PRE-SERVICE MIDDLE SCHOOL MATHEMATICS TEACHERS' PROPORTIONAL REASONING BEFORE AND AFTER A PRACTICE-BASED INSTRUCTIONAL MODULE

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I certify that this thesis satisfies all the requirements as a thesis for the degree of Doctor of Philosophy.

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ABSTRACT

PRE-SERVICE MIDDLE SCHOOL MATHEMATICS TEACHERS' PROPORTIONAL REASONING BEFORE AND AFTER A PRACTICE-BASED INSTRUCTIONAL MODULE

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The purpose of this study was to investigate pre-service middle school mathematics teachers' proportional reasoning before and after receiving a practice-based instructional module based on proportional reasoning. Data were collected from three pre-service teachers in the spring semester of 2012-2013. Pre-service teachers were junior students enrolled in elementary mathematics teaching program at a public university. A practice-based instructional module based on proportional reasoning was carried out during a five-week period. In this study, the researcher was also the teacher of the instruction at the same time. The Proportional Reasoning Test, semi-structured interviews and observations of student teachings were used to collect data about the participants' proportional reasoning.

The current study indicated that pre-service teachers improved their proportional reasoning by completing a practice-based instructional module. Before the instructional module, the pre-service teachers generally applied algebraic procedures without associating meaning and used limited number of strategies to solve problems. Furthermore, the result demonstrated that they had difficulties in distinguishing proportional from nonproportional situations and understanding mathematical relationships embedded in proportional situations. However, by the end of the instructional module, while they mostly preferred to use informal strategies, they relied less on formal strategies. Additionally, they utilized a broader range of strategies to solve problems and made sense of these strategies. Further, they could determine whether the quantities in a situation were related additively, multiplicatively, or in some other way. Moreover, they enhanced their understanding of the mathematical relationships embedded in proportional situations as a result of their participation in the instructional module.

Keywords: Proportional Reasoning, Pre-service Mathematics Teachers, Ratio and Proportion.

ÖZ

ORTAOKUL MATEMATİK ÖĞRETMEN ADAYLARININ UYGULAMAYA DAYALI BİR ÖĞRETİM MODÜLÜNÜN ÖNCESİNDE VE SONRASINDA ORANTISAL AKIL YÜRÜTMELERİ

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Bu çalışmanın temel amacı, ortaokul matematik öğretmen adaylarının, orantısal akıl yürütmeye yönelik uygulamaya dayalı bir öğretim modülüne katılımlarından önce ve sonra orantısal akıl yürütmelerini incelemektir. Çalışmanın verileri, 2012–2013 bahar döneminde, üç öğretmen adayından toplanmıştır. Öğretmen adayları, bir devlet üniversitesinde, matematik öğretmeni yetişme programına devam eden üçüncü sınıf öğrencileridir. Beş haftalık bir zaman dilimi içerisinde, orantısal akıl yürütmeye yönelik uygulamaya dayalı bir öğretim modülü yürütülmüştür. Yapılan uygulamada, araştırmacı aynı zamanda eğitim görevlisi olarak görev almıştır. Orantısal Akıl Yürütme Testi, yarı-yapılandırılmış mülakatlar ve öğretim deneyimlerinin gözlemleri, öğretmen adaylarının öğretim modülüne katılımlarından önce ve sonraki orantısal akıl yürütmelerine yönelik veri toplamak için kullanılmıştır.

Veri analizi sonucunda, öğretmen adaylarının, uygulamaya dayalı öğretim modülüne katılarak, orantısal akıl yürütmelerini geliştirdikleri görülmüştür. Öğretim modülünün öncesinde, öğretmen adayları, orantı problemlerini çözmek için genellikle içler-dışlar çarpımı gibi anlamdan yoksun cebirsel kuralları uygulamışlar ve sınırlı sayıda strateji kullanmışlardır. Bunun yanında, orantısal durumları orantısal olmayan durumlardan ayırt etmekte ve orantısal durumların içerdiği matematiksel ilişkileri anlamada zorluk çektikleri görülmüştür. Buna karşın, öğretim modülünün sonrasında, öğretmen adayları orantı problemlerini çözerken çoğunlukla, çarpımsal ilişkilerin kullanıldığı informal stratejileri (değişim çarpanı gibi) kullanmayı tercih ederken, içler-dışlar çarpımı ve diğer formal stratejileri kullanmayı pek tercih etmemişlerdir. Ayrıca, orantı problemlerini çözmek için farklı stratejiler kullanmışlar ve bu stratejileri anlamlandırabilmişlerdir. Buna ek olarak, verilen çoklukların aralarında toplamsal, çarpımsal veya başka ilişkilerin olup olmadığını belirleyebilmişlerdir. Bunun yanında, öğretmen adaylarının orantısal durumlardaki matematiksel ilişkileri anlayabildikleri görülmüştür.

Anahtar Kelimeler: Orantısal Akıl Yürütme, Matematik Öğretmen Adayları, Oran ve Orantı

To my husband Turgay and daughter Nehir whom I love boundlessly

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CHAPTER I

INTRODUCTION

1.1 Introduction

"One cannot teach what one does not know."
(Fennema & Franke, 1992, p. 147).

Teachers need a broad and deep knowledge of concepts, principles and strategies they teach (Ball & Cohen, 1999; Ma, 1999; National Board for Professional Teaching Standards [NBPTS], 2010). Effective teachers can use this knowledge to apply curricular goals and shape their instruction and assessment. Furthermore, they can understand the mathematical relationships between concepts and the applications of these concepts to problem solving in mathematics, in other disciplines, and in the world outside of school (NBPTS, 2010). In USA, the Standards of National Council of Teachers of Mathematics (2000) pays importance to build a broad and deep conceptual knowledge. Similarly, the mathematics curriculum in Turkey emphasizes conceptual learning that is required learning the meanings of the concepts instead of learning procedures, algorithms and rules in a rote manner (Ministry of National Education, 2013). Research has shown that the knowledge elementary and middle school mathematics teachers get their classrooms is procedurally based and largely misunderstood (Ball, 1990; Ma, 1999; Tirosh, 2000). This is also true with mathematics teachers' understanding of proportional situations and rational numbers (Ball, 1990; Chick, 2003; Cramer & Lesh, 1988; Harel & Behr, 1995; Lacampagne, Post, Harel, & Behr, 1988). However, teachers' knowledge and understandings have an important role in shaping the quality of their teaching (Ball, Bass, Sleep, &

Thames, 2005). In this context, both in-service and pre-service teachers' knowledge and understandings are important areas of research focus.

There have been many attempts to bridge the gap between practice and knowledge in the field of mathematics education. Most widely accepted model in mathematics education was raised by Ball, Thames, and Phelps (2008). The model conceptualizes knowledge for teaching mathematics differently and therefore link practice and knowledge differently, as well. Ball and her colleagues' aim is to further develop Shulman's (1986, 1987) notion of pedagogical content knowledge. To describe professional knowledge within mathematics Ball and her colleagues use the term "mathematical knowledge for teaching" (MKT). They define MKT as "the mathematical knowledge needed to carry out the work of teaching mathematics", where teaching is defined as "everything that teachers must do to support the learning of their students" (Ball et al., 2008, p.395). Similarly, this study investigated pre-service mathematics teachers' content knowledge which traces its roots to Shulman's work (1986, 1987).

In order to have an impact on pre-service teachers' knowledge and instructional practices, professional learning experiences should be closely tied to real classroom practices (Ball & Cohen, 1999; Smith 2001). However, traditional professional development programs do not serve to transform teachers' knowledge, beliefs and habits of practice (Smith, 2001). In order to develop mathematical knowledge for teaching, Ball and Cohen (1999) propose a professional development model that focuses to use practice as a site for professional learning. The teacher education model, which is commonly referred as practice-based professional development, aims to deepen and sharpen teachers' knowledge through a practice-based curriculum and to improve teachers' capacity for innovative practice (Ball & Cohen, 1999; Smith, 2001; Silver, 2009). Correspondingly in the current study, a practice-based instructional module was used to improve pre-service teachers' content knowledge and understanding in proportionality concepts.

Proportional reasoning has an important role in children's mathematical improvement because it is the mathematical base of a great number of concepts in the middle school mathematics curriculum. Actually, the NCTM (1989) asserts that proportional reasoning "is of such great importance that it merits whatever time and effort must be expended to assure its careful development. Students need to see many problem situations that can be modeled and then solved through proportional reasoning" (p. 82). Proportional reasoning is a measure of understanding of mathematical ideas in middle school mathematics curriculum. Moreover, it provides the mathematical foundation for more complex concepts in high school (Lamon, 2012). Proportional reasoning is labeled as "the capstone of elementary arithmetic and the cornerstone of all that is to follow." (Lesh, Post, & Behr, 1988, p.94). In other words, it is an important reasoning which is necessary to understand the elementary and high school mathematics concepts and beyond.

Proportional reasoning is defined as "detecting, expressing, analyzing, explaining, and providing evidence in support of assertions about proportional relationships" (Lamon, 2007, p.647). Ratio and proportion are concepts that fall under the general umbrella of proportional reasoning. Thus, a conceptual understanding in the ratio and proportion concepts requires being a proportional reasoner. Lamon (2012) claims that a proportional reasoner exhibits greater efficiency in problem solving and utilizes a range of strategies, sometimes unique strategies, for dealing with problems. According to Cramer and Post (1993) being a proportional reasoner is more than applying rote procedures such as the cross-product algorithm to solve the proportion problems. Giving a correct answer to a proportion problem does not indicate that proportional reasoning is taking place; moreover, assessing what students are thinking and how they solve the problem is more essential than a numeric answer (Cramer, Post, & Currier, 1993; Lamon, 2007). Therefore, the current research study aimed to gain in-depth understanding of pre-service teachers' proportional reasoning by using multiple sources of data.

1.2 Main and Sub-problems of the Study

1. What is the nature of pre-service mathematics teachers' proportional reasoning before receiving a practice-based instructional module based on proportional reasoning?

    1.1. What are pre-service mathematics teachers' approaches to different problem types before receiving a practice-based instructional module based on proportional reasoning?

    1.2. How do pre-service mathematics teachers distinguish proportional from nonproportional situations before receiving a practice-based instructional module based on proportional reasoning?

    1.3. How do pre-service mathematics teachers understand mathematical relationships embedded in proportional situations before receiving a practice-based instructional module based on proportional reasoning?

2. What is the nature of pre-service mathematics teachers' proportional reasoning after receiving a practice-based instructional module based on proportional reasoning?

    2.1. What are pre-service mathematics teachers' approaches to different problem types after receiving a practice-based instructional module based on proportional reasoning?

    2.2. How do pre-service mathematics teachers distinguish proportional from nonproportional situations after receiving a practice-based instructional module based on proportional reasoning?

    2.3. How do pre-service mathematics teachers understand mathematical relationships embedded in proportional situations after receiving a practice-based instructional module based on proportional reasoning?

3. What differences exist between pre-service mathematics teachers' proportional reasoning before and after receiving a practice-based instructional module based on proportional reasoning?

    3.1. What differences exist between pre-service mathematics teachers' approaches to different problem types before and after receiving a practice-based instructional module based on proportional reasoning?
    3.2. What differences exist between pre-service mathematics teachers' distinguishing proportional from nonproportional situations before and after receiving a practice-based instructional module based on proportional reasoning?
    3.3. What differences exist between pre-service mathematics teachers' understanding mathematical relationships embedded in proportional situations before and after receiving a practice-based instructional module based on proportional reasoning?

1.3 Significance of the Study

Teachers themselves had to have a conceptually based understanding of ratio and proportion content so as to teach their students the content in a conceptually based way (Cramer et al., 1993; Lamon, 2007). As Fennema and Franke (1992) said, "what a teacher knows is one of the most important influences on what is done in classrooms and ultimately on what students learn… One cannot teach what one does not know." (p. 147). That is to say, teachers need a broad and deep knowledge of concepts, principles and strategies they teach (Ball & Cohen, 1999; Ma, 1999). Therefore, while teachers are encouraged to teach toward deeper understandings of contents in the middle grades, it is needed to determine whether they have the necessary understanding and knowledge of the contents they teach (Cramer et al., 1993). With this in mind, this study intended to shed light on pre-service teachers' knowledge, conceptions, misconceptions and understandings in proportional reasoning. The study made investigation in proportional reasoning within a more comprehensive perspective than previous studies. Most of the previous research studies investigated teachers' and pre-service teachers' solution strategies in solving proportion problems (e.g., Akkuş-Çıkla, & Duatepe, 2002; Person et al., 2004), and some studies examined their capability to distinguish proportional from nonproportional relationships (e.g., Cramer et al., 1993; Ölmez, 2016), but there were limited number of studies that explored different components of proportional reasoning at the same time (Hillen, 2005). Thus, the current study examined the pre-service teachers' understandings in different components of proportional reasoning such as approaches to different problem types, distinguishing proportional from nonproportional relationships and understanding the mathematical relationships embedded in proportional situations before and after participation in a practice-based instructional module.

Proportional reasoning contains a network of understandings and relationships, and it plays a crucial role in solving ratio and proportion problems (Lamon, 2007). Further, this reasoning is the mathematical base of a great number of concepts in the middle school mathematics curriculum. Thus, the development of the ability to reason proportionally is one of the most important aims of the 5-8 grades curriculum (Van de Walle, Karp, & Bay-Williams, 2010). However, ratio and proportion are concepts that are generally difficult to understand for many school children (Behr, Lesh, Post, & Silver, 1983; Cramer, Post, & Behr, 1989; Hart, 1988; Lamon, 2007). Thus, mathematics teachers' knowledge in the instruction of ratio and proportion is critical. According to NCTM (1989), proportional reasoning ability develops in students throughout grades 5-8. Thus, pre-service middle school mathematics teachers, who will teach grades 5-8 in Turkey, are critical stakeholders whose conceptions of ratio and proportion need to be studied. Therefore, in this study, pre-service middle school mathematics teachers' knowledge and understandings were investigated within the concept of ratio and proportion.

Research showed that students and even teachers frequently use formal strategies, which are algebraic strategies in which rules and properties of algebra are used, to set up and solve proportion problems (Ben-Chaim, Keret, & Ilany, 2012; Cramer & Post, 1993). Undoubtedly, the most common formal strategy is *cross-multiplication strategy* (Cramer & Post, 1993; Lamon, 2012). However, the cross-multiplication procedure has no physical referent, and so, it has less meaning for students and teachers (Cramer & Post, 1993). Moreover, although it is an efficient strategy, it might cause confusion and lead to error due to the fact that it does not highlight multiplicative relationships between variables (Cramer & Post, 1993; Cramer et al., 1993). Furthermore, those who blindly apply an algorithm might have difficulties in determining whether a situation is proportional or not (Lamon, 2012). Similarly, research studies indicate that there is a strong tendency to over generalize proportionality; that is, many students and teachers incorrectly apply proportional reasoning in situations that have nonproportional relationships (Atabaş, 2014; Cramer et al., 1993; Van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2005; Van Dooren, De Bock, Depaepe, Janssens, & Verschaffel, 2003). Although the transition from additive to multiplicative thinking is an essential aspect of proportional reasoning, it has traditionally received little importance in the preparation of middle school mathematics teachers (Sowder et al., 1998). However, inaccurate additive strategies do not seem to disappear with maturation (Hart, 1988). Correspondingly, "proportional reasoning does not always develop naturally" (Hilton, Hilton, Dole, & Goos, 2013, p. 193). Additionally, some research studies argue that its development can be promoted with instruction (Berk, Taber, Gorowara, & Poetzl, 2009; Hillen, 2005; Sowder, Philipp, Armstrong, & Schappelle, 1998; Whitenack, & Ellington, 2013). Studies indicate the need for a targeted professional development program to develop pre-service teachers' proportional reasoning. The current study responds to the need through a practice-based instructional module which intended to improve pre-service teachers' flexibility in solving different problem types by using a broad range of strategies, increase their capability to distinguish proportional from nonproportional situations and enable them to better understand mathematical relationships in proportional situations. Thus, one of the purposes of this study was to examine the differences existed between pre-service mathematics teachers' proportional reasoning before and after participation in a practice-based instructional module

1.4 Definitions of Important Terms

In this section, some of the terms that were used in this study are defined to prevent any misunderstandings.

Pre-service Middle School Mathematics Teachers

Students in elementary school mathematics teacher education department in education faculties are called as pre-service middle school mathematics teachers. They are teacher candidates who are going to teach mathematics from fifth grade to eighth grade in middle schools after their graduations. In the present study, pre-service middle school mathematics teachers are junior students majoring in mathematics education department in a public university.

Proportional Reasoning

Proportional reasoning is defined as detecting, expressing, analyzing, explaining and providing evidence in support of assertions about proportional relationships (Lamon, 2007). In this study, the proportional reasoning refers to pre-service teachers' approaches to different problem types (i.e., missing value, numerical comparison, and qualitative reasoning), distinguishing proportional from nonproportional situations, and understanding the mathematical relationships embedded in proportional situations.

Proportional Relationship

Proportional relationship is defined as the relationship between two quantities when they have the same or a constant ratio or relation.

Missing Value Problem

A missing value problem is one in which three of the four quantities in the proportion a/b = c/d are provided and the aim is to determine the fourth quantity, which is the missing value in the proportion (Lamon, 2007).

Numerical Comparison Problem

In a numerical comparison problem, all of the four values that form two ratios (*a, b, c*, and *d*) are provided and the aim is to find out whether *a/b* is greater than, less than, or equal to *c/d* (Lamon, 2007; 2012).

Qualitative Reasoning Problem

A qualitative reasoning problem includes no numerical value; however, it requires the counterbalancing of variables in measure spaces (Cramer et al., 1993).

Practice-Based Instructional Module based on Proportional Reasoning

The practice-based instructional module based on proportional reasoning, which was a component of Methods of Teaching Mathematics II course, was given to the junior pre-service teachers during a five-week period. The aim of the module was to improve pre-service teachers' flexibility in solving different problem types by using a broad range of strategies, increase their capability to distinguish proportional from nonproportional situations and enable them to better understand mathematical relationships in proportional situations. The tasks in the module were closely tied to real classroom practices. It consisted of a variety of activities such as solving mathematical tasks, examining student work, discussions on video clips from the participants' student teachings, and analyzing a narrative case of teaching in a middle school classroom.

CHAPTER II

REVIEW OF RELATED LITERATURE

The primary purpose of this study was to investigate pre-service middle school mathematics teachers' proportional reasoning before and after receiving a practice-based instructional module based on proportional reasoning. This chapter describes the underlying theory that comprises the conceptual framework for this study, as well as previous studies that form the empirical framework of this study. The chapter includes two parts: a review of proportional reasoning and mathematical knowledge for teaching literature.

2.1 Proportional Reasoning

2.1.1 Being a Proportional Reasoner

The development of the ability to reason proportionally is one of the most important aims of the 5-8 grades curriculums (Van de Walle et al., 2010). Correspondingly, the NCTM (1989) notes that proportional reasoning "is of such great importance that it merits whatever time and effort must be expended to assure its careful development. Students need to see many problem situations that can be modeled and then solved through proportional reasoning" (p. 82). In fact, proportional reasoning is a measure of one's understanding of mathematical ideas in middle school mathematics curriculum; moreover, it provides the mathematical foundation for more complex concepts (Lamon, 2012). Proportional reasoning is defined as "detecting, expressing, analyzing, explaining, and providing evidence in support of assertions about proportional relationships" (Lamon, 2007, p.647). In addition, it is an ability of scaling up and down in situations that contain constant relationships between two quantities (Lamon, 2012). It is argued that this type of reasoning is both a qualitative and quantitative process (Lesh, et al., 1988). However, it goes beyond setting up a proportion and blindly applying rules and mechanical operations (Hoffer, 1988; Lamon, 2007). Furthermore, proportional reasoning is a mental process that involves argumentation and conscious analysis of the relationships between variables (Lamon, 2007, 2012). Similarly, Lesh, et al. (1988) defines proportional reasoning as "a form of mathematical reasoning that involves a sense of co-variation and of multiple comparisons, and the ability to mentally store and process several pieces of information." (p. 93). Accordingly, giving correct answers does not make sure that proportional reasoning is taking place because proportions may be solved by using mechanical knowledge about equivalent fractions or about numerical relationships, or by applying algorithmic procedures (e.g., cross-multiplication) without the understanding of proportional relationships (Lamon, 2007). In addition, Lamon (2007) asserts that an understanding of proportionality requires:

- expressing the meanings of quantities and variables and the constant of proportionality in the context in which they are used;
- the ability to use proportionality as a mathematical model to organize in appropriate real-world contexts;
- the ability to distinguish situations in which proportionality is not an appropriate mathematical model from situations in which it is useful;
- development and use of the language of proportionality;
- use of functions to express the covariation of 2 quantities;
- the ability to explain the difference between functions of the form y=mx and functions of the form y=mx+b. In the latter function, y is not proportional to x;
- knowing that the graph of a direct proportion situation is a straight line through the origin;
- knowing that the graph of y=mx+b is a straight line intersecting the y axis b units above the origin;
- the ability to distinguish different types of proportionality and to associate each of them with appropriate real-world situations in which they are applicable; and
- knowing that k is the constant ratio between two quantities in a direct proportional situation;

- knowing that the graph of an inversely proportional situation is a hyperbola (p. 639-640).

Similar to Lamon (2007), Cramer et al., (1993) claim that a proportional reasoner has to be able to solve different problem types (i.e., missing value problems, numerical comparison problems and qualitative reasoning problems), distinguish proportional from nonproportional situations, and understand the mathematical relationships embedded in proportional situations. In the current study, these three characteristics of a proportional reasoner were used as a framework to examine pre-service teachers' proportional reasoning. The characteristics are presented in a more detailed way in the following sections.

2.1.1.1 Solving Different Problem Types

In the literature, three different problem types are identified to assess proportionality (Cramer et al., 1993; Heller, Post, Behr, & Lesh, 1990; Post, Behr, and Lesh 1988). The problem types are missing value problems, numerical comparison problems and qualitative reasoning problems (i.e., qualitative prediction and qualitative comparison problems).

A missing value problem is a problem that "provides three of the four values in the proportion a/b = c/d and the goal is to find the missing value (Lamon, 2007, p.637). The following is a typical missing value problem: "A car is driven 175 km in 3 hours. How far will it travel in 12 hours at the same speed?" (Karplus, Pulos, & Stage, 1983, p. 220). In the problem, three pieces of information are given; these are distance (175 km) and travel time (3 hours), and travel time (12 hours) for an unknown distance. In this problem, the work is to find the unknown distance.

In a numerical comparison problem, all of the four values that form two ratios (*a*, *b*, *c*, and *d*) are provided and the aim is "to determine the order relation between the ratios *a/b* and *c/d*" (Lamon, 2007, p. 637). In other words, these types of problems require the comparison of two ratios in order to determine whether the two ratios are equal or which ratio is greater or smaller than the other one (Ben-Chaim et al., 2012). For instance, the problem, “Car A is driven 180 km in 3 hours. Car B is driven 400 km in 7 hours. Which car was driven faster?” (Karplus et al., 1983, p. 220) is a typical numerical comparison problem. In this problem, the work is to compare the ratios of kilometers to hours of each car to find out which car is faster.

Qualitative reasoning problems have two different types that are qualitative prediction problems and qualitative comparison problems. The problems do not include numerical values; however, they “require the counterbalancing of variables in measure spaces” (Cramer et al., 1993, p. 166). To illustrate, the problem, “Mary ran more laps than Greg. Mary ran for less time than Greg. Who was the faster runner?” (Cramer et al., 1993, p. 166) is a typical qualitative comparison problem. In addition, the problem, “If Devan ran fewer laps in more time than she did yesterday, would her running speed be (a) faster, (b) slower, (c) exactly the same, (d) not enough information to tell.” (Cramer et al., 1993, p. 166) is an example of qualitative prediction problem. The problems require qualitative comparisons that do not depend on numerical values (Ben-Chaimet al., 2012).

Researchers agree that being a proportional reasoner is more than applying rote procedures such as the cross-product algorithm to solve the proportion problems (Cramer, & Post, 1993; Lamon, 2007). Giving a correct answer to a proportion problem does not indicate that proportional reasoning is taking place; moreover, assessing what students are thinking and how they solve the problem is more essential than a numeric answer (Cramer et al., 1993; Lamon, 2007). Furthermore, according to Lamon (2007), proportional reasoning requires expressing the meanings of quantities, variables and the constant of proportionality in the context in which they are used. It is also important to note that solving multiple types of problems including missing value, numerical comparison, qualitative reasoning (i.e., prediction and comparison) problems with considerable flexibility, which meant to choose a strategy that is best suited to the problem, is an expected ability from a proportional reasoner (Cramer & Post, 1993; Cramer et al., 1993; Singh, 2000). Similarly, Lamon (2012) claims that a proportional reasoner exhibits greater efficiency in problem solving and utilizes a range of strategies, sometimes unique strategies, for dealing with problems.

2.1.1.2 Distinguishing Proportional from Nonproportional Situations

Understanding multiplicative relationships and distinguishing them from additive relationships is the heart of proportional reasoning (Sowder, Armstrong, et al., 1998). Lamon (2007) argues that a proportional reasoner can discriminate situations in which proportionality is an appropriate mathematical model from situations in which proportionality is not appropriate. In other words, a proportional reasoner can determine whether the quantities in a problem situation are related additively, multiplicatively, or in some other way. However, research studies indicate that there is a strong tendency to over generalize proportionality; that is, many students and teachers incorrectly apply proportional reasoning in situations that have nonproportional relationships (Atabaş, 2014; Cramer et al., 1993; Van Dooren et al., 2005; Van Dooren et al., 2003). For this reason, some researchers include the ability to distinguish proportional relationships from non-proportional relationships as a property of a proportional reasoner (Cramer et al., 1993; Lamon, 2007; 2012). Lamon (2012) states:

> Proportional thinkers can identify everyday contexts in which proportions are or are not useful. Proportions are not just mathematical objects or situations to which they know how to apply an algorithm. They can distinguish proportional from nonproportional situations and will not blindly apply an algorithm if the situation does not involve proportional relationships (p. 260).

Lamon (2012) claims that proportional reasoners can determine whether a situation is proportional or not, and so, they will not blindly apply an algorithm if the quantities in the situation are not proportional. To illustrate, Cramer et al. (1993) asked 33 prospective elementary teachers to solve the problem, “Sue and Julie were running equally fast around a track. Sue started first. When she had run 9 laps, Julie had run 3 laps. When Julie completed 15 laps, how many laps had Sue run?” Interestingly, 32 out of 33 prospective teachers solved this problem by setting up and solving a proportion: 9/3 = x/15; 3x = 135; x=45. That is, the prospective teachers did not realize the quantities in the problem were related additively instead of multiplicatively, and so, they used a proportional strategy that did not work. In a similar way, Van Dooren et al. (2005) asked students to solve the problem, “In the hallway of our school, 2 tables stand in a line. 10 chairs fit around them. Now, the teacher puts 6 tables in a line. How many chairs fit around these tables?” (p. 65). Students who could not distinguish proportional relationships from nonproportional relationships solved this problem by using a proportional strategy, namely; they multiplyed six by five. However, the correct solution is multiplying six by four and adding two; that is, the relationship between quantities in the situation is represented as a function in the form $f(x)=ax+b$, (with $b\neq 0$), which implicates a linear, but nonproportional relationship. Van Dooren et al. (2003) refer to the overreliance on proportionality as “illusion of linearity” (p.113). In fact, a proportional reasoner can explain the difference between functions of the form $y=mx$ and functions of the form $y=mx+b$; undoubtedly, in the latter function, y is not proportional to x (Lamon, 2007).

Lamon (2007) asserts that without knowledge of intensive quantities, which are the ones that cannot be measured directly, a student cannot be a proportional reasoner. An intensive quantity relates two extensive quantities (Simon & Blume, 1994a). For example, speed is an intensive quantity relating two extensive quantities, “distance” and “time”. Students need opportunities to analyze intensive quantities such as color intensity, oranginess of a drink, steepness of a hill and squareness of a rectangle, and to engage in argumentation and justification about how to measure these quantities (Lamon, 2007). Multiplicative comparisons are utilized to indicate the intensive quantities (Sowder, Sowder, & Nickerson, 2012). Correspondingly, the ability to recognize a ratio, which is a multiplicative comparison, as a proper measure of a given attribute is an indicator of the ability to reason multiplicatively (Simon & Blume, 1994b; Sowder et al., 2012). In a study conducted by Simon and Blume (1994b) to investigate pre-service teachers' ability to identify ratio as a measure, pre-service teachers had difficulty in recognizing ratio as a proper measure of steepness of ski ramps. Furthermore, some of them measured steepness of the ramps by using incorrect additive comparisons instead of utilizing multiplicative comparisons.

2.1.1.3 Understanding the Mathematical Relationships Embedded In Proportional Situations

A proportional reasoner is able to understand mathematical relationships embedded in proportional situations (Cramer et al., 1993). As mentioned earlier, Lamon (2007) asserts that a proportional reasoner is able to explain the difference between functions of the form y=mx and functions of the form y=mx+b; undoubtedly, in the latter function, y is not proportional to x. That is to say, while in the first function the relationship between the variables are multiplicative, in the second function they are not. Besides explaining and recognizing the difference between the two situations, a proportional reasoner is able to realize that proportional relationships are multiplicative in nature and can be expressed algebraically in the form y=mx (Cramer et al., 1993). In addition, a proportional reasoner knows that the graph of a directly proportional situation (y=mx) is a straight line through the origin and the graph of inversely proportional situation is a hyperbola; however, the graph of algebraic equation of the form y=mx+n, which is a nonproportional situation, is a straight line intersecting the y axis b units above the origin (Lamon, 2007). Moreover, a proportional reasoner knows that the "m" in the equation of direct proportion (y=mx) represents the slope of the line, and it is also the unit rate, or the constant factor that multiplicatively relates the quantities (Cramer et al., 1993, Lamon, 2007). For example, the mathematical relationships embedded in the problem, "3 U. S. dollars can be exchanged for 2 British pounds" is proportional because the relationship can be expressed algebraically as $y = 2/3x$ (y = British pounds and x = U. S. dollars). The constant factor 2/3 also describes the number of pounds per one dollar: 2/3 pound per 1 dollar that is called the unit rate. In contrast, the mathematical relationships embedded in the problem "A taxicab charges $1.00 plus 50 cents per kilometer", is not proportional because the relationship is expressed algebraically as $y = .50x + 1$ where y = cost, x = kilometers; in other words, it is defined by both multiplication and addition (Cramer et al., 1993).

In the studies by Smith, Silver, Leinhardt, and Hillen (2003) and Hillen (2005), the properties of a proportional reasoner mentioned above are grouped into four categories and referred as "*key understandings*". The term is also used in the current study. The four *key understandings* are described in Table 2.1.

Table 2.1 The four key understandings

Key understanding 1	Proportional relationships are multiplicative in nature
Key understanding 2	Proportional relationships are presented graphically by a line that contains the origin
Key understanding 3	The rate pairs (i.e., x, y pairs) in proportional relationships are equivalent
Key understanding 4	Proportional relationships can be represented symbolically by the equation $y = mx$, where the m is the slope, unit rate, and constant of proportionality

2.1.2 Different Solution Strategies

Research studies show that there are a wide range of strategies to solve proportion problems (Baroody & Coslick, 1998; Ben-Chaim et al., 2012; Cramer & Post, 1993; Cramer et al., 1993; Kaput & West, 1994; Lamon, 2007, 2012). Some researchers (Baroody & Coslick, 1998; Kaput & West, 1994) identified the strategies as informal and formal strategies. While some researchers called the "informal strategies" as "pre-formal strategies" (Ben-Chaim et al., 2012), some of them called them as "intuitive strategies" (Cramer & Post, 1993; Lamon, 2007), and some of them did not make any categorization of the strategies (Cramer et al., 1993; Lamon, 2012). In this research study categorization of the strategies was applied as informal and formal strategies.

In this section, first of all, some key aspects of proportional reasoning when examining ratios and solving proportion problems are mentioned. Secondly, the most commonly used informal strategies, which are *building-up, unit rate, factor of change* and *equivalent fractions strategies,* are presented. Thirdly, the formal strategies that are algebraic strategies in which rules and properties of algebra are used (e.g., *cross-multiplication*) are highlighted. Finally, additive strategy that yields incorrect answers to proportion problems is indicated.

Different units of measure, different sets of objects or different types of quantities are called as measure spaces (Lamon, 2007). In a proportional, there are two multiplicative relationships: the one is the relationship between quantities coming from the same measure space (i.e., *between-ratio*); the other one is the relationship between quantities coming from two different measure spaces (i.e., *within-ratio*) (Noelting, 1980; Vergnaud, 1983). On the other hand, Freudenthal (1973, 1978 as cited in Lamon, 2007) defines a *within-ratio* as a ratio, constituent magnitudes of which share the same measure space and a *between-ratio* as a ratio, constituent magnitudes of which are from different measure spaces. As can be seen, there are some inconsistencies between definitions of the terms. In this research study, the terms *within* and *between-ratios* were used parallel to Noelting's and Vergnaud's definitions. To illustrate, consider the following missing value problem, "A recipe calls for 2 scoops of sugar and 6 lemons. If we use 10 scoops of sugar, how many lemons do we need?" (Carpenter et al., 1999, p. 5). The ratio of the amount of sugar to lemon is a *within-ratio,* that is, it is the ratio "within" the first mixture. One who solves the problem with the aid of *within-ratio* would state that the number of lemons was 3 times the number of scoops of sugars, and so, multiplying the 10 scoops of sugar by 3 would give us the answer as 30 lemons. The ratio of the amount of sugar in the first mixture to second mixture and the ratio of the amount of lemons in the first mixture to second mixture are *between-ratios,* that is, they are the ratios "between" the first and second mixtures. One who solves the problem with the aid of *between-ratio* would state that the number of scoops of sugar in the second mixture is 5 times the number of scoops in the first mixture, and so, multiplying the 6 lemons by 5 would give us the answer as 30 lemons (Carpenter et al., 1999). Cramer (1993) claims that a proportional reasoner should realize both *within* and *between-ratios* in a proportional situation because "knowing that both within and between relationships exist offers students alternative strategies." (p. 163). Selecting *within* or *between-ratios* to solve problems might change with respect to numerical relationships between quantities. Based on research, students tend to look for the simplest numerical relationships between quantities, which include integer factors (Cramer et al., 1993). Moreover, it seems that a strategy involving the simplest numerical relations is a more efficient strategy. For instance, if there is an integer factor within a ratio and a non-integer factor between ratios (i.e., $4/20=x/35$), a *within-ratio strategy* would facilitate the computations, whereas for a ratio such as $2/7=6/x$ a *between-ratio strategy* would be easier to calculate. (Carpenter, et al., 1999). It is also important to note that the factor relating any two magnitudes within the same measure space; that is to say, the *between-ratios* for each pair of rates, is not a constant number; however, the factor relating magnitudes between the measure spaces; namely, the *within-ratio,* is a constant number (Cramer et al., 1993).

According to Lamon (1993a, 1993b, 1996), *unitizing* and *norming* are two important processes in the development of proportional reasoning. *Unitizing* is defined as “the process of constructing mental chunks in terms of which to think about a given quantity” (Lamon, 2012, p.104). *Norming* is described as “reinterpreting a situation in terms of some chosen unit” (Lamon, 2007, p.644). Lamon (1994) asserts that the ability to *unitize* “appears to be critical to the development of increasingly sophisticated mathematical ideas” (p. 133). Moreover, she points out the importance of flexibility in *unitizing* that means conceptualizing a quantity with regard to many pieces with different sizes (Lamon, 2012). For example, in the problem, “Suppose you go to the store and you see a sign that says kiwis are 3 for $0.67. You want to buy 9 kiwi fruits.” (Lamon, 2012, p.105), in order to find out how much money is needed to buy 9 kiwi fruits, one can consider the price of single kiwi and calculate one kiwi’s price as 22.33 cents, and then, multiply 22,33 by 9 to get the price of 9 kiwis. However, a person who is a flexible thinker can consider a group of 3 fruits and think the 9 kiwis as 3 (i.e., 3-packs), and then, he can easily find out the price of 9 kiwis by multiplying 0.67 by 3 (Lamon, 2012). In a similar way, Lamon (1993a, 1993b) argues that one of the most important difference between those who think proportionally and those who do not is, using composite units when the context suggests that using them is more effective than using singleton units.

One of the most commonly used informal strategy is *building-up strategy* in which one establishes a ratio and extends it to another ratio by using addition (Lamon, 2007, 2012). Consider the problem, “If two pencils cost 15 TL, how much will six pencils cost?” (Adapted from Lamon, 2007, p. 643). The problem can be solved by using *building-up strategy* as follows:

15 TL for 2 pencils
15 TL for 2 more gives 30 TL for 4 pencils
15 TL for 2 more gives 45 TL for 6 pencils.

In the *building-up strategy*, the main aim is to reason up to some desired quantity by using additive patterns (Lamon, 2007). Although it is a useful and an intuitive strategy children use spontaneously in solving many proportion problems, it cannot be regarded as a process in which proportional reasoning takes place without additional information (Lamon, 2007, 2012). The reason is that one who uses the strategy does not consider the multiplicative relationships between quantities. In other words, the constant ratio between the measure spaces is not taken into consideration in the process of solving a problem by using the *building-up strategy* (Lamon, 2007, 2012).

Unit rate strategy is another commonly used informal strategy that "tells how many units of one type of quantity correspond to one unit of another type of quantity (Lamon, 2012, p.52). That is to say, it is a strategy that asks the question, "How many for one?" (Cramer & Post, 1993). At this point, it is important to note that a rate is different from a ratio in such a way that while a ratio would not extent to other situations, a rate is extendible; in other words, it is not only used in a specific situation, but also used in many situations in which two quantities are connected in the same way (Lamon, 2012). The main aim of the strategy is finding the multiplicative relationship between measure spaces through division (Cramer et al., 1993). That is to say, it is a *within-ratio strategy*. For example, consider the problem, "Steve and Mark were driving equally fast along a country road. It took Steve 20 minutes to drive 4 miles. How long did it take Mark to drive 12 miles?" (Cramer & Post, 1993, p.406). There are always two unit rates corresponding to a given pair of quantities, each being the reciprocal of the other (Cramer & Post, 1993; Lesh et al., 1988). In the problem, the ratios can be shown in two ways:

$$\frac{20\ minutes}{4\ miles} \quad or \quad \frac{4\ miles}{20\ minutes}$$

The unit rate for the first one is 5 minutes per 1 mile because the result of the division of 20 minutes by 4 miles is 5. Similarly, the unit rate for the second one is 1/5 mile per 1 minute resulting from the division of 4 miles by 20 minutes. The first unit rate that describes the amount of time for one mile can be used to solve the problem as follows:

$$\frac{5\ minutes}{1\ mile} \times\ 12\ miles\ = 60\ minutes$$

If the problem asked the amount of miles Mark drove in 60 minutes, the second unit rate (1/5 mile per 1 minute) would be used (Cramer & Post, 1993). In order to solve a proportion problem, one should determine which unit rate must be used as a factor (Cramer & Post, 1993; Cramer et al., 1993; Lesh et al., 1988). However, students often have difficulty in determining the unit rate that is useful to solve the problem (Cramer et al., 1993).

Still another most commonly used informal strategy is *factor of change strategy* that asks the question "How many times greater?" (Cramer et al., 1993). In order to exemplify how the strategy can be utilized, consider the problem, "If three erasers cost 6 TL, how much will six eraser cost?" The *factor of change strategy* involves (a) comparing the number of erasers bought in each situation (b) determining the factor of change that indicates how many times as many erasers bought in the first situation as compared to the number of erasers bought in the second situation and (c) multiplying the factor times the price paid in the first situation (Bart, Post, Behr & Lesh, 1994). That is to say, a student utilizing *factor of change strategy* to solve the problem might reason that if the number of erasers is doubled, then the price must also be doubled (Cramer et al., 1993). The main aim of the strategy is finding the multiplicative relationship within a measure space (Cramer et al., 1993). In other words, it is a *between-ratio strategy*.

Factor of change strategy is also used when quantities are multiplied in a ratio table. While some students utilize two-column approach to solve proportion problems, some of others might use a ratio table or a proportion table (Lamon, 2012). A ratio table is a horizontal arrangement that shows how two variables are related. It is a suitable device to organize information (Lamon, 2012). Two solutions that use ratio tables to solve the problem, “A party planning guide says that 3 pizzas will serve about 7 people. How much pizza is needed for 350 people?” (Lamon, 2012, p.114) are presented below:

x 5 x 10

Number of people	7	35	350
Number of pizzas	3	15	150

Figure 2.1 A solution using a ratio table based on multiplication.

x 100 ÷ 2

Number of people	7	700	350
Number of pizzas	3	15	150

Figure 2.2 A solution using a ratio table based on multiplication and division.

The format of the tables can be changed; for instance, a student might only draw arrows and not use the table format as shown in the above figures. Moreover, there are likely to be several different correct ways of building ratio tables to solve proportion problems (Lamon, 2012). One can use multiplication (see Figure 2.1), division or the combinations of them (see Figure 2.2) in order to get the intended quantity. In addition to these operations, addition and subtraction might also be utilized in the ratio tables (Lamon, 2012). That is to say, ratio tables can also be used to present *building-up strategy*. As mentioned earlier, in the *building-up strategy* one establishes a ratio and extends it to another ratio by using addition (Lamon, 2007, 2012). For example, if the problem asks the number of pizzas for 35 people, the solution that uses *building-up strategy* in a ratio table might be shown as in Figure 2.3.

+7 +7 +7 +7

Number of people	7	14	21	28	35
Number of pizzas	3	6	9	12	15

+3 +3 +3 +3

Figure 2.3 A solution using a ratio table based on addition.

Equivalent fraction strategy, which is also called as *fraction strategy*, is another informal strategy. In this strategy, the labels are ignored and ideas of equivalence are used (Cramer & Post, 1993). That is, the rates are treated as fractions and the multiplication rule for obtaining equivalent fractions is applied as follows:

If $\frac{3}{60} = \frac{6}{?}$ then $\frac{3}{60} \times \frac{2}{2} = \frac{6}{120}$ (Cramer, et al., 1993, p. 167).

The main aim of the strategy is to find out a fraction with a term equal to the missing value, which is also equivalent to the given fraction. In order to do that, the given fraction (3/60) is multiplied by a particular fraction of the form n/n (2/2), which is equal to one, and then, the missing value (120) is found (Bart, Post, Behr & Lesh, 1994).

Cramer and Post (1993) argues that students should be encouraged to use the informal strategies to solve proportion problems; moreover, the formal strategies should not be taught until students internalize and use the informal strategies properly (Ben-Chaim et al., 2012; Cramer et al., 1993; Kaput & West, 1994; Post, Behr & Lesh, 1988). However, students and even teachers frequently use formal strategies, which are algebraic strategies in which rules and properties of algebra are used, to set up and solve proportion problems (Ben-Chaim et al., 2012; Cramer & Post, 1993). That is, a formal strategy is an equation-based approach that involves "the syntactic manipulation of formal algebraic equations (e.g., cross-multiplication or formal division to help isolate a variable)" (Kaput & West, 1994, p. 244). Undoubtedly, the most common formal strategy is *cross-multiplication strategy* (Cramer & Post, 1993; Lamon, 2012). The *cross-multiplication strategy* is a standard algorithm that involves setting up an equation of two ratios, one of which has an unknown quantity; cross-multiplying, and solving the equation for the unknown quantity (Van de Walle et al., 2010). In other words, it is the combination of several quantitative operations (Kaput & West, 1994). Although it is an efficient strategy, it might cause confusion and lead to error (Cramer & Post, 1993; Cramer et al., 1993). Furthermore, the ability to implement the cross-multiplication procedure does not include proportional reasoning because correct answers might be achieved without recognizing the structural similarities on both side of the proportion (Lamon, 1989; 1995; 2012). Moreover, the cross-multiplication procedure has no physical referent, and therefore, it has less meaning for students and for teachers (Cramer & Post, 1993). Correspondingly, Principles and Standards for School Mathematics argued:

> Facility with proportionality involves much more than setting two ratios equal and solving for a missing term. It involves recognizing quantities that are related proportionally and using numbers, tables,

> graphs and equations to think about the quantities and their relationship (NCTM 2000, p. 217).

As stated in the above quote, proportional reasoning is more than setting up two ratios and finding the answer. Similarly, according to Lamon (2012), one of the characteristics of a proportional thinker is "understand the relationships in simple proportional and inversely proportional situations so well that they have discovered for themselves the cross-multiply-and-divide algorithm" (p. 260).

