# CLASSIFICATION OF SKEW-SYMMETRIC FORMS CORRESPONDING TO CLUSTER ALGEBRAS WITH PRINCIPAL COEFFICIENTS

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# ABSTRACT

CLASSIFICATION OF SKEW-SYMMETRIC FORMS CORRESPONDING TO CLUSTER ALGEBRAS WITH PRINCIPAL COEFFICIENTS

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In this thesis, we study algebraic and combinatorial properties of the skew-symmetric forms that correspond to cluster algebras with principal coefficients. We obtain a classification of these forms under congruence and compute the Arf invariants for finite types.

Keywords: Cluster Algebra, skew-symmetric form, Arf invariant

# ÖZ

TEMEL KATSAYILI KLASTER CEBİRLERİNE KARŞILIK GELEN ANTİSİMETRİK FORMLARIN SINIFLANDIRILMASI

Mazı, Sedanur
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Bu tezde, temel katsayılı klaster cebirlerine karşılık gelen antisimetrik formların cebirsel ve kombinatoryal özelliklerini çalışıyoruz. Bu formları sınıflandırıp, sonlu tipler için Arf değişmezlerini hesaplıyoruz.

Anahtar Kelimeler: Klaster cebiri, antisimetrik form, Arf değişmezi

To My Mother Sevim Mazı and Father Naci Mazı

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# TABLE OF CONTENTS

# LIST OF FIGURES

FIGURES

# CHAPTER 1

# INTRODUCTION

Cluster algebras are commutative rings which have applications in many different areas of mathematics. For example; they provide a natural algebraic set-up to study recursively defined rational functions in combinatorics and number theory. In geometry, they introduce natural Poisson transformations. In topology, they formalise symmetries between triangulations of surfaces. In representation theory, they form a natural algebraic framework to study positivity.

A cluster algebra is uniquely determined by a class of skew-symmetric integer matrices, called a mutation class; here mutation is a certain operation on skew-symmetric matrices, it can also be viewed as an operation on certain graphs. Many important results and problems on cluster algebras can be described in terms of mutations, Therefore it is natural to study algebraic and combinatorial properties of these classes.

In this thesis, we study linear algebraic properties of the matrices in the mutation classes corresponding to the "cluster algebras with principle coefficients", which are the most fundamental type of cluster algebras. These matrices can be obtained from "principle extensions of skew-symmetric matrices" by a sequence of mutations (Definition 6.1). We obtain a classification of these matrices under congruence (Theorem 6.8). Furthermore, we consider the corresponding skew-symmetric forms over the two-element field and compute the Arf invariants for finite types (Theorem 6.11).

# CHAPTER 2

# SKEW-SYMMETRIC MATRICES AND BILINEAR FORMS

As we described in Section 1, the goal of this thesis is to establish some algebraic and combinatorial properties of skew-symmetric matrices with integer entries. These matrices have some special properties which are not commonly studied in standard linear algebra texts. Therefore, in this chapter, we collect some basic linear algebraic properties of skew-symmetric matrices over integers.

## 2.1 Basic definitions

In this thesis, we will only use matrices with integer entries. By a skew-symmetric matrix, we mean the following:

**Definition 2.1** *Let $B = (B_{i,j})$ be a $n \times n$ matrix over integers. $B$ is called skew-symmetric if $B^T = -B$ (so $B_{i,j} = -B_{j,i}$ for all $i, j$ and $B_{i,i} = 0$ for all $i$).*

To give an example, a $2 \times 2$ skew-symmetric matrix is a matrix of the form

$$\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$$

We will use the following terminology for the vectors in the nullspace of a skew-symmetric matrix:

**Definition 2.2** *Suppose that $B$ is a skew-symmetric $n \times n$ matrix. A vector $v \in \mathbb{Z}^n$ is called a "radical vector" for $B$ if $Bv = 0$.*

Note that, according to this definition, a radical vector is an integer vector. Also zero vector is a radical vector. The existence of non-zero radical vectors can be characterized by the determinant:

**Proposition 2.3** *Suppose that $B$ is a skew-symmetric integer matrix. Then $B$ has a nonzero radical vector if and only if* $\det(B) = 0$.

**Proof.** For the proof, we consider $B$ as a matrix over the field $\mathbb{Q}$. Then, it is well-known from linear algebra that, if $B$ has a radical vector, then $\det(B) = 0$. For the converse, suppose that $\det(B) = 0$. Then, by linear algebra, there exists a vector $X = (x_1, \ldots, x_n) \neq 0$ in $\mathbb{Q}^n$ such that $BX = O$. Let us assume that $x_i = p_i/q_i$, $i = 1, \ldots, n$ for some $p_i, q_i$ in $\mathbb{Z}$ and let $a = \operatorname{lcm}(q_1, \ldots, q_n)$. Then $v = aX$ in $\mathbb{Z}^n$ and it is a radical vector for $B$, because $Bv = B(aX) = a(BX) = 0$. $\square$

Let us also recall from standard linear algebra that if $B$ is a skew-symmetric $n \times n$ matrix and $n$ is odd, then $\det(B) = 0$, so it has a non-zero radical vector.

### 2.2 Congruence of skew-symmetric matrices

The definition of congruence for skew-symmetric matrices over a field is well-known in linear algebra: $B$ and $B'$ are "congruent" if $P^T BP = B'$ for some invertible matrix $P$. Since we work with matrices over integers, we require that $P$ is also an integer matrix; then $P$ is invertible if and only if $\det(P) = \mp 1$. Thus, in this thesis, we take the following as the definition of congruence:

**Definition 2.4** *Let $B$ and $B'$ be skew-symmetric $n \times n$ integer matrices. We say that $B$ and $B'$ are "congruent" if $P^T BP = B'$ for some invertible integer matrix $P$ with* $\det(P) = \mp 1$. *This defines an equivalence relation on $n \times n$ skew-symmetric integer matrices.*

Let us also note that rank is preserved under congruence, i.e. rank of $B$ is equal to rank of $B'$.

### 2.3 Classification of skew-symmetric matrices under congruence

Skew-symmetric matrices over integers are classified under congruence relation as follows:

**Theorem 2.5** *( Bourbaki, Algebra Ch. 9) Let $B$ be skew-symmetric integer matrix. Then there is an $n \times n$ integer matrix $P$ with $det(P) = \mp 1$ such that $P^T BP$ is of the form.*

$$\begin{pmatrix} 0 & d_1 & & & & & & & & \\ -d_1 & 0 & & & & & & & & \\ & & 0 & d_2 & & & & & & \\ & & -d_2 & 0 & & & & & & \\ & & & & \ddots & & & & & \\ & & & & & 0 & d_m & & & \\ & & & & & -d_m & 0 & & & \\ & & & & & & & 0 & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & 0 \end{pmatrix}$$

*where $d_i$ divides $d_{i+1}$, for $i = 1, 2, ..., n - 1$. Furthermore the integers $d_i$'s are uniquely determined. (Note that $2m$=rank of $B$.) We call this matrix* ”the normal form of $B$ with respect to congruence”.

As we described in Section 1, the goal of this thesis is to determine normal forms of a special class of matrices, called principal extensions of skew-symmetric matrices; we will obtain these normal forms in Chapter 6.

### 2.4 Skew-symmetric forms defined by skew-symmetric matrices

We will obtain the main results of this thesis (Chapter6) by using geometric properties of skew-symmetric matrices. To study these properties, we will use, as it is common in linear algebra, skew-symmetric forms defined as follows:

**Definition 2.6** *Suppose that $B$ is skew-symmetric integer matrix and let*

$$\mathscr{B} = \{e_1, e_2, ..., e_n\}$$

*be the standard basis for $\mathbb{Z}^n$. We denote by $\langle , \rangle$ the skew-symmetric bilinear form on $\mathbb{Z}^n$ defined on the standard basis vectors by $\langle e_i, e_j \rangle = B_{ij}$ for all $i, j = 1, \ldots, n$.*

Let us note that, for all $v, w$ in $\mathbb{Z}^n$, we have $\langle v, w \rangle = v^T B w$. Also $\langle v, w \rangle = -\langle w, v \rangle$ and $\langle v, v \rangle = 0$ for all v$\in \mathbb{Z}^n$. Furthermore, if $v$ is radical vector then $\langle v, w \rangle = 0$ for all $w \in \mathbb{Z}^n$.

## 2.5 Matrix of a basis

Once the form $\langle , \rangle$ is defined by its values on the standard basis vectors, we have an associated skew-symmetric matrix for any other basis as well:

**Definition 2.7** *In the set up of Definition 2.6, suppose $\xi = \{f_1, f_2, ..., f_n\}$ is a $\mathbb{Z}$-basis for $\mathbb{Z}^n$. We define the matrix of $\xi$ with respect to $\langle , \rangle$ as the skew-symmetric matrix*

$$\begin{bmatrix} \langle f_1, f_1 \rangle & \ddots & \langle f_1, f_n \rangle \\ & \ddots & \\ \langle f_n, f_1 \rangle & \ddots & \langle f_n, f_n \rangle \end{bmatrix}$$

Note that the matrix of the standard basis $\{e_1, \ldots, e_n\}$ with respect to $\langle , \rangle$ is $B$.

