INVESTIGATING MIDDLE SCHOOL MATHEMATICS TEACHERS' MATHEMATICAL KNOWLEDGE FOR TEACHING ALGEBRA: A MULTIPLE CASE STUDY

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ABSTRACT

INVESTIGATING MIDDLE SCHOOL MATHEMATICS TEACHERS' MATHEMATICAL KNOWLEDGE FOR TEACHING ALGEBRA: A MULTIPLE CASE STUDY

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The aim of this study was to examine middle school mathematics teachers' mathematical knowledge for teaching generalization of patterns and operations with algebraic expressions in planning and implementing processes. Data were collected from two middle school mathematics teachers who worked in the same public school throughout the instruction of algebra unit at 7th grade. In data collection process, lesson plans prepared by the teachers, pre-observation interviews, observations, field notes, and post-observation interviews were used as data collection tools. The teachers prepared lesson plans individually and implemented them in the instructions. The researcher observed the lessons and took notes. After the implementation, the researcher conducted the reflective interviews with the teachers. Data were analyzed qualitatively within the frame of Mathematical

Knowledge for Teaching (MKT) model. The descriptions and definitions of the knowledge domains of MKT were utilized for analysis. Findings indicated that the teachers had lack of specialized content knowledge about mathematical representations such as figural representation for patterns, or algebra tiles for algebraic expressions. The teachers' conceptual and adequate common content knowledge (CCK) and specialized content knowledge (SCK) had a positive impact on their pedagogical content knowledge (PCK). When the teachers had strong subject matter knowledge, they took into the students' thinking account (KCS), and they used teaching methods effectively (KCT). Thus, their strong knowledge also influenced positively classroom practices to be effective. However, the teachers need to have a good conceptual mathematical understanding and also knowledge of students' thinking in order to design effective lessons.

Keywords: Mathematical Knowledge for Teaching, Generalization of Patterns, Operations with Algebraic Expressions.

ÖZ

ORTAOKUL MATEMATİK ÖĞRETMENLERİNİN CEBİR ÖĞRETİMİNDEKİ MATEMATİKSEL BİLGİLERİNİN ARAŞTIRILMASI: ÇOKLU DURUM ÇALIŞMASI

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Bu çalışmanın amacı, ortaokul matematik öğretmenlerinin örüntü genellemesi ve cebirsel ifadelerle işlemleri öğretmek için planlama ve uygulama süreçlerindeki bilgilerini incelemektir. Veriler, aynı devlet okulunda çalışan iki ortaokul matematik öğretmeninden ve 7. sınıftaki cebir ünitesi öğretimi süresince toplanmıştır. Veri toplama sürecinde, öğretmenler tarafından hazırlanan ders planları, ders öncesi görüşmeler, gözlemler, alan notları ve ders sonrası görüşmeler veri toplama aracı olarak kullanılmıştır. Öğretmenler, bireysel olarak ders planlarını hazırlamış ve öğretimde uygulamışlardır. Araştırmacı dersleri gözlemiş ve alan notları almıştır. Her dersten sonra, araştırmacı öğretmenlerle yansıtıcı görüşmeler yapmıştır. Veriler, Öğretmek için Matematik Bilgisi (ÖMB) modeli çerçevesinde analiz edilmiştir. Bu analiz için ÖMB modelindeki bilgi ve alt bilgi alanlarının tanımlamaları ve betimlemelerinden yararlanılmıştır. Bulgular, öğretmenlerin gösterimler konusunda, örüntülerde şekilsel gösterim ya da cebirsel ifadelerde cebir karoları gibi, uzmanlık alan bilgilerinin eksik olduğunu göstermiştir. Öğretmenlerin kavramsal ve yeterli genel alan bilgileri ve uzmanlık alan bilgileri, onların pedagojik alan bilgisini olumlu bir şekilde etkilemiştir. Öğretmenler sağlam bir konu alan bilgisine sahip olduklarında, öğretim esnasında öğrencilerin düşünmesini dikkate almışlar (alan ve öğrenci bilgisi) ve öğretim yöntemlerini etkili bir şekilde kullanmışlardır (alan ve öğretme bilgisi). Ayrıca öğretmenlerin güçlü alan bilgisi, etkili öğretim uygulamalarında da olumlu bir etkiye sahiptir. Dolayısıyla, etkili ders tasarımları için, öğretmenler kavramsal olarak matematiği anlamaya ve öğrencilerin düşünme bilgisine sahip olmaya ihtiyaç duymaktadırlar.

Anahtar Kelimeler: Öğretmek için Matematik Bilgisi, Örüntü Genellemesi, Cebirsel İfadelerle İşlemler

To My Mother

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LIST OF ABBREVIATIONS

PISA	Programme for International Student Assessment
TEDS-M	Teacher Education and Development Study in Mathematics
LMT	Learning Mathematics for Teaching
NCTM	National Council of Teachers of Mathematics
MoNE	Ministry of National Education
MKT	Mathematical Knowledge for Teaching
SMK	Subject Matter Knowledge
PCK	Pedagogical Content Knowledge
CCK	Common Content Knowledge
SCK	Specialized Content Knowledge
HCK	Horizon Content Knowledge
KCS	Knowledge of Content and Students
KCT	Knowledge of Content and Teaching
KCC	Knowledge of Content and Curriculum

CHAPTER I

INTRODUCTION

There can be many factors that affect students' learning, understanding and achievement with respect to the perspectives of students and teachers in mathematics education. One of these aspects that comes from the perspective of teacher is teacher's knowledge. The research evidence proves that teachers' knowledge has positive effect on students' achievement (Fennema, Carpenter, Franke, Levi, Jacobs, & Empson, 1996; Hill, Ball, & Schilling, 2008; Hill, Rowan & Ball, 2005; Tchoshanov, Lesser, & Salazar, 2008). This idea is also supported with international exams such as PISA (Programme for International Student Assessment), studies such as TEDS-M (Teacher Education and Development Study in Mathematics) and LMT (Learning Mathematics for Teaching), that they emphasize that teachers' mathematics content knowledge is important for students' achievement (Blömeke & Delaney, 2012).

Teacher's knowledge as a term is first revealed and defined by Shulman (1986), and he claimed that *content knowledge* is a missing part of exams to certificate teachers by examining tests for teachers and he defined content knowledge as "the amount and organization of knowledge per se in the mind of the teacher" (p. 9). He divided content knowledge into three categories: subject matter knowledge, pedagogical content knowledge and curricular knowledge. Shulman (1986) indicated that "pedagogical content knowledge includes an understanding of what makes learning specific topics easy or difficult" (p. 9). Since then, many researchers have identified the components of teacher knowledge and elaborated this concept (Graeber & Tirosh, 2008). To illustrate, some researchers (e.g. Even & Tirosh, 1995; Fennema & Franke, 1992; Marks, 1990) added knowledge of students, some of them (e.g. Grossman, 1990) added knowledge of curriculum, and knowledge of instructional strategies as important components to pedagogical content knowledge. Actually, it has been shown that content knowledge is more effective for teaching with pedagogical knowledge (Ball, Thames, & Phelps, 2008; Ball, 2000; Hill, Rowan, & Ball, 2005; Hill, Schilling, & Ball, 2004).

1.1. Mathematical Knowledge

In recent years, several researchers have defined teachers' knowledge for mathematics, especially in mathematics education. Stacey (2008) explained what mathematical knowledge that teachers should know for secondary teaching by examining teacher education program. According to her, teachers should have knowledge of the content of mathematics, experience of doing mathematics with problem solving, investigations and modeling, knowledge about mathematics that includes history and developments of concepts, and knowledge of how to learn mathematics.

Based on Shulman's framework, several researchers developed frameworks for teacher knowledge in mathematics education. One of them is Ma's (1999) framework that includes the concept of profound understanding of fundamental mathematics. However, she does not only focus on knowledge of mathematics, she indicates that knowledge should be combined with knowing when and how to use for teaching. Another framework is An, Kulm and Wu's (2004) framework that includes content, teaching and curriculum components and there is knowledge of teaching in the center of this framework. Although Ma thinks that teachers should have profound understanding of fundamental mathematics, An et al. (2004) think profound content knowledge is not sufficient itself and suggest that the combination of content and pedagogy is an important aspect for teaching mathematics effectively. Thus, they define profound pedagogical content as "knowledge a deep and broad knowledge of teaching and curriculum" (p.148). In their framework, pedagogical content knowledge of students' thinking is explained as including four categories: building on students' ideas in mathematics, addressing students' misconceptions, engaging students in mathematics learning, and promoting student thinking about mathematics. These categories involve different components, for example, the category of engaging students in mathematics learning has six components such as manipulative activity, connecting to concrete model, using one representation and both representations, giving examples, and connecting to prior knowledge (An et al., 2004, p.155). They note that if teachers know students' mathematical thinking, it guides teachers for better mathematics teaching with developing their knowledge of content, curriculum and instruction. Similarly, based on Shulman's idea of PCK and other literature (Ball, 2000; Ma, 1999), Chick Baker, Pham, and Cheng (2006) proposed a detailed framework of explicit elements of PCK. There are three categories in this framework; clearly PCK, content knowledge in a pedagogical context, and pedagogical knowledge in a content context. The elements in *clearly PCK* category are about pedagogy and content, such as students' thinking, misconceptions; the second category is *content knowledge in a pedagogical context* about content knowledge for teaching such as conceptual and procedural knowledge, the third category *pedagogical knowledge in a content context* is about knowledge and strategies for particular content of mathematics, such as classroom techniques for teaching.

Ball, Thames, and Phelps (2008) proposed a model for mathematical knowledge for teaching. This model is a domain map that shows mathematical knowledge for teaching consisting of subject matter knowledge and pedagogical content knowledge. Subject matter knowledge has common content knowledge and specialized content knowledge components. Common content knowledge (CCK) is "mathematical knowledge that is used in teaching, but not directly taught to students" and it is used by people working with mathematics, and specialized content knowledge (SCK) is specific to mathematics teachers (Hill, Sleep, Lewis, & Ball, 2007, p. 132). According to Ball et al. (2008), PCK consists of knowledge of content and students (KCS), knowledge of content and teaching (KCT), and knowledge of content and curriculum (KCC). They separate KCS from subject matter knowledge and explain KCS with focusing on students' mathematical thinking and understanding and they also note that teachers should know KCS.

As it is seen, the theory of teacher knowledge was developed first by Shulman (1986), and then it has been continued to be evaluated, elaborated and investigated in other areas, particularly mathematics education. Researchers have been trying to explain teacher knowledge for particular mathematical topics, such as algebra (e.g. functions), probability, and statistics (Graeber & Tirosh, 2008). However, there has been lack of research about teachers' knowledge and practice of algebra in the literature (Doerr, 2004; El Mouhayar & Jurdak, 2013; Wilkie, 2014). Students' learning and understanding of algebra have been investigated mostly regarding students' perspective as students' misconceptions and conceptions, errors and difficulties; and there have been many studies which show middle and high school students have difficulties with algebra (Cooper, Boulton-Lewis, Atweh, Pillay, Wilss, & Mutch, 1997; Linchevski & Livneh, 1999; Sfard & Linchevski, 1994; Stacey & Macgregor, 1997; Warren, 1999). However, algebra teaching, considering the aspects of what and how algebra is taught, has not been examined much in the perspective of teacher when reviewing mathematics education literature. At that point, examining teachers' algebra knowledge may shed light on understanding how students learn algebra and why they usually have difficulty in learning it. Two of the issues that were important and difficult about algebra teaching and learning for teachers' development literature were defining the nature of knowledge of teachers for teaching algebra, and articulating teachers' knowing (Doerr, 2004). Considering these issues in learning and teaching algebra, it may be said that there is a need to do research for learning and teaching algebra regarding teachers' knowledge.

1.2. Algebra in Mathematics Education

According to Rakes, Valentine, McGatha, and Ronau (2010), algebra is core for developing of understanding of high school mathematics and so students' learning fundamental concepts of algebra is an important issue. These fundamental concepts are variables, writing algebraic expressions, and simplifying with algebraic expressions that are introduced to students at 6th grade in the curriculum developed by Ministry of National Education (MoNE, 2013). The concept of "early algebra" can be suitable for this situation that Carraher and Schliemann (2007) defined early algebra as "compass algebraic reasoning and algebra-related instruction among young learners-from approximately 6 to 12 years of age" (p.670). National Council of Teachers of Mathematics's (NCTM) (2000) endorsement has supported the idea that algebra should be taught in early grades, and The Rand Mathematics Study Panel Report (2003) indicates that algebra in elementary curriculum is a gatekeeper for K-12 schooling. Schmittau (2005) examines the Vygotskian Perspective which emphasizes that students should be taught initially the most abstract and general level of understanding to develop algebraic thinking. Vygotsky states that the students should have theoretical and empirical concepts to learn mathematics or algebra. Empirical concepts can be learned from everyday experiences, but theoretical concepts are given to students by teachers and so it is important to constitute the theoretical basic in the elementary school years (Schmittau, 2005). As another suggestion, Kieran (2007) developed a model about what should be done in early grades. Based on this model, working with unknowns, variables, and equality, such as equations that represent problem situations, expressions of generalization as generational activity; and factoring, substituting one expression for one another, adding and multiplying polynomial expressions, solving equations and inequalities as examples for transformational activity, can be done in early algebra.

Algebraic thinking develops with generalizing of patterns and using variables in elementary and middle school levels and this development continues from pre-kindergarten to high school (Van de Walle, Karp, & Bay-Williams, 2013). When our middle school mathematics curriculum is examined, in grade 6, generalization from patterns is given to build students' algebraic thinking, and simplifying algebraic expressions expressing verbal statements in algebraic form; in grade 7, solving simple equations and linear equations, drawing linear graphs; and in grade 8, identities and factorization are taught within the content of algebra learning domain (MoNE, 2013). If the students could learn the algebra topics regarding the objectives in these grades, they would have a conceptual algebraic thinking in later grades. Thus, supporting the students to think algebraically in elementary grades could prevent middle and high school students' difficulties in algebra (Cai, Ng, & Moyer, 2011). Since middle grade and lower secondary school students have difficulties in algebra, learning of algebra topics in early years has gained importance to prevent the difficulties in older grades (Kieran, 2007). This suggestion is also practicable since research results showed that students' algebra understanding in early grades could be developed with the approaches, such as tasks, or discourse that supported algebraic thinking (Ferrara & Sinclair, 2016; Malara & Navarra, 2009; Warren & Cooper, 2008; Warren, Cooper, & Lamb, 2006).

At this point, it may be said that teachers have important role in teaching algebra. Malara and Navarra (2009) emphasize the importance of teachers' role in the construction of children's knowledge of algebra in early grades. There is a didactic cut in the child's thought in the transition from arithmetic to algebra and the students have a resistance in operating with unknowns in the transition to algebraic thinking (Gallardo, 2000). Teachers may overcome this obstacle with their roles in the classroom in early grades while transforming students' understanding and knowledge of arithmetic to beginning algebra. In connection with this, the examination of the teaching of beginning algebra may provide an understanding on how students' learning improves conceptually since teachers have important role in teaching and a positive effect on students' achievement (Fennema, Carpenter, Franke, Levi, Jacobs, & Empson, 1996; Hill, Rowan & Ball, 2005). However, there have been few studies related to teaching of early algebra topics (Doerr, 2004). The studies about teacher subject matter knowledge focused on generally functions, slopes, and equations (e.g. Even, 1990; Stump, 1999; Even, Tirosh, & Robinson, 1993). Besides, more recent studies focused on both teachers' subject matter and pedagogical content knowledge for middle grade and secondary grade level algebra topics. Artigue, Assude, Grugeon and Lenfant (2001) described three dimensions, as epistemological, cognitive, and didactic dimension, for teachers' knowledge of inequalities; Ferrini-Mundy, Floden, and McCrory (2006) developed knowledge for teaching algebra framework for teaching expressions, equations inequalities, and function at middle school and secondary level; and Li (2007) proposed a framework for teachers' knowledge specific to solving algebraic equations. Especially, teachers' knowledge of students' thinking as pedagogical content knowledge was examined in algebra word problem solving (e.g. van Dooren, Verschaffel, & Onghena, 2002; Nathan & Koedinger, 2000). As it is seen, the proposed models and frameworks generally are about the algebra topics at secondary level (Doerr, 2004). Although there have been studies about algebra knowledge, Wilkie (2014) has asserted that there are few studies about teachers' knowledge and practice about algebra, particularly teachers' practice in teaching process. Thus, this study focused on both teacher knowledge and teacher practice for teaching early algebra topics to develop teacher algebra knowledge literature.

1.3. The Significance of the Study

Algebra is essential for students' understanding of high school mathematics (Rakes et al., 2010). However, students have resistance in the transition of arithmetic thinking to algebraic thinking (Gallardo, 2000). If this transition to algebra is not provided conceptually, students may have difficulty in mathematics since then, and there have been many studies which show middle and high school students have difficulties with algebra (Kieran, 2007; Sfard & Linchevski, 1994; Stacey & Macgregor, 1997; Warren, 1999). Especially, in this study, the investigation of early algebra topics may be useful to understand where students do or do not conceptualize fundamental concepts of algebra in transition to algebraic thinking. At that point, how algebra teaching in early grades should be is a critical issue and the teaching role belongs to the teachers in mathematics education. The teachers have this role, since they are the practitioners of what is suggested by the curriculum for effective algebra teaching (Dede & Argün, 2004). Considering the role of teachers in teaching algebra to students, examining teachers' pedagogical content knowledge, which is one of the components of MKT, may be useful for understanding the reasons of students' misconceptions and difficulties. It can provide teachers the chance of having the students to be prepare to transfer algebra, designing their lessons, and completing their knowledge effectively. If the problems are overcome in early algebra, it eases students' understanding of the topics in higher levels.

Moreover, research results have shown that teachers' knowledge has a positive effect on students' achievement (Ball, 2000; Hill, Rowan & Ball, 2004). Chazan et al. (2003) also assumes that if teachers have right knowledge, this provides an increase in students' achievement. Thus, if in-service teachers have misconceptions and lack of knowledge about algebra based on the findings of this study, trainings can be done to assist them for developing their knowledge. It may help to increase students' achievement in algebra.

More particularly, the scope of this research includes generalization of patterns and operations with algebraic expressions, and these are also important for developing students' algebra conceptualization. Lee (1996) indicated that "algebra, indeed all of mathematics is about generalizing patterns" (p.103). Generalizing patterns provides using arithmetical relationships between the input and output values and this is seen as one of the components of algebra (Katz, 1997; Usiskin, 1988). Generalization is important for developing the schemas about algebraic thinking (Hargreaves, Threlfall, Frobisher, & Shorrocks-Taylor, 1999; Steele & Johanning, 2004). The understanding of functional relationship between the position number and the term can support the learning of the concept of function in later grades (Usiskin, 1988). Thus, how the teachers teach and give the idea of generalization in the context of patterns can provide understanding the students' learning and help them to form meaningful schemas of algebra. After the conceptualization of variable and algebraic expression within generalization of patterns, the context of operations with algebraic expressions provides the procedural understanding of these concepts that the students manipulate algebraic expressions with adding, subtracting and multiplying in the learning of this topic (Capraro & Joffrinon, 2006).

This study which is also important for examining middle school mathematics teachers' knowledge about algebra, is expected to contribute to mathematics education literature. As the researchers (Baş, Erbaş, & Çetinkaya, 2011; Doerr, 2004; El Mouhayar & Jurdak, 2013; Kieran; 1992; Wilkie, 2014) indicated, mathematics education has lack of research about teachers' knowledge of algebra. Thus, there has been an increasing interest in investigating teacher content knowledge and pedagogical content knowledge of algebra (Saul, 2008). The current study focused on the teachers' knowledge in a process in the context of qualitative research. Since the qualitative research design can enable to gain detailed and rich information about the investigated phenomenon, this methodology is preferred for this study in order to examine teachers' knowledge rather than collecting static data once at a time (Cresswell, 2007). Two teachers participated in this study and each teacher's instruction for each topic was considered as a case and their knowledge was the unit of analysis within the context of this research. Actually, the quantitative studies, even large scale projects (TEDS-M, LMT, COACTIV) were conducted to investigate teacher knowledge. However, measuring teacher knowledge quantitatively with a survey and once at a time can limit the understanding of the concept (Hill et al., 2008). Kahan, Cooper, and Bethea (2003) suggest conducting qualitative research with different teachers to determine teacher knowledge with the aspect that affects the teaching process. With the current study, the teachers' actions in a process could give us more detailed information about their knowledge and also reliable patterns as the study was focused on a long process. In this process, the observation of the instructions and the interviews with the teacher could give an accurate picture of teacher knowledge. Besides analyzing the teachers' practices in their instruction, the process of their planning of the lessons were also analyzed. While the planning provided an understanding of the teachers' existing knowledge, the instruction provided an observation on how the teachers used their knowledge for teaching.

Thus, the data collection process of this study could give a holistic understanding about the teachers' teaching with planning and instructions.

Ball and her colleagues (2008) developed MKT model based on Shulman's (1986) theory of teacher knowledge as content knowledge and pedagogical content knowledge. This MKT model has been widely accepted and used by mathematics education researchers. Teacher knowledge is explained into sub-domains specific to mathematic teaching with this model (Hill et al., 2008). Thus, it is also considered to reveal existing teacher knowledge as specific to algebra teaching with using this model, which gives detailed descriptions of the knowledge domains. Another reason for using this model was that it was based on observations of teachers' instructions as qualitative aspect of the development of it. Since the main data of this study was observations of the instructions, the MKT model was found as appropriate. One of the aims of the study was to reveal the teachers' knowledge and how they used it in practice as Ball et al. (2008) indicated. The researchers also explained that MKT model provided to determine what teachers need to know to teach the content and how they need to use it in practice. In connection with this, the second aim of this study was to propose that the teachers need to know what and how to teach generalization patterns and operations with algebraic expressions. Stacey and Chick (2004) asserted that developing knowledge forms for teachers to learn and use with the knowledge of students' thinking and the knowledge of teaching was not an easy work. The current study attempted to contribute to this aim, to teacher knowledge literature, even particularly for algebra. Considering the aims of the study under these contributions, since MKT model includes content knowledge and pedagogical content knowledge together, it was preferred to be used for this research. Moreover, this study also attempted to explore the influence of subject matter knowledge (SMK) on pedagogical content knowledge (PCK) qualitatively in the context of the algebra topics. Even (1993) emphasized that conceptual subject matter knowledge is required for teaching mathematics effectively. Since this study also seeks what is needed for teaching algebra, the role of SMK in teaching is investigated. Several large-scale projects also found that content knowledge was required for pedagogical content knowledge (Ball et al., 2008; Blömeke & Kaiser, 2012; Hill et al., 2005; Krauss, Baumert, & Blum, 2008a). Besides, Depaepe et al. (2015) pointed out the need of the qualitative research about the relationship between content knowledge and pedagogical content knowledge for particular mathematical contents. Thus, this study might contribute to the literature of teacher knowledge by exploring the influence of SMK on PCK.

Examining teachers' knowledge can also be valuable for mathematics teacher education programs and if teachers have misconceptions and lack of knowledge about algebra, teacher educators can design their programs and method courses regarding developing prospective teachers' knowledge of algebra. Actually, mathematics teacher educators indicated that teachers need to have conceptual and connected knowledge about algebra to support students' learning of algebra. However, Magiera, van den Kieboom, and Moyer (2013) asserted that the suggestions on how to develop the teachers' knowledge by educators are a few. At that point, the current study, such studies, could reveal existing teacher knowledge and what it is lack of, and how the teachers use their existing knowledge. This situation may help the mathematics teacher educators to design training programs to develop teachers' knowledge. Under these considerations, this study is considered to contribute to teaching and learning mathematics, particularly algebra.

1.4. The Problem Statement

Algebra is one of the learning domains of our middle school mathematics curriculum and it is introduced to children at 6^{th} grade (MoNE, 2013). NCTM's (2000) endorsement and The Rand Mathematics Study Panel Report (2003) have supported that algebra should be taught in early grades because of its important role for understanding middle school and secondary school mathematics. Kaput (2000) advocated teaching algebra in early grades and emphasized the learning of "the study of functions, relations, and joint variation" (p. 19). In connection with this description, two important concepts are variable and algebraic expression

(Subramaniam & Banerjee, 2004). The development of these concepts as conceptually and procedurally is important for forming an equation and solving it later (Capraro & Joffrinon, 2006). The concept of variable is taught first in the context of generalization of patterns, and then in relation with the generalization, algebraic expressions are taught conceptually and procedurally in the context of operations with algebraic expressions in MoNE (2013). Thus, middle graders' conceptualization of algebra with the concept of variable begins to develop in the generalization of patterns first. Patterns can provide for analyzing the relationship between input and output values as numbers within the contexts or figures and making generalization using variables. It supports the students to transit arithmetic to algebra and to understand the function of variable in the generalization (English & Warren, 1998). According to Kieran's (2007) model of algebraic activity, generalization of patterns is a generalization activity and then comes transformational activity that requires manipulating the symbolic form of an expression or an equation in order to develop students' algebraic thinking in early grades. Operations with algebraic expressions as an algebra topic enables to collect like terms and multiply algebraic expressions to simplify algebraic expressions in the curriculum (MoNE, 2013). Thus, the scope of this study is teaching generalization of patterns and operations with algebraic expressions.

In the perspective of teaching, teachers' knowledge can be an important aspect of students' learning algebra as indicated in some studies and teachers' knowledge has positive impact on students' achievement (Fennema, Carpenter, Franke, Levi, Jacobs, & Empson, 1996; Hill, Rowan & Ball, 2005). However, there has been lack of research about teachers' knowledge and practice in algebra in the literature (Baş, Çetinkaya, & Erbaş, 2011; Doerr, 2004; El Mouhayar & Jurdak, 2013; Kieran; 1992; Wilkie, 2014). Considering the importance of algebra as a learning domain in mathematics and the lack of research about teachers' knowledge for teaching algebra, the purpose of this study is to reveal middle school mathematics teachers' mathematical knowledge for teaching generalization of patterns and operations with algebraic expressions. With this study, teacher knowledge was examined in planning and within the practices of implementing of lessons. The process of their planning of the lessons provided an understanding of the teachers' existing knowledge. The instruction provided an observation on how the teachers used their knowledge for teaching to reveal their practices. Investigating teacher practices as well as teacher knowledge can help to understand of students' understanding since teaching practices support students' learning (Saxe, Gearhart, & Seltzer, 1999). Lampert (2004) defined practice with action that "action is behavior with meaning, and practice is action informed by a particular organizational context (p. 2). Thus, it was focused on the teachers' actions in the process of the instructions throughout the teaching of the topics to extract the practices. In this process, the observations provided an examination of the teachers' reactions and responses at that time of the class, while the interviews provided a correct interpretation of what was observed with asking to the teachers and getting their explanations. As Wilkie (2014) indicated, the research about teaching algebra in middle school was few, especially, the study about teacher practice for teaching algebra as classroom research was scarce. Thus, this study also contributed to the literature with filling the void about practicing teacher algebra knowledge in classroom research.

For this study, the following research questions are framed:

1. What is the nature of middle school mathematics teachers' mathematical knowledge for teaching (MKT) generalization of patterns in planning and implementing lessons?

   1.a. What is the nature of middle school mathematics teachers' mathematical knowledge for teaching (MKT) for generalization of patterns in planning lessons?

   1.b. What is the nature of middle school mathematics teachers' mathematical knowledge for teaching (MKT) for generalization of patterns within the practices of implementing lessons?

2. What is the nature of middle school mathematics teachers' mathematical knowledge for teaching (MKT) operations with algebraic expressions in planning and implementing lessons?

2.a. What is the nature of middle school mathematics teachers' mathematical knowledge for teaching (MKT) for operations with algebraic expressions in planning lessons?

2.b. What is the nature of middle school mathematics teachers' mathematical knowledge for teaching (MKT) for operations with algebraic expressions within the practices of implementing lessons?

After, these research questions' answers are explored, the relationship between subject matter knowledge and pedagogical content knowledge as components of MKT is investigated with the following research question:

3. How does middle school mathematics teachers' subject matter knowledge (SMK) influence their pedagogical content knowledge (PCK) in the context of teaching generalization of patterns and operations with algebraic expressions?

To answer these questions, the researcher used Ball et al.'s (2005) Mathematical Knowledge for Teaching (MKT) theoretical framework in the study. This model has two main components as subject matter knowledge and pedagogical content knowledge, and sub-components as knowledge types. The components (SMK and PCK) and sub-domains (CCK, SCK, KCS, KCT, and KCC) of MKT model were utilized to analyze the teachers' knowledge for teaching generalization of patterns and operations with algebraic expressions in this study. Teacher knowledge is explained into domains and sub-domains specific to mathematic teaching within this model (Hill et al., 2008). Thus, it could be considered that teacher knowledge can be explained in detail with using this model's components. In this study, analyzing the teachers' knowledge using this model, it is aimed to provide a detailed description of their existing knowledge and also a proposal on what the teachers need to know for teaching these topics.

1.5. Definitions of Important Terms

It is necessary here to clarify exactly what is meant by the terms in the research questions. In this regard, the terms are defined as in the following.

Mathematical Knowledge for Teaching (MKT): Ball et al. (2008) describes mathematical knowledge for teaching stating “the mathematical knowledge needed to carry out the work of teaching” (p. 395). They centered the word of teaching in the definition and emphasized the tasks and mathematical demands of these tasks related to teaching. Ball et al. (2008) explained teaching as follows “showing students how to solve problems, answering students’ questions, and checking students’ work, it demands an understanding of the content of the school curriculum” (p. 395). Their developed model for mathematics knowledge for teaching (MKT) as a domain map consists of subject matter knowledge (SMK) and pedagogical content knowledge (PCK) components.

Subject Matter Knowledge (SMK): Shulman (1986) defined subject matter knowledge as "the amount and organization of knowledge per se in the mind of the teacher" (p. 9). According to Shulman (1986), teachers should have the knowledge of facts and procedures of subjects with reasoning underlying them. SMK includes the general mathematical knowledge as common content knowledge (CCK), the mathematical knowledge specific to teaching as specialized content knowledge (SCK), and the mathematical knowledge of the relations of the topics between grades as knowledge at the mathematical horizon (KMH) in the current study (Ball et al., 2008).

Pedagogical Content Knowledge (PCK): Shulman (1986) defined pedagogical content knowledge as “special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding” (p. 8). According to Shulman (1986), teachers should have the knowledge of different representations of the concepts, and the knowledge of students’ thinking. PCK includes the knowledge of students’ thinking as knowledge of content and students (KCS), the knowledge of teaching methods and techniques as knowledge of content and teaching (KCT), and the knowledge of mathematics curriculum contents and objectives as knowledge of content and curriculum (KCC) in the current study (Ball et al., 2008).

Practice: Practices are defined as "core activities (within mathematical domain and appropriate grade levels) that could and should occur regularly in the teaching of mathematics" (Franke, Kazemi & Battey, 2007, p. 249). Saxe et al. (1999) stated that the practices are formed between the teacher and the students so that classroom practice can support and provide a development for students' learning of mathematics.

Middle School Mathematics Teacher: Middle school where 5^{th}, 6^{th}, 7^{th}, and 8^{th} graders receive education after the primary school education in Turkey. Middle school mathematics teacher is the teacher who teaches mathematics with respect to the middle school mathematics curriculum to these grades.

Generalization of Patterns: This topic has the objective which is to "express using letters the relation in number patterns which are modelled" under the 'Patterns and Relations' sub-learning domain of algebra learning area in the 7^{th} grade mathematics curriculum (MoNE, 2009). This objective belongs to algebra learning area of 6^{th} mathematics curriculum in the new curriculum (MoNE, 2013).

Operations with algebraic expressions: This topic has the objectives which are to "perform addition and subtraction operations with algebraic expressions" and "multiply two algebraic expressions" under the 'Algebraic Expressions" sub-learning domain of algebra learning area in the 7^{th} grade mathematic curriculum (MoNE, 2009). These objectives belong to algebra learning area of 6^{th} and 8^{th} mathematics curriculum in the new curriculum (MoNE, 2013).

CHAPTER II

LITERATURE REVIEW

The purpose of this study was to examine middle school mathematics teachers' mathematical knowledge for teaching generalization of patterns and operations with algebraic expressions. In this context, frameworks about teacher knowledge in general and teachers' algebra knowledge, and studies related to algebra teaching and learning are reviewed in this chapter. Furthermore, this chapter includes the following sections; theoretical frameworks about teacher knowledge, models of teacher knowledge specific to mathematics teaching, the relationship between subject matter knowledge and pedagogical content knowledge, models of teacher knowledge for teaching algebra, early algebra, and studies related to teaching and learning algebra. A summary of literature review is presented at the end of the chapter.

2.1. Theoretical Frameworks about Teacher Knowledge

Shulman (1986) first defined the concept of teacher knowledge and since then this concept has been elaborated and expanded by many researchers (Cochran, DeRuiter, & King, 1993; Grossman, 1990; Magnusson, Krajcik & Borko, 1999). Especially, several researchers have described teacher knowledge for mathematics teachers (An, Wu, & Kulm, 2004; Ball, Thames, & Phelps, 2008; Fennema & Franke, 1992; Rowland, Turner, Thwaites, & Huckstep, 2009). In this section, the developed frameworks and models for teacher knowledge in general are explained.

Shulman (1986) stated that assessing teacher candidates regarding their subject matter knowledge and pedagogical knowledge could be a new issue addressing the research entitled "Knowledge growth in teaching". He examined the exams in 1800s to be a teacher, that had questions mostly based on subject matter knowledge and few of them was about pedagogical skills. Thus, he noted that pedagogical issues were not regarded as important firstly for the qualified teachers. Then, in 1980s, the contents in the examinations for being a teacher were about reading, writing, and solving problems. Although these examinations were based on research about designing lesson plans, assessment, the characteristics of children, and educational policies, Shulman (1986) queried where content knowledge to be taught. According to him, how subject matter knowledge was presented in the instruction should be questioned. In this regard, Shulman and colleagues proposed that “content knowledge” is a missing part of exams to certificate teachers by examining tests for teachers. Shulman (1986) also asked if content was more important than pedagogy, or if knowing pedagogy did not require content. He recognized the distinction between pedagogy and content. When he examined the studies about teaching, he noticed that the interest of the studies was pedagogical issues such as classroom management, preparing lesson plans, organizing time, and giving assignment. However, Shulman (1986) noted the necessity of answering these type of questions, “Where do teacher explanations come from? How do teachers decide what to teach, how to represent it, how to question students about it and how to deal with problems of misunderstanding?” (p. 8). Thus, he and his colleagues aimed to set up a balance among content and pedagogy in their study. In this context, they studied with secondary English, biology, and mathematics teachers in their first year of teaching. The researchers conducted interviews with the teachers about their planning and interpretations about materials they used, and observed their instructions. They collected data about their teacher education program. They observed one of the problems arise during the study that the teachers did not learn some topics to teach, and they found the textbooks or curriculum materials insufficient. With this research, Shulman and his colleagues developed a model for defining teachers’ knowledge. They proposed three categories for teacher knowledge as subject matter knowledge, pedagogical content knowledge and curricular knowledge.

Shulman (1986) defined subject matter knowledge as "the amount and organization of knowledge per se in the mind of the teacher" (p. 9). According to

Shulman (1986), knowing subject with facts and procedures is not enough, teachers need to understand the reason how it is so. Shulman identified the content knowledge with the structures of subject matter knowledge as substantive and syntactic referring Schwab (1978). He explained the substantive aspect as the basic concepts and principles of discipline, and the syntactic structure as the validity issues of the rules. According to Shulman (1986), teachers should explain the facts with warrants, their value for learning, and the relation with other disciplines based on theory and practice.

The second category of teacher knowledge is pedagogical content knowledge. This term is first seen as a component of the theory of teacher knowledge by Shulman (1986). Pedagogical content knowledge (PCK) is about teaching and a particular form of content knowledge (Shulman, 1986). Pedagogical content knowledge is defined by Shulman (1987) as "special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding" (p. 8). According to Shulman (1986), pedagogical content knowledge provides teachers to present subjects with different representations, analogies, illustrations, examples, explanations, and demonstrations to teach learners effectively. Thus, he also indicated that "pedagogical content knowledge includes an understanding of what makes learning specific topics easy or difficult" (p. 9). Teachers should know the conceptions and preconceptions that different leveled and aged children have, since teachers could handle possible difficulties and misconceptions in designing their lessons by using different methods and techniques based on the knowledge of the students' thinking (Shulman, 1986).

Curricular knowledge is the third category of content knowledge in Shulman's framework (1986). He defined curricular knowledge as "with particular grasp of the materials and programs that serve as "tools of the trade" for teachers" (Shulman, 1987, p. 8). He also suggested two aspects of the curricular knowledge that were lateral and vertical knowledge. The lateral curriculum knowledge is about the connection the topics that is taught with the other topics in different disciplines. On the other hand, the vertical curriculum knowledge is about knowing and being aware of before and after the topics that is taught in previous and will be taught in future years. Teachers with this knowledge could connect the topics with prior concepts and next concepts of the same topic.

Shulman (1987) developed a model knowledge base for teaching with categorizing teacher knowledge into seven groups after one year of the proposal of teacher knowledge concept. These categories are: content knowledge; general pedagogical knowledge that is about the strategies of classroom management; curriculum knowledge that is about materials and programs; pedagogical content knowledge; knowledge of learners and their characteristics; knowledge of educational contexts; and knowledge of educational ends, purposes, and values, and their philosophical and historical grounds (Shulman, 1987, p. 8). Shulman indicated the importance of pedagogical content knowledge concept particularly for teaching. For this knowledge, he explained as the combination of content and pedagogy to teach a topic with representations and organizations specific to this topic. Thus, he asserted that "pedagogical content knowledge is the category most likely to distinguish the understanding of the content specialist from that of the pedagogue" (p. 8).

Based on Shulman's (1986) characterization of teachers' knowledge, several components for teacher's knowledge have been identified and elaborated by different researchers in teacher education (e.g. Cochran, DeRuiter, & King, 1993; Grossman, 1990; Magnusson, Krajcik, & Borko, 1999). Grossman (1990), was one of the students of Shulman, studied with secondary English teachers. Grossman (1990) proposed teacher knowledge model with four categories, 1) subject matter knowledge, 2) general pedagogical knowledge, 3) pedagogical content knowledge, and 4) knowledge of context. The subject matter knowledge category is formed with the knowledge of content, knowledge of the substantive, and knowledge of the syntactic structures. The aspects of the substantive, and the syntactic structures knowledge explained by Schwab (1964, as cited in Shulman, 1986). While the substantive structure is related with the content as facts and principles, the syntactic structure is related with the process ways of the accuracy and validity of the rules and principles. In this context, Grossman's (1990) description of subject matter knowledge is related with the content that teachers know and present in teaching. The general pedagogical knowledge category is formed with learners and learning, classroom management, curriculum and instruction, and other pedagogical issues about teaching and learning. The third category of this model is pedagogical content knowledge. This knowledge is affected and developed by other three categories. One of the components of pedagogical content knowledge category is knowledge of students' understanding that consists of the knowledge of students' conceptions and misconceptions as Shulman (1986) indicated. Other components of this knowledge is knowledge for curriculum including teachers' decisions about the appropriateness of resources and materials for teaching a specific topic. This component also requires knowing of the relations of the topic with other disciplines, and the connections with previous and future topics with the same content. Although Shulman (1986) defined curricular knowledge separately explaining these properties, Grossman (1990) included this component to pedagogical content knowledge. Other components of this knowledge is conceptions of purposes for teaching subject matter. This component refers to the conceptions of objectives for teaching a topic and related beliefs about teaching the topic. Grossman (1990) expanded teacher knowledge concept with including teachers' beliefs. The fourth component of pedagogical content knowledge is knowledge of instructional strategies. These strategies are method and techniques, representations, models to provide students to learn a topic conceptually. The knowledge of context category is formed with the knowledge of the school such as the culture of school, the characteristics of district that the school is placed, the structure of families of students. This category and pedagogical content knowledge category interact with themselves.

In the perspective of constructivist learning, Cochran, DeRuiter, and King (1993) expanded Shulman's model to explain teacher knowledge. The researchers suggested "knowing" word instead of using "knowledge" in pedagogical content knowledge since knowing indicated the development in a process that was related with constructive approach. Cochran et al. (1993) redefined PCK as Pedagogical

Content Knowing (PCKg). In their model, PCKg is the center of it and is defined as "a teacher's integrated understanding of four components of pedagogy, subject matter content, student characteristics and the environmental context of learning" (p. 266). Cohran et al. (1993) indicated the environmental context of learning and teacher's knowledge of students when compared Shulman's model. Since students construct their learning in the constructive learning, teacher's knowledge of students is important component of PCKg. The other indicated component is the environmental context of learning about understanding the structure of teaching and learning such as school culture, parents' involvement. Cochran et al. (1993) stated the effect and contribution of four components on PCKg in the model could change in time. Thus, they suggested that teacher education programs present opportunities to experience four components to develop pre-service teachers' knowledge.

2.2. Models of Teacher Knowledge Specific to Mathematics Teaching

The researchers in mathematics education also proposed models for teacher knowledge especially in mathematics based on the frameworks (Cohran et al., 1993; Grossman, 1990; Shulman, 1986) about teacher knowledge as mentioned above. One of the models of teachers' knowledge for mathematics was proposed by Fennema and Franke (1992). They reviewed the literature about teacher knowledge critically and indicated knowledge of mathematics teaching is considered with subject matter knowledge, representation of subject matter knowledge, knowledge of students' thinking and teachers' beliefs. Teachers must know issues such as real-life situations, manipulatives and present to them students by relating mathematical ideas to promote students' understanding. The other point that the researchers concluded from review is knowledge of students that teachers would use for decision making in the instruction to improve students' understanding. With this review, Fennema and Franke (1992) proposed a research model for teachers' knowledge of mathematics. Thus, they asserted that teacher knowledge has dynamic and interactive structure that this knowledge is developed in time with experiences of teaching. Their proposed model is formed with knowledge of mathematics, pedagogical knowledge, knowledge of learners' cognitions, and beliefs. Knowledge of mathematics as referred by Shulman, as content knowledge is the knowledge of concepts, procedures, and problem-solving procedures. This content knowledge requires the conceptual understandings of concepts, understanding the relationships among concepts, and knowing the use of the concepts and procedures in mathematical situations. Pedagogical knowledge is the knowledge of pedagogical issues for teaching and learning such as classroom management, methods and techniques for planning, classroom organization is similar to Shulman's explanation for pedagogical knowledge. Knowledge of learners' cognitions in mathematics is about how students think and learn mathematics topics. It requires also knowing of acquisition of mathematical knowledge, possible difficulties, and expectation about students' good performance in learning process. These three knowledge components are in an interaction with context specific knowledge as the center of this model. Fennema and Franke (1993) emphasized the importance of beliefs and knowledge's taking part in a context. The researchers explained the role of context as "within a given context, teachers' knowledge of content interacts with knowledge of pedagogy and students' cognitions and combines beliefs to create a unique set of knowledge that drives classroom behavior" (p. 162). Fennema and Franke (1993) pointed out the transform in the development of teacher knowledge by interacting with content knowledge and students throughout of the instruction. According to Fennema and Franke (1993), teachers must transform content knowledge into teaching to enable students to learn. In this transformation, teachers adapt their knowledge regarding the students learning and so teacher knowledge can change and develop.

Another model about mathematics knowledge for teachers is entitled "Knowledge Quartet", and proposed by Rowland, Huckstep, and Thwaites (2005). The researchers studied with pre-service elementary teachers, and observed their lessons based on their planning. Their purpose was to reveal pre-service teachers' mathematics knowledge, and mathematics knowledge in teaching. Rowland et al. (2005) proposed the model using grounded theory for analysis. The categories of this model are foundation, transformation, connection, and contingency. The first category, foundation, consists of mathematical theoretical knowledge and mathematical knowledge for teaching, and beliefs about this knowledge. The researchers asserted that teachers learn the knowledge by their own in school and in pre-service training. The following three categories, except the first category, include the knowledge about planning and teaching mathematics. The second category, transformation, is teachers' transformation of content knowledge into teaching based on Shulman's (1987) definition. As Fennema and Franke (1993) indicated above, teachers can use pictures, concrete materials, and multiple representations to transform the knowledge that they have to improve students' learning. The third category, connection, includes choices and decisions for planning and implementing lessons. For this, teachers set connections between concepts, procedures, and topics; order contents, questions, tasks among lessons coherently. Connection knowledge is consistent with Shulman's curriculum knowledge description. The fourth category, contingency, is about teachers' knowledge of handling unexpected situations which occurr during teaching. This category includes teachers' responses to students' questions or statements, readapting the lessons including the unexpected situations to the lesson plan, and using students' emerged ideas for the benefit of students' learning. This model is different with including contingency category from the frameworks and models that were mentioned up to now. Herewith, Rowland et al. (2005) suggested using this model for lesson observations of teaching mathematics.

More particularly, the models specific to mathematics teachers' PCK are explained in the following. These frameworks categorized PCK especially for mathematics (An, Kulm, & Wu, 2004; Chick, Baker, Pham, & Cheng, 2006). Chick, Baker, Pham, and Cheng (2006) proposed one of the frameworks for PCK. Based on Shulman's (1986) theory of PCK and other literature (Ball, 2000; Ma, 1999), Chick et al. (2006) proposed a detailed framework to explicit elements of PCK. There are three categories in this framework; clearly PCK, content knowledge in a pedagogical context, and pedagogical knowledge in a content context. The elements in *clearly PCK* category are about pedagogy and content, that are discussing and using teaching strategies; identifying and addressing students' thinking, misconceptions, and understanding; describing representations of concepts; using resources, and curriculum knowledge. The researchers emphasized the knowledge of misconceptions in separate element of the first category. The second category is *content knowledge in a pedagogical context* about content knowledge for teaching. The elements in this category are using conceptual and procedural knowledge, making connections in mathematical concepts and structures, solving a mathematical problem. The third category *pedagogical knowledge in a content context* is about knowledge and strategies for particular content of mathematics. The elements are describing goals for students, focusing students' attention on content, and classroom techniques for teaching. Chick et al. (2006) proposed this framework specific to decimals with detailed descriptions from based on literature (Ball, 2000; Ma, 1999; Shulman, 1986), and suggested using it to examine for other mathematics topics.

Another framework for PCK in mathematics is An, Kulm and Wu's (2004) framework. Based on Shulman's (1987) concept of PCK, An et al. (2004) developed the network of pedagogical content knowledge. The researchers suggested the concepts of profound pedagogical content knowledge based on Ma's (1999) concept of profound understanding of fundamental mathematics. An et al. (2004) indicated that content knowledge should be connected with curriculum and teaching for effective mathematics teaching. In the model, PCK has three components, knowledge of content, knowledge of curriculum, and knowledge of teaching. Especially, knowledge of teaching component is the center of the model. There are interactive relationships between each component. An et al. (2004) studied with mathematics teachers in China and in U.S to compare the teachers' pedagogical content knowledge. They collected data from questionnaires about mathematics teaching and beliefs, interviews, and observations. The topics they included to the mathematics teaching questionnaire were fractions, ratios, and proportions. The researchers concluded that while Chinese teachers aimed to develop conceptual understanding depending on effective traditional methods, U.S. teachers aimed to develop students' understandings using creativity and inquiry without connection of activities and abstraction, and procedures. Based on the analysis of data, An et al. (2004) proposed four components and categories in the components for pedagogical content knowledge. According to their model of PCK, "deep and broad pedagogical content knowledge is important and necessary for effective teaching" (p. 169). Teachers should connect prior knowledge, concrete materials with conceptual knowledge to build on students' math ideas. It is important to teachers' addressing misconceptions using picture, table, and concrete model to correct them. Teachers also should use representations to engage students in math learning, and questions, activities, and tasks to promote students' mathematical thinking.

More broadly and detailed than these models, Ball, Thames, and Phelps (2008) developed a model that is entitled mathematical knowledge for teaching (MKT) that includes SMK and PCK specific to mathematics teachers. This model is explained in the following.

Ball and Bass (2002) described the term "mathematical knowledge for teaching" building of Shulman's concept of PCK. After the studies and research conducted to investigate and explain PCK for many years, this term has been acknowledged and used for mathematics education recently. According to Ball et al. (2008), PCK concept has not been developed well to show its usage in teaching as empirical evidence. Since there is not empirical testing, the concept of PCK does not function to improve teaching and learning, to revise the curriculum for teachers, to guide teacher development, and to understand the relationship between teacher knowledge and students' learning. The concept of PCK had a theoretical structure that teachers need to have. From this point of view, Ball and her colleagues have conducted a project to provide empirical base for knowledge for teaching.

Ball (1990) investigated prospective teachers' subject matter knowledge that had when they entered teacher education program. She asked questions to teacher candidates that were formed with classroom scenarios. She found that teacher candidates' knowledge as superficial and consists of rules. Ball (1990) asserted that teachers should have deep knowledge to understand the concepts and procedures conceptually and knowledge of learning of mathematics of students with the ways that they learn. Teachers also should have deep knowledge to respond students' questions reasonable, use multiple representations, and improve students' learning interpreting their ideas. Ball (1990) called this knowledge as substantive knowledge that requires correct knowledge of concepts and procedures, underlying principles of them, and connection between them. Ball and her colleagues emphasized that teachers must have deep mathematical knowledge to teach mathematics from the investigations about teacher knowledge in 1980 and 1990s (Ball, 1990; Ball & Bass, 1993). Thus, Ball and colleagues conducted project that is entitled Mathematics Teaching and Learning to Teach (MTLT) for investigating teachers' implementations in class and what mathematical knowledge teachers must have to teach mathematics (Ball & Bass, 2003). They examined a third grade teaching of mathematics in a year. They collected data from video records of lessons, transcriptions of audio records, students' works, and teachers' plans. They analyzed the practice of teaching mathematics to propose a framework for mathematical knowledge. Ball and Bass (2003) indicated the work of teaching as what happens in the class such as using representations to enable students' understanding, interpreting and responding to the students, and using mathematical language appropriately (Ball, 1990; Ball & Bass, 2003; Hill, Rowan, Ball; 2005). Ball and Bass (2003) made three implications for the work of teaching based on the analysis in their study. The first is teaching has substantial mathematical work to solve problems in mathematics. The second implication is unpacking of mathematical knowledge for teaching. For example, to develop children's conception of rational numbers, instead of giving the definition of rational numbers, teachers should develop fraction concept and then decimals concept connection with fraction in the process of teaching rational numbers. The third implication is the connectedness of mathematical topic. To illustrate, showing the difference of x^2+y^2 and $(x+y)^2$ with explaining their areas in geometric representations. According to Ball and Bass (2003), teachers also should predict students' thinking as they are learning and their knowledge are growing.

With examining teacher knowledge in this project qualitatively, Mathematical Knowledge for Teaching (MKT) model was developed based on the work of teaching (Ball et al., 2008). Then, the measures were developed to assess teachers' MKT to support the model quantitatively (Hill, Ball, & Schilling, 2008). They investigated the answers for how the organization of teachers' mathematical knowledge, and realibility of the questions. The researchers developed survey items based on number and operations, patterns and function at elementary level as topics, and *content knowledge* and *knowledge of content and student* as teacher knowledge domains. They concluded that content knowledge requires more than subject matter knowledge of mathematics in order to teach (Hill, Schilling, & Ball, 2004). Schilling, Blunk, and Hill (2007) stated that measuring content knowledge for teaching using validity argument approach contributed to differentiate the teachers with content knowledge and the teachers who did not have adequate content knowledge in their teaching and students' learning.

Up to this point, the concerns about subject matter knowledge and pedagogical content knowledge for teachers to teach mathematics are mentioned. The development of constructs of mathematical knowledge for teaching (MKT) and the measurement of these constructs have provided the model of MKT. Ball et al. (2008) describes MKT as "the mathematical knowledge needed to carry out the work of teaching" (p. 395). Teaching involves everything that teachers most do to support students' learning that are the instructions and all tasks in class. In addition to this, it includes planning lessons, assessing and grading students, assigning tasks and homework, informing parent about the works, providing equity, and having responsibility to principal. Ball et al. (2008) analyzed teachers' practice and determined mathematical demands of teaching. According to them, teachers need more mathematical knowledge than others. They exemplified it like this, everyone can do subtraction in three digit numbers and teachers also must know and do. However, this is not sufficient for teaching. Teachers also recognize students' errors in this operation, beyond the reasoning of this error, recognize the different procedures of the different errors to help students' learning. Teaching also requires explaining procedures, terms, and concepts with reasoning. While selecting examples to teach a procedure, teachers should know what critical numbers to use to improve students' understanding. With this analysis, Ball et al. (2008) have described mathematical demands of teaching and expanded Shulman's concept of teacher knowledge in their model.

Their proposed model for mathematics knowledge for teaching is a domain map that shows mathematical knowledge (PCK) for teaching consists of subject matter knowledge (SMK) and pedagogical content knowledge. Ball et al. (2008) divided these domains into subdomains in the model. Subject matter knowledge (SMK) is divided into three subdomains that are common content knowledge (CCK), specialized content knowledge (SCK), and horizon content knowledge (HCK). Pedagogical content knowledge (PCK) is divided into three subdomains that are knowledge of content and students (KCS), knowledge of content and teaching (KCT), and knowledge of content and curriculum (KCC).

The first subdomain of subject matter knowledge, common content knowledge (CCK) is defined as "the mathematical knowledge and skill used in settings other than teaching" (Ball et al., 2008, p. 399). This knowledge is not unique for teachers; this knowledge can be used by everyone who deals with mathematics. To illustrate, knowing the sides of rectangle are perpendicular, or multiplication of a number with zero yields 0. Teachers with this knowledge must recognize incorrect explanations, questions, definitions of textbooks, incorrect students' answers or solutions. The understanding of mathematics is necessary in planning and implementing the instruction. Otherwise, teaching can be interfered with the lack of CCK. The second subdomain of subject matter knowledge, specialized content knowledge (SCK) is defined as "the mathematical knowledge and skill unique to teaching" (Ball et al., 2008, p. 400). This knowledge is related with teaching. To illustrate, understanding the students' errors with their reasons, or representing division of 3 by 2/5 is about SCK. According to Ball et al. (2008), teachers must have decompressed and unpacked knowledge for teaching. Teachers can develop students' complex mathematical knowledge in time by teaching decompressed knowledge. However, this knowledge is beyond of the conceptual understanding. Teacher use this knowledge with pedagogical purposes for teaching mathematics. On the other hand, teachers should unpack the knowledge with presenting content as available for students to visualize and understanding. As a whole, the demands of mathematics teaching require specialized mathematical knowledge. The third subdomain of subject matter knowledge, horizon content knowledge (HCK) is defined as “an awareness of how mathematical topics are related over the span of mathematics included in the curriculum” (Ball et al., 2008, p. 403). Teachers know with this knowledge that how the topic that students learn is related in the next grades’ topics. Thus, teachers can prepare the students considering the topics that they learn in next years. Horizon content knowledge can guide teachers to connect the topics between grades and to make decisions about how the content is presented. To illustrate, teachers can prepare students to learn rational numbers while using number line with emphasizing the number line is filled with other numbers throughout the grades. However, Ball et al. (2008) are not sure about including HCK as a component to SMK or other categories.

The other domain of MKT is pedagogical content knowledge (PCK). The first subdomain of PCK is knowledge of content and students (KCS). KCS is defined as “the knowledge that combines knowing about students and knowing about mathematics” (Ball et al., 2008, p. 401). Teachers must know students’ common conceptions and misconceptions, errors, difficulties specific to a mathematical topic as the focus of KCS. KCS is a component of Shulman’s PCK concept, and this is apart from subject matter knowledge. The second subdomain of PCK is knowledge of content and teaching (KCT). KCT is defined as “the knowledge that combines knowing about teaching and knowing about mathematics” (Ball et al., 2008, p. 401). Teachers need to know how to design instruction for teaching mathematics. KCT involves the combination of mathematical knowledge and pedagogical issues specific to mathematics. Teacher have KCT can make decisions about the instruction. The third subdomain of PCK is knowledge of content and curriculum (KCC). KCC is explained as the knowledge of the contents regarding the curriculum order, suggested activities, and important explanations for teaching.

Ball and colleagues have detailed the components of teacher knowledge specific to mathematics. However, they consider that this model should be revised. They have indicated three problems about this model. One of them is about the difficulty to determine which knowledge teachers use while responding to some situations. For example, while one teacher is analyzing the errors using SCK, another teacher can know that is a common misconception based on previous experiences using KCS. The second problem is the structure about categories, which is not considered as dynamic. Ball et al. (2008) have indicated that they are interested in using knowledge in practice. The third problem is the difficulty to separate the categories each other in some situations, common content knowledge and specialized content knowledge, or specialized content knowledge and knowledge of content and students.

2.3. The relationship Between Subject Matter Knowledge and Pedagogical Content Knowledge

The models proposed the concept of teacher knowledge as including subject matter knowledge and pedagogical content knowledge as explained above. Besides, several studies also examine the relationship between these two knowledge types. To illustrate, Even (1993) investigated how prospective teachers' subject matter knowledge related with pedagogical content knowledge in the context of the function concept. Based on the qualitative analysis, Even (1993) concluded that most of the prospective teachers' subject matter knowledge of the function concept was not adequate. Therefore, the researcher noted that prospective teachers should have subject matter knowledge with a relational understanding since it affects teachers' pedagogical reasoning for teaching mathematics. Thus, the researcher suggested developing prospective teachers' subject matter knowledge within the constructive perspective and then the relational subject matter knowledge should be used in pedagogical decisions.

The large-scale projects, LMT (Learning Mathematics for Teaching), COACTIV (Cognitively ACTIVating instruction, and development of students' mathematical Literacy), and TEDS-M (Teacher Education and Development Study in Mathematics) were also conducted to investigate teachers' content knowledge (CK) and pedagogical content knowledge (PCK) of mathematics teachers.

The researchers (Ball et al., 2008; Hill et al., 2005) in LMT project investigated the knowledge that mathematics teachers have as Mathematics Knowledge for Teaching (MKT). The researchers developed a test to examine subject matter knowledge, CCK and SCK, and concluded that content knowledge affected students' achievement positively. It was not investigated in this project the relationship between CK and PCK since the instrument had the items related to subject matter knowledge. On the other hand, COACTIV project investigated the influence of PCK on students' learning. It was found that CK and PCK were correlated strongly (Krauss et al., 2008a). Similarly, TEDS-M study was conducted to examine CK and PCK, and it was concluded that prospective teachers who took the training special to their teaching area had better CK and PCK (Blömeke & Kaiser, 2012). Consistent with this finding, Krauss et al. (2008b) conducted a study within the context of COACTIV project and they concluded that the connection of CK and PCK was an indicator for expertise of mathematics. Based on the quantitative analysis in these projects, it was concluded that "CK and PCK are related positively and CK is necessary but not sufficient for PCK" (Depaepe et al., 2015, p. 82).

However, the items related with CK and PCK was in different mathematics learning areas in the last two studies. Thus, Depaepe et al. (2015) emphasized the need of research to investigate the relationship between CK and PCK in particular mathematical topics and they examined this relationship in rational numbers area quantitatively. They also suggested investigating this relationship supporting with qualitative data as further research. Considering this need in teacher knowledge research, this study also aimed to investigate the influence of subject matter knowledge (SMK) on pedagogical content knowledge (PCK) for teaching generalization of patterns and operations with algebraic expressions. Thus, it can be considered that this study can contribute with the findings related this aim to the literature on teachers' knowledge of algebra.

2.4. Classroom Practices and Teacher Knowledge in Mathematics

Franke et al. (2007) point out the need of the explanation of *routines of practice* for teaching mathematics with respect to content and grade levels. They also assert that how the practices assisted teachers' teaching of mathematics and affect students' learning mathematics should be investigated. There are several studies that showed appropriate practices which provide doing mathematics within the context of problem solving, constructions, and discourse supported students' learning of mathematics (Arcavi, Kessel, Meira, & Smith, 1998; Chapman, 2006; Schoenfeld, Minstrell, & van Zee, 1999; Silver & Smith; 1996). Thus, Gearhart et al. (1999) suggested developing teachers' knowledge to reveal effective practices.

One of the factors that has an influence on teaching practices is teacher knowledge (Hiebert, 1997). However, Ball (2000) asserted that there was a gap between content knowledge and the practice of a teacher. Teachers' content knowledge is important to interpret students' ideas, and it helps to present different and valuable opportunities for students about mathematics, but the presentation of content knowledge in practice can provide students to learn. Teachers should think whether the tasks or activities are good, appropriate for the level of the students, interesting for students or not; and if they have important mathematical ideas. Teachers should manage the discussions in class, know which ideas to use, which probing questions to ask, when explanations are needed to continue discussion to help the development of students' understanding. In this context, Ball (2000) suggested three issues to prepare teachers to teach, first is what content teachers should know for teaching, second is how they know this knowledge, and third is how teachers learn to use this knowledge in practice. Ball (2000) also presented solutions for these issues. She indicated that teachers could analyze the role of content knowledge in their work (p. 244). Especially, they should aware of their content knowledge specific to mathematics teaching in their practices. To illustrate, presenting their knowledge with using multiple representations and models supported effective mathematics teaching (Tchoshanov, 2011). Besides having strong and conceptual content knowledge, teachers also should have connectedness of contents to understand students' thinking, and combine of content knowledge with pedagogy as pedagogical content knowledge to transform the knowledge to the students. To do this, the opportunities such as using students' work, or videotaping the lessons should be presented for teachers and pre-service teachers to improve their knowledge to use in practice (Ball, 2000). As Franke et al. (2007) stated the importance of the knowledge of students' thinking to develop the teachers' practices. Hence, the knowledge of students' understanding as well as conceptual knowledge is necessary to organize effective lessons pedagogically (Fennema & Franke, 1992; Hiebert, 1997). In sum, teachers should have connected content knowledge, and they should use their knowledge pedagogically to develop their teaching practices (Doerr, 2004).

As Ball (2000) pointed out the gap between teacher knowledge and practice in general above, Doerr (2004) supported this claim for algebra teaching practices. Doerr (2004) defined this situation as a dilemma that was referred to the contradiction of the knowledge with the practice in teaching algebra. Since teachers did not have conceptual perceptions and experiences from their learning, it caused their practice not to be effective. Thus, the researchers give suggestions to support and develop teachers' practice of algebra. To illustrate, Brown and Smith (1997) suggested the use of questioning technique to help the students to think algebraically. Particularly, the presenting of tasks with algebraic thinking features, supporting teachers with experiences that included using models or representations of algebraic expressions, and showing how arithmetic and algebra can be connected can help to improve teachers' knowledge and practices of algebra (Asquith, Stephens, Knuth, & Alibali, 2007; Ayalon & Even, 2013; Blanton & Kaput, 2001). However, Wilkie (2014) pointed out the lack of research about teacher practice for teaching algebra. Thus, the aim of the current study is to examine teacher knowledge within the practices of teaching algebra, and to contribute the literature of teacher knowledge and practice of algebra.

2.5. Models of Teacher Knowledge for Teaching Algebra

As there have been models of teacher knowledge for mathematics teaching, more particularly there have been also models of teacher knowledge for teaching algebra as a domain in mathematics (Artigue, Assude, Grugeon, & Lenfant, 2001; Even, 1990, 1993; Ferrini-Mundy et al., 2006; Kieran, 2007; Li, 2007). Even (1990) proposed a framework about teachers' subject matter knowledge of function concept. Even (1990) described seven features of subject matter for function concept. *Essential features* are the knowledge of a concept such as knowing of its examples and non-examples. *Different representations* are the knowledge of the representation of a concept that includes using different representation, and making connection among them. *Alternative ways of approaching* are the knowledge of alternative approaches and the use of them for teaching a concept. *The strength of a concept* is the knowledge of understanding a concept relating to subtopics or sub-concepts that provides learning of new topics. *Basic repertoire* is the knowledge of important principles, procedures, and examples with conceptual understanding. *Knowledge and understanding a concept* is the knowledge of conceptual and procedural knowledge with relating them. *Knowledge about mathematics* is the general knowledge of mathematics to get conceptual and procedural knowledge.

Artigue et al. (2001) focused on inequalities and described three dimensions for teachers' knowledge of algebra as epistemological, cognitive, and didactic dimension. There are two properties of epistemological dimension. One of them is "the complexity of the algebraic symbolic system and the difficulties of its historical development" (p. 26). This knowledge can help teachers to understand students' difficulties. Other property is the extension and diversity of algebra. It includes the understanding of how algebra is hold in the curriculum with its function in solving problems. The cognitive dimension is about the knowledge of learning algebra that is about students' algebraic thinking. On the other hand, the didactic dimension is about the knowledge about objectives of algebra in the curriculum. Different from Even's (1990) model, Artigue et al.'s (2001) model had the knowledge about students' thinking and curriculum. Even (1990) had proposed the model for only subject matter for algebra teaching.

Consistent with the elements of Even's framework, Ferrini-Mundy, Floden, and McCrory (2006) proposed knowledge for teaching algebra framework, especially for expressions, equations inequalities, and function at middle school and secondary level. The framework formed as two-dimensional matrix that the rows have categories of knowledge of algebra, and the columns have tasks of teaching. As the third category of this framework, Ferrini-Mundy et al. (2006) defined three overarching categories; decompressing, trimming, and bridging. The researchers formed the categories of knowledge of algebra teaching (KAT) based on literature and their study. The categories are core content knowledge, representation, content trajectories that is the connection with the prior concepts and the concepts learned in future topics, applications and contexts is about the use of algebra in solving problems, language and conventions, and mathematical reasoning and proof. The other component of the framework is tasks of teaching that involves teachers' actions in using algebra knowledge in practice. Ferrini-Mundy et al. (2006) derived from the categories of tasks of teaching component from the discussion for knowledge of algebra teaching. The categories are analyzing students' work and thinking, designing, modifying and selecting mathematical tasks; establishing and revising mathematical goals for students; accessing and using tools and resources for teaching; explaining mathematical ideas and solving mathematical problems; building and supporting mathematical community and discourse. Ferrini-Mundy et al. (2006) described the overarching categories as mathematical practices with using knowledge of algebra in tasks of teaching. Decompressing is getting new and complex knowledge using existing knowledge, as similar to Ball et al. (2008) indicated connecting concepts to unpack the knowledge. Trimming, is the transformation of complex knowledge to a mathematical situation with including the mathematical idea or content to enable students to understand. On the other hand, Ferrini-Mundy et al. (2006) explained bridging as connecting the teachers' purposes with students' understanding; relating school algebra to abstract algebra; and making connections among mathematics domains. The researchers suggested using this framework as an analytical tool for examining knowledge of algebra teaching.

Li (2007) proposed a framework for teachers' knowledge of teaching specific to solving algebraic equations based on the literature about knowledge of algebra teaching as mentioned above. Li (2007) defined three categories for teachers' knowledge as knowledge of the mathematical subject matter, knowledge of learners' conceptions, and knowledge of didactic representations. Knowledge of the mathematical subject matter is the subject matter of concepts, rules, theories, principles, facts, and methods. More particularly, Lin (2007) examined this knowledge into categories, concepts such as structure, reasoning methods such as induction, mathematical activities such as proofing, and products such as definitions. Knowledge of learners' conceptions is based on PCK concept in the literature and consists of the knowledge of learners' levels, conceptions, misconceptions, errors, difficulties, and learning process. Knowledge of didactic representations is presenting the content by using teaching methods and strategies, and tools such as manipulatives, textbooks.

The existing frameworks about teachers' algebra knowledge mentioned up to this point are generally about the algebra topics at secondary level. Thus, the purpose of this study is to investigate teachers' knowledge for teaching generalization of patterns and operations with algebraic expressions as algebra topics at middle school grades, and Ball et al.'s (2008) framework is utilized to carry out this aim. Using MKT framework, this study also aims to propose what knowledge mathematics teachers need to have for teaching early algebra topics.

2.6. Early Algebra

Algebra has been seen as one of the important branches of mathematics that Cai, Ng and Moyer (2011) state algebra is a “gatekeeper” in mathematics. National Council of Teachers of Mathematics (NCTM, 2000) standards determine four goals for teaching algebra that are, Goal 1-understand patterns, relations, and functions; Goal 2-represent and analyze mathematical situations and structures using algebraic symbols; Goal 3-use mathematical models to represent and understand quantitative relationships; and Goal 4-analyze change in various contexts.

Children are introduced with algebra in elementary grades and then they continue to learn and use algebra throughout middle school and high school (MoNE, 2013). According to Rakes et al. (2010), algebra is a core element for developing understanding of high school mathematics and thus students’ learning of fundamental concepts of algebra is important issue. Howe (2005) defines algebra as in the following:

> Working with variables, and in particular, arithmetic with variables, so the formation of polynomial and rational expressions. This also includes representing, or “modeling” concrete situations with expressions, and setting up equations. It is also often extended to include extracting roots. (If these processes are iterated, they can produce highly complicated expressions. But school algebra does not go very far down this road.) It also includes manipulating expressions and equations, to simplify, solve and interpret (p, 1).

The fundamental concepts such as variables, generalizations of patterns, and algebraic expressions are introduced students at 6^{th} grade in Turkey (MoNE, 2013). Carraher and Schliemann (2007) call the process that involves 6-12 aged students’ algebraic thinking as “Early Algebra” stating algebraic reasoning among young students. They define algebra in elementary levels as early algebra that “compass algebraic reasoning and algebra-related instruction among young learners-from approximately 6 to 12 years of age” (p. 670). Van de Walle et al. (2013) indicate that understanding algebra develops in elementary and middle school levels, since children do many things about algebra such as generalization from patterns and using variable in these levels. Thus, algebraic thinking improves from pre-kindergarten to high school (Van de Walle et al., 2013). Cai, Ng, and Moyer (2011) suggest assisting students to think algebraically in elementary grades in order to prevent middle and high school students' difficulties in algebra. They also suggest that teachers and students work on arithmetic and algebra in the first five or six years of elementary school. Warren and Cooper (2008) assert that developing elementary students' algebraic thinking can prevent the difficulties in algebra of adolescents. To support this, the researchers designed two lessons for aged about 8 years and a teacher. They used tasks about extending the pattern and exploring the relationship between the position number and pattern. The results supported the suggestion of the researchers that students' understandings were developed with the experiment. Elementary students' functional thinking can be developed with giving the sense of the relationship between input and output values, and they can also express their thinking symbolically (Warren, Cooper, & Lamb, 2006). With similar purpose, Ferrara and Sinclair (2016) proposed the algebra discourse approach that was emerged with communication and interaction in the classroom context to examine early graders' understanding of variable concept. They asked both figural and numerical patterns to seek using algebra in the recursive and functional strategies. As conclusion, the researchers asserted that the discourse approach with focusing on pattern generalization developed early grade (2^{nd} and 3^{rd} grade) students' functional reasoning.

One of the approaches about algebra teaching in early grades is an elaborated Davydov approach. In this approach, algebra is introduced to pupils at the beginning of primary school, and as not a different topic, it is given to students within all topics and concepts related with the development of algebraic thinking. To illustrate, students can learn quantitative and abstract thinking with equality and inequality activities (Sutherland, 2004). Schmittau (2005) also states based on the Vygotskian Perspective, empirical concepts can be learned from everyday experience, but theoretical concepts are given to students by teachers. So, it is important to constitute the theoretical basics of algebra in the elementary school years.

Whereas some mathematicians think algebra should be given in early grades, some mathematicians think that students should spend about 6 years to learn the basics of arithmetic. Thus, when algebra should be given is an issue for teaching and learning algebra in mathematics education. Research related to algebra implies that algebra is difficult to learn for many adolescents and algebraic thinking development is important in early grades. Early algebra does not mean to give all algebraic notations and structures in early grades. For example; it includes the ability of comparing quantities, interpreting graphs and tables. If students gained these abilities, they can comprehend algebraic notations and structures easier in next grades. In this point, it is necessary to separate pre-algebra and early approaches. The perspectives of pre-algebra approaches are about facilitating of the transition from arithmetic to algebra. On the other hand, early algebra approaches assert that mathematical symbols are used both in arithmetic and algebra. (Carraher and Schliemann, 2007). Malara and Navarra (2009) handle the issue of how teachers can promote of children' algebraic thinking in early grades. They developed a theoretical linguistic model that consist of categories for early algebra based on the constructivist perspective. The categories are general, mathematical, linguistic, social-educational, psychological aspects to approach to early algebra (p. 244). Malara and Navarra (2009) gave examples for each category like this, the use of relational thinking for general, the use of variable for mathematical, the use of letter for linguistic, discussing for social-educational, and perception of pupils for psychological aspects. The researchers based on this approach, developed a task related to pattern generalization and had parts as teaching sequence to enable pre-service teachers to analyze students' productions. Malara and Navarra (2009) concluded that teachers could notice the importance of their roles in the discussion of pattern generalization to construct children's knowledge. One of the advocators of early algebra approaches is Kaput (2000) that explains algebra "the study of functions, relations, and joint variation" (p. 19). In connection with this description, the current study includes pattern generalization and operation with algebraic expressions as content in the investigation of teachers' knowledge of these early algebra topics.

2.7. Studies Related to Teaching and Learning Algebra

More studies in algebra learning and teaching have been seen recently, when literature is examined. Kieran (2007) described the main reasons to focus on in this domain are; firstly, the influence of Piaget's ideas and cognitive development psychology on mathematics education. Second, the studies based on skills-based approach did not show positive effect on students' performance. Then, the research on algebra had been influenced by government policies such as 'algebra for all'. Lastly, developing technology has affected the content and implementation of school algebra. Besides these influences on the studies, the interest of algebra topics in the research to the present day has been changed. In ancient times, algebra was considered just about as manipulating symbols and using of algebra to solve problems. Thus, research was interested in students' errors in solving equations, and application of rules. Then, psychologists with a behaviorist perspective affected this view, and the issues about skills and memorization were gained attention. Towards the end of 1970s, algebra education researchers came together and they focused on students' learning and understanding of algebra. Then, the empirical studies about application of algebraic activities and the effect on students' understanding were conducted. After with constructivism effects on algebra, how learners construct their conceptions about algebraic concepts, and the interactions on algebra in class with sociocultural approach while learning were investigated. Since 1990s, technology has been begun to gain importance in studies about algebra teaching and learning (Kieran, 2007).

In the perspective of teaching algebra, Doerr (2004) examined the studies about teacher knowledge of algebra, and their practice in algebra teaching in detailed perspective. She reviewed the studies up to 2004, and concluded that there was lack of research of teaching algebra. Doerr (2004) also presented four dilemmas about teacher knowledge and practice of algebra based on implications of Working Group. The first dilemma was about teachers' experiences. Pre-service teachers considered that the methods their teachers used were effective since they thought that they could learn with these ways. Thus, their perceptions about teaching to children was limited and not conceptually. The second dilemma was about the content of the instructions of algebra. Since teachers' knowledge and its development were not adequate and thus they were needed to understand. The third dilemma was about the structure of teachers' knowledge. It was need to be investigate to the development of teachers' knowledge in a process. The last dilemma was about the lack of large studies about teacher knowledge and this situation prevented to reach conclusions about teachers' knowledge and teaching algebra.

Doerr (2004) also stated that the studies about teacher subject matter knowledge focused on generally functions, slopes, and equations (e.g. Even, 1990; Stump, 1999; Even, Tirosh, & Robinson, 1993); on the other hand, the studies about teachers' knowledge of students' thinking as pedagogical content knowledge focused on algebra word problem solving (e.g. Nathan & Koedinger, 2000; van Dooren, Verschaffel, & Onghena, 2002;). As it is seen, there had been few studies related to early algebraic topics, and since that time, more studies have been conducted about teachers' knowledge of algebra at elementary and middle school grade levels, that will be explained in the following sections. However, Wilkie (2014) has asserted that there are few studies about teachers' knowledge and practice about algebra.

In the perspective of learning algebra, Kieran (2007) concluded that many studies' results have shown that most students have difficulty with algebra at different levels when the studies about algebra learning and teaching are examined. Considering these studies, Kieran developed a model for conceptualizing algebraic activity called GTG model that has three components: generational, transformational and global-meta-level activity. In the GTG model, generational activity includes working with unknowns, variables, and equality, such as equations that represent problem situations, expressions of generalization. This activity is usually used to begin formal algebra. Transformational activity is rule-based and requires the symbolic form of an expression or equation. Collecting like terms, operations with algebraic expressions, factoring, substituting expressions in other ones, simplifying algebraic expressions, solving equations and inequalities can be example for transformational activity. These examples are generally about manipulating algebraic expressions. The global-meta-level activity deals with constructing and working with algebraic objects and processes. It is thought as a tool for algebra and requires high-level skills. Problem solving, modeling in generalization of patterns, justifying, and proving can be exemplified for global-meta-level activity (Kieran, 2007). This study particularly interested in pattern generalization as generational activity, and simplification of algebraic expressions and operations with them as transformational activity regarding the scope of this research.

In learning algebra, the concepts of variable and algebraic expressions are important (Subramaniam & Banerjee, 2004). The development of these concepts both procedurally and conceptually is substantial for writing and solving equations later (Capraro & Joffrinon, 2006). The concept of variable is taught firstly when formal algebra begins. Arcavi and Schoenfeld (1988) assert that “the concept of variable is a basis for transition from arithmetic to algebra” (p.420). However, there have been many studies about students’ misconceptions about variables (Arcavi & Schoenfeld, 1988; Küchemann, 1978, 1981; MacGregor & Stacey, 1997; Wagner, 1983). The variable concept is given in the context of patterns and generalizations in middle grades, and then in algebraic expressions (MoNE, 2013). In this context, the studies about these algebraic topics are presented in the following sections.

After a brief history of research in teaching and learning algebra, more particularly this section presents the studies related to algebra within teacher knowledge and students’ conceptualization of algebra. The studies particularly are examined based on two algebra topics, generalization of patterns and algebraic expressions, under two main parts.

2.7.1. Studies Related to Generalization of Patterns

The studies related to generalization of patterns are presented under two sections as teachers' knowledge and students' conceptions in the following.

2.7.1.1. Teachers' Knowledge on Generalization of Patterns

Teachers' knowledge has been investigated in the studies within the context of subject matter knowledge (SMK) and pedagogical content knowledge (PCK). In general, when the interest is in-service teachers, the studies investigate the concept of PCK (Baş, Çetinkaya, & Erbaş, 2011; Blanton & Kaput, 2001; El Mouhayar & Jurdak, 2013; Kutluk, 2011; Wilkie, 2014); whereas the focus is pre-service teachers, the studies investigate the both concept of SMK and PCK about pattern generalization (Akyüz, Coşkun, & Hacıömeroğlu, 2009; Barbosa & Vale, 2015; Callejo & Zapatera, 2016; İmre & Akkoç, 2012; Kirwan, 2015; Magiera, van den Kieboom, Moyer, 2013; Rivera & Becker, 2007; Tanışlı & Köse, 2011; Zazkis & Liljedahl, 2002). The studies are presented in the following sections.

The opinions of teachers to pattern generalization is different in somehow. Bishop and Stump (2000) concluded that elementary and middle school teachers' conceptualizations for pattern generalization task were as problem solving merely. They did not consider pattern generalization as facilitating of transition to algebraic thinking. Consistent with this, Kutluk (2011) found that elementary mathematics teachers did not regard pattern generalization as important for algebra because of that the teachers did not have adequate CK, PCK and curriculum knowledge for teaching pattern generalization. Especially, they had lack of knowledge to identify students' difficulties in pattern generalization, and could not explain the reasons of them when they encountered while teaching. The teachers had also difficulties in generalization that some teachers considered the relationship only among between input values or output values. In their instruction, they used only numerical reasoning to find the relationship and they did not focus on the features of figures in figural patterns

(Kutluk, 2011). However, the studies suggest finding relationship in the context of figural patterns can develop students' reasoning for generalization (Barbosa & Vale, 2015; Rivera & Becker, 2005; Walkowiak, 2014).

Teachers' conceptions can change in time with seeing the ability of students' thinking algebraically. Blanton and Kaput (2001) shared the reflections of a 3rd grade teacher on algebraic activities, and examined the development of conceptualization about generalization of this teacher. The researchers presented tasks to this teacher with aiming to improve students' algebraic reasoning such as generalizing the rule of multiplication 1 by a number, or a context, handshake problem that created a pattern. As the teacher were using these tasks in an order, he realized that the students' algebraic reasoning was improved and students could think algebraically to find a number in the blank, or generalizing a rule, or generalizing a pattern. The researchers pointed out the teachers' practices for teaching algebra could be improved with the tasks which had algebraic thinking, and they decided the implementation of this program for other teachers' development.

Rivera (2010) described the generalization process with abductive-inductive action and symbolic action. Rivera (2010) explains as investigating the relationship in the pattern to propose a hypothesis based on the given steps of the pattern, and extending the pattern regarding the relationship is abductive-inductive action. This relationship is transformed as a rule to algebraic representation in symbolic action. El Mouhayar and Jurdak (2013) acknowledged Rivera's (2010) definition of generalization process and formed the questions based on this definition. They conducted a large study that they studied with 83 middle school mathematics teachers. They investigated how teachers' explain and identify students' actions in generalization process of patterns. In this context, they seek the answers for how teachers identify and explain students' abductive-inductive and symbolic actions. The researchers presented to the teachers two pattern tasks, one linear and one non-linear growth pattern, with expected students' answers for generalization of patterns. The teachers were asked to identify students' actions, and explain students' thinking while finding a term of pattern as near generalization, and getting the general rule as far generalization tasks. As the results of the study, teachers could identify students' strategies, but they did not have adequate knowledge to explain them for the reason of their strategies. Moreover, Baş, Çetinkaya, and Erbaş (2011) found that teachers did not identify adequately the possible strategies that students used for generalization of patterns. The teachers' expectations did not match with students' strategies for generalization of patterns. To illustrate, the teachers expected from the students to use functional thinking with co-variational strategies easily, but the students listed the terms to find near term and they were forced to use functional thinking when the generalization was asked. Another thing was that students mostly used numerical reasoning, although their teachers stated using figural reasoning. After they were presented actual students' answers and solutions, they could explain students' thinking and determine the strategies better. The researchers asserted that the reasons for the discrepancy of teachers' predictions and students thinking at the beginning might be about the limitations of their strategies they used and the lack of content knowledge.

In a broader perspective to knowledge, Wilkie (2014) examined upper primary teachers' content knowledge and pedagogical content knowledge of algebra focusing on functions, relations, and joint variation. The researcher developed an instrument with open-ended questions based on four domains of MKT framework, SCK, KCS, KCT, and KC. This survey was applied to 105 teachers that used Australian curriculum in order to investigate their knowledge for teaching algebra for 8-12 aged students. The contents and purposes of survey items are explained in detail in the following paragraphs since the relation with this study.

For SCK knowledge domain, Wilkie (2014) prepared the open-ended items about generalizing a geometric pattern, writing a functional relationship, and explaining generalization approaches. She expected from teachers as generalization approaches with two methods that co-variational (recursive) and correspondence (the relationship between input and output values based on functional thinking). For these questions in the survey, teachers were asked to write possible correct students answers, and comparing different tables including input and output values to determine teachers' generalization strategies and knowledge about students' approaches. The researcher examined teachers' answers using the framework about determining the development of functional thinking adapting from Markworth's (2010) study. For KCS knowledge domain, Wilkie (2014) aimed to determine teachers' suggesting different correct generalizations, teachers' recognition of students' level in generalization answers, and their knowledge about recursive and explicit generalization that students used. The researcher presented an incorrect student's answer to determine how teachers explain the reasoning of this answer. She identified KCT for pattern generalization as addressing a student's mistake in generalization and identifying appropriate strategies for conceptualizing functional thinking. Teachers' answers for these items were analyzed using a 4-level rubric that was about assessing the knowledge for teaching and learning functional thinking. Especially, for identifying appropriate strategies, teachers were presented terms related with functional thinking and they were asked to prepare an activity about function machine with using these terms, and they were also asked to explain how they used and select input-output values for teaching functional relationship as an example for KCT knowledge domain. For KC knowledge domain, the researcher presented the teachers content descriptions of curriculum and they were asked to scale this wording as easy or difficult to understand.

In general, Wilkie (2014) concluded that teachers have adequate SCK for generalizing figural patterns, but their KCS and KCT were not adequate and conceptual for teaching functional thinking in pattern generalization. That is, although teachers have knowledge of content for pattern generalization, they do not have adequate pedagogical content knowledge for teaching this algebraic topic. More particularly, teachers could generalize patterns with words or calculation, and exemplify correct students' answers; but they could not explain strategies as recursive or explicit of students' answers, they had difficulty in using algebraic symbols for general rule, and they did not have adequate experiences to create activities for teaching pattern generalization. Wilkie (2014) explained the situation that teachers' weaknesses for the reason of their lack of relational understanding for functional thinking. Thus, teachers had difficulty to explain students' misconceptions and difficulties in generalization as (KCS), and they could not use effectively function machine, or tables, or input-output values for teaching functional thinking (KCT). The researcher also investigated the relationship between the knowledge domains specific to functions, relations, and joint variation. She concluded that SCK and KCS is distinct that although teachers have strong SCK, they can have weak KCS. Since the teachers have weak relational understanding that requires getting generalization with understanding the procedures conceptually rather than writing a general rule with rules, they could not use effective teaching strategies and they had less KCT knowledge than SCK.

The studies that have focused on pre-service teachers examined their knowledge both SCK and PCK generally. Furthermore, the researchers presented examples to them from students' answers to investigate their PCK in research (Callejo & Zapatera, 2016; İmre & Akkoç, 2012; Magiera, van den Kieboom, & Moyer, 2013). Carpenter and Fennema (1992) indicated the importance of developing pre-service teachers' PCK with analyzing students' understanding and conceptions. The following studies generally use representative or actual students' responses while learning pattern generalization. Several studies also whose aims to develop pre-service teachers' PCK conducted their research in the courses related to teaching experiences.

İmre and Akkoç (2012) examined pre-service teachers' PCK about pattern generalization in the context of school practice course in teacher education program. The researchers also aimed to develop prospective teachers' PCK based on two elements of PCK, knowledge of students' understanding and difficulties, and knowledge of topic specific strategies and representations. They redefined the two components of PCK specific to pattern generalization using Radford's (2008) architecture of algebraic pattern generalization. Radford (2008) describes the generalization process as including abduction, transforming, and deducing phases. Abduction phase is recognizing the relationship and commonality in the pattern, transforming is using this relationship to find other terms by extending the pattern, and deducing phase is generalizing pattern and finding a general rule algebraically. According to İmre and Akkoç (2012), the first component includes students' misconceptions about using the difference between consecutive terms, and their difficulty in using algebra to generalize in abduction and transforming phases. The second component includes using representations for generalization patterns such as arithmetic, algebraic, pictorial and tabular in deducing phase. With the aiming of developing PCK particularly for two components, the researchers studied with three pre-service teachers using their observations of teaching practices in schools and their instructions for pattern generalization in the context of the course. The researchers and the pre-service teachers in the course discussed and evaluated the instructions based on Radford's model. Before the course, pre-service teachers did not consider students' difficulties in recognizing the relationship of pattern and writing arithmetic relationship for the first terms of the pattern. The pre-service teachers could not use different representations such as pictorial reasoning. After the course, the researchers concluded that pre-service teachers' understanding of students' thinking was developed that they took into students' conceptions account. They could guide the students to find a general rule to find each term in the pattern, and their use of algebra in generalization was improved. The pre-service teachers also used of tabular and partially pictorial representations while teaching generalization in their second instruction.

Similarly, in a field-based course, Magiera, van den Kieboom and Moyer (2013) investigated pre-service teachers' knowledge of algebraic thinking of their selves and students' algebraic thinking. They used multiple data sources that were pre-service teachers' solutions and interviewing with them, students' solutions, and pre-service teachers' analyzing students' solution and interviewing with the students. The researchers suggested pattern generalization tasks for pre-service teachers to solve and use in the interviews with students. The tasks could provide students to use algebraic thinking features that were organizing information, identifying a pattern, describing a rule, and justifying the rule based on Driscoll's (2001) description (p. 107). The results of this study showed that pre-service teachers could not identify or notice all features of algebraic thinking tasks. However, their ability in algebraic thinking related with identifying of students' algebraic thinking ability. That was, they did not investigate the features that they did not use in generalizing pattern. The another finding was consistent with the findings of İmre and Akkoç's study (2012) that pre-service teachers did not use different representations for generalizing patterns and thus they did not seek this feature in students' solutions. Akyüz, Coşkun, and Hacıömeroğlu (2009) also investigated pre-service teachers' use of translation among different representations such as tabular, graphical, algebraic, and numerical in generalizing pattern. In consistent with the mentioned studies, they found that using different representations could support students' generalization reasoning. Other finding was according to Magiera et al. (2013), pre-service teachers had difficulty in justifying a general rule as one of the algebraic thinking features that many studies also concluded it (Barbosa & Vale, 2015; İmre & Akkoç, 2012; Kirwan, 2015; Rivera & Becker, 2007; Tanışlı & Köse, 2011).

Callejo and Zapatera (2016) conducted the study with similar purpose as focusing on pre-service teachers' PCK. The researchers aimed to determine the characteristics of pre-service teachers' noticing students' mathematical thinking based on their identification and interpretation of students' answers for generalization of patterns. They have proposed three mathematical elements and related with three stages in pattern generalization based on literature. The first element is numerical and spatial structure that is "the number of elements of a term and the physical location of each element of this term in relation to the other elements in the term" (p. 6). It is related with stage 1 that students can do near generalizations, but cannot relate numerical and spatial features. The second element is functional relationship that is finding the relationship between the position of term and corresponding number. It is related with stage 2 that students can connect with numerical and spatial features, and generalize verbally or algebraically of the relationship. The third element is inverse process that determines the position number for a given term. It is related with stage 3 that students can connect with numerical and spatial features, use functional relationships and invert the term in position number using functional thinking. According to the findings of this study, although pre-service teachers could identify these elements in students' answers, they had difficulty in interpreting and explaining students' generalization understanding, as El Mouhayar and Jurdak (2013) stated in their findings.

Besides these studies, there have been studies that seek how pre-service teachers generalize different types of patterns. Some pre-service teachers tried to find the general rule of numerical pattern using recursive strategy (Zazkis & Liljedahl, 2002). They considered the rule without algebra was not adequate. On the other hand, some pre-service teachers used figural reasoning that was about using the features of figures to get relationship in figural patterns (Barbosa & Vale, 2015; Rivera & Becker, 2007). The studies suggest finding relationship in the context of figural patterns can develop students' reasoning for generalization. As pre-service teachers can use both numerical and figural reasoning in pattern generalization, but pre-service teachers who had figural reasoning could use functional thinking (Tanışlı & Köse, 2011). However, Kirwan (2015) asserted that pre-service teachers who got generalization could use figural features in generalizing, and numerical features for verifying the generalization.