Some strategies might yield an incorrect answer to a proportion problem. The most common used incorrect strategy is additive strategy (Ben-Chaim et al., 2012; Karplus et al., 1983). Incorrect additive strategy can be defined as "calculating the difference or sum of two parts of the ratio, and an attempt to divide the whole by the difference or the sum" (Ben-Chaim et al., 2012, p. 53). It is important to note that students often use incorrect additive strategies when the proportion problem has noninteger ratios (Karplus et al., 1983, Cramer et al., 1993; Post, Cramer, Behr, Lesh, & Harel, 1993; Singh, 2000; Sowder, Philipp, et al., 1998).

2.1.3 Teaching and Learning Ratio and Proportion

Ratio and proportion are concepts that fall under the general umbrella of proportional reasoning. Therefore, the following section is about ratio and proportion concepts. First, the definitions of ratio and proportion are given, and then, the research on learning and teaching of ratio and proportion are presented.

Ratio is defined as "a comparative index that conveys the abstract notion of relative magnitude" (Lamon, 1995, p. 169). A ratio can be represented in different ways (i.e., a to b, a:b and a/b). There are two types of ratio that compare quantities in the same measure space: part-to-whole ratios and part-to-part ratios. While part-to-whole ratios are comparisons of a part to a whole (e.g., the ratio of the number of boys in a school bus to the number of students in the school bus), part-to-part ratios are comparisons of one part of a whole to another part of the same whole (e.g., the ratio of the number of boys to the number of girls in the school bus) (Ben-Chaim et al., 2012). A fraction is also a part-to-whole ratio; therefore, it can be concluded that every fraction is also a ratio (Lamon, 2012). Yet, every ratio is not a fraction. To illustrate, if a ratio compares one part of a whole to another part of the same whole (part-to-part ratio) or compares quantities from different measure spaces (rate), the ratio (or rate) will not be a fraction (Lamon, 2012). Both part-to-part and part-to-whole ratios are comparisons of quantities in the same measure space. A ratio can also compare two quantities with different measuring units, which is called a rate. As noted before, a rate is different from a ratio in such a way that while a ratio would not extent to other situations, a rate is extendible; in other words, it is not only used in a specific situation, but also used in many situations in which two quantities are connected in the same way (Lamon, 2012).

It is also important to note that every ratio is not a rational number. For example, Pi (π), which is the ratio of the circumference of a circle to its diameter, is not a rational number because it is impossible to express it as a quotient of two whole numbers (Ben-Chaim et al., 2012; Lamon, 2012). In addition, a ratio may have a zero as its second component (Lamon, 2012). For instance, let's assume five apples are divided between two children. If the first child gets nothing, and the second child gets all of the apples, we can write the ratio as *0*:*5* to describe the proportional situation. However, if the first child gets all of the apples, and the second child gets nothing, the ratio becomes *5*:*0*, which is mathematically meaningless, but it describes a real life proportional situation (Ben-Chaim et al., 2012).

Freundenthal (1978) argues that a proportion consists of two equivalent ratios, further it is a linear, ratio-conserving mapping of one magnitude upon another (as cited in Lamon, 1989). Similarly, Lamon (1995) defines a proportion as "the statement of equality between one ratio and another in the sense that both convey the same relationship" (p. 171). In addition to these definitions, Ben-Chaim et al. (2012)

mentions another aspect of the proportion and states that corresponding elements of two sets are in a proportional relationship if there is a constant ratio (either direct or indirect) between them. In other words, the multiplicative relationship between two quantities has to be constant, either in the same, or opposite direction. For instance, according to gas laws, since pressure (*P*) is directly proportional to temperature (*T*), the quotient derived from pressure and temperature (*P/T*) is a constant. However, since pressure (*P*) is inversely proportional to volume (*V*), the products of volume and pressure (*P x T*) is a constant (Ben-Chaim et al., 2012). Correspondingly, Lamon (2007, 2012) identified two major types of invariance in proportional relationships: one is invariance of the ratio of two quantities, which means preserving the ratio of the two quantities when quantities are directly related and the other one is invariance of the product of two quantities, which means preserving the product of the quantities when the quantities are inversely related. Proportional reasoning requires recognizing these major types of invariance (Lamon, 2012). Furthermore, Lamon (2007) emphasizes that "the two quantities might be increased or decreased as long as the relationship between the quantities is preserved, that is, as long as their original ratio is maintained." (p.648). Additionally, in proportional situations, "two quantities are linked to each other in such a way that when one changes, the other one also changes in a precise way with the first quantity" (Lamon, 2012, p. 6), which is referred as covariance (Lamon, 2007, 2012). In a direct proportion, the direction of change in the related quantities is the same; that is, both quantities increase or decrease. Nonetheless, the critical point is that both quantities increase or decrease by the same factor (Lamon, 2012). However, in an inverse proportion, although two quantities change together in a synchronized way, the direction of change is not the same for both (Lamon, 2007). That is, when one quantity increases by a certain factor, the other quantity decreases by the inverse of that factor (Lamon, 2012).

The development of proportional reasoning has been studied over the fifty-five years. Inhelder and Piaget (1958) were the first researchers who constructed a theory about the development of proportional reasoning (as cited in Lamon, 2007). Many research about proportional reasoning have been conducted in several different disciplines (e.g., science education, psychology, mathematics education) (Lamon, 2007). However, one of the most comprehensive and longitudinal research attempts belongs to the Rational Number Project (RNP) from a mathematics education perspective. In 1979, the RNP began to investigate ratio and proportion concepts (Behr et al., 1983). Several research belonging to the project and some other studies argue that ratio and proportion are concepts which are generally difficult to understand for many school children (Behr et al., 1983; Cramer et al., 1989; Hart, 1988; Lamon, 2007). In such a context, mathematics teachers' roles in the instruction of ratio and proportion become critical. However, research shows that teachers' knowledge and understanding of these concepts are problematic as well (Chick, 2003; Cramer, & Lesh, 1988; Harel, & Behr, 1995; Lacampagne et al., 1988; Sowder et al., 1998). Some research on children's, pre-service teachers' and in-service teachers' knowledge and understanding of ratio and proportion concepts are presented below.

Within the scope of the RNP, Cramer and Post (1993) asked 913 seventh and eighth grade students to solve missing value, numerical comparison, qualitative prediction and comparison problems in different context: speed, scaling, mixture, and density. The main aim of the study was investigating students' facility with proportional reasoning. Cramer and Post (1993) found that success rates of missing value and numerical comparison problems were low. When students correctly solved these types of problem, they utilized four distinct solution strategies: *unit rate*, *factor of change*, *fraction*, and *cross-product*. The analysis of the results suggested that while the seventh graders mostly used the *unit rate strategy,* the eighth graders mostly used the *cross-product algorithm.* The researchers asserted that the *unit rate strategy* was an intuitive strategy, which built on students' real life experiences. Moreover, they claimed that the more intuitive *unit rate* and *factor of change* approaches related more meaningfully to the situation. Another important result of the study was that in a nonproportional problem, the seventh grade students, who had not been taught the *cross-product algorithm,* were more successful than the eighth graders, who had been taught the algorithm. This implies that overreliance on the *cross-multiplication strategy* might cause confusion. While the eighth graders incorrectly applied the *cross-product algorithm* to solve the problem, seventh graders solved it by using other problem solving strategies. In the light of these results, the researchers suggested that being a proportional reasoner was more than applying the *cross-product algorithm*. Furthermore, a proportional reasoner had to be able to solve multiple types of problems including missing value, numerical comparison, qualitative prediction and comparison problems. The researchers also found that scaling context was significantly more difficult than the other contexts. In other words, they argued that the context of the problem influenced difficulty of the problem. Similar findings were found in another study of the RNP by Heller, Ahlegren, Post, Behr, &Lesh (1989). The researchers examined the effects context on the performance of seventh grade students on proportional reasoning tasks. They determined that when students encountered with a less familiar problem context, the difficulty of the problem increased.

Cramer et al., (1993) conducted another study in the RNP. The study was a teaching experiment that aimed to investigate the seventh grade students' learning of ratio and proportion concepts. The researchers collected data from detailed lesson plans, activities, written tests, and student interviews. The RNP lessons aimed to reflect the belief that proportional reasoning was more than setting up two ratios and finding the answer. For that purpose, multiple strategies (i.e., building tables, unit rate, factor of change, fraction strategy) to solve proportion problems were taught to students. Additionally, in the lessons, cross-multiplication strategy was not taught until students developed and internalized more meaningful, although less efficient strategies. The results suggested that students were able to learn to use different strategies. Moreover, unlike the findings in Cramer and Post's (1993) study, there was not any one strategy seemed to be preferred by all students. However, it was also found that students had difficulty in problems that had noninteger relationships. Further, incorrect additive strategies were often utilized to solve the problems with noninteger relationships. In a similar way, Sowder, Philipp et al. (1998) asserted that additive reasoning had an important role in the early grades, and so, students returned to additive reasoning in multiplicative situations when they encountered a difficulty. For example, they reported that in a situation, "if one candy bar weighs 4 ounces and another weighs 8½" (Sowder, Philipp, et al., 1998, p.23), students would sometimes determine that the second bar weighs 2½ times as much as the first bar. In that case, the students combined multiplicative and additive reasoning in an incorrect way. Similarly, in another study of the RNP, Lesh, Post, and Behr (1987) found that varying the number size or the numerical relationships of quantities dramatically affected students' performance on proportion problems.

Singh (2000) interviewed with two sixth grade students to construct an understanding of their proportional reasoning schemes. The results suggested that while Karen had constructed multiplication schemes in solving proportion problems, Alice had not, although she correctly solved some problems. The researcher argued that Alice's strategy was procedural rather than conceptual because she solely relied on the *unit rate strategy* in a memorized manner. In such a way that she used one unit on all occasions. That is, she was unable to unitize the composite units to find a ratio unit. Therefore, once her strategy did not work, she used a wrong additive strategy. Similar to findings of Cramer and Post (1993), overreliance on a strategy (i.e., *unit rate strategy*) appeared to lead confusion. In addition, when the relationship between quantities was not integer, she again used an additive strategy. On the other hand, Karen was able to think in terms of composite units and iterating a composite unit to its referent point by preserving the invariance of the ratio. In the light of these findings, the researcher suggested that the *unit rate strategy* should not be taught students until unitizing and composite units were internalized. Further, he stressed the importance of solving a range of problems with considerable flexibility, which meant to choose a strategy that was best suited to the problem.

In Turkey, research studies indicate that students and teachers lack a deep understanding of ratio and proportion concepts and frequently rely on rote procedures such as the cross-product algorithm. According to the study by Çelik (2010), which was designed to investigate 204 seventh grade and 188 eight grade students' proportional reasoning skills, more than half of the students (60 %) lacked ability to think proportionality. In another study by Avcu and Avcu (2010), which was conducted to determine sixth grade students' strategies in solving ratio and proportion problems, students used six different strategies: *cross-product algorithm*, *factor of change*, *equivalent fractions*, *equivalent class*, *unit rate*, *and building-up* strategies. However, the results revealed that the most frequently used strategy was *cross-product algorithm.* In the same way, Duatepe, Akkuş-Çıkla and Kayhan (2005) examined 295 elementary school students' solution strategies on different types of proportional and nonproportional problems. The findings of the study suggested that the most commonly used strategy for missing value problems was *cross-multiplication algorithm;* for quantitative comparison problems (i.e., numerical comparison problems) was *unit rate*; for nonproportional problems was *additive strategy,* and for inverse proportion problems was *inverse proportion algorithm*. Moreover, it was concluded that students' solution strategies were affected by problem types. Similar results were found in the study by Kayhan (2005). The researcher investigated solution strategies of 143 sixth and seventh grade students on items required proportional reasoning skills in terms of their grade level, gender and problem types. The results showed that the students used different strategies for different problem types. Further, the results indicated that the most commonly used strategy for missing value problems were *cross-multiplication algorithm* and *unit rate,* and for quantitative comparison problems were *equivalence class* and *additive strategies.*

In a research study, Atabaş (2014) investigated 120 fifth grade and 101 sixth grade students' understanding of proportional and nonproportional situations. The researcher also examined the reasons of incorrect solutions depending on the numerical structures of the problems. Data were collected by an instrument that included twelve proportional and nonproportional problems: additive, constant, numerical comparison, and missing value, each with an integer and noninteger ratio. The results showed that missing value problems were solved with the highest success rate, whereas nonproportional constant problems were solved with the lowest success rate in both grade levels. Moreover, the researcher found that numerical structures of problems affected students' success and choice of strategies. To illustrate, in nonproportional additive problems, students' success rates in problems with noninteger ratios were increased significantly than problems with integer ratios. The researcher argued that noninteger numbers might warn students to read the problem more careful since the numbers did not evoke to use cross product algorithm. In contrast with the findings of Cramer et al., (1993), students in the study did not show a tendency to use incorrect additive strategies in proportional problems with noninteger ratios.

A study with first grade pre-service elementary mathematics teachers was conducted by Akkuş-Çıkla and Duatepe (2002). In order to investigate pre-service teachers' proportional reasoning abilities and solution strategies in ratio and proportion problems, the researchers conducted interviews with 12 pre-service teachers. In the interviews, eight open-ended questions, which included numerical comparison and missing value problems, asked to pre-service teachers. In addition to these problems, the definitions of ratio and proportion, the difference between them and daily life examples of the concepts were asked. The results showed that pre-service teachers had difficulty in defining ratio and proportion concepts and explaining the difference between them. Additionally, although pre-service teachers solved the ratio and proportion problems procedurally, they did not have conceptual knowledge required to solve and understand the problems. In other words, they preferred to utilize algebraic strategies in which algorithmic procedures (e.g., cross-multiplication) were applied without the understanding of proportional relationships. Similar findings were found in Person, Berenson, and Greenspan's (2004) study. The researchers examined a pre-service high school teacher's lesson plans on the two proportional reasoning concepts: rate of change and right triangle trigonometry and interviewed with him in order to investigate his beliefs and understanding concerning proportional reasoning. The findings of the study suggested that although the pre-service teacher felt extremely comfortable with numerical manipulations, he had difficulty in making connections between concepts. Moreover, when the interviewer asked him how to introduce the concept of ratio to middle grade students, he mentioned techniques and number operations without giving a sense of proportions. It was asserted that the pre-service teacher would probably present to his students an abstract and technical world in which connections were harder to make, but the correct answer was found.

Recently, Ölmez (2016) performed a research to investigate how pre-service teachers' formation of additive and multiplicative relationships supported and constrained their understandings of ratios and proportional relationships. Semi-structured interviews were conducted with six pre-service teachers. The results indicated that pre-service teachers' formation of multiplicative and additive relationships in proportional situations played an important role in their ability to solve proportional tasks. Moreover, pre-service teachers who formed multiplicative relationship between quantities were found to have a robust understanding of proportional relationships. On the other hand, pre-service teachers who mostly relied on additive relationships such as repeated addition and subtraction, focusing the differences rather than the multiplicative comparisons, and the use of the phrase "for every" were found to struggle in their reasoning about proportional relationships. Based on the findings of the study, the researcher proposed that multiplicative relationships had a critical role in ensuring a powerful understanding of ratios and proportional relationships. Similarly, Cramer et al. (1993) proposed that a proportional thinker can realize that proportional relationships have a multiplicative nature.

Understanding the concept of ratio is crucial in making the transition from additive to multiplicative thinking, and it should be given more attention than it has traditionally received in the preparation of middle school mathematics teachers (Sowder et al., 1998). The fact that both students and teachers frequently use additive strategies where multiplicative comparisons are required (Hart, 1988; Lesh et al., 1988). Further, incorrect additive strategies do not seem to disappear with maturation (Hart, 1988). However, Sowder et al., (1998) asserted understanding multiplicative relationships and distinguishing them from additive relationships is the heart of proportional reasoning

In the same manner, another important point highlighted in research studies is that there is a strong tendency to over generalize proportionality; that is, many students and teachers incorrectly apply proportional reasoning in situations that have nonproportional relationships (Atabaş, 2014; Cramer et al., 1993; Van Dooren et al., 2005; Van Dooren et al., 2003). Van Dooren et al. (2003) refer to the overreliance on proportionality as “illusion of linearity” (p.113). For instance, one of the findings of Atabaş’s (2014) study, which was mentioned earlier, was that when the incorrect strategies of fifth and sixth grade students were considered, there was a tendency to overuse proportional strategies in nonproportional situations. It implies that students had difficulty in distinguish proportional from nonproportional situations. Another study by Van Dooren et al. (2005) examined the development of misapplication of proportional reasoning with the age and the educational experience. A test consisting of proportional and nonproportional problems was administered to 1.062 students from Grades 2 to 8. The test had eight items with four categories. The first category was proportional problems (i.e., proportional missing value problems). The other three categories had different types of nonproportional problems: additive, linear, and constant problems. One of the linear problems was “In the hallway of our school, 2 tables stand in a line. 10 chairs fit around them. Now, the teacher puts 6 tables in a line. How many chairs fit around these tables?” (p. 65). An incorrect solution that used a proportional strategy was multiplying six by five and obtaining 30 (i.e., 30 = 6 x 5); on the other hand, the correct solution was multiplying six by four and adding two, and obtaining 26 (i.e., 26 = 4 x 6 + 2). In the problem situation, the relationship between quantities can be represented as a function in the form $f(x)=ax+b$, (with $b\neq0$), which implicates a linear, but nonproportional relationship, rather than a function in the form $f(x)=cx$, which implicates a linear and proportional relationship. The findings suggested that students had difficulty in recognizing the mathematical relationships embedded in the two relationships and distinguishing one relationship from the other. Therefore, they tended to apply proportional strategies in nonproportional situations. The number of errors of overreliance on proportionality increasing considerably second grade to fifth grade in parallel with the growing proportional reasoning capacity of the students. From sixth grade to eighth grade, students began to distinguish more often between proportional and nonproportional situations. Yet, even eighth grade, a considerable number of errors of overreliance on proportionality were made. Similarly, Cramer et al. (1993) asked 33 prospective elementary education teachers to solve the problem: "Sue and Julie were running equally fast around a track. Sue started first. When she had run 9 laps, Julie had run 3 laps. When Julie completed 15 laps, how many laps had Sue run?" to investigate their ability to distinguish proportional from nonproportional situations. Interestingly, 32 out of 33 prospective elementary education teachers solved this problem by setting up and solving a proportion. That is, they did not realize the quantities in the problem were related additively instead of multiplicatively, and thus they used a proportional strategy that was not work.

Some studies revealed that in-service and pre-service teachers might benefit from professional development opportunities that focus on developing their proportional reasoning. A study with five in-service middle grade teachers was conducted by Sowder, Philipp et al. (1998) as a project of the Teaching and Learning Rational Numbers and Quantities Working Group. One of the aims of the research was investigating the development of the middle school teachers' content knowledge through a two-year teacher professional development program focused on rational number, quantity, and proportional reasoning concepts. In addition, the researchers examined the influence of the development in the teachers' content knowledge on their teaching and student's learning. The teacher professional development program included solving mathematical tasks, examining student work and presentations of different mathematics educators about the concepts and their teaching. The findings of the study suggested that teachers' content knowledge and instructional practices improved through the process. Specifically, they constructed a deeper understanding on proportional and nonproportional situations and their difference. Moreover, they claimed that while teachers' understanding of proportional reasoning was developing, the students' learning of the concepts related proportional reasoning enhanced. Similarly, Berk et al. (2009) carried out a study with 148 pre-service elementary teachers, which aimed to investigate pre-service teachers' flexibility in the domain of proportional reasoning before and after an intervention that engaged the participants in comparing different solutions to proportion problems. The findings showed that the intervention led to significant gains in participants' flexibility in the use of multiple solution methods across a set of problems, solve the same problem using multiple methods, and choose strategically from among methods so as to reduce computational demands. Correspondingly, another study was conducted by Hillen (2005) to investigate pre-service secondary mathematics teachers' understandings about proportional reasoning before and after completion of a practice-based methods course that focused on proportional reasoning. A total of 16 pre-service teachers participated in the researcher's quasi-experimental design. The findings of the study revealed that teachers learned important aspects of proportional reasoning with the help of the course. For example, pre-service teachers who enrolled in the course used a broader range of strategies, significantly improved their capacity to distinguish proportional from nonproportional situations, and significantly improved their understanding of the nature of proportional relationships after the course, whereas those who did not enroll in the course did not. In a similar manner, in the study by Whitenack and Ellington (2013), they presented one whole-class discussion that took place in a middle school mathematics Rational Number and Proportional Reasoning course. Statistical measures indicated that teachers made gains in their understanding of proportionality concepts and substantial gains in their views of teaching with the help of class discussions on some proportional reasoning tasks.

As a result, students and teachers frequently use formal strategies, which are algebraic strategies in which rules and properties of algebra are used, to set up and solve proportion problems (Akkuş-Çıkla & Duatepe, 2002; Avcu & Avcu, 2010; Cramer & Post, 1993; Duatepe et al., 2005; Person et al., 2004). Moreover, research shown that there is a tendency to use only one strategy (e.g., *cross-multiplication*, *unit rate*) to solve all types of problems, which might lead confusion (Cramer & Post, 1993; Singh, 2000). In addition, the context and numerical structures of the problem influence difficulty of the problem (Atabaş, 2014; Cramer & Post, 1993; Cramer et al., 1993; Heller et al., 1989; Lesh et al., 1987; Singh, 2000; Sowder, Philipp et al., 1998). Furthermore, research revealed that many students and teachers incorrectly apply proportional reasoning in situations that have nonproportional relationships, which is referred as over generalization of proportionality (Atabaş, 2014; Cramer et al., 1993; Van Dooren et al., 2005; Van Dooren et al., 2003). Additionally, it was found that pre-service teachers frequently solved ratio and proportion problems procedurally and they did not have conceptual knowledge required to solve and understand the problems (Akkuş-Çıkla & Duatepe, 2002; Person et al., 2004).

2.2 Mathematical Knowledge for Teaching

In this research study, pre-service middle school mathematics teachers' content knowledge about proportionality concepts before and after receiving a practice-based instructional module based on proportional reasoning was investigated. Content knowledge is an important part of mathematical knowledge for teaching as described by Ball et al. (2008); therefore, in this section, firstly, theoretical framework of mathematical knowledge for teaching is presented. Then, a practice-based professional development model, which suggested by Ball and Cohen (1999) in order to develop mathematical knowledge for teaching, is described.

2.2.1 Theoretical Framework of Mathematical Knowledge for Teaching

As Munby et al. (2001) said, "many bridges remain to be built in both directions between practice and knowledge as we seek to understand the nature of teachers' knowledge and its development" (p.885). There have been many attempts to bridge the gap between practice and knowledge in the field of mathematics education. Most widely accepted model in mathematics education was raised by Ball et al. (2008) (See Figure 2.4). The model conceptualizes knowledge for teaching mathematics differently, and so, link practice and knowledge differently, as well. Ball and her colleagues aim is to further develop Shulman's (1986, 1987) notion of pedagogical content knowledge.

In the mid-1980s, when content and pedagogy were viewed separately and the emphasis was on pedagogy, Shulman asked that "How are content knowledge and general pedagogical knowledge related?" (Shulman, 1986, p.9). Eventually, Shulman and his colleagues introduced the field of teacher education to a new construct, "pedagogical content knowledge" (Shulman, 1986, 1987). One of the aims of Shulman's (1986, 1987) work was to focus the need to conceptualize content knowledge unique to teaching specific professional knowledge. To this end, Shulman (1986) offered three categories for the content knowledge, which are subject matter knowledge, pedagogical content knowledge, and curricular knowledge. This subject matter knowledge goes beyond recalling and demonstrating facts and procedures of the domain: "The teacher need not only understand that something is so; the teacher must further understand why it is so, on what grounds its warrant can be asserted, and under what circumstances our belief in its justification can be weakened and even denied." (p. 9), which implies that teachers' content knowledge should represent a deep understanding of the material to be mastered by the students (Krauss, Brunner, Kunter, et al., 2008). Pedagogical content knowledge, on the other hand, includes knowledge on how best to represent and formulate the subject to make it comprehensible to others, as well as knowledge on students' subject-specific conceptions and preconceptions that students of different ages and backgrounds bring with them, and if those preconceptions are misconceptions, which they so often are and what are the strategies to cope with them. In addition, curricular knowledge is knowledge of the full range of programs designed for teaching of particular subjects and topics at a given level, as well as the knowledge of alternative curriculum materials for a given specific subject within a grade (Shulman, 1986).

Shulman's (1986, 1987) work focused on types of knowledge necessary for teaching and the need to conceptualize domain-specific knowledge for teaching. He suggested a way of examining knowledge that only teachers know and only teachers can do (Berry, Loughran, & Van Driel, 2008). His work called attention of researchers, policymakers, and teacher educators to the need for subject-specific development of teacher knowledge. Furthermore, various researchers have taken up Shulman's project in their own content areas, including mathematics, social studies, science, physical education, communication, religion, chemistry, engineering, music, special education, English and others (Ball et al., 2008; Munby, Russell, & Martin, 2001).

To describe professional knowledge within mathematics Ball and her colleagues use the term "mathematical knowledge for teaching" (MKT). They define MKT as "the mathematical knowledge needed to carry out the work of teaching mathematics", where teaching is defined as "everything that teachers must do to support the learning of their students" (Ball et al., 2008, p.395).

As shown in the Figure 2.4, the first domain within MKT is common content knowledge (CCK), which is a sub-domain of Shulman's subject-matter knowledge category. This is defined as "the mathematical knowledge and skill used in settings other than teaching" (Ball et al., 2008, p.399) and involves the ability to perform calculations, solve mathematical problems, give mathematical definitions, and use terms and notation correctly. By calling this knowledge "common" Ball et al. mean that it is the knowledge that common with other professions.

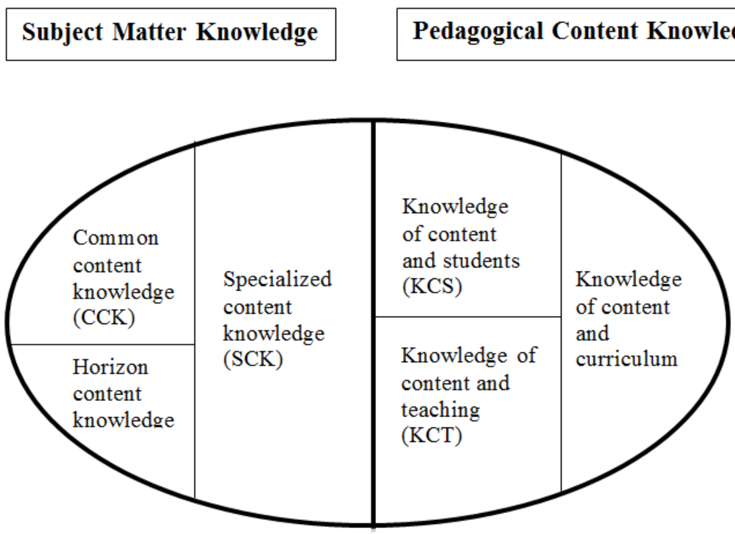

Figure 2.4 Mathematical knowledge for teaching model adapted from (Ball et al., 2008)

The second domain of MKT, which is also a sub-domain of Shulman's (1986) subject-matter knowledge category, is specialized content knowledge (SCK). It is mathematical knowledge which is "not typically needed for purposes other than teaching" (Ball et al., 2008, p. 400). In other words, it includes the ability to work with mathematical content in a way that is unique to teaching. Ball et al. (2008) identify some tasks that comprise this knowledge type:

Presenting mathematical ideas
Responding to students "why" questions
Finding an example to make a specific mathematical point
Recognizing what is involved in using a particular representation

Linking representations to underlying ideas and to other representations
Connecting a topic being taught to topics from prior or future years
Explaining mathematical goals and purposes to parents
Appraising and adapting the mathematical content of textbooks
Modifying tasks to be either easier or harder
Evaluating the plausibility of students' claims (often quickly)
Giving or evaluating mathematical explanations
Choosing and developing useable definitions
Using mathematical notation and language and critiquing its use
Asking productive mathematical questions
Selecting representations for particular purposes
Inspecting equivalencies (p. 400).

Consequently, the SCK domain constitutes knowledge beyond that teachers taught to their students. In brief, it is the knowledge of representing mathematical ideas, providing mathematical explanations for common rules and procedures and examining and understanding unusual solution methods or problems (Hill, Ball, & Schilling 2008). In addition, CCK and SCK are both mathematical knowledge, which do not comprise knowledge of students and teaching (Hill et al., 2008).

The third domain of MKT, which is also a sub-domain of Shulman's (1986) subject-matter knowledge category, is horizon content knowledge. Ball et al. (2008) define the knowledge as "an awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (p. 403).

The fourth domain of MKT, which is also a sub-domain of Shulman's (1986) pedagogical content knowledge category, is knowledge of content and students (KCS). It is "knowledge that combines knowing about students and knowing about mathematics" (Ball et al., 2008, p. 401). Shulman (1986) emphasized the importance of KCS as a primary element of pedagogical content knowledge. Shulman (1986) specified that the knowledge covers "an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons" (p. 9). In addition, Hill et al. (2008) defining

KCS as “knowledge of how students think about, know, or learn a topic and teachers’ understanding of how students learn a particular content” (p. 6). Moreover, Ball et al. (2008) asserted that KCS requires teachers to anticipate what students are likely to think and what they will find interesting or challenging. Most importantly, KCS entails an interaction between specific mathematical understanding and familiarity with students and their mathematical thinking (Ball et al., 2008).

The last two domains of MKT, which are also a sub-domain of Shulman’s (1986) pedagogical content knowledge category, is knowledge of content and teaching (KCT) and knowledge of content and curriculum. It comprises knowledge about instructional sequencing of particular content, about selecting useful examples for taking students deeper into the content, about evaluating the instructional advantages and disadvantages of representations used to teach a concept, and determining what different approaches and methods could be used to teach the concept (Ball, Bass, Sleep, & Thames, 2005; Ball et al., 2008).

Teachers’ mathematics knowledge for teaching has an important role in shaping the quality of their teaching (Ball et al., 2005). In this respect, both in-service and pre-service teachers’ mathematical knowledge for teaching is important areas of research focus. Moreover, besides their general mathematical knowledge for teaching, their knowledge in a specific content has also been investigated in various research studies. This study was parallel to such research in that its main aim was to investigate pre-service middle school mathematics teachers’ content knowledge in proportionality concepts before and after receiving a practice-based instructional module based on proportional reasoning. This knowledge is certainly a prerequisite for teaching ratio and proportion concepts.

2.2.2 Practice-Based Approach

In order to have an impact on pre-service teachers' knowledge and instructional practices, professional learning experiences should be closely tied to real classroom practices (Ball & Cohen, 1999; Smith 2001). According to Ball and Cohen (1999), teachers need opportunities to review their current practices and to examine others' practices; moreover, they need to learn more about mathematics contents and students they teach. However, traditional professional development programs do not serve to transform teachers' knowledge, beliefs and habits of practice (Smith, 2001). Ball and Cohen (1999) propose a professional development model that focuses to use practice as a site for professional learning. The teacher education model, which is commonly referred as practice-based professional development (PBPD), aims to deepen and sharpen teachers' knowledge through a practice-based curriculum and to improve teachers' capacity for innovative practice (Ball & Cohen, 1999; Smith, 2001; Silver, 2009). According to Ball, Sleep, Boerst, and Bass, (2009) practice-based approach "entails analyzing and naming aspects of the work of teaching and identifying the key demands of that work, including the content knowledge needed." (p. 460). The practice-based curriculum is closely connected to practice in such a way that it includes the following components:

- Documents of practice such as videotapes of classroom lessons, samples of student work, and written cases of teaching.
- Field-based assignments such as teaching a mathematics content, conducting a student interview (Ball, et al., 2009; Sleep, Boerst, & Ball, 2007).

The documents of practice can be drawn from teachers' own teaching or can be specifically collected from other's practice. Then, the documents would be utilized to develop teachers' knowledge of content, students' learning and teaching (Ball & Cohen, 1999). Tasks used in a practice-based professional development program are commonly called professional learning tasks (PLTs) and described as “activities that are situated in and organized around components and artifacts of instructional practice that replicate or resemble the work of teaching.” (Silver, 2009, p. 245). Similar to Ball et al. (2009) and Sleep et al. (2007), Silver (2009) states that professional learning tasks involves artifacts of practice such as curriculum materials, video or narrative records of classroom teaching cases, and examples of student work. However, Silver (2009) does not regard field-based assignments as professional learning tasks. Particularly, narrative cases of classroom mathematics lessons have been commonly used in practice-based professional development programs (Silver, 2009). In a book by Smith, Silver and Stein (2005) some narrative cases are presented to improve instruction in rational numbers and proportionality. Each case describes a middle school mathematics classroom in which a teacher and students engaging with a cognitively complex mathematics task. Smith et al. (2005) argues that by analyzing the narrative cases, “readers can wrestle with key issues of practice, such as what students appear to be learning and how the teaching supports or inhibits students’ learning opportunities.” (p. xii-xiii). With a similar purpose, one of the narrative cases (i.e., The Case of Marry Hanson) of the book by Smith et al. (2005) was used in the current study.

Ball and Cohen (1999) highlight three features of practice-based curriculum for professional education. The first one is that professional learning tasks such as real artifacts, records, moments and events permits a kind of study and analysis and using these tasks centers professional inquiry in practice. In a similar way, Silver (2009) highlights that using professional learning tasks in teacher education makes the work of teaching available for ongoing investigation and thoughtful inquiry. The second feature is that practice-based curriculum allows comparative perspectives on practice. Ball and Cohen (1999) claims, “In the traditionally individualistic structure of teaching, teachers rarely see teaching other than their own. Looking closely at student work produced in a different classroom offers teachers a chance to learn from others’ practice.” (p. 24). In a similar way, in the current study, short video records of pre-service teachers' students teaching were watched in order to give them opportunities to learn from others' and their own practice. And the last feature is that practice-based curriculum contributes to collective professional inquiry.

Smith (2009) refers to the materials taken from real mathematics classrooms as "samples of authentic practice". Moreover, she asserts that these materials provide opportunities for critique, inquiry, and investigation. Smith (2009) argues that a mathematical task along with carefully selected sorts of students' work could be an example of such material. In addition, she states,

> Teachers could be asked to complete the task, share various approaches that could be used to solve the task, and identify the mathematical ideas that are central to the task… Such a discussion is likely to enhance teachers' knowledge of mathematics content and of students as learners of mathematics (p.8).

The mathematical tasks provide pre-service teachers opportunities to construct or reconstruct their own knowledge of mathematics content. Similarly, in the current study, some mathematical tasks about proportional reasoning were used to enhance pre-service teachers' content knowledge. Besides, students' work, video clips of teaching and narrative cases were utilized as practice-based materials.

Studies revealed that using the practice-based materials in a teacher education program would help pre-service teachers to improve their content knowledge in mathematics concepts (Hillen, 2005; Silver, Clark, Ghousseini, Charalambous, & Sealy, 2007; Steele, 2006). To illustrate, a study with middle grade teachers was conducted by Silver et al. (2007) in the "Beyond Implementation: Focusing on Challenge and Learning (BIFOCAL)" project. The purpose of the research was investigating how the professional learning tasks used in a practice-based professional development program made opportunities for teachers to work on and learn about mathematical ideas. The findings of the study suggested that practice-based professional learning tasks provided many opportunities for teachers to learn mathematics such as building connections among related mathematical ideas and to rethink and reorganize the mathematics that they would encounter in their practice. Similarly, Steele (2006) investigated changes in pre-service and practicing teachers' knowledge needed for teaching geometry and measurement concepts through engagement in a practice-based professional development course. The findings of the study revealed that teachers improved their content knowledge and pedagogy through the practice-based teacher education. In particular, teachers' ability to use multiple solution methods, utilize multiple representations, produce mathematically sophisticated solutions, and identify key aspects of definitions enhanced. Correspondingly, another study was conducted by Hillen (2005) to investigate pre-service secondary mathematics teachers' understandings about proportional reasoning before and after completion of a practice-based methods course that focused on proportional reasoning and their ability to apply what was learned in a new setting. A total of 16 pre-service teachers participated in the researcher's quasi-experimental design. The findings of the study suggested that teachers learned some aspects of common and specialized content knowledge through practice-based methods course. Specifically, they constructed a deeper understanding of proportional and nonproportional situations and what it means for a relationship to be proportional. In addition, teachers learned a broader range of strategies, made sense of these strategies, and made connections among different representations of proportional situations during the practice-based teacher education. Moreover, teachers could draw upon their enhanced understandings of proportional relationships in the new setting which was a subsequent course on algebra.

CHAPTER III

METHODOLOGY

Within this chapter, first, the research design is presented. Then, data collection procedure, data collection tools and the practice-based instructional module focused on proportional reasoning embedded mathematics teaching method course are described. Finally, data analysis procedures, trustworthiness of the study and role of the researcher are stated.

3.1 The Research Design

The primary purpose of this study was to investigate pre-service middle school mathematics teachers' proportional reasoning before and after a practice-based instructional module based on proportional reasoning. In order to explore the nature of pre-service teachers' knowledge and understandings in proportionality concepts, the case study research methodology was used. Creswell (2007) defines the case study as "a qualitative approach in which the investigator explores a bounded system (a case) or multiple bounded systems (cases) over time, through detailed, in-depth data collection involving multiple sources of information" (p. 73). According to Yin (2009), the case study strategy consists "multiple sources of evidence, with data needing to converge in a triangulating fashion" (p. 18). Considering the statements, a case study design was employed for the study because the attempt in the current study was to gain in-depth understanding about pre-service teachers' proportional reasoning by using multiple sources of information. In addition, Sanders (1981) stated that "case studies help us to understand processes of events, projects, and programs and to discover context characteristics that will shed light on an issue or object" (p. 44). Similarly, Merriam (1998) argued that the interest in a case study design is “in process rather than outcomes, in context rather than a specific variable, in discovery rather than confirmation” (p. 19). Correspondingly, the main interest in this study was pre-service teachers’ processes in solving different problem types, distinguishing proportional from nonproportional situations and understanding mathematical relationships embedded in proportional situations. By this was, the researcher tried to shed light on pre-service teachers’ difficulties and misconceptions about proportionality concepts and differences in their conceptions between before and after the instructional module.

In order to gain a better understanding of the phenomenon, more than one case was included into the study. When more than one case is selected to illustrate the same issue or concern, researchers used the term multiple case study (Creswell, 2007; Yin, 2009). Thus, the current study utilized multiple case study design. The Elementary Mathematics Education program was chosen as the context of the study and the three pre-service mathematics teachers receiving the practice-based instructional module as a component of method course constituted the case of the study. Detailed information about the context and the participants of the study are presented in the following sections.

3.1.1 Context of the Study

This study was conducted in Elementary Mathematics Education program at a public university in the Western Black Sea Region, Turkey. The graduates of the teacher education program are qualified to work as middle school mathematics teachers at middle grades (Grades 5-8) in middle schools. This program is a four-year undergraduate degree program. Detailed information about all of the courses offered by the Elementary Mathematics Education program is given in the Appendix G. As a requirement of the program, in the first two years, pre-service teachers take mathematics content courses such as General Mathematics, Calculus, Discrete Mathematics, Geometry, Linear Algebra and take general educational science courses such as Introduction to Education, Educational Psychology, Instructional Principles and Methods. In addition, the program also includes the courses such as Turkish, Computer, Foreign Language, History and Physics. In the following two years, besides mathematics content and general educational science courses, the teacher education program offers method courses and practicum courses, in which pre-service teachers are engaged in mathematics teaching and learning process.

During the data collection process of the study, which covered the spring semester of 2012-2013 academic year, the participants of the study were in their third year of the program. They had taken Methods of Teaching Mathematics I course in the fall semester of the same academic year and they enrolled in Methods of Teaching Mathematics II course during the data collection process. In the Elementary Mathematics Education program, the methods course, which lasts 14 weeks (4 hours in each week) in each semester, is offered in the third year of the program. The purposes of the course are to provide pre-service teachers opportunities to develop a perspective for teaching mathematics with regard to the three components of mathematics education, which are mathematics, learning and teaching, and to develop their pedagogical knowledge and skills to teach middle school mathematics. The course content includes the basic concepts related to the field and the relationships of these concepts with teaching, and the purpose, principles and applications of Turkish middle school mathematics curriculum.

In the Elementary Mathematics Education Program, Community Service is one of the courses offered in the spring semester of the third year of the program. The aims of the course are understanding the importance of community service, identifying current problems of society, preparing a project in order to find solutions, participating scientific activities such as panel, conference, congress, and symposium as an audience, a viewer, and an organizer, getting involved as a volunteer in various projects within the framework of social responsibility, participating in community service activities for the implementation of the basic skills and knowledge in schools.

Pre-service teachers carried out different projects within the framework of social responsibility as a requirement of the course. The participants of the current study conducted a project which aimed to help poor students in a middle school in learning some mathematics topics for six-week period. One of the mathematics topics that the pre-service teachers taught students was the ratio and proportion. This was their first time in teaching mathematics in a real classroom environment because they did not yet take any practicum courses (i.e. School Experience and Teaching Practice courses), which are offered in the fourth year of the program.

3.1.2 Participants

Participants of the study were selected in two phases. In the first phase, convenience sampling was used. Forty junior pre-service middle school mathematics teachers, 28 women and 12 men, who enrolled in both Methods of Teaching Mathematics II and Community Service courses, at which the researcher was an instructor, were selected. The pre-service teachers studied at an elementary mathematics education program in the spring semester of 2012-2013 at a public university in Western Black Sea Region, Turkey. They had almost the same elementary mathematics major background as they had taken the same courses in the department. All of the pre-service teachers in the first phase took the Proportional Reasoning Test (PRET).

For the second phase of the sampling, maximum variation sampling, which is a type of purposeful sampling, was used as the method of choice since it aims at capturing and describing the central themes that cut across a great deal of participant variation (Patton, 2002). Furthermore, Patton (2002) stated that power of purposive sampling lies in selecting information-rich cases in order to get in depth information. Three pre-service teachers were selected with respect to their scores in the PRET, which aimed to examine pre-service teachers' proportional reasoning. Firstly, pre-service teachers were ordered according to their scores from high to low in the PRET. Secondly, they were divided into three groups in such a way that thirteen participants with highest scores were the first group, following fourteen participants were the second group, and the last thirteen participants were the third group. Finally, one of the participants in the first group (Ela), one of them in the second group (Mine), and one of them in the third group (Gaye) were invited to participate to the study. The researcher emphasized that participation in the study was voluntary and the findings were not used for grading purposes. Furthermore, the pre-service teachers were informed about the social responsibility project which aimed to help poor students in a middle school in learning some mathematics subjects and they were asked to whether they wanted to carry out the project as a requirement of Community Service Course. All of the participants agreed to conduct the project and participate in the study. In the study, the real names of the participants were not indicated; instead pseudonyms were utilized to ensure anonymity.

As a result, the subjects of the study were three pre-service middle school mathematics teachers who studied at an elementary mathematics education undergraduate program. The participants of the study were selected among junior pre-service teachers who enrolled in both Methods of Teaching Mathematics II and Community Service courses. The participants of the study had to enroll in Methods of Teaching Mathematics Course because the practice-based instructional module based on proportional reasoning was given to the junior pre-service teachers as a component of the course. Moreover, they had to take Community Service Course because their teachings of proportionality concepts to students in a middle school, as a part of social responsibility project in the course, were used to collect data.

Some information about the participants is given in the Table 3.1. All of the participants of the study were female pre-service teachers. While Ela and Gaye were 21 years old, Mine was 20 years old. Gaye and Mine graduated from Anatolian High School, and Ela graduated from high school. All of the participants had teaching experience as tutoring. However, they did not teach any subject in a real classroom environment. In fact, Gaye and Ela were tutoring only their cousins or acquaintances' children, whereas Mine had been tutoring middle school and high school students for two years by the time of the study.