Let us also recall from linear algebra that changing the basis affects the corresponding matrices as follows:

**Proposition 2.8** *Suppose $B$ is the matrix of the basis $\mathscr{B} = \{f_1, \ldots, f_n\}$ and $B'$ is the matrix of the $\mathscr{B}' = \{f'_1, \ldots, f'_n\}$. Let $T$ be the linear transformation defined by $T(f_i) = f'_i$ for all $i$ and $P$ be the matrix of $T$ with respect to $\mathscr{B}$. Then $B' = P^T B P$ (or $B = (P^{-1})^T B' (P^{-1})$), so $B$ is congruent to $B'$.*

This property will be the basis of our approach to study congruence of skew-symmetric matrices, i.e. we will view skew-symmetric matrices as matrices of a skew-symmetric bilinear form and study the effects of certain linear transformations on them. The main type of transformations we use will be the following:

**Proposition 2.9** *Let $V = \mathbb{Z}^n$ and $\mathscr{B} = \{f_1, ..., f_n\}$ be a $\mathbb{Z}$-basis for $V$ and $k \in \mathbb{Z}$. Let $T : V \to V$ be the linear transformation defined on the basis vectors as follows: $T(f_i) = f_i + kf_j$ for some $i, j = 1, \ldots, n$ and $k \in \mathbb{Z}$; $T(f_r) = f_r$ for $r \neq i$ . Then $T$ is invertible over $\mathbb{Z}$.*

**Proof.** Let us assume without loss of generality $i < j$. Let us note that the matrix of $T$ with respect to $\mathscr{B}$ is the lower triangular matrix $A$ with $A_{ii} = 1$ for all $i$ , $A_{ji} = k$ and the rest of the entries is zero. Then $\det(A) = 1$, so it is invertible over $\mathbb{Z}$. □

## 2.6 Symplectic bases

As it is well known in linear algebra, congruent skew-symmetric matrices can be viewed as the matrices of a skew-symmetric bilinear form with respect to some bases (Proposition 2.8). In particular, the matrices given in Theorem 2.5 can also be viewed as matrices of certain bases; these are called symplectic bases. More precisely:

**Definition 2.10** *Let $B$ be skew-symmetric matrix of size $n$ and $\langle , \rangle$ be the corresponding form on $\mathbb{Z}^n$ (Definition 2.6). Let*

$$\mathcal{C} = \{u_1, v_1, ...u_m, v_m, w_1...w_p\}, (p = n - 2m),$$

*be a basis such that the matrix of $\mathcal{C}$ with respect to $\langle , \rangle$ is as in Theorem 2.5. Then $\mathcal{C}$ is called a* symplectic basis *for $\langle , \rangle$.*

In Chapter 6, we will determine normal forms of the matrices called principal extensions of skew-symmetric matrices by establishing symplectic bases.

# CHAPTER 3

# SKEW-SYMMETRIC MATRICES AND GRAPHS

In this section, we will give some definitions that we will use to study combinatorial properties of skew-symmetric matrices. The main idea of this approach is to view a skew-symmetric matrix as a graph [3].

## 3.1 Graph of a skew-symmetric matrix

**Definition 3.1** *Let $B$ be a skew-symmetric $n \times n$ integer matrix. We define the graph of $B$ as the graph $\Gamma(B)$ with no loops or 2-cycles such that*

- *vertices are $1, \ldots, n$,*
- *there is an edge $i \xrightarrow{b} j$ if and only if $B_{ji} = b > 0$.*

*We call $b$ the weight of the edge; if $b = 1$, we do not specify it on the edge.*

**Example 3.2** *The following are some skew-symmetric matrices and the corresponding graphs:*

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 2 \\ 1 & -2 & 0 \end{bmatrix}$$

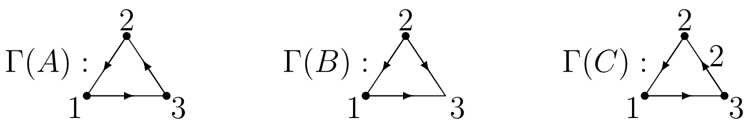



Figure 3.1: Examples of graphs of skew-symmetric matrices

## 3.2 Graph of a basis with respect to a skew-symmetric form

We consider skew-symmetric matrices as matrices of a skew-symmetric forms with respect to bases, therefore ,for convenience, we use the following analogue of Definition 3.1:

**Definition 3.3** *Let$B$ be a skew-symmetric $n \times n$ integer matrix and let $\langle , \rangle$ be the skew-symmetric form on $\mathbb{Z}^n$ given in Definition 2.6 . Let $\mathscr{B} = \{f_1, ..., f_n\}$ be a $\mathbb{Z}$-basis for $\mathbb{Z}^n$ . We define $\Gamma(\mathscr{B})$ as the graph with no loops or 2-cycles such that*

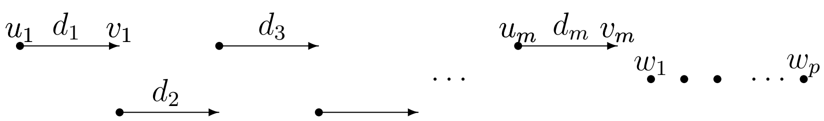
- *vertices are the basis vectors $f_1, \dots, f_n$,*
- *there is an edge $i \xrightarrow{b} j$ if and only if $\langle f_j, f_i \rangle = b > 0$.*

*if $b = 1$, we do not specify on the edge.*

Note that if $\mathscr{B} = \{e_1, ..., e_n\}$ is the standard basis, then $\Gamma(\mathscr{B})$ is equal to $\Gamma(B)$ (after relabeling each vertex $i$ by $e_i$).

## 3.3 Graphs of symplectic bases

Let us note the following special case of Definition 3.3: if $\mathcal{C} = \{u_1, v_1, ...u_m, v_m, w_1...w_p\}$ $(p = n - 2m)$ is a symplectic basis (Definition 2.10), then $\Gamma(\mathcal{C})$ is as follows:



# CHAPTER 4

# SKEW-SYMMETRIC MATRICES OVER $\mathbb{F}_2$ AND THE ARF INVARIANT

In this thesis, we study skew-symmetric matrices over integers also by considering their modulo 2 reductions. This allows us to use linear algebra over the two-element field $\mathbb{F}_2$ and obtain one of our main results (Theorem 6.11).

As it is well known, linear algebra over the field $\mathbb{F}_2$ requires some care because of its some peculiar properties. One of these properties is the skew-symmetry of a matrix: if we consider the definition in Section 2.1 (Definition 2.1) over $\mathbb{F}_2$, the diagonal elements of a skew-symmetric matrix need not be equal to 0. Therefore, to consider modulo 2 reductions of skew-symmetric integer matrices (which are the main objects of study in this thesis), we use linear algebra of *alternating* matrices:

**Definition 4.1** *Let $B$ be a matrix of size $n$ over $\mathbb{F}_2$={0,1}. We call $B$* alternating *if*

- $B_{ii} = 0$ *for* $i = 1, \ldots, n$,
- $B_{ij} = -B_{ji}$ *for all* $i \neq j$.

*Thus, if $A$ is a skew–symmetric matrix over integers, then $A \mod 2$ is an alternating matrix.*

**Example 4.2** *Let $A$ be the matrix*

$$\begin{bmatrix} 0 & 3 & -5 \\ -3 & 0 & 1 \\ 5 & -1 & 0 \end{bmatrix}$$

*Then* $A \mod 2$ *is the matrix*

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

The definitions and statements given in Chapter 2 hold for alternating matrices over $\mathbb{F}_2$ (so we will not restate them in this chapter). In particular, the "classification theorem", Theorem 2.5 holds with $d_1 = ... = d_m = 1$ (where $2m$=rank). However, to assign graphs to bases, we modify Definition 3.3 as follows:

**Definition 4.3** *Suppose that* $B$ *be* $n \times n$ *alternating matrix over* $\mathbb{F}_2$ *and let* $\langle, \rangle$ *be the alternating form defined by* $B$ *on* $\mathbb{F}_2^n$*. Let* $\mathscr{B} = \{f_1, ..., f_n\}$ *be a basis for* $\mathbb{F}_2^n$*. We define* $\Gamma(\mathscr{B})$ *as the undirected graph such that*

- *vertices of* $\Gamma(\mathscr{B})$ *are* $v_1, \ldots v_n$*,*
- *there is an edge* $v_i \bullet\!\!-\!\!-\!\!-\!\!\bullet v_j$ *if and only if* $\langle v_i, v_j \rangle = 1$*.*

It may be noted from Definition 4.1 that an alternating matrix is also symmetric, therefore there is an associated quadratic form defined as follows:

**Definition 4.4** *Let* $B$ *be alternating matrix over* $\mathbb{F}_2$ *and* $\langle, \rangle$ *be alternating bilinear form defined by* $B$ *on* $\mathbb{F}_2^n = V$ *(Definition 4.3 ). The quadratic form associated with* $\langle, \rangle$ *(or* $B$*) is the* $\mathbb{F}_2$*-valued function* $q : V \times V \to \mathbb{F}_2$ *such that*

- $q(u + v) = q(u) + q(v) + \langle u, v \rangle$ *for all* $u, v \in V$*,*
- $q(e_i) = 1$ *for all* $i = 1, \ldots, n$*.*

It is well known that quadratic forms over $\mathbb{F}_2$ are classified by a numerical invariant, called "Arf invariant", defined as follows:

**Definition 4.5** *Let* $B$ *be alternating matrix over* $\mathbb{F}_2$ *and* $\langle, \rangle$ *be alternating bilinear form defined by* $B$ *on* $\mathbb{F}_2^n = V$*. Let* $q$ *be quadratic form associated with* $B$ *and* $\mathscr{B} = \{u_1, v_1, \ldots, u_m, v_m, w_1, \ldots w_p\}$ *be a symplectic basis for* $\langle, \rangle$ *(Definition 2.10).*

*Let* $V_0 = span\{w_1, \ldots w_p\}$ *and suppose that* $q(V_0)$*=0, that is, for any* $v \in V_0$*, we have* $q(v) = 0$*. Then, we define*

$$Arf(q) = \sum_{i=1}^{m} q(u_i)q(v_i) \mod 2.$$

*It is well known that* $Arf(q)$ *is independent of the choice of the symplectic basis.*

In Section 6.4 , we will determine Arf invariants of matrices called principal extensions of skew-symmetric matrices.