2.7.1.2. Students' Conceptualization of Generalization of Patterns

The studies related with students' conceptions of pattern generalization are generally about students' strategies and reasoning for generalizing patterns (Amit & Neria, 2008; Becker & Rivera, 2005; El Mouhayar & Jurdak, 2016; Lannin, Barker, & Towsend, 2006; Rivera, 2010; Rivera & Becker, 2008; Steele & Johanning, 2004; Walkowiak, 2014; Warren & Cooper, 2008), and the development of using algebra for generalization at different grades (Jurdak & El Mouhayar, 2014; Walkowiak, 2014).

Steele and Johanning (2004) examined 7th graders' schemas for solving generalization problems, and aimed to develop students' schemas with a teaching experiment. The findings of the study showed that students who had well-connected schemas could generalize symbolically that was one of the essential components of algebraic thinking. They checked particular cases when they reached generalization in contrast to students with partial formed schemas. These students got help from tables with diagrams to interpret the relationship in the pattern. On the other hand, students with partial formed schemas had difficulty in generalizations and using algebraic notations. With a more particular perspective on conceptions that students have, their strategies for generalization are examined. Healy and Hoyles (1999) define recursive and explicit strategies for generalization. Students using recursive rule that explain the relationship focusing on the difference among consecutive output values in the pattern. Explicit rule is finding a rule relating input and output values in the pattern. Even students use both recursive and explicit strategies, students are expected to think explicitly to conceptualize generalization. Lannin, Barker, and Towsend (2006) asserted that using spreadsheets to support students' use of explicit strategies after thinking recursively.

Rivera and Becker (2008) have extended the students' conceptions literature about pattern generalization with particularly figural patterns. Figures change from one figure to the next one based on a relationship in figural patterns (Billings, 2008). Studies suggest the use of figural patterns for developing students' generalization strategies (Moss, Beatty, McNab, & Einsband, 2005; Rivera & Becker, 2008; Walkowiak, 2014; Warren & Cooper, 2008) Rivera and Becker (2008) proposed constructive and deconstructive strategy for middle school students' use for generalization of figural linear patterns. Constructive strategy is defined as "cognitively perceiving figures that structurally consist of non-overlapping constituent gestalts or parts" (Rivera & Becker, 2008, p. 70). It is about constructing the relationship with counting the part separately in figures. On the other hand, deconstructive strategy is explained as "the basis of initially seeing overlapping sub-configurations in the structure of the cues" (Rivera & Becker, 2008, p. 70). It is about constructing the relationship with considering the overlapping parts in the figures. To illustrate, in square toothpick pattern (4, 7, 10, 13, …), the toothpicks are added separately in constructive strategy and one possible generalization is 4+3(n-1), since the pattern is considered as 4, 4+3, 4+3+3, … in using this strategy. On the other hand, the whole toothpicks are added and subtract the overlapping toothpicks in deconstructive strategy and one possible generalization is 4n-(n-1), since the pattern is considered as 4 (no overlap), 4+4-1 (take away 1 overlapping side), 4+4+4-2 (take away 2 overlapping side) … in using this strategy. Rivera and Becker (2008) concluded that using deconstructive strategy was difficult for students to establish the relationship and generalization than constructive strategy in their teaching experiment.

Besides students' conceptions about figural pattern generalization, Rivera (2010) defines students' actions in generalization process with two interdependent actions as in the following:

> (1) abductive–inductive action on objects, which involves employing different ways of counting and structuring discrete objects or parts in a pattern in an algebraically useful manner; and (2) symbolic action, which involves translating (1) in the form of an algebraic generalization (p. 300).

Students seek the relationship in the pattern and try to propose a hypothesis based on the given steps of the pattern in abductive-inductive action. Then, they can extend the pattern regarding the hypothesis. To illustrate, exploring a relationship based on the total number of toothpicks in the 1^{st}, 2^{nd}, and 3^{rd} step of the pattern and finding the number of toothpicks in the 10^{th} step. This exploration is transformed as a rule to algebraic representation in symbolic action.

Several studies have examined students' conceptions about generalization across grade level. One of them is Jurdak and El Mouhayar's (2014) study that investigated the trend of generalization throughout the grades (4^{th} to 11^{th} grades), and the effects of features of tasks on students' generalization reasoning. The researchers applied a test that consisted of four pattern generalization questions to a large sample of students, as in their study with teachers. The questions were classified in pattern generalization type, function type, and degree of complexity. In this regard, one of the findings of the study was the facilitation effect of the generalization type and function type to students' reasoning level. For example, students could generalize near generalization type of tasks in linear patterns; however, they had difficulty with getting a general rule for n^{th} term in far generalization type of tasks. Other findings from the result of the study were that the development of students' level across grade and the variation of students' strategies in same grade. Similarly, for different grades, Walkoviak (2014) aimed to explore students' strategies for generalization of pictorial patterns for 2^{nd}, 5^{th}, and 8^{th} grade students. The researcher concluded that students used both figural and numerical reasoning for generalization; however, younger students used more figural reasoning. The findings show that the use of algebraic notations increase across the grade that younger students can use their invented notations (e.g. using a circle for representing start), and older students can use formal notations for generalization of patterns. Different from this study, El Mouhayar and Jurdak, (2016) found that lower graders (grade 4 and 5) used mostly numerical reasoning, and upper graders (grade 10 and 11) used mostly figural reasoning in pattern generalization. The researchers also suggest that students who used functional strategy had figural reasoning for pattern generalization as Markworth (2010) stated.

However, high-level students have difficulty with representing generalization of patterns algebraically too (Çayır & Akyüz, 2015). Becker and Rivera (2005) examined 9^{th} graders' analysis of patterns and functions. The students could extend the figural pattern, but few of them could generalize and represent the general rule with algebraic formula. MacGregor and Stacey (1996) asserted that older students could not relate the position number and the term in the pattern. When students only consider the consecutive terms in the pattern not the relationship between the position number and corresponding term of the entry pattern, they can have difficulty in making generalization (Harel, 2001).

2.7.2. Studies Related to Algebraic Expressions

The studies related to algebraic expressions are presented under two sections as teachers' knowledge and students' conceptions in the following.

2.7.2.1. Teachers' Knowledge on Algebraic Expressions

In the aspect of PCK, knowing students' misconceptions, difficulties, understanding is one of the components of teacher knowledge. Tirosh, Even and Robinson (1998) examined this aspect in the context of algebraic expressions. They examined teachers' awareness of students' tendency to conjoin or 'finish' open expressions, since this tendency caused difficulty for learning algebraic expressions and operations with them. The students with this tendency can add 4x+5 as 9 or 9x as an example. The researchers studied with two novice and two expert seventh grade teachers based on the years of experiences. The data were collected from lesson observations in that teachers were teaching algebraic expressions, teachers' lesson plan, and interviews with teachers after lessons. Tirosh et al. (1998) found that two novice teachers were not aware of this tendency while two experienced teachers expected that students had this tendency in dealing with algebraic expressions. The researchers observed that one novice teacher was not aware of the misconception that students could have, and thus he emphasized adding the numbers and letters separately as a rule when teaching simplification of algebraic expressions and the students gave incorrect answers. Whereas, the other novice teacher used rules, adding "like terms" indicating the terms had x as like terms. She used "fruit salad" techniques while explaining operations in algebraic expressions with representing apple and pear for variables. These strategies were used mostly in teaching algebraic expressions, but the researchers observed that these strategies caused some confuses in student learning. Thus, students had difficulty to understand the reasoning of algebraic expressions. One of the experienced teachers, firstly, explained what "like terms" and "unlike terms" were by considering students' difficulties, and continued the lesson with using this concept. Other experienced teacher provided students' conceptualization with using challenging strategies such as substitution, order of operations, and going backward. Tirosh et al. (1998) suggested that teachers should know students' difficulties and design their lessons using different approaches regarding students' conceptions.

The knowledge of students' conceptions also is one of the important elements of pedagogical content knowledge. In this context, Hallagan (2004) provided teachers to get students' works. The researcher focused on one teacher's algebra instruction particularly for teaching equivalent expressions. The researcher developed tasks that included modeling the equivalent expressions, and the teacher implemented the tasks and constructed a library selecting students' different solutions and works. The teacher used this modeling for the first time and gave lots of time for the instruction. Students worked on to show and explain 4s+4 expression equaled with 4(s+1), s+s+s+s+4, 2s+2(s+2), and 4(s+2)-4 within different pictures. This task had a context that asked the border of a square pool. As a result, the teacher recognized the usefulness of using the visual strategies based on area modeling to improve students' conceptual understanding than using only distributive property as procedural. Hallagan (2004) suggests getting a library that includes students' exemplary works to develop teachers' knowledge and improve algebra instruction.

2.7.2.2. Students' Conceptualization of Algebraic Expressions

The studies about algebraic notations and manipulations of algebraic expressions show that students do not have adequate conceptual knowledge for understanding the structure of the expressions and they have difficulty with manipulating them (Banerjee & Subramaniam, 2012; Booth, 1984; Gunnarsson, Sönnerhed, & Hernell, 2015; Küchemann, 1981; Livneh & Linchevski, 2007; Seng, 2010). MacGroger and Stacey (1997) proposed four reasons for students' difficulties about using algebraic notations: "intuitive assumptions and pragmatic reasoning about a new notation, analogies with familiar symbol systems, interference from new learning in mathematics, and the misleading teaching materials" (p. 1). Knowing and taking into the reasons account can support teaching and learning algebra. Furthermore, Seng (2010) identified students' errors about simplifying algebraic expressions in detail. These errors are incorrect order of operation (e.g. 2×a+a+15=30+a+a), addition of integers incorrectly, ignoring coefficients of 1 or -1 in front of the variable, multiplication of negative integer before the bracket incorrectly, ignoring the multiplication of second term in the bracket in using distributive property, addition of exponential form of expressions (e.g. $3a^2+4a^2=7a^2$ and writing the result 14a), conjoining expression incorrectly (e.g. 3a+3=7a or 7). Consistent with the reasons that were proposed by MacGroger and Stacey (1997), Seng (2010) also asserted possible causes for these errors as following; interference from new learning, difficulty in operating with the negative integers, misconceptions of algebraic expressions and misapplication of rules. As stated, new learnt concepts can lead students to inference some incorrect rules such as in the exponential algebraic expressions. Students can operate negative integers when they are coefficients of terms incorrectly that they could add $-6x+3x$ as $-9x$. Besides, students did not conceptualize algebraic expressions and had misconceptions. For example, they can think *ab* and *ba* are unlike terms, or the coefficient of *a* is 0. MacGroger and Stacey (1997) explained this situation with intuition about new notation. Last, students made errors in application of rules mostly in multiplication such as a×a=2a, and particularly application of distributive property with ignoring the multiplication of second term in the bracket such as 2(4a+3) = 8a+3. Seng (2010) concluded that students do not have conceptualization of algebraic expressions and thus students' understanding should be developed for simplifying algebraic expressions.

One of the commonly suggested approaches of many studies related with teaching algebraic expressions is transformation arithmetic to algebra (Livneh & Linchevski, 2007; Subramaniam & Banarjee, 2004; Warren, 2003). In this context, Banerjee and Subramaniam (2012) implemented a teaching approach to support 6th grade students' understanding for transition of arithmetic to algebra in the beginning algebra. They investigated the development of students' algebraic thinking throughout the approach over two years. They focused on particularly understanding rules and procedures in operations, simplifications of algebraic expressions, and equivalence of expressions. Thus, they aimed to give students the idea of similarity of the structure of arithmetic expressions and algebraic expressions connecting them with rules and properties used. One of the examples from the tasks used was that “which of the expressions given is equal to 23+17×15+12” and “simplify of 5×x+16+7×x-11”. This study was conducted as a design research and the teaching trials were revised during the study. Banerjee and Subramaniam (2012) found that students could use the rules and procedures in addition and subtraction of algebraic expressions, and they could also understand the reasoning of the equivalence of expressions. As a result, the researchers emphasized the importance of connecting arithmetic with algebra at the beginning algebra to help the improvement of students’ understanding of algebraic expressions. Particularly, teachers should be supplied the experiences about algebraic expressions with proving the equivalence of them to promote students’ understanding (Ayalon & Even, 2013).

With similar purpose, Livneh and Linchevski (2007) implemented an intervention as a direct instruction including numerical contexts that could address future algebraic structures such as order of operations, collecting like terms and the use of equal sign. To illustrate, “Is 75-25+25 equal or not equal to 75-50?” could address “Is 16-4x+3x equal or not equal to 16-7x?” (p. 219). This study showed that arithmetic teaching including corresponding algebraic purposes can support students’ understanding of algebra. Similarly, Subramaniam and Banarjee (2004) found that the students who learnt algebra connecting with arithmetic were better on writing algebraic expressions of verbal statements, simplifying algebraic expression, and applying rules in operations with bracket expressions than other students learnt algebra without arithmetic.

With a more particular perspective on algebraic expressions, the order of expressions and the role of brackets are examined in studies (Hoch & Dreyfus, 2004; Marchini & Papadopoulos; 2011; Livneh & Linchevski, 1999). The studies suggested that elementary level students could do operations more correctly with emphasizing the brackets. However, Gunnarsson, Sönnerhed and Hernell (2015) investigated whether the brackets helped students to operate such expressions a±b×c correctly. They concluded that use of brackets did not assist students to apply the rules in learning the order of operations in expressions. Livneh and Linchevski

(1999) suggest giving the structure sense of expressions to improve students' understanding.

2.8. Summary of the Literature Review

The literature review began with the frameworks of teacher knowledge in general. In the section, the components of teacher knowledge were explained, and the different models of teacher knowledge were compared within themselves. The models have common components like subject matter knowledge, pedagogical content knowledge, and curriculum knowledge (Shulman, 1986). Whereas, curricular knowledge was examined in the context of pedagogical content knowledge in some models (Grossman, 1990), some of them added new components to Shulman's categorization (Cochran, DeRuiter, & King, 1993; Magnusson, Krajcik & Borko, 1999). These components were like knowledge of students such as knowledge of learners and abilities, and knowledge of context such as school culture were emphasized and included to some models. Then, the models specific to mathematics teaching were reviewed and they put forward PCK concept that mathematics teachers must have (An et al., 2004; Chick et al., 2006; Fennema & Franke, 1992; Rowland et al., 2005). From these models, Ball et al.'s model for mathematical knowledge for teaching (MKT) was examined especially as the conceptual framework of the study. Ball et al. (2008) proposed sub-domains for content knowledge and pedagogical content knowledge for MKT. Furthermore, since the aim of the study is to examine teacher knowledge of algebra, the models for algebra knowledge also reviewed. These models were also based on Shulman's teacher knowledge concept and they presented features related with to specific algebra topics such as functions, inequalities, and linear equations at secondary level (Artigue, Assude, Grugeon, & Lenfant, 2001; Even, 1990, 1993; Ferrini-Mundy et al., 2006; Kieran, 2007; Li, 2007).

The studies have emphasized the importance of teaching algebra in early years to facilitate students' transformation of arithmetic knowledge to algebra (Cai, Ng, & Moyer, 2011; Van de Walle et al., 2013). Moreover, the studies showed that students' algebraic thinking can be developed with appropriate approaches such as tasks or discourse (Ferrara & Sinclair, 2016; Warren & Cooper, 2008; Warren, Cooper, & Lamb, 2006). At that point, examining teachers' teaching early algebra that is the beginning of algebraic thinking can be more advantageous to enhance students' learning conceptually. As teachers have important role in teaching and a positive effect on students' achievement (Fennema, Carpenter, Franke, Levi, Jacobs, & Empson, 1996; Hill, Ball, & Schilling, 2008; Hill, Rowan & Ball, 2005; Tchoshanov, Lesser, & Salazar, 2008). Particularly for algebra teaching, Malara and Navarra (2009) pointed out this issue and showed the importance of teachers' roles in discussion of pattern generalization for construction of children' knowledge in their study.

The studies about learning algebra show that students generally have difficulty in generalizing patterns algebraically and manipulating algebraic expressions as early algebra (Amit & Neria, 2008; Becker & Rivera, 2005; El Mouhayar & Jurdak, 2016; Kieran, 2007; Lannin, Barker, & Towsend, 2006; Rivera, 2010; Rivera & Becker, 2008; Steele & Johanning, 2004; Walkowiak, 2014; Warren & Cooper, 2008). Actually, as Capraro and Joffrinon (2006) stated that the development of the concepts of variable and algebraic expressions both procedurally and conceptually is important for writing and solving equations later. Thus, it may be more important to investigate middle school students' difficulties and misconceptions in these topics as the beginning algebra. Especially, understanding functional relationship between the position number and the term in pattern generalization can support the learning the concept of function in later grades (Usiskin, 1988). However, Jurdak and El Mouhayar (2014) examined students' (4^{th} to 11^{th} graders) generalization conceptions and concluded that the students had difficulty in generalization of patterns algebraically more than extending the pattern. This finding might show that the wide range as elementary, middle school and secondary school level students' difficulty in the concept of generalization. This finding was consistent with Becker and Rivera's (2005) conclusion that older students also had difficulty in pattern generalization. Similarly, Seng (2010) presented different misconceptions about algebraic expressions that students had in manipulating them. As well as Malara and Navarra (2009) indicated the importance of the role of teachers in teaching generalization of patterns, Ayalon and Even (2013) suggested the development of teachers in order to promote students' conceptions of algebraic expressions.

In addition to the students' difficulties in early algebra, the teachers also had lack of knowledge for teaching these topics. El Mouhayar and Jurdak (2013) explored that the teachers did not have adequate knowledge to explain the reason of the students' strategies for generalization. Besides, Wilkie (2014) found that upper primary teachers' knowledge of content and students and knowledge of content and teaching of functional thinking were not adequate and conceptual. Even, they had difficulty in using algebra for generalization as subject matter knowledge. Similarly, for teaching algebraic expressions, Tirosh et al. (1998) suggested improving teachers' knowledge of students' thinking in order to design their lessons effectively.

Considering the importance of teacher knowledge for students' learning, revealing teachers' existing knowledge and what knowledge they need to have in generalization of patterns and operations with algebraic expressions might provide teachers to assist developing students' understanding, as the purpose of the current study. Moreover, as it has been indicated that there are few studies about teachers' knowledge and practice of these topics in literature (Baş, Erbaş, & Çetinkaya, 2011; Doerr, 2004; El Mouhayar & Jurdak, 2013; Kieran; 1992; Wilkie, 2014). At this point, this study can be considered to contribute to mathematics education literature with this investigation to the lack of research about knowledge of algebra.

CHAPTER III

METHODOLOGY

The purpose of this study is to examine the middle school mathematic teachers' mathematical knowledge for teaching generalization of patterns and operations with algebraic expressions. More particularly, teachers' subject matter knowledge and pedagogical content knowledge that form MKT are investigated in planning and implementing the lesson for teaching generalization of patterns and operations with algebraic expressions. Besides, it is also aimed to examine how teachers use their MKT in teaching these algebraic topics in their instructions. In this context, the following research questions are framed:

1. What is the nature of middle school mathematics teachers' mathematical knowledge for teaching (MKT) generalization of patterns in planning and implementing lessons?

1.a. What is the nature of middle school mathematics teachers' mathematical knowledge for teaching (MKT) for generalization of patterns in planning lessons?

1.b. What is the nature of middle school mathematics teachers' mathematical knowledge for teaching (MKT) for generalization of patterns within the practices of implementing lessons?

2. What is the nature of middle school mathematics teachers' mathematical knowledge for teaching (MKT) operations with algebraic expressions in planning and implementing lessons?

2.a. What is the nature of middle school mathematics teachers' mathematical knowledge for teaching (MKT) operations with algebraic expressions in planning lessons?

2.b. What is the nature of middle school mathematics teachers' mathematical knowledge for teaching (MKT) operations with algebraic expressions within the practices of implementing lessons?

3. How does middle school mathematics teachers' subject matter knowledge (SMK) influence their pedagogical content knowledge (PCK) in the context of teaching generalization of patterns and operations with algebraic expressions?

This chapter presents the research design and the characteristics of case study first. Then, for this case study, participants and the procedures of data collection are explained. The framework and the procedures for data analysis are presented in the following sections. Finally, the trustworthiness of this study is explained.

3.1. Research Design

In this study, qualitative research design was used to reveal middle school mathematics teachers' mathematical knowledge for teaching in the context of generalization of patterns and operations with algebraic expressions. Patton (1985) defined qualitative research as "an effort to understand situations in their uniqueness as part of a particular context and the interactions there" (p. 1). For this effort, qualitative researchers investigate situations in their natural settings with the meanings of people attribute (Denzin & Lincoln, 2005).

Creswell (2007) explains about conducting a qualitative research, that if a problem or issue needs to be explored, complex and detailed understandings of the issue and the interpretations of the participants are placed. Qualitative research begins with assumptions and a theoretical perspective for a problem and inquiries the meanings of individuals or groups about this problem. This qualitative inquiry continues in a natural setting, with inductive data analysis and forming themes from data. In conclusion, there are interpretations of participants and researcher about the problem (Creswell, 2007). The concern of the researchers in qualitative research is the interpretations and meanings that people have about their experiences (Merriam, 2009).

Merriam (2009) describes four characteristics for the qualitative research. One of them is the focus on the process, understanding, and meaning that the researcher investigates in natural setting. For this study, the researcher's focuses are what the nature of the mathematical knowledge for teaching (MKT) is and how the teachers attend their MKT in class in the process of teaching of generalization of patterns and operations with algebraic expressions. As Merriam (2009) indicated that the researcher did not intervene to this process. The researcher attempted to understand the teachers' MKT with examining their teaching, experiences and the meanings that they attributed without intervention in the current study.

The second characteristics is that the researcher is the primary instrument of data collection and analysis (Merriam, 2009). The researcher can adapt herself/himself regarding to what is investigated with considering the purpose of the study. The researcher also can interpret the data with holistic perspective by collecting the data with acquiring unexpected situations, the correctness of responses, and the interactions at the time as well as the planned and expected process. Since the classrooms has complex structure, the qualitative research provides to examine this atmosphere (Lagemann & Shulman, 1999). This study has also this type of context and the researcher had opportunity to observe the teaching process in class as natural setting with expected and unexpected situations, the responses and reactions of teachers in teaching and students in learning, and their interactions at the time. Thus, the researcher adapted herself regarding considering the possibilities and also investigated the teachers' knowledge with several perspectives. In connection with the data collection process, the researcher could analyze the data taking the teaching process on the whole and in depth into consideration. Especially, in this study, the researcher's observation of the lessons, and communication with the teachers throughout the teaching process provided and eased forming the themes and codes to analyze the data.

The third characteristics is the inductive process that the collected data is used to develop concepts or theories, or to explain the concepts within the theory (Merriam, 2009). In this study, it is aimed to explain the teacher knowledge of algebra within the model of Mathematical Knowledge for Teaching. In analysis process, forming themes and codes also was inductive and they depended on the data that was gathered from observations, interviews and written documents. Based on the model of MKT components and their descriptions, the codes were formed for this study. The findings also cannot be determined before the investigation; they are explored in the process of research.

The fourth characteristics is the descriptive product that words are used to explain what is investigated than the use of numbers (Merriam, 2009). The researcher used the transcribed videotapes, interviews, field notes, and lesson plans as written documents to give detailed description for teachers' MKT in this study.

In sum, the qualitative research design, especially case study, is preferred for this study in the light of the explanations regarding the characteristics of the qualitative research. In the following section, the characteristics of case study that is used for the current study are explained.

3.2. Case Study

Creswell (2007) states that case study approach includes a case/cases that is explored by the researcher, detailed and in-depth data collection from multiple sources (e.g. observations, interviews), and reporting the themes for the case. According to Yin (2003), case studies are used to answer "why" and "how" questions about a phenomenon in real-life. Yin (2003) gives a technical definition for case study as "a case study is an empirical inquiry that investigates phenomenon within its real-life context, especially when the boundaries between phenomenon and context are not clearly evident" (p. 13). In case studies, what is investigated is considered with their variables related with the context (Yin, 2008).

Merriam (2009) considers that the case is a bounded system regarding the purpose of the study. The researchers (Fraenkel, Wallen, & Hyun, 2012; Merriam, 2009; Miles & Huberman, 1994; Stake, 2006) define the case as a phenomenon which is bounded within context. They explain that the case might be an individual, an event or a situation, an activity or process. Regarding to the case within context, the concept of the unit of analysis is proposed for case studies. The unit of analysis would be “one particular program, classroom of learners, one older learner” (Merriam, 2009, p. 41). The qualitative researchers determine the unit of analysis by considering the purpose of the study as what is aimed to investigate.

Merriam (2009) asserts that the phenomenon is must be bounded to be case. This bounding can be provided with limiting the time for data collection, the number of people to participate to the study, or the topic that is investigated. These boundaries also used for the current qualitative research. The time and topics are limited with the duration of instruction of algebra units, particularly generalization of patterns and operations with algebraic expressions. Because the purpose of this study is to examine the nature of mathematical knowledge for teaching algebra of middle school mathematics teachers. Thus, the instruction of these topics were lasted for four weeks and the research was carried out for these weeks as time was limited. The participants also were limited with two middle school teachers, since the teachers taught these topics based on the curriculum at the same time.

In case studies, investigating the case within the context provides to evaluate the variables that interact with the case. Thus, the researcher presents a holistic description with case study (Merriam, 2009). For this study, the observations of the instructions provided to consider the whole teaching process including the teachers’ teaching and the students’ learning in class time. The pre and post interviews also helped to understand the teaching process in the perspective of the teachers. Thus, these several data sources considering together can yield a holistic explanation about teachers’ mathematical knowledge for teaching algebra.

Yin (2008) describes four types of case study design that single-case design with single unit of analysis-holistic, multiple-case design with single unit of analysis-holistic, single-case design with multiple units of analysis-embedded, and multiple-case design with multiple units of analysis-embedded. In this study, multiple-case study design with single unit of analysis was used from types of case study designs to investigate mathematical knowledge for teaching based on teaching process of two middle school mathematics teachers by examining teacher's lesson plans, teaching process, and their reflections about lessons. In the design of this study, the context is algebra topics (generalization of patterns and operations with algebraic expressions) instruction, the cases are two middle school mathematics teachers, and the unit of analysis is teachers' mathematical knowledge for teaching as shown in the below figure. The teachers' MKT both is explained in holistically by itself, and with using compare-contrast technique in findings.

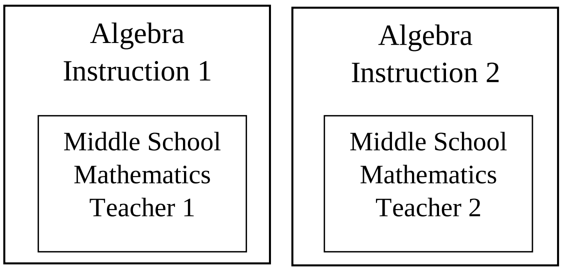


Figure 1 Multiple case study design with single unit of analysis in this study

According to Yin (2008), two or more cases are selected in multiple-case study design to show similar results for literal replications or contrasting results for theoretical replications. For this study, the aim is to reveal outcomes for teachers' knowledge based on Ball et al.'s (2005) MKT model. In this context, this study can be considered for literal replication, that the MKT model also can be evaluated by exemplary outcomes from using cases as two teachers' instructions.

3.3. Participants

This section describes the sampling method used for this study with rationale, and the participants with their demographic information. In this study, the participants were two middle school mathematics teachers whose mathematical knowledge for teaching were examined based on their instructions of algebra. They have been working in the same public school in Ankara. They were teaching algebra topics to 7th grade students at the same time during the data collection. For the selection of these teachers, convenient sampling method was used to provide the accessibility to the teachers, since they taught algebra at the same time in the same school. When random sampling is difficult as in this study, the researchers use convenient sampling and select the participants that are available in terms of location, and time (Fraenkel et al, 2012; Merriam, 2009). One of the reasons for selecting these teachers was that they volunteered to participate to this study. Because this study required a long time and the participants were expected to give time for interviews and discussions out of the class time. Thus, considering these availabilities, these participants were selected for this study. On the other hand, this selection could be purposive/purposeful sampling for the reason that giving rich and detailed information about teaching process of teachers. Creswell (2007) suggests using purposive sampling for qualitative case studies. Sample which can give rich information about the case in depth is selected to understand the phenomenon regarding the purpose of the study in purposeful sampling (Merriam, 2009). Patton (2002) states that "information - rich cases are those from which one can learn a great deal about issues of central importance to the purpose of the inquiry" (p. 230). Thus, the researchers select the sample based on their personal knowledge about participants (Fraenkel et al., 2012). The teachers in this study had a master degree from elementary mathematics education, and they have also been in a doctorate program in elementary mathematics education. According to the researcher opinion, they can have more knowledgeable about research and mathematical content than other teachers who had only bachelor degree, and this situation can provide give rich and detailed information about existing and required teacher knowledge in connection with the purpose of this study.

This research was conducted in 2014-2015 Fall Semester with two female middle school mathematics teachers in the same public school. Their names of the participants were changed to provide confidentiality of them, and used pseudonyms as A and B to represent them. Their demographic information is presented in the table below:

Table 1 The demographics information of the teachers

Demographics	Teacher A	Teacher B
Gender	Female	Female
Bachelor’s degree	Elementary Mathematics Education	Elementary Mathematics Education
Education Level	PhD student in doctorate program	PhD student in doctorate program
Experience in Teaching	3 years	9 years

The detailed characteristics of participants are described in the following sections in detail.

3.3.1. Teacher A

Teacher A was 27 years old during the data collection. She had a bachelor degree from elementary mathematics education department and she had also a master degree in elementary mathematics education. Her master’s thesis was about sixth grade students’ mathematical thinking in problem solving. She examined students’ solution strategies using Cai’s (2000) mathematical thinking scale. In this scale, there are also problems about pre-algebra that are figural and numerical patterns, and writing and solving first degree equations. Thus, she was considered to have knowledge and experiences about analyzing students’ strategies and thinking for algebraic problems. She has also been in a doctorate program in elementary mathematics education. She took course called “Development of algebraic thinking in elementary grades” in doctorate program. The aim of this course was informing the doctoral students about algebraic thinking literature and it also required doing a project as practical aspect. Thus, this teacher has literature knowledge about early algebra and she can be expected to collaborate with the researcher throughout the data collection. Teacher A has experienced in teaching middle school mathematics and has been working in a middle socio-economic level school for 3 years. This school was her second school that she worked. She has been teaching 5th, 6th, and 7th grade students and thus she has been teaching algebraic topics for each level of the middle school for 3 years. The researcher observed this teacher's lessons and talked to her before. The researcher explained the aim of the study to her, and she volunteered to participate to this study. She also expressed that she needed assistance for teaching some topics and wanted to develop herself. Thus, when she was explained the purpose of the study, she was willing to participate to this research.

3.3.2. Teacher B

Teacher B was 32 years old during the data collection. She had a bachelor degree from elementary mathematics education department and she had also a master degree in elementary mathematics education. Her master's thesis was about 5^{th} grade mathematical classroom's discourse in terms of the teacher and student aspects. The contents that Teacher B focused on were about numbers and geometry in her master thesis. Thus, she had knowledge and experiences about research, but not particularly in algebra topics. On the other hand, Teacher B can be expected to collaborate with the researcher throughout the data collection. She has also been in a doctorate program in elementary mathematics education. She did not take any courses related with algebra teaching in graduate classes. She has been teaching 5^{th}, 6^{th}, and 7^{th} grade students for 9 years and thus she has been teaching algebraic topics for each level of the middle school for 9 years. The researcher observed this teacher's lessons and talked to her before the research. The researcher explained the aim of the study, and she volunteered to participate to this study.

3.4. The Context of the Study

In Turkey, the new Middle School Mathematics Curriculum was proposed in 2013 for 4(elementary school) +4(middle school) +4(secondary school) education system. However, the implementation of this curriculum has been reflected on the textbooks gradually grade by grade. Thus, at the time of the data collection of this study, since the textbook was not published regarding of the implementation of the new curriculum, the teachers were using the books based on the old curriculum (MoNE, 2009). The old curriculum was designed for 6-8 grade levels. The books also were prepared based on the objectives and topics in this curriculum (MoNE, 2009). This textbook was written and published by the institution of Ministry of National Education in 2014. The teachers followed this textbook and student's workbook throughout the instruction of algebra topics for 7th grade level. The unit that was selected for this study was entitled as "Integer, Algebra and Geometry". This study focused on the two topics that were generalization of patterns and operations with algebraic expressions. The objectives, the sample of activities and the explanations for teaching these topics as in the curriculum (MoNE, 2009, p. 280-283) translated by the researcher are in the Appendix A. The objectives in the new curriculum (MoNE, 2013) that correspond with these objectives of the old curriculum as in the following:

6.2.1.1. Represent the relationship in the number patterns with letters, find the asked terms of the pattern which is represented with letters.

6.2.1.5. Add and subtract algebraic expressions.

6.2.1.6. Multiply a whole number and an algebraic expression.

8.2.1.2. Multiply two algebraic expressions.

These objectives are in the new 6th and 8th grade mathematics curriculum and under algebra learning area. The difference between the old and new curriculum regarding the objectives is that all of the objectives above belong to 7th grade in the old curriculum.

Beside the mathematical content of the instructions, the general description of the instruction pedagogically and the physical structure of the classrooms can be explained in order to describe the context. The teachers used the same textbook by MoNE (2014) including teacher's guidebook, students' textbook and students' workbook. They designed their lessons by focusing on the textbook order with its examples and activities, and the teachers used direct instruction method. Besides, the desks were set as to see the board, and the teachers usually were in front of the board and wrote what they explained on the board. There was also a projector, but the teachers did not use it the throughout the observations.

The research was conducted in two 7th grade classrooms and there were about 35 students aged 12-13 years old in each class. The teachers indicated that the students came from middle socio-economic level families. In the classrooms, the students sit in pairs in one desk as the teacher arranged. Although there were bulletin boards in the classrooms, there were not any mathematics works of students.

3.5. Data Collection Procedures

The data collection procedure had two main phases for this study: before the instruction, and during the instruction. In before the instruction phase, the teachers prepared the lesson plans individually, the researcher interviewed with each teacher, and the researcher suggested examples, activities, methods and techniques, and shared the findings of the questions in the tests (e.g. the strategies used by the students, misconceptions, errors that arose in the solutions). The teachers revised their lesson plans with what was suggested by the researcher. The setting of the suggestions and the teachers' preferences for revising their lesson plans were explained in detail in 3.6.1 section. In the second phase, during the instruction, the researcher observed the lessons in two different 7th grade classes, took field notes, and video-recorded by the camera. After each class session, the researcher conducted post-observation interviews with the teachers.

In sum, Figure 2 shows the process of data collection with the flow diagram below:

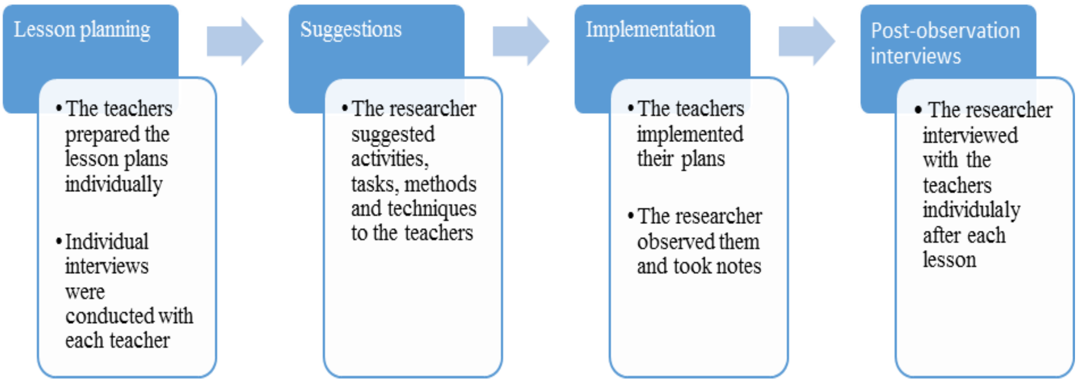


Figure 2 The process of data collection

The researcher gathered data from multiple sources (e.g. lesson plans, interviews, and observations) and this could provide understanding the actual classroom environment. Each type of data was collected to support other data from different instrument. Lesson plans with teachers' responses from pre-observation interviews; teachers' responses with observations; and observations with teachers' responses from post-observation interviews were supported and completed each other.

3.6. Data Sources

Multiple data sources were utilized to get rich and depth information about teachers' mathematical knowledge for teaching in this study. As Creswell (2007) states that "qualitative researchers typically gather multiple forms of data, such as interviews, observations, and document, rather than rely on a single data source" (p. 38). For this study, the data were collected from prepared the lesson plans by the teachers before the instruction, examining lesson plans with the teachers before the instruction and interviewing with the teachers (pre-observation interviews), observation the class during the instruction of algebraic topics, and reflective interviews (post-observation interview) with the teachers after the instruction. The data sources are explained in detail with rationale in the following sections.

3.6.1. Prepared Lesson Plans by the Teachers

The purpose of preparing lessons plans individually is to examine existing teachers' mathematical knowledge for teaching algebra. With preparing lesson plans, it is aimed to investigate what teachers' knowledge for preparing the lesson plan for teaching algebra (subject matter knowledge) is, and how teachers take into account students' thinking (pedagogical content knowledge) while they are designing their lessons. Preparing the lesson plans individually also provided the teachers to review the content which they would teach to the students.

In preparing lesson plan process, the researcher presented the objectives of algebra topics as in the 7^{th} grade mathematics curriculum for the teachers before the preparation of the lesson plans first. The algebra unit has 4 topics and 9 objectives. Thus, the teachers were expected to prepare 4 lesson plans for each topic. For this study, teachers' mathematical knowledge for two topics, generalization of patterns and operations with algebraic expressions, were examined. Thus, two lesson plans were examined in this study. The objectives that were included in the two lesson plans are shown in Appendix A. The teachers prepared the lesson plan for generalization of patterns first, and then for operations with algebraic expressions. The first lesson plan had one objective "Express the relation in number patterns which are modelled by using letters", the second lesson plan had objectives "Perform addition and subtraction operations with algebraic expressions" and "Multiply two algebraic expressions". The lesson plans were prepared by each teacher individually before the instruction. After they prepared the lesson plans, the researcher interviewed with each teacher about their lesson plans. This was an opportunity for the teachers to revise and develop the lesson plans, since their explanations verbally provided them to realize the strong and weak aspects of the lesson plans. In these pre-observation interviews, the researcher asked questions about preparation process to the teachers why they designed the lessons in this way. The structure of interviews is explained in detail in the following section.

After the interviews, the researcher and the two teachers came together. The researcher suggested activities, methods and techniques from the literature related with the objectives in the lesson plans for the teachers. If the teachers considered which of them could be useful for improving students' understanding, they revised their lesson plans and involved them in their lesson plans. The settings that includes the researcher's suggested activities, methods and techniques from literature are explained detailed in the Appendix B and Appendix C. The aim of these suggestions from literature was to understand of what the reasons of the teachers to select the activities or questions, and to observe of how they implement them in their instruction. It was considered that the selection and the implementation could provide to explore the teachers' knowledge to teach the algebra topics. Since the suggested activities, methods and techniques were from the literature, the teachers were suggested using them to improve the students' understanding and learning.

The setting of suggestions for generalization of patterns

The researcher made suggestions for the teachers based on the research related to teaching and learning of generalization of patterns and algebraic expressions. The researcher prepared the suggestions before the data collection. Especially, she took the activities, examples, and methods that were suggested for the teachers to develop their instruction. The suggestions were implemented and found useful to support the students' learning in research. The setting of suggestions for generalization patterns was based on the literature (Blanton & Kaput, 2003; Healy & Hoyles, 1999; Herskowitz, et al. 2002; Lannin, Barker & Townsend, 2006; Magiera, van den Kieboom & Moyer, 2013; Moss, Beatty, McNab, & Einsband, 2005; Moss, Beatty, Barkin, & Shillolo, 2008; Rivera & Becker, 2005; Smith, Silver & Stein, 2005; Walkowiak, 2014; Warren & Cooper, 2008). The researcher suggested a pattern test, three activities, and a representation for supporting the instruction of generalization of patterns (Appendix B). The researcher developed a pattern test using the questions in the literature and implemented this test to 6^{th}, 7^{th} and 8^{th} grade students in the same school in the previous semester before this study. The test contained numeric, pictorial, and tabular representations of linear growth patterns. The aim of the implementation of the test was to share the findings about middle school students' reasoning and strategies for generalizing different represented patterns with the teachers. The researcher pointed out the students' conceptions, misconceptions, errors, and difficulties while generalizing patterns in this sharing. The teachers involved the pattern test to their lesson plans and revised the lesson plans.

The suggested activities were presented and explained in detail in Appendix B. However, the teachers decided not use these activities in their lessons. As the representation, the researcher suggested the use of table from the examination of students' solutions in the pattern test as the students who got the correct generalization generally used tabular representations. In addition to this, the researcher suggested using the reasoning within figures or pictures in figural patterns as the studies (Healy & Hoyles, 1999; Moss, Beatty, McNab, & Einsband, 2005; Rivera & Becker, 2005; Walkowiak, 2014; Warren & Cooper, 2008) also stated that the students used numerical and figural reasoning, even younger students used figural reasoning more in pattern generalizations. The teachers used tabular representation of the patterns to underlie the relationship in the patterns in the instruction, and they used figures to provide visuality.

The setting of suggestions for operations with algebraic expressions

The setting of suggestions for operations with algebraic expressions was based on the literature (Gay & Jones, 2008; Kindt, 2014; Van de Walle et al., 2013). The researcher suggested several activities from the course book by Martin Kindt and the textbook (Appendix C). The teachers involved these activities to their lesson plans and revised them, and they used them in the instructions.

The researcher prepared "Algebraic expression" test which included the suggested activities and questions (see Appendix C). The researcher implemented this test for 8th graders in the same school. This test had four parts: operating with expressions, equivalent expressions, people at the amusement park activity, and correcting the error in subtraction operation that had two parenthesis algebraic expressions. The first reason for preparing this test was to have an idea about the practicality of suggested activities in teaching operation with algebraic expressions before the teachers' implementations. The second reason for implementing this test was to warn or prepare the teachers for possible conceptions, misconceptions, errors and difficulties that students can have while teaching. In this setting, the solution strategies, misconceptions, errors and difficulties of students were shared with the teachers. The results showed that the students had difficulty with remembering to multiply each term in the parenthesis while using the distributive property. The teachers agreed with this difficulty of students, and they indicated that overcoming these types of errors were difficult by giving several examples while teaching.

As the method and technique, the researcher suggested explaining the properties of addition and multiplication properties in algebraic expressions with connecting the properties of operations in arithmetic for the students as Van de Walle et al. (2013) indicated. The textbook published by MoNE also included the activity that provided this connection and these activities also were suggested to the teachers. The researcher explained that the use of algebra tiles with modeling for teaching multiplication was suggested representation in the curriculum (MoNE, 2009; 2013) and literature. The teachers already included the use of them in lesson plans.

3.6.2. Pre-Observation and Post-Observation Interviews

Yin (2003) states that interviews are important tools to collect data in case studies since these types of studies generally are about humans. The interviewed person can explain and make interpretations about what is investigated, and it can provide information beyond what is observed and understood. One of the interview types used in case studies that Yin (2003) explains is focused interviews that they include the questions from what is investigated as case, focus on specific situation, and lasted for short time. In this study, interviews were conducted before (pre-observation interviews) and after the instruction (post-observation interview). Pre-observation interviews focused on the lesson plans were prepared by the teachers, and post-observation interviews focused on the instruction for one or two class times.

According to Fraenkel et al. (2012), interviews can provide to understand what is on people's mind and also whether people's thoughts support the researcher's observations. The purpose of the pre-observation interviews was to understand teachers' knowledge and preparedness for teaching algebra topics. The pre-observation interviews were conducted after the teachers prepared the lesson plans. In these interviews, it was aimed to investigate how teachers decide to design the lessons like in the plans, and what the teachers take into account about students' thinking while preparing the lesson plans. Pre-observation interviews could also provide to check whether the researchers understood teachers' reasoning properly.

The researcher used semi-structured type of questions in the interviews. Corbetta (2003) defines the semi-structured interviews as in the following:

> Within each topic, the interviewer is free to conduct the conversation as he thinks fit, to ask the questions he deems appropriate in the words he considers best, to give explanation and ask for clarification if the answer is not clear, to prompt the respondent to elucidate further if necessary, and to establish his own style of conversation (p. 270).

The researcher prepared the same questions for the teachers to explain their lesson plans in their words based on the questions. The questions were semi-structured in nature and thus the expected answers can depend on the teacher's knowledge and the prompt questions can be asked if necessary. The questions were asked to the teachers in the pre-observation interviews as in the following:

1. How can you explain your lesson plan briefly?
2. How did you plan your lesson in this way?
3. What are the methods or techniques do you plan to use for supporting students' learning?

4. What prior or prerequisite knowledge that the students should have to learn this topic?
5. What possible misconceptions that students have while learning this topic?
6. What possible difficulties that students have while learning this topic?
7. What do you plan to do when the students have difficulty during the instruction?
8. What do you plan to measure and assess the students' learning?

The pre-observation interviews provided the researcher to understand the lesson plans as written documents in detail as the teachers explained the design of the lesson plans with rationale for each question and activity in lesson plans. These interviews also provided the researcher to evaluate the lesson plans correctly by teachers' explanations.

After each class, the researcher made post-observation interviews as reflective interviews. In a week, mathematics classes were carried out for three days and total five class hours. Thus, three post-interviews were conducted by each teacher after each class in a week, and in total 24 post interviews were made with the two teachers. These interviews lasted about 5-10 minutes. The reason to conduct these interviews right after the instruction was to prevent the teachers to forget what and how the lesson went on considering the implementation of questions or activities, the students' responses, difficulties and the teachers' reactions to the situations happened during the instruction. One of the purposes of these interviews in this study was to provide to evaluate teacher's lessons by herself. It was aimed to understand what the teachers thought about implementation of the lesson plans. The other purpose of the post-observation interviews was to check whether the researcher understood teacher's instruction properly or not from observing the lessons by comparing the teachers' responses. The researcher prepared the same questions for the teachers to evaluate their lessons in their words based on the questions. The questions were semi-structured in nature and thus the expected answers could depend on the teacher's opinions and the prompt questions could be asked if necessary. The questions were asked to the teachers in post-observation interviews as in the following:

1. How was the lesson in general?
2. What do you think of the implementation of the lesson plans in the instruction?
3. What do you think about the implementation of questions and activities?
4. How was the level of the students in this class?
5. Where did the students have difficulty in the instruction?
6. What did the students learn in this class?
7. Did you encounter any unexpected situations during the class? If yes, what were they, and how did you handle these situations?

The post-observation interviews provided the researcher to understand the lessons in the eye of the teachers. Thus, the researcher asked the questions about how the teachers perceived the flow of the lesson. These interviews also provided the researcher to evaluate the classes correctly by teachers' explanations.

The interview questions were prepared by the researcher examining the used interview protocols in related literature and then reviewed by a mathematics education researcher. Pilot interviews were conducted with the teachers for another topic before the main study to determine if the questions can serve for the purpose of the study. All interviews were audio-recorded with the permission of the teachers to transcribe later for analyzing. The interviews were conducted in the school library or in a classroom if which of the place was suitable to carry on the conversation comfortably and without interrupting.

3.6.3. Observations

Creswell (2012) defines observation as "the process of gathering open-ended, firsthand information by observing people and places at a research site" (p. 213). Observations also are one of the data sources in case studies (Yin, 2003). McDonough and Clarke (2002) state that observing lessons is a good method to examine teachers' knowledge. Thus, one of the data sources for this study was observations. The researcher observed total 33 mathematics lesson hours that included the instructions of generalization of patterns and operations with algebraic expressions. The purpose of the observations was to explore how teachers actually used their knowledge. These observations could also provide comparing teachers' practice in class with teacher's thinking and what was written in lesson plans.

During the instructions, the researcher acted as non-participant observer. Creswell (2012) indicates the role of non-participant observer as taking notes in the research setting and not to participate in the activities. The researcher sat at the back of the classroom, and watched the instructions. All observations were video-recorded with the official permission. The researcher focused the camera at the board and the teacher, and changed the direction of the camera not to catch teacher's actions. The researcher also took field notes throughout of the instructions. Denzin (1989) describes the observation field notes that are about participants, interactions, routines, and interpretations. Thus, the researcher took notes about how the lesson was going on and her interpretations about the instructions.

3.7. Duration of the Study

This study was conducted during the instruction of algebra topics at 7^{th} grade. The algebra unit had 9 objectives, and 16 lesson hours were advised as time by MoNE (2009). In the new curriculum (MoNE, 2013), it is advised to allow about 13 lesson hours for 4 objectives to teach these topics. However, the data collection process for two topics from the algebra unit was about 2 weeks for preparing lesson plans, and 4 weeks for the instructions. The timeline for data collection for this study as in the table below:

Table 2 Timeline for data collection

Date	Events
July 2014	Permissions from Research Center for Applied Ethics
August 2014	Permissions from Ankara Provincial Directorate for National Education and Ankara Yenimahalle District National Education Directorate
September 2014	Classrooms were determined, Pilot lesson planning and interviews, Main study lesson planning and interviews
October 2014	Pilot observations and Post-observation interviews
November 2014 - December 2014	Main study observations and Post-observation interviews
January 2015	General interview about the whole research process

Before data collection, the researcher prepared necessary official forms and an interview protocol, and applied Research Center for Applied Ethics of Middle East Technical University to get permissions. After getting the permission from ethics committee (see Appendix D), the researcher applied to Ankara Provincial Directorate for National Education to get permission for conducting the study in determined public school in Yenimahalle. For this particular school, the permission was taken from Ankara Yenimahalle District National Education Directorate (see Appendix E for the document). Getting this permission, the researcher explained the purpose of the study and the data collection process to the school management. The teachers already was informed about the study and thus they became volunteer. The school management adjusted the teachers' schedules in terms of teaching in 7th grade and not to cross the teachers' classes at the same hour. Since the researcher wanted to observe all classes for two teachers. According to the teachers' schedules, pre and post-observation interviews' times were determined. During the data collection necessary changes were made in these times.

In September 2014, the first two weeks of the 2014-2015 Fall semester were seminar weeks for the teachers before the lessons began. The working hours in these weeks finished at noon. Then, the researcher interviewed with the teachers about the lesson plans. The teachers prepared the lesson plans before they came school. After interviewing with the teachers individually, the researcher suggested examples, activities, methods and techniques for the topic for which they prepared the lesson plan. Thus, before the lessons began, one lesson plan for the pilot study and four lesson plans for the instruction of algebra unit were prepared and revised regarding the suggestions by the teachers.

After the semester began, the pilot study with observations and post-observation interviews were conducted for a week in October 2014. After the teachers experienced the research process for short time, the main study began in November 2014. The main study lasted to the mid of December 2014 included the two teachers' lessons. The main study included the instruction of generalization of patterns and operations with algebraic expressions. The observations and post-observation interviews were made about in 6 weeks. The researcher continued to observe the instructions and interview with the teachers for another 6 weeks. The all instructions were completed at the last week of January 2015, the end of the semester. At last, the researcher conducted a general interview with the teachers individually to get their opinions, interpretations, and suggestions about the research process.

3.8. Pilot Study

Before data collection for the main study, a pilot study was conducted in order to provide the teachers to get used to the research process of the study. To do this, before designing lessons for algebra unit, a lesson plan was prepared for rational numbers. This topic had 3 objectives that "explain the rational numbers and show on the number line, represent the rational numbers in different representations, compare and order of rational numbers". The pilot lesson plans were prepared for 5 lesson hours in a week. The teachers were interviewed individually about their lesson plans. The researcher suggested activities, tasks, methods and techniques that could be used for the teaching of this topic. The teachers wanted to use them and they involved them to their lesson plans. This pre-study was expected to provide the teachers to experience the lesson plan preparation, and inform and guide the researcher about what and how was examined in this preparation process.

After planning the lesson, the pilot observation was made in the class sessions. The researcher took field notes about teachers' actions throughout the teacher's lesson and recorded the instructions by the camera. This process provided the teachers and students to get used the camera in the classroom. After this lesson, the researcher determined the points in students' responses in teaching, which could be discussed as difficulty, misconceptions or conceptions by reviewing the videos and field notes. Then, the researcher interviewed with the teachers to reflect and evaluate their lessons.

3.9. Data Analysis Procedure

In this study, the data sources were lesson plans, interviews, and observations. In data analysis process, first, the teacher's responses from the pre-observation interviews about preparing lesson plans, the instructions as video-records, and teachers' responses from post-observation interviews about the instruction were transcribed and read by the researcher. Then, the transcriptions, and the lessons plans and observation notes as written work were gathered together to give a holistic picture about teachers' MKT, and organized regarding the procedure of data collection.

Creswell (2007) explained the data analysis procedure in qualitative research as that "consists of preparing and organizing the data for analysis, then reducing the data into themes through a process of coding and condensing the codes, and finally representing the data in figures, tables, or a discussion" (p. 148). Particularly, Creswell (2007) indicated cross-case analysis for multiple case studies. In this analysis, each case and themes are described in detail and then analyzed the cases within themes by comparing. Yin (2003) also suggested cross-case sythesis as an analytic technique for the analysis of two or more cases. The researcher can compare and contrast of the cases based on the framework that is used for analysis. In this study, the cross-case analysis technique that was suggested by Yin (2003) was used for analyzing and interpreting of the findings of the two cases.

The analysis of the qualitative research designs was begun with forming the initial and tentative codings, and then was continued with grouping them in themes with respect to similarities, and was ended with reporting the data (Merriam, 2009). For this study, the cases as the two teachers' planning and instructions were analyzed and described independently first. The data from the cases were analyzed and coded as a statement, an explanation, a dialogue, or a question that considered to be meaningful within itself. Then, the extracted codes were put into one of themes and then sub-themes considering the descriptions and definitions of themes and sub-themes in the MKT model. SMK and PCK as knowledge domains were considered as themes, their components as knowledge sub-domains (CCK, SMK, KCS, KCT, and KCC) were considered as sub-themes for this study, since the purpose of the study is to examine teachers' MKT. This process provided to analysis of cases individually within itself. Merriam (2009) stated that "a within - case analysis is followed by a cross - case analysis" (p. 205) for the analysis of multiple case studies. Thus, after forming these tentative codes, the codes which were categorized by comparing and contrasting and had a pattern in one case were also investigated for other case. From the analysis of two cases, the codes which were occurred in the two cases within a pattern were determined for the current study (Appendix F). The extracted codes were presented and interpreted by stating positive sign (+) for the teachers' use of appropriate and adequate knowledge, and negative sign (-) for the teachers' use of inappropriate and inadequate knowledge within tables for each case in findings independently. This analysis was carried out for the observed knowledge types. Thus, the negative sign did not mean the absence of knowledge. To illustrate, SCK7(+) indicates that the teacher had the knowledge to choose, make and use the tabular representation with focusing on the arithmetical relationships in tabular representation to underlie the relationship in pattern to conceptualize generalization and she used her knowledge effectively in instruction. Or, the students had difficulty in applying distributive property in learning multiplication, but the teacher stated that she did not understand what they needed to learn and apply it correctly. At this point, KCS3(-) indicates that the teacher's knowledge to understand the difficulties and needs of students with application of distribution property appeared inadequately. After the data were coded with this method, the results were compared and contrasted in discussion under the knowledge sub-domain headings in the discussion at the end. The comparison of two cases could provide making interpretation about the knowledge that the teachers should have by emphasizing the existing and lack of knowledge in two cases.

3.9.1. The Framework Used for the Analysis in This Study

In this study, Ball et al.'s (2008) MKT model was used to examine the nature of middle school mathematics teachers' MKT in planning and implementing of teaching generalization of patterns and operations with algebraic expressions. Although the related data set of planning and implementing were different, MKT model used to analyze them. The framework that was used for analysis of data with related research questions and data set are as in the following table:

Table 3 The framework used for analysis of data with related research questions and data set

Research questions	Related data set	Framework for Analysis
1.a. What is the nature of middle school mathematics teachers' mathematical knowledge for teaching (MKT) for generalization patterns in planning lessons?	Interviews about lesson planning Lesson plans	MKT framework

Table 3 (Continued)

Research questions	Related data set	Framework for Analysis
1.b. What is the nature of middle school mathematics teachers' mathematical knowledge for teaching (MKT) for generalization patterns within the practices of implementing lessons?	Observations and field notes Post-Observation interviews	MKT framework
2.a. What is the nature of middle school mathematics teachers' mathematical knowledge for teaching (MKT) for operations with algebraic expressions in planning lessons?	Interviews about lesson planning Lesson plans	MKT framework
2.b. What is the nature of middle school mathematics teachers' mathematical knowledge for teaching (MKT) for operations with algebraic expressions within the practices of implementing lessons?	Observations and field notes Post-Observation interviews	MKT framework
3. How does middle school mathematics teachers' subject matter knowledge (SMK) influence their pedagogical content knowledge (PCK) in the context of teaching generalization of patterns and operations with algebraic expressions?	Interviews about lesson planning Lesson plans Observations and field notes Post-Observation interviews	MKT framework

Ball, Thames and Phelps's (2008) Model of Mathematical Knowledge for Teaching (MKT)

Ball, Thames, and Phelps (2008) proposed a model for mathematics knowledge for teaching. This model is a domain map that shows mathematical knowledge for teaching consists of subject matter knowledge and pedagogical content knowledge (Figure 3). Subject matter knowledge (SMK) has common content knowledge (CCK), specialized content knowledge (SCK), and horizon content knowledge (HCK) components. Pedagogical content knowledge (PCK) consists of knowledge of content and students (KCS), knowledge of content and teaching (KCT), and knowledge of curriculum (KC) as in the following figure:

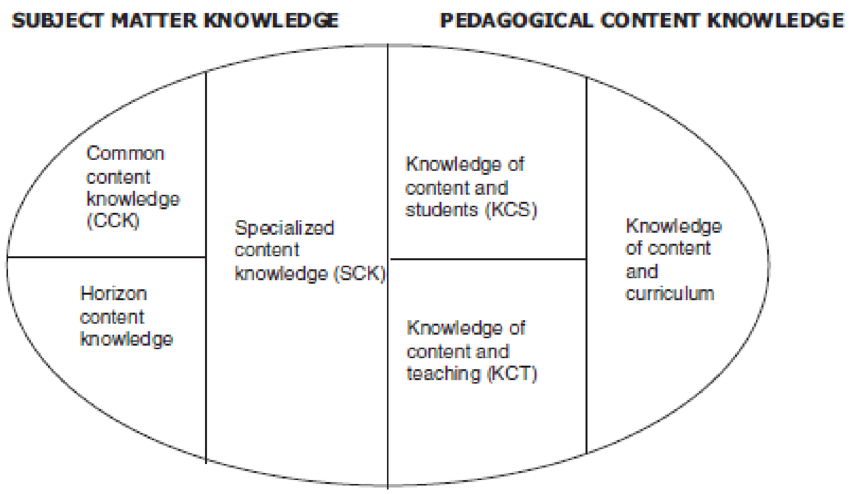


Figure 3 Domain Map for Mathematical Knowledge for Teaching (MKT)

The first component of subject matter knowledge, common content knowledge (CCK) is defined as "the knowledge that is used in the work of teaching in ways in common with how it is used in many other professions or occupations that also use mathematics" (Hill, Ball, & Schilling, 2008, p. 377). Ball et al. (2008) indicated that this mathematical knowledge can be used any settings which require mathematics. Thus, this knowledge is not specific for teaching and therefore mathematic teachers. The occupational groups who use mathematics in their working area have this knowledge. For teaching settings, Hill et al. (2008) state that CCK refers to subject matter knowledge proposed by Shulman (1986). Ball et al. (2008) also explained CCK that mathematics teachers should have by examplifying such as "*simply calculating an answer or, more generally, correctly solving mathematics problems, using terms and notation correctly writing on the board"* (p. 399). In addition to teachers' CCK knowledge and skills, they must be aware of incorrect answers, solutions of students or incorrect definitions, questions, and explanations of the textbooks. If the teachers did not have CCK or adequate CCK, they can make errors, use the terms incorrectly, or have difficulty in solving problems. These deficiencies in teacher's CCK can impede the development of students' learning and cause wasting time from the instruction (Ball et al., 2008). Beside these definitions, Sosa (2011, as cited in Carreño, Rojas, Montes, & Flores, 2013) proposed the descriptors for CCK in her dissertation that CCK includes the knowledge of "*definitions, rules, properties, and theorems related to a specific topic*".

Based on the definitions, descriptions, and examples from the literature and reviewing the data in this study, these three codes are framed for CCK component in this study:

> CCK1: The knowledge of definitions, rules, properties, and theorems related to a specific topic
> CCK2: The knowledge to use terms and notation correctly
> CCK3: The knowledge to simply calculating an answer or, more generally, correctly solving mathematics problems.

The second component of subject matter knowledge, specialized content knowledge (SCK) is defined as "the mathematical knowledge and skill unique to teaching" (Ball et al., 2008, p. 400). This knowledge requires mathematical understanding and reasoning. With this knowledge, teachers can explain and justify the mathematical ideas. However, this knowledge is beyond of the conceptual understanding. SCK is also about explaining the content and making decisions pedagogically. Ball et al. (2008) explain SCK by giving such examples that the knowledge is about "*how mathematical language is used*; *how to choose, make, and use mathematical representations effectively*, and *how to explain and justify one's mathematical ideas*" (p. 400). The researchers also describe the mathematical tasks for teaching, that require SCK, such as "*linking representations to underlying ideas and to other representations, connecting a topic being taught to topics from prior or future years, giving or evaluating mathematical explanations*, and *choosing and developing useable definitions"* (p. 400).

Based on the definitions, descriptions, and examples from the literature and reviewing the data in this study, these seven codes are framed for SCK component in this study:

SCK1: The knowledge to connect a topic being taught to topics from prior or future years
SCK2: The knowledge to link representations to underlying ideas and to other representations
SCK3: The knowledge to choose/give usable definition or explanations
SCK4: The knowledge of how to explain and justify one's mathematical ideas
SCK5: The knowledge of how mathematical language is used
SCK6: The knowledge of how to provide mathematical explanations for common rules and procedures
SCK7: The knowledge of how to choose, make, and use mathematical representations effectively.

The third component of subject matter knowledge, horizon content knowledge (HCK) is defined as "an awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (Ball et al., 2008, p. 403). Teachers know with this knowledge as the topic that they taught how is related in the next grades' curriculums. Thus, teachers can prepare the students what more they know in next years about the topic that they learn now. However, for this study, the codes could not be formed based on HCK, since the data did not give any examples for this knowledge type.

The other category of MKT is pedagogical content knowledge (PCK). This category has three components. The first component of PCK is knowledge of content and students (KCS). KCS is defined as "the knowledge that combines knowing about students and knowing about mathematics" (Ball et al., 2008, p. 401). Hill et al. (2008) asserted that knowledge of students refer to how students think about, know, or learn particular content. Thus, they indicated that this knowledge is not about "knowledge of teaching moves". The teachers with this knowledge can *anticipate or predict the mistakes or misconceptions that commonly arise during the instruction* (p.375), and know *conceptions and preconceptions that students have*. KCS can be a component of Shulman's PCK concept, and this is apart from subject matter knowledge. Ball et al. (2008) explain KCS with giving several examples. To illustrate, "when choosing an example, teachers need to *predict what students will find interesting and motivating*, or when assigning a task, teachers need to *anticipate what students are likely to do with it and whether they will find it easy or hard*" (p.

401). Sosa (2011, as cited in Carreno et al., 2013) describes KCS as *understanding of the needs and difficulties of students with mathematics topics*.

Based on the definitions, descriptions, and examples from the literature and reviewing the data in this study, these six codes are framed for KCS component in this study:

> KCS1: The knowledge to anticipate where and how students have difficulty
> KCS2: The knowledge to anticipate the misunderstandings that might arise with specific items being studied in class
> KCS3: The knowledge to understand the needs and difficulties of students with mathematics
> KCS4: The knowledge to hear and interpret students' emerging and incomplete thinking as expressed in the ways that pupils use language
> KCS5: The knowledge of common student conceptions and misconceptions about particular mathematical content
> KCS6: The knowledge to predict what students will find interesting and motivating.

The second component of PCK is knowledge of content and teaching (KCT). KCT is defined as "the knowledge that combines knowing about teaching and knowing about mathematics" (Ball et al., 2008, p. 401). According to Hill et al. (2008) KCT is formed with teaching moves considering of "*how to build on students' thinking* or *how to address and remedy student errors effectively*" (p. 378). Ball et al. (2008) explain KCT with giving such examples that the knowledge is about managing the order of topics for teaching or selecting examples for improving students' learning. Teachers with this knowledge can interpret advantages and disadvantages of representations or methods for teaching a particular concept. They also must guide the classroom discussions with making decisions about when to explain, to use students' ideas, and to ask a new question. It might be understood that these all examples are about teachers' instructional decisions or actions for supporting and improving students' learning.

Based on the definitions, descriptions, and examples from the literature and reviewing the data in this study, these eight codes are framed for KCT component in this study:

> KCT1: The knowledge to choose which examples to start with and which examples to use to take students deeper into the content
> KCT2: The knowledge to sequence particular content for instruction

KCT3: The knowledge to choose a particular representation or certain material for learning a concept or mathematical procedure
KCT4: The knowledge to evaluate the instructional advantages and disadvantages of representations used to teach a specific idea
KCT5: The knowledge to decide when to pause for more clarification, when to use a student's remark to make a mathematical point, and when to ask a new question or pose a new task to further students' learning in classroom discussion
KCT6: The knowledge to identify what different methods and procedures afford instructionally
KCT7: The knowledge of how to build on students' thinking
KCT8: The knowledge of how to address student errors effectively, remedy student errors.

The third component of PCK is knowledge of content and curriculum (KCC). KCC is explained as *the knowledge of the contents regarding the curriculum order, suggested activities, important explanations for teaching*. Sosa (2011, as cited in Carreno et al., 2013) described as "content in textbooks and the relation of previous and forthcoming mathematical topics". This knowledge requires to connect the relation with the topics for different grades. Teachers should know *the development of what is taught related with particular topic between grades*. Besides, they should know the materials, and assessment techniques proposed by national authorities.

Based on the definitions, descriptions, and examples from the literature and reviewing the data in this study, these two codes are framed for KCC component in this study:

KCC1: The knowledge to know the content and objectives in the curriculum
KCC2: The knowledge to judge how to utilize it to present, emphasize, sequence and instruct.

In sum, all codes used for analyzing MKT of teachers, for SCK and PCK respectively, are presented within tables in the Appendix F.

As considering Ball et al.'s (2008) explanations of four sub-domains for teaching decimals order, it can be exemplified in algebra teaching, particularly generalization of patterns lie that; the knowledge to generalize patterns and write the generalization algebraically is CCK, the knowledge to use different representation such as table to support the students to explore the relation between the position number and the terms is SCK, the knowledge to recognize students have difficulty with writing the general rule algebraically is KCS, and the knowledge to remedy the students' errors that arise while generalizing is KCT.

One of the reasons for using MKT model for this study is that this MKT model is widely accepted and used in mathematics education research. The authors developed this model based on the conceptualization of teacher knowledge of Shulman's (1986) theory as content knowledge and pedagogical content knowledge. This model also was developed as the result of qualitative and quantitative research in a longitudinal process. Another reason is that this model was based on observations of teachers' instructions in qualitative research context. The major data in this study are observations of the instructions. For that reason, the use of MKT model could be considered appropriate for the current study. Ball et al. (2008) indicate their purpose of developing MKT model is to determine what teachers know to teach content and how they use it in practice (p. 395). One of the aims of this study is to put forward what teachers need to know for teaching generalization of patterns and operations with algebraic expressions with the revealing of the existing knowledge. Thus, MKT model is used in this study under these considerations.

3.9.2. Analysis of Planning and Instruction

The knowledge of the teachers in planning is examined in the context of topics; generalization of patterns and operations with algebraic expressions and with respect to the phases of data collection. To analyze the teachers' knowledge, the interviews about their lesson plans and the meetings for suggestion by the researcher were transcribed and they were examined with their written lesson plans. These written documents were used to explore what the existing knowledge of the teachers was by using the codes based on the MKT framework. The extracted knowledge types were explained and interpreted based on MKT framework.

The mathematical knowledge for teaching of the teachers was extracted from her actions throughout the instruction by focusing on common patterns in observation data based on MKT framework. The mathematical knowledge for teaching of teachers that was extracted from her actions throughout the instruction was examined and interpreted within their practices. Based on the common patterns in the teacher's actions that they performed throughout the instructions, the practices of the teachers were grouped. Teacher A's purposeful actions to teach the topics were grouped into six practices: 1) choosing an example or activity to start teaching the topic with connecting to topics from prior years, 2) discussing on the activity related to the topic, 3) choosing the examples or activities to use to take students deeper into the topic, 4) implementing the suggested activities, 5) doing exercises related to the topic from textbook and workbook, and 6) presenting problems that combine knowledge related to other topics. On the other hand, Teacher B's purposeful actions to teach the topics were grouped into five practices: 1) connecting the topic being taught to topics from prior years, 2) discussing on the activity related to the topic, 3) choosing the examples or activities to use to take students deeper into the topic, 4) implementing the suggested activities, and 5) solving the questions and problems related to the topic from different resources. The practices of the teachers were examined within the context of topics; generalization of patterns and operations with algebraic expressions. It is important to note that the extracted knowledge types from instruction was also examined with planning before the instruction, reflections of the instruction in the post-observation interviews, and evaluated together to conclude the teachers' knowledge for teaching the algebra topics.

3.10. Trustworthiness

In qualitative research, findings and interpretations are needed to be accurate (Creswell, 2012). Merriam (2009) indicates that validity and reliability issues are necessary for research in collecting, analyzing and interpreting the findings to conceptualize the study. "Validity refers to the appropriateness, meaningfulness, correctness, and usefulness of the inferences a researcher makes. Reliability refers to the consistency of scores or answers from one administration of an instrument to another, and from one set of items to another" (Fraenkel et al., 2012, p. 147).

In order to provide the trustworthiness of the study, several methods (e.g. triangulation, member checking, peer examination, and cross-checking) are used. “Triangulation is the process of corroborating evidence from different individuals (e.g., a principal and a student), types of data (e.g., observational field notes and interviews), or methods of data collection (e.g., documents and interviews) in descriptions and themes in qualitative research” (Creswell, 2012, p. 259). In this study, data were collected from several sources that were preparing lesson plans, interviews, and observations in order to provide trustworthiness.

Member checking is providing accuracy by taking participant’s interpretations about findings, and initial data analysis by asking them (Guba & Lincoln, 1981). In this study, the interviews were conducted with the teachers to get their comments on findings and initial analysis of the data after transcribing and coding the data as member checking.

Another validating strategy is peer examination. According to Merriam (2009), peer examination means criticizing research findings together with researcher(s) who are familiar with the study or new to the study (Merriam, 2009). One researcher from who was a PhD student in mathematics education, was asked to examine the coding of the data in this study.

To provide reliability of coding the data, cross-checking can be utilized. According to Creswell (2009) “cross checking is developed by different researchers by comparing results that are independently derived” (p, 190). In this study, an expert in research and mathematics education, and also familiar with the research was asked to code the data for cross-checking. Then, the level of the coding in agreement was calculated to determine the consistency in coding. Wiersma (2000) stated that the importance of analyzing the data by several researchers and getting similar results in order to provide internal consistency. For this study, the researcher and the expert coded the data independently and then reached a full agreement with discussing the coding.

Last, this study was conducted in about a six-week-long period. This situation can also provide reliability of the study for the researchers to gain patterns in data accurately by collecting data in a long process.

CHAPTER IV

FINDINGS

This chapter documents and explains Teacher A's and Teacher B's mathematical knowledge for teaching algebra in planning and instruction. The planning phase includes the teachers' existing MKT for designing lessons in lesson plans and interviews. The instruction phase includes how the teachers use their MKT in teaching. The two phases are examined within the context of generalization of patterns and operations with algebraic expressions. Then, the chapter also presents how the teachers' SMK influences their PCK in the context of the instruction of generalization of patterns and operations with algebraic expressions.

4.1. The Case of Teacher A

This section documents and explains Teacher A's mathematical knowledge for teaching algebra in planning and instruction. The two phases are presented within two topics of algebra unit, generalization of patterns and operations (addition, subtraction and multiplication) with algebraic expressions.

4.1.1. Planning

The mathematical knowledge for teaching of Teacher A extracted from her lesson plans and the interview about planning with her by focusing on common patterns in data based on MKT framework. The two lesson plans were prepared before the instruction of the topics. The knowledge of Teacher A in planning was examined in the context of topics; generalization of patterns and operations with algebraic expressions; and in three main groups with respect to the phases of data collection: preparation of lesson plans individually, explanation of the lesson plans to the researcher, and revision of the lesson plan based on the researcher's suggestions. These phases were conducted before the instruction. The planning process of Teacher A was as in the following figure:

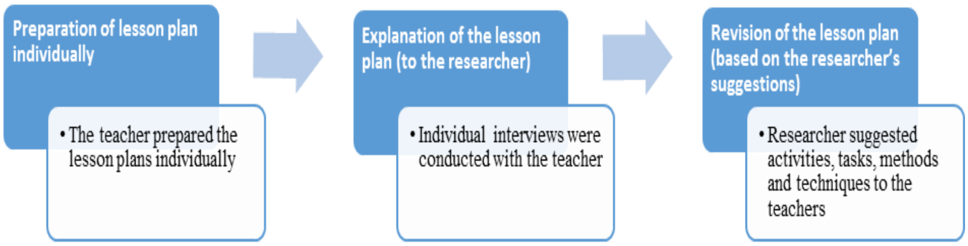


Figure 4 The process of planning

As seen in the figure, Teacher A prepared her lesson plan as written document for the instructions individually. Then, the individual interview with Teacher B was conducted to explain of her lesson plan to the researcher. Last, the researcher and two teachers came together, and the researcher suggested activities, tasks, methods and techniques about the topics to the teachers. The teachers selected one of the questions or activities that they wanted to use in the instruction and added them to their lesson plan, and revised the lesson plans with these changes. It is important to note that the extracted knowledge types from planning is also examined with reflecting in the instruction and evaluated together to conclude Teacher A's knowledge for teaching the algebra topics.

4.1.1.1. Planning for Teaching Generalization of Patterns

The planning process of teaching generalization of patterns included the lesson plans that the teacher used for the instruction, and the teacher's responses and anticipations in the interview about preparing the lesson plan. Teacher A explained the structure of her lesson plan with rationale and stated anticipations about students' thinking throughout the instruction.

Teacher A planned the lesson with respect to the objective for teaching generalization of patterns in the curriculum as "Students should be able to represent the relationship in the number patterns with letters by modelling the pattern". This objective is under patterns and relations sub-learning domain and algebra learning area. The teacher's knowledge of objective that belonged to content and curriculum for designing lesson was essential (KCC1+). She indicated integers, operations with integers, and getting the generalization rule as the prior concepts and knowledge that students should have for learning pattern generalization. She allowed five lesson hours for the instruction in general. She did not detail the organization of time for what she would do during the instruction. For the instruction, she planned to use question-answer technique and direct instruction method. As materials, she stated to use matchsticks to represent the patterns in activities.

Teacher A made explanation for the lesson plan following the order of it. Considering the objective at the beginning of the lesson plan, Teacher A stated that teaching of patterns began at 2nd grade with counting rhythmically and then pictorial patterns, and continuing generalization of patterns algebraically at 6th and 7th grades based on the objectives in the curriculum:

> A: The students learn the patterns in pictorial forms.
> Researcher: Then, do not they write the general rule algebraically, do they?
> A: Algebraic expressions are taught at 6th grade. Even, the teaching of patterns begins at 2th grade simply.
> Researcher: Can counting rhythmically be a pattern?
> A: This is also a pattern. 7th graders will learn linear and non-linear growth pattern in addition to the pattern knowledge in lower grades.

In this situation, Teacher A's knowledge of curriculum appeared when she emphasized what was taught about patterns in lower grades and judged of how to utilize the patterns at 7th grade in this lesson planning. Thus, she planned to remind the students what they learnt about patterns in previous year by asking to the students "what do you know and remember about patterns?" with aiming to reveal students' prior knowledge, and then to teach generalization of non-linear growth patterns. She planned to recall what was learnt about patterns at 5th grade, and to tell them linear and non-linear patterns they would learn in addition to their existing knowledge about patterns. Teacher A judged her knowledge of curriculum between grades to organize instruction of generalization of patterns. To do this, it is important to note that the teacher used the knowledge to present patterns with recalling students' prior knowledge, and emphasize what was learnt in previous grades and what would be learnt in this grade (KCC2+).

Teacher A planned to teach the generalization of patterns as she sequenced that first was linear growth patterns and then non-linear growth patterns. To do this, she chose examples and activities to start and to take the students deeper into the pattern generalization appropriately (KCT1+). She planned to start with teaching the generalization in linear growth pattern even though it was taught at 6th grade. She believed that if the students conceptualize the generalization of linear pattern, they can generalize the non-linear patterns easily. With respect to this, she sequenced the content for the instruction to be effective as she thought. She used her knowledge in organizing the contents in a reasonable way that the knowledge of generalization of linear pattern was a prerequisite concept for learning generalization of non-linear patterns (KCT2+). To start the instruction, she planned to begin with simple linear growth figural patterns that they could be represented with numbers as 5, 10, 15… or 4, 8, 12… from the book "Elementary and Middle School Mathematics" by Van de walle et al. (2013, p. 269). Teacher A aimed to remind the pattern concept and make the students feel the need of a relationship with these examples as introduction to the topic. The examples were linear growth figural patterns that their general rules were 3n, 5n, and 4n, and could be written using variable only and not require any constants. Thus, they were simple examples and appropriate for the grade level and beginning of the instruction. One of the examples for linear patterns that Teacher A planned to give from the book as in the figure:

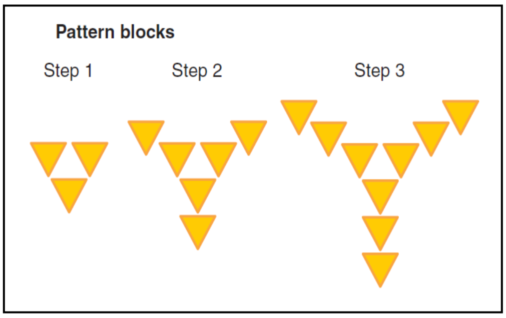


Figure 5 The example for linear growth figural pattern (Van de walle et al., 2013, p. 269)

The beginning pattern examples had the first three steps (as seen in the figure for one example), and Teacher A planned to ask students to work in pairs or small groups to represent the number of units in figures in these patterns as numerical within a table. After using the table, Teacher A planned to draw a graph with respect to the position number and the number of triangles in figures. She planned to show other three patterns with the same procedures as tabular and graphical representation. The teacher's knowledge to link tabular and graphical representations to underlie the relationship in pattern to conceptualize generalization would be effective as she planned to use multiple representations and connect them to represent the concept of general rule (SCK2+). She planned to complete the recalling process with these examples and finish the introduction part of the lesson plan.

In the middle of the lesson plan, she planned to focus on generalization patterns algebraically and she planned to do three pattern generalization activities to take the students deeper into the content. The two activities had linear growth patterns and it was asked to represent the pattern with a table and a graph, and explore the relationship between the position number and the number of units in figures. The first pattern was a linear figural pattern constructed with matchsticks and its terms were presented as 3, 5, 7, 9 … with numbers. This pattern was different from the beginning patterns as its general rule was 2n+1 and formed with a variable (2n) and constant (+1). Although this example was a bit more difficult to get the general rule than the starting activities, it was appropriate for the sequence of the activities and the level of the students in this grade regarding the objective of the curriculum. Teacher A planned to represent the number of matchsticks in the figures in a table and then with a graph. Her knowledge appeared in using tabular and graphical representations and linking them once more that she planned to teach pattern generalization with providing these connections throughout the lesson (SCK2+).

The second activity was similar to the previous pattern activities. It was asked to generalize the linear growth numerical pattern and also to find the 20^{th} term particularly. In the activity, it was asked to model the terms (3, 6, 9, 12 …) with matchsticks first and to fill the blanks in the given table as in Table 4:

Table 4 The table to represent the figural pattern with numbers

The position number	The number of matchsticks	The relation between the position number and the number of matchsticks
1	3	3x1
2	…	3x2
3	…	…
4	…	…
:	:	:
n	…	…

The activity had three questions: 1) explain the relationship between the position number and the number of matchsticks, 2) find how many matchsticks are used for the 20^{th} term, 3) find the 2^{nd}, 5^{th}, and 8^{th} term of the pattern whose general rule is 6n-2. In the activity, it was asked to find 20^{th} term after exploring the relationship. Asking the 20^{th} term might be more appropriate before getting the relationship since it also could provide to think on the need of a general rule to find the other terms. Besides this, the third question was related to another pattern (6n-2), but it was used in the context of this activity. Thus, this question was not appropriate since it was not related to 3, 6, 9, 12 … pattern. The teacher did not indicate how she would implement this activity in lesson plan. However, changing the sequence of the first two activities could be more appropriate that generalizing 3, 6, 9, 12 ... pattern before 3, 5, 7, 9 ... pattern was more reasonable regarding the difficulty of them. Because generalizing the pattern as 3n could be easier for students than getting 2n+1 for the reason that using a constant in general rule might be difficult to understand. Besides, 3, 6, 9, 12 ... pattern was similar to the examples in recalling phase. Thus, the order of generalization of these patterns should be changed to provide the development of students' understanding.

The third activity called "Modeling pattern" was non-linear growth figural pattern with formed unit cubes. It was asked to show the terms in a table and to generalize the relationship algebraically. The pattern was given as 2, 6, 12, 20 … with figures to the fourth term. First, it was asked to continue the pattern to the next two steps with numbers and figures. In the second part of the activity, it was asked to fill the blanks in the Table ***5*** 5 and write the term for the n^{th} number algebraically.

Table 5 The table to represent the figural pattern with numbers

The position number	The number of unit cubes	The relation between the position number and the number of matchsticks	
		1^{st} option	2^{nd} option
1	2	1.(1+1)	1^2+1
2	6	2.(2+1)	2^2+2
3	12	3.(3+1)	3^2+3
4	20	4.(4+1)	4^2+4
.	.	.	.
.	.	.	.
.	.	.	.
n		…?	…?

Presenting two options to show the relationship of the pattern could be useful for the students' understanding. Especially, the 2^{nd} option was related to exponential number that the students had learnt before the pattern generalization could provide students to connect the topics. It was asked to express the written numerical relationships verbally in the third part of the activity. However, Teacher A included this activity to the lesson plan since it was in the textbook and it exemplified a non-linear pattern. She did not make any explanations about the reason of selecting it and how she would implement this activity. She planned to ask the questions in the activity and wanted the students to answer them. This was the last activity in the lesson plan and this pattern was the only non-linear growth pattern as an example. This non-linear growth pattern was generalized as n.(n+1) or n^2+1 and this expression required exponential forms of the variable (n^2) and also a constant (+1). Thus, this example could be more difficult than linear growth patterns for students and the teacher gave the table as seen above to the student instead of asking them to draw it by herself. In this regard, Teacher A sequenced the examples and activities to teach pattern generalization from simple to complex ones using her knowledge and they also were appropriate for the level of the students (KCT1+).

Although Teacher A stated the requirement of teaching non-linear growth pattern for 7th graders, she also explained students' difficulty in generalizing patterns algebraically based on the previous experiences. To facilitate understanding the relationships in patterns, she planned to make links among representations to give the ideas about the relationships in the patterns by representing figural pattern with numbers in a table and showing the slope in graphic:

> A: I will ask the students to be a group of 2-3 people, and to examine the pattern and answer the questions. I will want them to represent the figures in pictorial pattern as numbers in a table. Then, there is also the graph of function.
> Researcher: What is the graph of function?
> A: For example, it is about how the terms in the patterns change regarding their position numbers. The change of the number of triangles with respect to the position number. But I did not examine for this pattern (in the Figure 5) how I can use the graph of function. I will look before the lesson.

In this situation, Teacher A used her knowledge to link representations among figural, tabular, and graphical representation to underlying the relationship between the position number and the number of triangles as functional thinking (SCK2+). Especially, she emphasized that the graphical representation could be useful for students to show the change of the terms regarding their position numbers and she chose to use it for improving students' learning of the relationship in patterns. However, her knowledge of particular representation for learning the relationships between input and output values in pattern based on functional thinking appeared inadequately (KCT3-) as she could not explain the use of this representation for the pattern in the script above.

Teacher A also indicated the difference between linear and non-linear pattern pointing out these last two activities:

> A: In 3, 6, 9 ... pattern, the difference between terms is constant, that is 3. Additionally, there is an example for non-linear pattern in the textbook. The "Modelling the pattern" activity is an example. The terms of this pattern goes 2, 6, 12, 20 ....

She stated the difference was constant in linear growth patterns, while she was giving Modeling Activity pattern as an example for the non-linear growth pattern. The teacher A had the knowledge of the patterns types and their properties and used technical language with calling the name of the patterns as linear growth pattern and non-linear growth pattern correctly by giving examples (CCK1+). At this point, she expressed her concerns and anticipations about students' difficulties to find the general rule of patterns:

> A: It is too difficult for students to identify the relationship. Students have difficulty with writing the general rule of the pattern. For example, in 3, 5, 7 … pattern, they can say "it increases by 2". But, they have difficulty with writing the general rule algebraically. … The generalization in non-linear patterns is more difficult for students. Even though I give the figures for representing the pattern, it is difficult to understand the growth of the figures for students… Indeed, the students were taught patterns and generalization of patterns at 6th grade. But, I cannot know how they learnt … Perhaps, if the students see the numbers one under the other in a table and then in a form like that for example 3.1, 3.2, 3.3 … for 3, 6, 9… pattern, they can write the general rule in algebraic form easier. …but this activity can be extra. If the students model the patterns by using cubes or matchsticks themselves, they can see the relationship easily. It can be better to model by themselves.

In her explanations, her knowledge to anticipate where and how students had difficulty in generalizing pattern algebraically appeared (KCS1+). She explained that although students expressed the relationship between the terms as difference (it increased by 2) verbally, they could not represent the relationship algebraically. She especially emphasized students' difficulty of generalizing non-linear pattern. To support students' understanding, she suggested using figural patterns, tabular representation, and manipulatives for modelling the pattern. With respect to the suggestions, she predicted about students' thinking and her knowledge about what the students would find interesting and motivating for understanding the relationship in the pattern to improve their understanding was appeared appropriately (KCS6+). She explained that using figural pattern could help the students to show the growth between figures. In a table, students could see the arithmetical relationship such as 3x1, 3x2, 3x3 ... one under the other in rows and it could guide students to write the general rule algebraically easier by recognizing what changes and where n should be written. The other suggestion that she proposed was using manipulatives such as cubes and matchsticks, and modeling the pattern by students. Teacher A thought modeling would be better for students. She made more explanations on using figures and modeling to represent the pattern:

> A: The models can be used, but it is important that how students investigate the relationship and what perspective they look the models or figures. At that point, the teacher should guide the students to investigate the increment between the figures. My lecturer marked the extra added ones in figure… but I cannot remember now, I should examine it before the instruction… If the manipulatives are given only, and the students are asked to find the relationship by themselves, it will not be useful. Modeling should support with other representations. After modeling, the translation of the terms to graph, and then to table is also important.

Teacher A gave emphasis to the use of figures and manipulatives to represent the pattern effectively while teaching generalization patterns. She noted the importance of guiding students for where and how they should look these models to explore the increment or growth to get the relationship. However, she could not explain more how to use the manipulatives concretely or figures in figural pattern because she could not remember. She could not have adequate knowledge about using the representations or models for guiding students to explore the relationship in figural patterns (SCK7-). She also thought that these figural representations should be supported with tabular as numerical and graphical representations (SCK2+).

After Teacher A expressed her anticipations for students' possible difficulties, she continued to tell about the flow of the planning. She stated giving the definition of general term after the generalization of 3, 5, 7, 9 … pattern algebraically, and the following two patterns that were 3, 6, 9 ... as linear growth pattern and 2, 6, 12, 20 …

as non-linear growth pattern in the lesson plan. The definition was from the textbook and Teacher A selected to give it in the lesson. This definition was:

> ‘n’ is the letter that is used to represent the general rule of the pattern. This is a sign, notation, symbol that indicates the position number of the numbers in the pattern. Thus, n is called as n^{th} term, the representative number, or the general term of the pattern. This n is a variable.

This definition was from the textbook and the teacher selected to give it to the students. Thus, her knowledge about usable definition appeared as merely choosing it from the textbook (SCK3⊥). This definition included the function of n as general term that it was a letter and was used to represent general rule. It is important to note that “n” was defined as a variable, since it varied with respect to the position number. The teacher wanted to give this definition that the textbook presented it and the teacher tried to present in the instruction what the textbook gave.

In general, Teacher A planned to teach the generalization of patterns as she sequenced that first was linear growth patterns and then non-linear growth patterns. To start the instruction, she planned to begin with simple linear growth figural patterns and then, she planned to focus on generalization patterns algebraically and she planned to do three pattern generalization as activities to take the students deeper into the content. In this regard, Teacher A sequenced the examples and activities to teach pattern generalization from simple to complex ones and they also were appropriate for the level of the students with her knowledge (KCT1+). However, she did not state the answers of the questions in the lesson plan and any expected answers that the students would give. However, involving the expected answers could help the teacher to prepare herself how she would handle or overcome the possible situations. This preparation also could prevent the waste of the time from the instruction. Besides, she did not determine the homework and assessment questions from textbook and workbook and so she did not indicate them in the lesson plan.

After the interview with the teacher, revision was the part where the teachers made the final version of their lesson plans with suggestions of the researcher. For the suggestions, the setting for pattern generalization was explained in Appendix B.