Table 3.1 Information about the participants of the study

	Gaye	Mine	Ela
Age	21	20	21
Gender	Female	Female	Female
Graduated High School	Anatolian High School	Anatolian High School	High School
Grade of Methods of Teaching Mathematics I	3.00	4.00	3.50
CGPA	2.48	2.72	2.30
Teaching Experience	Tutoring	Tutoring	Tutoring

3.2 Data Collection Procedure

Data collection for the study was conducted from March 2013 to May 2013 (see Table 3.2). First of all, Proportional Reasoning Test (PRET) was administered to the all of the pre-service teachers enrolled in Methods of Teaching Mathematics II course. Pre-service teachers completed the test in about 60 minutes. As mentioned earlier, three pre-service teachers were selected with respect to their scores in the first administration of the PRET. Then, the participants started to carry out the social responsibility project, which aimed to help poor students in a middle school in learning some mathematics subjects, as a requirement of Community Service course. The middle school students were selected by their mathematics teachers according to their willingness and families' social statues. Actually, the students who did not have opportunity to go any private teaching institution or receive tutoring were selected to the project. The students and their parents were informed about the project and asked if they would like to participate in. The students were agreed to participate in the lessons and permissions were obtained from their parents. An informed consent form (see Appendix E) was presented to students to ask their parents to read and sign. The lessons lasted approximately 80 minutes (40+40) and were conducted once a week. The pre-service teachers taught mathematics contexts in an empty class of the school after the students' lessons ended through six-week period. Gaye and Ela conducted the lessons with four 8th grade students and Mine carried out the lessons with eight 7th grade students; totally 16 students participated in the lessons. One of the mathematics subjects that the pre-service teachers taught students was the ratio and proportion. Each pre-service teacher taught the same subject (i.e., ratio and proportion) to different student groups twice, before and after the proportional reasoning instructional module. Gaye and Ela conducted the second teachings of the ratio and proportion concepts with each other's student groups. However, Mine divided her students into two groups of four persons and she conducted the first teaching with one group and the second teaching with the other group of students. The participants were observed during they were teaching the ratio and proportion subjects for the first and second time. After the observations of student teachings, interviews were conducted with the participants.

Table 3.2 Timeline of data collection

Date	**Events**
12 March 2013	First administration of the PRET
28 March-03 April 2013	First student teachings
02-03 April 2013	Pre-interviews
04 April-09 May 2013	Practice-based instructional module focused on proportional reasoning
10 May 2013	Second administration of the PRET
14-16 May 2013	Second student teachings
16-17 May 2013	Post-interviews

As seen in the Table 3.2, practice-based instructional module focused on proportional reasoning started after data collected about participants' proportional reasoning prior to the instructional module. The instructional module lasted five weeks. Since the instructional module was carried out as a component of the method course, all of the pre-service teachers enrolled in the method course attended the module. It was explained in a detailed way in the following section. Immediately after the instructional module, the PRET was administered to the participants of the study, who were three pre-service teachers. Then, the second student teachings and post-interviews were conducted. Before each student teaching, the participants prepared lesson plans and before the second student teaching, the researcher wanted them to write a revision report of the second lesson plan if they made any revision.

3.3 Data Collection Tools

In order to get deep information from the pre-service teachers, different data collection procedures were used. Creswell (2007) referred this type of data collection as “multiple source of information”. The data sources of the study were the PRET, interviews and observations of student teachings. Additional information was gathered from the pre-service teachers’ lesson plans and revision reports of the second lesson plans in the student teachings. The following sections represent detailed information about the data sources of the study.

3.3.1 Proportional Reasoning Test (PRET)

The PRET (shown in Appendix A), which had 24 mathematical tasks selected and/or adapted from literature by Hillen (2005), was used in the study as pretest and posttest so as to examine pre-service teachers’ proportional reasoning. First of all, the original instrument was translated into Turkish by the researcher. Secondly, two mathematics education professors, who are fluent in English, asked to determine whether the Turkish version of the instrument was identical in meaning with the original version in English. Thirdly, a Turkish education professor checked and edited the Turkish version of the instrument to prevent grammatical errors. Then, the

PRET was administered to four pre-service teachers who were not involved in the study. Finally, necessary modifications for some items were carried out and the last version of the PRET was obtained.

The aim of the PRET was to examine pre-service teachers' approaches to different problem types (i.e., missing value, numerical comparison, and qualitative reasoning), distinguishing proportional from nonproportional situations, and understanding the mathematical relationships embedded in proportional situations. The questions in the PRET included different problem types, contexts, and numeric features in order to assess pre-service teachers' proportional reasoning. Furthermore, the questions prompted pre-service teachers to explain their thinking or justify rationales of their answers. Table 3.3 shows the match between the PRET questions and what the questions intended to measure.

Table 3.3 The match between the PRET questions and what intended to measure

	Approaches To Different Problem Types				
Question Number	Missing value	Numerical comparison	Qualitative Reasoning	**Distinguishing proportional from nonproportional situations**	**Understanding mathematical relationships embedded in proportional situations**
1-4	X				
5				X	
6		X			
7-8			X		
9				X	
10				X	
11-22				X	X
23	X				
24		X		X	

Each question in the PRET is worth 4 points. However, the question 6 is an exception because it is worth 8 points. It asks participants to solve a problem in more than one way and they received 4 points for each solution. Thus, possible scores on the PRET range from 0 to 100.

In the PRET, there were some questions that measured pre-service teachers' approaches to different problem types. PRET questions from 1 to 4 were missing value problems that did not have any context. Pre-service teachers were asked to solve these four problems in more than one way if they could. Moreover, it was expected from pre-service teachers to use efficient strategies, which involved the simplest numerical relations between quantities, to solve the problems. To illustrate, in question 1 (i.e., $4/20=x/35$), since there was an integer *within-ratio* (i.e., 4 x 5=20), a *within-ratio strategy* would facilitate the computations; whereas, in question 2 (i.e., $2/7=6/x$), since there was an integer *between-ratio* (i.e., 2 x 3=6), a *between-ratio strategy* would be easier to calculate. In addition, question 23 of PRET was also a missing value problem that had a context involving similar figures. PRET questions 6 and 24 were numerical comparison problems which asked to determine the mixture with the stronger orange taste and the most square rectangular cloth, respectively. Furthermore, PRET questions 7 and 8 were qualitative problems which did not contain numerical values but required qualitative comparisons. PRET questions from 11 to 22 presented pre-service teachers with 12 relationships (3 presented in written language, 3 presented in graphs, 3 presented in equations, and 3 presented in tables) and asked to identify whether the given situations were proportional or nonproportional and explain their rationales for each situation. The questions measured pre-service teachers' distinguishing proportional from nonproportional situations. Further, pre-service teachers' explanations in the questions also provided evidence about their understandings of the mathematical relationships embedded in proportional situations.

Moreover, question 5 asked pre-service teachers to provide a word problem which could be solved by the given equation. In order to create the problem, pre-service teachers needed to write a word problem in which the quantities were related multiplicatively; in other words, the quantities should have proportional relationships. Furthermore, since the answer of the problem was a decimal number ($x = 7.5$), pre-service teachers needed to make sure that a noninteger answer made sense in the context (e.g., an answer of 7.5 cm would make sense, but an answer of 7.5 cars would not).

In PRET questions 6, 9, 10 and 24, pre-service teachers required to realize that ratio was a proper measure for the given attributes. For example, question 6 asked to compare two mixtures of orange juice made from different amounts of orange juice concentrates and water, and decide which had a "stronger orange taste". Moreover, pre-service teachers were asked to solve the problem in more than one way if they could. Additionally, question 9 was, "Murat mixed blue and white paints until he had a shade of blue paint that he liked. He needed another quart of paint, so he wanted to increase the amount of paint without changing the color. In order to do that, he added one glass of blue and one glass of white paint (2 glasses = 1quart)." The pre-service teachers were wanted to comment on the effectiveness of Murat's strategy about "shade of paint". Furthermore, question 10 asked to determine "relative steepness of ski ramps" if the height, the length of the base, and the width of the base of the ramps were given and question 24 asked to determine the cloth that was "the most square" if three rectangular clothes with dimensions were presented.

3.3.2 Interviews

Interviews were another important data collection tool for the study because it enabled the researcher to investigate the pre-service mathematics teachers' proportional reasoning in a more detailed way. After each student teaching, semi-structured interviews were conducted with the participants of the study. The interview questions' main aim was to get information on pre-service teachers' approaches to different problem types, distinguishing proportional from nonproportional situations and understanding mathematical relationships embedded in proportional situations. Moreover, some questions in the interviews were designed to give participants an opportunity to rethink their answers in the PRET. These questions asked participants to explain their answers of some questions in the PRET in a more detailed way. Therefore, the pretest and posttest of the participants were provided to them during the pre-interview and post-interview, respectively.

More specifically, in the pre-interview, first of all, some demographic information such as age, type of graduated high school, grades of mathematics teaching method courses they had taken, cumulative grade point average (CGPA), and teaching experiences were asked to the participants to have a general view of the participants. Then, 14 open-ended questions about the participants' knowledge and understanding in proportionality concepts were asked. In addition to these questions, probes and follow-up questions were prepared to deepen the responses of the questions, to increase the richness of the data and to give clues to the participants about the level of the response desired. The post-interview questions were parallel to pre-interview questions. However, in the post-interview, while some questions were removed, some additional questions were added about the contributions of the instructional module. The examples of questions are presented in Table 3.4 and complete interview protocols for both interviews are given in Appendix B.

To ensure the validity of the interview questions, three professors in mathematics education were asked to judge whether they were matched with the research questions and aim of the study. Then, the questions were revised until there was an agreement. All the interviews were tape-recorded and transcribed verbatim. They lasted approximately 90 minutes and were conducted in the meeting room of the university.

Table 3.4 Examples of interview questions

Can you explain the meaning of ratio and proportion?
Are ratio and proportion the same concepts? Why? - What are the differences between them?
How do you know whether two quantities are proportional or not? - How did you decide whether the quantities in the PRET questions 11 to 22 are proportional? Can you explain each of them? Why do you think so?
Can you give examples of proportional and nonproportional situations in real life? Why are the quantities in the situations proportional or not?
While teacher Turgay was teaching his students distinguishing proportional from nonproportional situations, one of the students Atacan said “All linear relationships are proportional. In other words, if two variables have a linear relationship, they are also proportional.” Do you think Atacan is right? Why? Can you explain?
What is the constant of proportionality? Can you explain by giving an example? - In the PRET, was there any question required to be used the constant of proportionality? Why did you use it? Which questions were they? What is the constant of proportionality in the questions?
Which strategies do you prefer to solve missing value problems? - Which strategy will you teach to your students as the most efficient strategy? - What will you do if your students cannot understand this strategy?

3.3.3 Observations of Student Teachings

In qualitative inquiry, observation is a fundamental and highly important method to collect invaluable data (Marshall, & Rossman, 2006). According to Creswell (2002), observation is “gathering of first-hand information by observing people and places at a research site” (p. 199). In the current study, in addition to the data collected through PRET and interviews, the participants were observed while they were teaching ratio and proportion concepts in the student teachings. The main purpose of the observations was to enhance the findings of the study about pre-service teachers’ approaches to different problem types, distinguishing proportional from nonproportional situations and understanding mathematical relationships embedded in proportional situations. Another reason to observe participants’ student teachings was to check data collected from PRET and interviews about participants’ knowledge and understanding in proportionality concepts because as Fennema and Franke (1992) said, “One cannot teach what one does not know” (p. 147). That is to say, it was expected that observations of student teachings would give additional information whether participants had a deep understanding on ratio and proportion concepts or not.

A structured observation form was not utilized while observations were conducted. However, the student teachings were video-recorded for the whole period of each lesson. There were two cameras: one camera recorded the pre-service teacher and the other camera recorded the students in each class. Moreover, the researcher took field notes while she was observing the teachings of participants. The field notes contained nonjudgmental, concrete descriptions of what had been observed (Marshall & Rossman, 2006). During the observations, a researcher's role can vary from a complete observer to complete involvement (Gold, 1958, as cited in Marshall & Rossman, 2006). In this study, the researcher was a complete observer which means "the researcher does not participate in activities at the setting." (Marshall & Rossman, 2006, p. 81). Through data analysis process, the observations of student teachings shed light on the participants' underlying difficulties and misconceptions about proportional reasoning and clarified the conclusions drawn from other data collection instruments.

3.3.4 Lesson Plans and Revision Reports

The participants of the study prepared lesson plans for each of the student teachings. As mentioned earlier, while Gaye and Ela prepared lesson plans for teaching proportionality concepts to 8^{th} grade students, Mine prepared for teaching proportionality concepts to 7^{th} grade students in the student teachings. Before the first student teachings, the researcher came together with the pre-service teachers to decide on which objectives in the "Ministry of National Education Middle School Mathematics Curriculum" document for grades 5 to 8 were taught by each participant. The objectives in the mathematics curriculum that each participant taught in the student teachings are provided in the Appendix D. The participants were expected to prepare lesson plans in line with the new middle school mathematics curriculum and applied them in their student teachings.

The participants prepared two lesson plans, in which the same objectives of the mathematics curriculum were intended to teach. After the first lesson plan was applied in the first student teaching, participants were told that they could revise their lesson plans which would be applied in the second student teaching if they wanted. Moreover, it was expected from the pre-service teachers that they wrote a revision report in which they explained the reasons why they made revisions in the first lesson plan if they did.

3.4 Practice-based Instructional Module Focused on Proportional Reasoning

A practice-based instructional module based on proportional reasoning, which was a component of Methods of Teaching Mathematics II course, was given to the junior pre-service teachers during a five-week period in the spring semester of 2012-2013 academic year. The instructional module was taught by the researcher, who was also the instructor of the method course. All of the pre-service teachers enrolled in the method course participated in the instructional module. However, only three pre-service teachers participated in this study since the purpose of the study was to gain in-depth understanding about pre-service teachers' proportional reasoning. The researcher and the pre-service teachers met once per week for three or four sections during five weeks. In total, 16 sessions, each of which was 50 minutes, were conducted with the pre-service teachers. The instructional module consisted of a variety of activities such as solving mathematical tasks, examining student work, discussions on video clips from the participants' student teachings, and analyzing a narrative case of teaching in a middle school classroom. The activities were developed through a process of reviewing resources from literature focusing on teaching ratio and proportion concepts and improving proportional reasoning. The narrative case was drawn from the book by Smith et al. (2005) and was translated into Turkish by the researcher. The activities were conducted in each week are shown in Table 3.5. Sample activities in the practice-based instructional module are presented in Appendix H.

Table 3.5 Activities in each week of the practice-based instructional module

	Activities
Week I	Flower Problem and Basketball Problem (Getting valid answers by using both additive and multiplicative comparisons)
(50+50+50 minutes)	Let's discuss! What is the difference between additive and multiplicative comparisons?
	Copy Machine, Science Club and Field Problems (Multiplicative situations, identifying ratio as a measure of squareness)
	Jogging Truck Problem (Additive situation)
	Let's discuss! In which problems do you use multiplicative reasoning? In which problems do you use additive reasoning? Is there any problem that can be solved by utilizing both of the reasoning?
	Discussion on possible misconceptions of students in the Copy Machine, Science Club, Field and Jogging Truck Problems (Inaccurate additive or visual comparisons, and over generalize proportionality)
	Examine a student work in the Field Problem and discussion on the meanings of the quantities calculated
	What is proportional reasoning? Why is it important? What is ratio?
	Let's discuss! Are all ratios rational numbers? Are all rational numbers ratios? Are all ratios part-whole comparisons? Are all part-whole comparisons ratio?
Homework I	Snake Problem (Multiplicative comparisons)
Week II	Diet Problem (Additive and multiplicative comparisons)
	Examine three students' work in the Diet Problem and discussion on the meanings of the quantities calculated
(50+50+50 minutes)	Examine a student's work in the Photo Enlargement Problem and discussion on the student's misconception (Inaccurate additive strategy)
	Examine a student's work in Age Problem and discussion on the student's misconception (Over generalize proportionality)
	What is ratio and rate? What are the differences between ratio and rate?
	Provide examples of real life situations for ratio and rate
	Find ratio and rates in Ball Problem
	Let's discuss! In which contexts are ratios used? (Similarity, measurement, etc.) In which contexts are rates used? (Density, speed, etc.)
Homework II	Assigned reading: Doğan & Çetin (2009). "Seventh and ninth grade students' misconceptions about ratio and proportion."
Week III	What is invariance in proportional relationships? Provide examples of real life situations for invariance.
	Rectangle and Ball Problems (Solving by using building-up strategy)
	Examine a student's work in the Rectangle Problem and discussion on the student's misconception (Ignoring invariance and using additive reasoning)
	Labor Problem (using strategies highlighting multiplicative relationships)
	Examine a student's work in Labor Problem (Limitation of building-up strategy)
	Fraction and Ratio: Similarities and differences
	What is proportion? Direct and inverse proportion: Graphs and algebraic expressions Provide examples of real life situations for direct and inverse proportion
	A video clip from Gaye's student teaching: Discussion on her inaccurate definition of direct proportion
	Discussion on a teacher's strategy to solve the Tractor Problem (Memorized algorithm)

Table 3.5 (continued)

	Nonproportional Situations: Additive and Constant Relationships
	A video clip from Mine's student teaching: Discussion on a student's misconception (Over generalize proportionality)
	Determining proportional relationship between the quantities in eight different situations presented in language
	Tea, Jogging Track and Tree Planting Problems (Qualitative problems)
Homework III	Assigned reading: Duatepe, Akkuş-Çıkla and Kayhan (2005). "An investigation of students' solution strategies for different proportional reasoning items."
Week IV	Within and Between-Ratios
	Similar Rectangles Problem (Realizing within and between-ratios and the relationships between within-ratio, slope and the constant of proportionality)
(50+50+50 minutes)	A video clip from Ela's student teaching: Discussion on her misconception that the multiplication of slope and constant of proportionality is equal to one.
	Tree Problem (Solving by building a ratio table and determining the rule for relating the number pairs in the table)
	Let's discuss! Are all linear relationships proportional?
	A video clip from Gaye's student teaching: Discussion on her linearity misconception
	A video clip from Gaye's student teaching: Discussion on a student's linearity misconception
	Determining proportional relationship between the quantities in four different situations presented in ratio tables, driving the graphs and writing the algebraic expressions
	A video clip from Mine's student teaching: Discussion on a student's misconception about determining proportionality in a ratio table
	Computer, Cake, and Apartment Problems (Solving by building ratio tables)
	Different Solution Strategies (Explanations and examples)
Homework IV	Running, Cheese and Birthday Problems (Solving by building ratio tables and determining the rules for relating the number pairs in the tables, and drawing the graphs of the relationships by using a computer applet)
	Candy Problem (Solving by using six different strategies)
Week V	Pizza Problem (Finding single and composite units)
	What is unit? Activities related unitizing and reunitizing
	Cornflakes Problem (Using a composite unit)
(50+50+50 minutes)	Drink and Bird Problems (Solving numerical comparison problems by using different strategies)
	Inaccurate Strategies
	Examine a student's work in Ship Problem (overreliance on cross-multiplication)
	Examine a student's work in Paperclip Problem (Inaccurate additive strategy)
	Examine students' work in Apple Juice, Drink Problems (Inaccurate strategies)
	Let's discuss! What is the difference between numerical comparison and missing value problems? Find the numerical comparison problems from among problems solved today
	The Case of Meral Teacher Solve the Candy Jar Problem in the case Discussion on the students' solution strategies, and the teacher's instructional strategies
Homework V	Write a reflection paper about what Meral Teacher do to improve students' proportional reasoning.

The main purpose of the instructional module was to deepen and sharpen teachers' knowledge and understanding in proportionality concepts and proportional relationships through practice-based professional learning tasks. In particular, the instructional module intended to improve pre-service teachers' flexibility in solving different problem types, missing value, numerical comparison and qualitative reasoning problems, by using a broad range of strategies. Additionally, the instructional module aimed to increase pre-service teachers' capability to distinguish proportional from nonproportional situations. Further, the module intended to enable pre-service teachers to better understand mathematical relationships in proportional relationships.

In line with these purposes, in the practice-based instructional module, pre-service teachers solved problems that contained both proportional and nonproportional situations. Moreover, they were encouraged to use a broad range of strategies to solve different problem types. Particularly, some mathematical tasks wanted pre-service teachers to use strategies highlighting multiplicative relationships. Additionally, the discussions were conducted about alternative solution strategies, the meanings of quantities calculated and possible misconceptions that students might have in solving the problems.

In addition to the problems, pre-service teachers examined the cases which contained students' work. In some cases, students solved proportional problems by using inaccurate strategies such as additive strategy. Moreover, students' misconceptions and difficulties were revealed in their solutions or speech. There were also some other cases that included accurate solutions of students. In these activities, the researcher wanted pre-service teachers to express the rationale of the solutions and the meaning of quantities students calculated.

Furthermore, video clips from the participants' student teachings were watched and analyzed in the instructional module. While some video clips revealed the participants' misconceptions and difficulties in proportionality concepts such as linearity misconception and inaccurate definitions, some other video clips indicated the students' misconceptions and difficulties in the student teachings.

In the practice-based instructional module, there were also discussions on some statements about proportional relationships that might lead confusion. Some example statements were, "All ratios are rational numbers", "All ratios are part-whole comparisons" and "All linear relationships are proportional". After the discussions, the researcher specified accurate and inaccurate statements and explained why. Moreover, in the instructional module, the researcher highlighted some important concepts and definitions about proportional reasoning such ratio, rate, proportion, *within* and *between-ratios*, invariance, and unit.

In the last week, a written narrative case, in which a teacher taught ratio and proportion concepts in a middle school classroom, was analyzed. The narrative case was drawn from the book by Smith et al. (2005) and was translated into Turkish by the researcher. The case described a middle school mathematics classroom in which a teacher, Meral Teacher, and her students engaging with a cognitively complex mathematics task of proportional reasoning. In the instructional module, first of all, pre-service teachers tried to solve the task, and then, they discussed and analyzed the students' solution strategies, and the teacher's instructional strategies in the case. Further, as homework, they wrote a reflection paper about what Meral Teacher do to improve students' proportional reasoning. Similarly, in each week, there was some homework for the pre-service teachers such as reading assigned articles and solving mathematical tasks.

3.5 Data Analysis Procedures

Data analysis is described by Bogdan and Biklen (1998) as "the process of systematically searching and arranging the interview transcripts, filed notes and other materials that you accumulate to increase your own understanding of them and to enable you to present what you have discovered to others." (p. 157). Further, the researchers emphasized that the data analysis includes organizing data, breaking them into manageable units, synthesizing them, searching for patterns, discovering important concepts to answers research questions and deciding what to tell others.

In this study, data analysis was performed in order to investigate pre-service mathematics teachers' proportional reasoning before and after a practice-based instructional module based on proportional reasoning. In order to answer the research questions, data gathered from the first and second administrations of the PRET, pre and post-interviews, observations of student teachings, lesson plans and revision reports of the second lesson plan were analyzed. In the current study, content analysis was used to break the data into manageable units on the basis of the codes created (Patton, 2002; Yıldırım & Şimşek, 2013). According to Patton (2002), the main aim of the content analysis is to organize and simplify the complexity of data into meaningful and manageable themes or categories.

As an initial attempt, the literature was reviewed in order to determine the understanding and knowledge that is required to be a proportional reasoner. Then, the themes were derived from the related literature as appearing in the research questions. These themes were "approaches to different problem types", "distinguishing proportional from nonproportional situations" and "understanding mathematical relationships embedded in proportional situations". Afterwards, categories and subcategories for each theme were formed. Sometimes, the categories and subcategories were constituted by using the recurring patterns in the data set, but in other occasions they were drawn from the literature based on proportional reasoning. In other words, before the data analysis, a tentative list of codes (e.g., utilizing a broad range of strategies, flexibility in unitizing and reunitizing quantities, over generalize proportionality, utilizing key understandings) based on proportional reasoning literature were developed. During the data analysis process, if a concept appeared in the data did not quite fit the predefined codes, new codes (e.g., making qualitative comparisons not depending on numerical values, recognizing within and between-ratios, making connections among different representations) were added. Additionally, possible relationships among the subcategories were detected and similar subcategories were grouped into same categories. Furthermore, irrelevant categories derived from the literature were eliminated. After all of the modifications of the coding categories finished, the researcher went back the data set and recoded the materials. The ultimate list of the coding categories is presented in the Appendix C.

In order to manage the data obtained from the first and second administrations of the PRET, the grouping of the PRET questions under the three themes were utilized (see Table 3.4). For example, the participants' responses to the questions under the first theme, which was "approaches to different problem types", were coded based on their solution strategies and processes in solving the different problem types. Moreover, the responses of the questions under the second theme, which was "distinguishing proportional from nonproportional situations", were coded so as to demonstrate whether the participants classified relationships as proportional or nonproportional, identified ratio as measure and provided an example and nonexample of proportional relationships. Furthermore, the participants' responses to the questions under the third theme, which was "understanding the mathematical relationships embedded in proportional situations", were coded to indicate the four *key understandings,* which were identified in the literature chapter, if any, the participants used to justify rationales of their classifications about proportional relationships. Afterwards, the codes were placed under the related categories and subcategories derived from the literature and the data set.

For the analysis of data obtained from the interviews and observations of student teachings, all audio recordings of interviews and video recordings of student teachings were transcribed verbatim by the researcher. Besides the transcribed data of observations, the researcher's field notes, the participants' lesson plans and revision reports of the second lesson plan were also inserted into the transcripts of the video recordings to get a detailed and precise description of the participants' proportional reasoning in the student teachings.

First of all, the researcher read the transcribed data several times to gain general sense. Then, from each transcript, significant phrases or expressions directly related to the themes were identified. For instance, if the pre-service teachers were solving a problem in the student teaching or interview, their solution strategies and explanations were coded under the first theme, which was "approaches to different problem types". Additionally, their expressions about classifying relationships as proportional or nonproportional, identifying ratio as measure and providing examples of proportional and nonproportional situations were coded under the second theme, which was "distinguishing proportional from nonproportional situations". Further, when the participants defined proportional concepts and determine proportional relationships, their explanations and justifications were coded under the third theme, which was "understanding the mathematical relationships embedded in proportional situations". Then, these expressions were placed under the categories and subcategories derived from the literature and the data set. This was an interactive process between the data and the codes because the researcher went back and forth among the data and the codes so as to locate the codes under the related categories and subcategories.

3.6 Trustworthiness

Patton (2002) stated that validity and reliability are two important concepts that any researcher should consider while designing a study, analyzing results, and judging the quality of the study. However, reliability and validity issues are perceived differently by some qualitative researchers who think that the qualitative investigation fails to adhere to canons of reliability and validity (LeCompte, & Goetz, 1982). Thus, many qualitative researchers prefer to use different terminology instead of using the terms validity and reliability (Erlandson, Harris, Skipper, & Allen, 1993). Establishing trustworthiness in qualitative research is important in judging the quality of the study. Erlandson et al. (1993) described credibility, transferability, dependability, and confirmability as indicators of trustworthiness in qualitative studies. For this reason, instead of using the term validity and reliability, in this study, the term trustworthiness is used. Certain criteria for judging the quality of the research study is described below.

In this study, person and data triangulation was used for increasing the credibility of the study. The participants were three junior pre-service middle school mathematics teachers; that is to say, more than one individual was used as a source of data. In addition, different data collection tools such as pre/posttest, pre/post-interview, videos, and written artifacts were collected to give answers for the research questions. Moreover, for increasing the credibility, member checking that refers to taking the narrative report back to the participants for checking the accuracy of the findings and interpretations (Creswell, 2007) was also used. Furthermore, peer debriefing that refers to external check of the research process by peers and discussing its accuracy with them (Creswell, 2007) was used. In this sense, at every stage the researcher asked her colleagues who were qualified in qualitative research to comment on the research process. Their interpretations made the researcher revise her methods, and develop greater explanation of the research design and strength her arguments in light of comments made.

Thick descriptions that allow the researcher to elaborate on the research setting in detail for establishing transferability were used (Erlandson et al., 1993). In this sense, enough description and details about the content, data collection procedures, physical setting and participants were provided. Moreover, the raw data was transferred to the readers in a reorganized way by considering concepts and themes appeared without any interpretation, this was utilized by giving quotations and paraphrasing. In addition to thick description, purposive sampling was used to maximize the range of specific information that can be obtained from data.

In order to establish dependability and reduce bias, coding of the data was independently conducted by the researcher and a second coder who was informed about the proportional reasoning and data analysis framework of the study. Both coders analyzed the ten percent of the data set with pseudonym names for the participants. Once coding was completed individually, there were eighty-seven percent agreements. Then, the researcher and the second coder discussed the discrepancies until a consensus was reached. Additionally, dependability audit that is one of the strategies that establish dependability was also used in the study (Erlandson et al., 1993). One mathematics professors in an educational faculty reviewed and checked the activities and procedures during the research process.

3.7 Role of the Researcher

In a qualitative study, the researchers play an important role on collecting and analyzing data (Merriam, 1998). The researchers should give information about their role in the study in order to avoid discrediting the study (Patton, 2002). Thus, the researcher tries to explain her role in this study. The researcher of the present study was also the instructor of the "Methods of Mathematics Teaching II" and "Community Service" courses. Since the researcher was an instructor in the same department, she had a strong relationship with the pre-service teachers. Moreover, the researcher was the instructor of the pre-service teachers' another course during third year in the education program. Being the instructor of the course and the researcher at once provided some advantages and disadvantages. For instance, the researcher became aware of the participants' knowledge and understanding in proportionality concepts and recognized them. As an advantage, this enabled the researcher to understand the participants' ideas better and to make more sensitive analysis of the data. Moreover, since the participants had known the researcher earlier; they might be willing to participate in the interview and explain their views without hesitation and thus, the interviews were conducted in a sincere atmosphere. Yet, the responses of the interviewees might not always reflect the reality because of the sincere relationship with the researcher. It is also worth noting that as the researcher knew the participants before the interviews, she might tend to be subjective while asking the interview questions (Coghlan & Brannick, 2001). To prevent bias in interview questions, interview schedules were prepared for both pre and post-interviews. Furthermore, in order to reduce the effect of being the instructor of the course, during whole data collection process, the researcher tried to make them feel comfortable by stressing that the aim of the study was to understand their conceptions, rather than judging or grading them.

CHAPTER IV

RESULTS

Within this chapter, the findings of the research study are presented. This study was designed to investigate pre-service middle school mathematics teachers' proportional reasoning before and after receiving a practice-based instructional module based on proportional reasoning. Pre-service teachers' proportional reasoning was classified under three major categories emerging from the literature. The major categories were approaches to different problem types, distinguishing proportional from nonproportional situations and understanding mathematical relationships embedded in proportional situations. Then, subcategories for each theme were formed by analyzing the research data. The chapter includes four main sections. The first section presents the results about pre-service teachers' proportional reasoning before receiving a practice-based instructional module based on proportional reasoning. The second presents the results about pre-service teachers' proportional reasoning after receiving the instructional module. The third deals with the results about the differences between pre-service mathematics teachers' proportional reasoning before and after the instructional module based on proportional reasoning. The fourth section presents the summary of the research findings.

4.1 Pre-service Teachers' Proportional Reasoning before the Proportional Reasoning Instructional Module

In this section, the findings about pre-service teachers' approaches to different problem types, distinguishing proportional from nonproportional situations, and understanding of mathematical relationships embedded in proportional situations before receiving a practice-based instructional module based on proportional reasoning are presented for each pre-service teacher.

4.1.1 Approaches to Different Problem Types

In order to investigate pre-service teachers' approaches to different problem types, which were missing value, numerical comparison and qualitative reasoning problems, before participation in a practice-based instructional module, the solution strategies and processes of pre-service teachers in solving the different problem types were analyzed. Since solution strategies and processes might change with respect to problem types, results of different problem types are presented under different subcategories.

4.1.1.1 Gaye's Approaches to Different Problem Types

Missing Value Problems: Solution Strategies and Processes

Gaye could find correct answers of missing value problems in all relevant instruments. However, her solution strategies were limited. Indeed, she used only four different strategies (*cross-multiplication, isolating the unknown, factor of change, unit rate*). Moreover, she mostly used formal strategies, in which rules and properties of algebra were used, to solve missing value problems. To illustrate, in the PRET, there were some missing value problems (questions 1-4) which did not have any context. In the pretest, Gaye used *cross-multiplication* as a leading strategy to solve the problems. As a second strategy, she used another formal strategy, *isolating the unknown,* which meant multiplying or dividing both sides of the equation by a value to leave alone the missing value, and no other strategy was used by her to solve these problems. An example of the *isolating the unknown* strategy was her second solution for the first question. In her solution, she wrote the ratio, 4/20, in its simplest form, 1/5, and then, she multiplied both sides of the equation ($1/5=x/35$) by 35 to leave alone the missing value. Similarly, in the pre-interview, Gaye mostly used *cross-multiplication strategy* in solving missing value problems. For instance, when the researcher asked her to find the density of a mixture, she set up a proportion (3/5=x/100), cross multiplied (100x3=5x(x)) and solved it for x. Correspondingly, for another missing value problem in the interview, which was "If 8 balloons are 12 TL, how much 6 balloons will cost?" she set up the algebraic expression as 8/12=6/x and found x=9 by *cross-multiplication*. In the same way, she solved a proportion problem, which was exemplified by her, by using *cross-multiplication* in the pre-interview as follows:

> Gaye: For example, in an exam, if you get 40 points for 20 true questions, you should get more points for 40 true questions. This is a proportional situation.
> Researcher: How many points will you get?
> Gaye: When I cross multiply hmm… Multiply 40 by 40, divide the result by 20, and the answer is 80.
> Researcher: How?
> Gaye: It is proportional to the twice of points. If we get 40 points for 20 questions, we should get 80 points for 40 questions because 40 questions are equal to 2 times 20 questions and so we should find 2 times 40 points.

Although identifying the scale factor, 2, and multiplying the number of true questions by it was very easy for the above problem, Gaye firstly, solved it by using *cross-multiplication*. Afterwards, she explained it by utilizing the scale factors. It can be concluded that Gaye had difficulty in recognizing efficient strategies, which facilitated the computations required to find missing values and used formal strategies even though there was an integer *between* or *within-ratio*. Another evidence for the claim is that in some PRET questions, though there were integer *within* and *between-ratios* between quantities, she did not use them and solved the problems by using formal strategies.

Data indicated that Gaye saw *cross-multiplication* as an indispensable and the most important strategy. To illustrate, in the first student teaching, she noted that "In a direct proportion, we have to use cross-multiplication, don't forget this!" after she solved a problem by using *cross-multiplication*. Similarly, in the pre-interview, when the researcher asked her which strategies she used to find a missing value in a proportion, she said *cross-multiplication* as a leading strategy as shown in the following quotation:

> I generally use cross-multiplication. Another strategy could be multiplying or dividing both side of the equation by a value to leave alone the "x". But it takes too much time, so I mostly use cross-multiplication. Hmm… I did not use any other strategy.

As seen in the above quote, Gaye saw *cross-multiplication* as an indispensable strategy. In addition, she assumed a formal strategy, *isolating the unknown* (i.e., multiplying or dividing both sides by a value to leave alone the missing value), as an alternative strategy. However, although *cross-multiplication* was indispensable for Gaye in her solutions, she was aware of its difficulty of understanding by students. Accordingly, when the researcher asked her which strategy she would teach to her students, she told a formal strategy, *isolating the unknown*. In addition, unless her students understood the formal strategies (i.e., *cross-multiplication* or *isolating the unknown*), she preferred to find a unit rate (i.e., 3 coins for 1 ball) with the help of drawing:

> If they did not understand, I would teach by drawing. For example, for this question (showing PRET question 2), I will draw 2 balls and 6 coins. Then, I will ask, "If I pay 6 coins for 2 balls, find the number of coins that I will pay for 7 balls." In the similar way, I will also draw 7 balls and I will say if I pay 6 coins for 2 balls, I should divide by two both of them to find the number of coins for one ball (drawing on a sheet). By this way, I can find that I should pay 3 coins for 1 ball. Finally, I can find the number of coins for 7 balls by drawing 3 coins for each ball.

She preferred to use an informal strategy, *unit rate strategy,* if her students did not understand formal strategies. In a similar manner, Gaye sometimes used informal strategies that highlighted multiplicative relationships, instead of formal strategies (e.g., *cross-multiplication*) to solve missing value problems. For example, she solved a missing value problem in the PRET (question 23) by utilizing *factor of change strategy* although neither the *within* nor the *between-ratios* were integer in the problem. In her solution, she could accurately identify 7/2, *between-ratio,* which was also the scale factor, and multiplied 7/2 with 3 and found 21/2. Similarly, in her first student teaching, Gaye solved some problems by using *factor of change strategy* in such a way that she simplified one of the ratios in order to find an integer scale factor (i.e., *between-ratio*) and then found the missing value by multiplying the scale factor with the corresponding quantity. For instance, one of the problems was "If an F-16 plane can travel 4800 meters in 12 seconds, how many meters does the plane travel in 2 seconds?" In order to solve the problem, first of all, she wrote the proportion (4800m/12s=?/2s), and then, simplified the first ratio (400m/1s) to express it in the simplest form. Later, she found the *between-ratio* as 2 and multiplied 400 by 2 to find out the meters traveled in two seconds. Although she highlighted multiplicative relationships, she tended to use the *between-ratio* and not mentioned the *within-ratio*. For example, in the above problem, there were both integer *between* and integer *within-ratio.* However, she did not mention the *within-ratio,* which was 400. It can be concluded that while she recognized *between-ratio* in a proportion, she had difficulty in recognizing *within-ratio*. Another evidence for the claim is that she did not use *within-ratio* in the first question of PRET though there was an integer *within-ratio* between quantities.

Gaye had difficulty in providing meaningful explanations for her solutions that highlighted proportional relationships between variables and using proportional reasoning language in her explanations. For example, a problem in the student teaching was "In a shiny day, a boy's father's height is 180 cm and his shadow's height is 240 cm. What is the boy's shadow's height if his height is 150 cm?" In order to solve the problem, Gaye set the algebraic expression (180x?=150x240) by using *cross-multiplication* without giving any meaningful explanation and found the missing value. In other words, she did not provide an explanation for her solution that went beyond a description of the steps she had taken to determine the solution. Similarly, another problem in the student teaching, which was plane problem presented above, she could not explain the meanings of the quantities in her solution. To illustrate, for the problem, she could make an explanation that the meters the plane travel were 400 times the seconds it took, but she did not provide such an explanation. Likewise, in PRET question 23, she could not explain her solution.

Numerical Comparison Problems: Solution Strategies and Processes

Gaye used two accurate strategies, which were *converting decimal expressions* and *building-up*, to solve numerical comparison problems. To illustrate, PRET had a numerical comparison problem (question 6) that asked to compare two mixtures of orange juice concentrates and water. In her first solution, Gaye wrote ratios of orange juice concentrates to water in fractional form, converted them into decimal expressions and compared them. She accurately concluded that mixture B had stronger orange taste than mixture A. But she did not make any other explanation about why she chose the larger number. In her second solution, she found the amount of water in each mixture for the same amount of orange juice concentrates. To do this, she used *building-up strategy*. Firstly, for mixture A, she increased the amount of orange juice concentrate by twos and water by threes until 6 cups of orange juice concentrate and 9 cups of water were reached. Secondly, for mixture B, she increased the amount of orange juice concentrate by threes and water by fours until 6 cups of orange juice concentrate and 8 cups of water were reached. Finally, she compared the amount of water and concluded that since mixture B had less water, it had stronger orange taste than mixture A. She could accurately determine the mixture with a stronger orange taste and tried to explain her answer. Yet, she could not explain the quantities that were calculated clearly.

Gaye used an inaccurate additive strategy to solve a numerical comparison problem. The problem (PRET question 24) presented three rectangular clothes with dimensions and asked pre-service teachers to decide the cloth that was "the most square". In her solution, firstly, Gaye calculated the differences between dimensions of each rectangle as 35-23=12, 155-139=16, and 75-56=19. Then, she concluded that since twelve was the smallest difference, the first cloth was "the most square". Her explanation for the answer was, "For being a square, all dimensions of a rectangle should be the same length, so the cloth which has less difference among its dimensions is the most square." She used an inaccurate additive strategy; thus, she could not find the correct answer of the problem.

Gaye had difficulties in making sense of quantities in a numerical comparison problem. For example, an item in the pre-interview detected Gaye's some difficulties on making sense of quantities in a proportional situation. The problem was "Selin mixed 3 ounces of chocolate syrup with 5 ounces of milk to make chocolate milk. Emre mixed 5 ounces of the same chocolate syrup with 8 ounces of milk. Who did prepare the drink with a stronger chocolate flavor? Explain how you know." Pre-service teachers were asked to evaluate five students' solutions to the problem. Gaye explained clearly the quantities which were used to make comparisons in the first three solutions of students. However, she was unsure about the fourth student's solution strategy and struggled to make sense of the numbers 8 and 13 (the total amount of each mixture) in the solution. She said:

> The student compared the ratio of chocolate to chocolate and the ratio of milk to milk, separately. But, where did the 13 come from … I did not exactly understand what he compared here. I think it is not a valid solution.

Gaye was unable to make sense of the quantities in the fourth student's solution in which the student utilized *part-to-whole ratio* by finding the ratio of chocolate syrup to total chocolate milk. It can be concluded that Gaye had difficulty in realizing *part-*

to-whole ratios. The solutions of Gaye to question 6 in PRET (orange juice problem) would be additional evidences for the same claim because, as mentioned earlier; she solved the problem in two different ways, both of which were based on *part-to-part ratios* (i.e., the ratios of orange juice concentrates to water). In addition, for the fifth student's solution, Gaye was initially unable to make sense of the quantities in the solution. In particular, she had difficulty in determining where the 24 and 25 came from. However, after a while, she accurately explained the solution.

In the student teaching, Gaye did not ask any numerical comparison problem to her students. Moreover, in the pre-interview, she did not mention any situation that could be a numerical comparison problem while she was giving examples of proportional situations in real life.

Qualitative Reasoning Problems: Solution Strategies and Processes

Gaye used an inaccurate additive strategy to solve a qualitative reasoning problem. To illustrate, in a qualitative reasoning problem of PRET (question 8), although she accurately found the answer of the question, she made an additive comparison for explaining her answer. Her explanation was as follows:

> Not enough information to tell. Because if the amount of apple and orange used for the fruit juice decreased in the same amount, the fruit juice would taste the same. But it is said only less, so we cannot say anything accurately.

As seen in the quote above, she did not mention the ratios of the apples and oranges in the fruit juice. In addition, she made an additive comparison by saying, "decreased in the same amount".

Moreover, she had difficulty in making qualitative comparisons that did not depend on numerical values. For instance, she solved a qualitative reasoning problem of PRET (question 7) by giving numerical examples. The problem was, "Esra ran more laps than Gonca. Esra ran for less time than Gonca. Who was the faster runner?". She explained her answer as follows: "Let's assume Gonca ran 2 laps and Esra ran 3 laps since Esra ran in a shorter amount of time she was the faster runner." Although she made a multiplicative comparison to solve the qualitative problem, she could not make qualitative comparisons that did not depend on numerical values because she quantitatively compared the number of laps.

In the student teaching, she did not ask any qualitative reasoning problem to her students. Moreover, in the pre-interview, she did not mention any situation that could be a qualitative reasoning problem while she was giving examples of proportional situations in real life.

4.1.1.2 Mine's Approaches to Different Problem Types

Missing Value Problems: Solution Strategies and Processes

Mine could find correct answers of missing value problems in all relevant instruments. However, Mine's solution strategies were very limited. In fact, she used only three different strategies (*cross-multiplication, isolating the unknown, factor of change*). Moreover, she mostly used formal strategies, in which rules and properties of algebra were used, instead of informal strategies highlighting multiplicative relationships. For instance, in the PRET, there were some missing value problems (questions 1-4) which did not have any context. In the pretest, Mine used *cross-multiplication* as a leading strategy to solve these problems. As a second strategy, she used another formal strategy, *isolating the unknown.* She used the strategy in the third question by equating the denominators and in the fourth question by equating the numerators to leave alone the missing value. No other strategy was used by her to solve these problems. In a similar way, she solved another missing value problem of PRET (question 23) by using *cross-multiplication*. Likewise, she used *cross-multiplication* to solve some problems in the pre-interview. For example, one of the problems was, “If 8 balloons are 12 TL, how much 6 balloons will cost?”. In order to solve the problem, she set up the algebraic expression (8x(x)=6x12) by using *cross-multiplication* and found the answer as 9. Similarly, she solved another missing value problem, which asked to find the density of a mixture by utilizing *cross-multiplication*. First of all, she set up a proportion (3/5=x/100); secondly, she cross multiplied (100x3=5x(x)) and then, she solved it for x. In addition to the evidence, in the pre-interview, she stated that she preferred to use formal strategies (e.g., *cross-multiplication, isolating the unknown*) to solve missing value problems as follows:

> Researcher: Which strategies do you use to find a missing value in a proportion?
> Mine: First of all, I use cross multiplication. Then, maybe I equate denominators or numerators. That is to say, I use the two strategies.
> Researcher: Which strategy will you teach your students as the most effective strategy?
> Mine: The most effective …er… Firstly, I will tell my students to find the simplest form of the fractions in order to ease the operations. Then, I will say that the most effective strategy is cross-multiplication. I also use it generally.
> Researcher: What will you do if your students cannot understand this strategy?
> Mine: If they do not understand cross-multiplication, I will equate the denominators. Can I give an example?
> Researcher: Okay.
> Mine: For example, let’s assume the question is 2/7=6/x, I will multiply the first ratio by x and the second one by 7 (writing on the sheet: 2x/7x=42/7x). “7x” goes from each side and 2x(x)=42 remains. Similarly, in the cross-multiplication, I multiply x by 2 and 6 by 7 (writing on the sheet: 2x(x)=6x7). That is, I get the same thing. In other words, these are not so different from each other. I will teach like that.
> Researcher: If they are the same, why do you prefer this strategy?
> Mine: I prefer the strategy because I think that they will easily understand why we cross multiply with the help of this strategy.

