AN APPLICATION OF THE BLACK LITTERMAN MODEL IN BORSA ISTANBUL USING ANALYSTS' FORECASTS AS VIEWS

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ABSTRACT

AN APPLICATION OF THE BLACK LITTERMAN MODEL IN BORSA ISTANBUL USING ANALYSTS' FORECASTS AS VIEWS

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The optimal number of stocks to include in a portfolio in order to achieve the maximum diversification benefit has been one of the issues in which investors have focused on since Markowitz introduced fundamentals of the Modern Portfolio Theory. Each stock included in an investor's portfolio decreases the portfolio risk, while increasing the transaction costs incurred by the investor to create this portfolio. In this thesis, the size of a well-diversified portfolio consisting of stocks included consistently in the Borsa İstanbul-50 (BIST-50) Index during every calendar year from 2005 to 2015 are determined. Due to the differences in the number of stocks consistently included in the BIST-50 index and the correlation between these stocks from one year to another, the range for the optimal number of stocks varies between 8-10 to 16-18 stocks for the years in the sample period analyzed. The results in this thesis indicate that the average size of the portfolio for the years examined in this thesis is between 11 and 13 stocks. The same analysis is repeated by using posterior variance-covariance matrix derived from Black-Litterman (B-L) portfolio optimization model, which is another subject of this thesis. When the results derived from both the prior and the posterior variance-covariance matrices are compared, no remarkable differences are observed.

Another issue that investors have been interested in is the allocation of funds across stocks included in a portfolio to earn the maximum return. Although Markowitz made a significant contribution to portfolio optimization in theory, he was criticized in practice for his model's high sensitivity to inputs and disregard to investors' views. To resolve some of the problems with Markowitz Model, B-L, developed a model which uses the market returns derived from the Capital Asset Pricing Model (CAPM) as its first estimate, and updates this first estimate with investors' views. In this thesis, market returns of each stock included consistently in the BIST-50 Index during the whole one year are combined with average return expectations of Bloomberg financial analysts for the corresponding stock to incorporate investor views. It is observed that the weights of stocks on which Bloomberg analysts did not state any opinion do not change that much from their market capitalization weights. It is observed that the posterior weights of some stocks on which Bloomberg analysts specified views do not change in the same direction as the views expressed on them. For example, it is possible to observe a negative change in the weight of a stock when the analysts express a positive view on this stock or vice versa. Possible reasons for these counterintuitive changes in the weights of stocks are the covariance structure of the stocks and the way the analyst views are defined. These explanations are shown to be instrumental by using an example of a portfolio with 3 stocks. Furthermore, first the budget constraint and then the short selling constraint in addition to the budget constraint are imposed on B-L portfolio optimization, and the results are analyzed. Finally, optimal B-L portfolios obtained by incorporating average views of Bloomberg analysts and the portfolios constructed from the CAPM are compared in terms of the Sharpe ratio and efficient frontier. As a result of these comparisons, it is seen that under certain conditions the portfolios based on the B-L Model perform better than the portfolios based on the CAPM. However, under some other conditions the portfolios based on the CAPM perform better than the portfolios based on the B-L Model.

Keywords: Portfolio diversification, Black-Litterman model, BIST-50

ÖZ

YATIRIMCI GÖRÜŞÜ OLARAK ANALİST TAHMİNLERİNİ KULLANAN BLACK LITTERMAN MODELİNİN BORSA İSTANBUL ÜZERİNE BİR UYGULAMASI

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Markowitz'in Modern Portföy Kuramı'nın temellerini ortaya koymasından itibaren, yatırımcıların odaklandığı konulardan bir tanesi portföy çeşitlendirmesinden en iyi verimi alabilmek için portföye dâhil edilmesi gereken en uygun hisse senedi sayısını belirlemek olmuştur. Portföye eklenen her bir hisse senedi portföyün riskinde azalış sağlarken yatırımcının ödemesi gereken toplam işlem maliyetinde artışa sebep olmaktadır. Bu yüksek lisans tezinde, 2005'ten 2015'e kadar olan sürede her 1 yıl boyunca BIST-50 endeksinde sürekli olarak yer alan hisse senetlerinden oluşturulan iyi çeşitlendirilmiş bir portföyün büyüklüğünün ne olması gerektiği belirlenmeye çalışılmaktadır. Araştırma döneminde yer alan her bir yıllık dönemde incelemeye dâhil edilen hisse senedi sayısı ve hisse senetlerinin birbirleriyle olan korelasyonu değiştiği için iyi çeşitlendirilmiş bir portföyde yer alması gereken en uygun hisse senedi sayısının 8-10 ile 16-18 arasında değişkenlik gösterdiği görülmektedir. Gözlemlenen tüm yıllar için ortalama en uygun hisse senedi sayısı 11 ila 13 hisse senedi arasında seyretmektedir. Ayrıca, bu tezin bir diğer konusu olan B-L portföy optimizasyonundan elde edilen varyans-kovaryans matrisi kullanılarak aynı çalışma tekrarlanmakta ve elde edilen sonuçlar ilk çalışmanın bulgularıyla karşılaştırılmaktadır. Bu iki uygulamanın bulguları arasında dikkate değer bir fark gözlenmemektedir.

Yatırımcıların ilgilendiği konulardan bir diğeri de en yüksek getiriyi elde edebilmek için bir potföyde yer alan her bir hisse senedine ne kadar yatırım yapılması gerektiğinin belirlenmesidir. Markowitz, portföy optimizasyonu konusuna teorik olarak önemli katkılar sağlamış olmakla birlikte, geliştirmiş olduğu modelin yüksek girdi hassasiyeti ve yatırımcı görüşlerinin dâhil edilemesine imkân tanımaması nedeniyle pratikte eleştirilere maruz kalmıştır. Markowitz modelinin sorunlarını çözmek amacıyla B-L, hisse senedi getirilerini hesaplamak için ilk tahmin olarak Finansal Varlık Fiyatlandırma Modeli'nden (FVFM) elde edilen piyasa getirilerini kullanan ve çıkan sonuçları yatırımcının görüşleriyle güncelleyen bir model geliştirmişlerdir. Bu çalışmada, güncellenmiş getirileri elde etmek için 1 yıl boyunca BIST-50 endeksinde sürekli olarak işlem gören her bir hisse senedinin piyasa getirisi ile Bloomberg finans analistlerinin aynı hisse senedi için ortalama getiri beklentisi birleştirilmektedir. Bloomberg finans analistlerinin görüş vermediği hisse senetlerinin portföy içindeki ağırlıklarının hisse senetlerinin piyasa ağırlıklarına çok yakın kaldığı gözlemlenmektedir. Bloomberg finans analistlerinin görüş verdiği bazı hisse senetlerinin ağırlıklarının ise analistlerin verdiği görüşlerle uyumlu olarak değişmediği durumlar olmaktadır. Örneğin finansal analistlerin olumlu görüş verdiği bir hisse senedinin portföy içindeki ağırlığı düşebilmektedir. Sezgilere aykırı bu bulguların olası nedenlerinin hisse senetleri arasındaki kovaryans yapısı ve görüşlerin tanımlanma şekli olabileceği 3 hisse senedinden oluşan bir portföy örnek gösterilerek açıklanmaktadır. Bunlara ek olarak, B-L portföy optimizasyonuna ilk olarak bütçe kısıtı, daha sonra da bütçe kısıtına ilaveten açığa satış yapılmaması kısıtı getirilmekte ve çıkan sonuçlar analiz edilmektedir. Son olarak, Bloomberg analistlerinin ortalama görüşleri ile elde edilen B-L portföyleri FVFM'den elde edilen portföylerle Sharpe oranı ve risk-getiri eğrisi yaklaşımı kullanılarak karşılaştırılmaktadır. Bu karşılaştırmalar sonucunda B-L Modeli'ne dayanan portföyün bazı koşullarda FVFM' ye dayanan portföyden daha iyi performans gösterirken bazılarında da daha kötü performans gösterdiği görülmektedir.

Anahtar Kelimeler : Portföy çeşitliliği, Black-Litterman modeli, BIST-50

To My Beloved Parents

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AEFES	Anadolu Efes Biracılık ve Malt Sanayii A.Ş.
AKBNK	Akbank T.A.Ş.
AKCNS	Akçansa Çimento Sanayi ve Ticaret A.Ş.
AKENR	Akenerji Elektrik Üretim A.Ş.
AKGRT	Aksigorta A.Ş.
AKSA	Aksa Akrilik Kimya Sanayii A.Ş.
ALARK	Alarko Holding A.Ş.
ANSGR	Anadolu Anonim Türk Sigorta Şirketi
APT	Arbitrage pricing theory
AR(1)	Autoregressive Model of order 1
ARCLK	Arçelik A.Ş.
ASELS	Aselsan Elektronik Sanayi ve Ticaret A.Ş.
ASYAB	Asya Katılım Bankası A.Ş.
AYGAZ	Aygaz A.Ş.
BAGFS	Bağfaş Bandırma Gübre Fabrikaları A.Ş.
BEKO	Beko Elektronik A.Ş.
BIMAS	Bim Birleşik Mağazalar A.Ş.
BIST-30	Borsa İstanbul - 30
BIST-50	Borsa İstanbul - 50
BIST-100	Borsa İstanbul - 100
BJKAS	Beşiktaş Futbol Yatırımları Sanayi ve Ticaret A.Ş.
B-L	Black-Litterman
BRISA	Bridgestone Sabancı Lastik Sanayi ve Ticaret A.Ş.
CAL	Capital allocation line
CAPM	Capital Asset Pricing Model
CCOLA	Coca-Cola İçecek A.Ş.
CML	Capital market line
DEVA	Deva Holding A.Ş.
DISBA	Dışbank Türk Dış Ticaret Bankası A.Ş.
DOAS	Doğuş Otomotiv Servis ve Ticaret A.Ş.

DOHOL	Doğan Şirketler Grubu Holding A.Ş.
DYHOL	Doğan Yayın Holding A.Ş.
ECILC	Eczacıbaşı İlaç Sanayi ve Ticaret A.Ş.
EFES	Efes Holding A.Ş.
EGARCH	Exponential Generalized Autoregressive Conditional Heteroskedasticity
ENKAI	Enka İnşaat ve Sanayi A.Ş.
EREGL	Ereğli Demir ve Çelik Fabrikaları T.A.Ş.
FENER	Fenerbahçe Futbol A.Ş.
FINBN	Finansbank A.Ş.
FORTS	Fortis Bank A.Ş.
FROTO	Ford Otomotiv Sanayi A.Ş.
GARAN	Türkiye Garanti Bankası A.Ş.
GDP	Gross Domestic Product
GLYHO	Global Yatırım Holding A.Ş.
GOLTS	Göltaş Göller Bölgesi Çimento Sanayi ve Ticaret A.Ş.
GOZDE	Gözde Girişim Sermayesi Yatırım Ortaklığı A.Ş.
GSDHO	Gsd Holding A.Ş.
GSRAY	Galatasaray Sportif Sınai ve Ticari Yatırımlar A.Ş.
GUBRF	Gübre Fabrikaları T.A.Ş.
HALKB	Türkiye Halk Bankası A.Ş.
HURGZ	Hürriyet Gazetecilik ve Matbaacılık A.Ş.
IHLAS	İhlas Holding A.Ş.
IPEKE	İpek Doğal Enerji Üretim A.Ş.
ISCTR	Türkiye İş Bankası A.Ş.
ISGYO	İş Gayrimenkul Yatırım Ortaklığı A.Ş.
IZMDC	İzmir Demir Çelik Sanayi A.Ş.
KARTN	Kartonsan Karton Sanayi ve Ticaret A.Ş.
KCHOL	Koç Holding A.Ş.
KONYA	Konya Çimento Sanayii A.Ş.
KOZAA	Koza Anadolu Metal Madencilik İşletmeleri A.Ş.
KRDMD	Kardemir Karabük Demir Çelik Sanayi ve Ticaret A.Ş.
MGROS	Migros Ticaret A.Ş.
MIGRS	Migros Türk T.A.Ş.
MPT	Modern Portfolio Theory
MV	Mean-Variance

NETAS	Netaş Telekomünikasyon A.Ş.
NTHOL	Net Holding A.Ş.
OTKAR	Otokar Otomotiv ve Savunma Sanayi A.Ş.
PETKM	Petkim Petrokimya Holding A.Ş.
PRKTE	Park Elektrik Üretim Madencilik Sanayi ve Ticaret A.Ş.
PRKME	Park Elektrik Üretim Madencilik Sanayi ve Ticaret A.Ş.
PTOFS	Omv Petrol Ofisi A.Ş.
SAHOL	Hacı Ömer Sabancı Holding A.Ş.
SISE	Türkiye Şişe ve Cam Fabrikaları A.Ş.
SKBNK	Şekerbank T.A.Ş.
SML	Security market line
SNGYO	Sinpaş Gayrimenkul Yatırım Ortaklığı A.Ş.
TAVHL	Tav Havalimanları Holding A.Ş.
TCELL	Turkcell İletişim Hizmetleri A.Ş.
TEBNK	Türk Ekonomi Bankası A.Ş.
THYAO	Türk Hava Yolları A.O.
TIRE	Mondi Tire Kutsan Kağıt Ve Ambalaj Sanayi A.Ş.
TKFEN	Tekfen Holding A.Ş.
TNSAS	Tansaş Perakende Mağazacılık T.A.Ş.
TOASO	Tofaş Türk Otomobil Fabrikası A.Ş.
TRKCM	Trakya Cam Sanayii A.Ş.
TSKB	Türkiye Sınai Kalkınma Bankası A.Ş.
TTKOM	Türk Telekomünikasyon A.Ş.
TTRAK	Türk Traktör ve Ziraat Makineleri A.Ş.
TUPRS	Tüpraş-Türkiye Petrol Rafineleri A.Ş.
ULKER	Ülker Bisküvi Sanayi A.Ş.
U.S.	United States
VAKBN	Türkiye Vakıflar Bankası T.A.O.
VESTL	Vestel Elektronik Sanayi ve Ticaret A.Ş.
YAZIC	Yazıcılar Holding A.Ş.
YKBNK	Yapı ve Kredi Bankası A.Ş.
ZOREN	Zorlu Enerji Elektrik Üretim A.Ş.

CHAPTER 1

INTRODUCTION

Investors are concerned about the optimal number of securities to invest in from a wide range of securities. Diversification is a risk management approach that enables an investor to reduce the effect of any one asset on the overall performance of a portfolio by including a wide variety of assets in the portfolio. In other words, diversification lessens the risk of a portfolio without effecting its return. By increasing the number of securities in the portfolio, an investor can achieve the maximum diversification benefit while incurring considerable amount of transaction costs. Therefore, an investor has to set the balance between the reduction in the risk of the portfolio because of diversification and the increase in the transaction costs due to the large number of securities included in the portfolio to achieve that diversification benefit [11].

The issue of the optimal number of securities needed to construct a well-diversified portfolio, which is one of the subjects of this thesis, has been debated among researchers since the middle of the nineteenth century. In the literature, the study of Evans and Archer (1968, [12]) is one of the first attempts to determine the optimal number of stocks to have from New York Stock Exchange's Standard and Poor's (S&P) Index to construct a well-diversified portfolio. They start with a single stock portfolio and then increase the number of securities included in the portfolio iteratively. By simulating the risk of each of these portfolios that successively have more securities in them, they show that approximately 10 randomly selected stocks are needed to construct a well-diversified portfolio. Furthermore, they support their results by conducting a t-test on the risks of portfolios that have successively more securities in them. Results of these tests reveal that no statistically significant decrease in the average risk of the portfolio is achieved by increasing the number of stocks included in it once there are already 10 stocks in the portfolio. In this thesis, the simulation technique of Evans and Archer is used to demonstrate risk-reduction benefits of holding more than one stock in a portfolio for Borsa Istanbul (BIST). Following their methodology, a t-test is also used in order to determine the optimal portfolio size for BIST to achieve the maximum diversification benefit.

Another research on this subject belongs to Beck, Perfect and Peterson (1996, [1]). They focus on the effect of number of portfolio replications on the sensitivity of the statistical test used. According to their research, portfolios should be replicated a substantial number of times in order to detect a significant change in the portfolio risk as the number of securities in the portfolio increases. This is because the curve showing the relationship between the number of securities in the portfolio and the mean standard deviation of the portfolio becomes flatter as the number of securities in the portfolio increases. They show that high number of replications also affects the sensitivity of statistical tests and hence probability of rejecting the null hypothesis. In this thesis, number of replications is chosen to be 1000. In addition to this, successive portfolio sizes are determined by increasing the number of securities in the portfolio by 2. Then, the standard deviation of these successive portfolios are compared by using the t-test.

Besides these studies on stocks trading on the U.S. Stock Exchanges mentioned in the previous paragraphs, there are some studies conducted on BIST as well. For example, Gökçe and Cura (2003, [10]), Tosun and Oruç (2010, [28]) and Demirci and Keskintürk (2007) examine the stocks from BIST-30 index, and Atan and Duman (2007) study the stocks from the BIST-100 index. This thesis is the first study that analyzes stocks from BIST-50 Index.

In this thesis, the optimal number of stocks from BIST-50 index to form a well-diversified portfolio for each year in the sample period from 2005 to 2015 are determined. The optimal number of stocks to have for a well-diversified portfolio are examined by using the classical variance-covariance matrix obtained from historical stock returns and the posterior variance-covariance matrix as defined in the Black-Litterman (B-L) Method.

In addition to determining the optimal number of stocks to form a well-diversified portfolio, researchers are also concerned about how to allocate money from their budget across stocks in a portfolio since Markowitz's (1952, [19]) seminal paper on Modern Portfolio Theory. According to his approach, expected returns and variance of each stock and covariance of each pair of stocks in the portfolio should be estimated.

In the early 1990s, Fisher Black and Robert Litterman, two researchers at Goldman Sachs Company, introduce a new model of asset allocation. This model, which is known as the B-L model, is first introduced in the paper of B-L (1990, [4]), and extended in following studies of B-L (1991, [17], 1992, [5]). The B-L method enables investors to adjust equilibrium returns on stocks to reflect their views on these stocks by using the Bayesian approach. There are two key features of this model which differs from the classical mean-variance approach. One of them is that investors can define views on the expected returns of as many securities as they wish. In the classical mean-variance approach, expected returns on each asset in a portfolio must be forecast and a little change in the estimation of return on a security results in large change in the portfolio weight allocated to that security. However, in the B-L method, this large change in the portfolio weight as a result of changes in estimated returns is not observed. The second one is that the equilibrium market returns are taken as prior estimation of security returns and investor views are blended with this prior information set. Taking into account investor views in determining optimal portfolio choices of investors is not possible in the classical mean-variance approach.

In the first paper of B-L (1990, [4]), market views are mixed with the equilibrium returns estimated by using the International Capital Asset Pricing Model (ICAPM) instead of CAPM since the data set includes currencies, bonds and forward contracts of different countries. In the second paper of B-L (1991, [17]), equity securities are included in the universe of assets that can be invested in a portfolio in addition to the currencies and the bonds from their first paper. In the third paper of B-L (1992, [5]), effectiveness of different investment strategies are analyzed for global portfolio optimizers. Their original papers do not provide an intuitive explanation of their model in detail. However, the paper by He and Litterman (1999, [14]) reveal the intuition behind the B-L model. Similarly, details on mathematical aspects of the B-L model are not provided in the original papers of B-L. However, Satchell and Scowcroft (2000, [23]) explain the Bayesian portfolio construction for incorporating the investors' views into prior information. In addition to this, Walters (2011, [29]) clarifies Theil's Mixed Estimation approach which is another method of blending prior returns with the views.

There is just one article applying the B-L method to stocks trading on the BIST by Çalışkan (2012, [9]) and there are couple of master thesis written on this subject. In these studies, either the subjective opinion of researchers or the pseudo stock returns estimated by using quantitative methods such as EGARCH and AR(1) are considered as view returns. All of these studies acknowledge the difficulty in obtaining subjective views on the stocks listed on BIST. In this thesis, the analysts' average target price estimates, available on Bloomberg database, on the stocks included in the BIST-50 Index are taken as investors' views. And these views are blended with the equilibrium returns of these stocks. Moreover, real world constraints such as budget and short selling constraints, are imposed iteratively on the optimization model while maximizing the utility of the investors. Finally, the CAPM and the B-L portfolios constructed for each of the years in the sample period from 2005 to 2015 are compared by using the Sharpe ratios and the efficient frontiers. The results show that the B-L portfolios do not always perform better than the CAPM portfolios in terms of these comparison methods.

The empirical findings in this thesis indicate that the range of the optimal number of BIST-50 stocks needed to form a well-diversified portfolio varies between 8-10 and 16-18 stocks from one estimation year to another. Moreover, there is not much difference between the range of optimal number of stocks based on the classical variance-covariance matrix and the posterior variance-covariance matrix for each estimation year. Furthermore, the results show that the B-L portfolios do not always perform better than the CAPM portfolios in terms of Sharpe ratio and efficient frontier comparisons.

The remaining parts of thesis is organized as follows. Chapter 2 describes literature review on the subjects regarding the optimal number of stocks needed to form a well-diversified portfolio, portfolio theories before B-L model, their mathematical aspects and an intuitive explanation of the B-L method. Chapter 3 has the mathematical derivations of the B-L Model. Chapter 4 discusses the methodology used in the thesis and the data sources. Chapter 5 summarizes the empirical findings of this thesis. Finally, Chapter 6 provides our conclusions.

CHAPTER 2

LITERATURE REVIEW

A key factor of the Modern Portfolio Theory (MPT) is the principle of diversification. The idiom "Do not put all your eggs in one basket" explains the concept of diversification very well. It gives the advice of investing one's money in a variety of financial securities instead of only one security. The portfolio's total risk is composed of two parts: non-diversifiable (systematic) risk and diversifiable (firm-specific) risk. The macroeconomic factors such as the business cycle, inflation, interest rates and exchange rates generate non-diversifiable risk which is an inherent risk regardless of the operating activities of the firm. This risk thus remains even if one holds all the securities in the market in his portfolio. On the other hand, diversifiable risk is the unique risk of an asset that comes from firm-specific effects such as personnel changes, and the achievements of a department like research and development or marketing. Since firm specific effects are not correlated across firms, this risk can be eliminated by adding assets to a portfolio. In other words, diversification enables us to minimize the firm-specific risk [6].

Markowitz (1952, [19]) who is known as the father of the MPT, shows that one can decrease an individual asset's risk to which investors are exposed to by holding a diversified portfolio of assets. In his study, in order to indicate the risk-reduction benefit of holding more than one security, the mathematical formula for calculating the risk of a portfolio of assets is constructed. Markowitz analyzes the expected portfolio return, portfolio risk, correlation of assets in the portfolio and the impact of diversification on the portfolio risk. The expected return on the portfolio is calculated as the sum of weighted average of the expected returns on the securities in the portfolio. On the other hand, the portfolio risk, which is described as the variance or the standard deviation of returns on the portfolio, is shown to be not equal to a weighted average of variances of assets in the portfolio. The variance of returns on the portfolio is shown to be the weighted sum of the terms in the variance-covariance matrix of securities included in the portfolio. Markowitz demonstrates the benefits from diversification for portfolios that have less than perfectly positively correlated assets. Furthermore, it is shown that the lower the correlation between the assets in a portfolio, the higher the gains from diversification [6].

After the pioneering work of Markowitz, diversification is discussed in a variety of papers. Researchers show that when the number of assets in a portfolio increases, the portfolio's risk declines because of elimination of the firm-specific risk. However, they wonder how many randomly selected assets are required to create a well-diversified portfolio. Additional securities enhance the degree of diversification benefits at a decreasing rate and bring more transaction costs at the same time. The optimal number of stocks needed for a well-diversified portfolio has been debated among many researchers such as Evans and Archer (1968, [12]), Fisher and Lorie (1970, [13]), Elton and Gruber (1977, [11]), Statman (1987, [25]), Beck, Perfect and Peterson (1996, [1]), Gökçe and Cura (2003, [10]), Tosun and Oruç (2010, [28]) among others. The next section summarizes the findings of these studies.

2.1 Literature related to Optimal Number of Stocks for a Well-diversified Portfolio

First of all, Evans and Archer (1968, [12]) analyze the effect of additional number of assets in the portfolio on the risk (the standard deviation) of the portfolios. The data used in this study are coming from 470 securities included in the Standard and Poor's (S&P) 500 index in 1958. The securities are selected randomly and for each security semi-annual observations are collected for the period from January 1958 to July 1967. The return of each security for each period and the geometric mean return of each security as the average return for entire period is calculated and the standard deviation of each stock is calculated as the dispersion of the logarithms returns around the average return. First, only one stock is selected randomly from 470 securities. This is the one stock portfolio and the average standard deviation of this portfolio is used to determine the benefit of diversification from adding more stocks to a portfolio. Then equally weighted two stock portfolios are formed without replacement which means that the first stock is selected from 470 securities, and the second stock is selected from the remaining 469 securities. This process is continued until the portfolio has randomly selected 40 stocks in it. Each of these portfolios are formed 60 times. The portfolio return and the standard deviation of the portfolio return are calculated and recorded in a table. The average standard deviation of the portfolios are regressed on the inverse of the number of stocks in the portfolio to determine the optimal number of stocks needed to form a well-diversified portfolio. They show that the average standard deviation of the portfolio asymptote to the average non-diversifiable risk estimated by calculating the standard deviation of a portfolio containing all of 470 securities for the period they examined. Finally, by using a t-test and an F-test at the significance level of 0.05, they show that in order to achieve a statistically significant decrease in the average portfolio standard deviation, a substantial increase in the number of securities included in a portfolio is required once the portfolio already has 8 securities in it.

Evans and Archer (1968, [12]) deal with the change in the average standard deviation of a portfolio returns as the number of stocks in the portfolio increases, but Fisher and Lorie (1970, [13]) concentrate on the frequency distributions of returns on portfolios, and also on individual stocks. Furthermore, although Evans and Archer keep the holding period of investments constant, Fisher and Lorie conduct their analysis for different holding periods. They use the wealth ratio of the portfolio calculated as the ratio of the ending value of the portfolio to the beginning value of the portfolio as their return measure. Fisher and Lorie concentrate on three issues related to the variability of portfolio returns containing stocks from the New York Stock Exchange. First, they analyze the frequency distribution of returns over one to forty years during a sample period from 1926 to 1965. Therefore, they provide a statistical view of diversification effects in the literature. Second, they analyze the aggregated distribution of stock returns in the portfolio for holding periods of 1, 5, 10, 20, and 40 years. Finally, the return distributions of portfolios having only one stock to 128 stocks are determined. They conclude that the frequency distribution of returns on portfolios with 8 stocks are almost same as the frequency distribution of returns on portfolios with more than 8 stocks. According to the findings of their research, an investor can hold 8 stocks in a portfolio instead of dealing with a large number of stocks in a portfolio and have the same frequency distribution of portfolio returns.

Contrary to simulation techniques used in studies such as Evans and Archer (1968, [12]), Elton and Gruber (1977, [11]) focus mostly on obtaining an analytical expression for the relationship between the number of securities in the portfolio and the portfolio risk. Elton and Gruber (1977, [11]) use the total risk due to not only mean differences but also the second moment of the variance of returns as the portfolio risk, and support the study of Fisher and Lorie (1970, [13]) in which the distribution of all possible portfolio returns instead of dispersion of mean portfolio return is found. The exact formula for the expected variance of N-asset portfolio is first shown in the Markowitz's study [19]. In addition to this exact formula, Elton and Gruber derive an approximation to this formula by using simulated data. They use their approximation formula in their empirical analysis. The data used in the analysis consists of weekly returns from 150 to 3,290 securities from the New York and the American Stock Exchanges for the period from June 1971 to June 1974. They expect the parameter estimates like portfolio return and variance of that return to differ across samples with different number of stocks. However, values of the parameters estimated for the sample of 150 are very close to those for the sample of 3,290. Securities are selected randomly without replacement and they calculate the variance of an equally weighted portfolio of all securities in the population. They find that the expected annualized standard deviation is 23.701% for 1-stock portfolios, 11.506% for 10-stock portfolios and 9.211% for a portfolio including all 3,290 stocks. Therefore, they conclude that the majority of the possible decrease in the portfolio risk is attained once there are 10 stocks in the portfolio. According to the results of previous papers mentioned in this article, most of the risk of an individual security is diversified away by holding portfolios with 10 to 20 securities, whereas the risk-reduction benefits from adding stocks beyond 15 are still significant.

Statman (1987, [25]) compares the risk of complete portfolios (passive portfolios) that are on the Security Market Line constructed from the risk free asset and the S&P 500 (taken as the market portfolio) to the risk of one constructed by randomly selecting 10 to 50 stocks to form an active portfolio with the same return as the complete portfolio. While making this comparison between returns of actively and passively managed portfolios, he also takes into account the transaction and administrative costs of an actively managed portfolio. These costs are estimated as the difference between the returns of an actively and a passively managed portfolios. The passively managed portfolio is taken as the S&P 500index and the actively managed portfolio is proxied by the Vanguard Index Trust. Finally, the optimal number of stocks required for a well-diversified portfolio by taking into consideration pros and cons of diversification is found for both a borrowing and a lending investor. They allow for a difference between the borrowing (approximated by the Treasury bill rate) and the lending rates (approximated by the Call Money rate which is the interest rate charged on margin loans having the stock as a collateral by brokers). He demonstrates that at least 30 stocks from the point of view of a borrowing investor, and 40 stocks from the point of view of a lending investor are needed to form a well-diversified portfolio. The reason behind the difference between the optimal number of stocks for a lending and a borrowing investor is that a borrowing investor pays the spread between the borrowing rate and the lending rate which is estimated to be 2 percent per year.

Beck, Perfect and Peterson (1996, [1]) deliberate a different aspect of the analysis carried out to determine the required number of stocks for a well-diversified portfolio than the previous studies. They state that in most of the diversification studies, number of portfolio replications carried out to determine the portfolio expected return and the risk are not taken into account. When a high number of replications is used, the curve representing the relationship between the portfolio risk and the portfolio size becomes more flat. However, the number of portfolio replications also affect the precision of the statistical tests conducted in these studies. Therefore, a large number of replications leads to more sensitive test statistics so that statistically significant differences between the variance of the sample portfolios and the market portfolio are always observed. Monthly returns of 1,221 securities from the New York Stock Exchange and the American Stock Exchange with data available in the University of Chicago's Center for Research in Securities Prices database are taken into account and an equally weighted portfolio containing all of these stocks is considered as a proxy for market portfolio. They conduct some hypothesis tests for portfolio replications ranging from 50 to 2000 at the 5% level of significance. The null hypothesis states that the portfolio's variance is equal to the market portfolio's variance, and the alternative hypothesis states that the portfolio's variance is greater than the market portfolio's variance. Additionally, they conduct the same test with different alternative hypothesis which states that the portfolio's variance is equal to $1+\epsilon$ (ϵ: the percentage away from the market variance) times the market portfolio's variance. If an investor wants to hold a portfolio within 2% of the risk of the market portfolio, according to the power curves of chi-square test in their study, 200 replications and 48 securities are the optimal numbers in order to minimize the diversifiable risk by using this test. Increase in the number of replication raised the precision of the test. They ask researchers to use more robust statistical test and to be aware of the effect of the number of portfolio replications on the sensitivity of the statistical test.

Gökçe and Cura (2003, [10]) study stocks from the Borsa Istanbul-30 (BIST-30) Index in order to determine the optimal number of stocks to hold for a well-diversified portfolio. They work with weekly returns for these stocks over a period from January 1999 to June 2000. Their algorithm enables them to analyze all possible portfolios of a specific size that can be created from a universe of certain number of securities. For instance, if one wants to analyze all portfolios of 2 stocks that can be created from a universe of 30 stocks, then one needs to look at 435 (30×29/2) different 2 stock portfolios. Then, the geometric and the arithmetic portfolio returns, and the minimum and the maximum portfolio standard deviation for both equally weighted and market value weighted portfolios are calculated. To determine the optimal number of stocks for a well-diversified portfolio, three different criteria are specified. The first one is the ratio of average risk of the portfolio of a certain size to the market (BIST-30 index) risk. As the number of stocks used to form the portfolio increases, this ratio is expected to get closer to 1. However, after having certain number of securities in the portfolio, incremental decrease in the risk of the portfolio as a result of an incremental increase in the number of stocks included in the portfolio becomes negligible. The number of stocks included in the portfolio at that point is taken as the optimal number of stocks to have in order to form a well-diversified portfolio according to this criterion. The second criteria is the reduction in the diversifiable risk by comparison to 1-stock portfolio's diversifiable risk as the number of stocks in the portfolio increases. To calculate the second criteria, first the average non-systematic risk of N-stock portfolios is subtracted from the average non-systematic risk of 1-stock portfolios. Then this difference is divided by the average non-systematic risk of 1-stock portfolios and multiplied by 100. When incremental percentage decrease in the non-systematic risk is less than 1%, the number of stocks included in the portfolio at that point is taken to be necessary and sufficient number of stocks to form a well-diversified portfolio according to this criterion. The third and the last criteria is the reduction in the total risk by comparison to 1-stock portfolio's total risk as the number of stocks in the portfolio increases. To calculate the last criteria, first, the ratio of the average total risk of N-stock portfolio to the average total risk of 1-stock portfolio is calculated. Then, this ratio is subtracted from 1, and the resulting value is multiplied by 100. As the number of stocks in the portfolio increases, the average total risk decreases due to decrease in the non-systematic risk of the portfolio. When incremental percentage decrease in the average total risk is less than 1%, the number of stocks included in the portfolio at that point is the required number of stocks for a well-diversified portfolio according to this criterion. The minimum and the maximum number of stocks indicated by these criteria are defined as the range of stocks needed to have a well-diversified portfolio. For a well-diversified equally weighted portfolio, the optimal number of stocks needed is determined to be between 6 and 13 stocks, whereas for a well-diversified market value weighted portfolio, it is determined to be between 7 and 14 stocks.

Tosun and Oruç (2010, [28]) analyze the monthly returns and the standard deviation of returns on 20 stocks that are consistently included in the BIST-30 Index during the whole period from January 2001 to December 2008. Markowitz Mean-Variance (MV) model is used in this study. Three criteria proposed by Gökçe and Cura (2003, [10]) are also utilized to determine the optimal number of stocks to have in a well-diversified portfolio. Equally weighted portfolios consisting of 5 to 7 stocks are indicated to be well-diversified portfolios. Some other studies on the BIST are cited in this paper as well. However, author of this thesis can not find these studies electronically. The work of Demirci and Keskintürk (2007) is one of those studies cited. Demirci and Keskintürk use weekly returns of the stocks included in the BIST-30 Index instead of monthly returns as in the study of Tosun and Oruç (2010), and determine the optimal portfolio size to be between 3 and 17 stocks. The work of Atan and Duman (2007) is also cited in Tosun and Oruç (2010). Atan and Duman (2007) use monthly returns of the stocks included in the BIST-100 index instead of those included in the BIST-30 index as in Tosun and Oruç (2010) and Gökçe and Cura (2003), and they conclude that 11 stocks are needed to form a well-diversified portfolio.

The number of stocks needed to have a diversified portfolio is different for the American and the Turkish Stock Exchanges. The reason behind this difference is the correlation structure of returns on stocks in these markets. As the correlation between securities decreases, the number of stocks needed to form a well-diversified portfolio should also decrease. However, even though the correlation of the returns on Turkish stocks is higher than that for the American stocks, less number of securities are needed to form a well-diversified portfolio in Turkey. This finding is counterintuitive.

As pointed out above, diversification has been one of the main topics discussed in various papers. The effect of diversification on a portfolio variance is explained mathematically in the following subsection. The remaining sections of this chapter are allocated to the review of the literature on identification of assets to be included in a portfolio and the allocation of portfolio value across these assets, another debated issue in the portfolio theory literature.

2.1.1 Mathematical Preliminaries of Diversification

Markowitz (1952, [19]) defined the expected return on the portfolio as a weighted sum of average returns of each security in the portfolio. The expected return on the portfolio consisting of N assets is described analytically as:

$$E(r_p) = \sum_{i=1}^{N} w_i E(r_i), \tag{2.1}$$

where

w_i : the weight of the security i,

$E(r_i)$: the expected return on the security i.

The variance of the portfolio is defined as a weighted sum of covariances, and is described analytically as:

$$\sigma_p^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \text{Cov}(r_i, r_j), \tag{2.2}$$

where

$\text{Cov}(r_i, r_j)$: the covariance between the returns of security i and j.

In order to investigate risk reduction, let us give a simple and clear example of a portfolio containing only stocks A and B. The variance of the portfolio can be shown as:

$$\sigma_p^2 = w_A w_A \text{Cov}(r_A, r_A) + w_B w_B \text{Cov}(r_B, r_B) + 2 w_A w_B \text{Cov}(r_A, r_B). \tag{2.3}$$

Since the covariance of the return on an asset with itself is the variance of the return on that asset, the Equation (2.3) can be rewritten as:

$$\sigma_p^2 = w_A^2 \sigma_A^2 + w_B^2 \sigma_B^2 + 2 w_A w_B \text{Cov}(r_A, r_B). \tag{2.4}$$

It can easily be deduced from the above Equation (2.4) of the portfolio variance that the risk is eliminated when the covariance term is negative. On the other hand, even if the covariance term is positive, some of the risk in individual securities is still eliminated when these securities are combined in a portfolio. Just under the circumstance that returns of these two securities are perfectly positively correlated, there will be no diversification of individual security risks when these two securities are combined in a portfolio.

To indicate this statement above, the covariance term can be rewritten in terms of the correlation coefficient. Correlation coefficient is the measure of the extent to which two securities move up and down together. It takes values between -1 and +1. If the correlation coefficient is equal to -1, then the securities are perfectly negatively correlated. If the correlation coefficient is equal to +1, then the securities are perfectly positively correlated. Correlation coefficient is the normalized version of the covariance term:

$$\rho_{AB} = \frac{\text{Cov}(r_A, r_B)}{\sigma_A \sigma_B}. \tag{2.5}$$

After substituting Equation (2.5) into the Equation (2.4), the portfolio variance equation becomes:

$$\sigma_p^2 = w_A^2 \sigma_A^2 + w_B^2 \sigma_B^2 + 2 w_A w_B \sigma_A \sigma_B \rho_{AB}. \tag{2.6}$$

Consider the situation in which the correlation coefficient of the two stocks is equal to +1. In that case, the variance of the portfolio becomes the perfect square of a weighted sum of standard deviations of two securities:

$$\sigma_p^2 = (w_A \sigma_A + w_B \sigma_B)^2. \tag{2.7}$$

After taking the square root of both sides of the Equation (2.7), it becomes:

$$\sigma_p = w_A \sigma_A + w_B \sigma_B. \tag{2.8}$$

Thus, the standard deviation of the portfolio consisting of two perfectly positively correlated securities is equal to the weighted sum of their standard deviations indicating that there is no diversification benefit from combining these two securities in a portfolio. Apart from the case in which the correlation coefficient of these two stocks is equal to +1, the standard deviation of the portfolio is smaller than the weighted average of the standard deviations of two securities indicating that there is some diversification benefit from combining these two securities in a portfolio. In other words, portfolios, apart from the ones having all perfectly positively correlated securities, enable investors to benefit from diversification to some extent. As the correlation between securities decreases, an investor can gain more from diversification [6].

Now, consider the situation in which the correlation coefficient of these two stocks is equal to -1. In this instance, the variance of the portfolio becomes:

$$\sigma_p^2 = (w_A\sigma_A - w_B\sigma_B)^2. \tag{2.9}$$

After taking the square root of both sides, the standard deviation is:

$$\sigma_p = |w_A\sigma_A - w_B\sigma_B|\,. \tag{2.10}$$

Since the portfolio standard deviation can be equal to zero in this case, the maximum gain from diversification is obtained. Let us give a simple example of the portfolio consisting of stock A with 20% standard deviation and stock B with 12% standard deviation taken from [6]. For three cases, ρ = -1, ρ = 0, and ρ = +1, the relationship between the standard deviation of the portfolio and the weight of stock A is represented in Figure 2.1:

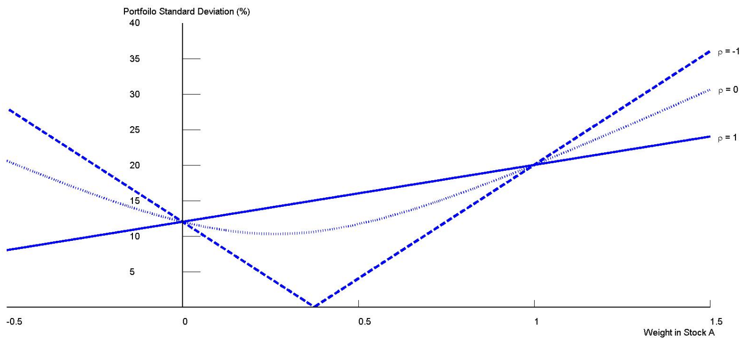


Figure 2.1: Portfolio Standard Deviation as a Function of the Weights of Stock A.

This figure shows the standard deviation of the portfolio including stocks A and B against the weight of stock A. The standard deviation of stock A is 20% and the standard deviation of stock B is 12%. The straight line in this graph shows the relationship between the standard deviation of the portfolio and the weight of stock A when the correlation between these two stocks equals to 1. The dotted curve represents this relation when the correlation between these stocks equals to 0. The dashed line illustrates this relation when the correlation between these stocks equals to -1.

This graph illustrates that the largest decrease in the standard deviation of the portfolio is captured when ρ = -1. Apart from the case of perfect positive correlation, as the weight of stock A rises from 0 to 1, firstly the portfolio standard deviation declines due to the effect of diversification. However, as the weight of stock A increase, portfolio standard deviation increases as well since stock A has a higher standard deviation than stock B. Finally the portfolio is undiversified when an investor puts all of his money in stock A or stock B [6].

2.1.2 Naive Diversification Strategy

First of all, let us remark the variance equation of the portfolio consisting of N stocks:

$$\sigma_p^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \text{Cov}(r_i, r_j). \tag{2.11}$$

Naive diversification strategy recommends constructing a portfolio of these N stocks by investing equal amounts in each of them. It is also known as $1/N$ strategy. When the stocks are equally weighted, the portfolio standard deviation Equation (2.11) becomes:

$$\sigma_p^2 = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{N} \sigma_i^2 + \sum_{\substack{j=1 \\ j \neq i}}^{N} \sum_{i=1}^{N} \frac{1}{N^2} \text{Cov}(r_i, r_j). \tag{2.12}$$

Markowitz (1976, [20]) described the portfolio variance as a function of average variance and, the average covariance of the securities in that portfolio. This relationship can be written as:

$$\bar{\sigma}^2 = \frac{1}{N} \sum_{i=1}^{N} \sigma_i^2, \tag{2.13}$$

$$\overline{\text{Cov}} = \frac{1}{N(N-1)} \sum_{\substack{j=1 \\ j \neq i}}^{N} \sum_{i=1}^{N} \text{Cov}(r_i, r_j), \tag{2.14}$$

$$\sigma_p^2 = \frac{1}{N} \bar{\sigma}^2 + \frac{N-1}{N} \overline{\text{Cov}}. \tag{2.15}$$

Markowitz (1976, [20]) also clarifies the effect of diversification. There are two main factors contributing to the decrease in the portfolio variance towards 0. As can be easily seen from the formula, one of them is reducing the average covariance to 0, and the other one is raising the number of stocks (N) in a portfolio. As more securities added to the portfolio, the risk (standard deviation) of the portfolio decreases towards

0. However, when N increases, the second term on the right-hand side of the equation converges to $\overline{\mathrm{Cov}}$ since the term $\frac{N-1}{N}$ converges to 1. This part is called the undiversifiable risk. As can be easily seen from the formula, this undiversifiable risk is due to the covariance of the returns on securities, and is considered to be caused by macroeconomic factors [6].

The relationship between undiversifiable risk and the correlations between securities is displayed clearly when an identical standard deviation, σ for all securities, and an identical correlation coefficient, ρ, between returns on all securities, are assumed. Then the Equation (2.15) can be written as:

$$\sigma_p^2 = \frac{1}{N}\sigma^2 + \frac{N-1}{N}\rho\sigma^2. \tag{2.16}$$

2.2 Literature related to Portfolio Theories

The pioneering work on this subject belongs to Markowitz. Markowitz (1952, [19]) formulates the decision-making problem as a trade-off between portfolio's expected return and its variance. Markowitz's Mean-Variance optimization model under certain assumptions (i.e. rational and informed investors and efficient markets) allows us to construct portfolios which have the maximum possible expected return for a given level of risk or the minimum possible risk for a given level of expected return. These portfolios form the efficient frontier. When a risk free asset is added to this environment, combinations of this risk free asset and a risky asset creates a capital allocation line (CAL). The slope of a CAL is the ratio of excess return on the portfolio to the risk of that portfolio (standard deviation). The slope is also called as Sharpe ratio. The capital allocation line with the highest slope in a market is called as the capital market line (CML) and it is the locus of all portfolios created from the risk free asset and a broad index of stocks in the market [6].

In Markowitz model, to determine the best possible portfolio, an investor should consider the expected returns of all stocks, the variances and the covariances of returns on these stocks in the portfolio. Hence, when one's portfolio includes a large number of stocks, the large number of input data is needed for calculation and a larger variance-covariance matrix needs to be estimated. For instance, if one's portfolio consists of N securities, then N expected returns, N variance and $N(N$-1)/2 covariance terms, that is, a total of $N(N$+3)/2 inputs will be required. If the security returns satisfy the assumptions of the Single-Index Model, then the number of parameters to be estimated can be reduced significantly. This model requires regressing the excess return of a stock (return on a stock minus risk-free rate) on the excess return of the market index. Most observations reveal that stock prices move in the same direction with the market, that is, prices of the stocks increase when the market goes up and prices of the stocks decrease when the market goes down. This suggests that the stock returns might be correlated with the market returns, and this correlation might be estimated by the regression analysis of a stock's returns on a stock market index returns (as a proxy for the market). Therefore, using the Single-Index model, N market expected excess returns (called as alpha), N sensitivity coefficients (called as beta), N residual variances can be estimated. The market risk premium and the variance of the returns on the market index can also be determined easily. Using these inputs the variance-covariance matrix of security returns can be constructed. Therefore, in this case, the total number of variables to estimate is $3N$+2 in order to construct the parameters needed for the Markowitz optimization instead of $N(N$+3)/2.

The Capital Asset Pricing Model is developed by Sharpe (1964, [24]), Treynor (1962) and Lintner (1965). The CAPM is a single-index model. It is the pioneering equilibrium model (the model in which the demand for a stock is equal to the supply of that stock). It has a set of strict assumptions such as no transaction costs and taxes, investors with homogeneous expectations, mean-variance optimizing investors. Under these assumptions, it is shown that all investors hold the same portfolio of risky assets which is the market portfolio. Each stock in this market portfolio has a weight equal to the market value of that asset divided by the total market value of all risky assets in the market. An investor can borrow and lend an unlimited amount at the risk-free rate in order to adapt the risk of the market portfolio to her preferred risk level. All combinations of the risk-free and risky assets are located on the Security Market Line (SML). The Security Market Line is the same as the Capital Market Line with one difference. The relevant risk measure for the SML is the beta of a security whereas the relevant risk measure for the CML is the standard deviation of security returns. The CAPM defines the relationship between the expected return and the beta, systematic risk, of a security.

Although the CAPM is an important equilibrium model, Stephen Ross (1976, [26]) criticizes the CAPM on its assumption of a single factor determining the security returns. Arbitrage Pricing Theory (APT), a multifactor model, is first suggested by Ross (1976, [26]) as an alternative to the CAPM. In APT, unlike the CAPM, stock prices can be affected from several macro factors such as anticipated growth or decline in gross domestic product (GDP), and changes in interest rates. Underlying principle of the APT is the law of one price, that is, two identical items can not be sold at different prices. It states that an arbitrage opportunity arises only if an investor can gain a riskless profit without spending any money out of his/her pocket [6].

Although Markowitz Mean-Variance approach has a significant effect on the portfolio theory, most money managers believe that there are some shortcomings of this model in practice [18]. The first one is that the model obliges investors to assign quite large weights to stocks with large historical expected returns, or quite low weights to stocks with low historical returns. The second and the more important one is that the model does not allow investors to embed their current views with respect to current conditions into the model. Here, the Black-Litterman (B-L) portfolio optimization model, which is the fundamental subject of this thesis comes to rescue. Before the literature of the B-L model is reviewed, the mathematics behind the Markowitz portfolio optimization and the Capital Asset Pricing Model is described in the following subsections.

2.2.1 Markowitz Mean-Variance Optimization

Markowitz (1952, [19]) describes the risk-return trade-off analytically as:

$$\text{maximize} \quad \sum_{i=1}^{N} w_i E(r_i)$$

subject to

$$\sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \text{Cov}(r_i, r_j) \leq c_1,$$

$$\sum_{i=1}^{N} w_i = 1, \quad w_i \geq 0 \quad \forall i \in N.$$

where c_1 is the specified level of portfolio risk.

In this optimization model, the portfolio expected return is maximized for a given level of portfolio risk while satisfying the budget and no short-selling constraints. A budget constraint represents that investors must spend all of their money on the securities included in the portfolio. Short selling is the selling of a security that an investor does not own. To be able to sell this security, an investor needs to borrow that security from another investor and, promise to deliver the security back to its original owner on demand. This type of trading is prevented by no short-selling constraint, i.e. by requiring security weights to be positive.

The risk-return trade-off can also be modeled by minimizing the portfolio risk for a given level of the portfolio expected return with similar constraints above:

$$\text{minimize} \quad \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \text{Cov}(r_i, r_j)$$

subject to

$$\sum_{i=1}^{N} w_i E(r_i) \geq c_2,$$

$$\sum_{i=1}^{N} w_i = 1, \quad w_i \geq 0 \quad \forall i \in N.$$

where c_2 is the specified level of portfolio expected return.

Markowitz optimization generates the efficient frontier of investment opportunity set. The portfolios are formed by varying the proportion of each security in the portfolio, thus the portfolios' expected returns and standard deviations are represented by a graph. The minimum-variance frontier is constructed from all points that satisfies the minimum variance for any given portfolio expected return. The point that lies on the minimum-variance frontier which has the lowest variance among all minimum variance portfolios is defined as the global minimum-variance portfolio. The part of the minimum variance frontier that lies below the global minimum-variance portfolio is inefficient, since for any portfolio on the inefficient part of the frontier, there is another one on the efficient part that has the same standard deviation but higher expected return. Therefore, risk averse utility maximizing investors will not be interested in holding these inefficient portfolios. When this inefficient part is removed from the minimum-variance frontier, the remaining part of this frontier is defined as the efficient frontier. In Figure 2.2 obtained from He and Litterman (1999, [14]), minimum-variance frontier, global minimum-variance portfolio and efficient frontier are marked clearly.

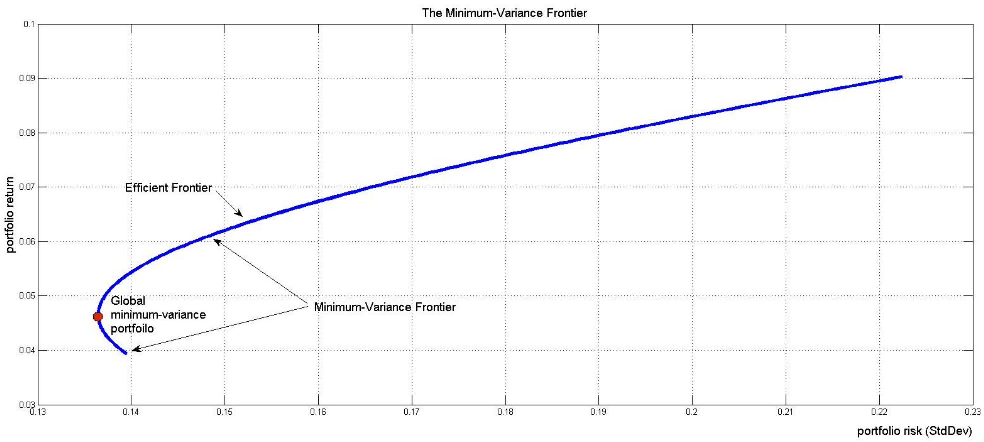


Figure 2.2: The Minimum-Variance Frontier.

This figure shows the minimum-variance frontier of the portfolio consisting of the assets in the study of He and Litterman (1999, [14]). One can plot the expected portfolio returns on the Y axis and the standard deviations (risks) of the portfolio having these assets on the X axis of a graph. The minimum variance-frontier is the set of all possible portfolios with the minimum risk for each portfolio expected return. The lowest risk portfolio among all minimum risky portfolios is known as the global minimum-variance portfolio. The efficient frontier represents all efficient portfolios above the global minimum-variance portfolio.

One of the other expansions of the portfolio optimization is investing in the risk-free asset along with the risky assets. Capital allocation line (CAL) indicates all combinations of the risk-free asset and a risky portfolio. The slope of the capital allocation line, which is known as Sharpe ratio, is equal to incremental expected return of the portfolio consisting of the risk-free asset and the risky portfolio per incremental standard deviation of the portfolio. Sharpe ratio can be written as:

$$S = \frac{E(r_p) - r_f}{\sigma_p},$$

where r_f is the expected return of the risk-free asset.

Many capital allocation lines can be drawn by combining the risk-free asset and a different risky portfolio on the efficient frontier. One of the lines, Capital Market Line (CML), is the combination of the risk-free asset and the market portfolio which contains all the assets in the market. This CAL has the highest Sharpe ratio. Therefore, all investors prefer the portfolios that lies on the CML when compared to portfolios on other CALs.

The choice of an optimal portfolio for each investor from the efficient portfolios is another issue. The amount of the risk-free asset and the risky portfolio an investor is willing to hold must be determined. At this point, the concept of risk aversion needs to be introduced. Investors are categorized into groups based on their level of risk aversion as risk lover, risk neutral and risk averse. Risk lover investors take a risk even if there is no compensation for bearing that risk. Risk averse investors are willing to invest in risky assets only if they offer a higher return than a safe alternative. Risk neutral investors care only about the return of their investments, and are indifferent to the risk. Investors are assumed to be risk averse in optimization problems. However, there can be differences in their risk aversion levels. A less risk averse investor is willing to take more risk to gain higher return. A more risk averse investor on the other hand, prefer less risky investments. To convince a more risk averse investor to hold higher risk investments, we need to offer much higher return. For instance, among two investments with the equal expected return, but different risks, a risk averse investor will always choose the one with the lower risk.

In order to determine the optimal portfolio weights for a preferred level of risk aversion, the utility maximization can be used. Utility of an investor can be calculated based on a utility function. The most commonly used utility function by financial theorists has the following form:

$$U = E(r) - \frac{1}{2}\lambda\sigma^2,$$

U : utility value,
λ : the investor's risk aversion coefficient ($\lambda < 0$ for risk lovers, $\lambda = 0$ for risk neutrals, and $\lambda > 0$ for risk averse investors).

When the amount of w is invested in the risky portfolio, and the remaining part (1-w) is invested in the risk-free asset, the optimal weight of the risky portfolio can be found by maximizing the utility function of the investor:

$$\text{maximize}_w\, U = E(r_p) - \frac{1}{2}\lambda\sigma_p^2 = r_f + w[E(r_{rp}) - r_f] - \frac{1}{2}\lambda w^2\sigma_{rp}^2,$$

$E(r_{rp})$: the expected return of the risky portfolio,
σ_{rp}^2 : the variance of the risky portfolio.

In order to solve this maximization problem, we need to take the first derivative of this expression and then equate it to 0. The optimal weight for the risky portfolio, $w^\star$, is obtained as follows:

$$w^\star = \frac{E(r_{rp}) - r_f}{\lambda \sigma_{rp}^2}.$$

2.2.2 The Capital Asset Pricing Model

The Capital Asset Pricing Model is one of the most important innovations in modern portfolio theory. It was developed by Sharpe (1964, [24]) and similar works were observed in the papers of Treynor (1962) and Lintner (1965). The concept of the risk-return trade-off is the main idea behind the modern portfolio theory. To put it more explicitly, the more risk an investor takes, the higher the expected return that will be gained from that investment to compensate the investor for risk taken. The Capital Asset Pricing Model is one way to determine the potential return that an investor will obtain by taking risk. Using the CAPM, the required rate of return on a security can be calculated as a function of the systematic risk of that security. The model can be represented as follows:

$$E(r_i) = r_f + \beta_i[E(r_M) - r_f], \tag{2.17}$$

$E(r_i)$: expected return on security i,
r_f : risk-free rate,
β_i : beta of security i,
$E(r_M)$: expected return on market portfolio.

Before analyzing the CAPM explicitly, we will give the assumptions of the model. Some of the assumptions are about an investor's behavior and the others are about the structure of the market. The main assumptions about the investors' behaviors are, that investors are rational, risk-averse and mean-variance optimizers. It means that all investors behave judiciously in market transactions, their risk aversion coefficients are greater than 0, they follow Markowitz mean-variance optimization model. Another assumption, which is related to market structure, is that investors are price-takers. In other words, their own trading activities do not affect the stock prices, i.e. markets are competitive. The other assumption is that investors have homogeneous expectations; they have same input lists including identical expected returns, variances and covariances of the securities. Final assumption of this part is that investors focus on a single investment period. The main assumption about the market structure is that, there is no taxes and transaction costs. Another one is that all information is publicly available, that means there is no private information that is kept secret. Final one is that all securities are publicly exchanged, short-selling is possible, and investors can lend and borrow any amount at the certain risk-free rate [6].

By means of these assumptions, the nature of the equilibrium in security market can be examined. This equilibrium market model states that the supply and the demand of the securities are equal to each other. When the demand exceeds the supply of a particular security, the excess demand raises the price of the security, and decreases the expected return of the security. Thus, the demand will decline and become equal to the supply of the security, and the market will reach an equilibrium. In this market, each lender can match with a corresponding borrower. Since all investors own the same input list, their investment horizon is identical and they all apply the Markowitz method on the same securities, all investors will hold the same portfolio of risky assets, which is the market portfolio, M. Market portfolio contains all tradable securities with proportions equal to the market value of a security divided by the total market value of all securities.

According to the CAPM, the risk premium on the market portfolio equals to the multiplication of its risk and the average degree of the risk aversion, it can be shown as:

$$E(r_M) - r_f = \bar{\lambda}\sigma_M^2.$$

The risk premium on a security is assessed by its contribution to the risk of the market portfolio. The contribution of a security to the risk of the market portfolio, thus, can be measured by the covariance of that security's returns with returns on all the assets that form the market portfolio.

The Sharpe ratio for investments in security i is:

$$\frac{\text{security } i\text{'s contribution to risk premium}}{\text{security } i\text{'s contribution to variance}} = \frac{E(r_i) - r_f}{\text{Cov}(r_i, r_M)}.$$

The Sharpe ratio for investment in the market portfolio is:

$$\frac{\text{Market risk premium}}{\text{Market variance}} = \frac{E(r_M) - r_f}{\sigma_M^2}.$$

The equilibrium approach states that the Sharpe ratios of all investments must be equal. If the Sharpe ratio for one investment is greater than the Sharpe ratio for another investment, then investors' portfolios would be rearranged toward the portfolio with greater Sharpe ratio. This would indicate that security prices would be pressured until the ratios would again be equal. In equilibrium, the Sharpe ratio of any security i and the market portfolio must be equal:

$$\frac{E(r_i) - r_f}{\text{Cov}(r_i, r_M)} = \frac{E(r_M) - r_f}{\sigma_M^2}.$$

and the risk premium for security i is:

$$E(r_i) - r_f = \frac{\text{Cov}(r_i, r_M)}{\sigma_M^2}[E(r_M) - r_f].$$

The ratio $\frac{\text{Cov}(r_i, r_M)}{\sigma_M^2}$ is called as beta and it is the measure of the contribution of security i to the variance of the market portfolio divided by the variance of the market portfolio. By putting the β_i instead of this ratio, the Equation (2.17) is reached.

The expected return-beta relationship can be illustrated graphically as the security market line (SML). Since the contribution of the market portfolio to the variance of the market portfolio is equal to the total variance of the portfolio, the market has a beta of 1, and the slope of the SML is equal to the market risk premium. To demonstrate this visually, Figure 2.3 taken from [8] is drawn.

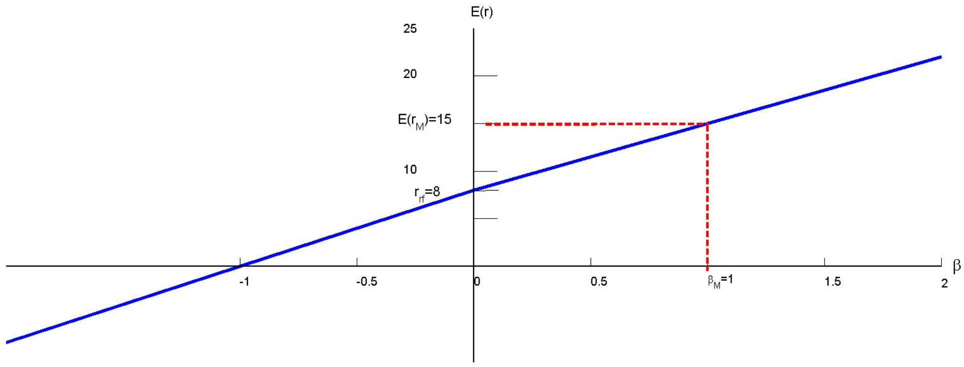


Figure 2.3: Security Market Line.

This figure shows the relation between the expected return and the risk measured by beta. The expected return on a security or a portfolio is shown on the Y axis and the risk (beta) is shown on the X axis. Beta measures the sensitivity of the asset to the market portfolio. The data was obtained from Financial Management [8]. The expected return of the risk free asset is 8% and the expected return of the market portfolio is 15%. Since the market portfolio is perfectly positively correlated with itself, it has a beta of 1.

Security Market Line can also be used to analyze whether securities are fairly priced or not. SML provides investors the required rate of return necessary in order to compensate them for the risk that they are taking. If the expected return of a security is greater than the required rate of return on this security, in other words, this security lies above the SML, then the security will be defined as underpriced. On the other hand, if the expected return of a security is less than the required rate of return on this security, that means this security lies below the SML, then the security will be defined as overpriced. The securities that lies on the SML are described as fairly priced.

2.3 Literature related to Black-Litterman Method

The B-L model is first developed by Fisher Black who was a partner in Goldman Sachs Asset Management group of companies and Robert Litterman who was a vice president in the Fixed-Income Research Department at Goldman Sachs Asset Management group of companies [4]. The B-L method allows investors to update the equilibrium expected returns based on their views. On the research of B-L (1990, [4]), the weights from International Capital Asset Pricing Model (ICAPM) are used as a starting point and these weights are adjusted to reflect their market views. Currencies, bonds and forward contracts of different countries are taken as the assets investors can have in their portfolios. Their observation for the Markowitz model is that small changes in the expected returns result in large changes in the optimal weights of securities in the portfolio. In their new model, views can be defined in a relative sense as well as in an absolute sense and the level of confidence can be assigned to each view as well. On the other hand, in Mean-Variance approach the views should be assigned to all securities in the portfolio and the views have to be specified in an absolute sense. After defining their current views on bonds and currencies in a manner formulated mathematically, the optimal portfolio weights based on Mean-Variance approach are compared to the optimal portfolio weights based on the B-L approach. Since the views are defined on each security in an absolute sense, the optimal weights based on the Mean-Variance approach are shown to be quite extreme when compared to the optimal weights based on the B-L approach. In the B-L model, the posterior weights are different from the market weights to the extent of defined views. In order to demonstrate this, they use an example. Views of investors on certain assets are expressed and then these views are used to update the market equilibrium excess returns. When investors are more confident about their views, a higher than 50% is assigned to their views in updating the equilibrium excess returns. The optimal weights of the portfolios created by assigning 100% weight to the views are a little bit extreme than the optimal weights of the ones formed by assigning less weight to the views. Therefore, B-L state that because of the ability of the model to assign different weights to the views, placing artificial limits on investment amounts in order to construct balanced portfolios becomes unnecessary.

Even though B-L (1990, [4]) create an innovative model for asset allocation problem, lots of question marks about their model are left in the minds of researchers. Therefore, the papers of B-L (1991, [17]) and B-L (1992, [5]) are published within 2 years to answer some of these equations. On the paper of B-L (1991, [17]), investment universe is expanded to include equities as well as currencies and bonds. More details on this new model and its application are presented in this article. The data is the monthly expected returns of stocks, currencies and bonds for the 7 countries (United States, Japan, Germany, France, the United Kingdom, Canada, and Australia) from January 1975 to August 1991. Calculation of currency hedged excess returns on an international bond or stock, the mean and the standard deviation of these returns are explained. Comparisons similar to the ones in B-L (1990, [4]) are done by adding stocks. Lastly, how to mix the equilibrium excess returns with the investor's views by using the Theil's mixed estimation method (1971, [27]), how to define the views as vectors, and the formula for the new combined return vector as a function of the excess equilibrium return vector and the views are presented in details. In this new formulation, apart from the equilibrium returns and the views, τ, which is a constant and scaling factor, is seen as a multiplier of the variance. This scaling factor is assumed to have a value between 0 and 1, since B-L assume that uncertainty of the expected equilibrium returns is smaller than the uncertainty of equilibrium returns.

In addition to the work of B-L (1991, [17]), the work of B-L (1992, [5]) has another extension of the original model. This paper serves as a guide for global portfolio managers. They use the same data as B-L (1991, [17]). They take the considerable increase in expected portfolio returns as an indication of the benefit from global diversification. To show how their model performs in simulations, three strategies consisting of investing in higher-yielding currencies, higher-yielding bonds of countries and stocks of the countries having higher dividend to bond yield ratios are tested by using rolling window approach. They conclude that the two strategies, which are investing in high-yielding currencies and stocks of the countries having higher dividend to bond yield ratios performed considerably better and the remaining strategy performed poorly.

He and Litterman (1999, [14]) clarify some issues of the B-L model known as a "black box" model. The posterior expected returns based on the equilibrium returns and the views are defined as random variables, and just their probability distributions are derived. They clarify that in the B-L method the prior distribution is the equilibrium distribution coming from the CAPM, and the additional information is the investor views. In order to find the posterior distributions of expected returns, the CAPM equilibrium returns and the investor views are blended by using the Bayesian approach. Three intuitive features of the B-L model are verified analytically and numerically. First of all, they show that the optimal risky portfolio for an unconstrained problem is the weighted sum of the equilibrium portfolio and portfolios representing the investor's views. Secondly, when a view return is more optimistic than the implied equilibrium return of that security and the view returns on other securities, a positive weight is assigned to the portfolio representing that view. Lastly, the weights increase not only due to the optimistic views, but also an increase in the level of confidence in the views. Moreover, how to change the optimal weight vector when one new view is added, is shown analytically.

Satchell and Scowcroft (2000, [23]) show each step needed to extract the posterior distribution from the CAPM prior and the investor views by using Bayes' theorem. In Chapter 3 of this thesis, most of the derivations in this article are utilized and verified. They develop an alternative reference model and prove that results similar to the original model can be obtained with this alternative model as well. In their alternative model, posterior returns follow a multivariate t distribution. Walters (2014, [29]) explains that, in this reference model, there is no information on the precision of the estimate due to employing point estimates instead of distributions. Thus, the historical covariance matrix without the scaling factor (τ) is in the formula. Moreover, there is no need to consider the posterior variance since the variance is not updated.

Application of the B-L method to different problems is explained in the article of Idzorek (2004, [15]). In the literature, there are a few examples of preparing essential inputs of the model in the form of a matrix. On the other hand, in Idzorek, comprehensive examples of constructing the vector of returns coming from the views, covariance matrix of the views and the pick matrix of the views are given in order to produce inputs for the B-L model. Historical monthly returns for the eight assets including U.S.

Bonds, International Bonds, U.S. Large Growth, U.S. Large Value, U.S. Small Growth, U.S. Small Value, International Developed Equity, and International Emerging Equity over a 5 year period are used in these examples. The views expressed are formed in the same way as in B-L (1990, [4]). He develops a new approach in order to integrate confidence levels assigned by the investors to their views. For each view, the implied confidence in the view are calculated by using the B-L formula with 100% certainty of the view. In light of the structure of the implied confidence, the tilts due to the views on the portfolio weights are found, and sum of the squared differences between the weights with tilts and without tilts is minimized. Hence, an investor can control the extent of the tilts by changing confidence level ranging from 0% to 100%. While B-L take a quantitative investor's perspective, Idzorek (2004, [15]) provides information to a qualitative investor to utilize the B-L model via his new method.

Meucci (2006, [22]) develops a new method in order to determine posterior distribution arising from the mixture of the equilibrium and the view distributions by using the principle of opinion pooling instead of Bayesian approach in the B-L model. Thus, unlike in B-L approach, the views are not needed to be normal and uncorrelated with each other. The marginal distribution of each view is found out separately. However, the views can be jointly co-dependent. The copula directly comes up from the market structure. Finally, a joint distribution for the market can be extracted from the joint distribution of the views by allowing an appropriate change in coordinates. In another article of Meucci (2010, [21]), the B-L model and its extensions are explained with related proofs.

Almost all derivations and transformations related to the B-L model are presented in the article of Walters (2014, [29]). His work is like a literature survey of the B-L method. By using Theil's Mixed Estimation Model and Bayes' Theory the derivation of the posterior mean and variance arising from the blending of the equilibrium mean with the views, variance with the view's mean, and confidences in views are shown. Furthermore, differences between the canonical model which is proposed by B-L (1992, [5]) and the reference model which is proposed by Satchell and Scowcroft (2000, [23]), and Meucci (2006, [22]) are revealed. Some analysis are done by changing the parameters of the model. Thanks to this paper, one can access the whole history of the B-L model.

The B-L method is applied to several financial instruments and securities traded on different stock exchanges. There is only one article which is an application of the B-L method to securities of Turkish Stock Exchange Market. Çalışkan (2012, [9]) uses the data of daily returns of 17 stocks included in the BIST-30 Index over the period from 2003 to 2009. Portfolios constructed from Mean-Variance method and the B-L method are compared in terms of their unsystematic and total risks. In terms of systematic risks, in each period the portfolios constructed from the B-L method have lower beta factors than the portfolios constructed from Mean-Variance method. With regard to unsystematic risk, in 11 periods over 13 periods analyzed, the portfolios constructed from the B-L method give better results than the portfolios constructed from Mean-

Variance method. From the point of total risks, in 9 periods over 13 periods analyzed, the portfolios formed from the B-L method have lower total risks than the portfolios formed from Mean-Variance method.

Bozdemir (2011, [7]) has a master thesis on the application of B-L method to Turkish Stock Market. He deals with returns on Turkish industrial price indexes for the period from July 2000 to November 2010. Two information sets used in this thesis in order to apply the B-L Model are the returns derived from the Capital Asset Pricing Model and returns from an AR(1) Model used to represent investor views. These two information sets are blended by Theil's Mixed Estimation Method. However, the posterior weights obtained are not consistent with the expectations. By giving a simple 2 assets example he explains that the reason behind the unexpected relationship between the expected return and the corresponding posterior weight is the covariance structure. Finally, performances of the B-L and the CAPM strategy returns are compared with regards to compounded returns, mean variance analysis, utility values and unpaired student-t test. In the end, he concludes that B-L portfolio derived from the combination of CAPM and AR(1) returns is not superior to the market portfolio, because there is not any statistically significant difference between the mean returns of the B-L and the CAPM strategies.

CHAPTER 3

MATHEMATICAL DERIVATIONS FOR THE BLACK-LITTERMAN MODEL

It has been shown by Becker and Gürtler (2009, [2]) that Black and Litterman (B-L) (1991, [4]) assume return vector of N specified assets is normally distributed with $N \times 1$ expected return vector μ and $N \times N$ variance-covariance matrix Σ :

$$r \sim N(\mu, \Sigma).$$

It is also assumed that the variance-covariance matrix is known and is estimated from historical return data of specified assets. Moreover, Becker and Gürtler (2009, [2]) indicate that B-L (1991, [4]) determine the vector of expected returns as a random vector which is distributed normally with known parameters Π, τ and Σ :

$$\mu \sim N(\Pi, \tau\Sigma).$$

Π is the $N \times 1$ equilibrium return vector and is supposed to be a neutral reference point. In the B-L model, "equilibrium" returns are considered as a starting point by Idzorek (2004, [15]). Equilibrium returns are derived from the Capital Asset Pricing Model (CAPM). These returns are the equilibrium returns that clear the market of all assets. The equilibrium returns are calculated via a reverse optimization method in which the vector of implied excess equilibrium returns is derived from the equation given as:

$$\Pi = \lambda \Sigma w_{\text{MKT}}, \tag{3.1}$$

where

Π is the implied excess equilibrium return vector ($N \times 1$ column vector),

λ is the risk aversion coefficient,

Σ is the covariance matrix of excess returns ($N \times N$ matrix), and

w_{MKT} is the market capitalization weight ($N \times 1$ column vector) of the assets.

The risk aversion coefficient of all investors in the market (λ) describes a trade-off between the expected return and the risk. B-L assign 2.5 to λ which represents the world average risk tolerance described in Black (1989, [3]). Idzorek (2004, [15]) assigns 3.07 to λ which is derived from dividing excess returns by the variance of excess returns. The risk aversion coefficient serves as a scaling factor for the estimate of excess returns from the reverse optimization. These estimated excess returns get higher as the risk aversion coefficient increases.
To obtain the Formula (3.1), the following utility function is maximized:

$$U = w^T \Pi - \left(\frac{\lambda}{2}\right) w^T \Sigma w.$$

U is a concave function; therefore, it has a single global maximum. In order to find this global maximum, the first derivative of the utility function with respect to w is taken. After setting this derivative equal to zero, equation presented below is obtained:

$$\frac{dU}{dw} = \Pi - \lambda \Sigma w = 0,$$

$$\Pi = \lambda \Sigma w.$$

The variance-covariance matrix of the expected return vector μ is supposed to be the multiplication of the variance-covariance matrix of returns with a scaling factor $\tau > 0$. B-L (1992, [5]) assume that the uncertainty in the expected return is smaller than the uncertainty of returns themselves. Therefore they assign a quite small number to τ.

In the B-L model, investors' views are expressed as either in absolute terms such as "asset A will have a return of X." or in relative terms such as "asset A will outperform asset B by X." Idzorek (2004, [15]) uses 8 asset classes consisting of US Bonds (p1), International Bonds (p2), US Large Growth (p3), US Large Value (p4), US Small Growth (p5), US Small Value stocks (p6), International Developed Equity (p7) and International Emerging Equity (p8). Three sample views on these asset classes specified in the article of Idzorek (2004, [15]) are restated below in the format of B-L (1990):
View 1: The excess return of International Developed Equity (p7) will be 5.25%.

View 2: The excess returns of International Bonds (p2) will be 25 basis points greater than the excess returns of U.S. Bonds (p1).

View 3: The excess returns of U.S. Large Growth (p3) and U.S. Small Growth (p5) will be 2% greater than the excess returns of U.S. Large Value (p4) and U.S. Small Value (p6).[1]

[1] Idzorek (2004, [15]) states that the portfolio including U.S. Large Growth and U.S. Small Growth assets is the market-capitalization weighted portfolio and the portfolio including U.S. Large Value and U.S. Small Value is the market-capitalization weighted portfolio as well.

View 1 is an absolute view; on the other hand, views 2 and 3 are relative views.

Incorporation of these views into the B-L model is tricky. It is stated in the model that there is no need to specify views on all assets. If we continue with Idzorek's eight assets example, the number of views (k) is 3; therefore, the View vector (q) is a 3×1 column vector. The uncertainty of views results in a random, unknown, independent, normally-distributed Error Term Vector (ϵ) with a mean of 0 and a covariance matrix Ω. Therefore, a view has the form of $q+\epsilon$.

General Case: Idzorek's Example:

$$q+\epsilon = \begin{bmatrix} q_1 \\ \vdots \\ q_k \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \vdots \\ \epsilon_k \end{bmatrix}, \qquad q+\epsilon = \begin{bmatrix} 5.25 \\ 0.25 \\ 2 \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \vdots \\ \epsilon_k \end{bmatrix}.$$

The Error Term Vector (ϵ) itself is not part of the B-L formula directly. However, the variance of each error term (ω), which is the absolute difference from the error terms (ϵ) expected value of 0, is incorporated into the formula by Idzorek (2004, [15]). The variances of error terms (ω) constitute Ω, where Ω is a diagonal variance-covariance matrix. Since views expressed on different assets are assumed to be independent of one another, the off-diagonal elements of Ω are 0. The variances of error terms ω indicate the uncertainty associated with these views.

General case:

$$\Omega = \begin{bmatrix} \omega_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \omega_k \end{bmatrix}.$$

Views stated in the column vector q are assigned to the related assets by a weight matrix P. Each stated view is shown as a $1\times N$ row vector. Therefore, K views are shown in a $K \times N$ matrix. If we look at Idzorek's example based on the work of Satchell and Scowcrof (2000, [23]), in which there are 8 assets and 3 views, P is a 3×8 matrix.

General case: Idzorek's Example (Equally Weighted Method)

$$P = \begin{bmatrix} p_{1,1} & \ldots 0 & p_{1,n} \\ \vdots & \ddots & \vdots \\ p_{k,1} & \cdots & p_{k,n} \end{bmatrix}, P = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & -0.5 & 0.5 & -0.5 & 0 & 0 \end{bmatrix}.$$

The first row of Matrix P represents view 1, an absolute view. Views 2 and 3, relative views, are expressed by Rows 2 and 3, respectively.

In constructing the weight matrix depicted above, Satchell and Scowcroft (2000, [23]) assign a weight of zero to any asset that is not mentioned in a view. On the other hand, if there are more than one assets mentioned in a view, all of these assets are assigned equal weights. This is one of the methods of identifying the values of Matrix

P. Idzorek (2004, [15]), on the other hand, uses another method in order to construct pick or weight matrix, P. Idzorek (2004, [15]) assigns market value weights to all the assets mentioned in a view. Assets that are not mentioned in a view still have a weight of 0. More precisely, the relative weighting of each individual asset is calculated by dividing that asset's market capitalization by the total market capitalization of either the outperforming or underperforming assets in that specific view.

Matrix P (Market Capitalization Method)

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.9 & -0.9 & 0.1 & -0.1 & 0 & 0 \end{bmatrix}.$$

When Matrix P is constructed, the variance of each individual view portfolio can be calculated. The variance of an individual view portfolio is $p_k \Sigma p_k^T$, where p_k is a single $1 \times N$ row vector from Matrix P that corresponds to the k^{th} view and Σ is the covariance matrix of excess returns.

3.1 Blending Prior and Posterior Returns with Bayes Theorem

Bayes Theorem allows investors to blend implied equilibrium returns with investors' views and states that:

$$f(E(r)|\Pi) = \frac{f(\Pi|E(r))f(E(r))}{f(\Pi)},$$

$f(E(r))$ is the prior density function that expresses the (prior) beliefs of the investor,

$f(\Pi)$ represents the marginal probability density function (p.d.f.) of CAPM equilibrium returns,

$f(\Pi|E(r))$ is the conditional p.d.f. of the CAPM equilibrium return, given the forecasts declared by the investor.

Satchell and Scowcroft (2000, [23]) explain comprehensive mathematics behind the B-L model in detail. They make two normality assumptions:

Assumption 3.1.
Let $\Upsilon = PE(r)$. Υ follows a normal distribution with mean matrix q and covariance matrix Ω:

$$\Upsilon \sim N(q, \Omega).$$

As it is mentioned before in Idzorek's example, Ω is a diagonal covariance matrix having elements of 0's on off-diagonals. Each diagonal element ω_{ii} indicates the level of certainty the investor has on that view. For example $\omega_{ii} = 0$ indicates absolute certainty. q and Ω are called as Bayesian hyper parameters.

Assumption 3.2.
Π given $E(r)$ follows a normal distribution with mean matrix $E(r)$ and covariance matrix $\tau\Sigma$ where τ is a scaling factor:

$$\Pi|E(r) \sim N(E(r), \tau\Sigma).$$

One can deduce from this assumption that the equilibrium excess returns given the investor's estimates equal to the investor's estimates on average. On the other hand, in CAPM approach, where all investors have common beliefs and hold the same portfolio, then Π indicates the equilibrium returns given the investors' common beliefs.

Theorem 3.1. *The p.d.f. of $E(r)$ given Π is distributed normally such that:*

$$f(E(r)|\Pi) \sim N([(\tau\Sigma)^{-1}+P^T\Omega^{-1}P]^{-1}[(\tau\Sigma)^{-1}\Pi+P^T\Omega^{-1}q], [(\tau\Sigma)^{-1}+P^T\Omega^{-1}P]^{-1}),$$

where

$$\mu_{BL} = [(\tau\Sigma)^{-1} + P^T\Omega^{-1}P]^{-1}[(\tau\Sigma)^{-1}\Pi + P^T\Omega^{-1}q,$$

$$\Sigma_{BL} = [(\tau\Sigma)^{-1} + P^T\Omega^{-1}P]^{-1}.$$

Proof. [2]

By Assumption (3.2), we can write

$$f(\Pi|E(r)) = \frac{1}{\sqrt{(2\pi)^N|\tau\Sigma|}} e^{\{-\frac{1}{2}(\Pi-E(r))^T(\tau\Sigma)^{-1}(\Pi-E(r))\}}.$$

By Assumption (3.1), the p.d.f. of $E(r)$

$$f(E(r)) = \frac{1}{\sqrt{(2\pi)^K|\Omega|}} e^{\{-\frac{1}{2}(PE(r)-q)^T\Omega^{-1}(PE(r)-q)\}}.$$

Let us define a constant $c = \frac{1}{\sqrt{(2\pi)^{N+K}|\tau\Sigma||\Omega|}}$.

By Bayes Formula

$$f(E(r)|\Pi) = \frac{ce^{\{-\frac{1}{2}[\{(\Pi-E(r))^T(\tau\Sigma)^{-1}(\Pi-E(r))\}-\{(PE(r)-q)^T\Omega^{-1}(PE(r)-q)\}]\}}}{f(\Pi)}. \tag{3.2}$$

[2] The major steps of the proof are obtained from the appendix of the paper by Satchell and Scowcroft (2000, [23]). All formulas of distributions, some matrix operations and multiplications are inserted between steps in order to clarify the transition from one step to another.

Since p.d.f. of Π does not include any $E(r)$ term, one can consider $f(\Pi)$ as a normalizing factor. After realizing that, we just need to deal with the exponential term in Equation (3.2). By using some basic matrix operations and multiplications, this term can be written in the following form:

$$-\frac{1}{2}\{(E(r))^T(\tau\Sigma)^{-1}E(r)-(E(r))^T(\tau\Sigma)^{-1}\Pi-\Pi^T(\tau\Sigma)^{-1}E(r)+\Pi^T(\tau\Sigma)^{-1}\Pi$$
$$+(E(r))^TP^T\Omega^{-1}PE(r)-q^T\Omega^{-1}PE(r)-(E(r))^TP^T\Omega^{-1}q+q^T\Omega^{-1}q\}.$$

Remark 3.1.

$$(E(r))^T(\tau\Sigma)^{-1}\Pi=\Pi^T(\tau\Sigma)^{-1}E(r),$$

$$(E(r))^TP^T\Omega^{-1}q=q^T\Omega^{-1}PE(r).$$

By using Remark (3.1), the exponential term in Equation (3.2) becomes:

$$\begin{aligned}
= \; & -\frac{1}{2}\{(E(r))^T(\tau\Sigma)^{-1}E(r)-2\Pi^T(\tau\Sigma)^{-1}E(r)+\Pi^T(\tau\Sigma)^{-1}\Pi+(E(r))^TP^T\Omega^{-1}PE(r)\\
& -2q^T\Omega^{-1}PE(r)+q^T\Omega^{-1}q\}\\
= \; & -\frac{1}{2}\{(E(r))^T[((\tau\Sigma)^{-1}+P^T\Omega P)((\tau\Sigma)^{-1}+P^T\Omega P)^{-1}((\tau\Sigma)^{-1}+P^T\Omega^{-1}P)]E(r)\\
& -2[(\Pi^T(\tau\Sigma)^{-1}+q^T\Omega^{-1}P)((\tau\Sigma)^{-1}+P^T\Omega^{-1}P)^{-1}((\tau\Sigma)^{-1}+P^T\Omega^{-1}P)]E(r)\\
& +q^T\Omega^{-1}q+\Pi^T(\tau\Sigma)^{-1}\Pi\}.
\end{aligned}$$

Let

$C=(\tau\Sigma)^{-1}\Pi+P^T\Omega^{-1}q,$

$H=(\tau\Sigma)^{-1}+P^T\Omega^{-1}P$, note that H is symmetric.

$A=q^T\Omega^{-1}q+\Pi^T(\tau\Sigma)^{-1}\Pi.$

Remark 3.2.
H is symmetric which means $H=H^T$.

Σ is an $N\times N$ covariance matrix. Therefore, it is a square matrix. If Σ is invertible, then Σ^T is invertible and $(\Sigma^{-1})^T=(\Sigma^T)^{-1}$. Furthermore, by the definition of covariance matrix, this is also a symmetric matrix which indicates that $\Sigma^T=\Sigma$.

Ω is an $K\times K$ square matrix. If Ω is invertible, then Ω^T is invertible and $(\Omega^{-1})^T=(\Omega^T)^{-1}$. Furthermore, Ω is a diagonal matrix indicating that $\Omega=\Omega^T$.

$H^T=[(\tau\Sigma)^{-1}+P^T\Omega^{-1}P]^T=((\tau\Sigma)^T)^{-1}+P^T(\Omega^T)^{-1}P=(\tau\Sigma)^{-1}+P^T\Omega^{-1}P=H.$

Using the property of H, we can rewrite the exponential terms as follows:

$$\begin{aligned}
& -\frac{1}{2}\{(E(r))^T H^T H^{-1} H E(r) - 2C^T H^{-1} H E(r) + A\} \\
= & -\frac{1}{2}\{(HE(r) - C)^T H^{-1}(HE(r) - C) + A - C^T H^{-1} C\} \\
= & -\frac{1}{2}\{(E(r) - H^{-1}C)^T H(E(r) - H^{-1}C) + A - C^T H^{-1} C\}.
\end{aligned}$$

Since there is no $E(r)$ term in $A - C^T HC$, terms like $A - C^T HC$ appear to be constant in the equation above. Later, when this equation is integrated in terms of $E(r)$, these terms disappear. Hence,

$$f(E(r)|\Pi) \sim e^{\{-\frac{1}{2}[(E(r)-H^{-1}C)^T H(E(r)-H^{-1}C)]\}},$$

so that $E(r)|\Pi$ has a mean of $H^{-1}C = [(\tau\Sigma)^{-1} + P^T\Omega^{-1}P]^{-1}[(\tau\Sigma)^{-1}\Pi + P^T\Omega^{-1}q]$

and the conditional variance of $Var(r|\Pi) = [(\tau\Sigma)^{-1} + P^T\Omega^{-1}P]^{-1}$.

□

Satchell and Scowcroft (2000, [23]) also explain an alternative formulation of the B-L model in detail by making two more assumptions:

Assumption 3.3.

$$f(\Pi|E(r), \tau) \sim N(E(r), \tau\Sigma),$$

$$f(E(r)|\tau) \sim N(q, \tau\Omega).$$

Assumption 3.4.
The marginal (prior) p.d.f. of $\omega = \frac{1}{\tau}$ is given by the following

$$f(\omega) = \frac{(\frac{\lambda}{2})^{K/2}\omega^{(K/2)-1}e^{(-\frac{\lambda\omega}{2})}}{\Gamma(K/2)}, \qquad 0 < \omega < \infty.$$

This p.d.f. has two hyper parameters K and λ, and these parameters are assumed to be independent of $f(\Pi)$.

Theorem 3.2. *By Assumptions (3.3) and (3.4)*

$$f(E(r)|\Pi) \sim [K + (E(r) - \theta)^T \lambda^\star V(E(r) - \theta)]^{-\frac{N+K}{2}},$$

which is a multivariate t distribution. The vector θ is $E(r|\Pi)$ given in Theorem (3.1), the matrix V is the $Var(r|\Pi)$ given in Theorem (3.1) while

$$\lambda^{\star} = \frac{K}{\lambda + A - C^T H^{-1} C},$$

where A, C, H are defined in the Proof of Theorem (3.1).

Proof. [3]

By Bayes Formula

$$f(E(r), \omega|\Pi) = \frac{f(\Pi|E(r), \omega) f(E(r), \omega)}{f(\Pi)} = \frac{f(\Pi|E(r), \omega) f(E(r)|\omega) f(\omega)}{f(\Pi)}$$

$$f(\Pi|E(r), \omega) = \frac{\omega^{N/2}}{\sqrt{(2\pi)^N |\Sigma|}} e^{\{-\frac{1}{2}(\Pi - E(r))^T \omega \Sigma^{-1}(\Pi - E(r))\}}$$

$$f(E(r)|\omega) = \frac{1}{\sqrt{(2\pi)^K |\Omega|}} e^{\{-\frac{1}{2}(PE(r)-q)^T \omega \Omega^{-1}(PE(r)-q)\}}.$$

If we define $c = \frac{1}{(2\pi)^{(N+K)/2}|\Sigma|^{1/2}|\Omega|^{1/2}}$

$$f(\Pi|E(r), \omega) f(E(r)|\omega) = c\omega^{N/2} e^{\{-\frac{\omega}{2}G\}},$$

where $G = (\Pi - E(r))^T \Sigma^{-1}(\Pi - E(r)) + (PE(r) - q)^T \Omega^{-1}(PE(r) - q)$.

By Assumption (3.4)

$$\begin{aligned} f(E(r), \omega|\Pi) &= \frac{c\omega^{N/2} e^{\{-\frac{\omega}{2}G\}} \left(\frac{\lambda}{2}\right)^{K/2} \omega^{(K/2)-1} e^{\{-\frac{\lambda\omega}{2}\}}}{\Gamma(\frac{K}{2}) f(\Pi)} \\ &= \frac{c\omega^{(\frac{N+K}{2})-1} e^{\{-\frac{\omega}{2}(G+\lambda)\}} \left(\frac{\lambda}{2}\right)^{K/2}}{\Gamma(\frac{K}{2}) f(\Pi)}. \end{aligned}$$

The probability distribution function shown above is integrated in terms of ω in order to find $f(E(r)|\Pi)$:

$v = \frac{\omega}{2}(G + \lambda), \qquad \omega = \frac{2v}{(G+\lambda)}, \qquad d\omega = \frac{2}{(G+\lambda)} dv,$

[3] The major steps of this proof are obtained from the appendix of the paper by Satchell and Scowcroft (2000, [23]). A few formulas of distributions are added in order to clarify the whole structure of the proof.

$$f(E(r)|\Pi) = c'\left(\frac{\lambda}{2}\right)^{K/2}\int_0^\infty e^{-v}\left(\frac{2v}{G+\lambda}\right)^{\left(\frac{N+K}{2}\right)-1}\left(\frac{2}{G+\lambda}\right)dv.$$

then

$= \frac{c'\left(\frac{\lambda}{2}\right)^{K/2}2^{(N+K)/2}\Gamma\left(\frac{N+K}{2}\right)}{\Gamma\left(\frac{K}{2}\right)(G+\lambda)^{(N+K)/2}}.$

A multivariate t distribution is defined for matrices $\theta(l\times 1)$ and $V(l\times l)$. Furthermore, a positive constant υ is defined:

$$f(x|\theta,V,\upsilon,l) = \frac{\upsilon^{\upsilon/2}\Gamma\left(\frac{\upsilon+l}{2}\right)|V|^{1/2}[\upsilon+(x-\theta)^TV(x-\theta)]^{(\upsilon+l)/2}}{\pi^{1/2}\Gamma\left(\frac{\upsilon}{2}\right)}.$$

If $G+\lambda$ is rewritten in terms of A, C, and H as shown in the proof of Theorem (3.1), we can obtain:

$$\begin{aligned} G+\lambda &= (E(r)-H^{-1}C)^TH(E(r)-H^{-1}C)+A-C^TH^{-1}C+\lambda \\ &= (E(r)-H^{-1}C)^T\frac{KH}{\lambda+A-C^TH^{-1}C}(E(r)-H^{-1}C)+K. \end{aligned}$$

This shows that $f(E(r)|\Pi)$ is multivariate t ,

$\theta = H^{-1}C$ (as before),

$\upsilon = \frac{K}{\lambda+A-C^TH^{-1}C}H$, $l=N$ and $\upsilon = K$.

□

3.2 Some Extensions of Black-Litterman Model

He and Litterman (1999, [14]) bring a new perspective to the B-L model. In their model, uncertainties about investors' views are defined as follows:

$$\omega_{ii} = P_i(\tau\Sigma)P_i^T.$$

Since τ is a quite small number, diagonal elements of Ω become much smaller than those in the case described above. As a result, weights assigned to securities with views increase more.

They state that stock returns are distributed normally as follows:

$$r \sim N(\overline{\mu}, \overline{\Sigma}),$$

where

$$\overline{\mu} = [(\tau\Sigma)^{-1} + P^T\Omega^{-1}P]^{-1}[(\tau\Sigma)^{-1}\Pi + P^T\Omega^{-1}q],$$

$$\overline{M}^{-1} = [(\tau\Sigma)^{-1} + P^T\Omega^{-1}P]^{-1},$$

$$\overline{\Sigma} = \Sigma + \overline{M}^{-1}.$$

Inputs of the standard mean-variance optimization method with security returns having the mean $\overline{\mu}$ and the covariance matrix $\overline{\Sigma}$ are different from those in B-L method. Apart from this difference, following the same logic of B-L method, the utility function of the investor is being maximized in the optimization:

$$\text{maximize}\,\{w^T\overline{\mu} - \frac{\lambda}{2}w^T\overline{\Sigma}w\}.$$

Let us take the first derivative of this objective function with respect to w, and obtain

$$\overline{\mu} = \lambda\overline{\Sigma}w^\star,$$

or, equivalently,

$$w^\star = \frac{1}{\lambda}\overline{\Sigma}^{-1}\overline{\mu},$$

where $w^\star$ is the vector of optimal portfolio weights. These optimal portfolio weights can be written in the form of

$$w^\star = \frac{1}{\lambda}\overline{\Sigma}^{-1}\overline{M}^{-1}[(\tau\Sigma)^{-1}\Pi + P^T\Omega^{-1}q].$$

Remark 3.3.

$$(A+B)^{-1} = A^{-1} - A^{-1}B(I + A^{-1}B)^{-1}A^{-1}.$$

By Remark (3.3)

$$\overline{\Sigma}^{-1} = (\Sigma + \overline{M}^{-1})^{-1} = \overline{M} - \overline{M}(\overline{M} + \Sigma^{-1})^{-1}\overline{M},$$

$$\overline{\Sigma}^{-1}\overline{M}^{-1} = \frac{\tau}{1+\tau}\left(I - P^T A^{-1} P \frac{\Sigma}{1+\tau}\right),$$

$$A = \frac{\Omega}{\tau} + \frac{P\Sigma}{1+\tau}P^T \qquad w^{\star} = \frac{1}{\lambda}\overline{\Sigma}^{-1}\overline{\mu},$$

$$w^{\star} = \frac{1}{1+\tau}(w_{eq} + P^T \times \Lambda),$$

where $w_{eq} = (\lambda\Sigma)^{-1}\Pi$

$$\Lambda = \frac{1}{\lambda}\tau\Omega^{-1}q - A^{-1}P\frac{\Sigma}{1+\tau}w_{eq} - \frac{1}{\lambda}A^{-1}P\frac{\Sigma}{1+\tau}P^T\tau\Omega^{-1}q.$$

It can be easily understood from He and Litterman (1999, [14]) that Λ indicates tilts of the posterior weights due to specified views. In other words, Λ is the measure of impact of the investors' views on portfolio weights assigned to securities. It can be easily seen from the first term in the formula of Λ that Λ moves in the same direction with q, and moves in the opposite direction with Ω. More precisely, if the investor state bearish views, then Λ decreases. However, if the investor is less confident about these bearish views, then Λ decreases less. The minus sign in front of the second term in the equation indicates that the effect of investors views on security weights decreases when the covariance between a view portfolio and the market equilibrium portfolio is high. The minus sign in front of the third term in the equation implies that the effect of views on security weights declines when the covariance between a view portfolio and other view portfolios is high.

Meucci (2010, [21]) assumes that μ is estimated without any estimation error. In other words, when $\tau \equiv 0$, then B-L model boils down to the reference model, and

$$r \sim N(\Pi, \Sigma).$$

B-L state that the investors assert the views on the parameter μ. In addition to this, the $K \times N$ "pick" matrix P indicates K views in the linear form, and this matrix's k-th row determines the relative weight of each expected return in the corresponding view. To incorporate the level of uncertainty with the corresponding view, B-L assume the views' returns follows a normal distribution:

$$P\mu \sim N(v, \Omega), \tag{3.3}$$

v is the vector of the views' returns,

Ω is the matrix of the uncertainty of the views.

According to Meucci, if the investors have just qualitative views, they should set the entries of v in terms of the volatility of the market:

$$v_k \equiv (P\Pi)_k + \eta_k \sqrt{(P\Sigma P^T)_{k,k}} \qquad (k = 1, ..., K),$$

$\eta_k \in \{-\beta, -\alpha, +\alpha, +\beta\}$.

Here, η_k allows investors to define views as "very bearish" (is indicated by $-\beta$), "bearish" (is indicated by $-\alpha$, "bullish" (is indicated by $+\alpha$, and "very bullish" (is indicated by $+\beta$). α and β are two constants. Meucci states that the values of these parameters can possibly be chosen as $\alpha \equiv 1$ and $\beta \equiv 2$. However, it can also be defined as in Meucci (2005)

$$\Omega = \frac{1}{c} P\Sigma P^T,$$

where the part of $P\Sigma P^T$ comes from the market volatilities and correlations and $c \in (0, \infty)$ indicates an overall level of confidence in the views. Additionally, to specify a scale-independent and different relative uncertainty level to each view, the equation above should be reconstructed as follows:

$$\Omega = \frac{1}{c} diag(u) P\Sigma P^T diag(u),$$

where $u \in (0, \infty)^K$

Meucci analyzes the uncertainty of the views in two ways. First, when the levels of confidence in the views are relatively low, i.e. $\Omega \to \infty$ in 3.3. Then, the posterior becomes the reference model:

$$X \sim N(\Pi, \Sigma).$$

Second, when investors are quite sure of their views, i.e. $\Omega \to 0$. Then, the posterior becomes the reference model conditioned on the stated views. In that case, conditional distribution is still normal:

$$X|v \sim N(\mu|v, \Sigma|v),$$

$$\mu|v \equiv \Pi + \Sigma P^T (P\Sigma P^T)^{-1}(v - P\Pi),$$

$$\Sigma|v \equiv \Sigma - \Sigma P^T (P\Sigma P^T)^{-1} P\Sigma.$$

As it is mentioned before, in the B-L method, the investors assert their views on the parameter μ, not on the return r, and hence, the confidence in the views just affects the estimation risk part ($\tau\Sigma$) of the covariance ($(1+\tau)\Sigma$) , not its volatility-correlation component Σ.

To start with the reference model, μ is not taken as a random variable. Thus, we set $\mu \equiv \Pi$ in order to obtain the reference model while not taking the estimation risk into consideration. Investors can specify their views on linear functions of the market $V = PX$, where P is a pick matrix. The views, V, which is defined by the reference model as a random variable given the market are distributed normally

$$V|x \sim N(Px, \Omega),$$

$$X \sim N(\mu^m_{BL}, \Sigma^m_{BL}) \begin{array}{l} \nearrow \; X \sim N(\Pi, \Sigma) \qquad \text{(no confidence: } \Omega \to \infty) \\ \\ \searrow \; X \sim N(\mu|v, \Sigma|v) \quad \text{(full confidence: } \Omega \to 0) \end{array}$$

Proof. [4]

$$X \sim N(\Pi, \Sigma),$$

$$f(x) = \frac{1}{\sqrt{(2\pi)^N |\Sigma|}} e^{\{-\frac{1}{2}(x-\Pi)^T \Sigma^{-1}(x-\Pi)\}},$$

$$v|x \sim N(Px, \Omega),$$

$$f(v|x) = \frac{1}{\sqrt{(2\pi)^K |\Omega|}} e^{\{-\frac{1}{2}(v-\Pi x)^T \Omega^{-1}(v-\Pi x)\}}.$$

By Bayes' Rule:

$$f(X|v;\Omega) = \frac{f(v|x) f(x)}{f(v)}.$$

[4] The major steps of this proof are obtained from the technical appendix in the paper of Meucci (2010, [21]). Meucci also cites the study of Satchell and Scowcroft (2000, [23]) in this technical appendix. All formulas of distributions and some matrix operations and multiplications are inserted between steps in order to clarify the transition from one step to another.

We can define $c = \frac{1}{\sqrt{(2\pi)^{(N+K)}|\Sigma||\Omega|}}$,

$$f(x|v) = \frac{ce^{\{-\frac{1}{2}\left[(x-\Pi)^T\Sigma^{-1}(x-\Pi)+(v-\Pi x)^T\Omega^{-1}(v-\Pi x)\right]\}}}{f(v)},$$

$f(v)$ can be considered as a normalizing factor. Then we just need to deal with the exponential term

$$\begin{aligned}
& (x-\Pi)^T\Sigma^{-1}(x-\Pi) + (v-\Pi x)^T\Omega^{-1}(v-\Pi x) \\
= \; & x^T\Sigma^{-1}x - \Pi^T\Sigma^{-1}x - x^T\Sigma^{-1}\Pi + \Pi^T\Sigma^{-1}\Pi + v^T\Omega^{-1}v - v^T\Omega^{-1}Px - x^TP^T\Omega^{-1}v \\
& + x^TP^T\Omega^{-1}Px.
\end{aligned}$$

Remark 3.4.
It should be stated that:

$$x^T\Sigma^{-1}\Pi = \Pi^T\Sigma^{-1}x,$$

$$x^TP^T\Omega^{-1}v = v^T\Omega^{-1}Px.$$

By Remark (3.4)

$$\begin{aligned}
= \; & x^T\Sigma^{-1}x - 2\Pi^T\Sigma^{-1}x + \Pi^T\Sigma^{-1}\Pi + v^T\Omega^{-1}v - 2v^T\Omega^{-1}Px + x^TP^T\Omega^{-1}Px \\
= \; & x^T[(\Sigma^{-1} + P^T\Omega P)(\Sigma^{-1} + P^T\Omega P)^{-1}(\Sigma^{-1} + P^T\Omega^{-1}P)]x \\
& -2[(\Pi^T\Sigma^{-1} + v^T\Omega^{-1}P)(\Sigma^{-1} + P^T\Omega^{-1}P)^{-1}(\Sigma^{-1} + P^T\Omega^{-1}P)]x + \Pi^T\Sigma^{-1}\Pi \\
& + v^T\Omega^{-1}v,
\end{aligned}$$

$C = \Sigma^{-1}\Pi + P^T\Omega^{-1}v,$

$H = \Sigma^{-1} + P^T\Omega^{-1}P$, note that H is symmetric.

$A = \Pi^T\Sigma^{-1}\Pi + v^T\Omega^{-1}v.$

Remark 3.5.
H is symmetric which means $H = H^T$.

Σ is an $N \times N$ covariance matrix. Therefore, it is a square matrix. If Σ is invertible, then Σ^T is invertible and $(\Sigma^{-1})^T = (\Sigma^T)^{-1}$. Furthermore, by definition of covariance matrix, this is also a symmetric matrix which indicates that $\Sigma^T = \Sigma$.

Ω is an $K \times K$ square matrix. If Ω is invertible, then Ω^T is invertible and $(\Omega^{-1})^T = (\Omega^T)^{-1}$. Furthermore, Ω is diagonal indicating that $\Omega = \Omega^T$.

$$H^T = [\Sigma^{-1} + P^T\Omega^{-1}P]^T = (\Sigma^T)^{-1} + P^T(\Omega^T)^{-1}P = \Sigma^{-1} + P^T\Omega^{-1}P = H$$

$$\begin{aligned} & x^T H^T H^{-1} Hx - 2C^T H^{-1} Hx + A = (Hx - C)^T H^{-1}(Hx - C) + A - C^T H^{-1} C \\ = \; & (x - H^{-1}C)^T H(x - H^{-1}C) + A - C^T H^{-1}C. \end{aligned}$$

Since there is no x term in $A - C^T H^{-1}C$, terms like $A - C^T H^{-1}C$ appear to be constant. Later when this equation is integrated in terms of x, these terms drop out of the equation. Hence,

$$f(x|v) \sim e^{\{-\frac{1}{2}(x-H^{-1}C)^T H(x-H^{-1}C)\}},$$

so that $x|v$ has a mean of $H^{-1}C = [\Sigma^{-1} + P^T\Omega^{-1}P]^{-1}[\Sigma^{-1}\Pi + P^T\Omega^{-1}v]$

and a variance of $Var(x|v) = [\Sigma^{-1} + P^T\Omega^{-1}P]^{-1}$.

□

Remark 3.6.
Using the matrix identity (A and D invertible)

$$(A - BD^{-1}C)^{-1} = A^{-1} - A^{-1}B(CA^{-1}B - D)^{-1}CA^{-1},$$

$$\begin{aligned} \mu_{BL}^m &= (\Sigma^{-1} + P^T\Omega^{-1}P)^{-1}(\Sigma^{-1}\Pi + P^T\Omega^{-1}v) \\ &= (\Sigma - \Sigma P^T(P\Sigma P^T + \Omega)^{-1}P\Sigma)(\Sigma^{-1}\Pi + P^T\Omega^{-1}v) \\ &= \Pi - \Sigma P^T(P\Sigma P^T + \Omega)^{-1}P\Pi + \Sigma P^T\Omega^{-1}v - \Sigma P^T(P\Sigma P^T\Omega)^{-1}P\Sigma P^T\Omega^{-1}v \\ &= \Pi + \Sigma P^T(\Omega^{-1} - (P\Sigma P^T + \Omega)^{-1}P\Sigma)(\Sigma^{-1}\Pi + P^T\Omega^{-1}v) \\ & \quad -\Sigma P^T(P\Sigma P^T + \Omega)^{-1}P\Pi. \end{aligned}$$

Note that

$$\Omega^{-1} - (P\Sigma P^T + \Omega)^{-1}P\Sigma P^T\Omega^{-1} - (P\Sigma P^T + \Omega)^{-1},$$

which can easily be checked. By left-multiplying both sides by $(P\Sigma P^T + \Omega)$, it can be written as

$$\mu_{BL}^m = \Pi + \Sigma P^T(P\Sigma P^T + \Omega)^{-1}(v - P\Pi),$$

$$\Sigma_{BL}^m = (\Sigma^{-1} + P^T\Omega^{-1}P)^{-1} = \Sigma - \Sigma P^T(P\Sigma P^T + \Omega)^{-1}P\Sigma.$$

3.3 Blending Prior and Posterior Returns with Theil's Mixed Estimation Approach

Theil's Mixed Estimation approach is another method used for blending prior returns with the views. Walters (2011, [29]) explains the details of this approach in the appendix of his paper. B-L (1992, [5]) assume that prior returns can be written as a linear model such as

$$\Pi = x\beta + u,$$

Π is the equilibrium return vector,

β is the expected return vector,

u is the residual vector which follows a normal distribution with mean 0 and variance Φ.

Walters also introduces views as a linear model such as:

$$q = P\beta + v,$$

where

q is the view vector,

v is residual vector which is normally distributed with mean 0 and variance Ω,

Walters assumes that both Ω and Σ are non-singular.