Revision included adding new examples, activities and acknowledging suggested method or techniques to lesson plans with the aim of supporting students' understanding. Teacher A decided to add the pattern task and revised the middle part of her lesson plan where she asked students to generalize several patterns from different resources. The implementation of the pattern test based on the teacher's knowledge was explained in the *implementing the suggested activities* practice in the instruction section. One of the suggested activities was dot patterns that she did not prefer to use it since her knowledge of objectives as grade level, both 7^{th} and 8^{th} grade, of the curriculum appeared appropriately without involving these activities (KCC1+).

4.1.1.2. The Extracted Knowledge Types from Planning for Generalization of Pattern

Table 6 The extracted knowledge types from planning for generalization of patterns

SMK		PCK		
CCK	SCK	KCS	KCT	KCC
CCK1(+)	SCK2(+,+,+) SCK3(⊥) SCK7(-)	KCS1(+) KCS6(+)	KCT1(+,+) KCT2(+) KCT3(-)	KCC1(+,+) KCC2(+)

Table 6 shows what type of knowledge of subject matter knowledge (SMK) and pedagogical content knowledge (PCK) that Teacher A had in planning phases. (+) sign indicates the teacher's existing knowledge was adequate or appropriate, while (-) sign indicates the teacher's existing knowledge was inadequate or inappropriate. Each sign (+ or -) in the same knowledge type refers to Teacher A's different intention of use this knowledge during planning. Thus, the knowledge which refers to same intention of use the knowledge was not presented in the table. Besides, for SCK3 knowledge type, (⊥) is used to indicate the teacher's knowledge to develop definition or explanation appeared as merely choosing it from the textbook and presenting to the students. Since this situation meets the code (SCK3) partially, (┴) sign is used.

In general, CCK1(+) indicates that her knowledge of the patterns types and their properties, and her knowledge to use of technical language with calling the name of the patterns as linear growth pattern and non-linear growth pattern correctly by giving examples was adequate. SCK2 indicates that Teacher A’s knowledge to link tabular (SCK2(+,+)) and graphical representations (SCK2(+) to underlie the relationship of pattern to conceptualize generalization was appropriate in order to use multiple representations and connect them to represent the concept of general rule. However, SCK7(-) indicates that her knowledge about using the representations or models for guiding the students to explore the relationship in figural patterns was inadequate that, she could not explain how to use the manipulatives concretely or figures in figural pattern. Besides, she emphasized that the graphical representation could be useful for students to show the change of the terms, but she could not explain the use of this representation for the pattern. Her knowledge of particular representation for learning the relationships between input and output values in pattern based on functional thinking appeared inadequately (KCT3(-)).

Related with the knowledge of students’ thinking, KCS1(+) and KCS6(+) indicates that her knowledge to anticipate the students had difficulty in generalizing pattern algebraically, and her knowledge to predict using figural patterns that the students would find interesting and motivating for understanding the relationship in the pattern was appeared appropriately. KCT1(+,+) indicates her knowledge to choose examples and activities to start with simple linear growth figural patterns was appropriate; and her knowledge to choose which examples and activities to take the students deeper into the pattern generalization with linear growth patterns and non-linear growth patterns as simple examples to complex ones was appropriate. In the curriculum perspective, KCC is related with curriculum knowledge and KCC1(+,+) indicates that her knowledge of objective that belonged to content and curriculum for designing lesson was essential and adequate. Besides, KCC2(+) indicates that her knowledge to judge and present the instruction appeared adequately with emphasizing counting rhythmically and then pictorial patterns in lower grades, and then generalization of patterns algebraically at 6^{th} and 7^{th}.

4.1.1.3. Planning for Teaching Operations with Algebraic Expressions

The planning process of operations with algebraic expressions included the lesson plans that the teacher used for the instruction and the teacher's responses and anticipations about preparing the lesson plan in the interview. Teacher A explained the structure of her lesson plan with rationale and stated anticipations about students' thinking while teaching.

Teacher A planned the lesson with respect to the objective for operations with algebraic expressions in the curriculum as "Students should be able to add and subtract algebraic expressions and students should be able to multiply two algebraic expressions". These objectives are under algebraic expressions sub-learning domain and algebra learning area. The teacher's knowledge of objective that belonged to content and curriculum for designing lesson was essential (KCC1+). Teacher A planned the lessons with respect to the order of the objectives in the curriculum. Thus, she planned to teach addition and subtraction of algebraic expressions first, and then multiplication of algebraic expressions. She indicated that variable, and addition and subtraction with integers are the prior concepts and knowledge that students should have for learning operations with algebraic expressions. She allowed five lesson hours for the instruction in general. She did not detail the organization of time for what she would do during the instruction. For the instruction, she planned to use question-answer technique, discussion and direct instruction method. As materials, she stated to use algebra tiles to model algebraic expressions.

Teacher A gave a general overview about how she designed the lessons at the beginning of the interview. First, she explained that she planned to remind term and coefficient concepts as prior knowledge with the *Beginning Activity*. She planned to recall term, coefficient, and unknown concepts from 6^{th} grade with this activity. Teacher A judged her knowledge of curriculum between grades to organize instruction of operations with algebraic expressions. To do this, it is important to note that the teacher used her knowledge to present the instruction with recalling students' prior knowledge, and emphasize what was learnt in previous grades (KCC2+). She planned to do the following activity in the figure:

Activity: There are 2 eggs and 4 olives in one of the plates, and there are 1 egg and 6 olives in the other plate. In total;
How many eggs are there?
How many olives are there?
Can you add olives and eggs?
Can you subtract eggs from olives?

Figure 6 Beginning Activity

This activity was from 6th grade textbook and its purpose was to make the feel of the addition or subtraction of similar foods in the plates. However, the teacher indicated to involve this activity in order to remind the concepts of term, coefficient, and unknown with questioning of why these concepts were needed and what they were. But, the information in the activity was known, it would be troublesome to represent the known situations with algebra concepts. She did not state any explanations about the implementation of the activity. After this activity, she planned to explain the concepts about algebraic expressions with examining in 8t+3, and 9x-7 expressions. To exemplify, she indicated in her lesson plan as 8t is a term, +3 is a constant, 8 is a coefficient, and t is variable (unknown). Her knowledge to connect the topic being taught to algebraic expression from prior year appeared appropriately as she planned to remind the concepts of term, constant, coefficient, and variable with using these examples (SCK1+). Beginning with the examples was appropriate since the students had learnt them in previous year and they would remember them easily. However, it could be more appropriate to change the sequence of the beginning activity and the examples since the first activity was about like term concept and it could provide the intuition of like term for learning of the addition and subtraction operations for the middle of the lesson. Thus, her knowledge to choose which examples to start appeared inefficiently at this point (KCT1-) as the examples were appropriate, but the order of activity and examples should be changed to provide the development of students' understanding. Then, Teacher A would ask the students to write the definitions of term, constant term, and coefficient concepts in their notebooks. The definitions were:

> Each of the addends that form the algebraic expressions is called term. The terms that do not have variable are constant term, the number that is written as factor before the variable is called as coefficient.

The definitions were the textbook definition and the teacher selected to give them to the students. Thus, her knowledge about usable definition appeared as merely choosing it from the textbook (SCK3⊥). The definitions included the explanations that term was described as each addend of the algebraic expression first, and constant term was defined as without variable using the term concept, and coefficient was called the number that was multiplied with the variable. The only thing might be a problem that was the use of addend concept. Because it could make the students to think that subtraction was ignored, and only addition was investigated to determine the term in the expressions. The teacher wanted to give this definition in order to remind the students what the concepts were.

In the middle of the lesson, she planned to teach "like term" concept to provide the students to connect the addition and subtraction of algebraic expressions. To do this, she planned to ask the following question from 6^{th} grade textbook to make the feel of the concept of like term:

> Question1 (Q1): There are 6 chickens and 2 cocks in one of the coops, and there are 4 chickens and 1 cock in the other coop. When we get them together in a third coop, write the algebraic expressions to represent them in this coop.

This question was similar to the first activity where she planned to ask the students to represent the number of eggs and olives as algebraically. Teacher A aimed with this question to show the number of these animals algebraically and then of how like terms were added when the two coops were gathered. To illustrate, the first coop was represented as 6t+2h; the second coop was represented as 4t+h; and lastly the third coop was represented as 10t+3h when the first two coops were got together (t represents chicken; h represents cock). However, as in the *Beginning Activity*, the number of animals was known, thus it would be troublesome to represent the known situations with algebra concepts.

Up to now, she planned to give the idea of like term with Beginning Activity and Q1 in a similar way. After these activities, she planned to define "like term" as in the textbook by using the addition and subtraction of algebraic expressions concepts. This explanation was:

> Like terms are the terms that have the variables with same or different coefficients in an algebraic expression. The coefficients of like terms are added or subtracted for addition and subtraction of algebraic expressions.

This definition was the textbook definition and the teacher selected to give it to the students. Thus, her knowledge about usable definition appeared as merely choosing it from the textbook once again (SCK3⊥). This definition included the explanations for like terms that had variables with same or different coefficients. However, this definition could be inadequate that x and x^2 are the expressions that have same variable but they are unlike terms. Thus, it should have been stated that the like terms had the same variable with the same exponential forms. Teacher A planned to use it for the reason that the textbook presented, but she did not examine it and think that it might cause any misunderstandings.

Afterwards, she indicated to continue the lesson with modeling the addition of algebraic expressions. While she was explaining the design of the lesson plan, she emphasized the use of algebra tiles also for subtraction and multiplication. When the researcher asked why she thought that the use of algebra tiles was useful for the students, she made following explanations:

> A: The students think that x is an abstract concept. What is x? What is 3x, 3x-2? When they model them with algebra tiles, they can understand that x is not a frightening thing. Even, 8^{th} graders have a fear about using algebra. Using modelling with tiles can provide the algebraic expression to be more concrete. The tiles also will be yellow. The students can understand that the same colored tiles can come together. They are visual materials and the students can visualize easily.

In this situation, Teacher A explained that the visualization of using algebra tiles in modelling, and the same colored tiles could assist the students to conceptualize the addition of algebraic expressions. Since the students had difficulty with the conceptualization of the variable concept, and related with addition of algebraic expressions, the teacher suggested that the visualization of abstract concepts with tiles could support students' understanding by using her knowledge to understand the needs and difficulties of students with working on this topic (KCS3+). The first example to model with algebra tiles was (3x-2)+(2x+6) in her lesson plan. While modeling, she planned to represent the expressions separately first using the tiles as in the following figure:

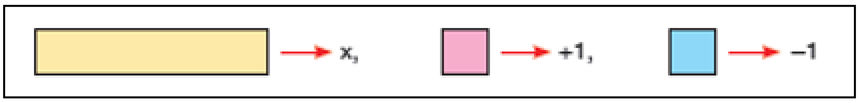


Figure 7 The representation of x, +1, and -1 with tiles

She explained the implementation of the modeling that she indicated to group the variables (3x and 2x), and constant terms (-2 and +6) separately to add the expressions and to get the result as 5x+4 in her lesson plan. Then, she planned to exemplify the subtraction of algebraic expressions. For teaching subtraction, she would use the same procedure as teaching addition that she planned to ask a question (see Q2) similar to Q1 first, and then use the modeling for (5x-3)-(2x-2) with algebra tiles.

Q2: A farmer had 3 cows and 9 sheep. The farmer sold 1 cow and 5 sheep. Write algebraic expression to represent the rest of the cows and sheep.

In general, Teacher A designed the instruction of addition and subtraction with algebraic expressions with beginning with the questions (Q1 and Q2) that had contexts required representing of given information algebraically and then doing operations with using the algebraic representations. The questions could be useful to teach the requirement of like terms to add and subtract, however representing the known situations with using variable next to the number of items (e.g. 6 chickens = 6t, 4 cocks= 4h) might be troublesome that the numbers of them were known and it was not needed to use variable. After solving these questions, she planned to model addition and subtraction operations with using algebra tiles. Teacher A's knowledge to choose which examples to use take into the students deeper into the content appeared appropriately (KCT1+) as she planned to connect the like term concept with real life situations first and then to use models to explain mathematical point that was how the like terms were added or subtracted. Thus, the sequence of the examples and activities to teach addition and subtraction with algebraic expressions were appropriate for the development of students' understanding.

For the second objective of topic as multiplication with algebraic expressions, she designed the lesson as in the instruction of addition and subtraction. First, presenting an activity that had a context about saving money, and then modeling several multiplication operations. The activity called as *Nermin's Money* was in the textbook and it was required to write algebraic expressions of verbal statements, and the multiplication of a number and an algebraic expression. This activity was as in the following:

> Nermin saves money from her allowance and has some in her moneybox. Nermin puts 5TL that her mother gave to her into her moneybox. Her father said that he would give money that 2 times of the saved money. Write the algebraic expression that Nermin would take from her father and explain how you would write it.

This question had a real life context and could connect the multiplication. It required to multiply 2 and n+5, which could be a simple example for the beginning of multiplication. Thus this activity was appropriate for the students as her knowledge to choose which examples to start appeared at this point (KCT1+). Later on, Teacher A planned to ask the following question to the students: "Write the algebraic expression that shows the perimeter of the below rectangle (Figure 8):

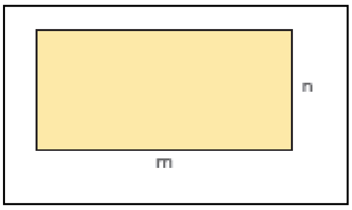


Figure 8 The rectangle with m and n length sides

Teacher A's aim for asking this question was to connect the repeated addition with multiplication, thus she planned to show 2m+2n as the multiplication of 2 and the total length of sides of the rectangle as 2.(m+n). This question might be useful for the connection with addition instead of directly using multiplication procedures. Teacher A's knowledge connect the topic to addition as previous topic appeared effectively as she used the concept of perimeter of rectangle (SCK1+). After this example, Teacher A planned to model 4x and 3.(x+2) with algebra tiles. She explained the use of tiles as in the textbook. The textbook presented the modeling with showing 4 items of x tiles and showed the rectangle with 4 and x length sides. For other modeling, the textbook showed the rectangle with 3 and x+2 length sides and presented the area of it. The teacher would model as in the textbook that she did not make more explanations for these examples. Then, she planned to ask what (4m-3)+(4m-3) was. This question was similar to the question that was asked the perimeter of the rectangle as it also provided the connection of repeated addition with multiplication. Since this question was similar to the perimeter of the rectangle, it could be asked after it and the sequence could be changed as before the modeling activities. This sequence could be more appropriate as connecting with addition for multiplication first could provide to understand the underlying idea of multiplication, and then modeling with tiles of multiplication for the development of students' understanding. Thus, the place of this question should be changed as after the rectangle example. As a different example, she planned to ask the students to write the expression whose model was given as in the Figure 9:

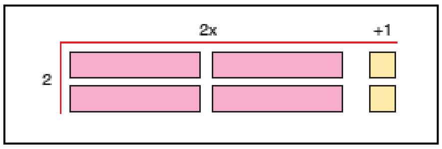


Figure 9 The representation of modeling of 2.(2x+1)

The teacher would expect the students to write 2.(2x+1) and it was inverse procedure of the modeling of given multiplied expressions.

She especially emphasized the use of algebra tiles to teach operations with algebraic expressions in the interview. However, she also indicated that the students would have difficulty with multiplication of algebraic expressions and the use of algebra tiles in the instruction. She expressed her concerns and anticipations about students' difficulties in applying the distributive property in multiplication and modeling with algebra tiles for multiplication:

> A: The students can understand easily addition and subtraction by modelling with tiles. Since they see the similar ones and add them to find the number of them. But, they have difficulty with the multiplication, especially while using distributive property such as 2.(3x+2). The students can get confused addition with multiplication. Perhaps, they can understand it by modelling with tiles. But they have difficulty using the tiles, also.

In this situation, Teacher A indicated that the students had difficulty in using distributive property in multiplications of algebraic expressions with the reason for confusing the addition and multiplication, although they could do addition and subtraction operations easily. She suggested the use of algebra tiles to overcome this difficulty, but she also considered that the students had difficulty with it as her knowledge to anticipate where and how the students had difficulty about application of distribution property appeared appropriately (KCS1+).

Therewith, when the researcher asked what possible misconceptions the students had while working on this topic, the teacher made explanation with focusing on 4x as an example as in the following script:

> A: For example, multiply 4 by x, it is 4x. The students can ask if a number substitute for x as considering 4x two-digit number. Typical error. They have difficulty with these types of representations of multiplications. However,

> they might understand the meaning of x as an unknown by modelling with tiles. Besides, I had also difficulty with teaching 4x as multiplication.

In this situation, Teacher A stated that the misconception about multiplication expressions such as 4x and explained that the students could consider 4x as two-digit number and they could not perceive as multiplication 4 by x. Her knowledge of common students' misconceptions about algebraic expressions appeared appropriately (KCS5+). She suggested using modelling with tiles to overcome this misconception. When she was asked how the tiles could provide the students to link to multiplication idea, she explained as in the script:

> A: I will make these tiles by myself or I will want the students to prepare them. They can make with using colorful papers. These tiles modelled the multiplication of algebraic expressions. For example, 4x, suppose this  represents "x". How many x are there? Let's add. We put four of them side by side. Then, I connect like this, the shortcut is 4.x of this addition operation. We can do like this.

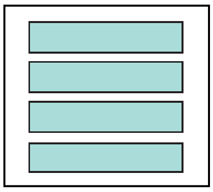


Figure 10 The representation of 4x with algebra tiles

Teacher A explained that the algebra tiles would provide the demonstration of the repeated addition by putting 4 tiles side by side. She planned to link the representations for multiplication operation with using algebra tiles in this way. However, she did not indicate the concept of area calculation in this representation to link algebraic and geometric representation since her knowledge to choose, make and use the algebra tiles was appeared inadequately. Focusing on repeated addition in this representation could not have provided this link to underlie the multiplication idea (SCK2-). Lastly, the teacher planned to ask the multiplications from the textbook exercises about the application of distributive property (see Figure 11). She did not indicate how she applied this property especially.

a. $5 \cdot y = 5y$	b. $7 \cdot (x + 1) = 7x + 7$
c. $12 \cdot (5x + 6) = 60x + 72$	ç. $6 \cdot (2x + 5) = 12x + 30$
d. $7 \cdot (3x + 2y + 4) = 21x + 14y + 28$	e. $8 \cdot (x + 2t + 3) = 8x + 16t + 24$

Figure 11 The multiplication exercises (6th grade textbook, Sevgi Publications, p. 192)

In general, Teacher A designed the instruction of multiplication with algebraic expressions by beginning with *Nermin's Money Activity* that had contexts required representing of given information algebraically and then doing operations with using the algebraic representations. Then, she planned to model multiplication operation with using algebra tiles. Teacher A's knowledge to choose which examples to use take into the students deeper into the content appeared effectively (KCT1). Since she planned to connect multiplication concept as 2 times of a money with real life situations first, and then to connect repeated addition with multiplication, and then to use models to explain mathematical point that was how the algebraic expression were multiplied based on the area of rectangle. Thus, the sequence of the examples and activities to teach multiplication with algebraic expressions was appropriate for the development of students' understanding.

After the interview with the teacher, revision was the part where the teachers made the final version of their lesson plans with suggestions of the researcher. For the suggestions, the setting for operations with algebraic expressions was explained in Appendix C. Revision included adding new examples, activities and acknowledging suggested method or techniques to lesson plans with the aim of supporting students' understanding. Teacher A appreciated and included the suggested activities by the researcher to revise her lesson plan. The implementation of the suggested activities was explained in *the implementing of the suggested activities practice* in the instruction section. Beside the activities, the researcher suggested connecting arithmetic with algebra for teaching the properties of addition and multiplication properties in algebraic expressions as a method, and using algebra tiles as manipulatives. Teacher B would use algebra tiles as she indicated in her lesson plan and appreciate to connect between arithmetic and algebra while teaching as a suggestion.

4.1.1.4. The Extracted Knowledge Types from Planning for Operations with Algebraic Expressions

Table 7 The extracted knowledge types from planning for operations with algebraic expressions

SMK		PCK		
CCK	SCK	KCS	KCT	KCC
	SCK1(+,+) SCK2(-) SCK3(⊥,⊥)	KCS1(+) KCS3(+) KCS5(+)	KCT1(-,+,+,+)	KCC1(+) KCC2(+)

Table 7 shows what type of knowledge of subject matter knowledge (SMK) and pedagogical content knowledge (PCK) that Teacher A had in planning phases. (+) sign indicates the teacher's existing knowledge was adequate or appropriate, while (-) sign indicates the teacher's existing knowledge was inadequate or inappropriate. Each sign (+ or -) in the same knowledge type refers to Teacher A's different intention of use this knowledge during planning. Thus, the knowledge which refers to same intention of use the knowledge was not presented in the table. Besides, for SCK3 knowledge type, (⊥) is used to indicate the teacher's knowledge to develop definition or explanation appeared as merely choosing it from the textbook and presenting to the students. Since this situation meets the code (SCK3) partially, (⊥) sign is used.

In general, SCK1(+,+) indicates that her knowledge to connect the topic with reminding the concepts of term, constant, coefficient, and variable with using these examples for teaching addition and subtraction, and to connect repeated addition with multiplication as previous topic appeared effectively. To remind the algebraic concepts, SCK3(⊥,⊥) indicates that her knowledge about usable definition of term, constant term, coefficient, and like term appeared as merely choosing it from the textbook. Especially, her knowledge to choose, make and use the algebra tiles as representation was appeared inadequately since focusing on repeated addition in using of algebra tiles could not have provided the link among algebraic and geometric representation to underlie the multiplication idea (SCK2(-)). However, KCS1(+) and KCS3(+) indicate that her knowledge about students' difficulty in working on variable concept related with addition of algebraic expressions and application of distribution property in learning multiplication was appropriate as she suggested using algebra tiles to overcome their difficulties. Besides, KCS5(+) indicates that her knowledge of students' common misconceptions was appropriate such as the students' thinking of 4.x as two-digit number. For the instructions, KCT1(-,+) indicate that her knowledge to choose which examples to start with was inappropriate and to use take into the students deeper into the content was appropriate. KCT1(+,+) indicates that her sequence for teaching addition and subtraction was that first connecting the like term concept with real life situations and then using models to explain how the like terms were added or subtracted. On the other hand, her sequence for teaching multiplication of algebraic expressions was that first connecting multiplication concept with real life situations, and then connecting repeated addition with multiplication, and then using models how the algebraic expressions were multiplied. In the curriculum perspective, KCC1(+) indicates that her knowledge of objectives that belonged to content and curriculum for designing lesson was essential and adequate as in the curriculum and KCC2(+) indicates that her knowledge to present the instruction appropriately with recalling students' prior knowledge, and emphasize what was learnt in previous grades.

4.1.2. Instruction

The mathematical knowledge for teaching of Teacher A was extracted from her actions throughout the instruction by focusing on common patterns in observation data. Based on the common patterns in the teacher's actions that she performed throughout the instructions, the practices of Teacher A were grouped as seen in the Figure 12. The teacher's purposeful actions to teach the topics were grouped into six practices: 1) choosing an example or activity to start teaching the topic with connecting to topics from prior years, 2) discussing on the activity related to the topic, 3) choosing the examples or activities to use to take the students deeper into the topic, 4) implementing the suggested activities, 5) doing exercises related to the topic from textbook and workbook, and 6) presenting problems that combine knowledge related to other topics. The two different instructions were conducted for teaching two algebra topics. The practices of Teacher A were examined within the context of topics; generalization of patterns and operations with algebraic expressions in the following sections. It is important to note that the extracted knowledge types from instruction are also examined with planning before the instruction, and the reflections of the instruction in the post-observation interviews, and then are evaluated together to conclude Teacher A's knowledge for teaching the algebra topics.

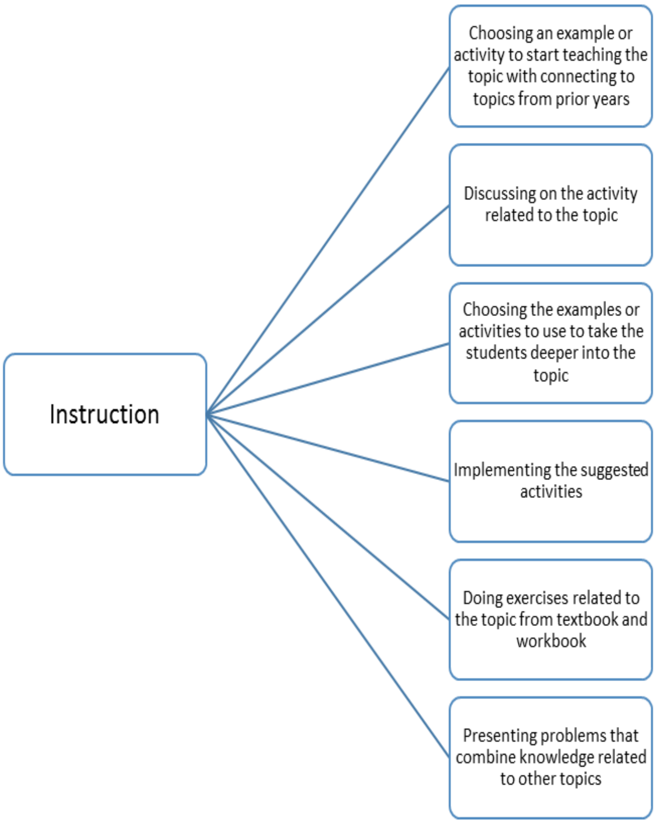


Figure 12 The practices of Teacher A during the instruction

4.1.2.1. Practices in the Instruction of Generalization of Patterns

Teacher A's purposeful actions for teaching generalization of patterns were grouped into six practices: 1) choosing an example or activity to start teaching generalization of patterns with connecting to topics from prior years, and 2) discussing on the activity related generalization of patterns, 3) choosing the examples or activities to use to take the students deeper into generalization patterns, 4) implementing the pattern test, 5) doing exercises related to generalization patterns from textbook and workbook, and 6) presenting problems that combine knowledge related to exponential numbers. The extracted teacher's knowledge based on MKT framework was analyzed within these practices. The reflection of the instruction after each lesson was also presented to provide the teacher to evaluate her instruction by herself. The interpretations of the teacher could give information about her knowledge about students' thinking and learning with respect to the instruction. The classroom dialogues that were most representative for knowledge type the teacher had, were selected from the instruction to illustrate how the teacher used her knowledge in teaching.

The first practice was choosing an example or activity to start teaching generalization of patterns with connecting to topics from prior years and this title is extracted from one of the descriptors of KCT and SCK. This practice examined that the teacher chose which example or activity to start teaching pattern generalization with rationale, and how she implemented it in the classroom. It included the teacher's recalling process for prior knowledge that students have to learn pattern generalization. To do this, the teacher first reminded the prior knowledge related with pattern concept that the students learnt in previous grades. Then, connecting with them she indicated what they would learn in this grade. For this connection, the teachers asked questions about pattern concept, such as what it was, and how it was formed. The second practice was discussing on the activity related generalization of patterns and this practice was also affected by the descriptors of KCT. This practice included a discussion for generalizing linear growth figural pattern with using its tabular representation. The teacher emphasized the generalization at this part of the lesson since it provided the first teaching of getting the general rule. She let the students to explain their answers and provided opputinities to discuss the answers if they worked for the entry pattern by encouraging the students to participate. The third practice was choosing the examples or activities to use to take the students deeper into generalization of patterns and this title also was extracted from one of the descriptors of KCT. This was also as a continuation of choosing an example or activity to start practice. This practice included how the teacher taught generalization of non-linear pattern to improve students' knowledge with getting deep the content using non-linear patterns that required the exponential formed general rule and thus it could be challenging for the students. The fourth practice was implementing the pattern test and differs from the non-linear activity with including only linear patterns. Since the teacher involved the pattern test that the researcher suggested and the teacher allocated about one lesson hour time, this part of the lesson was explained under a separate practice. The fifth practice was doing exercises related to generalization of patterns from textbook and workbook, and the teacher asked the exercises to the students as in the order of the book. This part was as questioning by the teacher and answering by the students. The sixth practice was presenting problems that combined knowledge related to exponential numbers as the teacher presented three problems from research to improve students' understanding of generalization of non-linear patterns and she had also given one example the beginning of the development. These practices are explained with examining how the teacher used her knowledge based on MKT framework in the following sections.

4.1.2.1.1. Practice One: Choosing an Example or Activity to Start Teaching Generalization of Patterns with Connecting to Topics from Prior Years

At the first lesson of the instruction, Teacher A began the instruction with asking questions about pattern concept. She asked the questions with connecting the pattern topic from prior years to recall students' knowledge about pattern. She asked students what they remembered about patterns from 6^{th} grade and 5^{th} grade, and how the pattern was defined. She used her knowledge to connect the knowledge about pattern topic between grades (SCK1+). As the students gave answers, she responded to students appropriately. When the students gave correct answers, she interpreted students' answers and made explanations with her own sentences. If the students gave answers incorrectly, she asked to the student a new question to make realize the error in their answers. Her knowledge to develop a usable definition for the pattern concept with using students' answers appeared as in the following script:

> A: We will remember the patterns first. What do you remember from 6^{th} grade and 5^{th} grade? What is the pattern?
> S: Continuing with the same figure.
> A: Does the figure not change?
> S: It changes.
> A: It can change. What is between them?
> S: Particular figures continue by forming a pattern.

S: But, the sizes of the figures can change. The figure does not change. It goes bigger or smaller.
S: It goes taller or shorter.
A: Yes, the patterns are the relations between the different figures, are not? We will do operations with numbers in patterns.

In this situation, the students responded to other students' answers to correct or complete their answers. Teacher A did not respond to the students' answers individually, but then she interpreted the answers and gave a definition. Her knowledge to develop usable definition appeared (SCK3-) and she defined the pattern as the relations between the figures, but the patterns were created also with numbers and had relationships in them. Thus, her developed definition could be inadequate and might cause lack of understanding as if patterns have always figures. At that point, she tried to connect this definition with numbers, but she stated the operations with numbers at this time. She would state using variables and algebra instead of arithmetic to increase students' awareness of what they would use for pattern generalization. As in the definition, she used the word "the relationship" instead of the pattern as if they had same meaning at some points of the flow of the lesson. However, the pattern has figures or numbers based on a relationship and her knowledge of definition of pattern appeared inappropriately (CCK1-).

Teacher A sequenced the examples for instruction as simple to more complex ones regarding students' learning. After reminding the pattern concept and giving the definition of it, she started to give the pattern examples to teach generalization. She chose to start figural patterns that could be represented with numbers such as 3, 6, 9 ..., 5, 10, 15 ..., and 4, 8, 12, 16 ... from the book entitled "Elementary and Middle School Mathematics" (Van de walle, 2013, p. 269). The general rules of these patterns were 3n, 5n, and 4n, and could be written using variable only and they did not require any constants. Her knowledge to choose these pattern examples to start appeared appropriately as the examples were simple ones and appropriate for the beginning and they supported to recall what the students learnt in prior year (KCT1+).

Teacher A generally asked the question to one student and focused on his/her answers later on throughout the instruction. She sometimes let the students to discuss the answers creating the discussion environment. For the first question in which the pattern had triangles (see Figure 5), the dialogue between Teacher A and the student was in the following script. She asked to the student what the relation could be in the terms of pattern and then she explained and justified this student's answer:

> A: How is the relationship for this pattern?
> S^*: It goes 3 by 3.
> A: Yes, add 3 here (3 to 6), add 3 here (6 to 9). It goes like this. It increases 3 by 3. What is the 4^{th} term? 5^{th} term?
> S^*: 12 and then 15.
> A: Okay, how can we find this relationship instead of counting one by one? Can we write a rule to find 20^{th} term? For example, if what the 100^{th} term is asked, do you write the terms to 100^{th} term?
> S^*: I will multiply 100 by 3.
> A: Since it increases 3 by 3, you say that you multiply 100 by 3. So, you multiply the position number by the increment. Is it true?
> S^*: Yes.
> A: For example, does it work for the 2^{th} term? The position number is 2, the increment is 3, 6. It is true. For the 3^{rd} term?
> S^*: It works.
> A: Multiply 3 by 3, 9. It works. Do you agree with your friend's idea? So, what do we do with the position number and the increment?
> Students: We will multiply.
> (S^* represents the same student)

In this situation, Teacher A's aim was guiding the students to get a rule to find the asked terms. Thus, she asked 20^{th} term, but without waiting an answer, she asked what the 100^{th} term was. In this dialogue, Teacher A generally explained the student's ideas and justified it by doing multiplication as her knowledge of how to explain and justify the student's mathematical ideas about finding 100^{th} term of the pattern appeared adequately (SCK4+). However, she did not ask other students what their ideas were and only used the student's answer to explain how the relationship could be explored. Lastly, Teacher A interpreted the meaning of this multiplication as the position number and the increment by herself without allowing the students find out it. She also exemplified this multiplication for the first three terms as in the following representation:

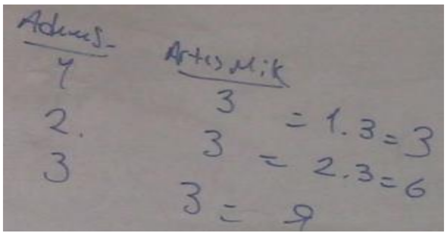


Figure 13 The representation of the multiplication of the position number and the increment by Teacher A

In this situation, Teacher A did not write multiplication symbol "x" or "." between numbers first, but she used the equal sign after the numbers and multiplied them. Her knowledge of how mathematical language was used appeared inappropriately (SCK5-) since she did not use required notation (multiplication sign) and this representation was not correct.

For the first question, while the student was answering for what the 4th and 5th term, the teacher filled the table and wrote the numbers in it. Teacher A used the table to show the relationship between terms in the patterns throughout the instruction. As in the first question, generally she wanted the students to form a table to represent the figural patterns and numerical patterns with numbers in the table. Teacher A asked what the terms were respectively and wrote these numbers in the table. She transferred the units of figures as numbers in the table and pointed out the difference between the terms in the table as seen in Figure 14:

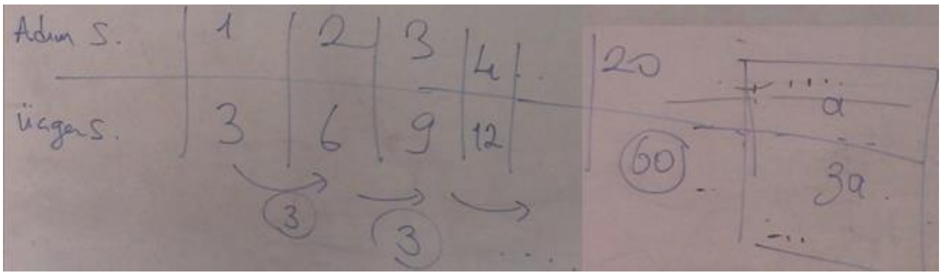


Figure 14 The tabular representation for the terms of the first pattern used by Teacher A

Her knowledge to choose, make and use the tabular representation appeared with focusing on the arithmetical relationships in tabular representation to underlie the relationship in pattern to conceptualize generalization could be effective (SCK7+). Actually, she had emphasized using graphs and linking among these representations in lesson her planning. But, she did not mention about it for this question. She asked what the 20th term and then 100th term of the pattern was in order to make feel the need of a rule to find these asked terms. After she stated the multiplication of the increment and the position number, she guided the students to write an unknown number using letters:

> A: When the position number is 20, and the increment is 3, then the term is 60. If we continue this, for example, we say, let's write an unknown number here. What can we write? We represented with letters, *a* or *b*.

She wrote *a* as the unknown number in the position number row and in the right column after the 20th term (see Figure 14). At this point, her knowledge of how mathematical language was used appeared inappropriately since she used "an unknown number" term instead of variables (SCK5-). The concept of variable is essential to understand the generalization of patterns as the general term is changed based on the position number. However, the unknown number term is used for equations, and different values cannot be written for the unknown numbers. The number which provide the equality in the equation are substituted for an unknown number. Then, she wrote *a* in the position number row and wrote 3*a* in the number of triangles row. This situation, representing the multiplication 3 and *a* as 3*a* without explaining might be troublesome for the students since this representation did not show clearly the multiplication operation and this was the first example. Teacher A should have explained the steps in procedures in clearer way.

Teacher A used the same procedure to get the general rule for the following three pattern examples and the following script exemplified how Teacher A made generalization for the linear growth pattern:

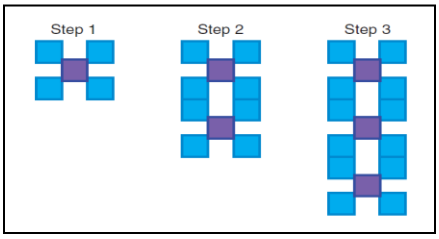


Figure 15 The example for linear growth figural pattern (Van de walle et al., 2013, p. 269)

A: Now, you are drawing a table. Write the position number, and what can you write the second row? The number of squares. How does the position number go on? 1, 2, 3, 4. (She is drawing the table). First, we will find the 20^{th} term, and then a^{th} term. Okay? How does the relationship between the position number and the number of squares change? Continue for the 4^{th} term, and then do the 20^{th} term last. Then, write "*a*". Examine the pattern and try to find out the relationship. How many squares in the 1^{st} picture? Then, find the 2^{nd} picture.
S^*: 5.
A: What is the difference between terms? Try to find out. We are trying to find the pattern between the terms in terms of numbers. Come to the board.
S^*: (She is writing 5, 10, 15, 20… 60, 5*a*)
S: She did wrong. (S^* is erasing 60)
A: How many squares in the 1^{st} picture? 5. For the 2^{nd} picture? 10., for the 3^{rd} picture, 15. So, what is the increment?
Students: 5.
A: Multiply the position number by the increment. What is the increment for 20^{th} term? 5. So, what is it? It is 100. Then, multiply a by 5. Is it 5*a*?
Students: Yes.
A: Is there something that is not understood, here?
Students: No.

In this situation, Teacher A guided the students to find the relationship in the pattern. She wanted them to draw a table, and to place the position numbers and the number of squares. For this, she asked the students to examine how the relationship changed. At that point, her knowledge of how mathematical language was used appeared incorrectly (SCK5-). She made error in using mathematical language since the relationship did not change in the linear growth pattern as in this question. Then, she wanted the students to find the 20^{th} term and then to write the a^{th} term after right as she used the similar procedure in the previous pattern generalization. The way was finding 20^{th} term first and then writing a for the next column as in the figure:

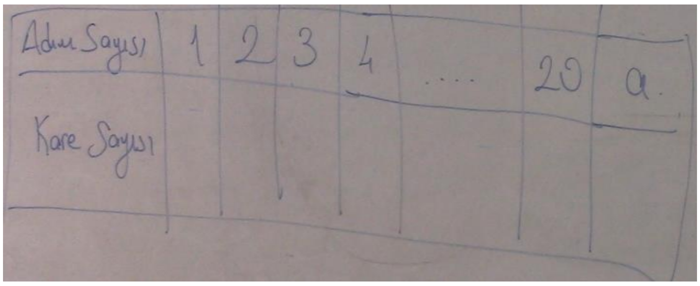


Figure 16 The representation of general term (a) used by Teacher A

Afterwards, Teacher A found the general rule for ath term. Teacher A used this way also for the next pattern questions. This situation might cause misconceptions and let the students to think that the pattern question can be solved only by using this way. For the following step, Teacher A wanted the students to find out the relationship in the pattern by multiplying the position number and the increment. The students could perceive this phrase as a rule for generalization and they might memorize it. This way could prevent the students to conceptualize the generalization process. Her knowledge to anticipate the misunderstandings that might arise with pattern generalization being studied in class appeared inadequately (KCS2-). Because the students can think that the general term always come after 20th term, and they may also consider that it is 21st term. This situation can prevent the understanding the function of variable and so general term. Thus, the teacher should have made explanations for the meaning and reasoning of the procedure and the general term conceptually.

4.1.2.1.2. Practice Two: Discussing On the Activity Related to Generalization Patterns

After Teacher A generalized the third pattern using the same procedure as explained above, she did not want the students to do the fourth pattern since she thought the students could understand the generalization for these type of patterns.

Then, she continued with the figural pattern that was formed with matchsticks and their terms were like 3, 5, 7, 9 ... as linear growth pattern. This pattern example was in the introduction part of the instruction and different from the first examples as the general rule of this pattern required to be written with using constant. For this discrepancy, Teacher A emphasized the generalization process more with creating the discussion environment. Actually, Teacher A generally used question and answer teaching method in the generalization process, and she focused on one or two students' answers, asked them to explain their answers. That was the discussion environment was not seen frequently in lessons. The discussion of generalization of this pattern that several students participated by giving answers, and the students and Teacher A tried to generalize the pattern together as in the following script:

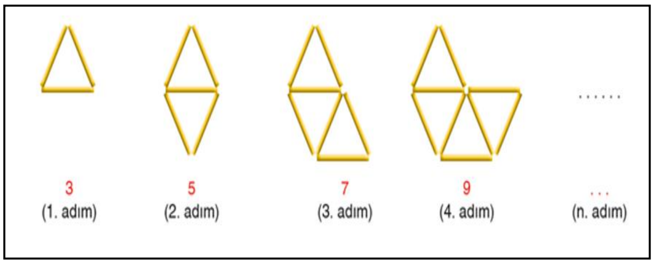


Figure 17 The matchsticks pattern

A: How many triangles are used for the first figure?
S:1.
A: How many matchsticks are used for the first figure? We will count the matchsticks in the figures. For example, there are 3 matchsticks in the first figure. How many matchsticks are used for the second figure?
S: 5.
A: In the 3rd figure?
S: 7. It is increasing by 2.
A: Yes.
S: 9.
S:11.
A: 11. It goes like this. How many matchsticks will we use for the nth term? How can we find it?
S: 13.
S: 2n.
A: Is it 2n? Let's say 2n for the relationship in the pattern. Then, what will we do to find the 1st term? What does "2n" mean? We say, multiplication the

position number by the increment. (She is writing 1 x 2 under this). Does it work?
S: It does not.

In this situation, Teacher A wanted the students to focus on the number of matchsticks and asked them what the next terms were after the given terms. While the students were answering, the teacher recorded their answers in a table. After Teacher A got answers; she asked directly what the n^{th} term was with using her knowledge of when to pose a new question to further learning of students in the discussion (KCT5+). First, the student came up with 13 as the answer, but Teacher A did not take into this answer account. This student may not have conceptualization of n^{th} term, since the student answered using number instead of algebraic notation. Then, Teacher A used 2n as another student's answer and checked whether it worked. She emphasized the multiplication of the position number and the increment, and she multiplied 1 by 2 regarding this formula (2n). when she did not get 3 as the 1^{st} term, the students realized 2n did not work. Then the discussion went on like this:

A: What can we do then?
S^*: Let's say 3.
A: We say the increment, 3. Does it work? Multiply 1 by 3, 3.
S^*: Yes, it works.
A: Let's try for the 2^{nd} term? Multiply 2 by 3, 6. But here is 5. It does not work.
S: Does '5n' work?
A: Multiply 5 by 2, 10; 5 by 3, 15. It does not work, too.
(S^* represents the same student)

Later in the discussion, S^* proposed to use 3 instead of 2. Thus, Teacher A multiplied 3 by 1, but, when she multiplied 3 by 2, she obtained 6 instead of 5. Another student proposed 5n as the general rule, Teacher A showed that the terms were 5, 10, and 15 with using 5n. Up to this point, Teacher A got the students' answers (2n, 3n, and 5n), and tried them if it worked, and her knowledge of when to use students' remarks to make a mathematical point appeared appropriately (KCT5+) as she guided them to need a different general rule from their answers. Thus, Teacher A led the students to add or subtract a number to algebraic expression in order to find the general rule of the pattern, and she asked what the multiplication of the increment and position number was for the 1st term:

> A: We try to find 3, 5, 7, 9 ... Will we do something different from previous examples? Will we add or subtract something? We have explained "the position number x the increment" once again. For example, if we add a number to this expression, can it be? The position number x the increment = 1x2 =2. To get 3, for the 1st term, what number do we add?
> S*: 1.
> A: (She is writing (2x1) + 1). Okay, does it work for 2nd term, too? What is the position number?
> S*: 2.
> A: When multiply 2 by 2 and add 1, will we get 5? (She is writing (2x2) +1 on the board).
> (S* represents the same student)

At this point of the discussion, Teacher A's knowledge to decide when to pause for more clarification for writing the arithmetical rule for the 1st and 2nd term appeared appropriately, and she paused the discussion and made explanation. First, she got 2 from 1x2 and she asked what was added to get 3 as the 1st term this time. She wrote (2x1) + 1, and then she showed that this rule worked for the 2nd term. At that point, most of the students had difficulty with understanding how this expression was written and asked to the teacher to explain again, and teacher explained again. Then, she wrote this representation for the 3rd and 4th term and got 7 and 9. She made explanation to make clarification for the arithmetical rule that she wrote with exemplifying for the first four terms (KCT5+). Lastly, she connected this representation to nth term and wrote (2xn) +1 algebraically. She explained that this was the general rule in algebraic representation. Her knowledge of how mathematical language was used appeared appropriately as she used notations such as parenthesis, operation signs, and algebraic notation (n) correctly (SCK5+). Throughout the instruction, she generally first wrote arithmetical rule using numbers to help the students to get the general rule and then she wrote the general rule algebraically by using appropriate notations.

After Teacher A generalized the 3, 5, 7, 9… pattern as 2n+1 algebraically, two students asked the questions as in the following script:

S: Teacher, can we only write “plus” in general rules? Can we subtract and write minus?
A: It is plus here. Minus can also be regarding the relationship of the pattern.
S*: Why did we find “plus 1”?
A: You said that the increment is 2, the position number 1, and we got 2. I asked, what is added to get 3? Thus, we add 1.
S*: So, this added 1 can be changed.
A: Yes, it can change regarding the pattern. We will examine different patterns also.
(S* represents the same student)

In this situation, Teacher A responded to the student who was wondering about the general rule was only written with addition that the general rule can be changed in terms of pattern relationship. In the second question, Teacher A made explanation to overcome the student’s difficulty about the generalization and explained how she found the general rule once again. Her knowledge to understand the needs and difficulties of students with writing the general rule appeared appropriately as she explained the missed points or responded the asked questions by students (KCS3+).

In this discussion process, Teacher A got the students’ answers (2n, 3n, and 5n), tried them if it worked, and guided them to get the general rule. When the dialogues were examined, it could be observed that Teacher A talked more than the students. The students answered only asked questions and proposed 2n, 3n, and 5n for the general rule. Teacher A tried these answers and did operations by herself and then she asked to the students if the rule worked or not. However, when this process was compared the other pattern generalization process, it could be said that several students could participate by explaining their answers. Teacher A’s knowledge to lead a discussion in the classroom appeared with this way that included getting several answers, showing they did not work, and explaining how the general rule for the pattern could be found. It is important to note another point in the generalization process that focusing on the arithmetical relationships in tabular representation to underlie the relationship in pattern to conceptualize generalization could be effective with her knowledge to choose, make and use the tabular representation (SCK7+). However, she used the figures (matchsticks) merely to provide visuality since she did not make any explanations about the change of the figures and the relationship among the numbers and her knowledge to link figural and numerical representations to underlying the idea of the relationship of the pattern appeared inadequately (SCK2-).

At the end of the lesson, Teacher A gave the following definition of general term, as she stated in her lesson plan:

> 'n' letter which is used in general rule, is a sign, symbol, or notation, determines the position number in the pattern. Thus, n is called n^{th} number, representing term or the general term. This letter is a variable.

Her knowledge to give the definition appeared at the end of the lesson with choosing from textbook and giving it to the students after the introduction activities about generalization of patterns (SCK3⊥). She was explaining the general term and the general rule as in the following:

> A: We used the letter "n". It determines the position number in the pattern. So, it can be a symbol, notation, or sign. We say that it is n^{th} number and the number for the n^{th} term. It is called representing term or the general term. What is n? It is a variable. You learned the concept of variable and unknown conceptions. "n" is not known, so it is unknown.

Her knowledge of how mathematical language was used about general term and general rule appeared problematic at some points of the flow of the lesson (SCK5-). In above explanations, she did not differentiate variable and unknown conceptions and used them as if they had the same meaning. Related with the concept of general term, another situation that Teacher A had troublesome was the usage of the general term and general rule concepts. She asked to the students "What is the n^{th} rule?", when she wanted the students to find the general rule. She should have asked what the general rule was since this question might cause the students to perceive the rule was in use for only n^{th} term.

4.1.2.1.3. Practice Three: Choosing the Examples or Activities to Use to Take the Students Deeper into Generalization of Patterns

During the second day of the instruction, Teacher A continued with the *Modeling Pattern activity* as in her lesson plan. She indicated not to generalize the other matchstick pattern (3, 6, 9, 12 …) in the post interview after the first day instruction. Thus, she wanted the students to open their books for doing the *Modeling Pattern activity*. This activity had a non-linear figural pattern formed with unit cubes and gave also the representation of the terms in a table (Table 5).

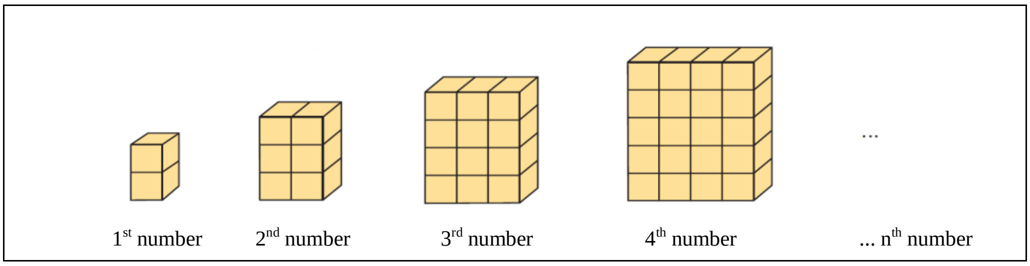


Figure 18 The non-linear growth figural pattern in modeling pattern activity

Teacher A asked the students to read the activity silently first, and then explained that this pattern was different from previous patterns. To explain this difference, she exemplified a linear pattern for the students first:

A: How many cubes are used for the first figure?
S: 2
A: 2. For the second figure?
S: 6
A: 6. (She is writing on the board like 2, 6, 12, 20 ...). Now, we seek the relationship again. How were the patterns in the previous lesson? For example, we used 5 for the first figure, how many of them we used for the second figure?
S: 10
A: 10. For the third figure?
S: 15.
A: It increases by 5 that is in a regular order.

Her knowledge to connect the concept of linear growth pattern and non-linear growth pattern appeared appropriately as she reminded the linear growth pattern with giving 5, 10, 15 ... pattern as an example for it and then she made connection by emphasizing the difference between these two types of patterns (SCK1+), and explained this non-linear growth pattern like this:

> A: Then, we look for this pattern. Does it increase in a regular order similarly? From 2 to 6, it increases 4. For the second difference (from 6 to 12) how does it increase?
> S: 6.
> A: 6. Here (from 12 to 20)?
> S: 8.
> A: That is, the increment is increasing. The pattern goes by increasing.
> S: It increases irregularly.
> A: Yes, it increases irregularly. There is not a certain order. 4, 6, 8. … hımm, actually, it increases regularly, the increment increases by 2. Does it continue with growing non-linearly, doesn't it?

Teacher A showed the difference between the terms and explained the increment increased by 2. Since she first explained the increment was regular for the linear pattern (5, 10, 15 …), the student answered as the increment was irregular for the non-linear pattern. First, she accepted this answer as increasing irregularly, but then she realized the increment had a non-linear growth and she corrected the statement that the increment increased by 2 regularly. That was a rule to provide this regularity. She should have emphasized the difference between the terms of a linear growth pattern was constant instead of using the word 'regular'. This explanation could cause misunderstanding about the concepts of linear and nonlinear growth patterns. However, her knowledge of how mathematical language was used appeared and she used inappropriate word to explain the property of non-linear growth pattern, but she then corrected it and explained it appropriately (SCK5+). Then, she continued with explaining the given table to show the relationship in the pattern:

> A: The relationship is given in the table… When the position number is 1, multiply 1 and 1 and add 1 (1. (1+1)). Is it the multiplication of the position number and the number of cubes? ... But, what it happened? The position number is 1, the number of cubes is 2, (she is examining the activity as in the book), it is not the number of cubes. For example, to find 1, 1+1; to find 6, 2.(2+1) is written. For the 3^{rd} term, 3.(3+1); for the 4^{th} term, 4.(4+1) and it goes like this. Then, if we want to write this representation using n, how is it?

First, Teacher A explained the relationship incorrectly that she indicated the multiplication of the position number and the number of cubes. However, the number of cubes was the result and not one the multipliers which formed the general rule. But then, she realized this definition was incorrect and told it was not the number of cubes. Her knowledge of how to provide mathematical explanation for general rule appeared with this way since she did not examine the table before the lesson and used the relationship as it was in the table (SCK6-). She did not ask the students to explore themselves first, and she gave it directly to them. While she was explaining the first four terms, she asked how they would write the rule using n. At that point, she did not use the n^{th} term, or ask how they could find the other terms, she only stated writing the rule with using n. Thus, her knowledge of how to provide mathematical explanations for procedures of getting the general rule appeared inappropriately and students could perceive the only purpose was using n in writing the rule (SCK6-). However, the students did not give any answers to this question of the teacher, then she explained once again the representations in the table:

> A: How can we express the multiplication of the position number and the number of cubes? For example, when the position number is 1, here (showing the 1^{st} option column) is 1.(1+1); when the position number is 2, it is 2.(2+1). When the position number is 3, it is 3.(3+1). When the position number is n, how can we write?
> S: n.(n+1).
> A: You have said that when the position number is n, multiply n and n and add 1. Here, n comes in the position number. Add the position number 1 to n (n+1). Then, multiply them (n.(n+1)).

In this explanation, Teacher A used the phrase "the multiplication of the position number and the number of cubes" again incorrectly to express the general rule as indicated above. This time, the student gave the correct answer, and the teacher explained as the multiplication of the position number and 1 more of it correctly using her knowledge to provide explanations for procedures (SCK6+). She also emphasized the 2^{nd} option to represent the relationship in the table:

> A: There is also 2^{nd} option here. We have learnt the distribution property of multiplication on addition, the second option is about it. For example, we distribute 1, 1.1 + 1.1, what happened then, 1^2+1. Let's look second one, 2.2+2.1, how can we represent 2.2? How many times 2 is written?

Students: 2.
A: Then, 2 power of 2, 2^2+2. For the 3rd term, 3.3+3.1=3^2+3. If we write this representation with using n, it is n^2+n.

She explained the second representation with showing the application of distribution property on the first representation. Her knowledge to provide explanations for this procedure was appropriate as the students have learnt this property and they could understand (SCK6+). She also wrote the distribution representation using the operation signs correctly such as 2.2+2.1 (SCK5+). However, she asked how many 2 was written in 2.2 to represent it as in exponential form. This question might be troublesome for the student since there are 2 items of 2 in 2+2, but this is an addition operation. She might have asked as how many 2 was multiplied to prevent misunderstanding. Thus, her knowledge of how mathematical language in this question sentence was not appropriate (SCK5-). Toward the end of this activity, another student came up with another idea:

S*: Teacher, I found the rule like this: I multiplied 1 with the next number, 1 and 2, then 2 and 3, 3 and 4. I could find the same results.
A: How did you find the general rule? What do you multiply by the position number?
S*: I multiply the position number and the next number after the position number.
A: It is the same thing. Actually we did like that, multiplying 1 and 2, 2 and 3, 3 and 4. That's good and correct.

S* explained his solution as multiplying the position number and the next number after the position number such as 1 and 2, 2 and 3, 3 and 4. In response to his proposed answer, Teacher A indicated that it had same reasoning with her explanation and accepted this answer. However, the student found a relationship between the position numbers as input values, but the relationship was between the position number and the terms, input and output values. Teacher A did not realize the student's incorrect reasoning and she accepted it as correct. At that point, Teacher A's knowledge of how to explain and justify the student's idea was inadequate that she accepted his explanation as so but his solution and explanation were incorrect (SCK4-).

4.1.2.1.4. Practice Four: Implementing the Pattern Test

After, the modeling pattern activity, Teacher A asked to students to do the questions in the pattern test that the researcher suggested. The major added task was the pattern test which Teacher A appreciated and included to her lesson plan. Teacher A examined the questions in the test and explained that the test could be useful for supporting students' learning of pattern generalization. Especially, she stated that the figural patterns could facilitate to get the relationship since the figures provide visuality for students. Her knowledge to predict figural patterns that students would find interesting and motivating appeared to select this test (KCS6+). Teacher A included this test to her lesson plan where she planned to do exercises after teaching generalization. In connection with the pattern test, the researcher suggested the use of table from examination of students' correct solutions in the test. Teacher A already planned to represent the patterns with table and also a graph.

While solving these questions in the test, Teacher A generally first draw a table and then she wrote the first, second, third and fourth term in the table by asking to the students as seen in the Figure 19:

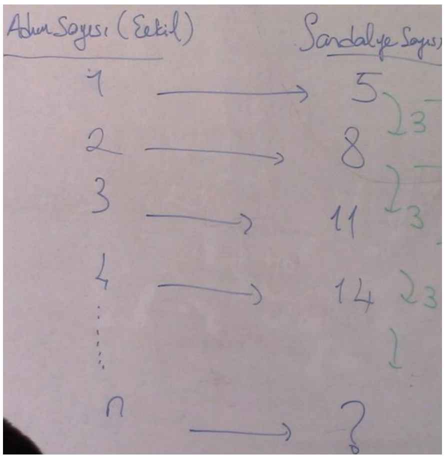


Figure 19 The tabular representation of terms of the pattern

Then, she guided the student to generalize pattern by asking what the n^{th} term with pointing out what the difference between terms was. Since the teacher taught exploring the relationship with emphasizing the difference, the students perceived the general rule as if adding the difference to n. For example, if the difference was 3 as in 5, 8, 11, 14… pattern that was the first question in the pattern test, then the students generally wrote the general rule as n+3 algebraically:

> A: … Then, let's find the relationship between the n^{th} figure and the number of chairs. What is the increment?
> S^*: Is it n+3?
> A: Does it work?
> S^*: Yes.
> A: Let's try. It increases by 3. When the position number is 1, add 3 to 1, I get 4. But the first term is 5. Then, it does not work.
> S: Teacher, n.3+2.
> (S^* represents the same student)

Since the teacher focused on the increment among the terms to explore the relationship, S^* answered n+3 for n^{th} term. The students could write the increment adding to n to get the general rule. Since they could not understand the generalization process conceptually, they tried to write general rule using the increment and n. Thus, Teacher A's emphasis on the difference between the terms might cause the misconception about generalization because some students answered the rest of the questions of the test with using this reasoning. Her knowledge to anticipate the misunderstanding that might arise with pattern generalization being studied in class appeared inadequately since she did not realize it even though the students continued to use same reasoning for other pattern generalizations (KCS2-).

Teacher A explained the generalization for the first question in detail and but she did not make explanation much for the rest of the questions in the test. She generally asked who solved the question and if the answer was true, she wanted to this student to show at the board how the student found the general rule. However, since the students had difficulty with the sixth question of the test especially, she discussed with students how the generalization could be made. Because the numbers of the pattern in this question were decreasing (60, 55, 50, 54, 40, 35… n), and the students did not have familiarity with this type of patterns. Teacher A and the students' generalization process of this pattern was as in the following script:

A: For this question, the numbers go from the big numbers to smaller ones.
S: nxn/2.
S: I did. The numbers go down 5 by 5.
A: Good, it is decreasing 5 by 5. So, we will multiply the position number by 5. Is that so?
S: nx5/2.
A: The 1st term is 60, then, 55, 50, 45. Write the position number and the number. Write the multiplication of 5 by the position number. What is the position number here (for 1st term)?
S: 1.
A: 1. Yes, 5xn, 5n.
S: Teacher, it does not work.
A: Yes, it does not. The position number is 1, it is 5. But, here is 60. Then, to get 60, what can we do? Let's try. For the 1st term, I multiply 1 by 5. What can we add to get 60?
S*: 55.
A: So, does it work for all terms?
S*: No.
A: Then, think it is related to 5n.
S*: Teacher, is it 5n/2?
A: 5n/2 does not work for 1st term. Put 1 instead of n, 5/2 is 2,5. It does not work. Think of other things. Do you agree on "5n"?
Students: Yes.
(S* represents the same student)

In this situation, Teacher A ignored the student's answer at the beginning, and did not respond to this student. However, in proceed of the discussion, she responded to S* appropriately and interpreted his wrong answer and made explanations by putting 1 in n why it was wrong. She wanted the students to use her formulization that was multiplying the position number and the increment. Teacher A had the tendency to answer to her questions without waiting the students' answers. She said 5n as the answer herself without waiting the students to think, and so she generally led the students by saying the answer. When the students did not have any ideas to get 60, Teacher A led the students to think the subtraction from a number:

A: For example, here, think 5n at the beginning of the rule. Determine a number and think of subtraction 5n from this number. Try for this. (She is going around the class). S* has approached to the rule. Write it using "n". Can you write on the board?
S: (She is writing (70-5)-5, (70-5)-10 ...)

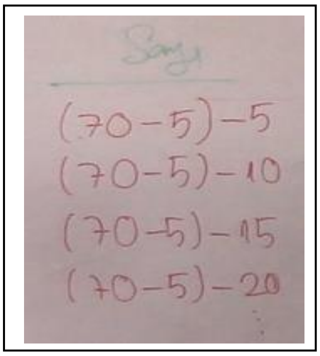


Figure 20 The representation by the student

A: Has she found the 1st term, 65? Yes. For the 2nd term, it is 60. Does it work? Actually, we can represent like this (The teacher is writing (70-5)-1.5, (70-5)-2.5, (70-5)-3.5 ... one under the other to show the position number). We can write 70-5, thus 65. The rule is 65-5n. You can try and see whether this rule works for all terms.

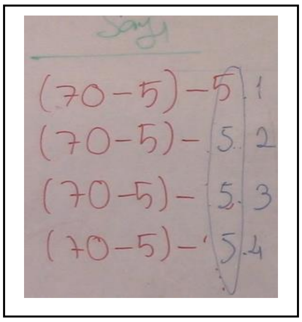


Figure 21 The representation by Teacher A

The general rules that the students found to this question were ax+b form in algebraically. In these type of questions, Teacher A also generally asked the students what was added to find the 1st term and guided them adding with a number to get the rule. Then, Teacher A used S*'s production for the generalization and her knowledge to interpret this student's emerging and incomplete thinking for the general rule as expressed in the way which was her explored representation appeared appropriately (KCS4+). Teacher A rearranged the student's expressions with showing the multiplications to emphasize the position number, such as (70-5)-10 as (70-5)-2.5 (Figure 21). Teacher A showed the relationships with these representations and connected to its algebraic form with writing n for the position number's place. She also subtracted 5 from 70 as 65 and wrote the general rule 65-5n finally.

4.1.2.1.5. Practice Five: Doing Exercises Related to Generalization Patterns from Textbook and Workbook

After solving the pattern test, she asked the questions in the textbook and student's workbook to the students. This part of the lesson was like the students' completing a worksheet at the same time. Teacher A gave some time to the students and then wanted the students who got the correct generalization to show at the board to other students. Teacher A did not make any explanations by herself. Teacher A asked the questions as in the order of the textbook. The questions asked to generalize a linear figural pattern (e.g. 4, 8, 12, 16 …, and 1, 3, 5, 7 …) or to find the asked term for the pattern whose general rule was given (e.g. find the 4^{th} term for the pattern that its general rule was 3a+1).

In answering these questions, it was important to note that some students asked to the teacher why they did not add something to 4n for writing the general rule of 4, 8, 12, 16 ... pattern. It seems as if the students perceived the teacher's explanation (multiply the difference and position number, and add a number to find the first term) as a rule and memorized it, and wanted to apply it for all generalization pattern questions. In response to this question, Teacher A explained that adding or subtracting some numbers for the generalization was not required for every generalization. At that point, her knowledge to anticipate the misunderstanding that raised with pattern generalization being studied in class appeared inadequately since she did not realize the students' misconceptions about exploring the relationship of the pattern (KCS2-).

It was asked to generalize four numerical patterns in the last question of the exercises from the textbook in this lesson. Two patterns were linear (3, 4, 5, 6 …, and 4, 6, 8, 10 …), and the other pattern were non-linear patterns (1, 4, 9, 16 …, and 2, 5, 10, 17 …). In the generalization process of these patterns, for the 3, 4, 5, 6… pattern, she guided the students with a different explanation from previous ones:

> A: Let's look, you can consider the relationship between the position number and the number. Not only between numbers (output values). Up to now, we

have investigated whether it is increasing by 1. 1st term is 3, 2nd term is 4. What is the difference between the position number and the term?
Students: 2.
A: It is increasing by 2. Then, we can add 2 to the position number. You can have the rule with this way. So, what is the general rule?
Students: n+2.

When this pattern was asked to generalize, the students gave answer as n+1 by considering the difference between the terms as the reason might be Teacher A's emphasis of the difference for generalization before. In the above dialogue, although she guided the students appropriately to investigate the relationship between the position number and the number, she did not do what she said. Actually, she should have investigated the relationship as between the position number and the number for pattern generalization, but she might have considered that the relationship between the numbers (output values) up to now as she generally emphasized the increment among the terms, and thus her explanation indicated her misconception about generalization of patterns. This explanation also differed from her explanation that was the general rule as the multiplication of the position number by the increment that she stated previous pattern examples. She added the increment to n for this pattern instead of multiplying it by n. This situation showed her lack of content knowledge that she did not provide mathematical explanations for procedure for functional thinking in generalization (SCK6-). For the 3, 4, 5, 6 … pattern, she found the difference as 2 from 3-1 (for the 1st term) and add it to n to represent the general rule algebraically. It might cause misconceptions since some students used this method for the next pattern 4, 6, 8 ..., and they found the difference between the position number and the term as 3 from 4-1 and add the difference to n, and generalized this pattern as n+3 incorrectly; or when the difference between the position number and the term as 4 from 6-2 and add the difference to n, and generalized this pattern as n+4 incorrectly. Her knowledge to anticipate the misunderstanding that might arise with pattern generalization being studied in class appeared inadequately since she did not realize it even though the students continued to use same reasoning for this pattern generalization (KCS2-).

After the generalization of two patterns, the students generalized 1, 4, 9, 16 … non-linear growth pattern as n^2 with the help of the teacher, and Teacher A gave the other 2, 5, 10, 17 … non-linear pattern as homework since the time was over for the lesson. She finished the lesson with giving this question as homework to the students.

4.1.2.1.6. Practice Six: Presenting Problems That Combine Knowledge Related to Exponential Numbers

Before doing the pattern test and the exercises in textbook and students' workbook, Teacher A introduced non-linear growth patterns with only generalizing 2, 6, 12, 20 … pattern. However, she did not continue with this type of patterns, instead of this, she gave many examples about linear growth patterns from the pattern test and the books. After these examples, she gave examples about non-linear patterns from suggested examples in literature in this lesson. She focused on non-linear patterns for two hour-lessons and she guided the students to find an algebraic expression using "n^2" for the non-linear growth patterns throughout the teaching non-linear patterns. Thus, the students wrote n^2 first and then used trial and error by adding or subtracting numbers to n^2. The students generally try to find the first term by putting 1 in the place of n. To illustrate, for 2, 5, 10, 17 ... pattern (see the below figure), the student generalized it as n^2+1 and she explained that she added 1 to n^2 and tried n^2+1 for the first five terms whether it worked. Teacher A made explanation about the solution:

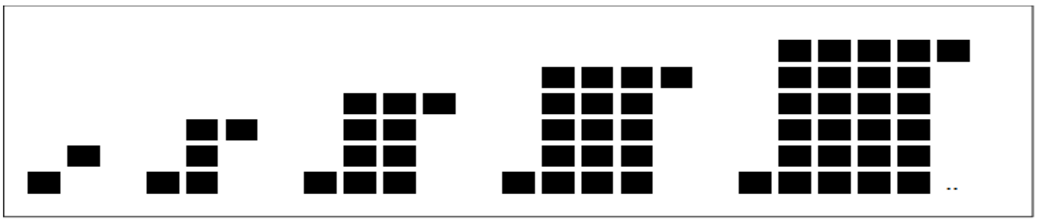


Figure 22 The first non-linear growth pattern example (Smith, Silver, & Stein, 2005, p.33)

> A: When the first term is 1, the number of squares is 2. When the second term is 2, the number of squares is 5. So, 3 is added. Then, 5 is added and the third term is 10. So, the differences are 3, 5, 7, 9 ... They go on increasing by 2. It goes on by changing, not linear. Then, this type of pattern is called non-linear pattern (increasingly go on). The distinction from previous patterns is that. The relationship between patterns goes on increasingly. How did your friend do? She multiplied the position number by itself and added 1 to it. Let's try: n.n+1. $1.1+1=1^2+1=2$, $2.2+1=2^2+1=5$, $3.3+1= 3^2+1=10$. The general rule works for all of them. Then, what can we say for n? That is n^2+1. It is the general rule algebraically.

In this situation, Teacher A first explained what the distinction of this type of patterns was from linear growth patterns. But, then she said that the relationship between patterns went on increasingly. In this mathematical explanation, she used mathematical language inappropriately. Because, one of the reasons is the relationship does not increase, and it is always same and it works for all terms in the pattern. The second reason is that she used "patterns" as the terms in the pattern. Her knowledge of how mathematical language was used appeared with this explanation incorrectly (SCK5-). Then, Teacher A explained the student's solution and showed that the rule worked for the first three terms. While writing the expressions with the numbers, she used notations (operation signs, equal sign and exponential form) appropriately with her knowledge to use notations (SCK5-). Then, she generalized the pattern as n.n+1 and represented it as n^2+1. Another similar pattern was 1, 4, 9, 16 … pattern constructed with triangles (see Figure 23).

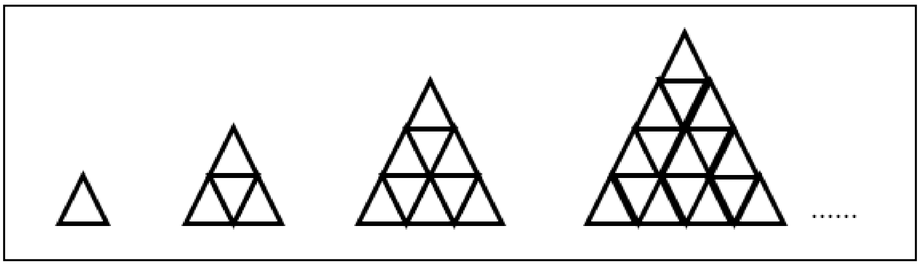


Figure 23 The second non-linear growth pattern example (Warren & Cooper, 2008, p. 176)

Teacher A wanted the students to use similar way as in the previous example to generalize of this pattern. One student explained his reasoning by explaining trial and error method using n^2. He showed that the rule worked for the first four terms. Teacher A did not make more explanations after the student's explanations. However, the students had difficulty with the pattern in Table 8:

Table 8 The tabular represented non-linear pattern (Steele & Johanning, 2004, p. 80)

The number of sides of polygon	The number of diagonals
3	0
4	2
5	5
6	9
…	…
n	?

In this question, the relationship between the number of sides of polygon and the number of diagonals of polygon in the pattern was asked. The number of diagonals for 3-sided polygon, 4-sided polygon, 5-sided polygon, and 6-sided polygon were given as 0, 2, 5, and 9. It was asked what the number of diagonals for n-sided polygon was. The questioning process between the student and Teacher A was as in the following script:

S^*: The difference is between 0 and 2 is 2; 2 and 5 is 3; 5 and 9 is 4; 9 and 14 is 5. Teacher, the differences go like +2, +3, +4, and +5. So, I have found (n+1).n, since there is 1 between the differences, I selected the choice that has (n+1).
A: Does it work? For example, triangle, write 3 in the place of n. Can you try?
S^*: It works for the first term, but it does not work for others.
A: What did you write in the place of n?
S^*: 0.
A: Then, did you take the number of diagonals?
S^*: Yes.
A: So, if you write 2 for n, what do you find?
S^*: Teacher, the choice is like that n.(n+1). Otherwise n+1.
(S^* represents the same student)

In this situation, Teacher A gave four different choices for the general rule, when the students had difficulty and they did not come up with any answers. The choices were in order of A) (n-3).n, B) (n+1).n, C) (n-3).n / 2, D) (n+1).n / 2. When the students were showed these choices, they used the trial and error method to find the correct choice. However, the student only considered the increment "+1" between the differences in the pattern, he selected the choice that had n+1. He also stated that the rule was n+1, but it was not included in choices, thus he selected (n+1).n. Teacher A's emphasizing the difference between terms to generalize could cause this student to think with this way. Thus, her knowledge to anticipate the possible misunderstanding with getting the general rule appeared inadequately (KCS2-). This student also used 0 (the number of polygons) in general rule by substituting 0 for n and got the number of polygons again as 0. The student did not understand the relationship between the position number and the term, she focused on getting the result as the number of diagonals. The teacher realized this fault, and she continued getting other students' answers:

> S^*: The answer is C.
> A: Can you show us your solution?
> S^*: I have tried the all choices one by one. (She is writing (n-3).n/2 = 3-3=0.3=0/2=0)
> A: You did for 3. (Teacher A is writing (3-3)=0, 0.3/0=0). You did for 4, (4-3).4/2, it is 2. I shouldn't have given the choices to you. Now, since this is an increasing non-linear pattern, you will get something with n^2. For example, we said that n-3, why? Here, the number of sides is 3, to get 0, we have to subtract from 3. So, it can be (n-3).n. Let's suppose that this rule works for the first term, 3. We continue. For the second term, 2, what will I do? I have to divide by 2. So, we will write the number of sides of polygon in place of n, and we will get the number of diagonals of polygon.
> (S^* represents the same student)

The student found the correct answer and explained that she tried the rule in C for n=3 and n=4. This student represented for n=3 incorrectly as (n-3).n/2=3-3=0.3=0/2=0. Because, the expressions in the both sides of the equal sign did not equal each other. Teacher A did not correct it, but she wrote correctly herself while explaining using her knowledge of parenthesis, operation sign, and equal sign (SCK5+). Teacher A sometimes corrected the students' incorrect using notations, but she missed some of them. However, Teacher A used the equal sign correctly; the students might misunderstand and have misconceptions about the equal sign if their writings were not corrected.

Teacher A accepted the student's solution and showed how the rule worked for n=3 and n=4. Teacher A emphasized finding "0" for n=3 as the first term in the pattern. According to Teacher A, it was important that 'n-3' algebraic expression was required to get 0, and also the rule had to have n^2. But, there were two choices that had (n-3).n expression. Thus, she also tried the rule for n=4 to find the second term. She explained that this expression must be divided by 2 to get 2 as in the pattern. For this pattern, Teacher A made explanations and justifications for the student's answer and her knowledge to explain and justify of the student's ideas as it was appeared inappropriately by using trial and error method (SCK4-). However, she led the students to try the choices and she also used the same way herself. This guidance might cause the students to memorize and prevent them to conceptualize generalization of the non-linear growth pattern and Teacher A's knowledge how to provide mathematical explanations appeared in this way appropriately (SCK6-).

After the lesson, in the post-observation interview, Teacher A asserted that the students had difficulty with the generalization of non-linear growth patterns and they did not understand it. She explained her impressions as in the script:

> A: They did not understand in this lesson, the non-linear growth pattern. The situation of multiplication of n by n. They did not make any efforts. They wanted to find the general rule immediately as in the linear growth pattern. So, the level of the students was not good.

Teacher A explained the reason of the students' difficulty of the generalization of the non-linear growth pattern was that the lack of efforts of the students for getting the generalization. In this regard, when she was asked how she could overcome this struggle, she suggested the design of lesson in a different way such as using the real-life examples. However, she explained that this type of lesson should be designed well and the use of real life examples in instruction takes time. Her knowledge to identify using real-life situations as a different method afford instructionally appeared as she suggested to design the lessons based on real life situations and she only explained that it required well-structured design and took time to implement (KCT6-).

Teacher A finished the instruction for teaching generalization of patterns with these examples at the third day. At the end, she did not summarize what she taught for generalization patterns. The only thing that she did not use was the graph of function. Although she indicated the use of it and she would examine before the lesson, she did not mention about in the lessons.

4.1.2.1.7. The Extracted Knowledge Types from the Instruction for Generalization of Patterns

Table 9 The extracted knowledge types from the instruction for generalization patterns

Practices	Extracted knowledge types				
	SMK		PCK		
	CCK	SCK	KCS	KCT	KCC
Choosing an example or activity to start teaching generalization of patterns with connecting to topics from prior years	CCK1(-)	SCK1(+) SCK3(-) SCK4(+) SCK5(-,-,-) SCK7(+)	KCS2(-)	KCT1(+)	
Discussing on the activity related generalization of patterns		SCK2(-) SCK3(⊥) SCK5(+,-) SCK7(+)	KCS3(+)	KCT5(+,+,+)	
Choosing the examples or activities to use to take students deeper into generalization of patterns		SCK1(+) SCK4(-) SCK5(+,+,-) SCK6(-,-,+,+)			

Table 9 (Continued)

Practices	Extracted knowledge types				
	SMK		PCK		
	CCK	SCK	KCS	KCT	KCC
Implementing the pattern test			KCS2(+) KCS4(+) KCS6(+)		
Doing exercises related to generalization of patterns from textbook and workbook		SCK6(-)	KCS2(-,-)		
Presenting problems that combine knowledge related to exponential numbers		SCK4(-) SCK5(-,-,+) SCK6(-)	KCS2(-)	KCT6(-)	

Table 9 shows what type of knowledge of subject matter knowledge (SMK) and pedagogical content knowledge (PCK) that Teacher A had in instruction practices. (+) sign indicates the teacher’s existing knowledge was adequate or appropriate and she used it effectively, while (-) sign indicates the teacher’s existing knowledge was inadequate or inappropriate and she used it ineffectively. Each sign (+ or -) in the same knowledge type refers to Teacher A’s different use of this knowledge during instruction. Besides, for SCK3 knowledge type, (⊥) is used to indicate the teacher’s knowledge to develop definition or explanation appeared as merely choosing it from the textbook and presenting to the students. Since this situation meets the code (SCK3) partially, (⊥) sign is used.

CCK1(-) indicates that her knowledge of definition of pattern appeared inappropriately since she defined pattern as the relationship. For SCK, SCK1(+) in the first practice indicates that her knowledge to connect the knowledge about pattern topic between grades. In connection with this, SCK3(-) indicates her knowledge to develop usable definition appeared inadequately since she defined the pattern as the relations between the figures, but the patterns were created also with numbers and had relationships in them. On the other hand, SCK3(⊥) in the second practice indicates the teacher's use of the definition of general term concept with choosing from the textbook and explaining to the students. SCK7(+) indicates that her knowledge to choose, make and use the tabular representation with focusing on the arithmetical relationships in tabular representation to underlie the relationship in pattern to conceptualize generalization could be effective, while SCK2(-) indicates that her knowledge to link figural and numerical representations to underlying the idea of the relationship of the pattern appeared inadequately with using the figures (matchsticks) merely to provide visuality and not making any explanations about the change of the figures and the relationship among the numbers. SCK5 knowledge type is related with the mathematical language the teacher used. Particularly, SCK5(-,-,-) in the first practice indicates respectively that were not using multiplication sign in arithmetical representation, using unknown and variable concept changeable without their function, and explaining the relationship changed for linear growth patterns incorrectly. Beside this, SCK5(-) in the second practice indicates that she used general rule as the n^{th} rule inappropriately. Especially, SCK4(+) indicates that the teacher explained the students' ideas for general rule by justifying with substituting 1 or 2 for n if they worked or not. The teacher used the knowledge of students' thinking and KCS3(+) indicates her knowledge to understand the needs and difficulties of students with writing the general rule appeared appropriately as she explained the missed points or responded the asked questions by students. However, KCS2(-) in the first practice indicates that the teacher's finding 20^{th} term first and then writing a for the next column as the method for generalization might cause misunderstanding and her knowledge to anticipate the students' misconceptions with pattern generalization appeared inadequately. KCT5 knowledge type is related with the teacher's leading of the discussion. KCT5(+,+,+) indicates that the teacher's asking n^{th} term to further learning of students, guiding the students to find a rule including a constant (e.g. 2n+1), and making clarification about arithmetical representation of the rule (e.g. 2.1+1=3).

SCK1(+) in the third practice indicates that her knowledge to connect the concept of linear growth pattern and non-linear growth pattern with pointing out the increment between the terms of these two types of patterns was appropriate. In connection with this, though SCK5(+) in the third practice indicates that her knowledge of how mathematical language was used with explaining the increment increased regularly as the property of non-linear growth patterns was appropriate. On the other hand, SCK5(-) in the sixth practice indicates that her knowledge of inappropriate use of mathematical language. Since she emphasized that the relation increased in non-linear growth pattern, and she used pattern and term concept as same. Although SCK6(+,+) in the third practice that her knowledge of how to provide explanation for the relationship between the position number and the output values as the multiplication of the position number and 1 more of it was adequate (for 2, 6, 12, 20 … pattern), SCK6(-) indicates that the teacher's explanation for the general rule of the non-linear pattern as using n^2 was inadequate in the third practice. She used notations (operation signs, equal sign and exponential form) appropriately with her knowledge (SCK5(+,+) in the sixth practice). However, she had difficulty in generalization of non-linear patterns and thus SCK6(-) in the fourth practice indicates her knowledge to provide explanation was inadequate. Besides, SCK4(-) in the third practice indicates that her knowledge of how to explain the student's answer was inappropriate since she accepted the incorrect answer that was about the multiplication of position numbers respectively such as 2 and 3, or 6 and 7. In addition to her difficulty in non-linear pattern generalization, she made incorrect explanations in generalization of linear growth patterns. SCK6(-) in the sixth practice indicates that her knowledge to provide mathematical explanation for the relationship in pattern was inappropriate since she had investigated the relationship among the output values. This situation showed her lack of content knowledge about functional thinking in generalization. Because of her explanations about generalization linear-growth patterns with emphasizing the difference between terms and adding a number to get the general rule throughout the instruction, the students had misunderstandings and they added the difference to n and tried to put a number for all generalization. Thus, KCS2(-,-) in the fifth practice indicates that her knowledge to anticipate the misunderstanding that might arise was inadequate. Nevertheless, when she realized the students' emerging ideas about generalization, KCS4(+) in the fourth practice indicates that her knowledge to interpret the student's thinking for generalization of the pattern whose terms were decreasing was appropriate. Lastly, KCT6(-) in the sixth practice indicates that her knowledge to identify using real-life situations as a different method afford the instruction of pattern generalization instructionally appeared inadequately since she could not suggest a design for this method.

4.1.2.2. Practices in the Instruction of Operations with Algebraic Expressions

There were two objectives for teaching operations with algebraic expressions as indicated planning section and the teacher designed her lessons respectively based on the two objectives within the same lesson plan. Since she taught addition and subtraction first and then multiplication of algebraic expressions, her instructions were explained and documented respectively in this section.

4.1.2.2.1. Practices in the Instruction of Addition and Subtraction (Simplification) of Algebraic Expressions

The teacher's purposeful actions for teaching addition and subtraction of algebraic expressions were grouped into six practices: 1) choosing an example or activity to start teaching addition and subtraction of algebraic expressions with connecting to topics from prior years, 2) discussing on the activity related to addition and subtraction of algebraic expressions, 3) choosing the examples or activities to use to take the students deeper into addition and subtraction of algebraic expressions, 4) implementing the suggested activities, 5) doing exercises related to addition and subtraction of algebraic expressions from textbook and workbook, and 6) presenting problems that combine knowledge related to fraction and geometry. The extracted teacher's knowledge based on MKT framework was analyzed within these practices. The reflection of the instruction after each lesson was also presented to provide the teacher to evaluate her instruction by herself. The interpretations of the teacher could give information about her knowledge about students' thinking and learning with respect to the instruction. The classroom dialogues that were most representative for knowledge types the teacher had were selected from the instruction to illustrate how the teacher used her knowledge in teaching.

The first practice was choosing an example or activity to start teaching addition and subtraction of algebraic expressions with connecting to topics from prior years and this title also was extracted from one of the descriptors of KCT and SCK. This practice examined that the teacher chose which example or activity to start teaching addition and subtraction of algebraic expressions with rationale, and how she implemented it in the classroom. It also included the teacher's recalling process for prior knowledge that students had to learn addition and subtraction of algebraic expressions. To do this, the teacher first reminded the concepts of term, coefficient, and variable as prior knowledge that the students learnt in previous grade. For this connection, the teacher illustrated the algebraic terms on two algebraic expressions. The second practice was discussing on the activity related to addition and subtraction of algebraic expressions and this practice was also affected by one of the descriptors of KCT. This practice included a small discussion about which terms in algebraic expression could be added and how they were added. The teacher let the students to explain their answers and responded their questions that they asked in order to understand. The two practices were to introduce of addition and subtraction of algebraic expressions and the introduction part lasted three-lesson hours. Then, she followed the middle part of the lesson plan to improve students' understanding in the development of the instruction. The third practice was choosing the examples or activities to use to take the students deeper into addition and subtraction of algebraic expressions and this title was also extracted from one of the descriptors of KCT. This was also as a continuation of choosing an example or activity to start practice. This practice included how the teacher taught addition and subtraction of algebraic expressions to improve students' knowledge with getting deep the content using algebra tiles and problems about real life situations. The fourth practice was implementing the suggested activities that were about writing algebraic expressions. The fifth practice was doing exercises related to addition and subtraction of algebraic expressions from textbook and workbook and the teacher asked the exercises to the students as in the order of the book. This part was as questioning by teacher and answering by students. The sixth practice was presenting problems that combined knowledge related to geometry that the teacher presented problems which required fractions and geometry knowledge. These practices are explained with examining how the teacher used her knowledge based on MKT framework in the following sections.

4.1.2.2.1.1. Practice One: Choosing an Example or Activity to Start Teaching Addition and Subtraction of Algebraic Expressions with Connecting to Topics from Prior Years

At the first lesson of the instruction, Teacher A reminded the prior knowledge that the students had learnt at 6th grade by explaining the term, unknown, and coefficient terms with exemplifying two algebraic expressions. After explaining the term concept, she connected the like term concept to teach addition and subtraction and her knowledge to connect the topic being taught to topics from prior years appeared with reminding algebra concepts (SCK1+). She gave and explained 8t+3 and -9*x*-7 as examples for recalling their prior knowledge as in the following:

> A: 8t+3 is given. What does the meaning of '8, t, and 3?' t is the unknown, 8 is the coefficient, and 3 is the constant term. 8t is the term. Let's examine with another example. -9*x*-7. What is *x*?
> S: Unknown.
> A: Yes. -9 is the coefficient, 7 is the constant term. -9*x* is the term.

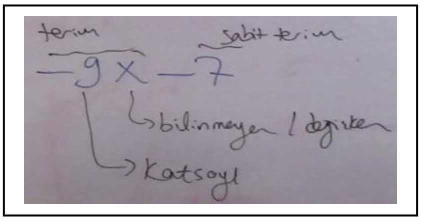


Figure 24 The representation of algebra concepts by Teacher A

In this situation, Teacher A reminded the concepts related with algebraic expressions that the students learnt them at 6^{th} grade. She used two examples: 8t+3 and -9x-7. She explained that 8, and -9 were the coefficients; t and x were the unknowns; 3 and 7 were the constant terms; and 8t and -9x were the terms. She used the unknown and variable concept together as they had the same meaning since she wrote as in the figure using slash. She also used with this way in the instruction of generalization of patterns. She might not have known the difference between these terms and thus her knowledge to use terms was problematic (CCK2-). However, she showed only 7 for the constant term without the sign of it as seen in the figure. This showing might be misunderstanding for the students since they might ignore the signs of the terms. At this point, her knowledge to anticipate the misunderstandings that might arise with the term concept being studied in class appeared inadequately (KCS2-). Then, the teacher gave the following definitions to the students:

> Each of the addends that form the algebraic expressions is called term. The terms that do not have variable are constant term, the number that is written as factor before the variable is called as coefficient.

The definitions were the textbook definition and the teacher gave them to the students as in the textbook. Thus, her knowledge about usable definition appeared as merely choosing it from the textbook (SCK3⊥) and the analysis of this definition was explained in planning section in detail.

Teacher A began the instruction with algebraic expressions using pattern generalization that the students learnt as the researcher's suggestion. She decided to use *the bacterial growth pattern* and *operating with number strips* to make this connection with her knowledge appropriately (SCK1+). First, Teacher A implemented the *Bacterial Growth Pattern* activity from the textbook. This activity had two patterns in the context of growing bacteria. The first bacteria type was growing as 2, 4, 6, 8…, and the second bacteria type was growing as 3, 6, 9, 12… It was asked to generalize patterns first, then to add and multiply the terms of the first pattern and the second pattern, and to generalize the added pattern and multiplied pattern in the activity. Finally, it was asked that how there was a relationship between the generalizations at the beginning and the generalizations after operated them. This question guided the learners to recognize and explore the addition and multiplication of the general rules of patterns as 2n and 3n as algebraic expressions. With implementing the *Bacterial Growth Activity*, her knowledge to choose which examples to start with appeared effectively (KCT1+). The dialogues between the teacher and the students in the implementation of the activity were as in the following:

> A: What is asked? It is asked to express the general rule of the patterns with using n. The first pattern (2, 4, 6, 8 ...) is increasing by 2. Who wants to say the general rule of the first pattern?
> S: n+2.
> S: n^2.
> A: Let's try. n+2. Add 2 to the position number, 1+2=3. It does not work. n^2? The squared of n. It does not work.
> S: n. Can it be the rule?
> A: Substitute 1 for n, it is 1. But, we must find 2.
> S: n times 2?
> A: Does it work? Multiply the position number by 2. Multiply 1 by 2, then 2 by 2, 3 by 2. They are correct. If we go on like this, n.2 that is 2n, is it correct? Yes. Look for the second pattern.
> S: n times 3.
> A: Let's try (She is substituting 1, 2, 3 for n). The rule is 3n. We have learnt generalization of pattern in previous lessons. It is asked to add these patterns. Add 2 and 3,..
> S: 5, 10, 15, 20.
> A: What will we get from the addition of 2n and 3n?
> S: 5n.
> A: Okay, can we find it without the addition of the terms? Can we find this rule from the pattern, generalization of the pattern (5, 10, 15, 20 ...)? Yes, it is 5n again. You have found 5n from addition.