Mine saw *cross-multiplication* as an indispensable and the most important strategy. She stated that her foremost strategy was *cross-multiplication* to solve missing value problems. Her second strategy was *isolating the unknown* (i.e., equating denominators or numerators to leave alone missing value). Furthermore, when the researcher asked her which strategy she would teach her students if they did not understand *cross-multiplication*, she told the same formal strategy, *isolating the unknown*. Moreover, she stated that she would teach the strategy to ease operations in *cross-multiplication*. Correspondingly, in the student teaching, she taught *cross-multiplication* as a rule that should be memorized to solve direct proportion problems. To illustrate, she dictated that "… In a direct proportion problem when we cross multiplied, the products should be equal" after she solved a problem by using *cross-multiplication*.

Correspondingly, in the student teaching, she solved two out of four missing value problems by utilizing *cross-multiplication.* For instance, one of the problems was "If an F-16 plane can travel 4800 meters in 12 seconds, how many meters does the plane travel in 4 seconds?" In order to solve the problem, she set the algebraic expression (12x?=4x4800) by using *cross-multiplication* and found the missing value. However, she provided an explanation for her solution by using *between-ratios*. For instance, she stated:

> In a direct proportion, when I multiplied the quantities (showing 4 seconds and 4800 meters) and the quantities (showing 12 seconds and "?"), the answers should be equal (writing on the board: 12x?=4x4800, ?=1600). Because I decrease 12 seconds by 1/3; in other words, I divide 12 by 3 and get 4. At the same time, I should divide 4800 by 3 in order to find x because they are directly proportional... Don't forget that in a direct proportion problem when we cross multiplied, they should be equal.

Although Mine preferred to solve the problem above by utilizing *cross-multiplication*, she explained how she found the missing value by using the multiplicative relationships between quantities. It can be concluded that she could provide a meaningful explanation for her solution by using multiplicative relationships and with a proportional reasoning language. Another evidence for the claim is her explanation for one of the PRET questions (question 23) which had a context involving similar figures. She stated, “To make sure that they do not distort any of the images, they should increase the length and width of the image by the same ratio, so I used direct proportion and cross multiplied.” She pointed out that the length and width of the image should be increased in the same ratio though she solved it by using *cross-multiplication.*

Mine sometimes used an informal strategy, *factor of change,* which highlighted multiplicative relationships to solve missing value problems. To illustrate, in the first student teaching, she solved two out of four problems by using *factor of change strategy*. In both problems, she highlighted multiplicative relationships. One of the problems was “If a student solved 60 questions in 40 minutes, how many questions would he solve in 10 minutes?” When she asked the problem, students tried to solve it by using *cross-multiplication*. However, she encouraged them to use multiplicative relationships and she said:

> I am solving the problem differently from you; namely, I do not use direct proportion. He solved 60 questions in 40 minutes; if I divided the number of minutes by 2, I should divide the number of questions by 2, too (writing on the board: 60 questions in 40 minutes; 30 questions in 20 minutes). The problem requires finding the number of questions in 10 minutes. I can get it by dividing 20 by 2 so I should also divide 30 by 2 and the answer is 15 questions. In that case, which quantities are proportional? Time and solved questions are proportional because as I decreased time by 1/4, the questions decreased by 1/4, too.

At the beginning of the above quote, she stated that she did not use “direct proportion”. In fact, she wanted to say “cross-multiplication*”* instead of “direct proportion” because she did not use *cross-multiplication*. However, she surely utilized *direct proportion* in order to solve the problem. In brief, although she did not use proportional reasoning language in her explanation; she realized the multiplicative relationship between two quantities in the problem. On the other hand, in both problems, since she used *factor of change strategy,* she utilized only *between-*

ratio and did not mention *within-ratio*. For example, another problem asked the ticket price for 4 persons if one ticket was 6 TL, there were both integer *between* and integer *within-ratio.* However, she did not mention the *within-ratio*, which was 6. It can be concluded that while she recognized *between-ratio* in a proportion, she had difficulty in recognizing *within-ratio*. Another evidence for the claim is that she did not use *within-ratio* in the first question of PRET although there was an integer *within-ratio* between quantities.

In addition, Mine used *factor of change strategy* to solve some problems in the pre-interview. For instance, she solved a problem that was exemplified by her by utilizing the strategy. The problem was "If a mixture has 200 gr water and 50 gr salt, how much salt will be in the mixture that had 400 gr water?" In order to solve the problem, she used the integer scale factor, 2, and multiplied 50 by 2 and found the answer as 100 gr salt. In a similar way, she solved another missing value problem, which was about buying fruit and its price, by using *factor of change* as follows:

> If 1 kg fruit was 2 TL, we should pay 4 TL for 2 kgs fruit, and similarly if we bought 3 kgs fruit, we should pay 6 TL. In other words, we should multiply kilograms of fruits and their price with the same number…

The problem was exemplified by her when the researcher asked her to create a real life proportional situation. Although, she could recognize the efficient strategy for the above problems, she sometimes had difficulty in recognizing efficient strategies, which facilitated the computations required to find missing values. To illustrate, in some PRET questions, though there were integer *within* and *between-ratios* between quantities, she did not use them.

Numerical Comparison Problems: Solution Strategies and Processes

In order to solve numerical comparison problems, Mine could use only one accurate strategy, *fraction strategy,* in which the rates were treated as fractions and a common denominator or numerator was found to make comparisons. To illustrate, she used the strategy in PRET question 6, which asked to compare two mixtures of orange juice concentrate and water. In her solution, she scaled up the amount of mixtures to a common amount by using the strategy. For this purpose, she wrote the ratios of orange juice concentrates to total mixtures as fractions and found a common denominator. Later, she multiplied 2/5 (the ratio for the mixture A) by 7/7 to produce 14/35, and then, she multiplied 3/7 (the ratio for the mixture B) by 5/5 to produce 15/35 which had the same denominator with the first ratio (14/35). Finally, she compared the amount of orange juice concentrates for the same amount of mixtures and concluded that mixture B had a stronger orange taste than mixture A. In conclusion, she used *part-to-whole ratios* by comparing orange juice concentrate to the total mixture. Moreover, Mine successfully chose mixture B as having a stronger orange taste than mixture A. Yet, she utilized only one strategy to solve the problem and could not exactly justify her rationale. Additionally, she did not explain the meaning of the quantities in the context in which they were used.

Mine used an inaccurate additive strategy to solve a numerical comparison problem. The problem (PRET question 24) presented three rectangular clothes with dimensions and asked pre-service teachers to decide the cloth that was "the most square". Mine calculated the differences between the dimensions of each rectangle and concluded that the cloth which had the less difference between its dimensions was "the most square". She used an inaccurate additive strategy; thus, she could not find the correct answer of the problem.

Mine sometimes could make sense of quantities in a numerical comparison problem. For example, an item in the pre-interview asked pre-service teachers to evaluate five students' solutions to the comparison problem. Mine explained clearly the quantities that were used to make comparisons in each of the five student's solutions to the problem. Moreover, she realized both *part-to-part* and *part-to-whole ratios* in the solutions.

In the student teaching, Mine did not ask any numerical comparison problem to her students. Moreover, in the pre-interview, she did not mention any situation that could be a numerical comparison problem while she was giving examples of proportional situations in real life.

Qualitative Reasoning Problems: Solution Strategies and Processes

Mine sometimes could make qualitative comparisons that did not depend on numerical values. To illustrate, in a qualitative reasoning problem of PRET (question 7), she found the correct answer and explained as, "The running speed is equal to the division of the distance by time (writing on the sheet: V=X/t). Therefore, if Esra ran more laps in a shorter amount of time than Gonca, she must run at a faster speed." She provided a valid explanation by making multiplicative comparisons between variables.

However, she sometimes used inaccurate additive strategies when she needed to use multiplicative strategies to solve proportional problems. To illustrate, in a qualitative reasoning problem of PRET (question 8), she made an additive comparison for explaining her answer as follows:

> The taste of today's fruit juice is the same with the yesterday's taste. If we used less orange and apple for the fruit juice that we prepared today, it is only about that we would drink less fruit juice today than yesterday.

As seen in the above explanation, Mine only took in the account the amount of the fruit juices and she did not consider the ratios of the apples and oranges in them. Therefore, she got an incorrect answer. In other words, she made an additive comparison between yesterday’s fruit juice and today’s fruit juice instead of a multiplicative comparison. Thus, it can be concluded that she had difficulty in recognizing multiplicative relationships in a qualitative reasoning problem.

In the student teaching, she did not ask any qualitative reasoning problem to her students. Moreover, in the pre-interview, she did not mention any situation that could be a qualitative reasoning problem while she was giving examples of proportional situations in real life.

4.1.1.3 Ela’s Approaches to Different Problem Types

Missing Value Problems: Solution Strategies and Processes

Ela could find correct answers of missing value problems in all relevant instruments. She used five different strategies (*cross-multiplication, isolating the unknown, factor of change, constant of proportionality, inverse proportion algorithm*). However, she mostly used formal strategies, in which rules and properties of algebra were used, instead of informal strategies highlighting multiplicative relationships. For instance, in the PRET, there were some missing value problems (questions 1-4) which did not have any context. In the pretest, Ela solved each problem in more than one way by using *cross-multiplication* and *isolating the unknown* strategies, both of which were formal strategies, and no other strategy was used by her to solve these problems. In order to solve the problems, she utilized the *isolating the unknown strategy* in two different ways: the one was multiplying or dividing both sides of the equation by a value to leave alone the missing value, and the other one was equating denominators, and then, multiplying and dividing both sides of the equation by the same values to leave alone the missing value. For example, in her second solution of the first question, she wrote the ratio, 4/20, in its simplest form, 1/5, and then, multiplied both sides of the equation (1/5=x/35) by 35 to leave alone the missing value. In addition, in her second solutions for the second, third and fourth questions, firstly, she equated the denominators, and then, multiplied and divided both sides of the equation by the same values to leave alone the missing values. Another evidence for the claim that she mostly used formal strategies is her solution to another missing value problem of PRET (question 23) in which she used *cross-multiplication.* Likewise, in the student teaching, she mostly used formal strategies. To illustrate, she solved all directly proportional problems by using *cross-multiplication strategy* and two out of four inversely proportional problems by using *inverse proportion algorithm*. Similarly, in the pre-interview, Ela only used *cross-multiplication strategy* in order to solve missing value problems. For instance, a problem in the interview was "If 8 balloons are 12 TL, how much 6 balloons will cost?" she utilized *cross-multiplication*. Firstly, she set up a proportion (8/12=6/*x*); secondly, she cross multiplied (12x6=8x(*x*)), and then, she divided the result of 12x6 by 8. In a similar way, she used *cross-multiplication* to solve another missing value problem which was exemplified by her. The problem was "If 10 books are 50 TL, how much 2 books will cost?" She set up the algebraic expression (10x(*x*)=2x50) by using *cross-multiplication* and found the answer as 10. Although identifying the *within-ratio,* 5, and multiplying the number of books (2) by it was very easy for the above problem, Ela solved it by using *cross-multiplication*. It seems that she had difficulty in using efficient strategies, which facilitated the computations, and used formal strategies even though there was an integer *between* or *within-ratio*. Another evidence for the claim is that in some PRET questions, though there were integer *within* and *between-ratios* between quantities, she did not use them and solved the problems by using formal strategies. In addition to these evidence, in the pre-interview, she stated that she preferred to use formal strategies (e.g., *cross-multiplication, isolating the unknown*) to solve missing value problems as follows:

Researcher: Which strategies do you use to find a missing value in a proportion?
Ela: I think the easiest way to solve such problems is cross-multiplication. Thus, I always use it.
Researcher: Any other strategy?
Ela: I solve by equating denominators. For example, in the equation (writing on the sheet: $\frac{5}{10} = \frac{?}{20}$), the first one can be multiplied by 2 to make it equal to the other one, then, they can be simplified and the answer will be 10.
Researcher: Which strategy will you teach your students as the most effective strategy?
Ela: Cross-multiplication is the easiest one, and so, it is effective.
Researcher: Why is it an effective strategy?
Ela: Because if I try to teach by using "the constant of *k*", it might confuse students' minds and complicate the operations. However, by using cross-multiplication, it becomes very easy to solve.
Researcher: What will you do if your students cannot understand this strategy?
Ela: If they do not understand with cross-multiplication, I will teach by equating denominators because it helps to leave alone the unknown and get the answer.

It might be concluded that Ela saw *cross multiplication* as the most effective and an indispensable strategy. Accordingly, she stated that her foremost strategy was *cross-multiplication* to solve missing value problems since it was easy. Her second strategy was *isolating the unknown* by finding a common denominator. In addition, when the researcher asked her which strategy she would teach her students if they did not understand *cross-multiplication*, she told the same formal strategy (i.e., *isolating the unknown* by equating denominators). Furthermore, she argued that *the constant of proportionality* was difficult to understand by students. In the same manner, in the student teaching, she stated that "In an inverse proportion, we have to use the strategy (*inverse proportion algorithm*), don't forget this!" after she solved an inversely proportional problem by using *inverse proportion algorithm.* Moreover, she taught the strategy as a leading and the most important strategy in solving inversely proportional problems and encouraged students to use it.

Ela sometimes used informal strategies that highlighted multiplicative relationships, instead of formal strategies (e.g., *cross-multiplication*). To illustrate, in the first student teaching, she solved some inversely proportional problems by utilizing *factor of change* and *the constant of proportionality*. In both problems, she highlighted multiplicative relationships. One of the problems was "A tractor plows a field in 12 days. How long will it take 3 tractors to plow the field?" She solved the problem as follows:

> As you can see, the number of tractors and field plowed are inversely proportional, so if you treble the number of tractors, the field plowed decreases to a third. I mean, we have to divide 12 by 3 to find the field plowed, and the answer is 4.

As seen in the above quote, Ela highlighted multiplicative relationships between the inversely proportional variables. Moreover, she taught *the constant of proportionality* as an alternative strategy and she said, "In an inverse proportion, the product of two quantities is a constant number. Thus, we can also solve the problem by using the constant number (writing on the board: 12x1=12, 3x(?)=12 and ?=4)" It seems that she utilized the functional relationships between inversely proportional variables (i.e., $x.y=k$).

Ela had difficulty in providing meaningful explanations for her solutions that highlighted proportional relationships between variables and using proportional reasoning language in her explanations. For example, in the student teaching, in order to solve directly proportional problems, Ela set the algebraic expressions by using *cross-multiplication* without giving any meaningful explanation and found the missing values. Similarly, in the student teaching, she solved some inversely proportional problems by using *inverse proportion algorithm* without giving any meaningful explanation. One of the problems was "Two taps fill a pool in 45 minutes. How much time will it take to fill the pool if 3 taps are open?" She solved the problem as follows:

> First of all, we have to determine whether the situation is inversely proportional or not. As the number of taps goes up, time goes down, so they are inversely proportional. Secondly, we must write the same quantities one under the other; then, multiply the number of taps by the minutes, and then, divide the answer by the number taps.

As can be seen, she did not highlight multiplicative relationships between variables. Further, she did not provide an explanation for her solution that went beyond a description of the steps she had taken to determine the solution. Likewise, in PRET question 23, she could not explain her solution.

Numerical Comparison Problems: Solution Strategies and Processes

Ela used three accurate strategies, which were *cross-multiplication* and *converting decimal expressions,* to solve numerical comparison problems. To illustrate, PRET had a numerical comparison problem (question 6) that asked to compare two mixtures of orange juice concentrate and water. In her first solution, she used *cross-multiplication strategy*. First of all, she wrote, “I will find the amount of orange juice concentrates in 7 cups of mixture, provided that 5 cups of mixture contains 2 cups of orange juice concentrate”. Secondly, she set up a proportion (5/2=7/x), cross multiplied (7x2=5x(x)) and found the answer as 2,8 cups of orange juice concentrates. Then, she explained her decision as:

> If mixture B has 2,8 cups of orange juice concentrates, the tastes of the mixtures will be the same. Yet, in mixture B, there are 3 cups of orange juice concentrates, so it has a stronger orange taste than mixture A.

Ela could accurately determine the mixture with a stronger orange taste and tried to explain her answer. Yet, as seen in the above quote, she could not clearly explain the quantities in the context in which they were used. In her second solution of the same problem, she used decimal expressions of the ratios and compared them. At first, she wrote the ratios of orange juice concentrates to total mixtures in fractional form, converted them into decimal expressions and compared them. Later, she concluded that mixture B had stronger orange taste than mixture A since the ratio of orange juice concentrate to total mixture was bigger. As a result, she had successfully made connection between the decimal expressions and how the mixtures would taste. Moreover, she could explain the meaning of the quantities she used to determine the relative strength of the mixtures explicitly. In conclusion, Ela was able to solved the problem in two different ways, both of which were based on *part-to-whole ratios* that compared orange juice concentrates to the total mixtures.

Ela used an inaccurate additive strategy to solve a numerical comparison problem. The problem (PRET question 24) presented three rectangular clothes with dimensions and asked pre-service teachers to determine the cloth that was "the most square". In her solution, firstly, Ela calculated the differences between the dimensions of each rectangle as 35-23=12, 155-139=16, and 75-56=19. Then, she concluded that since twelve was the smallest difference, the first cloth was "the most square". Her explanation for the answer was, "The one with the closest dimensions regarding length and width is the most square. Thus, the cloth that has the less difference between its dimensions, which is the first cloth, is the most square." She used an inaccurate additive strategy; thus, she could not find the correct answer of the problem.

Ela sometimes could make sense of quantities in a numerical comparison problem. For example, an item in the pre-interview asked pre-service teachers to evaluate five students' solutions to the comparison problem. She explained clearly the quantities that were used to make comparisons in each of the five student's solutions to the problem. Moreover, she realized both *part-to-part* and *part-to-whole ratios* in the solutions. In a similar way, as mentioned earlier, she could explain the meaning of the quantities she used to determine the relative strength of the mixtures in one of her solutions of PRET question 6.

In the student teaching, Ela did not ask any numerical comparison problem to her students. Moreover, in the pre-interview, she did not mention any situation that could be a numerical comparison problem while she was giving examples of proportional situations in real life.

Qualitative Reasoning Problems: Solution Strategies and Processes

Ela utilized two different strategies to solve qualitative reasoning problem. One of the strategies was giving numerical examples and the other one was making qualitative comparisons. For example, she solved a qualitative problem in the PRET (question 7) by giving numerical examples; in other words, she converted the qualitative problem to a numerical comparison problem. She explained her answer as follows:

> Esra is faster than Gonca because let's assume Esra runs 6 laps in 5 seconds and Gonca runs 3 laps in 10 seconds. If both of them run 6 laps, Esra will run in 5 seconds, but Gonca will run in 20 seconds.

Although she made multiplicative comparisons to solve the qualitative problem and found the correct answer, she could not make qualitative comparisons that did not depend on numerical values because she quantitatively compared the amount of time they run in the same number of laps. In other words, she did not interpret the qualitative relationship that existed between two quantities without giving numerical examples. In addition, as seen in the above explanation, she did not use proportional reasoning language.

On the other hand, she solved a qualitative problem in the PRET (question 8) by making qualitative comparisons that did not depend on numerical values. She wrote an explanation that accurately interpreted the variables in the problem. Moreover, she used proportional reasoning langue in her explanation. She explained her answer as:

> If the amounts of apple and orange used for the fruit juice decrease by the same ratio, the fruit juice will be the same taste. But if they decrease by different ratios, the taste will change, so there is not sufficient information to tell.

It is important to note that she considered the ratios of the apples and oranges in the fruit juices. Additionally, she provided a valid explanation by making multiplicative comparisons between variables without giving numerical examples.

In the student teaching, she did not ask any qualitative reasoning problem to her students. Moreover, in the pre-interview, she did not mention any situation that could be a qualitative reasoning problem while she was giving examples of proportional situations in real life.

4.1.2 Distinguishing Proportional from Nonproportional Situations

Pre-service teachers' work on classifying relationships as proportional or nonproportional, identifying ratio as measure and providing examples for proportional and nonproportional relationships were analyzed so as to reveal pre-service teachers' distinguishing proportional from nonproportional situations before receiving a practice-based instructional module based on proportional reasoning. In the following section, the results of the analyses are presented.

4.1.2.1 Gaye's Findings about Distinguishing Proportional from Nonproportional Situations

Classifying Relationships as Proportional or Nonproportional

Gaye had difficulty in classifying relationships as proportional or nonproportional. To illustrate, PRET had some questions (questions 11 to 22) which presented pre-service teachers 12 relationships (three presented in written language, three presented in graphs, three presented in equations, and three presented in tables) and asked to identify whether the given situations were proportional or nonproportional. In the pretest, Gaye could not accurately classify five out of 12 relationships as proportional or nonproportional. In particular, she inaccurately classified two of the relationships presented in language and table, and one of the relationship presented in graph. However, she accurately classified all the relationships presented in equation. It is interesting to note that Gaye's inaccurate classifications were not limited to a particular representation.

Gaye appears to have believed that all linear relationships were proportional. Thus, she sometimes over generalized proportionality; in other words, she inaccurately applied proportional reasoning in situations that had nonproportional relationships. To demonstrate, in the pretest, she classified linear relationships as proportional in three of the four representations (language, graph and table). Among these relationships, she inaccurately classified four linear and nonproportional relationships (questions 11, 13, 15 and 21) as proportional. Additionally, some explanations she produced for the questions support the claim about the linearity misconception. To illustrate, in question 15 of the pretest (a line does not pass through the origin), she determined that the relationship was proportional and explained as, "It does not matter that the graph goes through the origin or not, all linear relationships are proportional, therefore x and y have a proportional relationship." She concluded that the linear and nonproportional relationship between quantities in question 15 was proportional although the graph of the quantities did not pass through the origin. It is interesting to note that Gaye accurately classified the other linear and nonproportional relationship presented in equation (questions 17) as nonproportional. However, she could not provide evidence to support her claim.

Evidence supporting the claim that Gaye believed all linear relationships were proportional was her answer to an interview question in the pre-interview as shown in the following excerpt:

> Researcher: While teacher Turgay was teaching his students distinguishing proportional from nonproportional situations, one of the students Atacan said "All linear relationships are proportional. In other words, if two variables have a linear relationship, they are also proportional." Do you think Atacan is right? Why?
> Gaye: I think he is right. If there is a linear relationship between variables, they have to increase or decrease at the same time.
> Researcher: Can you explain?
> Gaye: It is very easy to understand because linearity means proportionality. Thus, Atacan is right.

As seen in the excerpt above, she held the misconception that all linear relationships were proportional. Moreover, she asserted that linearity ensured proportionality.

In addition, two examples given by Gaye for direct proportion in the first student teaching supported the same claim. The first example was, "A young tree's length increases 20 cm every year. If its first length is 50 cm, draw the length-time graph of the tree for three years." She solved the problem as presented in the following excerpt:

> Gaye: We will start to draw the graph with 50 cm because the length of the tree is 50 cm at the beginning. One year later, it will be 70; two years later, it will be 90 and three years later it will be 110 cm, it goes like that. (she found the points in the coordinate plane, then connecting the dots and drew the line through the points)
> Student: Is it a direct proportion?
> Gaye: Yes. Can you explain why it is a direct proportion?
> Student: As we said earlier when one thing increases the other thing has to increase, too.
> Gaye: What increases in comparison to what?
> Student: While time increases, the length increases, too.
> Gaye: Yes, correct. Since while time increases the length also increases, there is a direct proportion.

As seen in the excerpt above, she concluded that the linear and nonproportional relationship was directly proportional. Moreover, it seems that she believed that increasing or decreasing at the same time was enough for the variables to be proportional. The second example she created in her first student teaching asked the length-time graph of a baby who born 40 cm and lengthen 10 cm ever month. In a similar way, first of all, she drew the graph of a line through the points 40, 50, 60 and 70. Then, she concluded that since both variables increased at the same time, the graph was a graph of directly proportional variables. Similar to previous example, she regarded the linear and nonproportional relationship between variables as directly proportional. Further, in the pre-interview, when the researcher asked her to provide an example of a real life proportional situation, she exemplified a situation in which quantities had a linear and nonproportional relationship instead of a proportional relationship.

On the other hand, Gaye could not be sure about how to decide a proportional relationship in some situations. For example, in pretest question 13, which asked “the relationship between Sevil and Nehir’s positions on a marathon course if they ran at the same pace but Sevil had run 2 kilometers before Nehir started”, she indicated that there was a proportional relationship between quantities although there was not. She explained as, “They are proportional because they run at the same pace and the distance between them is always 2 km, it does not differ; therefore, Nehir never catch Sevil.” In the pre-interview, when the researcher wanted her to expand her answer, she could not decide whether the relationship between quantities was proportional or not. She said:

> For a proportional relationship, the ratios of the distances they taken at different times have to be the same. For example, if Sevil run 40 km, Nehir will run 38 km, the ratio will be 40/38. Similarly, if Nehir run 16 km, Sevil will run 18 km, the ratio will be 16/18. But they are not equal, so they cannot be proportional. But, how does this happen! The distance between them remains unchanged so they must be proportional …er... I am confused that which one is true? I am not sure.

She could not be sure how to decide proportionality. First of all, she thought additively and considered the constant distance between them, and then, she calculated the positions of the girls at different times and compared the ratios.

Another doubt of Gaye was about quadratic relationships. For example, in the pretest, while she accurately classified the quadratic relationships presented in graph (question 16) and equation (question 18) as nonproportional, she inaccurately classified the other quadratic relationship presented in table (question 22) as proportional. The reason might be that question 22 was presented as a table, which allowed her to realize a pattern between *x* variable and *y* variable for each row. In fact, she wrote in the pretest, "8 is 2 times 4; 18 is 3 times 6; 32 is 4 times 32. There is a ratio between multipliers (2, 3 and 4) because they increase one by one." As can be understood from her explanation, she assumed that the additive pattern between multipliers could be evidence of a proportional relationship. However, in the pre-interview when the researcher wanted her to expand her answer to question 22, she could not decide whether the relationship between quantities was proportional or not, and said, "When we look at the multipliers, there is a proportion; but when we look at the results (showing *x* and *y*), there is not any proportion. I do not exactly know which one is important." Her doubts about quadratic relationships revealed she did not know that all proportional relationships were linear, which could be evidence of nonproportionality of quadratic relationships. Moreover, in the quadratic relationships that she accurately classified as nonproportional, she could not provide any valid evidence to support her claims.

Identifying Ratio as Measure

Gaye could recognize ratio as a proper method to measure concentration of a mixture and steepness of ski ramps. To demonstrate, one of the PRET problems (question 6) asked pre-service teachers to compare two mixtures of orange juice concentrate and water. In this problem, Gaye used the ratios to measure the orange concentrations of the mixtures. However, she did not use proportional reasoning language in her explanation. Particularly, she did not use the word “ratio” though she used ratios. Another evidence for the claim is her statements about an item in the pre-interview, which asked pre-service teachers to evaluate five students’ solutions. The students compared two mixtures of chocolate milk made from different amounts of chocolate syrup and milk, and determined which had a stronger chocolate flavor. While she was making sense of students’ solutions in the pre-interview, she recognized that the students had used ratio as a measure of chocolate concentration of the mixtures. To illustrate, she said:

> The student compared the ratio of chocolate to milk for each mixture. He found the ratio of chocolate as 3/5 for Emre’s mixture, and as 5/8 for Selin’s mixture. Since she found a bigger ratio for Selin’s mixture, she correctly concluded that Selin’s mixture had a stronger chocolate flavor.

Although the word “ratio” was not mentioned in neither the problem nor the solutions of the students, Gaye preferred to use the word “ratio” to explain the relationship between the amount of chocolate syrup and milk in the mixture. It can be concluded that she noticed that ratio was as a proper method to measure chocolate concentration of a mixture. In addition, in a problem of PRET (question 10), which asked to determine relative steepness of ski ramps if the height, the length of the base, and the width of the base of the ramps were given, Gaye stated that one could rate the ramps from steepest to least steep by using the ratios of heights to lengths of the bases. In other words, she used ratio as a measure of steepness of ski ramps. Additionally, she noticed that the width of the base did not have an effect on the steepness of the ramps.

On the other hand, data revealed that Gaye did not see ratio as a proper method to measure some attributes (i.e., shade of paint, “squareness” of a rectangle). In fact, she used additive comparisons to measure these attributes instead of using ratio which was a multiplicative comparison. For example, in a PRET problem (question 9), a boy mixed blue and white paints until he had a shade of blue paint that he liked. He needed another quart of paint, so he wanted to increase the amount of paint without changing the color. In order to do that, he added one glass of blue and one glass of white paint (2 glasses = 1quart). The pre-service teachers were asked to comment on the effectiveness of the strategy. Gaye made an inaccurate decision and stated:

> Since the amounts of paints he added are the same, it does not differ that 1 gr white and 1 gr blue paint added or 10 gr white and 10 gr blue paint added, these are the same things. If you added equal amounts of paints, the strategy is always useful, so Murat's strategy is effective.

The above explanation indicated that Gaye's focus was on the equal amounts of the paints added, which was an additive approach. She asserted that if the same amount of the paints repeatedly added, larger amounts of the mixture that maintained the same color could be made. In fact, she did not mention the ratios of white and blue paints that made up the original and new mixtures. It can be concluded that she did not see ratio as a proper measure of the shade of the paint for the problem. In a similar way, in PRET question 24, which presented three rectangular clothes with dimensions and asked pre-service teachers to decide the cloth which was "the most square", she did not identify ratio as a proper method to measure "squareness" of a rectangle. Moreover, as explained earlier, she calculated the differences between the width and length of each rectangle and compared them. In other words, she made an additive comparison to determine the attribute.

Providing Examples for Proportional and Nonproportional Relationships

Gaye was not able to provide a valid example of a proportional relationship. To demonstrate, in the pre-interview, when the researcher asked her to give an example of a real life proportional situation, she stated:

> The amount of money and time are proportional when one invests money in a bank. For example, if one has 100 TL and the bank gives 3

> TL for each mount. The money will increase by 3%. By this was, there is a ratio between them.

As can be seen in the above quote, she exemplified a situation in which quantities had a linear and nonproportional relationship instead of a proportional relationship. Moreover, in the PRET, there was a question (question 5) that asked pre-service teachers to provide a word problem which could be solved by the given equation. Gaye could not provide a word problem in which the quantities related proportionally. Actually, she did not write a problem that could be solved. Furthermore, there was not any statement of her that implied multiplicative relationship between variables in the problem.

Gaye could provide a valid example of a nonproportional relationship. For instance, in the pre-interview, when the researcher asked her to give an example of a real life nonproportional situation, she said:

> Let's assume, we rent a car and pay 200 TL. The number of persons who get in the car does not change the given money. 5 persons pay 200 TL; similarly, 4 persons pay 200 TL, too. It does not differ.

As seen in the above, Gaye was able to provide a valid example of a nonproportional in which the variables had a constant relationship, but she could not exactly explain why her example was nonproportional.

4.1.2.2 Mine's Findings about Distinguishing Proportional from Nonproportional Situations

Classifying Relationships as Proportional or Nonproportional

Mine generally did not have difficulty in classifying relationships as proportional or nonproportional. To illustrate, PRET had some questions (questions 11 to 22) which asked pre-service teachers to identify whether the given situations were proportional or nonproportional. Mine accurately classified most of the relationships as proportional or not. In fact, she inaccurately classified only a relationship (question 11) as proportional, which was presented in language.

Mine seems to have assumed that increasing or decreasing at the same time was enough to classify a situation as proportional. Therefore, she sometimes over generalized proportionality; in other words, she inaccurately applied proportional reasoning in situations that had nonproportional relationships. For example, in PRET question 11, pre-service teachers were asked to classify "the relationship between number of kilometers for a customer's taxi ride and cost of the trip if the customer pays a 2,2 TL fee, plus 1,9 TL per kilometer for the taxi", as proportional or nonproportional. Mine stated that the relationship was proportional because while the number of kilometers increased, the cost of the trip increased also. In other words, she concluded that the linear and nonproportional relationship between quantities in the question was proportional although the quantities did not increase in the same ratio. Similarly, though she accurately classified the relationship in question 12, which asked "the relationship between the number of movie tickets purchased and the total cost", she gave the explanation, "They are proportional because as the number of tickets increases, the total cost also increases. The opposite is also true; as the number of tickets decreases, the total cost also decreases." She only considered that the direction of change of the related quantities was the same. Moreover, she did not mention the fact that the change between the quantities were in the same ratio. However, in the pre-interview, she realized her mistake when the researcher asked her to elaborate her answer to question 11 as follows:

> Mine: I said they were proportional because when the number of kilometers increased, the amount of money was increased, too. Actually, it is a direct proportion because when one variable increases the other also increases.
> Researcher: Can you explain what you mean? Imagine you are teaching it to your students?

> Mine: One will pay 1,9 TL + 2,2 TL for one km and pay 3,8 TL +2,2 TL (writing on a sheet) for 2 kms. Wait a minute! It is wrong. They are not proportional.
> Researcher: Why?
> Mine: 2,2 TL destroys the proportionality. If one did not pay 2,2 TL in the beginning, the paid money and kilometers would be proportional.
> Researcher: Why?
> Mine: Because 2,2 TL did not change with respect to kilometers.
> Researcher: But you said if both of them increase, they are proportional. And both of them increase in here, don't they?
> Mine: Yes, both of them increase but there is not a constant factor. I mean, for the variables to be proportional when one variable increases by n, the other has to increase by n, too.

During the pre-interview, Mine appears to have eliminated her misconception that a relationship was proportional because as x increased, y also increased, and all linear relationships were proportional. Moreover, she considered the constant ratio that defined the relationship between quantities. Her answer to an interview question in the pre-interview is the other evidence supporting the claim that Mine knew that all linear relationships were not proportional. The question asked pre-service teachers to determine whether a statement of a student about linear relationships was true or not, and explain why. The student stated, "All linear relationships are proportional. In other words, if two variables have a linear relationship, they are also proportional." Mine argued that the student was not right and said:

> I think the statement is not true. For example, $y=mx+n$ is an equation of a linear relationship. Yet, y and x are not proportional. There is not a constant factor of y when x increases by m because we add n.

It can be concluded that she could explain the difference between functions of the form $y=mx$ and functions of the form $y=mx+n$. Without doubt, in the latter function y was not proportional to x. In addition, she stated that for the quantities to be proportional, linearity was not enough.

In a similar way, in the student teaching, she emphasized that all linear relationships were not proportional. Furthermore, she asked problems that had linear and nonproportional relationships besides problems had linear and proportional relationships. Additionally, while she was teaching her students how they could distinguish proportional from non-proportional relationships, she said:

> For the quantities to be proportional, both of them have to increase or decrease by the same ratio. For instance, if one of the quantities increases by 2, the other one has to increase by 2, too. If you add a number to any side of the equation, you will not get proportional relationship similar to this one (writing on the board: y=3x+5). Since x increases by 3, but y does not increase by 3.

The explanation of Mine is another evidence supporting the claim that Mine eliminated her misconceptions about linearity after the pre-interview.

Mine had difficulties in classifying quadratic relationships as nonproportional. For example, in the pre-interview, she made inaccurate classifications for questions 16 and 18, which had quadratic relationships (e.g., $y=mx^2$), although she accurately classified them as non-proportional in the pretest. She stated:

> … Oops! I made one more mistake, I said 16 and 18 were not proportional, but they are not true… Question 16 was a graph of an equation like, $y=ax^2$, similarly the equation in question 18 was $y=3x^2$. Both of them are proportional because there is not any value that is added to x or y (writing on the sheet: $y=mx+n$).

As understood from the above quote, Mine inaccurately classified the quadratic relationships as proportional since there was not any constant number that was added to one of the variables. In other words, she concluded that the quadratic relationships, which were nonlinear, were proportional. It seems that she did not know that if variables had proportional relationships, they also should have linear relationships. Similarly, she probably did not see that the variables in the questions did not increase or decrease by the same ratio, which was required to be proportional.

Identifying Ratio as Measure

Mine could recognize ratio as a proper method to measure concentration of a mixture and steepness of ski ramps. For instance, one of the PRET problems (question 6) asked pre-service teachers to compare two mixtures of orange juice concentrate and water. In this problem, Mine used the ratios to measure the orange concentrations of the mixtures. But, she did not use the word "ratio", while she was explaining her solution. In brief, although she utilized ratio to solve the problem, she did not use proportional reasoning language in her explanation. Another evidence for the claim is her statements about an item in the pre-interview, which asked pre-service teachers to evaluate five students' solutions in which they compared two mixtures of chocolate milk made from different amounts of chocolate syrup and milk. As Mine made sense of students' solutions in the pre-interview, she noticed that the students had used ratio as a measure of chocolate concentration of the mixtures. For instance, she stated:

> Researcher: Is the student 1 right?
> Mine: Yes, he is right.
> Researcher: What do 3/5 and 5/8 mean? Namely, what do 0,6 and 0,625 represent in the problem?
> Mine: The student found the ratio of chocolate syrup to milk, and concluded that Selin's chocolate milk had a stronger chocolate flavor than the other one.
> Researcher: Why? Can you explain?
> Mine: Because when the student compared the ratios of chocolate syrup to milk, she found a bigger ratio for Selin's chocolate milk, and so, he concluded that it had more chocolate flavor.

Mine preferred to use the word "ratio" to clarify the relationship between the amount of chocolate syrup and milk in the mixtures although the word was not mentioned in neither the problem nor the solutions of the students. In conclusion, she appears to have recognized that ratio was as a proper method to measure chocolate concentration of a mixture. In addition, in a problem of PRET (question 10), which asked to determine relative steepness of ski ramps if the height, the length of the base, and the width of the base of the ramps were given, she made a multiplicative comparison to measure the attribute. That is to say, she used ratio as a measure of steepness of ski ramps. Moreover, she noticed that the width of the base did not have an effect on the steepness of the ramps.

However, data showed that Mine did not see ratio as a proper method to measure some attributes (i.e., shade of paint, "squareness" of a rectangle). In fact, she used additive comparisons to measure these attributes instead of using ratio which was a multiplicative comparison. For instance, in a PRET problem (question 9), which asked pre-service teachers to comment on the effectiveness of a boy's strategy about changing the amount of paint by saving its shade, Mine concluded that the boy's strategy was not effective, and said:

> We do not know the amounts of the blue and white paints in the beginning mixture. If we knew the amounts, the strategy could be effective, but since we do not know, the strategy is not effective. In other words, the new mixture's color will be bluer or whiter than the color of the beginning mixture, but not the same.

Although Mine accurately determined ineffectiveness of the strategy, she only mentioned the amounts of the paints in the original mixture, which was an additive approach. Actually, she did not consider the ratios of white and blue paints that made up the original and new mixtures. It seems that she did not realize ratio as a proper measure of the shade of the paint for this problem. Similarly, in PRET question 24, she did not identify ratio as a proper method to measure "squareness" of a rectangle. Furthermore, she used an additive comparison to determine the attribute.

Providing Examples for Proportional and Nonproportional Relationships

Mine sometimes could not provide a valid example of a proportional relationship. For instance, in the PRET, there was a question (question 5) that asked pre-service teachers to provide a word problem which could be solved by the given equation. In the pretest, Mine could not create a valid missing value word problem in which the quantities related multiplicatively. She wrote: “If we get 3/8, when we divide a number by 20, what is the number?” In fact, she wrote a problem that did not went beyond a description of the steps one had taken to solve the equation.

However, Mine sometimes could exemplify situations in which variables had proportional relationships. To illustrate, in the pre-interview, when the researcher asked her to provide an example of a real life proportional situation, Mine could create a valid example for a proportional real life situation. Furthermore, she explained why the situation was proportional by utilizing the statement that variables in a proportional situation had to increase or decrease in the same ratio.

Mine could provide a valid example of a nonproportional relationship. For example, in the pre-interview, when the researcher asked her to give an example of a real life nonproportional situation, the example of her as follows:

> Mine: If a person pays 5 TL for an hour parking fee and additional charge of 3TL, the relationship between paid money and hours will not be proportional.
> Researcher: Why? Can you explain?
> Mine: For example, if the person stayed an hour, he would pay 8 TL, and if he stayed two hours, he would pay 13 TL. As the hour increases, the paid money increases; yet, they do not increase by the same factor.

Mine accurately created a linear and nonproportional example in which the quantities had additive relationship. In addition to that, she could explain why her example was

nonproportional by emphasizing that the change between the quantities in a proportional situation had to be in the same ratio.

4.1.2.3 Ela's Findings about Distinguishing Proportional from Nonproportional Situations

Classifying Relationships as Proportional or Nonproportional

Ela did not have difficulty in classifying relationships as proportional or nonproportional. To demonstrate, PRET had some questions (questions 11 to 22) which asked pre-service teachers to identify whether the given situations were proportional or nonproportional. In the pretest, Ela accurately classified all of the relationships as proportional or not.

Data indicated that Ela knew all linear relationships were not proportional and all proportional relationships were linear. For example, in the pretest, Ela accurately classified all of the linear and proportional relationships (questions 12, 14, 19 and 20) as proportional, all of the linear and nonproportional relationships (questions 11, 13, 15, 17 and 21) as nonproportional and all of the quadratic relationships as nonproportional (questions 16, 18 and 22). Moreover, in the student teaching, while Ela was solving nonproportional problems with additive relationships, she noted that all linear relationships were not proportional and also emphasized that all proportional relationships were linear.

Her answer to an interview question in the pre-interview is another evidence supporting the claim that Ela knew all linear relationships were not proportional. The researcher asked Ela to determine whether a statement of a student about linear relationships was true or not and to explain why, as shown below:

> Researcher: While teacher Turgay was teaching his students distinguishing proportional from nonproportional situations, one of the students Atacan said "All linear relationships are proportional. In other words, if two variables have a linear relationship, they are also proportional." Do you think Atacan is right?
> Ela: No, he is not right.
> Researcher: Why? Can you explain?

Ela: If there is a linear relationship between variables, the equation showing the relationship could be y = mx+n or y=mx because both of them have linear graphs. Atacan is not right because while y=mx is proportional, y=mx+n is not proportional, so all linears are not proportional.

It can be concluded that she could explain the difference between functions of the form $y=mx$ and functions of the form $y=mx+n$. Without doubt, in the latter function y was not proportional to x. In addition, she stated that all linear relationships were not proportional. In a similar manner,

Ela had difficulties in providing evidence to support her claims about proportionality. To illustrate, in the pretest, while she was explaining her classifications for the situations that had proportional relationships (questions 12, 14, 19 and 20), she said that the direction of change of the related quantities was the same, but she did not mention the fact that the change between the quantities was in the same ratio. For example, for question 12 that asked “the relationship between the number of movie tickets purchased and the total cost”, she stated, “The relationship between the number of tickets and the money is proportional because as the number of tickets increases, the money increases, too.” However, while she was justifying her classification for the situations that did not have proportional relationships (questions 11, 13, 15, 16, 17, 18, 21 and 22), she ignored that the direction of change of the related quantities was the same. That is to say, although the related quantities in the questions increased or decreased at the same time, she did not determine that the quantities were proportional. The reason might be that she could distinguish proportional from nonproportional relationships, but she could not provide evidence to support her claims about proportionality. Another evidence for the claim is that in the student teaching, while she was teaching her students how they could distinguish proportional from non-proportional relationships, she said, “If one quantity increases, the other one also increases, and if one quantity decreases, the other one also decreases, these quantities are directly proportional.” She did not emphasize that the change between the quantities had to be in the same ratio.

Identifying Ratio as Measure

Ela could identify ratio as a proper measure in the context of determining concentration of a mixture and steepness of ski ramps. For instance, one of the PRET problems (question 6) asked to compare two mixtures of orange juice concentrate and water. In this problem, Ela utilized the ratios of the quantities to measure the concentrations of the mixtures. Furthermore, although she did not exactly utilize proportional reasoning language in her explanations, she used the word “ratio” in her solution. Another evidence for the claim is her statements about an item in the pre-interview, which asked pre-service teachers to evaluate five students’ solutions in which they compared two mixtures of chocolate milk made from different amounts of chocolate syrup and milk. She recognized that the students had used ratio as a measure of chocolate concentration of the mixtures. Further, although the word “ratio” was not mentioned in neither the problem nor the solutions of the students, Ela preferred to use the word to explain the relationships between quantities in the solutions. Moreover, in a problem of PRET (question 10), which asked to determine relative steepness of ski ramps if the height, the length of the base, and the width of the base of the ramps were given, she made a multiplicative comparison to measure the attribute. In other words, she used ratio as a measure of steepness of ski ramps. In addition, she noticed that the width of the base did not have an effect on the steepness of the ramps.

