STABLE ULRICH BUNDLES ON FANO 3-FOLDS WITH PICARD NUMBER 2

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

ÖZHAN GENÇ

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY
IN
MATHEMATICS

JULY 2016

Approval of the thesis:

STABLE ULRICH BUNDLES ON FANO 3-FOLDS WITH PICARD NUMBER 2

submitted by **ÖZHAN GENÇ** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy in Mathematics Department, Middle East Technical University** by,

Prof. Dr. Gülbin Dural Ünver
Dean, Graduate School of **Natural and Applied Sciences** _______________

Prof. Dr. Mustafa Korkmaz
Head of Department, **Mathematics** _______________

Assoc. Prof. Dr. Emre Coşkun
Supervisor, **Department of Mathematics, METU** _______________

Examining Committee Members:

Prof. Dr. M. Hurşit Önsiper
Department of Mathematics, METU _______________

Assoc. Prof. Dr. Emre Coşkun
Department of Mathematics, METU _______________

Assoc. Prof. Dr. Mustafa Kalafat
Department of Mechatronic Engineering, Tunceli University _______________

Assoc. Prof. Dr. A. U. Özgür Kişisel
Department of Mathematics, METU _______________

Assoc. Prof. Dr. Mesut Şahin
Department of Mathematics, Hacettepe University _______________

Date: _______________

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last Name: ÖZHAN GENÇ

Signature :

ABSTRACT

STABLE ULRICH BUNDLES ON FANO 3-FOLDS WITH PICARD NUMBER 2

Genç, Özhan
Ph.D., Department of Mathematics
Supervisor : Assoc. Prof. Dr. Emre Coşkun

July 2016, 64 pages

In this thesis, we consider the existence problem of rank one and two stable Ulrich bundles on imprimitive Fano 3-folds obtained by blowing-up one of $\mathbb{P}^3$, Q (smooth quadric in $\mathbb{P}^4$), V_3 (smooth cubic in $\mathbb{P}^4$) or V_4 (complete intersection of two quadrics in $\mathbb{P}^5$) along a smooth irreducible curve. We prove that the only class which admits Ulrich line bundles is the one obtained by blowing up a genus 3, degree 6 curve in $\mathbb{P}^3$. Also, we prove that there exist stable rank two Ulrich bundles with $c_1 = 3H$ on a generic member of this deformation class.

Keywords: Ulrich bundle, Fano variety

ÖZ

PİCARD SAYISI 2 OLAN ÜÇ BOYUTLU FANO VARYETELERİ ÜZERİNDE KARARLI ULRİCH DEMETLERİ

Genç, Özhan
Doktora, Matematik Bölümü
Tez Yöneticisi : Doç. Dr. Emre Coşkun

Temmuz 2016 , 64 sayfa

Bu tezde, $\mathbb{P}^3$, Q ($\mathbb{P}^4$ içinde pürüzsüz ikinci dereceden hiper yüzey), V_3 ($\mathbb{P}^4$ içinde pürüzsüz üçüncü dereceden hiper yüzey) veya V_4 ($\mathbb{P}^5$ içinde pürüzsüz ikinci dereceden iki tane hiper yüzeyin tam kesişimi) varyetelerinden herhangi birinin, indirgenemez ve pürüzsüz bir eğri boyunca patlatılmasıyla elde edilen imprimitif üç boyutlu Fano varyeteleri üzerinde, birinci ve ikinci mertebeden kararlı Ulrich demetlerinin varlığı üzerinde durduk. Birinci mertebeden Ulrich demetlerin sadece $\mathbb{P}^3$'ün cinsi 3 derecesi 6 eğri boyunca patlatılmasıyla elde edilen imprimitif üç boyutlu Fano varyeteleri üzerinde varolduğunu ispatladık. Ayrıca, $\mathbb{P}^3$'ün cinsi 3 derecesi 6 eğri boyunca patlatılmasıyla elde edilen imprimitif üç boyutlu Fano varyetelerinin genel bir temsilcisi üzerindei ikinci mertebeden, kararlı ve birinci Chern sınıfı $c_1 = 3H$ olan Ulrich demetleri olduğunu ispatladık.

Anahtar Kelimeler: Ulrich demetleri, Fano varyeteleri

To my mother, Gülhan Genç

ACKNOWLEDGMENTS

I would like to express my sincere gratitude to my advisor Professor Emre Coşkun for the continuous support of my Ph.D study and related research, for his patience, motivation, and immense knowledge. His guidance helped me in all the time of research and writing of this thesis.
Besides my advisor, I would like to thank the rest of my thesis committee: Professor M. Hurşit Önsiper, Professor Mustafa Kalafat, Professor A. U. Özgür Kişisel, Professor Mesut Şahin, Professor İbrahim Ünal and Professor Celalettin Kaya for their encouragement and insightful comments.
A very special thanks goes out to Professor M. Hurşit Önsiper, without whose motivation and encouragement I would not have considered a graduate career. M. Hurşit Önsiper is the one professor who truly made a difference in my life. He provided me with direction, technical support and became more of a mentor and friend, than a professor. I doubt that I will ever be able to convey my appreciation fully, but I owe him my eternal gratitude.
My sincere thanks also goes to Professor Yusuf Mustopa for giving me permission to reproduce his Proposition 2.1.15.
I would also like to thank all the members of Department of Mathematics of METU.
This work is also supported by TÜBTAK-BİDEB PhD scholarship (2211).
Last but not the least, I would like to thank my family: my parents Gülhan Genç and Yusuf Özkıran Genç, my sister Özge Genç and my fiance Hüsne Soylu for supporting me spiritually throughout my entire life.

TABLE OF CONTENTS

LIST OF TABLES

TABLES

LIST OF FIGURES

CHAPTER 1

INTRODUCTION AND PRELIMINARIES

1.1 Introduction

The existence of Ulrich bundles on smooth projective varieties is related to a number of geometric questions. For instance, the existence of rank 1 or rank 2 Ulrich bundles on a hypersurface is related to the representation of that hypersurface as a determinant or Pfaffian ([1]). Another question of interest is the Minimal Resolution Conjecture (MRC) ([23], [13]). In [7], the existence problem of Ulrich bundles on del Pezzo surfaces was related to the MRC for a general smooth curve in the linear system of the first Chern class of the Ulrich bundle. Also, in [10], it is proved that the cone of cohomology tables of vector bundles on a k-dimensonal scheme $X \subset \mathbb{P}^N$ is the same as the cone of cohomology tables of vector bundles on $\mathbb{P}^k$ if and only if there exists an Ulrich bundle on X.

It was conjectured in [11] that on any variety there exist Ulrich bundles. Although it is known that any projective curve ([9]), hypersurfaces and complete intersections ([18]), cubic surfaces ([5]), abelian surfaces ([2]), Veronese varieties ([11]) admit Ulrich bundles, such a general existence result is not known. The finer question of determining the minimal rank of Ulrich bundles (which do not contain bundles of lower rank as direct summands) on a given variety seems to be a quite difficult problem.

The problem that has attracted the most attention is the existence of stable Ulrich bundles with given rank and Chern classes. Stable Ulrich bundles are particularly interesting as they are the building blocks of all Ulrich bundles: Every Ulrich bundle is semistable, and the Jordan-Hölder factors are stable Ulrich bundles.

In this paper, we studied the construction of stable Ulrich bundles on imprimitive Fano 3-folds obtained by blowing-up one of $\mathbb{P}^3$, Q (smooth quadric in $\mathbb{P}^4$), V_3 (smooth cubic in $\mathbb{P}^4$) or V_4 (complete intersection of two quadrics in $\mathbb{P}^5$) along a smooth irreducible curve. There are 36 deformation classes of Fano 3-folds with Picard number $\rho = 2$ and 27 of these are imprimitive ([21, Table 12.3]). Among all imprimitive Fano 3-folds of Picard number $\rho = 2$, 21 deformation classes are obtained by blowing-up one of $\mathbb{P}^3$, Q, V_3 or V_4 along a smooth irreducible curve. We focus on these 3-folds and we consider rank 1 and 2 stable Ulrich bundles.

First, we prove that the only class which admits rank 1 Ulrich bundles is the one obtained by blowing up a genus 3, degree 6 curve in $\mathbb{P}^3$, which is [21, No:12 in Table 12.3]. These varieties admit two classes of rank 1 Ulrich bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ (Theorem 2.1.17).

The next step is to construct rank 2 stable Ulrich bundles on these varieties. To do this, we first construct rank 2 simple Ulrich bundles (Theorem 2.2.14). For this, we use extensions of rank 1 Ulrich bundles $\mathcal{L}_1$ and $\mathcal{L}_2$:

$$0 \to \mathcal{L}_1 \to \mathcal{E} \to \mathcal{L}_2 \to 0$$

or

$$0 \to \mathcal{L}_2 \to \mathcal{E} \to \mathcal{L}_1 \to 0.$$

Then $\mathcal{E}$ is Ulrich and simple; and it has first Chern class $3H$.

Then, to determine whether there exists a stable Ulrich bundle of rank 2 with $c_1 = 3H$, we use the Quot scheme. It is known that stable vector bundles are simple. We consider the local dimension of the Quot scheme at the simple Ulrich bundle with first Chern class $3H$ and find a lower bound to this dimension (Theorem 2.2.35). Then we find an upper bound to the dimension of the subset parametrizing the non-stable Ulrich bundles (Proposition 2.2.37 and Proposition 2.2.39). The latter dimension is strictly smaller than the former; that is, there are stable, rank 2 Ulrich bundles with first Chern class $3H$ (Theorem 2.2.41).

Notations and Conventions

We work over an algebraically closed field $\mathbb{K}$ of characteristic 0.

- X : Smooth projective variety of degree c and dimension k in $\mathbb{P}^N$.
- H_X: Hyperplane class of X.
- K_X: Canonical divisor of X.
- $\mathcal{E}(t)$: The vector bundle $\mathcal{E} \otimes \mathcal{O}_X(tH_X)$ where $\mathcal{E}$ is a vector bundle on X, and $t \in \mathbb{Z}$.
- C: Smooth, irreducible curve of degree d and genus g.
- Q: Smooth quadric in $\mathbb{P}^4$
- V_3: Smooth cubic in $\mathbb{P}^4$
- V_4: Complete intersection of two quadrics in $\mathbb{P}^5$
- $\widetilde{X}$: Blow-up of X along C.
- $\widetilde{Y}$: Non-hyperelliptic Fano 3-fold which is obtained by blowing up one of $\mathbb{P}^3$, Q, V_3 or V_4 along C.
- Y: Deformation class of Fano 3-folds which is obtained by blowing up $\mathbb{P}^3$ along a smooth irreducible space curve of degree 6 and genus 3, which is scheme theoretic intersection of cubics.

1.2 Preliminaries

1.2.1 Fano Varieties

Definition 1.2.1 *A smooth projective variety X is called a $Fano$ variety if its anti-canonical divisor $-K_X$ is ample.*

Definition 1.2.2 *A Fano 3-fold is $imprimitive$ if it is isomorphic to the blow-up of a Fano 3-fold along a smooth irreducible curve.*

The classification of Fano 3-folds with $\rho = 2$ has been completed and it can be found in [21, Table 12.3]. In this paper, we consider the question of existence of Ulrich bundles on $\widetilde{Y}$.

Upon blowing-up X along C, we have the following commutative diagram:

$$\begin{array}{ccc} E & \overset{j}{\hookrightarrow} & \widetilde{X} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ C & \overset{i}{\hookrightarrow} & X \end{array}$$

the map f is the blow-down map and $E = \mathbb{P}N$ is the exceptional divisor, where N is the normal bundle of C in X. Recall that $\widetilde{X}$ stands for $\widetilde{Y}$. Let h be the class of a plane in $A^1(X)$, and let $l = h^2$ be the class of a line in $A^2(X)$. We will denote $\widetilde{h}$ and $\widetilde{l}$ for the pullbacks of h and l to $\widetilde{X}$ respectively; and e denotes the class of the exceptional divisor. Also for any divisor $D \in Z^1(C)$, we denote by $F_D = g^*D \in Z^1(C)$ the corresponding linear combination of fibers $E \to C$, and similarly for divisor classes.

Theorem 1.2.3 *Let $D = a\widetilde{h} - be$ be a divisor on $\widetilde{Y} = \widetilde{\mathbb{P}}^3$, where $a, b \in \mathbb{Z}$. Let $D(t) = D + tH_{\widetilde{Y}}$. Then*

$$\begin{aligned} \chi(\widetilde{Y}, \mathcal{O}(D(t))) \;=\; & \frac{1}{6}[62 - 8d + 2g]t^3 \\ & +\frac{1}{6}[(48 - 3d)a + (6g - 12d - 6)b - 12d + 3g + 93]t^2 \\ & +\frac{1}{6}[12a^2 + (6g - 6)b^2 - 6dab + (48 - 3d)a \\ & \quad +(6g - 6 - 12d)b + 43 - 4d + g]t \\ & +\frac{1}{6}[a^3 + (4d + 2g - 2)b^3 + 6a^2 + (3g - 3da - 3)b^2 + 11a \\ & \quad +(g - 3da - 4d - 1)b + 6]. \end{aligned}$$

Proof 1.2.4 *It is well-known that*

$$K_{\mathbb{P}^3} = (-3 - 1)h = -4h$$

and

$$c(T_{\mathbb{P}^3}) = (1 + h)^{1+3} = 1 + 4h + 6h^2 + 4h^3.$$

So by [14, Example 15.4.3], we have

$$c_1(T_{\widetilde{\mathbb{P}^3}}) \;=\; f^*c_1(T_{\mathbb{P}}^3) + (1 - 2)[E]$$

$$
\begin{aligned}
&= f^*(4h) - e \\
&= 4\widetilde{h} - e \\
c_2(T_{\widetilde{\mathbb{P}}^3}) &= f^*c_2(T^3_{\mathbb{P}}) + f^*i_*[C] - f^*c_1(T^3_{\mathbb{P}})[E] \\
&= f^*(6h^2) + d\widetilde{l} - f^*(4h)e \\
&= (6+d)\widetilde{h}^2 - 4\widetilde{h}e \\
K_{\widetilde{\mathbb{P}}^3} &= f^*K_{\mathbb{P}^3} + (2-1)[E] \\
&= f^*(-4h) + e \\
&= -4\widetilde{h} + e.
\end{aligned}
$$

Then using [24, Lemma 2.1], we obtain

$$\widetilde{h}^3 = 1$$

$$e^3 = -(-K_{\mathbb{P}^3} \cdot C) + 2 - 2g = -(4h \cdot C) + 2 - 2g = -4d - 2g + 2$$

$$
\begin{aligned}
e^2 \cdot (-K_{\widetilde{\mathbb{P}}^3}) = 2g - 2 \;&\Rightarrow\; e^2(4\widetilde{h} - e) = 2g - 2 \\
&\Rightarrow\; 4\widetilde{h}e^2 - e^3 = 2g - 2 \\
&\Rightarrow\; 4\widetilde{h}e^2 - (-4d - 2g + 2) = 2g - 2 \\
&\Rightarrow\; \widetilde{h}e^2 = -d
\end{aligned}
$$

$$
\begin{aligned}
e \cdot (-K_{\widetilde{\mathbb{P}}^3})^2 = (-K_{\mathbb{P}^3} \cdot C) + 2 - 2g \;&\Rightarrow\; e(4\widetilde{h} - e)^2 = (4h \cdot C) + 2 - 2g \\
&\Rightarrow\; 16\widetilde{h}^2 e - 8\widetilde{h}e^2 + e^3 = 4d + 2 - 2g \\
&\Rightarrow\; 16\widetilde{h}^2 e + 4d - 2g + 2 = 4d + 2 - 2g \\
&\Rightarrow\; \widetilde{h}^2 e = 0.
\end{aligned}
$$

Since $\widetilde{Y} = \widetilde{\mathbb{P}}^3$ *is non-hyperelliptic Fano,*

$$H_{\widetilde{\mathbb{P}}^3} = -K_{\widetilde{\mathbb{P}}^3} = 4\widetilde{h} - e.$$

Let $D = a\widetilde{h} - be$ *be a divisor class on* $\widetilde{Y}$. *Then*

$$
\begin{aligned}
D(t) = D + tH_{\widetilde{Y}} &= (a\widetilde{h} - be) + t(4\widetilde{h} - e) \\
&= (a + 4t)\widetilde{h} - (b + t)e.
\end{aligned}
$$

Then, we apply Riemann-Roch theorem for line bundles on 3-folds to obtain

$$\begin{aligned}\chi(\widetilde{Y},\mathcal{O}(D(t))) &= \frac{1}{6}(D(t))^3+\frac{1}{4}c_1(T_{\widetilde{Y}})\cdot(D(t))^2+\frac{1}{24}c_1(T_{\widetilde{Y}})\cdot c_2(T_{\widetilde{Y}})\\ &\quad +\frac{1}{12}(c_1^2(T_{\widetilde{Y}})+c_2(T_{\widetilde{Y}}))\cdot(D(t)).\end{aligned}$$

Then we have

$$\begin{aligned}\chi(\widetilde{Y},\mathcal{O}(D(t))) &= \frac{1}{6}[(a+4t)\widetilde{h}-(b+t)e]^3\\ &\quad+\frac{1}{4}[4\widetilde{h}-e][(a+4t)\widetilde{h}-(b+t)e]^2\\ &\quad+\frac{1}{12}[(4\widetilde{h}-e)^2+(6+d)\widetilde{h}^2-4\widetilde{h}e][(a+4t)\widetilde{h}-(b+t)e]\\ &\quad+\frac{1}{24}[4\widetilde{h}-e][6\widetilde{h}^2+d\widetilde{l}-4\widetilde{h}e]\\ &= \frac{1}{6}[(a+4t)^3\widetilde{h}^3-3(a+4t)^2(b+t)\widetilde{h}^2e\\ &\qquad+3(a+4t)(b+t)^2\widetilde{h}e^2-(b+t)^3e^3]\\ &\quad+\frac{1}{4}[4\widetilde{h}-e][(a+4t)^2\widetilde{h}^2-2(a+4t)(b+t)\widetilde{h}e+(b+t)^2e^2]\\ &\quad+\frac{1}{12}[(22+d)\widetilde{h}^2-8\widetilde{h}e+e^2-4\widetilde{h}e][(a+4t)\widetilde{h}-(b+t)e]\\ &\quad+\frac{1}{24}[4\widetilde{h}-e][(6+d)\widetilde{h}^2-4\widetilde{h}e].\end{aligned}$$

Then,

$$\begin{aligned}\chi(\widetilde{Y},\mathcal{O}(D(t))) &= \frac{1}{6}[(a+4t)^3-3d(a+4t)(b+t)^2-(-4d-2g+2)(b+t)^3]\\ &\quad+\frac{1}{4}[4(a+4t)^2-4d(b+t)^2-2d(a+4t)(b+t)\\ &\qquad-(-4d-2g+2)(b+t)^2]\\ &\quad+\frac{1}{12}[22(a+4t)-12d(b+t)-(-4d-2g+2)(b+t)]\\ &\quad+\frac{1}{24}[24+4d-4d].\end{aligned}$$

Now, by expanding, we obtain

$$\begin{aligned}\chi(\widetilde{Y},\mathcal{O}(D(t))) &= \frac{1}{6}[a^3+12a^2t+48at^2+64t^3-3dab^2-6dabt-12db^2t\\ &\quad-3dat^2-24dbt^2-12dt^3+4db^3+12db^2t+12dbt^2+4dt^3\\ &\quad+2gb^3+6gb^2t+6gbt^2+2gt^3-2b^3-6b^2t-6bt^2-2t^3]\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{4}[4a^2+32at+64t^2-4db^2-8dbt-4dt^2-2dab-2dat \\
&\quad -8dbt-8dt^2+4db^2+8dbt+4dt^2+2gb^2+4gbt+2gt^2 \\
&\quad -2b^2-4bt-2t^2] \\
&+\frac{1}{12}[22a+88t-12db-12dt+4db+4dt+2gb+2gt \\
&\quad -2b-2t] \\
&+\frac{1}{24}24.
\end{aligned}
$$