# CHAPTER 5

# MUTATIONS

Mutation has been defined first in [2] as an operation on skew-symmetric matrices to define cluster algebras. Since then, it has been observed later mutation appears naturally in many other areas of mathematics [7]. Also several equivalent types of mutations have been formulated. In this chapter, we will discuss basic properties of three types of mutations: mutations of bases, matrices and graphs. We will first define mutations of bases and show how the other mutations can be obtained from it.

## 5.1 Mutations of bases

In this section, we define mutation as a special type of base change for $\mathbb{Z}^n$:

**Definition 5.1** *Suppose that* $\langle , \rangle$ *is a skew-symmetric form on* $\mathbb{Z}^n$ *(defined by a skew-symmetric matrix, Definition 2.6). Suppose also that* $\mathscr{B} = \{v_1, \ldots, v_n\}$ *is a* $\mathbb{Z}$*-basis of* $\mathbb{Z}^n$ *and let* $B$ *denote the matrix of* $\mathscr{B}$ *with respect to* $\langle , \rangle$ *(so* $\langle v_i, v_j \rangle = B_{i,j}$ *for all* $i, j$ *by Definition 2.7) .*

*For any* $\epsilon \in \{+, -\}$*, we define* $\epsilon$*-mutation of the basis* $\mathscr{B} = \{v_1, \ldots, v_n\}$ *as the basis*

$$\mu_k{}^{\epsilon}(\mathscr{B}) = \mathscr{B}' = \{v'_1, \ldots, v'_n\}$$

*such that*

- $v'_k = -v_k$*,*
- $v'_i = v_i + |B_{ik}|v_k$ *if* $sgn(B_{ik}) = -\epsilon$*,*

- $v'_i = v_i$ *otherwise.*

Let us now show that the inverse of a mutation is also a mutation:

**Proposition 5.2** $\mu_k^{-\epsilon}\mu_k^{\epsilon}(\mathscr{B}) = \mathscr{B}$.

**Proof.** Assume, without loss of generality, that $\epsilon = +$ and $\mu_k^+(\mathscr{B}) = \mathscr{B}' = \{v'_1, \ldots, v'_n\}$. Suppose that $v_i$ is connected to $v_k$ with $\langle v_i, v_k\rangle < 0$. Then we have $v'_i = v_i + |B_{ik}|v_k$, $v'_k = -v_k$, $v'_j = v_j$ and for any $j \neq i$ with $\langle v_j, v_k\rangle \geq 0$. Now let $\mathscr{B}'' = \mu_k^-(\mathscr{B}') = \{v''_1, \ldots, v''_n\}$ then we have $v''_k = -v'_k = -(-v_k) = v_k$. Note that $\langle v'_k, v'_i\rangle = \langle -v_k, v_i + |B_{ik}|v_k\rangle = -\langle v_k, v_i\rangle - |B_{ik}|\langle v_k, v_k\rangle = -\langle v_k, v_i\rangle$. Thus, $v''_i = v'_i + |B_{ki}|v'_k = v_i + |B_{ik}|v_k + |B_{ki}|(-v_k) = v_i$ and $v''_j = v'_j = v_j$. Hence $\mathscr{B} = \mathscr{B}''$. □

## 5.2 Mutations of matrices

The bases ${\mu_k}^+(\mathscr{B})$ and ${\mu_k}^-(\mathscr{B})$ in Definition 5.1 are different in general, however, their matrices turn out to be equal. More precisely:

**Proposition 5.3** *In the setup of Proposition 5.1, let $B'$ and $B''$ be the matrices of the bases $\mathscr{B}'$= ${\mu_k}^+(\mathscr{B})$ and $\mathscr{B}''$=${\mu_k}^-(\mathscr{B})$ with respect to the form $\langle , \rangle$ (Definition 2.7). Then $B' = B''$.*

**Proof.** Let $\mathscr{B} = \{v_1, \ldots, v_n\}$ and $\mathscr{B}' = \{v'_1, \ldots, v'_n\} = {\mu_k}^+(\mathscr{B})$. Let $\mathscr{B}'' = \{v''_1, \ldots, v''_n\} = {\mu_k}^-(\mathscr{B})$. We will show that for any $i$,$j$=$\{1, \ldots, n\}$ we have $B'_{i,j} = \langle v'_i, v'_j\rangle = \langle v''_i, v''_j\rangle = B''_{i,j}$. Let us first assume that $i = k$ or $j = k$. We may assume that $i = k$ and $B_{jk} = \beta \geq 0$ . Then we have $v'_j = v_j$, $v'_i = -v_i = -v_k$ and $v''_j = v_j + |B_{jk}|v_k = v_j + |B_{jk}|v_i$, $v''_i = -v_i = -v_k$, so, $\langle v'_i, v'_j\rangle = \langle -v_k, v_j\rangle$ and $\langle v''_i, v''_j\rangle = \langle -v_i, v_j + \beta v_k\rangle$=$\langle -v_k, v_j\rangle + \beta\langle -v_k, v_k\rangle$=$\langle -v_k, v_j\rangle$. Thus $\langle v'_i, v'_j\rangle = \langle v''_i, v''_j\rangle$.

For the rest of the proof, we assume that $i, j \neq k$. We continue considering in cases:

Case 1. Both of $v_i$ and $v_j$ are connected to $v_k$. We consider this case in two subcases:

*Subcase 1.1.* Both of $v_i$ and $v_j$ are connected to $v_k$ with the same direction. We may assume that $sgn(B_{jk}) = + = sgn(B_{ik})$ and $|B_{ik}| = \alpha$, $|B_{jk}| = \beta$ . Then, we have $v'_i = v_i + \alpha v_k$, $v'_j = v_j + \beta v_k$ and $v''_i = v_i$, $v''_j = v_j$. Notice that $\langle v'_i, v'_j\rangle = \langle v_i + \alpha v_k, v_j + \beta v_k\rangle = \langle v_i, v_j\rangle + \beta\langle v_i, v_k\rangle + \alpha\langle v_k, v_j\rangle + \alpha\beta\langle v_k, v_k\rangle = \langle v_i, v_j\rangle$ and $\langle v''_i, v''_j\rangle = \langle v_i, v_j\rangle$, so $\langle v'_i, v'_j\rangle = \langle v''_i, v''_j\rangle$.

*Subcase 1.2.* Both of $v_i$ and $v_j$ are connected to $v_k$ with opposite directions. We may assume that $sgn(B_{jk}) = -$, $sgn(B_{ik}) = +$ and $|B_{ik}| = \alpha$, $|B_{jk}| = \beta$. Then we have $v'_i = v_i + \alpha v_k$, $v'_j = v_j$ and $v''_i = v_i$, $v''_j = v_j + \beta v_k$, so $\langle v'_i, v'_j\rangle = \langle v_i, v_j\rangle + \alpha\langle v_k, v_j\rangle = \langle v_i, v_j\rangle + \alpha\beta$ and $\langle v''_i, v''_j\rangle = \langle v_i, v_j\rangle + \beta\langle v_i, v_k\rangle = \langle v_i, v_j\rangle + \beta\alpha$. Thus $\langle v'_i, v'_j\rangle = \langle v''_i, v''_j\rangle$.

Case 2. Suppose one of the $v_i$ and $v_j$ is not connected to $v_k$ in $\Gamma$. We may assume that $v_i$ is not connected to $v_k$.

*Subcase 2.1.* Suppose $v_j$ is connected to $v_k$ with opposite direction. Then we have $v'_j = v_j$, $v'_i = v_i$ and $v''_j = v_j + |B_{jk}|v_k$, $v''_i = v_i$, so, $\langle v'_i, v'_j\rangle = \langle v_i, v_j\rangle$ and $\langle v''_i, v''_j\rangle = \langle v_i, v_j\rangle + |B_{jk}|\langle v_i, v_k\rangle = \langle v_i, v_j\rangle$. Thus $\langle v'_i, v'_j\rangle = \langle v''_i, v''_j\rangle$.