In this situation, Teacher A responded to the students' answers about the general rule appropriately. She tried all of them and showed whether they worked or not. However, Teacher A carried out the activity as if the students knew the addition and multiplication of algebraic expressions. She wanted the students to find the addition and multiplication of the general rules first, then she found the general rule of the result pattern and used this pattern rule to check the result of operations. Nevertheless, Teacher A was expected to generalize the result patterns first, and then to make connection the general rules of added and multiplied patterns with the operations. It might more make sense that if she should have made guided the students to find the general rules of added and multiplied patterns then she could have asked them to operate the general rules. Thus, her knowledge of how to provide mathematical explanations for common rules and procedures appeared inappropriately (SCK6-). Instead of this, she added and multiplied the general rules and used the result pattern's general rule to check the result was correct or not. On the other hand, the part of the multiplication of patterns were explained in multiplication practices section in detail.

Related to the pattern generalization, the other suggested activity was *Operating with Strips activity* from in "Positive algebra – A collection of productive exercises" course book. Teacher B proceed the lesson with this activity. She did the first two questions in this activity: one addition and one subtraction of patterns questions. She gave the two questions as homework. There were three patterns in each question. The terms of two patterns were added and subtracted and the operation resulted the third pattern's terms. It was asked to generalize the patterns first, and then to do operations, addition or subtraction. The researcher suggested this activity to provide the students to recognize the connection between the resulted pattern rule and the addition of the rules of the addend patterns. Teacher A's knowledge to connect operations with algebraic expressions to generalization pattern that the students had learnt and known appeared appropriately (SCK1+) to select this activity and implemented it. There were four questions in this activity: two addition and two subtraction questions, and three patterns in one of the questions as in the following figure:

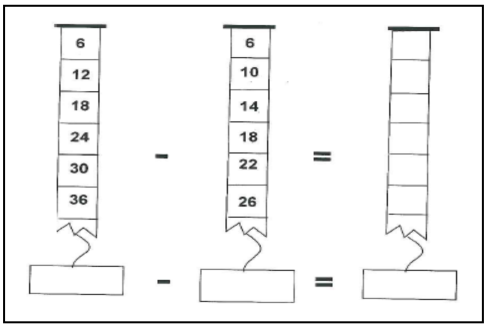


Figure 25 Operating with Number Strips

The terms of two patterns were added and subtracted, and the third pattern's terms were resulted. Then, the three patterns were generalized and the students were expected to explore the relationship between operation of the two patterns' generalizations and the third (result) pattern generalization. To illustrate, the answering of the question in the figure as in the following:

> A: There is a subtraction operation between these patterns. Let's generalize the first pattern (6, 12, 18, 24, 30 ...).
> S: 6n.
> A: This pattern increases by 6. (She is writing 6.1, 6.2, 6.3 ...) Multiplying 6 by the position number is 6n. Let's generalize the second pattern (6, 10, 14, 18 ...).
> S1: Multiply n by 4.
> A: Is it correct? When n=1, it is 4, but the first term in the pattern is 6.
> S: 6+4n.
> A: Substitute 1 for n, 4 plus 2, 6. Okay, write in the box 4+2n. It is asked to subtract this time (The students are subtracting the terms of the second pattern from the terms of the first pattern). Who wants to generalize the third pattern (0, 2, 4, 6, 8 ...)?
> S: 0+2n.
> S: 2n+2.
> A: Is it 2n-2? Substitute 1 for n. Then, 2. Okay. Can we find the same rule by subtracting the rule of the second pattern from the rule of the first pattern? 6n-(4n+2), distribute the minus sign to 4n and +2 in the parenthesis.

In this situation, Teacher A responded to the students appropriately as she tried the students' answers to check whether they were correct or not. However, she did not take into '0+2n' answer account. This answer was troublesome since the student found the increment as 2 and multiplied 2 by n and got 2n. Then, he added the first term (0) to 2n, 0+2n. The students were confused to find the first term and he added the first term instead of it. However, Teacher A did not respond to this student. She might have used the student's error to teach pattern generalization. Her knowledge to anticipate the misunderstandings that might arise with studying pattern generalization appeared inadequately (KCS2-). Finally, the teacher corrected other student's answer with putting minus sign instead of plus sign and answered by herself, and explained the subtraction operation applying the distribution property. At this point, her knowledge of how to provide mathematical explanations for subtraction procedures appeared correctly (SCK6+).

After the lesson had finished, the researcher interviewed with the teacher to get her ideas about the lesson. Teacher A stated that the students had difficulty with generalization patterns in operating with number strips activity at the beginning of the instruction. Teacher A explained this situation as in the following script:

> A: The bacterial growth pattern was good and the students' performance was good. But, they were confused in the operating with number strips activity. They had difficulty generalizing patterns. They forgot how they generalized, since they did not study on it after learning. So, I had also difficulty in connecting with operations since I explained the pattern generalization again.

Teacher A asserted that the students did not study individually adequate after learning, thus they had difficulty with generalizing of patterns in this activity. She had to explain how the pattern's rule was found for the students, since she realized the students' difficulty in generalization with her appropriate knowledge to understand the needs and difficulties of students with pattern generalization (KCS3+).

4.1.2.2.1.2. Practice Two: Discussing on the Activity Related to Addition and Subtraction of Algebraic Expressions

During the second day of the instruction, Teacher A implemented the activity that had 'eggs-olives' context to make the need of the requirement of like terms for addition. This activity had two plates that one of them had 2 eggs and 4 olives, the other plate had 1 egg and 6 olives. The teacher drew two plates on the board and asked how eggs and olives could be added:

> A: How can we express the addition of eggs and olives?
> S: 2y+4z.
> A: Is there anyone that thinks as 6yz?
> Students: No.
> A: That's, you have added eggs and olives separately. It is same for the second plate. 1y+6z. Can we not to write 1 as the coefficient of y?
> Students: Yes.
> A: y+6z.
> S: Why we express them as y and z?
> A: We used the letters to represent egg and olive.
> S: When we add them, will we express 7y+z?

A: Since we cannot add them, we wrote them separately. They could not be added.

In this situation, Teacher A used this activity to teach the requirement of like terms to do addition of algebraic expressions. Actually, she had indicated to remind the concepts of term, coefficient, and unknown with this activity in planning; but she used it appropriately in the instruction. Thus, she drew the plates on the board that had eggs and olives as different foods, and she asked how they could add the eggs and olives. Since this question provided the students to think the similar ones could be added, the teacher used this activity to connect the concept of like terms as mathematical ideas. At this point, her knowledge to decide when to ask a new question to further students' learning in a classroom discussion appeared appropriately (KCT5+). However, the student answered as 2y+4z, Teacher A accepted this answer and the teacher also expressed the second plate as 1y+6z algebraically. The number of eggs and olives were known in these plates and thus expressing them algebraically was troublesome. That was, one student did not understand why they represented egg and olive with y and z letter. Another student asked to the teacher if the sum of the number of food in the second plate was as 7y+z. The student might have added the coefficients of 1y+6z as 7 and wrote 7 at the beginning of y+z expression. Since Teacher A did not make adequate explanations about like terms, some students might not understand the procedures of addition of algebraic expressions. Indeed, the teacher's asking of how eggs and olives were added was crucial to connect the requirement of like terms for doing addition. However, using of variables to represent of the knowns was troublesome. Some students did not understand why eggs and olives were added separately since not making an explanation after getting the correct answer and her knowledge of to decide when pause for more clarification appeared inadequately (KCT5-). Then, the teacher asked how they would add the foods in the two plates:

A: For the first plate, it is 2y+4z; for the second plate, 1y+4z. How can we add them?
S: We have added egg and egg, and olive and olive.
A: Is everyone agree?
Students: Yes.

A: Then, 3y+10z. You have added the like terms (She is showing on the board as in the below figure).

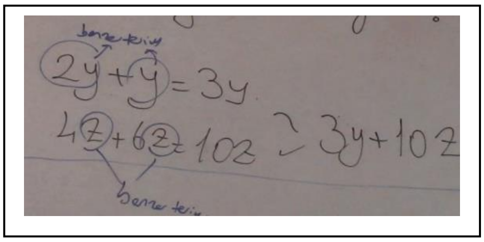


Figure 26 The addition of 2y+4z and y+4z by Teacher A

The teacher showed the addition of like terms as in the figure that she added 2y and y and then 4z and 6z respectively in order to point out the like terms. She put together their results as 3y+10z. Teacher A finished the lesson with giving the following explanation for addition and subtraction of algebraic expressions:

> The coefficients of like terms are added or subtracted for addition and subtraction of algebraic expressions.

Actually, she had stated the definition of like term concept with this explanation in her lesson plan, but she gave only this explanation part in the lesson. This explanation was about the procedure of addition and subtraction. It was explained that the coefficients of like terms were added or subtracted and so this explanation was essential for students' understanding. However, the teacher did not provide more explanations by herself after the students wrote it their notebooks. Her knowledge about usable explanation appeared as merely choosing it from the textbook (SCK3⊥).

4.1.2.2.1.3. Practice Three: Choosing the Examples or Activities to Use to Take the Students Deeper into Addition and Subtraction of Algebraic Expressions

After the teacher implemented the activity related with the number of eggs and olives in order to remind like term concept, she proceed the lesson with modeling activities to improve students' understanding and take them deeper into content. To do this, the students brought algebra tiles that they prepared with colorful papers before the lesson. Teacher A asked to the students to model 3*x*-2 expression first:

> A: Let's model 3*x*-2 with algebra tiles. Blue papers represent minus sign; green ones represent plus sign. How can we represent? How many '*x*' are there?
> S: 3.
> A: For modelling 3*x*, take 3 yellow papers; for modelling -2, take 2 blue papers (she is representing these papers on the board as in the below figure) When do you see this representation, what will you say? 3*x* and -2, 3*x*-2. Now, you model 2*x*+6.

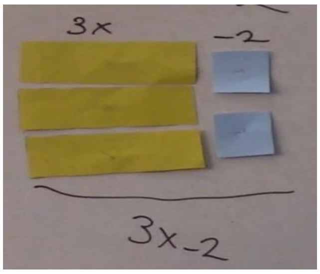


Figure 27 The modeling of 3*x*-2 with algebra tiles

> S: (He is modelling with 2 yellow papers and 6 green papers)
> A: Okay, if we add 3*x*-2 and 2*x*+6 by modeling. What happens when blue and green papers come together?
> S: First, I add 3*x* and 2*x* (She is sticking 5 yellow papers on the board). It is 5x. (Then, she is sticking 2 blue and 6 green papers on the board)
> A: Will we represent like this? What is the result of adding of -1 and +1?
> S: 0.
> A: -1 and +1 neutralize each other. 3*x*+2*x*=5*x*, -2+6=+4.

In this situation, Teacher A chose the algebra tiles for modelling 3*x*-2 and then addition it with 2*x*+6. She wanted the students to use their colorful papers on their desks. She did not make any explanations these papers about what represented and so she did not explain that these colorful rectangles and squares represent the areas of them. She should have explained to the students that x represented the area of the rectangle whose sides were x and 1, and 1 represented the area of the square whose side was 1. The students used the algebra tiles without knowing the area conceptualization that was what x and 1 represented. Without these explanations for the students' understanding, her knowledge to link representations to underlying ideas that was area concept appeared inadequately (SCK2-). Besides the modeling, Teacher A stated that -2+6=+4 since -1 and +1 neutralized each other. However, she stated that she did not use counters while teaching addition of integers in the post-interview. Thus, it might be asked if the students understand this rule. Her knowledge of how to choose, make and use mathematical representations effectively appeared (SCK7+), but she should have made more explanations for the first modeling example. After that, she asked to the students to model (2*x*+5)+(4*x*-9) with algebra tiles and this example was not in her lesson plan. She decided to model it at that time of teaching. Teacher A used the algebra tiles to represent *x*, +1 and -1 as explained in the previous example. She used the models to provide visualize for *x* as abstract concept particularly for the students.

In teaching subtraction, she asked Q2 (as indicated in planning section) to the students first and then she posed a similar question (Q3) at that time:

> Q3: Ali ate 10 anchovies and 20 scads; Ayşe 5 ate anchovies and 12 scads in dinner. How much more did Ali eat than Ayşe?

The teacher asked the questions with the aim to conceptualize subtraction in problem context. Q2 and Q3 were similar and they might cause misunderstanding for the students. Because, the problems had not any unknown situations and Teacher A selected them to represent the number of animals or fish algebraically. The students expressed for Q3 as (10h+20i) – (5h+12i) = 5h+8i algebraically. Teacher A expected this answer and approved it. However, the number of them was apparent, thus the representation of them was troublesome since her knowledge to choose which examples to use to take students deeper into subtraction of algebraic expressions appeared inappropriately (KCT1-). Then, the teacher used algebra tiles to model (5*x*-3)-(2*x*-2) as in the following script:

> A: Let's model (5*x*-3)-(2*x*-2), then with other way (The student is modeling 5*x*-3, then 2*x*-2 respectively. Last, he is modelling the result as 3*x*-1 as in Figure 28).
> S: 2*x* is subtracted from 5*x*, 3*x*. Two of -1 is subtracted from 3, it remains one -1.
> A: This is the first solution. Let's do with the other way.

S: 5x-2x=3x, and -3-(-2).
A: -3 and +2, how can we add them? What happens if the signs are different?
S: We will subtract; it is -1.
A: Okay, is there anyone that distribute the minus in the expressions in parenthesis? It would be better. You can do like this, 5x-3-2x+2=3x-1.

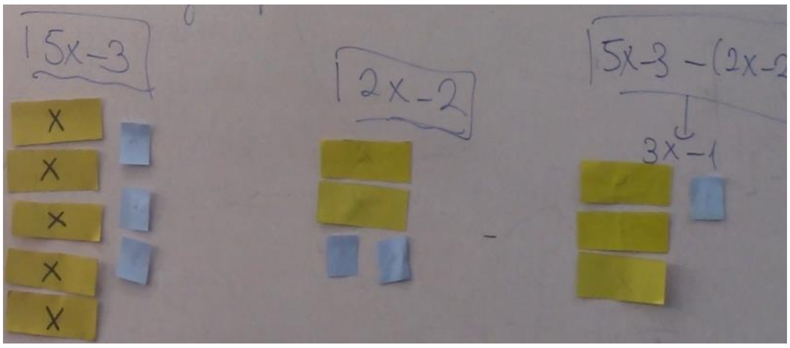


Figure 28 The modeling of (5x-3)-(2x-2) with algebra tiles

In this situation, Teacher A asked the students to model the subtraction operation, but she made errors with using this model. Since this is a subtraction operation, only the first expression is modelled and then the second expression is taken away from the first one. In this modelling, three expressions as minuend, subtrahend, and difference, were modelled separately. Teacher A's knowledge of how to use mathematical representations as algebra tiles for teaching subtraction appeared inappropriately (SCK7-). Then, while Teacher A was explaining the studentS's solution, she asked to the students what the sign of result if the different signed numbers were added. This question was about memorization of the rule of addition. Lastly, Teacher A asked the students to use the other way with multiplying the minus sign before the parenthesis by the expressions in the parenthesis that was the distribution property. She applied this property by herself and did not explain in detail.

However, Teacher A had difficulty in using algebra tiles for modelling the second subtraction example. She did not indicate this example in her lesson plan and she formed it at the time. She had difficulty with using algebra tiles for modeling of it and gave up to use tiles, and then she used the distributive property to do operation as in the following:

> A: Let's model 4*x*+3-(2*x*-1).
> S: Is the answer 2*x*+4?
> A: Yes. You did with using the other way. First, we will model with the tiles. You can model the last column.
> S: Subtract 2*x* from 4*x*, it is 2*x*. There is a minus sign. When minus is multiplied by minus, it is plus. That is +1. We will add +3 and +1, it is +4.
> A: You have done with the other way.
> S2: Why did this not happen in previous example?
> A: That's to say, their signs are different. Subtract -1 from +3 (She cannot model this operation). Hııı, I did not show you modeling by counters while teaching the addition and subtraction of integers. I did not confuse you anymore. It is difficult with modeling. Let's distribute the minus sign. Your friend has found 2*x* by subtracting 2*x* from 4*x*. +3-(-1), +3+1, it is +4.

In this situation, Teacher A asked the students to model 4*x*+3-(2*x*-1). The student gave the correct answer at once. Teacher A wanted him to show his solution and asked to show the modeling as in the first example by pointing out the last column. That was modeling of three expressions, as minuend, subtrahend, and difference separately. However, the student did the subtraction using the distributive property and explained with his way. In this example, since there was a minus before 1 in the parenthesis, when the parenthesis was removed, -1 was multiplied the minus and it became +1. Then, Teacher A tried to model this operation with the tiles, but she had difficulty in showing of the subtracting -1 from +3 in +3-(-1). She could not model this operation and gave up modeling and her knowledge of how to use mathematical representations as algebra tiles for doing subtraction appeared inadequately (SCK7-). She used the student's solution and explained it to the class.

After the lesson finished, the researcher interviewed with the teacher to get her ideas about the lesson. She indicated the difficulty with using modeling with algebra tiles for teaching subtraction as in the following:

> A: The beginning with modeling was good, but we did not use it for subtraction. I did not use counters for subtraction of integers anyway. I was confused while using it in the subtraction.
> Researcher: For this question, you mean the subtraction as +3-(-1)?
> A: Yes.

Researcher: For this, we have to subtract -1 from +3. You might say that you add something that does not change +3. For example, +1 and -1. You take -1, and you have +1. Then, add +1 and +3 as +4.
A: It is correct. If I had taught like this in integers. It might be easy with counters. I did not understand the modeling in subtraction while studying, thus I could not show.

When the researcher suggested the way that could be used in modeling of subtraction, Teacher A understood and admitted that she could not understand how she used also in integers before. She stated that she tried for algebraic expressions, but she could not show the subtraction with the tiles and her knowledge of how to choose and use algebra tiles as representations for teaching subtraction of algebraic expressions appeared inadequately (SCK7-).

4.1.2.2.1.4. Practice Four: Implementing the Suggested Activities

In this part of the lesson, Teacher A implemented some of the researcher's suggested activities. Most of her selected activities were about writing algebraic expressions. One of them was writing algebraic expressions for the given verbal statements such as 5 less of the addition of x and y, the addition of 2 times x and 3 times y. Although the students answered x^2+y^2 for addition of power 2 of x and power 2 of y, the students had difficulty only the power 2 of addition of x and y. Thus, Teacher A's knowledge of how to provide mathematical explanations for procedures appeared appropriately (SCK6+) and she explained it with adding first and then getting the power 2 of this addition and answered $(x+y)^2$.

After that, the similar activity related to the context as people in the amusement park was done. In the activity, the representations of the number of people with letters were given such as A=the number of woman workers in the amusement park; E=the number of people who were 18-aged and below visited the amusement park in a day; F=the price of adult ticket for a day. It was asked to represent given verbal statements using these letters. To illustrate, one of the statements was that the number of people who visited the amusement park in a day and its algebraic representation was A+B+C+D+E, or the total money that came from the tickets in a day (suppose each visitor took the ticket for a day and regarding his/her age) and its algebraic representation was (C+D).F+(E.G). Teacher A involved this activity to her lesson plan where she planned to do exercises after teaching operations. Actually, the researcher suggested this activity using before teaching operations in order to recall the students' prior knowledge about algebraic expressions, but the teacher put it as she preferred to the doing exercises part of the lesson. Her knowledge to predict that students would find the amusement park context interesting and motivating appeared to select this activity (KCS6+). In the implementation of the activity, Teacher A wanted the students who could write the statements algebraically to answer. Several students could write the algebraic representations and they answered. Teacher A did not make any explanations or discuss with the class about the statements. Although the students answered the previous question easily, they had difficulty with understanding when there was a context and thus her knowledge to understand the needs and difficulties of students with writing algebraic expressions appeared inadequately (KCS3-).

The next activity was about explaining how the given paired algebraic expressions were equal. These expressions were: n+n+n+n and 4xn; nxnxnxn and n^4; n+n+n+n and (2xn)+(2xn); $(m+1)^2$ and (m+1)x(m+1); $(m+1)^2$ and m^2+2m+1. The two students answered for the first two expressions as in the following:

S: n+n+n+n is 4n; 4xn is also 4n. So they are equal.
A: These are equal. The second one?
S: The first is n^4, so they are equal.
A: The others are equal and goes like that.

In this situation, Teacher A asked to the students whether the pair expressions were equal or not. However, it was asked to explain why they were equal and it was given that they were equal already. Teacher A might have missed what was asked in the activity and so she accepted the students' answers and she did not make any explanations herself. Thus, her knowledge to answer the question correctly appeared inadequately for this activity (CCK3-).

Finally, Teacher A wanted the students to do the rest of the questions operating with number strips activity. In these questions, the patterns were linear growth, and it was asked to find the pattern and its rule. The implementation of one of the questions was as in the following:

> A: Let's find the rule of the second pattern. The first is given as n^2 (1, 4, 9, 16…). The third pattern goes like 16, 25, 36 ... and its rule $(n+3)^2$.
> S^*: The second pattern is like that 15, 21, 27, 33, 39, 55...
> A: What is the rule of it?
> S^*: They are increasing by 6.
> A: Then the rule starts with 6n. To get 15?
> S^*: 6n+9.
> A: Then the third one's rule is n^2+6n+9.
> (S^* represents the same student)

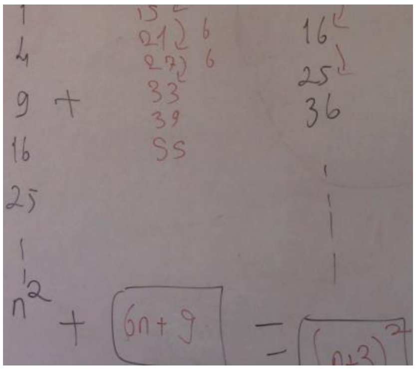


Figure 29 Operating with number strips activity

In this situation, the non-linear growth patterns' rules were given (n^2 and $(n+3)^2$) and it was asked to find and generalize the subtrahend pattern as the second one. The student found the terms of this pattern first, and then she got the rule of it as 6n+9 with the help of the teacher. Teacher A asked to him what should be added to 6n to get 15 when n=1. Finally, Teacher A said that the third pattern as the result of the subtraction was n^2+6n+9. However, she wrote $(n+3)^2$ without any explanations about how they were equal. This was about the concept of identities that it was in the 8th grade curriculum, thus she might not have selected this question. Her knowledge to know the content in the curriculum appeared inappropriately (KCC1-).

4.1.2.2.1.5. Practice Five: Doing Exercises Related to Addition and Subtraction of Algebraic Expressions from Textbook and Workbook

After the suggested activities, Teacher A continued with the exercises in the textbook. The exercises included addition and subtraction questions. The first question was about determining the properties of addition operation for the given similar arithmetic and algebraic expressions such as 2+(7+4)=(2+7)+4 and $a+(b+c)=(a+b)+c$:

> S: It shows the distributive property.
> A: No.
> S: Associative property.
> A: Yes, what happened in the distributive property? For example, if 2.(7+4) is given, we multiply 2 with the first number, and then with the second number in the parenthesis, that would be 2.7+2.4. Here, it is associated. All signs are positive.

In this situation, Teacher A made explanation for who considered the example was related the distributive property. However, the students could not decide this example was not an example for this property. Thus, Teacher A gave example as 2.(7+4) and explained the application of the distribution property on this example. She might have expressed the distribution property for multiplication operations, but she had explained that the signs were positive to refer addition. Her knowledge of how to provide mathematical explanations for distribution property with exemplifying on arithmetic appeared appropriately but she should have explained also what the all signs were positive meant (SCK6+). In this question, what the property for 9+0=9 and $a+0=a$ was asked in another option. For this question, one student answered neutral element, while another student answered unit element. Teacher A indicated that they were same and had the same meaning. Teacher A responded to the students appropriately with this explanation about neutral element and unit element as she used her knowledge of how mathematical language correctly (SCK5+).

In doing exercises, a problematic situation appeared in Teacher A's conception about constant term concept since she did not acknowledge the constant term as a term as in the script:

> A: What is the like term (in $9x+5x^2-9x-7$)? For example, $9x$ and $-5x$ are like?
> Students: Yes.
> A: Since they have x. For example, let's think $2x^2$ and $3x^2$ are like, is that so. There is x^2. In $9x+5x^2–9x–7$, which of them is like? $9x$ and $-9x$ are like. Are there any terms that like to $5x^2$? None. It is asked to add the coefficients. What is the coefficient?
> S: The numbers that are next to x.
> A: Yes, true. Then, 9, 5, and -9 are coefficients, is that so? Is -7 the coefficient?
> S: Constant term.
> A: Yes, it is the constant term. When we add the coefficients, 9 + 5 – 9 = 5.

Teacher A approved the student's answer for what the coefficient was. The student explained the coefficient as the numbers that were next to x. Teacher A did not accept -7 as a coefficient since she considered it was a constant term not a term with variables. Whereas the constant term was also a term, Teacher A could not consider $-7 = -7x^0$. She answered the other related questions below by considering this way incorrectly such as how many terms there were and what the addition of the coefficients was. She made errors in mathematical explanations about the constant term concept. She had the lack of knowledge the constant term concept and her knowledge to use terms appeared incorrectly (CCK2-). Thus, the students also learnt with this way incompetently. Another situation could be problematic was that Teacher A stated $9x$ and $-5x$ as like terms and explained for they had x. Then, she explained and exemplified $2x^2$ and $3x^2$ as like terms since they had x^2. At that point, she should have emphasized that the variable and its power must be the same as the requirement for like term. Because some students might have misunderstandings about determining the like terms with focusing on only variable without considering its power. Thus, her knowledge to anticipate the misunderstandings that might arise with studying on like terms appeared inadequately (KCS2-).

In another question, that was about deciding whether the given statement was true or false with determining the terms and like terms, adding the coefficients, and simplifying the given expressions, Teacher A did not take the constant term as a term and answered the questions incorrectly as explained in the above question. She also taught the term concept as she knew and the students had also lack of knowledge as in the following script:

> A: a^2-3a-4+2a+5a^2, it is asked how many terms there are. Is 'a^2' a term? Yes. Is '-3a' a term? Yes. Is -4 a term?
> Students: No.
> A: What is it? It is a constant term, not the term. 2a is a term, 5a^2 is a term. How many terms are there? 4.

In this situation, Teacher A explained that -4 was not a term since it was a constant term. The students also knew like this. Thus, Teacher A answered as there were 4 terms in this expression, but there were 5 terms including -4. She knew -4 as constant term correctly but she did not accept it as a term so that her knowledge to use terms appeared incorrectly (CCK2-). It showed her lack of content knowledge about term concept in algebraic expression. In the post-interview, the dialogue about constant term between the researcher and the teacher as in the following:

> R: You did not accept the constant as a term.
> A: Is it a term?
> R: The constant term is also a term.
> A: So, we take it as a term?
> R: Yes, the term is described as the part that is divided by plus or minus sign. You can examine like this. The constant is a special situation.
> A: It is also a term. That's right.

In this situation, Teacher A realized the error she made in the instruction and she learnt the constant was a term in this dialogue. The researcher explained for her that the term was the part which was divided by a sign in the expression. Then, she approved the constant term was also a term.

However, some students had misunderstanding about the coefficient concept, and the teacher recognized it and her knowledge of how to address and remedy students' errors appeared effectively (KCT8+). In the question related to simplifying the given algebraic expression ($7x^2$–$4x$+$5x$+x^2+1) with adding or subtracting the like terms, the student confused with the power and the coefficient of x^2 as in the following:

> A: What is the addition of $7x^2$ and x^2? What is the coefficient of x^2?
> S*: 2.

A: Is it? What is the coefficient of $7x^2$? What is the unknown?
S*: *x*.
A: The power 2 is exponent. Then, what is the coefficient?
S*: 7.
A: What is the coefficient of x^2?
S*: 2.
A: There is one unknown here.
(S* represents the same student)

In this situation, Teacher A guided the student to add $7x^2$ and x^2 by asking what the coefficient of x^2. The students answered 2 because of confusing the power and the coefficient. Since Teacher A realized that, she asked the student what the coefficient of $7x^2$. She explained that the power was 2 and thus the student said that the coefficient was 7. After all, the student answered again that the coefficient of x^2 was 2. Then, Teacher A said that there was one unknown referring x^2. The student did not conceptualize the concept of coefficient and he confused it with the power. Teacher A might showed and explained with indicating the coefficient and the power on the board.

Beside the problems in the use of terms, there were problems in using notations such as equal sign or minus sign. To illustrate, many students used the equal sign inappropriately. However, Teacher A did not correct these representations and she accepted them. The student did operations for 1-5m-9+m+12m as seen in the below figure. The dialogue between the student and Teacher A was as in the following script:

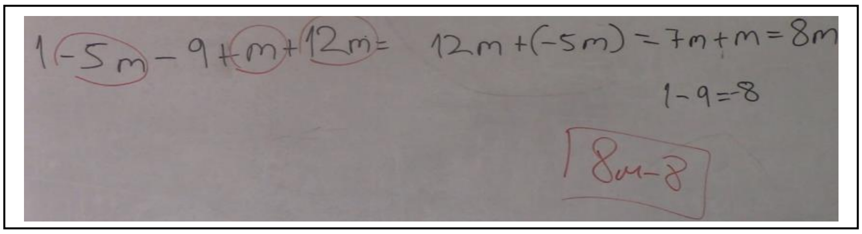


Figure 30 The use of equal sign by the student

S*: 7m+m, it is m^2.
A: But, it happens in multiplication.
S*: Then, 8m.

A: Now, add the numbers, (The student is writing 1-9 = -8), bring together the results.
(S^* represents the same student)

In this situation, Teacher A responded to the students inappropriately since the student might had misunderstanding about m^2. However, Teacher A explained that this operation was not a multiplication to correct the student's answer as m^2 for 7m+m. The explanation for this student's answer should have been that multiplication of a variable by itself yielded the power 2 of it. She had explained in the Bacterial Growth Activity at the beginning of the instruction. There were two items of the same variables and she represented the power of 2 it (e.g. $2n.3n=6n^2$ since there were two items of n). This situation was pointed out since it could cause the misunderstanding for students. Then, the student answered 8m correctly, but he used the equal sign inappropriately to show the result that he found as in the figure. The student added the variables and constant terms separately. He started to write the addition of variables as if it equaled the whole expression. He first added 12m + (-5m), got 7m. Then, he wrote 7m after the equal sign and add m, and got 8m. In this case, 8m was the result of the question regarding the using the equal sign. Teacher A did not intervene any of these representations. However, the student used the equal signs incorrectly. After Teacher A asked to add the constant term, the student added them (1-9) separately and found -8. Teacher A wrote the results together as 8m-8 at the bottom, not the place of the answer for the asked question and her knowledge of how mathematical language was used inappropriately (SCK5-).

Another notation that Teacher A used incompetently was minus sign before the expressions. Teacher A sometimes expressed the like terms without their signs to the students. It was asked to add the coefficients up, and Teacher A asked the students which one of them was like in $3y^2-4yx+5y^2-yx$ as in the following:

A: Which of them is like?
S: $3y^2$ and $5y^2$.
A: Yes, else?
S: 4*yx* and *yx*.
A: Yes, 4*yx* and *yx*.

In this situation, Teacher A approved the student's answer that was 4yx and yx were like terms. However, the student expressed these terms incompetently since she did not indicate their signs. Teacher A also did not express their minus signs and accepted the student's answer as so. Thus, she made errors in notations while expressing the like terms and her knowledge of how mathematical language was used appeared inadequately (SCK5-). Despite this, the teacher took their minus signs while adding the coefficients and answered the question correctly.

Some students could not understand how they added or subtracted the algebraic expressions and they could make errors. To illustrate, in finding *a*+*b* when *a* = (2*x*-1), *b* = 3(*x*+3), the student did operations as in the following script:

> S*: Since there is not a distribution in a, I am writing 2*x*-1=1 (The teacher corrected it, then the student is distributing 3 to (*x*+3) for b as in the Figure 31)
> A: Have you found 12*x* from adding 3x and 9? Are they like terms? (The student is writing 3x+9). Add 2x-1 and 3x+9.
> S*: Since there is not a distribution in a, I am writing 2x-1=1 (The teacher corrected it, then the student is distributing 3 to (*x+3*) for *b* as in the Figure 31). I will add the like terms. -1+9=10
> A: When you add -1 and 9, what is the result?
> S*: -10, since their signs are different.
> A: 5*x*+8
> (S* represents the same student)

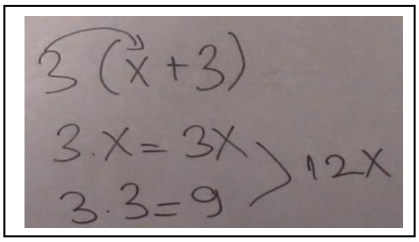


Figure 31 The addition by the student

In this situation, the student explained as 2*x*-1 =1 incorrectly. However, 2*x* and -1 were not like terms and thus they could not be added or subtracted. Teacher A did not make any explanations that these terms were not the like terms, and she erased 1. Then, the student used the distribution property for b, and he found 3*x* and 9, added them as 12*x* incorrectly. Teacher A asked the student to add 2*x*-1 and 3*x*+9.

This time, the student added the constant terms as -1 and +9 added incorrectly and found 10. When Teacher A asked what the result was again, the student answered -10 and explained their signs of the numbers were different. The reason might be the memorization of the rules about addition of the integers not the conceptualization of them. Lastly, Teacher A got the result herself when the student could not answer correctly. Although some students as observed could not conceptualize the like term concepts for addition and subtraction, Teacher A did not provide any explanations again. Her knowledge of how to address the student's errors and remedy them appeared inadequately (KCT8-) since she realized the student's errors while instruction, she only corrected them herself, but she did not ensure if the errors were remedied and the students learnt conceptually the concepts.

4.1.2.2.1.6. Practice Six: Presenting Problems that Combine Knowledge Related to Fraction and Geometry

After doing the suggested activities and exercises in the books, she presented the problems that required the prior knowledge about fractions and geometry for improving students' understanding. These problems required to combine the prior knowledge and operations with algebraic expressions being learnt, that might be more challenging for the students. Teacher A presented them but she could have difficulty in solving them in the instruction. The solving process of the two problems was exemplified in the following.

The following problem was related with fractions and could be appropriate for arithmetical solution methods. Since it was in the textbook, the teacher asked to the students to solve it. The problem and the solution of it by a student as in the following:

> 'Problem: One bus driver goes 2/9 of the distance that he should go in the first hour; 5/18 of it in the second hour; 1/6 of it in the third hour. What is the remaining distance over the total distance?'

S*3: I equated their denominators to 18, $\frac{4}{18} + \frac{5}{18} + \frac{3}{18} = \frac{12}{18}$.

A: For example, what do we do when it is asked 2 times of 12? We multiply 12 by 2. Do we know the distance that he should go? No. then, we add x in front of them. The driver went 12x/18 of the distance. How can we find the remaining distance? The whole is 18/18, then I will subtract from it.
S*: 18/18-12/18=6/18.
(S* represents the same student)

In this situation, the student solved the problem arithmetically that she knew before. The students could not consider the use of algebra and represent the unknown situation with a variable. Instead of this, she preferred to use fractions and arithmetical operations. Teacher A guided the student to use algebra by writing x in front of fractions. She explained that the distance the driver would go since they did not know it for representing with x. Her knowledge of how to provide mathematical explanations might have appeared inadequately and some students could not understand why x was written before the fractions (SCK6-). Thus, she might have made more explanations how she used algebra in solving this problem.

The problem related with geometry was as in the following figure:

Calculate the perimeter of the shape using the isosceles triangle.

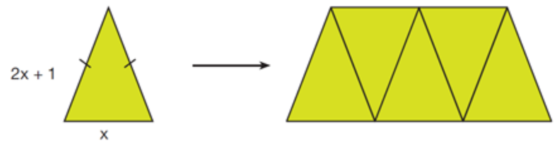


Figure 32 The problem related with the perimeter of triangle

One student solved this problem correctly and then the teacher explained the student's ideas and solving method. However, the student used the equal sign inappropriately as in the **Figure 33**. The student and the teacher explained the solution as in the following:

S: (2*x*+1)x2=4*x*+2, add the all *x*, 5*x*. In sum 9*x*+2.
A: The sides of the triangle are 2*x*+1, the base is *x*. How many is 2*x*+1 there? 2. Then, your friend multiplied 2 by 2*x*+1. He distributed 2 to the parenthesis. First multiply 2 by 2*x*, then 1. Then, he counted *x*, and found 5*x*. When they were added, it was 9*x*+2.

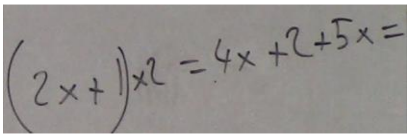


Figure 33 The use of equal sign by the student

In this situation, the student wrote the lengths of the sides of the shape first as seen in the figure and he solved the problem adding all the lengths to find the perimeter. The student first multiplied 2*x*+1 by 2 since there were two sides that the length was 2*x*+1. Then he counted the sides with length *x*, and found 5*x*. He wrote this 5*x* next to the result of the multiplication and he used the equal sign inappropriately. Teacher A did not correct this representation, and approved the result as correct and explained this student's solution to the class so that her knowledge of how mathematical language was used inappropriately (SCK5-).

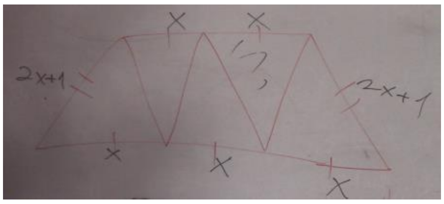


Figure 34 The lengths of the polygon

4.1.2.2.1.7. The Extracted Knowledge Types from the Instruction for Addition and Subtraction of Algebraic Expressions

Table 10 The extracted knowledge types from the instruction for addition and subtraction of algebraic expressions

Practices	Extracted knowledge types				
	SMK		PCK		
	CCK	SCK	KCS	KCT	KCC
Choosing an example or activity to start teaching addition and subtraction of algebraic expressions with connecting to topics from prior years	CCK2(-)	SCK1(+,+,+) SCK3(⊥) SCK6(-,+)	KCS2(-,-) KCS3(+)	KCT1(+)	
Discussing on the activity related addition and subtraction of algebraic expressions		SCK3(⊥)		KCT5(+,-)	
Choosing the examples or activities to use to take students deeper into addition and subtraction of algebraic expressions		SCK2(-) SCK7(+,-,-,-)		KCT1(-)	
Implementing the suggested activities	CCK3(-)	SCK6(+)	KCS3(-) KCS6(+)		KCC1(-)
Doing exercises related to addition and subtraction of algebraic expressions from textbook and workbook	CCK2(-)	SCK5(+,-,-) SCK6(-)	KCS2(-)	KCT8(+,-)	

Table 10 (Continued)

Practices	Extracted knowledge types				
	SMK		PCK		
	CCK	SCK	KCS	KCT	KCC
Presenting problems that combine knowledge related to fraction and geometry		SCK5(-) SCK6(-)			

Table 10 shows what type of knowledge of subject matter knowledge (SMK) and pedagogical content knowledge (PCK) that Teacher A had in instruction practices. (+) sign indicates the teacher's existing knowledge was adequate or appropriate and she used it effectively, while (-) sign indicates the teacher's existing knowledge was inadequate or inappropriate and she used it ineffectively. Each sign (+ or -) in the same knowledge type refers to Teacher A's different use of this knowledge during instruction. Besides, for SCK3 knowledge type, (┴) is used to indicate the teacher's knowledge to develop definition or explanation appeared as merely choosing it from the textbook and presenting to the students. Since this situation meets the code (SCK3) partially, (┴) sign is used.

SCK1(+) indicates that her knowledge to connect the topic with prior topic with reminding the concepts of term, coefficient, and constant term was appropriate for the beginning. To do this, she gave the definitions of them as in the textbook (SCK3(┴) in the second practice). However, while reminding these algebra concepts, CCK2(-) in the first practice indicates that her knowledge to use terms was inappropriate since she used the unknown and variable concept as they had the same meaning. Beside this, KCS2(-) in the first practice indicates that her knowledge to anticipate the misunderstandings that might arise with the term concept was inadequate as she showed the constant term without its sign of. KCT1(+) indicates that her knowledge to choose to start with the beginning activity appeared effectively in order to connect generalization of patterns as prior topic (SCK1(+,+) in the first practice). With similar purpose, she presented *Operating with Number Strips* activity for the students, but the implementation of it was inappropriate. Because SCK6(-) in the first practice indicates that her knowledge of how to provide mathematical connections between getting the general rule of result pattern and operating of the general rules was not as expected. However, SCK6(+) in the first practice indicates that her knowledge of how to provide mathematical explanations of applying of distribution in subtraction operation was adequate. Beside this, KCS3(+) indicates that her knowledge about students' difficulty so that she provided explanations how the pattern's rule was found. Nevertheless, her knowledge to anticipate the misunderstandings about getting the general rule that might arise was inadequate (KCS2(-) in the first practice). The discussion in implementing of the activity that had 'eggs-olives' context, while KCT5(+) indicates that her knowledge to decide when to ask a new question to further students' learning with asking how to add the terms was appropriate, KCT5(-) was inadequate that she did not pause to make clarification about how to add like terms and to represent the items with letters as algebraically. Although SCK3(⊥) indicates she provided usable explanation of the procedures for addition and subtraction operation, it was appeared as merely choosing it from the textbook she did not provide more explanations by herself.

Most of the extracted knowledge types in the third, fourth, fifth and sixth practices were about modeling with algebra tiles and using language related to algebra concepts and notations. SCK2(-) indicates that her knowledge to link algebraic representation with geometric representation by using area concept was inadequate so that she did not use with this link in modeling with algebra tiles. Instead of this, she used them as visual representation and showed addition examples with them adequately (SCK7(+)). However, SCK7(-,-,-) indicates that her knowledge of how to use mathematical representations as algebra tiles for teaching subtraction appeared inappropriately and she could not show the subtraction with the tiles. Beside modeling, she used problems related with real life situations for teaching but KCT1(-) indicates that her knowledge to choose which examples to use to take the students deeper into subtraction of algebraic expressions was inappropriate as she represented the number of known items with algebraic representations. On the other hand, SCK5(-,-,-) indicates the problems in using of equal sign and showing the minus sign. The students did not use equal sign appropriately or they ignored the minus sign in determining the term in expressions, but the teacher did not correct them and make any explanations. Actually, she used appropriately neutral element and unit element (SCK5(+)), and she provided adequate explanations for the distribution property (SCK6(+) in the fifth practice). But, she had the lack of knowledge of the constant term concept and her knowledge to use of it incorrectly so that she did not accept it as a term (CCK2(-,-) in the fifth practice). One of the important terms was like term concept, but KCS2(-) in the fifth practice indicates that her knowledge to anticipate the misunderstandings that might arise with studying on like terms was inadequate since she only emphasized the kind of variable without considering its power. Thus, some students had difficulty to determine the like terms for addition or subtraction later on, but her knowledge to address the student's errors about like terms and to remedy them appeared inadequately (KCT8(-)). However, she provided explanation to remedy the students' errors in confusing with the coefficient and power of the same term (KCT8(+)). Apart from these, in solving questions or implementing activities, she sometimes could not solve or answer the questions correctly by ignoring what was given or what was asked (CCK3(-)). Related with activities, although SCK6(+) in the fourth practice indicates her knowledge to explain how to write algebraic expression adequately, KCS3(-) in the same practice indicates her knowledge to understand the difficulties of students in writing algebraic expression in a context was inadequate.

4.1.2.2.2. Practices in the Instruction of Multiplication of Algebraic Expressions

The teacher's purposeful actions for teaching multiplication of algebraic expressions were grouped into six practices: 1) choosing an example or activity to start teaching multiplication of algebraic expressions with connecting to topics from prior years, 2) discussing on the activity related to multiplication of algebraic expressions, 3) choosing the examples or activities to use to take the students deeper into multiplication of algebraic expressions, 4) implementing the suggested activities, 5) doing exercises related to multiplication of algebraic expressions from textbook and workbook, and 6) presenting problems that combine knowledge related to geometry. The extracted teacher's knowledge based on MKT framework was analyzed within these practices. The reflection of the instruction after each lesson was also presented to provide the teacher to evaluate her instruction by herself. The interpretations of the teacher could give information about her knowledge about students' thinking and learning with respect to the instruction. The classroom dialogues that were most representative for knowledge type the teacher had, were selected from the instruction to illustrate how the teacher used her knowledge in teaching.

The first practice was choosing an example or activity to start teaching multiplication of algebraic expressions with connecting to topics from prior years and this title was also extracted from one of the descriptors of KCT and SCK. This practice examined that the teacher chose which example or activity to start teaching multiplication of algebraic expressions with rationale, and how she implemented it in the classroom. The activity provided to connect pattern generalization that the students had learnt before. The second practice was discussing on the activity related to multiplication of algebraic expressions and this practice was also affected by the descriptors of KCT. This practice included a small discussion about how an integer and an algebraic expression was multiplied in a problem context. She let the students to explain their answers and responded their questions that they asked to understand. The third practice was choosing the examples or activities to use to take the students deeper into multiplication of algebraic expressions and this title was also extracted from one of the descriptors of KCT. This was also as a continuation of choosing an example or activity to start practice. This practice included how the teacher taught multiplication of algebraic expressions to improve students' knowledge with getting deep the content using algebra tiles and problems that had real life situations. The fourth practice was doing exercises related to multiplication of algebraic expressions from textbook and workbook and the teacher asked the exercises to the students as in the order of the book. This part was as questioning by the teacher and answering by students. The fifth practice was presenting problems that combined the knowledge related to geometry. The teacher presented problems that required geometry knowledge from textbook and workbook to improve students' understanding of multiplication of algebraic expressions. The sixth practice was implementing the suggested activities about addition, subtraction, and multiplication of algebraic expressions. These practices are explained with examining how the teacher used her knowledge based on extracted knowledge from MKT framework.

4.1.2.2.2.1. Practice One: Choosing an Example or Activity to Start Teaching Multiplication of Algebraic Expressions with Connecting to Topics from Prior Years

At the first lesson of the instruction for operations with algebraic expression, Teacher A implemented the Bacterial Growth Activity to start operations with algebraic expressions and her knowledge to choose which examples to start with appeared effectively (KCT1+). This activity had two patterns in the context of growing bacteria and the first one was growing as 2, 4, 6, 8 …, and the second one was growing as 3, 6, 9, 12 … It was asked to recognize and explore the addition and multiplication of the general rules of patterns as 2n and 3n as algebraic expressions. The implementation of addition and subtraction part of this activity by Teacher A was explained in the addition and subtraction practices section in detail. Teacher A's knowledge to connect also multiplication of algebraic expressions to pattern generalization as previous topic appeared in implementing of this activity (SCK1+).

In the multiplication of patterns, Teacher A approved the student's explanation for the result of multiplication n by n in multiplication of 2n and 3n. Since some students had difficulty with conceptualization of n^2 as the result of multiplication n by n, Teacher A asked the student to explain why the result was n^2:

> A: Now, it is asked also to multiply the patterns (She is writing 6, 24, 54, 64 ... from multiplication of the terms). Multiply 2n by 3n.
> S: 6n.

A: How can we multiply? Multiply 2 by 3, 6. What can we do with ‘n’s?
S: n^2.
A: Why?
S: Since there are two ‘n’s.
A: Yes, there are two ‘n’s. That’s $6n^2$. Let’s check it with using the pattern terms.

In this situation, Teacher A’s response to the student was not adequate as mathematical explanation since the existence of two items did not always yield n^2. To illustrate, n+n expression had two items of n and yielded 2n. Thus, her knowledge of how mathematical language was used appeared inappropriately (SCK5-). Instead of this, the teacher might have stated that there were multiplied 2 items of n.

4.1.2.2.2.2. Practice Two: Discussing on the Activity Related to Multiplication of Algebraic Expressions

For the instruction of multiplication of algebraic expressions, Teacher A started with Nermin’s money activity. This activity was in the textbook and it required to write algebraic expressions of verbal statements, and to multiply a number and an algebraic expression. This activity was as in the following:

Nermin saves money from her allowance and has some in her moneybox. Nermin puts 5TL that her mother gave to her into her moneybox. Her father said that he would give money that 2 times of the saved money. Write the algebraic expression that Nermin would take from her father and explain how you would write it.

This question had a real life context and could provide to connect the multiplication of algebraic expressions. It required to multiply 2 by n+5, which could be a simple example for the beginning of multiplication as the teacher explained in the script:

A: Nermin has a money box and some money in it. What can we write for this money that we do not know? Let’s write n. Her mother said that she would give to her 5TL since Nermin got 5 in the exam. Let’s add 5TL more.
S: n+5
A: Then, her father said that he would give to her 2 times of the money in the money box. .... (n+5)x2, do you agree with this? Is there anyone has other ideas?

S: n^2+10.

However, the students had difficulty in multiplication since they had confused with multiplication of n by n and 2 by n. They had learnt $n.n=n^2$ in the bacterial growth activity, and thus they answered n^2 for the multiplication of 2 and n. Teacher A provided explanations to show the difference of these multiplications:

A: To be n^2, you have n money, and you will be given n more.
S: It is said like this, anyway.
A: Hımm, it is not then, if you are told that n times of it, n^2 will be. $(nxn)=n^2$. But, here, 2 times of the saved money, you think like that. So, multiplication by 2, and n times of n are different. n times of n will be n^2.
S: Multiplication 3 by 2 and 3^2 are different.
A: Yes.

In this situation, Teacher B explained n^2 as addition of n and n first, when one student answered n^2+10 for representing 2 times of n+5. In response to the teacher, the student indicated that it was asked like this addressing the answer (n+5)x2. Then, Teacher A realized that she made an incorrect explanation and she corrected it immediately so that her knowledge of how to provide mathematical explanations for procedures appeared with pointing out the difference of two multiplications (SCK6+). She explained n^2 as n times of n, and emphasized that 2 times of n was 2n and it differed from n^2. Her knowledge to decide when to pause for more clarification appeared appropriately at that point of the discussion (KCT5+). Teacher A finished to solve the problem as in the following explanation:

A: How did you multiply n+5 by 2? I take 2 in front of the parenthesis and distribute 2 to the parenthesis If I write 2 in front of the parenthesis, is it the same, 2x(n+5)? In multiplication operation, the number that is multiplied can be replaced in the distributive property of multiplication. 2x(n+5)=2n+2.5, what is the result? 2n+10. What the money that is saved last? 2n+10.

In this situation, Teacher A confused with the commutative property and the distributive property. Although she used the commutative property in multiplication by changing the place of 2, she expressed this change as related to the distributive property. Her writing of (n+5)x2 and 2x(n+5) as equally was correct, but her expression of it showed a problematic usage of mathematical language (SCK5-).

Finally, Teacher A solved the problem correctly but she did not consider what was asked in the problem that she expressed the result incorrectly. She found 2n+10 to represent the money that Nermin's father gave to her as if the asked money in the problem. However, she expressed 2n+10 as the total money in the money box. It was asked to express what her father gave to her algebraically not the total money in the money box. To find the total money, she should have added 2n+10 and n+5, and got 3n+15. She did not aware of it and thus she solved the problem correctly, but she expressed the result incorrectly and her knowledge to solve problems correctly appeared partially adequate in this problem (CCK3-).

4.1.2.2.2.3. Practice Three: Choosing the Examples or Activities to Use to Take the Students Deeper into Multiplication of Algebraic Expressions

After the teacher taught the multiplication of n+5 by 2, Teacher Ashe proceed the lesson with representing repeated additions and multiplying them, and modeling several multiplications with algebra tiles to take the students deeper into the content. Related to geometry, she asked to the students to express the perimeter of the rectangle whose sides were m and n length algebraically. One of the students solved this question as in the following:

> S*: m+m+n+n=2m+2n
> A: Okay, what is the common in two expressions?
> S*: 2.
> A: Then, we can take the parenthesis of 2, we take m and n from the expression. (She is writing 2(m+n)). The distribution property of multiplication to addition.
> (S* represents the same student)

In this situation, the student added the variables separately and he did not use multiplication. Teacher A guided him to find the common number in the expression and she wrote 2(m+n) herself. She explained that it was the distribution property of multiplication to addition. This question could be useful for students for the reason that it provided to connect the repeated addition with multiplication. It also required the students to use their prior knowledge about the rectangle concept so that her knowledge of connect the topic to prior topics effectively with her selection of this example (SCK1+). After this example, she asked to the students to model two multiplications. Then, she gave similar example again that she asked what (4m-3)+(4m-3) was. The dialogues between the students and the teacher as in the following:

> S: What is m?
> A: We can say *x* instead of m. They have same function.
> S: Add 4*x* and 4*x* first, 8*x*; then -3 and -3, -6.
> A: Yes, it is 8*x*-6. Now, think this operation as multiplication. How many 4*x*-3 is there? There are 2 of them. Can we represent as 2.(4*x*-3)? Let's do using the distribution property of multiplication to subtraction.
> S: I multiply first 2 by 4*x*, then 2 by 3. 2.4*x*=8*x*, 2.-3=-6, the result is 8*x*-6.

In this situation, it was observed that some students could not have conceptualized the concept of variable and one student asked what m was, and Teacher A decided to express it as *x* and explained they had same function. Although the teacher provided explanations, she might also have explained the variable concept. Then, the student first added the like terms and got the correct result as 8*x*-6. Therewith, Teacher A guided the students to write these two same expressions as multiplication with 2 in order to represent the repeated addition as multiplication. In response to this, another student multiplied 2 by 4*x*-3 using the distribution property as Teacher A wanted. Teacher A's aim for asking this question was to connect the repeated addition to multiplication. Thus, these questions might be useful for connection with addition instead of directly using multiplication and Teacher A's knowledge of how to provide mathematical explanations for multiplication procedures appeared effectively (SCK6+). After this example, the teacher wanted the students to write this following explanation about multiplication in their notebooks:

> When a whole number is multiplied by an algebraic expression, the whole number is multiplied by each term in the algebraic expression.

This explanation was from the textbook and had limited information. Because the algebraic expressions can be multiplied by integer or rational numbers also. Thus, the teacher could have revised this explanation, or made explanation as the other numbers also could be multiplied by the algebraic expressions but her knowledge about usable explanation appeared as merely choosing it from the textbook (SCK3⊥).

Before the last example, Teacher A modeled 4*x* and 3(*x*+2) with algebra tiles with the same way for teaching multiplication. She represented *x*, and integers as +1 and -1 with the different colored papers. Teacher A wanted the students to model 4x first:

S: We should paste 4 of *x* (The yellow papers represent x).
A: Is it the long way, *x*+*x*+*x*+*x*, to add 4 of *x*? We multiplied 4 by *x*, that's 4.*x*=4*x*.

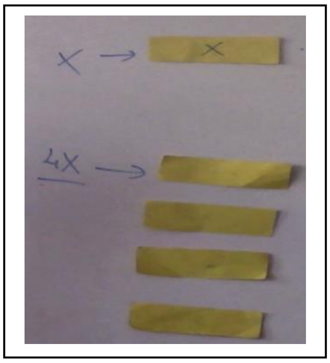


Figure 35 The representation of 4*x* with algebra tiles

In this situation, Teacher A represented 4*x* using the 4 algebra tiles of *x* as seen in the figure. She did not make explanations about the meaning of 4*x* that it was the area of the rectangle whose sides were 4 and *x*. However, Teacher A did not indicate the area concept. Instead of this, she used the tiles to represent *x*, and she explained 4*x* using 4 items of *x* tile as repeated addition. Similarly, then she explained the modeling of 3.(*x*+2) using algebra tiles as in the script:

A: How do you add x and 2?
S*: *x* plus 1 plus 1.
A: Yes, we first model the parenthesis algebraic expression (The student is pasting the board 1 yellow paper (to represent *x*) and 2 green papers (to represent +1)). How many (*x*+2) is there?
S*: 3.
A: Then, what will you do?
S*: I will do 2 more of (*x*+2).
A: Yes. How many x is there? How many +1 is there?
S*: 3 items of *x*. 6 items of +1. It is 3*x*+6.

(S* represents the same student)

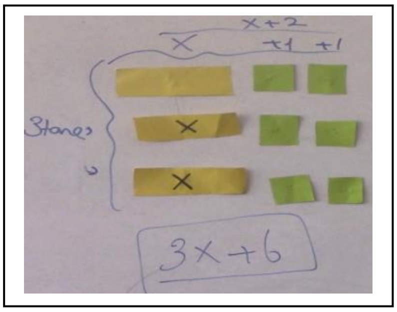


Figure 36 The representation of 3.(x+2) with algebra tiles

In this situation, Teacher A guided the student to consider repeated addition. She asked to the student how many (x+2) should be, after the student represented one (x+2) with algebra tiles. Then, Teacher A wanted the student to count all x and all +1 to get the result. The student answered 3x+6 counting all papers in the model as in the figure. Her knowledge of how to use mathematical representations appeared inadequately since she did not explain the multiplication using area calculation of rectangles for the last examples (SCK7-). Actually, she modelled the following multiplications with this way.

Teacher A asked the students to model 2.(3x+3) that was not indicated in her lesson plan. Before this example, Teacher A had explained the distribution property for multiplying 2.(4x-3) as explained above. Thus, the student used this property for this example first and he modelled 2.(3x+3) considering the result of the multiplication of 2 and 3x, and the multiplication of 2 and 3. She pasted the papers regarding these results (6 items of x and 6 items of +1). She should have multiplied using the algebra tiles in order to use algebra tile fit the purpose of it. Therewith, Teacher A asked if anyone put 2 items of (3x+3) (Figure 37). The conception of Teacher A's use of algebra tiles was reasoning with repeated addition as indicated above and she also guided the students to use the tiles with this way. This method was correct but it did not serve the purpose of using algebra tiles for teaching multiplication. Thus, she did not use the algebraic tiles effectively so that her knowledge of how to use tiles appeared inadequately (SCK7-).

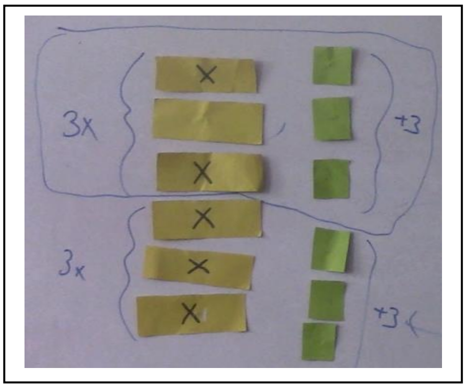


Figure 37 The representation of 2.(3*x*+3) with algebra tiles

After this example, Teacher A did not use modeling and she asked to the students to do the following six questions as she indicated in her lesson plan using the distribution property: 12.(5*x*+6), 7.(*x*+1), 6.(2*x*+5), 7.(3*x*+2*y*+4), and 8. (*x*+2t+3). Teacher A realized the students' errors and misconceptions from their answers while answering the questions. She addressed these errors by asking to the students whether they were true or not. The answering process of 12.(5*x*+6) in the instruction as in the following:

S*: (She is writing 5*x*+6=11*x*)
A: Your friend added 5x and 6, and got 11*x*. Is it true? Can they be added?
Students: No.
A: You should use the distributive property for this multiplication.
S*: (She is writing 12.5*x*=60*x*, 12.(+6)=+72, separately, and found 60*x*+72).

In this situation, Teacher A realized the student's misconception about addition of like terms, and she asked to the class if it was true and these terms could be added. Thus, her knowledge of how to address the student's error appeared appropriately with asking questions to point out the students' attention to the error (KCT8+). The students answered that they could not be added. The teacher might explain again like term concept to remedy the error effectively. Then, she directly told the student to apply the distributive property for multiplication and the student answered the question correctly using the property.

Some students could not understand what the distribution property though they could apply it. To illustrate, the implementation of 7.(x+1) was as in the following:

> A: Who understands only modeling? Can you come to the board?
> S: Multiply 7 and x. It is 7x. Multiply 7 and 1, 7 (see Figure 38).
> A: You have done it by using the distributive property. I thought that you would put 7 items of x+1 tiles.

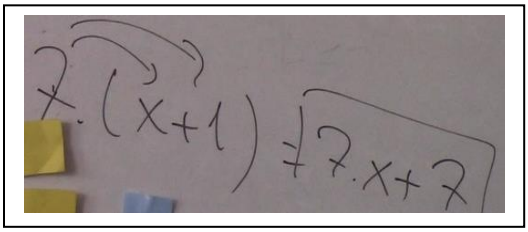


Figure 38 The application of distribution property for 7.(x+1)

In this situation, some students were confused with using the algebra tiles and the distributive property, since the modeling was not used effectively regarding the purpose as using the area concept. The student distributed 7 to the parenthesis expression, and he represented the result with colored papers as if modeling. Teacher A's conception of modeling with the algebra tiles was repeated addition as explained above. The other examples were answered using the distributive property procedurally assisting by the teacher.

After the lesson finished, the researcher interviewed with the teacher to get her ideas about the lesson. When the researcher asked what Teacher A considered about the use of algebra tiles, her knowledge to evaluate the instructional advantages and disadvantages of using algebra tiles for teaching operations with algebraic expressions appeared (KCT4-) as in the following:

> A: The modeling is good for the students. They could understand better with it. But, they modelled after they used distribution. It was interesting. So, I cannot understand whether the modeling is useful or not. It provides visuality for this reason it is good. But the students could not model the expression in my opinion.
> R: Do you think like that for addition?
> A: For addition, it was useful. I would teach addition with using the tiles again. But, in the multiplication, the students first distributed then they modelled with using tiles.

Teacher A stated that she was not ensure about the effectiveness of modeling with algebra tiles for teaching multiplication. Because she thought that the students applied distribution first, and they modelled regarding the result they found. She did not interpret of using algebra tiles for teaching multiplication adequately since she did not know how to use algebra tiles with connecting the area of rectangle concept. As she did not use the models appropriately, she could not understand if they supported students' learning or not. However, she considered using the algebra tiles for teaching addition in future since she believed that they were useful for learning addition.

4.1.2.2.2.4. Practice Four: Doing Exercises Related to Multiplication of Algebraic Expressions from Textbook and Workbook

After the teacher implemented what she indicated in her lesson plan, she started to do exercises in the textbook and workbook. It was asked in the questions to do addition, subtraction, and multiplication to simplify given algebraic expressions. The students especially had difficulty with using distributive property for the parenthesis expressions such as $4x^2–4(x+2)–x.(4x-5)$. Teacher A did this question herself with explaining the procedures for $–x.(4x-5)$ as in the script:

> A: Multiply $-x$ and $4x$, it is $-4x^2$, why? How many x do we have?
> S: 2.
> A: Then, how is it written? The power 2 of x. If there are 3 items of x, it is written the power 3 of x.

Teacher A had explained x^2 with stating there were two items of x as well as she explained n^2 in *Bacterial Growth Activity*. Later on the instruction, she continued to explain that the multiplication of a variable by itself; "to multiply $2a$ and a, since there are two items of a in multiplication, a^2, if there are 3 of them a^3 that we say". Her knowledge of how mathematical language was used appeared inappropriately (SCK5-). On the other hand, she did not use these type of questions that required the multiplication of two algebraic expressions. She indicated after the instruction that these type of multiplications were appropriate for 8th grade level so that her knowledge to know the content in the curriculum appeared appropriately (KCC1+).

Similarly, one of the examples was the following question from the textbook and it was asked to determine the expression of “The simplest form of -(*x*-9)+2.(4-3*x*)+8*x* is *x*+17” was true or false. In solving this question, Teacher A did not pay attention to the student’s solution process and thus she did not realize the result was incorrect. Teacher A helped the student to simplify -(*x*-9)+2.(4-3*x*)+8*x* as in the following:

> A: What will you do with the minus?
> S*: I will distribute it to the parenthesis. -*x*, multiply minus and minus, it is plus, +9*x*.
> A: Why +9*x*?
> S*: Multiply minus and minus, it is plus and so +9*x*.
> A: Distribute 2 for the other.
> S*: I will multiply 2 by 4, then -3*x*, then..
> A: Okay, it has finished. (But the student is writing 8*x* twice, the teacher has not realized it).
> S*: 16*x*, then 9*x*. Add 9 and 8, 17. 9*x*+17.
> (S* represents the same student)

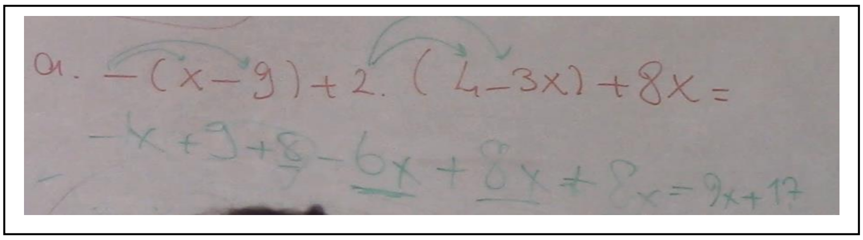


Figure 39 The simplification of the algebraic expression by the student

In this situation, Teacher A warned the student not to multiply 2 by 8*x*, the student did not write it, but he wrote 8*x* twice as in the figure. Teacher did not realize this error, and thus the result was incorrect as 9*x*+17. If the student had written correctly the expression, the result would be *x*+17. Thus, the class answered this question as false regarding the answer ‘9*x*+17’. For the teacher’s lack of attention, the question was answered incorrectly.

After the lesson finished, the researcher interviewed with the teacher to get her ideas about the lesson. Teacher A especially emphasized that the students had difficulty in applying the distributive property. She explained this situation as in the following:

> A: The distribution property is not understood conceptually. Actually, I have explained many times, also showed with modeling. They were confused. I do not know, if I should give more examples. If I will explain with modeling only, they might confuse again.

Teacher A indicated that the students could not understand the application of distributive property although she explained it with using algebra tiles. She considered to give more examples, however she did not ensure about the efficacy of using modeling for the later on. Thus, her knowledge to understand the difficulties of students with application of distribution property appeared inadequately (KCS3-) but she did not understand what they needed to learn and apply it correctly.

4.1.2.2.2.5. Practice Five: Presenting Problems that Combine Knowledge Related to Geometry

After doing exercises, Teacher A continued the lesson with the problems that required operating with algebraic expressions from the textbook and workbook. She presented the problems that required the prior knowledge about geometry for improving students' understanding. These problems required to combine the prior knowledge and operations with algebraic expressions being learnt, that might be more challenging for the students. Teacher A presented them but she also could have difficulty in solving them in the instruction. However, in these questions, Teacher A expressed what would do to find perimeter or area herself without discussing or reminding these concepts for the students. The one of the problems was as in the following figure:

For the next ABC triangle, IABI=(a-3) cm, IBCI = IABI + 7, and IACI = 2.IABI. What is the perimeter of this triangle?

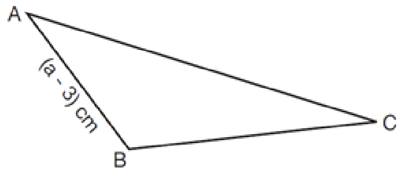


Figure 40 The problem related with the perimeter of triangle

The solving process of the problem was as in the following script:

A: Is there anyone solved this problem? The length of BC is 7 more than the length of AB. How can we express it?
S: We write a-3+7.
A: Yes, a-3+7, add -3 and 7, -4. That's IBCI=a+4. The length of AC is 2 times of the length of AB. What can we do?
S*: Multiply 2 with a-3. We distribute 2 to the parenthesis.
A: 2a-6. The perimeter is asked. The all lengths are explicit. We should add all of them. How can we do?
S*: We can add the terms that had 'a' with each other, and the numbers with each other. 4a-5.
(S* represents the same student)

In this situation, Teacher A asked the questions to the students, but she answered herself generally. The students told what would be done, and Teacher A did. Teacher A added a-3+7, multiplied 2 with a-3, and got the results. She also expressed that adding all the lengths of the triangle to find its perimeter. Her knowledge to connect the topic to geometry from prior years appeared in solving this problem adequately (SCK1+). But she might have wanted the students to do these operations, and also discussed with the class what the perimeter was and how it was calculated. However, Teacher A had difficulty with solving by herself in several problems related with geometry. One of them was as in the figure:

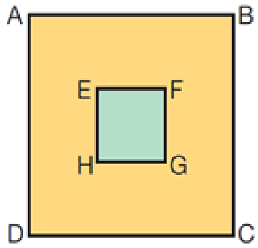
ABCD and EFGH squares are given as in the next shape. The perimeter of ABCD square is 3 times of the perimeter of EFGH square. Write the algebraic expression that shows the difference of areas of these two squares.	

Figure 41 The problem related with the area of square

The student had solved this problem before the lesson and the teacher asked her to explain her solution. While the student was explaining, the teacher's questions and explanations were as in the following:

> S*: I called *a* for the one side of EFGH square. The big square's side is x. 3*a* is equal to *x*.
> A: What is the perimeter of EFGH?
> S*: 4*a*.
> A: What is for ABCD?
> S*: 4*x*.
> A: (She is writing *b* instead of *x*). Then, (4*b*) = (4*a*)x3. How can we multiply this?
> S*: We distribute, multiply first 4 with 3, then *a*.
> A: There is not positive sign between them. There is multiplication here, but not addition. We don' have to apply distribution. Then, we multiply, 4*b*=12*a*. Then, we can say that 12*a* is the perimeter of ABCD. We wrote *b* in terms of *a*. It is asked the difference of areas. A(ABCD)=12*a*x12*a*. .. hımm, but, 12*a* is the perimeter, then one side of this square is 3*a*. To find the area, we multiply 3*a* two times, 3*a*x3*a*=9a^2. The other's area is *a*x*a*=a^2. The difference is 9a^2-a^2 = 8a^2.
> (S* represents the same student)

In this situation, Teacher A wanted S* student to solve the problem. This student used two different variables as *x* and *a*, and found the perimeters of the squares as 4*a* and 4*x* respectively. At this point, Teacher A intervened and wrote *b* instead of *x*. She tried to find a relationship between the perimeters as (4*b*)=(4*a*)x3. She asked how 4*a* and 3 were multiplied and one student proposed to use distributive property. Some students could consider using this property immediately when they saw the parenthesis so that it might be a misunderstanding. Teacher A explained that there must be an addition operation in the parenthesis to use the distribution (SCK6+) and she multiplied them herself as 12*a*. She considered 12*a* as the one side of the square and multiplied 12*a* and 12*a* to find the area, first. But, she realized that 12*a* was the perimeter and found the one side of ABCD square as 3*a*. She corrected this error at once. Then, she found the areas of squares in terms of a, and the difference between them last. She expressed that multiplying of 3*a* two times while finding the perimeter. It might be a problematic expression that it is difficult to understand what the multiplication two times and her knowledge of how mathematical language was used appeared inadequately (SCK5-). The more appropriate expression might be multiplying 3a with itself. On the other hand, Teacher A had difficulty with using the different variables in solving this problem. However, she had to continue with the student's way of solving. This solving method might also be difficult to understand for the students since it might be over their level. Then, Teacher A asked a similar problem to the class to develop students' understanding of this type of problems. The problem was "The perimeter of ABCD square is 4 times of the perimeter of EFGH square. Write the algebraic expression that represents the difference of the areas of these two squares". The only different part from to the previous problem was the ratio of the perimeters and it was 4 for this problem. One student solved the problem and explained as in the following:

> S: Suppose that's *x* (showing the length of one side of big square) and that's a (showing the length of one side of small square). Since 4 times of *a* length, the perimeter of ABCD is 4x. The other is 4a. 4x=(4a)x4, 4x=16a. A(ABCD)=4*a*x4*a*=16a^2, the other square's area is *a*x*a*=a^2. The difference is 16a^2-a^2=15a^2.
>
> A: One of the square's side is *x*, its perimeter is 4*x*; the other is *a*, its perimeter is 4*a*. 4*x* is equal to 4*a*x4, that's 4*x* is equal to 16*a*. If the perimeter is 16*a*, one side of it is 4*a*. The area of it is 16a^2; the other area is a^2. The difference is 15a^2.
>
> S: Teacher, I cannot understand 16a^2.
>
> A: Let's look, we multiply 4 and 4*a*, But, if it is 4.(4+*a*), we will multiply 4 first, then *a*, 16+4*a*. So, if there is an addition between the terms, we multiply with the first one and then the second one.

In this situation, Teacher A adapted this problem at that time since the students had difficulty with understanding the solution of previous similar problem and her knowledge to understand the needs and difficulties of students appeared appropriately (KCS3+). She made explanations about the student's solution for the problem. The student used two different variables, *a* and *x*. Then, the student represented *x* in terms of *a*, and she found that $x=4a$. She calculated the areas $16a^2$ and a^2, and the difference was $16a^2-a^2=15a^2$. However, another student could not understand how $16a^2$ was found. Teacher A showed the difference of the multiplication $4.4a$, and $4.(4+a)$. She explained that if there was addition, she would multiply 4 by each expression in the parenthesis respectively. She provided explanations to indicate the difference of the multiplication of a number and an algebraic expression, and a number and a parenthesis expression using the distributive property while solving the geometry problem as in the figure:

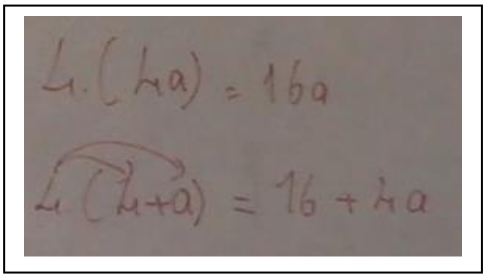


Figure 42 The explanation for the difference of multiplications by Teacher A

At that point, she might have asked what the student could not understand. Because the student might have not understood the representation of x in terms of a. However, she provided explanations about the difference between the multiplication of two expressions and the multiplication required distributive property considering only the student's having the difficulty with multiplication. Thus, her knowledge of how to provide mathematical explanations appeared might be inadequate (SCK6-).

Teacher A sometimes did not pay attention much to the given information if it was correct or sufficient in questions and problems. While solving problems related with multiplication of algebraic expressions and geometry, she did not recognize the missing information for the angle of the triangle. It was asked to calculate the area of the triangle as in the below figure. She made the following explanation and the students made calculation of the area:

A: How did we find the area of triangle? Multiply the two sides and divide by 2.
S: When multiply, $4a.\ 4a = 16a^2$. Then we divide it by 2, $16a^2{:}2=8a^2$.

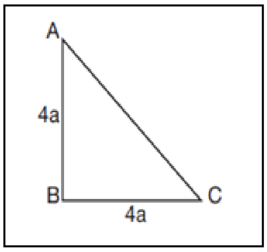


Figure 43 The problem related with the area of triangle

In this situation, Teacher A explained the area of triangle that was the multiplication of the two sides and then the division it by 2 without considering the triangle was right or not. This question was in the textbook and it was not indicated perpendicularity. Thus, this question had lack of information. However, Teacher A assumed it as the right triangle and asked to the students to calculate regarding the area of the right triangle. She should have realized this lack information and explained it to the students so that her knowledge to anticipate the misunderstandings that might arise with area of triangle being studied appeared inadequately (KCS2-). The students might have misunderstanding that all types of the triangles can be found with this formula.

After finishing solving of the other problems, Teacher A summarized operations with algebraic expressions by giving several examples: $(2x+4)+(3x-5)$, $(2x-4)-(2x-5)$, $(2x-5)-2.(3x-4)$, $x.(x-4)$, and $-2x.(x-5)$. She especially did these examples in order to remind the application of distributive property. These examples were preferred regarding the complexity and difficulty of them so that the first one was an addition operation; then the second one was a subtraction operation that included a parenthesis expression; the third had a subtraction operation that had a parenthesis expression with the coefficient 2; the fourth was a multiplication of *x* and algebraic expressions; and the last one was a multiplication of two algebraic expressions that one of them was with the coefficient -2. In answering these examples, Teacher A used equal sign inappropriately but this usage might cause errors in notations. To illustrate, while she was adding two algebraic expressions, she added them separately under the expression as in the figure and she did not write the result of the addition after the equal sign.

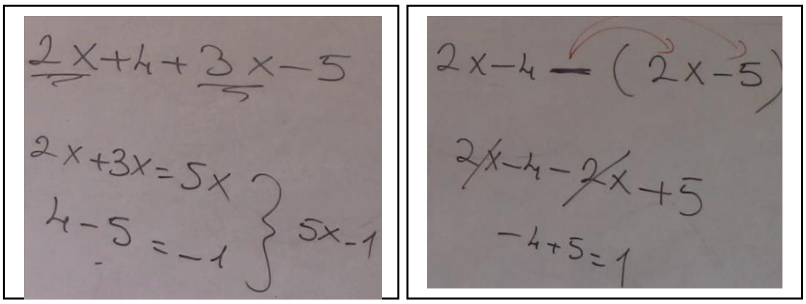


Figure 44 The addition and subtraction of two algebraic expressions by Teacher A

Similarly, her use of equal sign was problematic in subtraction and multiplication. Teacher A distributed the minus sign to the parenthesis expression and wrote the expressions without parenthesis in the second row as in the above figure. When the constant terms were remained (-4 and 5), she put the equal sign to show the result of -4+5. She did not use the equal sign to this operation. Teacher A’s inappropriate use of equal sign was observed also in multiplication of algebraic expressions teaching. She did not use any equal signs while multiplying of two expressions. She multiplied the algebraic expression with the expressions in the parenthesis one by one using the distribution property. She got the result at the third row, but she did not write it as the result using the equal sign as in the following figure.

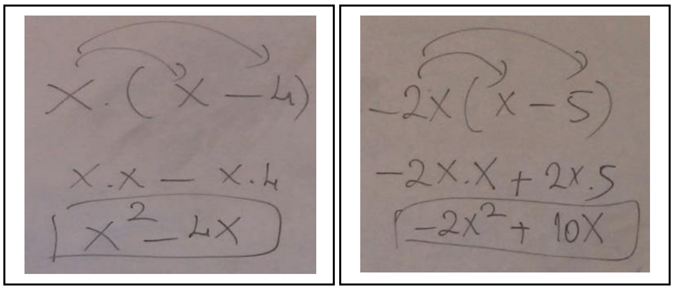


Figure 45 The multiplication of two algebraic expressions by Teacher A

In implementing of these activities, Teacher A's knowledge of how mathematical language was used appeared inappropriately that she did not use the equal sign to show the result after the given expressions (SCK5-).

4.1.2.2.2.6. Practice Six: Implementing the Suggested Activities

After doing the multiplications using distributive property, Teacher A continued with the researcher's suggested examples. The first examples had patterns, and it was asked to multiply a number with the terms and the general rule. One of the examples was multiplication 5 with 1, 3, 5, 7, 9, 11 ... pattern as in the following script:

> A: First, generalize the pattern and find the general rule. Then, multiply 5 with the terms and the general rule. What is the rule?
> Students: …
> A: The terms are increasing by 2, it is 2n. The first term is 1, when 1 is substituting for n, 2.1=2, to get the first term, subtract 1 from 2, 2.1-1=1, for the second term, 2.2-1=3, it goes like that. So, the rule is 2n-1. It is asked to multiply it by 5. Multiply 5 by 1, 5; 5 by 3, 15 ... This new pattern's terms are increasing by 10. What is the rule?
> S: 10n-5.
> A: The terms are decreasing by 10. To get the first term, subtract 5. 10n-5. Or, you can multiply 5 and 2n-1. First, multiply 5 and 2n, 10n; then multiply 5 and -1, -5. You can find with using both two ways.

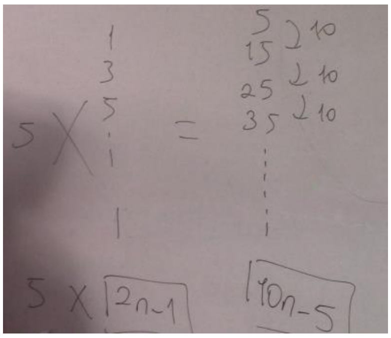


Figure 46 The multiplication of 5 and the pattern

In this situation, Teacher A found the pattern's rule first herself because the students could not answer. After she explained how she found the rule, the students could find the multiplied pattern's rule. She indicated that multiplying the terms and finding the general rule of new pattern, or multiplying 5 and 2n-1 in order to get the general rule of the new pattern were the acceptable ways to answer to this question. She explained the solution of the questions appropriately with her knowledge to connect the multiplication of algebraic expressions to pattern generalization (SCK1+).

Besides the students' difficulty in getting the general rule, some students' misunderstanding about generalization of patterns and substituting numbers for n in the general rule also were observed. One of the questions was "Multiply $\frac{1}{2}$ with 10, 16, 22, 28 ... pattern" from this activity. The dialogues between Teacher A and the students as in the following script:

A: For pattern, 10, 16, 22, 28 ... , What is the general rule?
S: 6n-10
A: Why -10? Substitute 1 for n. What is it?
S*: 61.
A: 61? What is the meaning of n here? It requires multiplication. What is the multiplication of 6 and 1?
S*: 6.
A: When 10 is subtracted, it is -4. Then, you should add. The terms are increasing by 6, it is 6n. You should add something to 6n. 6n is not like this: 6_, it is not a blank space. n is a variable, 1, 2, 3 ... can be. You should

multiply as 6x1, 6x2. It requires multiplication. Then, we write the rule 6n + 4.
S: Where 4 come from?
A: When we multiply 1 and 6, it is 6. To get 10, we must add 4.
(S* represents the same student)

In this situation, Teacher A asked to the students to generalize the pattern, first. One student generalized as 6n-10. The student found the difference between the terms as 6, and then he subtracted the first term 10 from 6n. Teacher A explained that this technique could be used for generalization in solving one question of exercises related pattern generalization. Thus, the conceptualization of generalization of patterns was still troublesome as observed. Teacher A wanted the student to substitute 1 for n to remark the incorrectness of 6n-10. However, the students wrote 1 in the place of n instead of multiplying 6 and 1. Teacher A responded to the student appropriately at that point that she asked what the meaning of n in the rule to address this student's error. She explained the function of n as multiplication by 6 to remedy this error so that her knowledge of how to address students' errors effectively and to remedy them appeared (KCT8+). Then, she provided more explanations for the students to differentiate the blank space next to 6 as 6_, and 6n. Then, Teacher A found the general rule as 6n+4. When the student asked where 4 was found, Teacher A explained that 4 was added for getting 10. Lastly, the terms of patterns were multiplied by $\frac{1}{2}$ and the results were found correctly as explained as in the following:

A: I will first multiply $\frac{1}{2}$ by 6n, suppose that there is 1 in the denominator of 6n, it is 3n. Then, I multiply $\frac{1}{2}$ and 4, suppose that there is 1 in the denominator of 4, it is 2. That's multiply $\frac{1}{2}$ by 6n first, and then 4 (see below figure).

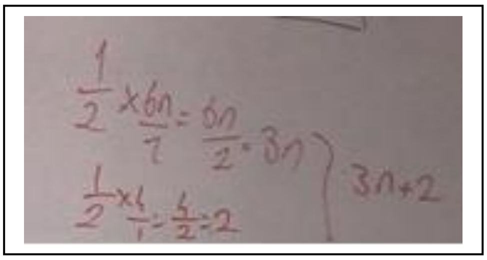


Figure 47 The multiplication of $\frac{1}{2}$ and 6n+4 by Teacher A

In this situation, Teacher A provided explanations for multiplication of a rational number as $\frac{1}{2}$ and an algebraic expression as 6n+4 using the distributive property. She explained the multiplication with $\frac{1}{2}$ with putting 1 in the denominator of 6n and 4 as in the figure. She got 3n+2 from these multiplications. Her knowledge of how to provide mathematical explanations for multiplication procedures for a rational number and a whole number (n is the position number) appeared adequately (SCK6+).

Then, Teacher A continued with the researcher's suggested activity that had different algebraic expressions that were added, subtracted or multiplied regarding the operation between them as branched from the "Positive algebra – A collection of productive exercises" course book. This activity also was considered enjoyable for the students as they would try to get the result for last bubble with gathering the results before the last one. Teacher A involved this activity to her lesson plan where she planned to do exercises after teaching operations as application hour of mathematics lesson. Her knowledge to predict that students would find interesting and motivating appeared appropriately to select this activity and she indicated doing operations to get the last bubble could be a different activity related with three operations for the students (KCS6+). The students could easily do these activities in enjoyment. They orderly answered for one of the bubbles in the figure:

S: I subtracted 14 from 21, 7+5n.
S: I subtracted 21+13m from 25+15m, it is 4+2m.
A: Something was subtracted from 21+13m to get 11+12m, what is it?
S: 10+m.
A: Okay, Then?

S: 20+10m.
S: 15+14m.
S: 35+24m.

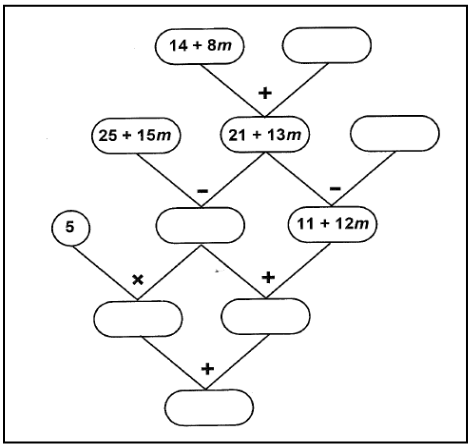


Figure 48 Operating with Expressions activity

The students expressed correct answers for the bubbles respectively and thus, the teacher's explanations were not required.

Although the students did the previous activity easily, they had difficulty in matching the identical expressions in the following activity (Figure 49). The "Equivalent II" activity was from the course book and had 9 algebraic expressions. It was asked to find the equivalent expressions and connect them with an arrow. The expressions had parenthesis and required doing multiplication by using the distributive property. It might make sense for the students since the expressions which seemed different at first glance were multiplied within themselves and they became equal. Teacher A involved this activity to her lesson plan where she planned to do exercises after teaching operations as application that and knowledge to predict that students would find it interesting and motivating appeared (KCS6+). However, Teacher A did not notice all the matched expressions and thus this question was answered partially correct.

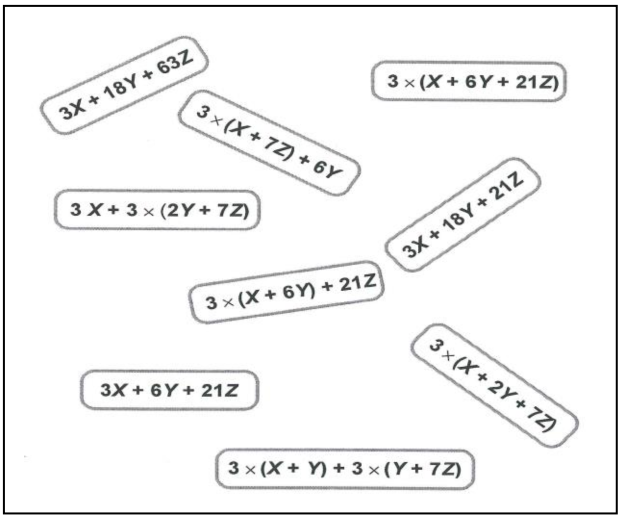


Figure 49 Equivalent activity

S: We will distribute 3 to the parenthesis (3x(X+7Z)+6Y).
A: You have chosen the expression that is not matched any of them.
S: 3x(X+6Y+21Z), distribute 3, 3X+18Y+63Z. This expression is identical to the first one, 3X+18Y+63Z.
S: 3X+18Y+21Z and 3x(X+6Y)+21Z are identical since distributing 3 to the parenthesis as 3X+18Y+21Z.
A: Another?
S: 3X+3x(2Y+7Z) and 3X+6Y+21Z.
S: There is one more with identical these two, 3x(X+2Y+7Z).
A: Okay.

In this situation, Teacher A responded to the student inappropriately who answered first and chose 3x(X+7Z)+6Y. Teacher A indicated that this expression was not matched any of them, and this was incorrect. There were four expressions that were matched identically with this expression. Actually, this expression was identical to the three expressions that the students answered. There was also one expression that was identical to them was 3x(X+Y) + 3(Y+7Z) as 3X+6Y+21Z. However, this expression was not operated and simplified and Teacher A did not realize that it could be matched. Thus, this question was solved partially correct that Teacher A's knowledge to solve mathematics problems appeared inadequately (CCK3-).

In the post-interviews, the teacher pointed out that the suggested questions and activities were easier and clearer than the questions in the textbook and workbook. She said that the students could perform well in these suggested activities, and they enjoyed while learning. She stated that she would start with them before the books' questions in future.

In general, Teacher A taught multiplication with using algebra tiles and applying the distributive property at the beginning of the instruction of multiplication. Then, she solved the problems and answered the questions in the textbook and workbook as in the order of the books. She did not select any examples about multiplication, and thus all the questions related with the operations with algebraic expressions were answered in the instruction. After these, Teacher A did the suggested activities that she selected. Thus, it might be said that there was not a reasonable sequence or connection within these examples.

4.1.2.2.2.7. The Extracted Knowledge Types from the Instruction for Multiplication of Algebraic Expressions

Table 11 The extracted knowledge types from instruction for multiplication of algebraic expressions

Practices	Extracted knowledge types				
	SMK		PCK		
	CCK	SCK	KCS	KCT	KCC
Choosing an example or activity to start teaching multiplication of algebraic expressions with connecting to topics from prior years		SCK1(+) SCK5(-)		KCT1(+)	
Discussing on the activity related multiplication of algebraic expressions	CCK3(-)	SCK5(-) SCK6(+)		KCT5(+)	

Table 11 (Continued)

Practices	Extracted knowledge types				
	SMK		PCK		
	CCK	SCK	KCS	KCT	KCC
Choosing the examples or activities to use to take students deeper into multiplication of algebraic expressions		SCK1(+) SCK3(⊥) SCK6(+) SCK7(-,-)		KCT4(-) KCT8(+)	
Doing exercises related to multiplication of algebraic expressions from textbook and workbook		SCK5(-)	KCS3(-)		KCC1(+)
Presenting problems that combine knowledge related to geometry		SCK1(+) SCK5(-,-) SCK6(+,-)	KCS2(-) KCS3(+)		
Implementing the suggested activities	CCK3(-)	SCK1(+) SCK6(+)	KCS6(+,+)	KCT8(+)	

Table 11 shows what type of knowledge of subject matter knowledge (SMK) and pedagogical content knowledge (PCK) that Teacher A had in instruction practices. (+) sign indicates the teacher's existing knowledge was adequate or appropriate and she used it effectively, while (-) sign indicates the teacher's existing knowledge was inadequate or inappropriate and she used it ineffectively. Each sign (+ or -) in the same knowledge type refers to Teacher A's different use of this knowledge during instruction. Besides, for SCK3 knowledge type, (⊥) is used to indicate the teacher's knowledge to develop definition or explanation appeared as merely choosing it from the textbook and presenting to the students. Since this situation meets the code (SCK3) partially, (⊥) sign is used.