Then, collecting the terms with same powers of t

$$
\begin{aligned}
\chi(\widetilde{Y},\mathcal{O}(D(t))) &= \frac{1}{24}[256-48d+16d+8g-8]t^3 \\
&+\frac{1}{24}[192a-12da-96db+48db+24gb-24b+384-24d \\
&\quad -48d+24d+12g-12]t^2 \\
&+\frac{1}{24}[48a^2-24dab-48db^2+48db^2+24gb^2-24b^2+192a \\
&\quad -48db-12da-48db+48db+24gb-24b+176-24d \\
&\quad +8d+4g-4]t \\
&+\frac{1}{24}[4a^3-12dab^2+16db^3+8gb^3-8b^3+24a^2-24db^2 \\
&\quad -12dab+24db^2+12gb^2-12b^2+44a-24db \\
&\quad +8db+4gb-4b+24].
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\chi(\widetilde{Y},\mathcal{O}(D(t))) &= \frac{1}{6}[62-8d+2g]t^3 \\
&+\frac{1}{6}[(48-3d)a+(6g-12d-6)b-12d+3g+93]t^2 \\
&+\frac{1}{6}[12a^2+(6g-6)b^2-6dab+(48-3d)a \\
&\quad +(6g-6-12d)b+43-4d+g]t \\
&+\frac{1}{6}[a^3+(4d+2g-2)b^3+6a^2+(3g-3da-3)b^2+11a \\
&\quad +(g-3da-4d-1)b+6].
\end{aligned}
$$

Theorem 1.2.5 *Let* $D = a\widetilde{h} - be$ *be a divisor on* $\widetilde{Y} = \widetilde{Q}$*, where* $a, b \in \mathbb{Z}$*. Let* $D(t) = D + tH_{\widetilde{Q}}$*. Then*

$$
\chi(\widetilde{Y},\mathcal{O}(D(t))) = \frac{1}{24}[208-24d+8g]t^3
$$

$$+\frac{1}{24}[(216-12d)a+(24g-36d-24)b-36d+12g$$
$$+312]t^2$$
$$+\frac{1}{24}[72a^2+(24g-24)b^2-24dab+(216-12d)a$$
$$+(24g-24-36d)b+152-6d+4g]t$$
$$+\frac{1}{24}[8a^3+(12d+8g-8)b^3+36a^2+(12g-12da-12)b^2$$
$$+52a+(4g-12da-12d-4)b+24+3d].$$

Proof 1.2.6 *Since Q is a smooth quadric*

$$K_Q=(2-4-1)=-3h$$

and

$$c(T_Q)=\frac{(1+h)^{1+4}}{1+2h}=1+3h+4h^2+2h^3.$$

So by [14, Example 15.4.3],

$$\begin{aligned}
c_1(T_{\widetilde{Q}}) &= f^*c_1(T_Q)+(1-2)[E]\\
&= f^*(3h)-e\\
&= 3\widetilde{h}-e\\
c_2(T_{\widetilde{Q}}) &= f^*c_2(T_Q)+f^*i_*[C]-f^*c_1(T_Q)[E]\\
&= f^*(4h^2)+d\widetilde{l}-f^*(3h)e\\
&= 4\widetilde{h}^2+d\widetilde{l}-3\widetilde{h}e\\
&= (4+d)\widetilde{h}^2-3\widetilde{h}e\\
K_{\widetilde{Q}} &= f^*K_Q+(2-1)[E]\\
&= f^*(-3h)+e\\
&= -3\widetilde{h}+e.
\end{aligned}$$

Then using [24, Lemma 2.1] and the fact that Q is a smooth quadric, we obtain

$$\widetilde{h}^3=2$$

$$e^3=-(-K_Q\cdot C)+2-2g=-(3h\cdot C)+2-2g=-3d-2g+2$$

$$e^2\cdot(-K_{\widetilde{Q}})=2g-2 \quad\Rightarrow\quad e^2(3\widetilde{h}-e)=2g-2$$

$$
\begin{aligned}
&\Rightarrow\ 3\widetilde{h}e^2 - e^3 = 2g-2\\
&\Rightarrow\ 3\widetilde{h}e^2 - (-3d-2g+2) = 2g-2\\
&\Rightarrow\ \widetilde{h}e^2 = -d
\end{aligned}
$$

$$
\begin{aligned}
e\cdot(-K_{\widetilde{Q}})^2 = (-K_Q\cdot C)+2-2g \ &\Rightarrow\ e(3\widetilde{h}-e)^2 = (3h\cdot C)+2-2g\\
&\Rightarrow\ 9\widetilde{h}^2e - 6\widetilde{h}e^2 + e^3 = 3d+2-2g\\
&\Rightarrow\ 9\widetilde{h}^2e + 6d - 3d - 2g + 2 = 3d+2-2g\\
&\Rightarrow\ \widetilde{h}^2e = 0.
\end{aligned}
$$

Since $\widetilde{Y}=\widetilde{Q}$ is non-hyperelliptic Fano,

$$
H_{\widetilde{Q}} = -K_{\widetilde{Q}} = 3\widetilde{h} - e.
$$

Let $D = a\widetilde{h} - be$ be a divisor class on $\widetilde{Y}$. Then

$$
\begin{aligned}
D(t) = D + tH_{\widetilde{Y}} &= (a\widetilde{h} - be) + t(3\widetilde{h} - e)\\
&= (a+3t)\widetilde{h} - (b+t)e.
\end{aligned}
$$

Then, applying the Riemann-Roch theorem for line bundles on 3-folds, we obtain

$$
\begin{aligned}
\chi(\widetilde{Y},\mathcal{O}(D(t))) &= \frac{1}{6}(D(t))^3 + \frac{1}{4}c_1(T_{\widetilde{Y}})\cdot(D(t))^2 + \frac{1}{24}c_1(T_{\widetilde{Y}})\cdot c_2(T_{\widetilde{Y}})\\
&\quad + \frac{1}{12}(c_1^2(T_{\widetilde{Y}}) + c_2(T_{\widetilde{Y}}))\cdot(D(t)).
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
\chi(\widetilde{Y},\mathcal{O}(D(t))) &= \frac{1}{6}[(a+3t)\widetilde{h} - (b+t)e]^3 + \frac{1}{4}[3\widetilde{h} - e][(a+3t)\widetilde{h} - (b+t)e]^2\\
&\quad + \frac{1}{12}[(3\widetilde{h}-e)^2 + (4+d)\widetilde{h}^2 - 3\widetilde{h}e][(a+3t)\widetilde{h} - (b+t)e]\\
&\quad + \frac{1}{24}[3\widetilde{h} - e][(4+d)\widetilde{h}^2 - 3\widetilde{h}e]\\
&= \frac{1}{6}[(a+3t)^3\widetilde{h}^3 - 3(a+3t)^2(b+t)\widetilde{h}^2e + 3(a+3t)(b+t)^2\widetilde{h}e^2\\
&\quad -(b+t)^3e^3]\\
&\quad + \frac{1}{4}[3\widetilde{h} - e][(a+3t)^2\widetilde{h}^2 - 2(a+3t)(b+t)\widetilde{h}e + (b+t)^2e^2]\\
&\quad + \frac{1}{12}[(13+d)\widetilde{h}^2 - 6\widetilde{h}e + e^2 - 3\widetilde{h}e][(a+3t)\widetilde{h} - (b+t)e]\\
&\quad + \frac{1}{24}[3\widetilde{h} - e][(4+d)\widetilde{h}^2 - 3\widetilde{h}e].
\end{aligned}
$$

Then,

$$\begin{aligned}\chi(\widetilde{Y},\mathcal{O}(D(t))) &= \frac{1}{6}[2(a+3t)^3-3d(a+3t)(b+t)^2\\ &\quad-(-3d-2g+2)(b+t)^3]\\ &+\frac{1}{4}[3.2(a+3t)^2-3d(b+t)^2-2d(a+3t)(b+t)\\ &\quad-(-3d-2g+2)(b+t)^2]\\ &+\frac{1}{12}[(26+d)(a+3t)-9d(b+t)-(-3d-2g+2)(b+t)]\\ &+\frac{1}{24}[24+6d-3d].\end{aligned}$$

Then, by expanding

$$\begin{aligned}\chi(\widetilde{Y},\mathcal{O}(D(t))) &= \frac{1}{6}[52t^3+2gt^3-6dt^3-6bt^2+6bgt^2-9bdt^2+54at^2\\ &\quad-3adt^2-6b^2t+6b^2gt-6abdt+18a^2t-2b^3+2b^3g\\ &\quad+3b^3d-3ab^2d+2a^3]\\ &+\frac{1}{4}[52t^2+2gt^2-6dt^2-4bt+4bgt-6bdt+36at\\ &\quad-2adt-2b^2+2b^2g-2abd+6a^2]\\ &+\frac{1}{12}[76t+2gt-3dt-2b+2bg-6bd+26a+ad]\\ &+\frac{1}{24}[3d+24].\end{aligned}$$

Then, collecting the terms with same powers of t

$$\begin{aligned}\chi(\widetilde{Y},\mathcal{O}(D(t))) &= \frac{1}{24}[208+8g-24d]t^3\\ &+\frac{1}{24}[-24b+24gb-36db+216a-12da+312+12g\\ &\quad-36d]t^2\\ &+\frac{1}{24}[-24b^2+24gb^2-24dab+72a^2-24b+24gb-36db\\ &\quad+216a-12da+152+4g-6d]t\\ &+\frac{1}{24}[-8b^3+8gb^3+12db^3-12dab^2+8a^3+12gb^2-12dab\\ &\quad-12b^2+36a^2-4b+4gb-12db+52a+2da+3d+24].\end{aligned}$$

Finally,

$$\begin{aligned}\chi(\widetilde{Y},\mathcal{O}(D(t))) &= \frac{1}{24}[208-24d+8g]t^3\\ &+\frac{1}{24}[(216-12d)a+(24g-36d-24)b-36d+12g\end{aligned}$$

$$
\begin{aligned}
&+312]t^2\\
&+\frac{1}{24}[72a^2+(24g-24)b^2-24dab+(216-12d)a\\
&\quad+(24g-24-36d)b+152-6d+4g]t\\
&+\frac{1}{24}[8a^3+(12d+8g-8)b^3+36a^2+(12g-12da-12)b^2\\
&\quad+(52+2d)a+(4g-12da-12d-4)b+24+3d].
\end{aligned}
$$

Theorem 1.2.7 *Let $D = a\widetilde{h} - be$ be a divisor on $\widetilde{Y} = \widetilde{V}_3$, where $a, b \in \mathbb{Z}$. Let $D(t) = D + tH_{\widetilde{V}_3}$. Then*

$$
\begin{aligned}
\chi(\widetilde{Y}, \mathcal{O}(D(t))) &= \frac{1}{12}[44-8d+4g]t^3\\
&+\frac{1}{12}[(72-6d)a+(12g-12d-12)b-12d+6g+66]t^2\\
&+\frac{1}{12}[36a^2+(12g-12)b^2-12dab+(72-6d)a\\
&\quad+(12g-12-12d)b+46+2g]t\\
&+\frac{1}{12}[6a^3+(4d+4g-4)b^3+18a^2+(6g-6da-6)b^2\\
&\quad+(24+2d)a+(2g-6da-4d-2)b+12+2d].
\end{aligned}
$$

Proof 1.2.8 *Since V_3 is a smooth cubic, we have*

$$K_{V_3} = (3-4-1) = -2h$$

and

$$c(T_{V_3}) = \frac{(1+h)^{1+4}}{1+3h} = 1+2h+4h^2-2h^3.$$

So by [14, Chapter 15.4.3],

$$
\begin{aligned}
c_1(T_{\widetilde{V}_3}) &= f^*c_1(T_{V_3}) + (1-2)[E]\\
&= f^*(2h) - e\\
&= 2\widetilde{h} - e\\
c_2(T_{\widetilde{V}_3}) &= f^*c_2(T_{V_3}) + f^*i_*[C] - f^*c_1(T_{V_3})[E]\\
&= f^*(4h^2) + d\widetilde{l} - f^*(2h)e\\
&= 4\widetilde{h}^2 + d\widetilde{l} - 2\widetilde{h}e\\
&= (4+d)\widetilde{h}^2 - 2\widetilde{h}e\\
K_{\widetilde{V}_3} &= f^*K_{V_3} + (2-1)[E]
\end{aligned}
$$

$$
\begin{aligned}
&= f^*(-2h) + e \\
&= -2\widetilde{h} + e.
\end{aligned}
$$

Then using [24, Lemma 2.1] and the fact that V_3 *is a smooth cubic,*

$$
\widetilde{h}^3 = 3
$$

$$
e^3 = -(-K_{V_3} \cdot C) + 2 - 2g = -(2h \cdot C) + 2 - 2g = -2d - 2g + 2
$$

$$
\begin{aligned}
e^2 \cdot (-K_{\widetilde{V}_3}) = 2g - 2 \;&\Rightarrow\; e^2(2\widetilde{h} - e) = 2g - 2 \\
&\Rightarrow\; 2\widetilde{h}e^2 - e^3 = 2g - 2 \\
&\Rightarrow\; 2\widetilde{h}e^2 - (-2d - 2g + 2) = 2g - 2 \\
&\Rightarrow\; \widetilde{h}e^2 = -d
\end{aligned}
$$

$$
\begin{aligned}
e \cdot (-K_{\widetilde{V}_3})^2 = (-K_{V_3} \cdot C) + 2 - 2g \;&\Rightarrow\; e(2\widetilde{h} - e)^2 = (2h \cdot C) + 2 - 2g \\
&\Rightarrow\; 4\widetilde{h}^2 e - 4\widetilde{h}e^2 + e^3 = 2d + 2 - 2g \\
&\Rightarrow\; 4\widetilde{h}^2 e + 4d - 2d - 2g + 2 = 2d + 2 - 2g \\
&\Rightarrow\; \widetilde{h}^2 e = 0.
\end{aligned}
$$

Since $\widetilde{Y} = \widetilde{V}_3$ *is Fano,*

$$
H_{\widetilde{V}_3} = -K_{\widetilde{V}_3} = 2\widetilde{h} - e.
$$

Let $D = a\widetilde{h} - be$ *be a divisor on* $\widetilde{V}_3$*. Then*

$$
\begin{aligned}
D(t) = D + tH_{\widetilde{Y}} &= (a\widetilde{h} - be) + t(2\widetilde{h} - e) \\
&= (a + 2t)\widetilde{h} - (b + t)e.
\end{aligned}
$$

Then, applying the Riemann-Roch theorem for line bundles on 3-folds, we obtain

$$
\begin{aligned}
\chi(\widetilde{Y}, \mathcal{O}(D(t))) &= \frac{1}{6}(D(t))^3 + \frac{1}{4}c_1(T_{\widetilde{Y}}) \cdot (D(t))^2 + \frac{1}{24}c_1(T_{\widetilde{Y}}) \cdot c_2(T_{\widetilde{Y}}) \\
&\quad + \frac{1}{12}(c_1^2(T_{\widetilde{Y}}) + c_2(T_{\widetilde{Y}})) \cdot (D(t)).
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
\chi(\widetilde{Y}, \mathcal{O}(D(t))) &= \frac{1}{6}[(a + 2t)\widetilde{h} - (b + t)e]^3 + \frac{1}{4}[2\widetilde{h} - e][(a + 2t)\widetilde{h} - (b + t)e]^2 \\
&\quad + \frac{1}{12}[(2\widetilde{h} - e)^2 + (4 + d)\widetilde{h}^2 - 2\widetilde{h}e][(a + 2t)\widetilde{h} - (b + t)e]
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{1}{24}[2\widetilde{h}-e][(4+d)\widetilde{h}^2-2\widetilde{h}e] \\
= \quad & \frac{1}{6}[(a+2t)^3\widetilde{h}^3-3(a+2t)^2(b+t)\widetilde{h}^2e+3(a+2t)(b+t)^2\widetilde{h}e^2 \\
& \quad -(b+t)^3e^3] \\
& +\frac{1}{4}[2\widetilde{h}-e][(a+2t)^2\widetilde{h}^2-2(a+2t)(b+t)\widetilde{h}e+(b+t)^2e^2] \\
& +\frac{1}{12}[(8+d)\widetilde{h}^2-4\widetilde{h}e+e^2-2\widetilde{h}e][(a+2t)\widetilde{h}-(b+t)e] \\
& +\frac{1}{24}[2\widetilde{h}-e][(4+d)\widetilde{h}^2-2\widetilde{h}e].
\end{aligned}
$$

Then,

$$
\begin{aligned}
\chi(\widetilde{Y},\mathcal{O}(D(t))) \quad = \quad & \frac{1}{6}[3(a+2t)^3-3d(a+2t)(b+t)^2 \\
& \quad -(-2d-2g+2)(b+t)^3] \\
& +\frac{1}{4}[2.3(a+2t)^2-2d(b+t)^2-2d(a+2t)(b+t) \\
& \quad -(-2d-2g+2)(b+t)^2] \\
& +\frac{1}{12}[3(8+d)(a+2t)-6d(b+t)-d(a+2t) \\
& \quad -(-2d-2g+2)(b+t)] \\
& +\frac{1}{24}[2.3(4+d)-2d].
\end{aligned}
$$

Then, by expanding

$$
\begin{aligned}
\chi(\widetilde{Y},\mathcal{O}(D(t))) \quad = \quad & \frac{1}{6}[22t^3+2gt^3-4dt^3-6bt^2+6bgt^2-6bdt^2+36at^2 \\
& \quad -3adt^2-6b^2t+6b^2gt-6abdt+18a^2t-2b^3+2b^3g \\
& \quad +2b^3d-3ab^2d+3a^3] \\
& +\frac{1}{4}[22t^2+2gt^2-4dt^2-4bt+4bgt-4bdt+24at \\
& \quad -2adt-2b^2+2b^2g-2abd+6a^2] \\
& +\frac{1}{12}[46t+2gt-2b+2bg-4bd+24a+2ad] \\
& +\frac{1}{24}[4d+24].
\end{aligned}
$$

Then, collecting the terms with same powers of t

$$
\begin{aligned}
\chi(\widetilde{Y},\mathcal{O}(D(t))) \quad = \quad & \frac{1}{12}[44+4g-8d]t^3 \\
& +\frac{1}{12}[-12b+12gb-12db+72a-6da+66+6g-12d]t^2 \\
& +\frac{1}{12}[-12b^2+12gb^2-12dab+36a^2-12b+12gb-12db
\end{aligned}
$$

$$
\begin{aligned}
&+72a - 6da + 46 + 2g]t \\
&+\frac{1}{12}[-4b^3 + 4gb^3 + 4db^3 - 6dab^2 + 6a^3 - 6b^2 + 6gb^2 - 6dab \\
&+18a^2 - 2b + 2gb - 4db + 24a + 2da + 2d + 12].
\end{aligned}
$$

Finally

$$
\begin{aligned}
\chi(\widetilde{Y}, \mathcal{O}(D(t))) &= \frac{1}{12}[44 - 8d + 4g]t^3 \\
&+\frac{1}{12}[(72 - 6d)a + (12g - 12d - 12)b - 12d + 6g + 66]t^2 \\
&+\frac{1}{12}[36a^2 + (12g - 12)b^2 - 12dab + (72 - 6d)a \\
&+(12g - 12 - 12d)b + 46 + 2g]t \\
&+\frac{1}{12}[6a^3 + (4d + 4g - 4)b^3 + 18a^2 + (6g - 6da - 6)b^2 \\
&+(24 + 2d)a + (2g - 6da - 4d - 2)b + 12 + 2d].
\end{aligned}
$$