*Subcase 2.2.* Suppose $v_j$ is not connected to $v_k$. Then we have $v'_i = v_i$, $v'_j = v_j$ and $v''_i = v_i$, $v''_j = v_j$, so, $\langle v'_i, v'_j\rangle = \langle v''_i, v''_j\rangle = \langle v_i, v_j\rangle$. This completes the proof. □

**Definition 5.4** *In the setup of Proposition 5.1, let $B'$ be the matrix of the form $\langle,\rangle$ with respect to $\mathscr{B}'= {\mu_k}^+(\mathscr{B})$ (or $\mathscr{B}''={\mu_k}^-(\mathscr{B})$, Proposition 5.3 ). We denote $B' = \mu_k(B)$ and call it "the mutation of $B$ at $k$".*

We can describe the matrix $B' = \mu_k(B)$ explicitly as follows:

**Proposition 5.5** *$B' = \mu_k(B)$ is equal to the following matrix:*

$$B'_{ij} = \begin{cases} -B_{ij} & if \text{ i=k or j=k} \\ B_{ij} + sgn(B_{ik})[B_{ik}B_{kj}]_+ & otherwise. \end{cases} \tag{5.1}$$

*where* $[B_{ik}B_{kj}]_+ = \max\{B_{ik}B_{kj}, 0\}$.

**Proof.** Let $B$ be a matrix with respect to the basis $\mathscr{B} = \{v_1, \ldots, v_n\}$ and $\mathscr{B}' = \{v'_1, \ldots, v'_n\}$ be the $\epsilon$-mutation of the basis at $k$.

Case 1. We first consider the situation for $i = k$ or $j = k$. Now for any $i, j \in \{1, \ldots, n\}$ we may assume that $i = k$, $j \neq k$.

*Subcase 1.1.* Assume $sgn(B_{jk}) = \epsilon$, then we have $v'_i = v'_k = -v_k = -v_i$ and $v'_j = v_j$. Note that $B'_{ij} = \langle v'_i, v'_j \rangle = \langle -v_k, v_j \rangle = -\langle v_k, v_j \rangle = -\langle v_i, v_j \rangle = -B_{ij}$. Thus, $B'_{ij}$ is as in the proposition.

*Subcase 1.2.* Assume now $sgn(B_{jk}) = -\epsilon$, then we have $v'_i = v'_k = -v_k = -v_i$ and $v'_j = v_j + |B_{jk}|v_k$. Note that $B'_{ij} = \langle v'_i, v'_j \rangle = \langle -v_k, v_j + |B_{jk}|v_k \rangle = -\langle v_k, v_j \rangle - \langle v_k, |B_{jk}|v_k \rangle = -\langle v_k, v_j \rangle - 0 = -\langle v_k, v_j \rangle = -\langle v_i, v_j \rangle = -B_{ij}$. Thus, $B'_{ij}$ is as in the proposition.

Case 2. Now assume that $i, j \neq k$ and $|B_{ik}| = \alpha$, $|B_{jk}| = \beta$ where $\alpha, \beta$ is greater than or equal 0. We continue with two cases.

*Case 2.1.* Consider the case $\alpha = 0$ or $\beta = 0$. Assume that $\alpha = 0$ and $sgn(B_{jk}) = \epsilon$. Now we have $v'_i = v_i$, $v'_j = v_j$ and $B'_{ij} = \langle v'_i, v'_j \rangle = \langle v_i, v_j \rangle = B_{ij}$ . Notice that $B_{ij} + sgn(B_{ik})[B_{ik}B_{kj}]_+ = B_{ij} + sgn(B_{ik}) \max\{B_{ik}B_{kj}, 0\} = B_{ij} + sgn(B_{ik}) \max\{0 \cdot B_{kj}, 0\} = B_{ij}$. So, $B'_{ij}$ is as in the proposition.

*Case 2.2.* Assume that $sgn(B_{ik}) = \epsilon$ and $sgn(B_{jk}) = \epsilon$, then $v'_i = v_i$, $v'_j = v_j$ and $B'_{ij} = \langle v'_i, v'_j \rangle = \langle v_i, v_j \rangle = B_{ij}$. Note that $B_{ij} + sgn(B_{ik})[B_{ik}B_{kj}]_+ = B_{ij} + sgn(B_{ik}) \max\{B_{ik}B_{kj}, 0\} = B_{ij} + (+) \max\{\alpha(-\beta), 0\} = B_{ij} + 0 = B_{ij}$. So, $B'_{ij}$ is as in the proposition.

*Case 2.3.* Now, assume without loss of generality that $sgn(B_{ik}) = \epsilon$ and $sgn(B_{jk}) = -\epsilon$, then we have $v'_i = v_i$, $v'_j = v_j + |B_{jk}|v_k$ and $B'_{ij} = \langle v'_i v'_j \rangle = \langle v_i, v_j + |B_{jk}|v_k \rangle = \langle v_i, v_j \rangle + |B_{jk}|\langle v_i, v_k \rangle$. Notice that in this case , $B_{ij} + sgn(B_{ik})[B_{ik}B_{kj}]_+ = B_{ij} + (+) \max\{B_{ik}B_{kj}, 0\} = B_{ij} + \max\{\alpha\beta, 0\} = B_{ij} + \alpha\beta$. So, $B'_{ij}$ is as in the proposition. This completes the proof. □

### 5.3 Mutations of graphs

Let us recall that we represent a skew-symmetric matrix by a graph as in Definition 3.1. Then the mutation operation on skew-symmetric matrices can be viewed as an operation on the graphs of skew-symmetric matrices. More precisely:

**Definition 5.6** *Suppose that $B$ is a skew-symmetric matrix with $\Gamma = \Gamma(B)$ and let $\Gamma' = \Gamma(\mu_k(B))$. We denote the graph $\Gamma' = \mu_k(\Gamma)$ and call it "the mutation of $\Gamma$ at $k$".*

We obtain $\mu_k(\Gamma)$ from $\Gamma$ in the following way:

- the orientations of all edges incident to $k$ are reversed, their weights unchanged,
- For any vertices $i$ and $j$ which are connected to $k$ in $\Gamma$, the direction and the weight of the edge $\{i, j\}$ in $\mu_k(\Gamma)$ are given in the Figure 5.1.

### 5.4 Basic properties of mutations

Some basic properties of the mutation operation are the following:

1. Mutation is involutive: $\mu_k^2$ =Identity.

2. If $\Gamma'$ subgraph of $\Gamma$ and $k \in \Gamma'$, then $\mu_k(\Gamma')$ is also subgraph of $\mu_k(\Gamma)$.

3. If $\Gamma$ is connected, then $\mu_k(\Gamma)$ is also connected.

*Proof of property 1*: Let $\mathscr{B}$ be a basis whose graph is $\Gamma = \Gamma(\mathscr{B})$ and $\mathscr{B}' = \mu_k^{\epsilon}(\mathscr{B})$, $\mathscr{B}'' = \mu_k^{-\epsilon}(\mathscr{B})$. By Proposition 5.2 and Proposition 5.3, we have $\mu_k^{-\epsilon}\mu_k^{\epsilon}(\mathscr{B}) = \mathscr{B}$ and $\Gamma(\mathscr{B}') = \Gamma(\mathscr{B}'')$. Now we will show that $\mu_k^2(\Gamma(\mathscr{B})) = \mu_k\mu_k(\Gamma(\mathscr{B})) = \Gamma(\mathscr{B})$. Now $\mu_k^2(\Gamma(\mathscr{B})) = \mu_k\mu_k(\Gamma(\mathscr{B})) = \mu_k(\Gamma(\mu_k^{\epsilon}(\mathscr{B}))) = \Gamma(\mu_k^{-\epsilon}\mu_k^{\epsilon}(\mathscr{B})) = \Gamma(\mathscr{B}) = \Gamma$.

**Definition 5.7** *We say that skew-symmetric matrices $B$ and $B'$ (or graphs $\Gamma$ and $\Gamma'$) are "mutation-equivalent" if $B$ can be obtained from $B'$ by a sequence of mutations.*

*This is an equivalence relation on skew-symmetric matrices (or graphs). Equivalence classes under this relation are called mutation classes.*

The general problem related with mutations is to find properties (of matrices, graphs,...) which are invariant under mutation-equivalence. In this thesis we study some linear algebraic properties which are invariant under mutations. For this, let us discuss how mutation is related to congruence.