KCT1(+) indicates that her knowledge to choose which examples to start with connecting pattern generalization as previous topic was effective (SCK1(+) in the first practice). In the implementation of this activity, the teacher expressed the multiplication of n and n as n^2 with explaining there were two items of n inappropriately (SCK5(-) in the first practice), but then in the second practice, SCK6(+) indicates that she provided correct explanations for the procedure of n.n with explaining n times of n and emphasized that 2 times of n was 2n and it differed from n^2. She made these explanations when she decided when to pause for more clarification for the multiplication of a variable with itself was appropriate (KCT5(+)). However, SCK5(-) in the second practice indicates that her knowledge of usage of mathematical language was inappropriate and she confused with the commutative property and the distributive property in explaining the equality of (n+5)x2 and 2x(n+5) expressions. As it was explained in addition and subtraction practices, CCK3(-) in the second practice indicates that her knowledge to answer question was partially correct that she missed what was asked.

Most of the extracted knowledge types in the third, fourth, fifth and sixth practices were about modeling with algebra tiles and providing mathematical explanations for the procedures of multiplication. SCK7(-,-) indicates that her knowledge of how to use algebra tiles as mathematical representations appeared inadequately so that she did not explain the multiplication using area calculation of rectangles. Thus, KCT4(-) indicates that her knowledge to evaluate the instructional advantages and disadvantages of using algebra tiles for teaching operations with algebraic expressions was inadequate since she did not use the models appropriately, and she could not understand if they supported students' learning or not. On the other hand, SCK6(+) in the third practice indicates that her knowledge to provide mathematical explanation for procedure of distribution property appropriately in application of it. But, the students had difficulty in applying it and her knowledge to understand the difficulties of students with application of distribution property appeared inadequately. Thus, she stated that she did not understand what they needed to learn and apply it correctly (KCS3(-)). Besides, SCK1(+) in the third practice indicates that her knowledge to connect repeated addition for teaching multiplication and geometry to improve the students' understanding of algebraic expression was appropriate. However, SCK5(-) in the fifth practice indicates that her knowledge to use mathematical language while expressing n^2 inappropriately as explained above.

Another problematic situation related to mathematical language was the use of equal sign, and SCK5(-) in the fifth practice indicates that she did not use the equal sign to show the result after the given expressions in operations with algebraic expressions. However, KCT8(+) in the sixth practice indicates that her knowledge of how to address students' errors effectively and to remedy them and she explained the function of n as multiplication by 6 in 6n expression. She also indicated that this situation as possible common misconception in planning interview. In the curriculum perspective, KCC1(+) indicates that her knowledge to know the content in the curriculum appeared appropriately since she did not use the questions that required to multiply of two algebraic expressions.

4.1.2.3. Summary of the Instructions

Teacher A generally used direct instruction and question-answer teaching method for teaching. Actually, she explained group work to the researcher in the interview, she did not use this method. As instruction format, she presented the lesson to the class as a whole group, as traditional teaching. Generally, she did the first question or task by herself, and then she asked the students to do the other questions. This process that the students work their own on questions was like student work time. At the beginning of the topic, she tried to create a discussion environment to support students' learning. However, her questions in the discussion was leading or she answered herself without waiting students' ideas. She did not give enough time for the students to think on the questions. However, after each question, she asked to the students whether they could understand the solution and if they had difficulty with understanding, she explained the solution once more time.

In teaching of generalization of patterns, the teacher permanently emphasized the multiplication of the position number and the increment and then adding number to find the first term. This was a rule or a formulization for the students in the instruction. Sometimes, the teacher had difficulty with solving some pattern generalization questions since she did not examine the questions before the lesson and she tried to understand them during the instruction. Actually, this situation wasted time from instruction.

In teaching operations with algebraic expressions, the teacher used real-life situations at the beginning to discuss the like term concept for addition and subtraction and of how the multiplication was represented with algebraic expressions. Then, she used algebra tiles to represent operations. Although she used the tiles appropriately for teaching addition and subtraction, she could not use them for teaching multiplication appropriately. Since she used algebra tiles only to provide visuality for the students without connecting with the calculation of area of square or rectangle. She used distribution property correctly to operate algebraic expressions procedurally. However, she had difficulty in solving several problems that the students used two kinds of variables.

4.2. The Case of Teacher B

This section documents and explains Teacher B's mathematical knowledge for teaching algebra in planning and instruction. The two phases are presented within two topics of algebra unit, generalization of patterns and operations (addition, subtraction and multiplication) with algebraic expressions.

4.2.1. Planning

The mathematical knowledge for teaching of Teacher B extracted from her lesson plans and the interview about planning with her by focusing on common patterns in data based on MKT framework. The two lesson plans were prepared before the instruction of the topics. The knowledge of Teacher B in planning was examined in the context of topics; generalization of patterns and operations with algebraic expressions; and in three main groups with respect to the phases of data collection: preparation of lesson plans individually, explanation of the lesson plans to the researcher, and revision of the lesson plan based on the researcher's suggestions.

These phases were conducted before the instruction. The planning process of Teacher B was as in the following figure:

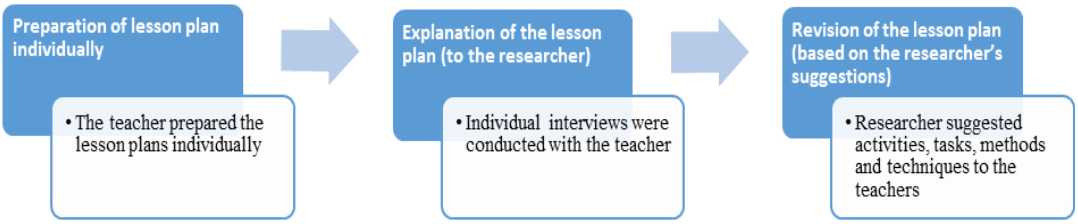


Figure 50 The process of planning

As seen in the figure, Teacher B prepared her lesson plan as written document for the instructions individually. Then, the individual interview with Teacher B was conducted to explain of her lesson plan to the researcher. Lastly, the researcher and two teachers came together, and the researcher suggested activities, tasks, methods and techniques about the topics to the teachers. The teachers selected one of the questions or activities that they wanted to use in the instruction and added them to their lesson plan, and revised the lesson plans with these changes. It is important to note that the extracted knowledge types from planning is also examined with reflecting in the instruction and evaluated together to conclude Teacher B's knowledge for teaching the algebra topics.

4.2.1.1. Planning for Teaching Generalization of Patterns

The planning process of teaching generalization of patterns included the lesson plans that the teacher used for the instruction and the teacher's responses and anticipations in the interview about preparing the lesson plan. Teacher B explained the structure of her lesson plan with rationale and stated anticipations about students' thinking throughout the instruction.

Teacher B made explanation for the lesson plan following the order of it. Teacher B explained that she designed her lesson plan considering the objective in the curriculum for teaching pattern generalization. Teacher B planned the lesson with respect to the objective for teaching generalization of patterns in the curriculum as "Students should be able to represent the relationship in the number patterns with letters modelling the pattern". This objective is under patterns and relations sub-learning domain and algebra learning area. The teacher's knowledge of objectives that belong to content and curriculum for designing lesson was essential (KCC1+). She indicated the knowledge of integers as the prior concept and knowledge that students should have for learning pattern generalization. She allowed two lesson hours for the instruction. For the instruction, she planned to use question-answer technique, discussion, problem-solving, and direct instruction method. She did not state any materials to use in the instruction.

For the beginning of the instruction, Teacher B planned to teach a non-linear growth pattern generalization. The pattern was in the first activity of pattern generalization topic of the textbook and it was given as 2, 6, 12, 20… Teacher B first planned to ask the students to model this pattern. For the activity, she aimed to use different representations to give the idea of the relationship in the pattern by modeling the pattern using manipulatives and representing the figural pattern with numbers in a table:

> B: In patterns, I will ask the students to model the patterns. First, I will give the numbers as terms in the patterns, then they can use any materials they want, such as unit cubes or dots. After that, I will draw a table with giving some terms. I will ask the students to find the following terms and fill the table.

The figural representation of this pattern with unit cubes was in the textbook, but the teacher did not plan to give this representation, instead of this, she planned to model this pattern with manipulatives. After the students modeled this pattern, she planned to select one of the models to use in generalizing the pattern. Then, she planned to ask the students to fill the blanks in the following table:

Table 12 The table to represent the figural pattern with numbers

The position number	The number of unit cubes	The relation between the position number and the number of matchsticks	
		1st option	2nd option
1	2	1.(1+1)	1^2+1
2	6	2.(2+1)	2^2+2
3	12	3.(3+1)	3^2+3
4	20	4.(4+1)	4^2+4
.	.	.	.
.	.	.	.
n		…?	…?

Teacher B indicated in her lesson plan that she asked the students to continue the next several steps of the pattern in the table and to think on what should be written for n. number. The teacher's knowledge to connect numerical and figural representations with using tabular representations and modeling was appeared as merely to show these multiple representations. However, especially focusing on the arithmetical relationships in tabular representation to underlie the relationship in the pattern to conceptualize generalization could be effective and her knowledge to choose, make and use the tabular representation was appeared (SCK7+). She explained the process of implementation of this activity by finding several next terms in the table first and then discussing the generalization of the pattern with using the arithmetical representations in this table. Actually, she aimed to underlie the relationship of the pattern with using the tabular representation since she planned to use numerical and figural representation as merely using different representation of the same pattern. She did not aim to use figural reasoning as how the figures change and to relate this change with the relationship between numbers. Although she would not use figural reasoning, her knowledge of how to choose, make, and use tabular representation was appeared in order to teach general rule of the pattern using arithmetical representations (e.g. 1.(1+1) and 1^2+1) effectively.

At that point, Teacher B also explained her anticipation based on her previous experiences about students' difficulty in making sense of the concept of n that was n^{th} term:

> B: In general, the problem that the students have is about writing the general rule. They have difficulty with it. After the given terms, they can find the following 2-3 terms. But, only successful students can ask "can we write n to find a rule?". But, it is few. I generally explain like this: 2 is used in 2^{nd} term, 3 is used in 3^{rd} term, 4 is used in 4^{th} term... If then, do we use n for the n^{th} term? If I connect the n^{th} term as the general term like this, the students can understand more easily. Then, the general rule of the pattern. I make the students feel the need of a general rule by asking what we use for the n^{th} term.

In her explanations, her knowledge to anticipate where and how the students had difficulty in understanding the concept of n appeared appropriately (KCS1+). In this situation, Teacher B stated that high achieved students could grasp the arithmetical relationship between the numbers as the terms of patterns, and these students could understand where n should be written easier than other students. As she stated, most of the students had difficulty with conceptualizing the n^{th} term and corresponding the general rule. At this point, Teacher B explained her method for connecting the relationship and the n^{th} term to build on students' thinking of general term and generalization. She planned to use inductive reasoning to provide the students to think and use n for n^{th} term in order to reach the generalization. To do this, she planned to explain using the position number to find the corresponding terms such as 2 for 2^{nd} term, 3 for the 3^{rd} term, 4 for 4^{th} term and then n for the n^{th} term. Teacher B's knowledge of building the students' understanding of generalization appeared effectively with connecting the general term with this inductive reasoning method (KCT7+).

Although Teacher B explained her method to help students' understanding the general term and generalization of patterns, she indicated the difficulty of students in writing the general rule algebraically, especially in realizing and generalizing the non-liner patterns' rules:

> B: When to come the n^{th} term, they have difficulty. Finding the general rule is not easy. More successful students can see the relationship easily. For example, they can catch the relationship by using the numbers in terms of the linear patterns such as 2 times and 1 more of it, 3 times and 2 more of it. But, they cannot see the quadratic form such as n^2 in patterns… When the students have difficulty with the general rule, I give some time the students to think on the pattern. Then, we discuss their ideas. They should solve different questions in patterns.

She indicated that the students could get the relationship in linear growth patterns easily. However, the students could not think represent the general rule using "n^2" for non-linear pattern generalization. Her knowledge to anticipate of students' difficulty in generalizing non-linear patterns appeared and she emphasized this difficulty with students' not to be able to think and use n^2 in generalization appropriately (KCS1+). However, she did not to use the term with its name as non-linear pattern in her explanations, instead of this, she stated the patterns with quadratic form as n^2 to point out the difference from linear patterns. Actually, non-linear patterns differ from linear patterns with the growth between terms, and thus the general rule of non-linear patterns are formed with exponential expressions such as n^2, n^3, or n^4 not only n^2. Her knowledge to use terms about patterns correctly was not adequate to explain it (CCK2-). As well as she explained her anticipations and concerns about students' difficulty, she suggested several methods and procedures that she used to support students' learning. She explained her suggestions as giving time students and then discussing what the rule of the pattern was, and providing students to experience writing the general rule for different patterns. Teacher B's knowledge to identify what different methods and procedures afford instructionally appeared effectively so that she suggested discussion as method and experiencing with different pattern types as procedures to improve students' understanding of generalization patterns (KCT6+).

After Teacher B expressed her anticipations for students' possible difficulties and suggestions to overcome them, she continued to tell about the flow of the instruction. After generalization of the first pattern algebraically, Teacher B planned to define the concept of general rule of the pattern using the textbook definition. This definition was:

> The representation with using n of the n^{th} position number is called the general rule of the pattern. For example: $2n+1$, n^2+3n ….

The definition was from the textbook and Teacher B selected to give it in the lesson. She did not develop this definition and she planned to use as it was. Thus, her knowledge about usable definition of general rule appeared as merely choosing it from the textbook (SCK3┴).

In the middle of the lesson plan, Teacher B planned to ask two questions from the textbook related to pattern generalization. First, similar to the beginning activity, Teacher B planned to give a linear numerical pattern to generalize with the same procedure in previous activity. This question was like that, "Model 3, 5, 7 … pattern and express with two different representations of the general rule of the pattern". After the generalization of this pattern, she planned to ask the second question to the students. It was asked to find the 7th term in the pattern whose general rule was $3n^2$. After the class answered these questions, Teacher B planned to give another definition from the textbook. The definition was:

> Let's show the position number of an unknown number in the pattern with "n". In this situation, the unknown number in the pattern is called nth number; n is called as the representative number, or the general term of the pattern.

She did not develop this definition as in the previous definition and she planned to use as it was. Thus, her knowledge about usable definition of general term appeared as merely choosing it from the textbook once more (SCK3┴). However, it would be more appropriate to give this definition with the definition of general rule after the first activity developmentally. The general term was also used in the general rule concept so that defining of it would be developmentally before giving the definition of general rule.

In general, Teacher B planned to teach the generalization of patterns as she sequenced that first was non-linear growth patterns, and then linear and non-linear growth patterns. She planned to carry out the lesson based on the sequence of the textbook. To start the instruction, she chose the non-linear pattern activity (2, 6, 12, 20…) from the textbook. She planned to use modeling to represent the pattern and to give a table to get the relationship of the pattern. Then, she planned to continue the instruction with two growth numerical patterns to take the students deeper into the pattern generalization. The first question was about generalizing 3, 5, 7… linear pattern, and the second question was about finding the 7th term of the non-linear pattern particularly whose general rule was $3n^2$. She planned to focus on generalization patterns algebraically and she planned to do these pattern generalizations to take the students deeper into the content. Her knowledge to choose which examples to start with and which examples to use to take the students deeper in to the content was appeared with this sequence of the lesson plan inappropriately (KCT1-). In this sequence, it would be more appropriate begin with the linear growth pattern (3,5,7…) example before the non-linear pattern generalization. Since this order might provide to teach pattern generalization from simple to complex ones developmentally and they also were appropriate for the level of the students. Beginning with linear growth pattern generalization could also provide to remind students to pattern generalization from 6th grade as prior concept. However, Teacher B did not determine the homework and assessment questions from textbook and workbook and so she did not indicate them in the lesson plan.

After the interview with the teacher, revision was the part where the teachers made the final version of their lesson plans with suggestions of the researcher. For the suggestions, the setting for pattern generalization was explained in Appendix B. Revision included adding new examples, activities and acknowledging suggested method or techniques to lesson plans with the aim of supporting students' understanding. Teacher B decided to add the pattern task and revised the middle part of her lesson plan where she asked students to generalize several patterns from different resources. The implementation of the pattern test based on the teacher's knowledge was explained in the *implementing the pattern test practice* in the instruction. One of the suggested activities was dot patterns that she did not prefer to use it and her knowledge of objectives as grade level, both 7th and 8th grade, of the curriculum appeared to not to involve these activities appropriately (KCC1+).

4.2.1.2. The Extracted Knowledge Types from Planning for Generalization of Pattern

Table 13 The extracted knowledge types from planning for generalization patterns

SMK		PCK		
CCK	SCK	KCS	KCT	KCC
CCK2(-)	SCK3 (⊥,⊥) SCK7(+)	KCS1(+,+)	KCT1(-) KCT6(+) KCT7(+)	KCC1(+,+)

Table 13 shows what type of knowledge of subject matter knowledge (SMK) and pedagogical content knowledge (PCK) that Teacher B had in planning phases. (+) sign indicates the teacher's existing knowledge was adequate or appropriate, while (-) sign indicates the teacher's existing knowledge was inadequate or inappropriate. Each sign (+ or -) in the same knowledge type refers to Teacher B's different intention of use this knowledge during planning. Besides, for SCK3 knowledge type, (⊥) is used to indicate the teacher's knowledge to develop definition or explanation appeared as merely choosing it from the textbook and presenting to the students. Since this situation meets the code (SCK3) partially, (⊥) sign is used.

In general, SCK7(+) indicates that her knowledge to choose, make and use the tabular representation was appeared effectively with focusing on the arithmetical relationships in tabular representation to underlie the relationship in pattern to conceptualize generalization. SCK3(⊥,⊥) indicates that her knowledge about usable definition of general term and general rule of the pattern appeared as merely choosing it from the textbook and it was not developed by the teacher or class. Her knowledge of students' thinking related with pattern generalization was adequate that KCS1(+,+) indicates that her knowledge to anticipate the students had difficulty in generalizing pattern algebraically and particularly in non-linear patterns. However, she did not to use the term with its name as non-linear pattern in her explanations, instead of this, she stated the patterns with quadratic form as n^2 to point out the difference from linear patterns since her knowledge to use terms about patterns correctly was not adequate to explain it (CCK2(-)). To facilitate the students' understanding of generalization patterns, she suggested discussion as method and experiencing with different pattern types as procedures so that her knowledge of different methods and procedures afford instructionally was appeared effectively (KCT6(+)). She also planned to explain using the position number to find the corresponding terms such as 2 for 2nd term, 3 for the 3rd term, 4 for 4th term and then n for the nth term and KCT7(+) indicates that her knowledge of building the students' understanding of generalization appeared effectively with connecting the general term with this inductive reasoning method. On the other hand, KCT1(-) indicates her knowledge to choose examples and activities to start with non-linear patterns and then continue with the linear growth patterns to take the students deeper into the pattern generalization was not appropriate as developmentally. In the curriculum perspective, KCC1(+,+) is related to curriculum knowledge and her knowledge of objective that belonged to content and curriculum for designing lesson was essential and adequate, and it also indicated that her knowledge of objectives as grade level, both 7th and 8th grade, of the curriculum appeared to not to involve the activities related to the patterns with special numbers since it was in 8th grade curriculum.

4.2.1.3. Planning for Teaching Operations with Algebraic Expressions

The planning process of operations with algebraic expressions included the lesson plans that the teacher used for the instruction and the teacher's responses and anticipations in the interview about preparing the lesson plan. Teacher B explained the structure of her lesson plan with rationale and stated anticipations about students' thinking throughout the instruction.

Teacher B planned the lesson with respect to the objective for operations with algebraic expressions in the curriculum as "Students should be able to add and subtract with algebraic expressions and students should be able to multiply two algebraic expressions". These objectives are under algebraic expressions sub-learning domain and algebra learning area. The teacher's knowledge of objective that belonged to content and curriculum for designing lesson was essential (KCC1+). Teacher B planned the lessons with respect to the order of the objectives in the curriculum. Thus, she planned to teach addition and subtraction of algebraic expressions first, and then multiplication of algebraic expressions. She indicated algebraic expression, operations with integers, and area of polygons as the prior concepts and knowledge that students should have for learning operations with algebraic expressions. She allowed three lesson hours for the instruction in general. She did not detail the organization of time for what she would do during the instruction. For the instruction, she planned to use question-answer technique, discussion, problem solving and direct instruction method. As materials, she stated to use paper and scissor in the instruction.

In the interview, she first emphasized that the students should have conceptualized the concept of algebraic expression at 6th grade for learning operation with algebraic expressions. For reminding it, she prepared a worksheet that included the students' prior knowledge. This worksheet was from a test book and included writing verbal statements as algebraic expressions and writing algebraic expressions as verbal statements, determining the coefficients, the number of terms, and like terms in algebraic expressions. Her knowledge to connect the topic being taught to topics from prior year appeared appropriately (SCK1+) and she planned to deliver the worksheet at the beginning of the lesson and wanted the students work individually and then she planned to discuss with the class when the students gave answers.

Then, she explained what the beginning activity that included rectangle papers was and how it would support students' learning of the properties of addition and subtraction with algebraic expressions. This activity was about commutative and associative property in operations with algebraic expressions using unit squares papers and connecting the properties of addition with integers. In this activity, rectangle papers whose one side was 1 cm, and the other side was changeable length. Firstly, it was asked to show the commutative and associative property of addition operation with integers by using rectangle papers whose sides were known. Then, it was asked to do same procedure to show the commutative and associative property of addition operation with algebraic expressions by using rectangle papers whose one side was 1 cm and other sides were x or y, with laying together first *x* and then *y*, or first *y* then *x*, represented the same paper as the result as in the figure:

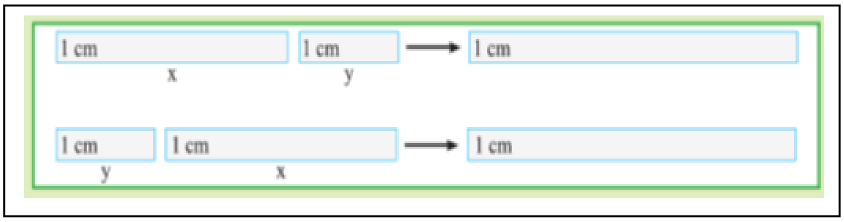


Figure 51 The representation of addition of *x* and *y* with rectangle papers

The same procedure was asked to do for associative property of addition operation with algebraic expressions by using *x*, 2*x*, and 3*x* length sided rectangle papers as in the figure:

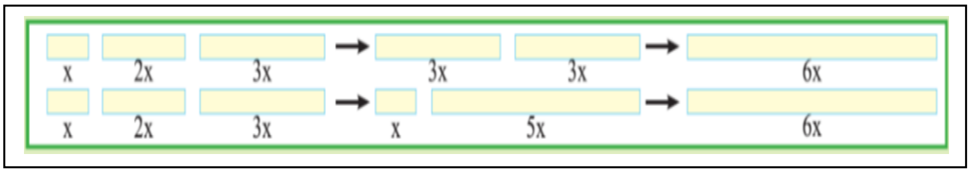


Figure 52 The representation of associative property with rectangle paper

Teacher B planned to explain associative property with this way. It was showed the associative property with papers as in the following: $x+2x+3x=3x+3x=6x$ and $x+2x+3x=x+5x=6x$. Lastly, it was asked to model x with paper by folding the paper in order to get 4 equal parts and then cutting these parts as in the floowing figure:

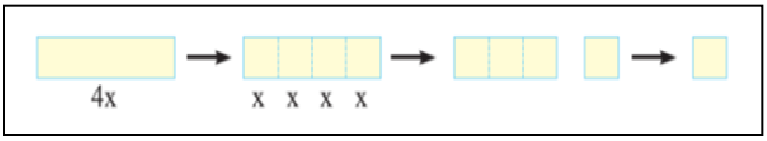


Figure 53 The representation of *x*

Teacher B's knowledge to choose which examples to start appeared efficiently at this point (KCT1+) that she involved this activity which provided to connect the properties of arithmetical operations that the students had learnt before. When the researcher asked to the teacher why she selected this activity, she made explanations as in the following script:

> B: Because the unit is equal, they are unit squares. Their sides are 1 cm length. Or, the ratio is constant, the students can use their notebooks since they are also divided into unit squares. At that point, the students can understand the result of addition is same clearly when the change of the unknown length of papers. The unit property here can provide the students to understand what x, $2x$, or $3x$ mean. They can see these expressions on the grid paper. They can recognize x or y. They can understand the addition of them.

In this situation, Teacher B made explanation about what this activity could provide the students to understand of algebraic expressions (e.g. x, $2x$) and addition of them (e.g. $x+y$). She explained that using unit squares provide one side was constant as 1cm while the other could change, and the different length of rectangle papers represented x or y. Getting these papers together yielded the result of $x+y$ and $y+x$, and since the length of papers was equal and this situation indicated the commutative property. Then, she planned to explain x using the paper, then 2 times of the length of x as $2x$, then 3 times of the length of x as $3x$. Teacher B preferred the papers as concrete materials to visualize the algebraic expressions for students as her knowledge to predict what students would find interesting and motivating appropriately (KCS6+). She stated that the papers whose one side was 1 cm and the other side was changeable length could give the idea of different variables (x and y), and the multiplication of a variable by integers (e.g. x, $2x$).

Then, she planned to provide explanations for addition and subtraction operations with algebraic expressions. To do this, she planned to use the textbook's representations. To illustrate for modeling of $4x$-$3x$ with connecting subtraction arithmetic as in the following figure:

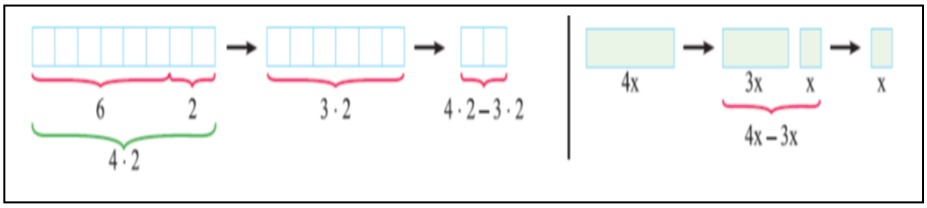


Figure 54 The representation of 4*x*-3*x* connecting with 4.2-3.2

The 2 units were modelled as *x* for algebra in the figure and 4.2 was represented with 4*x* and 3.2 was represented with 3*x*.

There were three representations of addition operations in the lesson plan as in the figure. The representations might support students' learning of operations in algebra as they could see the procedures together at the same time. Her knowledge of how to provide mathematical explanations for procedures appeared with these representations efficiently (SCK6+) since she connected the arithmetic with algebra in addition and subtraction in order to show the similar procedures for operations. Similarly, she also planned to show these representations in the table that included the arithmetical operations and the corresponding operations with algebraic expressions. She explained that "I have two 3, and if I add one 3, I get three 3. I can connect this situation with operations with algebraic expressions". This table included the arithmetical operations in first column, the algebraic representations of it in the second column, the terms in the third column, and the coefficients in the fourth column as exemplified for the first row as in the following:

3+2.3=3.3	x+2*x*=3x	*x*, 2*x*, 3*x*	1, 2, 3

Table 14 Operations and their algebraic representations

This table gave four operation examples to show the similarity of operations with algebraic expression and operations with integers. Teacher B preferred this representation and the beginning activity to explain the rules and procedures in operations with algebraic expressions and her knowledge of how to provide mathematical explanations for addition and subtraction rules and procedures appeared efficiently (SCK6+). Beside this, she planned to provide the explanation for like term concept. This explanation was:

> Like terms are the terms that have the variables with same or different coefficients in an algebraic expression. The coefficients of like terms are added or subtracted for addition and subtraction of algebraic expressions.

This definition was from the textbook and the teacher selected to give it to the students. Thus, her knowledge about usable definition appeared as merely choosing it from the textbook (SCK3⊥). This definition included the explanations for like terms that had variables with same or different coefficients. However, this definition could be inadequate since x and x^2 are the expressions that have same variable but they are unlike terms. Thus, it should have been stated that the like terms had the same variable with the same exponential forms. Teacher B planned to use it for the reason that the textbook presented, but she did not examine it and think that it might cause any misunderstandings.

Then, Teacher B planned to continue explaining the properties of operation in algebra with an activity. There were two columns that had arithmetical and algebraic expressions in the activity. It was asked to match the expressions that were equal and then to discuss the properties of addition operation were valid or not for algebraic expressions. With this activity, she planned to finish teaching addition and subtraction operation with algebraic expression. She planned mostly to focus on the properties of addition with activities and explanations for the instruction. She should have emphasized the like term concept in doing operations but she planned to only provide explanation for like term as indicated above. Thus, this explanation without applying in operations could not be adequate to understand the like term concept. However, conceptualizing this concept is important to do addition and subtraction operation with algebraic expression and so her knowledge to choose which examples to take the students deeper into the content appeared inadequately with this presentation of the lesson (KCT1-) since she planned to focus on the properties of operations.

For the second objective of topic as multiplication with algebraic expressions, she designed the lesson as in the textbook order. First, there was an activity related with modeling the multiplications by using algebra tiles. In the activity, the modeling of *x.x* was given as in the below figure, and then, it was asked to find respectively the areas of the rectangles whose sides were *x* and *x*+2, and *x*+2 and *x*+5.

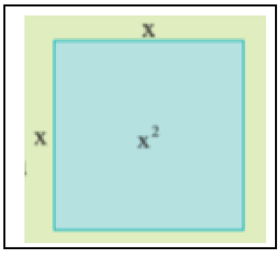


Figure 55 The representation of x^2 with modeling algebra tile

For the second rectangle area, it was asked to separate the area into two rectangles that one of them was *x* and *x*+5 length sided, and the other was 2 and *x*+5 length sided. With this guiding, the students were expected to explore the distribution property of multiplication on addition operation. Her knowledge to choose which examples to start appeared efficiently at this point (KCT1+) so that she planned to model the multiplications using algebra tiles and connecting the idea of the area of rectangles that the students knew with multiplication as in the activity. Then, she planned to model *x*.(*x*+3) with using algebra tiles. Doing these multiplications with the help of the area calculation, she planned to explain also the distribution property in addition to modeling with algebra tiles for the following examples in her lesson plan. She planned to make links among algebraic and geometrical representations to give the idea of multiplication of algebraic expressions by using algebra tiles:

> B: For multiplication, I will use algebra tiles, if there are, or I will prepare with papers, or I will draw on the board. The students can understand while using the area of square. Since they know the area of square well. They also can use the area of rectangle easily. The length of the sides is x and x, the area of this square is x^2. When *x*+3 is asked, 3 units are added. They can connect the multiplication with the area concept with this activity.

In this situation, Teacher B made explanations for $x.(x+3)$ multiplication. She planned to make connections this multiplication with the area of rectangle by using algebra tiles. She asserted that the students could understand the multiplication well with this way. She explained that the multiplication of x by x yielded x^2, and this could be represented with the area of the square whose sides were x and x. Multiplying x by 3 yielded $3x$ and this could be represented with the three rectangles whose sides x and 1, as 3 units as Teachers B stated. Her knowledge to link algebraic and geometrical representations to underlying the area concept appeared appropriately and she explained the area concept that underlying of modeling multiplication of algebraic expressions with algebra tiles appropriately (SCK2+). After teaching multiplication with modeling, she planned to apply the distribution property for multiplying algebraic expressions procedurally. At that point, she indicated the difficulty in theapplication of the property:

> B: In multiplication, they have difficulty with distributive property for multiplication of two algebraic expressions. They can make sign error particularly if there is subtraction operation in parenthesis. They can forget the minus sign. Especially, if there are two algebraic expressions within two parentheses, they have difficulty with these multiplications.

In this situation, Teacher B indicated the students' difficulty in using distributive property in multiplication of algebraic expressions. Especially, she pointed out the two examples. One of them was if the parenthesis operation was subtraction such as $x.(x-2)$, the students could not consider the minus sign. The other was the multiplication of two parenthesis algebraic expressions such as $(x+2).(x-3)$. Her knowledge to anticipate where and how students have difficulty appeared appropriately and she explained where the students had difficulty in learning multiplication and how they might make errors (KCS1+).

Therewith, when the researcher asked what the students had possible misconceptions while working on this topic, the teacher explained the misconceptions in addition and subtraction of algebraic expressions by giving examples as in the following script:

> B: For example, in $3x$-x, the students can forget 1 as the coefficient of x. They have difficulty, they cannot understand this '1' easily. .. Another

example is $4x+3y-x$. They can add or subtract all the coefficient without considering the variables are same or not.

In this situation, Teacher B indicated that the students did not consider the coefficients of the variables such as x, or not to recognize the variables were different such as x, y, and thus they did operations incorrectly. She also pointed out the common misconceptions in multiplication:

B: When x is multiplied by x, the students can get $2x$. Higher achiever students do not do this error. Other students are generally confused this. They know the result is x^2 from the area of square. However, they make errors. Especially, if there is a coefficient of x, such as $2x$. Multiplication of $2x$ and x. They can make errors with this multiplication more.

In this situation, Teacher B indicated the common misconception in multiplication of x by x and the students could get $2x$ instead of x^2, though the students knew the result was x^2 using the area of square. Especially, she gave another example that they got confused in multiplication of $2x$ by x. With these misconception examples related with operations, Teacher B's knowledge of common student conceptions and misconceptions about operations with algebraic expressions appeared appropriately (KCS5+). Teacher B suggested using the grid papers and algebra tiles to ease students' understanding of operations and overcome the misconceptions as in the following script:

B: In addition and subtraction, using the grid papers as indicated in the first activity is useful for the students. It eases students' understanding. In multiplication, modeling with algebra tiles is meaningful for the students, since the concept of the area provides to understand conceptually. However, they have difficulty with applying distributive property in multiplication. I try to solve lots of questions for multiplication.

In this situation, Teacher B indicated the using papers and algebra tiles to improve students' understanding. While she explained the algebra tiles to give the idea of area, she did not explain how the grid papers support students' learning of addition and subtraction. Especially, she emphasized using them at the beginning of teaching the operations as she indicated to use in her lesson plan. She also planned to solve lots of questions for improving of students' understanding. Her knowledge of how to address student errors effectively, and to remedy student errors appeared appropriately so that she suggested using manipulatives and involved them with in the beginning activities in the lesson plan in order to prevent possible misconceptions in the instruction (KCT8+).

She also explained in detail the prior knowledge that the students should have. She stated that the students should know what algebraic expressions were and the importance of operations with integers to understand operations with algebraic expressions:

> B: The students should know algebraic expressions, operations with integers. There is an integer and an unknown number in algebraic expression. They should recognize that it represents a number. They should know the area of polygons.

In this situation, Teacher B stated that the students should understand the meaning of algebraic expression and it represented a number with an unknown and a constant. Besides, she explained the knowledge of the area of polygons was required for learning multiplication of two algebraic expressions. Her knowledge to connect the topic to topics from prior years appeared and she stated connecting the algebraic expression concept and area of polygons with operations with algebraic expression adequately (SCK1+). She also explained that she would not give the examples that included rational numbers, and she planned to select the algebraic expression within integers that would be appropriate for the level of the students.

In general, Teacher B designed the lessons based on the two objectives to begin with the activities that require using material and manipulative such as papers for showing the property of addition and then algebra tiles for modeling for teaching multiplication of algebraic expressions so that her knowledge to choose a particular representation or certain material for mathematical procedure appeared appropriately (KCT3+). After the students understood the operations and could do addition, subtraction, and multiplication, she planned to ask them to do the exercises procedurally from textbook and workbook. In her lesson plan, Teacher B involved the implementation procedures of the beginning activities and solutions of questions related to operation with algebraic expressions as in the textbook, but she did not state how she would apply the beginning activities by herself in the lesson plan and any expected answers that the students would give. However, involving the expected answers could help the teacher to prepare herself how she would handle or overcome the possible situations.

After the interview with the teacher, revision was the part where the teachers made the final version of their lesson plans with suggestions of the researcher. For the suggestions, the setting for operations with algebraic expressions was explained in Appendix C. Revision included adding new examples, activities and acknowledging suggested method or techniques to lesson plans with the aim of supporting students' understanding. Teacher B appreciated and included the suggested activities by the researcher to revise her lesson plan. The implementation of the suggested activities was explained in the *implementing the suggested activities* practice in the instruction. Beside the activities, the researcher suggested connecting arithmetic with algebra for teaching the properties of addition and multiplication properties in algebraic expressions as a method, and using algebra tiles as manipulatives. Teacher B would use algebra tiles and connect between arithmetic and algebra while teaching as she indicated in her lesson plan.

4.2.1.4. The Extracted Knowledge Types from Planning for Operations with Algebraic Expressions

Table 15 The extracted knowledge types from planning for generalization patterns

SMK		PCK		
CCK	SCK	KCS	KCT	KCC
	SCK1(+,+) SCK2(+) SCK3(⊥) SCK6(+,+)	KCS1(+) KCS5(+) KCS6(+)	KCT1(+,-,+) KCT3(+) KCT8(+)	KCC1(+)

Table 15 shows what type of knowledge of subject matter knowledge (SMK) and pedagogical content knowledge (PCK) that Teacher B had in planning phases.

(+) sign indicates the teacher's existing knowledge was adequate or appropriate, while (-) sign indicates the teacher's existing knowledge was inadequate or inappropriate. Each sign (+ or -) in the same knowledge type refers to Teacher B's different intention of use this knowledge during planning. Besides, for SCK3 knowledge type, (⊥) is used to indicate the teacher's knowledge to develop definition or explanation appeared as merely choosing it from the textbook and presenting to the students. Since this situation meets the code (SCK3) partially, (⊥) sign is used.

In general, SCK1(+,+) indicates that her knowledge to connect the topic being taught to topics from prior year was appropriate so that she planned to deliver the worksheet in order to remind the term, constant, like term, and coefficient concepts with connecting teach addition and subtraction, and she planned to connect the area concept with multiplication. Besides, SCK6(+) indicates that her knowledge to provide mathematical explanations for the procedures of addition and subtraction with using papers to represent variables; for the properties of addition with connecting the properties of addition in integer within comparison in a table. In teaching addition and subtraction, KCS6(+) indicates in planning phase that the papers as concrete materials to visualize the algebraic expressions for students were appropriate and she predicted that the students would find them interesting and motivating. SCK2(+) indicates that her knowledge to link algebraic and geometrical representations to underlying the area concept and of how to choose, make, and use mathematical representations was appropriate. She explained the area concept that underlying of modeling multiplication of algebraic expressions with algebra tiles correctly. In connection with the use of papers and algebra tiles, KCT8(+) indicates that she suggested using these manipulatives to address and remedy the students' possible errors or misconceptions with her knowledge appropriately. Her knowledge of students' thinking related with operations with algebraic expressions was appropriate and KCS1(+) indicates that her knowledge of the students' difficulty in the application of distribution property and KCS5(+) indicates that her knowledge of common students' misconceptions was appropriate such as the students' thinking of x.x as 2x. For the addition and subtraction instruction, KCT1(+) indicates that her knowledge to choose which examples to start teaching addition and subtraction appeared efficiently with connecting the properties of arithmetical operations that the students had learnt before. However, KCT1(-) indicates that her knowledge to choose which examples to take the students deeper into the content appeared inadequately because of focusing on the properties of operations merely not the like term concept. On the other hand, for the instruction of multiplication, KCT1(+) indicates that her knowledge to choose which examples to start and to take the students to deeper into the multiplication appeared efficiently with modeling the multiplications using algebra tiles and connecting the idea of the area of rectangles. In the curriculum perspective, KCC1(+) indicates her knowledge of objectives that belonged to content and curriculum for designing lesson was essential and adequate.

4.2.2. Instruction

The mathematical knowledge for teaching of Teacher B extracted from her actions throughout the instruction by focusing on common patterns in observation data based on MKT framework. Based on the common patterns in the teacher's actions that she performed throughout the instructions, the practices of Teacher B were grouped as seen in the Figure 56. The teacher's purposeful actions to teach the topic were grouped into five practices: 1) connecting the topic being taught to topics from prior years, 2) discussing on the activity related to the topic, 3) choosing the examples or activities to use to take the students deeper into the topic, 4) implementing the suggested activities, and 5) solving the questions and problems related to the topic from different resources. The two different instructions conducted for teaching two algebraic topics. The practices of Teacher B were examined within the context of topics; generalization of patterns and operations with algebraic expressions in the following sections. It is important to note that the extracted knowledge types from instruction are also examined with planning before the instruction, the reflections of the instruction in the post-observation interviews, and then are evaluated together to conclude Teacher B’s knowledge for teaching the algebra topics.

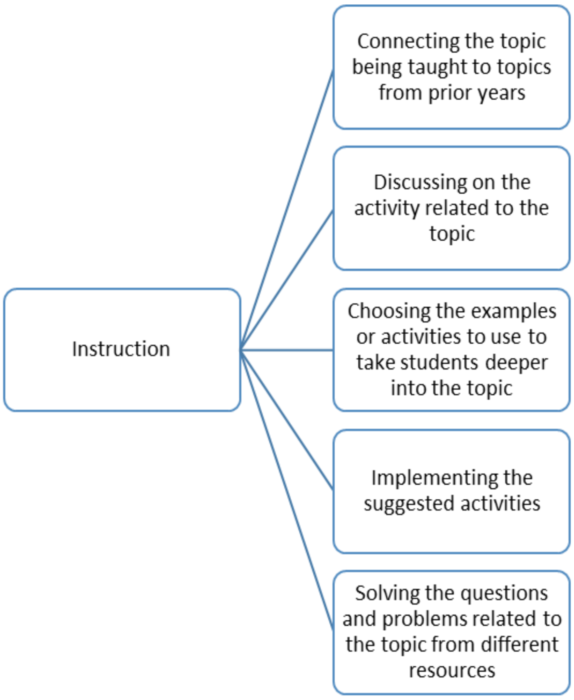


Figure 56 The practices of Teacher B during the instruction

4.2.2.1. Practices in the Instruction of Generalization of Patterns

Teacher B’s purposeful actions for teaching generalization of patterns were grouped into five practices: 1) connecting generalization of patterns to topics from prior years, 2) discussing on the activity related generalization of patterns, 3) choosing the examples or activities to use to take the students deeper into generalization of patterns, 4) implementing the pattern test, and 5) solving the questions related to generalization of patterns from the national exams. The extracted teacher’s knowledge based on MKT framework was analyzed within these practices. The reflection of the instruction after lessons was also presented to provide the teacher to evaluate her instruction by herself. The interpretations of the teacher could give information about her knowledge about students’ thinking and learning with respect to the instruction. The classroom dialogues that were most representative for knowledge type the teacher had, were selected from the instruction to illustrate how the teacher used her knowledge in teaching.

The first practice was connecting generalization pattern to topics from prior years and this title was extracted from one of the descriptors of SCK. This practice included the teacher's recalling process for prior knowledge that the students had to learn pattern generalization. To do this, the teacher first reminded the related concepts that the students learnt in previous grades. Then, connecting with them she indicated what they would learn in this grade. For this connection, the teachers asked questions about pattern concept, such as what it was, and how it was formed. The second practice was discussing on the activity related to generalization of patterns and this practice was also affected by the descriptors of KCT. This practice included a discussion for generalizing non-linear figural growth pattern with using its tabular representation. The teacher emphasized the generalization at this part of the lesson since it provided the first teaching of getting the general rule. She let the students to explain their answers and provided to discuss the answers if they worked for the entry pattern by participating the students. The third practice was choosing the examples or activities to use to take the students deeper into generalization of patterns and this title also was extracted from one of the descriptors of KCT. This was also as a continuation of choosing an example or activity to start category. This practice included how the teacher taught generalization of linear growth pattern differed from the beginning activity included non-linear growth pattern to improve students' knowledge as in the sequence of her lesson plan. The fourth practice was implementing the pattern test including only linear patterns. Since the teacher involved the pattern test that the researcher suggested and allocated about three lesson-hours' time, and so this part of the lesson was explained under a separate heading. The fifth practice was solving the questions and problems related to generalization patterns from the national exams from previous years. This part was as questioning by teacher and answering by students. These practices are explained with examining how the teacher used her knowledge based on MKT framework.

4.2.2.1.1. Practice One: Connecting Generalization Patterns to Topics from Prior Years

At the first lesson of the instruction, Teacher B began the instruction with asking questions about pattern concept. She asked the questions with connecting the pattern topic from prior years to recall students' knowledge about pattern. She asked students what they understood and remembered from previous years when pattern was told. She used her knowledge to connect the knowledge about pattern topic between grades appropriately (SCK1+). As the students gave answers, she responded to students appropriately. When the students gave correct answers, she interpreted students' answers and made explanations with her own sentences. Her knowledge to develop a usable definition for the pattern concept with using students' answers appeared as in the following script:

> B: What is a pattern? What do you remember about the pattern from previous years?
> S (Student): The order of a figure.
> S: Is it a number?
> S: The repeat of a figure.
> S: There is a certain order.
> B: Actually you say true things, there is an order, and it goes in a particular way. As you say, numbers or figures can be used. Then, we can say, the patterns are numbers or figures that continue based on a certain rule. For example, if the 2, 4, 6, 8 … pattern is given...
> S: Two by two.
> B: Okay, you have found the rule of this pattern. Now, we go a step further and we will find the rule of the patterns in this lesson.

In this situation, Teacher B asked to the students what the pattern was and then she accepted their answers. She developed the definition of the pattern with using students' answers. Her knowledge to develop usable definition appeared appropriately and she defined the pattern as the numbers or figures that continued within an order or a certain rule (SCK3+). At that point, she exemplified 2, 4, 6, 8 … pattern. When the student indicated "two by two" immediately, the teacher responded to this student that this was the rule of the pattern. However, Teacher B called "two by two" as the rule of this pattern, but she explained that they would find the rule of the pattern as a next step. This situation might confuse the students since some students might consider only verbal statements as the differences between the terms was the rule of the pattern when they were asked. Thus, her knowledge to anticipate the misunderstandings that might arise with pattern generalization being studied in class appeared inadequately (KCS2-). She should have made stated that the rule was writing algebraically using letters to prevent possible misunderstandings.

4.2.2.1.2. Practice Two: Discussing on the Activity Related to Generalization of Patterns

Teacher B taught patterns with centering the textbook and gave examples from the textbook as in its order. She started with the first example of the textbook that was 2, 6, 12, 20 … pattern at the first day of the instruction after reminding what the pattern was. This pattern was non-linear figural growth pattern in the textbook, and the teacher gave this pattern by modeling with unit cubes as manipulatives. After this non-linear growth pattern, she did not generalize any non-linear pattern types excluding this example throughout the instruction. This pattern example was in the introduction part of the instruction and was different from linear growth patterns and thus Teacher B emphasized the generalization process more with creating the discussion environment. Actually, she provided discussion environment for the students so that they could share their ideas and Teacher B generally responded to them appropriately in lessons. When one of the students came up with the correct answer, she did not accept the answer immediately, she gave some time to other students to think about the question.

At the beginning of the generalization of 2, 6, 12, 20 … pattern, Teacher B first modeled the pattern for the first four terms with unit cubes by herself as in the below figure. Her knowledge of how to use mathematical representations appeared in this point inadequately since she used figures to provide visuality for the students (SCK7-) and she indicated it as modeling. However, she did not use the change of figures to point out the increment or growth in order to guide the students to explore the relationship of the pattern. Thus, this modeling was not appropriate to support students' learning. Actually, she used figural representations only to facilitate the students' imaginary. She explained to the students that this was a figural pattern and asked them to represent it with using numbers. While she was asking to the student to do this, she also indicated that it would be a numerical pattern. Her knowledge to link figural and numerical representations to underlying the idea of the relationship of the pattern appeared inadequately and she used the tools merely to provide visuality since she did not make any explanations about the change of the figures and the relationship among the numbers (SCK2-).

Figure 57 The representation of the pattern with unit cubes

While the students were saying the numbers of unit cubes, she wrote them one under another for the first four terms of the pattern (2, 6, 12, 20 …). Then she asked to the students:

B: Okay, what is the next? Can you guess?
S: 29
S: 26
S*: 30. The difference between 2 and 6 is 4. Then, 2 is added to this difference.
B: That's is 4 from (6-2), it is 6 from (12-6).
S*: Yes, then the difference is 8, then we add 10 and get 30.
B: Other ideas?
S: Is it 29?
B: 29?
S: 32.
B: Okay, you have said 30.
(S* represents the same student)

Teacher B created a column with written number and called the number of cubes for this row. Then, she created a column to the left before the first column and called this column as the position number as seen in the below figure. She wrote 1, 2, 3, 4, and 5 in this new column:

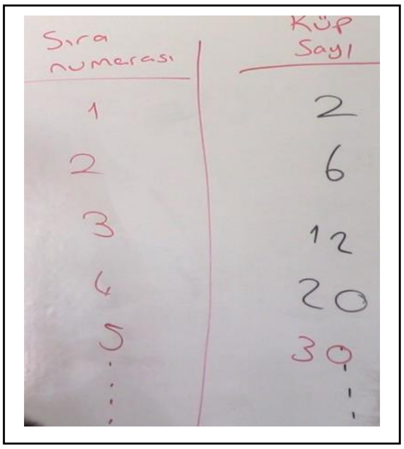


Figure 58 The representation of the terms in a table

At that point, Teacher B's knowledge of when to use students' remarks to make a mathematical point appeared appropriately (KCT5+) and the teacher asked new question as how to find 50th term to lead the students in order to find a general rule. After the teacher asked what the 2nd, 3rd, 4th, and 5th terms were and got the correct answers, she asked what the 50th term was:

> B: If I ask 50th term?
> S: We cannot find by counting one by one.
> B: Your friend said that we could not find with counting one by one. What can we do then? Will you write one by one to 50th term?
> S*: Multiply 50 by 2, and add the last term.
> B: What is the last term?
> S*: The number that we find last.
> B: Then, you must find 49th term.
> S: If you know 49th term, you can find 50th term also.
> B: Okay, can we find a relation between the position number and the number of cubes?
> S: Teacher, I find the relation. For example, when we multiply 1 by 2, we will get 2; 2 by 3, 6; 3 by 4, 12. Similarly, multiply 4 by 5, 20; 5 by 6, 30.
> (S* represents the same student)

In this situation, Teacher B made the students to think about 50th term asking how to find it. She wanted the students to find a way different from counting one by one to 50th term since she felt the students to need a rule for finding any terms in the pattern. Then, the student proposed multiplying 50 by 2 and adding the last term to find directly 50th term. When Teacher B asked what the last term was, the other student responded to this argument. This student stated that knowing 49th term provided to find 50th and in this case finding 49th term was a problem. At that point, Teacher B guided the students to think a relationship between the position number and the term and inserted a column to the right called the relationship between the position number and the number of cubes. The student in the dialogue proposed an arithmetic relation immediately. While this student was explaining his idea, the teacher was writing on the board 1.2=2, 2.3=6, 3.4=12, 4.5=20, 5.6=30 and she asked to the student to explain them as in the following script:

> B: Okay, which of them mean what?
> S: …
> B: For example, what 1 mean? Where do you take?
> S*: The position number
> B: What 2 mean? Where do you take?
> S*: The number of cubes?
> B: But, you will get the number of cubes. Let's call it anything else. Is it 1 more of the position number?
> S**: Yes, multiply the position number by 1 more of it.
> B: Let's look, to find the number of cubes, (She is writing 1.(1+1), 2.(2+1), 3.(3+1), 4.(4+1), 5.(5+1)). What do you tell about the second one in these writings?
> S**: Always, 1 more of the position number.
> (S* and S** represent the same students)

When the teacher asked to explain what 1 meant and what 2 meant in the first multiplication, S* explained that 1 was the position number and 2 was the number of cubes that the student explained the meaning of 2 incorrectly. At that point, the teacher led the students directly that it was 1 more of the position number without waiting any answers of students. Then, she represented the multiplications as 1.(1+1), 2.(2+1), 3.(3+1), 4.(4+1), 5.(5+1) (see the below figure). Her knowledge of how mathematical language was used appeared as she used parenthesis, equal sign, and operation signs (addition and multiplication) appropriately in writing these numerical representations (SCK5+).

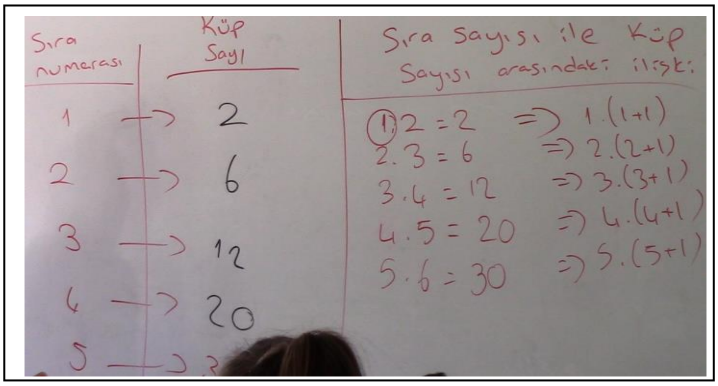


Figure 59 Tabular representation of the pattern by Teacher B

Her knowledge to choose, make and use the tabular representation appeared that focusing on the arithmetical relationships in tabular representation to underlie the relationship in pattern to conceptualize generalization could be effective (SCK7+). When the students realized that the relation was multiplying the position number by one more of it, the teacher asked what they could write for n^{th} term. However, the students had difficulty to answer to this question. Teacher B realized the difficulty of students and gave example by asking if the position number was 50 what would be multiplied, and if the position number was 100 what would be multiplied. The students could answer 51 for 50^{th} term, and 101 for 100^{th} term. Nevertheless, the students could not answer for n^{th} term again. When she asked 'What is 1 more of n? How can we present it?' and she did not get any answers. Then, the teacher asked how they wrote one more of a number algebraically referring 6^{th} grade, one student answered as n+1, and the teacher wrote n.(n+1) as the general rule. Then, the teacher wrote the general rule with another algebraic representation as ($n.(n+1)=n^2+1$) by using distributive property:

> B: … or we can use .. Remember, the distributive property of multiplication on addition. I multiply n by n (n.n), then n by 1. When we multiply two n, we can represent it as exponential. This is another representation of the general rule in this question. You can use it, also.

Her knowledge of how to provide mathematical explanations for the other algebraic representation appeared and she explained it with applying the distributive property of multiplication appropriately (SCK6+). She also used algebraic notations and operation signs and equal sign appropriately while explaining with using her knowledge of using mathematical language (SCK5+).

After the class got the general rule for the pattern, some students had difficulty with understanding some mathematical points, Teacher B paused for more clarification and made explanations about how they wrote the rule again:

> B: You have found the 5^{th} term and understood how the other terms will go on. But, we will not ask only the next terms. If what 20^{th}, 50^{th}, or 100^{th} term is asked, you cannot write to 100^{th} term one by one. So, you should reach a generalization. We will find a relationship between the position number and the numbers (the number of cubes). Can we find? Your friend said that, he could find if she multiply 1 by 2. Multiplying 2 by 3, and 6; multiplying 3 by 4, and 12. Be careful. We multiply the position number and 1 more of this number. Is this a pattern? Yes. Do these multiplications work?
> Students: Yes.
> B: Then, if we come 20^{th} term, to find the number of cubes, we multiply 20 by 21. If we come 40^{th} term?
> Students: 41.
> T: We come 65^{th} term?
> Students: 66.

She explained that a rule which worked for any terms was needed for finding asked terms of the pattern first such as 20^{th}, 50^{th}, or 100^{th} term. She also stated that this rule was used for finding any term in the pattern instead of counting to the asked term. Then, she explained that this rule was a relationship between the position number and the corresponding term and pointed out that the number of cubes yielded from the multiplication of the position number and one more of it. She exemplified that if the position number was 40, it was multiplying 40 and 41, or if the position number was 65, it was multiplying 65 and 66. Her knowledge to decide when to pause for more clarification appeared appropriately when the students had difficulty in understanding how she formed the general rule algebraically (KCT5+). In this situation, where the students had difficulty was generalization of the pattern, as Teacher B expected while planning. Thus, Teacher B explained how they wrote the rule again. Her knowledge to understand the difficulties of students with algebraic expressions appeared appropriately (KCS3+) so that she exemplified to guide the students to find the n^{th} term. Then, she described why the general rule was used. She explained as “What does the general rule mean? We put any number for n, we find the same thing”.

Teacher B realized the lack of knowledge of students about writing verbal statements algebraically, when she asked ‘What is 1 more of n? How can we present it?’ and she did not get any answers. Then, she gave some examples from writing algebraic representation of verbal statements in order to remind the students algebraic expressions since her knowledge to understand the needs of recalling algebraic expressions for students appropriately (KCS3+):

> B: … You learnt algebraic expressions last year. For example, 5 more of one number. You say x for the number since we do not know, or you can say a. 5 more of it, x+5. When 5 less of it asked, you say b-5. Now, we use n for unknown value, and 1 more of it is n+1.

In this situation, Teacher B connected the writing of general rule algebraically with algebraic expressions that was taught at 6^{th} grade. She reminded algebraic expressions by giving some examples such as 5 more of a number, 5 less of a number, and then she connected to 1 more of it as n+1 for using of generalization of 2, 6, 12, 20 … pattern. Thus, her knowledge to connect the topic being taught to topics from prior years appeared appropriately (SCK1+).

Teacher B finished the lesson during the first day of the instruction with giving the following definition of general term:

> The representation with using n of the n^{th} position number is called the general rule of the pattern.

Her knowledge to give the definition appeared at the end of the lesson with choosing from textbook and giving it to the students after the introduction activity about generalization of patterns (SCK3⊥). She was explaining the general term and the general rule as in the following:

> B: When we generalize the pattern, using letters and algebraic expressions, we will get a general rule. What do we call it? n. You can use another letter. We usually call “n” the general rule. It is entitled as the general term. What does it mean? You can substitute any number for n, you find the same thing.

When the explanations were examined, Teacher B's troublesome with the usage of the general term and general rule concepts appeared. Teacher B called 'n' both the general rule and the general term. Her knowledge to use terms about general term and general rule inappropriately since she confused with these terms while using (CCK2-). Then, her explanation about the function of the general term was also troublesome that getting the same thing by using the general term for any number can cause misunderstandings that mathematical language used inappropriately (SCK5-). Her knowledge to anticipate the misunderstandings that might arise with the concepts of general term and general rule being studied in class appeared inadequately (KCS2-). Thus, she should have made more detail about what 'same thing' meant.

In this discussion process, the teacher generally asked new questions to proceed the discussion for generalization and made clearer the points that students could not understand. She used students' answers and questioned them to explain their answers, and she responded to the students appropriately. Other students could also participate and explain their ideas about their friends' answers and proposed new ideas. At the end of the discussion, she summed up what the class talked and gave the definition of general term and general rule. In general, Teacher B used the discussions in lesson since she wanted the students to explore the solutions with brainstorming, as she stated in lesson planning.

After the lesson finished, the researcher interviewed with the teacher to get her ideas about the lesson. Teacher B stated that the lessons were carried out as she planned, and she did not encounter any unexpected situations. According to Teacher B, the level of the students, the students' participation and performance were good. **S**he made these explanations in the post-interview as in the following script:

> B: Students could continue the pattern easily. It is important to note that they realized getting 50th term one by one was not possible much. They thought about how they find with a short way. One student could find the relationship as arithmetically immediately. But, they had difficulty with generalization. They should solve different questions. This difficulty stems from algebraic expressions. They were taught in previous year, but they did not remember.

In this situation, Teacher B emphasized the importance of the students' need of finding a rule to find 50th term and she also guided the students using this way. Then, although one student found the relationship arithmetically, Teacher B stated that the students could not understand writing the general rule algebraically. In this regard, she suggested solving different questions for students' understanding. She thought as the reason was the lack of knowledge about algebraic expressions. Her knowledge to understand the difficulties of students with generalization of patterns appropriately and she asserted as the reason that the students forgot the concept of algebraic expressions (KCS3+).

4.2.2.1.3. Practice Three: Choosing the Examples or Activities to Use to Take Students Deeper into Generalization of Patterns

During the second day of the instruction, Teacher A continued with generalizing 3, 5, 7 … pattern from the textbook as in her lesson plan. This was a linear growth numerical pattern and the teacher asked the students to model this pattern first. The students' modeling of the pattern were as in the following figures:

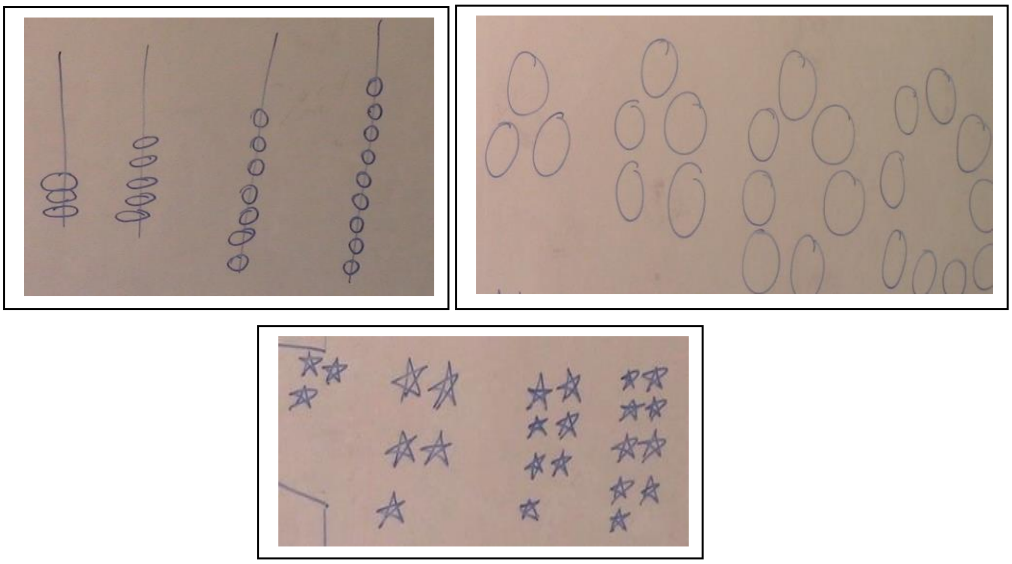


Figure 60 Modeling the pattern with different figures by the students

Teacher B's knowledge of how to use mathematical representations appeared in this point that she used figures to provide visuality for the students inadequately (SCK7-). She did not use the change of figures to point out the growth in order to guide the students to explore the relationship of the pattern. Thus, her use of figures in this way could not be called as modeling and this representation could not support the students' understanding of generalization patterns. After the teacher got the students' answers, Teacher B indicated that they could model the pattern and wanted to create a table. She also emphasized finding a general rule for the pattern. She drew a table and wrote the position number and the number of circles in each column. She filled the rows for the first four terms of the pattern, put dots and asked what the 100^{th} term was. Some students proposed multiplying 100 by 101 to find 100^{th} term. At that point, Teacher B realized the students' misconception in their answers. After the generalization of 2, 6, 12, 20 … pattern as n.(n+1) in the first lesson, some students overgeneralized that they considered this rule worked for other patterns. Thus, her knowledge of how to address the students' errors effectively and she tried to remedy these errors with making explanations (KCT8+). She stated that this rule worked for the 2, 6, 12, 20 … pattern and it might not work for this pattern (3, 5, 7 …). She also showed to the students that multiplying 1 by 2 did not yield 3 as the 1^{st} term. Then S proposed a rule as in the following script:

> S: For example, we multiply 1 by 2 and add 1; multiply 2 by 2 and add 1; multiply 3 by 2 and add 1.
> B: Does it work? Can we use this rule for all terms in the pattern? Your friend explained the relationship that 2 times of the position number and adding 1 to it. 1. 2 +1, is it 3?
> Students: Yes.

In this dialogue, Teacher B explained the student's ideas and justified it by showing multiplication of the position number and 2, and adding 2 to it for the 1^{st} term. Then, she explained with showing that the same procedure worked for 2^{nd}, 3^{rd}, 4^{th}, and 5^{th} term. Her knowledge of how to provide mathematical explanations for the procedure as 2 times of the position number and adding 1 to it appeared (SCK6+). While she was showing, she also asked to the students what the results of the application of the procedure in order to get approval of other students for the general rule. Teacher B wrote this arithmetical relationship as 2 times of the position number and adding 1 to it by herself using the mathematical language such as equal sign, operation signs and parenthesis appropriately (SCK5+). With the application of the procedure and arithmetical representations of it for the first four terms, Teacher B explained and justified the student's proposed answer for understanding of other students using her knowledge (SCK4+).

Another student proposed a different way to find 100^{th} term and Teacher B's explanation was as in the following script:

> S: When 1 is added with 2, it is 3. If we add 100 and 101, is it correct?
> B: Be careful, the relationship investigated between the position number and the terms. It is not between the terms. Because your concern is what the number for which order in the pattern. Investigate the corresponding relationship between two of them, not between terms.

In this situation, the student proposed adding the position number with the next position number to get corresponding term by exemplifying adding 1 and 2 to find 3 as the 1^{st} term. Teacher B realized the student's error and her knowledge of how to address students' errors effectively appeared (KCT8+) and she stated the relationship was investigated between the position number and the term not between the terms in the pattern in order to remedy this misunderstanding of the student.

The only thing that she did not do was the question "The general rule of a pattern is $3n^2$, n represents the position number. Find the 7^{th} term in this pattern" in her lesson plan and then she explained that she forgot to do it in the post-interview.

4.2.2.1.4. Practice Four: Implementing the Pattern Test

After the modeling pattern activity, Teacher B asked to students to do the questions in the pattern test that the researcher suggested. The major added task was the pattern test and Teacher B appreciated and included it to her lesson plan. Teacher B stated that experiencing with different represented patterns could improve students' learning of generalization concept. Her knowledge to predict the numerical, figural and tabular represented patterns that students would find interesting and motivating appeared to select this test appropriately (KCS6+). Teacher B included this test to her lesson plan where she planned to do exercises after teach generalization. In connection with the pattern test, the researcher suggested the use of table from examination of students' correct solutions in the test. Teacher B made the following explanations about the use of table:

> B: I teach patterns by using table directly. Because students could understand better with the table. They represent everything in the table, the position number, the number of figures… they could see the relationship clearly.
> Researcher: Of course, the student can understand well with using table. However, some of them might use the change of figures, they can continue the figures and explore the relationship with this way. Some of them use numerical reasoning. That's related to the way of students' thinking and understanding.
> B: They can see the relationship in table very clearly. I use always table in instruction.

Teacher B would use the tabular representations in her teaching of generalization patterns. However, she insisted on using only table to represent patterns based on previous teaching, although the researcher tried to explain that every students' understanding and thinking could be different and some students could understand easier with using figural reasoning or numerical reasoning.

In the instruction, she generally asked to the students to explain the question with their own words at the beginning of solution in order to understand their thinking by hearing as they expressed. To illustrate for the first pattern which was a linear growth figural pattern in the test:

> B: Now, you have read the first question of the pattern test. Is there someone to explain the question?
> S: There are trapezoid tables and chairs around them. The tables are increasing one by one; the chairs are increasing by 3.
> B: Which of them you should consider to solve the question, tables or chairs?
> Students: Chairs.
> B: Anyway, the number of tables and the position number of figures are same.

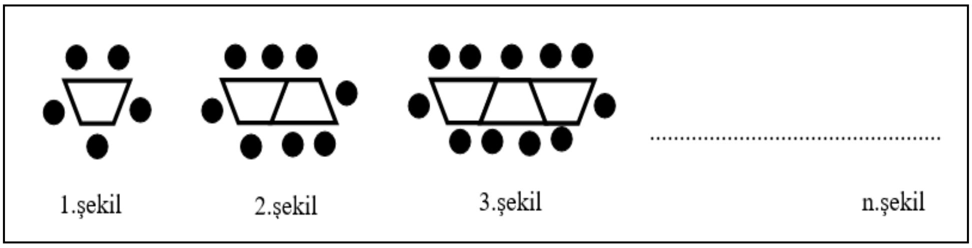


Figure 61 The first question of the pattern test

In this situation, Teacher B asked to the student to explain what he understood related to the pattern and she especially asked what should be considered to get generalization to guide the students to explore the relationship between the position number and the number of chairs so that her knowledge to interpret students' emerging thinking as they expressed appeared appropriately (KCS4+). Then, the teacher wanted the students to transform the numbers to a table. While solving the pattern questions, Teacher B first wanted the students to draw a table and to write the given terms in columns. In the pattern, there were 5 chairs for the 1st table, 8 chairs for the 2nd table, and 11 chairs for the 3rd table. It was asked to find the relationship between the nth table and corresponding number of chairs (

*Figure **61*** 61).

The teacher asked how many chairs were used for 4th table and the students answered as 14 with explaining the terms were increasing by 3. Then, another student proposed an arithmetic relationship among the terms of the pattern as in the following script:

> S: Multiply 1 and 3 and add 2, it is 5; multiply 2 and 3 and add 2, it is 8; multiply 3 and 3 and add 2, it is 11; multiply 4 and 3 and add 2, and it is 14. For the first row, 1.3+2=5, for the second row 2.3+2=8, for the third row 3.3+2=11, for the fourth row 4.3+2=14.
> B: Okay, let's generalize then. If we come to nth row, we put n table?
> S: n.3+2.
> B: Let's organize 3n+2. Then, if we put 100 tables, how many chairs will we put around them? Multiply 100 and 3, 300, and then add 2, it is 302.

The student could find and write the arithmetical relationship between the position number and the number of chairs, then the student could get the general rule algebraically as n.3+2. The teacher rewrote the expression as 3n+2 and showed how this rule was used to find 100th term. However, Teacher B explained a shortcut to find the general rule and showed it for the first question. Her explanations for the 5, 8, 11, 14 … pattern whose general rule was 3n+2 were as in the following script:

> B: If the differences between terms are constant, what is the difference here?
> Students: 3.
> B: It goes with increasing by 3. So the rule begins with '3n'. Then by, the first term is 5. To get 5, when 1 is substituted for n, and 1 is multiplied by 3, that is 3. To get 5, what is added? 2. You can find the general rule by using this way. Start with 3n and add 2. The general rule is 3n+2. … Let's try this way for the previous example, 3, 5, 7… pattern. This pattern continues by increasing 2. So, start with 2n. For n=1, 2n=2.1=2. But the first term is 3, what is added to get 3? 1 is required. Then, the general rule is 2n+1. Be careful, we can use this way for the patterns whose terms increase constantly. But, for the second question of the pattern test, there is not a constant increment. Can you see? 10, 16, 28, 64 …. You should find the relationship in these type of patterns.

In this situation, Teacher B presented the shortcut to the students to find the general rule of the patterns. According to her, if the difference between the terms was constant, this difference was multiplied by n first. Then, what was required to find the first term was added to or subtracted from algebraic expression that had 'n'. Teacher B exemplified for 5, 8, 11, 14… pattern and 3, 5, 7… pattern by applying the shortcut procedure. Hereafter, the teacher wanted the students to use this shortcut to find the general rule and used by herself for other pattern generalization. The students could perceive this phrase as a rule for generalization and they might memorize it. This way could prevent the students to conceptualize the generalization process. Additionally, using only the first term to complete the general rule might cause misunderstandings of students that they could consider the first term only while generalizing and it could prevent understanding the rule was valid for all the terms of the pattern. Her knowledge to anticipate the misunderstandings that might arise with pattern generalization being studied in class appeared inadequately (KCS2-). Thus, she should have made explanations for the meaning and reasoning of the procedure and the general rule conceptually. Another problematic point was about her explanation that investigating the relationship in which the pattern did not have constant difference between its terms for the second question of the pattern test (Table 16). All patterns had relationships in itself, and this explanation was troublesome about the pattern concept. This explanation also might cause misconceptions for the students since they could perceive that when the difference was constant, the shortcut was applied; when the difference was not constant, the relationship was investigated.

Table 16 The second question of the pattern test

The number of t-shirt	TL
3	10
5	16
9	28
21	64
:	:
:	:
n	?

Thus, her knowledge to anticipate the misunderstandings that might arise with pattern generalization for different linear growth patterns being studied in class appeared inadequately (KCS2-). The second day of the instruction finished at this point, and then she continued to solve the remained questions of the pattern test in the next lesson.

Teacher B used tables to show the relationship between terms for the first two patterns. She drew the table and wrote the numbers in it and guided the students to use the table. For the third day of the instruction, she started to solve the questions of the test with using the shortcut that she explained in previous lesson. For the third question of the test, the student explained the rule for 3, 7, 11, 15, 19 … numerical pattern as in the script:

> S: We multiply 1 by 2 and add 1; multiply 2 by 3 and add 1; multiply 3 by 4?
> B: Is the general rule is valid for all terms? But, you have increased the number multiplied by the position number.

In this situation, the student did not conceptualize the concept of general rule and thus she used different rules for the first three terms. Teacher B realized this misunderstanding and reminded the function of general rule as it must work for all terms. She remedied the student's misunderstanding at that point by making explanations for the meaning of the general rule so that her knowledge of how to address students' errors appeared effectively (KCT8+).

Then, the teacher guided the students to use shortcut to find the general rule and the following script exemplifies how Teacher B made generalization for the fourth question of the pattern test with using this shortcut:

> B: Is this a linear growth pattern? Are their terms increasing or decreasing?
> Students: Linear growth.
> B: So, to generalize, start with 2n since the difference between the terms is 2. The first term is 1. To get 1, what number should we subtract?
> Students: 1.
> B: Yes, we should subtract 1. 2n-1. Okay. Let's look if this rule works for other terms. Multiply by 2 and subtract 1, 1.2-1=1, 2.2-1=3, and 3.2-1=5, so on.

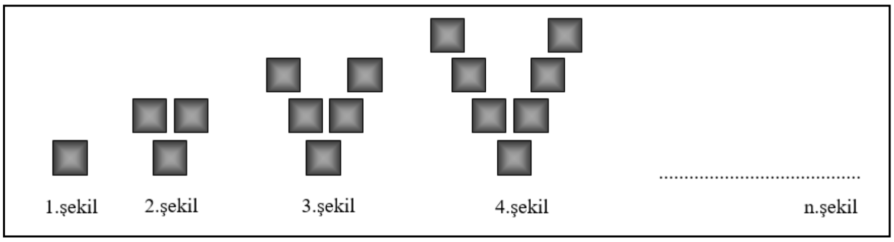


Figure 62 The fourth question of the pattern test

The fourth question was a linear figural growth pattern whose terms were 1, 3, 5, 7…. In this situation, Teacher A generalized it using the shortcut. She asked the students to use this way for generalization of other patterns in the test. Especially, the students had difficulty in generalizing of the pattern in the sixth question. It was asked to generalize 60, 55, 50, 54, 40, 35 … n pattern. The following script exemplifies how Teacher B made generalization for this pattern discussing with class:

> B: What is this pattern type?
> Students: It is decreasing by 5.
> S: Is the rule 'n-5'?
> B: Check this rule. Substitute numbers for n. Do you get 60, and then 55? If we substitute 1 for n-5, we get -4 from 1-5. It did not work. Remember, the

meaning of n is the position number. So, for the 6th term, we substitute 6 for n. Do you have any idea?
S: 1.n-6.
B: Let's check. If we substitute 1 for n-6, we get -5 from 1-6. But we should get 60. Be careful, when there is an increment, for the previous pattern (1, 4, 7, 10…) I always write +3. How does the general rule begin?
Students: 3n.
B: This patterns' numbers are decreasing by 5. Then, how does this pattern begin?
Students: -5n.
B: Okay, but, we must get 60. I have -5 for the first term, what can we do to get 60?
S 4: Add 65.
B: Does '+65' work? Add 65 to -5 for the first term, we get 60. For the second term, multiply -5 by 2, add 65, 55. (She is writing -5.1=-5+65=60, -5.2=-10+65=55).

In this situation, Teacher B responded to the students appropriately and she asked them to check their answers such as n-5, n-6. She used the students' proposed general rule by substituting 1 for n in these algebraic expressions and showed that they did not work. She led the students to use the shortcut and reminded the general rule of 1, 4, 7, 10… pattern beginning was 3n. When the students answered -5n, Teacher B asked how the first term was found as 60. The students proposed adding 65 and Teacher B checked it for the first and second term using -5n+65. But, she made errors in arithmetical expressions in using notations (-5.1=-5+65=60, -5.2=-10+65=55) and her knowledge how mathematical technical language was used appeared inappropriately since she used equal sign incorrectly (SCK5-). Because, the expressions in the both sides of the equal sign do not equal each other. She used the equal sign as it had the function of getting the result incorrectly. Excluding this expression, she generally used the equal sign appropriately in teaching patterns.

After the class generalized this pattern as 65-5n algebraically with helping of Teacher B and using the shortcut, the teacher created a new question as an example for the pattern with decreasing numbers to improve the students' understanding. Teacher B asked the students to generalize of 30, 27, 24, 21 … pattern and the student answered:

S: -3n+33. I can write like this: 3.1+33= 30, -3.2+33=27, -3.3+33=24, -3.4+33=21 …

> B: Since this pattern's terms are decreasing, the rule will start with -3n. What is the first term? 30. I should get 30. Thus, -3, so when I have 3 TL debt, I want to have net worth as 30. So, if 33 TL is added, 30 TL will be net worth. -3n+33. This is the general rule.

In this situation, the student generalized the pattern correctly and showed the rule worked for the first four terms. While Teacher B was explaining, she used borrowing analogy with debt and net worth for expressing of general rule and her knowledge of how mathematical language was used appeared appropriately (SCK5+).

At this point, the pattern test finished and then the teacher started to solve prior national exam questions as explained detailed in following section. It is important to note that Teacher B used numerical reasoning in generalizing patterns and wanted to students to think in this way. There were two figural growth patterns (first and fourth question) in test, she did not emphasize the change of figures to underlie the figural reasoning. She always wrote the numbers in the table and seek the relationship between the numbers as she indicated in lesson planning. This situation might make difficult for the students who could understand figural reasoning easier so that her knowledge to understand the need of students with pattern generalization process appeared inadequately (KCS3-). On the other hand, Teacher B started the generalization process with representing arithmetical relationship for the first four or five terms and then connecting the relationship she asked what the n^{th} term was. Teacher B used her knowledge of inductive reasoning (CCK1+) and she used this knowledge to provide mathematical explanation for generalization process. She started specific examples and then she reached the generalization with this inductive reasoning appropriately.

After the lesson finished, the researcher interviewed with the teacher to get her ideas about the lesson. Teacher B indicated that the pattern test was useful for improving the students' understanding:

> B: It was good. I studied on it before. This test includes both number and pictorial patterns. The students could see both of the examples with it. The sixth question was different. In general rules, unknown comes first. But, a number comes (65-5n) before the unknown for the general rule in the sixth

> question. They can see a different pattern type with this question, and it was useful for them, although they had difficulty with generalization of it.

Teacher B indicated that the importance of the students' need of experience with different types of patterns such as number and figural patterns and she considered that the pattern test provided these patterns. Her knowledge to understand the needs of the students appeared appropriately (KCS3+). Besides, the general rule of the pattern whose terms were decreasing in sixth question was not as in usual expressions in previous examples. According to her, this pattern's rule could provide students to consider the unknown also could be written after a constant in algebraic general rules.

4.2.2.1.5. Practice Five: Solving the Questions Related to Generalization of Pattern from the National Exams

At the end of this lesson, Teacher B asked following two questions. The first one was in the standard exam (SBS) from previous years, the second one was a question from a test book:

> 1. Which of the pattern do not have 103 as a term? (2009 SBS)
> A) 2n+1 B) 2n+2 C) n+2 D) n+1
> 2. For the pattern that its general rule is 4n-2, a) What is 30^{th} term? b) What is the position number for 22?

In this situation, Teacher B wanted the students to make aware of the types of questions about patterns that were asked in standard exams. Several students could understand and answer what was asked in the first question. These students proposed to use reverse operation by trying for all choices. As an example, for A option, they subtracted 1 from 103, and then divided it by 2. If the result was an integer, this rule could be, since the position number was always an integer. These students could understand the function of position numbers and terms in patterns and they could answer correctly. However, many students had difficulty to understand this method, and Teacher B's knowledge to understand the difficulties of students with what the position number and the term were appeared appropriately (KCS3+) so that Teacher

B made a different explanation for solving this question. She stated 103 is an odd number, if which of the expressions in the choices yielded an even number was the answer. She showed the application of the explanations for B option:

> B: For choice B, is the result of this expression an odd or even number? When an even number is multiplied by 2, is the result an odd or even number?
> Students: Even.
> B: Yes, then when this even number is added with any even number like 2 in this expression, it is an even number. But, our term 103 is an odd number. Thus, this '2n+2' is not the rule for the term 103.

However, the students did not understand again, and Teacher B realized it since her knowledge how to provide mathematical explanations for procedures appeared inadequately (SCK6-), even this procedure could seem more complex. It is important to note that the students had difficulty with these procedures since they did not understand how the position number and the general rule related with each other and they could not conceptualize the concept of generalization of patterns. Lastly, the teacher showed that C and D options could be the rule of the pattern whose term was 103 by applying the reverse operations.

In the second question, some students proposed to use the reverse operations for finding what was asked in question a. Since the students did not understand the procedure for previous question, they only considered applying the reverse operations. Therewith, Teacher B made these explanations:

> B: But, here, 30 is not the result. It is asked what the term is for the 30^{th} number. If the given number is the result, we will do the reverse operations. Whatever the rule is, I apply it for the 1^{st} term, 2^{nd} term ... 30^{th} term.

After the explanations, the students substituted 30 for n and got 118. Then, they applied the reverse operations to answer for b choice of the question and answered as 6. Teacher B emphasized the difference between these two questions in a and b. She indicated that while the term was asked for 30^{th} number in question a, the position number was asked for 22 in the pattern in question b. Her knowledge how to provide mathematical explanations for procedures for finding the position number and the term appeared adequately (SCK6+).

4.2.2.1.6. The Extracted Knowledge Types from the Instruction for Generalization of Patterns

Table 17 The extracted knowledge types from instruction for generalization of patterns

Practices	Extracted knowledge types				
	SMK		PCK		
	CCK	SCK	KCS	KCT	KCC
Connecting generalization of patterns to topics from prior years		SCK1(+) SCK3(+)	KCS2(-)		
Discussing on the activity related generalization of patterns	CCK2(-)	SCK1(+) SCK2(-) SCK3(⊥) SCK5(+,+,-) SCK6(+) SCK7(-,+)	KCS2(-) KCS3(+,+,+)	KCT5(+,+)	
Choosing the examples or activities to use to take students deeper into generalization of patterns		SCK4(+) SCK5(+) SCK6(+) SCK7(-)		KCT8(+,+)	
Implementing the pattern test	CCK1(+)	SCK5(-,+)	KCS2(-,-) KCS3(-,+) KCS4(+) KCS6(+)	KCT8(+)	
Solving the questions and problems related to generalization of patterns from the national exams		SCK6(-,+)	KCS3(+)		

Table 17 shows what type of knowledge of subject matter knowledge (SMK) and pedagogical content knowledge (PCK) that Teacher B had in instruction practices. (+) sign indicates the teacher’s existing knowledge was adequate or appropriate and she used it effectively, while (-) sign indicates the teacher’s existing knowledge was inadequate or inappropriate and she used it ineffectively. Each sign (+ or -) in the same knowledge type refers to Teacher B’s different use of this knowledge during instruction. Besides, for SCK3 knowledge type, $(\perp)$ is used to indicate the teacher’s knowledge to develop definition or explanation appeared as merely choosing it from the textbook and presenting to the students. Since this situation meets the code (SCK3) partially, $(\perp)$ sign is used.

SCK1(+) in the first practice indicates that her knowledge to connect the knowledge about pattern topic between grades was appropriate. In connection with this, SCK3(+) indicates her knowledge to develop usable definition was adequate and she defined the pattern as the numbers or figures that continued within an order or a certain rule. However, KCS2(-) indicates that her knowledge to anticipate the misunderstandings about the general rule concept was inadequate since she had accepted the verbal expression such as increasing two by two as the rule but then she stated that she would get the rule, and she wrote the general rule algebraically. This situation could confuse the students’ understanding of what the meaning of the general rule concept. In connection with the general rule concept, CCK2(-) indicates her knowledge to use terms about general term and general rule was inappropriate since she called ‘n’ both the general rule and the general term. Another point in her explanation was about the function of the general term that getting the same thing by using the general term might be troublesome. Thus, KCS2(-) in the second practice indicates that she could not anticipate possible misunderstandings about pattern generalization. SCK7(+) indicates that her knowledge to choose, make and use the tabular representation and figural representation with focusing on the arithmetical relationships in tabular representation to underlie the relationship in pattern to conceptualize generalization could be effective, while SCK2(-) indicates that her knowledge to link figural and numerical representations to underlying the idea of the relationship of the pattern appeared inadequately with using the figures (manipulatives or drawings) merely to provide visuality and not make any explanations about the change of the figures and the relationship among the numbers. SCK5 knowledge type is related with the mathematical language the teacher used and SCK5(+,+) indicates her knowledge of using mathematical language was appropriate with using algebraic notations and operation signs and equal sign correctly while explaining the generalization process. For the second practice, KCT5 knowledge type is related with the teacher's leading of the discussion and KCT5(+,+) indicates that the teacher's asking 50^{th} term to lead the students in order to find a general rule, and making clarification about getting the general rule when the students had difficulty in understanding it. She also showed the use of general rule with exemplifying 40^{th} and 65^{th} term appropriately, and she reminded writing algebraic representation of verbal statements since she understood the students' needs and difficulties (KCS3(+,+,+)).

SCK7(-) in the third practice indicates that her knowledge of how to use mathematical representations was inadequate that she used figures to provide visuality for the students without showing the change of figures to point out the growth in order to guide the students to explore the relationship of the pattern. Actually, her knowledge about the pattern generalization was adequate. Thus, SCK6(+) in the third practice indicates that her knowledge to provide mathematical explanations for the function of the general rule with showing that the same procedure worked for 2^{nd}, 3^{rd}, 4^{th}, and 5^{th} term. She also addressed and remedied the students' errors at that time of the generalization process and KCT8(+,+) indicates that she stated that the relationship was investigated between the position number and the term, not between the terms in the pattern, that was the functional thinking. Thus, it indicates that her knowledge to address the students' errors was appropriate with explaining the general rule since some students considered the rule of the first pattern was also valid for other patterns. Though her knowledge about pattern generalization was conceptual, she showed and asked to the students to use the shortcut that was multiplying the difference between the terms and n then adding or subtracting a number to find the first term if the difference was constant. This situation could cause the students to perceive it as rule and also to consider the first term only while generalizing, and it could prevent understanding the rule was valid for all the terms of the pattern. Thus, her knowledge to anticipate the misunderstandings that might arise with pattern generalization being studied in class was inadequate (KCS2(-,-) in the fourth practice). Although she guided the students to use this rule procedurally, she addressed and remedied all students' errors at that time and KCT8(+) in the fourth practice indicates that her knowledge of how to address students' errors effectively appeared. Nevertheless, she used the tabular representation and numerical reasoning between terms, thus this situation might make difficult for the students who could understand figural reasoning easier so that KCS3 (-) in the fourth practice indicates that her knowledge to understand the need of students with pattern generalization process was inadequate.

4.2.2.2. Practices in the Instruction of Operations with Algebraic Expressions

There were two objectives for teaching operations with algebraic expressions as indicated planning section and the teacher designed her lessons respectively based on the two objectives orderly within the same lesson plan. Since she taught addition and subtraction first and then multiplication of algebraic expressions, her instructions were explained and documented respectively in this section.

4.2.2.2.1. Practices in the Instruction of Addition and Subtraction (Simplification) of Algebraic Expressions

Teacher B's purposeful actions for teaching addition and subtraction of algebraic expressions were grouped into five practices: 1) connecting addition and subtraction of algebraic expressions to topics from prior years, 2) discussing on the activity related to addition and subtraction of algebraic expressions, 3) choosing the examples or activities to use to take the students deeper into addition and subtraction of algebraic expressions, 4) implementing the suggested activities, and 5) solving the questions and problems related to addition and subtraction of algebraic expressions from a test book, the textbook and workbook. The extracted teacher's knowledge based on MKT framework was analyzed within these practices. The reflection of the instruction after each lesson was also presented to provide the teacher to evaluate her instruction by herself. The interpretations of the teacher could give information about her knowledge about students' thinking and learning with respect to the instruction. The classroom dialogues that were most representative for knowledge type the teacher had, were selected from the instruction to illustrate how the teacher used her knowledge in teaching.

The first practice was connecting addition and subtraction of algebraic expressions to topics from prior years and this title was extracted from one of the descriptors of SCK. This practice included the teacher's connecting the topic being taught to pattern generalization and operations with integers, and recalling process for prior knowledge that the students have to learn addition and subtraction of algebraic expressions. To do this, she implemented two activities related to pattern generalization and a matching activity related to the properties of addition operations in integers first. Then, the teacher first wanted the students to work on a worksheet in order to remind the concepts of term, coefficient, like term, constant term and variable that the students learnt in previous grades. The second practice was discussing on the activity related to addition and subtraction of algebraic expressions and this practice was also affected by the descriptors of KCT. This practice included a discussion about which terms in algebraic expression could be added and how they were added. The teacher emphasized the like term concept with using different colored unit cubes to represent kinds of variables in this discussion. She let the students to explain their answers and provided to discuss the answers with challenging questions. The third practice was choosing the examples or activities to use to take the students deeper into addition and subtraction of algebraic expressions and this title was also extracted from one of the descriptors of KCT. This was also as a continuation of choosing an example or activity to start practice. This practice includes how the teacher taught addition and subtraction of algebraic expressions to improve students' knowledge with getting deep the content using analogies that were representing as apple-pear of variables and net worth concept for explaining procedures. The fourth practice was solving questions and problems related to addition and subtraction of algebraic expressions from a test book, the textbook and workbook. She wanted the students to solve the first questions procedurally and then for the following questions she asked the students to write the expressions that were modeled with algebra tiles. The teacher provided explanations and helped the students to make more clarification in this practice. The fifth practice was implementing the suggested activities that were about the justification of identical expressions. These practices are explained with examining how the teacher used her knowledge based on MKT framework.