Theorem 1.2.9 *Let $D = a\widetilde{h} - be$ be a divisor on $\widetilde{Y} = \widetilde{V}_4$, where $a, b \in \mathbb{Z}$. Let $D(t) = D + tH_{\widetilde{V}_4}$. Then*

$$
\begin{aligned}
\chi(\widetilde{Y}, \mathcal{O}(D(t))) &= \frac{1}{12}[60 - 8d + 4g]t^3 \\
&+\frac{1}{12}[(96 - 6d)a + (12g - 12d - 12)b - 12d + 6g + 90]t^2 \\
&+\frac{1}{12}[48a^2 + (12g - 12)b^2 - 12dab + (96 - 6d)a \\
&+(12g - 12 - 12d)b + 54 + 2g + 2d]t \\
&+\frac{1}{12}[8a^3 + (4d + 4g - 4)b^3 + 24a^2 + (6g - 6da - 6)b^2 \\
&+(28 + 3d)a + (2g - 6da - 4d - 2)b + 12 + 3d].
\end{aligned}
$$

Proof 1.2.10 *Since V_3 is a complete intersection of two quadrics,*

$$K_{V_4} = (2 + 2 - 5 - 1) = -2h$$

and

$$c(T_{V_4}) = \frac{(1 + h)^{1+5}}{(1 + 2h)^2} = 1 + 2h + 3h^2 + 0h^3.$$

So by [14, Example 15.4.3],

$$
\begin{aligned}
c_1(T_{\widetilde{V}_4}) &= f^*c_1(T_{V_4}) + (1 - 2)[E] \\
&= f^*(2h) - e
\end{aligned}
$$

$$
\begin{aligned}
&= 2\widetilde{h} - e \\
c_2(T_{\widetilde{V}_4}) &= f^*c_2(T_{V_4}) + f^*i_*[C] - f^*c_1(T_{V_4})[E] \\
&= f^*(3h^2) + d\widetilde{l} - f^*(2h)e \\
&= 3\widetilde{h}^2 + d\widetilde{l} - 2\widetilde{h}e \\
&= (3+d)\widetilde{h}^2 - 2\widetilde{h}e \\
K_{\widetilde{V}_4} &= f^*K_{V_4} + (2-1)[E] \\
&= f^*(-2h) + e \\
&= -2\widetilde{h} + e.
\end{aligned}
$$

Then using [24, Lemma 2.1] and that V_4 is a complete intersection of two quadrics,

$$\widetilde{h}^3 = 2.2 = 4$$

$$e^3 = -(-K_{V_4} \cdot C) + 2 - 2g = -(2h \cdot C) + 2 - 2g = -2d - 2g + 2$$

$$
\begin{aligned}
e^2 \cdot (-K_{\widetilde{V}_4}) = 2g - 2 \;&\Rightarrow\; e^2(2\widetilde{h} - e) = 2g - 2 \\
&\Rightarrow\; 2\widetilde{h}e^2 - e^3 = 2g - 2 \\
&\Rightarrow\; 2\widetilde{h}e^2 - (-2d - 2g + 2) = 2g - 2 \\
&\Rightarrow\; \widetilde{h}e^2 = -d
\end{aligned}
$$

$$
\begin{aligned}
e \cdot (-K_{\widetilde{V}_4})^2 = (-K_{V_4} \cdot C) + 2 - 2g \;&\Rightarrow\; e(2\widetilde{h} - e)^2 = (2h \cdot C) + 2 - 2g \\
&\Rightarrow\; 4\widetilde{h}^2 e - 4\widetilde{h}e^2 + e^3 = 2d + 2 - 2g \\
&\Rightarrow\; 4\widetilde{h}^2 e + 4d - 2d - 2g + 2 = 2d + 2 - 2g \\
&\Rightarrow\; \widetilde{h}^2 e = 0.
\end{aligned}
$$

Since $\widetilde{Y} = \widetilde{V}_4$ is non-hyperelliptic Fano,

$$H_{\widetilde{V}_4} = -K_{\widetilde{V}_4} = 2\widetilde{h} - e.$$

Let $D = a\widetilde{h} - be$ be a divisor on $\widetilde{V}_4$. Then

$$
\begin{aligned}
D(t) = D + tH_{\widetilde{Y}} &= (a\widetilde{h} - be) + t(2\widetilde{h} - e) \\
&= (a + 2t)\widetilde{h} - (b + t)e.
\end{aligned}
$$

Then, applying the Riemann-Roch theorem for line bundles on 3-folds, we obtain

$$\begin{aligned}\chi(\widetilde{Y},\mathcal{O}(D(t))) &= \frac{1}{6}(D(t))^3+\frac{1}{4}c_1(T_{\widetilde{Y}})\cdot(D(t))^2+\frac{1}{24}c_1(T_{\widetilde{Y}})\cdot c_2(T_{\widetilde{Y}})\\ &\quad +\frac{1}{12}(c_1^2(T_{\widetilde{Y}})+c_2(T_{\widetilde{Y}}))\cdot(D(t)).\end{aligned}$$

Then, we have

$$\begin{aligned}\chi(\widetilde{Y},\mathcal{O}(D(t))) &= \frac{1}{6}[(a+2t)\widetilde{h}-(b+t)e]^3+\frac{1}{4}[2\widetilde{h}-e][(a+2t)\widetilde{h}-(b+t)e]^2\\ &\quad +\frac{1}{12}[(2\widetilde{h}-e)^2+(3+d)\widetilde{h}^2-2\widetilde{h}e][(a+2t)\widetilde{h}-(b+t)e]\\ &\quad +\frac{1}{24}[2\widetilde{h}-e][(3+d)\widetilde{h}^2-2\widetilde{h}e]\\ &= \frac{1}{6}[(a+2t)^3\widetilde{h}^3-3(a+2t)^2(b+t)\widetilde{h}^2e+3(a+2t)(b+t)^2\widetilde{h}e^2\\ &\quad -(b+t)^3e^3]\\ &\quad +\frac{1}{4}[2\widetilde{h}-e][(a+2t)^2\widetilde{h}^2-2(a+2t)(b+t)\widetilde{h}e+(b+t)^2e^2]\\ &\quad +\frac{1}{12}[(7+d)\widetilde{h}^2-4\widetilde{h}e+e^2-2\widetilde{h}e][(a+2t)\widetilde{h}-(b+t)e]\\ &\quad +\frac{1}{24}[2\widetilde{h}-e][(3+d)\widetilde{h}^2-2\widetilde{h}e].\end{aligned}$$

Then,

$$\begin{aligned}\chi(\widetilde{Y},\mathcal{O}(D(t))) &= \frac{1}{6}[4(a+2t)^3-3d(a+2t)(b+t)^2\\ &\quad -(-2d-2g+2)(b+t)^3]\\ &\quad +\frac{1}{4}[2.4(a+2t)^2-2d(b+t)^2-2d(a+2t)(b+t)\\ &\quad -(-2d-2g+2)(b+t)^2]\\ &\quad +\frac{1}{12}[4(7+d)(a+2t)-6d(b+t)-d(a+2t)\\ &\quad -(-2d-2g+2)(b+t)]\\ &\quad +\frac{1}{24}[2.4(3+d)-2d].\end{aligned}$$

Then, by expanding

$$\begin{aligned}\chi(\widetilde{Y},\mathcal{O}(D(t))) &= \frac{1}{6}[30t^3+2gt^3-4dt^3-6bt^2+6bgt^2-6bdt^2+48at^2\\ &\quad -3adt^2-6b^2t+6b^2gt-6abdt+24a^2t-2b^3+2b^3g\\ &\quad +2b^3d-3ab^2d+4a^3]\\ &\quad +\frac{1}{4}[30t^2+2gt^2-4dt^2-4bt+4bgt-4bdt+32at\\ &\quad -2adt-2b^2+2b^2g-2abd+8a^2]\end{aligned}$$

$$+\frac{1}{12}[54t + 2gt + 2dt - 2b + 2bg - 4bd + 28a + 3ad]$$
$$+\frac{1}{24}[6d + 24].$$

Then, collecting terms with same powers of t

$$\begin{aligned}
\chi(\widetilde{Y}, \mathcal{O}(D(t))) &= \frac{1}{12}[60 + 4g - 8d]t^3 \\
&+\frac{1}{12}[-12b + 12gb - 12db + 96a - 6da + 90 + 6g - 12d]t^2 \\
&+\frac{1}{12}[-12b^2 + 12gb^2 - 12dab + 48a^2 - 12b + 12gb - 12db \\
&\quad +96a - 6da + 54 + 2g + 2d]t \\
&+\frac{1}{12}[-4b^3 + 4gb^3 + 4db^3 - 6dab^2 + 8a^3 - 6b^2 + 6gb^2 - 6dab \\
&\quad +24a^2 - 2b + 2gb - 4db + 28a + 3da + 3d + 12].
\end{aligned}$$

Finally

$$\begin{aligned}
\chi(\widetilde{Y}, \mathcal{O}(D(t))) &= \frac{1}{12}[60 - 8d + 4g]t^3 \\
&+\frac{1}{12}[(96 - 6d)a + (12g - 12d - 12)b - 12d + 6g + 90]t^2 \\
&+\frac{1}{12}[48a^2 + (12g - 12)b^2 - 12dab + (96 - 6d)a \\
&\quad +(12g - 12 - 12d)b + 54 + 2g + 2d]t \\
&+\frac{1}{12}[8a^3 + (4d + 4g - 4)b^3 + 24a^2 + (6g - 6da - 6)b^2 \\
&\quad +(28 + 3d)a + (2g - 6da - 4d - 2)b + 12 + 3d].
\end{aligned}$$

CHAPTER 2

ULRICH BUNDLES

The general references for this section are [7] and [20].

Definition 2.0.11 *Let $\mathcal{E}$ be a vector bundle on a nonsingular projective variety X. Then $\mathcal{E}$ is said to be $semistable$ if for every nonzero subbundle $\mathcal{F}$ of $\mathcal{E}$ we have the inequality*

$$\frac{P_{\mathcal{F}}}{rank(\mathcal{F})} \leq \frac{P_{\mathcal{E}}}{rank(\mathcal{E})},$$

where $P_{\mathcal{F}}$ and $P_{\mathcal{E}}$ are the respective Hilbert polynomials and comparison is based on the lexicographic order. It is $stable$ if one always has strict inequality above.

Definition 2.0.12 *Let $\mathcal{E}$ be a vector bundle on a nonsingular projective variety X. The $slope$ $\mu(\mathcal{E})$ of $\mathcal{E}$ is defined as $deg(\mathcal{E})/rank(\mathcal{E})$. We say that $\mathcal{E}$ is μ-$semistable$ if for every subbundle $\mathcal{F}$ of $\mathcal{E}$ with $0 < rank(\mathcal{F}) < rank(\mathcal{E})$, we have $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$. We say $\mathcal{E}$ is μ-$stable$ if strict inequality always holds above.*

Lemma 2.0.13 *The two definitions are related as follows:*

$$\mu - stable \Rightarrow stable \Rightarrow semistable \Rightarrow \mu - semistable.$$

Proof 2.0.14 *See [20, 1.2.13].*

Definition 2.0.15 *A vector bundle $\mathcal{E}$ on X is called ACM (arithmetically Cohen-Macaulay) if $H^i(\mathcal{E}(t)) = 0$ for all $t \in \mathbb{Z}$ and $0 < i < k$.*

Definition 2.0.16 *Let $\mathcal{E}$ be a vector bundle of rank r on X. Then $\mathcal{E}$ is **Ulrich** if for some linear projection $\pi : X \rightarrow \mathbb{P}^k$ we have $\pi_*\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^k}^{cr}$.*

Proposition 2.0.17 *Let $\mathcal{E}$ be a vector bundle of rank r on X. Then $\mathcal{E}$ is Ulrich if and only if it is ACM with Hilbert polynomial $cr\binom{t+k}{k}$.*

Proof 2.0.18 *See [7, Proposition 2.3].*

Theorem 2.0.19 *Let $\widetilde{Y}$ be one of the following Fano 3-folds:*

1. *the blow-up of $\mathbb{P}^3$ along an intersection of two cubics,*
2. *the blow-up of $\mathbb{P}^3$ along a curve of degree 7 and genus 5 which is an intersection of cubics,*
3. *the blow-up of $\mathbb{P}^3$ along a curve of degree 6 and genus 3 which is an intersection of cubics,*
4. *the blow-up of $\mathbb{P}^3$ along the intersection of a quadric and a cubic,*
5. *the blow-up of $\mathbb{P}^3$ along an elliptic curve which is an intersection of two quadrics,*
6. *the blow-up of $\mathbb{P}^3$ along a twisted cubic,*
7. *the blow-up of $\mathbb{P}^3$ along a plane cubic,*
8. *the blow-up of $\mathbb{P}^3$ along a conic,*
9. *the blow-up of $\mathbb{P}^3$ along a line.*

Then Ulrich line bundles can exist only on the class (3).

Proof 2.0.20 *Let $D = a\widetilde{h} - be$ be a divisor class on $\widetilde{Y}$. We can compute Hilbert polynomial of $\mathcal{O}_{\widetilde{Y}}(D)$ by Theorem 1.2.3. By Proposition 2.0.17, this must be equal to $deg(\widetilde{Y})\binom{t+3}{3}$. We will equate the coefficients of these two polynomials and try to find integer solutions for a and b in each case separately.*

1. *(This case is [21, No.4 in Table 12.3].) Since C is an intersection of two cubics, $d = 9$. By the adjunction formula, $g = 10$. Then $k = H^3 = 10$. Now, equate the coefficients of t^2:*

$$\frac{10.6}{6}t^2 = \frac{1}{6}[(48 - 3.9)a + (6.10 - 12.9 - 6)b - 12.9 + 3.10 + 93]t^2$$

which gives

$$a = \frac{18b + 15}{7}.$$

Next, equate the coefficients of t *and use the above relation between* a *and* b *to get*

$$\begin{aligned}\frac{10.11}{6}t &= \frac{1}{6}[12(\frac{18b+15}{7})^2 + (6.10-6)b^2 - 6.9(\frac{18b+15}{7})b \\ &+(48-3.9)(\frac{18b+15}{7}) + (6.10-6-12.9)b \\ &+43-4.9+10]t\end{aligned}$$

which gives

$$b = \frac{3}{2} \mp \frac{7}{30}\sqrt{65}.$$

There is no integer solution for a *and* b*, so there exists no Ulrich line bundle.*

2. *(This case is [21, No.9 in Table 12.3].) It is given that* $d = 7$ *and* $g = 5$*. Then* $k = H^3 = 16$*. Now, equate the coefficients of* t^2*:*

$$\frac{16.6}{6}t^2 = \frac{1}{6}[(48-3.7)a + (6.5-12.7-6)b - 12.7 + 3.5 + 93]t^2$$

which gives

$$a = \frac{20b + 24}{9}.$$

Next, equate the coefficients of t *and use the above relation between* a *and* b *to get*

$$\begin{aligned}\frac{16.11}{6}t &= \frac{1}{6}[12(\frac{20b+24}{9})^2 + (6.5-6)b^2 - 6.7(\frac{20b+24}{9})b \\ &+(48-3.7)(\frac{20b+24}{9}) + (6.5-6-12.7)b + 43 - 4.7 + 5]t\end{aligned}$$

which gives

$$b = \frac{3}{2} \mp \frac{9}{34}\sqrt{34}.$$

There is no integer solution for a *and* b*, so there exists no Ulrich line bundle.*

3. *(This case is [21, No.12 in Table 12.3].) It is given that* $d = 6$ *and* $g = 3$*. Then* $k = H^3 = 20$*. Then equate the coefficients of* t^2*:*

$$\frac{20.6}{6}t^2 = \frac{1}{6}[(48-3.6)a + (6.3-12.6-6)b - 12.6 + 3.3 + 93]t^2$$

which gives

$$a = 2b + 3.$$

Next, equate the coefficients of t *and use the above relation between* a *and* b *to get*

$$\begin{aligned}\frac{20.11}{6}t &= \frac{1}{6}[12(2b+3)^2 + (6.3-6)b^2 - 6.6(2b+3)b \\ &\quad +(48-3.6)(2b+3) + (6.3-6-12.6)b + 43 - 4.6 + 3]t\end{aligned}$$

which gives

$$b = 0 \text{ or } b = 3.$$

Then we have $(a, b) = (3, 0)$ *or* $(a, b) = (9, 3)$*. Both of these solutions satisfy also the equality of coefficients of* t^2 *and constant terms. So the divisors* $3\widetilde{h}$ *and* $9\widetilde{h} - 3e$ *yield possible Ulrich line bundles. (We note that to be Ulrich, they must also satisfy the ACM condition.)*

4. *(This case is [21, No.15(a) in Table 12.3].) Since* C *is the intersection of a quadric and a cubic,* $d = 6$*. By the adjunction formula* $g = 4$*. Then* $k = H^3 = 22$*. Then equate the coefficients of* t^2*:*

$$\frac{22.6}{6}t^2 = \frac{1}{6}[(48-3.6)a + (6.4-12.6-6)b - 12.6 + 3.4 + 93]t^2$$

which gives

$$a = \frac{18b + 33}{10}.$$

Next, equate the coefficients of t *and use the above relation between* a *and* b *to get*

$$\begin{aligned}\frac{22.11}{6}t &= \frac{1}{6}[12(\frac{18b+33}{10})^2 + (6.4-6)b^2 - 6.6(\frac{18b+33}{10})b \\ &\quad +(48-3.6)(\frac{18b+33}{10}) + (6.4-6-12.6)b + 43 - 4.6 + 4]t\end{aligned}$$

which gives

$$b = \frac{3}{2} \mp \frac{5}{66}\sqrt{627}.$$

There is no integer solution for a *and* b*, so there exists no Ulrich line bundle.*

5. *(This case is [21, No.25 in Table 12.3].) Since C is an elliptic curve of intersection of two quadrics, $d = 4$ and $g = 1$. Then $k = H^3 = 32$. Then equate the coefficients of t^2:*

$$\frac{32.6}{6}t^2 = \frac{1}{6}[(48 - 3.4)a + (6.1 - 12.4 - 6)b - 12.4 + 3.1 + 93]t^2$$

which gives

$$a = \frac{4b + 12}{3}.$$

Next, equate the coefficients of t and use the above relation between a and b to get

$$\begin{aligned} \frac{32.11}{6}t &= \frac{1}{6}[12(\frac{4b+12}{3})^2 + (6.1 - 6)b^2 - 6.4(\frac{4b+12}{3})b \\ &\quad +(48 - 3.4)(\frac{4b+12}{3}) + (6.1 - 6 - 12.4)b + 43 - 4.4 + 1]t \end{aligned}$$

which gives

$$b = \frac{3}{2} \mp \frac{3}{4}\sqrt{6}.$$

There is no integer solution for a and b, so there exists no Ulrich line bundle.

6. *(This case is [21, No.27 in Table 12.3].) Since C is a twisted cubic, $d = 3$ and $g = 0$. Then $k = H^3 = 38$. Then equate the coefficients of t^2:*

$$\frac{38.6}{6}t^2 = \frac{1}{6}[(48 - 3.3)a + (6.0 - 12.3 - 6)b - 12.3 + 3.0 + 93]t^2$$

which gives

$$b = \frac{13a - 57}{14}.$$

Next, equate the coefficients of t and use the above relation between a and b to get

$$\begin{aligned} \frac{38.11}{6}t &= \frac{1}{6}[12a^2 + (6.0 - 6)(\frac{13a-57}{14})^2 - 6.3a(\frac{13a-57}{14}) \\ &\quad +(48 - 3.3)a + (6.0 - 6 - 12.3)(\frac{13a-57}{14}) + 43 - 4.3 + 0]t \end{aligned}$$

which gives

$$a = 6 \mp \frac{21}{323}\sqrt{969}.$$

There is no integer solution for a and b, so there exists no Ulrich line bundle.

7. *(This case is [21, No.28 in Table 12.3].) Since C is a plane cubic, $d = 3$ and $g = 1$ by the degree genus formula. Then $k = H^3 = 40$. Then equate the coefficients of t^2:*

$$\frac{40.6}{6}t^2 = \frac{1}{6}[(48 - 3.3)a + (6.1 - 12.3 - 6)b - 12.3 + 3.1 + 93]t^2$$

which gives

$$b = \frac{13a - 60}{12}.$$

Next, equate the coefficients of t and use the above relation between a and b to get

$$\begin{aligned}\frac{40.11}{6}t &= \frac{1}{6}[12a^2 + (6.1 - 6)(\frac{13a - 60}{12})^2 - 6.3a(\frac{13a - 60}{12}) \\ &\quad +(48 - 3.3)a + (6.1 - 6 - 12.3)(\frac{13a - 60}{12}) + 43 - 4.3 + 1]t\end{aligned}$$

which gives

$$a = 6 \mp \frac{2}{5}\sqrt{35}.$$

There is no integer solution for a and b, so there exists no Ulrich line bundle.

8. *(This case is [21, No.30 in Table 12.3].) Since C is a conic, $d = 2$ and $g = 0$. Then $k = H^3 = 46$. Then equate the coefficients of t^2:*

$$\frac{46.6}{6}t^2 = \frac{1}{6}[(48 - 3.2)a + (6.0 - 12.2 - 6)b - 12.2 + 3.0 + 93]t^2$$

which gives

$$b = \frac{14a - 69}{10}.$$

Next, equate the coefficients of t and use the above relation between a and b to get

$$\begin{aligned}\frac{46.11}{6}t &= \frac{1}{6}[12a^2 + (6.0 - 6)(\frac{14a - 69}{10})^2 - 6.2a(\frac{14a - 69}{10}) \\ &\quad +(48 - 3.2)a + (6.0 - 6 - 12.2)(\frac{14a - 69}{10}) + 43 - 4.2 + 0]t\end{aligned}$$

which gives

$$a = 6 \mp \frac{5}{138}\sqrt{2139}.$$

There is no integer solution for a and b, so there exists no Ulrich line bundle.

9. *(This case is [21, No.33 in Table 12.3].) Since C is a line, $d = 1$ and $g = 0$. Then $k = H^3 = 54$. Then equate the coefficients of t^2:*

$$\frac{54.6}{6}t^2 = \frac{1}{6}[(48-3.1)a + (6.0-12.1-6)b - 12.1 + 3.0 + 93]t^2$$

which gives

$$b = \frac{5a-27}{2}.$$

Next, equate the coefficients of t and use the above relation between a and b to get

$$\begin{aligned}\frac{54.11}{6}t &= \frac{1}{6}[12a^2 + (6.0-6)(\frac{5a-27}{2})^2 - 6.1a(\frac{5a-27}{2}) + (48-3.1)a \\ &\quad +(6.0-6-12.1)(\frac{5a-27}{2}) + 43 - 4.1 + 0]t\end{aligned}$$

which gives

$$a = 6 \mp \frac{1}{9}\sqrt{105}.$$

There is no integer solution for a and b, so there exists no Ulrich line bundle.

Theorem 2.0.21 *Let $\widetilde{Y}$ be one of the following Fano 3-folds:*

1. *the blow-up of Q along the intersection of two divisors from $|\mathcal{O}_Q(2)|$,*
2. *the blow-up of Q along a curve of degree 6 and genus 2,*
3. *the blow-up of Q along an elliptic curve of degree 5,*
4. *the blow-up of Q along a twisted quartic,*
5. *the blow-up of Q along an intersection of two divisors from $|\mathcal{O}_Q(1)|$ and $|\mathcal{O}_Q(2)|$,*
6. *the blow-up of Q along a conic,*
7. *the blow-up of Q along a line.*

Then Ulrich line bundles can not exist on non of them.

Proof 2.0.22 *Let* $D = a\widetilde{h} - be$ *be a divisor class on* $\widetilde{Y}$. *We can compute Hilbert polynomial of* $\mathcal{O}_{\widetilde{Y}}(D)$ *by Theorem 1.2.5. By Proposition 2.0.17, this must be equal to* $deg(\widetilde{Y})\binom{t+3}{3}$. *We will equate the coefficients of these two polynomials and try to find integer solutions for* a *and* b *in each case separately.*

1. *(This case is [21, No.7 in Table 12.3].) It is given that* C *is obtained by intersection of two divisors from* $|\mathcal{O}_Q(2)|$; *so* $d = 8$. *By the adjunction formula* $g = 5$. *Then* $k = H^3 = 14$. *Then equate the coefficients of* t^2*:*

$$\frac{14.6}{6}t^2 = \frac{1}{24}[(216 - 12.8)a + (24.5 - 36.8 - 24)b - 36.8 + 12.5 + 312]t^2$$

which gives

$$b = \frac{10a - 21}{16}.$$

Next, equate the coefficients of t *and use the above relation between* a *and* b *to get*

$$\begin{aligned}\frac{14.11}{6}t &= \frac{1}{24}[72a^2 + (24.5 - 24)(\frac{10a - 21}{16})^2 - 24.8a(\frac{10a - 21}{16}) \\ &\quad +(216 - 12.8)a + (24.5 - 24 - 36.8)(\frac{10a - 21}{16}) \\ &\quad +152 - 6.8 + 4.5]t\end{aligned}$$

which gives

$$a = \frac{9}{2} \mp \frac{2}{7}\sqrt{161}.$$

There is no integer solution for a *and* b, *so there exists no Ulrich line bundle.*

2. *(This case is [21, No.13 in Table 12.3].) It is given that* $d = 6$ *and* $g = 2$. *Then* $k = H^3 = 20$. *Then equate the coefficients of* t^2*:*

$$\frac{20.6}{6}t^2 = \frac{1}{24}[(216 - 12.6)a + (24.2 - 36.6 - 24)b - 36.6 + 12.2 + 312]t^2$$

which gives

$$b = \frac{6a - 15}{8}.$$

Next, equate the coefficients of t *and use the above relation between* a *and* b *to get*

$$\frac{20.11}{6}t = \frac{1}{24}[72a^2 + (24.2 - 24)(\frac{6a - 15}{8})^2 - 24.6a(\frac{6a - 15}{8})$$

$$+(216-12.6)a+(24.2-24-36.6)(\frac{6a-15}{8})$$
$$+152-6.6+4.2]t$$

which gives

$$a=\frac{9}{2}\mp\frac{4}{5}\sqrt{10}.$$

There is no integer solution for a and b, so there exists no Ulrich line bundle.

3. *(This case is [21, No.17 in Table 12.3].) It is given that $d=5$ and $g=1$. Then $k=H^3=24$. Then equate the coefficients of t^2:*

$$\frac{24.6}{6}t^2=\frac{1}{24}[(216-12.5)a+(24.1-36.5-24)b-36.5+12.1+312]t^2$$

which gives

$$b=\frac{13a-36}{15}.$$

Next, equate the coefficients of t and use the above relation between a and b to get

$$\begin{aligned}\frac{24.11}{6}t &= \frac{1}{24}[72a^2+(24.1-24)(\frac{13a-36}{15})^2-24.5a(\frac{13a-36}{15})\\ &\quad +(216-12.5)a+(24.1-24-36.5)(\frac{13a-36}{15})\\ &\quad +152-6.5+4.1]t\end{aligned}$$

which gives

$$a=\frac{9}{2}\mp\frac{5}{4}\sqrt{3}.$$

There is no integer solution for a and b, so there exists no Ulrich line bundle.

4. *(This case is [21, No.21 in Table 12.3].) Since C is twisted quartic, $d=4$ and $g=0$. Then $k=H^3=28$. Then equate the coefficients of t^2:*

$$\frac{28.6}{6}t^2=\frac{1}{24}[(216-12.4)a+(24.0-36.4-24)b-36.4+12.0+312]t^2$$

which gives

$$b=a-3.$$

Next, equate the coefficients of t and use the above relation between a and b to get

$$\frac{28.11}{6}t = \frac{1}{24}[72a^2+(24.0-24)(a-3)^2-24.4a(a-3)+(216-12.4)a$$

$$+(24.0-24-36.4)(a-3)+152-6.4+4.0]t$$

which gives

$$a=\frac{9}{2}\mp\frac{1}{2}\sqrt{13}.$$

There is no integer solution for a and b, so there exists no Ulrich line bundle.

5. *(This case is [21, No.23(a) in Table 12.3].) It is given that C is obtained by intersection of two divisors from $|\mathcal{O}_Q(1)|$ and $|\mathcal{O}_Q(2)|$; so $d=4$. By adjunction formula $g=1$. Then $k=H^3=30$. Then equate the coefficients of t^2:*

$$\frac{30.6}{6}t^2=\frac{1}{24}[(216-12.4)a+(24.1-36.4-24)b-36.4+12.1+312]t^2$$

which gives

$$b=\frac{14a-45}{12}.$$

Next, equate the coefficients of t and use the above relation between a and b to get

$$\begin{aligned}\frac{30.11}{6}t &= \frac{1}{24}[72a^2+(24.1-24)(\frac{14a-45}{12})^2-24.4a(\frac{14a-45}{12})\\ &\quad+(216-12.4)a+(24.1-24-36.4)(\frac{14a-45}{12})\\ &\quad+152-6.4+4.1]t\end{aligned}$$

which gives

$$a=\frac{9}{2}\mp\frac{9}{10}\sqrt{5}.$$

There is no integer solution for a and b, so there exists no Ulrich line bundle.

6. *(This case is [21, No.29 in Table 12.3].) Since C is a conic, $d=2$ and $g=0$. Then $k=H^3=40$. Then equate the coefficients of t^2:*

$$\frac{40.6}{6}t^2=\frac{1}{24}[(216-12.2)a+(24.0-36.2-24)b-36.2+12.0+312]t^2$$

which gives

$$b=\frac{4a-15}{2}.$$

Next, equate the coefficients of t and use the above relation between a and b to get

$$\frac{40.11}{6}t = \frac{1}{24}[72a^2+(24.0-24)(\frac{4a-15}{2})^2-24.2a(\frac{4a-15}{2})$$

$$+(216-12.2)a+(24.0-24-36.2)(\frac{4a-15}{2})$$
$$+152-6.2+4.0]t$$

which gives

$$a=\frac{9}{2}\mp\frac{1}{2}\sqrt{6}.$$

There is no integer solution for a and b, so there exists no Ulrich line bundle.

7. *(This case is [21, No.31 in Table 12.3].) Since C is a line, $d=1$ and $g=0$. Then $k=H^3=46$. Then equate the coefficients of t^2:*

$$\frac{46.6}{6}t^2=\frac{1}{24}[(216-12.1)a+(24.0-36.1-24)b-36.1+12.0+312]t^2$$

which gives

$$b=\frac{17a-69}{5}.$$

Next, equate the coefficients of t and use the above relation between a and b to get

$$\begin{aligned}\frac{46.11}{6}t &= \frac{1}{24}[72a^2+(24.0-24)(\frac{17a-69}{5})^2-24.1a(\frac{17a-69}{5})\\ &+(216-12.1)a+(24.0-24-36.1)(\frac{17a-69}{5})\\ &+152-6.1+4.0]t\end{aligned}$$

which gives

$$a=\frac{9}{2}\mp\frac{10}{299}\sqrt{598}.$$

There is no integer solution for a and b, so there exists no Ulrich line bundle.

Theorem 2.0.23 *Let $\widetilde{Y}$ be one of the following Fano 3-folds:*

1. *the blow-up of V_3 along a plane cubic,*
2. *the blow-up of V_3 along a line.*

Then Ulrich line bundles can not exist on non of them.

Proof 2.0.24 *Let $D=a\widetilde{h}-be$ be a divisor class on $\widetilde{Y}$. We can compute Hilbert polynomial of $\mathcal{O}_{\widetilde{Y}}(D)$ by Theorem 1.2.7. By Proposition 2.0.17, this must be equal to $deg(\widetilde{Y})\binom{t+3}{3}$. We will equate the coefficients of these two polynomials and try to find integer solutions for a and b in each case separately.*

1. *(This case is [21, No.5 in Table 12.3].) Since* C *is a plane cubic,* $d = 3$ *and* $g = 1$ *by the adjunction formula. Then* $k = H^3 = 12$*. Then equate the coefficients of* t^2*:*

$$\frac{12.6}{6}t^2 = \frac{1}{12}[(72 - 6.3)a + (12.1 - 12.3 - 12)b - 12.3 + 6.1 + 66]t^2$$

which gives

$$b = \frac{3a - 6}{2}.$$

Next, equate the coefficients of t *and use the above relation between* a *and* b *to get*

$$\begin{aligned} \frac{12.11}{6}t &= \frac{1}{12}[36a^2 + (12.1 - 12)(\frac{3a-6}{2})^2 - 12.3a(\frac{3a-6}{2}) \\ &\quad +(72 - 6.3)a + (12.1 - 12 - 12.3)(\frac{3a-6}{2}) + 46 + 2.1]t \end{aligned}$$

which gives

$$a = 3 \mp \sqrt{3}.$$

There is no integer solution for a *and* b*, so there exists no Ulrich line bundle.*

2. *(This case is [21, No.11 in Table 12.3].) Since* C *is a line,* $d = 1$ *and* $g = 0$*. Then* $k = H^3 = 18$*. Then equate the coefficients of* t^2*:*

$$\frac{18.6}{6}t^2 = \frac{1}{12}[(72 - 6.1)a + (12.0 - 12.1 - 12)b - 12.1 + 6.0 + 66]t^2$$

which gives

$$b = \frac{11a - 27}{4}.$$

Next, equate the coefficients of t *and use the above relation between* a *and* b *to get*

$$\begin{aligned} \frac{18.11}{6}t &= \frac{1}{12}[36a^2 + (12.0 - 12)(\frac{11a-27}{4})^2 - 12.1a(\frac{11a-27}{4}) \\ &\quad +(72 - 6.1)a + (12.0 - 12 - 12.1)(\frac{11a-27}{4}) + 46 + 2.0]t \end{aligned}$$

which gives

$$a = 3 \mp \frac{2}{117}\sqrt{2145}.$$

There is no integer solution for a *and* b*, so there exists no Ulrich line bundle.*

Theorem 2.0.25 *Let $\widetilde{Y}$ be one of the following Fano 3-folds:*

1. *the blow-up of V_4 along an elliptic curve which is an intersection of two hyperplane sections,*
2. *the blow-up of V_4 along a conic,*
3. *the blow-up of V_4 along a line.*

Then Ulrich line bundles can not exist on non of them.

Proof 2.0.26 *Let $D = a\widetilde{h} - be$ be a divisor class on $\widetilde{Y}$. We can compute Hilbert polynomial of $\mathcal{O}_{\widetilde{Y}}(D)$ by Theorem 1.2.9. By Proposition 2.0.17, this must be equal to $deg(\widetilde{Y})\binom{t+3}{3}$. We will equate the coefficients of these two polynomials and try to find integer solutions for a and b in each case separately.*

1. *(This case is [21, No.10 in Table 12.3].) Since C is an elliptic curve which is an intersection of two hyperplane sections, $d = 4$ and $g = 1$. Then $k = H^3 = 16$. Then equate the coefficients of t^2:*

$$\frac{16.6}{6}t^2 = \frac{1}{12}[(96 - 6.4)a + (12.1 - 12.4 - 12)b - 12.4 + 6.1 + 90]t^2$$

which gives

$$b = \frac{3a - 6}{2}.$$

Next, equate the coefficients of t and use the above relation between a and b to get

$$\begin{aligned}\frac{16.11}{6}t &= \frac{1}{12}[48a^2 + (12.1 - 12)(\frac{3a-6}{2})^2 - 12.4a(\frac{3a-6}{2}) \\ &\quad +(96 - 6.4)a + (12.1 - 12 - 12.4)(\frac{3a-6}{2}) \\ &\quad +54 + 2.1 + 2.4]t\end{aligned}$$

which gives

$$a = 3 \mp \sqrt{3}.$$

There is no integer solution for a and b, so there exists no Ulrich line bundle.

2. *(This case is [21, No.16 in Table 12.3].) Since C is a conic, $d = 2$ and $g = 0$. Then $k = H^3 = 22$. Then equate the coefficients of t^2:*

$$\frac{22.6}{6}t^2 = \frac{1}{12}[(96 - 6.2)a + (12.0 - 12.2 - 12)b - 12.2 + 6.0 + 90]t^2$$

which gives

$$b = \frac{14a - 33}{6}.$$

Next, equate the coefficients of t and use the above relation between a and b to get

$$\begin{aligned}\frac{22.11}{6}t &= \frac{1}{12}[48a^2 + (12.0 - 12)(\frac{14a - 33}{6})^2 - 12.2a(\frac{14a - 33}{6}) \\ &\quad +(96 - 6.2)a + (12.0 - 12 - 12.2)(\frac{14a - 33}{6}) \\ &\quad +54 + 2.0 + 2.2]t\end{aligned}$$

which gives

$$a = 3 \mp \frac{3}{110}\sqrt{1265}.$$

There is no integer solution for a and b, so there exists no Ulrich line bundle.

3. *(This case is [21, No.19 in Table 12.3].) Since C is a line, $d = 1$ and $g = 0$. Then $k = H^3 = 26$. Then equate the coefficients of t^2:*

$$\frac{26.6}{6}t^2 = \frac{1}{12}[(96 - 6.1)a + (12.0 - 12.1 - 12)b - 12.1 + 6.0 + 90]t^2$$

which gives

$$b = \frac{15a - 39}{4}.$$

Next, equate the coefficients of t and use the above relation between a and b to get

$$\begin{aligned}\frac{26.11}{6}t &= \frac{1}{12}[48a^2 + (12.0 - 12)(\frac{15a - 39}{4})^2 - 12.1a(\frac{15a - 39}{4}) \\ &\quad +(96 - 6.1)a + (12.0 - 12 - 12.1)(\frac{15a - 39}{4}) \\ &\quad +54 + 2.0 + 2.1]t\end{aligned}$$

which gives

$$a = 3 \mp \frac{2}{221}\sqrt{5083}.$$

There is no integer solution for a and b, so there exists no Ulrich line bundle.

2.1 Ulrich Line Bundles on Y

We recall that Y is the Fano 3-fold which is obtained as the blow-up of $\mathbb{P}^3$ along a curve C of degree 6 and genus 3.

We also recall the following commutative diagram as in 'Preliminaries' section:

$$\begin{array}{ccc} E & \overset{j}{\hookrightarrow} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ C & \overset{i}{\hookrightarrow} & \mathbb{P}^3 \end{array}$$

Proposition 2.1.1 *The canonical map $\mathcal{O}_C \to g_*\mathcal{O}_E$ is an isomorphism.*

Proof 2.1.2 *Note that $g : E \to C$ is a ruled surface. Then the result follows from [16, Lemma 2.1 of Chapter V].*

Corollary 2.1.3 $f_*(\mathcal{O}_{\widetilde{\mathbb{P}^3}}(-mE)) = I_C^m$ *and* $R^i f_*\mathcal{O}_{\widetilde{\mathbb{P}^3}}(-me) = 0$ *for* $m \geq 0$ *and* $i > 0$.

Proof 2.1.4 *See [22, Lemma 4.3.16].*

Lemma 2.1.5 $f_*\mathcal{O}_E(mE) = 0$ *for* $m > 0$.

Proof 2.1.6 *Note that $g : E \to C$ is a ruled surface. So, by [16, Proposition 8.20 of Chapter II], we have*

$$\begin{aligned} w_E \cong w_Y \otimes \mathcal{O}_Y(E) \otimes \mathcal{O}_E \quad &\Rightarrow \quad w_E \cong \mathcal{O}_E(E)(-1) \\ &\Rightarrow \quad \mathcal{O}_E(E) \cong w_E(1) \\ &\Rightarrow \quad \mathcal{O}_E(E) \cong \mathcal{O}_E(K_E + H_E). \end{aligned}$$

Then we have

$$\begin{aligned} \mathcal{O}_E(mE) &\cong \mathcal{O}_Y(mE) \otimes \mathcal{O}_E \\ &\cong \mathcal{O}_Y(E)^{\otimes m} \otimes \mathcal{O}_E \\ &\cong [\mathcal{O}_Y(E) \otimes \mathcal{O}_E]^{\otimes m} \\ &\cong \mathcal{O}_E(E)^{\otimes m} \end{aligned}$$

$$\begin{aligned} &\cong \mathcal{O}_E(K_E + H_E)^{\otimes m} \\ &\cong \mathcal{O}_E(m(K_E + H_E)) \end{aligned}$$

where $D = mK_E + mH_Y$.
Also, by [17, Lemma 2.10 in Chapter V], we know that

$$K_E \cong -2C_0 + D_C \cdot F$$

where C_0 *is a section of the map* g, F *is the fiber of* g *and* D_C *is a divisor class on* C. *Then*

$$\begin{aligned} D \cdot F &= (-2mC_0 + mD_C \cdot F + mH_E) \cdot F \\ &= -2mC_0 \cdot F + mD_C \cdot F^2 + mH_E \cdot F \\ &= -2m + 0 + m \\ &= -m. \end{aligned}$$

Hence $D \cdot F$ *is negative. So, following the proof of [17, Lemma 2.1 in Chapter V], one can easily show that*

$$f_*\mathcal{O}_E(D) = f_*\mathcal{O}_E(mE) = 0.$$

Proposition 2.1.7 $f_*(\mathcal{O}_{\widetilde{\mathbb{P}}^3}(mE)) = \mathcal{O}_{\mathbb{P}^3}$ *for* $m > 0$.

Proof 2.1.8 *We have the exact sequence*

$$0 \to \mathcal{O}_{\widetilde{\mathbb{P}}^3}(-E) \to \mathcal{O}_{\widetilde{\mathbb{P}}^3} \to \mathcal{O}_E \to 0.$$

Now twist this exact sequence by E *and get*

$$0 \to \mathcal{O}_{\widetilde{\mathbb{P}}^3} \to \mathcal{O}_{\widetilde{\mathbb{P}}^3}(E) \to \mathcal{O}_E(E) \to 0. \qquad (*)$$

Then consider the long exact sequence

$$0 \to f_*\mathcal{O}_{\widetilde{\mathbb{P}}^3} \to f_*\mathcal{O}_{\widetilde{\mathbb{P}}^3}(E) \to f_*\mathcal{O}_E(E) \to \cdots.$$

By Lemma 2.1.5, $f_*\mathcal{O}_E(E) = 0$. *So*

$$f_*\mathcal{O}_{\widetilde{\mathbb{P}}^3}(E) \simeq f_*\mathcal{O}_{\widetilde{\mathbb{P}}^3} \simeq \mathcal{O}_{P^3}.$$

Similarly, now twist the exact sequence (*) by $2E$ and get

$$0 \to \mathcal{O}_{\widetilde{\mathbb{P}}^3}(E) \to \mathcal{O}_{\widetilde{\mathbb{P}}^3}(2E) \to \mathcal{O}_E(2E) \to 0.$$

Then consider the exact sequence

$$0 \to f_*\mathcal{O}_{\widetilde{\mathbb{P}}^3}(E) \to f_*\mathcal{O}_{\widetilde{\mathbb{P}}^3}(2E) \to f_*\mathcal{O}_E(2E) \to \cdots.$$

Again by Lemma 2.1.5 $f_*\mathcal{O}_E(2E) = 0$. Therefore

$$f_*\mathcal{O}_{\widetilde{\mathbb{P}}^3}(2E) \simeq f_*\mathcal{O}_{\widetilde{\mathbb{P}}^3}(E) \simeq \mathcal{O}_{\mathbb{P}^3}.$$

Hence, by induction on m, we have $f_*(\mathcal{O}_{\widetilde{\mathbb{P}}^3}(mE)) = \mathcal{O}_{\mathbb{P}^3}$ for $m > 0$.

Lemma 2.1.9 *Let $\mathcal{E}$ be an Ulrich bundle of rank r on $\widetilde{Y}$. Then $\mathcal{E}^\vee(3)$ is also Ulrich.*

Proof 2.1.10 *We use Proposition 2.0.17.*
First,

$$\begin{aligned}
(*)\ \ H^i(\widetilde{Y}, \mathcal{E}^\vee(3)(t)) &= H^i(\widetilde{Y}, \mathcal{E}^\vee(3+t)) \\
&= H^{3-i}(\widetilde{Y}, \mathcal{E}(-3-t) \otimes K_{\widetilde{Y}})^\vee \ (Serre\ Duality) \\
&= H^{3-i}(\widetilde{Y}, \mathcal{E}(-3-t) \otimes (-H))^\vee \ (\widetilde{Y}\ is\ Fano) \\
&= H^{3-i}(\widetilde{Y}, \mathcal{E}(-4-t))^\vee.
\end{aligned}$$

But we know that $\mathcal{E}$ is Ulrich, so it is ACM by Proposition 2.0.17. Then the middle cohomologies of all twists of $\mathcal{E}$ vanish; so $H^{3-i}(\widetilde{Y}, \mathcal{E}(-4-t))$ vanishes for $i = 1, 2$ and $t \in \mathbb{Z}$.
Hence $H^i(\widetilde{Y}, \mathcal{E}^\vee(3)(t)) = 0$; that is, $\mathcal{E}^\vee(3)$ is ACM.
Second,

$$\chi(\widetilde{Y}, \mathcal{E}^\vee(3)(t)) = \sum_{i=0}^{3}(-1)^i h^i(\widetilde{Y}, \mathcal{E}^\vee(3)(t))$$

$$
\begin{aligned}
&= \sum_{i=0}^{3}(-1)^i h^{3-i}(\widetilde{Y}, \mathcal{E}(-4-t)) \quad (by\ (*)) \\
&= -cr\binom{-4-t+3}{3} \\
&= -cr\frac{(-t-1)(-t-2)(-t-3)}{6} \\
&= cr\frac{(t+1)(t+2)(t+3)}{6} \\
&= cr\binom{t+3}{3}
\end{aligned}
$$

Therefore $\mathcal{E}^\vee(3)$ is Ulrich by Proposition 2.0.17.

Lemma 2.1.11 *Let C be a curve cut out scheme-theoretically in $\mathbb{P}^3$ by cubic hypersurfaces. Then*

$$H^i(\mathbb{P}^3, I^a_{C|\mathbb{P}^3}(k)) = 0 \textit{ for } i \geq 1 \textit{ provided } k \geq 3a.$$

Proof 2.1.12 *This is a special case of [3, Proposition 1].*

Lemma 2.1.13 *If C is an ACM curve in $\mathbb{P}^3$ with $d = 6$ and $g = 3$, then its ideal sheaf I_C in $\mathbb{P}^3$ has the minimal free resolution:*

$$0 \to \mathcal{O}_{\mathbb{P}^3}^{\oplus 3}(-4) \to \mathcal{O}_{\mathbb{P}^3}^{\oplus 4}(-3) \to I_C \to 0.$$

Proof 2.1.14 *Since C is ACM, by [12, p.2], it has a minimal free resolution of the form:*

$$0 \to \bigoplus_{j=1}^{k-1} \mathcal{O}_{\mathbb{P}^3}(-n_j) \to \bigoplus_{l=1}^{k} \mathcal{O}_{\mathbb{P}^3}(-m_l) \to I_C \to 0.$$

Since $I_C(3)$ is generated by global sections [17, Ex. 8.7(c)], $m_l = 3$ for all l and we have:

$$(*) \qquad 0 \to \bigoplus_{j=1}^{k-1} \mathcal{O}_{\mathbb{P}^3}(-n_j) \to \bigoplus_{l=1}^{k} \mathcal{O}_{\mathbb{P}^3}(-3) \to I_C \to 0.$$

We know that $h^0(I_C(3)) = 4$ and $h^i(I_C(3-i)) = 0$ for all $i > 0$ by [17, Ex. 8.7(c)].
Since $h^i(I_C(3-i)) = 0$ for all $i > 0$, we have $h^2(I_C(1)) = 0$.
Then $\sum_{j=1}^{k-1} h^3(-n_j+1) = \sum_{j=1}^{k-1} h^0(n_j-1-4) = 0$. Then $n_j \leq 4$. But, since (*) is a minimal free resolution, we have $n_j \geq 4$. So $n_j = 4$. Since $h^0(I_C(3)) = 4$, we have $k = 4$.

Proposition 2.1.15 (Yusuf Mustopa, written in private communication) *If C is an ACM space curve with $d = 6$ and $g = 3$, then $H^i(I_C^2(5)) = 0$ for all $i > 0$.*

Proof 2.1.16 *Twisting the sequence*

$$0 \to I_C \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_C \to 0$$

by $I_C(5)$ yields the long exact sequence

$$0 \to \mathrm{Tor}_1^{\mathcal{O}_{\mathbb{P}^3}}(I_C(5), \mathcal{O}_C) \to I_C \otimes I_C(5) \to I_C(5) \to \mathcal{N}^*_{C|\mathbb{P}^3}(5) \to 0$$

This can be broken into two short exact sequences, one of which is

$$0 \to \mathrm{Tor}_1^{\mathcal{O}_{\mathbb{P}^3}}(I_C(5), \mathcal{O}_C) \to I_C \otimes I_C(5) \to I_C^2(5) \to 0.$$

Since $\mathrm{Tor}_1^{\mathcal{O}_{\mathbb{P}^3}}(I_C(5), \mathcal{O}_C)$ has at most 1-dimensional support, we have $H^i(\mathrm{Tor}_1^{\mathcal{O}_{\mathbb{P}^3}}(I_C(5), \mathcal{O}_C)) = 0$ for all $i > 1$. It then suffices to show the vanishing of $H^i(I_C \otimes I_C(5))$ for all $i > 0$.

We know, by Lemma 2.1.13, that I_C has a minimal free resolution of the form

$$0 \to \mathcal{O}_{\mathbb{P}^3}^{\oplus 3}(-4) \to \mathcal{O}_{\mathbb{P}^3}^{\oplus 4}(-3) \to I_C \to 0.$$

As before, we consider the twist by $I_C(5)$. Then we have a long exact sequence

$$0 \to \mathrm{Tor}_1^{\mathcal{O}_{\mathbb{P}^3}}(I_C, I_C(5)) \to I_C(1)^{\oplus 3} \to I_C(2)^{\oplus 4} \to I_C \otimes I_C(5) \to 0.$$

Given that $\mathrm{Tor}_1^{\mathcal{O}_{\mathbb{P}^3}}(I_C, I_C(5))$ has at most 1-dimensional support and is a subsheaf of the torsion-free sheaf $I_C(1)^{\oplus 3}$, it is equal to 0; so we have

$$0 \to I_C(1)^{\oplus 3} \to I_C(2)^{\oplus 4} \to I_C \otimes I_C(5) \to 0$$

But, we know that, by Lemma 2.1.13, I_C has a minimal free resolution of the form

$$0 \to \mathcal{O}_{\mathbb{P}^3}^{\oplus 3}(-4) \to \mathcal{O}_{\mathbb{P}^3}^{\oplus 4}(-3) \to I_C \to 0.$$

So $H^i(I_C(k)) = 0$ for all $i > 0$ and $k > 0$. Then $H^i(I_C \otimes I_C(5)) = 0$ for all $i > 0$; so the result follows.

Theorem 2.1.17 *Suppose that C is ACM. Then there are only two Ulrich line bundles L_1 and L_2, and they correspond to divisors $D_1 = 9\widetilde{h} - 3e$ and $D_2 = 3\widetilde{h}$ on Y.*

Proof 2.1.18 *We will use Proposition 2.0.17 to show that L_1 and L_2 are Ulrich line bundles. In Theorem 2.0.19, we showed that L_1 and L_2 satisfy the Hilbert polynomial condition. So, it remains to show that L_1 and L_2 are ACM; i.e, to show that $H^1(Y, L_1(t)) = H^2(Y, L_1(t)) = 0$ and $H^1(Y, L_2(t)) = H^2(Y, L_2(t)) = 0$ for all $t \in \mathbb{Z}$.*

Consider L_1 first.

- $t \geq 0$:

Then

$$L_1(t) = \mathcal{O}_Y(9\widetilde{h} - 3e + t(4\widetilde{h} - e)) = \mathcal{O}_Y((4t + 9)\widetilde{h} + (-t - 3)e).$$

Since $-t - 3 < 0$, by the projection formula and Corollary 2.1.3, we have

$$\begin{aligned} f_* L_1(t) &= f_* \mathcal{O}_Y((4t + 9)\widetilde{h} + (-t - 3)e) \\ &= \mathcal{O}_{\mathbb{P}^3}(4t + 9) \otimes f_* \mathcal{O}_Y((-t - 3)e) \\ &= I_C^{t+3} \otimes \mathcal{O}_{\mathbb{P}^3}(4t + 9) \\ &= I_C^{t+3}(4t + 9). \end{aligned}$$

So $H^1(\mathbb{P}^3, f_* L_1(t)) = H^1(\mathbb{P}^3, I_C^{t+3}(4t + 9))$ and it is 0 by Lemma 2.1.11, since $4t + 9 \geq 3(t + 3)$.
Now we consider $H^0(\mathbb{P}^3, R^1 f_* L_1(t))$. By projection formula, we have

$$H^0(\mathbb{P}^3, R^1 f_* L_1(t)) = H^0(\mathbb{P}^3, R^1 f_* \mathcal{O}_{\mathbb{P}^3}((-t - 3)e) \otimes \mathcal{O}_{\mathbb{P}^3}(4t + 9))$$

But $R^1 f_* \mathcal{O}_{\mathbb{P}^3}((-t - 3)e) = 0$ by Corollary 2.1.3, since $-t - 3 \leq 0$.
Since $H^p(\mathbb{P}^3, R^q f_* L_1(t)) \Longrightarrow H^{p+q}(Y, L_1(t))$ by the Leray spectral sequence and we showed that $H^1(\mathbb{P}^3, f_* L_1(t)) = H^0(\mathbb{P}^3, R^1 f_* L_1(t)) = 0$, we have

$$H^1(Y, L_1(t)) = 0.$$

Now consider $H^0(\mathbb{P}^3, R^2 f_* L_1(t))$ and $H^1(\mathbb{P}^3, R^1 f_* L_1(t))$. Note that they are 0 by Corollary 2.1.3.

Also $H^2(\mathbb{P}^3, f_*L_1(t)) = H^2(\mathbb{P}^3, I_C^t(4t+3)) = 0$ *again by Lemma 2.1.11.*
Since $H^p(\mathbb{P}^3, R^q f_* L_1(t)) \Longrightarrow H^{p+q}(Y, L_1(t))$ *by the Leray spectral sequence and we showed that* $H^0(\mathbb{P}^3, R^2 f_* L_1(t)) = H^1(\mathbb{P}^3, R^1 f_* L_1(t)) = H^2(\mathbb{P}^3, f_* L_1(t)) = 0$*,*

$$H^2(Y, L_1(t)) = 0.$$

- $t < -4$*:*

Then

$$\begin{aligned} H^1(Y, L_1(t)) &= H^2(Y, L_1^\vee(-t) \otimes K_Y)^\vee \\ &= H^2(Y, \mathcal{O}_Y((-4t-9)\widetilde{h} + (t+3)e) \otimes \mathcal{O}_Y(-4\widetilde{h} + e))^\vee \\ &= H^2(Y, \mathcal{O}_Y((-4t-13)\widetilde{h} + (t+4)e))^\vee. \end{aligned}$$

Similarly,

$$H^2(Y, L_1(t)) = H^1(Y, \mathcal{O}_Y((-4t-13)\widetilde{h} + (t+4)e))^\vee.$$

So, if $H^i(Y, \mathcal{O}_Y((-4t-13)\widetilde{h} + (t+4)e))$ *for* $i = 1, 2$ *vanishes, the result will follow.*
Since $t + 4 < 0$*, by the projection formula and Corollary 2.1.3, we have*

$$\begin{aligned} f_*\mathcal{O}_Y((-4t-13)\widetilde{h} + (t+4)e)) &= \mathcal{O}_{\mathbb{P}^3}(-4t-13) \otimes f_*\mathcal{O}_Y((t+4)e) \\ &= I_C^{-t-4} \otimes \mathcal{O}_{\mathbb{P}^3}(-4t-13) \\ &= I_C^{-t-4}(-4t-13). \end{aligned}$$

So $H^1(\mathbb{P}^3, f_*\mathcal{O}_Y((-4t-13)\widetilde{h} + (t+4)e)) = H^1(\mathbb{P}^3, I_C^{-t-4}(-4t-13))$ *and it is* 0 *by Lemma 2.1.11, since* $-4t-13 \geq 3(-t-4)$*.*
Now consider $H^0(\mathbb{P}^3, R^1 f_*\mathcal{O}_Y((-4t-13)\widetilde{h} + (t+4)e))$*. By the projection formula, we have*

$$H^0(\mathbb{P}^3, R^1 f_*\mathcal{O}_Y((-4t-13)\widetilde{h} + (t+4)e)) = \\ H^0(\mathbb{P}^3, R^1 f_*\mathcal{O}_{\mathbb{P}^3}((t+4)e) \otimes \mathcal{O}_{\mathbb{P}^3}(-4t-13)).$$

But $R^1 f_*\mathcal{O}_{\mathbb{P}^3}((t+4)e) = 0$ *by Corollary 2.1.3, since* $t + 4 \leq 0$*.*
Since $H^p(\mathbb{P}^3, R^q f_*\mathcal{O}_Y((-4t-13)\widetilde{h} + (t+4)e)) \Longrightarrow H^{p+q}(Y, \mathcal{O}_Y((-4t-13)\widetilde{h} +$

$(t+4)e))$ *by the Leray spectral sequence and we showed that* $H^1(\mathbb{P}^3, f_*\mathcal{O}_Y((-4t-13)\widetilde{h}+(t+4)e))$ *and* $H^0(\mathbb{P}^3, R^1f_*\mathcal{O}_Y((-4t-13)\widetilde{h}+(t+4)e))$ *vanish. So* $H^1(Y, \mathcal{O}_Y((-4t-13)\widetilde{h}+(t+4)e)) = 0$. *So,*

$$H^2(Y, L_1(t)) = 0.$$

Now consider $H^0(\mathbb{P}^3, R^2f_*\mathcal{O}_Y((-4t-13)\widetilde{h}+(t+4)e))$ *and* $H^1(\mathbb{P}^3, R^1f_*\mathcal{O}_Y((-4t-13)\widetilde{h}+(t+4)e))$*, and note that they are* 0 *by Corollary 2.1.3. Also* $H^2(\mathbb{P}^3, f_*\mathcal{O}_Y((-4t-13)\widetilde{h}+(t+4)e)) = H^2(\mathbb{P}^3, I_C^{-t-4}(-4t-13)) = 0$ *again by Lemma 2.1.11.*

Since $H^p(\mathbb{P}^3, R^qf_*\mathcal{O}_Y((-4t-13)\widetilde{h}+(t+4)e)) \Longrightarrow H^{p+q}(Y, \mathcal{O}_Y((-4t-13)\widetilde{h}+(t+4)e))$ *by the Leray spectral sequence and we showed that* $H^0(\mathbb{P}^3, R^2f_*\mathcal{O}_Y((-4t-13)\widetilde{h}+(t+4)e))$, $H^1(\mathbb{P}^3, R^1f_*\mathcal{O}_Y((-4t-13)\widetilde{h}+(t+4)e))$, $H^2(\mathbb{P}^3, f_*\mathcal{O}_Y((-4t-13)\widetilde{h}+(t+4)e))$ *and* $H^2(Y, \mathcal{O}_Y((-4t-13)\widetilde{h}+(t+4)e))$ *vanish. So*

$$H^1(Y, L_1(t)) = 0.$$

- $t = -4$*:*

Then

$$H^i(Y, L_1(-4)) = H^i(Y, \mathcal{O}_Y(-7\widetilde{h}+e)).$$

Then by [3, Lemma 1.4], we have

$$H^i(Y, \mathcal{O}_Y(-7\widetilde{h}+e)) \quad = \quad H^i(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(7h)).$$

So,

$$H^1(Y, L_1(-4)) = H^2(Y, L_1(-4)) = 0.$$

- $t = -3$*:*

Then

$$H^i(Y, L_1(-3)) = H^i(Y, \mathcal{O}_Y(-3\widetilde{h})).$$

Then by [3, Lemma 1.4], we have

$$H^i(Y, \mathcal{O}_Y(-3\widetilde{h})) = H^i(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3h)).$$

So,

$$H^1(Y, L_1(-3)) = H^2(Y, L_1(-3)) = 0.$$

So far, we showed that $H^1(Y, L_1(t)) = H^2(Y, L_1(t)) = 0$ for all t except $t = -1, -2$. For the remaining two values of t, we assume that C is ACM.

- $t = -1$:

Again by the Leray spectral sequence, if $H^i(\mathbb{P}^3, f_*L_1(-1))$ for $i = 1, 2$, $H^j(\mathbb{P}^3, R^1f_*L_1(-1))$ for $j = 0, 1$ and $H^0(\mathbb{P}^3, R^2f_*L_1(-1))$ vanishes, then $H^1(Y, L_1(-1))$ and $H^2(Y, L_1(-1))$ vanish.
Note that $H^i(\mathbb{P}^3, f_*L_1(-1)) = H^i(\mathbb{P}^3, I_C^2(5))$ for $i = 1, 2$ by the projection formula and Corollary 2.1.3; and we know that $H^i(\mathbb{P}^3, I_C^2(5)) = 0$ for $i = 1, 2$ by Proposition 2.1.15. Also, we know that $H^j(\mathbb{P}^3, R^1f_*L_1(-1)) = 0$ for $j = 0, 1$ and $H^0(\mathbb{P}^3, R^2f_*L_1(-1)) = 0$ by the projection formula and Corollary 2.1.3. So,

$$H^1(Y, L_1(-1)) = H^2(Y, L_1(-1)) = 0.$$

- $t = -2$:

By the Leray spectral sequence, if all of $H^i(\mathbb{P}^3, f_*L_1(-2))$ for $i = 1, 2$, $H^j(\mathbb{P}^3, R^1f_*L_1(-2))$ for $j = 0, 1$ and $H^0(\mathbb{P}^3, R^2f_*L_1(-2))$ vanish, then $H^1(Y, L_1(-2))$ and $H^2(Y, L_1(-2))$ vanish.
Note that $H^i(\mathbb{P}^3, f_*L_1(-2)) = H^i(\mathbb{P}^3, I_C(1))$ for $i = 1, 2$ by the projection formula and Corollary 2.1.3. But $H^i(\mathbb{P}^3, I_C(1)) = 0$ for $i = 1, 2$ since C is ACM and by Lemma 2.1.13 I_C has a minimal free resolution

$$0 \to \mathcal{O}_{\mathbb{P}^3}^{\oplus 3}(-4) \to \mathcal{O}_{\mathbb{P}^3}^{\oplus 4}(-3) \to I_C \to 0.$$

Also, we know that $H^j(\mathbb{P}^3, R^1f_*L_1(-2)) = 0$ for $j = 0, 1$ and $H^0(\mathbb{P}^3, R^2f_*L_1(-2)) = 0$ by projection formula and Corollary 2.1.3. So,

$$H^1(Y, L_1(-2)) = H^2(Y, L_1(-2)) = 0.$$

Hence $H^1(Y, L_1(t)) = H^2(Y, L_1(t)) = 0$ for all $t \in \mathbb{Z}$, and the result follows for L_1.

Consider L_2 next.
L_2 is Ulrich by Lemma 2.1.9, since

$$L_1^\vee(3) = (-(9)\widetilde{h} + 3e) + 3(4\widetilde{h} - e) = 3\widetilde{h} = L_2.$$

Remark 2.1.19 *We know that $H_{6,3,3}$, which is the open subscheme of the Hilbert Scheme parametrizing the smooth irreducible curves of $d = 6$ and $g = 3$ in $\mathbb{P}^3$, is irreducible by [8, Theorem 4]. Also, we know that the property of being an ACM sheaf is an open condition by [5]. Hence, if we assume C is ACM, then the line bundles L_1 and L_2 exist on a generic element of the deformation class Y.*

2.2 Rank 2 Ulrich Bundles on Y

Let E be a vector bundle of rank r, and L a line bundle on X. Then, by [14, Ex. 3.2.2], for all $p \geq 0$,

$$c_p(E \otimes L) = \sum_{i=0}^{p} \binom{r-i}{p-i} c_i(E) \cdot c_1^{p-i}(L).$$

Then, if E is a rank 2 vector bundle, we have

$$\begin{aligned} c_1(E \otimes \mathcal{O}_X(tH)) &= \sum_{i=0}^{1} \binom{2-i}{1-i} c_i(E) \cdot c_1^{1-i}(\mathcal{O}_X(tH)) \\ &= 2c_0(E) \cdot c_1(\mathcal{O}_X(tH)) + c_1(E) \\ &= c_1(E) + 2tH \end{aligned}$$

and

$$\begin{aligned} c_2(E \otimes \mathcal{O}_X(tH)) &= \sum_{i=0}^{2} \binom{2-i}{2-i} c_i(E) \cdot c_1^{2-i}(\mathcal{O}_X(tH)) \\ &= \sum_{i=0}^{2} c_i(E) \cdot c_1^{2-i}(\mathcal{O}_X(tH)) \\ &= c_0(E) \cdot c_1^2(\mathcal{O}_X(tH)) + c_1(E) \cdot c_1(\mathcal{O}_X(tH)) + c_2(E) \\ &= (tH)^2 + tc_1(E) \cdot H + c_2(E). \end{aligned}$$

Theorem 2.2.1 *Let $(\widetilde{Y}, H)$ be a Fano threefold which is the blow-up of $\mathbb{P}^3$ along a smooth, irreducible curve of degree d and genus g. If $\mathcal{E}$ is a rank 2 Ulrich bundle on $\widetilde{Y}$, then we have*

1. $H^2 \cdot c_1(\mathcal{E}) = 3H^3$,
2. $H \cdot c_2(\mathcal{E}) = \frac{1}{2} H \cdot c_1^2(\mathcal{E}) - 2H^3 + 4$,
3. $2c_1^3(\mathcal{E}) - 6c_1(\mathcal{E}) \cdot c_2(\mathcal{E}) + c_1(\mathcal{E}) \cdot c_2(K_{\widetilde{Y}}) = 9H^3$.

Proof 2.2.2 *Let $c_i = c_i(\mathcal{E})$ and $d_i = c_i(\mathcal{K}_{\widetilde{Y}})$. Then, by Riemann-Roch theorem*

$$
\begin{aligned}
\chi(\widetilde{Y}, \mathcal{E}(t)) &= \frac{1}{6}[(2tH + c_1)^3 - 3(2tH + c_1)((tH)^2 + tc_1 \cdot H + c_2)] \\
&\quad + \frac{1}{4} H[(2tH + c_1)^2 - 2((tH)^2 + tc_1 \cdot H + c_2)] \\
&\quad + \frac{1}{12}(H^2 + d_2)(2tH + c_1) + \frac{1}{12} H d_2 \\
&= \frac{1}{6}(8H^3t^3 + 12H^2 \cdot c_1 t^2 + 6H \cdot c_1^2 t + c_1^3 - 6H^3t^3 - 6H^2 \cdot c_1 t^2 \\
&\qquad -6H \cdot c_2 t - 3H^2 \cdot c_1 t^2 - 3H \cdot c_1^2 t - 3c_1 \cdot c_2) \\
&\quad + \frac{1}{4}(4H^3t^2 + 4H^2 \cdot c_1 t + H \cdot c_1^2 - 2H^3t^2 - 2H^2 \cdot c_1 t - 2H \cdot c_2) \\
&\quad + \frac{1}{12}(2H^3 t + H^2 \cdot c_1 + 2H \cdot d_2 t + c_1 \cdot d_2) + \frac{1}{12}(H \cdot d_2) \\
&= \frac{1}{3} H^3 t^3 + (\frac{1}{2} H^2 \cdot c_1 + \frac{1}{2} H^3) t^2 \\
&\quad + (\frac{1}{2} H \cdot c_1^2 - H \cdot c_2 + \frac{1}{2} H^2 \cdot c_1 + \frac{1}{6} H^3 + \frac{1}{6} H \cdot d_2) t \\
&\quad + (\frac{1}{6} c_1^3 - \frac{1}{2} c_1 \cdot c_2 + \frac{1}{4} H \cdot c_1^2 - \frac{1}{2} H \cdot c_2 + \frac{1}{12} H^2 \cdot c_1 \\
&\qquad + \frac{1}{12} c_1 \cdot d_2 + \frac{1}{12} H \cdot d_2).
\end{aligned}
$$

Since $\mathcal{E}$ is a rank 2 Ulrich bundle, by Proposition 2.0.17, we have

$$
\chi(\widetilde{Y}, \mathcal{E}(t)) = 2H^3 \binom{t+3}{3} = H^3 \frac{(t^3 + 6t^2 + 11t + 6)}{3}.
$$

So, if we equate coefficients of t^2, we get

$$
\frac{1}{2} H^2 \cdot c_1 + \frac{1}{2} H^3 = 2H^3
$$

$$
\Rightarrow H^2 \cdot c_1 = 3H^3.
$$

If we equate coefficients of t, we get

$$\frac{1}{2}H \cdot c_1^2 - H \cdot c_2 + \frac{1}{2}H^2 \cdot c_1 + \frac{1}{6}H^3 + \frac{1}{6}H \cdot d_2 = \frac{11}{3}H^3$$

$$\Rightarrow \quad H \cdot c_2 = \frac{1}{2}H \cdot c_1^2 - 2H^3 + \frac{1}{6}H \cdot d_2 \qquad (by\ part\ (1))$$
$$\Rightarrow \quad H \cdot c_2 = \frac{1}{2}H \cdot c_1^2 - 2H^3 + 4.$$

If we equate constant terms, we get

$$\frac{1}{6}c_1^3 - \frac{1}{2}c_1 \cdot c_2 + \frac{1}{4}H \cdot c_1^2 - \frac{1}{2}H \cdot c_2 + \frac{1}{12}H^2 \cdot c_1 + \frac{1}{12}c_1 \cdot d_2 + \frac{1}{12}H \cdot d_2 = 2H^3$$

$$\Rightarrow \quad 2c_1^3 - 6c_1 \cdot c_2 + c_1 \cdot d_2 = 9H^3.$$

Theorem 2.2.3 *Let $\mathcal{E}$ be a rank 2 Ulrich bundle on Y with $c_1(Y) = x\widetilde{h} - ye$. Then there are 7 possibilities for $c_1(Y)$, which are*

- *6$\widetilde{h}$,*
- *8$\widetilde{h}$-e,*
- *10$\widetilde{h}$-2e,*
- *12$\widetilde{h}$-3e=3H,*
- *14$\widetilde{h}$-4e,*
- *16$\widetilde{h}$-5e,*
- *18$\widetilde{h}$-6e.*

Proof 2.2.4 *We know that $H_Y = 4\widetilde{h} - e$. By Theorem 2.2.1*

$$(4\widetilde{h} - e)^2(x\widetilde{h} - ye) = 3(4\widetilde{h} - e)^3$$
$$\Rightarrow \quad 16x + 8y(-6) + x(-6) - y(-28) = 3.20 \qquad (Theorem\ 1.2.3)$$
$$\Rightarrow \quad x = 2y + 6.$$

Since $\mathcal{E}$ is Ulrich, it is μ-semistable by Theorem 2.2.24. So, we can apply Bogomolov's Inequality [20, Theorem 7.3.1] and get

$$(2.2c_2(\mathcal{E}) - (2-1)c_1^2(\mathcal{E}))H \geq 0$$

$$
\begin{aligned}
&\Rightarrow \quad 4Hc_2(\mathcal{E}) - Hc_1^2(\mathcal{E}) \geq 0 \\
&\Rightarrow \quad 4(H\frac{c_1^2(\mathcal{E})}{2} - 2H^3 + 4) - Hc_1^2(\mathcal{E}) \geq 0 && (Theorem\ 2.2.1) \\
&\Rightarrow \quad Hc_1^2(\mathcal{E}) - 8H^3 + 16 \geq 0 \\
&\Rightarrow \quad (4\widetilde{h} - e)(x\widetilde{h} - ye)^2 - 8.20 + 16 \geq 0 \\
&\Rightarrow \quad 4x^2 + 4y^2(-6) + 2xy(-6) - y^2(-28) - 144 \geq 0 && (Theorem\ 1.2.3) \\
&\Rightarrow \quad 4(2y+6)^2 - 24y^2 - 12y(2y+6) + 28y^2 - 144 \geq 0 \\
&\Rightarrow \quad -4y^2 + 24y \geq 0 \\
&\Rightarrow \quad 0 \leq y \leq 6.
\end{aligned}
$$

2.2.1 Simple Ulrich Bundles on Y with $c_1 = 3H$

Proposition 2.2.5 *Let X be projective variety of dimension k in $\mathbb{P}^N$ and I_X be the ideal sheaf of X in $\mathbb{P}^N$. Then $H^i(\mathbb{P}^N, I_X^n(t))$ is upper semi-continuous for $i > k$.*

Proof 2.2.6 *Twisting the sequence*

$$0 \to I_X \to \mathcal{O}_{\mathbb{P}^N} \to \mathcal{O}_X \to 0$$

by $I_X^{\otimes n-1}(t)$ yields the long exact sequence

$$0 \to \mathrm{Tor}_1^{\mathcal{O}_{\mathbb{P}^N}}(I_X^{\otimes n-1}(t), \mathcal{O}_X) \to I_X^{\otimes n}(t) \to I_X^{\otimes n-1}(t) \to \mathcal{O}_X \otimes I_X^{\otimes n-1}(t) \to 0$$

This can be broken into two short exact sequences, one of which is

$$0 \to \mathrm{Tor}_1^{\mathcal{O}_{\mathbb{P}^N}}(I_X^{\otimes n-1}(t), \mathcal{O}_X) \to I_X^{\otimes n}(t) \to I_X^n(t) \to 0.$$

Since $\mathrm{Tor}_1^{\mathcal{O}_{\mathbb{P}^N}}(I_X^{\otimes n-1}(t), \mathcal{O}_X)$ has at most k-dimensional support, we have $H^i(\mathbb{P}^N, \mathrm{Tor}_1^{\mathcal{O}_{\mathbb{P}^N}}(I_X^{\otimes n-1}(t), \mathcal{O}_X)) = 0$ for all $i > k$. So, by long exact sequence of cohomology, we get $H^i(\mathbb{P}^N, I_X^{\otimes n}(t)) = H^i(\mathbb{P}^N, I_X^n(t))$ for all $i > k$. Since left hand side is upper semi-continuous, the right hand side is upper semi-continuous.

Theorem 2.2.7 *Let C be an smooth ACM space curve with $d = 6$ and $g = 3$. Then $h^2(\mathbb{P}^3, I_C^3(6)) = 0$ and $h^2(\mathbb{P}^3, I_C^2(2)) \leq 8$ for a generic such C.*

Proof 2.2.8 *Use Macaulay2 [15] for computations:*

```
i1 : k = ZZ/32467; R = k[x,y,z,w];
```

Then load the package RandomSpaceCurves [4] to produce explicit example of smooth ACM space curve C' of $d = 6$ and $g = 3$ with ideal J:

```
i2 : load"RandomSpaceCurves.m2";
i3 : J=(random spaceCurve)(6,3,R)
o3 : ideal (-2215x^3+10620x^2y+2508xy^2-15048y^3-5453x^2z-2767xyz
        +8885y^2z+2225xz^2+1759yz^2-9499z^3+3014x^2w+12412xyw
        -1419y^2w-11910xzw-3506yzw-831z^2w-1546xw^2+4414yw^2
        -10576zw^2+15249w^3, -6292x^3-10864x^2y+5626xy^2-8024y^3
        +10837x^2z-6966xyz+9956y^2z-9501xz^2-9538yz^2+9745z^3
        +15655x^2w-3220xyw-12116y^2w+11148xzw-3392yzw-1539z^2w
        -3915xw^2-5992yw^2+15589zw^2+7309w^3, 870x^3+9582x^2y
        -172xy^2+8082y^3-13952x^2z+1923xyz+13352y^2z+7141xz^2
        -13354yz^2+15747z^3+1042x^2w+1494xyw-11584y^2w+7730xzw
        -4628yzw+9837z^2w-4220xw^2+4893yw^2-15379zw^2-13719w^3,
        -15941x^3-8361x^2y-16223xy^2+12866y^3-4501x^2z+13591xyz
        -11196y^2z-6043xz^2-7842yz^2+11284z^3+1057x^2w-2552xyw
        +6508y^2w+15994xzw-2374yzw-10280z^2w+7766xw^2+15317yw^2
        -10555zw^2+7241w^3)
o3 : Ideal of R
```

Then check whether C' is a smooth ACM space curve of $d = 6$ and $g = 3$:

```
i4 : (degree J, genus J, resolution J)
o4 : (6, 3, R^1 <-- R^4 <-- R^3 <-- 0)
           0       1       2       3
o4 : Sequence
```

Then compute $h^2(J^3_{C'}(6))$ and $h^2(J^2_{C'}(2))$:

```
i5 : J3 = J*J*J;
     J2 = J*J;
     vJ3 = Proj(R/J3);
     vJ2 = Proj(R/J2);
     sJ3 = sheaf module ideal vJ3;
```

```
      sJ2 = sheaf module ideal vJ2;
o5 : Ideal of R
o6 : Ideal of R
i11: (HH^2 (sJ3(6)), HH^2 (sJ2(2)))
o11: (0, k^8)
o11: Sequence
```

But, we know that these cohomologies are upper semi-continuous functions by Proposition 2.2.5. Hence, we have $h^2(\mathbb{P}^3, I_C^3(6)) = 0$ and $h^2(\mathbb{P}^3, I_C^2(2)) \leq 8$ for a generic element of all smooth ACM space curves of $d = 6$ and $g = 3$.