### 5.5 Mutations and congruence

As we mentioned above, being a fundamental operation with important applications, it is natural to ask for properties which are invariant under mutations. One such property is congruence:

**Proposition 5.8** *Suppose $B$ and $B'$ are mutation equivalent matrices. Then, there is a matrix $P$ with $det(P) = \mp 1$ such that $P^T BP = B'$, so $B$ and $B'$ are congruent. In particular, they are congruent to the same matrix of the form in Theorem 2.5.*

**Proof.** It is enough to prove the statement for $B' = \mu_k(B)$. Let $P$ be the matrix whose entries are defined as follows: $P_{ki} = B_{ki}$ for any $i \neq k$; , $P_{kk} = -1$; for any $r \neq k$, $P_{rr} = 1$ and all the rest are 0. Then it follows that $P$ is as in the statement of the proposition (see Proposition 2.8 ). $\square$

Similarly Arf invariant (Definition 4.5) is also a mutation invariant:

**Proposition 5.9** *[6, Proposition 5.2] Suppose $B$ and $B'$ are mutation equivalent matrices. Let $q$ and $q'$ and be the quadratic forms associated with $B \mod 2$ and $B' \mod 2$ respectively (Definition 4.4). Then $Arf(q) = Arf(q')$.*

### 5.6 Skew-symmetric matrices of finite type and their graphs

Finite type skew-symmetric matrices are the most important type of matrices with respect to the mutation operation. They were introduced in [3] as a natural analogue

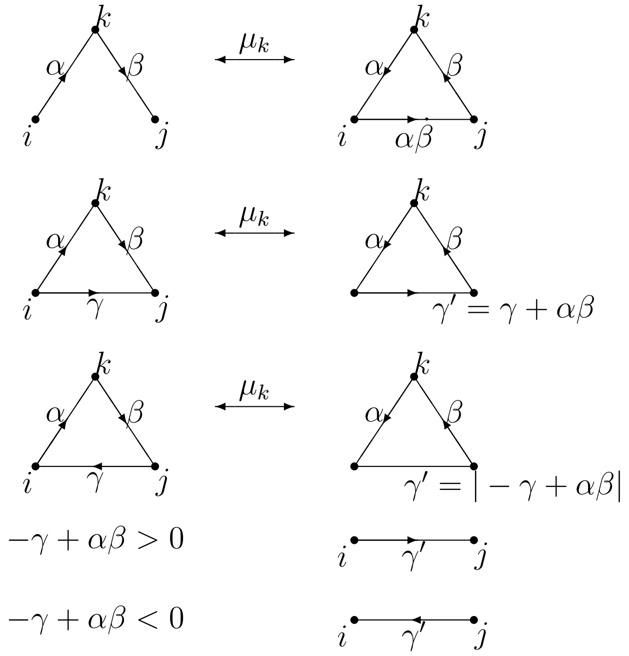



Figure 5.1: Mutation of graphs

of finite type cluster algebras, which are the most basic type of cluster algebras.

**Definition 5.10** *We say that a skew-symmetric matrix* $B$ *is of* finite type *if, for any* $B'$ *in the mutation class of* $B$*, we have* $|B'_{i,j}| \leq 1$*. Equivalently, we say that a graph* $\Gamma$ *is of finite type if, for any* $\Gamma'$ *in the mutation class of* $\Gamma$*, the weight of any edge in* $\Gamma'$ *is equal 1. We say that* $\Gamma$ *is of infinite type if it is not of finite type.*

The following are some properties of finite type graphs:

1. Subgraph of a finite type graph is also of finite type.

2. Mutation class of a finite type graph is finite.

*Proof of property 1*: Suppose $\Gamma$ is a finite type graph. Suppose it has a subgraph $\Gamma'$ which is not of finite type. Then, there is a sequence of vertices $k_1, \ldots, k_r$ in $\Gamma'$ such that $\Gamma_1 = \mu_{k_r} \ldots \mu_{k_1}(\Gamma')$ has an edge with weight is at least 2. On the other hand, $\Gamma_1$ is a subgraph of $\Gamma_2 = \mu_{k_r} \ldots \mu_{k_1}(\Gamma)$; since $\Gamma_2$ is also finite type, the weight of any edge in $\Gamma_2$ is equal to 1. We get a contradiction.

One of the main theorems of cluster algebras is the classification of skew-symmetric matrices (or graphs) of finite type:

**Theorem 5.11** *[3] A connected graph $\Gamma$ is of finite type if and only if it is mutation equivalent to a Dynkin graph in Figure 6.2.*

It is natural to ask how to recognize whether a given graph is of finite type. An effective method for this has been given in [6] using the following notion:

**Definition 5.12** *A graph $\Gamma$ is called minimal infinite type if it is of infinite type, however all of its proper subgraphs are of finite type.*

Minimal infinite graphs have been given in [6]; this can be used to determine finite type graphs as follows:

**Proposition 5.13** *A graph is of infinite type if and only if it has minimal infinite subgraph.*

**Proof.** Suppose $\Gamma$ is finite type graph with $n$ vertices. Then, any subgraph of $\Gamma$ is of finite type. Thus, $\Gamma$ does not have any minimal infinite subgraph.

Now, suppose $\Gamma$ is infinite type. Then, we will show that $\Gamma$ contains a minimal infinite subgraph by induction on the number $n$. The basis of induction is for $n = 2$. Then $\Gamma$ is a two vertex graph whose weight is greater than or equal 2, so $\Gamma$ itself is minimal. Thus, $\Gamma$ contains a minimal infinite subgraph as in the proposition. Now, suppose the statement is true for all infinite type graphs whose number of vertices is less than or equal $n - 1$, so we may assume that $\Gamma$ is of infinite type with $n$ vertices. If $\Gamma$ itself is minimal, then we can take $\Gamma = \Gamma'$, so we are done. We may assume now that $\Gamma$ is not minimal. Then it has a proper subgraph $\Gamma_1$ of infinite type with $n - 1$ vertices or less ,so, by induction assumption, the graph $\Gamma_1$ contains a subgraph $\Gamma_2$ which is minimal. Note that $\Gamma_2$ is also a subgraph of $\Gamma$. Thus, the statement is true for $\Gamma$ with $n$ vertices. This completes the proof. □

# CHAPTER 6

# PRINCIPLE EXTENSIONS OF SKEW-SYMMETRIC MATRICES AND THEIR CLASSIFICATION: MAIN RESULTS

Principal extensions of skew-symmetric matrices are the matrices that correspond to the "cluster algebras with principle coefficients", whose structural properties determine properties of the other cluster algebras as well [4]. Therefore, it is natural to study algebraic and combinatorial properties of the matrices which are mutation-equivalent to the principle extensions of skew-symmetric matrices. As we discussed in 5.5 the normal forms and Arf invariants are also mutation invariants. In this section, we will determine these two invariants for the matrices which are mutation-equivalent to principal extensions (Theorems 6.8 and 6.11).

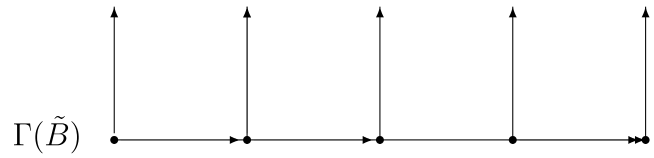
## 6.1 Basic definitions

**Definition 6.1** *Let $B$ be a skew-symmetric integer matrix of size $n$. We define* "principal skew-symmetric extension of $B$" *as the $2n \times 2n$ matrix $\tilde{B}$ of the form*

$$\tilde{B} = \left(\begin{array}{c|c} B & -I \\ \hline I & 0 \end{array}\right)$$

*Let us note that $\Gamma(\tilde{B})$ is obtained from $\Gamma(B)$ by introducing, for each vertex $i = 1, ..., n$ in $\Gamma(B)$, a new vertex $(n+i)$ and a new edge from $i$ to $(n+i)$.*

***Example 6.2*** *The following are the graphs of a skew-symmetric matrix and its principle extension:*




### 6.2 Principal extensions and the mutation operation: c-vectors

In this section, we recall some remarkable properties of the matrices mutation-equivalent to principal extensions (Definition 6.1). Let us first introduce some notation.

**Definition 6.3** *Let $\mathbb{T}_n$ be the $n$ regular tree with edges being labeled by the numbers $1, \ldots, n$; the $n$ edges incident to vertex have different labels. We write $t \underline{\;k\;} t'$ if $t, t' \in \mathbb{T}_n$ are connected by an edge whose label is $k$.*

We use the vertices of $\mathbb{T}_n$ as a location for the matrices that we obtain from principal extensions by mutations:

**Definition 6.4** *For any choice of an initial vertex $t_0$ in $\mathbb{T}_n$ and an initial skew-symmetric $n \times n$ matrix $B_0$, we assign each vertex $t$ in $\mathbb{T}_n$ with $t_0 \_\_ t_1 \_\_ \ldots \_\_ t_r$ (the edge $t_i \_\_ t_{i+1}$ is labeled by $k_{i+1}$) to an $n \times n$ matrix $C_t$ as follows: $C_{t_0} = I$, $C_t = [c_{1;t}, \ldots, c_{n;t}]$ where $C_t$ is the lower left part of $\mu_{k_r} ... \mu_{k_1}(\tilde{B}_0)$. We call $c_{i;t}$ c-vectors.*

The following is another striking result of the theory of cluster algebras:

**Theorem 6.5** *[1], [5] For each $t \in \mathbb{T}_n$, the corresponding c-vectors which are columns of the matrix $C_t$ are sign coherent i.e. all entries of $c_i$ are nonnegative or all are nonpositive; we write $sgn(c_i) = +$ or $sgn(c_i) = -$.*

Another interesting property of the c-vectors is that they can be obtained from the standard basis by mutations of bases (Definition 5.1):

