AN APPROACH FOR DETERMINING PROCESS ECONOMY PARAMETERS OF MULTIVARIATE LOSS FUNCTIONS

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ABSTRACT

AN APPROACH FOR DETERMINING PROCESS ECONOMY PARAMETERS OF MULTIVARIATE LOSS FUNCTIONS

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The aim of this study is to provide an effective method for determining parameters of multivariate loss functions, which are related with process economics. The loss functions are widely used in product and process design and other quality engineering applications. Although there are several studies about different types of loss functions, there is a lack of studies on determining cost matrix parameters of these functions. For this purpose, we propose a method based on multi-objective decision making tools. We illustrate use of the method on two example problems, and discuss their results.

Keywords: Multivariate Loss Functions, Parameter Determination, Quality Engineering

ÖZ

ÇOK DEĞİŞKENLİ KAYIP FONKSİYONLARININ SÜREÇ EKONOMİSİ PARAMETRELERİNİN BELİRLENMESİ İÇİN BİR YAKLAŞIM

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Bu tez çalışmanın amacı çok değişkenli kayıp fonksiyonlarının süreç ekonomisi parametrelerinin belirlenmesi konusunda bir etkili bir yöntem sağlamaktır. Kayıp fonksiyonları ürün ve süreç tasarımı ve diğer kalite mühendisliği uygulamalarında sıkça kullanılmaktadır. Literatürde farklı çok değişkenli kayıp fonksiyonları hakkında bir çok çalışma olduğu halde bu fonksiyonların maliyet matrisi parametrelerinin belirlenmesi konusunda yeterli bir çalışma bulunmamaktadır. Önerdiğimiz method bu parametrelerin belirlenmesi için çok amaçlı karar verme araçlarını kullanmaktadır. Önerilen methot iki örnek problem üzerinde açıklanmış ve sonuçları incelenmiştir.

Anahtar Kelimeler: Çok Değişkenli Kayıp Fonksiyonları, Parametre Belirlenmesi, Kalite Mühendisliği

To my family

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CHAPTER 1

INTRODUCTION

In a competitive environment arising from globalization, companies all around the world are forced to produce high quality products at low costs and as fast as possible in order to satisfy customers. Usually, overall quality perceived by the customers depends on several quality characteristics of the product. Hence, typically multiple quality characteristics need to be optimized simultaneously to create high quality products right at the first time, and minimize the quality loss, which is the loss to the society caused by low quality products. However, usually the quality characteristics are controlled by a common set of design parameters and improving a quality characteristic typically means worsening another, causing a trade-off between quality characteristics. Problems of finding the optimal design parameter setting that maximizes the overall quality of the products are called Multi-Response Design Parameter Optimization (MRDPO) problems and they are widely studied by researchers around the world.

According to Taguchi (1986), the quality loss of a product arises from the functional variation from its target value and he defines quadratic loss functions to represent the quality loss caused by a product. Using the quadratic quality loss functions, he combines multiple quality characteristics in one value function, which allows the optimization of a single objective to reach the optimal result. According to Suhr and Batson (2001), loss functions are mainly used in product and process design, and quality assurance programs.

The loss function developed by Taguchi (1986) considers only one quality characteristic of a product. Multivariate quality loss functions considering more than one quality characteristic are developed by Pignatiello (1993) and studied by several researchers including Ames et al. (1997), Tsui (1999), Wu and Chyu (2004), Vining (1998) and Ko et al. (2005). These functions are in the form of a value function or a statistical distance of a given product or process design point (i.e., design parameter settings) from a target set of design parameter values. This distance depends on variances and covariances of the quality characteristics at the given design point as well as process economics and decision maker preferences. The distance utilizes empirical models of relationships between the design parameters and expected values of the quality characteristics at these parameter levels. Although building these empirical models are studied in detail in the literature as well as estimating the variances and the covariances, determination of loss function parameters that are affected by process economics is not paid enough attention as pointed out by some researchers including Park and Kim (2005). Similarly, Kuhnt and Erdbrügge (2004) state that the method proposed by Pignatiello (1993), which is a widely used method in other studies, is impractical, since it requires perfect estimation of quality losses at chosen design points.

In this thesis, we study properties of multivariate loss functions and propose a method for determining the process economics related parameters of the multivariate quadratic loss functions. Based on a structural analysis of the loss functions, which we have performed to choose a proper approach, we develop an effective and robust method that does not require perfect estimation of loss at any stage. Our method can be used for building multivariate loss functions regardless of the number of quality characteristics under consideration. The purpose of the method we proposed is to find the best fitting function to the underlying value function of the decision maker, who typically is the quality engineer trying to find the best design parameter settings of a product and/or process.

The organization of the thesis is as follows. The background and related literature on quality loss functions and other related concepts are reviewed in Chapter 2. In Chapter 3, the structural analysis on different aspects of multivariate quality loss functions are provided. We propose a method for determining the parameters of multivariate loss functions in Chapter 4. In Chapter 5, two examples illustrating the method and discussions on the method and multivariate loss functions in general are provided. Our concluding remarks and future study directions are provided in Chapter 7.

CHAPTER 2

LITERATURE REVIEW AND BACKGROUND

In this chapter, the background and literature on related concepts to this study including the multi-response design parameter optimization problems and quality loss are presented together with Analytic Hierarchy Process. The chapter is divided into two sections, which are quality loss and relevant multi-objective decision making approaches.

2.1 QUALITY LOSS

DeVor et al. (2007) states that quality of a product is defined as its fitness for use. A quality product satisfies the requirements of the market. W.A. Shewhart (1927) brings the concept that increasing the quality of a product decreases the total cost of the product.

In the traditional view, quality loss is associated with the tangible costs related to the product or service under consideration. Lost sales due to low quality is considered as the indicator of the quality loss. In the traditional view, quality costs can be listed as follows according to DeVor et al. (2007):

- Prevention costs: As the name suggests, these costs result from the quality control system which aims to prevent production of low quality products.
- Appraisal costs: Similar to prevention costs, appraisal costs are related to the maintenance of quality assurance system. Both prevention costs and appraisal costs are hard to measure on product basis.

- Internal Failures: These costs arise from the low quality products before they are delivered to the customers and include scrap and rework, cost of transporting the low quality product, cost of adjusting the process, etc.
- External Failures: The costs related to external failures occur after the delivery of the product to the customer. Warranties, repairs, customer and product services, returns, product replacements, pollution, market share loss, increasing cost of marketing efforts can be included in the external failures which cause quality loss.

However, Taguchi (1989) brings a new definition to the quality loss and states that quality loss is the cost to the society. These losses can be listed as the losses due to the harmful effects on society and the losses due to the variation of the function of the product or service. This definition suggests that not only manufacturers, but also customers are affected from the loss resulting from a quality product.

Logothetis and Wynn (1989) claim that the losses caused by the intrinsic function of the product or service delivered should be excluded from the quality loss. They give the example of liquors. Although drunk people may harm themselves or other people, causing some losses, these losses cannot be thought as quality losses since removing alcohol from the liquors would result in a product which does not have the intrinsic function of the liquor. Hence, they suggest that the loss should be limited to the loss caused by variation in the function of product. Therefore, the quality loss caused by the product must be zero when all of the quality characteristics of that product are at their target values. Furthermore, the loss increases in an increasing manner as the quality characteristics deviates from their targets. If the target is a nominal value, then the response is called “nominal-the-best” type response. If the target is minimizing the value of quality characteristic, the type of the response is called as “smaller-the-better”. Finally, if the target is maximizing the response, then the type of the response is “larger-the-better”.

Using these definition of quality, Taguchi defines robust parameter design (RPD) problems according to Ross (1988) and suggests ways of solving them. In RPD problems the aim is to optimize quality characteristic such that its value is at the desirable level with minimum variation from the target value. The value of the quality characteristic depends on the values of design parameters, which are controllable and the noise factors, which cannot be controlled.

The robust parameter design problems with more than one quality characteristics are called MRDPO or robust design for multiple responses. If there is more than one quality characteristics under consideration, usually there is a trade-off between these quality characteristics according to Murphy et al. (2005), who provide a review on MRDPO problems and different approaches including the loss functions.

Taguchi et al. (1989) defines quality loss function for products with only one quality characteristic. The univariate loss function proposed by Taguchi is provided below.

$$\mathrm{L}(y(\mathbf{x}),t)=c(y(\mathbf{x})-t)^2 \tag{1}$$

In Equation (1), $y(\mathbf{x})$ represents the value of the quality characteristic, $\mathbf{x}$ array represents the design parameters determining the value of quality characteristics, t is the target value for the quality characteristic under consideration and c represents the proportionality coefficient. Figure 1 represents the graph of a univariate loss function example which is provided in Equation (2).

$$\mathrm{L}(y(\mathbf{x}),0.5)=(y(\mathbf{x})-0.5)^2 \tag{2}$$

The expected loss of the product, $\mathrm{E}(\mathrm{L}(y(\mathbf{x}),t))$, which needs to be minimized to minimize the loss to the society is calculated as follows.

$$\mathrm{E}(\mathrm{L}(y(\mathbf{x}),t))=c(\sigma^2+\mathrm{E}(y(\mathbf{x})-t)^2) \tag{3}$$

The term σ^2 represents the sample variance of quality characteristic $y(\mathbf{x})$.

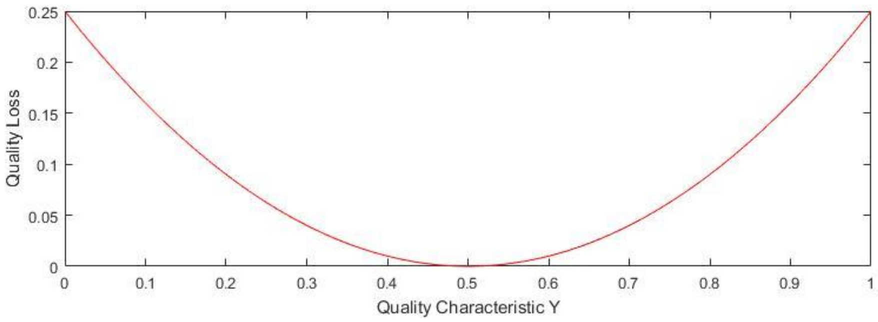

Figure 1 Example Univariate Loss Function

As mentioned above, only one quality characteristic can be analyzed using univariate quality loss functions. However, in real life applications usually there are more than one quality characteristic under consideration. Therefore, quality loss functions need to be adapted for cases where there are several quality characteristics.

Pignatiello (1993) suggests a multi-response loss function by extending Taguchi's univariate loss function. The function proposed by Pignatiello (1993) can be seen in Equation (4).

$$\mathrm{L}(\mathbf{y}(\mathbf{x}),\mathbf{t}) = (\mathbf{y}(\mathbf{x})\text{-}\mathbf{t})\ \mathbf{C}(\mathbf{y}(\mathbf{x})\text{-}\mathbf{t}) \tag{4}$$

The multivariate loss function is similar to the univariate loss function. However, since there are at least two quality characteristics under consideration, we have **t** array to represent the target values for all quality characteristics and **y**(**x**) array to represent values of quality characteristics when design factors are in the setting represented by the **x** array. Coefficient *c* is replaced with **C** matrix to reflect the loss incurred due to individual quality characteristics and the loss incurred due to pairwise interactions between quality characteristics. An example bivariate loss function can be seen in Figure 2.

The expected multivariate loss function is shown in Equation (5).

$$\mathrm{E}[\mathrm{L}(\mathbf{y}(\mathbf{x}),\mathbf{t})] = \left(\mathrm{E}\left(\mathbf{y}(\mathbf{x})\right)\text{-}\mathbf{t}\right)' \mathbf{C}\left(\mathrm{E}\left(\mathbf{y}(\mathbf{x})\right)\text{-}\mathbf{t}\right) + trace\left[\mathbf{C}\boldsymbol{\Sigma}_{\mathrm{y}}(\mathbf{x})\right] \tag{5}$$

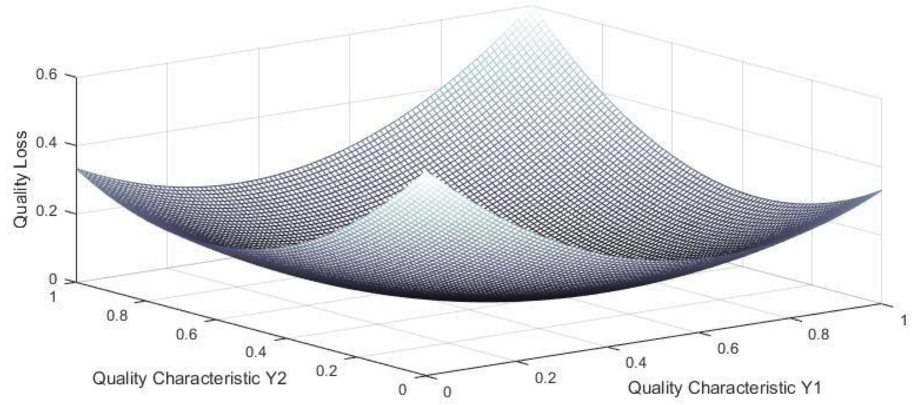

Figure 2 Example Multivariate Loss Function

Pignatiello (1993) also suggests a guideline for the determination of the **C** matrix. In his method, parameters are estimated using some reference points where the loss values are known.

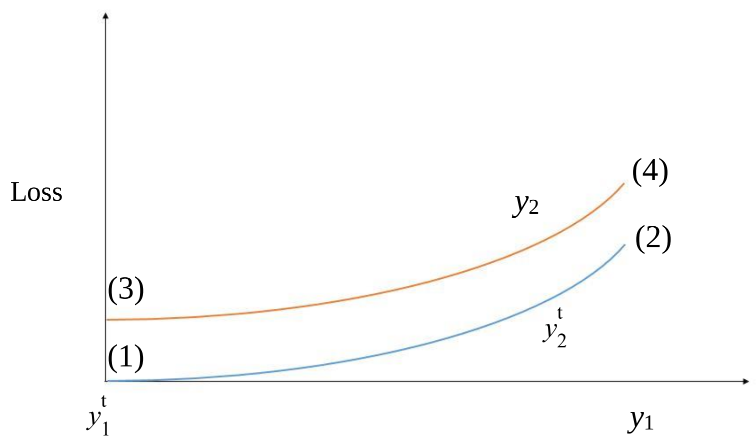

Figure 3 Reference Points Used in Loss Calculations

It is assumed that the loss values at points (1), (2), (3), and (4) are known. There are two levels determined for both quality characteristics. y_1 and y_2 represent the values of quality characteristics when they are at a chosen level which is greater than the target value while y_1^t and y_2^t represent the values of the quality characteristics when they are at their targets. For example, at point (4), both first and second quality characteristics have values greater than their target values, while at point (1) both quality characteristics are at their targets. Hence, at point (1), the quality loss is equal to zero whereas the quality loss is positive at the other three points. Pignatiello (1993) assumes that the **C** matrix is symmetric, hence there are three components of the **C** matrix which needs to be estimated for a bivariate loss function. They are estimated as in Equations (6), (7) and (8).

$$c_{11} = \frac{\mathrm{L}(2)}{(y_1(\mathbf{x})\text{-}t_1)^2} \tag{6}$$

$$c_{22} = \frac{\mathrm{L}(3)}{(y_2(\mathbf{x})\text{-}t_2)^2} \tag{7}$$

$$c_{12} = \frac{\mathrm{L}(4)\text{-}\mathrm{L}(2)\text{-}\mathrm{L}(3)}{(y_1(\mathbf{x})\text{-}t_1)(y_2(\mathbf{x})\text{-}t_2)} \tag{8}$$

L(2), L(3) and L(4) represent the loss values at points (2), (3) and (4), correspondingly. The diagonal elements of **C** matrix are represented by c_{11} and c_{22} while the off-diagonal elements are represented by c_{12}.

As mentioned above, this method requires the exact knowledge of loss values at the chosen points. However, as also acknowledged by Kuhnt and Erdbrügge (2004), perfect estimation of the loss value at the given points is almost impossible and the method proposed by Pignatiello (1993) is unrealistic. Park and Kim (2005) also state that since it is especially hard to determine the off-diagonal elements of **C** matrix, the practitioners may tend to ignore these elements and try to reach solutions using only the diagonal elements.