On the other hand, data indicated that Ela did not see ratio as a proper method to measure some attributes (i.e., shade of paint, “squareness” of a rectangle). In fact, she used additive comparisons to measure these attributes instead of using ratio which was a multiplicative comparison. To illustrate, in a PRET problem (question 9), which asked pre-service teachers to comment on the effectiveness of a boy’s strategy about changing the amount of paint by saving its shade, Ela had an incorrect answer and said, “The strategy is always useful because he added equal amounts of blue and white paints.” Ela paid attention to the equal amounts of the paints added, which was an additive approach. Indeed, she did not consider the ratios of white and blue paints that made up the original and new mixtures. It can be concluded that she did not see ratio as a proper measure of shade of paint for the problem. In a similar manner, in PRET question 24, she did not identify ratio as a proper method to measure "squareness" of a rectangle. Furthermore, she used an additive comparison to determine the attribute.

Providing Examples for Proportional and Nonproportional Relationships

Ela could exemplify situations in which variables had proportional relationships. To demonstrate, in the pre-interview, when the researcher asked her to give an example of a real life proportional situation, she could create a valid example for a proportional real life situation. Furthermore, she explained why the situation was proportional by utilizing the statement that if a variable in a proportional situation increased by a factor, the other variable had to increase by the same factor, too. Additionally, in the PRET, there was a question (question 5) that asked to provide a word problem which could be solved by the given equation. In the pretest, the problem written by Ela was as follows:

> A farmer divided a field into 8 equal parts and used 3 parts to grow tomato. The next year, the farmer will divide the same field into 20 equal parts. He wants to use the same amount of field to grow tomato as he used last year. Find the number of parts?

As can be seen, she could create a valid missing value problem in which the quantities had proportional relationships. In addition, the noninteger answer (7.5 parts of the field) made sense in the context.

Ela could provide a valid example of a nonproportional relationship. For example, in the pre-interview, when the researcher asked her to give an example of a real life nonproportional situation, the example of her was, "Let's think for a taxi ride of 2 kms, a customer pays 6 TL; however, another customer pays 8 TL for 5 kms. The number of kilometers and the cost of the trip are not proportional." Ela accurately created a linear and nonproportional example. Moreover, she could justify why her example was not proportional by using the statement that variables in a proportional situation had to increase or decrease by the same factor.

4.1.3 Understanding the Mathematical Relationships Embedded in Proportional Situations

In order to investigate pre-service teachers' understanding of mathematical relationships embedded in proportional situations, the four *key understandings*, if any, they used to define proportionality concepts and to determine proportional relationships were analyzed. Moreover, their difficulties in defining proportionality concepts and understanding the *key understandings* were presented. As aforementioned, the four *key understandings* were: (1) proportional relationships are multiplicative in nature; (2) proportional relationships are presented graphically by a line through the origin; (3) the rate pairs are equivalent in proportional relationships; (4) proportional relationships can be represented symbolically by the equation y = *m*x, where the *m* is the slope, unit rate, and constant of proportionality (Cramer et al., 1993 & Post et al., 1988). In the following section the results of the analyses before the proportional reasoning instructional module are presented under two subcategories: defining proportionality concepts and determining proportional relationships.

4.1.3.1 Gaye's Findings about Understanding the Mathematical Relationships Embedded in Proportional Situations

Defining Proportionality Concepts

Gaye had difficulties in defining the ratio and proportion concepts. For instance, in the first student teaching, Gaye defined the term ratio as, "the division of *a* by *b* is called the ratio of *a* to *b*". As can be seen, she asserted that ratio was the same thing with "division". However, in the pre-interview, Gaye almost accurately defined ratio as, "Ratio is relative conditions of two things when they are compared", but she did not use proportional reasoning language in her definition. Moreover, a misconception of Gaye was revealed when she taught ratio in the first student teaching. To illustrate, she said, "We represent ratio, "*a/b*", and read as, "ratio of *a* to *b*". The numerator can be zero, but the denominator never can be zero because if it is, the result will be undefined." As can be seen, Gaye assumed that a ratio could not have a zero as its second component. However, a proportional situation might contain the ratio of zero to zero (*0/0*). It can be concluded that she had some difficulties in *key understanding 2* which meant that proportional relationships were presented graphically by lines that passed through the origin.

Gaye did not exactly know the definition of proportion. For example, she defined proportion in the interview as follows:

> For example, if we travel 100 kilometers in 2 hours, how many hours does it take to travel 50 kilometers? There is a direct proportion, we should write it as, 100 x (x) = 2 x 50. This formalized equation is proportion.

She defined the term proportion as an equation, which was derived from *cross-multiplication,* without giving any meaningful explanation. In the student teaching, she was taken note to her students a true definition of proportion; however, similar to interview results, she used *cross-multiplication* while she was explaining the definition. To illustrate, she stated, "Proportion is *a/b=c/d*. How can we express this? Of course, by using cross-multiplication. It means *a* x *d* = *b* x *c*. This is the first rule for proportion, don't forget!" Similar to the pre-interview results, she did not utilize any *key understanding* to explain proportion in the student teaching. Furthermore, she taught some memorized rules about proportion to students without giving any meaningful explanation. The situation might demonstrate that she did not have enough understanding of why the multiplication of *a* by *d* was equal to *b* by *c* (i.e., $a \times d = b \times c$) in a proportion and under what conditions this procedure could be applied.

Gaye had difficulty to define direct proportion. In fact, she did not consider the constant ratio that defined the relationship between the variables in a direct proportion. Additionally, she did not utilize any *key understanding* to explain the concept. For example, she defined direct proportion in the first student teaching as, "If one variable increases, the other one also increases, and if one variable decreases, the other one also decreases, these variables are directly proportional." As seen in the definition, she did not mention multiplicative relationships between variables. Therefore, it can be concluded that she had some difficulties in *key understanding 1* which meant that proportional relationships were multiplicative in nature.

Moreover, Gaye had difficulty in distinguishing ratio from proportion. She used the terms ratio and proportion interchangeably. For example, in the first student teaching, she said "setting up a ratio" and "constant of ratio" instead of "setting up a proportion" and "constant of proportionality".

Gaye struggled to define the constant of proportionality. For instance, in the pre-interview, when the researcher asked her to define the constant of proportionality, she said:

> It is a ratio of relative change between two things when we compare them. The constant of proportionality never change though the quantities change. For example, the ratio of my mother's age and my age. Er… Sorry, the ratio changes. I confused. The constant of proportionality is the amount of change; namely, my mother's age will increase one by one and my age will increase one by one, so the ratio of one to one, 1/1, does not change.

As seen in the above quote, she assumed that the constant of proportionality was a constant additive relationship instead of a constant multiplicative relationship between quantities. Similarly, when the researcher asked her in which pretest questions she used the constant of proportionality, she said that she used it in questions 20 and 21, but she supposed that the constant of proportionality was a constant additive relationship between variables. On the other hand, she accurately determined the constants of proportionality in questions 6 and 23; however, she could not express the meanings of the constant of proportionality in the contexts in which it was used.

Determining Proportional Relationships

Gaye could not exactly explain the presence of proportional relationships by using the *key understandings*. To demonstrate, while she was teaching how to decide a proportional relationship in the first student teaching, she partially used *key understanding 3* and *key understanding 4* in a rote manner. She said:

> Gaye: The second rule is a:b:c=x:y:z. We can also represent as a/x=b/y=c/z. For the variables to be proportional, the fractions always have to be equal to the same thing. This thing is "*k*", that is constant of proportionality… a*/x=k, b/y=k, c/z=k* and from cross-multiplication it can be said *a=kx, b=ky, c=kz…* I want to ask a question: 10/15, 18/27 is that a proportion? Find the proportion?
> Student: How do we know?
> Gaye: For the variables to be proportional, you have to find a constant number like "*k*".
> Student: They are equal to 2/3, so it is a proportion.
> Gaye: Yes. It is a proportion because the ratios are equal.

As she noted in the above excerpt, she taught proportionality with the aid of some memorized rules without giving any meaningful explanation. In addition, she inadequately utilized the *key understandings*. To illustrate, she did not mention multiplicative relationships between quantities (*key understanding 1*). She mentioned the equality of the rate pairs in a proportion (*key understanding 3*), but she could not explain why it was true and not use proportional reasoning language in her speech (e.g., she said, "find the *proportion"* instead of "find the *constant of proportionality"*). Furthermore, she used the constant of proportionality to identify a proportion. However, she only stressed that proportional relationships were represented by $y=kx$, where k was the constant of proportionality. She did not explain that k was also slope and unit rate (*key understanding 4*) and not make connections between them. Moreover, she did not emphasize that both k and $1/k$ were the constant of proportionality; in other words, she probably did not realize that a constant multiplicative relationship existed between two quantities and it could be expressed in two ways.

In a similar manner, she could not adequately use the *key understandings* to justify her classifications about proportional relationships. In fact, she sometimes did not use any *key understanding* to determine proportional relationships. Further, she sometimes used inaccurate statements based on additive approach to decide proportionality. The justifications of Gaye about her classifications in PRET questions 11-22 in the pretest and pre-interview were evidence for the claims. In order to get in-depth information, in the pre-interview, the researcher asked Gaye to explain how she decided the relationships between the quantities in PRET questions 11-22. Table 4.1 shows the classifications and the explanations of Gaye for each question in the pretest and pre-interview. Although most of the results were similar, there were some different answers and detailed explanations in the pre-interview.

Table 4.1 Classifications and explanations about proportional relationships in PRET questions 11-22

	Pretest		Pre-interview	
Question	Classification	Explanation	Classification	Explanation
11 (Language)	*Proportional*	*Not Clear*	Not proportional	Cross-multiplication
12 (Language)	Proportional	Not Clear	Proportional	Key Understanding 1
13 (Language)	*Proportional*	*Additive Approach (Constant distance)*[a]	*Not Sure*	*Key Understanding 3*
14 (Graph)	Proportional	Constant Slope	Proportional	Key Understanding 4, Constant Slope
15 (Graph)	*Proportional*	*Constant Slope*	*Proportional*	*Constant Slope*
16 (Graph)	Not Proportional	Absence of constant slope	Not proportional	Absence of constant slope
17 (Equation)	Not Proportional	Key Understanding 1	Not Proportional	Key Understanding 1, Key Understanding 3
18 (Equation)	Not Proportional	Not Clear	Not Proportional	Absence of constant slope
19 (Equation)	Proportional	Key Understanding 1	Proportional	Key Understanding 1
20 (Table)	Proportional	Not Clear	Proportional	Key Understanding 3, Additive Approach (Finding pattern)[b]
21 (Table)	*Proportional*	*Additive Approach (Finding a pattern)*[c]	*Not Sure*	*Key Understanding 3, Additive Approach (Finding a pattern)*[c]
22 (Table)	*Proportional*	*Additive Approach (Finding a pattern)*[d]	*Not Sure*	*Key Understanding 3, Additive Approach (Finding a pattern)*[d]

Note: Italics indicates incorrect answers
[a] She considered the constant distance (2 km) between the positions of Nehir and Sevil.
[b] She found an additive pattern within *x* values (increasing by twos) and within *y* (increasing by threes) values.
[c] She found an additive pattern within *x* values (increasing by twos) and within *y* (increasing by fours) values.
[d] She found an additive pattern between the multipliers (increasing one by one) of *x* values that were multiplied to obtain *y* values for each row.

Gaye justified rationales of her classifications more explicitly in the pre-interview while some of her explanations (question 11, 12, 18 and 20) were not clear in the pretest. Moreover, while she explained her classifications by using only one of the *key understandings* (*key understanding 1*) in the pretest, she used three different types of *key understandings* (*key understandings 1, 3* and *4*) in the pre-interview. For example, for question 12, which asked "the relationship between the number of movie tickets purchased and the total cost of the tickets if each ticket costs 12 TL", she expressed the presence of proportional relationship in the pre-interview as follows:

> It is proportional because one ticket is 12 TL and 5 tickets are 12 x 5=60 TL. How many tickets you bought does not make any difference, in all cases, the same thing is done. That is, in order to find the amount of the money you will pay, you should multiply the number of tickets by 12TL.

Although she accurately classified the relationship as proportional, she could not write her rationale in the pretest. However, as seen in the above quote, she justified her rationale by using *key understanding 1* in the pre-interview.

Data indicated that there were some inconsistencies between the pretest and pre-interview results. To illustrate, she inaccurately classified the relationship between variables in question 11 as proportional in the pretest. However, in the pre-interview, she accurately classified the same relationship as nonproportional and explained as:

> Let's calculate. If the trip is 5 km, the cost will be 11,7 TL. If the trip is 2 km, the cost will be 6 TL. The same answer must be obtained from cross-multiplication, but when I cross multiply … er ... When 2 is multiplied by 11,7 and the product is divided by 2, the answer will be 4,68 TL not 6 TL, so I want to change my answer because they are not proportional.

As seen in the above quote, she explained the absence of proportional relationship by using *cross-multiplication* instead of using any *key understandings*.

Gaye sometimes used inaccurate statements based on additive approach to decide proportionality instead of *key understandings*. For instance, in the pretest, for question 13, which asked “the relationship between Sevil and Nehir’s positions on a marathon course if they run at the same pace, but Sevil ran 2 kilometers before Nehir started”, she wrote, “proportional” for the relationship between variables by using an additive approach in which she considered the constant distance (2 km) between Sevil and Nehir. However, in the pre-interview, she realized her mistake when the researcher asked her to elaborate her answer to the question and she said:

> For a proportional relationship, the ratios of the distances they taken at different times have to be the same. For example, if Sevil run 40 km, Nehir will run 38 km, the ratio will be 40/38. Similarly, if Nehir run 16 km, Sevil will run 18 km, the ratio will be 16/18. But they are not equal, so they cannot be proportional. But, how does this happen! The distance between them remains unchanged, so they must be proportional …er... I am confused which one is true? I am not sure.

Gaye explained that question 13 (presented in language) was proportional since the rate pairs were equivalent as seen in the above quote. While she did not utilize *key understanding 3* to justify any classification in the pretest, she used the *key understanding* to justify why the relationships presented in language (question 13), equation (question 17) and table (questions 20, 21 and 22) were proportional in the pre-interview. However, especially for the table representation, she did not exactly trust the *key understanding 3* to determine proportionality and tried to find some additive patterns between variables in the tables. When she found a pattern and was clear about *key understanding 3* (the rate pairs were equivalent) at the same time, she concluded that the relationship was proportional. On the other hand, when she found a pattern, but the rate pairs were not equivalent, she was not sure whether the relationship was proportional or not. For example, for question 22, she stated in the pre-interview:

> I compare 4/8 and 6/18; their simplest forms are 1/2 and 1/3. They are not equal, so they are not proportional. But there is a ratio like that: 8

> is 2 times 4; 18 is 3 times 6; 32 is 4 times 32. When we look at the multipliers, there is a proportion as 2 times, 3 times and 4 times; that is to say, they increase one to one, but when we look at the results (showing x and y), there is not any proportion. I do not exactly know which one is important.

She supposed that the relationship between *x* and *y* was proportional because of the additive pattern she found between the multipliers of *x* values to obtain *y* values in each row. As can be understood from her explanation, she assumed that the additive patterns between variables in a table representation could be evidence of a proportional relationship.

In addition, Gaye sometimes utilized the constant slope to explain why the relationship between variables was proportional. In other words, she supposed that a constant slope guaranteed proportionality. For example, in both pretest and pre-interview, she used the constant slope to justify why the relationships given in graph (questions 14 and 15) were proportional. In the pre-interview, her misconception about this argument was revealed more explicitly. For instance, she stated:

> The variables in questions 14 and 15 are surely proportional because they are linear graphs, and so, the graphs have constant slopes. We can say that if there is a constant slope, there is a proportional relationship, too. Question 14 is the graph of y=kx and its slope is k. It (k) never change… In question 15, the graph does not pass through the origin, but it does not matter since it has a slope and its equation is y=kx + c where the slope is k. Since the slope is k, the constant of proportionality is also k.

From Gaye's assertion in the quote above, it might be concluded that she did not know that proportional relationships were shown graphically by a line through the origin. Furthermore, although she understood the relationship between the slope and constant of proportionality in the equation $y=kx$, she supposed that a constant slope guaranteed proportionality. Additionally, she assumed that in the equation, $y=kx + c$, "*k*" was both slope and constant of proportionality though "*k*" was only the slope. In addition, it seems that she did not know the *key understanding 4*. To illustrate, she determined that the relationship between variables in question 15, which had an equation as $y=kx + c$, was proportional. Furthermore, she put forward that if the variables had a linear graph, they were proportional. That is other evidence supporting the claim about Gaye's linearity misconception. Correspondingly, for question 16, which asked the relationship between variables in a parabola graph, she accurately said that the relationship between variables was nonproportional, yet she explained her answer with lack of a constant slope.

4.1.3.2 Mine's Findings about Understanding the Mathematical Relationships Embedded in Proportional Situations

Defining Proportionality Concepts

Mine had difficulties in defining the ratio and proportion concepts. To illustrate, in both student teaching and pre-interview, Mine could not accurately defined the term ratio. In the first student teaching, she defined ratio as, "Ratio is the numbers which we can write as a/b; in other words, it is the division of a by b." As can be seen, she thought that ratio was the same thing with "division". Similarly, in the pre-interview, she defined ratio as, "If we get two numbers and write them as a/b, the fraction will be a ratio." She called the ratio "fraction" although they were different terms.

Data indicated that she did not exactly know the definition of proportion. For example, in the pre-interview, she struggled to define the proportion concept. Firstly, she stated that $a/b=c/d$ was a proportion, then she changed her opinion, and she argued that the algebraic expression of "$a=2k$" was a proportion since as "k" increased, "a" increased, too. She was not only unsuccessful in providing a valid definition of proportion, but she also provided an inaccurate statement in which increasing or decreasing in the same ratio was neglected. On the other hand, she defined the term proportion in the first student teaching as follows:

> A proportion is an equation stating that two or more ratios are equal. I mean, they are multiples of each other. In other words, when we compare two quantities, each of the quantities can be expressed by a multiplier of the other quantity.

Unlike the pre-interview, Mine accurately defined the term proportion in the first student teaching. Moreover, she clarified her definition by using proportional relationships' multiplicative nature (*key understanding 1*).

Mine had difficulty in explaining the differences between the terms ratio and proportion although she knew that they were different terms. To illustrate, in the pre-interview, she stated:

> Researcher: Are the ratio and proportion the same terms?
> Mine: No, they are not.
> Researcher: What is the difference?
> Mine: In the ratio, I only indicate a specific thing. However, there is not any comparison, but in the proportion, I compare two things.

Mine seems to have believed that a ratio did not compare any quantities though a ratio was a multiplicative comparison of two quantities.

Mine sometimes had difficulty in defining direct and inverse proportions because she did not consider the constant ratio that defined the relationship between the variables in direct and inverse proportions. To demonstrate, she defined the terms in the pre-interview as follows:

> If one variable increases, the other one also increases and if one variable decreases, the other one also decreases; these variables are directly proportional. However, if one variable increases, the other one decreases and if one variable decreases, the other one increases, these are inversely proportional.

Mine only pointed out that the direction of change of the related quantities had to be the same, but she did not emphasize that the change had to be in the same ratio.

Moreover, she did not utilize any *key understanding* to explain the direct and inverse proportions. Yet, in the first student teaching, she defined the terms by utilizing proportional relationships' multiplicative nature (*key understanding 1*) and emphasized that proportional relationships increased or decreased in the same ratio.

Mine struggled to define the constant of proportionality. For instance, in the pre-interview, when the researcher asked her to define the constant of proportionality, she did not clear which ratios (*within* or *between-ratio*) should be equal to the constant of proportionality. In other words, she probably did not know that the constant of proportionality was equal to *within-ratio* since it was a constant multiplicative relationship existed between two quantities in a proportional situation. Moreover, in the student teaching, while she was teaching the constant of proportionality, she did not emphasize that it could be expressed in two ways (e.g., 1/k and k). Furthermore, she did not use *key understanding 4* to define the constant of proportionality in both pre-interview and the first student teaching. In addition, in the pre-interview, when the researcher asked her in which pretest questions she used the constant of proportionality, she argued that she used it only question 20 and accurately identified the constant of proportionality for the question. However, she could not determine the constants of proportionality in the other questions although she used.

Determining Proportional Relationships

Mine could not adequately explain the presence of proportional relationships by using the *key understandings*. For example, in the pre-interview, she explained how she discriminated proportional quantities from non-proportional quantities as, "If one quantity increases, the other one also increases, and if one quantity decreases, the other one also decreases; these quantities will be proportional. That is to say, if they change correspondingly, we can call them proportional." It seems that she did not utilize any *key understanding* to determine proportional relationships. Further, she did not consider the *key understanding 1*. Conversely, in the first student teaching, she emphasized that for the quantities to be proportional; the change in the quantities had to be in the same ratio. For instance, she stated:

> Mine: In the table on the board, there is information about the number of pages that a printer printed in different minutes (she drew the following table on the board).
>
> | Minutes | 1 | 2 | 3 | 4 |
> |---|---|---|---|---|
> | Pages | 4 | 8 | 12 | 16 |
>
> Mine: In here, can you recognize proportional things? Or is there anything proportional to another thing?
> Student: Time and the number of pages are proportional.
> Mine: Yes, it is correct. As the minutes multiplied by two, the number of pages multiplied by two, (showing the first and second column). Similarly, as the minutes multiplied by three, the number of pages multiplied by three (showing the first and third column). Therefore, they are proportional.

As seen in the above excerpt, Mine explained the presence of proportionality by using the multiplicative nature of proportional relationships (*key understanding 1*). However, she did not utilize any other *key understanding*.

Similarly, she could not adequately use the *key understandings* to justify her classifications about proportional relationships. In fact, she sometimes did not use any *key understanding* to determine proportional relationships. Further, she sometimes used inaccurate statements to decide proportionality. The justifications of Mine about her classifications in PRET questions 11-22 in the pretest and pre-interview were evidence for the claims. In order to get in-depth information, in the pre-interview, the researcher asked Mine to explain how she decided the relationships between the quantities in PRET questions 11-22. Table 4.2 shows the classifications and the explanations of Mine for each question in the pretest and pre-interview. Although most of the results were similar, there were some different answers and detailed explanations in the pre-interview.

Table 4.2 Classifications and explanations about proportional relationships in PRET questions 11-22

	Pretest		**Pre-interview**	
Question	**Classification**	**Explanation**	**Classification**	**Explanation**
11 (Language)	*Proportional*	*As one variable increases, the other one also increases*	Not Proportional	Key Understanding 1
12 (Language)	Proportional	As one variable increases, the other one also increases	Proportional	Key Understanding 3
13 (Language)	Not Proportional	Cross-multiplication	Not Proportional	Key Understanding 1
14 (Graph)	Proportional	Key Understanding 3	Proportional	Key Understanding 4 Key Understanding 1
15 (Graph)	Not Proportional	Not Clear	Not Proportional	Key Understanding 2 Key Understanding 1
16 (Graph)	Not Proportional	Not Clear	*Proportional*	*Key Understanding 1*
17 (Equation)	Not Proportional	Key Understanding 3	Not Proportional	Key Understanding 1
18 (Equation)	Not Proportional	Key Understanding 3	*Proportional*	*Key Understanding 1*
19 (Equation)	Proportional	Key Understanding 3	Proportional	Key Understanding 3
20 (Table)	Proportional	Key Understanding 3	Proportional	Key Understanding 3 Key Understanding 1
21 (Table)	Not Proportional	Key Understanding 1 Key Understanding 3	Not Proportional	Key Understanding 1 Key Understanding 3
22 (Table)	Not Proportional	0/0 is undefined Key Understanding 3	Not Proportional	Key Understanding 3

Note. Italics indicates incorrect answers

Mine justified rationales of her classifications more explicitly in the pre-interview while some of her explanations were not clear and meaningful in the pretest. Furthermore, while she explained her classifications by using only two of the *key understandings* (*key understanding 1* and *3*) in the pretest, she explained her classifications by using all types of the *key understandings* in the pre-interview. To demonstrate, she explained the absence of proportional relationship in question 13 with the aid of *cross-multiplication* without giving any meaningful explanation in the pretest. On the other hand, she explained her rationale by using *key understanding 1* in the pre-interview. Another evidence for the claim was her explanation for question 15 (presented in graph) in the pre-interview, she stated:

> Question 15 is a graph of the equation, y=mx+n because the graph does not pass through the origin, and so, they are not proportional. Additionally, in here, when I increase the values of x, the values of y will not increase by the same ratio because of n. Therefore, I cannot say that x and y increase by the same multiplier, and so, they are not proportional.

Mine used *key understanding 2* and *key understanding 1* to justify her rationale in the pre-interview although she could not make a clear explanation in the pretest. Besides these arguments, she appears to have known that in the functions of the form $y=mx+n$, x was not proportional to y. For example, in question 14 (a graph of a proportional relationship), she explained her rationale by saying:

> Mine: It is a graph of the equation, $y=mx$. That is to say, as x increases, y increases, too. Thus, it is proportional.
> Researcher: Why?
> Mine: Because if you increase the x values by a multiplier, the y values will increase by the same multiplier. In the equation, *"m"* is already a constant.
> Researcher: What is "*m*"?
> Mine: The slope of the graph.

As seen in the above excerpt, Mine made use of proportional relationships' multiplicative nature (*key understanding 1*). Moreover, she mentioned that a proportional relationship could be represented symbolically as $y = mx$, where m was the slope, but she did not mention that m also the unit rate and the constant of proportionality. Therefore, it can be concluded that she inadequately utilized *key understanding 4*.

Data indicated that there were some inconsistencies between the pretest and pre-interview results. For example, she accurately classified the relationships between variables in question 16 and 18, both of which were quadratic relationships, as nonproportional in the pretest. However, in the pre-interview, she inaccurately classified the same relationships as proportional and explained her rationale for question 16 in the pre-interview as follows:

> The equation of the graph is $y=ax^2$. In the pretest, I said that x and y were not proportional, but I want to change my decision because as x increases, y increases and as x decreases, y decreases, too. That is to say, there is a multiplier, 'a'. Moreover, there is not any number that is added to any side of the equation, so they are proportional.

Mine tried to use *key understanding 1* to explain her statement that the quadratic relationship was proportional. However, she could not recognize that the variables in the quadratic relationships did not increase or decrease in the same ratio. In fact, in the situation, as x increased by a number, y increased by the square of the number, so they were not proportional. Moreover, it seems that she did not know that proportional relationships were shown by a line through the origin because although the graph in the question was a parabola, she classified it as proportional. In a similar way, she changed her classification for the variables in question 18, which was an equation ($y=3x^2$), and determined that they were proportional. It can be concluded that she did not exactly understand *key understanding 4*.

Mine sometimes used inaccurate statements to decide proportionality instead of *key understandings*. For example, in the pretest for questions 11 and 12, she only mentioned that the direction of change of the related quantities had to be the same, but she did not emphasize that the change between the quantities had to be in the same ratio. Moreover, she did not utilize any *key understanding* to explain the relationships in these questions. Yet, in the pre-interview, she used *key understanding 1* and *3* to justify her rationales in the questions. Furthermore, although she inaccurately classified the relationship between variables in question 11 as proportional in the pretest, she realized her inaccurate classification in the pre-interview. She supported her decision by saying, "Yes, both of them increase, but there is not a constant factor. I mean, for the variables to be proportional when one variable increases by *n*, the other has to increase by *n*, too." It seems that she made use of *key understanding 1* to explain why the relationships presented in the situation were nonproportional.

Another evidence for the claim that Mine sometimes did not use true statements to decide proportionality was her first explanation for question 22 (a table of a nonproportional relationship) in the pretest. She wrote "not proportional" for the relationship between variables and explained as, "Since *0/0* is undefined, the variables cannot be proportional." As can be seen, Mine assumed that a ratio could not have a zero as its second component. However, a proportional situation contains the ratio of zero to zero (*0/0*). It can be concluded that she had some difficulties in *key understanding 2*.

4.1.3.3 Ela’s Findings about Understanding the Mathematical Relationships Embedded in Proportional Situations

Defining Proportionality Concepts

Ela had difficulties in defining the ratio and proportion concepts. For instance, in the first student teaching, Ela could not define ratio and proportion terms. However, in the pre-interview, she defined the term ratio as, “Ratio is division of quantities. For example, let’s assume, my weight is 60 kgs, my friend’s weight is 70 kgs; and the ratio of the weights is division of 60 by 70.” She argued that ratio was the same thing with “division”, and so, she could not provide a valid definition of ratio. On the other hand, she defined proportion as the equality of the ratios. She could create a valid definition of proportion although she did not use proportional reasoning language in her definition. It can be said that she utilized *key understanding 3* to define proportion. Moreover, she knew that ratio and proportion were different terms and could explain the differences between them.

Ela struggled to define direct and inverse proportions. Indeed, she sometimes did not consider the constant ratio that defined the relationship between the variables in a proportion. Additionally, she mostly did not utilize any *key understanding* to explain the concepts. To illustrate, in both student teaching and pre-interview, Ela defined direct proportion as, “In a direct proportion, as one variable increases, the other one also increases, and as one variable decreases, the other one also decreases.” She only pointed out that the direction of change of the related quantities had to be the same, but she did not mention that the change between the quantities had to be in the same ratio. In fact, she did not emphasize multiplicative relationships (*key understanding 1*). In a similar way, in the pre-interview, she defined inverse proportion as, “As one variable increases, the other one decreases, and vice verse.” She only considered that the changes were in opposite directions. On the contrary, Ela could create a valid definition of inverse proportion in the first student teaching. The definition was, “If one variable increases in the same ratio as the other variable decreases, these variables are inversely proportional." She not only stated that the changes were in opposite directions, but also considered that the changes had to be in the same ratio. That is to say, she utilized *key understanding 1* to define inverse proportion in the student teaching.

Ela could define the constant of proportionality by using some *key understandings*. To demonstrate, in both student teaching and pre-interview, she accurately defined the constant of proportionality. To illustrate, in the first student teaching, she said, "The constant of proportionality is $\frac{x}{y} = \frac{z}{t} = k$. That is to say, all of the ratios are equal to a constant number, k, which we call the constant of proportionality." She made a true definition of the constant of proportionality by using the *key understanding 3*. However, she struggled to make connection between it and slope. To illustrate, in the first student teaching, she drew a graph of a relationship between time and distance of a bicycle travelling from one place to another. She found the constant of proportionality in the situation as seen in the following excerpt:

> Ela: In order to find the constant of proportionality, we should find the ratio of time to distance because in a direct proportion the division of the quantities is equal. Let's find. The bicycle travels 15 kms in an hour (writing on the board: 1/15) and 30 kms in two hours, 45 kms in three hours, 60 kms in four hours, 75 kms in five hours (writing on the board: 2/30=1/5, 3/45=1/5, 4/60=1/5, 5/75=1/5).
> Student: All of them is equal to 1/5.
> Ela: Yes. It is true. When we simplify them, they are all equal to 1/5. This is the constant of proportionality. Do you remember the slope?
> Student: Yes. It is x/y.
> Ela: No. The slope is the division of the change in the y axis to the change in the x axis. It is the same in each point. Let's find the slopes. (writing on the board: 15/1=15, 30/2=15) Find the slopes in the other points.
> Students: All of them are 15.
> Ela: Yes, the slope is 15. What is the relationship between the constant of proportionality and the slope?
> Students: … (No response from the students)

> Ela: 15 x 1/15 = 1, so we can say that the multiplication of the slope and the constant of proportionality is equal to one.

At the beginning of the excerpt, Ela accurately found the constant of proportionality by utilizing the equality of the rate pairs in the proportional situation (*key understanding 3*). Yet, it seems that Ela did not understand the relationship between the slope and the constant of proportionality and could not recognize that the constant of proportionality could be expressed in two ways (e.g., 1/15 and 15 for the situation). Moreover, she stated an inaccurate statement that the multiplication of the slope and the constant of proportionality were equal to one. In addition, she did not make any connection between table, graph and algebraic expression of the proportional situation. It can be concluded that she did not know the *key understanding 4*.

Determining Proportional Relationships

Ela sometimes could not adequately explain the presence of proportional relationships by using the *key understandings*. For example, in the pre-interview, when the researcher asked her how to determine presence of a proportional relationship between two variables, she said:

> The division of the variables has to be equal to each other. To illustrate, let's assume, there are some weights such as 40 kgs/60 kgs and 50/80... In order to determine proportionality, I must compare them. The first one is 4/6 and the second one is 5/8. They are not equal to each other, so they are not proportional.

She mentioned that the ratio pairs in a proportional situation had to be equal (*key understanding 3*) although she called the relationship "division" instead of "ratio". On the other hand, she did not mention multiplicative relationships between the quantities (*key understanding 1*).

In the same way, she could not adequately use the *key understandings* to justify her classifications about proportional relationships. Indeed, she sometimes did not use any *key understanding* to determine proportional relationships. Further, she sometimes used inaccurate statements to decide proportionality. The justifications of Ela about her classifications in PRET questions 11-22 in the pretest and pre-interview were evidence for the claims. In order to get in-depth information, in the pre-interview, the researcher asked Ela to explain how she decided the relationships between the quantities in PRET questions 11-22. Table 4.3 shows the classifications and the explanations of Ela for each question in the pretest and pre-interview. Although all of the classifications were the same in both pretest and pre-interview, there were some different and detailed explanations in the pre-interview.

Table 4.3 Classifications and explanations about proportional relationships in PRET questions 11-22

	Pretest		**Pre-interview**	
Question	**Classification**	**Explanation**	**Classification**	**Explanation**
11 (Language)	Not Proportional	Key Understanding 3	Not Proportional	Key Understanding 3
12 (Language)	Proportional	As one variable increases, the other one also increases Key Understanding 3	Proportional	Key Understanding 3
13 (Language)	Not Proportional	Not Clear	Not Proportional	Not Clear
14 (Graph)	Proportional	Key Understanding 3 As one variable increases, the other one also increases	Proportional	As one variable increases, the other one also increases
15 (Graph)	Not Proportional	Not Clear	Not Proportional	Key Understanding 3
16 (Graph)	Not Proportional	Key Understanding 3	Not Proportional	Key Understanding 3
17 (Equation)	Not Proportional	Key Understanding 3	Not Proportional	Key Understanding 3
18 (Equation)	Not Proportional	Key Understanding 3	Not Proportional	Key Understanding 3
19 (Equation)	Proportional	As one variable increases, the other one also increases Key Understanding 3	Proportional	Key Understanding 3 Key Understanding 4
20 (Table)	Proportional	As one variable increases, the other one also increases Key Understanding 3	Proportional	Key Understanding 3 Additive Approach (Finding a pattern)
21 (Table)	Not Proportional	Key Understanding 3	Not Proportional	Key Understanding 3 Additive Approach (Finding a pattern)
22 (Table)	Not Proportional	Key Understanding 3	Not Proportional	Key Understanding 3 Additive Approach (Finding a pattern)

Her foremost justification to explain whether a relationship was proportional or not was equality of the rate pairs in a proportional situation (*key understanding 3*). Actually, in the pretest, she did not use any other *key understanding.* Similarly, in the pre-interview, she mostly used *key understanding 3*, but she utilized *key understanding 4*, besides *key understanding 3*, to justify her classification in question 19 as seen in the following excerpt:

> Researcher: Why the variables in question 19 were proportional?
> Ela: Because the divisions of x values to y values are equal.
> Researcher: Can you explain your rationale by using any other statement?
> Ela: They are proportional because it is an equation (y=2,5x) that is similar to $y=mx$, which is the equation of a proportional relationship.
> Researcher: What does the meaning of "*m"* in the equation?
> Ela: "*m"* is the constant of proportionality.
> Researcher: Is there any other meaning?
> Ela: No.

At the beginning of the excerpt, Ela made use the *key understanding 3*. Later, she mentioned that a proportional relationship could be represented symbolically as $y = mx$, where m was the constant of proportionality, but she did not mention that m also the unit rate and slope. Thus, it can be concluded that she inadequately utilized *key understanding 4*.

Ela sometimes used inaccurate statements based on additive approach to decide proportionality. For example, in questions 12, 14, 19 and 20, she stated that the relationships of the variables in the questions were proportional because the variables increased or decreased at the same time, but she did not mention that the variables in a proportional situation had to increase or decrease in the same ratio. In a similar way, in the pre-interview, she did not mention that the change between the variables in a proportional situation had to be in the same ratio. Correspondingly, in both pretest and pre-interview, she never told about proportional relationships' multiplicative nature (*key understanding 1*). For example, in order to determine the relationship in question 14, she could generate the equation of the relationship presented in graph (y=x). Moreover, she explained the presence of proportionality by only saying, "x and y are proportional because as y increases or decreases, x also increases or decreases" in the pre-interview. Additionally, in the pre-interview, while Ela was justifying rationale of her classifications the relationships presented in table (questions 20, 21 and 22), she tried to find some additive patterns between quantities in the tables as follows:

> Researcher: Can you explain your rationale for question 20 by using any other statement from equality of ratios?
> Ela: We can look at the amount of increase of the quantities. For example, this column (showing x values: 4, 6, 8, 10, 12) increases in twos and that column (showing y values: 6, 9, 12, 15, 18) increases in threes.
> Researcher: Can you explain your rationale for question 21 by using the same statement?
> Ela: Let's look at the amount of increase of the x values and y values. x values increase in twos (showing 4, 6, 8, 10, 12) and y values (10, 14, 18, 22, 26) increase in fours. But, wait a minute, the divisions of x values by y values are not equal (writing on the sheet: $4/10 \neq 6/14$), so they cannot be proportional.
> Researcher: So what?
> Ela: It means that the amount of increase of the x values and y values does not always help us to decide proportionality.

In question 20, Ela supposed that the relationship between x and y was proportional because of the additive pattern she found within the x values and y values. In other words, she assumed that additive patterns between variables in a table representation could be evidence of a proportional relationship. Yet, in question 21, she recognized that the variables were not proportional although there were additive patterns within the x values and y values. However, she concluded that additive patterns could not always use to decide proportionality. In other words, she still saw additive patterns as one of the indicators of proportionality, but she thought that it could not be used in every situation.

4.2 Pre-service Teachers' Proportional Reasoning after the Practice-based Instructional Module

In this section, pre-service teachers' approaches to different problem types, distinguishing proportional from nonproportional situations, and understanding of mathematical relationships embedded in proportional situations after receiving a practice-based instructional module based on proportional reasoning are presented for each pre-service teacher.

4.2.1 Approaches to Different Problem Types

In order to investigate pre-service teachers' approaches to different problem types, which were missing value, numerical comparison and qualitative reasoning problems after participation in a practice-based instructional module, the solution strategies and processes of pre-service teachers in solving the different problem types were analyzed. Since solution strategies and processes might change with respect to problem types, results of different problem types presents under different subcategories.

4.2.1.1 Gaye's Approaches to Different Problem Types

Missing Value Problems: Solution Strategies and Processes

Gaye could find correct answers of missing value problems in all relevant instruments. Moreover, she used a broader range of strategies which were *factor of change, factor of change strategy* in a ratio table, *building-up, cross-multiplication, unit rate, building-up* in a ratio table, and *constant of proportionality* to solve missing value problems. Furthermore, she mostly used informal strategies that highlighted multiplicative relationships rather than formal strategies.

Gaye used *factor of change* as a leading strategy to solve missing value problems. To illustrate, in the PRET, there were some missing value problems (questions 1-4) which did not have any context. In the posttest, Gaye mostly used *factor of change strategy* to solve these problems. Similarly, in the student teaching, she utilized the strategy. For example, one of the problems was, "In a shiny day, a boy's father's height is 180 cm and his shadow's height is 240 cm. What is the boy's shadow's height if his height is 150 cm?" For the problem, Gaye used an adaptation of *factor of change strategy*, in which she simplified the ratio to identify the integer scale factor easier. Firstly, she simplified the ratio of 180 to 240 by dividing both sides by 6 and found the ratio of 30 to 40. Secondly, she equated 30/40 to 150/? (30/40=150/?), and then, identified the scale factor (i.e., *between-ratio*) as 5. It is important to note that she used an efficient strategy, which facilitated the computations required to find the missing value, for the problem. Further, she explained her solution by using multiplicative relationships and with a proportional reasoning language as follows:

> Gaye: First of all, I will find the ratio of the father's height to his shadow's height. It is 180/240. The ratio is saved as invariable.
> Student: We can cross multiply.
> Gaye: Yes, we can do but I will use a different strategy. Firstly, I divide 180 and 240 by 6. I should divide both denominator and numerator by the same number because I should increase or decrease them by the same ratio. The ratio (30/40) is never changed. If 30 goes to 150, how many times does it increase?
> Students: 5 times
> Gaye: If 30 increase 5 times, 40 should increase 5 times, too. So the answer is 200 cm.

As shown in the above quote, Gaye explained the problem by utilizing covariance of proportional relationships. That is to say, she highlighted that in a proportion ($a/b=c/d$), if a quantity (e.g., a) underwent a multiplicative change, the other quantity (e.g., b) should undergo the same change, so that the value of the ratio was left unchanged. In other words, she made multiplicative comparisons between quantities.

In a similar manner, in the post-interview, Gaye used *factor of change strategy* to solve missing value problems. One of the problems was “If 8 balloons are 12 TL, how much 6 balloons will cost?” In order to solve the problem, firstly, she found the scale factor, 3/2, and then, solved the problem by multiplying 12 TL by 3/2 and accurately found the answer as 9 TL. Likewise, she solved the other missing value problem, which was exemplified by her, by using the same strategy as seen in the following excerpt:

> Gaye: The relationship between bought petrol and given money can be an example for a proportional situation. For instance, if a liter of petrol costs 5 TL, we can found the amount of petrol that can be bought by 100TL.
> Researcher: How would you find?
> Gaye: 100 is 20 times 5, so a liter also has to be multiplied by 20 and the answer is 1x20=20 TL.

She easily recognized the integer scale factor (i.e., *between-ratio*) and solved the problem by using it. Moreover, it can be said that she tended to choose efficient strategies that facilitated the computations to solve proportional problems.

In addition, Gaye sometimes used *factor of change strategy* in a ratio table which was a strategy utilized multiplicative relationships between quantities. For instance, she solved a missing value problem in the PRET (question 23) by using the strategy. That is to say, she multiplied quantities by scale factors to build a ratio table, and then, added the values in each row to find out the missing value. As seen in the figure 4.1; first of all, she multiplied both 4 and 3 by 2; secondly, multiplied both 4 and 3 by 1/2 and finally, added the values to find 14 and determined the missing value as 21/2.

x 2

Length	4	8	2	4+8+2=14
Width	3	6	3/2	3+6+3/2=21/2

x 1/2

Figure 4.1 Solution of Gaye to PRET question 23

Gaye found the missing value by using *factor of change strategy* in a ratio table. This solution revealed that she viewed the ratio as reducible and increasable units, and by this way, she could solve the missing value problem with noninteger *between-ratio* by utilizing multiplicative relationships. Moreover, she explained the meanings of the quantities and used proportional reasoning language in her explanation. For instance, she said "…we can enlarge the photograph provided that the ratio of the length to width of the original photograph stays the same."

Building-up strategy was another strategy she used in some of her solutions. For instance, the second most used strategy by her was this strategy in PRET questions 1-4. Furthermore, she used the strategy in a ratio table in the student teaching. The problem was, "There is a box with 5 blue and 13 red balls, find the number of red balls if there are 15 blue balls in the box." In order to solve the problem, she utilized *building-up strategy* in a ratio table; that is to say, she increased the number of blue balls by fives and red balls by thirteens until 15 blue balls and 39 red balls were reached.

Gaye used *unit rate strategy*, which required determining *how many* or *how much for one* through fairly sharing and grouping, in solving some missing value problems. To illustrate, in the student teaching, she used this strategy to solve the problems that were “Find the number of sugars for one glass if there are 16 sugars for 4 glasses.” and “If there are 10 balls for 2 boys, how many balls will each boy get?” She used concrete materials to solve both of the problems. For the first problem, she brought cube sugars and plastics glasses to the classroom and matched the sugars and glasses to find out the number of sugars for each glass. In other words, she fairly shared the 16 sugars to 4 glasses by putting four cube sugars in each plastics glass. Similarly, for the second problem, she brought the pictures of two boys and ten balls to the classroom and matched the boys and balls to find out the number of balls for each boy. For both of the problems, she emphasized the meanings of the quantities and ratios while she was using concrete materials by asking questions as, “What did you find by dividing 16 by 4?”, “What does the ratio 10:2 mean?”