**Proposition 6.6** *In the setup of Definition 6.4, suppose $t \underline{\ k\ } t'$ in $\mathbb{T}_n$ and let $C_t = [c_1 \dots c_n]$, $C_{t'} = [c'_1 \dots c'_n]$. Let $\mathscr{B} = \{c_1, \dots, c_n\}$, $\mathscr{B}' = \{c'_1, \dots, c'_n\}$ and $sgn(c_k) = \epsilon$. Then $\mathscr{B}' = \mu_k^\epsilon(\mathscr{B})$.*

**Proof.** We denote $\hat{B} = \mu_{k_r} \dots \mu_{k_1}(\tilde{B}_0)$ and $\hat{B}' = \mu_k(\hat{B})$ and $C = C_t$, $C' = C'_t$ be the c-matrices of $\hat{B}$ and $\hat{B}'$ respectively. Let us note that $c_i = [\hat{B}_{n+1,i} \dots \hat{B}_{2n,i}]^T$ and $c'_i = [\hat{B}'_{n+1,i} \dots \hat{B}'_{2n,i}]^T$. Let $sgn(c_k) = \epsilon$; if $i = k$, then (Definition 5.1), we have the following: $\hat{B}'_{n+j,i} = \hat{B}_{n+j,i}$ for all $j = 1 \dots n$, so $c'_i = -c_i$ (as in the definition of basis mutation). Suppose now that $i \neq k$ and $sgn(B_{ik}) = -\epsilon$ then, for any $j = 1, \dots, n$, we have $\hat{B}'{}_{n+j,i} = \hat{B}_{n+j,i} + sgn(\hat{B}_{n+j,k}) \max\{\hat{B}_{n+j,k}\hat{B}_{ki}, 0\} = \hat{B}_{n+j,i} + \hat{B}_{n+j,k}\hat{B}_{ki} = \hat{B}_{n+j,i} + |\hat{B}_{ik}|c_k$, so $c'_i = c_i + |\hat{B}_{ik}|c_k$ as in the basis mutation. Suppose now $sgn(B_{ik}) = \epsilon$. Then $\max\{\hat{B}_{n+j,k}\hat{B}_{ki}, 0\} = 0$ and so, by definition of matrix mutation we have $\hat{B}'{}_{n+j,i} = \hat{B}_{n+j,i}$ and $c'_i = c_i$. Thus, $\mathscr{B}' = \mu_k(\mathscr{B})$. This completes the proof. □

In Definition 6.4, a c-matrix (whose columns are c-vectors) was defined as part of a matrix which is obtained from a principal extension by a sequence of mutations. That part determines the whole matrix as follows:

**Proposition 6.7** *In the setup of Definition 6.4 let $\langle , \rangle$ be the form defined by $B_0$ (Definition 2.7) and $B = \mu_{k_r} \dots \mu_{k_1}(B_0)$. Let $C = C_t = [c_1, \dots, c_n]$ be the c-matrix assigned to t in $\mathbb{T}_n$. Then $\langle c_i, c_j \rangle = B_{ij}$.*

**Proof.** Let $\hat{B} = \mu_{k_r} \dots \mu_{k_1}(\tilde{B}_0)$ and $\hat{B}' = \mu_k(\hat{B})$ where $B'$ is the upper left part of $\hat{B}'$ so ($B' = \mu_{k_r} \dots \mu_{k_1}(B)$. Let $C = C_t$ and $C' = C'_t$ be the $c$-matrices of $\hat{B}$ and $\hat{B}'$ respectively). Note that the statement is true for initial $B_0$ i.e. $\langle e_i, e_j \rangle = (B_0)_{ij}$.

Assume the statement is true for $B$, i.e. $\langle c_i, c_j \rangle = B_{ij}$ for all $i, j = 1 \ldots, n$. Now we will prove that the statement is true for $B'$, so $\langle c'_i, c'_j \rangle = B'_{ij}$ in cases. We assume without loss of generality $sgn(c_k) = +$ and $i, j \neq k$.

Case 1. Let $sgn(B_{ik}) = -$ and $sgn(B_{jk}) = -$. Then we have $c'_i = c_i + |B_{ik}|c_k$ and $c'_j = c_j + |B_{jk}|c_k$. Then $\langle c'_i, c'_j \rangle = \langle c_i + |B_{ik}|c_k, c_j + |B_{jk}|c_k \rangle = \langle c_i, c_j \rangle + |B_{jk}|\langle c_i, c_k \rangle + |B_{ik}|\langle c_k, c_j \rangle + |B_{ik}||B_{jk}|\langle c_k, c_k \rangle = B_{ij} + |B_{jk}|B_{ik} + |B_{ik}|B_{kj} = B_{ij} + sgn(B_{jk})B_{jk}B_{ik} + sgn(B_{ik})B_{ik}B_{kj} = B_{ij} + B_{ik}(-B_{jk} - B_{kj}) = B_{ij}$. But notice that $B'_{ij} = B_{ij} + sgn(B_{ik})\max\{B_{ik}B_{kj}, 0\}$ and $\max\{B_{ik}B_{kj}, 0\} = 0$, so $B'_{ij} = B_{ij} = \langle c_i, c_j \rangle = \langle c'_i, c'_j \rangle$.

Case 2. Let $sgn(B_{ik}) = -$ and $sgn(B_{jk}) = +$. Then we have $c'_i = c_i + |B_{ik}|c_k$ and $c'_j = c_j$. Then $\langle c'_i, c'_j \rangle = \langle c_i + |B_{ik}|c_k, c_j \rangle = \langle c_i, c_j \rangle + |B_{ik}|\langle c_k, c_j \rangle = B_{ij} + |B_{ik}|B_{kj} = B_{ij} + sgn(B_{ik})B_{ik}B_{kj}$. But notice that $B'_{ij} = B_{ij} + sgn(B_{ik})\max\{B_{ik}B_{kj}, 0\}$ and $\max\{B_{ik}B_{kj}, 0\} = B_{ik}B_{kj}$, so $B'_{ij} = B_{ij} + sgn(B_{ik})B_{ik}B_{kj} = \langle c'_i, c'_j \rangle$.

Case 3. Let $sgn(B_{ik}) = +$ and $sgn(B_{jk}) = -$. Then we have $c'_i = c_i$ and $c'_j = c_j + |B_{jk}|c_k$. Then $\langle c'_i, c'_j \rangle = \langle c_i, c_j + |B_{jk}|c_k \rangle = \langle c_i, c_j \rangle + |B_{jk}|\langle c_i, c_k \rangle = B_{ij} + |B_{jk}|B_{ik}$. But notice that $B'_{ij} = B_{ij} + sgn(B_{ik})\max\{B_{ik}B_{kj}, 0\}$ and $\max\{B_{ik}B_{kj}, 0\} = B_{ik}B_{kj}$, so $B'_{ij} = B_{ij} + sgn(B_{ik})B_{ik}B_{kj} = B_{ij} + sgn(B_{kj})B_{ik}B_{kj} = B_{ij} + |B_{kj}|B_{ik}$. Thus $B'_{ij} = \langle c'_i, c'_j \rangle$.

Case 4. Let $sgn(B_{ik}) = +$ and $sgn(B_{jk}) = +$. Then we have $\langle c'_i, c'_j \rangle = \langle c_i, c_j \rangle = B_{ij}$ but notice that $B'_{ij} = B_{ij} + sgn(B_{ik})\max\{B_{ik}B_{kj}, 0\}$ and $\max\{B_{ik}B_{kj}, 0\} = 0$, so $B'_{ij} = B_{ij} = \langle c_i, c_j \rangle = \langle c'_i, c'_j \rangle$. This completes the proof. $\square$

### 6.3 Classification of principal extensions (first main result of the thesis)

Our first result is the classification of principal extensions of skew-symmetric matrices under congruence:

**Theorem 6.8** *Let $B$ be skew-symmetric integer matrix of size $n$ and $\tilde{B}$ be the principal skew-symmetric extension of $B$. Then, the normal form of $\tilde{B}$ with respect to congruence (Theorem 2.5) is the following $2n \times 2n$ matrix:*

$$\begin{pmatrix} 0 & 1 & & & & & \\ -1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & -1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & -1 & 0 \end{pmatrix}$$

*(so, in the setup of Theorem 2.5, we have $d_1 = ... = d_n = 1$).*

**Proof.** Let $\langle , \rangle$ be the form defined by $\tilde{B}$ on $\mathbb{Z}^{2n}$ (we have $\langle e_i, e_j \rangle = \tilde{B}_{ij}$ ). Let us also note that, for $j, k = 1, \ldots, n$ with $k \neq j$, we have $\langle e_{n+j}, e_k \rangle = 0$. We will prove the theorem by showing the existence of a symplectic basis, whose matrix is as in the statement of the theorem (Definition 2.10). We will show that such a basis can be obtained by changing the initial basis using transformations of the following type:

**Lemma 6.9** *Suppose there is an edge from $e_i$ to $e_j$ with weight $\alpha$ in $\Gamma(B)$. Let $e'_i = e_i + \alpha e_{n+j}$. Then the graph of the basis $\tilde{\mathscr{B}}' = \{e_1, \ldots, e_{i-1}, e'_i, e_{i+1}, \ldots, e_j \ldots, e_n, e_{n+1}, \ldots, e_{2n}\}$ with respect to $\langle , \rangle$ is obtained from $\Gamma(\tilde{B})$ by removing only the edge from $e_i$ to $e_j$ and not introducing any new edges. Thus, the matrix of the basis $\tilde{\mathscr{B}}'$ with respect to $\langle , \rangle$ is the principal extension of the matrix $B'$ such that $B'_{i,j} = B'_{j,i} = 0$ and $B'_{k,l} = B_{k,l}$ for any $(k, l) \neq (i, j)$.*