Vining (1998) modifies the multivariate loss function proposed by Pignatiello (1993). He uses predicted responses (quality characteristics), $\hat{\mathbf{y}}(\mathbf{x})$, instead of exact values of the quality characteristics, $\mathbf{y}(\mathbf{x})$, in the multivariate loss function he proposed. The loss function suggested by Vining (1998) is as in Equation (9).

$$\mathrm{L}(\hat{\mathbf{y}}(\mathbf{x}),\mathbf{t})=(\hat{\mathbf{y}}(\mathbf{x})\text{-}\mathbf{t})'\mathbf{C}(\hat{\mathbf{y}}(\mathbf{x})\text{-}\mathbf{t}) \tag{9}$$

The expected loss function is presented in Equation (10).

$$E[L(\hat{\mathbf{y}}(\mathbf{x}),\mathbf{t})]=\left(E\left(\hat{\mathbf{y}}(\mathbf{x})\right)-\mathbf{t}\right)'\mathbf{C}\left(E\left(\hat{\mathbf{y}}(\mathbf{x})\right)-\mathbf{t}\right)+trace\left[\mathbf{C}\boldsymbol{\Sigma}_{\hat{\mathbf{y}}}(\mathbf{x})\right] \tag{10}$$

The term $trace[\mathbf{C}\boldsymbol{\Sigma}_{\hat{\mathbf{y}}}(\mathbf{x})]$ represents the loss caused by the prediction errors while term $\left(E\left(\hat{\mathbf{y}}(\mathbf{x})\right)-\mathbf{t}\right)'\mathbf{C}\left(E\left(\hat{\mathbf{y}}(\mathbf{x})\right)-\mathbf{t}\right)$ represents the loss caused by the deviations of predictions from their targets. The main difference between Pignatiello's approach and Vining's approach is the fact that Vining's approach considers quality of the predictions.

Vining (1998) also suggests a way to decompose the **C** matrix in order to increase the understanding of cost structure. The cost matrix, **C**, proposed by Vining for the loss function consists of the inverse of the variance-covariance matrix, $\boldsymbol{\Sigma}^{-1}$, and matrix **K**, which reflects the preference/cost structure for the quality characteristics. However, he does not specify the method for choosing matrix **K**. The open forms of matrices **K** and $\boldsymbol{\Sigma}^{-1}$ for bivariate case are provided in Equations (11) and (12) correspondingly.

$$\mathbf{K}=\begin{bmatrix} w_1 & k_1 \\ k_2 & w_2 \end{bmatrix} \tag{11}$$

In the **K** matrix, w_1 and w_2 correspond to the weights of outcomes, while k_1 and k_2 are the preferential dependency constants. Quality characteristics are preferentially dependent if preferences of some quality characteristics depend on the values of other quality characteristics. If the preferential dependency constants are positive, it indicates that the options where both quality characteristics have high or low values are preferred more than other alternatives.

$$\boldsymbol{\Sigma}^{-1}=\begin{bmatrix} \sigma_{22} & -\sigma_{12} \\ -\sigma_{12} & \sigma_{11} \end{bmatrix}\frac{1}{\sigma_{11}\sigma_{22}-\sigma_{12}^2} \tag{12}$$

The general form of cost matrix, **C**, is provided in Equation (13).

$$\mathbf{C}=\begin{bmatrix} w_1\sigma_{22}-k_1\sigma_{12} & -w_1\sigma_{12}+k_1\sigma_{11} \\ k_2\sigma_{22}-w_2\sigma_{12} & -k_2\sigma_{12}+w_2\sigma_{11} \end{bmatrix}\frac{1}{\sigma_{11}\sigma_{22}-\sigma_{12}^2} \tag{13}$$

Ko et al. (2005) extend further the multivariate loss functions proposed by Vining (1998) and Pignatiello (1993). They define $\tilde{\mathbf{y}}_{\text{new}}(\mathbf{x})$ to represent the vector of estimated future responses. The loss function they suggest is shown in Equation (14).

$$\mathrm{L}\left(\tilde{\mathbf{y}}_{\text{new}}(\mathbf{x}),\mathbf{t}\right)=\left(\tilde{\mathbf{y}}_{\text{new}}(\mathbf{x})\text{-}\mathbf{t}\right)^{'}\mathbf{C}\left(\tilde{\mathbf{y}}_{\text{new}}(\mathbf{x})\text{-}\mathbf{t}\right) \tag{14}$$

The expected loss function for a future observation is presented in Equation (15).

$$\mathrm{E}\left[\mathrm{L}\left(\tilde{\mathbf{y}}_{\text{new}}(\mathbf{x}),\mathbf{t}\right)\right]=\left(\mathrm{E}\left(\hat{\mathbf{y}}(\mathbf{x})\right)\text{-}\mathbf{t}\right)^{'}\mathbf{C}\left(\mathrm{E}\left(\hat{\mathbf{y}}(\mathbf{x})\right)\text{-}\mathbf{t}\right)+\mathit{trace}\left[\mathbf{C}\boldsymbol{\Sigma}_{\hat{\mathrm{y}}}(\mathbf{x})\right]+ \\ \mathit{trace}[\mathbf{C}\boldsymbol{\Sigma}_{\mathrm{y}}(\mathbf{x})] \tag{15}$$

The term $\left(\mathrm{E}\left(\hat{\mathbf{y}}(\mathbf{x})\right)\text{-}\mathbf{t}\right)^{'}\mathbf{C}\left(\mathrm{E}\left(\hat{\mathbf{y}}(\mathbf{x})\right)\text{-}\mathbf{t}\right)$ is identical to the first term of expected loss function proposed by Vining and represents the loss caused by the deviations of predictions from their target values. The second term, $\mathit{trace}[\mathbf{C}\boldsymbol{\Sigma}_{\hat{\mathrm{y}}}(\mathbf{x})]$, represents the loss due to low quality predictions. Finally, $\mathit{trace}[\mathbf{C}\boldsymbol{\Sigma}_{\mathrm{y}}(\mathbf{x})]$ represents the loss caused by the variation of quality characteristics at a given parameters setting, which is called as poor robustness. They also provide the **C** matrix suggested by Vining and do not bring any further explanation on the determination of **C** matrix.

There are several studies on multivariate loss functions based on the loss functions mentioned above. Multivariate loss functions are generally used in optimization of processes where there is more than one quality characteristic as mentioned by Ames et al. (1997) in their paper on an alternative multivariate loss function. They also suggest that to determine the weights of loss function, customers can be asked to compare the products, which is also similar to the approach used in thesis study. Chou and Chen (2001) extends the multivariate loss functions by providing a loss function which allows the changes in loss over time. They also suggest that it is appropriate to use loss functions to evaluate the quality of the product since other performance measures are more difficult to analyze. Maghsoodloo and Chang (2001) also suggest a new bivariate loss function and use the idea of dividing the design specification set into regions, which is also utilized in this study. They try to extend the parameter determination method proposed by Taguchi for univariate loss functions to be used in multivariate loss functions. However, their method also requires the exact knowledge of loss at given points. Bhamare et al. (2009) use a hybrid quality loss function based approach to handle also smaller-the-better and larger-the-better type of quality characteristics together with nominal-the-best type quality characteristics. However, their method does not consider the correlation between quality characteristics. Suhr and Batson (2001) suggest solution methods for constrained multivariate loss functions.

2.2 ANALYTIC HIERARCHY PROCESS

Analytic Hierarchy Process (AHP) is a decision making tool which aims to formalize the decision making process. It has been developed by Saaty (1980). It is based on the pairwise comparison of the elements to obtain their relative desirability. Saaty (1999) states that the comparison between two elements is made by answering the question how many times more strongly does one element contribute to the upper level element when compared to the other element. In order to express the preference, Saaty suggests a ratio scale between 1-9. The scale suggested by Saaty (1987) is presented in Table 1.

Table 1 The Scale Used in Comparisons

Verbal Pairwise Judgement	**Numerical Score**
Equally important	1
Moderately more important	3
Strongly more important	5
Very strongly more important	7
Extremely more important	9
Intermediate values	2, 4, 6, 8

Let x_{ijk} denote the numerical score obtained from the decision maker when he/she verbally compares alternatives j and k in criterion i. Then, x_{ikj} be calculated as in Equation (16).

$$x_{ikj}=\frac{1}{x_{ijk}} \tag{16}$$

Let x_{ik} be calculated as shown in Equation (17).

$$x_{ik}=\sum_{j} x_{ijk} \tag{17}$$

Then we obtain x'_{ijk}, which is an estimate of the relative value of alternative j over alternative k in criterion i, as shown in Equation (18).

$$x'_{ijk}=\frac{x_{ijk}}{x_{ik}} \tag{18}$$

Let X'_{ij} be an estimate of relative value of alternative j in criterion i and be calculated as shown in Equation (19) when there are n alternatives.

$$X'_{ij}=\frac{\sum_k x'_{ijk}}{n} \tag{19}$$

Similarly let y_{it} be score obtained from the decision maker when he/she verbally compares criteria i and t.

$$y_{ti}=\frac{1}{y_{it}} \tag{20}$$

Let y_t be calculated as shown in Equation (21).

$$y_t=\sum_{i} y_{it} \tag{21}$$

The estimated value of criterion i, y'_{it} is calculated as shown in Equation (22).

$$y'_{it} = \frac{y_{it}}{y_t} \tag{22}$$

Then the relative value of the criterion *i* when *m* criteria exist is calculated as shown in Equation (23).

$$Y'_i = \frac{\sum_t y'_{it}}{m} \tag{23}$$

Then the overall value of alternative *j*, X_j can be calculated as in Equation (24).

$$X_j = \sum_{\mathrm{i}} Y'_i X'_{ij} \tag{24}$$

The alternative with the greatest value is the most preferred alternative by the decision maker. X_j value is also the normalized preference value of the alternative *j*.

However, AHP is criticized several times as mentioned by Zahedi (1986). The author groups the criticism with respect to the four steps defined by Johnson (1980). The categories are the hierarchy of decisions, input data, estimation of relative weights and aggregation of relative weights of various levels into composite relative weights. The criticism includes the usage of additive value functions without justification, the restrictions due to the scale used. Jensen (1983) suggests that alternatives are compared simultaneously for all criteria to remove the aggregation step, which is a similar approach to the method used in this study.

CHAPTER 3

STRUCTURAL ANALYSIS OF LOSS FUNCTIONS

In this chapter, the quadratic bivariate quality loss functions are analyzed in different structural aspects. The structure of the cost matrix, the derivation of the cost matrix under different dependency conditions and resulting multivariate quality loss functions, special cases of the cost matrix and the representative strength of quality loss functions are examined in the following subsections.

3.1 DERIVATIONS OF EXPECTED LOSS FUNCTIONS UNDER DIFFERENT DEPENDENCY CASES

This subsection presents the analysis of the structure of the cost matrix and corresponding expected loss function under different settings in terms of preferential and statistical dependency. The cost matrices and corresponding expected loss functions are analyzed under four different cases.

3.1.1 NO STATISTICAL AND PREFERENTIAL DEPENDENCY BETWEEN RESPONSES

In this case there is no statistical and preferential dependency between responses; therefore, preferential dependency constants, k_1 and k_2, and correlation between responses σ_{12} are equal to zero. Hence, the cost matrix, **C**, becomes as in Equation (25).

$$\mathbf{C}=\begin{bmatrix} w_1\sigma_{22}(\mathbf{x}) & 0 \\ 0 & w_2\sigma_{11}(\mathbf{x}) \end{bmatrix}\frac{1}{\sigma_{11}(\mathbf{x})\sigma_{22}(\mathbf{x})} \tag{25}$$

The expected loss function using the cost matrix provided in Equation (25) is shown in Equation (26) and the open form of the expected loss function is provided in Equation (27).

$$\mathrm{E}[\mathrm{L}(\hat{\mathbf{y}}(\mathbf{x})\text{-}\mathbf{t})]=\begin{bmatrix} \hat{\bar{y}}_1(\mathbf{x})\text{-}t_1 & \hat{\bar{y}}_2(\mathbf{x})\text{-}t_2 \end{bmatrix}\begin{bmatrix} \frac{w_1}{\sigma_{11}(\mathbf{x})} & 0 \\ 0 & \frac{w_2}{\sigma_{22}(\mathbf{x})} \end{bmatrix}\begin{bmatrix} \hat{\bar{y}}_1(\mathbf{x})\text{-}t_1 \\ \hat{\bar{y}}_2(\mathbf{x})\text{-}t_2 \end{bmatrix}+trace\begin{bmatrix} w_1 & 0 \\ 0 & w_2 \end{bmatrix} \tag{26}$$

$$\mathrm{E}[\mathrm{L}(\hat{\mathbf{y}}(\mathbf{x})\text{-}\mathbf{t})]=\frac{w_1}{\sigma_{11}(\mathbf{x})}(\hat{\bar{y}}_1(\mathbf{x})\text{-}t_1)^2+\frac{w_2}{\sigma_{22}(\mathbf{x})}(\hat{\bar{y}}_2(\mathbf{x})\text{-}t_2)^2+w_1+w_2 \tag{27}$$

where $\hat{\bar{y}}_1(\mathbf{x})$ and $\hat{\bar{y}}_2(\mathbf{x})$ are the means of predicted responses one and two, respectively. It can be observed that multivariate expected loss function is reduced to the sum of two univariate expected loss functions; therefore, it becomes an additive function. Hence, determination of weights is trivial in this case. The methods proposed by Taguchi for univariate loss functions which are mentioned in Chapter 2 can be used to determine the weights.

3.1.2 NO PREFERENTIAL DEPENDENCY BETWEEN RESPONSES

In this case the responses have statistical dependency, however, there is not any preferential dependency between responses. The cost matrix in this case is shown in Equation (28).

$$\mathbf{C}=\begin{bmatrix} w_1\sigma_{22}(\mathbf{x}) & \text{-}w_1\sigma_{12}(\mathbf{x}) \\ \text{-}w_2\sigma_{12}(\mathbf{x}) & w_2\sigma_{11}(\mathbf{x}) \end{bmatrix}\frac{1}{\sigma_{11}(\mathbf{x})\sigma_{22}(\mathbf{x})\text{-}\sigma_{12}^2(\mathbf{x})} \tag{28}$$

The expected loss using the cost matrix given above is given in Equation (29) and its open form is given in Equation (30).

$$\mathrm{E}[\mathrm{L}(\hat{\mathbf{y}}(\mathbf{x})\text{-}\mathbf{t})]=\begin{bmatrix}\hat{\hat{y}}_1(\mathbf{x})\text{-}t_1 & \hat{\hat{y}}_2(\mathbf{x})\text{-}t_2\end{bmatrix}\begin{bmatrix}w_1\sigma_{22}(\mathbf{x}) & \text{-}w_1\sigma_{12}(\mathbf{x})\\ \text{-}w_2\sigma_{12}(\mathbf{x}) & w_2\sigma_{11}(\mathbf{x})\end{bmatrix}\frac{1}{\sigma_{11}(\mathbf{x})\sigma_{22}(\mathbf{x})\text{-}\sigma_{12}^2(\mathbf{x})} \tag{29}$$

$$\begin{bmatrix}\hat{\hat{y}}_1(\mathbf{x})\text{-}t_1\\ \hat{\hat{y}}_2(\mathbf{x})\text{-}t_2\end{bmatrix}+trace\begin{bmatrix}w_1 & 0\\ 0 & w_2\end{bmatrix}$$

$$\mathrm{E}[\mathrm{L}(\hat{\mathbf{y}}(\mathbf{x})\text{-}\mathbf{t})]=\frac{w_1\sigma_{22}(\mathbf{x})}{\sigma_{11}(\mathbf{x})\sigma_{22}(\mathbf{x})\text{-}\sigma_{12}^2(\mathbf{x})}(\hat{\hat{y}}_1(\mathbf{x})\text{-}t_1)^2+\frac{w_2\sigma_{11}(\mathbf{x})}{\sigma_{11}(\mathbf{x})\sigma_{22}(\mathbf{x})\text{-}\sigma_{12}^2(\mathbf{x})}(\hat{\hat{y}}_2(\mathbf{x})\text{-}t_2)^2 \tag{30}$$

$$\text{-}\frac{(w_1+w_2)\sigma_{12}(\mathbf{x})}{\sigma_{11}(\mathbf{x})\sigma_{22}(\mathbf{x})\text{-}\sigma_{12}^2(\mathbf{x})}(\hat{\hat{y}}_1(\mathbf{x})\text{-}t_1)(\hat{\hat{y}}_2(\mathbf{x})\text{-}t_2)+w_1+w_2$$

In the loss function provided in Equation (30), there are two terms which represent the effects of each quality characteristic separately in addition to the interaction term which represents the loss resulting from the interaction of two quality characteristics. The interaction term only represents the interaction due to statistical dependency.