Another strategy used by Gaye was *constant of proportionality*. To illustrate, in the student teaching, in order to solve a missing value problem, which had a context involving similar rectangles, she drew three similar rectangles on the board. The first rectangle had a length of 6 centimeters and a width of 2 centimeters, the second one had a length of 9 centimeters and a width of 3 centimeters, and the last one had a width of 6 centimeters and the length of the rectangle was asked. She found the *constant of proportionality* in the proportional situation and used it to find the missing value. By this way, she utilized the functional relationships between proportional variables (i.e., $x/y=k$). Moreover, while she was teaching, she stressed the meanings of the quantities by asking to students these questions, “What does the ratio of the first rectangle’s width to length mean?”, “Why are the ratios of the first and second rectangle’s width to length equal?” and “What should be the ratio of the third rectangle’s width to length if all the rectangles are similar?”.

Date revealed that Gaye did not see *cross-multiplication* as an indispensable strategy anymore. Moreover, she generally did not prefer to use the strategy. For instance, she used the strategy in solving only one missing value problem in the relevant instruments (PRET question 3). Moreover, her answers to interview questions indicated that she thought *cross-multiplication* was a memorized rule lacking meaning as follows:

> Researcher: Which strategy will you teach to your students as the most efficient strategy?
> Gaye: Indeed, it is difficult to tell that a strategy is the most efficient because it differs according to problem. However, building-up and factor of change strategies are good and effective…
> Researcher: What will you do if your students cannot understand these strategies?
> Gaye: I will teach by using equivalent fractions or unit rate strategies. But, in general, I do not prefer to use the unit rate strategy because the result might be a decimal number and it might confuse students' minds and complicate the operations. If I use all these strategies and students still do not understand, I will use cross-multiplication. Actually, I will teach cross-multiplication after these strategies because it enables to solve quickly in exams.
> Researcher: Does cross-multiplication provide understanding?
> Gaye: Never. It is only a memorized rule. There is not any operation that shows the ratios between quantities or their changing …well… increasing or decreasing at a constant rate. The operations in cross-multiply have not any meaning for students.

As seen in the above excerpt, although she said that she would teach *cross-multiplication* because of its quickness, she was aware that it consisted of memorized procedures lacking meaning. Additionally, she was aware that the most efficient strategy would be changed with regard to problem. Nevertheless, according to her, *building-up* and *factor of change* were effective strategies. Moreover, she thought that *unit rate strategy* would be difficult if there was a noninteger result for the unit. It might be concluded that she did not have flexibility in unitizing and reunitizing quantities. In other words, she had difficulties in using composite units when the quantities suggest that it was more convenient than using singleton units.

Gaye paid attention to the use of different strategies highlighting multiplicative relationships. For instance, in the second lesson plan, Gaye added some problems which allowed her to use different strategies rather than *cross-multiplication* without any guiding of the researcher. In her revision report of the lesson plan, she explained the reasons why she added the problems as follows:

> I added sugar problem and ball problem because they could be solved by a different strategy: grouping. In other words, in the first time, I gave rules and solved the problems with these rules, but in the second time, I solved the two problems by grouping without any rule.

It can be concluded that Gaye was aware that she solved the problems in the first student teaching by applying memorized rules and arithmetic procedures without giving any meaningful explanation. Moreover, she added one more missing value problem which could be solved by different strategies. The problem was "There is a box with 5 blue and 13 red balls, find the number of red balls if there are 15 blue balls in the box." In her revision report of the lesson plan, she explained the reasons why she added the problem:

> I deliberately asked the problem in my second student teaching. Firstly, I solved it by using unit rate strategy because I wanted to show that it was difficult to solve the problem by using unit rate because of noninteger numbers. And then, I solved the problem by using a ratio table as a more effective strategy.

She solved the problem in more than one way by using *unit rate* and *building-up strategies*. Her first strategy was *unit rate*. First of all, she found the number of red balls for one blue ball as 2,6 by fairly sharing. After that, she concluded that the ratio of blue balls to red balls was 1:2,6 and so, the number of red balls was 39 (2,6x15=39) if there are 15 blue balls in the box. Then, she mentioned the difficulty of noninteger numbers to calculate. However, she did not highlight that composite units could be used when the quantities suggest that it was more convenient than using singleton units. It can be another evidence for the claim that she did not have flexibility in unitizing and reunitizing quantities.

Data indicated that while Gaye recognized *between-ratio* in a proportion, she had difficulty in recognizing *within-ratio*. For instance, in the first question of PRET, the *within-ratio* was integer, and so, it was a more efficient strategy than *between-ratio* for the problem. But, Gaye could not recognize integer *within-ratio*, and used *factor of change*, which was a *between-ratio strategy,* to solve the problem. However, in the second question, she recognized the efficient strategy that was *between-ratio* and utilized integer *between-ratio* in order to solve the problem. Additionally, in the student teaching, she asked a missing value problem which did not have any context (i.e., 6/?=1/3) and solved it by using *between-ratio*. That is to say, she identified 6 as the scale factor and multiplied 3 by 6 and found 18. Although she highlighted multiplicative relationships, she used only the *between-ratio* and not mentioned the *within-ratio* which was also an integer number, 3. It can be additional evidence for the claim that while she recognized *between-ratios* in a proportion, she had difficulty in recognizing *within-ratios*.

Numerical Comparison Problems: Solution Strategies and Processes

Gaye used three different strategies, which were *building-up*, *fraction strategy*, and *converting decimal expressions* to solve numerical comparison problems. Her foremost strategies were *building-up* and *fraction strategies*. To illustrate, in her first solution of PRET question 6, which asked to compare two different mixtures of orange juice concentrates and water, she found the amount of water of each mixture for the same amount of orange juice concentrates by using *building-up strategy*. First of all, for mixture A, she increased the amount of orange juice concentrate by twos and water by threes until 6 cups of orange juice concentrate and 9 cups of water were reached. Then, for mixture B, she increased the amount of orange juice concentrate by threes and water by fours until 6 cups of orange juice concentrate and 8 cups of water were reached. And finally, she compared the amount of water and concluded that since mixture B had less water, it had stronger orange taste than mixture A. In brief, she could accurately determine the mixture with a stronger orange taste and clearly explained the quantities that were calculated.

Another strategy utilized by her was *fraction strategy*, in which the rates were treated as fractions and a common denominator or numerator was found to make comparisons. In her second solution of PRET question 6, she used the strategy by finding the amount of water in each mixture for the same amount of orange juice concentrates. She wrote the ratios of orange juice concentrates to water for each mixture as fractions and found a common numerator. To do this, at first, she multiplied 2/3 by 3/3 to produce 6/9 and then she multiplied 3/4 by 2/2 to produce 6/8 which had the same numerator with the first ratio (6/9). Finally, she compared the amount of water for the same amount of orange juice concentrates and concluded that since mixture B had less water, it had a stronger orange taste than mixture A. She could accurately determine the mixture with a stronger orange taste and explained the quantities that were calculated clearly. Similar to the pretest results, in her both solutions, Gaye preferred to use *part-to-part ratios* (the ratios of orange juice concentrates to water) instead of *part-to-whole ratios*. It might be asserted that Gaye still had difficulties in realizing *part-to-whole ratios*.

Correspondingly, interview data revealed that Gaye thought *building-up* and *fraction strategies* were the most efficient strategies in order to solve numerical comparison problems. She explained her idea as, "For example, in a mixture problem, finding a common amount for one of the variables in the mixture facilitates to compare them. The common amount may be found easily by *building-up* or finding common denominators (she meant *fraction strategy*)." According to Gaye, the strategies were effective since they facilitated comparing the ratios by holding one of the variables constant in the proportional situation.

Another strategy used by Gaye to solve numerical problems was *converting decimal expressions*. She used the strategy to solve a numerical comparison problem in the

PRET (question 24), which presented three rectangular clothes with dimensions. The problem asked pre-service teachers to determine the cloth that was "the most square". In her solution, firstly, Gaye determined the ratios of width to length of the three rectangular clothes. Secondly, she converted them into decimal expressions (23/35=0.657, 139/155=0.864, 56/75=0.746) and compared them. Finally, she accurately determined that the rectangular cloth with dimensions 139 and 155 was "the most square" since the ratio of its width to length was closest to 1.

In the student teaching, Gaye did not ask any numerical comparison problem to her students. Moreover, in the post-interview, she did not mention any situation that could be a numerical comparison problem while she was giving examples of proportional situations in real life.

Qualitative Reasoning Problems: Solution Strategies and Processes

Gaye solved qualitative reasoning problems by giving numerical examples. It seems that she made multiplicative comparisons between quantities, but she could not interpret the qualitative relationship that existed between two quantities without giving numerical examples. For instance, she solved both qualitative reasoning problems in the PRET by giving numerical examples; in other words, she converted the qualitative reasoning problems to numerical comparison problems. For instance, for question 7, she explained her answer as follows:

> For example, let's assume Esra runs 2 laps in 5 minutes and Gonca runs 1 lap in 10 minutes. If we find the number of laps they run in the same amount of time; it will be easy to solve the problem. For instance, in 10 minutes, Esra runs 4 laps and Gonca runs 1 lap. As a result, Esra runs more laps than Gonca in the same amount of time, so she runs at a higher speed than Gonca.

Although she made a multiplicative comparison to solve the qualitative problem, she could not make qualitative comparisons that did not depend on numerical values because she quantitatively compared the number of laps in the same amount of time.

In the student teaching, she did not ask any qualitative reasoning problem to her students. Moreover, in the post-interview, she did not mention any situation that could be a qualitative reasoning problem while she was giving examples of proportional situations in real life.

4.2.1.2 Mine's Approaches to Different Problem Types

Missing Value Problems: Solution Strategies and Processes

Mine could find correct answers of missing value problems in all relevant instruments. Moreover, she used a broader range of strategies which were *within-ratio, factor of change, equivalent fractions, factor of change strategy* in a ratio table, *building-up, unit rate,* and *constant of proportionality* to solve missing value problems. Furthermore, she used informal strategies that highlighted multiplicative relationships and avoided applying memorized rules such as *cross-multiplication*.

Mine used *factor of change* as a leading strategy to solve missing value problems. To illustrate, in the PRET, there were some missing value problems (questions 1-4) which did not have any context. In the posttest, Mine mostly used *factor of change strategy* to solve these problems. Similarly, she solved another missing value problem of PRET (question 23) by using the same strategy. In order to solve the problem, she could accurately identify 7/2 as the scale factor and multiplied 7/2 by 3 and found 21/2. She not only accurately found the answer, but she also explained why she multiplied 3 by 7/2 by using a proportional reasoning language. Additionally, in the student teaching, she utilized the strategy predominantly. For instance, she solved one of the problems with *factor of change* though the students persisted in solving the problem with *cross-multiplication* as seen in the following excerpt:

> Mine: Try to solve with a different strategy from cross-multiplication. For example, you can use factor of change.
> Student: But there is no factor.
> Mine: Be careful! There is a factor that is 3/2. 180 is equal to 3/2 multiplied by 120, and so, we should multiply 2 by 3/2, too. So, the answer is 3.
> Student: Why do we solve like that? Cross-multiplication is easier.
> Mine: When we solve by using cross-multiplication, there are lots of operations, and so, the possibility of making mistakes increases. Moreover, we cannot see which quantities are proportional or how many the quantities increase or decrease in the cross-multiplication.

Although students insisted on using cross-multiplication, especially when multiplies were noninteger, Mine discouraged them from applying rules and operations blindly and encouraged them to use strategies highlighting proportional relationships. In addition, she used proportional reasoning language in her speech.

In a similar manner, Mine solved most of the missing value problems in the post-interview by using *factor of change strategy*. For instance, one of the problems was "If 8 balloons are 12 TL, how much 6 balloons will cost?" She used the strategy to solve the problem. That is to say, first of all, she found the scale factor, 3/2, secondly, she multiplied 12 TL by 3/2, and then, she accurately found the answer as 9 TL.

Mine sometimes used *factor of change strategy* in a ratio table which was a strategy utilized multiplicative relationships between quantities. The strategy required viewing ratio as reducible and increasable units and by this way, one could solve the missing value problems that had noninteger *between-ratios*. For instance, she solved a missing value problem in the PRET (question 3) by using the strategy. The problem asked the missing value in the proportion, 3/8=x/20. She multiplied quantities by scale factors to build a ratio table, and then, added the values to find out the missing value as seen in the Figure 4.2.

x 2

3	6	3/2	6+3/2=15/2
8	16	4	16+4=20

x 1/2

Figure 4.2 Solution of Mine to PRET question 3

Another problem that she solved by using *factor of change strategy* in a ratio table, was in the student teaching. The problem was "If a printing house prints 60 books in 2 hours, how many books can it print in 80 minutes?" Mine built a ratio table to solve the problem as shown in the following excerpt:

> Student: I solved a bit differently, but I do not know whether it is true or not. 60 books in 120 minutes, it falls to half, and so, it should be 40 books in 80 minutes.
> Mine: (looking at the solution silently for a while) It is a good idea. In fact, I would not think of such a method.
> Student: Is my solution correct?
> Mine: Wait a minute. First of all, let me solve it (she drew the following table on the board and filled it).

Minutes	120	40	80
Books	60	20	40

> Mine: 60 books are printed in 120 minutes. Firstly, I try to find 80 minutes in the first row of the table. When I divide 120 by 3, the answer is 40; and then, I should divide 60 by 3, too; the answer is 20. I am seeking 80 minutes, so I should multiply 40 by 2, and so, I should multiply 20 by 2, too. Finally, the answer is 40. You can solve like that. As you see, we did not use cross-multiplication again.
> Student: Is my solution also correct?
> Mine: What did you do? …er… You said the half of 120 is 60, and so, the half of 80 is 40. Yes, your solution is also correct. In fact, it is an easier way.

Mine successfully found the missing value in the question by using *factor of change strategy* in a ratio table, which was a *between-ratio strategy,* highlighting multiplicative relationships. In the beginning, she did not utilize *within-ratio strategy* though it was a more effective strategy for the problem. But then, she realized the integer *within-ratio* in the proportional situation. Similarly, in the posttest, she realized and used both *within* and *between-ratios*. For example, she used *within-ratio* in the first question of PRET since there was an integer *within-ratio* and used *between-ratios* in the second and forth questions since there were integer *between-ratios*. Moreover, it can be said that she utilized efficient strategies that facilitated the computations to solve proportional problems.

In order to solve some missing value problems, Mine used *the constant of proportionality*. For instance, in the post-interview, she used it to solve two missing value problems which had inversely proportional relationships. One of the problems was "If a builder paints a wall in 3 days, how many days it takes to paint the same wall with two builders?" While she was solving the problem, she stayed, "we could find the days by solving the equation (writing on the sheet: dx2=3) since the multiplication of them should be equal to the constant of proportionality." She utilized the functional relationships between proportional variables (e.g., $x.y=k$). In addition, she used the strategy in the student teaching to solve a missing value problem with directly proportional relationship.

Equivalent fractions strategy was another strategy utilized by her to solve missing value problems. Within this strategy, the rates are treated as fractions and the multiplication rule for obtaining equivalent fractions is applied. To illustrate, she used the strategy to solve PRET question 2, which asked the missing value in the proportion, 2/7=6/x. She multiplied the given fraction, 2/7, by 3/3, which is equal to one, and then, found the missing value (21).

Another strategy Mine used was *unit rate*. She utilized it to solve a problem in the student teaching. The problem was "If an F-16 plane can travel 4800 meters in 12 seconds, how many meters will the plane travel in 4 and 8 seconds?" First of all, she stated that she would solve the problem by using *unit rate strategy,* and then, she found the number of meters that the plane travel in one second as 400, and then, multiplied it by 4 and 8. As can be seen, she used a singleton unit (one second) though a composite unit (4 seconds) would be more efficient in the context. Conversely, in the post-interview, she said that she might use composite units when it was more convenient than using singleton units as seen in the following excerpt:

> Researcher: How does unit rate strategy? Can you explain it?
> Mine: For example, let's assume the problem is, if 4 cookies are 20 TL, how much will I pay for 7 cookies? I can solve the problem first, by finding the paid money for 1 cookie, and then, multiplying it by 7.
> Researcher: What is unit?
> Mine: One cookie.
> Researcher: Does the unit have to be one all the time?
> Mine: No. it does not. The unit might be any number, for instance, 4.
> Researcher: How do you decide which number should be the unit?
> Mine: I can choose any number that suits my purpose. I mean, I can choose a number that eases my calculations.

Although she said that unit might be either one or a composite number, she did not use any composite number as a unit in student teaching. It might be concluded that she did not have flexibility in unitizing and reunitizing quantities.

Date revealed that Mine did not see *cross-multiplication* as an indispensable strategy anymore. Moreover, she generally did not prefer to use the strategy. For instance, she did not use the strategy in solving any problem in the relevant instruments. Correspondingly, in the post-interview, when the researcher asked her why she did not prefer to use *cross-multiplication strategy,* she stated:

> Because cross-multiplication is only about performing calculations such as multiplying and dividing. It seems like a memorized rule. There is no rationale behind it. I mean, there is nothing showing the ratios between variables. In the past, it was the only strategy that I used. But now, I know the other strategies, so there is not a valid reason to use it.

She thought that the strategy included applying memorized procedures without logic. Further, she stated that she would no longer use the strategy because she knew the other strategies. Additionally, according to Mine, *factor of change*, *building a ratio table* and *unit rate* were the leading strategies which she would prefer to solve missing value problems. Moreover, she expressed that the *factor of change* and *building a ratio table* were effective strategies since they facilitated realization of multiplicative relationships between variables.

Mine paid attention to the use of different strategies highlighting multiplicative relationships. For instance, in the second lesson plan, Mine added some problems which allowed her to use different strategies without any guiding of the researcher. In her revision report of the lesson plan, she explained the reasons why she added the problems as, “I added some problems to the second lesson plan because I wanted to solve more problems that could be solved with different solution strategies and, therefore, tried to improve students’ point of view.” Mine seems to have recognized the importance of using different strategies to solve the proportional problems. To illustrate, while she used *factor of change strategy* to solve a problem in the first student teaching, she used *building-up strategy* to solve the same problem in the second student teaching. In the revision report, she stated that she had used *building-up strategy* because she had wanted to teach a different strategy.

Numerical Comparison Problems: Solution Strategies and Processes

Mine used three different strategies, which were *fraction strategy, building-up*, and *converting decimal expressions* to solve numerical comparison problems. Her foremost strategy was *fraction strategy,* in which the rates were treated as fractions and a common denominator or numerator was found to make comparisons. To illustrate, in her first solution of PRET question 6, which asked to compare two mixtures of orange juice concentrate and water, in order to scale up the amount of the mixtures to a common amount, she utilized the strategy. First of all, she wrote the ratios of orange juice concentrates to total mixtures as fractions and found a common denominator. Afterwards, she compared the ratios of orange juice concentrates to total mixtures and concluded:

> Mixture B had a stronger orange taste than mixture A because mixture B’s ratio of orange juice concentrate to total mixture is 15/35 while the other one’s ratio is 14/35. The bigger the ratio is the more the orange taste gets.

As seen in the above quote, she could accurately determine the mixture with a stronger orange taste. Moreover, she clearly explained the quantities that were calculated and justified rationale of her solution. Additionally, she used proportional reasoning language in her explanation. Correspondingly, interview data revealed that Mine asserted *fraction strategy* was the most effective strategy to solve numerical comparison problems since it facilitated comparing the ratios by holding one of the variables constant.

Building-up strategy was another strategy utilized by her. In her second solution of PRET question 6, she used the strategy by finding the amount of water of each mixture for the same amount of orange juice concentrates. Firstly, for mixture A, she increased the amount of orange juice concentrate by twos and water by threes until 8 cups of orange juice concentrate and 12 cups of water were reached. Secondly, for mixture B, she increased the amount of orange juice concentrate by threes and water by fours until 9 cups of orange juice concentrate and 12 cups of water were reached. Finally, she compared the amounts of orange juice concentrates and concluded that since mixture B had more orange juice concentrate, it had a stronger orange taste than mixture A. In addition, she explained as:

> In the mixture A, there are 2 cups of orange juice concentrate corresponding to each 3 cups of water. In that case, if I added 2 cups of orange juice concentrate and 3 cups of water, its taste would not change since its ratio would not change…

She established a ratio and extended it to another ratio by using additive patterns. It is also important to note that while she used *part-to-whole ratios* (comparing orange juice concentrates to the total mixtures) in her first solution, she used *part-to-part ratios* (comparing orange juice concentrates to water) in her second solution. It might be asserted that Mine realized both *part-to-whole ratios* and *part-to-part ratios*.

Another strategy used by Mine to solve numerical problems was *converting decimal expressions*. She used the strategy to solve a numerical comparison problem in the PRET (question 24), which presented three rectangular clothes with dimensions. The problem asked pre-service teachers to determine the cloth that was “the most square”. First of all, Mine calculated the ratios of length to width of the three rectangular clothes. Secondly, she converted them into decimal expressions (35/23=1.52, 155/139=1.11, 75/56=1.33) and compared them. Finally, she accurately determined that the rectangular cloth with dimensions 155 and 139 was “the most square” since the ratio of its length to width was closest to 1.

In the student teaching, Mine did not ask any numerical comparison problem to her students. Moreover, in the post-interview, she did not mention any situation that could be a numerical comparison problem while she was giving examples of proportional situations in real life.

Qualitative Reasoning Problems: Solution Strategies and Processes

Mine sometimes could make qualitative comparisons that did not depend on numerical values. To illustrate, in a qualitative reasoning problem of PRET (question 7), which was “Esra ran more laps than Gonca. Esra ran for less time than Gonca. Who was the faster runner?” she found the correct answer of the problem as Esra was the faster runner. Moreover, she provided a valid explanation by making multiplicative comparisons between quantities.

However, Mine sometimes used an inaccurate additive strategy when she needed to use a multiplicative strategy to solve qualitative reasoning problems. To illustrate, in a qualitative problem of PRET (question 8), she could not find correct answer of the problem because she made an additive comparison between yesterday’s fruit juice and today’s fruit juice instead of a multiplicative comparison as she did in the pretest. Therefore, it can be concluded that she still had difficulty in recognizing multiplicative relationships in a qualitative reasoning problem.

In the second student teaching, she did not ask any qualitative reasoning problem to her students. Moreover, in the post-interview, she did not mention any situation that could be a qualitative reasoning problem while she was giving examples of proportional situations in real life.

4.2.1.3 Ela’s Approaches to Different Problem Types

Missing Value Problems: Solution Strategies and Processes

Ela could find correct answers of missing value problems in all relevant instruments. Moreover, she used a broader range of strategies which were *factor of change, cross-multiplication, factor of change strategy* in a ratio table, *constant of proportionality,* *unit rate* and *building-up* to solve missing value problems. Furthermore, she mostly used informal strategies that highlighted multiplicative relationships.

Ela used *factor of change* as a leading strategy to solve missing value problems. To illustrate, in the PRET, there was a missing value problem (questions 23) which had a context involving similar figures. Ela solved the problem by using *factor of change strategy*. She identified 14/4 as the scale factor and multiplied 14/4 by 3 and found 52/4. In addition, she explained why she multiplied 3 by 14/4 as follows:

> There is a multiplicative relationship. The length of the photograph increases by 14/4, so the width of the photograph has to increase by the same ratio. Thus, we should multiply 3 by 14/4 because the length and width of the photograph increase by the same ratio.

Ela explained the solution by highlighting multiplicative relationships and using a proportional reasoning language. Similarly, she utilized the same strategy in solving the other missing value problems of PRET (question 1-4). Additionally, in the post-interview, she utilized the *factor of change* predominantly. For example, one of the problems was "If 8 balloons are 12 TL, how much 6 balloons will cost?" She utilized an adaptation of *factor of change strategy*, in which she simplified the ratio to identify the integer scale factor (i.e., *between-ratio*) easier. Firstly, she simplified the ratio of 8 balloons to 12 TL by dividing both sides by 4 and found the ratio of 2 to 3. Secondly, she equated 2/3 to 6/? (2/3=6/?), and then, identified the scale factor as 3. It is important to note that she used an efficient strategy, which facilitated the computations required to find the missing value, for the problem.

Ela sometimes used *factor of change strategy* in a ratio table which was a strategy utilized multiplicative relationships between quantities. The strategy required viewing ratio as reducible and increasable units and by this way, one could solve the missing value problems that had noninteger *between-ratios*. For instance, she solved some missing value problems in the student teaching by using the strategy. In such a way that she multiplied quantities by scale factors to build ratio tables, and then, find out the missing values.

Building-up strategy was another strategy she used in some of her solutions. For instance, she used the strategy to solve a problem, which was exemplified by her, in the post-interview as follows:

> … For example, let's assume, we buy 2 balls and pay 7 TL. If we buy 6 balls, we can find the money paid by using building-up. We can increase the number of balls by twos and the amount of money by sevens until 6 balls are reached.

In order to solve the problem, she utilized *building-up strategy*; that is to say, she increased the number of balls and the amount of money by using additive patterns until desired quantity was reached.

Ela utilized *cross-multiplication strategy* in some of her solutions. To illustrate, she solved some of the missing value problems of PRET (question 1-4) by using the strategy. However, she did not see *cross-multiplication* as an indispensable strategy although she thought that it was one of the effective strategies. For instance, in the post-interview, she stated:

> Researcher: Which strategy will you teach your students as the most effective strategy to solve missing value problems?
> Ela: Cross-multiplication.
> Researcher: Why is it an effective strategy?
> Ela: Actually, cross-multiplication is a memorized procedure. That is to say, there is no rationale behind it. But, if a student learns the concepts correctly, he can use it because it enables to solve quickly.
> Researcher: What if a student does not learn ratio and proportion concepts exactly?
> Ela: I will use building-up strategy or factor of change.
> Researcher: What is the difference between the strategies and cross-multiplication?
> Ela: As I mentioned earlier, cross-multiplication is a memorized procedure. However, building-up is easy to understand. Moreover, in

> the cross-multiplication, there is nothing that shows the ratios between quantities. However, in factor of change strategy, students can see which quantities are proportional and the ratios of them.

Although she said that *cross-multiplication* was one of the effective strategies because of its quickness, she was aware that it consisted of some memorized procedures lacking meaning. In addition, she stated that she would teach *building-up strategy* to her students since it was an understandable strategy rather than a memorized procedure. In addition, she said that *factor of change* was also an understandable strategy since it highlighted multiplicative relationships between quantities.

Ela paid attention to use of different strategies highlighting multiplicative relationships. For instance, in the second lesson plan, Ela added some problems which allowed her to use different strategies without any guiding of the researcher. In her revision report of the lesson plan, she explained the reasons why she added the problems as follows:

> I changed most of the things in the first lesson plan. In the first plan, I gave memorized rules and solved the problems by using these rules, but in the second plan, I tried to solve the problems by using different strategies that made sense for students.

It can be concluded that Ela was aware that she solved the problems in the first student teaching by applying memorized rules and arithmetic procedures (e.g., *inverse proportion algorithm*) without giving any meaningful explanation. Moreover, she emphasized the importance of using different strategies. Furthermore, in order to teach inversely proportional relationships between variables, she added a problem that could be solved by concrete materials. She brought sugar candies to the classroom and told:

> I have 12 sugar candies. There are four students in the classroom. If a student takes all of the candies, he will get 12 candies. If two students fairly share them, each of them will get 6 candies. If three students fairly share them, each of them will get 4 candies. Let's make a table (drawing a table on the board and filling it with the number of students and the number of candies) … As you can see, when the number of students increases, the number of candies decreases in the same ratio.

As seen in the above quote, Ela tried to teach the meaning of being inversely proportional. In addition, she put emphasis on the relationships between quantities in an inversely proportional situation. Following, she asked, "If there were six students, how much sugar candy will each student get?" She solved the problem by using *factor of change strategy* in a ratio table and said, "A student gets 12 candies, and so, 6 students will get 2 candies because one is multiplied by 6; thus, we have to divide 12 by 6." She highlighted multiplicative relationships between the inversely proportional variables. In addition, she aimed at helping students recognize that the product of two quantities was a constant number in an inverse proportion, which was one types of invariance in proportional relationships. Moreover, she encouraged students to use *the constant of proportionality* as an alternative strategy to solve the problem. She utilized the functional relationships between inversely proportional variables (i.e., $x.y=k$). In addition, she did not use the *inverse proportion algorithm* to solve any inverse proportion problem. It is another evidence for the claim that she preferred to use strategies that highlighted multiplicative relationships (i.e., *factor of change*, *constant of proportionality*, and *unit rate*) instead of memorized procedures.

Data indicated that Ela did not have flexibility in unitizing and reunitizing quantities. In other words, she had difficulties in using composite units when the quantities suggest that it was more convenient than using singleton units. To illustrate, one of the problems that she solved by using *unit rate strategy* was, "If 4 workers paint a house in 10 days, how long will it take 8 workers to paint the same house?" First of all, she solved the problem by using *the constant of proportionality* (writing on the board: 4x10=40, 8x(?)=40 and (?)=5). Then, she utilized *unit rate strategy* by finding the number of days for a worker as 40 days. Later, she divided 40 by 8 in order to find the number of days if 8 workers painted the house. As can be seen, she used a singleton unit (one worker) though a composite unit (4 workers) would be more efficient for the problem.

It can be concluded that while Ela recognized *between-ratio* in a proportion, she had difficulty in recognizing *within-ratio*. For instance, in the first question of PRET, the *within-ratio* was integer, and so, it was a more efficient strategy than *between-ratio* for the problem. But, Ela could not recognize integer *within-ratio*, and used *factor of change*, which was a *between-ratio strategy,* to solve the problem. However, in the second question, she recognized the efficient strategy that was *between-ratio* and utilized integer *between-ratio* in order to solve the problem. Moreover, she did not utilize the *within-ratio* to solve any missing value problem in the relevant instruments even in cases where it was integer.

Numerical Comparison Problems: Solution Strategies and Processes

Elaused two different strategies, which were *fraction strategy* and *converting decimal expressions* to solve numerical comparison problems. One of the strategies was *fraction strategy,* in which the rates were treated as fractions and a common denominator or numerator was found to make comparisons. To illustrate, in her first solution of PRET question 6, which asked to compare two mixtures of orange juice concentrate and water, Ela found the amount of orange juice concentrates in each mixture for the same amount of water by using the strategy. She found the ratios that were equivalent to the ratios of the orange juice concentrates to water for each mixture. Firstly, she wrote the ratios of orange juice concentrates to total mixtures as fractions and found a common denominator. Secondly, she multiplied 2/3 (the ratio for the mixture A) by 4/4 to produce 8/12, and then, she multiplied 3/4 (the ratio for the mixture B) by 3/3 to produce 9/12 which had the same denominator with the first ratio (8/12). Finally, she compared the amount of orange juice concentrates for the same amount of water and concluded that since mixture B had more orange juice concentrate for the same amounts of water, it had a stronger orange taste than mixture A. Furthermore, in the post-interview, Ela stated that *fraction strategy* was the most effective strategy to solve numerical comparison problems since it facilitated comparing the ratios by holding one of the variables constant in a proportional situation.

Another strategy utilized by her was *converting decimal expressions*. For example, in her second solution of PRET question 6, she used the strategy. She wrote ratios of orange juice concentrates to water in fractional form, converted them into decimal expressions and compared them. She accurately concluded that mixture B had stronger orange taste than mixture A since the ratio of orange juice concentrates to water of mixture B was bigger. As a result, she had successfully made connection between the decimal expressions and the relative strength of the mixtures. In conclusion, in her both solutions, Ela could explicitly explain her rationale and the meaning of the quantities she used to determine the relative strength of the mixtures. Additionally, she used proportional reasoning language in her explanations. Moreover, she utilized *part-to-part ratios* by using the ratios of orange juice concentrates to water in her both solutions. In a similar way, she solved another numerical comparison of PRET (question 24), which presented three rectangular clothes with dimensions, by the same strategy. The problem asked pre-service teachers to determine the cloth that was "the most square". At first, Ela found the ratios of width to length and length to width of the three rectangular clothes. Secondly, she converted all of them into decimal expressions (widths to lengths; 23/35=0.657, 139/155=0.864, 56/75=0.746 and lengths to widths; 35/23=1.52, 155/139=1.11, 75/56=1.33) and compared them. Finally, she accurately determined that the rectangular cloth with dimensions 155 and 139 was "the most square" and explained as, "The second cloth was the most square because both the ratio of its width to length and length to width were closest to 1." Ela used a multiplicative strategy to solve the problem and explained it by using proportional reasoning language.

In the student teaching, Ela did not ask any numerical comparison problem to her students. Moreover, in the post-interview, she did not mention any situation that could be a numerical comparison problem while she was giving examples of proportional situations in real life.

Qualitative Reasoning Problems: Solution Strategies and Processes

In order to solve qualitative reasoning problems, Ela could make qualitative comparisons that did not depend on numerical values. Moreover, she could realize multiplicative relationships in a qualitative reasoning problem. To illustrate, in the PRET, there were two qualitative problems (questions 7 and 8) which contained no numerical value, but required the balancing of quantities. Ela solved both problems accurately by interpreting the qualitative relationship that existed between two quantities. Moreover, she provided valid explanations by making multiplicative comparisons between quantities and she used proportional reasoning langue in her explanations.

In the second student teaching, she did not ask any qualitative reasoning problem to her students. Moreover, in the post-interview, she did not mention any situation that could be a qualitative reasoning problem while she was giving examples of proportional situations in real life.

4.2.2 Findings about Distinguishing Proportional from Nonproportional Situations after the Proportional Reasoning Instructional Module

Pre-service teachers' work on classifying relationships as proportional or nonproportional, identifying ratio as measure and providing examples for proportional and nonproportional relationships were analyzed so as to reveal pre-service teachers' ability to distinguish proportional from nonproportional situations after participation in a practice-based instructional module based on proportional reasoning. In the following section, the results of the analyses are presented.

4.2.2.1 Gaye's Findings about Distinguishing Proportional from Nonproportional Situations

Classifying Relationships as Proportional or Nonproportional

Gaye did not have difficulty in classifying relationships as proportional or nonproportional. To demonstrate, PRET had some questions (questions 11 to 22) which asked pre-service teachers to identify whether the given situations were proportional or nonproportional. In the posttest, Gaye accurately classified all of the relationships as proportional or not.

Gaye appears to have eliminated the misconception that all linear relationships were proportional after the instructional module. For instance, in the posttest, she accurately classified all of the linear and proportional relationships (questions 12, 14, 19 and 20) as proportional and all of the linear and nonproportional relationships (questions 11,13,15,17 and 21) as nonproportional. Further, in the post-interview, when the researcher asked her to provide an example of a real life nonproportional situation, she could exemplify a situation in which quantities had a linear, but nonproportional relationship. Moreover, the explanations Gaye produced in the posttest and post-interview supported the same claim. For example, in the posttest, she supported her statement that question 21 (the table of a linear, nonproportional relationship) was nonproportional by writing, "It is linear, but there is not any variable that increases or decreases by the same ratio, so x and y are not proportional." It is also important to note that she considered multiplicative relationships to decide proportionality. In addition, in the post-interview, when the researcher asked her to explain how she decided the relationship between the quantities in question 16 of posttest (the graph of $y=x^2$), she stated:

> In a proportional situation, the relationship between variables has to be both linear and proportional. This graph goes through the origin, but it is not linear, and so y is not proportional to x. Since y increases or decreases in the same ratio of x^2, y is proportional to x^2 not x…

As seen in the above quote, Gaye knew that a proportional relationship required both linearity and proportionality; in other words, she knew that linearity at alone was not an indicator of proportional relationship, but it was a requirement for the variables to be proportional. Additionally, it is interesting to note that she not only determined the nonproportionality of the quadratic relationship, but she also realized the square proportionality ($y=kx^2$; y is proportional to the square of x) in the situation.

Another evidence for the claim that she did not believe all linear relationships were proportional is in her second student teaching. In the teaching, Gaye asked the same tree problem as the first student teaching. Similar to first time, firstly, she found the points in the coordinate plane; secondly, connecting the dots, and then, drew the line through the points. However, at this time, she accurately classified the variables in the problem as nonproportional instead of directly proportional. Moreover, in the revision report of the lesson plan she explained why she did not remove the problem as follows:

> In the first teaching, I gave incomplete information to students. In the second time, I made clear that the graph of proportional variables must go through the origin besides increasing or decreasing in the same ratio. For this purpose, I deliberately asked the tree problem to demonstrate that all linear graphs do not indicate direct proportion between variables.

The explanation seen in the above quote supports the claim that she did not believe all linear relationships were proportional anymore.

Gaye's answer to an interview question in the post-interview also revealed that she did not hold the misconception about linearity. The question asked pre-service teachers whether a student was right or not if he said that all linear relationships were proportional. Unlike she did in the pre-interview, Gaye did not agree with the student as follows:

> Gaye: He is not right… For example, posttest question 15 is a graph of linear relationship, but it is not proportional…
> Researcher: Another student Semih said "All proportional relationships are linear." Is Semih right?
> Gaye: He is right because the quantities having proportional relationships have to be linear.

As seen in the above excerpt, she appears to eliminate the misconception that all linear relationships were proportional. Moreover, she knew that if variables had proportional relationships, they also had linear relationships.

Identifying Ratio as Measure

Data revealed that Gaye could recognize ratio as a proper method to measure concentration of a mixture, steepness of ski ramps, shade of paint, and "squareness" of a rectangle. Further, she mostly utilized proportional reasoning language in her explanations. To illustrate, in PRET question 6, she used the ratios to measure the orange concentrations of the mixtures. Moreover, in another PRET problem (question 9), which asked to comment on the effectiveness of a boy's strategy about changing the amount of paint by saving its shade, Gaye accurately determined that the boy's strategy was not effective, and explained as follows:

> It does not work all the time because we do not know the ratios of white and blue paints in the beginning mixture. In order to increase the amount without changing the color, he will have to add the same ratio of white to blue paints as he used in the beginning. In other words, if he added one box white and one box blue in the beginning, then his strategy would be effective. For example, if he added 2 box

> blue and one box white in the beginning, then he would have to add 4/3 glass of blue and 2/3 glass of white paints in order to increase one quart.

Gaye used the word “ratio” to refer to the relationship between the amount of blue and white paints in the mixture. She asserted that the ratio of white to blue paints would not change if the same ratio of paints were added. The statement indicated that her criterion for whether the shade of paint would stay the same was based on whether the added mixture would maintain the ratio of white to blue paint. That is to say, Gaye seems to realize that ratio was a proper method to measure shade of paint.

In addition, in PRET question 10, which asked to determine relative steepness of ski ramps, she explained that one could rate the ramps from steepest to least steep by using ratios of the heights to lengths of the bases. That is to say, she used ratio as a measure of steepness of ski ramps and recognized that the width of the base did not have an effect on the steepness of the ramps. Furthermore, in PRET question 24, Gaye found the ratios of the widths to lengths of the three rectangular clothes and compared them to decide the rectangular cloth which was “the most square”. She used an accurate multiplicative strategy and saw ratio as a proper measure in the context of determining “squareness” of a rectangle.

Providing Examples for Proportional and Nonproportional Relationships

Gaye sometimes could not provide a valid example of a proportional relationship. To illustrate, in the PRET, there was a question (question 5) that asked to provide a word problem which could be solved by the given equation. In the posttest, Gaye could not create a missing value word problem in which the quantities related multiplicatively. She wrote, “A man has 8 cookies and gives his son three of them. If he has 20 cookies, how many cookies will he give to his son?” As seen in the problem, there was not any statement that implied a proportional relationship. It was unclear whether the ratio of given cookies to all cookies was kept the same between the two cases. Thus, it can be concluded that she could not write a problem that could be solved.

However, she sometimes could provide a valid example of a proportional relationship. To demonstrate, in the post-interview, when the researcher asked her to give an example of a real life proportional situation, she stated:

> The relationship between the amount of petrol bought and the amount of money given can be an example for a proportional situation. For instance, if a liter of petrol is 5 TL, we can found the amount of petrol that can be bought with 100 TL by using direct proportion…

She could justify why her example in the above quote was proportional by using the statement that variables in a directly proportional situation increased by the same ratio.

In addition, she was able to create valid examples of nonproportional relationships. For instance, in the post-interview, when the researcher asked her to give examples of real life nonproportional situations, she could provide two valid examples of nonproportional situations. First of all, she provided an example in which the variables had a constant relationship. Then, when the researcher asked her to create one more example, she stated:

> Ali and Ayşe went to a part. Each of them paid 100 TL for the entrance ticket, and they paid 5 TL for each drink. Ali drank 5 drinks and Ayşe drank 10 drinks… There is not any ratio between the number of drinks and the amount of money paid.

As seen in the above quote, she gave an example in which the variables had an additive relationship. Moreover, she could justify why her example was nonproportional.

4.2.2.2 Mine's Findings about Distinguishing Proportional from Nonproportional Situations

Classifying Relationships as Proportional or Nonproportional

Mine did not have difficulty in classifying relationships as proportional or nonproportional. To demonstrate, PRET had some questions (questions 11 to 22) which asked pre-service teachers to identify whether the given situations were proportional or nonproportional. In the posttest, Mine accurately classified all of the relationships as proportional or not.

Data revealed that Mine knew all linear relationships were not proportional and all proportional relationships were linear. For example, in the posttest, she accurately classified the linear and proportional relationships (questions 12, 14, 19 and 20) as proportional and the linear and nonproportional relationships (questions 11, 13, 15, 17 and 21) as nonproportional and the quadratic relationships as nonproportional (questions 16, 18 and 22). Moreover, in the post-interview, when the researcher asked whether a student was right or not if he said that all linear relationships were proportional. Mine did not agree with the student and she stated that for the variables to be proportional, linearity was not enough. When the researcher asked that whether another student was right or not if he said that all proportional relationships were linear. She gave right to the student and said, "For the variables to be proportional, they have to be linear." Additionally, in the student teaching, she exemplified the nonproportional situations that had additive relationships and emphasized that all linear relationships were not proportional.

The explanations Mine made in the posttest and post-interview support the claim that she knew that if variables had proportional relationships, they also had to have linear relationships. To illustrate, in the posttest, she supported her statement that question 16 (the graph of quadratic relationship) was nonproportional by writing, "The graph goes through the origin, but it is not linear. Therefore, the relationship between *x* and *y* cannot be proportional." Furthermore, in the post-interview, when the researcher asked her to tell how she decided the relationships between the quantities in posttest question 16 (the graph of $y=x^2$) and 18 (the equation of $y=3x^2$), she explained her answers as presented in the following quote:

> Questions 16 and 18 have similar relationships. The equation of them is $y=mx^2$. If I give one for x, y will be m; give two for x, y will be 4m. Namely, as I increase x by 2, y increases by 4. So, x and y are not proportional because they do not increase in the same ratio. Moreover, the graph (showing question 16) goes through the origin, but it is not a straight line.

Mine accurately classified the quadratic relationships as proportional. Furthermore, she stressed that the variables in the questions did not increase in the same ratio, which was required to be proportional. In addition, when the researcher asked her whether any variables were proportional in these questions, she noticed that the relationship between *y* and x^2 were proportional while *x* and *y* were not proportional for these questions.

Mine could distinguish situations in which proportionality was not an appropriate mathematical model from situations in which it was useful and explain why. For example, in the posttest, she mostly explained her claims about proportionality by using the statement that variables in a proportional situation had to increase or decrease in the same ratio. In fact, she not only considered that the direction of change of the related quantities had to be the same, but she also considered that the change between the quantities had to be in the same ratio. Furthermore, for some questions in the posttest, she explained her rationale by using multiplicative and additive relationships and their connections with proportionality. For instance, she justified her rationale for questions 11 and 15 in the post-interview:

> I stated that questions 11 and 15 were not proportional because both of them have additive relationships. Both of them have an expression of addition like "plus n". In question 11, adding 2.2 (writing on the sheet: y=1.9x+2.2) causes additive relationship and prevents multiplicative one which makes proportionality. Similarly, in question 15, the equation is y=mx+n in which adding "n" destroys the proportionality.

It seems that Mine distinguished multiplicative relationships from additive relationships. In addition, she stated that while proportional relationships had a multiplicative structure, additive structure did not establish a proportional relationship between quantities. Moreover, she used proportional reasoning language in her explanation.

Correspondingly, in the student teaching, she introduced multiplicative, additive and constant relationships and explained whether they were proportional or not. For example, while she was clarifying why a multiplicative relationship was proportional, she said:

> The problem asked if a student solves 60 questions in 40 minutes, how many questions will he solved in 10 minutes… If I divide the number of minutes (40) by 2, I will find 20 minutes. Be careful! I did not subtract 20 minutes, I divided by 2 because if I subtracted, the ratios will not be equal and the proportion will not be established...

It is important to note that Mine realized the multiplicative relationship between two variables. Moreover, she pointed out that additive relationships were not proportional since there were not equal ratios.

In addition, in the student teaching, she asked problems that had constant relationship between quantities. For instance, she asked, "If a T-shirt dries in 10 minutes, how many minutes will five T-shirts take to dry?" Her students answered as 50 minutes. She explained that the drying time of a T-shirt, 10 T-shirts or 100 T-shirts would not change; therefore, the drying time and the number of T-shirts were not proportional.

Furthermore, in the revision report of the lesson plan she explained why she added the problems that had constant relationships:

> In the first student teaching, I only gave examples of additive situations for nonproportional quantities. In the second time, I added two problems that had constant relationships between quantities because I think such problems will develop students' logical point of view and indicate that additive relationships are not the only situation that have nonproportional relationships.

As seen in the above quote, Mine knew that additive and constant relationships were nonproportional situations.

Identifying Ratio as Measure

Data indicated that Mine could recognize ratio as a proper method to measure concentration of a mixture, steepness of ski ramps, and "squareness" of a rectangle. Additionally, she mostly utilized proportional reasoning language in her explanations. For example, in PRET question 6, she used the ratios to measure the orange concentrations of the mixtures. Furthermore, in PRET question 10, which asked to determine relative steepness of ski ramps; she made a multiplicative comparison to measure the attribute. In other words, she used ratio as a measure of steepness of ski ramps and noticed that the width of the base did not have an effect on the steepness of the ramps. Moreover, in PRET question 24, Mine calculated the ratios of length to width of the three rectangular clothes and compared them to find the rectangular cloth which was "the most square". That is to say, she used an accurate multiplicative strategy and saw ratio as a proper measure in the context of determining "squareness" of a rectangle.

On the other hand, data showed that Mine did not see ratio as a proper method to measure shade of paint. In fact, she used an additive comparison to measure the attribute instead of a multiplicative comparison. For instance, in a PRET problem (question 9), which asked pre-service teachers to comment on the effectiveness of a boy's strategy about changing the amount of paint by saving its shade, Mine concluded that the boy's strategy was ineffective. However, she only mentioned the amounts of the paints in the original and new mixture. In addition, she argued that the color of new mixture would not change if the original mixture had the same amount of white and blue paints. Actually, she did not mention the ratios of white and blue paints that made up the original and new mixtures. It can be concluded that Mine still did not see ratio as a proper measure in the context of determining shade of paint.

Providing Examples for Proportional and Nonproportional Relationships

Mine was able to provide valid examples of proportional relationships. For instance, in the PRET, there was a question (question 5) that asked to provide a word problem which could be solved by the given equation. Mine created a missing value word problem, which had a context involving enlarging a rectangular shape, and asked to find how many centimeters the enlarging figure's length would be. Since the noninteger answer was 7.5 centimeters, it made sense in the context. Moreover, she created a problem in which the quantities had proportional relationships. Similarly, in the post-interview, when the researcher asked her to give an example of a real life proportional situation, she was able to create a valid example of a proportional situation.

Additionally, Mine could create valid examples of nonproportional relationships. For example, in the post-interview, when the researcher asked her to give examples of real life nonproportional situations, she was able to create two valid examples. First of all, she provided an example in which the variables had a linear and nonproportional relationship. Then, when the researcher asked her to create one more example, she gave an example in which the variables had constant relationship as follows:

> If one shirt dries in 10 minutes, 2 shirts will dry in 10 minutes, too. The time and the number of shirts are not proportional because the time for drying one shirt, 2 shirts or 20 shirts does not change when the number of shirts increases.

In conclusion, Mine could provide nonproportional real life examples in which the variables had additive and constant relationships. Furthermore, she could explain why the variables in the situation did not have proportional relationship.

4.2.2.3 Ela's Findings about Distinguishing Proportional from Nonproportional Situations

Classifying Relationships as Proportional or Nonproportional

Ela did not have difficulty in classifying relationships as proportional or nonproportional. For example, PRET had some questions (questions 11 to 22) which asked pre-service teachers to identify whether the given situations were proportional or nonproportional. In the posttest, Ela accurately classified all of the relationships as proportional or not.

Data indicated that Ela knew all linear relationships were not proportional and all proportional relationships were linear. For instance, in the posttest, she accurately classified the linear and proportional relationships (questions 12,14,19 and 20) as proportional and the linear and nonproportional relationships (questions 11,13,15,17 and 21) as nonproportional and the quadratic relationships as nonproportional (questions 16, 18 and 22). Some statements of Ela in the post-interview are evidence supporting the same claims. The researcher asked her to determine whether a student was right or not if he said that all linear relationships were proportional. Ela did not agree with the student. The related excerpt is as follows:

> Ela: He is not right. All linear relationships could not be proportional. For example, in the equation y=mx+n, y and x are not proportional, though the relationship between them is linear.
> Researcher: Another student Semih said "All proportional relationships are linear." Is Semih right?
> Ela: Yes, it is true. If they are proportional, they are also linear.
> Researcher: Are you sure? Why?
> Ela: Yes. Proportional variables must be linear because the equation is y=mx.

Ela could explain the difference between functions of the form $y=mx$ and functions of the form $y=mx+n$. Without doubt, in the latter function y was not proportional to x.

The explanations Ela made in the posttest and post-interview support the claim that she knew that all proportional relationships were linear. In the situations that had quadratic relationships (questions 16, 18 and 22), she stated that for the variables to be proportional, they had to be linear, not parabolic. It can be concluded that Ela knew that linearity at alone was not an indicator of proportional relationship, but it was a requirement for the variables to be proportional.

Ela could distinguish situations in which proportionality was not an appropriate mathematical model from situations in which it was useful and explain why. Moreover, she not only considered that the direction of change of the related quantities had to be the same, but she also considered that the change between the quantities had to be in the same ratio. To illustrate, she supported her decision that question 11 (a linear, nonproportional relationship presented in language) was nonproportional by writing, "The quantities do not increase by the same ratio because as the number of kilometers increase by two, the money given to taxi driver increase by 1.46" In addition, in the post-interview, when the researcher asked her to explain how she decided on the relationship between the quantities in the posttest question 22 (the table of a linear, nonproportional relationship), she clarified as follows:

> Researcher: Why did you conclude that the relationship in question 22 was not proportional?
> Ela: Because they do not increase by the same ratio.
> Researcher: Can you explain?
> Ela: Look! x is multiplied by 6/4 (showing 4 and 6 in the table), but in here, x is multiplied by 18/8 (showing 18 and 8 in the table). Similarly, as x is multiplied by 2 in here (showing 4 and 8 in the table), x is multiplied by 4 (showing 8 and 32 in the table). So, x and y are not proportional, but we can say that x^2 and y are proportional.
> Researcher: Why?
> Ela: Because as x^2, 4x4=16, is multiplied by 4 to obtain 8x8=64, y is also multiplied by 4 (showing 8 and 32 in the table).

She could provide evidence to support her claims about proportionality. Furthermore, it is interesting to note that she realized the square proportionality ($y=kx^2$; *y* is proportional to the square of *x*) in the situation.

In a similar manner, in the second student teaching, she emphasized that proportional relationships increased or decreased in the same ratio. In other words, she taught her students that increasing or decreasing at the same time did not enough for the variables to be proportional. Furthermore, she presented situations that had multiplicative, additive and constant relationships and explained whether they were proportional or not. For example, while she was clarifying why a situation that had a constant relationship between variables was not proportional, she stated:

> The problem asked if a T-shirt dries in ten minutes, how many minutes will five T-shirts take to dry. Time and the number of T-shirts are not proportional because the number of T-shirts does not make any difference in time that is required for the T-shirts to dry. Drying time of a T-shirt or 100 T-shirts does not matter. Remember! For the variables to be proportional they must have multiplicative relationships; in other words, they must increase by the same ratio. But, for the problem, as the number of T-shirts increases, drying time is constant.

She stated that while proportional relationships had a multiplicative structure, constant relationship did not establish a proportional relationship between quantities. Additionally, she utilized proportional reasoning language in her explanation.

In addition, in the posttest and post interview, for the linear and nonproportional relationships (questions 11,13,15,17 and 21), she explained her rationale by using multiplicative and additive relationships and their connections with proportionality. In fact, she argued that while proportional relationships had a multiplicative structure, additive structure did not establish a proportional relationship between quantities.

Identifying Ratio as Measure

Data revealed that Ela could recognize ratio as a proper method to measure concentration of a mixture, steepness of ski ramps, shade of paint, and "squareness" of a rectangle. Furthermore, she mostly utilized proportional reasoning language in her explanations. To illustrate, in PRET question 6, she used the ratios to measure the orange concentrations of the mixtures. Moreover, in another PRET problem (question 9), which asked to comment on the effectiveness of a boy's strategy about increasing the amount of paint without changing the color, Ela accurately determined that the boy's strategy was ineffective, and provided the following explanation:

> The strategy works provided that the ratio of white to blue paints in the beginning mixture is equal to the ratio of white to blue paints in the second mixture. Therefore, it is important to know the ratio of white to blue paints in the beginning mixture. If it is 1/1, the strategy works. Yet, we do not know the ratio. Therefore, it does not work all the time.

As seen in the above explanation, Ela utilized proportional reasoning language. Moreover, she used the word "ratio" to refer to the relationship between the amount of blue and white paints in the mixture. She asserted that the color of the new mixture would not change if the same ratio of paints was added. This assertion indicated that her criterion for whether the shade of paint would stay the same was based on whether the added mixture would maintain the ratio of white to blue paint. It can be concluded that Ela recognized that ratio was a proper method to measure shade of paint.

Additionally, in PRET question 10, which asked to determine relative steepness of ski ramps, she explained that one could rate the ramps from steepest to least steep by using ratios of the heights to lengths of the bases. Namely, she used ratio as a measure of steepness of ski ramps and recognized that the width of the base did not have an effect on the steepness of the ramps. Further, in PRET question 24, Ela found the ratios of the widths to lengths of the three rectangular clothes and compared them to determine the rectangular cloth which was "the most square". She used an accurate multiplicative strategy and saw ratio as a proper measure in the context of determining "squareness" of a rectangle.

Providing Examples for Proportional and Nonproportional Relationships

Ela was able to create valid examples of proportional relationships. For instance, in the PRET, there was a question (question 5) that asked to provide a word problem which could be solved by the given equation. Ela could write a missing value word problem, which was, "A farmer divides a field into 8 equal parts and irrigates 3 parts. If the farmer divides the same field into 20 equal parts, find the number of parts that can be irrigated by the same amount of water?" The noninteger answer (7.5 parts of the field) made sense in the context. Briefly, she created a valid problem in which the quantities had proportional relationships. Correspondingly, in the post-interview, when the researcher asked her to give an example of a real life proportional situation, she was able to create a valid example of a proportional situation.

Furthermore, Ela could create valid examples of nonproportional relationships. For example, in the post-interview, when the researcher asked her to give examples of real life nonproportional situations, she was able to create two valid examples. The first example of her was a situation that had linear and nonproportional relationships between variables. The second example was a situation that had constant relationship between variables. In brief, Ela could provide nonproportional real life examples in which the variables had additive and constant relationships. Moreover, she could justify why her examples were nonproportional by using the statement that variables in a proportional situation had to increase or decrease by the same ratio.

4.2.3 Understanding the Mathematical Relationships Embedded in Proportional Situations

In order to investigate pre-service teachers' understanding of mathematical relationships embedded in proportional situations, the four *key understandings,* if any, they used to define proportionality concepts and to determine proportional relationships were analyzed. Moreover, their difficulties in defining proportionality concepts and understanding the *key understandings* were presented. In the following section the results of the analyses after the proportional reasoning instructional module are presented under two subcategories: defining proportionality concepts and determining proportional relationships.

4.2.3.1 Gaye's Findings about Understanding the Mathematical Relationships Embedded in Proportional Situations

Defining Proportionality Concepts

Gaye knew true definitions of ratio and proportion. Additionally, she could explain the differences between the terms. Moreover, she used proportional reasoning language in her definitions. For instance, she defined the terms ratio and proportion in the post-interview as follows:

> Ratios are used to compare quantities. The ratio of 12 marbles to 8 marbles is an example of a ratio because these are two quantities and they are compared relatively. On the other hand, a proportion is a name we give to a statement that two or more ratios are equal.

She utilized one of the *key understandings* that was equality of rate pairs (*key understanding 3*) to define proportion. Furthermore, she appears to eliminate the misconception that a ratio could not have a zero as its second component. To illustrate, she stated in the second student teaching:

> Gaye: Ratio is relative conditions namely relative comparisons of two quantities. It can be represented "a/b"; a and b are real numbers. May these quantities a and b be zero?
> Students: No. "b" cannot be zero.
> Gaye: Are you sure? For instance, there are 5 cars in a garage and no cars in another garage, we can compare them.
> Student: Yes, it can be, but both of them cannot be zero.
> Gaye: Think about it. I will turn back here while teaching graphs.
> ….
> Gaye: I want to ask you a question, can we compare non-existent quantities, I mean comparing zero to zero?
> Student: … (No answer)
> Gaye: We can compare. If we consider the same garage example, we can compare the number of cars in two empty garages. By the same way, look at the graph of the proportional relationship, it passes from x=0 and y=0; that is, it passes from the origin.

Gaye not only eliminated her misconception that a ratio could not have a zero as its second component, but she also tried to prevent her students to believe the same misconception. Furthermore, she explained the possibility of comparing non-existent (e.g., zero car to zero car) quantities by using real life situations. Moreover, she utilized that proportional relationships were presented graphically by lines that passed through the origin (*key understanding 2*).

Gaye could define direct proportion by considering the constant ratio that defined the relationship between the variables in a direct proportion. For instance, in the second student teaching, she mentioned that there were two different types of proportional situation and she defined direct proportion by using proportional relationships' multiplicative nature (*key understanding 1*) and noted that directly proportional relationships increased or decreased by the same ratio.

Data indicated that Gaye could define the constant of proportionality for both direct and inverse proportions. In addition, she stressed invariance of the ratio of two quantities in a direct proportion and invariance of the product of two quantities in an inverse proportion by indicating the equations of directly and inversely proportional situations. Moreover, she emphasized that both k and $1/k$ were the constant of proportionality; in other words, she appears to realize that a constant multiplicative relationship existed between two quantities and it could be expressed in two ways. Furthermore, in the post-interview, when the researcher asked her in which posttest questions she used the constant of proportionality, she accurately determined the constants of proportionality in the proportional situations.

Determining Proportional Relationships

Gaye could explain the presence of proportional relationships by using the *key understandings*. For example, in the second student teaching, she explained how she discriminated proportional quantities from non-proportional quantities as, "…variables are proportional when one variable increases, the other increases in the same ratio, and when one variable decreases the other also decreases in the same ratio, simultaneously." As can be seen, she not only pointed out that the direction of change of the related quantities was the same, but she also emphasized that the change between the quantities had to be in the same ratio. In brief, she utilized the *key understanding 1*. Furthermore, in the second student teaching, while she was teaching proportional relationships, she set the algebraic expression of a proportional situation by using multiplicative nature of proportional relationships (*key understanding 1*) as follows:

> As mentioned earlier, the proportion is always equal to a number, we call it m; $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = m$. Let's take the first ratio, $\frac{a}{b} = m$, and multiply each side of the equation with "b". It will be a=mb. It means "a" increases by "m" factor of "b". In order to draw the graph of a proportion, we can write $\frac{y}{x} = m$. By the same way, y=mx. For example, in the equation, m would be 2; then, if x=1, y =2; if x=2, y=4. As you can see "y" increases by two factor of "x". That is to say, it always increases by the same ratio. I will show this in a graph… As you see, it is a line passes through the origin.

As seen in the above quote, Gaye made use of *key understanding 1* to identify proportional relationships. Moreover, she accurately drew the graph of the proportional relationship represented as an algebraic expression, which indicated that she knew the *key understanding 2*. In other words, she could make connections between the algebraic expression and the graph of a proportional situation. On the other hand, in another problem in the student teaching, she had difficulty in making connection among table and algebraic expression of a proportional situation. She asked students to draw the graph of a proportional situation presented in a table and found the algebraic expression of the relationship. In the table, there were the numbers of questions that a student solved at different days (e.g., 7 questions in 2 days, 14 questions in 4 days). She accurately drew the graph and pointed out the *key understanding 2*. However, she could not accurately write the algebraic expression of the situation, and could not explain how she found it. She stated:

> Gaye: I will write the equation of the proportional situation. For example, for the equation *y*=2*x* how many did "*y*" increase? Remember, it increased by twos. Now, look at the table and graph, we should find the value that the number of days increased by.

Days	2	4	8
Questions	7	14	28

> Students: It increases by twos.

> Gaye: Yes, true. So the equation is “2 x number of days = 7 x number of solved questions” because the number of day increases by twos and the number of solved question increases by sevens.
> Students: We do not understand the equation.
> Gaye: In order to ensure equality, the number of days should increase by twos and the number of solved questions should increase by sevens.
> Student: I do not understand.
> Gaye: I cannot explain anymore. You have got a problem in the equation part.

As can be understood from the excerpt, she had difficulty in finding the equation of the proportional relationship represented as a table. She wrote “2 x number of days = 7 x number of solved questions” instead of “7 x number of days = 2 x number of solved questions”. She did not realize her mistake; moreover, she said that her students had a problem. Furthermore, it can be concluded that she did not exactly understand the statement that a proportional relationship could be represented symbolically as $y = kx$, where k was the slope, the unit rate, and the constant of proportionality (*key understanding 4*) because if she understood it, she would try to find the slope of the line from the graph or the constant of proportionality by using ratios of the variables, and put one of them, which were the same, in the equation, $y=kx$.

Gaye could adequately use the *key understandings* to justify her classifications about proportional relationships. The justifications of Gaye about her classifications in PRET questions 11-22 in the posttest and post-interview were evidence for the claim. In order to get in-depth information, in the post-interview, the researcher asked Gaye to explain how she decided the relationships between the quantities in PRET questions 11-22.

Table 4.4 Classifications and explanations about proportional relationships in PRET questions 11-22

	Posttest		**Post-interview**	
Question	**Classification**	**Explanation**	**Classification**	**Explanation**
11 (Language)	Not Proportional	Key Understanding 1	Not proportional	Key Understanding 1
12 (Language)	Proportional	Not Clear	Proportional	Key Understanding 1
13 (Language)	Not Proportional	Key Understanding 1	Not Proportional	Key Understanding 1
14 (Graph)	Proportional	Key Understanding 1 Key Understanding 2	Proportional	Key Understanding 1, Key Understanding 2
15 (Graph)	Not Proportional	Key Understanding 1 Key Understanding 2	Not Proportional	Key Understanding 1, Key Understanding 2
16 (Graph)	Not Proportional	Key Understanding 1	Not proportional	Key Understanding 1, Key Understanding 2
17 (Equation)	Not Proportional	Key Understanding 1	Not Proportional	Key Understanding 1
18 (Equation)	Not Proportional	Key Understanding 1	Not Proportional	Key Understanding 1
19 (Equation)	Proportional	Key Understanding 1	Proportional	Key Understanding 1
20 (Table)	Proportional	Key Understanding 1	Proportional	Key Understanding 1, Key Understanding 4
21 (Table)	Not Proportional	Key Understanding 1	Not Proportional	Key Understanding 1
22 (Table)	Not Proportional	Not Clear	Not Proportional	Key Understanding 1, Key Understanding 3, Key Understanding 4

Table 4.4 shows the classifications and the explanations of Gaye for each question in the posttest and post-interview. Although all of the classifications were the same in both instruments, there were some additional *key understandings* and detailed explanations in the post-interview. In fact, some of Gaye's explanations were not clear in the posttest. However, when the researcher asked her to explain her rationale for her classifications she made use of the *key understandings* in the post-interview. For example, in question 22 (presented in table), she could write the equation of the variables in the table as $y=1/2x^2$, but she could not explain why the equation was not proportional in the posttest. On the other hand, in the post-interview, she stated:

> Gaye: The equation of the relationship is $y=1/2x^2$, y is proportional to x^2 not x…. For the variables to be proportional, x and y have to increase in the same ratio. In here, 4 increases by 2, but 8 increases by 4.
> Researcher: In order to determine whether the relationship is proportional or not what should we consider? Is there anything else?
> Gaye: We should consider the ratio of x to y, the result has to be a constant number which is called the constant of proportionality. It is also slope of the graph constructed by x and y. For this question, x_1/y_1 (4/8) is not equal to x_2/y_2 (6/18) and x_1/x_2 (4/6) is not equal to y_1/y_2 (8/18).
> Researcher: You said that for the variables to be proportional, x_1/x_2 has to be equal to y_1/y_2. Is it true?
> Gaye: Yes.
> Researcher: Does the result also have to be equal to x_3/x_4?
> Gaye: No it does not. It may increase by 2, 3 or any other number.

As seen in the above excerpt, Gaye utilized three different *key understandings* to confirm her claim that x and y were not proportional. Specifically, she made use of proportional relationships' multiplicative nature (*key understanding* 1) and noted that there was a proportional relationship between x^2 and y instead of x and y. Furthermore, she checked several rate pairs (*key understanding 3*) and concluded that there was not a constant of proportionality which was also the slope of the graph of x and y (*key understanding 4*). She seems to have understood the relationship between the slope and the constant of proportionality, but she did not mention the unit rate that was equal to both of them. In addition, she knew that the constant of proportionality was equal to *within-ratio* instead of *between-ratio* that was the ratio between *x* values (e.g., x_1/x_2, x_3/x_4).

As shown in the Table 4.4, Gaye used the four *key understandings* to justify her classifications. Her foremost justification to explain whether a relationship was proportional or not was that proportional relationships were multiplicative in nature (*key understanding 1*). Actually, in both posttest and post-interview, she made use of *key understanding1* for explaining all the relationships. However, while she used *key understanding 1* for some questions at alone, for some questions, she used the other *key understandings,* as well. To illustrate, in the post-interview, she used only *key understanding 1* to explain why the relationships presented in language (question 11, 12 and 13) and equation (question 17, 18 and 19) were proportional or not. However, she utilized key *understanding 2*, besides *key understanding 1*, to justify her classifications in questions 14, 15 and 16. Perhaps this is because the questions were presented as graph, which allowed her to utilize key *understanding 2*. For instance, in the post-interview, in order to justify her classification for question 16, which presented the graph of $y=x^2$, she said:

> It is not proportional because it is a graph of $y=x^2$. For x and y to be proportional, the equation should be, y=mx, in other words, y should be increased by a multiplier of x. But in here, it is increased by a multiplier of x^2 so x and y are not proportional despite the graph pass from (0,0). In a proportional situation, the relationship between variables has to be both linear and proportional. This graph goes through the origin but it does not linear, and so, y is not proportional to x.

As seen in the above quote, she used key *understandings 1* and *2* to justify her rationale. Besides these arguments, she appears to have known that proportional relationships could be expressed algebraically in the form *y=mx*. In addition, she did not provide any explanation based on a misconception. To show, as seen in the above quote, she appears to have understood that linearity at alone could not be a justification of proportional relationship.

Furthermore, she used key *understandings 1*, *3* and *4* to determine whether the relationships presented in table (question 20, 21 and 22) were proportional or not. To demonstrate, in order to determine the relationship in question 20, she generated the equation (y=3/2x) of the relationship presented in table and explained the presence of proportionality by saying, "*y* increases or decreases by 3/2 factor of *x*" in both posttest and post-interview (*key understanding 1*). Moreover, in the post-interview, she made connection between constant of proportionality and slope as she explained in the following excerpt:

> Gaye: By using the relationship between x and y in the table, I can generate a linear equation that is x=2/3y. This is an equation of proportional relationship.
> Interviewer: Can you show us algebraic expression and graph of two variables that are related proportionally.
> Gaye: The equation should be as y=mx or x=my because the ratio of y/x or x/y always should be equal to a real number like m. We call "m" constant of proportionality.
> Interviewer: Is "m" called anything else?
> Gaye: It is also slope.
> Interviewer: Why?
> Gaye: Look (drawing a line graph on a sheet). In order to find slope, one should look at the changes in the x and y axis. I am assumed the change for x is "a", the change for y is "b" and then the slope of the line would be "a/b". In that case, the equation of the line would be "y=a/b.x", a/b is equal to m as I said earlier. Additionally, the slope for a line should be the same for every point in the line. Similarly, the ratio in a graph of a proportional relationship should be the same at all the time and every points. In this way, we can make connection between them.

Gaye not only made connection between constant of proportionality and slope, but she also made connection among table, graph and algebraic expression of a proportional situation. Furthermore, she stated that "*m*" in the equation, "*y=mx*", represented the slope and the constant of proportionality (*key understanding 4*). As aforementioned, she appears to understand the relationship between the slope and the constant of proportionality, but she did not mention the unit rate that was equal to both of them.

4.2.3.2 Mine's Findings about Understanding the Mathematical Relationships Embedded in Proportional Situations

Defining Proportionality Concepts

Mine knew true definitions of the terms ratio and proportion. Additionally, she could explain the differences between the terms. Moreover, she used proportional reasoning language in her definitions. For example, in the second student teaching, she defined ratio as, "Ratio is a multiplicative comparison of two quantities. It tells us how much of one thing there is compared to another" and defined proportion as, "Proportion is equality of at least two ratios". It can be concluded that she utilized *key understanding 1* and *key understanding 3* to define ratio and proportion terms.

Mine could define the direct and inverse proportions by using proportional relationships' multiplicative nature (*key understanding 1*). For instance, she defined the terms in the post-interview as follows:

> If one variable increases, the other one also increases by the same ratio, and if one variable decreases, the other one also decreases by the same ratio; the variables have a directly proportional relationship. However, if one variable increases, the other one decreases by the same ratio and if one variable decreases, the other one increases by the same ratio; they have an inversely proportional relationship.

She emphasized that the change between the quantities had to be in the same ratio in a proportional situation. In other words, she considered the constant ratio that defined the relationships between the variables in direct and inverse proportions.

Mine could define the constant of proportionality for both direct and inverse proportions. To illustrate, in both post-interview and student teaching, she stressed that directly proportional relationships were represented by $y/x=k$ and inversely proportional relationships were represented by $x.y=k$, in which k was the constant of proportionality. Additionally, she explained that k was also the slope of the graph of the proportional variables and made connections between the constant of proportionality and the slope. Yet, she did not mention that k was also the unit rate. Therefore, it can be concluded that she inadequately utilized *key understanding 4*.

In addition, in the post-interview, as she defined the constant of proportionality, she exemplified the relationship between the amount of apples bought and the amount of money given and she argued that the constant ratio between the elements of the same measure space (i.e., the amount of apples/ the amount of money), *within-ratio*, was the constant of proportionality. In contrast, in the second student teaching, she said that the constant of proportionality was equal to both *within* and *between-ratio* while she was teaching her students how to solve a proportional problem. The problem was "If a car needs 20 liters of fuel to travel 140 kilometers, calculate the amount of fuel the car will need to travel 35 kilometers?" She explained it as:

> Mine: In order to solve the problem, let's find the constant of proportionality.
> Student: It is 7/1
> Mine: Yes, 7/1 or 1/7 is the constant of proportionality. But, can we find other constants of proportionality for the problem?
> Students: No.
> Mine: We can find because the ratios of kilometers to liters, liters to liters or kilometers to kilometers are all the constants of proportionality. For example, 20litres / 5litres = 140kilometers / 35kilometers = 4; since the ratios are equal to 4, it is also the constant of proportionality.

As seen in the excerpt above, she held the misunderstanding that the constant of proportionality was equal to both *within* and *between-ratios*. Unfortunately, it was the same mistake she did prior to instructional module. Unlike the post-interview results, she appears to keep the misunderstanding. It seems that she went back her inaccurate statement about constant of proportionality while she was teaching since she did not internalize the concept. However, she emphasized that the constant of proportionality could be expressed in two ways. Moreover, in the post-interview, when the researcher asked her in which posttest questions she used the constant of proportionality, she accurately determined the constants of proportionality in the proportional situations.

Determining Proportional Relationships

Mine could explain the presence of proportional relationships by using the *key understandings*. To demonstrate, in the post-interview, she explained how to decide whether two variables were proportional to each other as follows:

> If two variables, x and y, were proportional, there has to be a constant multiple between the measures of them. In other words, a constant number has to exist when the measure of the first variable, x, is divided by the measure of the second variable, y.

Mine made use of proportional relationships' multiplicative nature (*key understanding 1*) and the equality of the rate pairs in a proportional situation (*key understanding 3*) to determine the presence of proportionality.

Mine could adequately use the *key understandings* to justify her classifications about proportional relationships. The justifications of Mine about her classifications in PRET questions 11-22 in the posttest and post-interview were evidence for the claim. In order to get in-depth information, in the post-interview, the researcher asked Mine to explain how she decided the relationships between the quantities in PRET questions 11-22. Table 4.5 shows the classifications and the explanations of Mine for each question in the posttest and post-interview. Although all of the classifications were the same in both instruments, there were some additional *key understandings* and detailed explanations in the post-interview.

Table 4.5 Classifications and explanations about proportional relationships in PRET questions 11-22

	Posttest		**Post-interview**	
Question	**Classification**	**Explanation**	**Classification**	**Explanation**
11 (Language)	Not Proportional	Key Understanding 1	Not proportional	Key Understanding 1
12 (Language)	Proportional	Key Understanding 1	Proportional	Key Understanding 1
13 (Language)	Not Proportional	Key Understanding 1	Not Proportional	Key Understanding 1
14 (Graph)	Proportional	Key Understanding 1 Key Understanding 2	Proportional	Key Understanding 2 Key Understanding 4 Key Understanding 1
15 (Graph)	Not Proportional	Key Understanding 1 Key Understanding 2	Not Proportional	Key Understanding 1 Key Understanding 2
16 (Graph)	Not Proportional	Key Understanding 2 Key Understanding 1	Not proportional	Key Understanding 2 Key Understanding 1
17 (Equation)	Not Proportional	Key Understanding 1	Not Proportional	Key Understanding 1
18 (Equation)	Not Proportional	Key Understanding 1	Not Proportional	Key Understanding 1
19 (Equation)	Proportional	Key Understanding 1	Proportional	Key Understanding 1 Key Understanding 4
20 (Table)	Proportional	Key Understanding 1 Key Understanding 3	Proportional	Key Understanding 1 Key Understanding 4 Key Understanding 3
21 (Table)	Not Proportional	Key Understanding 1	Not Proportional	Key Understanding 1 Key Understanding 4
22 (Table)	Not Proportional	Key Understanding 1	Not Proportional	Key Understanding 2 Key Understanding 1

As shown in the Table 4.5, Mine used the four *key understandings* to justify her classifications about proportional relationships. In addition, her foremost justification to explain whether a relationship was proportional or not was that proportional relationships were multiplicative in nature (*key understanding 1*). Actually, in both posttest and post-interview, she made use of *key understanding1* for explaining all the relationships. However, while she used *key understanding 1* in some questions at alone, in some questions, she used the other *key understandings,* as well. For example, in the posttest, she utilized only *key understanding 1* to explain why the relationships presented in language (question 11, 12 and 13), equation (question 17, 18 and 19) and table (question 21 and 22) were proportional or not. However, she used *key understanding 2*, besides *key understanding 1*, to justify her classifications in questions 14, 15 and 16 which were presented in graphs. Nature of the representation seems to affect her selection of the *key understanding* in these questions because the questions were presented as graph which allowed her to utilize *key understanding 2*. In a similar way, in the post-interview, she used *key understanding 2* to justify her classifications for question 14, 15 and 16.

Mine did not provide any explanation based on a misconception. Additionally, she appears to eliminate her misconception about a ratio could not have a zero as its second component. To illustrate, in the post-interview, she explained why the relationship between variables in question 14, which was presented in a graph, was proportional as follows:

> Mine: They are proportional because the graph of proportional quantities passes through the origin.
> Researcher: Why? Can you explain?
> Mine: The equation of graph that passes through the origin is $y=mx$, and so, if x is two, y will be $2m$. I mean that x and y increases by the same ratio. Therefore, they are proportional.
> Researcher: You said the graph passed through the origin. Can you explain that?
> Mine: The graph of proportional quantities can pass through the point of (0,0) because I can compare zero pencil to zero pencil or anything else. Moreover, their equation is $y=mx$, so if x is zero, y will be zero.

Mine appears to know that a ratio might have a zero as its second component. Moreover, she explained the meaning of the ratio of zero to zero by giving an example. It can be concluded that she did not have any difficulty in *key understanding 2* which meant that proportional relationships were presented graphically by lines that passed through the origin. In addition, it seems that she knew that proportional relationships could be expressed algebraically in the form $y=mx$. Furthermore, she made connections between the algebraic expression and the graph of the proportional situation. In the meantime, she made use of *key understanding 1* and *key understanding 2* to explain proportionality.

Data revealed that Mine understood the relationship between the slope and the constant of proportionality. For instance, in the post-interview, she explained why the relationship between variables in question 20, which was presented in a table, was proportional as follows:

> They are proportional because the division of x by y is always equal to 2/3. Let's divide, 4/6, 6/9, etc. All of the ratios are equal to 2/3, so the equation of the variables is y=3/2x. It is similar to question 14, but in here, the slope of the graph is 3/2. It is also the division of y values by x values. That is to say, it is also the constant of proportionality.

Mine could write the equation ($y=3/2x$) of the relationship presented in table and explained the presence of proportionality by checking several rate pairs of x and y values in the table. She concluded that since all the rate pairs were equivalent, the variables were proportional (*key understanding 3*). Moreover, she argued that 3/2 was both constant of proportionality and slope of the graph of x and y values (*key understanding 4*). She understood the relationship between the slope and the constant of proportionality, but she did not mention the unit rate that was also equal to both of them. In addition, she appears to know that the constant of proportionality was equal to *within-ratio* instead of *between-ratio*.

Mine could explain why the quadratic relationships (e.g., $y=mx^2$) were not proportional by using the *key understandings*. To illustrate, in question 22, she could write the equation ($y=1/2x^2$) of the relationship presented in table and stated:

> Let's look at the table, once x is 4, y is 8 and once x is 8, y is 32. As you can see, as x increases by 2, y increases by 4. For the variables to be proportional, they have to increase or decrease by the same ratio, so they are not proportional. Yet, y and x^2 can be proportional because when x^2 is 16, y is 8 and when x^2 is 64, y is 32. Both of them increase by 4… the relationship in the question is similar to question 16; in other words, the graph goes through the origin, but it is not a straight line, so they are not proportional.

As seen in the above quote, Mine made use of the *key understanding* 1 and noted that there was a proportional relationship between x^2 and *y* instead of *x* and *y* since they increased by the same ratio. Additionally, she knew that proportional relationships were shown graphically by a line through the origin (*key understanding 2*).

4.2.3.3 Ela's Findings about Understanding the Mathematical Relationships Embedded in Proportional Situations

Defining Proportionality Concepts

Ela could provide valid definitions of ratio and proportion. Moreover, she used proportional reasoning language in her definitions. To illustrate, in the post-interview, she defined ratio as "Ratio is a comparison of two quantities". She appears to eliminate the misconception that ratio was the same thing with "division". On the other hand, she defined proportion as "equality of at least two ratios" by using *key understanding 3*. Furthermore, she knew that ratio and proportion were different terms and could explain the differences between them.

Ela could define direct and inverse proportion by utilizing proportional relationships' multiplicative nature (*key understanding 1*). Additionally, she not only pointed out that the direction of change of the related quantities had to be the same, but she also mentioned that the change between the quantities had to be in the same ratio. It seems that she considered the constant ratio that defined the relationship between the quantities.

Ela could provide a true definition of the constant of proportionality. In addition, she emphasized that both k and $1/k$ were the constant of proportionality; in other words, she probably realized that a constant multiplicative relationship existed between two quantities and it could be expressed in two ways. Furthermore, she could accurately find it in proportional situations. For example, in the second student teaching, she drew a graph of a relationship between time and distance of a bicycle travelling from one place to another. She found the constant of proportionality in the situation as follows:

> Ela: The problem gives us the graph of a relationship between time and distance of a bicycle travelling from one place to another. Since the relationship between the variables is directly proportional, the graph is a line that goes through the origin.
> Student: How did we understand the direct proportion?
> Ela: As I mentioned earlier, in a direct proportion the variables have to increase in the same ratio. Look at the graph, when the time increases by two (showing 1 and 2), the distance also increases by two (showing 15 and 30). Let's create a table for the same variables (she drew a table on the board and filled it). Look at the multipliers. Time and distance increase by the same multipliers. Does everybody agree that they are proportional?
> Students: Yes, we understood.
> Ela: Let's find the constant of proportionality and the slope in the situation. By using the graph, we can easily find it; the slope is (writing on the board: 15/1=30/2=45/3=60/4=15) 15. Since they are proportional, the constant of proportionality has to be 15, too. Find it.
> Students: It is 15.
> Ela: Yes, it is true. The constant of proportionality is 15 because all of the ratios are equal to a constant number that is 15 (writing on the board: 15/1=30/2=45/3=60/4=15).
> Student: They are equal.
> Ela: They are equal because the variables are directly proportion.

As seen in the above excerpt, Ela knew that proportional relationships were presented graphically by lines that passed through the origin (*key understanding 2*). Furthermore, she utilized proportional relationships' multiplicative nature (*key understanding 1*) to determine the presence of proportionality. In addition, she could make a connection between the graph and table of the proportional situation. Moreover, she accurately found the constant of proportionality by utilizing the equality of the rate pairs in the proportional situation (*key understanding 3*). Furthermore, she highlighted that in a proportional situation the slope and the constant of proportionality were the same. It can be concluded that she understood the relationship between the slope and the constant of proportionality. Yet, she did not note that a proportional relationship could be represented symbolically as $y = kx$, where k was the slope, the unit rate, and the constant of proportionality (*key understanding 4).* Thus, it can be concluded that she inadequately utilized *key understanding 4.*

Determining Proportional Relationships

Ela could explain the presence of proportional relationships by using the *key understandings*. To demonstrate, in the post-interview, when the researcher asked her how to determine presence of a proportional relationship between two variables, she said, "I look the multipliers, they must increase or decrease by the same multiplier." Similarly, in the second student teaching, Ela explained how she distinguished proportional quantities from non-proportional quantities as follows:

> In order to determine direct proportion, we should take multipliers into consideration. I mean, when one quantity increases by two, the other quantity has to increase by two, too. It cannot increase by three or any other multiplier. In other words, we should seek a multiplicative relationship between quantities.

As can be seen, in both post-interview and second student teaching, she made use of the *key understanding 1* to determine the presence of proportionality. Moreover, she used proportional reasoning language in her explanations. Additionally, in the second student teaching, she stated that in a proportional situation, the rate pairs had to be equal to each other (*key understanding 3*).

Ela could adequately use the *key understandings* to justify her classifications about proportional relationships. The justifications of Ela about her classifications in PRET questions 11-22 in the posttest and post-interview were evidence for the claim. In order to get in-depth information, in the post-interview, the researcher asked Ela to explain how she decided the relationships between the quantities in PRET questions 11-22. Table 4.6 indicates the classifications and the explanations of Ela for each question in the posttest and post-interview. Although all of the classifications were the same in both instruments, there were some additional *key understandings* and detailed explanations in the post-interview.

As shown in the Table 4.6, Ela explained how she classified each situation as proportional or nonproportional by using the four *key understandings.* In particular, Ela justified rationales of her classifications by using three of the *key understandings* in the posttest (*key understanding 1*, *2* and *4*) and all types of the *key understandings* in the post-interview. She mostly utilized the *key understanding 1* to decide proportionality. In fact, in both posttest and post-interview, her foremost justification was *key understanding 1.* Moreover, she not only considered that the direction of change of the related quantities had to be the same, but she also considered that the change between the quantities had to be in the same ratio.

Table 4.6 Classifications and explanations about proportional relationships in PRET questions 11-22

	Posttest		**Post-interview**	
Question	**Classification**	**Explanation**	**Classification**	**Explanation**
11 (Language)	Not Proportional	Key Understanding 1	Not proportional	Key Understanding 1
12 (Language)	Proportional	Key Understanding 1	Proportional	Key Understanding 1
13 (Language)	Not Proportional	Key Understanding 1	Not Proportional	Key Understanding 1 Key Understanding 3
14 (Graph)	Proportional	Key Understanding 2 Key Understanding 4	Proportional	Key Understanding 2 Key Understanding 1
15 (Graph)	Not Proportional	Key Understanding 2 Key Understanding 1	Not Proportional	Key Understanding 2 Key Understanding 1 Key Understanding 4
16 (Graph)	Not Proportional	Key Understanding 1 Key Understanding 2	Not proportional	Key Understanding 3 Key Understanding 2 Key Understanding 1
17 (Equation)	Not Proportional	Key Understanding 2 Key Understanding 1	Not Proportional	Key Understanding 1 Key Understanding 2
18 (Equation)	Not Proportional	Key Understanding 1	Not Proportional	Key Understanding 1 Key Understanding 2
19 (Equation)	Proportional	Key Understanding 2 Key Understanding 1	Proportional	Key Understanding 1 Key Understanding 2
20 (Table)	Proportional	Key Understanding 1	Proportional	Key Understanding 1 Key Understanding 3
21 (Table)	Not Proportional	Key Understanding 1 Key Understanding 4	Not Proportional	Key Understanding 1 Key Understanding 4
22 (Table)	Not Proportional	Key Understanding 1 Key Understanding 4	Not Proportional	Key Understanding 1 Key Understanding 4

It seems that the nature of the representations effected Ela's selection of the *key understandings*. For instance, in the questions (questions 14, 15 and 16) presented in graphs, she used *key understanding 2*. Perhaps this is because the questions were presented as graph which allowed her to utilize the *key understanding.*

Ela appears to have understood the relationship between the slope and the constant of proportionality. To illustrate, in the post-interview, she explained why the relationship between variables in question 15, which was presented in a graph, was proportional as follows:

> Ela: They are not proportional because the graph does not pass through the origin.
> Researcher: Why? Can you explain in a more detailed way?
> Ela: Because the equation of the graph is $y=mx+n$. "$+n$" destroys the proportion. It shows that there is an additive relationship between variables. Let's give numeric examples. If x is one, y will be m+n; if x is 2, y will be 2m+n. There is not a constant factor between x and y values. I mean, they do not increase by the same ratio. Thus, they are not proportional.
> Researcher: Can you explain your rationale by using any other strategy?
> Ela: We can also look the slope.
> Researcher: How?
> Ela: In the equation, $y=mx$, which is proportional, m is both slope and constant of proportionality. But, in the equation, $y=mx+n$, m is the slope, but it is not the constant of proportionality.

At the beginning of the excerpt, Ela justified her rationale by using *key understanding 2*. Then, she mentioned that for two variables to be proportional there had to be a multiplicative relationship between them (*key understanding 1).* Finally, she could write the equation of a proportional relationship as $y=mx$ and said that the slope and the constant of proportionality were equal to "m" (*key understanding 4)*. It seems that she understood the relationship between the slope and the constant of proportionality, but she did not mention the unit rate that was also equal to both of them. In addition to that, she appears to know that in the functions of the form

$y=mx$, x was proportional to y and in the functions of the form $y=mx+n$, x was not proportional to y.

Ela made connections among table, graph and algebraic expression of the given proportional situations. To illustrate, in the questions presented in graphs, firstly, she wrote the algebraic expressions of the situations, and then, she made decisions about proportionality. Additionally, in the questions presented in algebraic expressions, she drew the graphs of the situations to decide proportionality by using *key understanding 2*.

4.3 The Difference between Pre-service Teachers' Proportional Reasoning before and after the Practice-based Instructional Module

4.3.1 Differences in Approaches to Different Problem Types

In order to investigate pre-service teachers' approaches to different problem types before and after participation in a practice-based instructional module, the solution strategies and processes of pre-service teachers in solving the different problem types were analyzed.