**Proof.** Let us note that $\langle e'_i, e_j \rangle$=$\langle e_i + \alpha e_{n+j}, e_j \rangle$=$\langle e_i, e_j \rangle$+ $\alpha \langle e_{n+j}, e_j \rangle$=$-\alpha + \alpha \cdot 1 = 0$, so there is no edge between $e'_i$ and $e_j$ in $\Gamma(\tilde{B}')$. Also, for any $k = 1, \ldots, 2n$ with $k \neq j$, we have $\langle e'_i, e_k \rangle = \langle e_i + \alpha e_{n+j}, e_k \rangle = \langle e_i, e_k \rangle + \alpha \langle e_{n+j}, e_k \rangle = \langle e_i, e_k \rangle + 0 = \langle e_i, e_k \rangle$, so there is an edge between $e'_i$ and $e_k$ in $\Gamma(\tilde{B}')$ if and only if the same holds for $\Gamma(\tilde{B})$. □

To prove theorem, let us consider an ordering of the edges in $\Gamma(B)$ as

$$i_1 \xrightarrow{\alpha_1} j_1 \ , \ i_2 \xrightarrow{\alpha_2} j_2 \ , \ \cdots \ , \ i_r \xrightarrow{\alpha_r} j_r.$$

Applying Lemma 6.9 consecutively for this sequence of edges, we eliminate all edges of $\Gamma(B)$ and obtain a basis $\mathcal{C}$=$\{f_1, \dots, f_n, f_{n+1} = e_{n+1}, \dots, f_{2n} = e_{2n}\}$ such that the only edges in the graph of $\mathcal{C}$ are the edges from $e_i$ to $e_{n+i}$ for $i = 1, ..., n$, so $\mathcal{C}$ is a symplectic basis (Section 3.3) whose matrix is as in the statement of the theorem. The proof of the theorem has been completed. $\square$

## 6.4 Arf invariants of principal extensions of finite type skew-symmetric matrices (second main result of the thesis)

Our second result is a formula that gives Arf invariants of principal extensions of finite type skew-symmetric matrices. The formula computes the Arf invariant using combinatorial properties of the corresponding graphs, therefore we first give some related definitions.

**Definition 6.10** *Suppose $\Gamma$ is a graph, by a subgraph of $\Gamma$, we mean an induced (full) directed subgraph on some vertices of $\Gamma$ and we keep the edge weights as in $\Gamma$.*

*By a cycle in $\Gamma$ we mean an induced (full) subgraph whose vertices can be labeled by $\{1, 2, ..., r\}, r \geq 3$, such that there is an edge between $i$ and $j$ if and only if $|i-j| = 1$ or $\{i, j\} = \{1, r\}$.*

*Note that every vertex in a cycle must be connected to precisely two vertices in the cycle.*

Examples of subgraphs of a graph are the following:

Note that (2) is a subgraph of the graph (1) on the vertices {1,2,3} and it is a cycle, however, (3) is not cycle but still a subgraph of (1) on the vertices {2,3,4,5}.

We can now state our second main result:

**Theorem 6.11** *Let $B$ be a skew-symmetric matrix of finite type with graph $\Gamma(B)$ (Definition 3.1) . Let $\tilde{B}$ be the principal extension of $B$ and $\tilde{q}$ be the quadratic form associated with $\tilde{B} \mod 2$ (Definition 4.4). Then we have the following:*

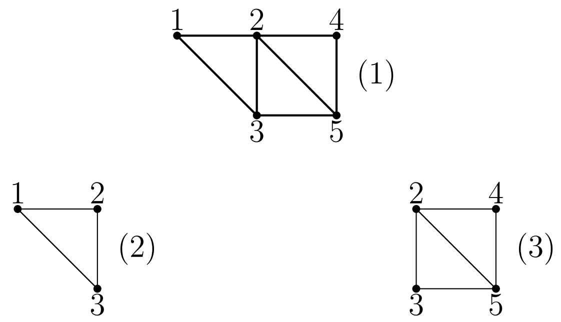


Figure 6.1: Examples of subgrahs

$$Arf(\tilde{q}) = \begin{cases} 0 & \textit{if } \Gamma(B) \textit{ has odd number of cycles} \\ 1 & \textit{if } \Gamma(B) \textit{ has even number of cycles.} \end{cases} \tag{6.1}$$

*(Note that $\tilde{B} \mod 2$ is non-singular by Theorem 6.8, so $Arf(\tilde{q})$ is defined (Definition 4.5 ))*

**Proof.** For the proof, we use the setup given in Chapter 4: we denote by $\langle , \rangle$ the alternating form defined by $\tilde{B} \mod 2$ on $\mathbb{F}_2^{2n}$, so $\langle e_i, e_j \rangle = \tilde{B}_{ij} \mod 2$, for all $i, j = 1, \ldots, 2n$, where $\{e_1, ..., e_{2n}\}$ is the standard basis of $\mathbb{F}_2^{2n}$. Since we work over $\mathbb{F}_2$ and each non-zero element of $\tilde{B}$ is either $1$ or $-1$ (because $B$ is of finite type), we use same notation $\Gamma(\tilde{B})$ for the undirected graph of the standard basis with respect to the form $\langle , \rangle$ (Definition 4.3). More generally, for a basis $\{f_1, ..., f_{2n}\}$ of $\mathbb{F}_2^{2n}$ whose matrix is $\tilde{B}'$ with respect to the form $\langle , \rangle$, $\Gamma(\tilde{B}')$ denotes the undirected graph as in Definition 4.3.

We first observe the following for $B$ as in the statement of the theorem:

**Lemma 6.12** *Suppose $\Gamma(B)$ has a leaf, i.e a vertex, say $e_i$, connected to exactly one other vertex, say $e_j$, in $\Gamma(B)$. Let $B'$ be the matrix of size $n-1$ such that $\Gamma(B')$ is the graph obtained from $\Gamma(B)$ by removing the vertex $e_i$. Let $\tilde{q}'$ be the quadratic form associated with $\tilde{B}' \mod 2$. Then $Arf(\tilde{q}) = Arf(\tilde{q}')$. Furthermore, $\Gamma(B)$ and $\Gamma(B')$ have the same number of cycles.*

**Proof.** We may assume that $i = n$. Then, for any $k = 1, \ldots, 2n-1$, we have the following: if $k = j$, then $\langle e_n + e_{n+j}, e_j \rangle$=$\langle e_n, e_j \rangle + \langle e_{n+j}, e_j \rangle$=1+1=0; if $k \neq j$, then

$\langle e_n + e_{n+j}, e_k\rangle$=$\langle e_n, e_k\rangle + \langle e_{n+j}, e_k\rangle$=0+0=0. Thus the graph $\Gamma(\tilde{B})$ is a disjoint union of $\Gamma(\tilde{B}')$ and the two vertex graph

$$e_{2n} \bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet\ (e_n + e_{n+j})$$

for all cases. Let now $\{u'_1, v'_1, \ldots, u'_{n-1}, v'_{n-1}\}$ be a symplectic basis for the restriction of the form $\langle,\rangle$ to the subspace $span\{e_1, .., e_{n-1}, e_{n+1}, ..., e_{2n-1}\}$ (note that the matrix of this basis with respect to $\langle,\rangle$ is $\tilde{B}' \mod 2$), so $Arf(\tilde{q}') = \sum_{i=1}^{n-1} q(u'_i)q(v'_i)$. Then

$$\{u'_1, v'_1, \ldots, u'_{n-1}, v'_{n-1}, e_n + e_{n+j}, e_{2n}\}$$

is a symplectic basis of $\langle,\rangle$ on the whole space $\mathbb{F}_2^{2n}$, so $Arf(\tilde{q}) = \sum_{i=1}^{n-1} q(u'_i)q(v'_i) + q(e_n + e_{n+j})q(e_{2n}) = Arf(\tilde{q}') + q(e_n + e_{n+j})q(e_{2n})$. We also have $q(e_n + e_{n+j}) = q(e_n) + q(e_{n+j}) + \langle e_n, e_{n+j}\rangle = 1 + 1 + 0 = 0$. Thus, $Arf(\tilde{q}) = 0 \cdot 1 + Arf(\tilde{q}') = Arf(\tilde{q}')$. Let us also notice that removing the vertex $e_i$ from $\Gamma(\tilde{B})$ doesn't change the number of cycles in $\Gamma(\tilde{B})$, because $e_i$ is connected to precisely one vertex, so it is not contained in any cycle in $\Gamma(B)$ (removing $e_i$ does not affect any cycle). Thus the number of cycles in $\Gamma(\tilde{B})$ and $\Gamma(\tilde{B}')$ are equal. □