4.2.2.2.1.1. Practice One: Connecting Addition and Subtraction of Algebraic Expressions to Topics from Prior Years

At the first lesson of the instruction, Teacher B began the instruction with algebraic expressions using patterns generalization that the students learnt, as the researcher's suggestion. She decided to use the bacterial growth pattern and operating with number strips to make this connection with her knowledge appropriately (SCK1+). First, Teacher B implemented the "Bacterial Growth Pattern" activity from the textbook. This activity had two patterns in the context of growing bacteria. The first bacteria type was growing as 2, 4, 6, 8…, and the second bacteria type was growing as 3, 6, 9, 12… It was asked to generalize patterns first, then to add and multiply the terms of the first pattern and the second pattern, and to generalize the added pattern and multiplied pattern in the activity. Finally, it was asked how there was a relationship between the generalizations at the beginning and the generalizations after operated them. This question guided the learners to recognize and explore the addition and multiplication of the general rules of patterns as 2n and 3n as algebraic expressions. The dialogues between the teacher and the students were in the implementation of the activity as in the following:

> B: Please, who wants to explain what is given and asked? (She is writing the terms for the first bacteria on the board as 2, 4, 6, 8 ...).

S: The number of bacteria.
B: It is given that the reproduction of bacteria periodically. How is the 1st one?
S: It increases by 2 (This student is writing 3, 6, 9 ... for the 2nd bacteria on the board)
B: Then, it is asked to express the patterns' rules using n. Let's find the general rule for the first pattern.
S: It increases by 2.
B: It is true, but we will find the general rule. You can draw a table. We have explained as if the increment or decreasing is constant.
S*: (She is writing 2n).
B: The rule will start with 2n. You have 2 for the 1st term, you will find already 2 when you substitute 1 for n. For n=1, multiply 1 by 2, 2. So, you do not need to add or subtract anything to 2n.
S*: 2n.
B: Only 2n. Let's find the general rule for the 2nd pattern, bacteria. Write the increment (The student is writing +3 between the terms, as 3, 6, 9 ...). Let's examine, for the 1st term, multiply 3 by 1, 3; for the 2nd term, multiply 3 by 2, 6. That's true. Adding and subtracting these patterns, let's write the resulted patterns (Teacher B is writing the patterns one under the other, with their rules, 2n and 3n). Let's do addition. What is it?
S**: 5, 10, 15, 20 ... 5n.
B: When we add, the terms are 5, 10, 15 ... okay, then, is the rule 5n? Let's check. What is in the 1st term?
S**: 5, then, 10, 15.
B: What is the increment?
S**: 5.
B: Then, 5n. The 1st term is, also.
(S* and S** represent the same student)

In this situation, Teacher B responded to the student partially appropriate when he answered as the pattern was increased by 2 for the question of what the general rule of 2, 4, 6, 8 ... pattern. The teacher expressed that it was true but she did not provide any explanations about it why she investigated a rule though the answer was true. Her knowledge of how to provide mathematical explanations for the meaning of the general rule appeared inadequately (SCK6-). Instead of this, the teacher explained that they would find the general rule, although the student's answer also was the rule but not with algebraic representation. Then, Teacher B found the general rule by explaining the solution with the shortcut as in the patterns. After she found 2n to represent the increment between the terms, she indicated that it was the rule since the 1st term was 2 and there was not need to add or subtract any numbers. Similarly, she explained to find the 2nd pattern rule as 3n. She asked to the students to add these patterns' terms, and they found 5, 10, 15, 20 ... 5n. She directly asked the students to add the terms and accepted the addition of 2n and 3n, as the students had known to do addition with algebraic expression from 6th grade. She used the resulted pattern rule to check whether the rule was correct or not. However, Teacher B was expected to generalize the result patterns first, and then to make connection the general rules of added and multiplied patterns with the operations. It might more make sense if she should have asked to add the terms as numbers that were known and found the general rule of this pattern and then, she would have connected this rule as 5n with the 2n+3n, the addition operation. Her knowledge of how to provide mathematical explanations for common rules and procedures appeared inappropriately (SCK6-).

Related with the pattern generalization, the other suggested activity was Operating with Strips activity from in "Positive algebra – A collection of productive exercises" course book. Teacher B proceed the lesson with this activity. She did the first two questions in this activity: one addition and one subtraction of patterns questions. She gave the two questions as homework. There were three patterns in each question. The terms of two patterns were added and subtracted and the operation resulted the third pattern's terms. It was asked to generalize the patterns first, and then to do operations, addition or subtraction.

The researcher suggested this activity to provide the students to recognize the connection between the resulted pattern rule and the addition of the rules of the addend patterns. Teacher B's knowledge to connect operations with algebraic expressions to generalization pattern that the students had learnt and known appeared appropriately (SCK1+) to select this activity and she implemented it. The implementation of one of the examples was in this activity as in the following:

> B: Who wants to generalize the first pattern? Let's do together.
> S: It goes like 6, 12, 18 ... (He is drawing a table and writing +6 between the terms).
> B: Then, we must to get the 1st term, that's 6. Do I have to need to add or subtract anything?
> Students: No.
> B: So, the rule is 6n. Let's look for the second pattern (6, 10, 14,, …).

S*: 4n+2. (She is writing +4 between the terms on the board).
B: You have to get 6 from 4n.
S*: 2.
B: For the 1st term, multiply 4 by 1, 4. To get 6, you have to add 2. Okay then, the last pattern, we will subtract the second pattern from the first pattern (The student is writing 0, 2, 4, 6, by subtracting and then generalized the pattern as 2n-2 as in the next figure) …
(S* represents the same student)

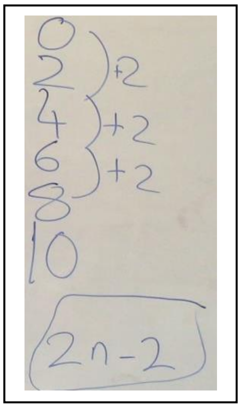


Figure 63 The generalization by the student

In this situation, Teacher B asked the students to find the general rule for the first two patterns. They could generalize the first pattern as 6n and the second pattern as 4n+2. The student on the board subtracted the terms of the first pattern from the terms of the second pattern and generalized the pattern that she found as 2n-2 for the result (third) pattern. Teacher B indicated that +2 became -2 when it was multiplied by the negative sign:

B: To get 0, we must subtract 2 from 2n. 6n-(4n+2) =2n-2. Be careful, do not write +2, remember integers, what did you do for 3-(-4)?
Students: Positive sign.
B: What did you do for 3-(+7)?
Students: Negative sign
B: Actually, you have subtracted 7 from 3. It is the same here. The negative sign affects the expressions in the parenthesis. Here, +2 becomes -2. We have crosschecked with subtraction, this rule (2n-2) is correct.

The teacher reminded the function of the negative sign in front of the parenthesis by giving examples of 3-(-4) and 3-(+7). She explained that the negative sign affected the signs of expressions in the parenthesis. Her knowledge to connect the topic to operations with integers as prior topic appeared appropriately (SCK1+). As in the Bacterial Growth Pattern activity, it was expected with this activity to generalize the third pattern using the numbers that were known first, and then to show this rule could be found by subtracting the rules of the second pattern rule from the first one. Thus, the teacher implemented this activity as it served the purpose of the connection so that Teacher B connected the generalization of the result pattern rule and the subtraction operation. She stated that they crosschecked the pattern rule with this subtraction so that her knowledge of how to provide mathematical explanations for common rules and procedures appeared effectively (SCK6+) in the implementation of this activity. Besides, Teacher B preferred to start operations with algebraic expressions using patterns generalization that the students learnt taking into consideration of the researcher's suggestion. She decided to use the bacterial growth pattern and operating with strips to make this connection and she chose the Bacterial Growth Activity to start operations with algebraic expressions and her knowledge to choose which examples to start with appeared effectively (KCT1+). On the other hand, the part of the multiplication of patterns were explained in multiplication practices section in detail.

After these beginning activities, Teacher B wanted the students to do "Matching" activity in the textbook. This activity had two columns (A and B) that included identical expressions, arithmetical and algebraic expressions. For example, there was (-5)+(-5)+(-5) in A column, while there was 3.(-5) as the corresponding expression in B column. To exemplify for the algebraic expressions, $a+a+a$ in A column, while there was $3a$ as the corresponding expression in B column. There were other examples for commutative, distributive and neutral element. Teacher B's aim was to connect the properties of addition operation in integers with operations in algebraic expressions. Since the students had learnt operations with integers, Teacher B wanted them to make connections for operations in arithmetic and algebra that and knowledge to connect the topic to the properties of addition operation in integers as prior topic appeared appropriately (SCK1+). The students could match these expressions easily and correctly.

After the activities about addition and subtraction, Teacher B distributed the worksheet to remind algebraic expressions that the students learnt at 6^{th} grade. Teacher B prepared this worksheet taking from a test book. The teacher reminded what the algebraic expression, term, coefficient, constant term, and like term were by working on the worksheet which had definitions and examples for these concepts in order to teach addition and subtraction. Thus, her knowledge to connect the topic being taught to topics from prior years appeared with reminding algebra concepts appropriately (SCK1+). She followed the worksheet and made explanations as in the following script:

> B: What do you remember about algebraic expressions?
> S: There is *x*.
> S: *x*, *y*.
> S: There is an unknown number.
> B: Anything else? Please read the definition (one student is reading) … We call algebraic expression as the representation of the relationships between the numbers, you learnt it the last year. Let's look at the examples. *x*, 3*y*, *a*+*b*, 2*x*/3 ... when the algebraic expression is told, I remember the unknown that's variable. What does 'variable' mean?
> S: It changes, *a*, *b*, *c*, it does not matter.
> S: The letters, figures, terms that substitute for a specific number.
> B: So, what do they represent? Numbers. Any number. That's we can substitute any number for them.

In this situation, Teacher B asked what the students remembered about algebraic expressions. When the students answered stating the unknowns such as *x* or *y*, Teacher B wanted the student to read the explanations in the worksheet first, and then she made explanations with her own words. Teacher B also stated unknowns for algebraic expressions. She used unknown and variable concepts as the same. She used the unknown and variable concept together as they had the same meaning so that her knowledge of how mathematical language was used appeared inappropriately (SCK5-). However, she explained the meaning of variable concept correctly. After she got the students' answers, she indicated that the variables represented the numbers so that her knowledge to develop a usable definition of variable with the students' answers appeared appropriately (SCK3+). After the explanation about variable concept, Teacher B made explanations about algebraic expression and emphasized the like term concept and gave several examples (a^2 and $4a^2$, $3x^2$ and $3y^2$, and $4x^2$ and $4x$) to teach it:

> B: When algebraic expression is told, the concept of term is come to the mind. Let's look for 2*x*-3*y*+5. The terms are considered with their signs. Here, the terms are 2*x*, -3*y*, +5. Thus, if how many the terms there are, look for the parts which is divided by + and – signs. Here, there are 3 terms. The coefficient is the other concept that we read. We call the coefficient as the

number which is multiplied by unknown in each term. The coefficient of *x* is 2 (2*x*), the coefficient of y is -3. Be careful, we take it with its sign as minus. We call the constant term as the term which is not multiplied by any unknowns. Are there any terms that do not have *x*, *a*, *b*, or *c*?
Students: 5.
B: The last concept is like term. The terms whose variables and powers are same are like terms. For example, 3*x* and -5*x*. Are they like, a^2 and $4a^2$?
Students: Yes.
B: $3x^2$ and $3y^2$?
Students: They are not.
B: Yes, look the variables. $4x^2$ and 4*x*? Are they like terms?
S: They are not, because one of them does not have a power.

In this situation, Teacher B made explanation about the concepts by giving example as 2*x*, -3*y*, +5. She stated the term was divided by signs and the coefficient was taken with its sign. She particularly emphasized the like terms and asked students whether a^2 and $4a^2$, $3x^2$ and $3y^2$, and $4x^2$ and 4*x* were like terms or not. She indicated that the same variable and power of this variable were required to be same. She emphasized the importance of like terms in order to do addition and subtraction with them. These explanations could show Teacher B's using of terms and notations about algebraic expressions correctly that her knowledge to use terms and notation correctly and adequately (CCK2+).

The following exercises in the worksheet were about writing verbal statements as algebraic expressions and writing algebraic expressions as verbal statements. The students could write the expressions such as 3 times of a number, or 8 more of a number. Teacher B drew the students' attention to prevent possible misunderstanding that might arise while reminding the prior concepts working on the worksheet. She anticipated these misunderstanding during the lesson and thus she made explanations pointing out them:

B: 12 less of a number.
S*: *a*-12.
B: Can it be 12-*a*?
S*: No.
B: Why? Because, you are not told to subtract from 12. 12 less of a number. The subtraction operation has not commutative property.
(S* represents the same student)

Teacher B was aware of possible errors while writing the expression within subtraction. Thus, she asked whether 12-*a* could be written or not to provide the students to recognize the difference and the students could understand the difference. However, they had difficulty with understanding the following exercise:

> B: 4 times of 6 more a number, how do you represent it algebraically?
> S*: *x* plus 6 times 4.
> B: Can you write on the board?
> S*: *x*+6.4
> B: Let's examine. Does everyone write like this?
> Students: Yes.
> B: When I look it, I understand this: add *x* to 4 times of 6. But you are told, 4 times of 6 more a number.
> S: 6*x*+4
> S: 4*x*-6
> S: 4*x*+6
> B: Its meaning is 6 more of 4 times of a number (She is writing *x*+6 in the parenthesis). You are told that there is a number, first find 6 more of it, and then multiply the result by 4. Did you notice the difference between your answers and the expected answer? We use the parenthesis. When we organize, 4.(*x*+6). This is the most common errors. Be careful. First, you add or subtract, then multiply it.
> (S* represents the same student)

In this situation, the students could not get the correct answer. Teacher B asked the student to write the first answer as *x*+6.4, and she explained what was understood from this expression. Then, other students answered as 6*x*+4, 4*x*-6, and 4*x*+6. Teacher B responded to the last answer and interpreted the meaning of 4*x*+6 in order to show the error. Then, she wrote the correct expression using the parenthesis. She stated the importance of the order of operations, and she explained that doing addition first, and then multiplication of this addition result. She also warned the student to be careful of doing these type of errors. Her knowledge to anticipate the misunderstandings that might arise with writing algebraic expression that required parenthesis appeared in class appropriately (KCS2+). Similarly, the students had difficulty with the expressions that were required parenthesis to represent the verbal statements as in the following:

> B: The half of 7 times of 3 less of a number. Can you write on the board?
> S: *x*-3.7:2
> B: Let's examine and understand. We will find 3 less of a number. You have subtracted, *x*-3. You multiplied the result by 7, (*x*-3).7. Then, it is asked to

get the half of it, (x-3).7/2. Let's skip the next column exercises. How do you express verbally 6(m-1)?
S5: 1 less of 6 times of a number?
B: First, 1 less of a number, then 6 times of it. (3k+2)/4?
S: 2 more of $\frac{3}{4}$ times of a number?
B: Are you sure?
S: The quarter of 2 more of 3 times of a number.
B: Or one fourth of as the quarter.

In this situation, Teacher B wanted the student to write the expression on the board to show the error in it. Then, she explained the algebraic expression representation step by step. Then, she skipped the next column that was asked verbal expressions of the given algebraic expressions. The students had difficulty to express the algebraic expressions that required parenthesis or fraction once more. Teacher B gave time the students to think and correct the expressions, and helped them to answer correctly and her knowledge to understand the needs and difficulties of students appeared appropriately (KCS3+). The next exercises were about determining the number of terms, the coefficients, the variables and the constant terms in algebraic expressions. Teacher B frequently emphasized the concept of term while working on the worksheet. Since the students had difficulty to determine the terms and answered incorrectly, Teacher B addressed these errors and made explanations to remedy them:

B: What are the terms in -3x? There is one term. What is it?
S: 3x.
B: -3x. We take the terms with their signs. What is the coefficient of it?
S: -3.
B: What is the variable?
S: x. There are not any constant terms.

In this situation, Teacher B corrected the student's incorrect answer at the time and made explanation. She explained that the terms were taken with their signs. As in the script, she generally emphasized the points that the students forgot or had difficulty to prevent from them making errors, and remedied the errors at the time. Similar situation about determining the terms happened for the example of *a.b.c*:

B: In the expression of *a.b.c*, what are the terms?
S: There are 3 terms, and their coefficients are 1.

B: Let's read the definition of the term again in the first page. How? The terms are divided by signs, positive and negative. Are there any terms that divided by signs in this expression?
S*: None.
B: Then, there is 1 term. That's *abc*, it does not matter whether the symbol of multiplication is between or not. What is the coefficient?
S*:1.
B: What are the variables?
S: *a*, *b*, *c*.
B: There is not any constant terms.

In this situation, Teacher B asked the students to read the definition of the term again when the students could not give the correct answer. She remedied this error by reminding the concept of the term. Then, she emphasized the signs divided the terms in an expression to address this error. Since a, b, and c was as multiplied representation, she explained that there was one term. The students could perceive *abc* as one term. Another example was as in the following:

B: What are the terms in *a*-2*b*+7?
S^*: 2 and 7.
B: The terms? Each of them is a term here: *a*, -2*b*, and 7.
S^*: Do we indicate +7 (positive)?
B: I said that the positive sign was not required to write before expressions. But, we take the negative sign. What are the coefficients?
S^*: -2 and 7.
B: Are you sure?
S: There is 1 in *a*.
B: How many '*a*' in this expression? 1. The other coefficients: 1 and -2. It does not matter to be *a* or *x*. If there is one variable, the coefficient of it is 1. What are the variables?
S: *a* and *b*.
B: The constant term?
S: None.
B: Be careful, there is 7.
(S^* represents the same student)

In this situation, Teacher B corrected the students' wrong answers while they were determining the terms. She emphasized that the negative sign was written several times as in this example during the lesson. The students could not consider that the coefficient of the variables was 1 if any number was not written. Teacher B explained that there was one a in the expression and so the coefficient of it as 1. Then, she said the correct answer as 7 for the constant term, since the students could not consider because it was the constant. Her knowledge to address the students' errors and to remedy them appeared effectively (KCT8+) in her explanation while doing these exercise.

The next exercises were about determining the like terms. One of the examples from these exercises was as in the following:

> B: Which of them is like to $-2x^2$? You are interested in x^2. Are there any terms that have x^2?
> S: There is.
> B: There is only x^2.
> S: There is one more that has x^2, $(-x^2y/5)$.
> B: There is y in it. The unknown parts must be the same. There is other unknown also. So, it is not like to $-2x^2$. If there is a number instead of *y*, it can be. So, two of them, $-2x^2$ and x^2 are like.

In this situation, the student confused that x^2 and $-x^2y/5$ were like since the second expression had x^2. Teacher B explained that the unknown parts must be the same, and the second expression had y and she responded to the student appropriately. She might have explained also the unknowns as multiplied representation could not be separated, such as y as multiplied with x^2 in $-x^2y/5$. She indicated that if there was a number instead of *y*, it could be similar. In general, these exercises were all about the previous topic as algebraic expressions at 6th grade. Teacher B reminded them working on the worksheet with explanations and doing exercises.

After the lesson finished, the researcher interviewed with the teacher to get her ideas about the lesson. Teacher B explained the points that the students had difficulty in the instruction as in the following:

> B: The students did not have difficulty in working on the worksheet. They could remember. This worksheet's aim was to remind the concepts about algebraic expressions. They had difficulty with writing "6 times of 5 more of a number" algebraically, for example. They had troubles when it is necessary to use the parenthesis.

The teacher reminded the concepts such as term, like term, coefficient that the students learnt in the previous year at the beginning of the instruction of operating with algebraic expressions. She stated that writing of verbal expressions algebraically that required to use parenthesis, was difficult for the students so that her knowledge to understand the difficulties of students appeared appropriately (KCS3+). Since these topics were in 6th grade curriculum, she explained that she did not work on them for a long time.

4.2.2.2.1.2. Practice Two: Discussing on the Activity Related to Addition and Subtraction of Algebraic Expressions

Teacher B created a discussion environment in lesson to teach the like term concept. She used manipulatives to represent the variables; orange unit cubes to represent *x*; green unit cubes to represent *y*, purple cylinder to represent 1 as integer; and red cylinder to represent 2 as integer. She started the discussion asking the students to choose 3 of the similar ones to explore the addition like that:

> B: Let's think, orange unit cubes represent *x*; green unit cubes represent y, purple cylinder represents 1 as integer; and red cylinder represents 2 as integer. If I want 3 items of *x*?
> S: I take 3 orange cubes.
> B: Why did you take from green cubes? All right, if I want to select 3 cubes from like ones?
> S*: I take 3 from green and orange ones.
> B: Are they like? Take 3 from like ones.
> S*: I take 3 from green and orange ones.
> B: But, their colors are different. What we said for the like term; both variable and powers must be the same. Let's suppose the color represent the power. For example, get together 3 green cubes and 5 orange cubes, add them. How do you call them? How are 8 cubes, green or orange?
> S: Mixed.
> B: Do we call them 8 mixed cubes? I have 3 green cubes and 5 oranges, how many cubes are there in total?
> S**: 8.
> B: How 8 cubes? What 8?
> S**: 8 cubes.
> B: How cubes?
> S**: Like cubes.
> B: Can you call them as green or orange cubes?
> S: No.
> (S* and S** represent the same student)

In this situation, Teacher B asked to the students to take 3*x*. Since Teacher B's aim was to feel the like term concept, she asked the students to take 3 cubes from like ones then. However, she could not guide the students with this question. She indicated that the color represented the power of the variables. But, she did not state this point once more. Then, the students answered as 8 cubes to the question of what 3 green cubes and 5 orange cubes were added. However, Teacher B persistently asked the students how 8 cubes were qualified to make the feel of the requirement of like terms for addition and her knowledge to decide when to use a student's remark to make a mathematical point, and when to ask a new question to further students' learning appeared appropriately (KCT5+). She used the student's answer of 8 and asked what color they were to show that they could not be not added since they were different. Then, she asked them to add 5 items of *x* and to represent the addition algebraically:

> B: Or you are told, there are 3 apples and 5 pears. What have I? Do I say 8 apples or pears? Listen, when is asked 5 items of *x*, what will you do? (She took 5 orange cubes) have you noticed? There are 5 cubes. Each of is *x*, 5 items of *x* in total. How can we represent it?
> S^*: $x+5$.
> B: You add 5 items of *x*. How many *x* do you write?
> S^*: 5.
> B: If I want you to add 4 items of 1, what will you do? 1+1+1+1. Do the same operation for x.
> S: 5 items of x are next to each other.
> B: That's $x+x+x+x+x=5x$ (she is writing on the board).
> S: If there are 50 items of *x*, is this too long to write?
> B: Instead of this addition representation, we say $5x$.
> (S^* represents the same student)

Teacher B provided explanations about how 4 items of 1 and then 5 items of x were added and her knowledge to decide when to pause for more clarification for teaching the addition of same variables and the representation of repeated addition as multiplication appeared appropriately (KCT5+). After Teacher B represented the addition as $x+x+x+x+x=5x$, she asked how 3 green cubes and 5 orange cubes were added once again:

> B: Let's think orange and green cubes again. You have 3 green cubes and 5 orange cubes. What is the total?
> Students: 8.
> B: But, what 8? I could not express, because I cannot have any common thing. While adding the units, you can express a common result. Thus, we explained that we could add or subtract the like terms. Look, we can add *x*

within themselves. If there are 10 items of *x*, you can say 10*x*. If there are 9 items of y, you can say 9*y*. It represents the unknown. When we get the cubes together, how can we call them? Mixed cubes? What is the number of them?
Students: 8.
B: What 8? Have you confused well enough?

The teacher could have expected $3x+5y$ as the answer, but the students answered 8 cubes again. The teacher indicated that 8 could not be used since the addends were not common. She exemplified that 10 items of *x* were represented as 10*x*, and 9 items of *y* were represented as 9*y*. The students answered as 8 cubes again, and Teacher A was pleased the students' confusion about what they called these cubes at the end of the lesson. She indicated this confusion could motivate the students to learn and not to forget the like term concept. However, she did not use the cubes for teaching operations anymore throughout the instruction. Thus, it might be asked whether this confusion was useful for the students or not.

After the lesson finished, the researcher interviewed with the teacher to get her ideas about the lesson. Teacher B used the different colored unit cubes to represent different variables for explaining the requirement of like terms for addition and subtraction. She explained the use of this method as in the following:

B: I confused their minds when transiting addition and subtraction. The mixed mind is good. We will overcome this confusion in the next lesson. Their participation was also good. I said before, I did not use cutting the papers. Instead of this, I used unit cubes. I wanted to use them as like terms. Because I thought that I would not manage cutting the papers. The cubes are ready.
Researcher: You have said that confusing their minds. How do the students respond to these situations?
B: Actually, they are making brainstorm. My purpose was to make them feel not to be able to add x and y. They said that there are 8 unit cubes. But how would they express their colors? Some interested students came after the lesson, and said that: but we could add cubes. I asked to them what the color of added cubes. Green or orange? I will connect with the like terms for addition.

First, she stated that she preferred to use unit cubes instead of cutting papers since it would be easier. Then, the teacher identified mixing the students' minds as brainstorming as explained in the dialogue that her knowledge to identify what different methods and procedures afford instructionally appeared (KCT6+). This method could be useful instructionally for providing students' wondering and thinking on how these cubes were added. Her aim of using the cubes was to show the requirement for addition and subtraction. She wanted to show that different colored cubes as different variables could not be added. However, some students still had difficulty with the like term concept throughout the instruction and thus confusing the students with using unit cubes to teach like term concept might be questioned whether it was good for the learning of students or not.

4.2.2.2.1.3. Practice Three: Choosing the Examples or Activities to Use to Take the Students Deeper into Addition and Subtraction of Algebraic Expressions

She proceeded teaching of addition and subtraction operations by using apple-pear to represent variables and net worth concept to explain the procedures for addition and subtraction as analogies. Her knowledge of how to provide mathematical explanations for the addition and subtraction procedures appeared effectively with the use of them (SCK6+). She used apple-pear analogy to make concrete the variables for the students for doing operations at the beginning of the teaching of simple addition examples, $3x+4x$ and $3x+4y$. While answering simple addition examples, she used the analogy as in the following:

> B: $3x+4x$, look at this addition, are they like terms?
> Students: Yes.
> B: Why are they like?
> S: Two of them have *x*.
> B: You can think as x represents a number. When I have 3 items of *x*, you can think as I have 3 apples. Is $4x$ an apple or a pear?
> Students: Apple.
> B: So, we can add them, let's add 3 apples and 4 apples. What is it? $7x$. Since we have added the like terms. Let's add $3x+4y$.
> S: They are not like.
> B: So, I cannot say anything, that's I have 3 apples and 4 pears. When I encounter this type of expressions, I give up as so, since I cannot add.

In this situation, Teacher B used to represent apple for *x* and pear for *y*, and she indicated that similar fruits could be added. She used $3x+4x$ and $3x+4y$ as addition examples for beginning. She explained the addition procedure with this way, apples and so x in $3x+4x$ could be added, while apples and pears could not be added in $3x+4y$. Her knowledge of how mathematical language was used appeared by expressing the kind of the variable with apple and pear analogy appropriately (SCK5+). Actually, she had stated to use rectangle and unit square papers to represent the variables in the lesson planning, she did not use them in the instruction. Instead of this, she used apple-pear analogy to represent the kind of variables concretely. Then, she went on the lesson with more complex addition and subtraction examples that had several variables, constant term, or exponential expressions in order to take the students deeper into the content. These examples were: $3x+4a-5x-2a-7$, $3x-4y+2x-y$, $n^2+2xy+7n^2-xy$, $2a+3-a-4$, $2x^2+5xy-x^2-2xy+5$, $x^2+2xy-1+x^2+9xy-4$. She made explanations for the simplification of $3x+4a-5x-2a-7$ as in the following:

> B: Let’s examine this example, $3x+4a-5x-2a-7$, which of them are like?
> S: $3x$ and $5x$.
> S: One more, $4a$ and $2a$.
> B: What is 7?
> S: The constant term.
> B: Be careful in operation, take the signs of the terms. You have 3 apples and you are told to give 5 apples to someone. Can you give?
> Students: No.
> B: What if? You are in debt 2 apples. That’s $-2x$. Then, I have 4 pears, you are told to give 2 pears to someone.
> S: $+2a$.
> B: $-2x+2a$, there is 7 but there is anything to add with it. The result is $-2x+2a+7$.

In this situation, Teacher B selected the addition of five expressions as example. The students could find the like terms as $3x$ and $5x$, and $4a$ and $2a$. Teacher B asked the students to add them by representing apples for x and pear for a, and net worth for operations. She emphasized that the signs of the terms were taken to do operation since the students answered $5x$ instead of $-5x$ and $2a$ instead of $-2a$. Then, she asked what if there were 3 apples and they were told to give 5 apples to someone. When the students answered as they could not give since there were not 5 apples, Teacher B stated the debt as 2 apples and represented it $-2x$. Similarly, she explained that she had 4 pears and they were told to give 2 pears to someone, Teacher B stated the rest 2 pears and represented it +2*a*. In this example, her knowledge of how mathematical language was used appeared by expressing the procedures with net worth concept by using assets and debts concepts as analogy appropriately (SCK5+). She indicated that there were not any constant terms to add with 7 and wrote the result as -2*x*+2*a*+7.

Teacher B corrected the students' writing while they were simplifying of four or more expressions. Because the students added the like terms under the expression separately and they did not write the results after the equal sign of the expression. She addressed this problematic representation as in the following:

> B: Let's add 3*x*-4*y*+2*x*-y. First, determine the like terms. (The student is writing separately, under the expression as in the figure) … You have 3*x* and you are told to add 2*x*.
> S*: 5*x* (He is writing 3*x*+2*x* =5*x*).
> B: You are in debt as 4*y*, and if you are in debt one more of *y*, how do you express this?
> S*: -5*y* (He is writing -4*y*-*y*=-5*y*).

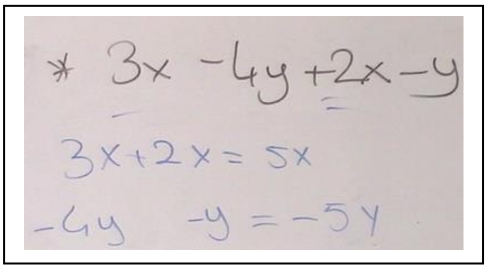


Figure 64 The addition of the like terms under the expression by the student

> B: You should not write like this. You do the operations separately and you leave them here, but you should write the results on the right side of the equal sign after finding the answers. Actually, you are not required to write them separately in another place. … We do not do operations with symbols, we do with the numbers, the coefficients. You do operation as in integers, then write the like term next to it.
> (S* represents the same student)

In this situation, Teacher B wanted the students to write the result that they found after the equal sign. Since the students added the like terms separately in another place, they did not write as the result. The teacher generally paid attention in using the notations correctly, and she also wanted the students to take care of it so that her knowledge of how mathematical language was used appeared with remedying the errors in the use of equal sign appropriately (SCK5+ and KCT8+). Teacher B also used the analogies in solving these questions when the students had difficulty to understand. The explanations of Teacher B's for solving '2*a*+3-*a*-4' as in the following:

B: I have 2 pears, if I give one of them, I have 1 pear. It is represented as 1a. I have 3 TL assets, 4 TL debts. Even if I get the assets, I will have 1 TL debt (She is erasing 1 in front of 1*a*). Look if the algebraic expression is alone, it means 1 of it. So, we do not need to write 1. You have to know 1 before it.

Teacher B generally used asset and debt concepts and apple-pear representations for explaining the solution of these questions as indicated above. The students could understand these analogies easier than only indicating addition or subtraction and variables.

After the class answered these questions, Teacher B proceeded with subtracting of two expressions within parenthesis. Firstly, she reminded the function of the negative sign before the parenthesis and used inductive reasoning to explain it to connect the subtraction of two algebraic expressions within parenthesis as in the following:

B: What did we do when we see this operation (4-(-3))?
S: Plus, +.

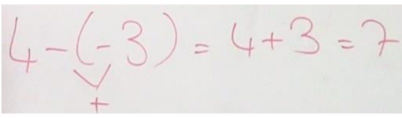


Figure 65 The example for the function of minus sign by Teacher B

B: (She is writing as in the figure) 4+(-3) =1, does not the sign change, does it? That's the plus is not affected. 4-(3) =1, it is 1 also. How can we interpret about the relationship between the sign and these operations?
S: Although negative sign and positive sign are replaced with each other, we can find the same result.
B: Other ideas?
S: There is not any changes both the negative sign in or out of the parenthesis.
B: Is that so? In the first operation, there is a negative sign and this becomes positive. In the second operation, the positive sign does not affect. But, in the third operation, the negative sign affects and I did subtraction. The negative

sign affects inside of the parenthesis. Where does this rule come? In integers, the multiplication of negative and negative yields positive. This rule also works for the multiplication of algebraic expressions.

In this situation, Teacher B made explanations about the effect of negative sign before the parenthesis by giving three examples to connect with subtraction of algebraic expressions. The first and the third example had subtraction operations as the negative sign before the parenthesis; the second example was an addition example. Teacher B asked the students to interpret the function of negative sign considering the examples. When she did not get the expected reasoning, she explained by herself. She emphasized the effect of the negative sign on the signs of expressions in the parenthesis. She reminded the rule that was about the multiplication of two negative signs resulted as the positive sign. Actually, her knowledge of how to provide mathematical explanations for the common rules appeared with exemplifying and connecting the operation in integers appropriately (SCK6+ and SCK1+) and she also expected from the students to deduce the function of the negative sign before the parenthesis. However, when she did not get the expected answer, she explained the rule about the signs in multiplication as in the figure. She indicated this rule worked for the operations with algebraic expressions and asked the following two examples to the students: $2a+3+5a+1$ and $2a+3-(5a+1)$. She used the same algebraic expressions with the positive and the negative sign between them to show how the sign affected the expression by the negative sign and so the result. The purpose of these examples was to show the function of the negative sign between the expressions within the parentheses. Her explanations were as in the following:

B: Let’s do $2a+3+5a+1$ together.
S: $7a+4$.
B: Okay, let’s do $2a+3-(5a+1)$, the negative sign comes before the parenthesis. Remember the properties of multiplication operation. What comes from multiplication of negative and positive sign?
Students: Negative.
B: So, you multiply the negative sign by the parenthesis. $-5a-1$, the result. You can think in a short way, if there is a negative sign in front of the parenthesis, the signs of the expressions inside all change inversely.

In this situation, Teacher B reminded the rule of “multiplication of negative and positive sign results negative sign”. She explained that the negative sign affected the parenthesis that while the result was $7a+4$ for the positive sign (addition), the result was $-5a-1$ for the negative sign (subtraction). She would have called this function as distributive property while explaining also and it would be more appropriate for the use of mathematical language. The selection of the examples that was simple to more complex, and also addition to subtraction operations was appropriate and her knowledge to choose examples to take the students deeper into appeared appropriately (KCT1+). However, she had not indicated any of these examples in her lesson plan.

4.2.2.2.1.4. Practice Four: Solving the Questions and Problems Related to Addition and Subtraction of Algebraic Expressions from the Test Book, Textbook and Workbook

After teaching addition and subtraction by answering several examples, Teacher B asked the following questions about addition and subtraction from a test book: $3x+4x-5x-x$, $7a-3a-a/2+a-a/2$, $(2x+1)+(x-6)$, $(2m-9)-(m-5)+(3m+7)$, $(-2y+3+y)–(-6-m)+3m +5$, $(-4+k)+6-3k+(8k+4)$. The students could answer for the first question as x easily. Then, the second question, $7a-3a-a/2+a-a/2$, had rational expressions. Teacher B indicated $a/2$ as half of a, and thus she asked what two halves of ‘*a*’ were and the students could answer one ‘*a*’ to this question. The other questions had the expressions with parenthesis. The students had difficulty in solving these questions. Thus, Teacher B solved them by making explanations especially for writing the expressions without the parentheses. The dialogues between the teacher and the students as in the following script:

B: What can you say for $(2x+1)+(x-6)$?
S: $4x-1$.
B: Other?
S: $-3x$
S: $+3x$
B: What confused you here?
S: $x-6$.

B: Why? If I erase the brackets, (2*x*+1+*x*-6), are you confused at the time?
Students: No.
S: 7.
S: 8.
S: 9.
S: 7*x*.
S: 8*x*.
B: While adding or subtracting, we operate the like ones. There is 2x, which of them can be added with 2*x*? Only *x*. Add 2*x* and *x*, it is 3*x*. You have 1 TL and 6TL debts,
S: -5.
B: Here, the parenthesis is not important since there is a positive sign before it. So, it does not matter there is a parenthesis or not. Even, you can write 2*x*+1+*x*-6= 3*x*-5. It is not a problem if there is not a negative sign before the parenthesis. You can do normal subtraction here.

In this situation, the students were confused when they saw the parentheses. However, there was not the negative sign in this question. Thus, the students answered different incorrect answers such as 4*x*-1, -3*x*, 3*x*, and *x*-6. In response to this situation, Teacher B asked what confused them and erased the brackets because of the positive sign before the parenthesis. However, the students could not solve it and gave answers such as 7, 8, 9, 7*x*, and 8*x*. Then, Teacher B solved the question by herself explaining with 2*x* and *x* were like terms and using the net worth concept. She indicated again that the positive sign did not affect the signs of expressions in parenthesis. Her knowledge to understand the difficulties of students in operating with the parenthesis algebraic expressions appeared appropriately so that she provided explanations about like term concept, the procedures of addition operation, and the sign before the parenthesis adequately considering the needs of the students (KCS3+).

After the teacher and the class solved all of the questions procedurally, the teacher asked the questions that had additions and were modelled with algebra tiles. The solving of one of the questions was as in the following:

B: The colors of the tiles are not so apparent (the below figure). Can you guess? Which of them +*x* or –*x*? or +1 or -1?
S*: There are +2*x* and +4 in the first box; +3*x* and -4 in the second box.
B: Why is there -1 in one box, +1 in other box?
S*: In the result, there are no squares.

B: There are 2 long and 4 squares in the first box. There are 3 long and 4 squares in the second box. In the result, there are 5 long, there are no squares. The squares in the first box is dark, +1. Or, there are no squares in the result, you can decide considering it. +1 neutralizes -1. Who wants to write the algebraic expression of it?
S: $2x+4 + 3x-4 = 5x$ (She is writing this expression on the board).
B: There are 2x and 3x and they have positive signs. Is there any change?
Students: No.
B: I have 4 TL assets, 4 TL debts. If I pay my debts with my assets, I have no money. We have crosschecked with this way, too.

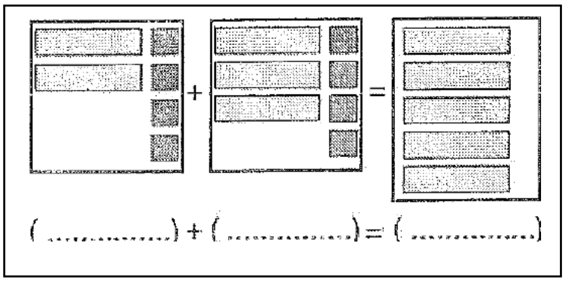


Figure 66 The asked addition expression that is modeled with algebra tiles

In this situation, Teacher B asked the students to estimate the expressions which were added to each other, since the colors of the tiles were not apparent enough in the paper. She expected the students to write the expressions by examining the tiles in result box. Since there weren't any squares in the result box, the students were expected to answer as there were +4 in one box and -4 in other box. These numbers could neutralize each other when they were added. One student could easily realize the expressions and write them correctly. Then, Teacher B made explanations to the class about the absence of squares in the result box using net worth concept. The problematic situation was about the use of mathematical language since Teacher B used "long" for rectangular shapes. She made errors in using the rectangle concept as technical mathematical language so that her knowledge of how mathematical language was used appeared inappropriately (SCK5-). Another point was about the use of algebra tiles. Teacher B did not prefer to use algebra tiles for teaching addition and subtraction. She used the rectangles to represent x, and the squares to represent 1 as integer only. She did not explain that x and 1 represented the areas of these quadrilaterals although she had indicated in lesson planning before, but her knowledge of how to use algebra tiles as mathematical representations appeared inadequately since she did not explain the area concept that was underlying idea of using algebra tiles (SCK7-).

The next step in the instruction was solving the geometry problems in the work book about the perimeter concepts using algebraic expressions. The problem related to geometry was as in the following figure:

Calculate the perimeter of the shape using the isosceles triangle.

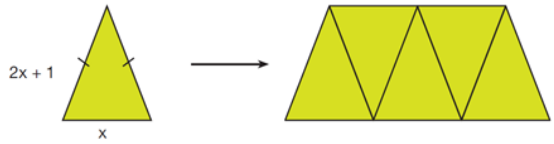
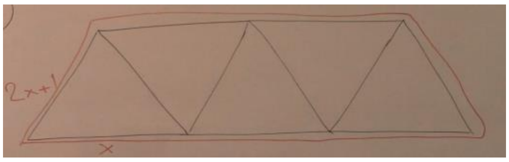


Figure 67 The problem related with the perimeter of triangle

The solving of the problem as in the following:

B: In the given shape, the triangles are put together, and the shape in the question is formed. It is asked to find the perimeter of this shape. Who wants to show it? (The student is surrounding the shape as in Figure 68). Be careful, sometimes, you can consider the segments inside of the shape. But you are asked to find the perimeter, not inside. Your friend has showed the perimeter correctly. What will we do then? You know the sides of the triangle. (The student is writing the length of the sides as in the Figure 69). Add all of the lengths.
S: $(2x+1)+(2x+1)=4x+2$, $x+x+x+x+x=5x$, $4x+2+5x=9x+2$.



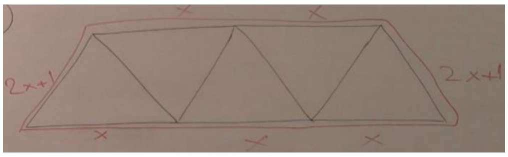


Figure 68 The drawing of the surrounding of the shape to show the perimeter by the student

Figure 69 The writing of the length of the sides by the student

In this situation, the teacher wanted the students to show the perimeter of the given shape by drawing first and the student could draw correctly. Therewith, the teacher warned the student not to take the segments inside of the shape as including to the perimeter. Her knowledge to anticipate the misunderstandings that might arise with the perimeter concept appeared effectively and she asked the student to show the perimeter and then she also provided explanations not to include the segments inside (KCS2+). Then, she asked to the student to write the lengths of the sides that formed the perimeter of the shape. The student wrote the length of the sides correctly, and she added the identical expressions with each other and then added the results of them as in the dialogue. After the student solved this problem, the teacher also pointed out the difference between the area and perimeter concept. She made these following explanations to prevent the students' possible errors about the confusion of area and perimeter:

> B: Have you differentiate the area and perimeter concepts? (She is taking the notebook in hand) Where is asked if the perimeter of this notebook is told?
> S: Its surround.
> B: Okay, where is asked if the area of this notebook is told? The area that is covered. So, be careful, we express the perimeter of the rectangle, when we say the area of region. Since there is a region in it. The area with its inside.

In this situation, the teacher asked to the students what they understood about the area and perimeter concepts since the students had generally misunderstandings about these concepts as the teacher indicated in lesson planning. She made explanations using the notebook as an example and she explained that the perimeter was the notebook's surround, while the area was the region inside of it. Her knowledge of common student misconceptions about the perimeter and area concept appeared adequately and she also emphasized possible misunderstanding with these concepts by providing explanations using the notebook (KCS5+). She also solved the questions and problem correctly with adequate explanations and her knowledge to calculate an answer and solve problem appeared effectively (CCK3+).

4.2.2.2.1.5. Practice Five: Implementing the Suggested Activities

Teacher B implemented "Operating with Number Strips" activity from the suggested examples to connect pattern generalization at the beginning of the instruction. Then, the suggested activity related to addition and subtraction of algebraic expressions was the following activity. It was about explaining how the given paired algebraic expressions were equal. These expressions were: n+n+n+n and 4xn; and n+n+n+n and (2xn)+(2xn). The dialogues between Teacher B and the students for justifying n+n+n+n and 4xn was equal:

> B: Here, 4 of n are added and the other is 4xn. Are they equal?
> S: Yes.
> S: They are not.
> B: How can we represent with the short way of 5+5+5+5?
> S: 4.5
> B: It is asked for n+n+n+n, it is 4.n; the other is 4xn. That's true.

Teacher B talked about the representation of the repeated addition as multiplication. She first asked how they could represent 5+5+5+5 as an example from arithmetic and one student represented it as 4.5. Then, the teacher showed n+n+n+n, as 4.n. One student used a different way and Teacher B explained by justifying this student's ideas as in the following:

> S: There are 4 of n. If we substitute 1 for n, the second expression is equal to 4.
> B: It is a different way. Your friend said that suppose 1 for n. It is asked to show how they are equal. You can show whatever you want. Is n represented a number?
> Students: Yes.
> B: Substitute 1 for n, are the results equal? The first is 1+1+1+1=4, the other is 4.1=4. They are equal.

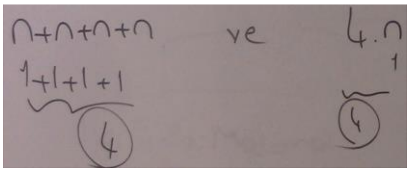


Figure 70 The justification of the equality of the expressions by substituting 1 for n

Teacher B explained the student's idea of showing and calculating 1+1+1+1=4 and 4.1=4. She justified by indicating n represented any number and this method also could be applicable. Her knowledge of how to explain and justify the student's mathematical ideas appeared appropriately (SCK4+). In general, when Teacher B got a different solution way from a student, she checked this way and explained and justified it to the class. Teacher B's explanations for the second pairs were as in the following:

B: Are n+n+n+n and (2xn) + (2xn) identical? How did you decide?
S: They are identical.
B: Why?
S: The first one is 4n. The second one is 2n+2n, 4n.
B: Anything else? Your friend said that the addition of variables in the first one is 4n; the second one is also 4n.
S: We can substitute 2 for n.
B: If we substitute 2 for n, 2+2+2+2=8. The other is 2.2=4, and if we add 4 and 4, it is 8.

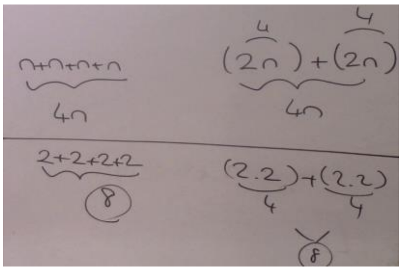


Figure 71 The justification of the equality of the expressions by substituting 2 for n

In this situation, Teacher B explained the equality of the two expressions with using two ways. The first way was the algebraic way that the variables were added in the first expression and 2 and n were multiplied as 2n, and then 2n and 2n were added for the second expression. The second way was substituting an integer and doing operations arithmetically. The student proposed to substitute 2 for n and the teacher showed the equality of the expressions with doing the operations by substituting 2 for n. Her knowledge of how to provide mathematical explanation for addition of variables with showing algebraically and arithmetically appeared appropriately (SCK6+).

4.2.2.2.1.6. The Extracted Knowledge Types from the Instruction for Addition and Subtraction of Algebraic Expressions

Table 18 The extracted knowledge types from instruction for addition and subtraction of algebraic expressions

Practices	Extracted knowledge types				
	SMK		PCK		
	CCK	SCK	KCS	KCT	KCC
Connecting addition and subtraction of algebraic expressions to topics from prior years	CCK2(+)	SCK1(+,+,+,+) SCK3(+) SCK5(-) SCK6(-,-,+)	KCS2(+) KCS3(+,+)	KCT1(+) KCT8(+)	
Discussing on the activity related to addition and subtraction of algebraic expressions				KCT5(+,+) KCT6(+)	
Choosing the examples or activities to use to take students deeper into addition and subtraction of algebraic expressions		SCK1(+) SCK5(+,+,+) SCK6(+,+)		KCT1(+) KCT8(+)	
Solving the questions and problems related to addition and subtraction of algebraic expressions from the test book, textbook and workbook	CCK3(+)	SCK5(-) SCK7(-)	KCS2(+) KCS3(+) KCS5(+)		
Implementing the suggested activities		SCK4(+) SCK6(+)			

Table 18 shows what type of knowledge of subject matter knowledge (SMK) and pedagogical content knowledge (PCK) that Teacher B had in instruction practices. (+) sign indicates the teacher's existing knowledge was adequate or appropriate and she used it effectively, while (-) sign indicates the teacher's existing knowledge was inadequate or inappropriate and she used it ineffectively. Each sign (+ or -) in the same knowledge type refers to Teacher B's different use of this knowledge during instruction. Besides, for SCK3 knowledge type, (⊥) is used to indicate the teacher's knowledge to develop definition or explanation appeared as merely choosing it from the textbook and presenting to the students.

KCT1(+) in the first practice indicates that her knowledge to choose to start with the beginning activities (Bacterial Growth Pattern and Operating with Number Strips activity) appeared effectively in order to connect generalization of patterns as prior topic (SCK1(+)). Although, the implementation of Bacterial Growth Pattern activity was inappropriate since SCK6(-) indicates that her knowledge of how to provide mathematical connections between getting the general rule of result pattern and operating of the general rules was not as expected. But she implemented the other activity as it was asked (SCK6(+)). Beside these activities, SCK1(+,+) in the first practice indicates that her knowledge to connect the properties of addition in integers in matching activity and her knowledge to connect the topic with prior topic with reminding the concepts of term, coefficient, and constant term with doing the worksheet was appropriate for the beginning. While the teacher was reminding these algebra concepts, CCK2(+) indicates that her knowledge to use terms (term, like term, and coefficient) was adequate, and KCT8(+) in the first practice indicates that her knowledge to address the students' errors about these concepts and to remedy them appeared effectively. Besides, SCK3(+) indicates that her knowledge to develop a usable definition of variable with the students' answers appeared appropriate that she indicated that the variables represented the numbers. While the class working on the worksheet, KCS2(+) in the first practice indicates that her knowledge to anticipate the misunderstandings that might arise with writing algebraic expressions that required parenthesis was appropriate, and KCS3(+,+)

indicates that her knowledge to understand the needs and difficulties of students so that she gave time to think and provided explanation for the students to correct the expressions. In the discussion, KCT5(+,+) indicates that her knowledge to decide when to use a student's remark to make a mathematical point, and when to ask a new question to further students' learning appeared appropriately and she persistently asked the students how 8 cubes (3 green cubes and 5 orange cubes) were qualified to make the students the feel of the requirement of like terms for addition. With this questioning, she used brainstorming that she identified mixing the students' minds as brainstorming so that her knowledge to identify what different methods and procedures afford instructionally appeared appropriately (KCT6(+)).

The prominent knowledge type in the third, fourth and fifth practice was SCK5 which was the knowledge related with the use of mathematical language. SCK5(+,+,+) indicates that her knowledge of how mathematical language was used appeared by expressing the kind of the variable with apple and pear analogy, by expressing the procedures with net worth concept by using assets and debts concepts appropriately in order to teach addition and subtraction. Besides, her knowledge of how mathematical language was used appeared with remedying the errors in the use of equal sign appropriately as she explained and corrected the students' incorrect writings when they added the like terms separately in another place, and they did not write as the result. However, SCK5(-) in the fourth practice indicates that her knowledge of how mathematical language was inappropriate that she used "long" as a word for rectangular shapes to explain "x" with algebra tile. In connection with this, SCK7(-) indicates that her knowledge of how to use algebra tiles as mathematical representations was inadequate since she did not explain the area concept that was underlying idea of using algebra tiles. In solving questions related to the topic, KCS3 (+) in the fourth practice indicates that her knowledge to understand the difficulties of students in operating with the parenthesis algebraic expressions was appropriate as she provided explanations about like term concept, the procedures of addition operation, and the sign before the parenthesis adequately considering the needs of the students. In solving the problems, she especially emphasized the difference between of area and perimeter concepts with her knowledge to anticipate the misunderstandings that might arise with studying these concepts (KCS2(+) and KCS5(+) in the fourth practice).

4.2.2.2.2. Practices in the Instruction of Multiplication of Algebraic Expressions

Teacher B's purposeful actions for teaching multiplication of algebraic expressions were grouped into five practices: 1) connecting multiplication of algebraic expressions to topics from prior years, 2) discussing on the activity related to multiplication of algebraic expressions, 3) choosing the examples or activities to use to take the students deeper into multiplication of algebraic expressions, 4) implementing the suggested activities, and 5) solving the questions and problems related to multiplication of algebraic expressions from the test book, textbook and workbook. The extracted teacher's knowledge based on MKT framework was analyzed within these practices. The reflection of the instruction after each lesson was also presented to provide the teacher to evaluate her instruction by herself. The interpretations of the teacher could give information about her knowledge about students' thinking and learning with respect to the instruction. The classroom dialogues that were most representative for knowledge type the teacher had, were selected from the instruction to illustrate how the teacher used her knowledge in teaching.

The first practice was connecting multiplication of algebraic expressions to topics from prior years and this title was also extracted from one of the descriptors of SCK. This practice examined how the teacher implemented the activity that provided to connect pattern generalization the students had learnt before. It also included how the teacher taught the multiplication with emphasizing the difference of the procedures in addition and subtraction. The second practice was discussing on the activity related to multiplication of algebraic expressions and this practice was also affected by the descriptors of KCT. This practice included a small discussion about how the distribution property was applied with connecting the application of it in the multiplication of integers. She also exemplified the application of this property in the multiplication of an integer and an algebraic expression procedurally. She let the students to explain their answers and responded their questions that they asked to understand. The third practice was choosing the examples or activities to use to take the students deeper into multiplication of algebraic expressions and this title was also extracted from one of the descriptors of KCT. This was also as a continuation of choosing an example or activity to start category. This practice included how the teacher taught the multiplication of two algebraic expressions by using the distribution property. The fourth practice was implementing the suggested activities about addition, subtraction, and multiplication of algebraic expressions. The fifth practice was solving the questions and problems related to multiplication of algebraic expressions from test book, textbook and workbook, which combined knowledge related to geometry from textbook and workbook to improve students' understanding of multiplication of algebraic expressions. These practices are explained with examining how the teacher used her knowledge based on MKT framework.

4.2.2.2.2.1. Practice One: Connecting Multiplication of Algebraic Expressions to Topics from Prior Years

At the first lesson of the instruction for operations with algebraic expression, Teacher B implemented the Bacterial Growth Activity to start operations with algebraic expressions so that her knowledge to choose which examples to start with appeared effectively (KCT1+. This activity had two patterns in the context of growing bacteria and the first one was growing as 2, 4, 6, 8 …, and the second one was growing as 3, 6, 9, 12 … It was asked to recognize and explore the addition and multiplication of the general rules of patterns as 2n and 3n as algebraic expressions. The implementation of addition and subtraction part of this activity by Teacher B was explained in the addition and subtraction practices section in detail. Teacher B's knowledge to connect multiplication of algebraic expressions to pattern generalization as previous topic appeared in implementing this activity (SCK1+), thus this activity required the multiplication of patterns. In the multiplication of patterns, the teacher asked the students to multiply these two patterns as in the following:

> B: Now, let's multiply the rules of the patterns, it is asked, too. It is a bit different. What is the result of this multiplication?
> S: 6n
> B: Is it 6n? You multiplied 3 by 2. But, how do we multiply n by n? We have learnt exponential numbers before. How did we multiply the two same numbers? For example; $3.3=3^2$, $5.5=5^2$, $10.10=10^2$, $6.6=6^2$, $4.4=4^2$, then what is n.n?
> Students: n squared (n^2).
> B: Look, we use 6 and n^2, $6n^2$. First, understand what the meaning of $6n^2$ is.
> S: Multiply by 2.

In this situation, Teacher B asked to the students the rules of the patterns as in the addition of them. She behaved as if the students had known to do multiplication with algebraic expression. Since the students did not multiply of algebraic expressions, one student answered as 6n. Teacher B responded 6 as multiplication of 3 by 2, but she asked what the multiplication of n by n. At that point, she reminded and connected the multiplication of the exponential numbers by giving examples as $3.3=3^2$, $5.5=5^2$, $10.10=10^2$, $6.6=6^2$, and $4.4=4^2$. She used this inductive reasoning and asked what the result of n.n was. The students could answer n^2 easily since the teacher talked about the way of reasoning of exponential numbers and her knowledge how to provide mathematical explanations for common rules appeared effectively (SCK6+). Then, she represented the result of the multiplication of the patterns as $6n^2$. When she asked what the meaning of $6n^2$ was, one student answered as multiplication by 2. Teacher B made explanations as multiplication by itself of a number and then by 6. The students could find the terms by multiplying the patterns' terms as 6, 24, 54, and 96. Teacher B wanted to check if this rule was correct or not and she tried to find the terms by substituting 1, 2, 3 for n in $6n^2$. However, she used the equal sign as notation incorrectly while explaining what $6n^2$ meant as operation. She explained how n was multiplied by n first, and then she substituted 1, 2, 3 for n in $6n^2$ as in the following script:

> B: In $6n^2$, it is expected to multiply by itself and also 6. Let's look for if it is so. What is the 1^{st} term?

Students: 6, then 24, 54, 96.
B: For the 1st term, to multiply by itself, that's 1.1=1.6=6, we have multiplied by 6 also. Let's look for the 2nd term, 2.2=4.6=24, 3.3=9.6=54. We have crosschecked the result.

In this situation, she used the equal sign incorrectly as 1.1=1.6=6. In this representation, 1 does not equal 6 as the result. She used the equal sign as the function of the result. Actually, she corrected when the students wrote in this way throughout the instruction, but she wrote with this way this time. Her knowledge of how mathematical language was used appeared inappropriately with using of equal sign (SCK5-).

In the implementation of this activity, she directly to asked the students to multiply the rules of the patterns, as if the students had known to do multiplication of algebraic expressions. She used the resulted pattern rule to check whether the rule was correct or not. However, Teacher B was expected to generalize the result patterns first, and then to make connection the general rules of multiplied patterns with the operations. It might more make sense if she should have asked to add the terms as numbers that were known and found the general rule of this pattern and then, she would have connected this rule as $6n^2$ with the 2n.3n, the multiplication operation. Her knowledge of how to provide mathematical explanations for common rules and procedures appeared inappropriately (SCK6-).

At the beginning of the teaching of multiplication with algebraic expressions, Teacher B wanted the students to write an explanation about multiplication of algebraic expressions in their notebooks. She first reminded what the requirement for doing addition and subtraction of algebraic expressions was and then connected this explanation with multiplication operation:

B: What is the requirement for doing addition and subtraction in algebraic expressions?
S: Variables.
B: That's the like terms that we operate. x and y, their symbols are equal, we could not operate when one is x the other is x^2, if there is power and it must be equal too. Then, we can do addition and subtraction. There is no requirement for multiplication. That is, the like terms are not required. Let's write the explanation:

> "Each of the expressions in the factors is multiplied each other. After, the like terms in the results of these multiplications are added and subtracted, and the simplest result is obtained".

In this situation, Teacher B asked to the students what the requirement for addition and subtraction was and one student answered as variables. Teacher B expressed it as like terms and reminded the variables and the power of them must be equal in the like terms. After that, she indicated that the like terms were not required for multiplication of algebraic expressions. She wanted the students to write the above explanation in the notebooks. This explanation was produced by Teacher B so that her knowledge to develop usable definition or explanation for explaining the procedure of multiplication operation appeared adequately (SCK3+). She explained that each of the expressions must be multiplied and then the results must be simplified to get the multiplication result.

After the lesson finished, the researcher interviewed with the teacher to get her ideas about the lesson. In general, Teacher B prepared well her lesson before teaching. She reviewed the lesson plan, and questions that she would ask. If she considered extra questions necessary, she brought to the class. She also considered her experiences in other classes for teaching. To illustrate, if the students in other classes had difficulty at some mathematical points, she used different methods or she gave up to use some activities while teaching this class. She made explanations about the use of algebra tiles by evaluating the experiences in other class as in the following:

> B: I used algebra tiles for teaching multiplication in other class, the students got confused completely. I was unable to work with them. So, I have decided not to use the tiles in this class. The students' knowledge of geometry also is weak. They forgot the area of square and rectangle. I tried to explain for a half hour on an example while using algebra tiles. I will take the use of algebra tiles out my lesson plan for this class.

In this situation, the teacher decided not to use the algebra tiles since she had difficulty with using them for teaching multiplication in other class. She indicated that the students' geometry knowledge was weak and thus using the algebra tiles was troublesome. Her knowledge to evaluate the instructional advantages and disadvantages of algebra tiles as representations used to teach multiplication appeared inadequately (KCT4-) since she did not use the algebra tiles to teach multiplication actually.

4.2.2.2.2.2. Practice Two: Discussing on the activity related to multiplication of algebraic expressions

In this part of the instruction, the teacher did not implement an activity, but since she wanted to emphasize the application of the distribution property, she preferred to discuss and teach this property using an example from integer. To do this, she reminded the property in multiplication of integers and then she connected it with the multiplication of algebraic expressions. The dialogues between Teacher B and the students as in the following:

> B: Remember integers. We explained the properties of operations. The properties of addition and multiplication. Can you remember the properties of multiplication in integers?
> S: 0, is the zero element.
> B: Anything else?
> S: Neutral element.
> B: Anything else?
> S: The commutative property.
> B: Anything else? … The distributive property. At that time, we said that we would use this property in algebraic expressions. For example; we did 3.(4-1). What do we do if there is a number before the parenthesis? Normally, you do the operations in the parenthesis. Also, we said another method.
> S: Multiply 3 by 4, and then 3 by -1.
> B: We said multiplication orderly. The distributive property, we multiply 3 by 4.
> S: 12.
> B: Yes, multiply 3 by -1.
> S: -3.
> B: The result is 12-3=9. If it is 3(*x*-1), compare two of them.
> S: *x*
> B: We write x in the place of 4. There are no more changes. When you see this kind of multiplication, use the distribution. Here, you cannot subtract 1 from *x*, in the previous example you can subtract 1 from 4. But it is not here. What can we do? We can distribute 3, multiply 3 by *x*, 3*x*; 3 by -1, -3. That's 3*x*-3. Are there any like terms? None. We do not need to do any operations.

In this situation, Teacher B explained the distribution property for multiplication by connecting with application of this property in integers. She explained it giving the example of 3.(4-1). One student responded correctly for this example that the multiplication 3 by 4 and then -1 respectively. Then, this student answered 12 and -3 correctly with the assist of the teacher. Then, Teacher B gave 3(x-1) as similar example to show the application of distributive property for algebraic expression. She indicated that this was similar since there was x instead of 4 in the expression. She explained to use the distributive property in order to do multiplication since 1 could not be subtracted from x. She multiplied 3 by x and -1 respectively by herself, and got 3x-3 as the result. She stated that the result was 3x-3 because of the absence of like terms that could be added or subtracted. Teacher B provided explanations about the distribution property and her knowledge of how to provide mathematical explanations for the procedure of multiplication in algebraic expression appeared appropriately (SCK6+) by connecting with the application of this property in integers (SCK1+). In general, she selected these two similar examples from arithmetic and algebra to explain the distribution property for multiplication of algebraic expressions connecting with arithmetic. Thus, this part of the lesson was not completely a discussion. Instead of this, the teacher made the students to recognize the use of distribution property with asking questions to the students in order to reveal their prior knowledge. In general, when Teacher B emphasized the concepts which she considered as important to learn the topic, she emphasized it by guiding the students with questions in order to think on it encouraging the use of their prior knowledge. Then, she went on the instruction with asking what the result of 4.(5-3x) and the students could answer correctly with the help of the teacher. The examples that the teacher presented required the multiplication of a number, particularly an integer, and an algebraic expression were explained in the following section.

4.2.2.2.2.3. Practice Three: Choosing the Examples or Activities to Use to Take the Students Deeper into Multiplication of Algebraic Expressions

After the teacher explained the application of the distribution property for the multiplication of algebraic expressions and exemplified the multiplication of an integer, and an algebraic expression, she went on the instruction with asking to multiply two algebraic expressions to take the students deeper into the topic. She asked to the students the following multiplications: x.(3-2x), 2a.(a+2), and (x+2).(x-1). It was important to note that the teacher emphasized the multiplication of the variable by itself in explaining these multiplications. The multiplication of x and (3-2x) was as in the following:

> B: Let's write the multiplications first, respectively, 3x-x.2x. Now, look here, for example 3.3, how we express this multiplication with exponential representation?
> S: 3 squared.
> B: What is 4.4?
> S: 4 squared.
> B: What is *a.a*?
> S: *a* squared.
> B: What is *x.x*?
> S: *x* squared.
> B: Then, x is multiplied with 3, 3x. After that, 2 is remained since we cannot do with it, -2x^2. 3x-2x^2, can you do any operation?
> Students: No.
> B: Because there are not any like terms.

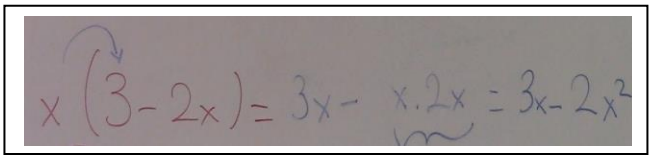


Figure 72 The application of the distribution for x.(3-2x)

In this situation, Teacher B reminded the multiplication of exponential numbers to show $x.x=x^2$, as she explained in the bacterial growth pattern activity. She used the same inductive reasoning there. She exemplified how an integer was multiplied by itself first, then she asked how x was multiplied by itself to the students. She showed the result to the students by multiplying x by 3, and x by -2x

respectively so that her knowledge of how to provide mathematical explanation for the procedures of the distribution property effectively (SCK6+) as she explained the procedures step by step as in the Figure 72. She finished the question stating that there were not any like terms required to do operation. She followed the same way to explain the multiplication of 2*a* and (*a*+2) as in the following:

> B: Let's look 2*a*.(*a*+2). If you cannot see the result of the multiplication, you can write the terms which are multiplied next to next in the right side of the equal sign. Which of them we multiply?
> S: 2*a* and *a*.
> B: First write like this: 2*a*.*a*, there is not any numbers that can be multiplied by 2, so 2 is written. Then, *a*.*a*=a^2. Then, multiply 2*a* by 2 and 2.2=4, there are not any variables that can be multiplied by a. So, it is 4*a*. Can we do any operations in a^2+4*a*?
> Students: Yes.
> S: No.
> B: You can consider the expressions (a^2 and 4*a*) are as apple and pear.

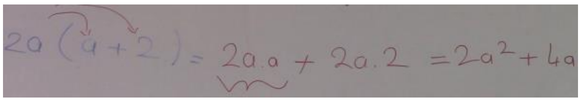


Figure 73 The application of the distribution for 2*a*.(*a*+2)

In this situation, Teacher B explained and showed the application of the distribution of multiplication in 2*a*.(*a*+2). She multiplied the terms respectively and wrote these multiplications separately after the equal sign as in the above figure. Then, she got the results and also suggested this writing for the students to do multiplication correctly as in the previous example. At last, she expressed the terms as apple and pear to explain that they could not be added, when many students considered that a^2 and 4*a* could be added.

Teacher B continued to provide explanations about the application of distributive property with exemplifying more complex example, (*x*+2).(*x*-1). This question required to multiply two parenthesis algebraic expressions. Her explanations for multiplying (*x*+2).(*x*-1) were as in the following:

> B: What is the difference between the previous examples and (*x*+2).(*x*-1)? There, we distributed one expression. Here, it is asked to multiply two expressions. If I can do the operations in the parenthesizes, we would do

operations and then multiply the result. But we cannot. There are no like terms in them. We use the distributive property. But how? Remember the explanation that we wrote. Each of the terms must be multiplied with each other. x and x; x and 1; 2 and x; 2 and 2. Let's look the results. Multiply x and x, x^2. Multiply plus and plus, it is plus. Then, x and -1?
S: -x
B: 2 and x?
S: +2x.
B: The last one is -2. What can we do now?
S: Like terms.
B: The like ones had x (She is writing x^2-x+2x-2 on the board). You have an apple debt, you also have 2 apples, what will leave to you after you pay the debt?
S: 1.
B: That's x. I cannot do any operations with 2. The result is x^2+x-2.

In this situation, Teacher B pointed out the difference of the example from previous examples with asking the multiplication of two parenthesis algebraic expressions. She provided explanations about the procedure of this multiplication with reminding the explanation that they wrote at the beginning of the instruction. She explained that each term was multiplied each other and showed it by multiplying the terms respectively as in the following figure.

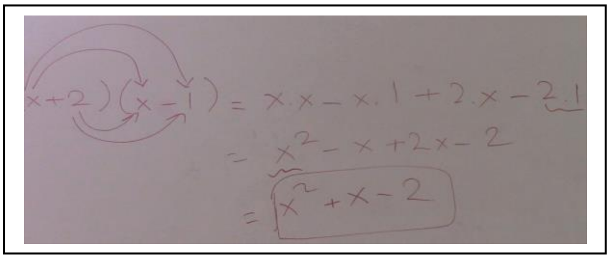


Figure 74 The application of the distribution for (x+2).(x-1)

Then, she asked what they would do to make the students think doing operation if the like terms existed. She explained that 2x and –x were like terms, and she added them using apple concept for x and net-worth analogy for their signs. In general, while doing these multiplications, her knowledge of how to provide mathematical explanations for the procedures of the distribution property with explaining the operation step by step and then using the analogies in addition and subtraction part appeared effectively (SCK6+). Beside this, although the teacher had not indicated these examples in her lesson planning, the sequence of them to improve the students' understanding of multiplication was appropriate since Teacher B sequenced the examples to teach multiplication from simple to complex ones and they also were appropriate for the level of the students with her knowledge (KCT1+).

After the lesson finished, the researcher interviewed with the teacher to get her ideas about the lesson. She stated the students' difficulty as in the following in the post interview:

> B: The students were good, but in comparison with learning of generalizing patterns, it was no so much. They have still difficulty with the multiplication. They could multiply a number and an algebraic expression, but there are problems in multiplication of x and x for example.

Teacher B explained the performance of the students in learning of multiplication of algebraic expressions with the comparison with pattern generalization. She considered that they learnt the generalization of patterns better. She stated that the students had difficulty with multiplication of two algebraic expressions such as x and x so that her knowledge to understand the difficulties of students with it appeared appropriately (KCS3+). Since she realized the students' difficulty, she planned to solve more questions to improve students' understanding of multiplication.

4.2.2.2.2.4. Practice Four: Implementing the Suggested Activities

After these multiplications, Teacher B continued with the researcher's suggested activity that had different algebraic expressions that added, subtracted or multiplied regarding the operation between them as branched from the "Positive algebra – A collection of productive exercises" course book. This activity also was considered enjoyable for the students that they would try to get the result for last bubble with gathering the results before the last one. Teacher B involved this activity to her lesson plan where she planned to do exercises after teaching operations as application hour of mathematics lesson. Her knowledge to predict that students would find interesting and motivating appeared appropriately to select this activity and she indicated doing operations to get the last bubble could be a different activity related with three operations for the students (KCS6+). The implementation of the second one in instruction was as in the following:

S: 6+5m is the first bubble.
B: There is 14+8m, to get 21, we should add 7 to 14. That's 7+5m.
S: 10+m is the second bubble.
B: Look, there is 21+13m and subtract something from it, the result is 11+12m. What do you subtract to get 11 from 21?
S: 10.
B: Okay, to get 12 from 13, we should subtract m.
S: 4+2m is the third bubble.
B: Is there a problem?
Students: No.
S: 20+10m...
S: 15 +14m...
S: 35+24m…

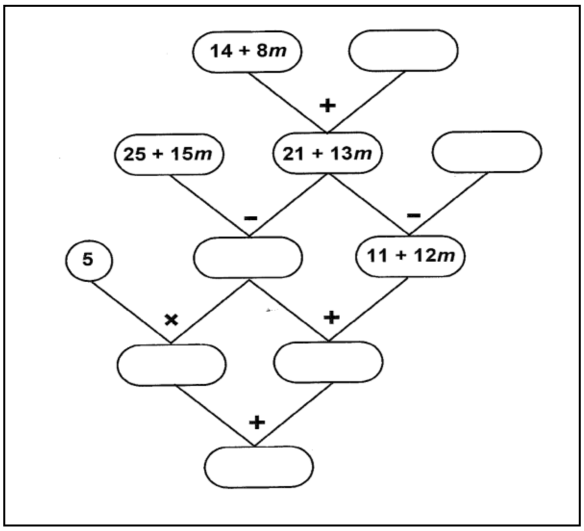


Figure 75 The operating with expressions activity

Besides doing operations to get the result in these questions, it was also asked to find addend or subtrahend as seen in the figure. In this situation, the student answered 6+5m incorrectly and the teacher explained that 7 was needed to get 21 from 14, and corrected the answer. The student's answer was correct for the second bubble, thus Teacher B explained the subtraction to the class by herself. The students gave the correct answers for other bubbles easily.

While the teacher was implementing these activities, one student answered as 15n for the 5+10n. Therewith, she addressed the student's error about the like term concept in addition. Some students also tried to add or subtract the terms that were not like, such as addition of an integer and an algebraic expression throughout the instruction. The teacher addressed the error when the student answered for 5.(5+10n) as in the following:

> S: I found 15n for 5+10n, multiply by 5, it is 75n.
> B: Let's examine, 5 is a constant term, 10n is an algebraic expression, so we cannot add them.
> S: Then, multiply 5 and 5 is 25; multiplying 5 by 10n is 50n.

In this situation, Teacher B realized the student's error in addition, and to address this error, she explained that 5 and 10n were not like terms and they could not be added. Her knowledge of how to address the student's error effectively, and to remedy it with reminding the requirement of like term to do addition (KCT8+).

Then, the teacher wanted the students to match the identical expressions in the researcher's suggested activity (Figure 76). The "Equivalent II" activity was from the course book and had 9 algebraic expressions. It was asked to find the equivalent expressions and connect them with an arrow. The expressions had parenthesis and required doing multiplication by using the distributive property. It might make sense for the students as the expressions which seemed different at first glance were multiplied within themselves and they became equal. Teacher B involved this activity to her lesson plan where she planned to do exercises after teaching operations as application andher knowledge to predict that students would find it interesting and motivating appeared (KCS6+). The teacher gave five minutes to the students to work on this activity individually. Then, the students showed the identical expressions with an arrow on the board, Teacher B guided the class to discuss whether the matching was correct or not. The dialogues between the students and Teacher B as in the following:

> B: Your friend said that she has found the identical ones more than two. We will check, when you finish.
> S: 3x(X+6Y+21Z) and 3X+18Y+21Z are identical.
> B: Let's check if it is true.

Students: No.
B: Why?
S6: It was not multiplied by 21.
B: Look, we are using the distribution property. Multiply 3 and X, 3X; 3 and 6, 18; 3 and 21, 63. So, 63Z must be. This matching is wrong.
S: 3X+6Y+21Z and 3X+18Y+21Z are identical.
B: What do you say for this?
Students: Wrong.
B: The first one has 6Y, but the other had 18Y.
S: 3X+18Y+63Z and 3X+18Y+21Z are identical.
Students: Wrong.
B: Here, the first one has 63Z, but the other had 21Z.

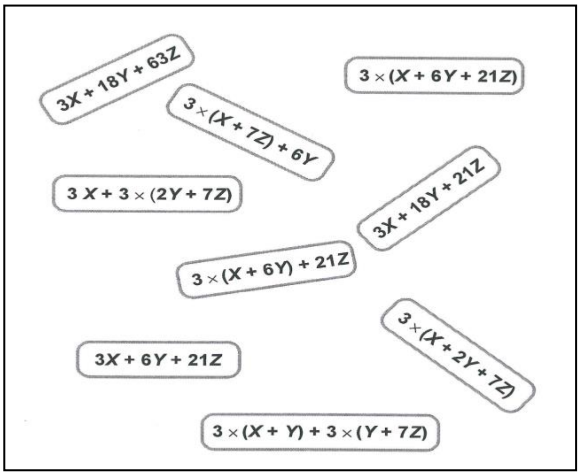


Figure 76 Equivalent activity II

In this situation, the students came up with their answers that they matched the expression incorrectly. Because they did not pay attention to all terms were not equal, or they did not distribute the parenthesis expression correctly. When these incorrect answers came, Teacher B asked to the class whether it was true or not before she explained by herself. The students realized the wrong matching, and Teacher B pointed out why the expressions were not identical so that her knowledge to decide when to pause for more clarification appeared effectively (KCT5+). At this point of the discussion, the correct matching was not made. After that, the discussion went on as in the following:

S: 3X+18Y+63Z and 3x(X+6Y+21Z) are identical.
B: We found shortly before 3x(X+6Y+21Z) is equal to 3X+18Y+63Z. True. Anything else?

S: There is one more to equal to them.
B: Your friend said that, 3x(X+6Y)+21Z is equal to these two. Let's examine, multiply 3 and X, 3X; 3 and 6Y, 18Y. This 3 is multiplied only these two, X and 6Y, not 21Z. There is difference in the terms with Z. If it is 63Z, it would be. But, it is equal to 3X+18Y+21Z. Have you noticed?
Students: Yes.
S: 3X+3x(2Y+7Z) and 3X+6Y+21Z are identical.
B: Let's examine. There is 3X, 6Y is needed, 3 is distributed to 2Y and 7Z, 6Y and 21Z are resulted. True, they are identical.

The students started to came up with correct matchings. The student proposed correctly hat two expressions were equal. Then, another student proposed one more expression that were equal to these two expressions. Teacher B's knowledge to decide when to use the student's answer to point out the equality of all the terms in the expression appeared appropriately (KCT5+). The teacher explained that 3x(X+6Y)+21Z did not have 63Z since 21Z did not multiplied by 3. Teacher B proposed one expression (3X+18Y+21Z) equal to this expression. After that, S matched the two expressions correctly. Up to now, three identical pair expressions were found. After that, more than two equal expressions were found and also equal to the last found two expressions:

B: Anything else?
S: 3x(X+7Z)+6Y and 3x(X+2Y+7Z) are identical.
B: Let's check. Distribute 3 to the parenthesis.
S: The first is 3X+21Z+6Y, the second is 3X+6Y+21Z.
B: Are they identical? Yes. There are 3X, 6Y, and 21Z. The place is different (6Y and 21Z), is it important? What the property of addition operation?
S: The commutative property.
B: Okay, what do you say for 3x(X+Y) + 3x(Y+7Z)?
S: 3X+3Y+3Y+21Z
B: Have you noticed 3Y+3Y?
S: 6Y.
B: That's 3X+6Y+21Z.

In this situation, Teacher B pointed out the commutative property of addition in the student's proposed expressions to show the equality of them. She used the student's remark as commutative property to make a mathematical point with this explanation with her knowledge (KCT5+). Then, Teacher B asked to the students what 3x(X+Y)+3x(Y+7Z) was. The student distributed 3 to the parenthesizes correctly, and when 3Y and 3Y were added, this expression simplified as 3X+6Y+21Z. These three expressions were equal to each other and also equal to the last found two expressions in previous dialogue. At that point, there were not any expressions that were not be matched. Finally, the teacher summarized all the students' answers as in the following:

> B: (She is circling the identical expressions) these two are identical among themselves, these other two expressions are identical among themselves, and 5 of them are identical among themselves. Is there anyone who could not understand?
> S7: Why did we match 5 of them?
> B: Because we found their results equal.
> S: How way do we use in matching?
> B: First, arrange the given algebraic expressions. If there is distribution, distribute; if there is addition, add them. Then, match regarding their results.

In the end of the activity, Teacher B showed the identical expressions by circling them. The two identical pairs were: 3X+18Y+63Z and 3(X+6Y+21Z); and 3x(X+6Y)+21Z and 3X+18Y+21Z, the rest were identical with each other. When one student asked why 5 expressions were equal, Teacher B explained that their simplified representation resulted likewise. Another student asked a general question about how matching questions should be solved. Teacher B suggested to do all operations first such as multiplication with using distribution, addition, and then match regarding their simplified presentation.

After that, similar activity was about determining whether the given pair algebraic expressions were identical or not. There were three expressions in this activity as in the below figure. While doing this activity, some students proposed the expressions were identical and other proposed they were not. The students who answered incorrectly forgot multiplying C by 2 in the first question, or they multiplied C by 2 although C was out of the parenthesis in the second question.

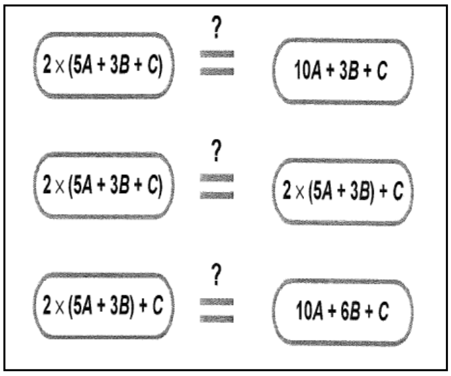


Figure 77 Equivalent activity I

The next similar suggested activity was about explaining how the given pair expressions were identical. These expressions were: nxnxnxn and n^4; $(m+1)^2$ and (m+1)x(m+1); $(m+1)^2$ and m^2+2m+1. The explanations of Teacher B's for nxnxnxn and n^4 were as in the following:

> B: How can you show the expressions are equal?
> Students: They are equal.
> B: Okay, but how did you decide? Your friend said that the multiplication of 4 of n. We learnt it in the concept of power. We explained that the multiplication of a number by itself as repeatedly can represent as exponential. So, how many n is there?
> Students: 4.
> B: Then, n^4 (see the figure). Or you can substitute a number for n. Give 2, 2.2.2.2 = 16. I will do the same thing for n^4, $2^4=16$. For that reason, they are equal.

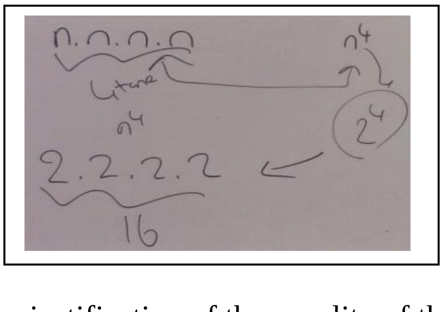


Figure 78 The justification of the equality of the expressions

In this situation, Teacher B provided explanations about the representation of exponential numbers that the repeated multiplication yielded them. Thus, she asked to the questions how many n was there and the students answered as 4. She showed the equality of the two expressions with this way algebraically. Then, she suggested the substitution of numbers for n to prove this equality, that the method produced by a student for the previous similar question in addition and subtraction practice. Her knowledge of how to provide mathematical explanation for the equality of these expressions appeared appropriately (SCK6).

After the lesson finished, the researcher interviewed with the teacher to get her ideas about the lesson. When the researcher asked to Teacher B to evaluate the effectiveness of the researcher's suggested activities for operation with algebraic expressions, she made these following expressions:

> B: The students could do easily the questions about doing operation as branched. Their performance was good. The matching activity was also good. They have difficulty when they have to apply the distribution, but when they saw the equal pair expressions, they could understand easily. So, the matching with corresponding equal expression is useful for the students. The questions were good, actually.

Teacher B stated that the suggested activities and questions were effective for learning of operations with algebraic expressions. Especially, she pointed out the effectiveness of the matching of identical algebraic expressions for learning application of the distribution property in multiplication. She explained that having the identical paired expressions could provide to the students to apply the distributive property correctly so that her knowledge to understand the needs and difficulties of students with the application of distribution property appeared appropriately (KCS3+).

4.2.2.2.2.5. Practice Five: Solving the Questions and Problems Related to Multiplication of Algebraic Expressions from the Test Book, Textbook and Workbook

Teacher B asked two questions from the test book, and problems that she formed at that time. The questions were about simplification of algebraic expressions that it was asked to add, subtract and multiply for the simplification. Teacher B used the analogies that were apple-pear and net-worth to explain of the simplification of these algebraic expressions. The answering of one of the examples in the instruction was as in the following:

> B: Let's do $3x$-($7x$+1)+4(x-1). First, look if there are the parenthesis and then, if there is a number before the parenthesis. Let's do, is there anything to do with $3x$?
> Students: No.
> B: I look before the parenthesis.
> S: There is a negative sign.
> B: There is no coefficient. So, the signs of the expressions in the parenthesis change as -$7x$-1. What will do for the next parenthesis expression? We must multiply 4 with each term in the parenthesis. $4x$-4 … Then, let's look the like terms. $3x$, -$7x$, $4x$. I have 3 apples, I cannot give 7 apples, so I have 4 apples debt. Then, I take 4 apples more, so there is no x. Or, I have 3 apples and 4 apples as 7 apples. For -$7x$, I will give 7 apples, last I have no apples. Look the constant terms. I have 1 TL debt, and then 4 debts more. I have 5 TL debts at last, as -5.

In this situation, Teacher B guided the students to do the parenthesis expressions operations first using the distribution property. After the students could write the expression without parenthesis, she used the analogies to explain doing operations with the like terms. She used apple concept for x, and then net-worth concept for the operations. She explained the addition of variables within two ways; firstly, she added $3x$, -$7x$, and $4x$ respectively. She indicated having 3 apples, and since not giving 7 apples and it would be 4 apples debt, and lastly taking 4 apples (+$4x$) and the debt was 0. Secondly, she added 3 apples and 4 apples as 7 apples, then gave these 7 apples to pay debt, and there was no apple at last. For the addition of the constant terms as integers (-1 and -4), she made similar explanation that having 1TL and 4TL debt yielded 5TL debt and she represented it -5. Her knowledge of how mathematical language was used appeared by expressing the kind of the variable with apple and pear analogy and the procedures with net worth concept by using assets and debts concepts as analogy appeared appropriately (SCK5+).

Teacher B addressed the students' errors about the like term concept while doing addition or subtraction. Some students tried to add or subtract the terms that were not like such as addition of an integer and an algebraic expression. A similar error was observed in simplifying of 3n-6-3(n+2), the teacher again explained the requirement of like terms to add or subtract as in the following:

S: I will distribute 3 to (n+2), Then I will add 3n and 3n, subtract from 6.
B: Your friend has said that he would add 3n and 3n and subtract from 6. Are there any unknowns with 6? Variable? None. We do operations only like terms.
S: 3n-3n, 0. -6 and -6, +12 since their signs are negative, it gives positive.
B: But this rule is valid for multiplication. Think that you have 6 TL debts, and then you have 6TL more debts.
S: -12.

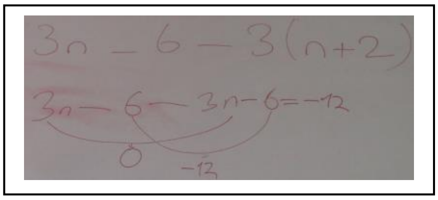


Figure 79 The simplification of the expression by the student

In this situation, Teacher B indicated that 6 was not an algebraic expression and so it could not be operated with algebraic expressions, when the student tried to do it. The other error of this student was the result of adding of -6 and -6 using the rule of multiplication, that was the negative signs yielded positive sign. Teacher B addressed this error with explaining this rule worked for multiplication, and asked to the student this addition using the debt concept. Then, the student answered correctly as -12 for addition of -6 and -6. Her knowledge of how to address the student's errors, that were adding unlike terms and adding negative integers incorrectly, appeared effectively. She remedied these errors with provided adequate explanations (KCT8+).

Then, the teacher continued the lesson with presenting and solving the problems related to geometry topics. While solving the geometry problems using algebraic expressions, Teacher B made explanations about the area and perimeter concept since she anticipated that the students confused them. The teacher selected this following problem to start solving the geometry problems and asked the questions as in the script:

> Problem: What is the area of the square whose side is x unit?
> Students: 4*x*.
> B: I have asked the area. What did you say?
> Students: Perimeter.
> B: Do you remember the area of the square?
> Students: Multiply two sides.
> B: I am drawing a square. One side is x unit. So, this is a square, the perimeter is 4.*x*=4*x*. But the area of it is asked. To find the area, multiply two sides. $x.x = x^2$ unit squares. For example, if the one side is 2 units, the area is 4 unit squares. If the one side is 5 units, what is the area?
> S: 25 unit squares.

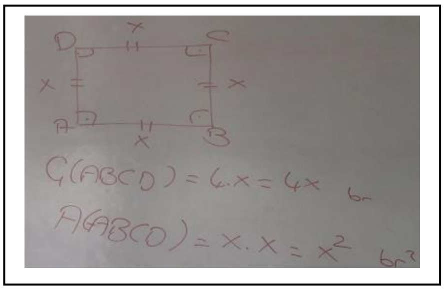


Figure 80 The calculation of the perimeter and area of the square by Teacher B

In this situation, Teacher B asked what the area of the square was, since she expected from the students to confuse the area and perimeter concepts. When the students answered 4*x* for the area, Teacher B reminded that the area was asked and the students answered 4*x* as perimeter. Then, Teacher B calculated the perimeter as 4*x* and the area of this square as x^2 with showing it on the board and explaining how they were found (see Figure 80). After that, she gave arithmetical examples about the calculation of square such as the area of 2 unit-sided square was 4 unit squares, or the area of 5 unit-sided square was 25 unit squares. Her knowledge of how to address the student's possible error that was confusing the area and perimeter concepts by asking this problem and reminding how the area and perimeter of square were calculated appeared effectively (KCT8+).

After the teacher exemplifying the perimeter and area of square, she continued with the rectangle concept. She asked the problem in which was asked to calculate the rectangle whose sides were x and $x+2$. She reminded the area concepts and the calculation of them to solve the geometry problems that required using algebraic expressions. Teacher B's explanations for the area of rectangle as in the following:

B: The rectangle whose sides are x and $x+2$ length. What do you imply from the shape?
S: The long side is 2 more than the short side.
B: Okay, what is the shape?
S: Rectangle.
B: How do you decide? The opposite sides are equal; did you decide regarding this? Can I say that is a rectangle only considering the equal sides?
S: The sides are 90^0.
B: The sides?
S: The vertexes.
B: The degrees must be 90^0. I ask you to calculate the perimeter and the area of this rectangle. First the perimeter. You know the one side of the rectangle, as well what do you know?
S: The other sides.
B: Then, I know all the sides, what can we do? Add all the sides. How do we add these algebraic expressions?
Students: Like terms.
B: The terms that had x are like.
S: $4x+4$.
B: 2 and 2 are added as 4. Now, let's find the area of this rectangle.
S: The multiplication of the long side and the short side. $x.(x+2)$.
B: Do we leave this expression like that? What can we do?
S^*: We can distribute.
B: Distribute 2 to the parenthesis.
S^*: $x.x+x.2 = x^2+2x$.
(S^* represents the same student)

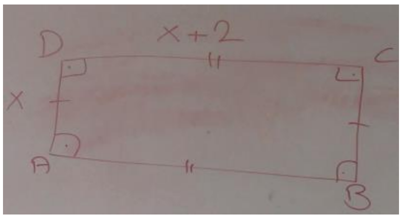


Figure 81 The rectangle that Teacher B asked to calculate its perimeter and area

In this situation, Teacher B asked the students what the properties of a rectangle with indicating whether the opposite equal sides were adequate. She reminded what the perimeter and the area concepts were and how they were calculated in the previous example in which was asked the area of a square. The student answered that the sides were 90^0 and the teacher asked if the sides were, and this student corrected this error and answered as the vertexes. The teacher asked the students to calculate the perimeter of the rectangle. She guided the students with asking what the unknown sides were. The students found them explaining the opposite equal sides and added the lengths of all sides with the help of the teacher. The perimeter was found as 4*x*+4. Then, the teacher asked the students to calculate the area of the rectangle. One student immediately answered as *x*.(*x*+2) with explaining the multiplication of the long side and the short side of the rectangle. When some students had difficulty with understanding the calculation of rectangle, Teacher B made explanations with a simple example by calculating as 2br-4br length rectangle:

> B: Okay, is there anyone who could not understand? Is it about area or perimeter?
> S: Area.
> B: When I have this kind of rectangle (see Figure 82), what can we do?
> S: We multiply the long side by the short side.
> B: For this?
> S: Multiply 2 by 4, 8 br^2.
> B: It does not a matter with operating with algebraic expressions. Here, there is an expression and a parenthesis expression and we should multiply it. I distribute x to the parenthesis expression. Respectively. Multiply *x* and, x^2; then *x* and 2, 2*x*.