Prior returns and the returns coming from the views can be combined in the following way:

$$\begin{bmatrix} \Pi \\ q \end{bmatrix} = \begin{bmatrix} x \\ P \end{bmatrix} \beta + \begin{bmatrix} u \\ v \end{bmatrix},$$

where the expected value of the residual is 0, and the expected value of the variance is

$$E\left(\begin{bmatrix} u \\ v \end{bmatrix} \begin{bmatrix} u^T & v^T \end{bmatrix} \right) = \begin{bmatrix} \Phi & 0 \\ 0 & \Omega \end{bmatrix}.$$

In order to estimate β, Walters uses the generalized least squares method:

$$\hat{\beta} = \left[\begin{bmatrix} x & P \end{bmatrix} \begin{bmatrix} \Phi & 0 \\ 0 & \Omega \end{bmatrix}^{-1} \begin{bmatrix} x^T \\ P^T \end{bmatrix} \right]^{-1} \begin{bmatrix} x^T & P^T \end{bmatrix} \begin{bmatrix} \Phi & 0 \\ 0 & \Omega \end{bmatrix}^{-1} \begin{bmatrix} \Pi \\ q \end{bmatrix}.$$

The above equation can be rewritten by using the matrix notation as follows:

$$\hat{\beta} = [x\Phi^{-1}x^T + P\Omega^{-1}P^T]^{-1}[x^T\Phi^{-1}\Pi + P^T\Omega^{-1}q].$$

By substituting formula of combination vector of equilibrium returns and the returns coming from the views into the $\hat{\beta}$ equation not written by using matrix notation, the following equation can be obtained.

$$\begin{aligned}
\hat{\beta} &= [x\Phi^{-1}x^T + P\Omega^{-1}P^T]^{-1}[x^T\Phi^{-1}(x\beta + u) + P^T\Omega^{-1}(P\beta + v)] \\
&= [x\Phi^{-1}x^T + P\Omega^{-1}P^T]^{-1}[x\beta\Phi^{-1}x^T + P^T\Omega^{-1}P\beta + x\Phi^{-1}u + P\Omega^{-1}v] \\
&= [x\Phi^{-1}x^T + P\Omega^{-1}P^T]^{-1}[x\Phi^{-1}x^T\beta + P\Omega^{-1}P^T\beta] + [x\Phi^{-1}x^T + P\Omega^{-1}P^T]^{-1} \\
&\quad [x\Phi^{-1}u + P\Omega^{-1}v],
\end{aligned}$$

$$\hat{\beta} = \beta + [x\Phi^{-1}x^T + P\Omega^{-1}P^T]^{-1}[x\Phi^{-1}u + P\Omega^{-1}v],$$

$$\hat{\beta} - \beta = [x\Phi^{-1}x^T + P\Omega^{-1}P^T]^{-1}[x\Phi^{-1}u + P\Omega^{-1}v],$$

The variance is the expectation of $(\hat{\beta} - \beta)^2$.

$$\begin{aligned}
E((\hat{\beta} - \beta)^2) &= ([x\Phi^{-1}x^T + P\Omega^{-1}P^T]^{-1}[x\Phi^{-1}u^T + P\Omega^{-1}v^T])^2 \\
&= [x\Phi^{-1}x^T + P\Omega^{-1}P^T]^{-2}[x\Phi^{-1}u^T u\Omega^{-1}x^T + P\Omega^{-1}v^T v\Omega^{-1}P^T \\
&\quad + x\Phi^{-1}u^T v\Omega^{-1}P^T + P\Omega^{-1}v^T u\Phi^{-1}x^T].
\end{aligned}$$

By Walters' assumptions, it can be easily deduced that $E(uu^T) = \Phi$, $E(vv^T) = \Omega$ and $E(uv^T) = 0$ since u and v are independent variables.

$$\begin{aligned}
E((\hat{\beta} - \beta)^2) &= [x\Phi^{-1}x^T + P\Omega^{-1}P^T]^{-2}[(x\Phi^{-1}\Phi\Phi^{-1}x^T) + (P\Omega^{-1}\Omega\Omega^{-1}P^T) + 0 + 0] \\
&= [x\Phi^{-1}x^T + P\Omega^{-1}P^T]^{-2}[x\Phi^{-1}x^T + P\Omega^{-1}P^T].
\end{aligned}$$

In order to obtain the B-L model, x should be identity matrix and $\Phi = \tau\Sigma$. Therefore after making these substitutions, Walters gets:

$$E((\hat{\beta} - \beta)^2) = [(\tau\Sigma)^{-1} + P\Omega^{-1}P^T]^{-1}.$$

To sum up, the B-L model helps investors with two specific features while computing the optimal weights for securities in their portfolios. The first one is that the market portfolio based on Capital Asset Pricing Model is defined as the initial portfolio before incorporating investors' views. Thus, the model has the substantial theoretical background and quantitative investors can easily calculate the initial portfolio returns and weights according to the CAPM framework. Moreover, in the B-L model the assets on which analysts do not specify any views still have their market-capitalization weights. In other studies in the literature, randomly weighted, equally weighted or global minimum variance portfolios are taken as initial portfolios [29]. The second attribute of this model is that investors are able to express their views on the security returns either in an absolute or a relative form. In Markowitz model, investors should estimate each asset's expected return in a portfolio, and this indicates that extreme weights can be assigned to each security in the portfolio due to the high sensitivity of model to small changes in inputs in the model. In the B-L model, the views can be expressed on the returns of stocks for which investors have information or have conducted security analysis. Furthermore, the confidence in these views can also be stated. Therefore, final weights assigned to securities with views change from the initial market capitalization weights in the direction of the views specified. These changes in weights are larger when investors are more confident about their views. Final weights assigned to securities without views do not change by much from the initial market capitalization weights.

CHAPTER 4

DATA AND METHODOLOGY

4.1 Data

In the analysis of this thesis, monthly continuously compounded returns on the stocks included in the Borsa Istanbul-50 (BIST-50) index are used. First, stocks that are consistently included in BIST-50 index in any given year during the sample period of this thesis are determined. Then monthly returns on these securities for the five years prior to the year of analysis are used to estimate the expected excess returns and variance-covariance matrix of these securities. For instance, for the year 2005, the stocks which are included in the BIST-50 index in the 1^{st}, 2^{nd}, 3^{rd}, and 4^{th} quarters of 2005 are identified. Then among those, the ones which have monthly return data available from January 2000 to December 2004 are selected. Which stocks included in the BIST-50 index for all periods in a particular year are found under the tab called "Constituent Equities of Indices" on the web site of BIST (www.borsaistanbul.com). Then, availability of monthly returns data for the prior 5 years are checked under the tab called "Monthly Price and Return of Equity Market Data" on the web site of BIST. List of the stocks analyzed for the periods from 2005 to 2015 are reported in the Table 4.1 on the next page. For each of the years from 2005 to 2015, monthly return data on these stocks for the prior 5 years, a total 60 observations, are downloaded. These returns are calculated from closing prices of the stocks adjusted for stock splits and dividends.

The excess returns on sample stocks, is calculated by subtracting the monthly risk-free rate from corresponding monthly stock returns. The effective annual return on the 3-month Turkish government securities is taken as a proxy for the risk-free rate since the data for 1-month Turkish treasury bills is not consistently available for the sample period analyzed in this thesis. Data for 3-month Turkish government securities are available under the tab called "Debt Securities Market Data" on the official website of the BIST. From the daily bulletins of government securities, effective annual yields on securities that have remaining time to maturity closest to 90 days in each month are selected as the proxy for the risk-free security. These effective annual returns are converted to monthly returns by using the following formula:

$$r_{f\text{monthly}} = (1 + r_{f\text{compounded}})^{1/12} - 1.$$

Table 4.1: Stock Codes Used in This Thesis for the Period from 2005 through 2015

This table presents the sample stocks analyzed in each of the years between 2005 and 2015. To determine this sample, first stocks that are consistently included in the BIST-50 index in each year are determined. After that, the existence of the monthly return data for the prior 5 years is controlled for each of these securities on the official web site of BIST. The stocks meeting both of these conditions are shown in this table.

The estimation years / The Codes of company	2005	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015
AEFES	X	X	X	X	X	X		X	X	X	X
AKBNK	X	X	X	X	X	X	X	X	X	X	X
AKCNS			X								
AKENR				X	X	X	X	X	X		
AKGRT	X	X	X	X	X	X					
AKSA	X										
ALARK	X		X	X							
ANSGR		X	X	X	X	X					
ARCLK	X	X	X	X	X	X	X	X	X	X	X
ASELS	X						X	X	X	X	X
ASYAB								X	X		
AYGAZ			X	X	X	X			X		
BAGFS					X	X	X	X			X
BEKO	X										
BIMAS							X	X	X	X	X
CCOLA										X	X
DEVA		X									
DISBA	X										
DOAS										X	X
DOHOL	X	X	X	X	X	X	X	X	X	X	X
DYHOL	X	X	X	X	X	X	X	X	X		
ECILC	X	X	X	X	X	X	X	X	X		
ENKAI				X	X	X	X	X	X	X	X
EREGL	X	X	X	X	X	X	X	X	X	X	X
FENER							X	X			
FINBN	X	X									
FORTS		X	X								
FROTO	X	X	X					X	X	X	X
GARAN	X	X	X	X	X	X	X	X	X	X	X
GLYHO	X	X	X	X							
GOLTS										X	X
GOZDE											X
GSDHO		X	X	X		X					
GSRAY								X			
GUBRF					X		X		X	X	X
HALKB									X	X	X
HURGZ	X	X	X	X	X	X					
IHLAS			X	X			X	X	X	X	
IPEKE							X		X	X	
ISCTR	X	X	X	X	X	X	X	X	X	X	X
ISGYO	X	X	X	X							
IZMDC								X			
KARTN		X									
KCHOL	X	X	X	X	X	X	X	X	X	X	X
KONYA								X			
KOZAA						X	X	X	X	X	X
KRDMD	X	X	X	X	X	X	X	X	X	X	X
MGROS								X	X	X	X
MIGRS	X	X	X	X							
NETAS								X			
NTHOL	X		X	X	X	X					
OTKAR										X	X
PETKM	X	X	X	X	X	X	X	X	X	X	X
PRKME								X			
PRKTE	X	X									
PTOFS	X	X	X	X	X						
SAHOL	X	X	X	X	X	X	X	X	X	X	X
SISE	X	X	X	X	X	X	X	X	X	X	X
SKBNK		X	X	X	X	X					
SNGYO									X		
TAVHL									X	X	X
TCELL		X	X	X	X	X	X	X	X	X	X
TEBNK					X	X	X				
THYAO		X	X	X	X	X	X	X	X	X	X
TIRE						X					
TKFEN										X	X
TNSAS	X										
TOASO	X	X	X	X	X	X	X	X	X	X	X
TRKCM			X	X	X				X	X	X
TSKB		X	X	X	X	X				X	X
TTKOM										X	X
TTRAK								X	X	X	
TUPRS	X	X	X	X	X	X	X	X	X	X	X
ULKER										X	X
VAKBN							X	X	X	X	X
VESTL	X	X	X	X	X	X					X
YAZIC										X	
YKBNK	X	X	X	X	X	X	X	X	X	X	X
ZOREN						X					

Finally, the natural logarithm of these excess returns are calculated by the formula $ln(1 + R)$ in Excel. As a result, monthly logarithmic excess returns of BIST-50 stocks are obtained to determine the optimal number of stocks needed to form a well-diversified portfolio from BIST-50 stocks.

For the application of the Markowitz and the Black-Litterman (B-L) models, a prior information set based on the CAPM approach is needed. Furthermore, for the B-L model, a second information set consisting of investor views should be constructed.

Let us start with the prior information set based on the CAPM approach. For the prior return estimation, the data used in this application are composed of monthly logarithmic excess returns on the stocks consistently included in the BIST-50 Index for the period from January 2000 to December 2014. The last data needed for the prior information set is the market capitalization of stocks on the month before the beginning of each 1 year period. This data is also collected from the tab called "Equity Market Data" on the website of the BIST for the December of each year from 2004 to 2014. Market capitalization weight of each stock is calculated by dividing market value of this security by the total market value of all the stocks that are available for inclusion in the portfolio for that year.

Investor view, in this thesis, is taken as the analysts' forecasts of target stock price for a company. This data is collected from the Bloomberg Analyst Recommendations module. From the menu of analyst recommendations, one can reach the buy/sell recommendations of analysts for a company, target price for that company shares and a time period over which this target price is expected to be observed. These recommendations and forecasts are made by Bloomberg analysts and analysts all around the world.

On the menu of analyst recommendations in the Bloomberg database, there is a list of target prices which are expressed by several analysts for a selected stock, and also there is an average target price for that stock. This average target stock price is used to represent the views of analysts, hence investors, in this thesis. Analyst recommendations page displays several analysts' target price estimates over the next 12 months from a specified date. Average target price is calculated by taking the mean of analysts' target price estimates in the last 3 months from a specified date.

The average target prices available in the Bloomberg for the i^{th} stock should be converted into an average target return for that stock by using the following equation:

$$r_i = \frac{\text{Average Target}P_i - \text{Current}P_i}{\text{Current}P_i}.$$

These target returns are for a 12-month period. However, the prior information set of this thesis contains monthly returns. Therefore, these annual target stock returns should be converted into monthly target returns as follows:

$$r_{\text{monthly}} = (1 + r_{\text{12months}})^{(1/12)} - 1.$$

Since our prior information set has logarithmic monthly excess returns on stocks, and the second information set should be consistent with the prior one, the views have to expressed as monthly excess returns as well. First, the excess monthly target returns on stocks are calculated by subtracting the corresponding risk free-rate for the estimation year from these returns. Second, the natural logarithms of these excess target returns are calculated by using the formula $ln(1+R)$ in Excel. Thus, the second set of information needed to apply the B-L model is obtained.

4.2 Methodology

4.2.1 Diversification

The excess returns of BIST-50 stocks are expected to be normally distributed. First, the normality of excess returns is checked by the Jarque-Bera test on MATLAB. The Jarque-Bera test is one of the normality test based on skewness and kurtosis. Skewness is a measure of how symmetric the returns are around their mean, and the sample skewness is calculated as:

$$S = \frac{\mu_3}{\sigma^3} = \frac{\frac{1}{N}\sum_{i=1}^{N}(r_i - \bar{r})^3}{\left(\frac{1}{N}\sum_{i=1}^{N}(r_i - \bar{r})^2\right)^{3/2}},$$

N : the number of observations (60 observations in our case).

Kurtosis is a measure of how thick the tails of the probability distribution of returns, and the sample kurtosis is defined as:

$$K = \frac{\mu_4}{\sigma^4} = \frac{\frac{1}{N}\sum_{i=1}^{N}(r_i - \bar{r})^4}{\left(\frac{1}{N}\sum_{i=1}^{N}(r_i - \bar{r})^2\right)^{2}}.$$

A normal distribution has the skewness of 0 and the kurtosis of 3. The Jarque-Bera test is now introduced. The null hypothesis states that the data follows a normal distribution, and the alternative hypothesis states that the data does not follow a normal distribution. The Jarque-Bera test statistic is:

$$JB = N\left[\frac{S^2}{6} + \frac{(K-3)^2}{24}\right],$$

which asymptotically follows a chi-squared distribution with two degrees of freedom. The test is proceeded as follows: when the calculated test statistic is greater than the critical value obtained from the table of χ^2_2 distribution at the particular significance level, the null hypothesis is rejected.

As mentioned in Subsection 2.1.2, a naive diversification strategy is performed. For each year, an Excel file is created with calculated logarithmic excess returns of the stocks. This Excel file is imported into MATLAB[1]. By means of the covariance function available in MATLAB, the variance-covariance matrix of these returns is calculated. After that, a random number ranging from 1 to total number of stocks to be analyzed in a given year is generated for the 1-stock portfolio. By taking the square root of corresponding variance of the chosen stock, standard deviation of the selected stock is calculated. This process is repeated 1000 times.

The standard deviation of each repeated 1-stock portfolio and the mean standard deviation of these portfolios are kept in the program of MATLAB. For 2 stock portfolios, a random number ranging from 1 to total number of stocks to be analyzed in a given year is generated. After that another random number ranging from 1 to total number of stocks to be analyzed excluding the number of the first stock selected is generated. Hence, stocks are chosen without replacement. Corresponding variances and covariances of these two chosen stocks are extracted from the variance-covariance matrix via the program in MATLAB. The standard deviations of each 2 stock portfolio and the mean standard deviation of these portfolios are calculated by the Equation 2.12 demonstrated in Subsection 2.1.2. This process is carried on with 3, 4, ..., total number of stocks in a given year stock portfolios.

To determine the optimal range of stocks to form a well-diversified portfolio, a t-test is performed. A statistically significant change in the mean standard deviation of two portfolios that have N (sample 1) and $N + 2$ (sample 2) stocks in them respectively are tested in this thesis by using a right tailed t-test at 5% significance level. Null hypothesis states that mean of the two samples is equal, and the alternative hypothesis states that mean of the first sample is greater than the mean of the second sample. T-test statistic to compare the sample mean of portfolio standard deviations are calculated as:

$$t = \frac{\bar{\mu}_1 - \bar{\mu}_2}{\sqrt{\frac{\bar{\sigma}_1}{N_1} + \frac{\bar{\sigma}_2}{N_2}}},$$

where

[1] All MATLAB codes used to run the analyses of this thesis are represented in the Appendix B.

$\bar{\mu}_1$, and $\bar{\mu}_2$: sample means,
$\bar{\sigma}_1$, and $\bar{\sigma}_2$: sample variances,
N_1, and N_2 : sample sizes.

When the calculated t statistic is greater than the critical t-value at 5% significance level, the null hypothesis is rejected. That range for which the null hypothesis is rejected for the first time is defined as the optimal number of stocks to have in a well-diversified portfolio.

To conduct this hypothesis testing, the program which is available in the Appendix B is written in MATLAB. First of all, the data of standard deviations of all repeated portfolios ranging from 1 to maximum number of stocks for a particular year is extracted from the excel file generated by the first program in the Appendix B. The sample consisting of standard deviations of 1 stock repeated portfolios and the sample including standard deviations of 2 stock repeated portfolios are tested in terms of means, and this process is continued with the data for 2-4 stock repeated portfolios, 4-6 stock repeated portfolios, and so on. The last range of stocks for which their mean standard deviations are statistically different from each other before no statistically significant difference is obtained for the first time is determined as the optimal range of stocks to construct a well-diversified portfolio.

4.2.2 Black-Litterman Method

In this Subsection, how the B-L model is constructed and applied in our study is explained in detail. There are four main steps to apply B-L model. First, the implied equilibrium returns based on the CAPM approach are found. Second, analyst views are formulated. Later, the posterior expected return and variance matrices are found by inserting the calculated values in the first two steps into B-L formula. Finally, the optimal posterior weight vector is found by the utility function optimization. Figure 4.1 represents each step of the B-L model. The details of each step are discussed in the following paragraphs.

Along with the collected data which is described in Section 4.1, the logarithmic excess equilibrium returns of stocks which are implied by the CAPM approach need to be calculated and used as prior returns of stocks in the B-L method. In order to obtain these equilibrium returns, the following utility function is required to be maximized:

$$U = w^T \Pi - \left(\frac{\lambda}{2}\right) w^T \Sigma w.$$

Here,

Π is the $N \times 1$ logarithmic excess equilibrium return vector,

λ is the risk aversion coefficient (scalar),

Σ is the $N \times N$ covariance matrix of logarithmic excess returns,

w is the $N\times$ 1 weight vector of the assets, and

N is the number of assets in the portfolio (For instance, for year 2015, N=39).

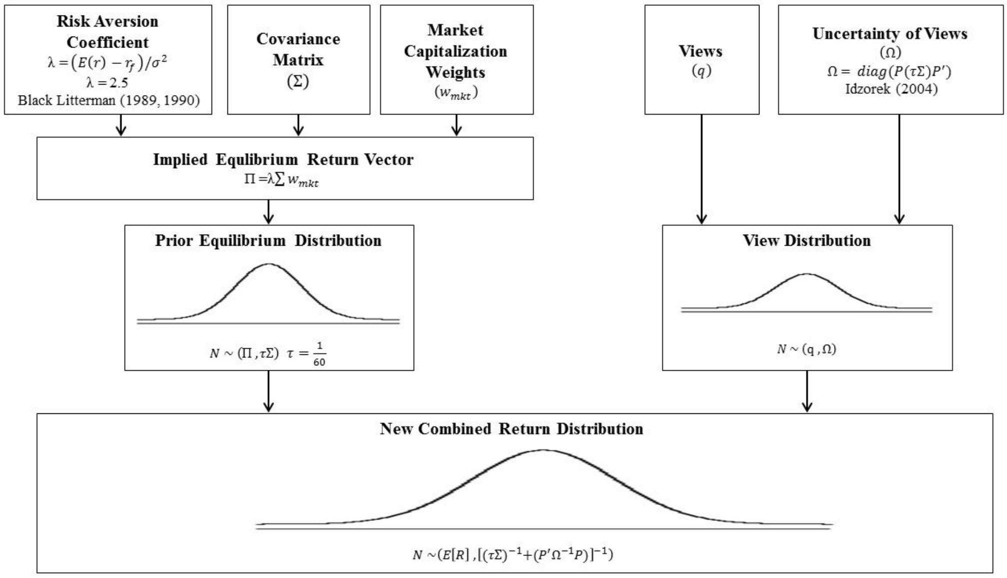


Figure 4.1: Steps of the Black-Litterman Model.

This figure shows each step of the B-L model taken from [16]. B-L model includes two information sets which are the prior information set derived from the CAPM and the second information set coming from Bloomberg analysts' views. Let us start explaining the prior information set consisting of risk aversion coefficient, covariance matrix, market capitalization weights. In this thesis, the world average risk aversion coefficient is taken as 2.5 in accordance with the study of Black (1989, [3]). The covariance matrix is the variance-covariance matrix of the 5-year historical returns. Each stock in the portfolio is weighted according to its market capitalization. After attaining the first information set, the implied equilibrium return vector can be calculated by taking the first derivative of the utility function of an investor with respect to the weight vector (w), then equating this to 0. The equilibrium returns are assumed to be distributed normally with mean Π and variance $\tau\Sigma$. τ is a scaling factor. It is determined as the inverse of the number of observations in this thesis. Let us continue explaining the second information set involving analysts' views. The weight vector q constitutes the stock returns derived from Bloomberg analysts' average target stock prices. The variance matrix of the views Ω is determined as historical variances of the stocks scaled by τ as the study of Idzorek (2004, [15]). Hence, the view returns are assumed to be distributed normally with mean q and variance Ω. Finally, an investor can blend the implied equilibrium returns with analyst views by Bayesian approach, and thus the posterior expected return and the posterior variance can be obtained.

Since U is a concave function, it has a single global maximum. To find this global maximum, the first derivative of this function with respect to w, has to be equated to 0.

The closed form of the solution is obtained as:

$$w = (\lambda\Sigma)^{-1}\Pi.$$

According to the CAPM approach, optimal weights are the market capitalization weights of the stocks in the portfolio. Market capitalization weight of k^{th} stock is calculated by dividing the market value of this security by the total market value of the portfolio constructed from all the securities available for investment:

$$w_{\text{MKT}}(k) = \frac{\text{Market Capitalization}(k)}{\sum_{k=1}^{N} \text{Market Capitalization}(k)}.$$

Variance-covariance matrix, Σ, is estimated from 5-year historical logarithmic excess returns of the stocks and is computed by using the covariance function in MATLAB.

In our study, the risk aversion coefficient is taken to be 2.5 which is estimated by Black (1989, [3]) as the world average risk aversion coefficient. Then using this constant risk aversion coefficient, the utility function of the investor with new expected returns (the CAPM returns combined with analysts' views) and variances of these returns is maximized.

Since the terms λ, Σ, and w_{MKT} are estimated from market data, they are known. The implied logarithmic excess equilibrium returns are obtained via a reverse optimization method from known information:

$$\Pi = \lambda\Sigma w_{\text{MKT}}.$$

Thus, the prior expected returns derived from the CAPM approach are obtained. Now it is time to formulate analyst views.

To construct the P "pick matrix", for each stock available for investment in any year analyzed in this thesis, the average target price estimates of analyst are collected from Bloomberg. As it is mentioned in Chapter 3, the views can be expressed either in an absolute or relative form. Since the Bloomberg database covers only absolute information about the stocks used in this thesis, we have just absolute views.

P is a $K \times N$ matrix. K represents the number of stocks on which analysts expressed a target price, and thus an average target price is available. N represents the total number of stocks available for investment in a portfolio for the given estimation year. For instance, for year 2015, the number of the stocks for which there is an average 12 month target price for the period from January 2014 to January 2015 is determined as

37 out of 39 stocks that are available for investment in a portfolio. Therefore, P is a 37×39 matrix for the year 2015. Each row of the matrix P represents one view. Since we have only absolute views, each view involves only one asset. In each row of the matrix P, there is only single 1 assigned to the stock on which a view is expressed, and all the other assets have 0s. Hence, the sum of the elements of each row equals to 1.

The view vector q is a K×1 column vector. In our study, q vector demonstrates the average target returns for stocks. Therefore, average target price available in the Bloomberg for the i^{th} stock is used to calculate the logarithmic monthly excess returns based on views as explained before.

After defining the expected logarithmic excess returns on the stocks, the uncertainty of these expected returns needs to be estimated. These uncertainties are represented by a K×K matrix, Ω. Ω is a diagonal matrix since the views are assumed to be independent from each other. Because of the fact that some of analysts' target prices are not reported in the Bloomberg database to everyone but, when calculating the average 12-month target price for a selected stock, these unreported estimates are also taken into account by the Bloomberg, the exact variability of these expected target prices from the mean target price can not be calculated. Hence, the historical variance of the actual returns on a selected stock is used to estimate the uncertainty of the expected return on this stock. The matrix Ω is constructed as mentioned in Idzorek (2004, [15]) by using the following formula:

$$\Omega = \text{diag}(P(\tau\Sigma)P^T).$$

Thus, the second information set is also ready to put in the B-L formula.

The main motivation of the B-L method is to combine a quantitative data and a qualitative view data into a joint framework. The equilibrium return vector (Π), the scalar (τ) and the historical variance-covariance matrix (Σ) are the components of the first information set. The pick matrix (P), the view vector (q), and the covariance matrix of the views (Ω) are part of the second information set. All of these terms are estimated in order to incorporate into the B-L formula. Blending of the CAPM returns with investor views using Bayesian as well as Theil's Mixed Estimation approaches are shown in Chapter 3 of this thesis. Both methods give the same result shown below:

$$f(E(r)|\Pi) \sim N([(\tau\Sigma)^{-1}+P^T\Omega^{-1}P]^{-1}[(\tau\Sigma)^{-1}\Pi+P^T\Omega^{-1}q], [(\tau\Sigma)^{-1}+P^T\Omega^{-1}P]^{-1}),$$

$E(r)$ is a N×1 new combined logarithmic excess return vector,

Π is a N×1 logarithmic excess equilibrium return vector,

τ is a scaling factor of historical covariance matrix (τ is constant and equals to 0.0167 in this thesis),

Σ is a $N \times N$ covariance matrix of historical logarithmic excess returns,

P is a $K \times N$ pick matrix of the stocks with views,

q is a $K \times 1$ logarithmic excess view returns vector,

Ω is a K×K the views' covariance matrix (Ω = diag($P(\tau\Sigma)P^T$) in this thesis).

The result is that the new expected returns are normally distributed with a mean of $\overline{\mu}$ and a variance of $\overline{M}^{-1}$, where

$$\overline{\mu} = [(\tau\Sigma)^{-1} + P^T\Omega^{-1}P]^{-1}[(\tau\Sigma)^{-1}\Pi + P^T\Omega^{-1}q],$$

$$\overline{M}^{-1} = [(\tau\Sigma)^{-1} + P^T\Omega^{-1}P]^{-1}.$$

These updated mean returns, namely $\overline{\mu}$, can be rewritten as:

$$\overline{\mu} = \Pi + \tau\Sigma P^T[(P\tau\Sigma P^T) + \Omega]^{-1}[q - P\Pi].$$

This formula is derived by some matrix operations. Interested readers can refer to Walters (2011, [29]) for the derivation of this formula.

As it is indicated in the beginning of the Chapter 3, the prior expected return vector is distributed normally with a mean μ and a variance Σ:

$$r \sim N(\mu, \Sigma).$$

Later, the mean returns are updated as $\overline{\mu}$. However, the distribution of the returns is not $N(\overline{\mu}, \Sigma)$. The posterior variance also needs to be updated. $\overline{M}^{-1}$ is the posterior variance, which is the variability of the posterior mean estimate around the actual mean. However, this, is not the variance of the posterior returns. In order to calculate the variability of posterior returns, the variance of the posterior mean estimate around the actual mean has to be added to the variance of the distribution of returns around the mean.

As He and Litterman (1999, [14]) states the expected returns are themselves random variables in the B-L model, therefore the distribution of the posterior returns becomes:

$$r \sim N(\overline{\mu}, \overline{\Sigma}),$$

$\overline{\Sigma} = \Sigma + \overline{M}^{-1}$.

This new combined return vector and the posterior variance are computed via MATLAB for each estimation year, since most of the inputs such as Π, Σ, P, q, Ω, K and N vary from one year to another.

In the final step, with the calculated $\overline{\mu}$ and the calculated covariance matrix $\overline{\Sigma}$, the optimal posterior weights can be calculated by the mean-variance optimization model. The utility function being maximized is as follows:

$$\underset{w}{\text{maximize}} w^T\overline{\mu} - \frac{\lambda}{2} w^T \overline{\Sigma} w.$$

Let us take the first derivative of this utility function with respect to w, and obtain

$$w^{\star} = \frac{1}{\lambda}\overline{\Sigma}^{-1}\overline{\mu},$$

where $w^{\star}$ is the vector of the optimal posterior weights.

This unconstrained optimization problem is solved using fminunc function in MATLAB. All codes are presented in the Appendix B.

4.2.2.1 Black-Litterman Method with Budget Constraint

In this case, the optimization problem to be solved can be formulated as follows:

$$\begin{gathered} \underset{w}{\text{maximize}} w^T\overline{\mu} - \frac{\lambda}{2} w^T \overline{\Sigma} w \\ \text{subject to} \\ \sum_{k=1}^{N} w_k = 1. \end{gathered}$$

This optimization problem with a budget constraint is solved by using fminconmaxutility function in MATLAB. MATLAB codes of this program are presented in the Appendix B.

4.2.2.2 Black-Litterman Method with Budget and No Short Selling Constraint

The mathematical formulation of the optimization problem in the case of budget and no short selling constrained B-L model can be shown as follows:

$$\underset{w}{\text{maximize}} w^T\overline{\mu} - \frac{\lambda}{2} w^T \overline{\Sigma} w$$
$$\text{subject to}$$
$$\sum_{k=1}^{N} w_k = 1,$$
$$w_k \geq 0 \quad \forall k.$$

This budget and no short selling constrained optimization problem is solved using fminconmaxutility2 function in MATLAB. MATLAB codes for this program are presented in the Appendix B.

CHAPTER 5

EMPIRICAL FINDINGS

5.1 Optimal Number of Stocks Needed to Create a Diversified Portfolio of BIST-50 Securities

In this section, the optimal number of stocks to have in a well-diversified portfolio of stocks included in the Borsa Istanbul-50 (BIST-50) Index are determined for the annual periods from 2005 to 2015 by using Markowitz optimization method. In addition to this, the same operation is performed for the posterior values of variance-covariance matrix in the Black-Litterman (B-L) model. The goal in this section is to analyze the change in the required number of stocks to create a well-diversified portfolio when investor views are taken into account.

The Jarque-Bera test described in Subsection 4.2.1 is performed on logarithmic and simple stock returns, and the results are reported in Table 5.1 on the next page. Since logarithmic returns of more stocks are shown to be distributed normally by Jarque-Bera test compared to simple returns for the majority of the years analyzed in this thesis, logarithmic excess returns are used in the analyses of this thesis.

As described in Section 4.1, for BIST-50 stocks listed in the Table 4.1 for each year of analysis, logarithmic excess returns data for the prior five years are prepared. After that, analyses are carried out by using the method explained in Subsection 4.2.1 for each year. 1000 portfolio replications are conducted for each portfolio sizes. According to the research of Beck, Perfect and Peterson (1996, [1]), as the number of portfolio replications increases, more smoother curves are obtained and also the results reflect the population much better. However, an increase in the number of replications also results in an increase in the sensitivity of the test statistic. As mentioned in their study, increase in the number of replications affects the rejection level of the null hypothesis of equal mean standard deviations for two successive portfolio sizes. By defining a smaller change in the number of stocks included in two consecutive portfolios used in the statistical test, the issue can be resolved. Therefore, statistical significance of a change in the mean standard deviation of two portfolios that have N (sample 1) and $N + 2$ (sample 2) stocks in them respectively are tested in this thesis by using a t-test to determine the optimal number of stocks needed to have well-diversified portfolio.

Table 5.1: The Results of Jarque-Bera Test

This table summarizes the results of the Jarque-Bera test on prior 5-year historical simple and logarithmic excess returns of all securities listed in the Table 4.1 for each year between 2005 and 2015 at 1% level of significance.

The estimation years	**2005**		**2006**		**2007**	
The return data used in J-B test	**Normal**	**Logarithmic**	**Normal**	**Logarithmic**	**Normal**	**Logarithmic**
The number of stocks normally distributed	23	28	23	28	28	32
The number of stocks not normally distributed	11	6	13	8	10	6
	2008		**2009**		**2010**	
	Normal	**Logarithmic**	**Normal**	**Logarithmic**	**Normal**	**Logarithmic**
The number of stocks normally distributed	28	34	29	27	33	28
The number of stocks not normally distributed	9	3	5	7	3	8
	2011		**2012**		**2013**	
	Normal	**Logarithmic**	**Normal**	**Logarithmic**	**Normal**	**Logarithmic**
The number of stocks normally distributed	27	27	30	27	32	33
The number of stocks not normally distributed	3	3	7	10	5	4
	2014		**2015**			
	Normal	**Logarithmic**	**Normal**	**Logarithmic**		
The number of stocks normally distributed	36	37	36	36		
The number of stocks not normally distributed	4	3	3	3		

Number of stocks, mean standard deviation of monthly logarithmic excess portfolio returns, and the ratio of mean standard deviation of an N-stock portfolio to mean standard deviation of a single stock portfolio for each year are shown in Table A.1 in the Appendix A. It is observed that as number of stocks increases in the portfolio, mean standard deviation of the portfolios declines due to diversification effect. For the year 2005, even if an investor holds the maximum number of stocks in the portfolio, only 17% of the total risk can be diversified away. On the other hand, for the year 2015, if all stocks are held in a portfolio, even though the percentage of the risk diversified away goes up to 31%, a significant portion of the total risk (69%) still remains in the portfolio. This indicates that the share of the systematic (undiversifiable) risk in total risk of a security in the Turkish financial market is quite high.

It is known that the more stocks an investor adds to a portfolio, the more risk-reduction benefit an investor gets, while, incurring higher transaction costs. In order to determine the optimal number of stocks to have in a well-diversified portfolio, the portfolio size for which there is no statistically significant reduction in the mean standard deviation of the portfolio has to be identified. If this analysis is done by comparing the mean standard deviation of two portfolios that have N (sample 1) and $N + 1$ (sample 2) stocks in them respectively, no change in the mean portfolio standard deviation requirement can be satisfied very quickly leading us to conclude that a few number of stocks are enough to create a diversified portfolio in Turkey. Thus, the statistically significant change in the mean standard deviation of two portfolios that have N (sample 1) and $N + 2$ (sample 2) stocks in them respectively are tested in this thesis.

The results including t-test statistics and t critical values for each year are shown in Table A.2 in the Appendix A. When no statistically significant difference in standard deviations of two portfolios with N and $N + 2$ securities is observed for the first time in a given year, the range of stocks prior to this, ($N - 2$ to N) is taken as the optimal number of stocks for that year. This range of optimal number of stocks for each year analyzed in this thesis are shown in Table 5.2. In year 2005, by holding the maximum number of stocks available, which is 34, at most 17% of the total risk of an average security can be diversified away as seen in Table A.1 in the Appendix A. By holding the optimal number of stocks which ranges from 10 to 12, approximately 16% of the total risk of an average security has already been diversified away. Therefore, a 1% reduction in the average standard deviation of the portfolio as we go from 10-12 stocks to 34 stocks is not considered as a statistically significant reduction in the portfolio risk. To gain a 1% risk-reduction benefit, high transaction costs may need to be paid. On the other hand, in year 2015, 29% of the total risk of an average security can be eliminated by holding 16 to 18 stocks in a portfolio. This reduction in portfolio risk increases to 31% of the total risk of an average security when the maximum number of stocks (39) available for investment are held in the portfolio. In the case of year 2015, the additional risk reduction of 2% in the average standard deviation of the portfolio as we increase the number of stocks in the portfolio from 16-18 to 39 is not statistically significant. As for the other years, the reduction in the mean portfolio standard deviation is at most 3% as we go from holding the optimal number of stocks to the maximum number of stocks available for investment. Again these reductions are not statistically significant and they may not be economically significant to justify high transaction costs associated with larger portfolio sizes.

The results in Table 5.2 show that the optimal number of stocks to hold in a well-diversified portfolio differs from one year to another due to change in correlation between the stocks. It is observed that an investor needs to hold more stocks in recent years compared to earlier years to create a diversified portfolio. While 10-12 stocks are needed to construct a well-diversified portfolio for the first observation year of 2005, the range of the optimal number of stocks needed for a well-diversified portfolio goes up to 16-18 for the last observation year of 2015.

Even though the sample stocks analyzed in this thesis are chosen from a different index and data from a different time period is analyzed, findings of this thesis are quite consistent with the findings from earlier studies on Turkish stock market. For example, Gökçe and Cura (2003, [10]) find the range for optimal number of stocks to be as 6-13 for equally weighted portfolios created from stocks included in the BIST-30 Index over the period from January 1999 to June 2000. Over the years analyzed in this thesis, the optimal number of stocks to hold ranges from 8-10 to 16-18. Whereas, an average range of the optimal number of stocks over the observed periods is approximately 11-13 which is consistent with the work of Gökçe and Cura (2003, [10]). Similarly, Atan

Table 5.2: The Range of Optimal Number of Stocks for each Year

This table presents ranges of the optimal number of stocks to create a well-diversified portfolio for each estimation year during the sample period from 2005 to 2015. The analyzed year is shown in the first column of this table and the second column reports the beginning and the end of the periods of historical monthly return data used in the analysis for that year.

Estimation year	**The period of historical monthly data used**	**The range of the optimal number of stocks**
2005	Jan 2000-Dec 2004	10-12
2006	Jan 2001-Dec 2005	8-10
2007	Jan 2002-Dec 2006	10-12
2008	Jan 2003-Dec 2007	12-14
2009	Jan 2004-Dec 2008	12-14
2010	Jan 2005-Dec 2009	10-12
2011	Jan 2006-Dec 2010	16-18
2012	Jan 2007-Dec 2011	8-10
2013	Jan 2008-Dec 2012	12-14
2014	Jan 2009-Dec 2013	14-16
2015	Jan 2010-Dec 2014	16-18

and Duman (2007) analyze stocks that are constituents of BIST-100 Index, and find 11, which is consistent with the findings in this thesis, as the optimal number of stocks to have in a well-diversified portfolio.

Even though the optimal number of stocks to have in a well-diversified portfolio changes from one year to another, according to the results reported in Table 5.2, over the period analyzed in this thesis on average, an investor needs to hold 11-13 stocks to gain most of the risk-reduction benefits from diversification.

5.1.1 Comparison of Results with Prior and Posterior Variance-Covariance Matrix in terms of Diversification

In Section 5.1, when determining the optimal number of stocks for a well-diversified portfolio, only the variance-covariance matrix of 5-year historical monthly returns is used. B-L define this variance-covariance matrix as a prior matrix. In order to determine the optimal weights for each stock in the portfolio while taking into account investor views, they develop a new model which is explained in detail in Chapter 3 of this thesis. In their model, the returns of stocks and the variances of these returns on stocks based on the CAPM approach form the first information set and investor views on the returns of stocks constitute the second information set. These two information sets are blended by Bayes' Formula. The posterior variance-covariance matrix consisting of the combination of the prior one and the variances of stock returns based on investor views is obtained. Mathematical background on the posterior variance-covariance matrix construction is also explained in Chapter 3. The prior one is denoted as Σ, and the formula for the posterior one as shown in the article of He and Litterman (1999, [14]) is as follows:

$$\bar{\Sigma} = \Sigma + ((\tau\Sigma)^{-1} + P^T\Omega^{-1}P)^{-1},$$

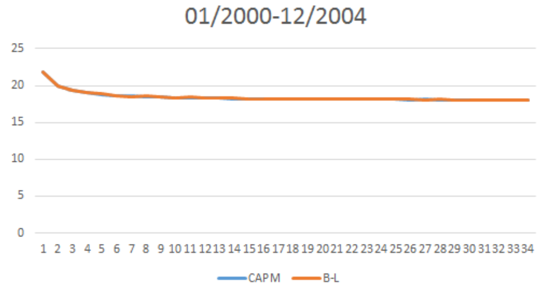
where τ is a scaling factor, P is the pick matrix which indicates the stocks on which investors express their views, and Ω is the matrix indicating the uncertainty associated with the views of investors.

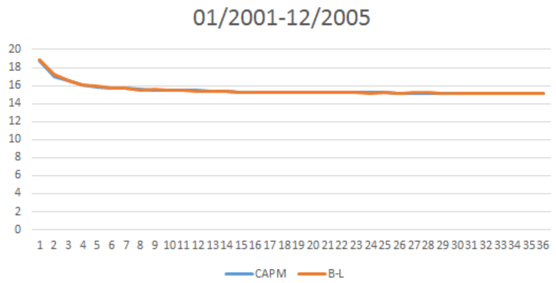
τ is determined as inverse of the total number of observations. Since we use 60 monthly historical returns for each year analyzed, τ equals to 0.0167 in this thesis. In addition to this, Ω is constructed as diag$(P(\tau\Sigma)P^T)$ in this thesis.

Analysis conducted in the previous section on the prior variance-covariance matrix is repeated here for the posterior variance-covariance matrix. The objective is to see how the optimal number of stocks change when investor views and the uncertainty associated with these views are taken into account. The posterior variance-covariance matrix based on the B-L method is used as an input. Table A.3 in the Appendix A shows the number of stocks in the portfolio, average standard deviation of the combination of logarithmic excess returns and investor views' returns, and the ratio of average standard deviation of the N-stock portfolio to average standard deviation of a single stock portfolio over the period analyzed in this thesis. The average standard deviation of the combined returns decreases as the numbers of stocks in the portfolio increases due to diversification effect as it does with the prior variance-covariance matrix. For the year 2005, the percentage of the risk diversified is just 17% by holding all of the stocks available for that period, whereas it increases to 31% for the year 2015. These results are quite similar to the ones obtained by using the prior variance-covariance matrix.

The relation between the average portfolio standard deviation and the number of stocks in the portfolio are shown in Figure 5.1 for each year in the sample when the prior variance-covariance matrix based on the CAPM and the posterior variance-covariance matrix based on the B-L method are used. There is no noticeable difference between these graphs for any of the years analyzed.

In order to find the optimal number of stocks for a well-diversified portfolio, a right tailed t-test at 5% significance level is also performed. The results are shown in Table A.4 in the Appendix A. Average variances, covariances, correlation coefficients and optimal range of stocks for a well diversified portfolio by using variance matrices based on both methods are summarized in Table 5.3. This table indicates that the average variances and covariances of the posterior matrix based on B-L approach are little bit higher than the average variances and covariances of the prior one. Moreover, for 2005, 2006, 2011 and 2015, less number of stocks are needed to create a well-diversified portfolio when the posterior variance matrix is used. For the remaining years, the range for the optimal number of stocks is the same as in the case of prior variance.


01/2000-12/2004
CAPM
B-L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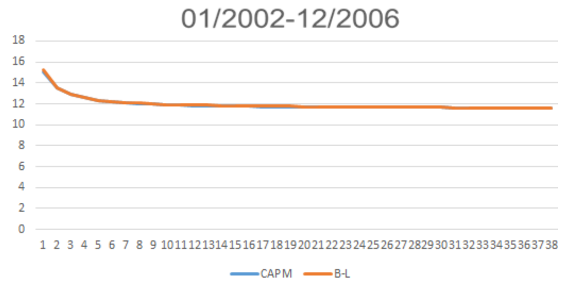
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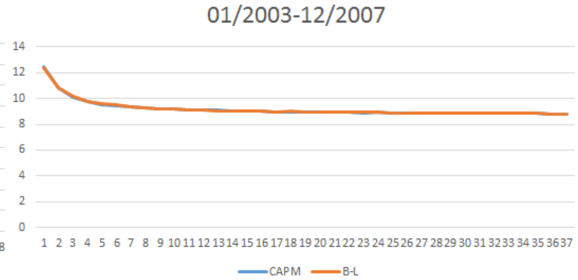
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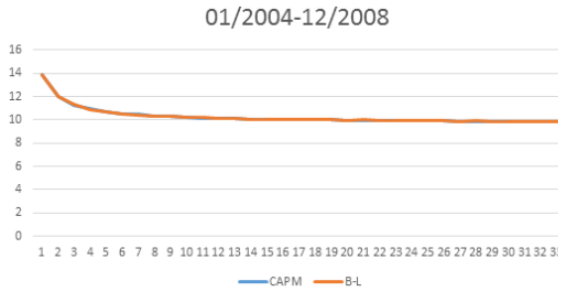
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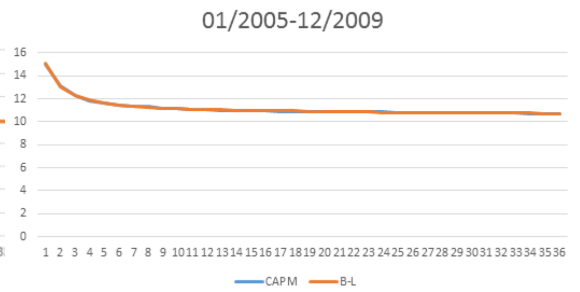
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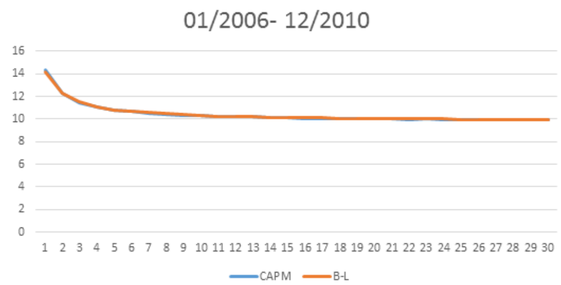
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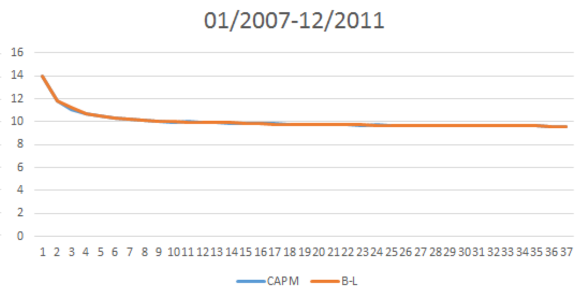
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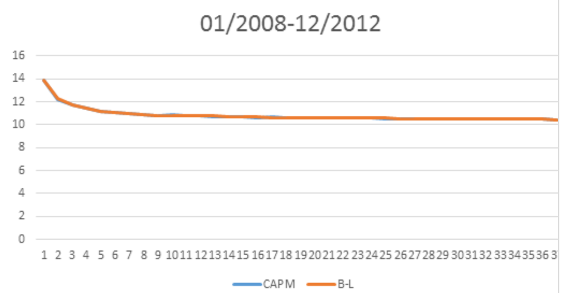
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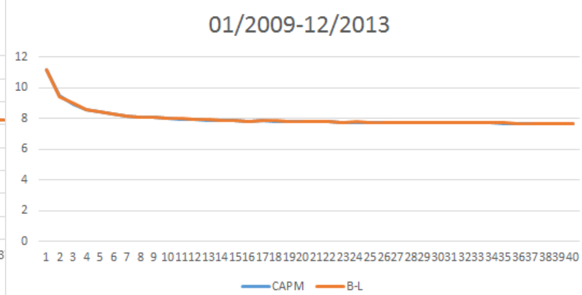
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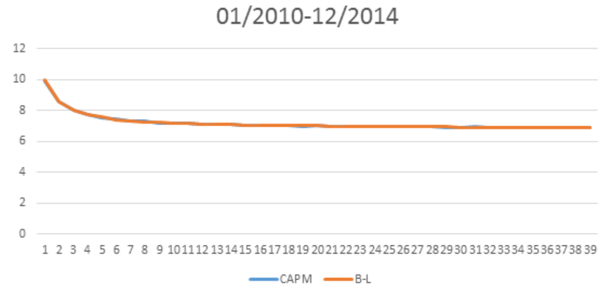


Figure 5.1: Risk-Reduction versus Number of Stocks in the Portfolio for both the Prior Variance Based on the CAPM and the Posterior Variance Based on Black-Litterman Method.

This figure shows the relation between the mean standard deviation of the portfolio and the number of stocks included in the portfolio for each year from 2005 to 2015 by using both the prior variance-covariance matrix (blue line) and the posterior variance-covariance matrix (red line). The prior matrix is the variance-covariance matrix of prior 5-year historical monthly returns of stocks consistently included in the BIST-50 index. The beginning and the end of these 5-year periods are shown above the graphs. After taking into account the investor views and the variance of these views, this prior matrix is converted into the posterior matrix by using the B-L formula.

Table 5.3: Comparison between the Prior and the Posterior Variance-Covariance Matrices in terms of Diversification Effect

This table presents the average variances, average covariances, average correlations and the ranges of the optimal number of stocks when both prior variance-covariance matrix based on the CAPM and the posterior variance-covariance matrix based on the B-L method are used for each year from 2005 to 2015. First, for each year, the prior matrix is estimated by using prior 5 year historical monthly return data. Then, the average variance and the average covariance of the stocks in the portfolio are calculated. Later, average correlation of these stocks in the portfolio is calculated by dividing average covariance by average variance. The same process is repeated for the posterior matrix derived from the B-L formula. For the purpose of attaining the range of the optimal number of stocks, first, the simulation technique of Evans and Archer (1968, [12]) is performed. By replicating this procedure for 1000 times, the portfolios consisting of stocks ranging from 1 to total number of stocks in any given year analyzed are generated and their mean standard deviations are found. In order to determine whether any statistically significant decrease in the mean standard deviation of two portfolios that have N and $N + 2$ stocks occurs or not, a t-test at 5% significance level is performed. Finally, the ideal range of stocks to create a diversified portfolio is selected as the last range of stocks whose mean standard deviations are statistically different from each other before the first range of stocks whose mean standard deviations are statistically indifferent from each other. The same process is repeated for posterior matrix based on the B-L method to obtain the range of the optimal number of stocks.

	Prior Variance(CAPM)				**Posterior Variance(Black Litterman)**			
	Average Variance	**Average Covariance**	**Average Correlation**	**The Range of the Optimal Number of Stocks**	**Average Variance**	**Average Covariance**	**Average Correlation**	**The Range of the Optimal Number of Stocks**
2005	491.53	321.12	0.65	10-12	494.03	321.46	0.65	8-10
2006	370.36	224.38	0.60	8-10	372.25	224.59	0.60	6-8
2007	236.73	131.28	0.55	10-12	238.00	131.38	0.55	10-12
2008	159.54	75.30	0.47	12-14	160.34	75.35	0.46	12-14
2009	196.14	93.32	0.47	12-14	197.07	93.39	0.47	12-14
2010	230.81	111.56	0.48	10-12	231.95	111.63	0.48	10-12
2011	210.45	95.03	0.45	16-18	211.56	95.11	0.44	14-16
2012	195.05	89.16	0.45	8-10	196.18	89.22	0.45	8-10
2013	197.18	106.75	0.54	12-14	198.04	106.81	0.53	12-14
2014	127.07	57.19	0.45	14-16	127.72	57.22	0.44	14-16
2015	102.72	45.99	0.44	16-18	103.26	46.02	0.44	14-16

As in the case of prior variance, the range of the optimal number of stocks differs from one year to another. However, taking into account investor views does not affect the range of optimal number of stocks to hold, and the average portfolio standard deviations that much.

In addition to this comparison, we consider the effect of scaling factor τ used to determine the posterior variance on our results. To see this effect, a sensitivity analysis for different τ value is conducted. The range of optimal number of stocks for a well diversified portfolio by using posterior variance matrix for different τs for 2015 are summarized in Table 5.4. Results in this table indicate that the range of the optimal number of stocks is not that sensitive to the value of τ chosen.

Table 5.4: The Range of Optimal Number of Stocks for Different Values of τ for 2015

This table presents the ranges of optimal number of stocks when the posterior matrices are constructed by using different values of the scaling factor τ for the estimation year 2015. τ is taken as the inverse of the number of observations (5 years = 60 observations) in this thesis to determine the range of optimal number of stocks when the posterior matrix is used. To study the sensitivity of our findings to τ values, a sensitivity analysis is conducted for τ values of $1/\sqrt{60}$, 0.25, 0.5 and 1.

The Values of τ	$\tau = \frac{1}{60}$	$\tau = \frac{1}{\sqrt{60}}$	$\tau = 0.25$	$\tau = 0.5$	$\tau = 1$
The Range of the Optimal Number of Stocks	14-16	14-16	14-16	16-18	18-20

5.2 An Empirical Application of the Unconstrained Black-Litterman Model to the Stocks Included in the BIST-50 Index

In the previous section, how many stocks from Borsa Istanbul-50 (BIST-50) Index are needed to form a well-diversified portfolio is determined for every year in the sample period analyzed. What portion of the portfolio should be invested in each stock is another issue considered by investors. Markowitz Mean-Variance approach has been used in determining optimal weights of each stock in the portfolio since the 1950's. Although it is a breakthrough theory in finance, it has some drawbacks such as the requirement of estimating all stock returns in a portfolio and high level of sensitivity of optimal portfolio weights to inputs. These problems result in extremely low or high weights assigned to stocks included in the portfolio. In addition to this, investors can not define their views about assets in a relative sense. After pointing out these problems, Black and Litterman (B-L) proposed a new model in the 1990's. Investors can specify their views on the stocks of their choice and also define the confidence level on their views in this model.

In this section, B-L model is applied to the stocks included in the BIST-50 Index. The data used, the parameters selected, and each step of the model are explained in detail in

Subsection 4.2.2. Initial optimal weights for each stock in the portfolio is taken as market capitalization weights according to the CAPM approach. The stock return based on views is calculated by using the financial analysts' average expected stock price which can be accessed from the Bloomberg database. After combining prior information with analysts' views, the B-L model is estimated without imposing any constraints. It is observed that view returns of some assets do not affect posterior weights of these assets in an expected way. First, reason behind this observation is explained. Secondly, B-L model is re-estimated once by imposing a budget constraint alone, and then a budget and no short selling constraints simultaneously on the model. Lastly, performances of portfolios generated by the B-L method and by the CAPM approach are analyzed in terms of two criteria: Sharpe ratio and efficient frontier.

The prior information set based on the CAPM approach is collected and prepared as described in Section 4.1. The risk aversion coefficient, which is one of the inputs of the prior information set, is generally estimated from the historical data and calculated as follows:

$$\lambda = \frac{E(r_M) - r_f}{\sigma^2(r_M)},$$

where

$E(r_M) - r_f$ is the estimated excess return of the market portfolio,

$\sigma^2(r_M)$ is the variance of the market portfolio.

In our study, market portfolio is the portfolio consisting of all stocks whose codes are presented in Table 4.1 for each estimation year. The excess return of this market portfolio is calculated as the market value weighted average of excess returns on the stocks included in the portfolio. The variance of the market portfolio is calculated by using matrix multiplication. The transpose of the market capitalization weight vector is multiplied by the estimated variance-covariance matrix, and the market capitalization weight vector. Finally, the risk aversion coefficient is found by dividing the calculated excess return of the market portfolio by the calculated variance of the market portfolio. The results are presented in Table 5.5.

Risk aversion coefficients estimated by this method are quite low and even negative for most of the estimated years. For only the years 2008 and 2014, the estimated risk aversion coefficients are consistent with expectations and the evidence from around the world. However, risk aversion coefficients estimated for the remaining years are quite low due to high variability of portfolio returns or negative average returns observed for the majority of the stocks in some years. Since these negative or very low estimates of risk aversion coefficients are not consistent with the expectations and the observed investor behavior, the risk aversion coefficient is taken to be 2.5 which is estimated by Black (1989, [3]) as the world average risk aversion coefficient.

Table 5.5: Risk Aversion Coefficients

This table presents the average logarithmic excess return of the market portfolio, the variance of the market portfolio and the risk aversion coefficient (λ) for each estimation year during the sample period of this thesis. The market portfolio is constructed as the market value weighted portfolio of all stocks shown in Table 4.1 for each estimation year during the sample period. The average 5-year historical logarithmic excess return and the variance of the market portfolio are calculated for each estimation year. The risk aversion coefficient is defined as excess return of the market portfolio per unit of the market risk (variance).

	Average Logarithmic Excess Return of The Portfolio (%)	**Variance of The Portfolio**	**Risk Aversion Coefficient (λ)**
2005	-2.58 %	291.00	-0.88
2006	-0.41 %	227.41	-0.18
2007	-0.23 %	127.07	-0.17
2008	1.47%	72.74	2.02
2009	-0.31 %	91.29	-0.33
2010	0.31%	108.85	0.28
2011	0.16%	111.03	0.14
2012	-0.12 %	98.77	-0.12
2013	0.1%	111.21	0.08
2014	1.45%	56.74	2.55
2015	0.66%	47.47	1.38

After the terms λ, Σ, and w_{MKT} are estimated from market data, the implied logarithmic excess equilibrium returns are computed for each year using MATLAB.

For year 2015, Figure 5.2 illustrates the historical average logarithmic returns and the implied equilibrium average logarithmic excess returns on all the stocks available for investment over the 5-year period from January 2010 to December 2014. As can be seen in this figure, the implied equilibrium excess returns are all positive. Furthermore, they are more stable than historical returns. Therefore, they are more consistent with the notion of expected excess returns. Hence, they can be good initial estimates of expected returns before taking investors' views into account.

The variance-covariance matrix of this implied logarithmic excess equilibrium returns of the stocks is calculated as $\tau\Sigma$. τ is the scaling factor for historical variance-covariance matrix. B-L (1992, [5]) state that because the uncertainty in the mean is less than the uncertainty in the returns, τ is close to 0. In Figure 5.2, it is also observed that the logarithmic excess equilibrium returns are less volatile than the historical average logarithmic excess returns. Hence, in this thesis τ is defined as in Walters (2014, [29]) as follows:

$$\tau = \frac{1}{T}.$$

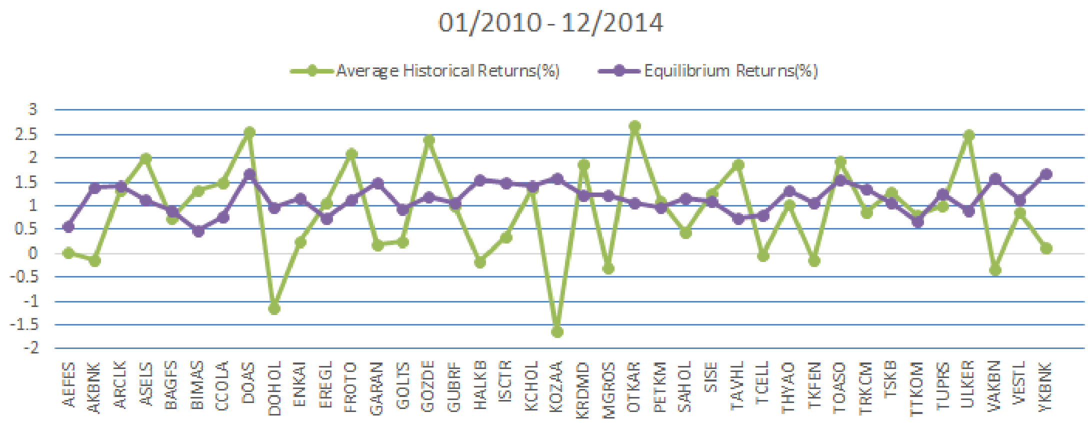


Figure 5.2: Comparison between Historical Average Returns and Equilibrium Returns.

This figure shows average 5-year historical and the implied equilibrium logarithmic excess returns of the stocks consistently included in the BIST-50 index for 2015. The green line indicates the average historical returns, whereas the purple line indicates the implied equilibrium returns of these stocks. The beginning and the end of the 5-year period are shown above the graph.

Here, T is the total number of observations. Since 60 monthly historical returns are used for each year analyzed, τ is constant and equals to 0.0167 in this thesis.

After determining the prior information set, it is time to gather the second information set including Bloomberg analysts' views. How these target stock prices are collected and converted into stock returns are explained in Section 4.1. Formulation of the pick matrix (P), the view vector (q), and the covariance matrix of the views (Ω) and blending of this second information set with the first information set by using B-L formula are explained step by step in Subsection 4.2.2. Finally, the posterior optimal weights are found by using the calculated posterior expected return vector and the posterior variance matrix as inputs in the Markowitz Mean-Variance Optimization.

Table 5.6 on the next page displays market capitalization (equilibrium) weights, the computed posterior weights and the difference between equilibrium weights and new weights resulting from the B-L method for the estimation year 2015. The results for other estimation years are available in Table A.5 in the Appendix A. The stocks whose codes are written in bold letters do not have a target stock price data available in the Bloomberg database. All the others have the average target stock price information.

As seen in Table 5.6, the sum of the market capitalization weights of the stocks is equal to 1 by definition. On the other hand, the sum of the new optimal posterior weights of the stocks is equal to -0.3215 since the optimization problem does not have a budget constraint. Minus sign indicates that an investor invest his own money and the money obtained from shorting this portfolio of stocks in the risk-free asset.

For GOLTS and VESTL Bloomberg analysts did not estimate a target stock price, i.e. they did not express an opinion. These stocks' optimal posterior weights slightly different from their market capitalization weights, however differences are not even noticeable. This is not surprising since the investor does not have any basis to change the weights of these securities. On the other hand, optimal posterior weights of securities for which analysts estimate a target stock price are significantly different from their market capitalization weights.

Table 5.6: Market Capitalization Weights, Optimal Posterior Weights and Differences for the Estimation Year 2015

This table presents the market capitalization weights, the optimal posterior weights of the stocks that are consistently included in the BIST-50 index in 2015 and the differences between them. First column shows the codes of the stocks that are consistently included in BIST-50 index during the whole year and have monthly return data from January 2010 to December 2014. In the second and third columns, the market capitalization and posterior weights of these securities are reported. The stock codes written in bold letters indicate the stocks for which the Bloomberg analysts did not give a price estimate. The difference between the market capitalization weights and the optimal posterior weights of both stocks with views and no views are shown in the last column of this table. Finally, in the last row of this table, the summation of both the market capitalization and the optimal posterior weights are presented.

	w_{MKT}	$w^\star$	$w_{\text{MKT}} - w^\star$
AEFES	0.0297	0.1840	−0.1544
AKBNK	0.0764	−0.1293	0.2058
ARCLK	0.0224	−0.1252	0.1475
ASELS	0.0132	−0.0529	0.0661
BAGFS	0.0014	−0.0748	0.0762
BIMAS	0.0335	−0.2791	0.3126
CCOLA	0.0283	0.1781	−0.1497
DOAS	0.0058	−0.1466	0.1524
DOHOL	0.0044	0.1329	−0.1286
ENKAI	0.0419	0.1274	−0.0855
EREGL	0.0344	−0.0658	0.1002
FROTO	0.0252	−0.2714	0.2966
GARAN	0.0873	−0.1276	0.2149
GOLTS	**0.0010**	**0.0010**	**0.0000**
GOZDE	0.0025	0.1229	−0.1205
GUBRF	0.0037	−0.2605	0.2642
HALKB	0.0383	0.4397	−0.4013
ISCTR	0.0669	−0.2730	0.3399
KCHOL	0.0694	−0.1678	0.2372
KOZAA	0.0014	0.3505	−0.3490
KRDMD	0.0035	0.3124	−0.3089
MGROS	0.0089	0.0934	−0.0845
OTKAR	0.0046	−0.3510	0.3556
PETKM	0.0087	−0.0978	0.1065
SAHOL	0.0457	0.4467	−0.4010
SISE	0.0137	−0.2372	0.2508
TAVHL	0.0153	0.0184	−0.0030
TCELL	0.0694	−0.3127	0.3821
THYAO	0.0293	−0.0475	0.0769
TKFEN	0.0047	0.4442	−0.4394
TOASO	0.0176	−0.0946	0.1122
TRKCM	0.0056	0.0100	−0.0044
TSKB	0.0067	0.4447	−0.4381
TTKOM	0.0561	−0.1686	0.2248
TUPRS	0.0305	−0.1821	0.2126
ULKER	0.0140	−0.1423	0.1563
VAKBN	0.0269	0.0388	−0.0119
VESTL	**0.0047**	**0.0046**	**0.0001**
YKBNK	0.0469	−0.0632	0.1101
SUM	1.000	−0.3215	

Table 5.7: View and Equilibrium Return Differences and Market versus Posterior Weight Differences for the Estimation Year 2015

This table presents the implied equilibrium returns, the view returns of the stocks calculated from Bloomberg analysts' average target price and their difference for the year 2015. Also in this table the market capitalization weights, the optimal posterior weights of these stocks and the difference between these two weights are reported for the same year.

	Π	q	$\Pi - q$		w_{MKT}	$w^{\star}$	$w_{\text{MKT}} - w^{\star}$
AEFES	0.0059	0.0068	−0.0009		0.0297	0.1840	−0.1544
AKBNK	0.0139	−0.0052	0.0191		0.0764	−0.1293	0.2058
ARCLK	0.0141	−0.0051	0.0192		0.0224	−0.1252	0.1475
ASELS	0.0114	−0.0043	0.0158		0.0132	−0.0529	0.0661
BAGFS	0.0090	−0.0029	0.0120		0.0014	−0.0748	0.0762
BIMAS	0.0048	−0.0030	0.0077		0.0335	−0.2791	0.3126
CCOLA	0.0076	0.0050	0.0027		0.0283	0.1781	−0.1497
DOAS	0.0169	−0.0072	0.0241		0.0058	−0.1466	0.1524
DOHOL	0.0098	0.0053	0.0045		0.0044	0.1329	−0.1286
ENKAI	0.0116	0.0053	0.0063		0.0419	0.1274	−0.0855
EREGL	0.0073	−0.0020	0.0094		0.0344	−0.0658	0.1002
FROTO	0.0112	−0.0129	0.0241		0.0252	−0.2714	0.2966
GARAN	0.0148	−0.0075	0.0223		0.0873	−0.1276	0.2149
GOZDE	0.0119	0.0072	0.0047		0.0025	0.1229	−0.1205
GUBRF	0.0106	−0.0130	0.0236		0.0037	−0.2605	0.2642
HALKB	0.0154	0.0143	0.0011		0.0383	0.4397	−0.4013
ISCTR	0.0148	−0.0105	0.0253		0.0669	−0.2730	0.3399
KCHOL	0.0143	−0.0067	0.0210		0.0694	−0.1678	0.2372
KOZAA	0.0159	0.0271	−0.0112		0.0014	0.3505	−0.3490
KRDMD	0.0122	0.0133	−0.0011		0.0035	0.3124	−0.3089
MGROS	0.0123	0.0028	0.0095		0.0089	0.0934	−0.0845
OTKAR	0.0106	−0.0213	0.0319		0.0046	−0.3510	0.3556
PETKM	0.0096	−0.0042	0.0138		0.0087	−0.0978	0.1065
SAHOL	0.0117	0.0071	0.0046		0.0457	0.4467	−0.4010
SISE	0.0111	−0.0088	0.0199		0.0137	−0.2372	0.2508
TAVHL	0.0075	−0.0034	0.0108		0.0153	0.0184	−0.0030
TCELL	0.0080	−0.0078	0.0158		0.0694	−0.3127	0.3821
THYAO	0.0132	−0.0041	0.0173		0.0293	−0.0475	0.0769
TKFEN	0.0108	0.0099	0.0008		0.0047	0.4442	−0.4394
TOASO	0.0155	−0.0065	0.0220		0.0176	−0.0946	0.1122
TRKCM	0.0135	−0.0034	0.0169		0.0056	0.0100	−0.0044
TSKB	0.0105	0.0072	0.0033		0.0067	0.4447	−0.4381
TTKOM	0.0067	−0.0055	0.0121		0.0561	−0.1686	0.2248
TUPRS	0.0127	−0.0079	0.0206		0.0305	−0.1821	0.2126
ULKER	0.0089	−0.0076	0.0165		0.0140	−0.1423	0.1563
VAKBN	0.0160	0.0008	0.0152		0.0269	0.0388	−0.0119
YKBNK	0.0167	−0.0043	0.0210		0.0469	−0.0632	0.1101

The posterior weights of the stocks are expected to change in the direction of the views. For example, if there is a positive view on a stock, its weight is expected to increase or vice versa. Results of this investigation for 2015 are presented in Table 5.7. The evidence is not consistent with this expectation for all the stock, however for about 68% it is consistent. Therefore, evidence is in general but not completely consistent with this expectation.

Bozdemir (2011, [7]) illustrates the nonexistence of a direct relationship between the views on expected returns and corresponding posterior weights for the case of 2 assets. In his example, the expected return on one of the assets is raised and everything else is kept constant. It is shown that an increase in the expected return of the asset results in a decrease in the posterior weight of the corresponding asset, a finding that is inconsistent with the expectation based on views. He concludes that the reason for this unexpected relationship between the view on expected return and the corresponding posterior weight is the covariance structure.

To understand the relationship between specified views on returns and the B-L posterior weights, first the relationship between the view return and the expected return of that asset needs to be examined.

To see the effect of the relationship between the expected and the view returns of assets on this unexpected relationship between specified views and the posterior weights, let's take a look at a simple example with 3 assets. Suppose that the equilibrium return on the first asset is equal to 1%, the second one is 2% and the third one is 0.5%. Suppose the view 1 states that the return of first asset will be 2% and the view 2 states that the return on the second asset will be 3%. Hence, view returns of these two assets are greater than their implied equilibrium returns. Let the covariance matrix of these assets be

$$\Sigma = \begin{bmatrix} 1 & -2 & 4 \\ -2 & 3 & 1 \\ 4 & 1 & 5 \end{bmatrix}.$$

Scaling factor τ and the uncertainty of view matrix Ω are taken as defined in this thesis earlier.

$$\tau = \frac{1}{T} = \frac{1}{60}, \qquad T: \text{total number of observations},$$

$$\Omega = \text{diag}(P(\tau\Sigma)P^T).$$

Expected posterior return that combines the equilibrium returns and investor views is calculated by the following formula:

$$E(r) = \Pi + \tau\Sigma P^T[(P\tau\Sigma P^T) + \Omega]^{-1}[q - P\Pi],$$

$$
\begin{aligned}
E(r) &= \begin{bmatrix} 1 \\ 2 \\ 0.5 \end{bmatrix} + \frac{1}{60}\begin{bmatrix} 1 & -2 & 4 \\ -2 & 3 & 1 \\ 4 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \\
&\times \left\{\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\frac{1}{60}\begin{bmatrix} 1 & -2 & 4 \\ -2 & 3 & 1 \\ 4 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\right) + \begin{bmatrix} 0.0167 & 0 \\ 0 & 0.05 \end{bmatrix}\right\}^{-1} \\
&\times \left(\begin{bmatrix} 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 0.5 \end{bmatrix}\right), \\
E(r) &= \begin{bmatrix} 1 \\ 1.5 \\ 5 \end{bmatrix}.
\end{aligned}
$$

The equilibrium return of the first asset is given as 1%, and the view return of the first asset is stated as 2% which is greater than the equilibrium return of this asset. As shown by the calculations above, the expected posterior return of this asset remains at 1%. Actually, given the view on this asset, its posterior return is expected to be greater than its equilibrium return. However, its posterior return is exactly equal to its equilibrium return. When the second asset is concerned, its posterior return (1.5%) is lower than its equilibrium (2%) and view (3%) returns. However, given the view on this asset, its posterior return is also expected to be greater than its implied equilibrium return. Contrary to the expectations, its posterior return is declined.

As it can be seen in this example, higher view return than the implied equilibrium return does not necessarily indicate higher posterior expected return than the equilibrium return. In other words, there is not a direct relationship between the view return of an asset and the posterior expected return of it. How the posterior expected return changes is determined by the covariance structure of the securities. In this example, there are only 3 assets, therefore it is a lot easier to see the effect of covariance structure on the posterior expected return. In the case of this thesis, there are approximately 50 assets with more than 30 views on these assets. Hence, analyzing the effect of covariance structure of securities on the posterior return updating mechanism is a lot harder to understand.

Let's continue work with this 3 assets example with different views on the assets. Suppose this time, there is only one relative view indicating that the return of the first asset is 1% higher than the return on the second asset. The rest of the data on these 3 assets are the same as before.

Then the posterior expected returns are calculated as follows:

$$
\begin{aligned}
E(r) &= \begin{bmatrix} 1 \\ 2 \\ 0.5 \end{bmatrix} + \frac{1}{60}\begin{bmatrix} 1 & -2 & 4 \\ -2 & 3 & 1 \\ 4 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \\
&\times \left\{ \left(\begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \frac{1}{60}\begin{bmatrix} 1 & -2 & 4 \\ -2 & 3 & 1 \\ 4 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right) + \begin{bmatrix} 0.1333 \end{bmatrix} \right\}^{-1} \\
&\times \left(\begin{bmatrix} 1 \end{bmatrix} - \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0.5 \end{bmatrix} \right), \\
E(r) &= \begin{bmatrix} 1.375 \\ 1.375 \\ 0.875 \end{bmatrix}.
\end{aligned}
$$

Expected returns of the first two assets are changed in the direction of the view stated; that is, while the first asset's expected return increases the second asset's expected return decreases. Thus, this example reveals that not only covariance structure but also the way views are defined affects the relation between view return and the posterior expected return of the corresponding asset.

Furthermore, confidence level of the view in addition to the covariance structure and the way views are stated is another factor affecting the relationship between the view returns and the posterior expected returns of an asset. Since there is no information on the confidence level of the views in Bloomberg database, this uncertainty of the view vector is defined in terms of the covariance matrix of security returns, and the view matrix. Therefore, it is adequate to say that covariance structure of the securities and the structure of the views defined might affect the relation between the view return and the posterior expected return of an asset in our case too.

5.2.1 An Empirical Application of the Black-Litterman Model with a Budget Constraint to the Stocks Included in the BIST-50 Index

In the empirical application of the unconstrained B-L Model, it is observed that the budget required to create the optimal portfolios ranges from -1.8017 in the estimation year of 2005 to 2.9325 in the estimation year of 2009 as reported in Table A.5 of the Appendix A. Since these budgets are quite extreme, putting a budget constraint on the B-L Model is considered. A budget constraint, which is a real world constraint, requires the sum of portfolio weights to be one.

The same data is used as in the application of the unconstrained B-L model up to the final Markowitz optimization step. In the final optimization model, the optimization problem is formulated with the budget constraint.

Results reported in Table 5.8 for 2015 show that posterior weights of stocks with no assigned views differ much more from their equilibrium weights. However, the value of the investor's utility function is not affected that much from the addition of this budget constraint to the optimization problem. If an investor desires to invest all of his/her money into the optimal portfolio of all the stocks consistently included in the BIST-50 index, his utility loss compared to the unconstrained B-L strategy is not that much from following this budget constrained B-L strategy.

5.2.2 An Empirical Application of the Black-Litterman Model with a Budget and a No Short Selling Constraints to the Stocks Included in the BIST-50 Index

In the application of both the unconstrained and the budget constrained B-L Models, there are several stocks that are sold short. However, there are restrictions on short selling in financial markets. In order to examine the effect of a short selling constraint on the optimal portfolio of the investor, no short selling constraint is added to the optimization problem with the budget constraint. No short selling constraint requires the weight of each security to be greater than or equal to 0.

The same data is used as in the application of both unconstrained and the budget constrained B-L model. This application only differs from previous applications of the B-L model in the final Markowitz optimization step. In the optimization problem, the utility function of the investor is maximized while satisfying the budget and the no short selling constraints simultaneously.

In Table 5.8, the maximum utility of the investor is achieved by investing in a small number of stocks. However, this utility is significantly lower than the utility investor has with either the unconstrained or only the budget constrained optimization. This finding is not surprising given the nature of the constrained optimization and it indicates that the no short selling constraint is more binding that the budget constraint. Note that in these optimizations, transaction costs associated with trading are not taken into account. Therefore, even though utility of the investor declines significantly after imposing the no short selling constraint, an investor might still be interested in applying this constrained optimization if the loss in utility because of short selling constraint is less than the loss in utility because of increased transaction costs. Furthermore, as in the case of budget constrained optimization, the optimal posterior weights of stocks with no assigned views deviate significantly from their equilibrium weights.

Table 5.8: Optimal Weights from only Budget constrained and Budget and No Short Selling Constrained B-L Portfolio Optimizations for the Estimation Year 2015

This table presents the market capitalization weights, the optimal posterior weights of stocks obtained from only budget constrained and both budget and short selling constrained portfolio optimizations. The stocks that are consistently included in the BIST-50 index throughout 2015 and have monthly return data from January 2010 to December 2014 are shown in the first column of this table. The second, third and fourth columns of this table show the market capitalization weights, posterior weights with budget constraint only and posterior weights with both budget and short selling constraints respectively. The stock for which Bloomberg analyst did not estimate a target price are shown in bold letters. Finally, the value of the utility function for both the unconstrained and the budget constrained portfolio optimization are shown in the last row of this table.

	w_{MKT}	$w^{\star}$	w_{OPT}(budget)	w_{OPT}(budget&no short selling)
AEFES	0.0297	0.1840	0.5497	0.6121
AKBNK	0.0764	−0.1293	−0.1588	0.0000
ARCLK	0.0224	−0.1252	−0.2357	0.0000
ASELS	0.0132	−0.0529	−0.1674	0.0000
BAGFS	0.0014	−0.0748	0.2495	0.0000
BIMAS	0.0335	−0.2791	0.1745	0.2303
CCOLA	0.0283	0.1781	0.2059	0.0070
DOAS	0.0058	−0.1466	−0.2888	0.0000
DOHOL	0.0044	0.1329	0.2865	0.0000
ENKAI	0.0419	0.1274	0.2485	0.0787
EREGL	0.0344	−0.0658	0.0589	0.0000
FROTO	0.0252	−0.2714	−0.4726	0.0000
GARAN	0.0873	−0.1276	0.1101	0.0000
GOLTS	**0.0010**	**0.0010**	**-0.1795**	**0.0000**
GOZDE	0.0025	0.1229	−0.0119	0.0000
GUBRF	0.0037	−0.2605	−0.2843	0.0000
HALKB	0.0383	0.4397	0.3426	0.0000
ISCTR	0.0669	−0.2730	−0.6788	0.0000
KCHOL	0.0694	−0.1678	−0.4000	0.0000
KOZAA	0.0014	0.3505	0.2606	0.0720
KRDMD	0.0035	0.3124	0.1798	0.0000
MGROS	0.0089	0.0934	0.1540	0.0000
OTKAR	0.0046	−0.3510	−0.1785	0.0000
PETKM	0.0087	−0.0978	0.2206	0.0000
SAHOL	0.0457	0.4467	0.6839	0.0000
SISE	0.0137	−0.2372	−0.1434	0.0000
TAVHL	0.0153	0.0184	−0.0588	0.0000
TCELL	0.0694	−0.3127	0.0866	0.0000
THYAO	0.0293	−0.0475	0.0067	0.0000
TKFEN	0.0047	0.4442	0.3046	0.0000
TOASO	0.0176	−0.0946	−0.2040	0.0000
TRKCM	0.0056	0.0100	−0.2022	0.0000
TSKB	0.0067	0.4447	0.5684	0.0000
TTKOM	0.0561	−0.1686	0.1308	0.0000
TUPRS	0.0305	−0.1821	−0.1081	0.0000
ULKER	0.0140	−0.1423	0.0561	0.0000
VAKBN	0.0269	0.0388	−0.1079	0.0000
VESTL	**0.0047**	**0.0046**	**-0.0838**	**0.0000**
YKBNK	0.0469	−0.0632	0.0859	0.0000
SUM		−0.3215	1.0000	1.0000
UTILITY		0.0097	0.0088	0.0001

5.2.3 Comparison of the CAPM and the Black-Litterman Models in terms of Sharpe Ratios

As described in Subsection 2.2.2 of this thesis, according to Capital Asset Pricing Model, every investor holds the market portfolio because everyone has access to the same information and uses the same decision rule in making their investment decisions. In other words, each investor holds a market value weighted portfolio of all the assets in the market. Market value weight of each stock, i.e. weight vector of the assets, logarithmic excess equilibrium return vector and covariance matrix of logarithmic excess returns are estimated for each year in the sample period from 2005 to 2015. Then, by multiplying the weight vector of the assets with the logarithmic excess equilibrium return vector, the excess return of the equilibrium portfolio is computed. Similarly, by multiplying the transpose of weight vector of the assets and the covariance matrix of logarithmic excess returns and the weight vector of the assets, the variance of the equilibrium portfolio is determined.

As explained mathematically in Chapter 3 of this thesis, the B-L Model combines the equilibrium expected returns for assets with investors' views using a Bayesian approach. In this thesis, expected stock return calculated from the analysts' average target stock price in the Bloomberg database is taken as a proxy for investors' views. The vector of the optimal posterior weights, new combined logarithmic excess return vector and the posterior covariance matrix of logarithmic excess returns are computed in MATLAB for each of the years in between 2005 and 2015. Therefore, by multiplying the vector of the optimal posterior weights with the new combined logarithmic excess return vector, the excess return on the B-L optimal portfolio is computed. Similarly, by multiplying the transpose of the vector of the optimal posterior weights, the posterior covariance matrix of logarithmic excess returns and the vector of the optimal posterior weights the variance of the optimal portfolios is calculated.

Performances of these two portfolios constructed from the CAPM and the B-L Model are compared by using a Sharpe ratio. Sharpe ratio is a measure of excess return of the portfolio per unit of the total risk of the portfolio (standard deviation of the portfolio). The portfolio with the higher Sharpe ratio has a better risk-adjusted performance. As given in the Subsection 2.2.1 of this thesis, the formula of the Sharpe ratio is:

$$S = \frac{E(r_p) - r_f}{\sigma_p}.$$

Sharpe ratios of the portfolios based on the CAPM and the B-L model for the 11 estimated years from 2005 to 2015 are presented in Table 5.9 along with the portfolio excess returns, the variance and the standard deviation of these returns.

Table 5.9: Sharpe Ratios of CAPM and B-L Strategies for each Estimation Years

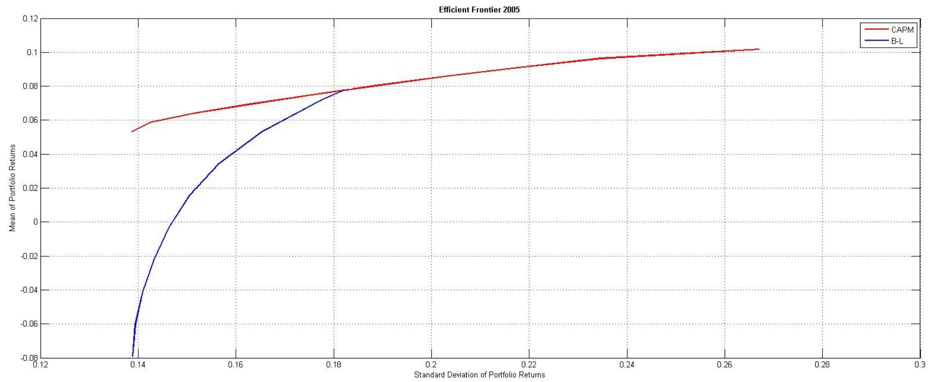
This table presents the excess expected return, the variance, and the standard deviation of the portfolios calculated with both the CAPM and the B-L methods for each estimation year between 2005 and 2015. Sharpe ratio measures the excess return of the portfolio per unit of the portfolio risk. This table is constructed to illustrate the portfolio which has better risk-adjusted performance.

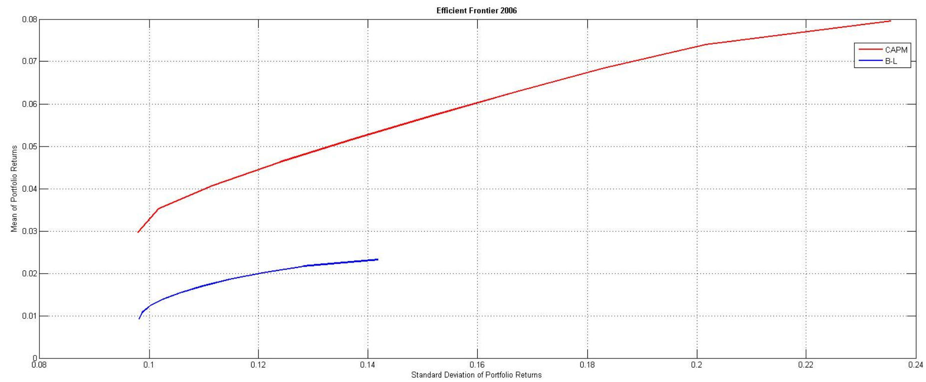
	2005		**2006**		**2007**		**2008**	
	CAPM	**B-L**	**CAPM**	**B-L**	**CAPM**	**B-L**	**CAPM**	**B-L**
E(R)	0.0727	1.9209	0.0569	0.0340	0.0318	0.0622	0.0182	0.0823
Var	0.0291	0.7684	0.0228	0.0136	0.0127	0.0249	0.0073	0.0329
Std.Dev.	0.1706	0.8766	0.1510	0.1166	0.1127	0.1578	0.0854	0.1814
Sharpe Ratio	0.4262	**2.1913**	**0.3768**	0.2915	0.2822	**0.3942**	0.2130	**0.4537**
	2009		**2010**		**2011**		**2012**	
	CAPM	**B-L**	**CAPM**	**B-L**	**CAPM**	**B-L**	**CAPM**	**B-L**
E(R)	0.0228	0.1810	0.0272	0.0262	0.0278	0.0235	0.0247	0.0461
Var	0.0091	0.0724	0.0109	0.0105	0.0111	0.0094	0.0099	0.0184
Std.Dev.	0.0954	0.2691	0.1044	0.1025	0.1054	0.0970	0.0995	0.1356
Sharpe Ratio	0.2390	**0.6727**	**0.2605**	0.2557	**0.2639**	0.2424	0.2482	**0.3399**
	2013		**2014**		**2015**			
	CAPM	**B-L**	**CAPM**	**B-L**	**CAPM**	**B-L**		
E(R)	0.0278	0.0116	0.0142	0.0354	0.0119	0.0194		
Var	0.0111	0.0047	0.0057	0.0141	0.0047	0.0078		
Std.Dev.	0.1054	0.0686	0.0755	0.1187	0.0686	0.0883		
Sharpe Ratio	**0.2639**	0.1692	0.1881	**0.2981**	0.1736	**0.2197**		

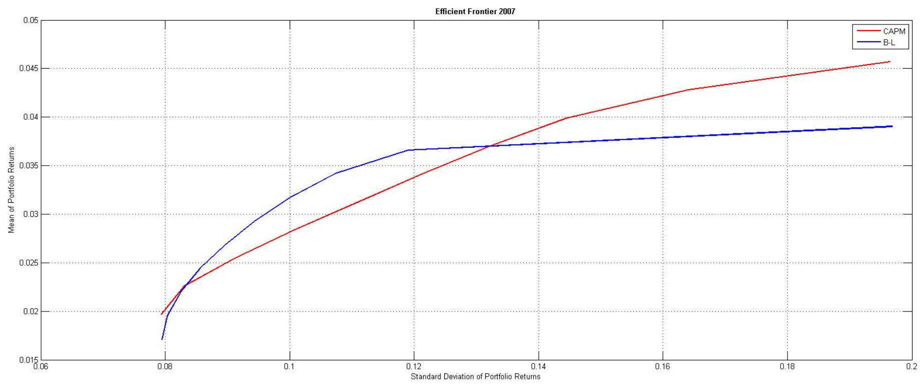
For years 2005, 2007, 2008, 2009, 2012, 2014 and 2015 the B-L strategy is slightly superior to the CAPM strategy when the Sharpe ratios are used for comparison. However, for the remaining years, the CAPM strategy slightly outperforms the B-L strategy according to this performance measure. Given these findings it is hard to conclude that the B-L strategy is always better than the CAPM strategy using this criteria.

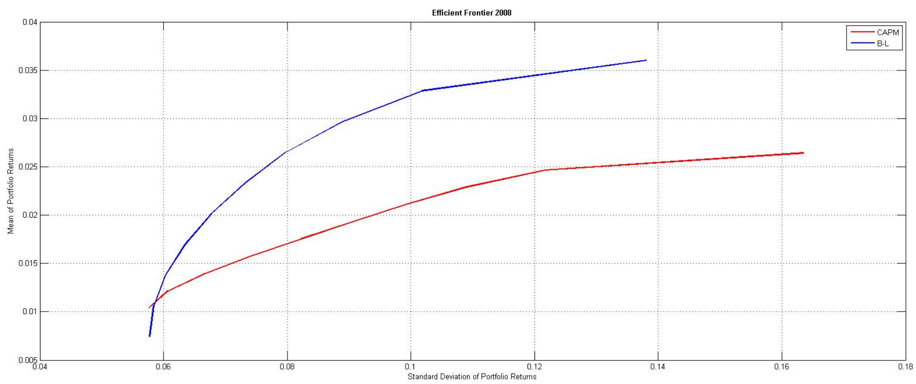
5.2.4 Comparison of the CAPM and the Black-Litterman Model Using Efficient Frontier Technique

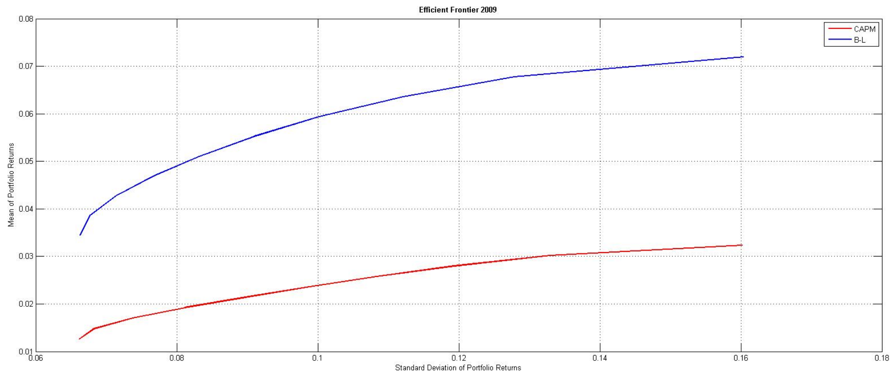
As presented in Subsection 2.2.1 of this thesis, the efficient frontier is a curve depicting the portfolios that give investors the greatest possible rate of return for a given value of risk they are willing to accept. The horizontal (X) axis of the efficient frontier graph is usually the portfolio risk measured by the standard deviations of portfolios and the vertical (Y) axis of this graph is the portfolio return. For the CAPM approach, the average of historical rate of returns for each stock and variances and covariance matrix of these average returns are calculated in MATLAB. For the B-L method, the posterior returns of each stock and the posterior variance-covariance matrix of these returns are calculated in MATLAB. By using plotEfficient function in MATLAB, the efficient frontiers of the portfolios according to the CAPM and the B-L approaches are estimated for each of the years in the sample period from 2005 to 2015 and displayed in Figure 5.3.

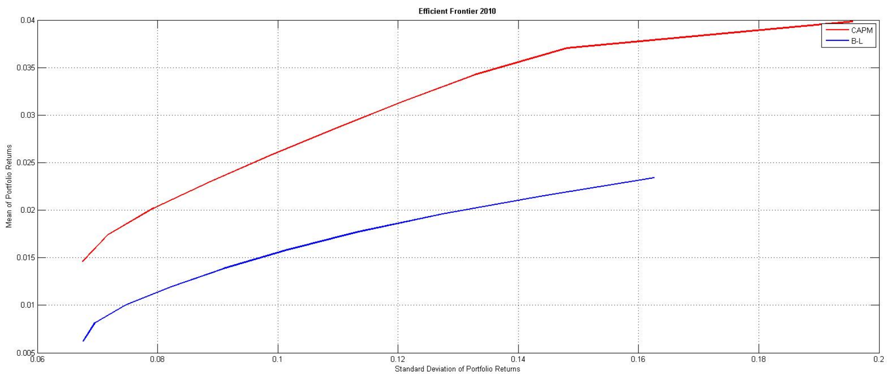

Efficient Frontier 2005
CAPM
B-L
Mean of Portfolio Returns
Standard Deviation of Portfolio Returns

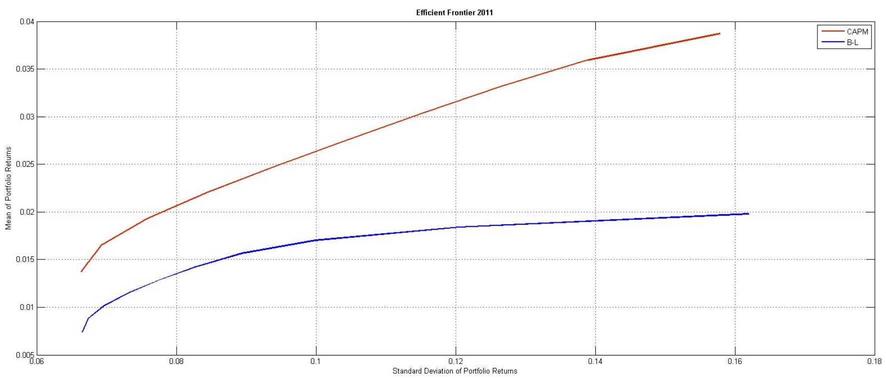

Efficient Frontier 2006
CAPM
B-L
Mean of Portfolio Returns
Standard Deviation of Portfolio Returns

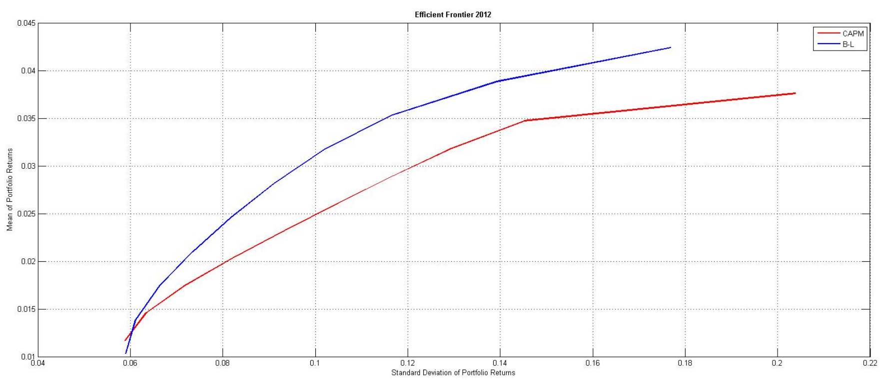

Efficient Frontier 2007
CAPM
B-L
Mean of Portfolio Returns
Standard Deviation of Portfolio Returns


Efficient Frontier 2008
CAPM
B-L
Mean of Portfolio Returns
Standard Deviation of Portfolio Returns


Efficient Frontier 2009
CAPM
B-L
Mean of Portfolio Returns
Standard Deviation of Portfolio Returns


Efficient Frontier 2010
CAPM
B-L
Mean of Portfolio Returns
Standard Deviation of Portfolio Returns


Efficient Frontier 2011
CAPM
B-L
Mean of Portfolio Returns
Standard Deviation of Portfolio Returns


Efficient Frontier 2012
CAPM
B-L
Mean of Portfolio Returns
Standard Deviation of Portfolio Returns

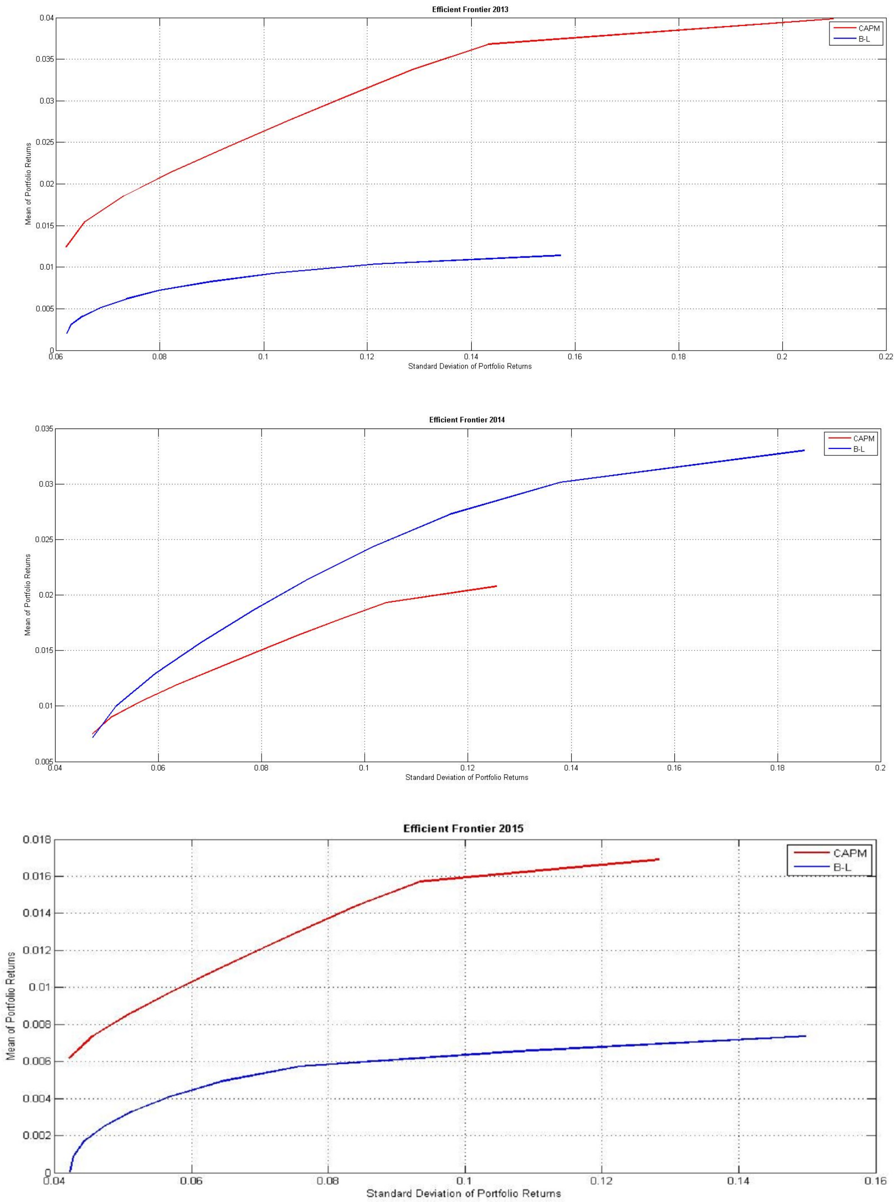


Figure 5.3: Efficient Frontiers for each Estimated Year.

This figure shows the efficient frontiers of the portfolios based on the CAPM and the B-L methods for each estimation year from 2005 to 2015. The efficient frontier is plotted on a graph with the standard deviation (risk) on the Y axis and the portfolio return on the X axis. The red line indicates the set of portfolios derived from CAPM approach, whereas the blue line indicates the set of portfolios derived from B-L approach.

When one of the efficient frontiers is above the other, that frontier has a higher expected return for the portfolio with a given standard deviation. It is seen from the Figure 5.3 that the B-L strategy outperforms the CAPM strategy for the years 2008, 2009, 2012, and 2014. On the other hand, the CAPM approach is better than the B-L approach for the years 2005, 2006, 2010, 2011, 2013 and 2015. For 2007, for portfolio standard deviations between 0.085 and 0.13, the B-L strategy is superior to the CAPM strategy. However, when the portfolio standard deviation is lower than 0.085 or higher than 0.13, the CAPM strategy is superior to the B-L strategy.

To find a possible explanation for this domination by the CAPM or the B-L approach in different years during our sample period, the average equilibrium return, average posterior return and the difference between two is calculated and reported in Table 5.10. An analysis of this table, reveals that when the average equilibrium return exceeds the average posterior return by 1% or more in any estimation year, the efficient frontier derived from the CAPM approach lies above the efficient frontier derived from the B-L approach for that year. In other words, when stock returns estimated from analysts' target stock price forecasts are underestimating the future stock returns by more than 1%, the efficient frontier of B-L is dominated by that of the CAPM. On the other hand, when the average equilibrium return exceeds the average posterior return by 0.01% or less, or the average posterior return exceeds the average equilibrium return regardless of the size of the difference, the efficient frontier derived from the B-L strategy lies above the efficient frontier derived from the CAPM strategy. In other words, when analysts are forecasting the future stock prices with small error or when they are overestimating the future stock prices, the efficient frontier of B-L dominates that of the CAPM. In the case that the average equilibrium return exceeds the average posterior return by more than 0.01% but less than 1%, which efficient frontier lies above the other is uncertain, because these two efficient frontiers intersect. For some levels of the portfolio standard deviation, efficient frontier derived from the B-L strategy can lie above the one from the CAPM strategy. However, for some other levels of the portfolio standard deviation, the opposite is true. These findings can be observed from the efficient frontiers depicted in Figure 5.3.

Table 5.10: Average Market Returns, Equilibrium Returns and Differences for each Estimation Year

This table presents the average view returns, the average implied equilibrium returns and the difference between these two values for all the stocks that are consistently included in the BIST-50 index and have target price estimated by the Bloomberg analysts.

	2005	**2006**	**2007**	**2008**	**2009**	**2010**
Avr. View Ret.	-0.0168	0.0077	0.0249	0.0192	0.0429	0.0088
Equilibrium Ret.	0.0731	0.0562	0.0315	0.0180	0.0220	0.0274
Difference	-0.0899	-0.0485	-0.0065	0.0011	0.0208	-0.0186
	2011	**2012**	**2013**	**2014**	**2015**	
Avr. View Ret.	0.0109	0.0242	0.0038	0.0132	-0.0014	
Equilibrum Ret.	0.0251	0.0237	0.0263	0.0141	0.0116	
Difference	-0.0142	0.0004	-0.0225	-0.0009	-0.0130	

CHAPTER 6

CONCLUSION

The question of how many stocks are needed to effectively diversify a portfolio has been burned in each investor's mind for many years. While adding more stocks to a portfolio is beneficial from the point of view of diversification, it results in higher transaction cost which is detrimental for an investor. This thesis reviews previous research on this subject for both the U.S. Stock Exchanges and the Turkish Stock Exchange. The mathematical aspect of the concept of diversification is discussed as well. Another question, an investor is concerned about is the allocation of his/her budget across stocks in a portfolio in order to achieve the maximum gain since the 1950s. Black and Litterman (B-L) (1990, [4]) brought a new perspective on Markowitz Mean-Variance approach. In their model, the returns of the securities in a portfolio obtained from CAPM approach provide a base case, and later these returns can be combined with an analyst or an investor's views. This thesis deals with the literature related to some finance theories starting from Markowitz Mean-Variance optimization to the B-L Model and mathematics behind these theories as well as B-L Model.

The optimal number of stocks for a well-diversified portfolio of the Borsa Istanbul-50 (BIST-50) Index constituents for the period from 2005 to 2015 is determined. In determining the optimal number of stocks, this thesis adopts the technique proposed by Evans and Archer (1968, [12]). The number of portfolio replication and the range of number of stocks in a portfolio to test whether statistically significant difference can occur are specified according to the study of Beck, Perfect and Peterson (1996, [1]). The optimal number of stocks is found to be as low as 8-10 for the years 2006 and 2012 and as high as 16-18 for the years 2011 and 2015. For the remaining years during the sample, the optimal number of stocks is found to be in between these extreme values. The average range of the optimal number of stocks over the sample period analyzed in this thesis is approximately 11-13. This finding is consistent with previous studies of the stocks included in the BIST-30 and the BIST-100. Moreover, for the year 2005, it is noticed that only 17% of the risk of a single stock portfolio is diversified by holding all the stocks included in the BIST-50 index in the portfolio, while 16% of the risk of a single stock portfolio is already diversified by holding the optimal number of stocks (10-12) for that year in the portfolio. In short, it can be concluded that a decrease of 1% in the portfolio risk may not be enough to compensate transaction costs due to investing in additional stocks. Elton and Gruber (1977, [11]) document a decrease of 51% in portfolio risk as the number of securities included in the portfolio increases from 1 to 10 securities for the New York and American Stock Exchanges. On the other hand, for the Turkish Stock Market, the maximum decline in the portfolio risk is found to be 31% in 2015. Comparison of these numbers might indicate the existence of much higher nondiversifiable risk in the Turkish Stock Exchange compared to American Stock Exchanges. Furthermore, the optimal number of stock for each year from 2005 to 2015 is determined for both prior variance-covariance matrix and posterior variance-covariance matrix derived from B-L method. Comparison of these values show that the optimal number of stocks obtained from the prior variance-covariance matrix does not differ significantly from the optimal number of stocks obtained from the posterior variance-covariance matrix. Moreover, for many years during the sample period, no difference in the optimal number of stocks obtained from these two different methods is observed.

B-L portfolio optimization method is applied to the stocks included in the BIST-50 Index for the estimation years from 2005 to 2015. The previous research on the Turkish Stock Exchange use either subjective views of authors or returns estimated by using an econometric model such as AR(1) and EGARCH as view returns due to the fact that it is hard to reach subjective analyst views for Turkish stocks. However, in this thesis, Bloomberg analysts' average price estimates are used to construct view returns which are hand collected from the Bloomberg database. In the unconstrained B-L optimization method, the optimal posterior weights of the stocks on which no views are expressed by a Bloomberg analyst are almost the same values as their equilibrium market capitalization weights. Surprisingly, B-L posterior weight of a stock with a view may not move in the same direction with the view expressed on it. In a simple example with 3 asset, the way the views are expressed (either relative or absolute) and the covariance structure of security returns are shown to be two of the reasons for the view return of a stock not being directly related to its B-L expected return and thus B-L posterior weight.

In addition to the unconstrained B-L portfolio optimization application mentioned in the previous paragraph, a budget constrained and both a budget and a no short selling constrained B-L models are implemented for the same set of stocks. Contrary to the study of B-L (1991, [17]), more deviations from equilibrium weights for stocks with no assigned view are observed in both the budget constrained, and the budget and short selling constrained optimizations. However, the value of the investor's utility function remains almost the same in the application of the budget constrained B-L optimization, whereas this value decreases significantly in the application of the budget and no short selling constrained B-L optimization. It can be deduced from this that investors gain more utility by short selling the stocks. Finally, whether the B-L strategy defeats the CAPM strategy or not is checked by using the Sharpe ratio and efficient frontier analysis. The portfolios derived from the B-L strategy perform better than the portfolios derived from the CAPM strategy based on the Sharpe ratio comparisons for most of the years in the sample period but not for all the years. The B-L strategy may not always be superior to the CAPM strategy because of the quality of analysts views on returns of the stocks used in this thesis. In contrast to the Sharpe ratio, the efficient frontier criteria show that the CAPM strategy beats the B-L strategy for most of years in the sample period analyzed in this thesis. According to the findings of this thesis, the efficient frontier derived from CAPM dominates the efficient frontier derived from B-L model when the average market return is 1% higher than or equal to the average view return. On the other hand, the efficient frontier derived from the CAPM is dominated by the one derived from the B-L model when the average market return is 0.01% lower than or equal to the average view return. However, it is impossible to determine which frontier dominates the other when the difference between the average market return and the view return is higher than 0.01% but lower than 1%. It can be concluded that when the average market return is sufficiently higher than the view return, the efficient frontier of CAPM dominates that of B-L model. On the other hand, when the average view return is sufficiently higher than the average market return, the efficient frontier of B-L Model dominates that of CAPM model.

This thesis can be extended in two different ways. First, the results of the optimal number of stocks obtained from the traditional models to form a well-diversified portfolio can potentially be added to the B-L model as a cardinality constraint. Such optimization problems can be solved by Heuristic methods as a future work. Second, the results of traditional portfolio optimization in the last step of the B-L model can be compared to the results of robust or resampled portfolio optimization.

This thesis also has some limitations. Some of the Bloomberg analyst's views are not available to users of this database. Therefore, it is impossible to calculate the variance of analyst's views. In this thesis, this variance is taken to be equal to the variance of historical returns on the stock for which any Bloomberg analyst expresses a view. This assumption is a limitation of this thesis and it definitely have an effect on the findings of this thesis. Another limitation of this thesis is that there does not exist any relative view on the Bloomberg database, therefore only the absolute views are taken into consideration in this thesis.

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APPENDIX A

Tables

Table A.1: Estimated Average Standard Deviations of Monthly Logarithmic Excess Portfolio Returns

This table presents the number of stocks in the portfolio, the mean standard deviation of these simulated portfolios and the ratio of mean standard deviation of the portfolio to mean standard deviation of a single stock for each estimation year during the sample period in this thesis. The prior variance-covariance matrix used is estimated from the prior 5-year historical return data. Initially, the replicated portfolios including stocks ranging from 1 to total number of stocks in the year analyzed are generated and their mean standard deviations are kept and shown in the second column of this table. In order to observe the percentage of risk which is distributed by holding large number of stocks in the portfolio the ratio is shown in the last column of this table.