3.1.3 NO STATISTICAL DEPENDENCY BETWEEN RESPONSES

The third instance is the case where the quality characteristics are not statistically dependent but there is preferential dependency between the responses which means the correlation term, σ_{12}, is equal to zero and the cost matrix becomes as in Equation (31).

$$\mathbf{C}=\begin{bmatrix}w_1\sigma_{22}(\mathbf{x}) & k_1\sigma_{11}(\mathbf{x})\\ k_2\sigma_{22}(\mathbf{x}) & w_2\sigma_{11}(\mathbf{x})\end{bmatrix}\frac{1}{\sigma_{11}(\mathbf{x})\sigma_{22}(\mathbf{x})} \tag{31}$$

The expected loss function resulting from the cost matrix in Equation (31) is provided in Equation (32) and open form of the expected loss function is provided in Equation (33).

$$E[L(\hat{\mathbf{y}}(\mathbf{x})\text{-}\mathbf{t})]=\begin{bmatrix}\hat{y}_1(\mathbf{x})\text{-}t_1 & \hat{y}_2(\mathbf{x})\text{-}t_2\end{bmatrix}\begin{bmatrix}w_1\sigma_{22}(\mathbf{x}) & k_1\sigma_{11}(\mathbf{x})\\ k_2\sigma_{22}(\mathbf{x}) & w_2\sigma_{11}(\mathbf{x})\end{bmatrix}\frac{1}{\sigma_{11}(\mathbf{x})\sigma_{22}(\mathbf{x})}\begin{bmatrix}\hat{y}_1(\mathbf{x})\text{-}t_1\\ \hat{y}_2(\mathbf{x})\text{-}t_2\end{bmatrix}+trace\begin{bmatrix}w_1 & k_1\\ k_2 & w_2\end{bmatrix} \tag{32}$$

$$E[L(\hat{\mathbf{y}}(\mathbf{x})\text{-}\mathbf{t})]=\frac{w_1}{\sigma_{11}(\mathbf{x})}(\hat{y}_1(\mathbf{x})\text{-}t_1)^2+\frac{w_2}{\sigma_{22}(\mathbf{x})}(\hat{y}_2(\mathbf{x})\text{-}t_2)^2-\frac{\sigma_{11}(\mathbf{x})k_1+\sigma_{22}(\mathbf{x})k_2}{\sigma_{11}(\mathbf{x})\sigma_{22}(\mathbf{x})}(\hat{y}_1(\mathbf{x})\text{-}t_1)(\hat{y}_2(\mathbf{x})\text{-}t_2)+w_1+w_2 \tag{33}$$

There are two terms representing the individual effects of quality characteristics on the quality loss, which are identical to the no dependency case. There is also the interaction term to represent the effect of preferential dependency between two quality characteristics on quality loss.

3.1.4 GENERAL FORM

The general form represents the case where both statistical and preferential dependency between quality characteristics are considered in the construction of the loss function. The general form of the cost matrix is provided in Equation (34).

$$\mathbf{C}=\begin{bmatrix}w_1\sigma_{22}(\mathbf{x})\text{-}k_1\sigma_{12}(\mathbf{x}) & \text{-}w_1\sigma_{12}(\mathbf{x})+k_1\sigma_{11}(\mathbf{x})\\ k_2\sigma_{22}(\mathbf{x})\text{-}w_2\sigma_{12}(\mathbf{x}) & \text{-}k_2\sigma_{12}(\mathbf{x})+w_2\sigma_{11}(\mathbf{x})\end{bmatrix}\frac{1}{\sigma_{11}(\mathbf{x})\sigma_{22}(\mathbf{x})\text{-}\sigma_{12}^2(\mathbf{x})} \tag{34}$$

The general form of the expected loss function can be derived as in Equation (35).

$$E[L(\hat{\mathbf{y}}(\mathbf{x})\text{-}\mathbf{t})]=\begin{bmatrix}\hat{y}_1(\mathbf{x})\text{-}t_1 & \hat{y}_2(\mathbf{x})\text{-}t_2\end{bmatrix}\begin{bmatrix}w_1\sigma_{22}(\mathbf{x})\text{-}k_1\sigma_{12}(\mathbf{x}) & \text{-}w_1\sigma_{12}(\mathbf{x})+k_1\sigma_{11}(\mathbf{x})\\ k_2\sigma_{22}(\mathbf{x})\text{-}w_2\sigma_{12}(\mathbf{x}) & \text{-}k_2\sigma_{12}(\mathbf{x})+w_2\sigma_{11}(\mathbf{x})\end{bmatrix}\frac{1}{\sigma_{11}(\mathbf{x})\sigma_{22}(\mathbf{x})\text{-}\sigma_{12}^2(\mathbf{x})}\begin{bmatrix}\hat{y}_1(\mathbf{x})\text{-}t_1\\ \hat{y}_2(\mathbf{x})\text{-}t_2\end{bmatrix}+trace\begin{bmatrix}w_1 & k_1\\ k_2 & w_2\end{bmatrix} \tag{35}$$

The derived expected loss function in open form is provided in Equation (36).

$$\mathrm{E}[\mathrm{L}(\hat{\mathbf{y}}(\mathbf{x})\text{-}\mathbf{t})]=\frac{(w_1\sigma_{22}(\mathbf{x})\text{-}k_1\sigma_{12}(\mathbf{x}))}{\sigma_{11}(\mathbf{x})\sigma_{22}(\mathbf{x})\text{-}\sigma_{12}^2(\mathbf{x})}(\hat{y}_1(\mathbf{x})\text{-}t_1)^2+\frac{(w_2\sigma_{11}(\mathbf{x})\text{-}k_2\sigma_{12}(\mathbf{x}))}{\sigma_{11}(\mathbf{x})\sigma_{22}(\mathbf{x})\text{-}\sigma_{12}^2(\mathbf{x})} \tag{36}$$

$$(\hat{y}_2(\mathbf{x})\text{-}t_2)^2+\frac{\sigma_{11}(\mathbf{x})k_1+\sigma_{22}(\mathbf{x})k_2\text{-}(w_1+w_2)\sigma_{12}(\mathbf{x})}{\sigma_{11}(\mathbf{x})\sigma_{22}(\mathbf{x})\text{-}\sigma_{12}^2(\mathbf{x})}(\hat{y}_1(\mathbf{x})\text{-}t_1)(\hat{y}_2(\mathbf{x})\text{-}t_2)+w_1+w_2$$

In the general form, terms reflecting the effects of individual quality characteristics and the interaction term become more complex when compared to the previous cases and the effects of individual parameters become harder to observe. The general form of loss function is similar for different numbers of quality characteristics. The general form of loss function when there are three quality characteristics under consideration is provided in Appendix A.

3.2 ANALYSIS ON THE STRUCTURE OF C MATRIX

Structure of **C** matrix is analyzed in this subsection. The analysis is done by evaluating the meanings of **K** and **Σ** matrices using an analogy between univariate loss functions and multivariate loss functions.

The univariate loss function proposed by Taguchi is shown in Equation (37).

$$\mathrm{L}(y(\mathbf{x}))=\mathrm{c}\ (y(\mathbf{x})\text{-}t)^2 \tag{37}$$

By rearranging the terms, it can be written as the Equation (38) which is similar to multivariate loss functions.

$$\mathrm{L}(y(\mathbf{x}))=(y(\mathbf{x})\text{-}t)\ c\ (y(\mathbf{x})\text{-}t) \tag{38}$$

By adopting the decomposition proposed by Vining, the coefficient c can be written as $c=k(\sigma^2)^{-1}$. Hence, the loss function becomes as in Equation (39).

$$\mathrm{L}(y(\mathbf{x}))=(y(\mathbf{x})\text{-}t)\ k\ (\sigma^2)^{-1}\ (y(\mathbf{x})\text{-}t) \tag{39}$$

In the univariate loss functions, the term c is calculated as described in Equations (40), (41) and (42) using the notions provided in Figure 4.

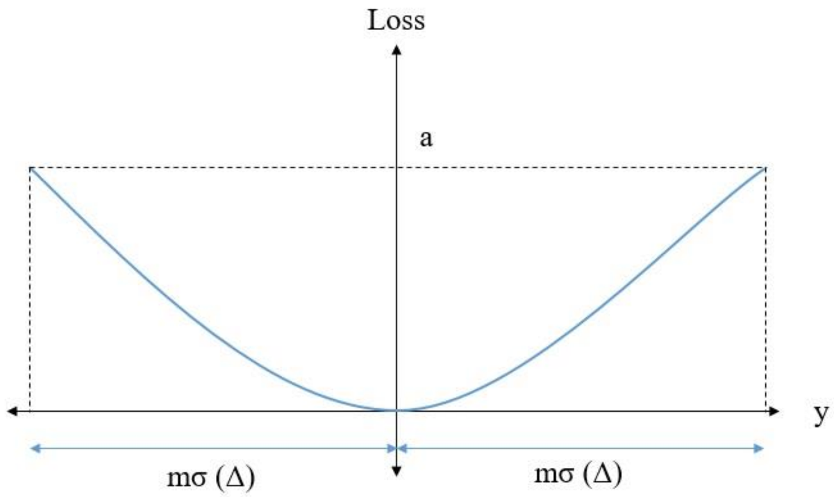

Figure 4 Univariate Loss Function

$$L(y(\mathbf{x}))=\frac{a}{\Delta^2}(y(\mathbf{x})-t)^2 \tag{40}$$

$$L(y(\mathbf{x}))=\frac{a}{(m\sigma)^2}(y(\mathbf{x})-t)^2 \tag{41}$$

$$L(y(\mathbf{x}))=\frac{a}{m^2}\left(\frac{y(\mathbf{x})-t}{\sigma}\right)^2 \tag{42}$$

In Equation (42), $\left(\frac{(y(\mathbf{x})-t)}{\sigma}\right)^2$ represents the normalized distance from target and c is $\frac{a}{(m\sigma)^2}$. Hence, we can derive k as shown in Equations (43) and (44).

$$\frac{k}{\sigma^2}=\frac{a}{(m\sigma)^2} \tag{43}$$

$$k=\frac{a}{m^2} \tag{44}$$

Since a and m values do not change through the function, losses at points with the same statistical distances from the target should be the same for a symmetric loss function in the univariate case. In order to check whether this is also true or not in the multivariate loss functions the following counterexample is studied.

Let $\boldsymbol{\mu} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ and $\boldsymbol{\Sigma} = \begin{pmatrix} 2 & -0.5 \\ -0.5 & 1 \end{pmatrix}$.

In the multivariate loss functions, the statistical distance is equivalent to the Mahalanobis distance. Most of the multivariate techniques are based on distances according to Johnson and Wichern (2002). The loss functions used in this study are also a statistical distance and are similar to the Mahalanobis distance. Maesschalck et al. (2000) state that the Mahalanobis distance between two vectors $\mathbf{y}_1$ and $\mathbf{y}_2$ is calculated as shown in Equation (45).

$$d(\mathbf{y}_1,\mathbf{y}_2)=\sqrt{(\mathbf{y}_1-\mathbf{y}_2)'\boldsymbol{\Sigma}^{-1}(\mathbf{y}_1-\mathbf{y}_2)} \tag{45}$$

$\boldsymbol{\Sigma}$ matrix corresponds to the variance-covariance matrix of vectors $\mathbf{y}_1$ and $\mathbf{y}_2$.

Let us choose four points which have the same Mahalanobis distance, 1 unit, from the mean.

These four points are:

- $P_1 = (0, 3)$
- $P_2 = (1, 1)$
- $P_3 = (3, 3)$
- $P_4 = (4, 1)$

Since these four points have the same Mahalanobis distances from the mean, their frequencies should be equal if the assumption is also valid for multivariate loss functions.

Let $\mathbf{T}=\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ and $\mathbf{K}=\begin{pmatrix} 2 & 0.5 \\ 0.5 & 3 \end{pmatrix}$

Then the losses calculated using $(\mathbf{Y}\text{-}\mathbf{T})'\mathbf{C}(\mathbf{Y}\text{-}\mathbf{T})$ where $\mathbf{C}=\mathbf{K}\boldsymbol{\Sigma}^{-1}$ are available below.

- $L(P_1) = 4.14$
- $L(P_2) = 7.14$
- $L(P_3) = 7.14$
- $L(P_4) = 4.14$

If we ask the decision maker not to consider the variation correlations we assume that $\boldsymbol{\Sigma}_I=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ we expect the answers below using $(\mathbf{Y}\text{-}\mathbf{T})'\mathbf{C}(\mathbf{Y}\text{-}\mathbf{T})$ where $\mathbf{C} = \mathbf{K}$.

- $L^I(P_1) = 9$
- $L^I(P_2) = 6$
- $L^I(P_3) = 6$
- $L^I(P_4) = 9$

Since all of the points have the same frequencies according to Mahalanobis distance we expect that the expected frequencies in the following equations to be almost equal.

- $9 \times E(frequency_1) = 4.14$
- $6 \times E(frequency_2) = 7.14$
- $6 \times E(frequency_3) = 7.14$
- $9 \times E(frequency_4) = 4.14$

The expected frequencies are:

- $E(frequency_1) = 0.46031746$

- E($frequency_2$) = 1.19047619
- E($frequency_3$) = 1.19047619
- E($frequency_4$) = 0.46031746

The expected frequencies are not equal meaning that losses at points with same statistical distances from the target are not equal to each other in multivariate loss functions. Hence, **K** matrix does not perfectly reflect the preference structure of the decision maker and the process economics excluding the variance-covariance structure of the quality characteristics.

3.3 ANALYSIS ON THE EFFECTS OF C MATRIX ON THE LOSS FUNCTION STRUCTURE

In this subsection, the effects of components of **C** matrix on the loss function structure is demonstrated using three examples.

In the first example the effect of off-diagonal terms, c_{12} and c_{21}, is illustrated in Figure 5 below.

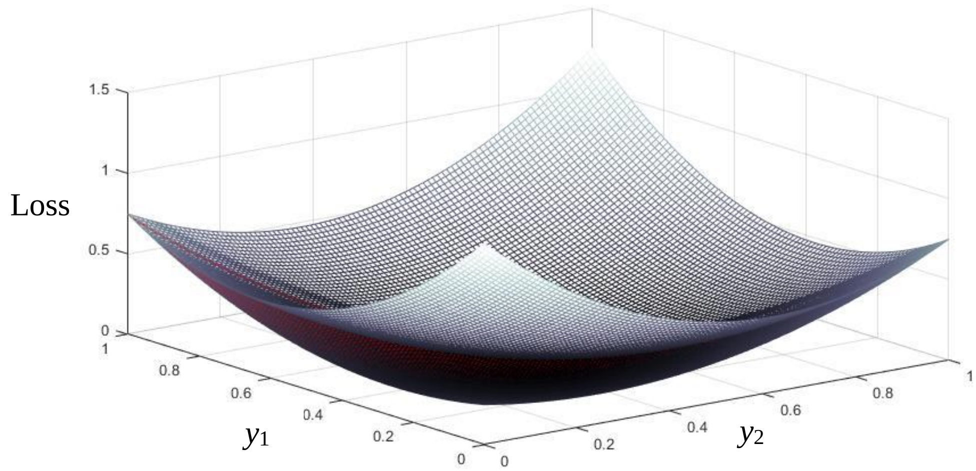

Figure 5 Effect of Diagonal Elements of **C**

The two loss functions presented in Figure 5 have the same **C** matrix parameters except c_{12}. The preference coefficients of the steeper loss function are as twice as the preference coefficients of the less steep loss function.

Figure 6 illustrates the effect of diagonal elements of **C** matrix. The off-diagonal elements of the **C** matrices of the two loss functions are equal to each other while the diagonal elements of the steeper loss function are as twice as the diagonal elements of the less steep loss function.

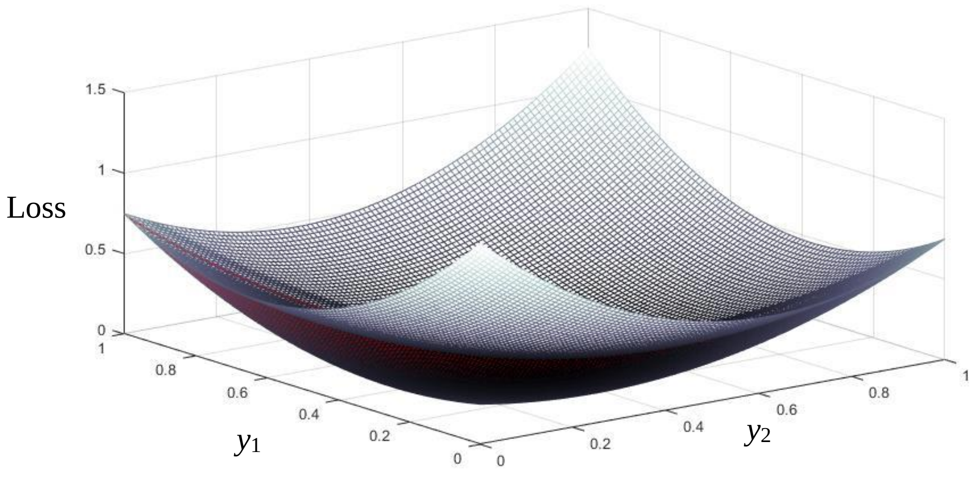

Figure 6 Effect of Off-Diagonal Elements of **C**

Finally, Figure 7 represents the simultaneous effects of both diagonal and off-diagonal elements of the **C** matrix. All elements of the cost matrix of steeper loss function are as twice as the elements of the cost matrix of the remaining loss function.

The examples provided show that all of the components of **C** matrix can significantly affect the structure of the loss function. As the components of **C** matrix increase, the loss function becomes steeper. It can be also concluded that although it is important to obtain the relative values of the components of **C** matrix, it is not sufficient to acquire the true underlying quality loss function of the decision maker. Hence, true estimation of the **C** matrix is crucial to obtain the real quality loss function of the decision maker.