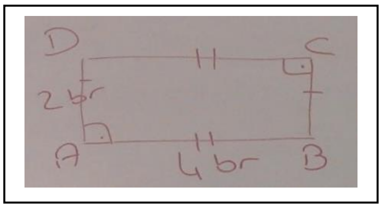


Figure 82 The rectangle that Teacher B exemplified

Then she also made explanations about the application of distributive property in the multiplication of *x* and (*x*+2). Teacher A connected the geometry concepts such as perimeter and area with the sides that were represented with algebraic expressions by asking this geometry problem so that her knowledge to connect the topic to prior topic appeared appropriately (SCK1+). She also reminded these concepts to the students firstly, and then she showed the application of multiplication of algebraic expression in calculating the perimeter and area of rectangle.

Then, the teacher continued the lesson with solving the problems related with geometry from the textbook and workbook. In solving the problems related with geometry using algebra, Teacher B understood the students' difficulties about the solution ways and she used similar simple arithmetic example as in the previous problem solving to support the students' understandings in the following problem. The problem that the teacher solved with this way as in the following:

> Problem: If the side of an 8-unit length square is reduced x unit, what is the perimeter of this square?
>
> B: You have a square, one side is 8 units. If you are told to reduce the side 3 units, what is the perimeter of it?
> S: 20.
> B: How did you find?
> S: I subtracted 3 from 8, 5. Then, I multiply 5 by 4, 20.
> B: In this question, you are told to reduce x unit instead of 3 units (The student is writing the operations on the board). If we reduce 3, we subtract 3 from 8, 5. Since all the sides are equal, multiply 4 by 5. What do you say while reducing *x* unit?
> S: We will subtract *x* from 8.
> B: Is it true?
> Students: Yes.
> B: (The student is writing 8-*x* on the board). Can you write the next column of the solution for reducing 3 units? (see the Figure 83). Is there a result of 8-*x*? There is not. To find the perimeter, multiply by 4.
> S: (8-*x*). 4, I should use the distributive property, (she is taking 4 before the parenthesis). 4.8, 4.*x*, then 32-4*x*.

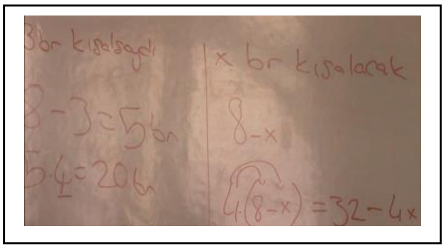


Figure 83 The solving of the problem on the board

In this situation, Teacher B asked to the students what if the one side of square was reduced 3 units to think on the solution. One student subtracted 3 from 8, and then multiply 4 by 5 to find the perimeter. In response to this, the teacher asked what if the one side of square was reduced *x* unit. The students could represent the one side as 8-*x* easily, and multiply by 4 using distribution to find the perimeter. The teacher wanted the student to write the algebraic solution next to the arithmetic solution on the board to show the same procedures. Her knowledge to understand the students' difficulties with calculating the perimeter of the square using algebraic expressions appeared appropriately and she considered their needs in order to solve the problem, and thus she used arithmetical example to guide the students to apply similar method for algebra by considering the students' prior knowledge (KCS3+).

The teacher used similar method for the explanation of solving another problem in the work book:

What is the area of the top surface with 4*x*-4 width and 6*x*+2 length rectangle desk?

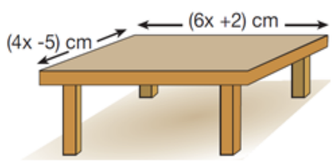


Figure 84 The problem related with the area of rectangle

B: You are given the information about the shape of the desk.
Students: Rectangle
B: What is the length of short side?
S: 4*x*-4.
B: What is the length of long side?
S: 6*x*+2.
B: Let's do this first. What is the area of this rectangle (she is drawing 2-5 cm units' rectangle on the board)? 10 or 14?
S: 10
B: 10 what?
S: 10 cm^2.
B: Be careful, do not confuse area and perimeter. What is the perimeter of it?
S: 14.
B: 14 cm. How did you find the area?
S: Multiplying the short side with the long side.

B: We do the same operation for this question. A(ABCD)=(4x-5).(6x+2), what did we do this type of multiplication?
S: Using the distribution. First distribute 4x.
B: Then, which we distribute?
S: 5.
B: -5. Multiply 4x and 6x, 24x^2; 4x and 2, 8x; -5 and 6x, -30x; -5 and 2; -10. There is nothing to do with x^2. I have 8 apples, if 30 apples are wanted?
S: I cannot give.
B: You have 22 apples debt. The result is 24x^2-22x-10.

In this situation, Teacher B asked what the area of a 2-5 cm length rectangle first. The students answered 10 cm^2 easily. The teacher warned the students not to confuse the concepts of area and perimeter. She showed the application of multiplication (4x-5) and (6x+2) using distribution. She explained the subtraction of 8x-30x by using apple to represent x and the debt concept to subtract the like terms. Lastly, she wrote correct answer on the board by herself. First, her knowledge to understand the possible students' difficulties of students with calculating the area of the rectangle using algebraic expressions appeared appropriately and she used arithmetical example to guide the students (KCS3+). Then, her knowledge of how to provide mathematical explanations for the procedure of the application of the distribution property appeared appropriately (SCK6+).

After these problems related with the concepts of square and rectangle, she continued with the problems related with the area of triangle. The teacher used the connection with calculating the area of the triangle and using algebraic expressions as sides' lengths. Her explanations for finding the area of triangle in the figure as in the following:

B: It is asked to find the area of the next triangle (Her drawing triangle on the board has 90^0 symbol). Have you noticed a difference between the shape in the book and on the board?
S: Perpendicular.
B: How do you find the area of the triangle? If you have a rectangle, how can you find the area of it?
B: You multiply these two (the long and short side).

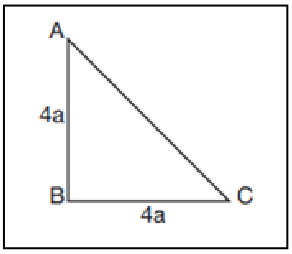


Figure 85 The problem related with the area of triangle

In this situation, Teacher B drew the triangle on the board adding the symbol of 90^0 to the triangle. Then, she explained that the rectangle has two equal right triangles with dividing by a diagonal. She talked about the reasoning for the area of the rectangle:

B: You multiply these two (the long and short side). If we draw this segment (diagonal – see Figure 86), what do we get?
Students: Triangle.
B: How many?
S: 2.
B: What is the area of one of these triangles?
S: The half of the rectangle.
B: Why? Remind that the diagonal divides rectangle as 2 equal triangles. That's the half of the area of the rectangle. How do you find the area of rectangle?
S: Multiplying the long side and the short side.
B: If I divide by 2, what do we get? The area of the triangle. But it works for only the right triangle. Multiply the right sides and divide it by 2. Have you understood why dividing by 2? While finding the formula of the area of the triangle, we generalize using the area of rectangle. We multiply the base length and the height, and divide by 2. If the triangle is right, it is easier since one of the right sides is the base, the other is the height. Then, what is the multiplication for this triangle? $4a.4a$ / 2
S: $16a$.
B: Keep in mind, $16a^2$, since we multiply 2 of a.
S: $8a^2$.

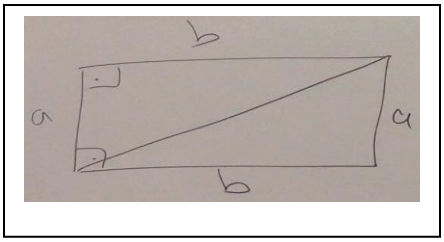


Figure 86 The drawing of the diagonal by Teacher B

Teacher B reminded the area of the rectangle in the previous problems and for that reason the students could describe the formula of the right triangle easily from the the teacher's reasoning. The teacher explained that the right sides were the base and the height by indicating the multiplication of them, and division it by 2 for the area of triangle. Finally, she wrote the expressions as $4a.4a/2$, and the student answered $16a$ for multiplication of $4a$ and $4a$. The teacher reminded that the multiplication of variable by itself required the power 2. Finally, the student answered $8a^2$ for the area of the triangle. In solving the problem, Teacher B connected with geometry as area of the triangle with algebra multiplication of the sides that were represented as and her knowledge to connect the topic to prior topic appeared appropriately (SCK1+).

In this situation, she realized and pointed out that the area of triangle could not be found with known two sides without knowing the angle since she anticipated the possible misunderstanding about calculation of the triangle as in the textbook. She drew the symbol of 90^0 to the triangle as in the Figure 87 that her knowledge to anticipate the misunderstandings that might arise with the area of triangle appeared appropriately (KCS2+). Since the students might have assumed that it was a right triangle since the shape was drawn like that in the textbook, and thus they might have the half of the multiplication of two sides.

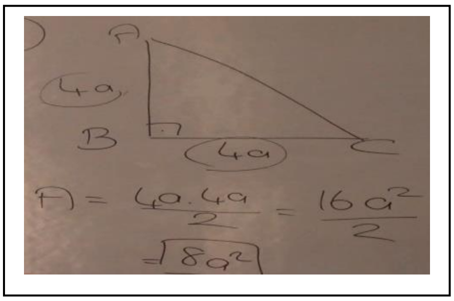


Figure 87 The solving of the problem by Teacher B

In general, Teacher B taught multiplication by applying the distributive property at the beginning of the instruction of multiplication. She selected the examples from simple to complex such as the multiplication of an integer and a parenthesis expression, a variable and a parenthesis algebraic expression, a variable with a coefficient (e.g. $2a$) and a parenthesis expression, and a multiplication of two parenthesis algebraic expressions. Then, the teacher implemented the suggested activities that she selected: the questions required doing operations as branched, the activity about finding the identical algebraic expressions, the activity about determining if the equal expressions were, and the activity about justification of the equality of identical expressions. After these activities about equality of expressions, Teacher B asked the students to calculate the area and the perimeter of a square and a rectangle as beginning of the problems. Then, Teacher B asked the students to solve the geometry problems in the textbook and workbook. She did not solve all the problems in the books, thus she selected which to solve for the instruction at the time since the lack of the time. However, she solved the questions and problems correctly with adequate explanations and her knowledge to calculate an answer and solve problem appeared effectively (CCK3+).

4.2.2.2.2.6. The Extracted Knowledge Types from the Instruction for Multiplication of Algebraic Expressions

Table 19 The extracted knowledge types from the instruction for multiplication of algebraic expressions

Practices	Extracted knowledge types				
	SMK		PCK		
	CCK	SCK	KCS	KCT	KCC
Connecting multiplication of algebraic expressions to topics from prior years		SCK1(+) SCK3(+) SCK5(-) SCK6(+,-)		KCT1(+) KCT4(-)	
Discussing on the activity related to multiplication of algebraic expressions		SCK1(+) SCK6(+)			
Choosing the examples or activities to use to take students deeper into multiplication of algebraic expressions		SCK6(+,+)	KCS3(+)	KCT1(+)	

Table 19 (Continued)

Practices	Extracted knowledge types				
	SMK		PCK		
	CCK	SCK	KCS	KCT	KCC
Implementing the suggested activities		SCK6(+)	KCS3(+) KCS6(+,+)	KCT5(+,+) KCT8(+)	
Solving the questions and problems related to multiplication of algebraic expressions from the test book, textbook, and workbook	CCK3(+)	SCK1(+,+) SCK5(+) SCK6(+)	KCS2(+) KCS3(+,+)	KCT8(+,+)	

Table 19 shows what type of knowledge of subject matter knowledge (SMK) and pedagogical content knowledge (PCK) that Teacher B had in instruction practices. (+) sign indicates the teacher's existing knowledge was adequate or appropriate and she used it effectively, while (-) sign indicates the teacher's existing knowledge was inadequate or inappropriate and she used it ineffectively. Each sign (+ or -) in the same knowledge type refers to Teacher B's different use of this knowledge during instruction. Besides, for SCK3 knowledge type, (⊥) is used to indicate the teacher's knowledge to develop definition or explanation appeared as merely choosing it from the textbook and presenting to the students.

KCT1(+) in the first practice indicates that her knowledge to choose which examples to start with connecting pattern generalization as previous topic was effective (SCK1(+) in the first practice). In the implementation of this activity, the teacher expressed the multiplication of n and n as n^2 with explaining how the exponential form of the numbers was written appropriately (SCK6(+) in the first practice). However, she used the equal sign as the function of the result in the explanation of the substitution of 1, 2, 3 for in $6n^2$ so that SCK5(-) indicates her knowledge of how mathematical language was used appeared inappropriately with using of equal sign though she corrected when the students wrote in this way throughout the instruction. The teacher did not use algebra tiles to teach multiplication of algebraic expressions and thus KCT4(-) indicates that her knowledge to evaluate the instructional advantages and disadvantages of algebra tiles as representations used to teach multiplication appeared inadequately. Instead of this, she developed usable explanation for explaining the procedure of multiplication operation appeared adequately (SCK3(+)) and she emphasized the distribution property. Her knowledge of how to provide mathematical explanations for the procedure of multiplication in algebraic expression was appropriate (SCK6(+)) by connecting with the application of this property in integers (SCK1(+)) in the second practice.

Most of the extracted knowledge types in the third, fourth and fifth practice were about the explanation of the application of the distribution property and the connection of multiplication of algebraic expressions with the area of polygons. Firstly, she asked to the students to do question procedurally and SCK6(+,+) in the third practice indicates that her knowledge of how to provide mathematical explanation for the procedures of the distribution property appeared effectively. She explained the procedures step by step and then she used the analogies to explain operations (SCK5(+) in the fifth practice). While doing the questions, she understood the difficulties of students with the multiplication of two algebraic expressions and the application of distribution property (KCS3(+) in the fourth practice). In solving geometry problems, her knowledge to connect algebra to the perimeter and area of square, rectangle, and triangle was appropriate (SCK1(+,+) in the fifth practice). In this process, her knowledge to address the student's error effectively, and to remedy it was effective. She explained that the requirement of like terms to be added or subtracted, and emphasized the difference of the perimeter and area concepts (KCT8(+) in the fifth practice). She also understood the needs and difficulties of the students with calculating the perimeter and the area of polygons and she used simple arithmetical example to guide the students to apply similar method for algebra by considering the students' prior knowledge (KCS3(+,+)) in the fifth practice).

4.2.2.3. Summary of the Instructions

Teacher B prepared herself well before teaching that she reviewed the lesson plan, and questions that she would ask. If she considered extra questions necessary, she brought to the class. She also considered her experiences from other classes while teaching. If the students in other classes had difficulty at some mathematical points, she tried to use different methods or she decided not to do same activities while teaching this class. At the beginning of the lesson, she reminded what was done in previous lesson and then started to the lesson. Teacher B generally used direct instruction, question and answer, and discussion teaching method for teaching. She did not have a computation at the end of the instruction as she explained to the researcher in lesson planning. As instruction format, she presented the lesson to the class as a whole group. Especially, she provided the discussion environment in order to participate the students actively to the learning process. She gave enough time for the students to think on the questions. After each question, she asked to the students whether they could understand the solution and if they had difficulty with understanding, she explained the solution once more time. Furthermore, she did not miss where and why the students had difficulty from their explanations and she realized the students' errors at that point and made necessary explanations.

In teaching generalization patterns, she generally used students' errors to explain the reason for why it was incorrect for learning of other students. To illustrate, the students generally had the tendency to find the general rule considering only the 1^{st} term, but the teacher showed that it did not work for other terms. However, in teaching generalization of patterns, the teacher permanently emphasized the multiplication of the position number and the increment and adding number to find the first term as shortcut. This was a rule or a formulization for the students in the instruction.

In teaching operations with algebraic expressions, she especially emphasized the like term concept for teaching addition and subtraction of algebraic expression, and the application of distribution property for teaching multiplication of algebraic expression procedurally. In the multiplication instruction, she also asked geometry problems to the students.

4.3. The Influence of the Teachers' Subject Matter Knowledge (SMK) on Their Pedagogical Content Knowledge (PCK)

This section presents the influence of subject matter knowledge (SMK) on pedagogical content knowledge (PCK) of Teacher A and Teacher B respectively based on their planning and instruction.

Teacher A's knowledge to choose examples and activities to start and take the students to deep into the content (KCT1) were generally appropriate in terms of the developmental order of content, the level of the students, and the suggestions by the curriculum in planning and designing the instructions. Moreover, she tried to improve the instructions with considering the suggestions from research as she was also a doctorate student in mathematics education. To illustrate, she involved the use of graphical representation to represent the relationship in the patterns, figural patterns, and algebra tiles to support the students' learning. This was one of the pedagogical content knowledge elements. However, she could not use these representations effectively since she did not know how to use them for the instruction. This situation was about the knowledge of using the representations and link among them related to specialized content knowledge (SCK2 and SCK7) of subject matter knowledge which was specific for teaching mathematics. Besides, her content knowledge about the important concepts that were generalization of patterns and application of distributive property in the multiplication was somehow troublesome. In the instruction of generalization of patterns, she had difficulty in generalizing patterns as she could not conceptualize functional thinking in the patterns. Thus, she could not anticipate possible misconceptions that might arise from her explaining about getting the general rule while she was teaching. The lack of conceptual understanding in her subject matter knowledge had a negative impact on her knowledge of content and students (KCS2) that was one of the elements of pedagogical content knowledge. Especially, she had difficulty in explaining how to generalize non-linear growth patterns and the reason of it might be the lack of knowledge of pattern generalization. Thus, she could not guide the students to generalize non-linear growth patterns appropriately and that was related to her pedagogical content knowledge. In the instruction of operations with algebraic expressions, although she knew that the students' difficulty in the application of distributive property to multiply algebraic expressions and algebra tiles was one of the suggested methods to teach it, she could not use algebra tiles in order to show the multiplication of algebraic expressions effectively. She knew what models, or representations were suggested to teach topics in general, but she did not know how to use them. She did not have a conceptual understanding about the reasoning of these representations and that showed her lack of specialized content knowledge. Though she had the knowledge of students' difficulties from her experiences in teaching previous years, her knowledge of students' thinking was inadequate that's why she could not anticipate misunderstanding that the students could have in the instruction. Sometimes, she used mathematical language, which was one of the elements of specialized content knowledge, inappropriately in order to explain the procedures or rules mathematically and this was also a possible reason for misunderstandings. More specifically, the teacher's common content knowledge was problematic about core concepts for teaching the algebra topics and her knowledge of definition of pattern and like term concepts was inadequate. This situation caused answering the questions and solving the problems incorrectly at some points of the instruction and the students had deficient or incorrect knowledge.

Table 20 The examples for the influence of Teacher A's SMK on her PCK

	SMK	PCK
Teacher A	She did not know how to use particular representations (e.g. graphical representation, figural patterns, or algebra tiles).	She could not use these representations effectively.
	She did not conceptualize functional thinking in the patterns.	She could not anticipate possible misconceptions that might arise from her explaining about getting the general rule.

Table 20 (Continued)

	SMK	PCK
Teacher A	She had difficulty in generalizing non-linear growth patterns.	She could not guide the students to generalize non-linear growth patterns appropriately.
	She could not use algebra tiles in order to show the multiplication of algebraic expressions effectively.	She could not build the multiplication understanding conceptually.
	She had lack of knowledge about like term, term and constant concept.	She could not anticipate possible students' difficulties and misconceptions in operating with algebraic expression.

Teacher B's knowledge to choose examples and activities to start and take the students to deep into the content (KCT1) was generally inadequate in terms of the developmental order of content for designing the instructions. Since she did not prepare lesson plans, she generally followed sequence in the textbook. When she considered that the examples and questions in the textbook were not adequate, she involved the questions from different resources such as test book or previous national exams to improve the instruction and the students' learning. Besides, she had strong conceptual content knowledge and thus her knowledge of content and students appeared effectively in the instructions. Her content knowledge about the important concepts for learning the algebra topics was adequate. She had functional thinking conceptually and emphasized it while teaching pattern generalization and she addressed and remedied the students' errors effectively at that time (KCT8). For learning operations with algebraic expressions, she permanently emphasized the concept of like term for addition and subtraction, and the distributive property for multiplication. Thus, she had the knowledge of leading a classroom discussion with critical questioning to further learning (KCT5). While she was explaining these core concepts with emphasizing, she pointed out the possible misconceptions that might arise and her knowledge about students' thinking was adequate. In addition to these anticipations, to prevent the students' misunderstandings, she had the knowledge of building on students' thinking to teach, although she did not plan the lesson effectively. To illustrate, she used inductive reasoning to make the students feel the need of a general rule in generalization of patterns (CCK1), and she connected the properties of integers of the students' prior knowledge to teach distributive property in multiplication algebraic expressions (SCK1). These situations showed that her subject matter knowledge (CCK and SCK) affected her knowledge of content and teaching positively and it was one of the elements of PCK. Her conceptual content knowledge also provided her with making appropriate mathematical explanations to support the students' understanding. For example, she used analogies such as apple-pear or asset-debt to explain operations with algebraic expressions or different colored cubes to explain the like term concept when the students had difficulty in understanding the content. These models showed her specialized content knowledge and she combined it with her knowledge of content and students. However, she only used her preferred representations in the instruction. She considered the tabular representation of the patterns was useful for the students and she always used it with explaining numerical reasoning. She did not consider that the use of algebra tiles was effective and she did not use it although she knew that the underlying idea of the algebra tiles was area concept. Actually she knew the suggestions in the literature to teach the algebra topics, but she used what she considered to be useful for the students. After she taught the content with explaining the reasoning behind it, she focused on developing the students' procedural knowledge.

Table 21 The examples for the influence of Teacher B's SMK on her PCK

	SMK	PCK
Teacher B	She had functional thinking conceptually and emphasized it while teaching pattern generalization.	She addressed and remedied the students' errors effectively at that time.
	She had the knowledge of analogies such as apple-pear or asset-debt to explain operations with algebraic expressions.	She predicted where and how the students have difficulty.
	She had the knowledge of core concepts (e.g. functional thinking, like term, and distributive property) conceptually.	She led the classroom discussions with critical questioning to further learning.

In sum, the common lack of knowledge for two teachers was using representations (SCK) and using mathematical language (SCK). When they could not use the representations effectively, they did not develop new teaching strategies or create activities (KCT). On the other hand, while they were using mathematical language inappropriately, they could not anticipiate possible misunderstandings (KCT).

CHAPTER V

DISCUSSION AND CONCLUSION

This study examined middle school mathematics teachers' planning lessons and instructions in order to reveal their mathematical knowledge for teaching (MKT). With this aim, two middle school mathematics teachers' planning and instructions for teaching generalization of patterns and operations with algebraic expressions were analyzed under the findings chapter. The focus in the first chapter was to describe Teacher A's knowledge for teaching generalization of patterns and operations with algebraic expressions, and to interpret it based on MKT model. Similarly, the focus in the second chapter was to describe Teacher B's knowledge for teaching generalization of patterns and operations with algebraic expressions, and to interpret it based on MKT model. In connection with the findings, this chapter presents the discussion of the findings under three main sections. In the first section of this chapter, the findings about teachers' MKT are discussed within the knowledge subdomains of MKT with respect to what knowledge the teachers had and how they used their knowledge in their instructions, and the influence of subject matter knowledge (SMK) on pedagogical content knowledge (PCK). In the second section of this chapter, the findings about teachers' practices in their instructions are discussed with respect to the relationship between knowledge and practices. In the third section of this chapter, the implications related to this study and the suggestions for future research are presented.

5.1. Mathematical Knowledge for Teaching (MKT)

This section presents the knowledge of Teacher A and Teacher B based on their planning and instruction by compare and contrast method. The findings of this comparison were discussed with literature, and the mathematical knowledge for teaching (MKT) generalization of patterns and operations with algebraic expressions that the middle school mathematics teachers should have is proposed for each sub-sections.

5.1.1. Subject Matter Knowledge (SMK)

The teachers' subject matter knowledge (SMK) is explained under two headings: common content knowledge (CCK) and specialized content knowledge (SCK) as the components of SMK of MKT model.

5.1.1.1. Common Content Knowledge (CCK)

This study revealed that the two middle school mathematics teachers' common content knowledge included mostly the knowledge of definition of the terms, using the terms, and solving or answering the questions related to generalization of patterns and operations with algebraic expressions.

In teaching of generalization of patterns, the teachers' knowledge of definition of the pattern concept appeared inappropriately or inadequately. To illustrate, Teacher A had called the names of the patterns as linear growth pattern and non-linear growth pattern correctly by giving examples appropriately in planning, however, her knowledge of definition of pattern appeared inappropriately in the instruction that she defined the pattern as the relations between the figures. On the other hand, Teacher B did not to use the term with its name as non-linear pattern in her explanations, instead of this, she stated the patterns with quadratic form as n^2 to point out the difference from linear patterns, and her knowledge to use terms about patterns correctly was not adequate to explain it. Another problematic situation related with pattern generalization was that Teacher B's knowledge of using terms about general term and general rule was inappropriate as she called 'n' both the general rule and the general term. Although Teacher B had inappropriate use of general term and rule concepts, she used her knowledge of inductive reasoning appropriately as she started specific examples and then she reached the generalization. Teacher B started the generalization process with representing arithmetical relationship for the first four or five terms and then connecting the relationship with asking what the n^{th} term was. This teacher's strategy for generalization of patterns is consistent with the literature related to pattern generalization. Radford (2008) describes the generalization process as including abduction, transforming, and deducing phases. Abduction phase is recognizing the relationship and commonality in the pattern, transforming is using this relationship to find other terms by extending the pattern, and deducing phase is generalizing pattern and finding a general rule algebraically. Teacher B asked the students what the difference was between the terms first, then she extended the patterns with getting the following terms and then she wrote the relationship algebraically.

In teaching of operations with algebraic expressions, while Teacher A's knowledge of using terms related with algebraic expressions was appeared inappropriately, Teacher B's knowledge to use term, like term, constant term, and coefficient concepts appeared correctly. Teacher A's knowledge of the constant term concept was incorrect as she did not accept it as a term in an algebraic expression. However, the two teachers used the unknown and variable concept as if they had the same meaning and they did not differentiate them because their knowledge of these terms was inadequate. There are different uses of variables in generalized arithmetic and in solving problems. Variables are parameters in the study of the relationship of the patterns, while they are unknowns or constants in solving equations in the problem contexts (Usiskin, 1988; 1995).

In general, as Teacher B solved the questions and problems correctly with adequate explanations, her knowledge of calculating an answer and solving problem appeared effectively, however, Teacher A could not solve or answer the questions related to algebraic expressions correctly since she ignored what was given or asked in the several points of the flow of the instruction. Teacher A answered what she found instead of what was asked in the question. To illustrate, she found 2n+10 correctly but she expressed what it referred to incorrectly. Another example was that she did not simplify 3x(X+Y) + 3(Y+7Z) expression and, thus she indicated that it did not match with any expressions equally. In other words, she missed some points, thus she did not answer the questions correctly and completely.

5.1.1.2. Specialized Content Knowledge (SCK)

This study revealed that the two middle school mathematics teachers' specialized content knowledge included mostly the knowledge to connect the topic to prior topic, to use representation, to provide mathematical explanations, and to use the mathematical language (technical language and notations) related with generalization of patterns and operations with algebraic expressions.

In teaching of generalization of patterns, the teachers used the tabular representation with focusing on the arithmetical relationships in the rows of the table to underlie the relationship in the pattern to conceptualize generalization effectively. As Warren and Cooper (2008) noted that finding the relationship between the position number and the term in the rows of the tables can provide the usage of the tabular representation of patterns effectively, rather than only writing the numbers at the row and seeking the difference between the previous one. However, the teachers used the figures or manipulatives merely to provide visuality and they did not make any explanations about the change of the figures and the relationship among the numbers. Teacher A also had stated to use graphical representation to show the change of the terms in planning, but she could not explain the use of this representation for the pattern and she could not use it in the instruction. Akyüz et al. (2009) concluded that transforming the relationship of the pattern between tabular, graphical, and numerical representations improved students' understanding of generalization. In this study, Teacher A especially had suggested using the graphical representation in pattern generalization, but she could not use it since she did not explain and know how to use it.

The teachers' explanations for the relationship of the patterns appeared differently. While Teacher A provided an explanation as she had investigated the relationship among the output values. This situation showed her lack of content knowledge about functional thinking in generalization. She also emphasized the difference between terms and adding a number to get the general rule throughout the instruction for the generalization of linear growth patterns. Consistent with this finding, Kutluk (2011) found that the teachers seek the relationship between input or output values, and it shows inadequate content knowledge. The reason for this lack of knowledge could be that the teacher did not have relational understanding of functional relationship (Wilkie, 2014). On the other hand, Teacher B always emphasized that the relationship between the position number and the terms appropriately for functional thinking, and she also explained the function of general rule with showing the same arithmetical procedure worked for the terms. Smith (2008) explained functional thinking as "representational thinking that focuses on the relationship between two (or more) varying quantities" (p. 143). Generalizing patterns required the functional thinking between the input and output values (Greenes, Cavanagh, Dacey, Findell, & Small, 2001; Moss, Beatty, Barkin, & Shillolo, 2008; Rivera, 2010). Beside this, Wilkie (2014) described generalizing a geometric pattern and explained generalization strategies as the components of teachers' SCK for teaching pattern generalization. Wilkie (2014) indicated that the teachers should have the knowledge of two methods that are co-variational based on recursive reasoning and correspondence based on the functional relationship. The teachers did not identify any of the strategies in this study, but Teacher B investigated the relationship between the position numbers and the terms based on recursive reasoning.

Besides the generalization of linear growth patterns, the teachers generalized non-linear growth patterns in the instruction. These types of patterns, which are appropriate for the middle school students level, could be generalized with quadratic expressions (MoNE, 2013) and writing quadratic expressions is one of the features of algebraic thinking (Blanton & Kaput, 2005). However, Teacher B gave one example, and Teacher A presented several examples about non-linear growth patterns in problem contexts from literature. Kieran (2007) stated that these generalizing activities in problem context are generational and transformational activities and important for developing middle school students' algebraic thinking. Although Teacher A presented several examples, she had difficulty in generalizing the non-linear growth patterns with inductive reasoning and she indicated the use of n^2 in order to get the general rule inadequately and she could not guide the students any more. Gierdien (2012) proposed a systematic way for generalizing matchsticks problems that formed as triangle, squares, and pentagons. The researcher asked to generalize the linear growth patterns respectively first, and then asked to sum all the matchsticks for the first two figures, then the first three figures and so on, and finally asked what the sum for m-gons in this systematic approach within exponential growth was. This method can also help linking linear growth patterns with non-linear growth patterns, and the teachers can use it in the instruction.

The teachers used notations (operation signs, equal sign and exponential form) appropriately with their adequate knowledge while explaining the generalization process. However, Teacher A's inappropriate use of mathematical language existed in the instruction. She used unknown and variable concepts changeable without their function, explained the relationship of the pattern changing for linear growth patterns incorrectly, and emphasized the relation increased in non-linear growth pattern. But, she corrected the last situation, and explained that the increment increased regularly as the property of non-linear growth patterns. On the other hand, Teacher B made errors in the representation of arithmetical expressions while using notations, her knowledge how mathematical technical language was used appeared inappropriately as she used the equal sign incorrectly. Carpenter, Franke and Levi (2003) indicated that the use of equal sign with integers as operations on the left and the result on the right could cause students' misconceptions about the equal sign. The understanding of the function of the equal sign as a relation between the expressions on the left side and right side can provide the students with transferring arithmetic to algebra and these students can have good performance in solving equations (Alibali, Knuth, & Hattikudur, 2007; Knuth, Stephens, McNeil, & Alibali, 2006).

In teaching addition and subtraction of algebraic expressions, the teachers' knowledge of how to use mathematical representations as algebra tiles was inadequate as they used them to provide visuality and concreteness. They did not explain the area concept that was underlying the idea of using algebra tiles. However, combining geometrical reasoning with other learning areas is one of the targets of mathematics teaching (NCTM, 2001). Besides, Teacher A could not show the subtraction operation with the tiles because of lack of her content knowledge.

Although Teacher A did not correct the students' incorrect use of the equal sign or determining the term without the sign, she used the notations appropriately by herself in the instruction of operations with algebraic expressions. In contrast to this, Teacher B remedied the errors in the use of the equal sign appropriately that she explained and corrected the students' incorrect writings when the students added the like terms separately in another place and they did not write as the result. This remediation can provide the students an understanding of the function of the equal sign as relationship between the first algebraic expression and the simplified expression (Alibali et al., 2007). Besides, Teacher B used mathematical language appropriately. She expressed the kind of the variable with apple and pear analogy, and the procedures with net worth concept by using assets and debts concepts properly in order to teach addition and subtraction. Similar apple-pear analogy was used by one of the novice teachers and this method was called as fruit salad in Tirosh et al.'s (1998) study and it provided an explanation of the addition and subtraction of different variables. However, Tirosh et al. (1998) observed this teacher's instruction and indicated that the students confused in simplifying algebraic expressions. In contrast to this finding, the teacher in this study used this analogy when the students had difficulty with understanding of simplification and then the students answered the questions correctly. But, Tirosh et al. (1998) asserted that this method did not provide relational understanding of operations with algebraic expressions. However, Ojose (2015) suggested the use of this type of real-life examples such as apple and orange as a model to teach adding unlike terms. To illustrate, Teacher B also had suggested using the papers as concrete materials as they could be interesting and motivating for students to visualize the algebraic expressions in planning, but she did not use them. Filloy and Sutherland (2006) asserted that the use of concrete models such as tiles, or apple-pear analogy provides translation to abstract level as algebraic expression to facilitate students' learning. The researchers described abstraction level as using of procedures and operations for simplification of algebraic expressions.

In teaching of multiplication algebraic expressions, the teachers' connection with prior topic in order to explain the distributive property of multiplication appeared appropriately in different ways. Teacher A connected multiplication with repeated addition to explain this property. Hallagan (2004) developed tasks to explore the equivalence of expressions using area modeling such as 4(s+1) and 4s+4 and the teacher implemented these types of activities in the instruction. The teacher indicated that the use of models to explain the distributive property developed the students' understanding of equivalent expressions. However, Teacher A did not use area modeling to explain multiplication of algebraic expressions. On the other hand, Teacher B connected with the application of this property in integers appropriately. Teacher B's method to show the application of the distributive property is also suggested by Ojose (2015) that exemplifying with integers first and then connecting algebraic expression help the students' learning. Actually, MoNE (2009, 2013) suggests the use of algebra tiles for teaching multiplication of algebraic expressions, but the teachers' conceptions about algebra tiles appeared differently and inadequately in the current study. While Teacher A used the tiles focusing on repeated addition with counting the number of tiles, Teacher B did not use them in her instruction though she emphasized the importance of using algebra tiles with connecting the area concept in planning. Thus, the two teachers did not use algebra tiles effectively and did not link algebraic and geometric representation to underlie the multiplication idea. The knowledge of using models (algebra tiles) is also important for multiplication to explain the distributive property in equivalent expressions and the teachers should teach this rule conceptually in the multiplication of algebraic expressions (Hallagan, 2004).

5.1.2. Pedagogical Content Knowledge

The teachers' pedagogical content knowledge (PCK) is explained under three headings: knowledge of content and students (KCS), knowledge of content and teaching (KCT), and knowledge of content and curriculum as the components of PCK of MKT model.

5.1.2.1. Knowledge of Content and Students (KCS)

This study revealed that the two middle school mathematics teachers' knowledge of content and students included mostly the knowledge to anticipate where and how the students have difficulty and possible misunderstandings, to understand the needs and difficulties of the students, to predict what the students find interesting and motivating, and to define common misconceptions about the contents.

In teaching of generalization of patterns, the teachers' knowledge to anticipate the difficulties the students have in generalizing pattern algebraically and particularly non-linear patterns appeared clearly. As Ebersbach and Wilkening (2007) showed that the students had difficulty with generalizing non-linear patterns rather than linear patterns. The students could extend the pattern as near generalization, but they had difficulty with getting the general rule algebraically as far generalization (Jurdak & El Mouhayar, 2014). Steele and Johanning (2004) found that the students with well-connected schemas could generalize algebraically, while the students with partial formed schemas had difficulty with using algebra in generalization. Even, older students, such as 9^{th} graders, also had difficulty in generalizing patterns algebraically too (Becker & Rivera, 2005; Çayır & Akyüz, 2015; Stacey, 1996). The reason of this difficulty might be the students' thinking of the relationship between the consecutive terms (Harel, 2001).

However, the explanations of generalizations were provided by the teachers in this study were appeared differently as in the following. Teacher A's explanations for generalization of patterns were troublesome as she could not anticipate possible misunderstandings. Teacher A guided the students to multiply the difference and position number, and to add a number to find the first term in order to get the general rule throughout the instruction. However, for another pattern, Teacher A explained that adding or subtracting some numbers for the generalization was not required for every generalization. She had first given a rule to get the general rule, but then she explained that it was not valid for all patterns. Her knowledge of anticipating the misunderstanding that raised with pattern generalization being studied in class appeared inadequately as she did not realize the students' misconceptions about exploring the relationship of the pattern. Another method that she proposed for the students was adding the difference between the terms to the general term. This situation also caused misconceptions in the learning of some students. These generalization ways of Teacher A can show her lack of knowledge of the content and students. The studies show that the teachers did not have adequate knowledge to explain the strategies that students used for generalization of patterns (Baş et al., 2011; Callejo & Zapatera, 2016; El Mouhayar & Jurdak, 2013). As indicated in the studies, Teacher A's inadequate knowledge of students' thinking could be derived from her strategies that showed her lack of content knowledge. On the other hand, Teacher B was seeking the relationship between input and output values in the pattern, she showed and asked the students to use the shortcut that was multiplying the difference between the terms and n, then to add or subtract a number to find the first term if the difference was constant throughout the instruction. This situation could cause the students to perceive it as a rule and also to consider the first term only while generalizing it, and this could prevent that understanding the rule was valid for all the terms of the pattern as Wilkie (2014) reminded us. Actually, the teachers' generalization methods were similar, they both extended the pattern based on the difference between the terms and found an algebraic rule considering the first term. However, the teachers in this study were able to reach the correct general rule algebraically in general. This finding did not coincide with the previous studies and they asserted that in-service teachers (Kutluk, 2011; Wilkie, 2014) and pre-service teachers (Barbosa & Vale, 2015; İmre & Akkoç, 2012; Kirwan, 2015; Magiera et al., 2013; Rivera & Becker, 2007; Tanışlı & Köse, 2011) had difficulty in using and justifying a general rule algebraically.

The teachers' knowledge to understand the needs and difficulties was adequate to explain the missing points or to respond the questions asked by students. However, the two teachers used the tabular representation and numerical reasoning to generalize the patterns, so that this situation might have made it difficult for the students who could understand figural reasoning easier. Because lower graders (grade 4 and 5) used mostly numerical reasoning, and upper graders (grade 10 and 11) used mostly figural reasoning in pattern generalization (El Mouhayar & Jurdak, 2016). Thus, encouraging the students to use both figural and numerical reasoning supports the teaching of the generalization of patterns. Although the teachers predicted using figural patterns, that the students found interesting and motivating for understanding the relationship in the pattern, would improve their understanding, they did not mention figural reasoning in the generalization of figural patterns. As Kutluk (2011), and Baş et al. (2011) indicated in their studies, the teachers used only numerical reasoning in generalization of patterns and did not use figural reasoning in figural patterns. However, many studies suggested using figural reasoning in order to develop students' understanding of relationship with considering the changes between the figures (Barbosa & Vale, 2015; Rivera & Becker, 2005; Walkowiak, 2014). Actually, before transforming the number of figures to the table, the use of figural reasoning as asking to the students how the units in the figures get together and how the relationship is based on provides to get the rule of the pattern conceptually and correctly (Thornton, 2001). The reason of the teachers' not using figural reasoning might be derived from their teacher training or lack of content knowledge. İmre and Akkoç (2012) stated that the pre-service teachers did not use different representations particularly figural representation before the evaluation and discussion of their teaching with their instructors. After they were guided to use different representations for the patterns, they started to use tabular and figural representations. For this study, the teachers used tabular and figural representations, but they always used numerical reasoning similar to pre-service teachers' use of recursive strategy for the general rule of numerical pattern in Zazkis and Liljedahl's (2002) study. The teachers in this study used the figures to provide visuality for the students merely. However, using figural features in generalizing, and numerical features for verifying the generalization support getting generalization (Kirwan, 2015).

Beside the use of figural patterns and figural reasoning to develop students' learning, Blanton and Kaput (2001) asserted that the teachers realized that the researchers' suggested tasks related with generalizing and required creating a pattern in a problem context provided an improvement for students' algebraic thinking of generalization. Similarly, for this study, the teachers thought that the suggested activities were interesting and motivating for the students since experiencing different activities could support students' understanding.

Furthermore, the teachers should have the knowledge of students' strategies to generalize patterns. Magiera, van den Kieboom and Moyer (2013) adapted the features of algebraic thinking based on Driscoll's (2001) description. They suggested that the teachers should have knowledge of students' thinking as "organizing information, predicting patterns, chunking information, describing change, and justifying a rule" (p. 95). More particularly for pattern generalization, Wilkie (2014) described one of the KCS items which was the knowledge to explain students' recursive or explicit (algebraic) strategies for generalization of patterns, in her study. Even, Towsend (2006) suggested using spreadsheets to facilitate the students' understanding of explicit strategies after they used recursive thinking. More particularly for figural pattern generalization, Rivera and Becker (2008) proposed constructive and deconstructive strategy for middle school students. While constructive strategy is about constructing the relationship with counting the part separately in figures, deconstructive strategy is about constructing the relationship with considering the overlapping parts in the figures. Thus, considering the knowledge to understand the students' needs of KCS, the teachers should have the knowledge of these strategies. Callejo and Zapatera (2016) also proposed three mathematical elements related with pattern generalization that pre-service teachers should identify. It could be suggested that in-service teachers could also identify these elements. These elements are; students' doing near generalization considering the number or the location of figure, generalizing algebraically based on functional relationship, and determining the position number for a given term inversely. The researchers found that the pre-service teachers identified these elements in students' generalization, but they could not explain their reasoning, as indicated in previous studies (Baş et al., 2011; El Mouhayar & Jurdak, 2013; İmre & Akkoç, 2012; Magiera, van den Kieboom, & Moyer, 2013).

In teaching of addition and subtraction of algebraic expressions, Teacher A's knowledge to anticipate the misunderstandings that might arise with studying on like terms was inadequate, since she only emphasized the kind of variable without considering its power. She explained the like term concept stating that it had variables with same or different coefficients, however, this definition was inadequate as x and x^2 were the expressions that had same variable but they were unlike terms and the definition could cause misconceptions. As Tirosh et al. (1998) indicated that the novice teacher who explained like term as having x was not aware of the students' misconceptions about adding unlike terms and the students had difficulty in simplification of algebraic expressions. Teacher A also did not show the signs of the constant term while determining it. On the other hand, Teacher B explained the like term concept, the procedures of addition operation, and the sign before the parenthesis adequately while studying on the worksheet at the beginning of the instruction with her knowledge of understanding the difficulties of students in operating with the parenthesis algebraic expressions and considering the needs of the students. The studies suggest emphasizing the brackets while teaching operations for elementary level students (Hoch & Dreyfus, 2004; Marchini & Papadopoulos, 2011), however, Livneh and Linchevski (1999) point out the importance of understanding the structure sense of expressions for conceptual understanding.

In teaching of multiplication of algebraic expressions, the teachers' knowledge to anticipate the students' difficulty in the application of distributive property appropriately. Since the students do not have adequate understanding of the structure of expressions, they had difficulty in understanding the concept of distributive property (Kieran, 1989). Teacher A had suggested using algebra tiles to overcome the students' difficulties in planning as they could provide for teaching the distributive property conceptually. However, Teacher B had instructed on how to do the exercises that required the application distributive property procedurally.

In planning process, the teachers also stated common misconceptions that the students could have based on their previous experiences, such as the students' thinking of 4.x as two-digit number, or x.x as 2x, or ignoring 1 as the coefficient of x in 3x-x, or adding unlike terms. This situation is one of the misconceptions that Seng (2010) identified. The other misconceptions are incorrect order of operations, addition of integers incorrectly, multiplication of negative integer before the bracket incorrectly, ignoring the multiplication of second term in the bracket in using distributive property, and writing the result of addition of exponential forms incorrectly (e.g. $7a^2$ as 14a) (Seng, 2010). MacGroger and Stacey (1997) and Seng (2010) proposed the reasons for these difficulties of students. Intuition about new notations, misapplication of rules, operating with negative integers and inadequate conceptions about the structure of algebraic expressions can cause the difficulties that the students encounter. The teachers should have the knowledge of these common misconceptions and possible reasons of them as KCS for teaching simplification of algebraic expressions. They should design their lessons with this knowledge. Lastly, in the instruction of multiplication of algebraic expressions, Teacher B especially emphasized the difference between of area and perimeter concepts. Her knowledge to anticipate the misunderstandings that might arise studying these concepts in solving problem related geometry were indicated in literature (Ashlock, 2001; Van de Walle et al., 2013). The students can confuse the perimeter and area concepts, and the reason might be their formulas or covering the same region (Van de Walle et al., 2013).

In general, Tirosh et al. (1998) pointed out that the teachers should design lessons for teaching algebraic expressions considering students' difficulties. The experienced teachers in their study explained like terms and unlike terms, and used challenging strategies such as substitution, order of operations, and going backward in teaching simplification of algebraic experiences. Thus, the knowledge of these strategies also could be one of descriptions of KCS for operations with algebraic expressions.

5.1.2.2. Knowledge of Content and Teaching

This study revealed that the two middle school mathematics teachers' knowledge of content and teaching included mostly choosing examples to start and take the students deeper into the content, deciding when to pause and asking a new question in classroom discussion, addressing the students' errors and remedying them, identifying different methods instructionally, and evaluating the advantages or disadvantages of representations.

In teaching of generalization of patterns, Teacher A's knowledge to choose examples and activities to start with simple linear growth figural patterns was appropriate, and then she took the students deeper into the pattern generalization with linear growth patterns and non-linear growth patterns from easy to difficult examples appropriately. On the other hand, Teacher B's knowledge to choose examples and activities to start with non-linear patterns and then to continue with the linear growth patterns to take students deeper into the pattern generalization were not appropriate developmentally. The teachers used the examples and activities from the textbook and they did not create the activities for the instruction. Thus, as Wilkie (2014) indicated, the teachers did not have adequate experiences to create activities related with generalization of patterns in order to give the idea of functional thinking such as function machine. This view is supported by Kaput (1999) who suggested using meaningful contexts from real life such as cost of a product and representing the relationship in them with diagrams, tables, graphs, and algebraic notations to teach functional thinking to the middle graders. Although Teacher B did not create new activities to teach pattern generalization, she knew how to build the students' understanding of generalization effectively with connecting the general term with the inductive reasoning method within the activities of the textbook. She also addressed and remedied the students' errors in the generalization process by stating the relationship which was investigated between the position number and the term. Wilkie (2014) described KCT item for generalization of patterns by addressing a student's mistake in generalization and identifying appropriate strategies for conceptualizing functional thinking, also Teacher B addressed the errors and explained the functional relationship in the pattern appropriately.

In teaching of addition and subtraction of algebraic expressions, the teachers' knowledge to choose examples and activities to start and then to take the students deeper into the topic were appropriate. Teacher A started with connecting generalization of patterns with the prior topic, and then connected the like term concept with real life situations. She used problems related to real life situations for teaching, however, her knowledge to choose which examples to use to take the students deeper into the content was inappropriate as she represented the number of known items with algebraic representations such as 3e for 3 eggs. Besides, she used models to explain how the like terms were added to take the students deep into the content, but she could not explain the subtraction operation with models. Similarly, Teacher B started with connecting generalization of patterns as prior topic with addition and then connected the properties of arithmetical operations that the students had learnt before. To support the students' learning of algebraic expressions, one of the commonly suggested approaches is connecting arithmetic with algebra (Livneh & Linchevski, 2007; Subramaniam & Banerjee, 2004; Warren, 2003). Teaching the procedures of operations with algebraic expressions connecting the arithmetic procedures based on the similarity of the structure of algebraic expressions and arithmetic expressions improves the students' understanding of rules and procedures in operations, simplifications of algebraic expressions, and equivalence of expressions (Banerjee & Subramaniam, 2012). Then, she emphasized the like term concept with using different colored unit cubes to take the students deep into addition and subtraction as Filloy and Sutherland (2006) suggested the use of concrete models to explain adding unlike terms in the transition to abstract algebra. In the development of the instruction, the knowledge type related to the teachers' leading of the discussions was appropriate to decide how to add the terms to further students' learning to make the students feel the requirement of like terms for addition within the activities.

In teaching of multiplication of algebraic expressions, the two teachers started the instruction with connecting generalization of patterns appropriately. Then, to take students deeper into the content, Teacher A chose repeated addition to represent multiplication by using models to show how the algebraic expressions were multiplied. However, she did not use the models appropriately, and thus she could not understand if they supported students' learning or not. On the other hand, Teacher B focused on the application of distributive property with connecting integers and answering the questions procedurally. She did not use algebra tiles to model multiplications although she emphasized the idea of the area of quadrilaterals in planning. Thus, similar to Teacher A, Teacher B's knowledge to evaluate the instructional advantages and disadvantages of algebra tiles as representations was inadequate. In solving geometry problems, Teacher B emphasized the difference of the perimeter and area concepts to address students' possible errors effectively and to remedy it in the instruction, and this method is also suggested by Ashlock (2001) to support the students' learning.

5.1.2.3. Knowledge of Content and Curriculum (KCC)

In the curriculum perspective, the teachers' knowledge of objective that belongs to content and curriculum for designing lesson was essential and adequate. Besides, Teacher A's knowledge to present the instruction properly with recalling students' prior knowledge, and to emphasize what was learnt in previous grades was adequate. In contrast with these findings, Kutluk (2011) showed the teachers' inadequate curriculum knowledge for teaching pattern generalization. However, Wilkie (2014) proposed that the teachers could evaluate the content descriptions with scaling as knowledge of curriculum. In this study, the teachers did not evaluate the objectives, they merely knew them and designed their lessons with centering them in their lesson plans. Thus, the knowledge of evaluating objectives, content descriptions, and sample activities related to the algebra topics in the curriculum could be considered as a component of KCC that the teachers should need to have.

5.1.3. The Influence of Subject Matter Knowledge (SMK) on Pedagogical Content Knowledge (PCK)

This section presents the influence of subject matter knowledge (SMK) on pedagogical content knowledge (PCK) of Teacher A and Teacher B based on their planning and instruction by compare and contrast method. The findings of this comparison were discussed with literature and within the sub-domains of MKT model in the context of generalization of patterns and operations with algebraic expressions.

The examples in the instructions showed that the teachers' conceptual and adequate CCK and SCK affected their teaching positively. When the teachers' had strong subject matter knowledge, they took into account the students' thinking (KCS) while teaching and they used teaching methods effectively (KCT). Even (1993) also supported this finding and she suggested developing prospective teachers' subject matter knowledge within the constructive perspective and using the relational subject matter knowledge in pedagogical decisions for teaching. Similarly, the large scale projects, COACTIV project (Krauss, Baumert, & Blum, 2008) and TEDS-M study (Blömeke & Kaiser, 2012) found that content knowledge and pedagogical content knowledge were correlated strongly and positively. It could be implied that when the teachers had strong SMK, they also had strong PCK. The quantitative results of these projects could give an overview about the relationship between SCK and PCK, and this study's findings were also consistent with them. However, (Depaepe et al., 2015, p. 82) asserted that "content knowledge is necessary but not sufficient for PCK". Besides, when the prospective teachers had the training special to their teaching area, they had better CK and PCK than other prospective teachers who did not have this type of training for teaching (Blömeke & Kaiser, 2012). Similarly, in Krauss et al.'s (2008b) study, the connection of CK and PCK was an indicator for expertise of mathematics was found. More particularly, they concluded that CK and PCK of the mathematics teachers, who took a deep mathematics content knowledge training, were interwoven, while CK and PCK of the teachers who did not take such a training were distinct. What caused the lack of PCK of the mathematics teachers was the lack of CK.

Moreover, this study examined the mathematics teachers' knowledge for teaching for the particular contents, generalization of patterns and operations with algebraic expressions, and within the subdomains of MKT in detail. Based on the findings, the teachers' SCK was the knowledge of using multiple representations and mathematical language was weak. They knew one or two representations and did not know other representations and they could not link among them, or they could make error in using mathematical language while explaining the procedures. In connection with SCK, when the teachers in this study had relational SCK, they could consider possible students' difficulties and misconceptions. This finding was consistent with one of the findings of Jacobs, Lamb, and Brown's (2010) study. They concluded that the lack of students' thinking or content knowledge could be the reason of the teachers' difficulty with understanding students' thinking. In contrast to it, Wilkie (2014) found that SCK and KCS did not support each other and there was not a relationship between them. Wilkie (2014) indicated that the teachers could find the general rule (SCK) but they could not consider the students' thinking of generalization pattern (KCS). In relation to SCK, Wilkie (2014) also asserted that the teachers' KCT was lower than their SCK. Most of the teachers could get the general rule correctly by themselves (SCK), but they could not understand the students' misunderstandings (KCS), thus they could not suggest an effective method to remedy the students’ errors with reasoning (KCT). The finding related to KCT was consistent with this study’s finding that the teachers did not have adequate SCK such as using figural reasoning for figural patterns or using algebra tiles to teach distributive property, thus they could not develop new teaching strategies or create activities to support the students who had difficulty in understanding the concepts (KCT). For KCS knowledge domain, especially Teacher B had adequate knowledge to understand the difficulties and anticipate possible misunderstandings of the students through her experiences, but she could not identify any strategies such as recursive or explicit that the students could use, similar to the teachers in Wilkie’s (2014) study. Thus, the teachers’ knowledge of algebraic language to interpret the students’ solutions was inadequate. The reason for this might be the lack of algebraic thinking ability of the teachers. Magiera et al. (2013) concluded that the pre-service teachers did not investigate the features that they did not use in generalizing pattern, thus their algebraic thinking ability affected their interpretation of the students’ algebraic thinking. In corresponding with the finding, El Mouhayar and Jurdak (2013) showed that the in-service teachers did not have adequate knowledge to explain the students’ strategies and they explained counting or drawing as strategies that were not related with the concept of variable. This might indicate that the lack of content knowledge of the teachers limited their knowledge of the students’ thinking (Baş et al., 2011).

5.2. Classroom Practices and Teacher Knowledge

This section presents teaching practices of Teacher A and Teacher B based on their planning and instruction by compare and contrast method. The findings of this comparison was discussed with literature in the context of generalization of patterns and operations with algebraic expressions. First, the practices of the teachers were explained respectively, and then the relationship between their knowledge and practices was explained based on the two teachers’ instructions in the following paragraphs.

Practice is defined as "core activities (within mathematical domain and appropriate grade levels) that could and should occur regularly in the teaching of mathematics" (Franke, Kazemi & Battey, 2007, p. 249). Considering this explanation, the practices included the teachers' purposeful actions and their knowledge was examined within these practices. However, the planning practices could not be extracted from the planning process, since the teachers did not prepare lesson plans before this study and they generally followed the textbook.

The extracted practices from Teacher A's instructions were; 1) choosing an example or activity to start teaching the topic with connecting the topics from prior years, and 2) discussing on the activity related to the topic, 3) choosing the examples or activities to use to take the students deeper into the topic, 4) implementing the suggested activities, 5) doing exercises related to the topic from textbook and workbook, and 6) presenting problems that combine knowledge related to other topics. On the other hand, the extracted practices from Teacher B's instructions were; 1) connecting the topic to the topics taught in prior years, 2) discussing on the activity related to the topic, 3) choosing the examples or activities to use to take the students deeper into the topic, 4) implementing the suggested activities, and 5) solving the questions and problems related to the topic from different resources. As it is seen, there are similarities and differences between the two teachers' practices. The number of practices from Teacher A's instruction is slightly more than Teacher B's. However, the number of practices did not indicate stronger knowledge in this study in general. Although Teacher B had less practice, she had stronger conceptual knowledge with connections within concepts and procedures (Skemp, 1978). Aslan-Tutak and Ertaş (2013) also noted that the pre-service teachers' conceptual understanding had a positive impact on their teaching style. Teachers' perceptions about teaching were not conceptual and they limited their teaching with the experiences in their learning. This situation was a contradiction of the knowledge with the practice in algebra teaching and it was one of the dilemmas about teacher knowledge and practice of algebra (Doerr, 2004). Thus, the reason for Teacher A's application of more practices might be the teaching style that she learnt or she could want to do what the textbook suggested. Nevertheless, this was not an indicator for having more knowledgeable for this study.

More particularly, the relationship between teachers' knowledge and practice was examined with respect to the knowledge types and use of the knowledge, and each practice for this study in detail. For this first practice, SCK1 (related with connecting the prior topics) and SCK3 (related with developing a usable definition) knowledge type were emerged mainly from the instructions. While Teacher A started the instruction with an example or activity and then connected to prior topics, Teacher B preferred to connect to prior topics first and then continue with an example or activity. It might be more appropriate that connecting to prior topics first since the students can understand the underlying idea or concept of the topic and it can facilitate the students' learning. Fennema and Franke (1992) emphasized that the teachers represent the concepts with the students' existing knowledge, so they can support students' learning. Thus, the connection between the concepts might provide conceptualization of the students at the beginning of teaching the new concepts.

SCK2 (related with linking among representations) and SCK7 (related with using the representation effectively) knowledge types were emerged in the second and third practice, as the development of the instruction with the classroom discussion and the examples took the students deep into the content. However, the knowledge of the teachers was not adequate, thus their use of it was not appropriate. The lack of knowledge about representations affected their use negatively. It might show that the knowledge and the practice is correlated positively. In consistent with this finding, Lloyd and Wilson (1998) showed that effective use of representations for functions improved teaching and students' learning. Besides, Tchoshanov (2011) asserted that teacher knowledge with connecting models or representations mathematically affected teaching mathematics in middle school.

The next practice was about the classroom discussion related to the knowledge to lead the discussion (KCT5) and Teacher B's knowledge generally emerged appropriately and adequately. Since her knowledge was conceptual and she knew the important concepts to teach, she guided the students with asking a new question, using the students' answers, and making explanation when necessary. This situation also supported the positive relationship between the knowledge and practice, and conceptual knowledge provided the teacher with leading the discussion effectively. Brown and Smith (1997) noted that using questioning made the students to think within practice in classroom supported students' understanding of algebra.

The practice of implementing the suggested activities was common for the two teachers and SCK6 (related with the knowledge to provide mathematical explanations) emerged mainly in this practice. The teachers did not have difficulty in implementing them and they also noted that the activities and examples expanded their repertoire of examples for teaching the algebra topics. The teachers' practices for teaching algebra could be improved with the tasks that require to think algebraically (Blanton & Kaput, 2001). In this study, suggesting the examples and activities from the literature to the teachers provided for creating a practice and experiencing them with different types of questions. To illustrate, Teacher B indicated the usefulness of the equivalent activity to teach the application of distributive property, as Ayalon and Even (2013) suggested providing experiences for the teachers about algebraic expressions that required proving the equivalence of them could support students' learning. Particularly, Asquith, Stephens, Knuth, and Alibali (2007) stated the development of teachers' knowledge and practices support the students' algebraic thinking with connecting arithmetic.

The other practices were about presenting and solving the questions and problems for the two teachers. While Teacher A's practices were examined into two groups as doing exercises and presenting problems, Teacher B's practice was examined into one group as solving questions and problems. Teacher A did exercises from the textbook and workbook. On the other hand, Teacher B presented and solved different questions or problems from different resources such as test book, the textbook and workbook, or the previous national exam questions. She decided by herself where to take the questions and what type of questions to ask taking the national exams into consideration. Actually, Teacher A also presented the questions and problems that combined knowledge related to other topics to improve the students' knowledge with challenging them. However, she also had difficulty in solving them or explaining them to the students, thus this practice was called as "presenting problems" not solving. In these practices, most extracted knowledge types were SCK5 (related with the use of mathematical language), SCK6 (related with the knowledge to provide mathematical explanations), KCS2 and KCS3 (related with the knowledge of students' thinking). While Teacher B's knowledge of content and students emerged appropriately, Teacher A's knowledge of content and students emerged inappropriately or inadequately. Franke et al. (2007) indicated that the knowledge of students' thinking helped the teachers to develop their practice. Similar to KCS, CCK3 related to the knowledge to answer questions and solve problems correctly emerged in Teacher B's instruction appropriately, but Teacher A sometimes solved the problems partially correct. As explained for Teacher A's instruction, the lack of KCS and CCK caused the deficiencies in the instruction and teaching.

In sum, the following table shows the mostly extracted knowledge types in terms of practices:

Table 22 The practices and the knowledge types

Practices		**Mostly extracted knowlege types in the practices**
Teacher A	**Teacher B**	
1) choosing an example or activity to start teaching the topic with connecting the topics from prior years	1) connecting the topic to the topics taught in prior years	SCK1 (connecting the prior topics) SCK3 (developing a usable definition)
2) discussing on the activity related to the topic	2) discussing on the activity related to the topic	KCT (leading the classroom discussion)
3) choosing the examples or activities to use to take the students deeper into the topic	3) choosing the examples or activities to use to take the students deeper into the topic	SCK2 (linking among representations) SCK7 (using the representation effectively)
4) implementing the suggested activities	4) implementing the suggested activities	SCK6 (providing mathematical explanations)

Table 22 (Continued)

Practices		Mostly extracted knowlege types in the practices
Teacher A	**Teacher B**	
5) doing exercises related to the topic from textbook and workbook	5) solving the questions and problems related to the topic from different resources	SCK5 (using mathematical language) and SCK6 KCS2 and KCS3 (the knowledge of students' thinking)
6) presenting problems that combine knowledge related to other topics		

In general, the teachers had same practices and also same types of knowledge, but their practices were not effective in the same manner since the extracted knowledge emerged positively for Teacher B and negatively for Teacher A. To illustrate quantitatively, the most extracted knowledge types in all practices were SCK5 and SCK6, but about a quarter of these knowledge types emerged negatively in the instructions of Teacher B, while about a half of these knowledge types emerged negatively in the instructions of Teacher A. Teachers need to have a good conceptual mathematical understanding and also knowledge of students' thinking in order to design their lessons with effective tasks (Hiebert, 1997). Thus, teacher knowledge has an important role for effective teaching actions related with students' learning. This idea was consistent with the example from the instructions about knowledge of content and teaching (KCT8) related to addressing and remedying the students' errors and the knowledge which was extracted in Teacher B's instructions was adequate. Teacher knowledge including students' understanding and needs is important for effective teaching (Anderson, White, & Sullivan, 2005). Teachers' using their knowledge in pedagogical organization is important for instruction (Fennema & Franke, 1992). The appropriate and adequate knowledge has an impact on the teachers' practice to be effective and thus teachers' knowledge should be transferred for effective teaching to develop teacher practice (Doerr, 2004).

5.3. Implications and Suggestions

This study analyzed two middle school mathematics teachers' planning and instruction and revealed the mathematical knowledge for teaching generalization of patterns and operations with algebraic expressions. Considering the findings and literature, the implications and suggestions were presented in this section.

One of the implications of this study was the proposed knowledge types specific to algebra topics. The knowledge types that mathematics teachers should have for teaching generalization of patterns and operations with algebraic expressions were proposed in Appendix G and Appendix H. The proposed knowledge types can be used to develop algebra knowledge models and thus this study might contribute to mathematics education literature, and particularly teacher knowledge of algebra. One of the aims of the study was to fill a void of middle school teachers' algebra knowledge and the study proposed the knowledge types specific to generalization of patterns and operations with algebraic expressions that the teachers should have. In addition to these topics, teacher knowledge can be examined in other algebra topics such as equations, slope, and linear equations in middle grades in order to give a general overview for algebra knowledge of teachers.

The proposed knowledge types can also be used to train pre-service teachers and in-service teachers. Mathematics teacher educators can take into consideration the importance of teacher knowledge for developing teachers and their teaching practice, as the finding of the current study, to improve the teacher training programs. The mathematics teacher educators can design the method courses with specifying to learning areas such as algebra, numbers or geometry. Thus, they can give more time for each topic and the courses can include the use of specific representations, models, mathematical language, instructional methods and techniques to develop pre-service teachers' knowledge. These courses also can include cases from classrooms and students' solutions for questions and problems to improve pre-service teachers' PCK. The school practicum courses also can be designed to develop pre-service teachers' teaching practices. The teacher educators can observe their practices, evaluate with other pre-service teachers and support new methods and techniques when necessary as İmre and Akkoç (2012) implemented and suggested. Mutually, developing teaching practices can enhance pre-service teachers' specialized content knowledge (Aslan-Tutak & Ertaş, 2013).

Similarly, in-service teacher training can also be prepared to develop in-service teachers' knowledge of algebra. It was found that the teachers need to have content knowledge for teaching these topics, especially SCK, in this study. When they have this knowledge, their knowledge of content and students (KCS) and knowledge of content and teaching (KCT) can also develop. In connection with it, teachers' connected and conceptual subject matter knowledge has a positive impact on their teaching. Teacher knowledge has an important role in teaching practice as it was seen in the findings. Beside well connected subject matter knowledge, appropriate pedagogical content knowledge is required for effective practices. As another implication, the necessity of developing teachers' knowledge of using particular representations or materials, and the necessity of developing teachers' knowledge of students' thinking were emerged for effective teaching of mathematics. As a suggestion to develop in-service teachers, the teachers can come together and share good examples and experiences from their classes, ask for suggestions what they can do where they have difficulty in teaching. The mathematics education researchers also can inform the teachers about the results of the studies and suggested activities or examples and methods from the literature in these trainings. In this study, the researcher suggested the examples, activities, tasks, and methods from the literature to inform and develop the teachers. Based on the suggested activities that the teachers involved while they were revising their lesson plans, the practice of *implementing the suggested activities* was emerged. Thus, in order to enhance teachers' practices of teaching algebra, the activities, tasks, methods and techniques from literature can be presented to teachers.

In this study, the teachers prepared the lesson plans individually since they did not have enough time for preparing them together. Thus, lesson studies can be conducted to support in-service teachers, as a further research. Because these studies provide the teachers with preparing the lessons together and learning from each other, and then they can give feedbacks with observing their instructions to improve their teaching. Another suggestion can be conducting design research experiment in order to use the revision part effectively. In design research, the teachers can work together with the researchers, they design a learning trajectory for specific to mathematics topics and revise their lessons based on teaching and learning of the students. Thus, it provides teachers with expanding their teaching repertoire with examples, activities, and teaching techniques and so they improve their knowledge, as well. Actually, this study provided a need assessment for teacher algebra knowledge as a first step. As a next step, designing and implementing a learning trajectory about teaching algebra for pre-service teachers will be planned.

The proposed knowledge types can also inform the curriculum developers that the objectives may be detailed and developed in relation with the expectations from the teachers as knowledge types. To illustrate, as a NCTM standard (2000, p. 222), “represent, analyze and generalize a variety of patterns with tables, graphs, words and, when possible, symbolic rules’’ corresponds to SCK2 and SCK7 knowledge types for teaching generalization patterns in the current study. The arrangement of the objectives with this way may guide the teachers what and how to teach effectively. The definitions and descriptions of knowledge types for teaching generalization of patterns and operations with algebraic expressions from the current study also may shed light on forming the new objectives.

In this study, the knowledge types were proposed for teaching generalization of patterns and operations with algebraic expressions. There might be also different knowledge types within different context such as numbers or geometry. Because, there are other descriptions for knowledge sub-domains (CCK, SCK, KCS, KCT, and KCC) based on MKT model (Ball et al., 2008). On the other hand, Horizon Content Knowledge (HCK) for teaching algebra topics was not observed in the current study. Therefore, future research should also concentrate on the investigation of teachers’ knowledge at the mathematical horizon of algebra to help the development of students’ understanding of algebra within middle school grades.

In relation with MKT model, some undetermined situations were come across in using of the model in the analysis of data. As Ball et al. (2008) indicated, there was a boundary problem within the knowledge types in the current study. To illustrate, it was difficult to decide whether CCK2 (related to using terms) or SCK5 (related to using mathematical language) represented the data. Another problematic situation was that SCK6 was related to providing mathematical explanations and this knowledge was emerged in each data segment. Additionally, SCK5 was related to using mathematical language and notations, and this knowledge was also emerged in each data segment. Actually, there could be several knowledge types in same data segments. Thus, the knowledge types might be combined when they had same meaning, or they might be explained clearer when they refer different knowledge in MKT model.

As following investigations, teachers' planning practices also can be explored and in relation with it, the connection of planning practices and instructional practices can be examined. Since this study focused on teachers, the impact of teacher knowledge types on students' learning can be investigated to give an idea from student's aspect. Thus, experimental studies can be conducted and it also provides quantitative evidence for the research of teacher knowledge.

This study limited to two teachers' instructions to investigate teacher knowledge as the unit of analysis. The number of teachers can be increased, and quantitative research with developing CK and PCK test to get a generalization for teachers' knowledge of algebra.

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APPENDICES

APPENDIX A: THE OBJECTIVES IN THE CURRICULUM (2009)

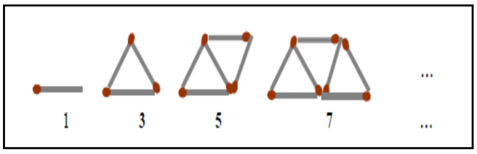
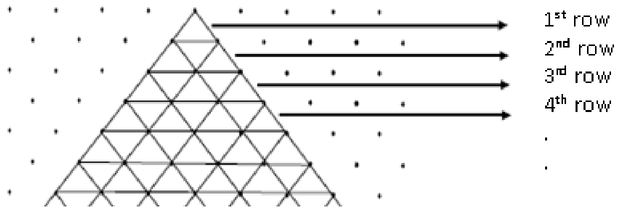
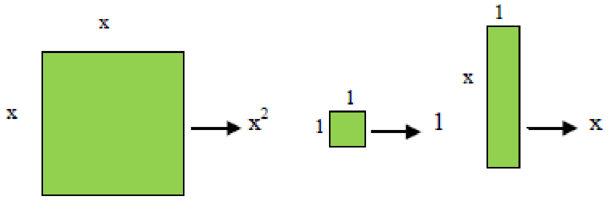
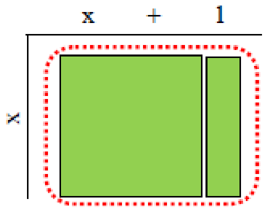
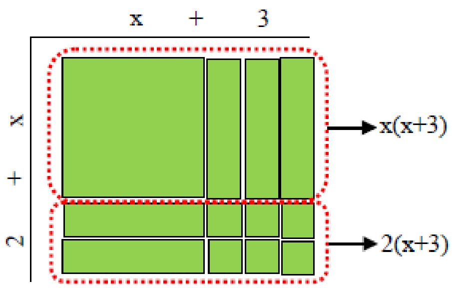
<table>
<tr><td>Sub-learning Domain</td><td>Objectives</td><td>The examples of activities</td></tr>
<tr><td>Patterns and Relations</td><td>Express using letters the relation in number patterns which are modelled.</td><td>One number pattern is selected: 1 3 5 7 …
The three models for this pattern and the tables in which the relation of the position number and the number of materials used is explained below.
1st model: Model with matchsticks representing 1 with 1 matchstick.


Table: The numerical relations between the position number and the number of matchsticks
<table>
<tr><td>The position number</td><td>The number of matchsticks</td><td colspan="2">The relation between the position number and the number of matchsticks</td><td></td></tr>
<tr><td></td><td></td><td>1st option</td><td>2nd option</td><td></td></tr>
<tr><td>1</td><td>1</td><td>1+(1-1)=1</td><td>2·1-1=1</td><td></td></tr>
<tr><td>2</td><td>3</td><td>2+(2-1)=3</td><td>2·2-1=3</td><td></td></tr>
<tr><td>3</td><td>5</td><td>3+(3-1)=5</td><td>2·3-1=5</td><td></td></tr>
<tr><td>4</td><td>7</td><td>4+(4-1)=7</td><td>2·4-1=7</td><td></td></tr>
<tr><td>...</td><td>...</td><td>...</td><td>...</td><td></td></tr>
<tr><td>n</td><td>...</td><td>n+(n-1)</td><td>2n-1</td><td></td></tr>
</table>
Ask students to express the numerical relations for 1st and 2nd options verbally. (For example; the addition of the position number and 1 less of it, or 2 less of 2 times of the position number)</td></tr>
<tr><td></td><td></td><td>2nd model: Take the terms that is equal to the number of equilateral triangles between the consecutive rows on the grid paper.

</td></tr>
</table>

<table>
<tr><td>Sub-learning Domain</td><td>Objectives</td><td>The examples of activities</td></tr>
<tr><td>Patterns and Relations</td><td>Express using letters the relation in number patterns which are modelled.</td><td>Table: The relation between the position number of row and the number of equilateral triangles
<table>
<tr><td rowspan="2">The position number</td><td rowspan="2">The number of equilateral triangles</td><td colspan="3">The relation between the position number of row and the number of equilateral triangles</td></tr>
<tr><td>1st option</td><td>2nd option</td><td></td></tr>
<tr><td>1</td><td>1</td><td>1+(1-1)=1</td><td>2·1-1=1</td><td></td></tr>
<tr><td>2</td><td>3</td><td>2+(2-1)=3</td><td>2·2-1=3</td><td></td></tr>
<tr><td>3</td><td>5</td><td>3+(3-1)=5</td><td>2·3-1=5</td><td></td></tr>
<tr><td>4</td><td>7</td><td>4+(4-1)=7</td><td>2·4-1=7</td><td></td></tr>
<tr><td>...</td><td>...</td><td>...</td><td>...</td><td></td></tr>
<tr><td>n</td><td>...</td><td>n+(n-1)</td><td>2n-1</td><td></td></tr>
</table></td></tr>
<tr><td>Algebraic expressions</td><td>1.Perform addition and subtraction operations with algebraic expressions.</td><td>The properties of addition operation are reminded before teaching operations with algebraic expressions. The concepts of term, coefficient, and similar term are emphasized.

Table: The properties of addition operation and their algebraic representations
<table>
<tr><td>The properties of addition operation</td><td>Algebraic representation</td></tr>
<tr><td>4+5 = 5+4</td><td>a+b = b+a</td></tr>
<tr><td>2+(4+5)= (2+4)+5</td><td>a+(b+c) = (a+b) = (a+b)+c</td></tr>
<tr><td>2+0 = 2</td><td>a+0 = a</td></tr>
</table>
The operations with algebraic expressions are modelled with numbers first.

Table: Operations and their algebraic representations
<table>
<tr><td>Operations</td><td>Algebraic representation</td></tr>
<tr><td>5+5+5 = 3.5</td><td>a+a+a = 3a</td></tr>
<tr><td>3.6+4.6 = 7.6</td><td>3c+4c = 7c</td></tr>
<tr><td>5.4-2.4 = 3.4</td><td>5d-2d = 3d</td></tr>
<tr><td>2+3.2+2 = 5.2</td><td>b+3b+b = 5b</td></tr>
</table>
The distributive property or grouping method are used while operating with algebraic expressions.
• 2x +3-x-4 = (2x-x) + (3-4) = x-1
• 4xy+9xy-2xy+5 = (4+9-2)xy+5 = 11xy+5
• $3y-6y^2+2y^2-5y = (3-5)y+(-6+2)y^2 = -2y-4y^2$</td></tr>
</table>

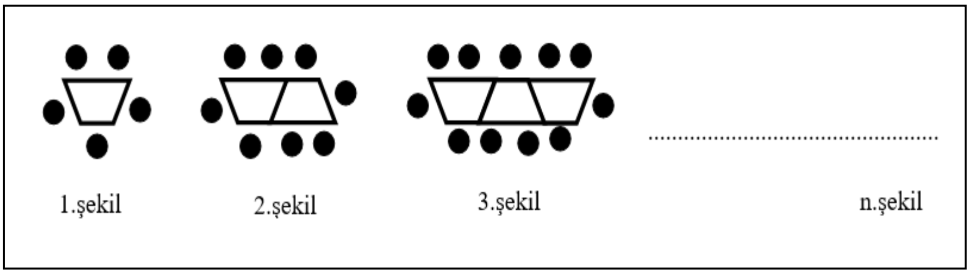
Sub-learning Domain	Objectives	The examples of activities
Algebraic expressions	2.Multiply two algebraic expressions.	The multiplication of x.(x+1) and (x+2).(x+3) using algebraic tiles;  are represented and the operations are modeled orderly as below.  The two algebraic expressions are multiplied by using the distributive property. x.(x+1) = x.x + x.1  The two algebraic expressions are multiplied by using the distributive property. $(x+2)\,(x+3) = x(x+3) + 2(x+3)$ $= x^2+3x+2x+6$ $= x^2+5x+6$

APPENDIX B: THE SUGGESTED ACTIVITIES RELATED WITH GENERALIZATION OF PATTERNS

The researcher suggested a pattern test, and three activities for supporting the instruction of generalization of patterns:

1. **Pattern test**

1.



Yukarıdaki yamuk şeklindeki masalara şekildeki gibi sandalyeler yerleştirilecektir. Her şekil, önceki şekle bir yamuk masa daha eklenerek devam etmektedir. Bu örüntüde 1.,2., ve 3., şekiller, örüntünün ilk üç şeklidir.

Örüntüdeki n.şekil ile çevresine yerleştirilebilecek sandalye sayısı arasında nasıl bir ilişki vardır?

Çözümünüzü açıklayınız:

2.Aşağıdaki tablo belli sayıdaki t-shirtin ne kadar olduğunu göstermektedir.

t-shirt sayısı	TL
3	10
5	16
9	28
21	64
: :	: :
n	?

Yukarıda verilen tabloya göre t-shirt sayısı ile değeri arasında bir ilişki vardır. Bu ilişkiye göre, n tane t-shirt için kaç TL ödenmelidir? Çözümünüzü açıklayınız:

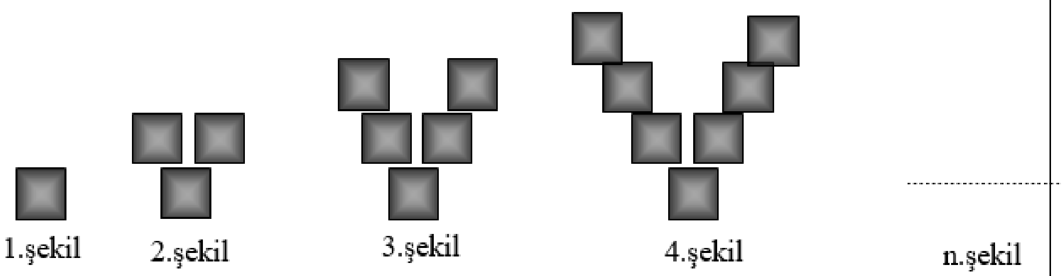
3.

1.sayı	2.sayı	3.sayı	4.sayı	5.sayı	....	n.sayı
3	**7**	**11**	**15**	**19**	**..........**	**?**

Yukarıdaki örüntüde yer alan sayılar bir kuralla oluşturulmuştur. Örüntüdeki sayıların oluşum kuralını bulunuz.

Çözümünüzü açıklayınız:

4.



Yukarıdaki örüntüde "V" harfinin değişik boyutları küçük kareler kullanarak oluşturulmuştur. Örüntüdeki herhangi bir "V" harfi ile küçük kare sayıları arasında nasıl bir ilişki vardır?

Çözümünüzü açıklayınız:

5. Zeynep'in doğum günü partisinde, zil ilk kez çaldığında bir arkadaşı gelmiştir. Bundan sonra çalan her zilde, gelen gruptaki kişi sayısı, bir önceki gelen gruptan 3 kişi fazladır. Aşağıdaki tabloda gelen kişi sayısı gösterilmiştir. Herhangi bir zil çalışını n kabul edersek, gelen kişi sayısını bulmak için kullanılacak genel ifade ne olmalıdır?

Zil sayısı	Gelen kişi sayısı
1	1
2	4
3	7
4	10
: :	: :
n	?

Çözümünüzü açıklayınız:

6.

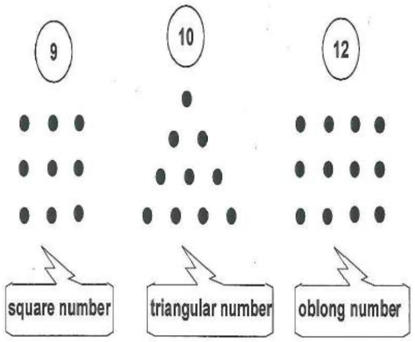
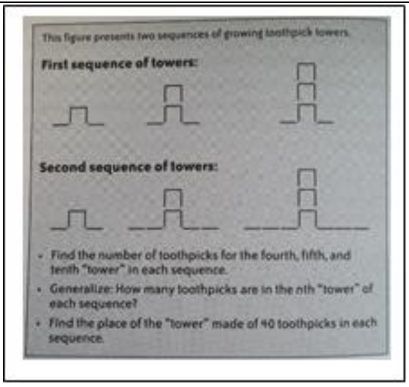
1.sayı	2.sayı	3.sayı	4.sayı	5.sayı	6.sayı	....	n.sayı
60	**55**	**50**	**45**	**40**	**35**	**..........**	**?**

Yukarıdaki örüntüde yer alan sayılar bir kuralla oluşturulmuştur. Örüntüdeki sayıların oluşum kuralını bulunuz.

Çözümünüzü açıklayınız:

2. Suggested Activities

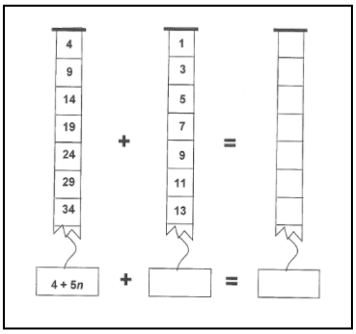
Name	Activity	The purpose of the activity
Handshake Problem	The handshake problem: Imagine you are at a huge party. Everyone starts to shake hands with other people who are there. If 2 people shake hands, there is 1 handshake. If 3 people are in a group and they each shake hands with the other people in the group, there are 3 handshakes. If 4 people are in a group and they each shake hands with the other people in the group, there are 6 handshakes. How many handshakes would there be if there were 10 people in the group? How many handshakes would there be if there were 100 people in the group? (Moss, Beatty, Barkin, & Shillolo, 2008).	The purpose of suggesting this problem was to enable writing the quadratic formula for generalization. This type of question was appropriate for 7th graders as the curriculum indicated. This problem has a context like shaking hands in a party that it can be a real – life situation and the students also could act out in the classroom while exploring the relation in the pattern. The teachers did not use this problem in their lessons.

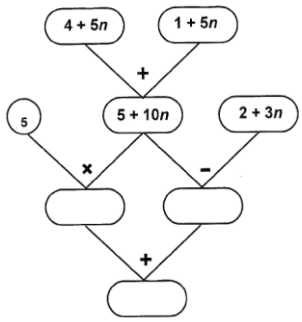
Dot patterns	 There are questions about the number patterns above. For example; Is there a rule? How you see this rule? Is 144 a square number?	The researcher took course about algebra teaching by Martin Kindt in Utrecht Summer School 2014 before data collection. Martin Kindt prepared the activities about algebra in "Positive algebra – A collection of productive exercises". This book has lots of activities that require think and reason about algebra for 12-16 aged-groups. The researcher explained the purpose of this study to the author and got permission from him for the use of these activities in this research. The researcher selected the activities related algebraic topic and appropriate for 7th grade level, and suggested to the teachers for using in their instructions.
The toothpick towers problem		This problem is a context-based and has two different patterns which their first term is same. In the first pattern, 3 toothpicks are added to at the top of the next figure, while 3 toothpicks are added to at the top and 1 toothpick is added the left and right sides of the next figure in the second pattern. The purpose of suggesting this problem to show different two patterns' generalization in the same context.

The suggested activities were "Handshake Problem" in entitled 'Algebra and Algebraic Thinking in School Mathematics' book (Moss, Beatty, Barkin, & Shillolo, 2008), "The toothpick towers problem" used by Herskowitz et al. (2002) in the algebra course, and "Dot Patterns" activity with special number patterns (e.g. square numbers, triangle numbers, oblong numbers) in "Positive algebra – A collection of productive exercises" by Martin Kindt. The "Handshake Problem" has a context like shaking hands in a party that it can be a real – life situation, and it requires the quadratic formula for generalization that is appropriate for the level of 7^{th} graders. However, the teachers did not use this problem in their lessons. "The toothpick towers problem" is a context-based and has two different patterns which their first term is same. The purpose of suggesting this problem to show different two patterns' generalization in the same context. Teacher A involved this problem to her lesson plan, but the two teachers did not use this problem in their lessons. In “Dot Patterns” activity, there are questions about the patterns such as if there is a rule, how you see this rule, or if 144 is a square number. However, the teachers indicated that this topic was in 8^{th} grade curriculum and thus they decided not to use these activities.

APPENDIX C: THE SUGGESTED ACTIVITIES RELATED WITH OPERATIONS WITH ALGEBRAIC EXPRESSIONS

The researcher suggested several activities from different resources for supporting the instruction of operations with algebraic expressions:

Name	Activity	The purpose of the activity
Bacterial Growth Pattern	There are two patterns in the context of growing of bacteria: 2, 4, 6, 8…, 3, 6, 9, 12… 1) Generalize patterns. 2) Add and multiply the terms of the first pattern and the second pattern, and generalize the added pattern and multiplied pattern in the activity. 3) How is the relationship between the generalizations at the beginning and the generalizations after operated them?	The reason for suggesting this activity is to teach operations with algebraic expressions using the pattern generalization that the students have learnt and known. This activity was considered to connect the previous topic that was pattern generalization.
Operating with Number Strips		The reason for suggesting this activity was to provide facility for the connection with patterns and operations with algebraic expressions. The teachers were suggested to use this activity with providing the students to recognize the connection between the resulted pattern rule and the addition of the rules of the addend patterns.
People at the Amusement Park	It is asked to represent the number of people or the total price about the amusement park using the given letters such as F or E. For example; the following information is given, C represents the number of males, who are 18-aged and over, visit the amusement park in a day; D represents	The reason for suggesting this activity is to remind the students to write verbal statements as algebraically in amusement park concept, and also make sense of the written expressions algebraically within the context.

<table>
<tr><td></td><td>the number of females, who are 18-aged and over, visit the amusement park in a day; E represents the number of teenagers, who are 18-aged and below, visit the amusement park in a day. It is asked to represent of the number of people who visit the amusement park in a day by using given information, and the expected answer is C+D+E.</td><td></td></tr>
<tr><td>Operating with Expressions</td><td>
</td><td>The reason for suggesting this activity is provide exercises that require to do three operations with using algebraic expressions in the same question, which functions summarize for three operations. This activity also was considered enjoyable for the students that they would try to get the result for last bubble with gathering the results before the last one.</td></tr>
<tr><td>The Price of Algebra I
and
The Price of Algebra II</td><td>I) There is a price list that shows how much point for which action. For example,

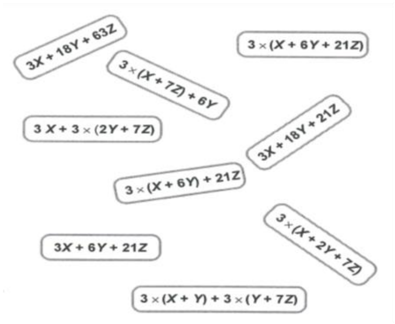
doing operations is 1 point each time,

taking a square is 2 points each time,

using variables 1 point each time. The algebraic expressions are given such as 3n+m, and it is asked to find the price of them such as what the price of 3n+m. To illustrate;

<table>
<tr><td>3</td><td>number</td><td>free</td></tr>
<tr><td>n</td><td>Using variable</td><td>1 point</td></tr>
<tr><td>3xn</td><td>multiplication</td><td>1 point</td></tr>
<tr><td>m</td><td>variable</td><td>1 point</td></tr>
<tr><td>3xn + m</td><td>addition</td><td>1 point</td></tr>
<tr><td></td><td>Total price</td><td>4 points</td></tr>
</table>

II)There are pairs of equivalent expressions and it is asked how they are equivalent and which pair is the cheapest. To illustrate, nxnxnxn and 4xn are one pair.</td><td>The reason for suggesting this activity is to make sense of using variables, operations, parenthesis, and taking power in algebraic expressions for students. This activity might be considered to provide this understanding for students with an enjoyable way.</td></tr>
</table>

<table>
<tr>
<td>Equivalent II</td>
<td>

Find the equivalent expressions and connect them with an arrow.</td>
<td>The reason for suggesting this activity was to use distributive property in multiplication with algebraic expressions. It might make sense for the students the expressions which seemed different at first glance were multiplied within themselves and they became equal.</td>
</tr>
</table>

The researcher suggested the bacterial growth pattern activity at the beginning of teaching operations with algebraic expressions. This activity is in the textbook that is published by MoNE (2014, p. 55). The teachers used this activity at the beginning of the instruction, they already would use since they followed the order of the textbook. Another activity that the researcher suggested was “People at the Amusement Park” in entitled ‘Algebra and Algebraic Thinking in School Mathematics’ book (Gay & Jones, 2008, p. 213). Teacher A wanted to use this activity, but she indicated some questions were difficult for students. Thus, the researcher adapted this activity regarding the level of the students with the teacher’s suggestions, and translated it into Turkish. This teacher used this activity in their lesson. The rest of the activities were from “Positive algebra – A collection of productive exercises” course book by Martin Kindt (2004). The teachers did not use “The Price of Algebra I” only, and they used other activities in their lessons. Besides the activities, the researcher suggested connecting arithmetic with algebra for teaching the properties of addition and multiplication properties in algebraic expressions as a method, and using algebra tiles as manipulatives. Teacher A would use algebra tiles as she indicated in her lesson plan while she involved to explain the connection between arithmetic and algebra while teaching.