2005		
Number Of Stocks in the Portfoilo	**Mean Standard Deviation of Returns**	**Ratio of Mean Standard Deviation to Mean Standard Deviation of a Single Stock**
1	21.8390	1
2	19.9393	0.9130
3	19.3732	0.8870
4	19.0139	0.8706
5	18.7270	0.8575
6	18.6538	0.8541
7	18.5521	0.8494
8	18.4704	0.8457
9	18.4661	0.8455
10	18.3895	0.8420
11	18.3471	0.8401
12	18.2799	0.8370
13	18.3250	0.8390
14	18.2533	0.8358
15	18.2304	0.8347
16	18.2468	0.8355
17	18.1932	0.8330
18	18.1970	0.8332
19	18.1905	0.8329
20	18.1484	0.8310
21	18.1599	0.8315
22	18.1594	0.8315
23	18.1530	0.8312
24	18.1557	0.8313
25	18.1083	0.8291
26	18.0857	0.8281
27	18.1118	0.8293
28	18.1076	0.8291
29	18.0850	0.8281
30	18.0815	0.8279
31	18.0912	0.8283
32	18.0701	0.8274
33	18.0696	0.8274
34	18.0593	0.8269

2006		
Number of Stocks in the Portfolio	**Mean Standard Deviation of Returns**	**Ratio of Mean Standard Deviation to Mean Standard Deviation of a Single Stock**
1	18.7262	1
2	17.0503	0.9105
3	16.5168	0.8820
4	16.1095	0.8602
5	15.8705	0.8475
6	15.7247	0.8397
7	15.6763	0.8371
8	15.5994	0.8330
9	15.5181	0.8286
10	15.4456	0.8248
11	15.4571	0.8254
12	15.4256	0.8237
13	15.3428	0.8193
14	15.3444	0.8194
15	15.2912	0.8165
16	15.2833	0.8161
17	15.2580	0.8147
18	15.2431	0.8139
19	15.2333	0.8134
20	15.2366	0.8136
21	15.2580	0.8147
22	15.2152	0.8125
23	15.1994	0.8116
24	15.1945	0.8114
25	15.1932	0.8113
26	15.1847	0.8108
27	15.1857	0.8109
28	15.1729	0.8102
29	15.1793	0.8105
30	15.1707	0.8101
31	15.1663	0.8098
32	15.1683	0.8100
33	15.1369	0.8083
34	15.1404	0.8085
35	15.1298	0.8079
36	15.1142	0.8071

2007		
Number of Stocks in the Portfolio	**Mean Standard Deviation of Returns**	**Ratio of Mean Standard Deviation to Mean Standard Deviation of a Single Stock**
1	15.0908	1
2	13.5741	0.8994
3	12.8681	0.8527
4	12.5656	0.8326
5	12.2877	0.8142
6	12.2064	0.8088
7	12.1337	0.8040
8	12.0179	0.7963
9	11.9767	0.7936
10	11.9282	0.7904
11	11.8766	0.7870
12	11.8011	0.7820
13	11.8412	0.7846
14	11.8349	0.7842
15	11.7787	0.7805
16	11.7823	0.7807
17	11.7444	0.7782
18	11.7434	0.7781
19	11.7308	0.7773
20	11.7103	0.7759
21	11.7185	0.7765
22	11.6965	0.7750
23	11.6973	0.7751
24	11.6951	0.7749
25	11.6781	0.7738
26	11.6642	0.7729
27	11.6510	0.7720
28	11.6597	0.7726
29	11.6452	0.7716
30	11.6495	0.7719
31	11.6278	0.7705
32	11.6329	0.7708
33	11.6218	0.7701
34	11.6093	0.7692
35	11.6169	0.7698
36	11.6038	0.7689
37	11.5930	0.7682
38	11.5785	0.7672

2008		
Number of Stocks in the Portfolio	**Mean Standard Deviation of Returns**	**Ratio of Mean Standard Deviation to Mean Standard Deviation of a Single Stock**
1	12.4628	1
2	10.8187	0.8680
3	10.0672	0.8077
4	9.7424	0.7817
5	9.5147	0.7634
6	9.4595	0.7590
7	9.3274	0.7484
8	9.2843	0.7449
9	9.2045	0.7385
10	9.2015	0.7383
11	9.1146	0.7313
12	9.1234	0.7320
13	9.0794	0.7285
14	9.0412	0.7254
15	9.0379	0.7251
16	9.0139	0.7232
17	8.9753	0.7201
18	8.9828	0.7207
19	8.9675	0.7195
20	8.9597	0.7189
21	8.9230	0.7159
22	8.9479	0.7179
23	8.9038	0.7144
24	8.9127	0.7151
25	8.8968	0.7138
26	8.8932	0.7135
27	8.8832	0.7127
28	8.8868	0.7130
29	8.8761	0.7122
30	8.8633	0.7111
31	8.8503	0.7101
32	8.8550	0.7105
33	8.8492	0.7100
34	8.8375	0.7091
35	8.8298	0.7084
36	8.8197	0.7076
37	8.8078	0.7067

2009		
Number of Stocks in the Portfolio	**Mean Standard Deviation of Returns**	**Ratio of Mean Standard Deviation to Mean Standard Deviation of a Single Stock**
1	13.8431	1
2	12.0105	0.8676
3	11.3038	0.8165
4	10.9585	0.7916
5	10.7038	0.7732
6	10.5449	0.7617
7	10.4677	0.7561
8	10.3529	0.7478
9	10.3162	0.7452
10	10.2043	0.7371
11	10.1771	0.7351
12	10.1396	0.7324
13	10.1097	0.7303
14	10.0860	0.7285
15	10.0594	0.7266
16	10.0576	0.7265
17	10.0299	0.7245
18	10.0017	0.7225
19	9.9973	0.7221
20	9.9757	0.7206
21	9.9647	0.7198
22	9.9605	0.7195
23	9.9523	0.7189
24	9.9586	0.7193
25	9.9012	0.7152
26	9.9026	0.7153
27	9.9005	0.7151
28	9.8986	0.7150
29	9.8916	0.7145
30	9.8741	0.7132
31	9.8705	0.7130
32	9.8595	0.7122
33	9.8374	0.7106
34	9.8159	0.7090

2010		
Number of Stocks in the Portfolio	**Mean Standard Deviation of Returns**	**Ratio of Mean Standard Deviation to Mean Standard Deviation of a Single Stock**
1	15.0133	1
2	13.1390	0.8751
3	12.2590	0.8165
4	11.8560	0.7897
5	11.6220	0.7741
6	11.4713	0.7640
7	11.3620	0.7567
8	11.3155	0.7536
9	11.1849	0.7450
10	11.1360	0.7417
11	11.0948	0.739
12	11.0511	0.7360
13	10.9921	0.7321
14	11.0145	0.7336
15	10.9515	0.7294
16	10.9386	0.7285
17	10.9109	0.7267
18	10.9150	0.7270
19	10.8802	0.7247
20	10.8920	0.7254
21	10.8506	0.7227
22	10.8523	0.7228
23	10.8633	0.7235
24	10.8402	0.7220
25	10.8180	0.7205
26	10.8086	0.7199
27	10.8028	0.7195
28	10.8063	0.7197
29	10.7886	0.7186
30	10.7793	0.7179
31	10.7795	0.7179
32	10.7584	0.7165
33	10.7531	0.7162
34	10.7428	0.7155
35	10.7331	0.7149
36	10.7180	0.7139

2011		
Number of Stocks in the Portfolio	Mean Standard Deviation of Returns	Ratio of Mean Standard Deviation to Mean Standard Deviation of a Single Stock
1	14.3096	1
2	12.2796	0.8581
3	11.4936	0.8032
4	11.0981	0.7755
5	10.8186	0.7560
6	10.6691	0.7455
7	10.5049	0.7341
8	10.4417	0.7297
9	10.3564	0.7237
10	10.3291	0.7218
11	10.2602	0.7170
12	10.2104	0.7135
13	10.2169	0.7139
14	10.1405	0.7086
15	10.1182	0.7070
16	10.0870	0.7049
17	10.0817	0.7045
18	10.0537	0.7025
19	10.0497	0.7023
20	10.0288	0.7008
21	10.0193	0.7001
22	9.9916	0.6982
23	10.0097	0.6995
24	9.9785	0.6973
25	9.9906	0.6981
26	9.9753	0.6971
27	9.9716	0.6968
28	9.9564	0.6957
29	9.9503	0.6953
30	9.9441	0.6949

2012		
Number of Stocks in the Portfolio	Mean Standard Deviation of Returns	Ratio of Mean Standard Deviation to Mean Standard Deviation of a Single Stock
1	13.9394	1
2	11.8672	0.8513
3	11.0712	0.7942
4	10.6545	0.7643
5	10.5208	0.7547
6	10.3258	0.7407
7	10.2530	0.7355
8	10.1446	0.7277
9	10.0541	0.7212
10	9.9613	0.7146
11	10.0042	0.7176
12	9.9206	0.7117
13	9.9114	0.7110
14	9.8516	0.7067
15	9.8437	0.7061
16	9.8232	0.7047
17	9.8142	0.7040
18	9.7839	0.7018
19	9.7588	0.7000
20	9.7543	0.6997
21	9.7299	0.6980
22	9.7418	0.6988
23	9.6938	0.6954
24	9.7196	0.6972
25	9.6904	0.6951
26	9.6896	0.6951
27	9.6887	0.6950
28	9.6714	0.6938
29	9.6666	0.6934
30	9.6564	0.6927
31	9.6531	0.6925
32	9.6434	0.6918
33	9.6321	0.6910
34	9.6348	0.6911
35	9.6201	0.6901
36	9.6056	0.6890
37	9.5928	0.6881

2013		
Number of Stocks in the Portfolio	Mean Standard Deviation of Returns	Ratio of Mean Standard Deviation to Mean Standard Deviation of a Single Stock
1	13.8466	1
2	12.2056	0.8814
3	11.7484	0.8484
4	11.4172	0.8245
5	11.2043	0.8091
6	11.1106	0.8024
7	11.0072	0.7949
8	10.9142	0.7882
9	10.8142	0.7810
10	10.8563	0.7840
11	10.7797	0.7785
12	10.7551	0.7767
13	10.7204	0.7742
14	10.6720	0.7707
15	10.6632	0.7700
16	10.6454	0.7688
17	10.6550	0.7695
18	10.6457	0.7688
19	10.5917	0.7649
20	10.6054	0.7659
21	10.5763	0.7638
22	10.5631	0.7628
23	10.5667	0.7631
24	10.5618	0.7627
25	10.5376	0.7610
26	10.5343	0.7607
27	10.5437	0.7614
28	10.5140	0.7593
29	10.5333	0.7607
30	10.5191	0.7596
31	10.5126	0.7592
32	10.5007	0.7583
33	10.4890	0.7575
34	10.4866	0.7573
35	10.4818	0.7570
36	10.4670	0.7559
37	10.4497	0.7546

2014		
Number of Stocks in the Portfolio	Mean Standard Deviation of Returns	Ratio of Mean Standard Deviation to Mean Standard Deviation of a Single Stock
1	11.1659	1
2	9.5173	0.8523
3	8.9355	0.8002
4	8.5658	0.7671
5	8.4128	0.7534
6	8.2965	0.7430
7	8.1974	0.7341
8	8.1120	0.7265
9	8.0800	0.7236
10	8.0072	0.7171
11	7.9569	0.7126
12	7.9590	0.7127
13	7.9013	0.7076
14	7.8852	0.7061
15	7.8613	0.7040
16	7.8381	0.7019
17	7.8489	0.7029
18	7.8385	0.7020
19	7.8343	0.7016
20	7.7993	0.6984
21	7.7897	0.6976
22	7.8009	0.6986
23	7.7770	0.6964
24	7.7616	0.6951
25	7.7717	0.6960
26	7.7496	0.6940
27	7.7454	0.6936
28	7.7434	0.6934
29	7.7341	0.6926
30	7.7317	0.6924
31	7.7111	0.6905
32	7.7233	0.6916
33	7.7135	0.6908
34	7.7082	0.6903
35	7.7027	0.6898
36	7.7019	0.6897
37	7.6964	0.6892
38	7.6953	0.6891
39	7.6865	0.6883
40	7.6770	0.6875

2015		
Number of Stocks in the Portfolio	Mean Standard Deviation of Returns	Ratio of Mean Standard Deviation to Mean Standard Deviation of a Single Stock
1	9.9162	1
2	8.6147	0.8688
3	8.0327	0.8101
4	7.7318	0.7797
5	7.5384	0.7602
6	7.4808	0.7544
7	7.3374	0.7399
8	7.3149	0.7377
9	7.2104	0.7271
10	7.1910	0.7252
11	7.1707	0.7231
12	7.1129	0.7173
13	7.0979	0.7158
14	7.0791	0.7139
15	7.0692	0.7129
16	7.0476	0.7107
17	7.0230	0.7082
18	7.0219	0.7081
19	7.0019	0.7061
20	7.0108	0.7070
21	6.9887	0.7048
22	6.9799	0.7039
23	6.9754	0.7034
24	6.9572	0.7016
25	6.9597	0.7018
26	6.9523	0.7011
27	6.9442	0.7003
28	6.9367	0.6995
29	6.9308	0.6989
30	6.9209	0.6979
31	6.9344	0.6993
32	6.9186	0.6977
33	6.9206	0.6979
34	6.9098	0.6968
35	6.9083	0.6967
36	6.9017	0.6960
37	6.9030	0.6961
38	6.8948	0.6953
39	6.8883	0.6947

Table A.2: The Results of T-test for each Year

This table presents the results of the t-test at 5% significance level for each estimation years during the sample period. T-test is performed to determine whether any change in the mean standard deviation of two portfolios that have N and $N + 2$ stocks occurs or not. The prior variance-covariance matrix used is estimated from the prior 5-year historical return data. Initially, the 1000 replicated portfolios including stocks ranging from 1 to total number of stocks in the year analyzed are generated and their mean standard deviations are kept. The statistical decrease in the mean standard deviation of the portfolios that have N and $N + 2$ stocks are tested. T-test statistic and the critical t values are shown in this table.

Estimation year	**The period of historical monthly data used**	
2005	**(Jan 2000-Dec 2004)**	
Portfolio size	**T-test statistic**	**Critical t value**
1 and 2	11.0577	1.6463
2 and 4	8.6463	1.6463
4 and 6	5.1252	1.6463
6 and 8	3.3184	1.6463
8 and 10	1.7105	1.6463
10 and 12	2.7862	1.6463
12 and 14	0.7410	1.6463
14 and 16	0.2088	1.6463
16 and 18	1.7218	1.6463
18 and 20	1.9371	1.6463
20 and 22	−0.5058	1.6463
22 and 24	0.1903	1.6463
24 and 26	4.1627	1.6463
26 and 28	−1.5591	1.6463
28 and 30	2.3176	1.6463
30 and 32	1.3302	1.6463
32 and 34	2.2397	1.6463

Estimation year	**The period of historical monthly data used**	
2006	**(Jan 2001-Dec 2005)**	
Portfolio size	**T-test statistic**	**Critical t value**
1 and 2	12.0661	1.6463
2 and 4	9.8776	1.6463
4 and 6	5.5598	1.6463
6 and 8	2.1795	1.6463
8 and 10	3.2723	1.6463
10 and 12	0.4854	1.6463
12 and 14	2.2558	1.6463
14 and 16	1.8541	1.6463
16 and 18	1.3587	1.6463
18 and 20	0.2438	1.6463
20 and 22	0.9029	1.6463
22 and 24	1.0231	1.6463
24 and 26	0.5334	1.6463
26 and 28	0.7244	1.6463
28 and 30	0.1560	1.6463
30 and 32	0.2151	1.6463
32 and 34	3.1963	1.6463
34 and 36	5.1311	1.6463
Estimation year	**The period of historical monthly data used**	
2007	**(Jan 2002-Dec 2006)**	
Portfolio size	**T-test statistic**	**Critical t value**
1 and 2	15.2200	1.6463
2 and 4	16.1687	1.6463
4 and 6	8.6698	1.6463
6 and 8	5.2866	1.6463
8 and 10	3.0659	1.6463
10 and 12	4.9719	1.6463
12 and 14	−1.4646	1.6463
14 and 16	2.5568	1.6463
16 and 18	2.1382	1.6463
18 and 20	2.0254	1.6463
20 and 22	0.9338	1.6463
22 and 24	0.1027	1.6463
24 and 26	2.5837	1.6463
26 and 28	0.4112	1.6463
28 and 30	1.0657	1.6463
30 and 32	2.0154	1.6463
32 and 34	3.5295	1.6463
34 and 36	1.0231	1.6463
36 and 38	7.9131	1.6463

Estimation year	The period of historical monthly data used	
2008	(Jan 2003-Dec 2007)	
Portfolio size	**T-test statistic**	**Critical t value**
1 and 2	18.0575	1.6463
2 and 4	19.0737	1.6463
4 and 6	7.1007	1.6463
6 and 8	5.2715	1.6463
8 and 10	2.7442	1.6463
10 and 12	3.0498	1.6463
12 and 14	3.6629	1.6463
14 and 16	1.3226	1.6463
16 and 18	1.6928	1.6463
18 and 20	1.4093	1.6463
20 and 22	0.8462	1.6463
22 and 24	2.7722	1.6463
24 and 26	1.6521	1.6463
26 and 28	0.6280	1.6463
28 and 30	2.6025	1.6463
30 and 32	1.0601	1.6463
32 and 34	2.8545	1.6463
34 and 37	8.0434	1.6463

Estimation year	The period of historical monthly data used	
2009	(Jan 2004-Dec 2008)	
Portfolio size	**T-test statistic**	**Critical t value**
1 and 2	21.3480	1.6463
2 and 4	17.7142	1.6463
4 and 6	9.0575	1.6463
6 and 8	5.0300	1.6463
8 and 10	4.4586	1.6463
10 and 12	2.1105	1.6463
12 and 14	1.9944	1.6463
14 and 16	1.2349	1.6463
16 and 18	2.7129	1.6463
18 and 20	1.3401	1.6463
20 and 22	0.9017	1.6463
22 and 24	0.1367	1.6463
24 and 26	4.3243	1.6463
26 and 28	0.3608	1.6463
28 and 30	2.7272	1.6463
30 and 32	2.1062	1.6463
32 and 34	10.9984	1.6463

Estimation year	The period of historical monthly data used	
2010	(Jan 2005-Dec 2009)	
Portfolio size	**T-test statistic**	**Critical t value**
1 and 2	18.0618	1.6463
2 and 4	17.4811	1.6463
4 and 6	6.9859	1.6463
6 and 8	3.4776	1.6463
8 and 10	4.4644	1.6463
10 and 12	2.4085	1.6463
12 and 14	1.1989	1.6463
14 and 16	2.7054	1.6463
16 and 18	0.9242	1.6463
18 and 20	1.0107	1.6463
20 and 22	2.0053	1.6463
22 and 24	0.6732	1.6463
24 and 26	1.9328	1.6463
26 and 28	0.1609	1.6463
28 and 30	2.3090	1.6463
30 and 32	2.1580	1.6463
32 and 34	2.1283	1.6463
34 and 36	5.6806	1.6463
Estimation year	**The period of historical monthly data used**	
2011	**(Jan 2006-Dec 2010)**	
Portfolio size	**T-test statistic**	**Critical t value**
1 and 2	20.4009	1.6463
2 and 4	18.6419	1.6463
4 and 6	8.7208	1.6463
6 and 8	5.6582	1.6463
8 and 10	3.2880	1.6463
10 and 12	3.8262	1.6463
12 and 14	2.6203	1.6463
14 and 16	2.3945	1.6463
16 and 18	1.6982	1.6463
18 and 20	1.4363	1.6463
20 and 22	2.4698	1.6463
22 and 24	1.0700	1.6463
24 and 26	0.3247	1.6463
26 and 28	2.6029	1.6463
28 and 30	3.1401	1.6463

Estimation year	The period of historical monthly data used	
2012	(Jan 2007-Dec 2011)	
Portfolio size	**T-test statistic**	**Critical t value**
1 and 2	21.2283	1.6463
2 and 4	18.0050	1.6463
4 and 6	6.7391	1.6463
6 and 8	4.4399	1.6463
8 and 10	5.4880	1.6463
10 and 12	1.3433	1.6463
12 and 14	2.5108	1.6463
14 and 16	1.1535	1.6463
16 and 18	1.8040	1.6463
18 and 20	1.5017	1.6463
20 and 22	0.7144	1.6463
22 and 24	1.3788	1.6463
24 and 26	2.0490	1.6463
26 and 28	1.5168	1.6463
28 and 30	1.4462	1.6463
30 and 32	1.5392	1.6463
32 and 34	1.1771	1.6463
34 and 37	9.6869	1.6463

Estimation year	The period of historical monthly data used	
2013	(Jan 2008-Dec 2012)	
Portfolio size	**T-test statistic**	**Critical t value**
1 and 2	17.5239	1.6463
2 and 4	12.0699	1.6463
4 and 6	6.1792	1.6463
6 and 8	4.9383	1.6463
8 and 10	1.7108	1.6463
10 and 12	3.4526	1.6463
12 and 14	3.0985	1.6463
14 and 16	1.1134	1.6463
16 and 18	−0.0164	1.6463
18 and 20	2.1828	1.6463
20 and 22	2.3893	1.6463
22 and 24	0.0814	1.6463
24 and 26	2.0245	1.6463
26 and 28	1.6947	1.6463
28 and 30	−0.4971	1.6463
30 and 32	2.0513	1.6463
32 and 34	1.9652	1.6463
34 and 37	8.5489	1.6463

Estimation year	**The period of historical monthly data used**	
2014	**(Jan 2009-Dec 2013)**	
Portfolio size	**T-test statistic**	**Critical t value**
1 and 2	18.3781	1.6463
2 and 4	16.8095	1.6463
4 and 6	6.8343	1.6463
6 and 8	5.7312	1.6463
8 and 10	4.0075	1.6463
10 and 12	2.0563	1.6463
12 and 14	3.5772	1.6463
14 and 16	2.5959	1.6463
16 and 18	−0.0276	1.6463
18 and 20	2.5679	1.6463
20 and 22	−0.1221	1.6463
22 and 24	3.2119	1.6463
24 and 26	1.0735	1.6463
26 and 28	0.6075	1.6463
28 and 30	1.3244	1.6463
30 and 32	1.0829	1.6463
32 and 34	2.2426	1.6463
34 and 36	1.1631	1.6463
36 and 38	1.5518	1.6463
38 and 40	7.2456	1.6463
Estimation year	**The period of historical monthly data used**	
2015	**(Jan 2010-Dec 2014)**	
Portfolio size	**T-test statistic**	**Critical t value**
1 and 2	19.0225	1.6463
2 and 4	20.5641	1.6463
4 and 6	8.2171	1.6463
6 and 8	6.7042	1.6463
8 and 10	5.9132	1.6463
10 and 12	4.1572	1.6463
12 and 14	1.9810	1.6463
14 and 16	2.1472	1.6463
16 and 18	1.8848	1.6463
18 and 20	0.8912	1.6463
20 and 22	2.7124	1.6463
22 and 24	2.2426	1.6463
24 and 26	0.5296	1.6463
26 and 28	1.9615	1.6463
28 and 30	2.1840	1.6463
30 and 32	0.3574	1.6463
32 and 34	1.6224	1.6463
34 and 36	1.8291	1.6463
36 and 39	4.9319	1.6463

Table A.3: Estimated Average Standard Deviations of the Combined Monthly Logarithmic Excess Portfolio Returns with Investor Views

This table presents the number of stocks in the portfolio, the mean standard deviation of these simulated portfolios and the ratio of mean standard deviation of the portfolio to mean standard deviation of a single stock for each estimation year during the sample period in this thesis. The posterior variance-covariance matrix based on B-L model is used. Initially, the replicated portfolios including stocks ranging from 1 to total number of stocks in the year analyzed are generated and their mean standard deviations are kept and shown in the second column of this table. In order to observe the percentage of risk which is diversified by holding large number of stocks in the portfolio the ratio is shown in the last column of this table.

2005		
Number of Stocks in the Portfolio	**Mean Standard Deviation of Returns**	**Ratio of Mean Standard Deviation to Mean Standard Deviation of a Single Stock**
1	21.8241	1
2	19.9101	0.9122
3	19.2962	0.8841
4	18.9955	0.8703
5	18.8612	0.8642
6	18.6747	0.8556
7	18.5106	0.8481
8	18.5601	0.8504
9	18.4671	0.8461
10	18.3924	0.8427
11	18.4101	0.8435
12	18.3584	0.8412
13	18.3240	0.8396
14	18.2818	0.8376
15	18.2518	0.8363
16	18.1986	0.8338
17	18.2193	0.8348
18	18.2263	0.8351
19	18.1845	0.8332
20	18.1884	0.8334
21	18.1831	0.8331
22	18.1630	0.8322
23	18.1706	0.8325
24	18.1619	0.8321
25	18.1427	0.8313
26	18.1087	0.8297
27	18.1066	0.8296
28	18.1180	0.8301
29	18.0992	0.8293
30	18.1073	0.8296
31	18.0987	0.8292
32	18.0914	0.8289
33	18.0788	0.8283
34	18.0704	0.8280

2006		
Number of Stocks in the Portfolio	**Mean Standard Deviation of Returns**	**Ratio of Mean Standard Deviation to Mean Standard Deviation of a Single Stock**
1	18.8776	1
2	17.2443	0.9134
3	16.5571	0.8770
4	16.1145	0.8536
5	15.9064	0.8426
6	15.7585	0.8347
7	15.6588	0.8294
8	15.4794	0.8199
9	15.5674	0.8246
10	15.4501	0.8184
11	15.4565	0.8187
12	15.4030	0.8159
13	15.3791	0.8146
14	15.3335	0.8122
15	15.2936	0.8101
16	15.3023	0.8106
17	15.2882	0.8098
18	15.2762	0.8092
19	15.2660	0.8086
20	15.2603	0.8083
21	15.2192	0.8062
22	15.2383	0.8072
23	15.2397	0.8072
24	15.1810	0.8041
25	15.2153	0.8059
26	15.1842	0.8043
27	15.2051	0.8054
28	15.2026	0.8053
29	15.1601	0.8030
30	15.1521	0.8026
31	15.1505	0.8025
32	15.1646	0.8033
33	15.1553	0.8028
34	15.1455	0.8023
35	15.1367	0.8018
36	15.1227	0.8010

2007		
Number of Stocks in the Portfolio	Mean Standard Deviation of Returns	Ratio of Mean Standard Deviation to Mean Standard Deviation of a Single Stock
1	15.2868	1
2	13.5435	0.8859
3	12.8829	0.8427
4	12.5909	0.8236
5	12.3405	0.8072
6	12.2381	0.8005
7	12.1299	0.7934
8	12.0621	0.7890
9	11.9835	0.7839
10	11.9345	0.7807
11	11.9314	0.7805
12	11.8477	0.7750
13	11.8652	0.7761
14	11.8159	0.7729
15	11.8031	0.7721
16	11.7861	0.7709
17	11.7809	0.7706
18	11.7646	0.7695
19	11.7483	0.7685
20	11.7378	0.7678
21	11.7170	0.7664
22	11.7068	0.7658
23	11.6935	0.7649
24	11.7026	0.7655
25	11.7016	0.7654
26	11.6713	0.7634
27	11.6750	0.7637
28	11.6651	0.7630
29	11.6470	0.7618
30	11.6583	0.7626
31	11.6408	0.7614
32	11.6288	0.7607
33	11.6262	0.7605
34	11.6192	0.7600
35	11.6048	0.7591
36	11.6075	0.7593
37	11.5967	0.7586
38	11.5841	0.7577

2008		
Number of Stocks in the Portfolio	Mean Standard Deviation of Returns	Ratio of Mean Standard Deviation to Mean Standard Deviation of a Single Stock
1	12.4085	1
2	10.8011	0.8704
3	10.1576	0.8186
4	9.8051	0.7902
5	9.6444	0.7772
6	9.4857	0.7644
7	9.3537	0.7538
8	9.2551	0.7458
9	9.2181	0.7428
10	9.1947	0.7410
11	9.1393	0.7365
12	9.1111	0.7342
13	9.0674	0.7307
14	9.0218	0.7270
15	9.0332	0.7279
16	9.0014	0.7254
17	8.9615	0.7222
18	8.9997	0.7252
19	8.9845	0.7240
20	8.9565	0.7218
21	8.9282	0.7195
22	8.9431	0.7207
23	8.9144	0.7184
24	8.9124	0.7182
25	8.9029	0.7174
26	8.8848	0.7160
27	8.8923	0.7166
28	8.8769	0.7153
29	8.8679	0.7146
30	8.8609	0.7141
31	8.8600	0.7140
32	8.8544	0.7135
33	8.8495	0.7131
34	8.8447	0.7127
35	8.8320	0.7117
36	8.8213	0.7109
37	8.8119	0.7101

2009		
Number of Stocks in the Portfolio	Mean Standard Deviation of Returns	Ratio of Mean Standard Deviation to Mean Standard Deviation of a Single Stock
1	13.8532	1
2	12.0511	0.8699
3	11.3838	0.8217
4	10.9302	0.7890
5	10.7103	0.7731
6	10.5566	0.7620
7	10.4356	0.7532
8	10.3582	0.7477
9	10.2980	0.7433
10	10.2699	0.7413
11	10.2206	0.7377
12	10.1796	0.7348
13	10.1342	0.7315
14	10.0777	0.7274
15	10.0359	0.7244
16	10.0832	0.7278
17	10.0355	0.7244
18	10.0082	0.7224
19	10.0027	0.7220
20	9.9792	0.7203
21	9.9976	0.7216
22	9.9599	0.7189
23	9.9468	0.7180
24	9.9288	0.7167
25	9.9369	0.7173
26	9.9245	0.7164
27	9.8996	0.7146
28	9.9011	0.7147
29	9.8889	0.7138
30	9.8784	0.7130
31	9.8677	0.7123
32	9.8659	0.7121
33	9.8474	0.7108
34	9.8203	0.7088

2010		
Number of Stocks in the Portfolio	Mean Standard Deviation of Returns	Ratio of Mean Standard Deviation to Mean Standard Deviation of a Single Stock
1	15.1070	1
2	13.0642	0.8647
3	12.2967	0.8139
4	11.9258	0.7894
5	11.6412	0.7705
6	11.4305	0.7566
7	11.3520	0.7514
8	11.2481	0.7445
9	11.1637	0.7389
10	11.1449	0.7377
11	11.0814	0.7335
12	11.0540	0.7317
13	11.0313	0.7302
14	11.0204	0.7294
15	10.9502	0.7248
16	10.9453	0.7245
17	10.9493	0.7247
18	10.9543	0.7251
19	10.8963	0.7212
20	10.8866	0.7206
21	10.8732	0.7197
22	10.8643	0.7191
23	10.8570	0.7186
24	10.8137	0.7158
25	10.8175	0.7160
26	10.7965	0.7146
27	10.8002	0.7149
28	10.8127	0.7157
29	10.7936	0.7144
30	10.7916	0.7143
31	10.7801	0.7135
32	10.7712	0.7129
33	10.7553	0.7119
34	10.7495	0.7115
35	10.7426	0.7111
36	10.7226	0.7097

2011		
Number of Stocks in the Portfolio	**Mean Standard Deviation of Returns**	**Ratio of Mean Standard Deviation to Mean Standard Deviation of a Single Stock**
1	14.2133	1
2	12.2979	0.8652
3	11.5155	0.8101
4	11.0579	0.7780
5	10.8059	0.7602
6	10.6986	0.7527
7	10.5883	0.7449
8	10.4757	0.7370
9	10.3733	0.7298
10	10.3199	0.7260
11	10.2629	0.7220
12	10.2143	0.7186
13	10.1845	0.7165
14	10.1618	0.7149
15	10.1306	0.7127
16	10.1133	0.7115
17	10.1058	0.7110
18	10.0874	0.7097
19	10.0389	0.7063
20	10.0394	0.7063
21	10.0136	0.7045
22	10.0194	0.7049
23	10.0065	0.7040
24	10.0041	0.7038
25	9.9852	0.7025
26	9.9834	0.7024
27	9.9730	0.7016
28	9.9556	0.7004
29	9.9547	0.7003
30	9.9498	0.7000

2012		
Number of Stocks in the Portfolio	**Mean Standard Deviation of Returns**	**Ratio of Mean Standard Deviation to Mean Standard Deviation of a Single Stock**
1	13.9696	1
2	11.8206	0.8461
3	11.2194	0.8031
4	10.7136	0.7669
5	10.5055	0.7520
6	10.3382	0.7400
7	10.2150	0.7312
8	10.1452	0.7262
9	10.0565	0.7198
10	10.0136	0.7168
11	9.9674	0.7135
12	9.9732	0.7139
13	9.9017	0.7088
14	9.9186	0.7100
15	9.8662	0.7062
16	9.8593	0.7057
17	9.8034	0.7017
18	9.7963	0.7012
19	9.7789	0.7000
20	9.7622	0.6988
21	9.7610	0.6987
22	9.7396	0.6972
23	9.7451	0.6975
24	9.7105	0.6951
25	9.6895	0.6936
26	9.6891	0.6935
27	9.6782	0.6928
28	9.6681	0.6920
29	9.6627	0.6916
30	9.6611	0.6915
31	9.6730	0.6924
32	9.6490	0.6907
33	9.6417	0.6901
34	9.6271	0.6891
35	9.6221	0.6887
36	9.6138	0.6881
37	9.5978	0.6870

2013		
Number of Stocks in the Portfolio	**Mean Standard Deviation of Returns**	**Ratio of Mean Standard Deviation to Mean Standard Deviation of a Single Stock**
1	13.8413	1
2	12.2761	0.8869
3	11.7309	0.8475
4	11.4105	0.8243
5	11.1560	0.8059
6	11.0826	0.8006
7	10.9684	0.7924
8	10.8864	0.7865
9	10.8269	0.7822
10	10.8216	0.7818
11	10.7645	0.7777
12	10.7502	0.7766
13	10.7478	0.7765
14	10.7000	0.7730
15	10.6723	0.7710
16	10.6649	0.7705
17	10.6460	0.7691
18	10.6217	0.7673
19	10.6303	0.7680
20	10.6078	0.7663
21	10.6179	0.7671
22	10.5728	0.7638
23	10.5823	0.7645
24	10.5611	0.7630
25	10.5645	0.7632
26	10.5479	0.7620
27	10.5463	0.7619
28	10.5346	0.7611
29	10.5206	0.7600
30	10.5153	0.7597
31	10.5191	0.7599
32	10.5014	0.7586
33	10.5008	0.7586
34	10.4943	0.7581
35	10.4798	0.7571
36	10.4731	0.7566
37	10.4538	0.7552

2014		
Number of Stocks in the Portfolio	**Mean Standard Deviation of Returns**	**Ratio of Mean Standard Deviation to Mean Standard Deviation of a Single Stock**
1	11.1695	1
2	9.4589	0.8468
3	8.9748	0.8035
4	8.6047	0.7703
5	8.4523	0.7567
6	8.3341	0.7461
7	8.1892	0.7331
8	8.1223	0.7271
9	8.0881	0.7241
10	8.0432	0.7201
11	8.0118	0.7172
12	7.9586	0.7125
13	7.9505	0.7118
14	7.8892	0.7063
15	7.8842	0.7058
16	7.8396	0.7018
17	7.8718	0.7047
18	7.8518	0.7029
19	7.8109	0.6993
20	7.8213	0.7002
21	7.7951	0.6978
22	7.8015	0.6984
23	7.7720	0.6958
24	7.7793	0.6964
25	7.7553	0.6943
26	7.7574	0.6945
27	7.7537	0.6941
28	7.7450	0.6934
29	7.7375	0.6927
30	7.7243	0.6915
31	7.7273	0.6918
32	7.7164	0.6908
33	7.7142	0.6906
34	7.7129	0.6905
35	7.7119	0.6904
36	7.7014	0.6895
37	7.6994	0.6893
38	7.6898	0.6884
39	7.6854	0.6880
40	7.6804	0.6876

2015		
Number of Stocks in the Portfolio	**Mean Standard Deviation of Returns**	**Ratio of Mean Standard Deviation to Mean Standard Deviation of a Single Stock**
1	9.9723	1
2	8.5912	0.8615
3	8.0280	0.8050
4	7.7517	0.7773
5	7.5704	0.7591
6	7.4175	0.7438
7	7.3534	0.7374
8	7.2752	0.7295
9	7.2267	0.7247
10	7.1873	0.7207
11	7.1704	0.7190
12	7.1321	0.7152
13	7.1209	0.7141
14	7.0779	0.7098
15	7.0665	0.7086
16	7.0308	0.7050
17	7.0473	0.7067
18	7.0285	0.7048
19	7.0050	0.7024
20	7.0054	0.7025
21	6.9913	0.7011
22	6.9937	0.7013
23	6.9792	0.6999
24	6.9760	0.6995
25	6.9568	0.6976
26	6.9519	0.6971
27	6.9526	0.6972
28	6.9403	0.6960
29	6.9363	0.6956
30	6.9292	0.6948
31	6.9194	0.6939
32	6.9298	0.6949
33	6.9173	0.6937
34	6.9160	0.6935
35	6.9118	0.6931
36	6.9073	0.6927
37	6.9042	0.6923
38	6.8980	0.6917
39	6.8915	0.6911

Table A.4: The Results of T-test performed on Posterior Variance Matrices for each Year

This table presents the results of the t-test at 5% significance level for each estimation years during the sample period. T-test is performed to determine whether any change in the mean standard deviation of two portfolios that have N and $N+2$ stocks occurs or not. The posterior variance-covariance matrix based on B-L model is used. Initially, the 1000 replicated portfolios including stocks ranging from 1 to total number of stocks in the year analyzed are generated and their mean standard deviations are kept. The statistical decrease in the mean standard deviation of the portfolios that have N and $N+2$ stocks are tested. T-test statistic and the critical t values are shown in this table.