**Lemma 6.13** *Suppose $\Gamma(B)$ has a vertex $e_i$ such that $e_i$ is contained in exactly one cycle $C$ in $\Gamma(B)$ and $e_i$ is not connected to any vertex which is not in $C$. Let $B'$ be the $(n-1) \times (n-1)$ matrix such that $\Gamma(B')$ is the graph obtained from $\Gamma(B)$ by removing the vertex $e_i$. Let $\tilde{q}'$ be the quadratic form associated with $\tilde{B}'$. Then $Arf(\tilde{q})$=1+$Arf(\tilde{q}')$. Furthermore, the number of cycles in $\Gamma(B')$ is one less than the number of cycles in $\Gamma(B)$.*

**Proof.** Let us assume, without loss of generality, that $i = n$. Let $e_j$ and $e_k$ be the vertices in $C$ which are connected to $e_i$. Note that these are the only vertices which are connected to $e_i$. Let us denote by $B'$ the basis $\{e_1, \ldots, e_{n-1}, e_n + e_{n+j} + +e_{n+k}, e_{n+1}, \ldots, e_{2n}\}$. Let us also note that for any $m = 1, \ldots, 2n-1$ we have the following: if $m = j$, then $\langle e_n + e_{n+j} + e_{n+k}, e_j\rangle$=$\langle e_n, e_j\rangle + \langle e_{n+j}, e_j\rangle + \langle e_{n+k}, e_j\rangle = = 1 + 1 + 0 = 0$; if $m = k$, then $\langle e_n + e_{n+j} + e_{n+k}, e_k\rangle$=$\langle e_n, e_k\rangle + \langle e_{n+j}, e_k\rangle + \langle e_{n+k}, e_k\rangle = 1 + 0 + 1 = 0$; if $m \neq k, j$, then $\langle e_n + e_{n+j} + e_{n+k}, e_m\rangle$=$\langle e_n, e_m\rangle + \langle e_{n+j}, e_m\rangle + \langle e_{n+k}, e_m\rangle = 0 + 0 + 0 = 0$. Thus, the graph of $\Gamma(\tilde{B})$ is disjoint union of $\Gamma(\tilde{B}')$ and the two vertex graph

$$e_{2n} \bullet\!\!\longrightarrow\!\!\bullet (e_n + e_{n+j} + e_{n+k})$$

Let now $\{u'_1, v'_1, \ldots, u'_{n-1}, v'_{n-1}\}$ be a symplectic basis for the restriction of the form $\langle , \rangle$ to the subspace $span\{e_1, .., e_{n-1}, e_{n+1}, ..., e_{2n-1}\}$ (note that the matrix of this basis with respect to $\langle , \rangle$ is $\tilde{B}' \mod 2$), so $Arf(\tilde{q}') = \sum_{i=1}^{n-1} q(u'_i)q(v'_i)$. Then

$$\{u'_1, v'_1, \ldots, u'_{n-1}, v'_{n-1}, e_n + e_{n+j}, e_{2n}\}$$

is a symplectic basis of $\langle , \rangle$ on the whole space $\mathbb{F}_2^{2n}$, so $Arf(\tilde{q}) = \sum_{i=1}^{n-1} q(u'_i)q(v'_i) + q(e_n + e_{n+j} + e_{n+k})q(e_{2n}) = Arf(\tilde{q}')+q(e_n + e_{n+j} + e_{n+k})q(e_{2n})$. Notice that, $q(e_n + e_{n+j} + e_{n+k}) = q(e_n) + q(e_{n+j}) + q(e_{n+k}) + \langle e_n, e_{n+j}\rangle + \langle e_n, e_{n+k}\rangle + \langle e_{n+j}, e_{n+k}\rangle = 1 + 1 + 1 + 0 + 0 + 0 = 1$. Thus $Arf(\tilde{q}) = 1 \cdot 1 + Arf(\tilde{q}') = 1 + Arf(\tilde{q}')$. Notice that removing the vertex $e_i$ from $\Gamma(\tilde{B})$ eliminates the cycle $C$, but does not affect any other cycle in $\Gamma(B)$, because $C$ is the only cycle that contains $e_i$, so the number of cycles in $\Gamma(\tilde{B})$ is one more than that of $\Gamma(\tilde{B}')$.

This completes the proof of the lemma. □

The following lemma follows from the description given in [6] for finite type graphs:

**Lemma 6.14** *Let $\Gamma$ be a finite type graph. Then it has a vertex as in Lemma 6.12 or in Lemma 6.13.*

To complete the proof of the theorem, suppose $\Gamma(B)$ is a finite-type graph with $n$ vertices. We will show that $Arf(\tilde{q})$ is as in the Theorem by induction on $n$. The basis of induction is for $n = 1$. Then, $\Gamma(B)$ is a single vertex $e_1$ and so, $\{e_1, e_2\}$ is a symplectic basis for $\Gamma(\tilde{B})$. Therefore, $Arf(\tilde{q}) = q(e_1)q(e_2) = 1$ which is as in the statement of the theorem because $\Gamma(B)$ has zero number of cycles for $n = 1$. For the inductive hypothesis, we will assume that the theorem is true for finite-type graphs with $n - 1$ vertices. We will show that $Arf(\tilde{q})$ is as in the theorem for any finite-type graph $\Gamma$ with $n$ vertices. Since $\Gamma$ is a finite type graph, by Lemma 6.14, it has a vertex as in the Lemma 6.12 or Lemma 6.13. Let us suppose first that (by Lemma 6.14) $\Gamma$ has a leaf as in the Lemma 6.12. Let $\Gamma'$ be the graph with $n - 1$ vertices obtained from $\Gamma$ by removing the leaf. Then, by Lemma 6.12, we have $Arf(\tilde{q})$=$Arf(\tilde{q}')$. Note that $\Gamma'$ is also of finite-type because it is a subgraph of $\Gamma$. By induction assumption, $Arf(\tilde{q}')$ is as in the theorem. Note also that, by Lemma 6.12, $\Gamma(B)$ and $\Gamma(B')$ have the same number of cycles, so, $Arf(\tilde{q})$ is also as in the theorem. To continue, let us now suppose (by Lemma 6.14) that $\Gamma$ has a vertex as in Lemma 6.13, say $e_n$. Let $\Gamma'$ be the graph with $n-1$ vertices obtained from $\Gamma$ by removing the vertex $e_n$. Then, by Lemma 6.13, we have $Arf(\tilde{q}) = 1 + Arf(\tilde{q}')$. Note that $\Gamma'$ is also of finite type because it is a subgraph of $\Gamma$. Furthermore, by induction assumption, $Arf(\tilde{q})$ is as in the statement of the theorem (Theorem 6.11), so we have

$$Arf(\tilde{q}) = \begin{cases} 1+0=1 & \text{if } \Gamma(\tilde{B}') \text{ has odd number of cycles} \\ 1+1=0 & \text{if } \Gamma(\tilde{B}') \text{ has even number of cycles.} \end{cases} \tag{6.2}$$

Also the graph $\Gamma(\tilde{B})$ has one more cycle than $\Gamma(\tilde{B}')$ (Lemma 6.12), so if $\Gamma(\tilde{B}')$ has odd number of cycles, then $\Gamma(\tilde{B})$ has even number of cycles and if $\Gamma(\tilde{B}')$ has even number of cycles, then $\Gamma(\tilde{B})$ has odd number of cycles. Thus

$$Arf(\tilde{q}) = \begin{cases} 1+0=1 & \text{if } \Gamma(\tilde{B}) \text{ has even number of cycles} \\ 1+1=0 & \text{if } \Gamma(\tilde{B}) \text{ has odd number of cycles,} \end{cases} \tag{6.3}$$

Also note that the cycles in $\Gamma(B)$ and $\Gamma(\tilde{B})$ are the same, so $Arf(\tilde{q})$ is as in the statement of the theorem. This completes the proof of theorem. $\square$

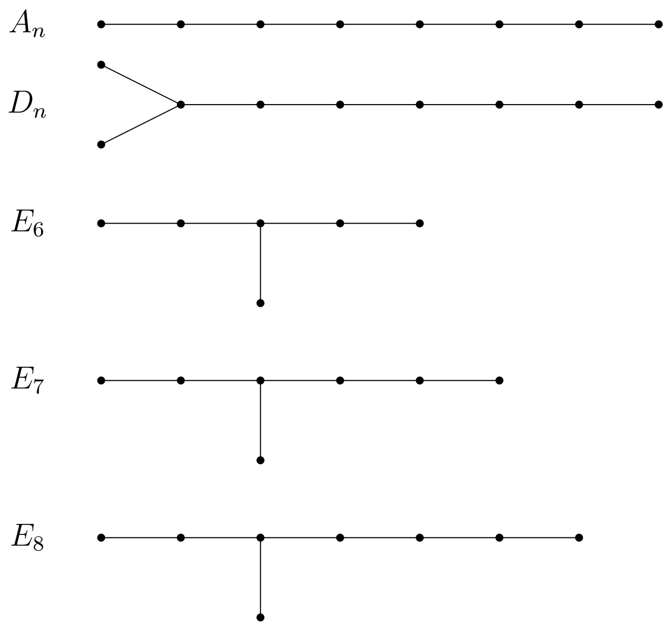



Figure 6.2: Dynkin graphs

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