The researcher prepared "Algebraic expression" test which includes the suggested activities and questions. The researcher implemented this test for 8^{th} graders in the same school. This test has four parts: operating with expressions, equivalent expressions, people at the amusement park activity, and correcting the error in subtraction operation that has two parenthesis algebraic expressions. The first reason for preparing this test is to have an idea about the practicality of suggested activities in teaching operation with algebraic expressions before the teachers' implementations. The second reason for implementing this test is to warn or prepare the teachers for possible conceptions, misconceptions, errors and difficulties that students have while teaching. In this setting, the solution strategies, misconceptions, errors and difficulties of students were shared with the teachers. The results showed that the students generally could do the operations with expressions part correctly.

APPENDIX D: APPROVAL OF THE ETHICS COMMITE OF METU RESEARCH CENTER FOR APPLIED ETHICS

UYGULAMALI ETİK ARAŞTIRMA MERKEZİ
APPLIED ETHICS RESEARCH CENTER

ORTA DOĞU TEKNİK ÜNİVERSİTESİ
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06.06.2014

Gönderilen : Y. Doç. Dr Didem AKYÜZ
İlköğretim Bölümü

Gönderen : Prof. Dr. Canan Özgen
IAK Başkanı

İlgi : Etik Onayı

Danışmanlığını yapmış olduğunuz İlköğretim Bölümü öğrencisi Dilek Girit'in "Investigating Middle School Mathematics Teachers' Pedagogical Content Knowledge about Algebra / Ortaokul Matematik Öğretmenlerinin Cebir ile İlgili Pedagojik Alan Bilgisinin Araştırılması" isimli araştırması "İnsan Araştırmaları Komitesi" tarafından uygun görülerek gerekli onay verilmiştir.

Bilgilerinize saygılarımla sunarım.

Etik Komite Onayı

Uygundur

06/06/2014

Prof.Dr. Canan Özgen
Uygulamalı Etik Araştırma Merkezi
(UEAM) Başkanı
ODTÜ 06531 ANKARA

APPENDIX E: PERMISSON OBTAINED FROM MINISTRY OF NATIONAL EDUCATION

T.C.
ANKARA VALİLİĞİ
Milli Eğitim Müdürlüğü

ÖĞRENCİ İŞLERİ
DA[illegible]

Sayı : 14588481/605.99/2872190 08/07/2014
Konu: Araştırma izni

ORTA DOĞU TEKNİK ÜNİVERSİTESİNE
(Öğrenci İşleri Daire Başkanlığı)

İlgi: a) MEB Yenilik ve Eğitim Teknolojileri Genel Müdürlüğünün 2012/13 nolu Genelgesi.
b) 26/06/2014 tarihli ve 7268 sayılı yazınız.

Üniversiteniz Eğitim Fakültesi Doktora Öğrencisi Dilek GİRİT' in **"Ortaokul matematik öğretmenlerinin cebir ile ilgili pedagojik alan bilgisinin araştırılması"** başlıklı araştırması kapsamında çalışma yapma talebi Müdürlüğümüzce uygun görülmüş ve araştırmanın yapılacağı İlçe Milli Eğitim Müdürlüğüne bilgi verilmiştir.

Görüşme formunun (1 sayfa) araştırmacı tarafından uygulama yapılacak sayıda çoğaltılması ve çalışmanın bitiminde iki örneğinin (cd ortamında) Müdürlüğümüz Strateji Geliştirme Şubesine gönderilmesini arz ederim.

Zafer YILMAZ
Müdür a.
Şube Müdürü

Güvenli Elektronik İmzalı
Aslı İle Aynıdır.
09.07.2014

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APPENDIX F: THE EXTRACTED CODES FOR ANALYSIS

Knowledge domains	Knowledge sub-domains	Codes
Subject Matter Knowledge (SMK)	CCK	CCK1: The knowledge of definitions, rules, properties, and theorems related to a specific topic CCK2: The knowledge to use terms and notation correctly CCK3: To simply calculating an answer or, more generally, correctly solving mathematics problems
	SCK	SCK1: The knowledge to connect a topic being taught to topics from prior or future years SCK2: The knowledge to link representations to underlying ideas and to other representations SCK3: The knowledge to choose/give usable definition or explanations SCK4: The knowledge of how to explain and justify one's mathematical ideas SCK5: The knowledge of how mathematical language is used SCK6: The knowledge of how to provide mathematical explanations for common rules and procedures SCK7: The knowledge of how to choose, make, and use mathematical representations effectively
Pedagogical Content Knowledge (PCK)	KCS	KCS1: The knowledge to anticipate where and how students have difficulty KCS2: The knowledge to anticipate the misunderstandings that might arise with specific items being studied in class KCS3: The knowledge to understand the needs and difficulties of students with mathematics KCS4: The knowledge to hear and interpret students' emerging and incomplete thinking as expressed in the ways that pupils use language KCS5: The knowledge of common student conceptions and misconceptions about particular mathematical content KCS6: The knowledge to predict what students will find interesting and motivating
	KCT	KCT1: The knowledge to choose which examples to start with and which examples to use to take students deeper into the content KCT2: The knowledge to sequence particular content for instruction KCT3: The knowledge to choose a particular representation or certain material for learning a concept or mathematical procedure KCT4: The knowledge to evaluate the instructional advantages and disadvantages of representations used to teach a specific idea KCT5: The knowledge to decide when to pause for more clarification, when to use a student's remark to make a mathematical point, and when to ask a new question or pose a new task to further students' learning in classroom discussion KCT6: The knowledge to identify what different methods and procedures afford instructionally

Knowledge domains	**Knowledge sub-domains**	**Codes**
	KCT	KCT7: The knowledge of how to build on students' thinking KCT8: The knowledge of how to address student errors effectively, remedy student errors
	KCC	KCC1: The knowledge to judge how to utilize it to present, emphasize, sequence and instruct KCC2: The knowledge to know the content and objectives in the curriculum

APPENDIX G: THE PROPOSED KNOWLEDGE TYPES FOR TEACHING GENERALIZATION OF PATTERNS

Knowledge sub-domain of SMK	Knowledge types	Description
CCK	CCK1	The knowledge of definition and properties of linear growth pattern and non-linear growth pattern The knowledge of inductive reasoning to generalize linear growth pattern
	CCK2	The knowledge to use the concept of non-linear growth pattern and exponential forms in explaining non-linear growth pattern correctly The knowledge to use the concepts of general term and general rule correctly
	CCK3	The knowledge to calculate an answer, or solve mathematics problems related with generalization of patterns correctly
SCK	SCK1	The knowledge to connect generalization of patterns between grades The knowledge to connect generalization of patterns with writing algebraic expression to get the general rule
	SCK2	The knowledge to link representations (figural, numerical, tabular, and graphical) to underlying the functional relationship
	SCK3	The knowledge to develop or choose usable definition of the pattern, general term, and general rule concepts
	SCK4	The knowledge of how to explain and justify the students' proposed general rules with substituting the position numbers in the general rule
	SCK5	The knowledge of how notations (operation signs, equal sign and exponential form) are used The knowledge of how mathematical language is used for explaining the relationship in linear and non-linear growth patterns
	SCK6	The knowledge of how to provide mathematical explanations for functional thinking in patterns The knowledge of how to provide mathematical explanations for the function of general rule
	SCK7	The knowledge of how to choose, make, and use tabular, figural, and graphical representations of patterns effectively

Knowledge sub-domain of PCK	Knowledge types	Description
KCS	KCS1	The knowledge to anticipate the students' difficulty in generalizing pattern algebraically
	KCS2	The knowledge to anticipate the misunderstandings that might arise with pattern generalization methods being studied
	KCS3	The knowledge to understand using figural and numerical reasoning as the needs of students with generalization of patterns The knowledge to understand writing algebraic expression as the difficulty of the students with generalization of patterns
	KCS6	The knowledge to predict using figural patterns that the students find interesting and motivating
KCT	KCT1	The knowledge to choose to start with simple linear growth patterns and to take students deeper into the pattern generalization with non-linear growth patterns
	KCT3	The knowledge to choose the graphical representation for the students to show the relationship between the position numbers and the terms
	KCT5	The knowledge to decide when the students have difficulty in understanding generalization of patterns in classroom discussion The knowledge to decide when to ask n^{th} term to further learning of students in classroom discussion
	KCT6	The knowledge to identify experiencing with different pattern types afford instructionally
	KCT7	The knowledge to build students' thinking with inductive reasoning for generalization of patterns
	KCT8	The knowledge of how to address students' errors in generalization and remedy them with explaining functional thinking
KCC	KCC1	The knowledge to know the content and objectives in the curriculum related to generalization of patterns
	KCC2	The knowledge to sequence, present the instruction developmentally with recalling students' prior knowledge, and emphasize what was learnt in previous grades

APPENDIX H: THE PROPOSED KNOWLEDGE TYPES FOR TEACHING OPERATIONS WITH ALGEBRAIC EXPRESSIONS

Knowledge sub-domain of SMK	Knowledge types	Description
CCK	CCK2	The knowledge to use the concepts of term, like term, constant term, and coefficient correctly
	CCK3	The knowledge to calculate an answer, solve mathematics problems related with operations with algebraic expressions correctly
SCK	SCK1	The knowledge to connect addition and subtraction of algebraic expressions with reminding the concepts of term, like term, constant, coefficient, and variable The knowledge to connect multiplication of algebraic expression with repeated addition as prior topic / The knowledge to connect the distributive property in multiplication with the application of it in integers
	SCK2	The knowledge to link algebraic representation to the concept of area to geometric representation with algebra tiles
	SCK3	The knowledge to develop or choose usable definition of the term, like term, coefficient, constant term, and variable concepts
	SCK5	The knowledge of how notations (equal sign and minus sign) are used The knowledge of how analogies as mathematical language are used for explaining the kind of variables and the operations The knowledge of how mathematical language is used for explaining the multiplication of a variable by itself
	SCK6	The knowledge of how to provide mathematical explanations for the procedures of operations with connecting generalization of patterns The knowledge of how to provide mathematical explanations for the distribution property in multiplication
	SCK7	The knowledge of how to choose, make, and use algebra tiles for teaching operations with algebraic expressions effectively

Knowledge sub-domain of PCK	Knowledge types	Description
KCS	KCS1	The knowledge to anticipate the students' difficulty in applying of distributive property in multiplication
	KCS2	The knowledge to anticipate the misunderstandings that might arise with studying on like term concept
	KCS3	The knowledge to understand the difficulties of students in operating with the parenthesis algebraic expressions The knowledge to understand the difficulties of students in applying distributive property with suggesting the use of algebra tiles
	KCS5	The knowledge of common students' misconceptions about multiplication of a number and a variable or a variable by itself The knowledge of common students' misconception about the perimeter and area concepts of quadrilateral
KCT	KCT1	The knowledge to choose the example related to pattern generalization and activities related to like term concept to take students deeper into addition and subtraction of algebraic expressions The knowledge to choose the example related to pattern generalization and examples related to modeling with algebra tiles to take students deeper into multiplication of algebraic expressions
	KCT4	The knowledge to evaluate advantages or disadvantages of using algebra tiles to teach multiplication
	KCT5	The knowledge to decide when to ask how to add terms
	KCT8	The knowledge of how to address the students' errors in determining like terms and remedy them The knowledge of how to address possible the students' errors in calculating the area and perimeter of squares or rectangles using algebraic expressions and remedy them
KCC	KCC1	The knowledge to know the content and objectives in the curriculum related to operations with algebraic expressions

APPENDIX I: TURKISH SUMMARY / TÜRKÇE ÖZET

ORTAOKUL MATEMATİK ÖĞRETMENLERİNİN CEBİR ÖĞRETİMİNDEKİ MATEMATİKSEL BİLGİLERİNİN ARAŞTIRILMASI: ÇOKLU DURUM ÇALIŞMASI

GİRİŞ VE ALAN YAZIN

Matematik eğitiminde öğrencinin öğrenmesini, anlamasını ve başarısını etkileyen faktörlerden biri de öğretmenin sahip olduğu bilgidir. Öğretmen bilgisinin öğrenci başarısı üzerinde olumlu etkiye sahip olduğunu gösteren bir çok araştırma (Fennema, Carpenter, Franke, Levi, Jacobs, & Empson, 1996; Hill, Ball, & Schilling, 2008; Hill, Rowan & Ball, 2005; Tchoshanov, Lesser, & Salazar, 2008), PISA gibi uluslararası sınav sonuçları ve büyük ölçekli araştırmalar (e.g. LMT, TEDS-M) bulunmaktadır. Öğretmen bilgisi kavramı ilk olarak Shulman (1986) tarafından tanımlanmıştır. Shulman'ın (1986) önerdiği öğretmen bilgisi modeli; konu alan bilgisi (KAB), pedagojik alan bilgisi (PAB) ve müfredat bilgisi kategorilerini içermektedir. Daha sonra birçok araştırmacı, öğretmen bilgisi kavramını geliştirmiş ve detaylandırmışlardır.

Grossman (1990) müfredat bilgisini pedagojik alan bilgisi (PAB) içinde incelerken, diğer modeller de Shulman'ın kategorilerine, öğrenci bilgisi, okul kültürü bilgisi gibi yeni bileşenler eklemiştir (Cochran, DeRuiter, & King, 1993; Magnusson, Krajcik & Borko, 1999). Öğretmen bilgisiyle ilgili genel modellerin yanında, matematik bilgisi ve matematik öğretmenlerinin sahip olması gereken bilgilerle ilgili modeller de geliştirilmiştir (An et al., 2004; Ball et al., 2008; Chick et al., 2006; Fennema & Franke, 1992; Rowland et al., 2005). Bu modellerin çoğu matematik öğretimi için PAB kavramına vurgu yaparken, Ball ve arkadaşları matematik öğretmenlerinin sahip olması gereken bilgiyi konu alan bilgisi ve pedagojik alan bilgisini birlikte ele alarak detaylı bileşenlerle incelemiş ve Öğretmek için Matmatiksel Bilgi (ÖMB) kavramını ortaya atmıştır. ÖMB'deki konu alan bilgisi, genel alan bilgisi, uzmanlık alan bilgisi ve kapsamlı alan bilgisi bileşenlerinden oluşmaktadır. Genel alan bilgisi, matematikle uğraşan herkesin sahip olduğu bilgi türü iken, uzmanlık alan bilgisi ise matematik öğretmenlerinin matematik öğretimi için sahip olması gereken bilgi olarak tanımlanmaktadır. Örneğin, 3'ü ½' ye bölmek genel alan bilgisi; öğrencilerin anlamasını desteklemek için bu bölmeyi alan modeli ile göstermek uzmanlık alan bilgisi kapsamında değerlendirilir. Kapsamlı alan bilgisi ile öğretmenler müfredata bütüncül bakarak öğrettikleri konuların önceki ve sonraki yıllardaki konularla olan bağlantısını görebilmektedir (Ball et al., 2008). Pedagojik alan bilgisi ise alan ve öğrenci bilgisi, alan ve öğretme bilgisi, ve alan ve müfredat bilgisinden oluşmaktadır. Alan ve öğrenci bilgisi, "öğrenci bilgisi ile matematik bilgisini birleştirir" (Ball et al., 2008, p. 401). Öğretmenler, öğrencilerin algılarını, bilgilerini, kavram yanılgılarını, hata ve zorluklarını bilmelidir. Alan ve öğretme bilgisi ise "öğretme bilgisi ve matematik bilgisini birleştirmeyi" gerektirmektedir (Ball et al., 2008, p. 401). Öğretmen bu bilgi ile, matematik öğretimi için derslerini tasarlamaktadır ve kararlar vermektedir. Üçüncü bileşen olan alan ve müfredat bilgisi ise, konuları müfredata göre sıralama, müfredat tarafından önerilen etkinlikler ve açıklamalar bilgisi ile ilgilidir.

Matematik eğitimi araştırmacıları, matematik öğrenme alanlarına ve konularına göre de öğretmenlerin sahip olması gereken bilgi ile ilgili modeller de geliştirmişlerdir. Bunlara örnek olarak, cebir, istatistik ya da olasılık verilebilir (Graeber & Tirosh, 2008). Buna rağmen, alanyazında öğretmenlerin cebir bilgisi ve uygulamaları ile ilgili az sayıda araştırma bulunmaktadır (Doerr, 2004; El Mouhayar & Jurdak, 2013; Wilkie, 2014). Özellikle cebir bilgisi ile ilgili geliştirilen modeller fonksiyon, eşitsizlikler, ve doğrusal denklemlerle ilgili olup genellikle ortaöğretim matematik konularıyla ilgilidir (Artigue, Assude, Grugeon, & Lenfant, 2001; Even, 1990, 1993; Ferrini-Mundy et al., 2006; Kieran, 2007; Li, 2007). Aslında, her sınıf seviyesindeki öğrencilerin cebir öğrenmesi ile ilgili zorluk yaşadığını gösteren, öğrencilerin bilgilerini, kavram yanılgılarını, hata ve zorluklarını araştıran çok sayıda çalışma bulunmaktadır (Cooper, Boulton-Lewis, Atweh, Pillay, Wilss, & Mutch, 1997; Linchevski & Livneh, 1999; Sfard & Linchevski, 1994; Warren, 1999; Stacey

& Macgregor, 1997). Bu noktada, öğretmenlerin bigisinin öğrencilerin öğrenmesi üzerinde etkili olduğu düşünüldüğünde, öğretmenlerin cebir bilgisini incelemek öğrencilerin nasıl öğrenebildiğini ve neden genellikle zorluk çektiklerini anlamamıza ışık tutabilir. Ama, cebir öğretimi açısından hangi cebirin öğretilmesi ve cebirin nasıl öğretilmesi ile ilgili öğretmenleri ilgilendiren konular çok araştırılmamıştır (Doerr, 2004). Bu konudaki araştırmaların azlığı düşünüldüğünde, öğretmen bilgisini araştıran çalışmalara ihtiyaç duyulduğu da söylenebilir.

Öğrencilerin ortaöğretim matematiğini anlaması için cebirin temel kavramlarını bilmesi çok önemlidir (Rakes, Valentine, McGatha, & Ronau, 2010). Bu temel kavramlar değişken, cebirsel ifade yazma, ve cebirsel ifadeleri sadeleştirmedir. Bu kavramlar öğrencilere 6.sınıfta örüntü genellemesi ve cebirsel ifadelerle işlemler konuları kapsamında öğretilmeye başlanır (MEB, 2013). Hatta, araştırmacılar erken yaşlarda cebir öğreniminin öğrencilerin aritmetikten cebire geçişteki önemine vurgu yapmaktadır (Cai, Ng, and Moyer, 2011; Carraher and Schliemann, 2007; Schmittau, 2005; Van de Walle et al., 2013). Ayrıca, çalışmalar erken yaşlardaki öğrencilerin cebirsel düşünmesinin, uygun yaklaşımlar, etkinlik ve görevlerle geliştirilebileceğini göstermiştir (Ferrara & Sinclair, 2016; Warren & Cooper, 2008; Warren, Cooper, & Lamb, 2006). Erken yaşlardaki cebir öğrenimi ile ilgili yapılan çalışmalar, öğrencilerin genellikle örüntüleri cebirsel olarak genelleme ve cebirsel ifadelerle işlemlerde zorlandığını göstermektedir (Amit & Neria, 2008; Becker & Rivera, 2005; El Mouhayar & Jurdak, 2016; Kieran, 2007; Lannin, Barker, & Towsend, 2006; Rivera, 2010; Rivera & Becker, 2008; Steele & Johanning, 2004; Walkowiak, 2014; Warren & Cooper, 2008). Aslında, değişken ve cebirsel ifadelerin kavramsal ve işlemsel olarak gelişmesi daha sonra denklem yazımı ve çözümü için önemlidir (Capraro & Joffrinon, 2006). Dolayısıyla, ortaokul öğrencilerinin genellikle zorlandığı ve kavram yanılgıları olduğu ve ayrıca cebirin öğretilmeye başlandığı bu konuları ele almak daha sonraki matematik öğretimi için de daha önemli olabilir. Özellikle, örüntülerde adım sayısı ve terim arasındaki fonksiyonel ilişkiyi anlamak sonraki yıllarda öğrenilecek olan fonksiyon kavramını da destekleyebilir (Usiskin, 1988). Örüntüler, bir bağlamda ya da şekiller arasındaki girdi ve çıktı değerleri arasındaki ilişkiyi analiz etmeyi ve değişkenleri kullanarak genelleme yapmayı sağlar. Böylece öğrencilerin genellemedeki değişkenin fonksiyonunu anlamasına yardımcı olarak, onların aritmetikten cebire geçişini destekler (English & Warren, 1998). Bununla birlikte, Jurdak ve El Mouhayar (2014) 4.sınıftan 11.sınıfa kadar olan öğrencilerin genellemeye dair bilgilerini incelemiş ve öğrencilerin örüntüyü devam ettirebildiklerini fakat cebirsel olarak genellemede zorluk yaşadıkları sonucuna varmışlardır. Bu sonucun, geniş bir aralıkta yani hem ilkokul, hem ortaokul ve hem de lise düzeyinde öğrencilerin genelleme kavramında zorluk yaşadıklarını gösterdiği söylenebilir. Bu sonuç ayrıca Becker ve Rivera'nın (2005) ulaştığı yaşı büyük olan öğrencilerin de örüntü genellemesinde zorlandığı sonucuyla da örtüşmektedir. Benzer şekilde öğrenciler cebirsel ifadelerler uğraşırken de sıkıntı yaşamaktadır ve Seng (2010) bu konuyu öğrenirken öğrencilerin sahip olduğu farklı kavram yanılgılarını tanımlamıştır.

Kieran'ın (2007) cebirsel etkinlik modeline göre, örüntü genellemesi bir genelleme etkinliğidir ve ardından dönüşümsel etkinlik gelir. Bu modele göre, erken yaşlarda öğrencilerin cebirsel düşünmesini geliştirmek için cebirsel ifade ya da denklem ile ilgili işlem yapmanın gerekli olduğu belirtilir. Cebirsel ifadelerle işlemler konusu da ifadeleri sadeleştirmek için benzer terimleri toplama ve cebirsel ifadeleri çarpma kazanımlarını içermektedir (MoNE, 2013). Bu noktada öğretmenlerin erken cebir konuları öğretimini incelemek öğrencilerin kavramsal bir şekilde öğrenmelerini geliştirmek için daha avanatajlı olabilir. Çünkü öğretmenlerin, öğretimde önemli bir rolü vardır ve öğrenci başarısı üzerinde olumlu bir etkiye sahiptir (Fennema, Carpenter, Franke, Levi, Jacobs, & Empson, 1996; Hill, Ball, & Schilling, 2008; Hill, Rowan & Ball, 2005; Tchoshanov, Lesser, & Salazar, 2008). Özellikle cebir öğretiminde, Malara ve Navarra (2009) çalışmalarında öğrencilerin örüntü genellemesi bilgilerini yapılandırmak için sınıf tartışmasında öğretmenlerin rolünün önemine dikkat çekmişlerdir. Örüntü genellemesinde olduğu gibi, Ayalon ve Even (2013) öğrencilerin cebirsel ifadelerle ilgili bilgilerini ve anlamalarını geliştirmek için öğretmenlerin geliştirilmesini önermişlerdir.

Cebirde öğrencilerin yaşadığı zorlukların yanında, öğretmenlerde de bu konuların öğretimi ile ilgili bilgi eksikliği bulunmaktadır. El Mouhayar ve Jurdak (2013), öğretmenlerin, öğrencilerin genelleme stratejilerini açıklayabilecekleri yeterli bilgiye sahip olmadığını görmüşlerdir. Ayrıca, Wilkie (2014) de öğretmenlerin fonksiyonel düşünme ile ilgili alan ve öğrenci bilgilerinin, ve alan ve öğretme bilgilerinin yeterli olmadığı ve kavramsal olmadığını bulmuştur. Hatta, öğretmenler genelleme için cebir kullanımında zorluk yaşamışlar ve bu durum onların konu alan bilgilerinin de eksik olduğunu göstermiştir. Dolayısıyla, Tirosh vd.'nin (1998) de önerdiği gibi, etkili dersler tasarlamak için öğretmenlerin öğrencilerin düşünmesiyle ilgili bilgilerini geliştirmek gerekmektedir.

Öğrencilerin öğrenmesinde öğretmen bilgisinin önemi düşünüldüğünde, bu çalışmanın amacı öğretmenlerin örüntü genellemesi ve cebirsel ifadelerle işlemler hakkında var olan bilgilerini ve hangi bilgiye ihtiyaç duyduklarını ortaya çıkarmaktır. Bunun yanında, alanyazında bu konularla ilgili öğretmenlerin bilgi ve uygulamalarını araştıran az sayıda çalışma olduğu belirtilmektedir (Baş, Çetinkaya, & Erbaş 2011; Doerr, 2004; El Mouhayar & Jurdak, 2013; Kieran; 1992; Wilkie, 2014). Bu noktada da bu çalışmanın öğretmenlerin cebir bilgisi konusunda matematik eğitimi alan yazınına katkı sağlayacağı düşünülebilir. Bununla birlikte özel olarak örüntü genellemesi ve cebirsel ifadelerle ilgili öğretmen bilgisisine dair bir modele rastlanmamış olup yukarıda da belitildiği gibi cebir bilgisi ile ilgili ortaya konulan modeller genellikle ortaöğretim düzeyinde yer almaktadır. Dolayısıyla bu çalışmada matematik eğitimi araştırmacıları tarafından yaygın bir şekilde kabul edilen ve kullanılan ÖMB modeli kullanılmıştır. Bu modelin sahip olduğu detaylı bilgi türleri ile açıklamaları, öğretmenlerin var olan bilgisini ortaya koymada etkili olacağı düşünülmüştür. Bu modeli kullanmanın diğer bir gerekçesi de araştırmacıların bu modeli geliştirirken nitel boyut olarak öğretim gözlemlerini kullanmış olmalarıdır. Bu çalışmanın da temel veri kaynağı ders gözlemi olduğu için bu modelin kullanımının uygun olacağı düşünülmüştür. Ayrıca, Ball ve arkadaşlarının (2008) bu modeli geliştirmekteki amaçları ile bu çalışmanın amacı da örtüşmektedir. Ball vd. (2008) ÖMB modelinin, öğretmenlerin konuları öğretmek için hangi bilgiye ihtiyaçları olduğu ve bu bilgiyi öğretimde nasıl kullandıklarını ortaya koymayı sağladığını belirtmişlerdir. Bu bağlamda, bu çalışmanın diğer bir amacı da, örüntü genellemesi ve cebirsel ifadelerle işlemler konusunda öğretmenlerin sahip olması gereken bilgi türleri ile ilgili önerilerde bulunabilmektir. Stacey ve Chick (2004), öğretmenler için öğrenebilecekleri ve öğrenci bilgisi ve öğretim bilgisiyle birlikte kullanabilecekleri bilgi yapıları oluşturmanın kolay bir şey olmadığını ileri sürmektedir. Mevcut çalışma, cebirdeki belli konular için bilgi yapıları oluşturma amacıyla yola çıkmış olup öğretmen bilgisi alan yazınına bu şekilde katkı sağlamayı planlamaktadır.

Öğretmenlerin bilgisini incelemek ayrıca matematik öğretmeni eğitim programları için de değerli sayılabilir. Öğretmenlerin bilgi eksikliği varsa ya da kavram yanılgıları bulunuyorsa, öğretmen eğitimcileri programlarını ve özel öğretim yöntemleri derslerini bu çalışmanın sonuçlarını da dikkate alarak tasarlayabilirler. Aslında, matematik öğretmeni eğitimcileri öğretmenlerin kavramsal ve bağlantılı bir cebir bilgisine ihtiyaç duyduğunu fakat öğretmenlerin bilgilerinin geliştirme konusundaki önerilerin azlığına dikkat çekmektedir (Magiera, van den Kieboom, & Moyer, 2013). Bu noktada, bu çalışma, ve bu gibi çalışmalar, öğretmenlerin var olan bilgisini ve bilgilerindeki eksikleri ortaya çıkararak, öğretmen eğitimcilerine yardımcı olabilir.

Belirtilen bütün bu değerlendirmeler ışığı altında bu çalışmanın cebir öğrenimi ve öğretimine katkı yapacağı düşünülmektedir. Bu bağlamda, öğretmenlerin örüntü genellemesi ve cebirsel ifadelerle işlemlerle ilgili bilgilerini araştırmak için bu çalışma yürütülmüştür. Bunun için aşağıdaki araştırma soruları oluşturulmuştur:

1. Ortaokul matematik öğretmenlerinin örüntü genellemesini öğretmek için planlama ve uygulamadaki matematiksel bilgilerinin yapısı nedir?

   1.a. Ortaokul matematik öğretmenlerinin örüntü genellemesini öğretmek için planlamadaki matematiksel bilgilerinin yapısı nedir?

   1.b. Ortaokul matematik öğretmenlerinin örüntü genellemesini öğretmek için uygulamadaki matematiksel bilgilerinin yapısı nedir?

2. Ortaokul matematik öğretmenlerinin cebirsel ifadelerle işlemleri öğretmek için planlama ve uygulamadaki matematiksel bilgilerinin yapısı nedir?

   2.a. Ortaokul matematik öğretmenlerinin cebirsel ifadelerle işlemleri öğretmek için planlamadaki matematiksel bilgilerinin yapısı nedir?

   2.b. Ortaokul matematik öğretmenlerinin cebirsel ifadelerle işlemleri öğretmek için uygulamadaki matematiksel bilgilerinin yapısı nedir?

Bu araştırma soruları cevaplandıktan sonra, ÖMB modelinin bileşenleri olan konu alan bilgisi ve pedagojik alan bilgisi arasındaki ilişki de şu araştırma sorusuyla incelenmiştir:

3. Ortaokul matematik öğretmenlerinin, örüntü genellemesi ve cebirsel ifadelerle işlemlerin öğretimi bağlamında, konu alan bilgileri pedagojik alan bilgilerini nasıl etkiler?

Bu araştırma sorularına cevap vermek için Ball vd.'nin (2008) Öğretmek için Matematiksel Bilgi (ÖMB) modeli teorik çerçeve olarak kullanılmıştır. Bu modelin temel bileşenleri, konu alan bilgisi ve pedagojik alan bilgisi, ve alt bileşenleri kullanılarak öğretmenlerin bilgisi analiz edilmiştir. Bu analizle birlikte, öğretmen bilgisi ile ilgili detaylı bir tanımlama ve ayrıca bu konuları öğretmek için öğretmenlerin ihtiyaç duyduğu bilgileri önermek amaçlanmıştır.

YÖNTEM

Araştırmanın Deseni

Bu çalışmanın amacı, örüntü genellemesi ve cebirsel ifadelerle işlemler konularının öğretimi için ortaokul matematik öğretmenlerinin sahip olduğu matematik bilgilerini incelemektir. Bunun için nitel araştırma yöntemi kullanılmıştır. Merriam (2009) nitel çalışma için dört özellik tanımlamaktadır. Bunlardan biri, araştımacının doğal ortamında araştırdığı süreci anlamaya odaklanmaktır. Bu çalışmada da araştırmacının odağı ÖMB'nin yapısı ve öğretmenlerin bu bilgilerini öğretim sürecinde nasıl kullandıklarını tanımlamaktır. Merriam'ın (2009) belirttiği gibi, araştırmacı bu sürece müdahele etmeden, öğretmenin öğretiminin, deneyimlerinin ve uygulamalarının anlamını anlamaya çalışmıştır. Nitel çalışmanın ikinci özelliği ise araştırmacının veri toplama ve analizinin ilk aracı olmasıdır. Araştırmacı, çalışmanın amacına ilişkin araştırılan olguya göre kendini adapte eder. Ayrıca, beklenmeyen durumları, verilen cevapların doğruluğu, ve zamanındaki etkileşimleri ele alarak, veriyi bütünsel bakış açısıyla yorumlar. Sınıf ortamları karmaşık bir yapıya sahip olduğu için, nitel araştırma bu bakış açısını sağlar (Lagemann & Shulman, 1999). Bu çalışma da böyle bir bağlama sahip olup, gözlemler yoluyla öğretimi doğal ortamında, beklenen ve beklenmeyen durumlarıyla, öğretmenin cevapları ve tepkileriyle ve öğrencilerle olan etkileşimleriyle ortaya koyma potansiyeline sahiptir. Böylece, araştırmacı beklenmeyen ihtimallere karşı uyum sağlamış ve öğretmenin bilgisini değişik açılardan incelemiştir. Veri toplama süreciyle ilişkili olarak, araştırmacı, bütün öğretim sürecini ele alarak ve bu süreçte öğretmenle de görüşmeler yaparak veriyi analiz etmek için kodlar ve temalar oluşturmuştur. Nitel araştırmanın üçüncü özelliği de toplanan verilerin kavram ve teoriler geliştirmektir, ya da kavramları teoriler içinde açıklamak için kullanılmasıdır. Bu çalışmada da, öğretmenelerin cebir bilgisi ÖMB modeli çerçevisinde açıklanmaya çalışılmıştır. Dördüncü özellik de nitel araştırmaların araştırılan olguya ilişkin sözcüklerle betimsel bir ürün ortaya çıkarmasıdır. Araştırmacı mevcut çalışmada, yazıya dökülmüş ders videolarını, görüşmeleri, alan notlarını ve ders planlarını öğretmenlerin bilgisini detaylı bir şekilde tanımlayabilmek için kullanmıştır.

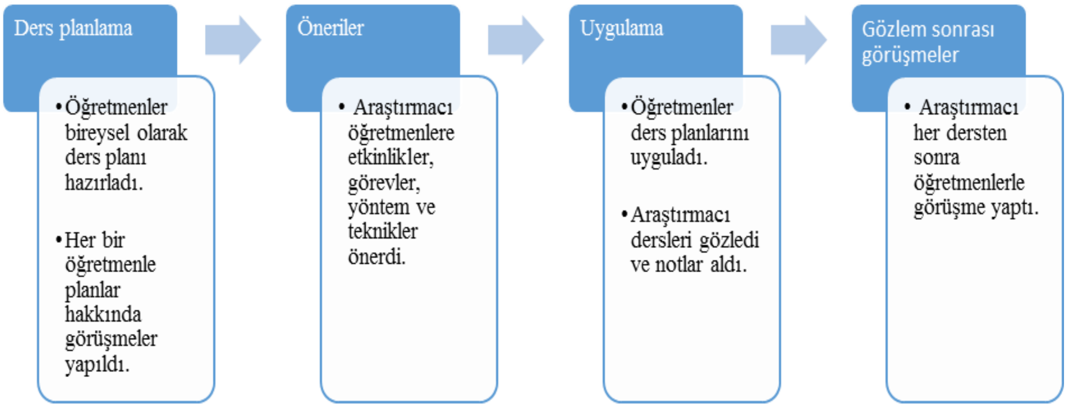
Özetle, yukarıdaki açıklamalar ışığında, nitel araştırma yöntemi ve özellikle durum çalışması, bu çalışma için tercih edilmiştir. Creswell (2007) durum çalışmasının, araştırmacının araştırdığı bir durum/durumlar, birçok kaynaktan (gözlemler, görüşmeler gibi) detaylı ve derinlemesine veri toplama, ve durum için temaları raporlamaktan oluşur. Durum çalışmalarında, bağlam içinde bir durumu araştırmak durumla etkileşimde olan değişkenleri değerlendirmeyi sağlar. Böylece, araştırmacı bütünsel bir tanımlama sunmaktadır (Merriam, 2009). Bu çalışmada araştırılan durum için, Yin'in (2008) tanımladığı bütüncül çoklu durum deseni kullanılmıştır. Bu tasarımda, bağlam cebir konularının öğretimi, durumlar iki matematik öğretmeni ve analiz edilen birim öğretmenin bilgisidir. Öğretmenin öğretmek için matematiksel bilgisi bütünsel olarak ve kıyaslama ve karşılaştırma tekniği kullanılarak açıklanmıştır. Bu çoklu durum çalışmasının iki farklı öğretim durumlarından gelen örnekleyici çıktılarla ÖMB modelini değerlendirmeye katkı sağlayacağı düşünülmüştür.

Katılımcılar

Bu çalışmada katılımcı olarak iki ortaokul matematik öğretmeni ile çalışılmıştır. Bu öğretmenler aynı devlet okulunda çalışmaktadır. Bu öğretmenlerin seçiminde uygun örnekleme kullanılmıştır. Öğretmenlerin aynı anda konuları öğreteceği düşünüldüğünde onlara ulaşım bu örneklemenin seçiminde etkili olmuştur. Öğretmenlerin bu çalışmada yer almaya gönüllü olması da onların seçiminde bir etken olmuştur. Çünkü, bu çalışma uzun bir süreç gerektirmiş ve öğretmenlerin ders saatleri dışında görüşmelere vakit ayırmaları beklenmiştir. Böylece bu elverişli durumlar düşünülerek, bu iki öğretmenle çalışılmıştır. Diğer yandan, bu örnekleme amaçsal örnekleme olarak da düşünülebilir, çünkü katılımcıların zengin ve detaylı veri verebilecek olması da dikkate alınmıştır. Bu öğretmenlerin matematik eğitimi programında doktora yaptıklarından dolayı daha bilgili ve araştırma konusunda deneyimli olduğu düşünülmüş ve detaylı veri elde edilebileceği beklenmiştir. Bu şekilde iki maematik öğretmeniyle, 2014-2015 eğitim ve öğretim yılında bu çalışma gerçekleştirilmiştir.

Veri Toplama Süreçleri

Veri toplama süreci, öğretimden önce ve öğretim süreci olmak üzere iki temel aşamadan oluşmaktadır. Öğretimden önceki aşamada öğretmenler ders planlarını bireysel olarak hazırlamıştır, araştırmacı herbir öğretmenle görüşme yapmıştır ve daha sonra öğretmenlere örnekler, etkinlikler, görevler, yöntem ve teknikler önermiş, ve öğrencilerin testlerde verdiği cevapları paylaşmıştır. Öğretmenler bu önerilere göre ders planlarını revize etmişlerdir. Öğretim aşamasında ise, araştırmacı dersleri gözlemiş, video kaydı yapmış ve notlar almıştır. Her dersten sonra da öğretmenlerle bireysel olarak görüşmeler yapmıştır. Aşağıdaki akış diyagramı veri toplama sürecini göstermektedir:



Şekil 1 Veri toplama süreci

Veri Toplama Araçları

Bu çalışmada veriler, öğretimden önce öğretmenlerin hazırladığı ders planlarından, ders planlarını öğretmenle birlikte inceleyerek yapılan görüşmelerden, öğretim esnasında yapılan gözlemlerden, ve öğretimden sonra öğretmenle yapılan yansıtıcı görüşmelerden elde edilmiştir.

Ders planlarını hazırlatmanın amacı öğretemenlerin var olan bilgisini araştırmaktır. Ders tasarımı yaparken, öğretmenlerin konu alan bilgisini ve pedagojik alan bilgisi olarak da öğrencilerin düşünmelerini nasıl dikkate aldıkları araştırılmıştır. Bu çalışmada öğretmenlerin hazırladığı iki ders planı kullanılmıştır. Birinci ders planı "Sayı örüntülerini modelleyerek bu örüntülerdeki ilişkiyi harflerle ifade eder.", ikinci ders planı ise "Cebirsel ifadelerle toplama ve çıkarma işlemleri yapar, ve iki cebirsel ifadeyi çarpar." kazanımlarına göre hazırlanmıştır.

Ders gözleminden önceki görüşmelerde, ders planlarına odaklanılmıştır; ders gözleminden sonraki görüşmelerde ise bir ya da iki ders saati gerçekleşen öğretime odaklanılmıştır. Görüşmelerde yarı yapılandırılmış sorular kullanılmıştır.

Araştırmacı iki öğretmenin derslerini cebir ünitesi boyunca gözlemiştir. Gözlemlerin amacı, öğretmenlerin bilgilerini uygulamada nasıl kullandıklarını araştırmaktır. Bu gözlemler ayrıca, ders planları ile uygulamaları karşılaştırmaya olanak sağlamıştır. Bu çalışma için örüntü genellemesi ve cebirsel ifadelerle işlemeler konularını içeren toplamda 33 saatlik ders gözlenmiş ve kayda alınmıştır.

Veri Analizi

Veri analizi sürecinde ilk olarak, öğretmenin ders planları hakkında yapılan görüşmeler, video kaydına alınmış ders anlatımları, ve ders gözleminden sonra yapılan görüşmeler yazıya dökülmüş ve araştırmacı tarafından okunmuştur. Daha sonra, öğretmenlerin ÖMB'si için bütüncül bir bakış açısını elde etmek amacıyla bütün bu yazılı belgeler bir araya getirilmiş ve birlikte değerlendirilmiştir. İki öğretmeni durum olarak birlikte analiz etmek ve yorumlamak için karşılaştırmalı durum analizi kullanılmıştır. Analiz için Ball ve arkadaşlarının (2008) geliştirdiği ÖMB modeli kullanılmıştır. Verideki anlamlı olduğu düşünülen bir cümle, bir açıklama, bir diyalog ya da soru analiz edilmiş ve kodlanmıştır. Bu geçici kodlar, benzerlik ve farklılıklarına göre, ÖMB modelinin açıklarıma dikkate alınarak, tema ve alt temaların altında gruplandırılmıştır. Bu modele dayanarak, konu alan bilgisi (KAB) ve pedagojik alan bilgisi (PAB) temalar; alt bilgi alanları da (genel alan bilgisi (GAB), uzmanlık alan bilgisi (UAB), kapsamlı alan bilgisi, alan ve öğrenci bilgisi (AÖB), alan ve öğretme bilgisi (AÖtB), ve alan ve müfredat bilgisi (AMB) alt temalar olarak düşünülmüştür. Böylece, herbir durumdan gelen veri, görüşmeler ve gözlemler, bu temeları ve alt temaları kullanılarak analiz edilmiştir. Oluşturulan geçici kodlardan belli bir tekrarı ve düzeni olanlar ve her iki durumda da ortaya çıkanlar, bu çalışmanın kodlarını oluşturmuştur. Çıkarılan kodlar aşağıdaki tabloda verilmiştir.

Tablo 1 Öğretmek için matematik bilgisi analizi için çıkarılan kodlar

Bilgi alanları	Alt-bilgi alanları	Kodlar
Konu alan bilgisi	Genel alan bilgisi	GAB1: Belirli bir konuya ilişkin tanımlar, kurallar ve teoremler bilgisi GAB2: Terim ve notasyonları doğru bir şekilde kullanma bilgisi GAB3: Basit bir şekilde bir cevapı hesaplama ya da daha genel olarak matematik problemlerini doğru çözme bilgisi
	Uzmanlık alan bilgisi	UAB1: Öğretilen bir konu ile önceki yıllarda ya da gelecek yıllardaki konular arasında bağlantı kurma bilgisi UAB2: Önemli fikirleri vermek için farklı gösterimler arasında ilişki kurma bilgisi UAB3: Kullanılabilir tanımlar ya da açıklamalar geliştirme ya da seçme bilgisi UAB4: Birine ait matematiksel fikirleri açıklama ve gerekçelendirme bilgisi UAB5: Matematiksel dilin kullanımı bilgisi UAB6: Genel kurallar ve işlemler için matematiksel açıklama sağlama bilgisi UAB7: Matematiksel gösterimleri seçme, yapma ve etkili bir şekilde kullanma bilgisi
Pedagojik alan bilgisi	Alan ve öğrenci bilgisi	AÖB1: Öğrencilerin nerede ve nasıl zorluk yaşayabileceği bilgisi AÖB2: Derste belirli kavramlarla çalışılırken oluşabilecek kavram yanılgılarını tahmin etme bilgisi AÖB3: Öğrencilerin matematiğe ilişkin ihtiyaçlarını ve zorluklarını anlama bilgisi AÖB4: Öğrenciler dili kullanırken onların ortaya çıkan ve tamamlanmamış düşüncelerini fark etme ve yorumlama bilgisi AÖB5: Belli matematik konularıyla ilgili genel öğrenci algıları ve kavram yanılgıları bilgisi AÖB6: Öğrencilerin ilginç bulduğu ve motivasyon sağlayacak şeyleri tahmin etme bilgisi
	Alan ve öğretme bilgisi	AÖtB1: Konu öğretimine başlamak ve öğrencileri konunun derinine götürmek için örnekler ve teknikler seçme bilgisi AÖtB2: Öğretim için konuları sıralama bilgisi AÖtB3: Bir kavramın ya da matematiksel işlemlerin öğrenime ilişkin belirli bir gösterim ya da materyal seçme bilgisi AÖtB4: Belirli bir kavramı öğretmek için kullanılan gösterimlerin avantajlarını ve dezavantajlarını değerlendirme bilgisi

	Alan ve öğretme bilgisi	AÖtB5: Bir sınıf tartışmasında, açıklama yapmak için durma, öğrencinin düşüncesindeki matematiksel noktayı kullanma,ve öğrencilerin daha ileri öğrenmesi için yeni bir soru ya da etkinlik oluşturmanın ne zaman olacağına karar verme bilgisi AÖtB6: Öğretimsel olarak farklı yöntem ve işlemleri tanımlama bilgisi AÖtB7: Öğrencinin düşünmesini nasıl oluşturacağını bilmesi AÖtB8: Öğrencilerin hatalarına ele alma ve onları düzeltme bilgisi
	Alan ve müfredat bilgisi	AMB1: Müfredattaki konuların ve kazanımların bilgisi AMB2: Konuyu sunma, vurgulama, sıralama ve öğretme bilgisi

Çıkarılan bu kodlar, var olan ve etkili kullanılan bilgiyi temsil etmek için artı işareti (+), yeterli olmayan ve etkili kullanılmayan bilgiyi temsil etmek için eksi işareti ile gösterilmiştir (Örn. AÖB1-, UAB2+). Bu analiz gözlenebilen bilgi türleri için yapılmıştır. Bu yöntemle kodlanan veriler her bir durum için bulgular kısmında, ve iki durum karşılaştırılarak da tartışma ve sonuç kısmında sunulmuştur. Bu karşılaştırma, ayrıca bir durumda ortaya çıkan fakat diğer durumda ortaya çıkmayan bilgi türlerini de karşılaştırmayı sağlamıştır.

Planlama ve uygulama aşamalarını analiz etmek ve yorumlamak için çıkarılan bu kodlar kullanılmıştır. Planlama aşaması, konulara göre ve veri toplama sürecine göre değerlendirilmiş ve rapor edilmiştir. Uygulama aşaması için ise öğretim boyunca öğretmenin belli bir düzen içinde tekrar eden eylemleri pratik olarak gruplanmıştır. Öğretmen A'nın öğretiminden çıkarılan pratikler altı grupta sunulmuştur: 1) önceki konularla bağlantı kurarak konuya başlamak için bir örnek ya da etkinlik seçmek, 2) konuyla ilgili bir etkinlik üzerinde tartışmak, 3) öğrencileri konunun daha derinine götürmek için örnekler ya da etkinlikler seçmek, 4) önerilen etkinlikleri uygulamak, 5) konuyla ilgili ders ve çalışma kitabından alıştırmalar yapmak, ve 6) diğer konularla ilgili bilgiyi birleştirmeyi gerektiren problemler sunmak. Diğer yandan, Öğretmen B'nin öğretiminden çıkarılan pratikler beş grupta sunulmuştur: 1) önceki konularla bağlantı kurmak, 2) konuyla ilgili bir etkinlik üzerinde tartışmak, 3) öğrencileri konunun daha derinine götürmek için örnekler ya da etkinlikler seçmek, 4) önerilen etkinlikleri uygulamak, ve 5) konuyla ilgili farklı kaynaklardan sorular ve problemler çözmek. Öğretmenlerin bilgisi, bu pratikler kapsamında kodlar kullanılarak yorumlanmıştır. Uygulamadan çıkarılan bilgi türleri planlama ve ders sonrası görüşmelerle birlikte ele alınarak değerlendirilmiştir.

BULGULAR

Bu çalışmanın bulguları, ele alınan matematik konularına göre, örüntü genellemesi ve cebirsel ifadelerle işlemler olarak, ve Öğretmen A ve Öğretmen B olarak iki durumu karşılaştırmalı şekilde açıklanmıştır. Öğretmenlerin bilgileriyle ilgili bulgular konu alan bilgisi ve pedagojik alan bilgisi başlıkları altında sunulmuştur.

Konu Alan Bilgisi

Örüntü genellemesi öğretiminde, öğretmenlerin örüntü kavramı tanımlama bigisi uygun olmayan ya da yetersiz bir şekilde ortaya çıkmıştır (GAB1). Öğretmen A planlamada sabit değişen ve artarak değişen örüntüleri doğru bir şekilde doğru örneklerle açıklamasına rağmen, öğretimde şekiller arasındaki ilişkiler şeklinde tanımlamış ve bu tanım da yeterli olmamıştır. Çünkü örüntüler hem şekillerden hem de sayılardan oluşabilmektedir. Diğer yandan, Öğretmen B artarak değişen örüntü için isim kullanmadan, n^2'li örüntü şeklinde bir isimlendirme ile derste açıklamalarda bulunmuştur. Dolayısıyla, bu öğretmenin terimleri kullanma bilgisinin yeterli olmadığını göstermiştir (GAB2). Diğer problemli durumlardan biri de genel terim ve genel kural kavramlarını ilişkin Öğretmen B'nin n'yi hem genel terim hem de genel kural olarak doğru bir şekilde kullanamamış olmasıdır. Buna rağmen, sadece bu öğretmen genelleme yaparken belli örneklerden başlayarak genellemeye ulaşmış ve tümevarım yöntemini etkili bir şekilde kullanmıştır (GAB1). İlk dört ya da beş terim için aritmetik ilşkiyi göstermiş ve bu ilişkiyi n.terimi sorarak genellemeye bağlamıştır.

Öğretmenlerin örüntü genellemesi konusunu öğrencilerin önceki yıllardaki bilgilerini hatırlatarak bağlantı kurması yeterli olmuştur. Öğretmen B ayrıca genel kuralı cebirsel olarak yazmayı öğretmek için 6. sınıftaki cebirsel ifadeler konusunu hatırlatararak bağlantı kurmuştur (UAB1). Bu bağlantıyı kurma sürecinde, Öğretmen B öğrencilerle birlikte örüntü kavramı için tanım geliştirmiştir ve belirli bir kurala dayalı olarak devam eden sayılar ve şekiller olarak tanımlamışlardır (UAK2). Öğretmen A yukarıda da belirtildiği gibi yeterli bir tanım oluşturamamıştır. Bununla birlikte her iki öğretmenin de genel terim ve genel kural tanımlamaları sadece ders kitabından seçip öğrencilere ifade etmek olarak görülmüştür, yani öğretmen ya da öğrenciler tarafından geliştirilmemiştir.

Öğretmenler örüntüdeki ilişkiyi öğretmek için, tablonun satırlarında adım sayısı ve terim arasındaki aritmetik ilişkiyi göstererek tablo gösterimini etkili bir şekilde kullanmıştır. Bununla birlikte, öğretmenler şekilleri ve materyalleri de örüntü gösteriminde kullanmışlardır; fakat şekillerin değişimi ile örüntüdeki ilişki bağlantısını kurmamışlardır (UAK2). Aslında şekillerin etkili kullanımında, öğretmenlerin şekillerin değişiminden yola çıkarak ilişkiyi öğretmeleri beklenmektedir. Bunun yanında, planlama aşamasında Öğretmen A grafik gösterimini de kullanacağını belirtmiş, ama kullanımını açıklayamamış ve derste de kullanamamıştır.

Öğretmenlerin örüntüdeki ilişkiyi açıklama şekilleri birbirinden farklı şekillerde ortaya çıkmıştır. Öğretmen A öğrencilere örüntüdeki ilişkinin çıktı değerleri arasında araştırıldığını açıklamıştır. Bu durum, bu öğretmenin fonksiyonel düşünmeye ilişkin konu alan bilgisinin eksikliğini göstermiştir. Öğretmen A sabit değişen örüntü genellemesini öğretirken sürekli terimler arasındaki farkı vurgulamış, ve bu fark ile n'yi çarpıp bu ifadeye bir sayı ekleme olarak genel kural bulmanın formülünü vermiştir. Diğer yandan, Öğretmen B genellikle adım sayısı ve terim arasındaki ilişkiye vurgu yapmıştır ve ayrıca genel kuralın fonksiyonunun tüm terimler için geçerli olduğunu açıklamıştır (UAK6). Sabit değişen örüntüleri genellemenin yanında, öğretmenler artarak değişen örüntüleri genelleme ile ilgili örnekler vermiştir. Bununla birlikte, Öğretmen B yalnızca bir örnek verirken, Öğretmen A öğrenciler için alan yazından değişik örnekler sunmuştur. Öğretmen A farklı örnekler sunmasına rağmen, artarak değişen örüntüleri genelleme de zorlanmış ve öğrencileri de sürekli n^2'yi kullanmaya yönlendirmiştir. Bu öğrenciler için kavramsal olmadığı için yeterli ve uygun bir öğretim olmamıştır.

Öğretmenler işlem işaretleri, eşitlik işareti ve üstel gösterimleri genellikle uygun bir şekilde kullanmışlardır (UAK5). Bununla birlikte, Öğretmen A'nın matematiksel dili yanlış kullanımları da ortaya çıkmıştır. Örneğin, Öğretmen A bilinmeyen ve değişken kavramlarını fonksiyonlarını düşünmeden birbirinin yerine kullanmış, sabit değişen örüntülerde ilişkinin değiştiğini ve artarak değişen örüntülerde de ilişkinin arttığını açıklamıştır. Ama, daha sonra artarak değişen örüntüler için yaptığı bu açıklamayı düzeltmiş ve artışın düzenli bir şekilde arttığını açıklamıştır. Diğer yandan Öğretmen B ise aritmetik gösterimlerde eşitlik işaretini uygun bir şekilde kullanmamıştır. Örneğin, -5.1=-5+65=60, ve -5.2=-10+65=55 ifadelerinde eşitlik işaretini sonuç bulma aracı olarak kullanmış, ilk yazdığı ifade ile eşitliğin en sonundaki ifadenin eşit olmadığını fark etmemiştir.

Cebirsel ifadelerle işlemlerin öğretiminde, Öğretmen A'nın cebirsel ifadelerle ilgili terimleri kullanma bilgisi yeterli bir şekilde ortaya çıkmazken, Öğretmen B'nin terim, benzer terim, sabit terim, ve katsayı kavramlarını uygun bir şekilde kullandığı görülmüştür. Öğretmen A sabit terimi bir terim olarak kabul etmemiş ve öğrencilere de bu şekilde öğretmiştir. Ayrıca iki öğretmen de bilinmeyen ve değişken kavramlarını fonksiyonlarını düşünmeden birbirinin yerine kullanmıştır (GAB2). Bu ortaya çıkan durumlar öğretmenlerin konu alan bilgi yetersizliğini ya da eksikliğini göstermiştir.

Cebirsel ifadelerde toplama ve çıkarma öğretiminde, başlangıç için öğretmenler örüntü genellemesiyle bağlantı kurdukları etkinliği kullanmıştır. Ancak, bu etkinliği beklenen şekilde uygulayamamışlardır. Aslında önce örüntüleri genelleyip daha sonra işlemler ile sonuç örüntüsünü göstermeleri gerekirken, önce işlem yapmışlar ve sonra örüntü genellemişlerdir. Bu yüzden iki konu arasındaki geçişi etkili bir şekilde yapamamışlardır. Bununla birlikte öğretmenlerin terim, sabit terim, katsayı ve değişken terimlerini hatırlatmak için kullandıkları örneklerle önceki konularla bağlantıyı etkili bir şekilde kullanmışlardır. Bu konunun öğretiminde öğretmenlerin önemli eksikliğinden biri de cebir karolarını uygun ve etkili bir şekilde kullanmamaları olmuştur. Öğretmenler bu karoların, alanları gösterdiğini açıklamamış ve sadece görsellik amacıyla karoları kullanmışlardır. Bu yüzden cebirsel gösterim ile geometrik gösterim arasında olması gereken bağlantı kurulamamıştır. Hatta görsellik amacıyla kullanırken bile, Öğretmen A çıkarma işlemini karolarla modelleyememiştir. Bu durum öğretmenlerin uzmanlık alan bilgilerindeki eksikliği göstermektedir (UAK7). Öğretmen B matematiksel dili uygun bir şekilde kullanarak, işlemleri açıklamak için borç ve net değer, ve farklı değişkenleri temsil etmek için elma-armut benzetmelerini kullanmıştır.

Cebirsel ifadelerle çarpma işlemi öğretiminde ise öğretmenlerin çarpmanın öğrenimi için önemli olan dağılma özelliğini açıklamaları farklılık göstermiştir. Çarpma işlemini açıklarken Öğretmen A tekrarlı çarpımla ilişki kurarken, Öğretmen B ise tamsayılarda dağılma özelliğini hatırlatarak ilişki kurmuştur. Ancak, öğretmenler bu konuda da cebir karolarını etkili ve amacına hizmet edecek şekilde kullanmamışlardır. Öğretmen A sadece tekrarlı toplamadan yararlanarak karoları kullanmış, Öğretmen B ise planında alan kavramı ile bağlantı kurulması gerektiğini açıklamışken cebir karolarını hiç kulanmamıştır.

Pedagojik Alan Bilgisi

Örüntü genellemesi öğretiminde, öğretmenlerin öğrencilerin örüntüyü cebirsel olarak genelleme konusunda zorluk yaşayacakları konusundaki tahminleri derste de görülmüştür. Ancak, öğretmenlerin genellemeyi açıklarken kullandıkları gerekçelendirmeler öğrencilerde bazı kavram yanılgılarına sebep olmuştur. Özellikle, Öğretmen A muhtemel kavram yanılgılarını dikkate almamıştır. Örneğin, Öğretmen A başlangıç örneklerinde sürekli 20.terimden hemen sonra genel terimi göstermiştir. Bu durum, öğrencilerin genel terimin hep 20.terimden sonra geldiğini düşünmelerine sebep olabilir ve genel terimin bir değişken olduğunu analamalarını engelleyebilir. Diğer bir örnek, Öğretmen A genel kuralı bulmak için, öğrencileri genellikle önce terimler arasındaki fark ile genel terimi çarpma ve 1.terimi verecek sayıyı bu ifadeye ekleme şeklinde yönlendirmiştir. Fakat, 4, 8, 12, 16 ... örüntüsünde her zaman bir sayı eklenmesine gerek olmadığını belirtmiştir. Bu öğretmenin kullandığı diğer bir yöntem de adım sayısı ile terim arasındaki farkı genel terime eklemek olarak görülmüştür. Örneğin, 3, 4, 5, 6,… örüntüsünde 3'ten 1'i çıkarıp elde ettiği fark 2'yi n'ye eklemiş ve genel kuralı n+2 olarak bulmuştur. İlk yöntem ezbere yönlendirmesine rağmen doğru kabul edilebilir, fakat diğer yöntemlerin matematiksel bir açıklaması görülmemektedir. Sadece bu örüntülere özgü bu yöntemleri kullanmıştır. Öğretmen A bu yüzden bu yöntemlerin öğrencilerde örüntü genellemesi ile ilgili kavram yanılgılarına sebep olabileceğini tahmin edememiştir ve bu da onun alan ve öğrenci bilgisinin yetersizliğine işaret etmiştir. Diğer yandan Öğretmen B sürekli ilk yöntemi kısa yol olarak kullanmış ve örüntü ilişkisinin girdi ve çıktı değerleri arasında araştırılması gerektiğini vurgulamıştır. Ancak öğrenciler bunu bir kural olarak algılayabilir ve özellikle bu yöntemin uygulanışında sadece 1.terimin dikkate alınması öğrencilerin genel kuralın tüm terimler için geçerli olduğunu anlamasına engel olabilir. Bu noktada, öğretemenin öğrenci bilgisi konusundaki yetersizliği görülmüştür. Yine de, iki öğretmenin de örüntülerin kuralını cebirsel olarak doğru bir şekilde ifade ettikleri söylenebilir. Ayrıca, öğretmenlerin öğrencilerin zorluklarını ve ihtiyaçlarını anlama bilgilerinin yeterli olduğu görülmüştür. Öğrencilerin kaçırdığı noktaları açıklamış ve öğrencilerin sorularını cevaplamışlardır. Bununla birlikte öğretemenler sürekli tablo gösterimini ve dolayısıyla sayısal akıl yürütmeyi kullanmışlardır. Halbuki bazı öğrencilerin şekilsel akıl yürütmeleri daha gelişmiş olabilir. Bu çalışmadaki öğretmenler şekilleri sadece görsellik amacıyla kullanmıştır. Bu noktada öğrencilerin ihtiyaçlarını dikkate almadıkları söylebilir.

PAB'ın diğer bir bileşeni olan alan ve öğretme bilgisi açısından ise öğretmenlerin dersleri için seçtikleri örnekler ve etkinlikler, öğretim yöntemleri, ve öğrencilerin yanlışlarını düzeltmek için kullandıkları yöntemler incelenmiştir. Genellikle Öğretmen A örüntü genellemesi için örneklerini ve ekinliklerini gelişimsel olarak uygun bir şekilde seçmiştir. Ama Öğretmen B'nin önce artarak değişen örüntü ile başlaması ve sonra sabit değişen örüntü ile devam etmesi gelişimsel olarak uygun görülmemiştir. Ancak, Öğretmen B'nin örüntü genellemesine ulaşırken sınıf tartışmasını etkili bir şekilde yönettiği görülmüştür. Bu öğretmen, 50.terimi sorarak öğrencileri genel bir kural bulmaya yönlendirmiş, öğrenciler zorluk yaşadığında tartışmayı durdurarak açıklama yapmış, ve n.terimi bulmada öğrencilere yardımcı olmuştur (AötB5). Bu şekilde tümdengelim yöntemiyle öğrencilerin genelleme kavramını yapılandırmaya çalışmıştır. Öğretmen B ayrıca öğrencilerin hatalarına zamanında müdahele ederek, onları düzeltmek için fonksiyonel düşünmeyi tekrar vurgulamıştır. Öğretmen B öğrencilerin öğrenmesini desteklemek için tartışma yöntemini ve farklı örnekler sunmayı önermiştir ve uygulamıştır. Ancak Öğretmen A ise gerçek hayat problemlerini kullanmayı önermesine rağmen bununla ilgili bir tasarım önerememiş ve uygulamamıştır.

Cebirsel ifadelerde toplama ve çıkarma öğretiminde, benzer terim kavramı önemli olmasına rağmen, Öğretmen A bu kavrama ilşkin öğrencilerin kavram yanılgısına sebep olabilecek açıklama yapmıştır. Öğretmen A benzer terim kavramını aynı tip değişkene sahip olma ile açıklamış ve üstel bir değişkenin de olabileceğini dikkate almamıştır. Örneğin x ve x^2 aynı tip değişkene sahip olmalarına rağmen üsleri farklı olduğu için benzer terim değildir. Ayrıca bu öğretmen bir cebirsel ifadede terimleri ifade ederken işaretlerini belirtmemektedir. Diğer yandan Öğretmen B kavram yanılgılarına sebep olabilecek durumları yeterli bir şekilde tahmin ettiğinden benzer terim kavramını, işlemleri, ve terimlerin işaretleriyle birlikte alındığını doğru bir şekilde açıklamıştır.

Cebirsel ifadelerle çarpma işlemi öğretimi için planlama aşamasında ise öğretmenlerin öğrencilerin dağılma özelliğini uygularken zorluk yaşabileceklerini doğru ifade etmişlerdir. Bunun yanında 4.x ifadesini iki basamaklı olarak düşünme ya da x ile x'in çarpımının 2x olduğunu düşünme gibi genel kavram yanılgılarını da doğru tahmin etmişlerdir.

Alan ve öğretme bilgisi açısından ise öğretmenlerin cebirsel ifadelerle işlemleri öğretmek için dersleri için seçtikleri örnekler ve etkinlikler uygun görülmüştür. Öğretmen A başlangıç için örüntü genellemesi ile ilişki kurmuş, ve gerçek hayat problemi kullanmıştır; fakat gelişme kısmında kullandığı örneklerde bilinen sayıları cebirsel olarak ifade ettiiği için bu kullanım uygun bulunmamıştır.

Öğretmen B ise başlangıçta diğer öğretmen gibi örüntü genellemesi ile ilişki kurmuş ve daha sonra tamsayılarda işlemlerin özelliklerini ilişki kurarak cebirsel ifadelerde işlemleri açıklamıştır. Öğretmen B ayrıca öğretim boyunca öğrencilerin yanlışlarını düzeltmek için uygun bir şekilde benzer terim kavramını vurgulamış ve açıklamıştır. Cebirsel ifadelerle çarpma işlemini öğretmek için ise Öğretmen A cebir karolarını kullanmış ama alan kavramı ile ilişkilendirmemiştir. Diğer yandan, Öğretmen B genelde dağılma özelliğinin uygulanmasına odaklanmış ve işlemsel olarak sürekli dağılma özelliğini gerektiren örnekler yapmıştır. Dolayısıyla, iki öğretmen de ders sonrası görüşmelerde cebir karolarının kullanımını bir yöntem olarak yeterli ve uygun olarak değerlendirememişlerdir (AötB4).

PAB'ın üçüncü bileşeni alan ve müfredat bilgisi açısından ise öğretmenlerin ders tasarımı için konu ve kazanım bilgilerinin yeterli olduğu görülmüştür (AMB1). Bunun yanında, Öğretmen A'nın öğrencilere önceki bilgilerini hatırlatarak ve önceki konuları vurgulayarak öğretimi sunması da uygun görülmüştür (AMB2).

TARTIŞMA VE ÖNERİLER

Bu çalışmada ortaokul matematik öğretmenlerinin öğretmek için matematik bilgisini araştırmak amacıyla öğretmenlerin örüntü genelleme ve cebirsel ifadelerle işlemler konusunu öğretmek için yaptıkları planlama ve uygulama analiz edilmiştir. Bu bölümde de elde edilen bulgular alan yazınla ilişkilendirilerek tartışılmıştır.

Öğretmek için Matematik Bilgisi

Örüntü genelleme öğretiminde, Öğretmen B tümdengelim yöntemini kullanmıştır. Öğretmen B önce ilk dört ya da beş terim için bir aritmetik ilişki bulmuş ve sonra bu ilişkiyi n.terimi sorarak genel kural ile ilişkilendirmiştir. Bu öğretmenin yöntemi alan yazında Radford'un (2008) tanımladığı genelleme süreciyle de örütşmektedir. Radford (2008) genellemeyi dışa çekme, dönüştürme, ve çıkarım yapma aşamalarından oluşacak şekilde açıklamıştır. Dışa çekme, örüntüdeki ilişkiyi fark etme; dönüştürme, ilişkiyi kullanarak örüntü devem ettirme, ve çıkarım yapma, cebirsel olaral genel bir kural bulmadır. Öğretmen B de önce terimler arsındaki farkı öğrencilere sormuş, sonra örüntüyü devamındaki birkaç terime genişletmiş ve son olarak da ilişkiyi cebirsel olarak ifade etmiştir. Ancak, öğretmenlerin genel terimi hem değişken hem de bilinmeyen olarak ifade ettikleri görülmüştür. Usiskin'e (1988; 1995) göre değişkenler örüntülerdeki ilişkiyi ararken kullanılan paametreler iken, denklem çözümlerinde ise bilinmeyendir. Dolasıyla öğretmenlerin bu kavramların anlamlarındaki farkı ile ilgili bilgilerinin yetersiz olduğu söylenebilir. Kavramlarla ilişkili olarak, Öğretmen B bazen eşitlik işaretini uygun kullanmamıştır. Ancak bu kullanım öğrencilerde eşitlik işaretinin fonksiyonuna ilişkin kavram yanılgısına sebep olabilir (Carpenter, Franke & Levi, 2003). Ayrıca, eşitlik işaretinin doğru kullanımının gelişimi, öğrencilerin denklem çözümünde de iyi bir performans göstermeleri için önemlidir (Alibali, Knuth, & Hattikudur, 2007).

Öğretmenlerin örüntüleri genellemek için kulandıkları tablo gösterimi de, satırlardaki adım sayısı ve terim arasındaki ilişkiyi fark ettirmek için etkili bir şekilde kullanıldığında Warren ve Cooper'ın (2008) da belirttiği gibi ilişkiyi bulmaya yardımcı olur. Bununla birlikte öğretmenler şekilsel gösterimi de kullanmış olmalarına rağmen, bu şekillerin arasındaki değişimi ilişki bulmada kullanmamışlardır. Hatta, Öğretmen A grafiksel gösterimi de kulanmayı planlamasına rağmen, bu gösterimi hiç kulanmamıştır. Aslında, Akyüz, Coşkun, ve Hacıömeroğlu (2009) çalışmalarında ulaştıkları sonuca göre, örüntüdeki ilişkinin tablo, grafik ve sayısal olarak gösterimleri arasında geçiş yapılması öğrencilerin genelleme kavramını anlamalarını geliştirdiği görülmüştür. Dolayısıyla, öğretmenler, gösterimleri amacına uygun bir şekilde kullanarak örüntüdeki ilişkiyi öğretme bilgilerini geliştirmelidir.

Örüntü genellemesi ile ilgili önemli konulardan biri de öğretmenlerin genel kurala ulaşırken yaptıkları açıklamalar olmuştur. Özellikle Öğretmen A örüntüdeki ilişkinin terimler arasında araştırılmasına vurgu yapmıştır. Benzer şekilde, Kutluk (2011) da yaptığı çalışmada öğretmenlerin, ya adım sayıları arasında ya da terimler arasında bir ilişki aradığını tespit etmiştir. Bu durumun öğretmenlerin konu alan bilgisindeki eksikliğe işaret ettiğini belirtmiştir. Bu eksikliğin nedeni olarak da öğretmenlerin fonksiyonel düşünme ile ilgili kavramsal bilgilerinin olmaması gösterilebilir (Wilkie, 2014). Smith (2008) fonksiyonel düşünmeyi "iki ya da daha fazla değişen nicelikler arasındaki ilişkiyi temsili düşünme" olarak açıklamıştır (s. 143). Örüntüyü genelleyebilme, girdi ve çıktı değerleri arasındaki fonksiyonel düşünmeyi gerektirmektedir (Greenes, Cavanagh, Dacey, Findell, & Small, 2001; Moss, Beatty, Barkin, & Shillolo, 2008; Rivera, 2010). Sabit değişen örüntüler yanında, öğretmenler artarak değişen örüntüleri de genellemiştir. Bu tip örüntüler müfredata göre 7.sınıf öğrencilerinin seviyelerine uygundur (MEB, 2013). Ayrıca ikinci dereceden ifadelerle cebirsel ifade yazmak cebirsel düşünmenin özelliklerinden biri olarak tanımlanmaktadır (Blanton & Kaput, 2005). Bununla birlikte, Öğretmen B bu örüntülere bir örnek vermiş, Öğretmen A ise farklı örnekler sunmuştur. Ancak, Öğretmen A bu örüntüleri genellerken zorlanmıştır. Öğretmenler, kendilerini artarak değişen örüntüleri genelleme stratejileri konusunda kendilerini geliştirebilirler ve Gierden'in (2012) de çalışmasında açıkladığı ve örneklediği gibi bir bağlam içinde sistematik bir yol izleyebilirler.

Öğretmenlerin öğrenci bilgileri düşünüldüğünde ise, genellikle öğrencilerin yaşayabileceği zorluklarla ilgili tahminleri daha önce yapılan çalışmalarda da ortaya çıkmıştır. Birçok çalışma, öğrencilerin örüntüleri cebirsel olarak genellemede zorlandıklarını göstermiştir (Çayır & Akyüz, 2015; Harel, 2001; Jurdak & El Mouhayar, 2014; Stacey, 1996; Steele & Johanning, 2004). Ancak, özellikle Öğretmen A yaptığı açıklamaların öğrencilerde kavram yanılgısına sebep olabileceğini öngörememiştir. Öğretmen A'nın örüntü genellemesine ulaşmak için kullandığı ve fonksiyonel düşünmeye uygun olmayan yöntemler, alan ve öğrenci bilgisinin eksikliğini göstermiştir. Zaten yapılan çalışmalar da öğretmenlerin, öğrencilerin stratejilerini tanımlayabildikleri fakat açıklayamadıkları sonucuna ulaşmıştır (Baş, Erbaş, & Çetinkaya, 2011; Callejo & Zapatera, 2016; El Mouhayar & Jurdak, 2013). Diğer yandan, Öğretmen B fonksiyonel ilişkiyi doğru bir şekilde göstermiş ama kısa yöntemi kullanmıştır. Bu yöntemde sadece 1. terime odaklanmak Wilkie'nin (2014) de uyardığı gibi öğrencilerin genel kuralın tüm terimler için geçerli olduğunu anlamasına engel olabilir. Yine de bu çalışmadaki öğretmenler, genel kurala ulaşmışlardır. Halbuki çalışmalar, öğretmenlerin ve öğretmen adaylarının çoğunun genel kural bulmada ve gerekçelendirmede zorlandığını göstermiştir. (Barbosa & Vale, 2015; İmre & Akkoç, 2012; Kirwan, 2015; Kutluk, 2011; Magiera et al., 2013; Rivera & Becker, 2007; Tanışlı & Köse, 2011; Wilkie, 2014). Öğrencilerle ilgili diğer bir nokta da şekil örüntülerinin kullanımıyla ilgilidir. Araştırmacılar, şekillerdeki birimlerin nasıl bir araya geldiğini ve şekillerin hangi ilişkiye göre devam ettiğini açıklamanın ve böylece öğrencilerin şekilsel düşünmelerini sağlamanın genel kuralı kavramsal olarak elde etmeyi sağladığını vurgulamaktadır (Barbosa & Vale, 2015; Rivera & Becker, 2005; Thornton, 2001; Walkowiak, 2014). Ancak mevcut çalışmada da olduğu gibi, öğretmenler genellikle sayısal akıl yürütme ile genellemeyi öğretmekte, şekilsel akıl yürütmeyi kullanmamaktadır (Baş et al., 2011; Kutluk, 2011).

Alan ve öğretme bilgisi açısından incelendiğinde, öğretmenlerin sadece kitaplardaki örnekleri kullandıkları görülmüştür. Wilkie (2014) de yaptığı çalışmada, öğretmenlerin fonksiyonel düşünmeyi içeren etkinlik oluşturmada yeterli bilgiye sahip olmadığı sonucuna ulaşmıştır.

Cebirsel ifadelerle işlemler öğretiminde önemli problemlerden biri ise öğretmenlerin cebir karolarını uygun bir şekilde kullanamamalarıdır. Öğretmenler cebir karolarında çarpmayı öğretmek için alan kavramını açıklamamışlardır. Aslında, geometrik düşünme ile matematiğin öğrenme alanları arasında bağlantılar kurmak matematik öğretiminin amaçlarından biridir (NCTM, 2001). Özellikle toplama ve çıkarma işlemlerini yaparken Öğretmen B’nin borç-net değer ve elma-armut benzetmelerini kullanması Ojose’nin (2015) de önerdiği günlük hayattan somut örneklerle benzer terim kavramını öğretme yöntemiyle örtüşmektedir. Ancak, bu önerilerin tersine olarak Tirosh ve arkadaşları (1998) bu tarz benzetmeleri kullanarak meyve salatası yöntemini kullanmanın öğrencilerin kafasını karıştırdığı sonucuna varmışlardır. Çünkü bu yöntemin bağlantısal ve kavramsal olmadığını ileri sürmüşlerdir. Çarpma işleminde ise önemli durumlardan biri dağılma özelliğinin uygulanması olmuştur. Alan modellemesi ile dağılma özelliğinde cebirsel ifadelerin eşitliğini keşfettirmek öğrencilerin anlamasını geliştirir (Hallagan, 2004). Ancak öğretmenler cebir karolarını bu şekilde kullanmamıştır. Yine de, Öğretmen B'nin dağılma özelliğini anlatırken tamsayılarla ilişki kurması da öğrencilerin öğrenmesine yardımcı olan yöntemlerden biridir (Ojose, 2015).

Aynı şekilde cebirsel ifadelerde de benzer terim ve sabit terim ile ilgili yaptığı açıklamaların öğrencilerde yanlış anlamalara sebep olabilceğini Öğretmen A düşünememiştir. Fakat Tirosh vd.'nin (1988) çalışmasında da görüldüğü gibi öğrencilerin cebirsel ifadeleri sadeleştimede zorlanmasına sebep olabilir. Diğer yandan Öğretmen B'nin sürekli benzer terimi vurgulaması ve öğrencilerin anlamasını kolaylaştırmak için parantezleri doğru bir şekilde kullanması da önerilen yöntemlerden biridir (Hoch & Dreyfus, 2004; Marchini & Papadopoulos, 2011). Özellikle çarpma işleminde öğretmenlerin planlamada belirttiği kavram yanılgıları önemli olmakla birlikte, diğer birçok muhtemel kavram yanılgılarının da farkında olmalı ve derslerinde bunlara dikkat etmelidirler (MacGroger & Stacey, 1997; Seng, 2010).

Alan ve öğretme bilgisi açısından bakıldığında, öğretmenlerin cebirsel ifadelere başlarken benzer terim kavramını hissettirmek için günlük hayattan örnekler (yumurta ve zeytin gibi) ya da somut modeller (farklı renklerde birim küpler gibi) kullanmıştır. Benzer şekilde, Filloy ve Sutherland (2006) cebirde benzer olmayan terimlerin toplamasını açıklamak için somut modellerin kullanımı önermektedir. Özellikle, Ashlock'un (2001) öğrencilerin öğrenmesini desteklemek için önerdiği gibi, çarpma işleminde geometri ile ilgili problemleri çözerken Öğretmen B öğrencilerin alan ve çevre kavramları ile ilgili multemel kavram yanılgılarını düşünerek, açıklamalar yapmış ve bu hataları düzeltmiştir.

Genel olarak, öğretimden verilen örnek durumlar öğretmenlerin yeterli kavramsal genel alan bilgisi ve uzmanlık alan bilgisine sahip olmaları, öğretimlerini olumlu bir şekilde etkilemektedir. Öğretmenler güçlü bir konu alan bilgisine sahip olduğunda, öğretim esnasında öğerencilerin düşünmelerini dikkate almışlar (AÖB) ve öğretim yöntemlerini etkili bir şekilde kullanmışlardır (AötB). Even (1993) de bu bulguyu destekleyecek şekilde aday öğretemenlerin konu alan bilgisini yapısalcı yaklaşımla geliştirmeyi ve öğretimde pedagojik kararlar için bu kavramsal bilgiyi kullanmayı önerir. Benzer şekilde, büyük ölçekli çalışmalar da, alan bilgisi ile pedagojik alan bilgisinin güçlü ve olumlu bir şekilde ilişkili olduğunu bumuşlardır (Blömeke & Kaiser, 2012; Krauss, Baumert, & Blum, 2008). Mevcut çalışmanın sonuçları da nitel olarak sağlam bir alan bilgisinin pedagojik alan bilgisi üzerinde olumlu bir etkiye sahp olduğunu göstermiştir. Özel olarak bu çalışmalardaki cebir konuları için, öğretmenlerin yeterli bir gösterim bilgisine sahip olmamaları (UAK2 ve UAK7), örüntü için şekilsel ve grafiksel; cebirsel ifadeler için cebir karoları gibi gösterim bilgilerinin eksikliği, onların öğretim yöntemlerini kısıtlamış ve etkili olmasını engellemiştir (AÖtB4). Ya da öğretmenlerin yeterli ve kavramsal bir fonksiyonel düşünmeye sahip olmaları (KAB), sınıf tartışmalarında öğrencileri doğru ve etkili bir şekilde yönlendirebilme, muhtemel kavram yanılgılarını fark edip düzeltebilme, ve soru ve problemleri doğru bir şekilde çözebilmelerini sağlamıştır (PAB).

Öğretim Pratikleri

Genel olarak öğretmenlerin uygulamada ortaya çıkan pratikleri ve bilgi türleri benzerlik göstermekle birlikte, ortaya çıkan bilgilerde eksiklik ve yetersizlik olduğu için pratikleri aynı şekilde etkili olmamıştır. Özellikle Öğretmen A'nın sayı olarak daha fazla pratiği görünmesine rağmen, kavramsal bilgi eksiklikleri uygulamadaki pratikleri olumsuz etkilemiştir. Öğretmenlerin kavramsal bilgilerindeki eksiklikler onların öğretimini sınırlamaktadır (Aslan-Tutak & Ertaş, 2013). Bu durum, cebir öğretiminde bilgi ile pratik arasındaki çelişkilerden birine de işaret etmektedir (Doerr, 2004). Etkili olmayan fakat sayı olarak daha fazla olan pratiklerin sebebi olarak, öğretmenin kendisinin de böyle öğrenmiş olması ya da ders kitabının önerdiği herşeyi öğretmenin yapması olarak düşünülebilir.

Bu çalışmada, öğretmenlerin gösterimler ile ilgili bilgi eksikliğinin, öğretimlerindeki etkililiği azalttığı görülmüştür. Halbuki, model ve temsillerle matematiksel olarak bağlantı kurulan bilgi ortaokul matematiğini öğrenmeyi olumlu etkilemektedir (Tchoshanov, 2011). Ya da yeterli ve kavramsal bilgiye sahip olan öğretmen, tartışma pratiğinde öğrencileri doğru bir şekile yönledirebilmiş ve tartışmayı etkili bir şekilde yönetebilmiştir. Öğrencilerin düşünmesini sağlamak için sorgulamak, öğrencilerin cebir öğrenmesini desteklemektedir (Brown & Smith, 1997). Özellikle, soru ve problem çözümlerini içeren pratiklerde, bilgisi kavramsal olan öğretmen, doğru matematiksel açıklamalar yapabilmiş ve öğrencilerin hata ve kavram yanılgılarını zamanında fark ederek düzeltmiştir. Franke vd. (2007) öğrencilerin düşünme bilgisinin, öğretemenlerin uygulamalarını geliştirmede onlara yardımcı olduğunu belirtmektedir. Öğretmenlerin etkili görevlerle derslerini tasarlamaları için, iyi bir kavramsal bilgiye ve öğrenci bilgisine ihtiyaçları vardır (Hiebert, 1997). Öğretim uygulamalarını geliştirmek için öğretmenelerin uygun ve yeterli bilgi sahip olmaları sağlanmalıdır (Doerr, 2004).

Öneriler

Bu çalışmanın bulgularının, matematik öğretmeni bilgisi alanına, özellikle de öğretmenlerin cebir bilgisi alanına katkı yaptığı düşünülebilir. Örüntü genellemesi ve cebirsel ifadelerle işlemler konularına özgü öğretmenlerin sahip olması gereken bilgi türlerini önererek, ortaokul matematik öğretmenlerinin cebir bilgisindeki alan yazına katkı sağladığı söylenebilir.

Bu çalışmanın sonuçları, öğretmenlerin kendisini ve dolayısıyla öğretim uygulamalarını geliştirmesi için öğretmen bilgisinin önemini göstermiştir. Matematik öğretmeni eğitimcileri de bu bulguyu dikkate alarak programlarındaki özel öğretim yöntemleri ve okul uygulamaları derslerini öğretmen bilgisini daha çok dikkate alarak tasarlayabilirler. Hatta, cebir, sayı, geometri gibi belirli matematik alanlarına özgü dersler de planlayabilirler. Benzer şekilde, okullarda çalışan öğretmenlerin de bilgilerini geliştirmeye yönelik eğitimler düzenlenebilir. Ayrıca, öğretmenlerin bir araya gelmesi, deneyimlerinden iyi örnekler paylaşmaları, ve birbirlerine zorlandıkları yerlerde neler yapılabileceğine dair öneriler sunmaları sağlanabilir. Hatta, matematik eğtimi araştırmacıları da alan yazından örnekler, etkinlikler, yöntem ve teknikler sunarak öğretmenlere yardımcı olabilir. İleri araştırmalarda, öğretmenlere yardımcı olmak amacıyla ders imecesi şeklinde çalışmalar yapılabilir. Diğer bir öneri de, tasarım tabanlı araştırmalar kapsamında öğretmenlerle işbirliği içerisinde, konu öğretiminde kullanabilecekleri öğrenme döngüsü oluşturulabilir.

Gelecek araştırmalarda, öğretmenlerin planlama pratikleri de araştırılarak, öğretim pratikleriyle birlikte incelenebilir. Bu çalışmanın odağı sadece öğretmenler olduğu gibi, ayrıca öğrenci açısından da öğrencilerin öğrenmesini araştırmak için deneysel çalışmalar yapılabilir. Böylece nicel boyutundan da bu konu araştırılmış olur. Bu çalışma, araştırmanın amacı nedeniyle iki öğretmenin öğretimiyle sınırlandırılmıştır. Dolayısıyla nicel olarak öğretmen sayısı arttırılarak KAB ve PAB odaklı testlerle öğretmenlerin cebir bilgisi konusunda genel sonuçlar da elde edilebilir. Hatta, denklemler, eğim, doğrusal denklemler gibi diğer cebir konuları da kapsayarak öğretmen bilgisine ilişkin daha geniş bir bakış açısı da sunulabilir. İleride yapılacak araştırmalarda, ortaokul sürecinde öğrencilerin cebiri anlamasını geliştirmeye yardımcı olmak için, öğretmenlerin kapsamlı alan bilgisi de incelenebilir.

APPENDIX J: CURRICULUM VITAE

PERSONAL INFORMATION

Surname, Name: Girit, Dilek
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Date and Place of Birth: 30 October 1986
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dilekgirit@gmail.com

EDUCATION

Degree	**Institution**	**Year of Graduation**
MS	Eskişehir Osmangazi University, Elementary Mathematics Education	2011
BS	Anadolu University, Elementary Mathematics Education, Eskişehir	2008
High School	Tuğlacılar High School, Tekirdağ	2004

WORKING EXPERIENCE

Year	**Place**	**Enrollment**
2012- Present	METU Department of Elementary Education	Research Assistant
2011 - 2012	Trakya University Department of Elementary Mathematics Education	Research Assistant
2008 - 2011	Tekirdağ 13 Kasım Middle School	Mathematics Teacher

LANGUAGES:

English (advanced).

COMPUTER SKILLS:

SPSS, Microsoft Office Programs, Dynamic Geometry Software (GeoGebra, Capri, Capri3D, Geometer's Sketchpad)

ACADEMIC INTERESTS:

Teacher knowledge, teacher development, teaching and learning algebra and decimals.

AWARDS

Tuğlacılar High School Student Achievement Award, Second Prize, 2004

Anadolu University Student Achievement Award (BS), First Prize, 2008

Eskişehir Osmangazi University Student Achievement Award (MS), First Prize, 2011

TUBITAK MSc Scholarship (2009 –2011)

TUBITAK Phd Scholarship, (2012 – 2016)

THESIS

Girit, D. (2011). The effect of quantum learning on middle school students' academic success, anxiety and attitude towards mathematics, Unpublished Master's Thesis, Eskişehir Osmangazi University, Eskişehir.

PRESENTATIONS

Girit, D. & Akyüz, D. (2013). Students' conceptions about decimals and connection with fractions. *ECER*, İstanbul.

Girit, D. & Akyüz, D. (2014). Prospective elementary mathematics teachers' opinions and recommendations about students' knowledge of the relationship among fraction, decimal representations and percentage of the concept. *13th Mathematics Sympozium*, Karabük, Turkey.

Girit, D. & Akyüz, D. (2014). The strategies for generalization of patterns of middle school students at different grades. *XI. UFBMEK Ulusal Fen Bilimleri ve Matematik Eğitimi Kongresi (International Congress of Science and Mathematic Education),* Adana, Turkey.

Girit, D. & Akyüz, D. (2015). An examination of teachers' mathematical knowledge for teaching patterns in lesson planning. *ISER (International Society of Educational Research) – World Conference on Education,* İstanbul, Turkey.

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ARTICLES

Girit, D. & Yenilmez, K. (2013). Mathematics Teachers' Opinions about New Sub Learning Domains in Elementary Mathematics (6-8) Curriculum. *Ondokuz Mayıs University Education Faculty Journal, 32*(2), 385-419.

Girit, D. & Akyüz, D. (2016). Pre-Service middle school mathematics teachers' understanding of students' knowledge: Location of decimal numbers on a number line. *International Journal of Education in Mathematics, Science, and Technology*, *4*(2), 84-100.

Girit, D. & Akyüz, D. (in press). Algebraic thinking in middle school students at different grades: Conceptions about generalization of patterns. *Necatibey Faculty of Education Electronic Journal of Science and Mathematics Education.*

APPENDIX K: TEZ FOTOKOPİSİ İZİN FORMU

ENSTİTÜ

Fen Bilimleri Enstitüsü [ ]

Sosyal Bilimler Enstitüsü [x]

Uygulamalı Matematik Enstitüsü [ ]

Enformatik Enstitüsü [ ]

Deniz Bilimleri Enstitüsü [ ]

YAZARIN

Soyadı : Girit
Adı : Dilek
Bölümü : İlköğretim

TEZİN ADI (İngilizce) : Investigating Middle School Mathematics Teachers' Mathematical Knowledge for Teaching Algebra: A Multiple Case Study

TEZİN TÜRÜ : Yüksek Lisans [ ] Doktora [x]

1. Tezimin tamamından kaynak gösterilmek şartıyla fotokopi alınabilir. [ ]
2. Tezimin içindekiler sayfası, özet, indeks sayfalarından ve/veya bir bölümünden kaynak gösterilmek şartıyla fotokopi alınabilir. [ ]
3. Tezimden bir bir (1) yıl süreyle fotokopi alınamaz. [x]

TEZİN KÜTÜPHANEYE TESLİM TARİHİ: