12

WRITE(6,*)'c13:',C13,'c22:',C22

WRITE(6,*)'c23:',C23,'c33:',C33

WRITE(6,*)'C1: ',K1, 'C2:',K2

WRITE(6,*)'External Pressure(MPa):',PEX

WRITE(6,*)'Dimensionless Pex:',PEX2

WRITE(6,*)'Dimensionless Fz:',FZ

PAUSE

100 FORMAT (3X,5(F11.6,3X))

STOP

END PROGRAM

A.3   Both Internally and Externally Pressurized Cylinders

The given program solves both internally and externally pressurized cylinders made of orthotropic and isotropic materials at the elastic limits. According to R1, R2 and Rh values, transformation from orthotropy to isotropy is valid. To be able to use the given program, one should assign the below inputs.

INPUTS:

Er: Elasticity Modulus in radial direction. It should be inserted in MPa unit.

Vrt, Vtz, Vrz: Poisson ratios

R1, R2, Rh : Orthotropy parameters

N: Ratio between internal and external pressure N=Pex/Pin

A, B: Inner and outer radius of the cylinders. These values should be inserted in meters.

SYR: Radial yield limit

OUTPUTS:

cij: Compliance matrix terms

sij: Stiffness matrix terms

C1 and C2: Arbitrary constants

Pin: Dimensionless elastic limit internal pressure

Pex: Dimensionless elastic limit external pressure

Fz: Dimensionless axial stress

U: Dimensionless radial displacement

SR, ST, SZ, Hill: Radial, tangential, axial and Hill's stress distributions

PROGRAM COM

IMPLICIT NONE

DOUBLE PRECISION::E1,V12,V13,V23,R1,R2,R3,SYR,PEX,SRD,STD,SZD,N

DOUBLE PRECISION::S11,S12,S13,S22,S23,S33,DEN,C11,C12,C13,C22,C23,C33

DOUBLE PRECISION::F,G,H,P,CR,CT,CZ,PRE,K1,K2,U,SR,ST,SZ,PH,PRE2,FZ

REAL::R,A,B

WRITE(6,*)' Please enter Er(MPa),Vrt,Vrz,Vtz,R1,R2,N'

READ(5,*) E1, V12,V13,V23,R1,R2,N

WRITE(6,*)'Please insert inner(meters) and outer radius(meters) of the cylinder'

READ(5,*) A,B

WRITE(6,*)'Please enter radial yielding strength(MPa) and Rh'

READ(5,*) SYR,R3

DEN= R1**2 * V12**2 + R2 * V23**2 + R1 * (-1 + R2* V13 * (V13 + 2*V12 * V23))

S11=E1*(-R1+R2*V23 **2)/DEN

S12=-E1*R1*(R1*V12+R2*V13*V23)/DEN

S13=-E1*R1*R2*(V13+V12*V23)/DEN

S22=E1*R1**2 * (-1+R2*V13**2) / DEN

S23=-E1*R1*R2*(R1*V13*V12+V23)/DEN

S33=E1*R1*R2*(-1+R1*V12**2)/DEN

C11=1/E1

C12=-V12/E1

C13=-V13/E1

C22=1/(R1*E1)

C23=-V23/(R1*E1)

C33=1/(R2*E1)

P=SQRT(S22/S11)

CR=-1.0

CT=(A*B**(2*P)*(P*S11+S12)*(P*S12-S22)-A**(1+2*P)*(P*S11-S12)

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*(P*S12+S22)-2*A**P*B**(1+P)*N*P*(S12**2-S11*S22))

/(A*(A**(2*P)-B**(2*P))*(P**2*S11**2-S12**2))

CZ=-(C13/C33)*CR - (C23/C33)*CT

PRE=(Sqrt(1 + R3)*SYR)/Sqrt(CR**2 + CT**2 - 2*CR*CZ - 2*CT*CZ + 2*CZ**2 + CR**2*R3 - 2*CR*CT*R3 + CT**2*R3)

PRE2=PRE/SYR

PEX=-N*PRE2

K1=(A**P*B**P*(-(A*B**P) + A**P*B*N)*PRE)/((A**(2*P) - B**(2*P))*(P*S11 - S12))

K2=-(((A**(1 + P) - B**(1 + P)*N)*PRE)/((A**(2*P) - B**(2*P))*(P*S11 + S12)))

IF (R1 .EQ. 1.0) THEN

FZ=(3.141592* (-(A-B)*(A+B)*K2*(S13+S23) +2*K1*(S13-S23)

*(Log(A) - Log(B))))/(SYR*B**2)

ELSE

FZ=((2*(-A**(1-P)+B**(1-P))*K1*3.141592*(P*S13-S23))/(-1+P) +(2*(-A**(1+P)

+B**(1+P))*K2*3.141592*(P*S13+S23))/(1+P))/(SYR*B**2)

END IF

DO 10 R=A,B,0.0025

U=(K1*R**(-P) + K2*R**(P)) *(E1/(SYR*B))

SR=(K1 * R**(-1-P) * (-P*S11+S12) + K2 * R**(-1+P) * (P*S11+S12))

SRD=SR/SYR

ST=(K1 * R**(-1-P) * (-P*S12+S22) + K2 * R**(-1+P) * (P*S12+S22))

```fortran
      STD=ST/SYR

      SZ=(K1 * R**(-1-P) * (-P*S13+S23) + K2 * R**(-1+P) * (P*S13+S23))

      SZD=SZ/SYR

      PH=((ST-SZ)**2 + (SR-SZ)**2 + R3*(SR-ST)**2)/((R3 +1)*SYR**2)

      WRITE(6,*)' U SR ST SZ Hill'

      WRITE(6,*)'————————————————————————————'

      WRITE(6,100) U,SRD,STD,SZD,PH

10    CONTINUE

      WRITE(6,*)'————————————————————————————'

      WRITE(6,*)

      WRITE(6,*)' CONSTANTS '

      WRITE(6,*)

      WRITE(6,*)'s11:',S11, 'S12:',S12

      WRITE(6,*)'s13:',S13, 'S22:',S22

      WRITE(6,*)'s23:',S23, 'S33:',S33

      WRITE(6,*)'c11:',C11, 'C12:',C12

      WRITE(6,*)'c13:',C13, 'C22:',C22

      WRITE(6,*)'c23:',C23, 'C33:',C33

      WRITE(6,*)'C1: ',K1, 'K2: ',K2

      WRITE(6,*)'Dimensionless Pin:',PRE2

      WRITE(6,*)'Dimensionless Pex:',PEX

      WRITE(6,*)'Dimensionless Fz:',FZ
```

PAUSE

100 FORMAT (3X,5(F11.6,3X))

STOP

END PROGRAM

A.4 Annularly Rotating Cylinders

This fortran program solves annularly rotating cylinders made of orthotropic and isotropic materials at the elastic limits. If R1 and R2 are taken as R1=1, R2=1 and Rh=1 then program solves isotropic structures. Other values give solutions to orthotropic structures.

Program Inputs:

Er: Elasticity Modulus in radial direction. It should be inserted in MPa unit.

Vrt, Vtz, Vrz: Poisson ratios

R1, R2, Rh: Orthotropy parameters

Density: Should be written in kg/m^3 unit.

A, B: Inner and outer radius of the cylinders. These values should be inserted in meters.

SYR: Radial yield limit(MPa)

Program Outputs:

cij: Compliance matrix terms

sij: Stiffness matrix terms

C1 and C2: Arbitrary constants

Omega: Dimensionless elastic limit annular speed

Fz: Dimensionless axial stress

U: Dimensionless radial displacement

SR, ST, SZ, Hill: Radial, tangential, axial and Hill's stress distributions

```
PROGRAM ROT

IMPLICIT NONE

DOUBLE PRECISION::E1,V12,V13,V23,R1,R2,R3,SYR,DE,SRD,STD

DOUBLE PRECISION::S11,S12,S13,S22,S23,S33,DEN,C11,C12,C13,C22

DOUBLE PRECISION::F,G,H,P,CR,CT,CZ,OM,K1,K2,K3,U,SR,ST,SZ

DOUBLE PRECISION::OM2,PH,C23,C33,SZD,FZ

REAL::R,A,B

WRITE(6,*)' Please enter E1(MPa),V12,V13,V23,R1,R2,Density(kg/m^3)'

READ(5,*) E1, V12,V13,V23,R1,R2,DE

WRITE(6,*)'Please insert inner(meters) and outer radius(meters) of the cylin-
der'

READ(5,*) A,B

WRITE(6,*)'Please enter radial yielding strength(MPa) and Rh'

READ(5,*) SYR,R3

DEN= R1**2 * V12**2 + R2 * V23**2 + R1 * (-1 + R2* V13 * (V13 + 2*V12
* V23))

S11=E1*(-R1+R2*V23 **2)/DEN

S12=-E1*R1*(R1*V12+R2*V13*V23)/DEN

S13=-E1*R1*R2*(V13+V12*V23)/DEN

S22=E1*R1**2*(-1+R2*V13**2)/DEN

S23=-E1*R1*R2*(R1*V13*V12+V23)/DEN
```

S33=E1*R1*R2*(-1+R1*V12**2)/DEN

C11=1/E1

C12=-V12/E1

C13=-V13/E1

C22=1/(R1*E1)

C23=-V23/(R1*E1)

C33=1/(R2*E1)

P=SQRT(S22/S11)

CR=(DE/(S22-9*S11))*(A**(3+2*P) *(-3+P) *(P*S11-S12) +2*A**P *B**(3+P) *P *(3*S11+S12) -A**3 *B**(2*P) *(3+P) *(P*S11+S12)) *(-S12**2 +S11*S22)

CT=CR / (A *(A**(2*P) - B**(2*P)) *(P*S11 -S12) *(P*S11+S12))

CZ=(-C23/C33)*CT

OM=(SYR**2+R3*SYR**2)**0.25/(CT**2-2*CT*CZ+2*CZ**2+CT**2*R3)**0.25

OM2=OM*B*SQRT(DE/SYR)

K3=(DE * OM**2) / (S22 - 9*S11)

K1=((A**(2*P) *B**(3+P) - A**(3+P) *B**(2*P)) *(3*S11 + S12) *K3) / ((A**(2*P) - B**(2*P)) *(-S12+S11*P))

K2=-(((A**(3+P) - B**(3+P)) *(3*S11+S12) *K3) / ((A**(2*P) -B**(2*P)) *(S12+S11*P)))

IF (R1 .EQ. 1.0) THEN

FZ=((A**4 *C13*3.141592 *DE*OM**2)/(2.*C33)-(B**4*C13*3.141592*DE*OM**2)

/(2.*C33))/(SYR*B**2)

ELSE

```
FZ=((3.141592*(-(A*((4*K1*(P*S13 -S23))/(A**P*(-1+P))

+A**3*K3*(3*S13+S23)+(4*A**P*K2*(P*S13+S23))/(1+P)))

+B*((4*K1*(P*S13-S23))/(B**P*(-1+P))

+B**3*K3*(3*S13+S23)+(4*B**P*K2*(P*S13+S23))

/(1+P))))/2.)/(SYR*B**2)

END IF

DO 10 R= A,B,0.0025

U=(K1*R**(-P) + K2*R**(P) + K3* R**(3))*E1/(SYR*B)

SR=(K1 * R**(-1-P) * (-P*S11+S12) + K2 * R**(-1+P) * (P*S11+S12) +
K3* R**2 * (3*S11+S12))

SRD=SR/SYR

ST=(K1 * R**(-1-P) * (-P*S12+S22) + K2 * R**(-1+P) * (P*S12+S22) +
K3* R**2 * (3*S12+S22))

STD=ST/SYR

SZ=(K1 * R**(-1-P) * (-P*S13+S23) + K2 * R**(-1+P) * (P*S13+S23) + K3*
R**2 * (3*S13+S23))

SZD=SZ/SYR

PH=((ST-SZ)**2 + (SR-SZ)**2 + R3*(SR-ST)**2)/((R3 +1)*SYR**2)

WRITE(6,*)' U SR ST SZ Hill'

WRITE(6,*)'————————————————————'

WRITE(6,100) U,SRD,STD,SZD,PH

10 CONTINUE

WRITE(6,*)'————————————————————'
```

WRITE(6,*)

WRITE(6,*)' CONSTANTS '

WRITE(6,*)

WRITE(6,*)'s11:',S11,'s12:',S12

WRITE(6,*)'s13:',S13,'s22:',S22

WRITE(6,*)'s23:',S23,'s33:',S33

WRITE(6,*)'c11:',C11,'c12:',C12

WRITE(6,*)'c13:',C13,'c22:',C22

WRITE(6,*)'c23:',C23,'c33:',C33

WRITE(6,*)'C1: ',K1, 'C2:',K2

WRITE(6,*)'Omega:',OM2

WRITE(6,*)'Dimensionless axial force:',FZ

PAUSE

100 FORMAT (3X,5(F11.6,3X))

STOP

END PROGRAM

A.5 Combination of Annular Rotation and Internal Pressure

At the elastic limits, computation program gives results to annularly rotating and internally pressurized cylinders. After inputs are declared, program gives the following outputs.

INPUTS:

Er: Elasticity Modulus in radial direction. It should be inserted in MPa unit.

Vrt, Vtz, Vrz: Poisson ratios

R1, R2, Rh: Orthotropy parameters

Density: Should be written in kg/m^3 unit.

A, B: Inner and outer radius of the cylinders. These values should be inserted in meters.

SYR: Radial yield limit(MPa)

OUTPUTS:

C1 and C2: Arbitrary constants

Omega: Dimensionless elastic limit annular speed

U: Dimensionless radial displacement

SR, ST, SZ, Hill: Dimensionless radial, tangential, axial and Hill's stress distributions

PROGRAM ROT

IMPLICIT NONE

DOUBLE PRECISION::E1,V12,V13,V23,R1,R2,R3,SYR,DE,SRD,STD,,C33

DOUBLE PRECISION::S11,S12,S13,S22,S23,S33,DEN,C11,C12,C13,C22,C23

DOUBLE PRECISION::F,P,CR,CT,CZ,OM,K1,K2,K3,U,SR,ST,SZ,OM2,PH,FZ

DOUBLE PRECISION::SZD,PIN,SRN,STN,SZN,PHN,SRND,STND,SZND

REAL::R,A,B,T

WRITE(6,*)'Please enter E1(MPa),V12,V13,V23,R1,R2,Density(kg/m^3)'

READ(5,*) E1, V12,V13,V23,R1,R2,DE

WRITE(6,*)'Please insert inner(meters) and outer radius(meters) of the cylinder'

```
READ(5,*) A,B

WRITE(6,*)',*)'Please enter radial yielding strength(MPa) and Rh'

READ(5,*) SYR,R3

DEN= R1**2 * V12**2 + R2 * V23**2 + R1 * (-1 + R2* V13 * (V13 + 2*V12
* V23))

S11=E1*(-R1+R2*V23 **2)/DEN

S12=-E1*R1*(R1*V12+R2*V13*V23)/DEN

S13=-E1*R1*R2*(V13+V12*V23)/DEN

S22=E1*R1**2*(-1+R2*V13**2)/DEN

S23=-E1*R1*R2*(R1*V13*V12+V23)/DEN

S33=E1*R1*R2*(-1+R1*V12**2)/DEN

C11=1/E1

C12=-V12/E1

C13=-V13/E1

C22=1/(R1*E1)

C23=-V23/(R1*E1)

C33=1/(R2*E1)

P=SQRT(S22/S11)

PIN=50

DO 10 OM= 40, 50, 0.00001

K3=(DE * OM**2) / (S22 - 9*S11)

K1=(A**P*B**P*(A**P*B**3*K3*(3*S11+S12)-A*B**P
```

$*$(PIN $+$A$**$2$*$K3$*$(3$*$S11$+$S12))))/((A$**$(2$*$P)-B$**$(2$*$P))$*$(P$*$S11-S12))

K2$=$(B$**$(3$+$P)$*$K3$*$(3$*$S11$+$S12)-A$**$(1$+$P)$*$(PIN$+$A$**$2$*$K3$*$(3$*$S11$+$S12)))

/((A$**$(2$*$P) -B$**$(2$*$P))$*$(P$*$S11$+$S12))

SR$=$(K1$*$A$**$(-1-P)$*$(-P$*$S11$+$S12)$+$K2$*$A$**$(-1$+$P) $*$(P$*$S11$+$S12)

$+$K3$*$A$**$2$*$(3$*$S11$+$S12))

ST$=$(K1$*$A$**$(-1-P)$*$(-P$*$S12$+$S22)$+$K2$*$A$**$(-1$+$P)$*$(P$*$S12$+$S22)

$+$K3$*$A$**$2$*$(3$*$S12$+$S22))

SZ$=$(K1$*$A$**$(-1-P)$*$(-P$*$S13$+$S23)$+$K2$*$A$**$(-1$+$P)$*$(P$*$S13$+$S23)

$+$K3$*$A$**$2$*$(3$*$S13$+$S23))

PH$=$((ST-SZ)$**$2 $+$ (SR-SZ)$**$2 $+$ R3$*$(SR-ST)$**$2)/((R3 $+$1)$*$SYR$**$2)

OM2$=$OM$*$B$*$SQRT(DE/SYR)

IF (PH .GE. 1.0) EXIT

10 END DO

GO TO 20

20 DO 30 T$=$A,B ,0.0025

U$=$(K1$*$T$**$(-P)$+$K2$*$T$**$P $+$K3$*$T$**$3)$*$(E1/(B$*$SYR))

SRN$=$(K1$*$T$**$(-1-P)$*$(-P$*$S11$+$S12)$+$K2$*$T$**$(-1$+$P)$*$(P$*$S11$+$S12)

$+$K3$*$T$**$2$*$(3$*$S11$+$S12))

SRND$=$SRN/SYR

STN$=$(K1$*$T$**$(-1-P)$*$(-P$*$S12$+$S22)$+$K2$*$T$**$(-1$+$P)$*$(P$*$S12$+$S22)

$+$K3$*$T$**$2$*$(3$*$S12$+$S22))

STND$=$STN/SYR

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SZN=(K1*T**(-1-P)*(-P*S13+S23)+K2*T**(-1+P)*(P*S13+S23)

+K3*T**2*(3*S13+S23))

SZND=SZN/SYR

PHN=((STN-SZN)**2+(SRN-SZN)**2+R3*(SRN-STN)**2)/((R3 +1)*SYR**2)

WRITE(6,*)' U SR ST SZ HILL '

WRITE(6,*)'————————————————————————————————————'

WRITE(6,200) U,SRND,STND,SZND,PHN

30 CONTINUE

WRITE(6,*)'C1:',K1, 'C2:',K2, 'OMEGA:',OM2

200 FORMAT (4X,5(F11.6,4X))

PAUSE

STOP

END PROGRAM

A.6 Combination of Annular Rotation and External Pressure

At the elastic limits, the given fortran program responds to annularly rotating
and externally pressurized cylinders. After inputs assigned, following outputs
are displayed.

INPUTS:

Er: Elasticity Modulus in radial direction. It should be inserted in MPa unit.

Vrt, Vtz, Vrz: Poisson ratios

R1, R2, Rh: Orthotropy parameters

Density: Should be written in kg/m^3 unit.

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A, B: Inner and outer radius of the cylinders. These values should be inserted in meters.

SYR:Radial yield limit

OUTPUTS:

C1 and C2: Arbitrary constants

Omega: Dimensionless elastic limit annular speed

U: Dimensionless radial displacement

SR, ST, SZ, Hill: Dimensionless radial, tangential, axial and Hill's stress distributions

PROGRAM ROT

IMPLICIT NONE

DOUBLE PRECISION::E1,V12,V13,V23,R1,R2,R3,SYR,DE,SRD,STD,SZD

DOUBLE PRECISION::S11,S12,S13,S22,S23,S33,DEN,C11,C12,C13,SZND

DOUBLE PRECISION::P,CR,CT,CZ,OM,K1,K2,K3,U,SR,ST,SZ,OM2,PH,FZ

DOUBLE PRECISION::PEX,SRN,STN,SZN,C22,C23,C33,PHN,SRND,STND

REAL::R,A,B,T

WRITE(6,*)' Please enter E1(MPa),V12,V13,V23,R1,R2,Density(kg/m^3)'

READ(5,*) E1, V12,V13,V23,R1,R2,DE

WRITE(6,*)'Please insert inner(meters) and outer radius(meters) of the cylinder'

READ(5,*) A,B

WRITE(6,*)'Please enter yielding strength for r, tetha and z directions in Pascal Unit'

```
READ(5,*) SYR,R3

DEN= R1**2 * V12**2 + R2 * V23**2 + R1 * (-1 + R2* V13 * (V13 + 2*V12
* V23))

S11=E1*(-R1+R2*V23 **2)/DEN

S12=-E1*R1*(R1*V12+R2*V13*V23)/DEN

S13=-E1*R1*R2*(V13+V12*V23)/DEN

S22=E1*R1**2*(-1+R2*V13**2)/DEN

S23=-E1*R1*R2*(R1*V13*V12+V23)/DEN

S33=E1*R1*R2*(-1+R1*V12**2)/DEN

C11=1/E1

C12=-V12/E1

C13=-V13/E1

C22=1/(R1*E1)

C23=-V23/(R1*E1)

C33=1/(R2*E1)

P=SQRT(S22/S11)

PEX=50

DO 10 OM= 70, 80, 0.00001

K3=(DE * OM**2) / (S22 - 9*S11)

K1=(A**P*B**P*(-(A**3*B**P*K3*(3*S11+S12))+A**P*B*(PEX + B**2*K3*(3*S11
+ S12))))/((A**(2*P)-B**(2*P))*(P*S11-S12))

K2=(-(A**(3 + P)*K3*(3*S11 + S12)) + B**(1 + P)*(PEX + B**2*K3*(3*S11
+ S12)))/((A**(2*P) - B**(2*P))*(P*S11 + S12))
```

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SR=(K1 * A**(-1-P) * (-P*S11+S12) + K2 * A**(-1+P) * (P*S11+S12) + K3* A**2 * (3*S11+S12))

ST=(K1 * A**(-1-P) * (-P*S12+S22) + K2 * A**(-1+P) * (P*S12+S22) + K3* A**2 * (3*S12+S22))

SZ=(K1 * A**(-1-P) * (-P*S13+S23) + K2 * A**(-1+P) * (P*S13+S23) + K3* A**2 * (3*S13+S23))

PH=((ST-SZ)**2 + (SR-SZ)**2 + R3*(SR-ST)**2)/((R3 +1)*SYR**2)

OM2=OM*B*SQRT(DE/SYR)

IF (PH .GT. 1.0) EXIT

10 END DO

GO TO 20

20 DO 30 T=A,B ,0.0025

U=(K1*T**(-P)+K2*T**P +K3*T**3)*(E1/(B*SYR))

SRN=(K1 * T**(-1-P) * (-P*S11+S12) + K2 * T**(-1+P) * (P*S11+S12) + K3* T**2 * (3*S11+S12))

SRND=SRN/SYR

STN=(K1 * T**(-1-P) * (-P*S12+S22) + K2 * T**(-1+P) * (P*S12+S22) + K3* T**2 * (3*S12+S22))

STND=STN/SYR

SZN=(K1 * T**(-1-P) * (-P*S13+S23) + K2 * T**(-1+P) * (P*S13+S23) + K3* T**2 * (3*S13+S23))

SZND=SZN/SYR

PHN=((STN-SZN)**2 +(SRN-SZN)**2 +R3*(SRN-STN)**2)/((R3 +1)*SYR**2)

WRITE(6,*)' U SR ST SZ HILL '

WRITE(6,*)'————————————————————————'

WRITE(6,200) U,SRND,STND,SZND,PHN

30 CONTINUE

WRITE(6,*)'C1:',K1, 'C2:',K2, 'OMEGA:',OM2

200 FORMAT (4X,5(F11.6,4X))

PAUSE

STOP

END PROGRAM

A.7   Elastic Limit Internal Pressure Variation

This routine gives elastic limit internal pressure alteration with respect to R1 and R2. It asks several questions and give elastic limits as as output. If Do loop is set as R1=R1, program holds R2 constant and R1 varies. On the other hand, If Do loop is set as R2=R2, R1 stays constant and R2 varies.

PROGRAM INTERNAL

IMPLICIT NONE

DOUBLE PRECISION::E1,V12,V13,V23,SYR,R3

DOUBLE PRECISION::S11,S22,DEN,C13,C23,C33

DOUBLE PRECISION::F,G,H,P,CR,CT,CZ,PIN,A,B

REAL::R1,R2

WRITE(6,*)' Please enter Er(MPa),Vrt,Vrz,Vtz,R1,R2'