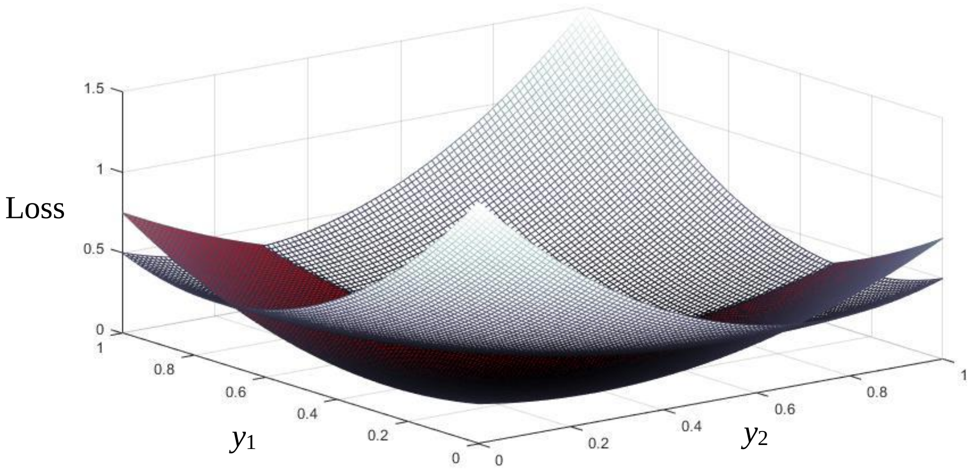

Figure 7 Effect of **C** Matrix

3.4 ANALYSIS OF THE EFFECTS OF K MATRIX ON THE LOSS FUNCTION STRUCTURE

In this subsection, the **K** matrix which includes the effects of individual weights of quality characteristics and preferential dependency between them to the cost matrix **C,** is analyzed. Vining (1998) and Ko (2005) defines **K** as a typically diagonal matrix. In order to show whether the **K** matrix must be diagonal or not, structures of the cost matrix and variance-covariance matrix and their relationship to the **K** matrix should be examined.

The cost matrix, **C**, and variance-covariance matrix, **Σ**, are symmetric matrices by definition.

$$\mathbf{C}=\mathbf{K}\,\mathbf{\Sigma}^{-1} \tag{46}$$

By multiplying both sides of the equation with **Σ** we obtain Equation (47)

$$\mathbf{C}\,\mathbf{\Sigma}=\mathbf{K}\,\mathbf{\Sigma}^{-1}\,\mathbf{\Sigma} \tag{47}$$

Hence,

$$\mathbf{C}\,\mathbf{\Sigma} = \mathbf{K} \tag{48}$$

Both **C** and **Σ** are symmetric matrices. However, this condition does not guarantee that **K** will be also symmetric as it can be seen in the example below.

Let **C** be $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and **Σ** be $\begin{bmatrix} 3 & 1 \\ 1 & 5 \end{bmatrix}$.

Then, $\mathbf{C}\,\mathbf{\Sigma} = \mathbf{K} = \begin{bmatrix} 7 & 7 \\ 5 & 11 \end{bmatrix}$ which is not a symmetric matrix. Therefore, we cannot conclude that **K** is a symmetric matrix.

If **K** matrix is not diagonal, it indicates that there is a preferential dependency between quality characteristics. **K** matrix is expected to be symmetric if it is not diagonal. The off-diagonal terms reflect the preferential dependency between quality characteristic pairs and they should be equal to each other for each and every pair regardless of the order of comparison. However, as it can be seen from the example above, **K** matrix is not necessarily symmetric. **K** matrix is not a perfect measure of the decision maker as it is shown in Subsection 3.2. It only provides insight on the preferences of the decision maker excluding the variance-covariance structure of the quality characteristics. Hence, **K** matrix can also be asymmetric.

Pignatiello (1993) uses the **C** matrix given in Equation (49) in his paper.

$$\mathbf{C} = \begin{bmatrix} 1.0 & 0.5 \\ 0.5 & 2.5 \end{bmatrix} \tag{49}$$

When all of the three controllable variables are at their minimum level the **Σ** matrix in Equation (50) is obtained.

$$\mathbf{\Sigma} = \begin{bmatrix} 0.20 & 0.17 \\ 0.17 & 0.21 \end{bmatrix} \tag{50}$$

Using the **C** $\boldsymbol{\Sigma}$ =**K**, we obtain the **K** matrix in Equation (51) which is an asymmetric matrix.

$$\mathbf{K}= \begin{bmatrix} 0.29 & 0.28 \\ 0.53 & 0.61 \end{bmatrix} \tag{51}$$

Hence, to examine all the possible cases, an analysis is made considering three cases of **K** matrix which are diagonal, symmetric and asymmetric. For each case of **K** matrix, three different $\boldsymbol{\Sigma}$ matrices reflecting the cases where there is no correlation, positive correlation and negative correlation between two quality characteristics correspondingly are analyzed. $\boldsymbol{\Sigma}$ and corresponding $\boldsymbol{\Sigma}^{-1}$ matrices for each correlation type is presented in Table 2 below.

Table 2 Variance-Covariance Matrices for Different Correlation Types

	$\boldsymbol{\Sigma}$	$\boldsymbol{\Sigma}^{-1}$	**Correlation coefficient (ρ)**
No correlation	$\begin{bmatrix} 1.00 & 0.00 \\ 0.00 & 1.00 \end{bmatrix}$	$\begin{bmatrix} 1.00 & 0.00 \\ 0.00 & 1.00 \end{bmatrix}$	0.0
Positive correlation	$\begin{bmatrix} 1.00 & 0.50 \\ 0.50 & 1.00 \end{bmatrix}$	$\begin{bmatrix} 1.33 & -0.67 \\ -0.67 & 1.33 \end{bmatrix}$	0.5
Negative correlation	$\begin{bmatrix} 1.00 & -0.50 \\ -0.50 & 1.00 \end{bmatrix}$	$\begin{bmatrix} 1.33 & 0.67 \\ 0.67 & 1.33 \end{bmatrix}$	-0.5

For these three different correlation cases, the effect of different types of **K** matrix on quality loss function is analyzed in the following subsections.

3.4.1 DIAGONAL K MATRIX

In the first case, the **K** matrix is assumed to be diagonal meaning that there is no preferential dependency between quality characteristics. The diagonal **K** matrix is chosen as:

$$\mathbf{K} = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}$$

Figure 8 represents the contour plots of quality loss functions under different correlation cases.

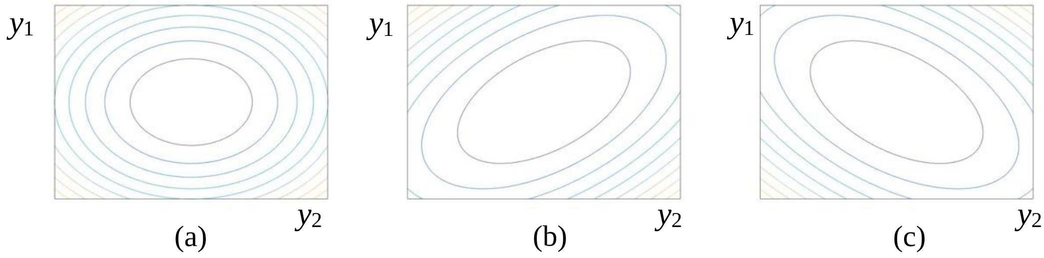

Figure 8 Contour Plots of Loss Functions with Diagonal **K** Matrix (a) rho=0.0, (b) rho=0.5, (c) rho= -0.5

As it can be perceived from the Figure 8(a), when there is no correlation between responses, contours of the loss function are not skewed since there is no linear dependency between the responses. On the other hand, when there is positive correlation between the quality characteristics that can be seen in Figure 8(b), quality loss increases at a slower rate when both quality characteristics increase or decrease at the same time. When the quality characteristics are negatively correlated which is illustrated in Figure 8(c), the loss function increases with an increasing rate when the quality characteristics simultaneously increase or decrease. Hence, the orientation of the loss function depends on the correlation when the **K** matrix is diagonal.

3.4.2 SYMMETRIC K MATRIX

K matrix is symmetric in the second case. There can be both positive or both negative preferential dependency between the quality characteristics. Hence two different **K** matrices are analyzed here which are:

$$\mathbf{K_1} = \begin{bmatrix} 1.0 & 0.5 \\ 0.5 & 1.0 \end{bmatrix} \qquad \mathbf{K_2} = \begin{bmatrix} 1.0 & -0.5 \\ -0.5 & 1.0 \end{bmatrix}$$

Figures 9 and 10 represents the contours of loss functions for different correlation types for $\mathbf{K_1}$ and $\mathbf{K_2}$ matrices respectively.

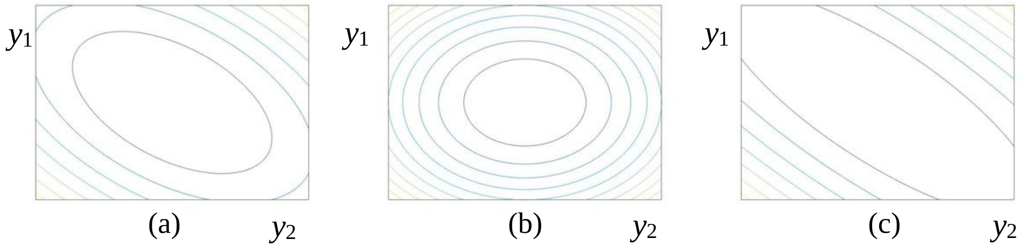

Figure 9 Contour Plots of Loss Functions with Symmetric $\mathbf{K_1}$ Matrix (a) rho=0.0, (b) rho=0.5, (c) rho= -0.5

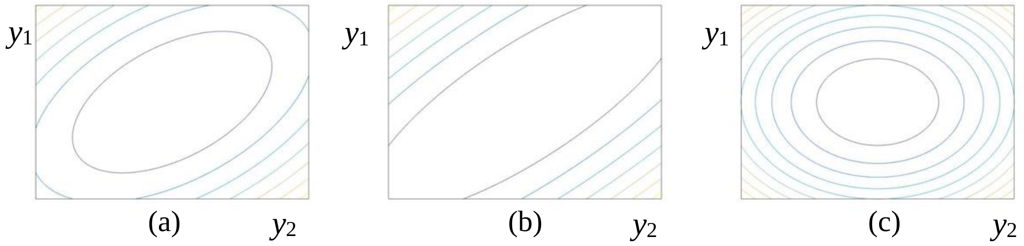

Figure 10 Contour Plots of Loss Functions with Symmetric $\mathbf{K_2}$ Matrix (a) rho=0.0, (b) rho=0.5, (c) rho= -0.5

As it can be observed from the Figures 9(a) and 10(a), when there is no correlation between the quality characteristics, the orientation of the loss function depends on the **K** matrix for both cases. The rate of increase escalates when exactly one of the preferential dependency and correlation is positive while the other one is negative, which are illustrated in 9(c) and 10(b). Conversely, the effects of preferential dependency and statistical dependency cancel each other when they are both negative or positive, as can be seen in Figure 9(b) and 10(c). We can also observe that identical **K** and **Σ** matrices have contrary effects on the loss function. When there is positive correlation between responses, the rate of increase of the loss decreases when both quality characteristic increase or decrease. However, when there is positive preferential dependency between responses, the quality loss becomes larger more rapidly when both quality characteristics simultaneously increase or decrease.

3.4.3 ASYMMETRIC K MATRIX

The third and final type of **K** matrix is asymmetric. Two different **K** matrices which are transpose of each other are analyzed. The **K** matrices are:

$$\mathbf{K_3} = \begin{bmatrix} 1.0 & 0.5 \\ -0.5 & 1.0 \end{bmatrix} \qquad \mathbf{K_4} = \begin{bmatrix} 1.0 & -0.5 \\ 0.5 & 1.0 \end{bmatrix}$$

The contours of the loss function under different type of correlations are represented in Figures 11 and 12 for matrices $\mathbf{K_3}$ and $\mathbf{K_4}$ respectively.

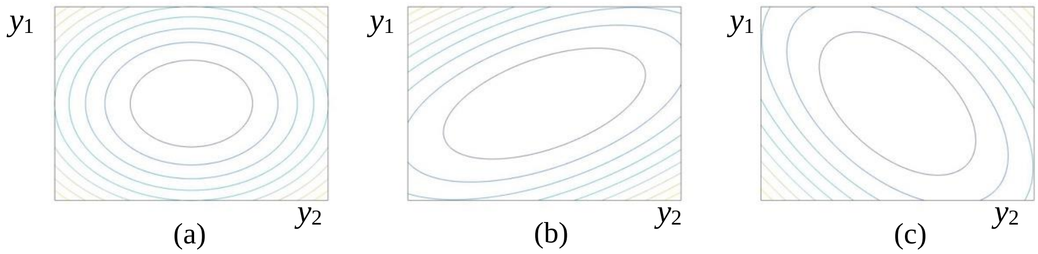

Figure 11 Contour Plots of Loss Functions with Asymmetric $\mathbf{K_3}$ Matrix (a) rho=0.0, (b) rho=0.5, (c) rho= -0.5

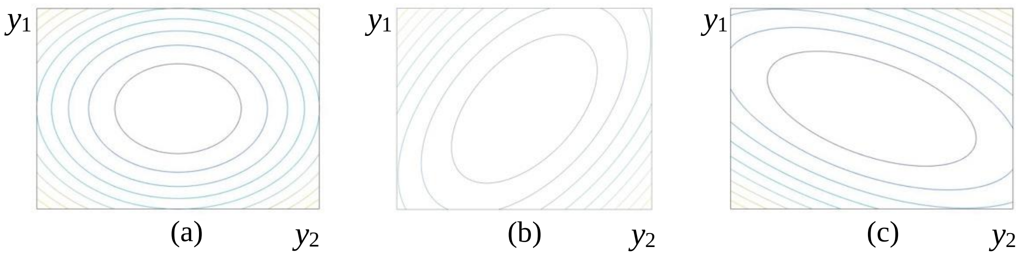

Figure 12 Contour Plots of Loss Functions with Asymmetric $\mathbf{K_4}$ Matrix (a) rho=0.0, (b) rho=0.5, (c) rho= -0.5

The loss function behaves similar to the loss function with diagonal **K** matrix in both cases. If there is no correlation between the quality characteristics, there is no statistical dependency. Hence, from section 3.1.3 we can observe that only the interaction term includes the preferential dependency constants k_1 and k_2. Since we assumed variances of both quality characteristics are equal, the coefficient of interaction term becomes zero and loss function becomes identical to the no correlation loss function with diagonal **K** matrix in both cases. For positive correlation and negative correlation examples, the orientation of quality loss function slightly differs from the loss function in 3.4.1 because individual effects of quality characteristics also depend on the preferential dependency constants.

3.5 THE REPRESENTATIONAL POWER OF BIVARIATE QUADRATIC LOSS FUNCTIONS

The quadratic multivariate loss functions are monotonous functions and cannot represent all possible value functions a decision maker can have. The real value functions of the decision makers may have different shapes which are not compatible with quadratic loss functions. In this subsection, the fit to the closest loss function for different types of value functions are examined to illustrate the representational power of quadratic loss functions considering two quality characteristics. The examples which have four different correlation types among responses and eight different value functions. It is assumed that there exist two quality characteristics ($y_1(\mathbf{x})$ and $y_2(\mathbf{x})$) and their values depend on 3 factors (x_1, x_2 and x_3). Design parameter x_1 is uniformly distributed between 0 and 1 and other factors are assumed to be constant at their chosen values for the sake of simplicity.

The different types of correlations included in the study can be shown as:

Strong Positive Correlation (SPC)

$$y_1(\mathbf{x})=4x_1+4x_2+3x_3 \tag{52}$$

$$y_2(\mathbf{x})=8x_1+8x_2+6x_3 \tag{53}$$

Strong Negative Correlation (SNC)

$$y_1(\mathbf{x})=x_1-x_2+x_3 \tag{54}$$

$$y_2(\mathbf{x})=-x_1+x_2-x_3 \tag{55}$$

Weak Negative Correlation (WNC)

$$y_1(\mathbf{x})=x_1x_2x_3+x_2x_3 \tag{56}$$

$$y_2(\mathbf{x})=x_1^2x_2x_3-x_2x_3^2 \tag{57}$$

Weak Positive Correlation (WPC)

$$y_1(\mathbf{x})=x_1+2x_2-3x_3 \tag{58}$$

$$y_2(\mathbf{x})=2x_1+x_2+3x_3 \tag{59}$$

The variance- covariance information for the four different cases are available in Table 3.

Table 3 Variance-Covariance Information for Different Correlation Cases

	Variance of y_1	**Variance of y_2**	**Covariance**	**Pearson Correlation Coefficient**
SPC	5.161	20.644	10.322	1.000
SNC	0.378	0.378	-0.378	-1.000
WPC	1.762	0.755	0.126	0.109
WNC	0.161	0.0698	-0.040	-0.375

Since the value function of the decision maker may affect the performance of loss function, following value functions are constructed.

Linear Value Functions

$$v_1(\mathbf{y}(\mathbf{x}))=0.1y_1(\mathbf{x})+0.9\,y_2(\mathbf{x}) \tag{60}$$

$$v_2(\mathbf{y}(\mathbf{x}))=0.5y_1(\mathbf{x})-0.5\,y_2(\mathbf{x}) \tag{61}$$

Multiplicative Value Functions

$$v_3(\mathbf{y}(\mathbf{x}))=y_1(\mathbf{x})^2+y_1(\mathbf{x})\, y_2(\mathbf{x})+ y_2(\mathbf{x})^2 \tag{62}$$

$$v_4(\mathbf{y}(\mathbf{x}))=y_1(\mathbf{x})^2\, y_2(\mathbf{x})^3 \tag{63}$$

Tchebycheff Value Functions

$$v_5(\mathbf{y}(\mathbf{x}))=max\{0.1|t_1-y_1(\mathbf{x})|,\ 0.9|t_2- y_2(\mathbf{x})|\} \tag{64}$$

$$v_6(\mathbf{y}(\mathbf{x}))=max\{0.5|t_1-y_1(\mathbf{x})|,\ 0.5|t_2- y_2(\mathbf{x})|\} \tag{65}$$

According to the value functions, following objectives are determined for the responses.

Table 4 Objectives of the Selected Value Functions

Value Functions	$\boldsymbol{y_1}$	$\boldsymbol{y_2}$
$v_1(\mathbf{y}(\mathbf{x}))$	Maximize	Maximize
$v_2(\mathbf{y}(\mathbf{x}))$	Maximize	Minimize
$v_3(\mathbf{y}(\mathbf{x}))$	Maximize	Maximize
$v_4(\mathbf{y}(\mathbf{x}))$	Maximize	Maximize
$v_5(\mathbf{y}(\mathbf{x}))$	Target is the best	Target is the best
$v_6(\mathbf{y}(\mathbf{x}))$	Target is the best	Target is the best

It is assumed that the decision maker is able to provide perfect information when he/she is asked questions about his/her preferences.

The loss value at the target is assumed to be equal to zero. For the parameter determination purposes, three other points are asked to the decision maker. His/her answers are available in Table 5.

Table 5 Decision Maker's Answers for Strong Correlation Cases

	SPC			**SNC**		
	y_1, y_2^t	y_1^t, y_2	y_1, y_2	y_1, y_2^t	y_1^t, y_2	y_1, y_2
$v_1(\mathbf{y}(\mathbf{x}))$	1.1	19.8	20.9	0.3	2.7	3.0
$v_2(\mathbf{y}(\mathbf{x}))$	5.5	11.0	165.0	1.5	1.5	3
$v_3(\mathbf{y}(\mathbf{x}))$	363.0	726.0	847.0	6.0	3.0	0.0
$v_4(\mathbf{y}(\mathbf{x}))$	1288408.0	1288408.0	1288408.0	3.0	36.0	12.0
$v_5(\mathbf{y}(\mathbf{x}))$	1.1	19.8	19.8	0.3	2.7	2.7
$v_6(\mathbf{y}(\mathbf{x}))$	5.5	11.0	11.0	1.5	1.5	1.5

Table 6 Decision Maker's Answers for Weak Correlation Cases

	WPC			**WNC**		
	y_1, y_2^t	y_1^t, y_2	y_1, y_2	y_1, y_2^t	y_1^t, y_2	y_1, y_2
$v_1(\mathbf{y}(\mathbf{x}))$	0.6	5.4	6.0	0.2	1.2	1.4
$v_2(\mathbf{y}(\mathbf{x}))$	3.0	3.0	6.0	1.0	0.7	1.7
$v_3(\mathbf{y}(\mathbf{x}))$	36.0	54.0	54.0	3.9	1.9	3.9
$v_4(\mathbf{y}(\mathbf{x}))$	0	1944.0	1944.0	0.2	4.2	0.2
$v_5(\mathbf{y}(\mathbf{x}))$	0.6	5.4	5.4	0.2	1.2	1.2
$v_6(\mathbf{y}(\mathbf{x}))$	3.0	3.0	3.0	1.0	0.7	1.0

In order to determine the components of the cost matrix, the method proposed by Pignatello (1993), which is available in Chapter 2, is used.

The loss function coefficients obtained are available in Table 7 and Table 8.

In order to test the fit of the loss functions to underlying value functions, a comparison between the optimal values obtained by real value functions (V) and multivariate loss functions (L) is provided in Table 9.

Table 7 Loss Function Coefficients for Strong Correlation Cases

	SPC			**SNC**		
	c_{11}	c_{22}	c_{12}	c_{11}	c_{22}	c_{12}
$v_1(\mathbf{y}(\mathbf{x}))$	0.009	0.041	0.000	0.033	0.300	0.000
$v_2(\mathbf{y}(\mathbf{x}))$	0.045	0.023	0.000	0.167	0.167	0.000
$v_3(\mathbf{y}(\mathbf{x}))$	3.000	1.500	-0.500	0.667	0.333	-0.500
$v_4(\mathbf{y}(\mathbf{x}))$	10648.000	2662.000	-2662.000	0.333	4.000	-1.500
$v_5(\mathbf{y}(\mathbf{x}))$	0.009	0.041	-0.002	0.033	0.300	-0.017
$v_6(\mathbf{y}(\mathbf{x}))$	0.045	0.023	-0.011	0.167	0.167	-0.0833

Table 8 Loss Function Coefficients for Weak Correlation Cases

	WPC			**WNC**		
	c_{11}	c_{22}	c_{12}	c_{11}	c_{22}	c_{12}
$v_1(\mathbf{y}(\mathbf{x}))$	0.017	0.150	0.000	0.050	0.652	0.000
$v_2(\mathbf{y}(\mathbf{x}))$	0.083	0.083	0.000	0.250	0.362	0.000
$v_3(\mathbf{y}(\mathbf{x}))$	1.000	1.500	-0.750	1.190	1.000	-0.500
$v_4(\mathbf{y}(\mathbf{x}))$	0.000	54.000	0.000	0.055	2.216	-2.776
$v_5(\mathbf{y}(\mathbf{x}))$	0.017	0.150	-0.008	0.050	0.652	-0.132
$v_6(\mathbf{y}(\mathbf{x}))$	0.083	0.083	-0.042	0.250	0.362	-0.454

In the strong correlation cases, for seven case out of twelve cases, loss function is able to find the true optimal solution. In the weak correlation cases, there are two cases where the results obtained from the value function and loss function match. The loss function gives better results in two cases due to the solver capabilities used in the solutions.

Table 9 Comparison of Optimal Values

		$v_1(\mathbf{y}(\mathbf{x}))$	$v_2(\mathbf{y}(\mathbf{x}))$	$v_3(\mathbf{y}(\mathbf{x}))$	$v_4(\mathbf{y}(\mathbf{x}))$	$v_5(\mathbf{y}(\mathbf{x}))$	$v_6(\mathbf{y}(\mathbf{x}))$
SPC	**V**	**20.90**	**0.00**	**847.00**	**1288408.00**	**0.00**	**0.00**
	L	**20.90**	-1.83	**847.00**	**1288408.00**	**0.00**	**0.00**
SNC	**V**	**0.80**	**2.00**	**4.00**	**1.00**	**0.27**	**0.75**
	L	0.56	**2.00**	0.69	0.02	0.32	**0.75**
WNC	**V**	**0.47**	**1.13**	**4.00**	**0.00**	**0.55**	0.75
	L	0.32	1.12	3.53	**0.00**	0.72	**0.28**
WPC	**V**	**5.40**	**0.50**	**36.00**	**500.00**	**0.10**	1.50
	L	5.16	**0.50**	30.00	0.00	1.48	**0.75**

Loss functions perform poorly and are not able to find the optimal points for half of the cases although it is assumed that the decision maker is perfectly consistent in his/her answers. Performance of the loss functions is expected to decrease further when the decision maker is not consistent while answering the questions and also is not able to tell the exact loss values of points under consideration. Hence loss functions are not appropriate to use if the underlying value function differs from a quadratic loss function.

CHAPTER 4

A METHOD FOR PARAMETER ESTIMATION

This chapter presents the general approach proposed for the estimation of multivariate quadratic loss function parameters associated with process economics and DM preferences. Firstly, an overview of the method is provided. In the following subsections, details of each step of the method are presented, which are initialization, selection of products to compare quality losses, estimation and comparison of the quality losses for selected products, optimization, and evaluation of fit.

4.1 OVERVIEW OF THE METHOD

This subsection represents the overview of the proposed method. Firstly, decision makers that will be included in the study are identified. Then, the products that will be used in the current iteration are selected according to the rules described in Subsection 4.3. The decision maker estimates the quality losses associated with selected products using the available data relevant to quality losses of products under consideration. Using the estimated loss values, products are compared in terms of their desirability. The comparison results are converted to model parameters and a mathematical model is constructed and solved in order to find optimal parameters of the multivariate loss function. The results obtained from the mathematical model are evaluated using some test products. The steps are repeated until the algorithm converges. The flow of the method can also be seen in Figure 13.

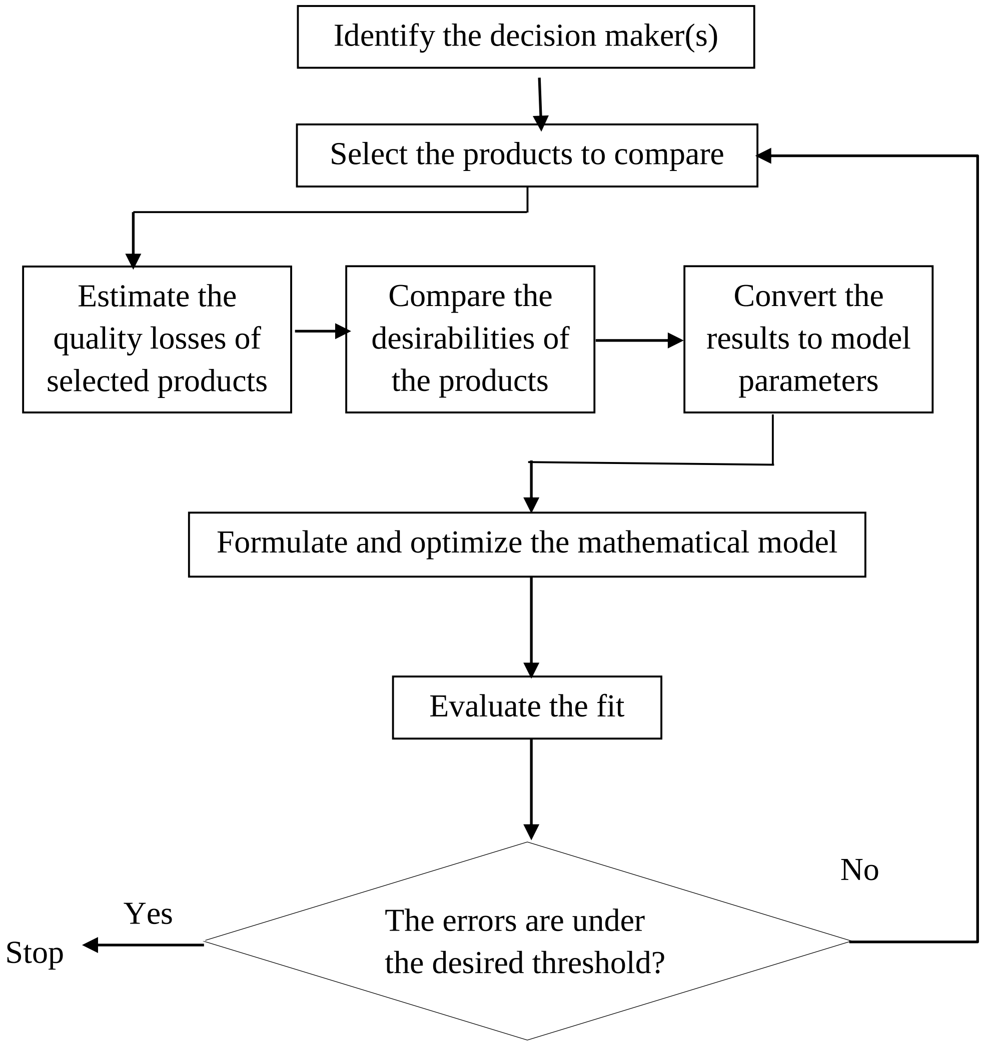

Figure 13 Flowchart of the Method

4.2 INITIALIZATION

In the initialization step, the decision makers of the problem should be determined. For different cases, the decision makers can include employees of the producer playing the role of internal customers for the product under consideration, customer representatives, and experts in the related field.

The study is usually initiated by the producer. Hence, the producer intrinsically takes part in the decision making process. However, as mentioned in Chapter 2, the quality loss is the loss to the society, which includes not only producer loss but also customer loss. Hence, if possible, customers should also take part in the decision making process. If the product is an end product, the customers are the end customers. On the other hand, if the product is a component (semi-finished product) the customer is the immediate successor of the current production process such as the chief of the assembly. If customers cannot participate in the process, experts such as quality and marketing managers may be included in the study to reflect the opinions of the customers.

4.3 SELECTION OF PRODUCTS TO COMPARE QUALITY LOSSES

The first step of the approach is selection of the products. Although there may be some restrictions in the product selection, the following rules are suggested to increase the speed of convergence of the approach.

Firstly, the products or services which cause extremely high quality loss should be excluded in the product selection process. The utilization of multivariate quadratic loss functions may be improper in the situations where consequences of the usage of product are not tolerable. For example, if the product or service with a certain set of quality characteristic values is very likely to cause the death of a user, the cost of quality of that particular product cannot be represented properly by the loss functions. Hence, it is suggested that all characteristics of the products selected should be in or around the specification limits.

As proposed by Taguchi (1986), the products at the upper specification limit (USL) or the lower specification limit (LSL) are selected to construct the univariate quality loss function since the calculation of quality loss is relatively easy at the specification limits. A similar approach is suggested here in order to limit the quality loss to repair and replacement costs which can be observed relatively easily.

The products should be selected in the neighborhood of specification limits if it is possible.

Maghsoodloo and Chang (2001) divide the quality loss into areas with respect to the design specifications and the products positions according to these specifications. For the bivariate case where both of the responses are nominal the best type, there are four regions, where both characteristics are below the target, first characteristic is above the target while second characteristic is below the target, both characteristics are above the target and finally first characteristic is below the target while second characteristic is above the target, respectively. Figure 14 demonstrates the approach mentioned above.

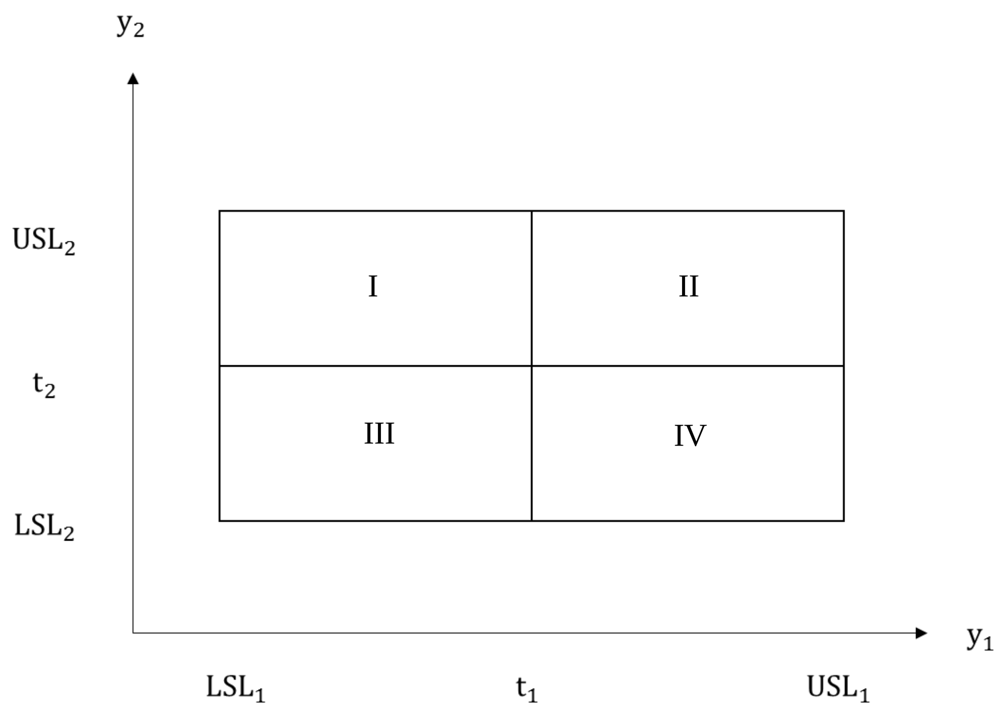

Figure 14 The Regions Constructed from Specification Limits and Targets

If we generalize this approach for a product with *n* different quality characteristics, we obtain 2^n regions. As shown in Chapter 3, **K** is not necessarily a symmetric matrix. Hence, for the n quality characteristic case, there are n^2 unknown parameters in the **K** matrix.

If there are more than four quality characteristics, $n^2 < 2^n$. We suggest n^2 different regions to be selected from 2^n areas in the first iteration of algorithm and one product to be selected from that particular areas. If the algorithm does not converge in the first iteration, then new products should be selected from the regions which are not used in the previous step.

In the cases where there are less than five quality characteristics, 2^n regions will not be sufficient for point selection, since $n^2 \geq 2^n$. 2^n areas may also not be sufficient until convergence for the cases where there are more than 4 quality characteristics. Under these circumstances, more than one product should be selected from each and every area at each step.

If only one product is to be selected from each region, the selection can be done randomly in the region. However, if more than one product needs to be selected from a region, the products should not be too similar in order to enable comparison between them. The decision makers may also suggest some sample products from the region under consideration to compare which they can compare easily. Finally, some random products may be selected and the decision maker may choose the product closest to the random point suggested.

If the time is a concern in the process, then the test points used in the previous iteration may be used as the selected points in the current iteration.

4.4 ESTIMATION AND COMPARISON OF THE QUALITY LOSSES FOR SELECTED PRODUCTS

The quality losses at the chosen points need to be estimated in order to guide the decision maker in the normalization process or comparisons. As mentioned in Chapter 2, quality loss may include various quality cost components. The decision maker may choose the costs which will be included in the cost calculations from the costs mentioned. Also he/she may add additional costs. However, the decision maker should be consistent while selecting the costs and use the same costs for the calculation of each point. In this study repair and replacement costs are considered in the estimated cost calculations which is a similar approach to univariate loss function studies mentioned in Chapter 2.

The estimation procedures for the quality loss may differ due to nature of the data available. There are different sources which are a part of customer value chain. For possible sources of data, corresponding strategies are presented below.

The service records in the warranty period of the product can be used to estimate losses if the customers cannot be included in the decision making process and past service records of the considered product are available. This method can be applied to an existing product or a derivative of an existing product. The service records should include all the related information to be proper to use. In this case, data mining tools can be useful to construct a proper data set in order to use in the subsequent steps. The number of repaired products and returned products can be analyzed to understand which failures cause the necessity of repair/return action. If a product is sold but is not returned or requested to be repaired, then it can be assumed that the customers have chosen to 'do nothing'. Pareto charts can be constructed to understand critical failures according to customers and quality loss of a product can be assigned accordingly.

Expert opinions can be used if the service records are not available and the customers cannot be included in the decision making process. Sales department of the producer can perform conjoint analysis to determine the quality values associated with the products which will be used in the study. Also expert opinions may be useful if the products require a test period or the defects are hard to recognize in the short term.

If the customers can be included in the study, a focus group which consists of ten to thirty customers can be formed in order to gather data. Their opinions for the quality loss of the given products should be collected and combined for each and every product which will be used in this study to obtain their estimated quality loss values. Also, the opinion of the producer can be included to obtain more realistic values.

When only the repair and replacement costs are considered the question that must be asked to the customers is "Would you request a replacement, ask for the considered product to be repaired or do nothing and continue to use the product?". If the decision makers are not customers then the question becomes "How many customers out of 100, on the average would request a replacement, ask for the considered product to be repaired or do nothing and continue to use the product?".

When the quality loss values for all products are determined, next step is performing a comparison study. The decision maker needs to compare each product pair considering his/her preferences and the quality loss values of the products using a method similar to AHP. While making the comparisons, other quality costs which cannot be calculated can also be included in the quality loss to a certain extent. For example, although the loss of goodwill is hard to calculate, the decision maker may have an opinion on the relative magnitudes of the costs caused by loss of goodwill when individual product pairs are considered. If there no additional costs which cannot be calculated, normalization can be used instead of using a comparison matrix.

If the number of products to compare is high, the practitioners may only calculate the first row of the comparison matrix and assume consistency in other rows in order to reduce the number of comparisons.

However, the results of comparison reflect the desirability of the products while the estimated quality losses are needed for the optimization step. Hence, the results need to be converted to reflect the relative quality losses of the products.

Let c_i be the priority vector value corresponding to product i, l_i be the loss coefficient of product i which is obtained by conversion of c_i, and b be the benchmark product in the current iteration. Benchmark product can be any product evaluated in the current iteration. The l_i values are calculated as shown in Equation (66).

$$l_i=\frac{c_b}{c_i} \tag{66}$$

4.5 OPTIMIZATION

Optimization is the step where parameters of the quadratic multivariate loss function are estimated using the data from the previous steps of the algorithm. A nonlinear mathematical model is used in the parameter estimation procedure.

The objective of the model can be stated as providing the best fit to the multivariate quadratic loss function by minimizing square errors which is a similar approach to ordinary least squares (OLS) method. The constraints of the model include the constraints which relate the comparison results to the quadratic loss function under consideration and other constraints which provide additional bounds on the parameters.

For the n quality characteristics case we have at least n^2 products in each iteration of the algorithm if the decision maker does not have a time constraint.

The mathematical model is defined as in Equations (67), (68), (69) and (70).

$$\textit{Minimize } \sum_{i=1}^{n} \epsilon_i^2 \tag{67}$$

Subject to

$$l_i Q=\left(\mathbf{y_i}(\mathbf{x})-\mathbf{t}\right)^{\mathrm{T}}\mathbf{C}\left(\mathbf{y_i}(\mathbf{x})-\mathbf{t}\right)+\epsilon_i \qquad \forall i=1,\ldots,n \tag{68}$$

$$\mathbf{C} = \mathbf{K}\mathbf{\Sigma}^{-1} \tag{69}$$

$$Q \geq Q_b \tag{70}$$

where l_i is the converted comparison result of i^{th} product, Q_b is the lower bound on the loss of benchmark product, $\mathbf{y_i} = \{y_{i1}, y_{i2},\ldots,y_{in}\}$, $\mathbf{t}=\{t_1, t_2,\ldots,t_n\}$ and $\mathbf{\Sigma}^{-1}$ is the inverse of variance-covariance matrix. The decision variables are represented with $\mathbf{C}$ and $\mathbf{K}$ matrices and Q and ϵ_i.

Since the loss cannot be negative by definition, **C** must be positive definite. Hence additional constraints can be added to ensure **C** is positive definite.

In order to make **C** positive definite, for the two quality characteristics case, the bound suggested by Maghsoodloo and Chang (2001) can be considered which is given in Equation (71).

$$4c_{11}c_{22}>c_{12}^2 \tag{71}$$

4.6 EVALUATION OF FIT

The fit obtained in the optimization step needs to be evaluated. There are various methods that can be used to test the fit of the quality loss function predicted in the optimization step to the underlying value function of the decision maker, which includes selecting random products from the convex hull of already evaluated products and selecting products near the products which have the highest error values in the optimization step of current iteration according to Burr and Tobin (2015) in the "Measuring the effect of data mining on inference" entry of Encyclopedia of Information Science and Technology. Also selecting products from the most important area according to the decision maker may be considered if the number of questions that can be answered by the decision maker is limited.

The number of products which will be included in the evaluation of the fit may depend on the situation. Since the quality loss values for the test products needs to be calculated, it is suggested that the points which will be used in the next iteration may be used as test points. However, a cross validation approach can be used to determine the test products if the number of products that can be tested is limited due to time concerns. Cross validation means dividing the available data to *k* sections and using some sections as the test data (Ross et al., 2009). The number of sections that the data will be divided into and number of sections that will be used as test data can be determined depending on the situation.

When the points that will be used are determined, the error threshold needs to be specified and the quality loss values of all of the test products need to be estimated. If the difference between the estimated values and values obtained from estimated loss function is less than the desired threshold value for all test products, then the estimated quality loss function represents the underlying value function of decision maker and algorithm stops. Otherwise, next iteration of the algorithm starts from the second step, which is selections of the products to compare quality losses.

CHAPTER 5

ILLUSTRATIVE EXAMPLES AND DISCUSSIONS

In this chapter, two examples are provided to illustrate the method under different conditions together with the discussion on the method proposed and multivariate loss functions in general.

5.1 METAL WORKING CASE

The proposed approach is illustrated on a metal working example. According to Dolgun (2014) surface roughness and wall thickness are among the quality characteristics of metal parts produced by turning operation.

In our case, we assume that we are performing a study for an aircraft company. The customer in this case is a department which produces engines for aircrafts, which is an internal customer. Since the quality characteristics need to be tested for a time period, expert opinions are more useful to determine the quality losses. Hence the decision makers are quality engineers or experts from the engine department in our example.

After the decision makers are determined, the next step of the algorithm is the selection of the products. In the example by Dolgun (2014), the target for surface roughness (y_1) is 2 μm and the target for wall thickness (y_2) is 4 millimeters. Moreover, the specification limits are 0 μm and 8 μm for surface roughness and the wall thickness should be between 3.99 millimeter and 4.01 millimeters. Finally, since it is not provided, we assume that the variance-covariance matrix of surface roughness and wall thickness is as shown in Equation (72).

$$\boldsymbol{\Sigma}=\begin{bmatrix}0.2000 & 0.0005\\ 0.0005 & 0.0010\end{bmatrix} \tag{72}$$

This assumption is made according to the properties of data provided by Dolgun (2014).

Let the cost matrix of underlying loss function be as shown in Equation (73).

$$\mathbf{C}=\begin{bmatrix}5 & 0\\ 0 & 90000\end{bmatrix} \tag{73}$$

The answers of experts are consistent with the cost matrix given in Equation (73) through the example.

We need to select one product from each area mentioned in Subsection 4.3. There are four areas since there are two quality characteristics. The selected four products are:

- *Product* 1 = (1.0 µm, 4.005 mm)
- *Product* 2 = (1.5 µm, 3.995 mm)
- *Product* 3 = (2.5 µm, 4.005 mm)
- *Product* 4 = (3.0 µm, 3.995 mm)

The next step is estimating the quality loss values of the selected products using the expert opinion as mentioned above. We assume that the experts state how many times on the average the engine department requests a replacement or repair or does nothing about the selected products out of a hundred times they received the product and their answers are stated in Table 10.

Table 10 Experts' Actions on Selected Products

	Replacement	**Repair**	**Do nothing**
***Product* 1**	10	20	70
***Product* 2**	5	10	85
***Product* 3**	5	10	85
***Product* 4**	10	20	70

Let the replacement cost be \$50 and the repair cost be \$10. The answers of the experts are created using the cost matrix given in Equation (73) and the replacement and repair costs. It is assumed that the number of customers that would request a repair is as twice as the number of customers that would request a replacement. For example, the percentage of customers that would request a replacement or repair for *Product* 1 is estimated using the following method.

$$L(Product\ 1) = \begin{pmatrix} 1.000\text{-}2.000 & 4.005\text{-}4.000 \end{pmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 90000 \end{bmatrix} \begin{pmatrix} 1.000\text{-}2.000 \\ 4.005\text{-}4.000 \end{pmatrix}$$

$$L(Product\ 1) = 7.25$$

Let the percentage of customers that would request a replacement be x and the percentage of customers that would request repair be $2x$.

Then $50x+10(2x) = 7.25$ and $x = 0.1035$

By rounding the x value, we obtain the percentage of customers that would request replacement as 10% and the percentage of customers that would request repair as 20%.

Using the answers of experts, the estimated average losses of selected products are calculated as given in Equations (74), (75), (76) and (77).

$$\mathrm{L}(Product\ 1) = 0.10\times 50 + 0.20\times 10 = 7.0 \tag{74}$$

$$\mathrm{L}(Product\ 2) = 0.05\times 50 + 0.10\times 10 = 3.5 \tag{75}$$

$$\mathrm{L}(Product\ 3) = 0.05\times 50 + 0.10\times 10 = 3.5 \tag{76}$$

$$\mathrm{L}(Product\ 4) = 0.10\times 50 + 0.20\times 10 = 7.0 \tag{77}$$

We assume that the experts' preferences are inversely proportional with the quality loss values of the products. Since the customer is an internal customer, the other costs are ignored and the parameters of the model can be calculated using normalization. If we normalize the costs ensuring they add up to one, the costs and model parameters in Table 11 are obtained.

Table 11 Normalized Costs and Model Parameters of Example 1

	Normalized Costs	**Model Parameters (l_i)**
***Product* 1**	0.333	1.000
***Product* 2**	0.167	0.500
***Product* 3**	0.167	0.500
***Product* 4**	0.333	1.000

Since the benchmark product is *Product* 1, the estimated loss value of *Product* 1, which is 7, becomes the lower bound Q_b. Then the mathematical model is:

Minimize $\sum_{i=1}^{4} \epsilon_i^2$

Subject to

$l_i Q = \left(\mathbf{y}_i(\mathbf{x})\text{-}\mathbf{t}\right)^{\mathrm{T}} \mathbf{C} \left(\mathbf{y}_i(\mathbf{x})\text{-}\mathbf{t}\right) + \epsilon_i \qquad \forall i=1,\ldots,4$

$\mathbf{C} = \mathbf{K}\mathbf{\Sigma}^{-1}$

$Q \geq 7$

All error variables are equal to zero in the optimal solution. The **C** and **K** matrices are obtained as given in Equations (78) and (79) correspondingly.

$$\mathbf{C}=\begin{bmatrix} 4.667 & 174.440 \\ -174.440 & 93333.340 \end{bmatrix} \tag{78}$$

$$\mathbf{K}=\begin{bmatrix} 1.021 & 0.177 \\ 11.779 & 93.246 \end{bmatrix} \tag{79}$$

We assume the decision maker has a time limit. Hence the products that will be used in the next iteration are selected as the test points in this iteration. The products which are selected from the four areas are given below.

- *Product* 5 = (1.0 µm, 3.995 mm)
- *Product* 6 = (1.5 µm, 4.005 mm)
- *Product* 7 = (2.5 µm, 3.995 mm)
- *Product* 8 = (3.0 µm, 4.005 mm)

The experts estimate the number of times the products are requested to be replaced or repaired out of a hundred times they are received by the engine department as shown in Table 12 according to the cost matrix provided in Equation (71).

Table 12 Experts' Actions on Test Products in Example 1

	Replacement	**Repair**	**Do nothing**
***Product* 5**	10	20	70
***Product* 6**	5	10	85
***Product* 7**	5	10	85
***Product* 8**	10	20	70

Then the estimated average losses which are calculated using replacement and repair costs are provided in Equations (80), (81), (82) and (83).

$$L(\textit{Product } 5) = 0.10\times 50 + 0.20\times 10 = 7.0 \tag{80}$$

$$L(\textit{Product } 6) = 0.05\times 50 + 0.10\times 10 = 3.5 \tag{81}$$

$$L(\textit{Product } 7) = 0.05\times 50 + 0.10\times 10 = 3.5 \tag{82}$$

$$L(\textit{Product } 8) = 0.10\times 50 + 0.20\times 10 = 7.0 \tag{83}$$

The desired threshold is 10% in this case. The loss values estimated from expert opinions, the losses calculated using the obtained loss function and percentage errors are given in Table 13.

Table 13 Test Results of Example 1

	Expert Opinion	**Calculated Loss**	**Percentage Error**
***Product* 5**	7.0	7.0	0.0
***Product* 6**	3.5	3.5	0.0
***Product* 7**	3.5	3.5	0.0
***Product* 8**	7.0	7.0	0.0

Hence the loss function obtained is satisfactory and the process stops. The **C** matrix, **K** matrix and loss function obtained are available in Equations (84), (85) and (86) correspondingly.

$$\mathbf{C}=\begin{bmatrix} 4.667 & 174.440 \\ -174.440 & 93333.340 \end{bmatrix} \tag{84}$$

$$\mathbf{K}=\begin{bmatrix} 1.021 & 0.177 \\ 11.779 & 93.246 \end{bmatrix} \tag{85}$$

$$\mathrm{L}(\mathbf{y}(\mathbf{x}),\mathbf{t}) = (\mathbf{y}(\mathbf{x})-\mathbf{t}) \begin{bmatrix} 4.667 & 174.440 \\ -174.440 & 93333.340 \end{bmatrix} (\mathbf{y}(\mathbf{x})-\mathbf{t}) \tag{86}$$

Open form of the loss function is provided in Equation (87).

$$\begin{gathered}\mathrm{L}(\mathbf{y}(\mathbf{x}),\mathbf{t})=4.667\left(y_1(\mathbf{x})\text{-}2.000\right)^2 \\ +(174.440\text{-}174.440)\left(y_1(\mathbf{x})\text{-}2.000\right)\left(y_2(\mathbf{x})\text{-}4.000\right) \\ +93333.340\left(y_2(\mathbf{x})\text{-}4.000\right)^2\end{gathered} \tag{87}$$

The interaction term cancels out and the loss function becomes as shown in Equation (88).

$$\mathrm{L}(\mathbf{y}(\mathbf{x}),\mathbf{t})=4.667\left(y_1(\mathbf{x})\text{-}2.000\right)^2+93333.340\left(y_2(\mathbf{x})\text{-}4.000\right)^2 \tag{88}$$

The open form of underlying loss function which we try to find is provided in Equation (89).

$$\mathrm{L}(\mathbf{y}(\mathbf{x}),\mathbf{t})=5\left(y_1\text{-}2.000\right)^2+90000\left(y_2\text{-}4.000\right)^2 \tag{89}$$

Figure 15 illustrates the loss functions provided in Equations (88) and (89).