Remark 2.2.9 *Since cohomology is an upper semi-continuous function, as stated in the proof of Theorem 2.2.7, smooth ACM space curves of $d = 6$ and $g = 3$ satisfying $h^2(I_C^3(6)) = 0$ form an open subset of all smooth ACM space curves of $d = 6$ and $g = 3$. Also by Remark 2.1.19, we know that $H_{6,3,3}$ is irreducible and smooth ACM space curves of $d = 6$ and $g = 3$ form an open subset in $H_{6,3,3}$. So, smooth ACM space curves of $d = 6$ and $g = 3$ satisfying $h^2(\mathbb{P}^3, I_C^3(6)) = 0$ form an open subset of all smooth space curves of $d = 6$ and $g = 3$. Hence, $h^2(\mathbb{P}^3, I_C^3(6)) = 0$ for a generic element of the deformation class Y. By a similar argument, $h^2(\mathbb{P}^3, I_C^2(2)) \leq 8$ for a generic element of the deformation class Y.*

Corollary 2.2.10 *For a generic element of the deformation class of Y, we have $ext^1(L_2, L_1) = 8$.*

Proof 2.2.11 *We know that*

$$ext^1(L_2, L_1) = h^1(Y, L_2^\vee \otimes L_1),$$

where $L_2^\vee \otimes L_1 = \mathcal{O}_Y(-(3\widetilde{h}) + (9\widetilde{h} - 3e)) = \mathcal{O}_Y(6\widetilde{h} - 3e)$.
By Theorem 1.2.3, $\chi(Y, L_2^\vee \otimes L_1) = -8$. So, we have

$$h^0(L_2^\vee \otimes L_1) - h^1(L_2^\vee \otimes L_1) + h^2(L_2^\vee \otimes L_1) - h^3(L_2^\vee \otimes L_1) = -8$$

$$\begin{aligned}\Rightarrow h^1(L_2^\vee \otimes L_1) &= 8 + h^0(L_2^\vee \otimes L_1) + h^2(L_2^\vee \otimes L_1) - h^3(L_2^\vee \otimes L_1) \\ &= 8 + \hom(L_2, L_1) + h^2(L_2^\vee \otimes L_1) - \hom(L_1(1), L_2)\end{aligned}$$

where $\hom(L_2, L_1) = \hom(L_1(1), L_2) = 0$ *by [20, Proposition 1.2.7]. So*

$$h^1(Y, L_2^\vee \otimes L_1) \;\; = \;\; h^2(Y, L_2^\vee \otimes L_1) + 8.$$

Use the Leray spectral sequence to compute $h^2(Y, L_2^\vee \otimes L_1)$*:*

$$H^p(\mathbb{P}^3, R^q f_* L_2^\vee \otimes L_1) \Longrightarrow H^{p+q}(Y, L_2^\vee \otimes L_1)$$

where f *is the blow-down map.*
We know that $H^0(\mathbb{P}^3, R^2 f_* L_2^\vee \otimes L_1) = H^0(\mathbb{P}^3, R^2 f_* \mathcal{O}_Y(-3e) \otimes \mathcal{O}_{\mathbb{P}^3}(6))$ *by the projection formula and* $R^2 f_* \mathcal{O}_Y(-3e) = 0$ *by Corollary 2.1.3.*
So, $H^0(\mathbb{P}^3, R^2 f_* L_2^\vee \otimes L_1) = 0$*. Similarly,* $H^1(\mathbb{P}^3, R^1 f_* L_2^\vee \otimes L_1) = 0$*.*
Also by the projection formula, we know that

$$\begin{aligned} f_*(L_2^\vee \otimes L_1) &= f_* \mathcal{O}_Y(6\widetilde{h} - 3e) \\ &= \mathcal{O}_{\mathbb{P}^3}(6) \otimes f_* \mathcal{O}_Y(-3e) \\ &= I_C^3 \otimes \mathcal{O}_{\mathbb{P}^3}(6) \\ &= I_C^3(6). \end{aligned}$$

So $H^2(\mathbb{P}^3, f_* L_2^\vee \otimes L_1) = H^2(\mathbb{P}^3, I_C^3(6))$*. Hence, by the Leray spectral sequence,*

$$H^2(Y, L_2^\vee \otimes L_1) = H^2(\mathbb{P}^3, I_C^3(6)).$$

So

$$h^1(Y, L_2^\vee \otimes L_1) \;\; = \;\; h^2(\mathbb{P}^3, I_C^3(6)) + 8.$$

But, $h^2(\mathbb{P}^3, I_C^3(6)) = 0$ *by Remark 2.2.9, for a generic element of deformation class* Y*. Hence,* $ext^1(L_2, L_1) = 8$ *for a generic element of deformation class* Y*.*

Corollary 2.2.12 *For a generic element of deformation class* Y*,* $ext^1(L_1, L_2) \leq 8$*.*

Proof 2.2.13 *We know that*

$$ext^1(L_1, L_2) = h^1(Y, L_1^\vee \otimes L_2) = h^2(Y, L_2^\vee \otimes L_1 \otimes K_Y)$$

where $L_2^\vee \otimes L_1 \otimes K_Y = \mathcal{O}_Y(-(3\widetilde{h}) + (9\widetilde{h} - 3e) + (-4\widetilde{h} + e)) = \mathcal{O}_Y(2\widetilde{h} - 2e)$*.*
Use the Leray spectral sequence to compute $h^2(Y, L_2^\vee \otimes L_1 \otimes K_Y)$*:*

$$H^p(\mathbb{P}^3, R^q f_* L_2^\vee \otimes L_1 \otimes K_Y) \Longrightarrow H^{p+q}(Y, h^2(Y, L_2^\vee \otimes L_1 \otimes K_Y))$$

where f is the blow-down map.
We know that $H^0(\mathbb{P}^3, R^2 f_* L_2^\vee \otimes L_1 \otimes K_Y) = H^0(\mathbb{P}^3, R^2 f_* \mathcal{O}_Y(-2e) \otimes \mathcal{O}_{\mathbb{P}^3}(2))$ by the projection formula and $R^2 f_* \mathcal{O}_Y(-2e) = 0$ by Corollary 2.1.3.
So, $H^0(\mathbb{P}^3, R^2 f_* L_2^\vee \otimes L_1 \otimes K_Y) = 0$.
Similarly, $H^1(\mathbb{P}^3, R^1 f_* L_2^\vee \otimes L_1 \otimes K_Y) = H^1(\mathbb{P}^3, R^1 f_* \mathcal{O}_Y(-2e) \otimes \mathcal{O}_{\mathbb{P}^3}(2))$ by the projection formula and $R^1 f_* \mathcal{O}_Y(-3e) = 0$ by Corollary 2.1.3.
So, $H^1(\mathbb{P}^3, R^1 f_* L_2^\vee \otimes L_1 \otimes K_Y) = 0$.
Also, by the projection formula, we know that

$$\begin{aligned} f_*(L_2^\vee \otimes L_1 \otimes K_Y) &= f_* \mathcal{O}_Y(2\widetilde{h} - 2e) \\ &= \mathcal{O}_{\mathbb{P}^3}(2) \otimes f_* \mathcal{O}_Y(-2e) \\ &= I_C^2 \otimes \mathcal{O}_{\mathbb{P}^3}(2) \\ &= I_C^2(2). \end{aligned}$$

So $H^2(\mathbb{P}^3, f_* L_2^\vee \otimes L_1 \otimes K_Y) = H^2(\mathbb{P}^3, I_C^2(2))$. Hence, by the Leray spectral sequence,

$$H^2(Y, L_2^\vee \otimes L_1 \otimes K_Y) = H^2(\mathbb{P}^3, I_C^2(2)).$$

But, $h^2(\mathbb{P}^3, I_C^2(2)) \leq 8$ by Remark 2.2.9, for a generic element of deformation class Y.
Hence, $ext^1(L_1, L_2) \leq 8$ for a generic element of deformation class Y.

Theorem 2.2.14 *Let $\mathcal{E}$ be a rank 2 vector bundle on Y obtained by a non-split extension*

$$0 \to L_1 \to \mathcal{E} \to L_2 \to 0$$

or

$$0 \to L_2 \to \mathcal{E} \to L_1 \to 0$$

where $L_1 = \mathcal{O}_Y(9\widetilde{h} - 3e)$ and $L_2 = \mathcal{O}_Y(3\widetilde{h})$. Then $\mathcal{E}$ is a simple Ulrich bundle with $c_1(\mathcal{E}) = 12\widetilde{h} - 3e$ and $c_2(\mathcal{E}) = 27\widetilde{h}^2 - 9\widetilde{h}e$.

Proof 2.2.15 *By Theorem 2.1.17, L_1 and L_2 are Ulrich line bundles. Since they are Ulrich, they have the same slope by Proposition 2.0.17. Since they are line bundles, they are trivially stable. Clearly, they are non-isomorphic. Hence $\mathcal{E}$ is a simple vector bundle by [5, Lemma 4.2].*

Since L_1 and L_2 are Ulrich bundles, $\mathcal{E}$ is an Ulrich bundle by [7, Proposition 2.8]. Moreover, we have

$$\begin{aligned} c_1(\mathcal{E}) &= c_1(L_1) + c_1(L_2) \\ &= (9\widetilde{h} - 3e) + (3\widetilde{h}) \\ &= 12\widetilde{h} - 3e \end{aligned}$$

and

$$\begin{aligned} c_2(\mathcal{E}) &= c_1(L_1)c_1(L_2) \\ &= (9\widetilde{h} - 3e)(3\widetilde{h}) \\ &= 27\widetilde{h}^2 - 9\widetilde{h}e. \end{aligned}$$

Theorem 2.2.16 *Let $\mathcal{E}$ be a rank 2 simple Ulrich bundle on Y with $c_1(\mathcal{E}) = 12\widetilde{h} - 3e$ and $c_2(\mathcal{E}) = 27\widetilde{h}^2 - 9\widetilde{h}e$. Then $h^1(\mathcal{E} \otimes \mathcal{E}^\vee) - h^2(\mathcal{E} \otimes \mathcal{E}^\vee) = 15$.*

Proof 2.2.17 *Note that the Chern polynomial of $\mathcal{E}$ is*

$$c_t(\mathcal{E}) = (1 + (9\widetilde{h} - 3e)t)(1 + (3\widetilde{h})t) = \prod_{i=1}^{2}(1 + a_i t)$$

where $a_1 = 9\widetilde{h} - 3e$ and $a_2 = 3\widetilde{h}$.
Also,

$$\begin{aligned} c_1(\mathcal{E}^\vee) &= (-1)^1 c_1(\mathcal{E}) \\ &= -12\widetilde{h} + 3e \end{aligned}$$

and

$$\begin{aligned} c_2(\mathcal{E}^\vee) &= (-1)^2 c_2(\mathcal{E}) \\ &= 27\widetilde{h}^2 - 9\widetilde{h}e. \end{aligned}$$

Then the Chern polynomial of $\mathcal{E}^\vee$ is

$$c_t(\mathcal{E}^\vee) = (1 + (-9\widetilde{h} + 3e)t)(1 + (-3\widetilde{h})t) = \prod_{i=1}^{2}(1 + b_i t)$$

where $b_1 = -(9\widetilde{h} - 3e)$ and $b_2 = -3\widetilde{h}$. Then we have

$$c_t(\mathcal{E} \otimes \mathcal{E}^\vee) = \prod_{i,j=1}^{2}(1 + (a_i + b_j)t)$$

$$
\begin{aligned}
&= (1+0t)(1+(6\widetilde{h}-3e)t)(1+(-6\widetilde{h}+3e)t)(1+0t) \\
&= 1+0t+(-36\widetilde{h}^2+36\widetilde{h}e-9e^2)t^2+0t^3+0t^4.
\end{aligned}
$$

So, $c_2(\mathcal{E}\otimes\mathcal{E}^\vee) = -36\widetilde{h}^2+36\widetilde{h}e-9e^2$ *and* $c_i(\mathcal{E}\otimes\mathcal{E}^\vee)=0$ *for* $i=1,3,4$.

By Theorem 1.2.3, we have

$$c_1(\mathcal{T}_Y) = 4\widetilde{h}-e$$

$$c_2(\mathcal{T}_Y) = 12\widetilde{h}^2-4\widetilde{h}e$$

and

$$\deg(\widetilde{h}^3) = 1$$

$$\deg(\widetilde{h}^2e) = 0$$

$$\deg(\widetilde{h}e^2) = -6$$

$$\deg(\widetilde{e}^3) = -28.$$

Apply the Riemann-Roch theorem for $\mathcal{E}\otimes\mathcal{E}^\vee$ *on* Y *if* $c_i = c_i(\mathcal{E}\otimes\mathcal{E}^\vee)$ *and* $d_i = c_i(\mathcal{T}_{\widetilde{Y}})$*:*

$$
\begin{aligned}
\chi(\widetilde{Y},\mathcal{E}\otimes\mathcal{E}^\vee) &= \frac{1}{6}(c_1^3-3c_1c_2+3c_3)+\frac{1}{4}d_1(c_1^2-2c_2)+\frac{1}{12}(d_1^2+d_2)c_1 \\
&\quad +\frac{4}{24}d_1d_2 \\
&= \frac{1}{4}(4\widetilde{h}-e)(-2(-36\widetilde{h}^2+36\widetilde{h}e-9e^2)) \\
&\quad +\frac{4}{24}(4\widetilde{h}-e)(12\widetilde{h}^2-4\widetilde{h}e) \\
&= \frac{1}{4}(-72)+\frac{1}{6}(24) \\
&= -14.
\end{aligned}
$$

Then we have

$$h^0(\mathcal{E}\otimes\mathcal{E}^\vee)-h^1(\mathcal{E}\otimes\mathcal{E}^\vee)+h^2(\mathcal{E}\otimes\mathcal{E}^\vee)-h^3(\mathcal{E}\otimes\mathcal{E}^\vee) = -14$$

$$
\begin{aligned}
\Rightarrow h^1(\mathcal{E}\otimes\mathcal{E}^\vee)-h^2(\mathcal{E}\otimes\mathcal{E}^\vee) &= 14+h^0(\mathcal{E}\otimes\mathcal{E}^\vee)-h^3(\mathcal{E}\otimes\mathcal{E}^\vee) \\
&= 14+\hom(\mathcal{E},\mathcal{E})-\hom(\mathcal{E}(1),\mathcal{E})
\end{aligned}
$$

where $\hom(\mathcal{E},\mathcal{E})=1$ *since* $\mathcal{E}$ *is simple. So*

$$h^1(\mathcal{E}\otimes\mathcal{E}^\vee)-h^2(\mathcal{E}\otimes\mathcal{E}^\vee) = 14+1-\hom(\mathcal{E}(1),\mathcal{E})$$

where $\hom(\mathcal{E}(1),\mathcal{E})=0$ *by [20, Proposition 1.2.7]. So*

$$
\begin{aligned}
h^1(\mathcal{E}\otimes\mathcal{E}^\vee)-h^2(\mathcal{E}\otimes\mathcal{E}^\vee) &= 14+1-0 \\
&= 15.
\end{aligned}
$$

2.2.2 Quot Scheme

The general reference for this section is [20, Section 2.2].
The Quot scheme $Quot_X(F,P)$ parametrizes quotient sheaves of a given $\mathcal{O}_X$-module F with Hilbert polynomial P. In this subsection, we briefly review some properties of the Quot scheme, including properties about its local dimension.

Let κ be a field, S be κ-scheme of finite type and Sch/S be the category of S-schemes. Let $\phi : X \to S$ be a projective morphism and $\mathcal{O}_X(1)$ an ϕ-ample line bundle on X. Let $\mathcal{H}$ be a coherent $\mathcal{O}_X$-module and $P \in \mathbb{Q}[z]$ a polynomial. The functor

$$\mathcal{Q} := \underline{Quot}_{X/S} : (Sch/S)^o \to (Sets)$$

is defined as follows:
If $T \to S$ is an object in Sch/S, let $\mathcal{Q}(T)$ be the set of all T-flat coherent quotient sheaves $\mathcal{H}_T = \mathcal{O}_T \otimes \mathcal{H} \to F$ with Hilbert poynomial P. And if $h : T' \to T$ is an S-morphism, let $\mathcal{Q}(h) : \mathcal{Q}(T) \to \mathcal{Q}(T')$ be the map that sends $\mathcal{H}_T \to F$ to $\mathcal{H}_{T'} \to h_X^* F$.

Theorem 2.2.18 *The functor $\underline{Quot}_{X/S}(\mathcal{H},P)$ is represented by a projective S-scheme $\pi : Quot_{X/S}(\mathcal{H},P) \to S$.*

Proof 2.2.19 *See [20, Theorem 2.2.4].*

Proposition 2.2.20 *Let X be a projective scheme over a field κ and $\mathcal{H}$ a coherent sheaf on X. Let $[q : \mathcal{H} \to F] \in Quot(\mathcal{H},P)$ be a κ-rational point and $K = ker(q)$. Then*

$$hom(K,F) \geq dim_{[q]} Quot(H,P) \geq hom(K,F) - ext^1(K,F).$$

If equality holds at the second place, $Quot(\mathcal{H},P)$ is a local complete intersection near $[q]$. If $ext^1(K,F) = 0$, then $Quot(\mathcal{H},P)$ is smooth at $[q]$.

Proof 2.2.21 *See [20, Proposition 2.2.8].*

2.2.3 Stable Ulrich Bundles on Y with $c_1 = 3H$

We review some well-known facts.

Proposition 2.2.22 *Let $\mathcal{E}$ be a stable bundle on X. Then $\mathcal{E}$ is simple; i.e, $End(\mathcal{E}) \cong \mathbb{K}$.*

Proof 2.2.23 *Since $\mathbb{K}$ is algebraically closed, it follows from [20, Corollary 1.2.8].*

Theorem 2.2.24 *Let $\mathcal{E}$ be an Ulrich bundle of rank r on a nonsingular projective variety X. Then,*

- *$\mathcal{E}$ is semistable and μ-semistable,*
- *If $\mathcal{E}$ is stable, then it is also μ-stable.*

Proof 2.2.25 *See [5, Theorem 2.9].*

Hence, (semi)stability and μ-(semi)stability are equivalent for an Ulrich bundle $\mathcal{E}$ by Lemma 2.0.13 and Theorem 2.2.24.

Proposition 2.2.26 *Let $\mathcal{E}$ be an Ulrich bundle of rank r on a nonsingular projective variety X. Then $\mathcal{E}$ is globally generated.*

Proof 2.2.27 *See [7, Corollary 2.5].*

Lemma 2.2.28 *Let $\mathcal{E}$ be an Ulrich bundle on X. Then for any Jordan-Hölder filtration*

$$0 = \mathcal{E}_0 \subseteq \mathcal{E}_1 \subseteq \cdots \subseteq \mathcal{E}_{m-1} \subseteq \mathcal{E}_m = \mathcal{E}$$

we have that $\mathcal{E}_i$ is an Ulrich bundle for $1 \leq i \leq m$. In particular, if $\mathcal{E}$ is a $strictly$ $semistable$ Ulrich bundle of rank $r \geq 2$, then there exist a subbundle $\mathcal{F}$ of $\mathcal{E}$ having rank $s < r$ which is Ulrich.

Proof 2.2.29 *See [6, Lemma 2.15].*

Definition 2.2.30 *Let E be a nontrivial locally free sheaf on X. The trace map tr : $End(E) \to \mathcal{O}_X$ induces $tr^i : Ext^i(E,E) \cong H^i(End(E)) \to H^i(\mathcal{O}_X)$. These homomorphisms are surjective. Let $Ext^i(E,E)_o$ denote the kernel of tr^i.*

Proposition 2.2.31 *If E is locally free sheaf on Y, then $Ext^i(E,E)_o = Ext^i(E,E)$ for $0 < i < 3$.*

Proof 2.2.32 *Note that $H^i(Y, \mathcal{O}_Y) = 0$ for $0 < i < 3$. So the kernel of tr^i is $Ext^i(E,E)$ for $0 < i < 3$.*

We want to analyze the local dimension of Quot scheme. For this, we will follow the discussion and the notation of [20, Section 4.3].
Let F be semistable sheaf on X. Let m be a sufficiently large integer such that $F(m)$ is globally generated, V be a vector space of dimension $P_X(m)$ and $\mathcal{H} := V \otimes_k \mathcal{O}_X(-m)$. Let $R \subset Quot(\mathcal{H}, P)$ be the open subscheme of those quotients $[\rho : \mathcal{H} \to \mathcal{E}]$ where $V \to H^0(\mathcal{E})$ is an isomorphism.

Proposition 2.2.33 $H^i(Y, \mathcal{O}_Y) \cong 0$ *for* $i > 0$.

Proof 2.2.34 *See [19, p. 153].*

Theorem 2.2.35 *Let $\mathcal{E}$ be a rank 2 simple Ulrich bundle on Y, with $c_1(\mathcal{E}) = 12\widetilde{h} - 3e$ and $c_2(\mathcal{E}) = 27\widetilde{h}^2 - 9\widetilde{h}e$. Then $\dim_{[\rho]} R \geq 1614$ for a fixed $[\rho : \mathcal{H} \to \mathcal{E}]$.*

Proof 2.2.36 *We will follow the construction in [20, p.115].*
First, note that $\mathcal{E}$ is semistable by Theorem 2.2.24. Second, $\mathcal{E}$ is globally generated by Proposition 2.2.26.
So V is a vector space of dimension 40, since $P_Y(0) = 20 \cdot 2\binom{3+0}{3} = 40$.
Then $\mathcal{H} := V \otimes_{\mathbb{K}} \mathcal{O}_Y = \mathcal{O}_Y^{\oplus 40}$.
Fix $[\rho : \mathcal{H} \to \mathcal{E}] \in R$.