READ(5,*) E1, V12,V13,V23,R1,R2

WRITE(6,*)'Please insert inner(meters) and outer radius(meters) of the cylinder'

```fortran
READ(5,*) A,B

WRITE(6,*)'Please enter radial yield limit(MPa) and Rh'

READ(5,*) SYR,R3

CR=-1

DO 10 R3=R3 , 2 , 0.1

DEN= R1**2 * V12**2 +R2 *V23**2 + R1 *(-1 + R2*V13 * (V13 + 2*V12
*V23))

S11=E1*(-R1+R2*V23 **2)/DEN

S22=E1*R1**2*(-1+R2*V13**2)/DEN

C13=-V13/E1

C23=-V23/(R1*E1)

C33=1/(R2*E1)

P=SQRT((R1**2 * (-1+ R2 * V13**2)) / (-R1 + R2 * V23**2))

CT=-((A**(2*P)+B**(2*P))*P) / (A**(2*P)-B**(2*P))

CZ=-(C13/C33)*CR - (C23/C33)*CT

PIN=(Sqrt(1 + R3)*SYR)/Sqrt(CR**2 + CT**2 - 2*CR*CZ - 2*CT*CZ +
2*CZ**2 + CR**2*R3 - 2*CR*CT*R3 + CT**2*R3)/SYR

WRITE(6,100),PIN

10 CONTINUE

100 FORMAT (3X,5(F11.8,3X))

PAUSE

STOP

END PROGRAM
```

A.8   Elastic Limit External Pressure Variation

The given fortran code displays the external pressure alteration according to R1 and R2 which are the orthotropy parameters. If DO loop is set as R1=R1, program holds R2 constant and R1 varies. On the other hand, If DO loop is set as R2=R2, R1 stays constant and R2 varies. The variation of the limits can easily be altered by changing the range of the DO loop.

```
PROGRAM EXTERNAL1

IMPLICIT NONE

DOUBLE PRECISION::E1,V12,V13,V23,SYR,R3

DOUBLE PRECISION::S11,S22,DEN,C13,C23,C33

DOUBLE PRECISION::P,CR,CT,CZ,PEX,A,B

REAL::R1,R2

WRITE(6,*)' Please enter E1(MPa),V12,V13,V23,R1,R2'

READ(5,*) E1, V12,V13,V23,R1,R2

WRITE(6,*)'Please insert inner(meters) and outer radius(meters) of the cylin-
der'

READ(5,*) A,B

WRITE(6,*)'Please enter radial yield limit(MPa) and Rh'

READ(5,*) SYR,R3

CR=0

DO 10 R2=R2 , 2, 0.1

DEN= R1**2 * V12**2 + R2 * V23**2 + R1 * (-1 + R2* V13 * (V13 + 2*V12
* V23))

S11=E1*(-R1+R2*V23 **2)/DEN
```

S22=E1*R1**2*(-1+R2*V13**2)/DEN

C13=-V13/E1

C23=-V23/(R1*E1)

C33=1/(R2*E1)

P=SQRT((R1**2 * (-1+ R2 * V13**2)) / (-R1 + R2 * V23**2))

CT=(2*A**(-1 + P) * B**(1 + P) *P) / (A**(2*P) - B**(2*P))

CZ=-(C13/C33)*CR - (C23/C33)*CT

PEX=(Sqrt(1 + R3)*SYR)/Sqrt(CR**2 + CT**2 - 2*CR*CZ - 2*CT*CZ + 2*CZ**2 + CR**2*R3 - 2*CR*CT*R3 + CT**2*R3)/SYR

WRITE(6,100),PEX

10 CONTINUE

100 FORMAT (3X,4(E15.6,3X))

PAUSE

STOP

END PROGRAM

A.9 Elastic Limit Combined Pressure Variation

The below fortran code shows the elastic limit alteration of both internally and externally pressured cylinders. If DO loop is set as R1=R1, program holds R2 constant and R1 varies. On the other hand, If DO loop is set as R2=R2, R1 stays constant and R2 varies. The variation of the limits can easily be altered by changing the range of the DO loop.

PROGRAM PRE2

IMPLICIT NONE

```
DOUBLE PRECISION::E1,V12,V13,V23,SYR,N

DOUBLE PRECISION::S11,S12,S22,DEN,C13,C23,C33

DOUBLE PRECISION::P,CR,CT,CZ,PRE,A,B,SM,R3

REAL::R1,R2

WRITE(6,*)' Please enter E1(MPa),V12,V13,V23,R1,R2,N'

READ(5,*) E1, V12,V13,V23,R1,R2,N

WRITE(6,*)'Please insert inner(meters) and outer radius(meters) of the cylin-
der'

READ(5,*) A,B

WRITE(6,*)'Please enter radial yield limit(MPa) and Rh'

READ(5,*) SYR,R3

CR=-1

DO 10 R2=R2 , 2 , 0.1

DEN= R1**2 * V12**2 + R2 * V23**2 + R1 * (-1 + R2* V13 * (V13 + 2*V12
* V23))

S11=E1*(-R1+R2*V23 **2)/DEN

S12=-E1*R1*(R1*V12+R2*V13*V23)/DEN

S22=E1*R1**2*(-1+R2*V13**2)/DEN

C13=-V13/E1

C23=-V23/(R1*E1)

C33=1/(R2*E1)

P=SQRT((R1**2 * (-1+ R2 * V13**2)) / (-R1 + R2 * V23**2))

CT=(A*B**(2*P))*(P*S11 + S12)*(P*S12-S22)
```

-A**(1+2*P)*(P*S11-S12)*(P*S12+S22)

-2*A**P*B**(1+P)*N*P*(S12**2-S11*S22))

/(A*(A**(2*P)-B**(2*P))*(P**2*S11**2-S12**2))

CZ=-(C13/C33)*CR - (C23/C33)*CT

PRE=(Sqrt(1 + R3)*SYR)/Sqrt(CR**2 + CT**2 - 2*CR*CZ - 2*CT*CZ + 2*CZ**2 + CR**2*R3 - 2*CR*CT*R3 + CT**2*R3)/SYR

WRITE(6,100),PRE

10 CONTINUE

100 FORMAT (3X,4(E15.7,3X))

PAUSE

STOP

END PROGRAM

A.10   Elastic Limit Annular Rotation Variation

Elastic limit annular rotation can be investigated with the below code. Elastic limit alteration with respect orthotropy parameters is found with the following. If DO loop is set as R1=R1, program holds R2 constant and R1 varies. On the other hand, If DO loop is set as R2=R2, R1 stays constant and R2 varies. The variation of the limits can easily be altered by changing the range of the DO loop.

PROGRAM OMEGA1

IMPLICIT NONE

DOUBLE PRECISION::E1,V12,V13,V23,SYR,OM2

DOUBLE PRECISION::S11,S12,S22,DEN,C13,C23,C33

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```fortran
DOUBLE PRECISION::P,CR,CT,CZ,OM,A,B,DE,R3

REAL::R1,R2

WRITE(6,*)' Please enter E1(MPa),V12,V13,V23,R1,R2,Dencity(kg/m^3)'

READ(5,*) E1, V12,V13,V23,R1,R2,DE

WRITE(6,*)'Please insert inner(meters) and outer radius(meters) of the cylinder'

READ(5,*) A,B

WRITE(6,*)'Please enter radial yield limit(MPa) and Rh'

READ(5,*) SYR,R3

DO 10 R1=R1 , 2 , 0.1

DEN= R1**2 * V12**2 + R2 * V23**2 + R1 * (-1 + R2* V13 * (V13 + 2*V12 * V23))

S11=E1*(-R1+R2*V23 **2)/DEN

S12=-E1*R1*(R1*V12+R2*V13*V23)/DEN

S22=E1*R1**2*(-1+R2*V13**2)/DEN

C13=-V13/E1

C23=-V23/(R1*E1)

C33=1/(R2*E1)

P=SQRT((R1**2 * (-1+ R2 * V13**2)) / (-R1 + R2 * V23**2))

CR=(DE/(S22-9*S11))*(A**(3+2*P) *(-3+P) *(P*S11-S12) +2*A**P *B**(3+P)
*P *(3*S11+S12) -A**3 *B**(2*P) *(3+P) *(P*S11+S12)) *(-S12**2 +S11*S22)

CT=CR / (A *(A**(2*P) - B**(2*P)) *(P*S11 -S12) *(P*S11+S12))

CZ=(-C23/C33)*CT
```

```
OM=(SYR**2 + R3*SYR**2)**0.25

/(CT**2 - 2*CT*CZ + 2*CZ**2 + CT**2*R3)**0.25

OM2=OM*B*SQRT(DE/SYR)

WRITE(6,100),OM2

10 CONTINUE

100 FORMAT (3X,4(E15.7,3X))

PAUSE

STOP

END PROGRAM
```