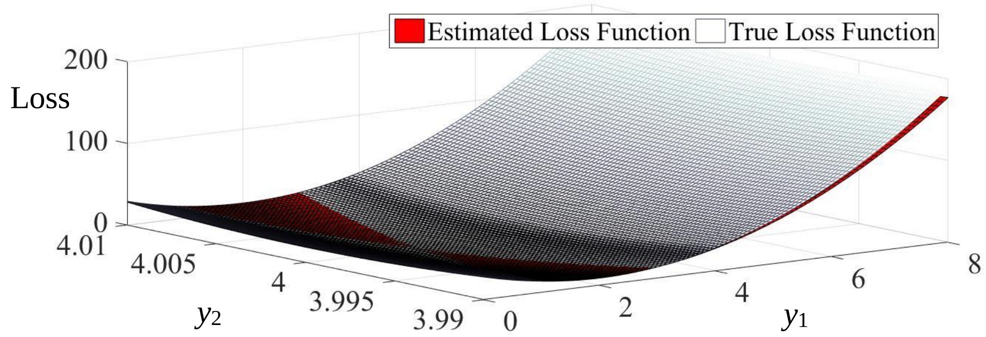

Figure 15 Illustration of Real Loss Function and Estimated Loss Function for Example 1

5.2 SHIRT CASE

The second case which is used to demonstrate the proposed method is determination of the quality loss function for luxury shirts with stripes. Two quality characteristics, fabric pattern matching, which is measured using mismatch between centers of meeting stripes (y_1) and stitch density (y_2), are chosen to be used in the study.

Fabric pattern matching is evaluated using the difference between the start of a pattern in sleeves and the main body of the shirt and its target value is 0.00 millimeters. The acceptable values of difference in terms of fabric pattern matching are between -0.005 millimeters and 0.005 millimeters. The average stitch density should be between 2.9 stitches per centimeter and 3.1 stitches per centimeter and its target value is 3 stitches per centimeter. The estimated variance-covariance matrix of two quality characteristics is provided in Equation (90).

$$\mathbf{\Sigma}=\begin{bmatrix} 0.001 & -0.0005 \\ -0.0005 & 0.05 \end{bmatrix} \tag{90}$$

The quality manager of the manufacturer firm is the decision maker of the process. However, in luxury items customers have high quality expectations and their opinions should also be included in the study. We assume that the quality manager answers are consistent with cost matrix $\mathbf{C_1}$ and customers' answers are consistent with cost matrix $\mathbf{C_2}$ which are provided in Equations (91) and (92) correspondingly.

$$\mathbf{C_1} = \begin{bmatrix} 30000000 & 200000 \\ 200000 & 130000 \end{bmatrix} \tag{91}$$

$$\mathbf{C_2} = \begin{bmatrix} 10000000 & 100000 \\ 100000 & 100000 \end{bmatrix} \tag{92}$$

Four products should be selected in the first iteration since there are two quality characteristics under consideration. The products which are selected from the four different areas can be listed as:

- *Product* 1 = (-0.002 mm, 3.05 stitches per cm)
- *Product* 2 = (0.001 mm, 2.95 stitches per cm)
- *Product* 3 = (0.002 mm, 3.07 stitches per cm)
- *Product* 4 = (-0.001 mm, 2.98 stitches per cm)

The selected products can be represented as in Figure 16.

After the product selection is completed a survey is assumed to be made to a focus group consisting of 20 customers. Due to the nature of the product, repair is not an option and the customers may do nothing about the product or would request a replacement. Let the cost of replacement be 1500$ per product. The results of the survey are as shown in Table 14.

Table 14 Customers' Actions on Selected Products for First Iteration of Example 2

	Replacement	**Do nothing**
***Product* 1**	4	16
***Product* 2**	3	17
***Product* 3**	7	13
***Product* 4**	1	19

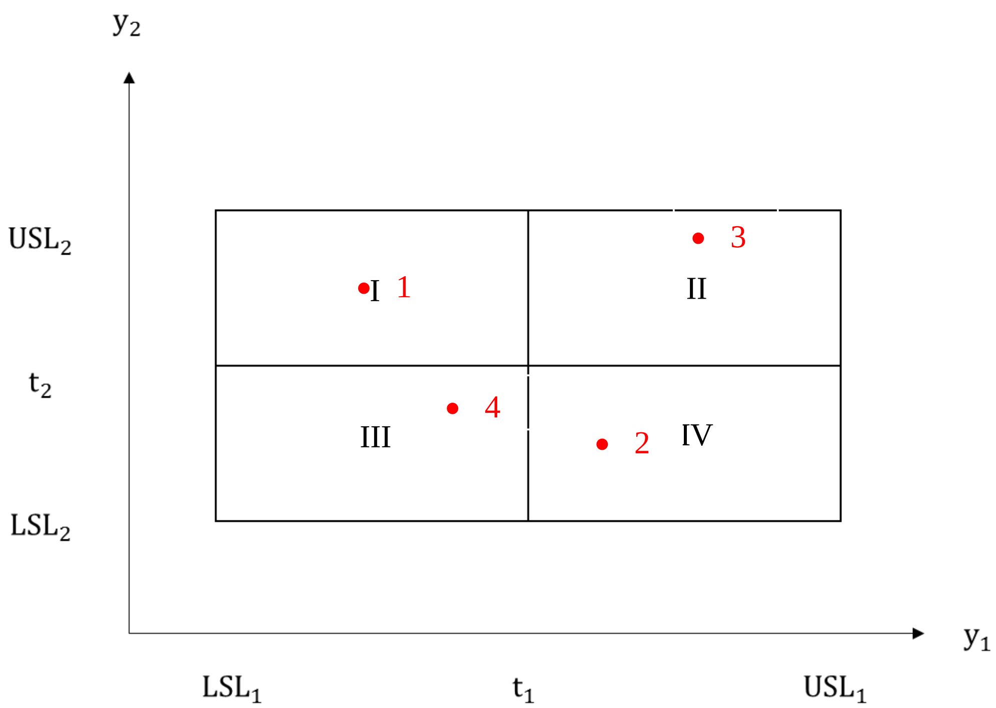

Figure 16 Selected Products

The results in Table 14 are calculated according to the **C_2** matrix and replacement cost. For example, the number of customers that request a new product if they received *Product* 1 is calculated as follows.

$$L(\textit{Product } 1) = (-0.002\text{-}0.000 \quad 3.050\text{-}3.000) \begin{bmatrix} 10000000 & 100000 \\ 100000 & 100000 \end{bmatrix} \begin{pmatrix} -0.002\text{-}0.000 \\ 3.050\text{-}3.000 \end{pmatrix}$$

$$L(\textit{Product } 1) = 270$$

Let the number of customers that would request a replacement be x.

Then $75x = 270$ and $x = 3.6$

By rounding the x value, we obtain the number of customers that would request replacement as 4.

Using the answers from customers, the average replacement costs (RC) can be calculated as:

$$RC(\textit{Product } 1) = 0.20\text{x}1500 = 300\$ \quad (93)$$

$$\text{RC}(\textit{Product } 2) = 0.15\text{x}1500 = 225\$ \tag{94}$$

$$\text{RC}(\textit{Product } 3) = 0.35\text{x}1500 = 525\$ \tag{95}$$

$$\text{RC}(\textit{Product } 4) = 0.05\text{x}1500 = 75\$ \tag{96}$$

When the preferences are calculated inversely proportional to the quality losses, the comparison matrix in Table 15 is obtained.

Table 15 Initial Comparison Matrix for First Iteration of Example 2

	***Product* 1**	***Product* 2**	***Product* 3**	***Product* 4**
***Product* 1**	1.00	0.75	1.75	0.25
***Product* 2**	1.33	1.00	2.33	0.33
***Product* 3**	0.57	0.43	1.00	0.14
***Product* 4**	4.00	3.00	7.00	1.00

For example, in the comparison matrix in Table 15 the inverse replacement cost ratio of *Product* 2 to that of *Product* 1 is calculated as

$$\frac{RC\ (Product\ 1)}{RC\ (\text{Benchmark } Product\ 2)} = \frac{300}{225} = 1.33$$

However, the quality manager of the manufacturing firm is assumed to believe that these preferences should be modified to reflect other external costs such as loss of goodwill. Let the modified comparison matrix be as shown in Table 16.

Table 16 Modified Comparison Matrix for First Iteration of Example 2

	***Product* 1**	***Product* 2**	***Product* 3**	***Product* 4**
***Product* 1**	1.00	0.75	2.00	0.25
***Product* 2**	1.33	1.00	2.67	0.33
***Product* 3**	0.50	0.38	1.00	0.13
***Product* 4**	4.00	3.00	8.00	1.00

The modified comparison matrix is obtained by the quality manager by adding estimated additional costs such as loss of goodwill in this example. For example, the inverse overall cost ratio of *Product 2* to that of *Product 1* is calculated as:

$$\frac{RC\ (Product\ 1) + Estimated\ loss\ of\ goodwill\ of\ Product\ 1}{RC\ (Benchmark\ Product\ 2) + Estimated\ loss\ of\ goodwill\ of\ Product\ 2} =$$

$$\frac{300+100}{225+75} = 1.33$$

In order to calculate the total loss estimated by the quality manager, we used the $\mathbf{C_1}$ matrix and to obtain loss of goodwill we subtracted the repair replacement cost from the total estimated loss. For example, the estimated loss of goodwill due to Product 1 is calculated as follows.

$$L(Product\ 1) = \begin{pmatrix} -0.002-0.000 & 3.050-3.000 \end{pmatrix} \begin{bmatrix} 30000000 & 200000 \\ 200000 & 130000 \end{bmatrix} \begin{pmatrix} -0.002-0.000 \\ 3.050-3.000 \end{pmatrix}$$

L(*Product* 1)=405$

By rounding the expected loss to the nearest multiple of hundred we obtain 400$ and by subtracting the replacement cost we obtain the estimated loss of goodwill as 100$.

Table 17 presents the priority vector constructed using the procedure suggested in AHP and the converted parameters obtained from priority vector when the benchmark product is chosen as *Product* 2.

Table 17 Priority Vector Values and Model Parameters for First Iteration of Example 2

	Eigenvector	**Model Parameters (l_i)**
***Product* 1**	0.146	1.333
***Product* 2**	0.195	1.000
***Product* 3**	0.073	2.667
***Product* 4**	0.585	0.333

The lower bound on the loss of the benchmark product is 300. Using these parameters, the mathematical model is constructed as presented below:

Minimize $\sum_{i=1}^{4} \epsilon_i^2$

Subject to

$l_i Q = (\mathbf{y}_i(\mathrm{x}) - \mathbf{t})^{\mathrm{T}} \mathbf{C} (\mathbf{y}_i(\mathrm{x}) - \mathbf{t}) + \epsilon_i \qquad \forall i = 1, \ldots, 4$

$\mathbf{C} = \mathbf{K}\boldsymbol{\Sigma}^{-1}$

$Q \geq 300$

The resulting values of the decision variables in **C**, **K**, and **ϵ** are given in Equations (97), (98) and (99) correspondingly.

$$\mathbf{C} = \begin{bmatrix} 31600397 & 379363 \\ 18217 & 126130 \end{bmatrix} \tag{97}$$

$$\mathbf{K} = \begin{bmatrix} 31411.1 & 3168.0 \\ -44.8 & 6297.4 \end{bmatrix} \tag{98}$$

$$\boldsymbol{\epsilon} = [-2.06 \quad -27.04 \quad 0.00 \quad 9.90] \tag{99}$$

Let us assume that the decision maker has a time limit, hence we are using one test product. Since the quality loss function has the highest percentage error when *Product* 4 is calculated, a product close to *Product* 4 is chosen as the test product. Let the test product be *Product* 5 (-0.001 mm, 2.97 stitches per cm) and 1 out of 20 customers state that they would request a replacement. Then the estimated average loss is given in Equation (100).

$$L(\mathit{Product}\ 5) = 0.05 * 1500 = 75\$ \tag{100}$$

The manufacturer modifies the estimated loss as 200. When we use the **C** matrix estimated, we calculate the loss as 157$, which means percentage error is 21.50%. If we assume that the desired threshold is 10%, the fit test fails and the algorithm moves on to the next iteration.

Since we have a time limit the test product is used as the selected product in second iteration. The comparison matrix when the fifth product is added becomes the one in Table 18.

Table 18 Initial Comparison Matrix for Second Iteration of Example 2

	***Product* 1**	***Product* 2**	***Product* 3**	***Product* 4**	***Product* 5**
***Product* 1**	1.00	0.75	1.75	0.25	0.25
***Product* 2**	1.33	1.00	2.33	0.33	0.33
***Product* 3**	0.57	0.43	1.00	0.14	0.14
***Product* 4**	4.00	3.00	7.00	1.00	1.00
***Product* 5**	4.00	3.00	7.00	1.00	1.00

The decision maker modifies the comparison matrix as in Table 19.

Table 19 Modified Comparison Matrix for Second Iteration of Example 2

	***Product* 1**	***Product* 2**	***Product* 3**	***Product* 4**	***Product* 5**
***Product* 1**	1.00	0.75	2.00	0.25	0.50
***Product* 2**	1.33	1.00	2.67	0.33	0.67
***Product* 3**	0.50	0.38	1.00	0.13	0.25
***Product* 4**	4.00	3.00	8.00	1.00	2.00
***Product* 5**	2.00	1.50	4.00	0.50	1.00

The priority vector and the converted parameters obtained from priority vector when the benchmark product is chosen as *Product* 2 are presented in Table 20.

Table 20 Priority Vector Values and Model Parameters for Second Iteration of Example 2

	Priority Vector	**Model Parameters (l_i)**
***Product* 1**	0.11	1.33
***Product* 2**	0.15	1.00
***Product* 3**	0.06	2.64
***Product* 4**	0.45	0.33
***Product* 5**	0.23	0.66

Using the parameters obtained from comparison study, the mathematical model is constructed as can be seen below.

Minimize $\sum_{i=1}^{5} \epsilon_i^2$

Subject to

$l_iQ = (\mathbf{y}_i(\mathbf{x})-\mathbf{t})^T \mathbf{C} (\mathbf{y}_i(\mathbf{x})-\mathbf{t}) + \epsilon_i \qquad \forall i=1,\ldots,5$

$\mathbf{C} = \mathbf{K\Sigma}^{-1}$

$Q \geq 300$

In the optimal solution the values of decision variables are obtained as in Equations (101), (102) and (103).

$$\mathbf{C}=\begin{bmatrix} 31600396 & 379366 \\ 18220 & 123620 \end{bmatrix} \tag{101}$$

$$\mathbf{K}=\begin{bmatrix} 31411.1 & 3168.1 \\ -43.6 & 6171.9 \end{bmatrix} \tag{102}$$

$$\boldsymbol{\epsilon}=\begin{bmatrix} 4.21 & -20.77 & 0.00 & 10.90 & 45.01 \end{bmatrix} \tag{103}$$

In order to test the fit of quality loss function obtained to underlying cost function of decision maker, a new test point which is close to the product with largest percentage error value should be selected. Let *Product* 6 be selected as (-0.001 mm, 2.96 stitches per cm) and two customer say that she would request a change meaning the estimated quality loss is 150$ and it is modified by the decision maker as 250$.

When we calculate the quality loss using the function we obtained in the second iteration, we find the quality loss is 245.30$. Hence the error is 1.88% which is less than the desired threshold and the algorithm stops. The **C** matrix, **K** matrix and loss function obtained are available in Equations (104), (105) and (106) correspondingly.

$$\mathbf{C}=\begin{bmatrix} 31600396 & 379366 \\ 18220 & 123620 \end{bmatrix} \tag{104}$$

$$\mathbf{K}=\begin{bmatrix} 31411.1 & 3168.1 \\ -43.6 & 6171.9 \end{bmatrix} \tag{105}$$

$$\mathrm{L}(\mathbf{y}(\mathbf{x}),\mathbf{t}) =(\mathbf{y}(\mathbf{x})\text{-}\mathbf{t}) \begin{bmatrix} 31600396 & 379366 \\ 18220 & 123620 \end{bmatrix} (\mathbf{y}(\mathbf{x})\text{-}\mathbf{t}) \tag{106}$$

Open form of the loss function is provided in Equation (107).

$$\begin{gathered}\mathrm{L}(\mathbf{y}(\mathbf{x}),\mathbf{t})=31600396\left(y_1(\mathbf{x})\text{-}0.00\right)^2+123620\left(y_2(\mathbf{x})\text{-}3.000\right)^2 \\ +(379366+18220)\left(y_1(\mathbf{x})\text{-}0.00\right)\left(y_2(\mathbf{x})\text{-}3.00)\right)\end{gathered} \tag{107}$$

When we simplify the interaction term the loss function becomes as shown in Equation (108).

$$\begin{gathered}\mathrm{L}(\mathbf{y}(\mathbf{x}),\mathbf{t})=31600396\left(y_1(\mathbf{x})\text{-}0.00\right)^2+397586\left(y_1(\mathbf{x})\text{-}0.00\right)\left(y_2(\mathbf{x})\text{-}3.00\right) \\ +123620\left(y_2(\mathbf{x})\text{-}3.00\right)^2\end{gathered} \tag{108}$$

The open form of underlying loss function which we try to find is provided in Equation (109).

$$\begin{gathered}\mathrm{L}(\mathbf{y}(\mathbf{x}),\mathbf{t})=30000000\left(y_1(\mathbf{x})\text{-}0.00\right)^2+400000\left(y_1(\mathbf{x})\text{-}0.00\right)\left(y_2(\mathbf{x})\text{-}3.00\right) \\ +130000\left(y_2(\mathbf{x})\text{-}3.00\right)^2\end{gathered} \tag{109}$$

Figure 17 illustrates the loss functions provided in Equations (108) and (109).