1. *Let K be the kernel of ρ; that is, we have*

$$0 \to K \to \mathcal{H} \to \mathcal{E} \to 0.$$

Then we have the long exact sequence of cohomology

$$\begin{aligned} 0 &\to H^0(Y,K) \to H^0(Y,\mathcal{H}) \to H^0(Y,\mathcal{E}) \\ &\to H^1(Y,K) \to H^1(Y,\mathcal{H}) \to H^1(Y,\mathcal{E}) \\ &\to H^2(Y,K) \to H^2(Y,\mathcal{H}) \to H^2(Y,\mathcal{E}) \\ &\to H^3(Y,K) \to H^3(Y,\mathcal{H}) \to H^3(Y,\mathcal{E}) \to 0. \end{aligned}$$

Since $\mathcal{H} = \mathcal{O}_Y^{\oplus 40}$ *and* $\mathcal{E}$ *is globally generated by Proposition 2.2.26,* $H^0(Y,\mathcal{H}) \cong H^0(Y,\mathcal{E})$. *So* $H^0(Y,K) \cong 0$. *Then, since* $Hom(\mathcal{H},K) \cong Hom(\mathcal{O}_Y,K)^{\oplus 40} \cong H^0(Y,K)^{\oplus 40}$, $Hom(\mathcal{H},K) \cong 0$.
Since $H^1(Y,\mathcal{H}) \cong H^1(Y,\mathcal{O}_Y)^{\oplus 40} \cong 0$ *by Proposition 2.2.33 and* $H^0(Y,\mathcal{H}) \cong H^0(Y,\mathcal{E})$, $H^1(Y,K) \cong 0$. *Then, since* $Ext^1(\mathcal{H},K) \cong Ext^1(\mathcal{O}_Y,K)^{\oplus 40} \cong H^1(Y,K)^{\oplus 40}$, $Ext^1(\mathcal{H},K) \cong 0$.
Since $H^2(Y,\mathcal{H}) \cong H^2(Y,\mathcal{O}_Y)^{\oplus 40} \cong 0$ *by Proposition 2.2.33 and* $H^1(Y,\mathcal{E}) \cong 0$ *by being that* $\mathcal{E}$ *is Ulrich,* $H^2(Y,K) \cong 0$. *Then, since* $Ext^2(\mathcal{H},K) \cong Ext^2(\mathcal{O}_Y,K)^{\oplus 40} \cong H^2(Y,K)^{\oplus 40}$, $Ext^2(\mathcal{H},K) \cong 0$.
Since $H^3(Y,\mathcal{H}) \cong H^3(Y,\mathcal{O}_Y)^{\oplus 40} \cong 0$ *by Proposition 2.2.33 and* $H^2(Y,\mathcal{E}) \cong 0$ *by being that* $\mathcal{E}$ *is Ulrich,* $H^3(Y,K) \cong 0$. *Then, since* $Ext^3(\mathcal{H},K) \cong Ext^3(\mathcal{O}_Y,K)^{\oplus 40} \cong H^3(Y,K)^{\oplus 40}$, $Ext^3(\mathcal{H},K) \cong 0$.
Hence $Hom(\mathcal{H},K) \cong 0$ *and* $Ext^i(\mathcal{H},K) \cong 0$ *for* $i > 0$.

2. *Consider the short exact sequence*

$$0 \to K \to \mathcal{H} \to \mathcal{E} \to 0.$$

Then take the functor $Hom(\mathcal{H},-)$

$$\begin{aligned} 0 &\to Hom(\mathcal{H},K) \to Hom(\mathcal{H},\mathcal{H}) \to Hom(\mathcal{H},\mathcal{E}) \\ &\to Ext^1(\mathcal{H},K) \to Ext^1(\mathcal{H},\mathcal{H}) \to Ext^1(\mathcal{H},\mathcal{E}) \\ &\to Ext^2(\mathcal{H},K) \to Ext^2(\mathcal{H},\mathcal{H}) \to Ext^2(\mathcal{H},\mathcal{E}) \\ &\to Ext^3(\mathcal{H},K) \to Ext^3(\mathcal{H},\mathcal{H}) \to Ext^3(\mathcal{H},\mathcal{E}) \to 0. \end{aligned}$$

By step (1), we know that $Hom(\mathcal{H},K) \cong 0$ *and* $Ext^i(\mathcal{H},K) \cong 0$ *for* $i > 0$. *So,* $Hom(\mathcal{H},\mathcal{H}) \cong Hom(\mathcal{H},\mathcal{E})$ *and* $Ext^i(\mathcal{H},\mathcal{H}) \cong Ext^i(\mathcal{H},\mathcal{E})$ *for* $i > 0$.
On the other hand, $Ext^i(\mathcal{H},\mathcal{H}) \cong Ext^i(\mathcal{O}_Y^{\oplus 40},\mathcal{O}_Y^{\oplus 40}) \cong H^i(Y,\mathcal{O}_Y)^{\oplus 1600}$ *for* $i > 0$. *Since* $H^i(Y,\mathcal{O}_Y) \cong 0$ *for* $i > 0$ *by Proposition 2.2.33,* $Ext^i(\mathcal{H},\mathcal{H}) \cong 0$.
Hence $Hom(\mathcal{H},\mathcal{H}) \cong Hom(\mathcal{H},\mathcal{E})$ *and* $Ext^i(\mathcal{H},\mathcal{E}) = 0$, $i > 0$.

3. *Again consider the short exact sequence*

$$0 \to K \to \mathcal{H} \to \mathcal{E} \to 0.$$

Then take the functor $Hom(-, \mathcal{E})$ *of it*

$$\begin{aligned} 0 &\to Hom(\mathcal{E},\mathcal{E}) \to Hom(\mathcal{H},\mathcal{E}) \to Hom(K,\mathcal{E}) \\ &\to Ext^1(\mathcal{E},\mathcal{E}) \to Ext^1(\mathcal{H},\mathcal{E}) = 0 \to \dots \end{aligned}$$

leads to equality $hom(K,\mathcal{E}) = hom(\mathcal{H},\mathcal{E}) + ext^1(\mathcal{E},\mathcal{E}) - hom(\mathcal{E},\mathcal{E})$*.*
Since $Ext^i(\mathcal{H},\mathcal{E}) = 0$ *for* $i > 0$*,* $Ext^i(K,\mathcal{E}) \cong Ext^{i+1}(\mathcal{E},\mathcal{E})$ *for* $i > 0$*.*

4. *The boundary map* $Ext^1(K,\mathcal{E}) \to Ext^2(\mathcal{E},\mathcal{E})$ *maps the obstruction to extend* $[\rho]$ *onto the obstructions to extend* $[\mathcal{E}]$ *(see [20, 2.A.8]). The latter is contained in the subspace* $Ext^2(\mathcal{E},\mathcal{E})_o$*. This gives the dimension bound, using Proposition 2.2.20,*

$$\dim_{[\rho]} R \geq hom(K,\mathcal{E}) - ext^2(\mathcal{E},\mathcal{E})_o.$$

Then, by step (3), we have

$$\dim_{[\rho]} R \geq hom(\mathcal{H},\mathcal{E}) + ext^1(\mathcal{E},\mathcal{E}) - hom(\mathcal{E},\mathcal{E}) - ext^2(\mathcal{E},\mathcal{E})_o.$$

Then, by Proposition 2.2.31

$$\dim_{[\rho]} R \geq hom(\mathcal{H},\mathcal{E}) + ext^1(\mathcal{E},\mathcal{E}) - hom(\mathcal{E},\mathcal{E}) - ext^2(\mathcal{E},\mathcal{E}).$$

Then, by step (2), we have

$$\dim_{[\rho]} R \geq hom(\mathcal{H},\mathcal{H}) + ext^1(\mathcal{E},\mathcal{E}) - hom(\mathcal{E},\mathcal{E}) - ext^2(\mathcal{E},\mathcal{E}).$$

Since $\mathcal{E}$ *is simple and* $\mathcal{H} = \mathcal{O}_Y^{\oplus 40}$*, we have*

$$\dim_{[\rho]} R \geq 1600 + ext^1(\mathcal{E},\mathcal{E}) - 1 - ext^2(\mathcal{E},\mathcal{E}).$$

Then, by Theorem 2.2.16 and the equality $h^i(\mathcal{E} \otimes \mathcal{E}^\vee) = ext^i(\mathcal{E},\mathcal{E})$*, we have*

$$\dim_{[\rho]} R \geq 1600 + -1 + 15 = 1614.$$

Let $R' \subset Quot(\mathcal{H}, P)$ be the subset parametrizing the quotients $[\rho : \mathcal{H} \to \mathcal{E}]$ where $\mathcal{E}$ is obtained as an extension of L_2 by L_1.

Proposition 2.2.37 $\dim_{[\rho]} R^{'} = 1606$ *for a fixed* $[\rho : \mathcal{H} \to \mathcal{E}]$.

Proof 2.2.38 *The projectivization of* $Ext^1(L_2, L_1)$ *has dimension* $8-1=7$ *by Corollary 2.2.10.* $R^{'}$ *is the union of all orbits of extensions of* L_2 *by* L_1 *under the action of* $PGL(V)$*, so around each fixed* $[\rho : \mathcal{H} \to \mathcal{E}]$, $\dim_{[\rho]} R^{'} = 1599 + 7 = 1606$.

Let $R^{''} \subset Quot(\mathcal{H}, P)$ be the subset parametrizing the quotients $[\rho : \mathcal{H} \to \mathcal{E}]$ where $\mathcal{E}$ is obtained as an extension of L_1 by L_2.

Proposition 2.2.39 $\dim_{[\rho]} R^{''} \leq 1606$ *for a fixed* $[\rho : \mathcal{H} \to \mathcal{E}]$.

Proof 2.2.40 *The projectivization of* $Ext^1(L_1, L_2)$ *has dimension* $\leq 8-1=7$ *by Corollary 2.2.12.* $R^{''}$ *is the union of all orbits of extensions of* L_1 *by* L_2 *under the action of* $PGL(V)$*, so around each fixed* $[\rho : \mathcal{H} \to \mathcal{E}]$, $\dim_{[\rho]} R^{''} \leq 1599+7 = 1606$.

Theorem 2.2.41 *There exist rank 2 stable Ulrich bundles with* $c_1(\mathcal{E}) = 12\widetilde{h} - 3e$ *on a generic element of the deformation class* Y.

Proof 2.2.42 *By Theorem 2.2.14, there are rank 2 simple Ulrich bundle* $\mathcal{E}$ *with the given Chern classes.*
We know that the property of being Ulrich is an open condition. So there is an open subset U *of* R *around* $[\rho : \mathcal{H} \to \mathcal{E}]$ *containing Ulrich bundles. By Theorem 2.2.35,* U *has dimension at least 1614.*
We also know that every Ulrich bundle is semistable by Theorem 2.2.24. If all elements of U *were strictly semistable, then by Lemma 2.2.28 and [7, Proposition 2.8], they would be extensions of Ulrich line bundles. But there are only two Ulrich line bundles* L_1 *and* L_2 *on* Y*. So they would be extensions of* L_2 *by* L_1 *or extensions of* L_1 *by* L_2.
However, the dimension of $R^{'}$ *at the points that are extensions of* L_2 *by* L_1 *is 1606 by Proposition 2.2.37 and the dimension of* $R^{''}$ *at the points that are extensions of* L_1 *by* L_2 *is at most 1606 by Proposition 2.2.39. Since both these dimensions are strictly smaller than 1614, not all Ulrich bundles with the given Chern classes are obtained by extensions. In other words, not all Ulrich bundles in* U *are strictly semistable. Hence there are rank 2 stable Ulrich bundles with* $c_1(\mathcal{E}) = 12\widetilde{h} - 3e$.

REFERENCES

[1] A. Beauville. Determinantal hypersurfaces. *Michigan Math. J.*, 48:39–64, 2000. Dedicated to William Fulton on the occasion of his 60th birthday.

[2] A. Beauville. Ulrich bundles on abelian surfaces. *arXiv preprint arXiv:1512.00992*, 2015.

[3] A. Bertram, L. Ein, and R. Lazarsfeld. Vanishing theorems, a theorem of Severi, and the equations defining projective varieties. *J. Amer. Math. Soc.*, 4(3):587–602, 1991.

[4] H.-C. G. v. Bothmer, F. Geiss, and F.-O. Schreyer. Randomspacecurves.m2, http://www.math.uiuc.edu/macaulay2/doc/macaulay2-1.9/share/doc/macaulay2/randomspacecurves/html/index.html.

[5] M. Casanellas, R. Hartshorne, F. Geiss, and F.-O. Schreyer. Stable Ulrich bundles. *Internat. J. Math.*, 23(8):1250083, 50, 2012.

[6] E. Coskun, R. S. Kulkarni, and Y. Mustopa. On representations of Clifford algebras of ternary cubic forms. In *New trends in noncommutative algebra*, volume 562 of *Contemp. Math.*, pages 91–99. Amer. Math. Soc., Providence, RI, 2012.

[7] E. Coskun, R. S. Kulkarni, and Y. Mustopa. The geometry of Ulrich bundles on del Pezzo surfaces. *J. Algebra*, 375:280–301, 2013.

[8] L. Ein. Hilbert scheme of smooth space curves. *Ann. Sci. École Norm. Sup. (4)*, 19(4):469–478, 1986.

[9] D. Eisenbud and F.-O. Schreyer. Betti numbers of graded modules and cohomology of vector bundles. *J. Amer. Math. Soc.*, 22(3):859–888, 2009.

[10] D. Eisenbud and F.-O. Schreyer. Boij-Söderberg theory. In *Combinatorial aspects of commutative algebra and algebraic geometry*, volume 6 of *Abel Symp.*, pages 35–48. Springer, Berlin, 2011.

[11] D. Eisenbud, F.-O. Schreyer, and J. Weyman. Resultants and Chow forms via exterior syzygies. *J. Amer. Math. Soc.*, 16(3):537–579, 2003.

[12] P. Ellia and M. Idà. Some connections between equations and geometric properties of curves in $\mathbf{P}^3$. In *Geometry and complex variables (Bologna, 1988/1990)*,

volume 132 of *Lecture Notes in Pure and Appl. Math.*, pages 177–188. Dekker, New York, 1991.

[13] G. Farkas, M. Mustaţă, and M. Popa. Divisors on $M_{g,g+1}$ and the minimal resolution conjecture for points on canonical curves. *Ann. Sci. École Norm. Sup. (4)*, 36(4):553–581, 2003.

[14] W. Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1984.

[15] D. R. Grayson and M. E. Stillman. Macaulay2, a software system for research in algebraic geometry, http://www.math.uiuc.edu/macaulay2/.

[16] R. Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[17] R. Hartshorne. *Deformation theory*, volume 257 of *Graduate Texts in Mathematics*. Springer, New York, 2010.

[18] J. Herzog, B. Ulrich, and J. Backelin. Linear maximal Cohen-Macaulay modules over strict complete intersections. *J. Pure Appl. Algebra*, 71(2-3):187–202, 1991.

[19] H. Hironaka. Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II. *Ann. of Math. (2)* **79** *(1964), 109–203; ibid. (2)*, 79:205–326, 1964.

[20] D. Huybrechts and M. Lehn. *The geometry of moduli spaces of sheaves*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, second edition, 2010.

[21] V. A. Iskovskikh and Y. G. Prokhorov. Fano varieties. In *Algebraic geometry, V*, volume 47 of *Encyclopaedia Math. Sci.*, pages 1–247. Springer, Berlin, 1999.

[22] R. Lazarsfeld. *Positivity in algebraic geometry. I*, volume 48 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2004. Classical setting: line bundles and linear series.

[23] A. Lorenzini. The minimal resolution conjecture. *J. Algebra*, 156(1):5–35, 1993.

[24] S. Mori and S. Mukai. Classification of Fano 3-folds with $B_2 \geq 2$. I. In *Algebraic and topological theories (Kinosaki, 1984)*, pages 496–545. Kinokuniya, Tokyo, 1986.

CURRICULUM VITAE

PERSONAL INFORMATION

Surname, Name: Genç, Özhan
Nationality: Turkish (TC)
Date and Place of Birth: 24.06.1987, Ankara
Marital Status: Single
Phone: 0535 370 05 03

EDUCATION

Degree	Institution	Year of Graduation
B.S.	Middle East Technical University	2009
High School	Arı Sicence High School Name	2004

PROFESSIONAL EXPERIENCE

Year	Place	Enrollment
2009-2016	Department of Mathematics, METU	Research Assistant
2008-2009 Spring	Department of Mathematics, METU	Student Assistant

PUBLICATIONS

- O. Genc, Stable Ulrich vector bundles on Fano 3-folds with Picard number $\rho = 2$, submitted.

Research Interests

- Ulrich bundles on Fano varieties, Geometry of moduli spaces of vector bundles.

Seminar Talks

- Middle East Technical University, General Seminar, May 2016.
- Hacettepe University, General Seminar, May 2016.
- Middle East Technical University, Algebraic Geometry and Arithmetic Seminar, November 2011, October 2012, March 2013.

Conference Talks

- GAeL XXIV, Izmir, Turkey, June 2016.
- XVIII. Antalya Algebra Days, Izmir, TURKEY, May 2016.

Awards and Fundings

- BAP (Scientific Research Projects, supported by Middle East Technical University), January 2014 – December 2016.
- TUBITAK grant 2211, October 2009 - October 2014.

Conferences and Workshops Organized

- ACM Bundles on Algebraic Varieties, June 2015, Middle East Technical University, Ankara, Turkey.

Conferences and Workshops Attended

- Mediterranean Complex Projective Geometry, May 2016, Carry-le-Rouet, France.

- Ankara-Istanbul Algebraic Geometry and Number Theory Meetings, Fall 2014, Spring 2015, Fall 2015, Spring 2015, Middle East Technical University and Istanbul Center for Mathematical Sciences, Turkey.
- Summer School "Moduli Spaces and Derived Categories", August 2015, Palazzone of the Scuola Normale Superiore, Cortona, Italy.
- VBAC 2014 Algebraic Varieties: Bundles, Topology, Physics, September 2014, Freie Universität Berlin, Berlin, GERMANY.
- Algebraic Geometry and Number Theory, June 2014, Galatasaray University, Istanbul, TURKEY.
- Gokova Geometry/Topology Conference, May 2012, Gokova, TURKEY.
- Gokova Geometry/Topology Conference, May 2011, Gokova, TURKEY.
- Izmir Algebraic Geometric Topology Days II, January 2011, Dokuz Eylul University, Izmir, TURKEY.
- Gokova Geometry/Topology Conference, May 2010, Gokova, TURKEY.
- Number Theory, Arithmetic and Algebraic Geometry Conference, April 2010, Middle East Technical University Northern Cyprus Campus, Guzelyurt, CYPRUS.

References

Research

- Emre Coskun
  Assistant Professor
  Middle East Technical University
  +90-312 210 5394
  `emcoskun@metu.edu.tr`
- M. Hursit Onsiper
  Professor
  Middle East Technical University

+90-312 210 5383
hursit@metu.edu.tr

- Rajesh S. Kulkarni
  Professor
  Michigan State University
  (517) 353-8145
  kulkarni@math.msu.edu

- A. U . Ozgur Kisisel
  Associate Professor
  Middle East Technical University
  +90-312 210 5388
  akisisel@metu.edu.tr

Teaching

- Muhiddin Uguz
  Vice Chairman
  Middle East Technical University
  +90-312 210 5364
  muhid@metu.edu.tr