Estimation year	**The period of historical monthly data used**	
2005	**(Jan 2000-Dec 2004)**	
Portfolio size	**T-test statistic**	**Critical t value**
1 and 2	11.7139	1.6463
2 and 4	8.8969	1.6463
4 and 6	4.5063	1.6463
6 and 8	2.0013	1.6463
8 and 10	3.5213	1.6463
10 and 12	0.8200	1.6463
12 and 14	2.2214	1.6463
14 and 16	2.6911	1.6463
16 and 18	−0.9894	1.6463
18 and 20	1.5667	1.6463
20 and 22	1.2002	1.6463
22 and 24	0.0568	1.6463
24 and 26	3.1969	1.6463
26 and 28	−0.6360	1.6463
28 and 30	0.9628	1.6463
30 and 32	1.8920	1.6463
32 and 34	4.4025	1.6463
Estimation year	**The period of historical monthly data used**	
2006	**(Jan 2001-Dec 2005)**	
Portfolio size	**T-test statistic**	**Critical t value**
1 and 2	11.7939	1.6463
2 and 4	12.1232	1.6463
4 and 6	5.4236	1.6463
6 and 8	5.0317	1.6463
8 and 10	0.6267	1.6463
10 and 12	1.1444	1.6463
12 and 14	1.9486	1.6463
14 and 16	0.9385	1.6463
16 and 18	0.8757	1.6463
18 and 20	0.6210	1.6463
20 and 22	0.9601	1.6463
22 and 24	2.7895	1.6463
24 and 26	−0.1709	1.6463
26 and 28	−1.1383	1.6463
28 and 30	3.5915	1.6463
30 and 32	−1.1087	1.6463
32 and 34	2.2295	1.6463
34 and 36	4.6152	1.6463

Estimation year	The period of historical monthly data used	
2007	(Jan 2002-Dec 2006)	
Portfolio size	T-test statistic	Critical t value
1 and 2	17.4608	1.6463
2 and 4	15.1451	1.6463
4 and 6	8.5874	1.6463
6 and 8	5.1089	1.6463
8 and 10	4.4163	1.6463
10 and 12	3.4191	1.6463
12 and 14	1.4068	1.6463
14 and 16	1.4814	1.6463
16 and 18	1.1873	1.6463
18 and 20	1.6370	1.6463
20 and 22	2.1046	1.6463
22 and 24	0.3151	1.6463
24 and 26	2.6004	1.6463
26 and 28	0.5572	1.6463
28 and 30	0.7318	1.6463
30 and 32	3.5571	1.6463
32 and 34	1.3690	1.6463
34 and 36	2.0767	1.6463
36 and 38	7.4008	1.6463
Estimation year	The period of historical monthly data used	
2008	(Jan 2003-Dec 2007)	
Portfolio size	T-test statistic	Critical t value
1 and 2	18.2338	1.6463
2 and 4	17.5559	1.6463
4 and 6	7.9068	1.6463
6 and 8	7.0662	1.6463
8 and 10	2.1009	1.6463
10 and 12	3.2173	1.6463
12 and 14	3.9064	1.6463
14 and 16	1.0271	1.6463
16 and 18	0.0923	1.6463
18 and 20	2.5932	1.6463
20 and 22	0.8920	1.6463
22 and 24	2.3048	1.6463
24 and 26	2.3264	1.6463
26 and 28	0.7461	1.6463
28 and 30	1.7730	1.6463
30 and 32	0.8626	1.6463
32 and 34	1.6035	1.6463
34 and 37	8.8228	1.6463

Estimation year	The period of historical monthly data used	
2009	(Jan 2004-Dec 2008)	
Portfolio size	T-test statistic	Critical t value
1 and 2	20.7922	1.6463
2 and 4	18.8026	1.6463
4 and 6	7.8394	1.6463
6 and 8	5.3122	1.6463
8 and 10	2.8109	1.6463
10 and 12	3.1407	1.6463
12 and 14	3.8802	1.6463
14 and 16	−0.2481	1.6463
16 and 18	3.6049	1.6463
18 and 20	1.5647	1.6463
20 and 22	1.1505	1.6463
22 and 24	2.1266	1.6463
24 and 26	0.3426	1.6463
26 and 28	2.0824	1.6463
28 and 30	2.4404	1.6463
30 and 32	1.8851	1.6463
32 and 34	11.413	1.6463

Estimation year	The period of historical monthly data used	
2010	(Jan 2005-Dec 2009)	
Portfolio size	T-test statistic	Critical t value
1 and 2	20.0641	1.6463
2 and 4	16.6292	1.6463
4 and 6	9.3226	1.6463
6 and 8	4.1521	1.6463
8 and 10	2.6907	1.6463
10 and 12	2.7069	1.6463
12 and 14	1.0685	1.6463
14 and 16	2.7680	1.6463
16 and 18	−0.3540	1.6463
18 and 20	3.0043	1.6463
20 and 22	1.1014	1.6463
22 and 24	2.7844	1.6463
24 and 26	1.0758	1.6463
26 and 28	−1.1753	1.6463
28 and 30	1.7471	1.6463
30 and 32	2.0203	1.6463
32 and 34	2.8310	1.6463
34 and 36	6.1885	1.6463
34 and 36	6.1885	1.6463

Estimation year	The period of historical monthly data used	
2011	(Jan 2006-Dec 2010)	
Portfolio size	T-test statistic	Critical t value
1 and 2	19.1147	1.6463
2 and 4	19.0495	1.6463
4 and 6	7.4948	1.6463
6 and 8	5.4274	1.6463
8 and 10	4.4428	1.6463
10 and 12	3.6531	1.6463
12 and 14	2.0740	1.6463
14 and 16	2.1442	1.6463
16 and 18	1.3185	1.6463
18 and 20	2.8161	1.6463
20 and 22	1.3949	1.6463
22 and 24	1.2453	1.6463
24 and 26	2.0446	1.6463
26 and 28	3.8377	1.6463
28 and 30	1.4264	1.6463

Estimation year	The period of historical monthly data used	
2012	(Jan 2007-Dec 2011)	
Portfolio size	T-test statistic	Critical t value
1 and 2	22.4464	1.6463
2 and 4	17.5932	1.6463
4 and 6	7.6384	1.6463
6 and 8	4.7627	1.6463
8 and 10	3.8114	1.6463
10 and 12	1.3113	1.6463
12 and 14	1.9904	1.6463
14 and 16	2.4319	1.6463
16 and 18	2.8451	1.6463
18 and 20	1.7335	1.6463
20 and 22	1.2942	1.6463
22 and 24	1.8766	1.6463
24 and 26	1.4966	1.6463
26 and 28	1.6840	1.6463
28 and 30	0.6649	1.6463
30 and 32	1.3053	1.6463
32 and 34	3.0338	1.6463
34 and 37	6.8273	1.6463

Estimation year	The period of historical monthly data used	
2013	(Jan 2008-Dec 2012)	
Portfolio size	T-test statistic	Critical t value
1 and 2	16.1151	1.6463
2 and 4	12.9447	1.6463
4 and 6	6.6047	1.6463
6 and 8	4.8851	1.6463
8 and 10	1.8753	1.6463
10 and 12	2.3791	1.6463
12 and 14	1.9300	1.6463
14 and 16	1.4624	1.6463
16 and 18	2.0373	1.6463
18 and 20	0.7280	1.6463
20 and 22	2.0649	1.6463
22 and 24	0.7554	1.6463
24 and 26	0.9666	1.6463
26 and 28	1.0842	1.6463
28 and 30	1.8232	1.6463
30 and 32	1.5688	1.6463
32 and 34	1.0140	1.6463
34 and 37	9.5991	1.6463

Estimation year	The period of historical monthly data used	
2014	(Jan 2009-Dec 2013)	
Portfolio size	T-test statistic	Critical t value
1 and 2	18.5598	1.6463
2 and 4	15.3562	1.6463
4 and 6	6.9359	1.6463
6 and 8	6.9304	1.6463
8 and 10	3.0242	1.6463
10 and 12	3.6215	1.6463
12 and 14	3.3675	1.6463
14 and 16	2.7684	1.6463
16 and 18	-0.7251	1.6463
18 and 20	2.0801	1.6463
20 and 22	1.5063	1.6463
22 and 24	1.8416	1.6463
24 and 26	1.9795	1.6463
26 and 28	1.2548	1.6463
28 and 30	2.2795	1.6463
30 and 32	1.0166	1.6463
32 and 34	0.5190	1.6463
34 and 36	2.0488	1.6463
36 and 38	2.6946	1.6463
38 and 40	3.6728	1.6463

Estimation year	The period of historical monthly data used	
2015	(Jan 2010-Dec 2014)	
Portfolio size	T-test statistic	Critical t value
1 and 2	19.4502	1.6463
2 and 4	19.3480	1.6463
4 and 6	10.6767	1.6463
6 and 8	5.6327	1.6463
8 and 10	3.9419	1.6463
10 and 12	2.9388	1.6463
12 and 14	3.2758	1.6463
14 and 16	3.1661	1.6463
16 and 18	0.1766	1.6463
18 and 20	1.8700	1.6463
20 and 22	1.0784	1.6463
22 and 24	1.7716	1.6463
24 and 26	2.6452	1.6463
26 and 28	1.4090	1.6463
28 and 30	1.5756	1.6463
30 and 32	−0.0958	1.6463
32 and 34	2.4989	1.6463
34 and 36	1.9604	1.6463
36 and 39	5.8677	1.6463

Table A.5: Market Capitalization Weights, Optimal Posterior Weights and Differences for the Estimation Years from 2005 to 2014

This table presents the market capitalization weights, the optimal posterior weights of the stocks that are consistently traded on BIST-50 index for each year from 2005 to 2014 and the differences between them. First column shows the codes of the stocks that are consistently included in BIST-50 index during whole the year and in addition to this monthly return data for former 5-year is available. Second column shows the weights of these stocks according to market capitalization. By following each step of the B-L model described in Figure 4.1 the expected posterior returns and the posterior variances of these stocks can be calculated. The investor utility function is maximized by using these calculated values. By taking the derivative of this function with respect to weight vector the optimal posterior weights of these stocks are obtained and shown in the third column of this table. Bloomberg analysts did not estimate any target stock price for the stock codes written in bold letters. The difference between the market capitalization weights and the optimal posterior weights of both stocks with views and no views are shown in the last column of this table. Finally, the last row of this table shows the summation of both the market capitalization weights and the optimal posterior weights.

2005			
	w_{MKT}	$w^{\star}$	$w_{\text{MKT}} - w^{\star}$
AKBNK	**0.1563**	**0.1537**	**0.0026**
AKGRT	0.0103	0.5039	-0.4936
AKSA	0.0052	-0.1996	0.2048
ALARK	0.0080	0.3840	-0.3760
ARCLK	0.0412	0.6378	-0.5966
ASELS	**0.0042**	**0.0041**	**0.0001**
BEKO	0.0067	-4.4986	4.5053
DISBA	**0.0115**	**0.0114**	**0.0001**
DOHOL	**0.0262**	**0.0258**	**0.0004**
DYHOL	0.0228	-0.2696	0.2924
ECILC	**0.0055**	**0.0055**	**0.0000**
EFES	**0.0024**	**0.0024**	**0.0000**
EREGL	0.0371	0.5624	-0.5253
FINBN	0.0182	-0.2010	0.2192
FROTO	0.0390	1.0287	-0.9897
GARAN	**0.0638**	**0.0627**	**0.0011**
GLYHO	**0.0010**	**0.0010**	**0.0000**
HURGZ	0.0165	-0.2210	0.2375
ISCTR	0.1525	0.1056	0.0469
ISGYO	0.0079	0.0719	-0.0640
KCHOL	0.0867	0.1575	-0.0708
KRDMD	**0.0028**	**0.0028**	**0.0000**
MIGRS	0.0192	-7.6724	7.6916
NTHOL	**0.0017**	**0.0016**	**0.0001**
PETKM	**0.0167**	**0.0165**	**0.0002**
PRKTE	**0.0040**	**0.0040**	**0.0000**
PTOFS	**0.0193**	**0.0190**	**0.0003**
SAHOL	0.0786	0.8077	-0.7291
SISE	0.0197	0.2033	-0.1836
TNSAS	0.0064	0.1730	-0.1666
TOASO	0.0154	0.2947	-0.2793
TUPRS	0.0428	1.3425	-1.2997
VESTL	0.0103	4.7412	-4.7309
YKBNK	0.0398	-0.0642	0.1040
SUM	1.0000	-1.8017	

2006			
	w_{MKT}	$w^{\star}$	$w_{\text{MKT}} - w^{\star}$
AEFES	0.0288	-0.2137	0.2425
AKBNK	0.1337	0.1584	-0.0247
AKGRT	0.0106	0.2602	-0.2496
ANSGR	0.0037	-0.1272	0.1309
ARCLK	0.0254	0.2324	-0.2070
DEVA	**0.0030**	**0.0029**	**0.0001**
DOHOL	0.0218	0.0296	-0.0078
DYHOL	0.0208	-0.1428	0.1636
ECILC	**0.0054**	**0.0053**	**0.0001**
EREGL	0.0295	-0.3625	0.3920
FINBN	0.0389	-0.0490	0.0879
FORTS	0.0173	-0.3416	0.3589
FROTO	0.0280	0.7547	-0.7267
GARAN	0.0695	0.0329	0.0366
GLYHO	**0.0014**	**0.0013**	**0.0001**
GSDHO	**0.0018**	**0.0018**	**0.0000**
HURGZ	0.0149	-0.1916	0.2065
ISCTR	0.1555	0.0014	0.1541
ISGYO	**0.0066**	**0.0065**	**0.0001**
KARTN	**0.0025**	**0.0025**	**0.0000**
KCHOL	0.0493	-0.0874	0.1367
KRDMD	**0.0014**	**0.0014**	**0.0000**
MIGRS	0.0122	0.0758	-0.0636
PETKM	0.0109	-0.4303	0.4412
PRKTE	**0.0023**	**0.0023**	**0.0000**
PTOFS	0.0176	-0.0795	0.0971
SAHOL	0.0620	0.0862	-0.0242
SISE	0.0134	-0.1268	0.1402
SKBNK	**0.0042**	**0.0041**	**0.0001**
TCELL	0.1027	0.1277	-0.0250
THYAO	0.0100	-0.1788	0.1888
TOASO	0.0096	0.1943	-0.1847
TSKB	**0.0062**	**0.0061**	**0.0001**
TUPRS	0.0419	0.9485	-0.9066
VESTL	**0.0054**	**0.0053**	**0.0001**
YKBNK	0.0320	-0.0428	0.0748
SUM	1.0000	0.5676	

2007			
	w_{MKT}	$w^{\star}$	$w_{\mathrm{MKT}} - w^{\star}$
AEFES	0.0345	-0.2465	0.2810
AKBNK	0.1323	-0.0603	0.1926
AKCNS	0.0114	0.3299	-0.3185
AKGRT	0.0114	0.3726	-0.3612
ALARK	0.0044	-0.2422	0.2466
ANSGR	0.0033	0.0563	-0.0530
ARCLK	0.0234	0.0951	-0.0717
AYGAZ	0.0067	0.2604	-0.2537
DOHOL	0.0234	0.1129	-0.0895
DYHOL	0.0212	-0.3033	0.3245
ECILC	0.0070	0.2719	-0.2649
EREGL	0.0307	-0.1615	0.1922
FORTS	**0.0144**	**0.0142**	**0.0002**
FROTO	0.0280	0.9262	-0.8982
GARAN	0.0687	-0.1152	0.1839
GLYHO	0.0011	-0.1100	0.1111
GSDHO	**0.0017**	**0.0016**	**0.0001**
HURGZ	0.0109	-0.1193	0.1302
IHLAS	**0.0014**	**0.0013**	**0.0001**
ISCTR	0.1253	0.1708	-0.0455
ISGYO	0.0067	-0.0324	0.0391
KCHOL	0.0487	-0.0412	0.0899
KRDMD	**0.0014**	**0.0013**	**0.0001**
MIGRS	0.0226	-0.5916	0.6142
NTHOL	**0.0015**	**0.0014**	**0.0001**
PETKM	0.0073	-0.1618	0.1691
PTOFS	0.0134	-0.0235	0.0369
SAHOL	0.0699	-0.0844	0.1543
SISE	0.0148	-0.0917	0.1065
SKBNK	**0.0042**	**0.0041**	**0.0001**
TCELL	0.1100	-0.0203	0.1303
THYAO	0.0075	-0.0753	0.0828
TOASO	0.0171	0.1201	-0.1030
TRKCM	0.0080	0.4007	-0.3927
TSKB	0.0054	0.0381	-0.0327
TUPRS	0.0424	0.7822	-0.7398
VESTL	0.0041	-0.3522	0.3563
YKBNK	0.0541	-0.1097	0.1638
SUM	1.000	1.0187	

2008			
	w_{MKT}	$w^{\star}$	$w_{\mathrm{MKT}} - w^{\star}$
AEFES	0.0290	-0.6174	0.6464
AKBNK	0.1209	0.0621	0.0588
AKENR	0.0032	-0.2581	0.2613
AKGRT	0.0098	0.7944	-0.7846
ALARK	0.0037	-0.4089	0.4126
ANSGR	0.0025	0.3541	-0.3516
ARCLK	0.0151	0.2759	-0.2608
AYGAZ	0.0061	0.3142	-0.3081
DOHOL	0.0154	0.1491	-0.1337
DYHOL	0.0136	-0.1846	0.1982
ECILC	0.0044	0.6530	-0.6486
ENKAI	0.0854	-0.7513	0.8367
EREGL	0.0403	-0.0319	0.0722
GARAN	0.1021	-0.2344	0.3365
GLYHO	0.0019	-0.0619	0.0638
GSDHO	0.0014	0.0514	-0.0500
HURGZ	0.0071	-0.0883	0.0954
IHLAS	**0.0019**	**0.0019**	**0.0000**
ISCTR	0.0938	0.1750	-0.0812
ISGYO	0.0036	0.2944	-0.2908
KCHOL	0.0513	-0.1506	0.2019
KRDMD	0.0024	-0.1027	0.1051
MIGRS	0.0189	-0.5796	0.5985
NTHOL	0.0018	0.0775	-0.0757
PETKM	0.0082	-0.5130	0.5212
PTOFS	0.0138	0.1855	-0.1717
SAHOL	0.0538	0.0205	0.0333
SISE	0.0104	-0.0260	0.0364
SKBNK	0.0096	-0.2366	0.2462
TCELL	0.1304	0.1841	-0.0537
THYAO	0.0070	-0.0354	0.0424
TOASO	0.0142	0.2066	-0.1924
TRKCM	0.0062	0.1646	-0.1584
TSKB	0.0036	-0.0050	0.0086
TUPRS	0.0397	0.6776	-0.6379
VESTL	**0.0021**	**0.0021**	**0.0000**
YKBNK	0.0654	-0.1507	0.2161
SUM	1.000	0.2076	

2009			
	w_{MKT}	$w^{\star}$	$w_{\text{MKT}} - w^{\star}$
AEFES	0.0419	-0.0848	0.1267
AKBNK	0.1297	-0.2821	0.4118
AKENR	0.0036	0.1523	-0.1487
AKGRT	0.0079	0.1306	-0.1227
ANSGR	0.0029	0.0568	-0.0539
ARCLK	0.0075	-0.1203	0.1278
AYGAZ	0.0055	0.5542	-0.5487
BAGFS	0.0017	0.2750	-0.2733
DOHOL	0.0140	0.2698	-0.2558
DYHOL	0.0037	-0.1530	0.1567
ECILC	0.0044	0.6465	-0.6421
ENKAI	0.0570	-0.1540	0.2110
EREGL	0.0428	-0.3695	0.4123
GARAN	0.0988	-0.2436	0.3424
GUBRF	0.0046	0.3757	-0.3711
HURGZ	0.0027	-0.0493	0.0520
ISCTR	0.1022	-0.0059	0.1081
KCHOL	0.0477	0.1014	-0.0537
KRDMD	0.0017	0.2112	-0.2095
NTHOL	0.0009	-0.0056	0.0065
PETKM	0.0086	-0.5300	0.5386
PTOFS	0.0137	0.0354	-0.0217
SAHOL	0.0570	-0.1567	0.2137
SISE	0.0099	-0.1283	0.1382
SKBNK	0.0039	-0.1623	0.1662
TCELL	0.1741	0.6965	-0.5224
TEBNK	0.0082	-0.0856	0.0938
THYAO	0.0090	0.0310	-0.0220
TOASO	0.0052	0.5491	-0.5439
TRKCM	0.0045	0.3598	-0.3553
TSKB	0.0042	-0.1816	0.1858
TUPRS	0.0367	1.2377	-1.2010
VESTL	0.0013	0.2400	-0.2387
YKBNK	0.0826	-0.2779	0.3605
SUM	1.0000	2.9325	

2010			
	w_{MKT}	$w^{\star}$	$w_{\text{MKT}} - w^{\star}$
AEFES	0.0370	-0.0143	0.0513
AKBNK	0.1386	-0.1206	0.2592
AKENR	0.0045	0.1567	-0.1522
AKGRT	0.0078	0.5608	-0.5530
ANSGR	0.0028	0.2348	-0.2320
ARCLK	0.0193	0.0498	-0.0305
AYGAZ	0.0083	0.3104	-0.3021
BAGFS	0.0014	-0.1374	0.1388
DOHOL	0.0123	0.0512	-0.0389
DYHOL	0.0055	-0.3319	0.3374
ECILC	0.0067	-0.1053	0.1120
ENKAI	0.0607	-0.1566	0.2173
EREGL	0.0352	0.0753	-0.0401
GARAN	0.1304	0.0076	0.1228
GSDHO	0.0012	0.0568	-0.0556
HURGZ	0.0050	-0.0612	0.0662
ISCTR	0.0949	0.2370	-0.1421
KCHOL	0.0522	0.0157	0.0365
KOZAA	0.0046	-0.1209	0.1255
KRDMD	0.0019	-0.1055	0.1074
NTHOL	0.0010	0.1087	-0.1077
PETKM	0.0073	-0.1361	0.1434
SAHOL	0.0534	-0.0262	0.0796
SISE	0.0101	-0.1185	0.1286
SKBNK	0.0063	-0.1485	0.1548
TCELL	0.1140	0.1023	0.0117
TEBNK	0.0151	-0.2602	0.2753
THYAO	0.0244	-0.2333	0.2577
TIRE	**0.0021**	**0.0020**	**0.0001**
TOASO	0.0115	0.1809	-0.1694
TSKB	0.0053	0.0991	-0.0938
TUPRS	0.0364	0.5353	-0.4989
ULKER	0.0046	0.1231	-0.1185
VESTL	0.0043	-0.1528	0.1571
YKBNK	0.0697	0.0422	0.0275
ZOREN	0.0043	-0.1165	0.1208
SUM	1.0000	0.6039	

2011			
	w_{MKT}	$w^{\star}$	$w_{\text{MKT}} - w^{\star}$
AKBNK	0.1345	-0.0444	0.1789
AKENR	0.0053	-0.0048	0.0101
ARCLK	0.0207	0.0537	-0.0330
ASELS	0.0076	-0.1287	0.1363
BAGFS	0.0019	-0.0025	0.0044
BIMAS	0.0312	-0.1683	0.1995
DOHOL	0.0108	0.0362	-0.0254
DYHOL	0.0077	-0.4614	0.4691
ECILC	0.0054	0.3738	-0.3684
ENKAI	0.0497	-0.0633	0.1130
EREGL	0.0320	0.1548	-0.1228
FENER	**0.0052**	**0.0052**	**0.0000**
GARAN	0.1287	0.1314	-0.0027
GUBRF	0.0056	0.0866	-0.0810
IHLAS	**0.0028**	**0.0028**	**0.0000**
IPEKE	0.0019	-0.0873	0.0892
ISCTR	0.0970	0.3160	-0.2190
KCHOL	0.0712	-0.0806	0.1518
KOZAA	0.0035	0.0271	-0.0236
KRDMD	0.0018	-0.1178	0.1196
PETKM	0.0093	-0.0504	0.0597
SAHOL	0.0576	0.0702	-0.0126
SISE	0.0122	0.0172	-0.0050
TCELL	0.0910	-0.0782	0.1692
TEBNK	0.0096	-0.1206	0.1302
THYAO	0.0212	0.1801	-0.1589
TOASO	0.0156	0.1856	-0.1700
TUPRS	0.0379	0.4009	-0.3630
VAKBN	0.0383	-0.0105	0.0488
YKBNK	0.0828	0.1740	-0.0912
SUM	1.0000	0.7968	

2012			
	w_{MKT}	$w^{\star}$	$w_{\text{MKT}} - w^{\star}$
AEFES	0.0486	-0.1896	0.2382
AKBNK	0.1141	-0.1585	0.2726
AKENR	0.0033	0.1900	-0.1867
ARCLK	0.0196	-0.0172	0.0368
ASELS	0.0091	-0.1347	0.1438
ASYAB	0.0068	-0.2151	0.2219
BAGFS	0.0022	0.0977	-0.0955
BIMAS	0.0378	-0.2499	0.2877
DOHOL	0.0062	0.3511	-0.3449
DYHOL	0.0047	-0.1264	0.1311
ECILC	0.0047	0.4485	-0.4438
ENKAI	0.0486	-0.0965	0.1451
EREGL	0.0334	0.0284	0.0050
FENER	**0.0047**	**0.0047**	**0.0000**
FROTO	0.0254	-0.3096	0.3350
GARAN	0.1174	-0.0527	0.1701
GSRAY	**0.0022**	**0.0021**	**0.0001**
IHLAS	**0.0029**	**0.0028**	**0.0001**
ISCTR	0.0706	0.1829	-0.1123
IZMDC	0.0062	-0.2245	0.2307
KCHOL	0.0650	-0.0910	0.1560
KONYA	**0.0071**	**0.0070**	**0.0001**
KOZAA	0.0024	0.3251	-0.3227
KRDMD	0.0022	0.0315	-0.0293
MGROS	0.0107	0.1839	-0.1732
NETAS	**0.0038**	**0.0037**	**0.0001**
PETKM	0.0093	0.1229	-0.1136
PRKME	0.0024	0.2301	-0.2277
SAHOL	0.0522	0.1726	-0.1204
SISE	0.0176	-0.0717	0.0893
TCELL	0.0925	-0.1830	0.2755
THYAO	0.0121	0.2028	-0.1907
TOASO	0.0140	0.1593	-0.1453
TTRAK	0.0085	0.1111	-0.1026
TUPRS	0.0475	-0.0513	0.0988
VAKBN	0.0290	0.0048	0.0242
YKBNK	0.0554	0.1389	-0.0835
SUM	1.0000	0.8302	

2013			
	w_{MKT}	$w^{\star}$	$w_{\mathrm{MKT}} - w^{\star}$
AEFES	0.0422	0.1404	-0.0982
AKBNK	0.0976	0.0040	0.0936
AKENR	0.0033	-0.0861	0.0894
ARCLK	0.0219	-0.0619	0.0838
ASELS	0.0117	-0.2557	0.2674
ASYAB	0.0055	-0.0850	0.0905
AYGAZ	0.0079	0.3713	-0.3634
BIMAS	0.0367	-0.3605	0.3972
DOHOL	0.0063	0.1336	-0.1273
DYHOL	0.0043	-0.0386	0.0429
ECILC	0.0030	0.3361	-0.3331
ENKAI	0.0412	-0.1644	0.2056
EREGL	0.0210	0.0982	-0.0772
FROTO	0.0208	-0.1878	0.2086
GARAN	0.1079	-0.0238	0.1317
GUBRF	0.0033	0.2167	-0.2134
HALKB	0.0607	0.1545	-0.0938
IHLAS	**0.0027**	**0.0027**	**0.0000**
IPEKE	0.0043	-0.1138	0.1181
ISCTR	0.0771	0.0590	0.0181
KCHOL	0.0651	-0.1641	0.2292
KOZAA	0.0061	0.1670	-0.1609
KRDMD	0.0021	0.0345	-0.0324
MGROS	0.0106	0.0655	-0.0549
PETKM	0.0077	-0.4984	0.5061
SAHOL	0.0555	0.0142	0.0413
SISE	0.0123	-0.0532	0.0655
SNGYO	0.0023	0.1301	-0.1278
TAVHL	0.0092	0.3108	-0.3016
TCELL	0.0705	0.1305	-0.0600
THYAO	0.0208	-0.0960	0.1168
TOASO	0.0145	0.0978	-0.0833
TRKCM	0.0047	0.1400	-0.1353
TTRAK	0.0086	-0.1064	0.1150
TUPRS	0.0358	-0.1084	0.1442
VAKBN	0.0320	-0.0481	0.0801
YKBNK	0.0627	0.0422	0.0205
SUM	1.0000	0.1969	

2014			
	w_{MKT}	$w^{\star}$	$w_{\mathrm{MKT}} - w^{\star}$
AEFES	0.0382	-0.4562	0.4944
AKBNK	0.0743	-0.1105	0.1848
ARCLK	0.0228	-0.1542	0.1770
ASELS	0.0120	-0.2078	0.2198
BIMAS	0.0365	-0.1808	0.2173
CCOLA	0.0365	-0.0737	0.1102
DOAS	0.0041	0.2944	-0.2903
DOHOL	0.0049	0.3448	-0.3399
ENKAI	0.0534	-0.3397	0.3931
EREGL	0.0250	0.2811	-0.2561
FROTO	0.0221	0.0720	-0.0499
GARAN	0.0811	-0.0249	0.1060
GOLTS	**0.0010**	**0.0009**	**0.0001**
GUBRF	0.0026	0.2390	-0.2364
HALKB	0.0421	0.4161	-0.3740
IHLAS	**0.0009**	**0.0009**	**0.0000**
IPEKE	0.0022	0.0294	-0.0272
ISCTR	0.0580	0.0436	0.0144
KCHOL	0.0619	-0.3068	0.3687
KOZAA	0.0026	0.3535	-0.3509
KRDMD	0.0021	0.3407	-0.3386
MGROS	0.0079	0.0888	-0.0809
OTKAR	0.0037	-0.5140	0.5177
PETKM	0.0076	-0.1247	0.1323
SAHOL	0.0489	0.1002	-0.0513
SISE	0.0119	-0.0340	0.0459
TAVHL	0.0156	-0.1704	0.1860
TCELL	0.0693	0.3161	-0.2468
THYAO	0.0247	0.1593	-0.1346
TKFEN	0.0051	0.0459	-0.0408
TOASO	0.0186	-0.2880	0.3066
TRKCM	0.0050	0.1400	-0.1350
TSKB	0.0066	0.1958	-0.1892
TTKOM	0.0579	0.3926	-0.3347
TTRAK	0.0091	0.0522	-0.0431
TUPRS	0.0298	-0.1250	0.1548
ULKER	0.0144	-0.1240	0.1384
VAKBN	0.0265	-0.0464	0.0729
YAZIC	0.0083	-0.2318	0.2401
YKBNK	0.0449	0.0570	-0.0121
SUM	1.0000	0.4514	

APPENDIX B

MATLAB Codes

- Diversification (ALL m.file)

```
% A= the matrix of monthly excess logarithmic returns
%    of the stocks
% n= # of iterations
% filename= name of the excel file will be created
%           (including average standard deviations
%            of the portfolios consisting of ranging
%            from 1 to maximum number of stocks
%            available on BIST-50 in particular year
%             for each iterations)
function [] = ALL(A,n,filename)
    ExcelData=zeros(size(A,2),n+3);
    % excel file includes first column consisting of
    % number of stocks, second column consisting of
    % average standard deviation of all iterative portolios,
    % third column consisting of -1's in order to
    % distinguish between average standard deviation of
    % all portfolios and standard deviations of
    % each iterative portfolios
    totalstdMatrix=zeros(size(A,2),n);
    for selectedrandomnumber=1:size(A,2)
        warehouse=size(A,2); % # of stocks used for
                             %   particular period
        selectedrandom=zeros(selectedrandomnumber,n);
        eachcolumn=zeros(1,selectedrandomnumber);
        rand1=0;
        if(warehouse>=selectedrandomnumber)
            for i=1:n
                for j=1:selectedrandomnumber
                  selectedrandom(j,i)=1+round((warehouse-1)...
                    *rand(1,1));
                    rand1=selectedrandom(j,i);
          % for without replacement ismember function is used
                      while (ismember(rand1, eachcolumn)...
                                   == 1)
                              rand1=1+round((warehouse-1)...
                                  *rand(1,1));
                           end
                    selectedrandom(j,i)=rand1;
```

```
                eachcolumn(j)=rand1;
            end
            eachcolumn=zeros(1,selectedrandomnumber);
            rand1=0;
        end
    end

    B= cov ( A );
    totalstd=0;
    innertotal=0; %covariance total
    selectedwarehouses=size(selectedrandom,1);
    % # of selected warehouse
    selection=zeros(1,selectedwarehouses);

    for i=1: n
        for j=1: selectedwarehouses
            selection(j)=selectedrandom(j,i);
            % populate via random selected warehouses
        end

    allposibilities = combvec(selection,selection);
    %find all combinations of stocks
    allposibilitiesnumber =size(allposibilities,2);
    % # of combination

        for k=1:allposibilitiesnumber
                % calculate variance of the portfolios for each
                % combinations
                innertotal=innertotal+ ...
                    ((selectedwarehouses^2)^-1) ...
                *B(allposibilities(1,k),allposibilities(2,k));
        end
    totalstdMatrix(selectedrandomnumber,i)=sqrt(innertotal);
    totalstd=totalstd+sqrt(innertotal);
    innertotal=0;
    selection=zeros(1,selectedwarehouses);
    end
    averagestd=totalstd/n;

   ExcelData(selectedrandomnumber,:)=[selectedrandomnumber ...
       [averagestd -1 totalstdMatrix(selectedrandomnumber,:)]];
  end
  xlswrite(strcat(filename,'.xlsx'),ExcelData);
  % the excel file is created with the name specified before
end
```

- Ttest (hypothesisTtest m.file)

```
% filename= the name of the excel file
% which was created by previous program
% range= increase in the number of stocks
% in order to compare average standard deviation
% of the portfolios consisting these number of stocks
% side = tail of t-test (both,left or right)
```

```
function [] = hyphothesisTtest(filename,range,side)
 Test=xlsread(strcat(filename,'.xlsx'));
 i=0;j=0;k=1;
 b=zeros(floor(size(Test,1)/range),8);
 while i<size(Test,1)
        if(i==0)
                Test2=Test(i+1,:);
                Test3=Test(i+range,:);
                [h,p,ci,stats] =ttest(Test2(4:size(Test,2)), ...
                Test3(4:size(Test,2)),'Tail',side);
                % h= hypothesis test result
                % p= p-value
                % ci= confidence interval
                % stats= test statistics containing the following
                %        tstat |> Value of the test statistic
                %        df |> Degrees of freedom of the test
                %        sd |> Estimated population standard deviation
        else
                Test2=Test(i,:);
                Test3=Test(i+range,:);
                j=i+range;
                if (size(Test,1)-j)<range
                        Test3=Test(size(Test,1),:);
                        [h,p,ci,stats] =ttest(Test2(4:size(Test,2)),...
                        Test3(4:size(Test,2)),'Tail',side);
                else
                        [h,p,ci,stats] =ttest(Test2(4:size(Test,2)),...
                                Test3(4:size(Test,2)),'Tail',side);
                end
    end
b(k,:)=[i,h,p,ci,stats.tstat,stats.df,stats.sd];
        i=i+range;
    k=k+1;
    if (size(Test,1)-j)<range
        break;
    end
 end
 b
 xlswrite(strcat('Ttest-',filename,'-',side,'.xlsx'),b);
end
```

- Objective function (objfun2 m.file)

```
function f = objfun2(x)
global er1 %the posterior return vector
global ps1 %the posterior covariance matrix
  for i=1:39 % number of stocks in the portfolio for 2015
  c1(i)=x(i);
 end
f=[c1*er1-0.5*(2.5)*c1*ps1*c1']; %max utility
f=-f; %to find max
end
```

- Unconstrained optimization (fminunc m.file)

```
options = optimoptions(@fminunc,'MaxFunEvals',1000000, ...
'MaxIter',1000000,'TolFun',1e-10);
[x, fval] = fminunc(@objfun2,zeros(1,39),options)
```

- Budget constrained optimization (fminconmaxutility m.file)

```
function [] = fminconmaxutility(N,b)
x0=zeros(1,N); %starting point
Aeq=ones(1,N);
beq=b; %b=1 budget constraint
options = optimoptions(@fmincon,'MaxFunEvals',1000000,...
    'MaxIter',1000000,'TolFun',1e-10,'TolCon',1e-10);
[x, fval] = fmincon(@objfun2, x0, [], [],Aeq , beq,...
    [],[],[],options)
end
```

- Budget and no short selling constrained optimization (fminconmaxutility2 m.file)

```
function [] = fminconmaxutility2(N,b)

x0=zeros(1,N); %starting point
Aeq=ones(1,N);
beq=b; %b=1 budget constraint
lb=zeros(1,N); %lower bounds
options = optimoptions(@fmincon,'MaxFunEvals',1000000,...
    'MaxIter',1000000,'TolFun',1e-10,'TolCon',1e-10);
[x, fval] = fmincon(@objfun2, x0, [], [],Aeq , beq, ...
    lb,[],[],options)
end
```