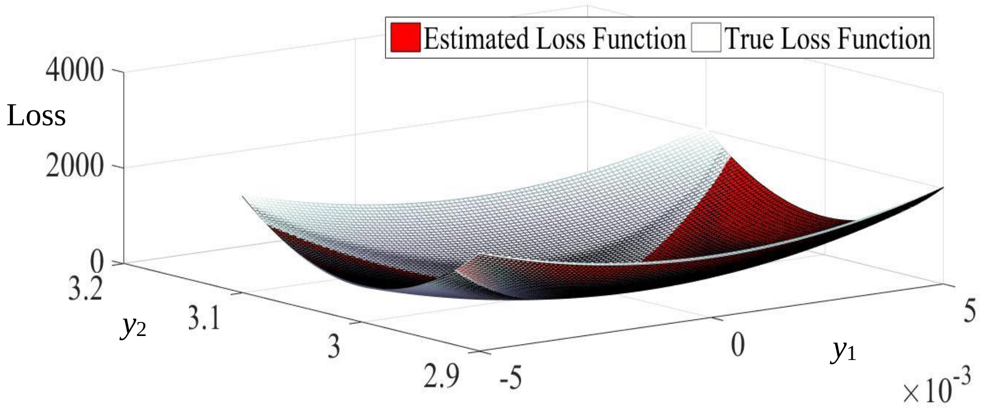

Figure 17 Illustration of Real Loss Function and Estimated Loss Function for Example 2

5.3 DISCUSSIONS

In this thesis, we have studied the process economy related parameters of multivariate loss functions and provided a practical approach for estimating the parameters. The purpose of our approach is providing the closest fit to the loss function regardless of the type of underlying value function of decision maker. However, we observed that multivariate quadratic loss functions are not appropriate to represent all types of value functions that a decision maker can have. Multivariate quadratic loss functions may perform poorly if decision maker's value function is nonconvex or highly non-monotonous. If the algorithm does not converge after a reasonable number of steps, the true value function may be significantly different from a quadratic loss function and it might be useful to consider another type of function to represent the true value function. However, the perfect representation might be achieved if the value function is a quadratic loss function.

Another issue about multivariate loss function is the difficulty of estimating the quality loss value of a given product or process. Estimation of tangible costs which affect the quality loss is relatively simple and structured. On the other hand, intangible costs such as loss of goodwill and overhead costs for a given product are hard to include in the estimation of quality loss. Both tangible and intangible costs are significant for construction of cost matrix **C**. As shown in Chapter 3, the correct estimation of **C** matrix is crucial for multivariate loss functions. Hence, practitioners should be careful about the estimation of intangible costs.

Even if the **C** matrix is estimated correctly, the **K** matrix estimation can be wrong when **Σ** estimation is not adequate. Appropriate methods should be used for estimating **Σ** in order to obtain meaningful inferences from **K** matrix.

Estimating the cost matrix, **C**, and its elements might improve understanding of decision maker's preferences and, allow usage of univariate methods in predictive modelling. However, if loss function is to be used for optimization, using interactive multi-criteria decision making approaches supported with statistical inference might prove to be a better choice.

CHAPTER 6

CONCLUSIONS

In this thesis, we propose a method for determining the parameters of multivariate loss functions by developing an algorithm to obtain the best fit to the underlying value function of the decision maker which utilizes the quadratic loss function in product or process design and quality assurance processes.

To the best of our knowledge, there is only one approach in the literature for parameter determination, which is proposed by Pignatiello (1993). However, his approach requires an exact knowledge of quality loss at certain points, which is not applicable in practice. Instead of using exact loss values, our algorithm takes rough estimates of the loss values of some selected products, and uses optimization to find the best fitting quadratic loss function. The method we propose is applicable for all problems where the underlying value function is a quadratic loss function. Even if the underlying value function is not a quadratic loss function, our algorithm aims to provide the closest quadratic loss function fit to the true value function.

Results of this research can be utilized by practitioners who need to compare quality of products or find optimal design parameter settings of products and processes especially when the decision makers do not have enough time to interact with the analyst in the prediction and optimization process. Such cases occur when quality inspection and control need to be made by inexperienced people or by automated systems. However, if it is possible to interact with the decision maker, one may prefer more effective approaches for quality prediction and optimization, than using the multivariate loss functions.

Both univariate and multivariate quadratic loss functions are useful tools for quality engineering purposes and worth studying by researchers. However, there is a tradeoff between the practicality of the method to predict the parameters of the quadratic loss functions and accuracy of the results. Therefore, future research may consider other loss estimation processes to improve the results of the algorithm.

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APPENDIX A

The preference matrix, **K**, when there are three quality characteristics is given below.

$$\mathbf{K}=\begin{bmatrix} w_1 & k_{12} & k_{13} \\ k_{21} & w_2 & k_{23} \\ k_{31} & k_{32} & w_3 \end{bmatrix}$$

The variance-covariance matrix, **Σ,** is provided below.

$$\mathbf{\Sigma}=\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix}$$

Inverse of variance-covariance matrix can be represented as shown below.

$$\mathbf{\Sigma}^{-1}=\begin{bmatrix} -\left(-\sigma_{23}^2+\sigma_{22}\sigma_{33}\right) & -(\sigma_{13}\sigma_{23}-\sigma_{12}\sigma_{33}) & -(\sigma_{12}\sigma_{23}-\sigma_{13}\sigma_{22}) \\ -(\sigma_{13}\sigma_{23}-\sigma_{12}\sigma_{33}) & -\left(-\sigma_{13}^2+\sigma_{11}\sigma_{33}\right) & -(\sigma_{12}\sigma_{13}-\sigma_{11}\sigma_{23}) \\ -(\sigma_{12}\sigma_{23}-\sigma_{13}\sigma_{22}) & -(\sigma_{12}\sigma_{13}-\sigma_{11}\sigma_{23}) & -\left(-\sigma_{12}^2+\sigma_{11}\sigma_{32}\right) \end{bmatrix}$$

$$\times\frac{1}{\left(\sigma_{33}\sigma_{12}^2-2\sigma_{12}\sigma_{13}\sigma_{23}+\sigma_{22}\sigma_{13}^2+\sigma_{11}\sigma_{23}^2-\sigma_{11}\sigma_{22}\sigma_{33}\right)}$$

The cost matrix **C** and its components are presented below.

$$\mathbf{C}=\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

$$c_{11}=\frac{-(w_1\left(-\sigma_{23}^2+\sigma_{22}\sigma_{33}\right)+k_{12}(\sigma_{13}\sigma_{23}-\sigma_{12}\sigma_{33})+k_{13}(\sigma_{12}\sigma_{23}-\sigma_{13}\sigma_{22}))}{\left(\sigma_{33}\sigma_{12}^2-2\sigma_{12}\sigma_{13}\sigma_{23}+\sigma_{22}\sigma_{13}^2+\sigma_{11}\sigma_{23}^2-\sigma_{11}\sigma_{22}\sigma_{33}\right)}$$

$$c_{12}=\frac{-(w_1(\sigma_{13}\sigma_{23}-\sigma_{12}\sigma_{33})+k_{12}\left(-\sigma_{13}^2+\sigma_{11}\sigma_{33}\right)+k_{13}(\sigma_{12}\sigma_{13}-\sigma_{11}\sigma_{23}))}{\left(\sigma_{33}\sigma_{12}^2-2\sigma_{12}\sigma_{13}\sigma_{23}+\sigma_{22}\sigma_{13}^2+\sigma_{11}\sigma_{23}^2-\sigma_{11}\sigma_{22}\sigma_{33}\right)}$$

$$c_{13}=\frac{-(w_1(\sigma_{12}\sigma_{23}-\sigma_{13}\sigma_{22})+k_{12}(\sigma_{12}\sigma_{13}-\sigma_{11}\sigma_{23})+k_{13}\left(-\sigma_{12}^2+\sigma_{11}\sigma_{32}\right))}{\left(\sigma_{33}\sigma_{12}^2-2\sigma_{12}\sigma_{13}\sigma_{23}+\sigma_{22}\sigma_{13}^2+\sigma_{11}\sigma_{23}^2-\sigma_{11}\sigma_{22}\sigma_{33}\right)}$$

$$c_{21}=\frac{-(k_{21}\left(-\sigma_{23}^2+\sigma_{22}\sigma_{33}\right)+w_2(\sigma_{13}\sigma_{23}-\sigma_{12}\sigma_{33})+k_{23}(\sigma_{12}\sigma_{23}-\sigma_{13}\sigma_{22}))}{\left(\sigma_{33}\sigma_{12}^2-2\sigma_{12}\sigma_{13}\sigma_{23}+\sigma_{22}\sigma_{13}^2+\sigma_{11}\sigma_{23}^2-\sigma_{11}\sigma_{22}\sigma_{33}\right)}$$

$$c_{22} = \frac{-(k_{21}(\sigma_{13}\sigma_{23}-\sigma_{12}\sigma_{33})+w_2(-\sigma_{13}^2+\sigma_{11}\sigma_{33})+k_{23}(\sigma_{12}\sigma_{13}-\sigma_{11}\sigma_{23}))}{(\sigma_{33}\sigma_{12}^2-2\sigma_{12}\sigma_{13}\sigma_{23}+\sigma_{22}\sigma_{13}^2+\sigma_{11}\sigma_{23}^2-\sigma_{11}\sigma_{22}\sigma_{33})}$$

$$c_{23} = \frac{-(k_{21}(\sigma_{12}\sigma_{23}-\sigma_{13}\sigma_{22})+w_2(\sigma_{12}\sigma_{13}-\sigma_{11}\sigma_{23})+k_{23}(-\sigma_{12}^2+\sigma_{11}\sigma_{32}))}{(\sigma_{33}\sigma_{12}^2-2\sigma_{12}\sigma_{13}\sigma_{23}+\sigma_{22}\sigma_{13}^2+\sigma_{11}\sigma_{23}^2-\sigma_{11}\sigma_{22}\sigma_{33})}$$

$$c_{31} = \frac{-(k_{31}(-\sigma_{23}^2+\sigma_{22}\sigma_{33})+k_{32}(\sigma_{13}\sigma_{23}-\sigma_{12}\sigma_{33})+w_3(\sigma_{12}\sigma_{23}-\sigma_{13}\sigma_{22}))}{(\sigma_{33}\sigma_{12}^2-2\sigma_{12}\sigma_{13}\sigma_{23}+\sigma_{22}\sigma_{13}^2+\sigma_{11}\sigma_{23}^2-\sigma_{11}\sigma_{22}\sigma_{33})}$$

$$c_{32} = \frac{-(k_{31}(\sigma_{13}\sigma_{23}-\sigma_{12}\sigma_{33})+k_{32}(-\sigma_{13}^2+\sigma_{11}\sigma_{33})+w_3(\sigma_{12}\sigma_{13}-\sigma_{11}\sigma_{23}))}{(\sigma_{33}\sigma_{12}^2-2\sigma_{12}\sigma_{13}\sigma_{23}+\sigma_{22}\sigma_{13}^2+\sigma_{11}\sigma_{23}^2-\sigma_{11}\sigma_{22}\sigma_{33})}$$

$$c_{33} = \frac{-(k_{31}(\sigma_{12}\sigma_{23}-\sigma_{13}\sigma_{22})+k_{32}(\sigma_{12}\sigma_{13}-\sigma_{11}\sigma_{23})+w_3(-\sigma_{12}^2+\sigma_{11}\sigma_{32}))}{(\sigma_{33}\sigma_{12}^2-2\sigma_{12}\sigma_{13}\sigma_{23}+\sigma_{22}\sigma_{13}^2+\sigma_{11}\sigma_{23}^2-\sigma_{11}\sigma_{22}\sigma_{33})}$$

Using the C matrix and its components the general form of expected loss function when there are three quality characteristics is calculated as follows.

$$\mathrm{E}[\mathrm{L}(\mathbf{y}(\mathbf{x}),\mathbf{t}]=\frac{-(w_1(-\sigma_{23}^2+\sigma_{22}\sigma_{33})+k_{12}(\sigma_{13}\sigma_{23}-\sigma_{12}\sigma_{33})+k_{13}(\sigma_{12}\sigma_{23}-\sigma_{13}\sigma_{22}))}{(\sigma_{33}\sigma_{12}^2-2\sigma_{12}\sigma_{13}\sigma_{23}+\sigma_{22}\sigma_{13}^2+\sigma_{11}\sigma_{23}^2-\sigma_{11}\sigma_{22}\sigma_{33})}\left(y_1(\mathbf{x})-t_1\right)^2$$

$$+\frac{-(k_{21}(\sigma_{13}\sigma_{23}-\sigma_{12}\sigma_{33})+w_2(-\sigma_{13}^2+\sigma_{11}\sigma_{33})+k_{23}(\sigma_{12}\sigma_{13}-\sigma_{11}\sigma_{23}))}{(\sigma_{33}\sigma_{12}^2-2\sigma_{12}\sigma_{13}\sigma_{23}+\sigma_{22}\sigma_{13}^2+\sigma_{11}\sigma_{23}^2-\sigma_{11}\sigma_{22}\sigma_{33})}\left(y_2(\mathbf{x})-t_2\right)^2$$

$$+\frac{-(k_{31}(\sigma_{12}\sigma_{23}-\sigma_{13}\sigma_{22})+k_{32}(\sigma_{12}\sigma_{13}-\sigma_{11}\sigma_{23})+w_3(-\sigma_{12}^2+\sigma_{11}\sigma_{32}))}{(\sigma_{33}\sigma_{12}^2-2\sigma_{12}\sigma_{13}\sigma_{23}+\sigma_{22}\sigma_{13}^2+\sigma_{11}\sigma_{23}^2-\sigma_{11}\sigma_{22}\sigma_{33})}\left(y_3(\mathbf{x})-t_3\right)^2$$

$$+\frac{-((w_1+w_{2)}(\sigma_{13}\sigma_{23}-\sigma_{12}\sigma_{33})+k_{12}(-\sigma_{13}^2+\sigma_{11}\sigma_{33})+k_{13}(\sigma_{12}\sigma_{13}-\sigma_{11}\sigma_{23})+k_{21}(-\sigma_{23}^2+\sigma_{22}\sigma_{33})+k_{23}(\sigma_{12}\sigma_{23}-\sigma_{13}\sigma_{22}))}{(\sigma_{33}\sigma_{12}^2-2\sigma_{12}\sigma_{13}\sigma_{23}+\sigma_{22}\sigma_{13}^2+\sigma_{11}\sigma_{23}^2-\sigma_{11}\sigma_{22}\sigma_{33})}$$

$$\left(y_1(\mathbf{x})-t_1\right)\left(y_2(\mathbf{x})-t_2\right)$$

$$+\frac{-((w_1+w_3)(\sigma_{12}\sigma_{23}-\sigma_{13}\sigma_{22})+k_{12}(\sigma_{12}\sigma_{13}-\sigma_{11}\sigma_{23})+k_{13}(-\sigma_{12}^2+\sigma_{11}\sigma_{32})(k_{31}(-\sigma_{23}^2+\sigma_{22}\sigma_{33})+k_{32}(\sigma_{13}\sigma_{23}-\sigma_{12}\sigma_{33}))}{(\sigma_{33}\sigma_{12}^2-2\sigma_{12}\sigma_{13}\sigma_{23}+\sigma_{22}\sigma_{13}^2+\sigma_{11}\sigma_{23}^2-\sigma_{11}\sigma_{22}\sigma_{33})}$$

$$\left(y_1(\mathbf{x})-t_1\right)\left(y_3(\mathbf{x})-t_3\right)$$

$$+\frac{-((w_2+w_3)(\sigma_{12}\sigma_{13}-\sigma_{11}\sigma_{23})+k_{21}(\sigma_{12}\sigma_{23}-\sigma_{13}\sigma_{22})+k_{23}(-\sigma_{12}^2+\sigma_{11}\sigma_{32})+k_{31}(\sigma_{13}\sigma_{23}-\sigma_{12}\sigma_{33})+k_{32}(-\sigma_{13}^2+\sigma_{11}\sigma_{33}))}{(\sigma_{33}\sigma_{12}^2-2\sigma_{12}\sigma_{13}\sigma_{23}+\sigma_{22}\sigma_{13}^2+\sigma_{11}\sigma_{23}^2-\sigma_{11}\sigma_{22}\sigma_{33})}$$

$$\left(y_2(\mathbf{x})-t_2\right)\left(y_3(\mathbf{x})-t_3\right)$$

$$+w_1+w_2+w_3$$