4.3.1.1 Differences in Gaye's Approaches to Different Problem Types

Missing Value Problems: Solution Strategies and Processes

Gaye could find correct answers of missing value problems in all relevant instruments before and after the instructional module. While her solution strategies were limited before the instructional module (i.e., *cross-multiplication, isolating the unknown, factor of change, unit rate*), she utilized a broad range of strategies after the instructional module (i.e., *factor of change*, *factor of change strategy* in a ratio table, *building-up, cross-multiplication, unit rate, building-up* in a ratio table, and *constant of proportionality*). Moreover, unlike the results prior to instructional module, she mostly utilized informal strategies that highlighted multiplicative relationships between variables rather than formal strategies in which rules and properties of algebra were used in a rote manner. To illustrate, in the PRET, there were four missing value problems (questions 1-4) which did not have any context. In the pretest, while she used only two strategies, both of which were formal strategies (i.e., *cross-multiplication*, *isolating the unknown*), in the posttest, she used four different strategies (i.e., *between-ratio, building-up, factor of change, cross-multiplication*) to solve these questions. Similarly, in the first student teaching, Gaye used two different strategies, *factor of change* and *cross-multiplication*; however, in the second time, she used four different strategies, all of which were informal strategies, *unit rate, building-up, constant of proportionality* and *factor of change* (see Table 4.7). Another evidence for the claim that she mostly utilized informal strategies after the instructional module is that while she solved a problem about cost of balloons by using *cross-multiplication* in the pre-interview, in the post-interview she found the scale factor and solved the same problem by using *factor of change strategy*. Moreover, in the pre-interview, when the researcher asked her which strategy she would teach to her students as the most efficient strategy, she told a formal strategy, *isolating the unknown strategy*, whereas she stated the informal strategies, *building-up* and *factor of change* in the post-interview.

Table 4.7 Gaye's solution strategies to missing value problems

	Pre	Post
PRET question 1	Cross-multiplication & Isolating the unknown	Factor of change & Factor of change (Ratio table)[a]
PRET question 2	Cross-multiplication & Cross-multiplication	Factor of change & Building-up
PRET question 3	Cross-multiplication & Isolating the unknown	Factor of change & Cross-multiplication
PRET question 4	Isolating the unknown & Cross-multiplication	Factor of change & Building-up
PRET question 23	Factor of change	Factor of change (Ratio table)[a]
Student Teaching	Factor of change Factor of change Factor of change Cross-multiplication Cross-multiplication	Unit Rate Unit Rate Unit Rate & Building-up (Ratio table)[b] Constant of proportionality Factor of change Factor of change
Interview	Cross-multiplication Cross-multiplication Cross-multiplication & Factor of change Isolating the unknown & Unit rate	Factor of change Factor of change

[a]Factor of change strategy is used in a ratio table.
[b] Building-up strategy is used in a ratio table.

Gaye had difficulty in recognizing efficient strategies, which facilitated the computations required to find missing values, before the instructional module, whereas she mostly recognize and use efficient strategies after the instructional module. For example, she mostly used efficient strategies to solve missing value problems in the posttest. In contrast, she could not use efficient strategies to solve any of the problems in the pretest. Likewise, in the pre-interview, she solved a missing value problem by using *cross-multiplication* although there was an integer scale factor. However, in the post-interview, in order to solve a similar problem, she easily recognized the integer scale factor and solved the problem by using *factor of change strategy*, which was an efficient strategy for the problem.

Unlike the results prior to instructional module, Gaye did not see *cross-multiplication* as an indispensable and the most important strategy anymore.

Moreover, she generally did not prefer to use the strategy. For instance, she used the strategy to solve only one missing value problem in the relevant instruments. In addition, she tried to teach her students different strategies from *cross-multiplication* in the second student teaching, while she taught *cross-multiplication* as a leading and the most important strategy in the first teaching.

In contrast to results before the instructional module, Gaye could provide valid explanations for her solutions by using proportional reasoning language and expressed the meanings of the quantities in the proportional situations.

It is sad to say that she did not have flexibility in unitizing and reunitizing quantities even after the proportional reasoning instructional module. That is, she could not use composite units when the quantities suggest that it was more convenient than using singleton units.

Furthermore, after the instructional module, while she recognized *between-ratios* in a proportion, she had difficulty in recognizing *within-ratios* as she did earlier. For instance, in both of the tests, she could not recognize and use *within-ratio* though there was an integer *within-ratio* in the first question of PRET. Further, in the student teachings and interviews, she mostly utilized the *between-ratio* in a proportion to solve missing value problems and did not mention the *within-ratio* even in situations where it was integer.

Numerical Comparison Problems: Solution Strategies and Processes

In order to solve numerical comparison problems, Gaye used two accurate strategies, which were *converting decimal expressions* and *building-up*, before the instructional module. However, after the instructional module, she utilized three different strategies, *building-up, fraction strategy*, and *converting decimal expressions,* all of which were accurate strategies.

Unlike the results prior to instructional module, Gaye did not use an inaccurate additive strategy to solve numerical comparison problems. To illustrate, in the PRET, there was a numerical comparison problem (question 24) which presented three rectangular clothes with dimensions and asked the cloth that was “the most square”. In the pretest, Gaye used an inaccurate additive strategy; thus, she inaccurately solved the problem. On the other hand, in the posttest, she accurately solved the problem by utilizing a multiplicative strategy, *converting decimal expressions*, and explained her solution by using proportional reasoning language. Moreover, while Gaye had difficulties in making sense of quantities in a numerical comparison problem prior to the instructional module, she clearly explained the quantities that were calculated to solve problems after the instructional module.

Gaye appears to have difficulties in realizing *part-to-whole ratios*even after the instructional module. For example, in order to solve one of the numerical comparison problems of PRET (question 6), she did not use *part-to-whole rations* in any solution of her in both administrations of the PRET. Moreover, in the pre-interview, she struggled to make sense of the quantities in a student’s solution in which *part-to-whole ratio* was utilized.

In both student teachings, Gaye did not ask any numerical comparison problem to her students. Moreover, in both interviews, she did not mention any situation that could be a numerical comparison problem while she was giving examples of proportional situations in real life.

Qualitative Reasoning Problems: Solution Strategies and Processes

Contrary to the results prior to instructional module, it is good to say that she could make multiplicative comparisons to solve qualitative problems, but she could not interpret the qualitative relationship that existed between two variables without giving numerical examples. For instance, in the PRET, there were two qualitative problems (questions 7 and 8) which contained no numerical value but required the balancing of quantities. In both pretest and posttest, Gaye accurately solved the problems. However, in the pretest, she made an additive comparison instead of multiplicative one in the question 8. In the posttest, although she made multiplicative comparisons to solve the both problems, she could not make qualitative comparisons that did not depend on numerical values because she solved the problems by giving numerical examples.

In both student teachings, Gaye did not ask any qualitative reasoning problem to her students. Moreover, in both interviews, she did not mention any situation that could be a qualitative reasoning problem while she was giving examples of proportional situations in real life.

4.3.1.2 Differences in Mine's Approaches to Different Problem Types

Missing Value Problems: Solution Strategies and Processes

Mine could find correct answers of missing value problems in all relevant instruments before and after the instructional module. While her solution strategies were limited before the instructional module (i.e.., *cross-multiplication, isolating the unknown, factor of change*), she utilized a broader range of strategies after the instructional module (i.e., *within-ratio, factor of change, equivalent fractions, factor of change* in a ratio table, *building-up, unit rate*, and *constant of proportionality*). Moreover, unlike the results prior to instructional module, she utilized informal strategies that highlighted multiplicative relationships between variables rather than formal strategies in which rules and properties of algebra were used in a rote manner. To illustrate, in the PRET, there were four missing value problems (questions 1-4) which did not have any context. In the pretest, while she used only two strategies, both of which were formal strategies (i.e., *cross-multiplication, isolating the unknown*), in the posttest, she used four different strategies (i.e., *within-ratio, factor* *of change, equivalent fractions, factor of change* in a ratio table), all of which were informal strategies, to solve these problems (see Table 4.8). Correspondingly, PRET had one more missing value problem (question 23) which had a context involving similar figures. While she solved the problem by applying algorithmic procedures (i.e., *cross-multiplication*) in the pretest, she solved it by using *factor of change strategy,* which highlighted multiplicative relationships, in the posttest. In a similar way, in the first student teaching, Mine used only two different strategies, *factor of change* and *cross-multiplication*. However, in the second time, she used six different strategies, *building-up, factor of change, unit rate, constant of proportionality, equivalent fractions* and *factor of change* in a ratio table, all of which were informal strategies.

Another evidence for the claim that she utilized informal strategies after the instructional module is her solution strategies in the post-interview, all of which were informal strategies. For instance, while she solved a problem about cost of balloons by using *cross-multiplication* in the pre-interview, she found the noninteger scale factor and solved the same problem by using an informal strategy, *factor of change,* in the post-interview. Additionally, in the pre-interview, when the researcher asked her which strategy she would teach to her students as the most efficient strategy, she told a formal strategy, *cross-multiplication strategy*. However, in the post-interview, she told the informal strategies, *factor of change* and *building ratio tables,* and she stated that these strategies were effective since they facilitated realization of multiplicative relationships between variables.

Before the instructional module, Mine sometimes had difficulty in recognizing efficient strategies, which facilitated the computations required to find missing values, whereas she recognized and used efficient strategies after the instructional module. For example, in the posttest, she could use efficient strategies while she was solving missing value problems. In specific, she recognized and used both *within* and *between-ratios* in the posttest, whereas she did not use *within-ratio* even though there was an integer *within-ratio* in the pretest.

Table 4.8 Mine's solution strategies to missing value problems

	Pre	Post
PRET question 1	Cross-multiplication & Cross-multiplication	Within-ratio & Factor of change
PRET question 2	Cross-multiplication & Cross-multiplication	Equivalent fractions & Factor of change
PRET question 3	Cross-multiplication & Isolating the unknown	Factor of change (Ratio table)[a] & Factor of change
PRET question 4	Isolating the unknown & Cross-multiplication	Factor of change & Within-ratio
PRET question 23	Cross-multiplication	Factor of change
Student Teaching	Factor of change Factor of change Cross-multiplication Cross-multiplication	Building-up Factor of change Unit Rate Equivalent fractions & Constant of Proportionality Factor of change Factor of change Factor of change Factor of change (Ratio table)[a]
Interview	Factor of change Factor of change Cross-multiplication Cross-multiplication	Factor of change Factor of change & Constant of Proportionality Constant of Proportionality

[a]Factor of change strategy is used in a ratio table.

Data after the instructional module revealed that Mine did not see *cross-multiplication* as an indispensable and the most important strategy, anymore. Moreover, she did not prefer to use the strategy. For instance, in the post-interview, she stated that *cross-multiplication* included applying memorized procedures without logic. Moreover, in contrast to pre-interview, her foremost strategy was not *cross-multiplication* in the post-interview. In fact, *factor of change* and *building a ratio table* were the leading strategies which she would prefer to solve missing value problems. Additionally, in the second student teaching, she tried to teach her students informal strategies highlighting multiplicative relationships, while in the first teaching, she taught *cross-multiplication* as a rule that should be memorized to solve direct proportion problems.

Before the instructional module, Mine could realize the multiplicative relationship between quantities. However, she did not use proportional reasoning language in her explanations and could not explain the meanings of the quantities in the proportional situations. In contrast, after the instructional module, she explained both multiplicative relationships and the meanings of the quantities with a proportional reasoning language.

It is sad to say that she did not have flexibility in unitizing and reunitizing quantities even after the proportional reasoning instructional module. That is, she had difficulties in using composite units when the quantities suggest that it was more convenient than using singleton units.

Numerical Comparison Problems: Solution Strategies and Processes

In order to solve numerical comparison problems, Mine used an accurate strategy, *fraction strategy,* before the instructional module. However, after the instructional module, she utilized three different strategies, *fraction strategy, building-up,* and *converting decimal expressions,* all of which were accurate strategies.

Unlike the results prior to instructional module, Mine did not use an inaccurate additive strategy to solve any numerical comparison problem. To illustrate, in the PRET, there was a numerical comparison problem (question 24) which presented three rectangular clothes with dimensions and asked the cloth that was “the most square”. In the pretest, Mine used an inaccurate additive strategy; thus, she inaccurately solved the problem. On the other hand, in the posttest, she accurately solved the problem by utilizing a multiplicative strategy, *converting decimal expressions,* and explained her solution by using proportional reasoning language.

Moreover, while Mine sometimes had difficulties in making sense of quantities in a numerical comparison problem prior to the instructional module, she clearly explained the quantities that were calculated to solve problems after the instructional module.

Results both before and after the instructional module indicated that Mine could realize *part-to-whole* and *part-to-part ratios* in a numerical comparison problem. For example, in order to solve one of the numerical comparison problems of PRET (question 6), she used both of the ratios in the posttest. Moreover, in the pre-interview, she could make sense of the quantities in student's solutions in which *part-to-whole* and *part-to-part ratios* were utilized.

In both student teachings, Mine did not ask any numerical comparison problem to her students. Moreover, in both interviews, she did not mention any situation that could be a numerical comparison problem while she was giving examples of proportional situations in real life.

Qualitative Reasoning Problems: Solution Strategies and Processes

Results both before and after the instructional module revealed that Mine sometimes could make multiplicative comparisons to solve qualitative problems. Moreover, she could interpret the qualitative relationship that existed between two variables without giving numerical examples. For instance, in the PRET, there was a qualitative problem (questions 7) which contained no numerical value but required the balancing of quantities. In both pretest and posttest, Mine accurately solved the problem by making multiplicative comparisons.

Mine had difficulty in recognizing multiplicative relationships in a qualitative reasoning problem even after the instructional module. Therefore, she sometimes used an inaccurate additive strategy when she needed to use a multiplicative strategy to solve qualitative reasoning problems. To illustrate, in both pretest and posttest, in a qualitative problem (PRET question 8), she could not find correct answer of the problem because she made an additive comparison instead of a multiplicative comparison.

In both student teachings, Mine did not ask any qualitative reasoning problem to her students. Moreover, in both interviews, she did not mention any situation that could be a qualitative reasoning problem while she was giving examples of proportional situations in real life.

4.3.1.3 Differences in Ela's Approaches to Different Problem Types

Missing Value Problems: Solution Strategies and Processes

Ela could find correct answers of missing value problems in all relevant instruments before and after the instructional module. Before the instructional module, although she used a considerable number of strategies (i.e.., *cross-multiplication, isolating the unknown, factor of change, constant of proportionality, inverse proportion algorithm*) to solve missing value problems, she mostly utilized formal strategies in which rules and properties of algebra were used in a rote manner. However, after the instructional module, she mostly used informal strategies that highlighted multiplicative relationships between variables and utilized a broader range of strategies (i.e., *cross-multiplication*, *factor of change, factor of change* in a ratio table, *constant of proportionality, building-up, unit rate*). In the PRET, there were four missing value problems (questions 1-4) which did not have any context. In both pretest and posttest, Ela used only two strategies: *cross-multiplication* and *isolating the unknown* in the pretest, and *cross-multiplication* and *factor of change* in the posttest (see Table 4.9). Yet, while in the pretest, she used only formal strategies, in the posttest, she used both a formal and an informal strategy. Similarly, PRET had one more missing value problem (question 23) which had a context involving similar figures. While Ela solved the problem by applying algorithmic procedures (i.e., *cross-multiplication*) in the pretest, she solved it by using *factor of change strategy* in the posttest. Additionally, in the posttest, she explained her answer by using multiplicative relationships and proportional reasoning language, whereas in the pretest, she did not provide an explanation for her solution that went beyond a description of the steps she had taken to determine the solution.

Table 4.9 Ela's solution strategies to missing value problems

	Pre	Post
PRET question 1	Cross-multiplication & Isolating the unknown	Cross-multiplication & Between-ratio
PRET question 2	Cross-multiplication & Isolating the unknown	Cross-multiplication & Between-ratio
PRET question 3	Cross-multiplication & Isolating the unknown	Cross-multiplication & Between-ratio
PRET question 4	Cross-multiplication & Isolating the unknown	Cross-multiplication & Between-ratio
PRET question 23	Cross-multiplication	Factor of change
Student Teaching	Cross-multiplication Cross-multiplication Factor of change Factor of change & Constant of Proportionality Inverse proportion algorithm Inverse proportion algorithm	Factor of change (Ratio table)[a] & Constant of proportionality Factor of change Factor of change (Ratio table)[a] & Constant of proportionality Factor of change (Ratio table)[a] & Constant of proportionality Constant of proportionality & Unit rate Constant of proportionality & Unit rate Factor of change Factor of change Unit rate
Interview	Cross-multiplication Cross-multiplication Cross-multiplication	Factor of change Factor of change Building-up & Constant of proportionality

[a]Factor of change strategy is used in a ratio table.

Another evidence for the claim that she utilized informal strategies after the instructional module is her solution strategies in the second student teaching. For instance, although the number of strategies she used to solve missing value problems in the student teachings did not change, in the second student teaching, she used informal strategies highlighting multiplicative relationships and tried to teach the meanings of concepts (e.g. inverse proportion) and quantities. However, in the first time, she taught *cross-multiplication* and *inverse proportion algorithm* as memorized rules without highlighting multiplicative relationships, and encouraged students to use these strategies. Moreover, she was aware that she solved the problems in the first student teaching by applying memorized rules and arithmetic procedures. For example, while she utilized *cross-multiplication strategy* to solve a directly proportional problem in the first teaching, she solved the same problem by using *factor of change strategy,* which highlighted multiplicative relationships, in the second time. Similarly, in the first teaching, she solved two inverse proportion problems by *inverse proportion algorithm* whereas in the second teaching, she solved the same problems by using *the constant of proportionality* and *unit rate strategies*. It is good to say that she did not use the *inverse proportion algorithm* or any other memorized procedure to solve any problem in the second student teaching. Unlike her first teaching, in the second teaching, Ela tried to teach the meaning of being inversely proportional and put emphasis on the relationships between quantities in an inversely proportional situation. Furthermore, in both student teachings, Ela utilized the functional relationships between inversely proportional variables (i.e., $x.y=k$). However, in the second teaching, she placed more importance on it and aimed at helping students recognize that the product of two quantities was a constant number in an inverse proportion, which was one types of invariance in proportional relationships.

In a similar manner, in the pre-interview, Ela used only a formal strategy, *cross-multiplication*, in order to solve missing value problems. In contrast, in the post-interview, she used informal strategies (e.g., *factor of change*) and avoided applying memorized rules. For instance, while she solved a problem about cost of balloons by using *cross-multiplication* in the pre-interview, she utilized an adaptation of *factor of change strategy*, in which she simplified the ratio to identify the integer scale factor, in the post-interview.

Data after the instructional module revealed that Ela did not see *cross-multiplication* as an indispensable and the most important strategy, anymore. To illustrate, in the post-interview, although Ela said that *cross-multiplication* was one of the effective strategies because of its quickness, she was aware that it consisted of some memorized procedures lacking meaning. Further, while in the pre-interview, she told that she would teach only formal strategies (i.e., *cross-multiplication*, *isolating the unknown*) to her students, in the post-interview, she stated that she would teach informal strategies (i.e., *building-up, factor of change*) to her students besides formal strategies.

Ela had difficulty in recognizing efficient strategies, which facilitated the computations required to find missing values, before the instructional module, whereas she mostly recognize and use efficient strategies after the instructional module. For example, in the posttest, she mostly used efficient strategies while she was solving missing value problems. In contrast, in the pretest, she could not use efficient strategies to solve any of the problems. Correspondingly, in the pre-interview, she solved missing value problems by using *cross-multiplication* although there was integer *between* or *within-ratios*. However, in the post-interview, in order to solve similar problems, she easily recognized the integer scale factors and solved the problem by using *factor of change strategy,* which was an efficient strategy for the problems.

It is sad to say that she did not have flexibility in unitizing and reunitizing quantities even after the proportional reasoning instructional module. That is, she had difficulties in using composite units when the quantities suggest that it was more convenient than using singleton units.

Additionally, after the instructional module, while she recognized *between-ratio* in a proportion, she had difficulty in recognizing *within-ratio* as she did earlier. For instance, in both of the tests, she could not recognize and use *within-ratio* though there was an integer *within-ratio* in the first question of PRET. Further, in the student teachings and interviews, she mostly utilized the *between-ratio* in a proportion to solve missing value problems and did not mention the *within-ratio* even in situations where it was integer.

Numerical Comparison Problems: Solution Strategies and Processes

In order to solve numerical comparison problems, Ela used two accurate strategies, *cross-multiplication*, *converting decimal expressions*, and an inaccurate strategy, additive strategy, before the instructional module. However, after the instructional module, she utilized two different strategies, *fraction strategy* and *converting decimal expressions*, both of which were accurate strategies.

Unlike the results prior to instructional module, Ela did not use an inaccurate additive strategy to solve any numerical comparison problem after the instructional module. To illustrate, in the PRET, there was a numerical comparison problem (question 24) which presented three rectangular clothes with dimensions and asked the cloth that was "the most square". In the pretest, Ela used an inaccurate additive strategy; thus, she inaccurately solved the problem. However, in the posttest, she accurately solved the problem by utilizing a multiplicative strategy, *converting decimal expressions*, and explained her solution by using proportional reasoning language. Furthermore, while Ela sometimes had difficulties in making sense of quantities in a numerical comparison problem prior to the instructional module, she clearly explained the quantities that were calculated to solve problems after the instructional module.

Results both before and after the instructional module indicated that Ela could realize *part-to-whole* and *part-to-part ratios* in a numerical comparison problem. To illustrate, in order to solve one of the numerical comparison problems of PRET (question 6), while in the pretest, she used *part-to-whole ratios,* in the posttest, she used *part-to-part ratios.* Moreover, in the pre-interview, she could make sense of the quantities in student's solutions in which *part-to-whole* and *part-to-part ratios* were utilized.

In both student teachings, Ela did not ask any numerical comparison problem to her students. Moreover, in both interviews, she did not mention any situation that could be a numerical comparison problem while she was giving examples of proportional situations in real life.

Qualitative Reasoning Problems: Solution Strategies and Processes

Results both before and after the instructional module revealed that Ela could realize multiplicative relationships in a qualitative reasoning problem. To illustrate, in the PRET, there were two qualitative problems (questions 7 and 8) which contained no numerical value, but required the balancing of quantities. In both pretest and posttest, Ela solved both problems accurately by making multiplicative comparisons. However, in the pretest, she sometimes utilized numerical examples to solve the numerical comparison problems. On the other hand, by the end of the instructional module, she could interpret the qualitative relationship that existed between two quantities without giving numerical examples. Moreover, in both tests, she provided a valid explanation by making multiplicative comparisons between quantities.

In both student teachings, Ela did not ask any qualitative reasoning problem to her students. Moreover, in both interviews, she did not mention any situation that could be a qualitative reasoning problem while she was giving examples of proportional situations in real life.

4.3.2 Differences in Distinguishing Proportional from Nonproportional Situations

Pre-service teachers' work on classifying relationships as proportional or nonproportional, identifying ratio as measure and providing examples for proportional and nonproportional relationships were summarized so as to compare pre-service teachers' distinguishing proportional from nonproportional situations before and after receiving a practice-based instructional module based on proportional reasoning. In the following section, the results of the analyses are presented.

4.3.2.1 Differences in Gaye's Findings on Distinguishing Proportional from Nonproportional Situations

Classifying Relationships as Proportional or Nonproportional

Unlike the results prior to instructional module, Gaye did not have difficulty in classifying relationships as proportional or nonproportional after the instructional module. For example, for PRET questions 11-22, while she inaccurately classified five of the relationships as proportional or not in the pretest, she accurately classified all of them in the posttest.

The misconception that all linear relationships were proportional appears to have been eliminated after the instructional module. To illustrate, in the posttest, she accurately classified all of the linear, proportional relationships (questions 12, 14, 19 and 20) as proportional and all of the linear, nonproportional relationships (questions 11, 13, 15, 17 and 21) as nonproportional whereas in the pretest, she inaccurately classified four linear and nonproportional relationship (questions 11, 13, 15 and 21) as proportional. Additionally, in both student teachings, Gaye asked a problem, which was, "A young tree's length increases 20 cm every year. If its first length is 50 cm, draw the length-time graph of the tree for three years." In the first student teaching, she argued that the linear and nonproportional relationship between the length and time was directly proportional. In contrast, in the second time, she accurately classified the relationship between the variables as nonproportional.

Unlike the results before to instructional module, by the end of the module, Gaye knew that a proportional relationship required both linearity and proportionality. For instance, the researcher asked Gaye whether a student was right or not if he said that all linear relationships were proportional in the interviews. Unlike the pre-interview, in the post-interview, Gaye did not agree with the student; in other words, she knew that linearity at alone was not an indicator of proportional relationship, but it was a requirement for the variables to be proportional.

Before the instructional module, Gaye sometimes could not be sure how to decide whether a relationship was proportional or not especially if the relationship was nonproportional. Moreover, she had difficulty to provide evidence to support her claims about proportional relationships. She assumed additive patterns or constant differences as evidence of proportionality. On the other hand, after the instructional module, it was seen that her doubts were removed, and she utilized multiplicative relationships to decide and explain proportional relationships.

Identifying Ratio as Measure

Before the instructional module, Gaye sometimes could identify ratio as a proper method to measure the attributes in relevant instruments. However, after the instructional module, she always could recognize ratio as a proper method to measure the attributes. For instance, in the pretest, she could recognize ratio as an appropriate measure in the context of determining concentration of a mixture and steepness of ski ramps. In the posttest, she could identify ratio as a proper method to measure concentration of a mixture, steepness of ski ramps, shade of paint, and "squareness" of a rectangle. Moreover, unlike the pretest, in the posttest, she used proportional reasoning language in her explanations; specifically, she used the word "ratio" while she was explaining her solutions in which she utilized ratio.

Result prior to instructional module revealed that Gaye used additive comparisons to measure some attributes (i.e., shade of paint, "squareness" of a rectangle) instead of using ratio. But, after the instructional module she could use ratio, which was a multiplicative comparison, to measure the attributes. For instance, in a PRET problem (question 9), which asked to comment on the effectiveness of a boy's strategy about changing the amount of paint by saving its shade, she did not use and realize ratio as a proper measure of shade of the paint in the pretest. However, in the posttest, her criterion for whether the shade of paint would stay the same was based on whether the added mixture would maintain the ratio of white to blue paint. In addition, in another PRET problem (question 24), she did not see ratio as a proper method to measure "squareness" of a rectangle and made an additive comparison to determine the attribute in the pretest. On the other hand, in the posttest, she used a multiplicative strategy and identified ratio as a proper measure in the context of determining "squareness" of a rectangle.

Providing Examples for Proportional and Nonproportional Relationships

Results both before and after the instructional module revealed that Gaye sometimes could not provide valid examples of proportional relationships. To illustrate, in PRET question 5, which asked to provide a word problem which could be solved by the given equation, Gaye could not create a problem in which the quantities related multiplicatively in both pretest and posttest. Moreover, in the pre-interview, when the researcher asked her to give an example of a real life proportional situation, Gaye was not able to create a valid example for a proportional real life situation. She created a linear and nonproportional example instead of a proportional example.

According to results after the instructional module, Gaye sometimes could create valid proportional examples. For example, in the post-interview, when the researcher asked her to give an example of a real life proportional situation, she was able to provide a valid example. Furthermore, she could justify why her example was proportional by using the statement that variables in a directly proportional situation increased by the same ratio.

Gaye could provide valid examples of a nonproportional relationship before and after the instructional module. For instance, in both interviews, when the researcher asked her to give examples of real life nonproportional situations, she was able to give valid examples. However, unlike the pre-interview, she could explain why her examples were nonproportional in the post-interview. In both interviews, one of the relationships exemplified by her was constant relationship. Moreover, in the posttest, she provided an additional nonproportional example in which the relationship between variables was linear.

4.3.2.2 Differences in Mine's Findings on Distinguishing Proportional from Nonproportional Situations

Classifying Relationships as Proportional or Nonproportional

Results both before and after the instructional module indicated that Mine generally did not have difficulty in classifying relationships as proportional or nonproportional. For example, for PRET questions 11-22, in the pretest, she inaccurately classified only a relationship (question 11) as proportional, and in the posttest, she accurately classified all of them.

Mine sometimes assumed that increasing or decreasing at the same time was enough to classify a situation as proportional before the instructional module. Therefore, she sometimes over generalized proportionality; in other words, she inaccurately applied proportional reasoning in situations that had nonproportional relationships like she did in the pretest. However, in the pre-interview, she realized her mistakes in the pretest and appears to have eliminated her misconceptions that a relationship was proportional because as *x* increased, *y* also increased and all linear relationships were proportional. Similarly, in the posttest and post-interview, she mostly explained her classification by using the statement that variables in a proportional situation had to increase or decrease in the same ratio. In addition, she used multiplicative and additive relationships and their connections with proportionality to support her claims.

Moreover, in both student teachings, Mine emphasized that all linear relationships were not proportional. Furthermore, she asked problems that had non-proportional relationships besides problems with proportional relationships. Yet, in the first student teaching, she asked only nonproportional problems with additive relationships and could not exactly explain why the problems were not proportional. On the other hand, in the second time, she introduced both additive and constant relationships and could explicitly explain why they were not proportional.

Unlike the results prior to instructional module, by the end of the module, Mine accurately classified the quadratic relationships in the PRET and provided evidence to support her claims. Her explanations in the post-interview indicated that she learned that if variables had proportional relationships, they also had to have linear relationships. Furthermore, she knew that proportional relationship increase or decrease with the same ratio. In both interviews, Mine stated that all linear relationships were not proportional. For instance, when the researcher asked her whether a student was right or not if he said that all linear relationships were proportional, she said that she did not agree with the student in both interviews.

Identifying Ratio as Measure

Results both before and after the instructional module revealed that Mine sometimes could identify ratio as a proper method to measure the attributes in relevant instruments. For example, in the pretest, she could recognize ratio as an appropriate measure in the context of determining concentration of a mixture and steepness of ski ramps. Furthermore, in the posttest, she could identify ratio as a proper method to measure concentration of a mixture, steepness of ski ramps, and "squareness" of a rectangle. However, unlike the pretest, in the posttest, she used proportional reasoning language in her explanations; specifically, she used the word "ratio" while she was explaining her solutions in which she utilized ratio.

According to results both before and after the instructional module, Mine used additive comparisons to measure some attributes instead of using ratio. To illustrate, in a PRET problem (question 9), in both tests, she used additive comparisons to measure the attribute instead of multiplicative comparisons; in other words, she did not see ratio as an appropriate measure of the shade of paint.

Additionally, while she could identify ratio as an appropriate measure in the context of determining "squareness" of a rectangle in the posttest, she could not do in the pretest. For instance, in PRET question 24, she made an additive comparison to determine the attribute in the pretest. On the other hand, in the posttest, she used a multiplicative comparison and used ratio to determine "squareness" of a rectangle.

Providing Examples for Proportional and Nonproportional Relationships

Before the instructional module, Mine sometimes was not able to provide a valid example of a proportional relationship. For example, in PRET question 5, which asked to provide a word problem which could be solved by the given equation, she could not create a problem in which the quantities related multiplicatively in the pretest. In contrast, in the posttest, she was able to provide a valid missing value problem in which the quantities related multiplicatively.

Moreover, in both interviews, when the researcher asked her to give an example of a real life proportional situation, she could create valid examples of proportional real life situations and explained why the situation was proportional by utilizing the statement that variables in a proportional situation had to increase or decrease in the same ratio.

Results both before and after the instructional module indicated that Mine could exemplify situations in which variables had nonproportional relationships. To demonstrate, in both interviews, when the researcher asked her to give an example of a real life nonproportional situation, she could give valid examples and provide evidence in support of assertions about the nonproportional relationship. Further, in both interviews, one of the relationships exemplified by her was additive relationship in which the variables had a linear and nonproportional relationship. Moreover, in the posttest, she provided an additional nonproportional example in which the variables had constant relationship.

4.3.2.3 Differences in Ela's Findings on Distinguishing Proportional from Nonproportional Situations

Classifying Relationships as Proportional or Nonproportional

Results both before and after the instructional module indicated that Ela did not have difficulty in classifying relationships as proportional or nonproportional. For example, for PRET questions 11-22, in both tests, Ela accurately classified all of the relationships as proportional or nonproportional.

Prior to the instructional module, she sometimes could not provide evidence to support her claims about proportionality, whereas she always could do after the instructional module. Moreover, before the module, when she provided evidence to support her classifications, she considered that the direction of change of the related quantities was the same, but she did not mention the fact that the change between the quantities was in the same ratio. In contrast, by the end of the instructional module, she not only could distinguish proportional from nonproportional relationships, but she also stated that proportional relationships increased or decreased with the same ratio. In addition, she explained her rationale by using multiplicative and additive relationships and their connections with proportionality. To illustrate, in the posttest and post-interview, she argued that while proportional relationships had a multiplicative structure, additive structure did not establish a proportional relationship between quantities. Additionally, in the posttest questions that had quadratic relationships (questions 16, 18 and 22), she clarified that the relationships were not proportional because for the variables to be proportional, they had to be linear not parabolic. However, in the pretest, although she accurately determined that the the relationships were not proportional, she could not exactly explain her rationale.

According to results both before and after the instructional module, Ela could explain the difference between functions of the form $y=mx$ and functions of the form $y=mx+n$. Furthermore, she knew that all linear relationships were not proportional and all proportional relationships were linear. For instance, in both student teachings, she emphasized both of the statements about proportional relationships. Furthermore, she asked problems that had non-proportional relationships besides problems with proportional relationships. Yet, in the first student teaching, she asked only nonproportional problems with additive relationships and could not exactly explain why the problems were not proportional. On the other hand, in the second time, she introduced both additive and constant relationships and could explicitly explaine why they were not proportional.

Identifying Ratio as Measure

Before the instructional module, Ela sometimes could identify ratio as a proper method to measure the attributes in relevant instruments. However, after the instructional module, she always could recognize ratio as a proper method to measure the attributes. To illustrate, in the pretest, she could recognize ratio as a proper measure in the context of determining concentration of a mixture and steepness of ski ramps. In the posttest, she could identify ratio as a proper method to measure concentration of a mixture, steepness of ski ramps, shade of paint, and "squareness" of a rectangle. Furthermore, unlike the pretest, in the posttest, she used proportional reasoning language in her explanations; specifically, she used the word "ratio" while she was explaining her solutions in which she utilized ratio.

Result prior to instructional module revealed that Ela used additive comparisons to measure some attributes instead of using ratio. But, after the instructional module she could use ratio, which was a multiplicative comparison, to measure the attributes. For instance, in a PRET problem (question 9), she used an additive comparison to measure the attribute instead of multiplicative comparisons; in other words, she did not see ratio as a proper measure of shade of paint. However, in the posttest, she made multiplicative comparisons and used ratios of paints to determine the shade of the paint. Additionally, in another PRET problem (question 24), she did not realize ratio as a proper method to measure "squareness" of a rectangle and made an additive comparison to determine the attribute in the pretest. In contrast, in the posttest, she used a multiplicative strategy and identified ratio as a proper measure in the context of determining "squareness" of a rectangle.

Providing Examples for Proportional and Nonproportional Relationships

According to results both before and after the instructional module, Ela was able to provide valid examples of proportional relationships. For instance, in PRET question

5, which asked to provide a word problem which could be solved by the given equation, she could not create a missing value problem in which the quantities related multiplicatively in both tests.

Additionally, in both interviews, when the researcher asked her to give an example of a real life proportional situation, she could create valid examples of proportional real life situations and explained why the situation was proportional by utilizing the statement that variables in a proportional situation had to increase or decrease in the same ratio.

Results both before and after the instructional module indicated that Ela was able to exemplify situations in which variables had nonproportional relationships. To illustrate, in both interviews, when the researcher asked her to give an example of a real life nonproportional situation, she could give valid examples and provide evidence in support of assertions about the nonproportional relationship. In both interviews, one of the relationships exemplified by her was additive relationship in which the variables had a linear and nonproportional relationship. Moreover, in the posttest, she provided an additional nonproportional example in which the variables had constant relationship.

4.3.3 Differences in Understanding the Mathematical Relationships Embedded in Proportional Situations

In this section, pre-service teachers' work before and after the proportional reasoning instructional module were compared, so as to specify the four *key understandings,* if any, they used to define proportionality concepts and to determine proportional relationships were analyzed. Moreover, their difficulties in defining proportionality concepts and understanding the *key understandings* before and after the instructional module were presented.

4.3.3.1 Differences in Gaye's Findings on Understanding the Mathematical Relationships Embedded in Proportional Situations

Defining Proportionality Concepts

Before the proportional reasoning instructional module, Gaye had difficulties to define ratio and proportion concepts. She asserted that ratio was the same thing with "division". In addition, she defined the term proportion as an equation, which was derived from *cross-multiplication*, without giving any meaningful explanation. Moreover, she did not make use of any *key understanding* to define ratio and proportion. Furthermore, she could not distinguish ratio from proportion and used the terms interchangeably. Additionally, she did not use proportional reasoning language in her definitions. In contrast, after the instructional module, Gaye could make true definitions of ratio and proportion, and used proportional reasoning language in her definitions. Moreover, she utilized *key understanding 3* and *key understanding 1* to define the terms. In addition, she could distinguish ratio from proportion. The results prior to instructional module showed that she did not know proportional relationships were shown graphically by a line through the origin (*key understanding 2*). For example, in the first student teaching, while she was defining the ratio concept, she assumed that a ratio could not have a zero as its second component. Conversely, after the instructional module, she appears to eliminate the misconception and understand the *key understanding 2*.

In the pre-interview, she struggled to define the constant of proportionality. Furthermore, she assumed that the constant of proportionality was a constant additive relationship instead of a constant multiplicative relationship between quantities. In contrast, she accurately defined the constant of proportionality for both direct and inverse proportion in the post-interview. Moreover, by the end of the instructional module, she emphasized that both *k* and *1/k* were the constant of proportionality; in other words, she appears to learn that a constant multiplicative relationship existed between two quantities and it could be expressed in two ways.

Determining Proportional Relationships

Prior to the instructional module, Gaye could not exactly explain the presence of proportionality by using the *key understandings.* To illustrate, in the first student teaching, while she was explaining how to determine proportional relationships, she mentioned the equality of the rate pairs in a proportion (*key understanding 3*), but she could not explain why it was true and not use proportional reasoning language in her speech. On the other hand, in the second student teaching, she utilized multiplicative nature of proportional relationships (*key understanding 1*) to decide proportionality. Moreover, in the second student teaching, she set the algebraic expression of a proportional situation by using multiplicative nature of proportional relationships (*key understanding 1*) instead of using memorized rules as she did in the first student teaching.

By the end of the instructional module, Gaye generally could make connection among table, graph and algebraic expression of a proportional situation. Yet, in the second student teaching, she had difficulty in finding the algebraic equation of a proportional relationship presented as a table although she accurately drew the graph of the relationship.

Prior to the instructional module, Gaye could not adequately use the *key understandings* to justify her classifications about proportional relationships. In fact, she sometimes did not use any *key understanding* to determine the proportional relationships. Further, she sometimes used inaccurate statements based on additive approach to decide proportionality. In contrast, after the instructional module, she could adequately use the *key understandings* to justify her classifications about proportional relationships. For example, before the instructional module, especially for the table representation, she did not exactly trust the *key understanding 3* and she tried to find some additive patterns between variables in the tables. She assumed that additive patterns between variables in a table representation could be evidence of a proportional relationship. Furthermore, although she understood the relationship between the slope and constant of proportionality in the equation $y=mx$, she supposed that a constant slope guaranteed proportionality; namely, she concluded that in the equation $y=mx+n$, y is proportional to x. It can be concluded that she had a misconception that all linear relationships were proportional and she did not exactly understand the *key understanding 4.* In contrast, after the instructional module, she learned that proportional relationships could be expressed algebraically in the form $y=mx$ and did not provide any explanation based on a misconception. However, she still did not exactly understand the *key understanding 4.* For instance, in the post-interview, she made connection between the slope and the constant of proportionality, but she did not mention the unit rate that was equal to both of them.

4.3.3.2 Differences in Mine's Findings on Understanding the Mathematical Relationships Embedded in Proportional Situations

Defining Proportionality Concepts

Before the proportional reasoning instructional module, Mine had difficulties to define ratio and proportion concepts. She asserted that ratio was the same thing with "division" and "fraction". Moreover, she could not explain the differences between the terms ratio and proportion although she knew that they were different terms. However, in the first student teaching, she used *key understanding 1* to define direct and inverse proportions. On the other hand, in the post-interview and second student teaching, Mine stated true definitions of ratio and proportion and used proportional reasoning language in her definitions. Furthermore, she utilized *key understanding 1* and *key understanding 3* to define ratio and proportion concepts. Additionally, unlike the results prior to the instructional module, she could explain the differences between the terms ratio and proportion and used proportional reasoning language in her definitions.

In the pre-interview, Mine did not clear which ratios (*within* or *between-ratio*) should be equal to the constant of proportionality. In other words, she probably did not know that a constant multiplicative relationship existed between two quantities in a proportional situation, which meant *within-ratio*. In a similar way, in the second student teaching, she held the misunderstanding that the constant of proportionality was equal to both *within* and *between-ratios* although she clearly indicated that it was equal to *within-ratio* in the post-interview. However, unlike the results prior to the instructional module, after the instructional module, she emphasized that the constant of proportionality could be expressed in two ways (*k* and *1/k*).

Before the instructional module, she did not use the *key understanding 4* in her definitions. However, by the end of the module, she used the *key understanding* but, she did not mention the unit rate although she could make connection between the slope and the constant of proportionality.

Determining Proportional Relationships

Before the instructional module, Mine explained the presence of proportionality by using the multiplicative nature of proportionality (*key understanding 1*) and the equality of the rate pairs in the proportion (*key understanding 3*), yet she could not clearly explain which rate pairs would be equal to be proportional. Similarly, after the instructional module, she made use of *key understanding 1* and *key understanding 3* to determine the presence of proportionality, but this time, she could clearly explain which rate pairs would be equal to be proportional.

According to results both before and after the instructional module, Mine utilized all of the key understandings to justify her classifications about proportional relationships. Yet, prior to instructional module, she could not explain her rationales in all of the situations by adequately utilizing the *key understandings*. In fact, she sometimes used inaccurate statements based on additive approach to decide proportionality instead of *key understandings*. To illustrate, in the pretest, for questions 11 and 12, she only mentioned that the direction of change of the related quantities had to be the same, but she did not emphasize that the change between the quantities had to be in the same ratio. On the other hand, in the posttest and post-interview, she did not use any inaccurate statement and utilized the *key understandings* to decide proportionality of the situations.

The results prior to instructional module showed that Mine had some difficulties in *key understanding 2*. For instance, in the pretest, she assumed that a ratio could not have a zero as its second component though a proportional situation might contain the ratio of zero to zero (*0/0*). Conversely, after the instructional module, she appears to eliminate the misconception that a ratio could not have a zero as its second component and understand the *key understanding 2*. Additional evidence was her classification of question 16 in the pre-interview. It seems that she did not know that proportional relationships were shown graphically by a line through the origin (*key understanding 2*) because although the graph in the question was a parabola, she classified it as proportional. In contrast, after the instructional module, Mine accurately classified the quadratic relationship as nonproportional and adequately utilized the *key understanding 2*. It can be concluded that she did not have any difficulty in *key understanding 2* after the instructional module.

According to results both before and after the instructional module, Mine did not have any difficulty in *key understanding 1* and *3*. Moreover, she preferred to use the two statements in most of her justifications of proportionality. Additionally, both of the results indicated that she knew that in the functions of the form $y=mx+n$, x was not proportional to y. However, the results of the pre-interview showed that she did not know that in the functions of the form $y=mx^2$, x was not proportional to y. It can be concluded that she had difficulties in *key understanding 4*. By the end of the instructional module, she understood the relationships in the functions of the form $y=mx^2$. But, she still did not exactly understand the *key understandings 4.* To illustrate, in the post-interview, she made connection between the slope and the constant of proportionality, but she did not mention the unit rate that was equal to both of them.

4.3.3.3 Differences in Ela's Findings on Understanding the Mathematical Relationships Embedded in Proportional Situations

Defining Proportionality Concepts

Prior to the instructional module, Ela could not provide a valid definition of ratio because she argued that ratio was the same thing with "division". On the other hand, she could define proportion by utilizing *key understanding 3* although she did not use proportional reasoning language in her definition. After the instructional module, she appears to eliminate the misconception that ratio was the same thing with "division" and provided valid definitions of ratio and proportion by using proportional reasoning language. Similar to results prior to module, she defined proportion by using *key understanding 3* after the instructional module. The results both before and after the instructional module showed that she knew ratio and proportion were different terms and could explain the differences between them. Additionally, she could utilize proportional relationships' multiplicative nature (*key understanding 1*) to define and explain inverse proportion.

According to results both before and after the instructional module, Ela provided a true definition of the constant of proportionality and accurately found it in a proportional situation by utilizing *key understanding 3*. However, results from the first student teaching revealed that she did not understand the relationship between the slope and the constant of proportionality and could not recognize that the constant of proportionality could be expressed in two ways. Moreover, she stated an inaccurate statement that the multiplication of the slope and the constant of proportionality was equal to one. In conclusion, she had difficulties in *key understanding 4*. On the other hand, in the second student teaching, she highlighted that in a proportional situation the slope and the constant of proportionality were the same. In addition, she emphasized that both *k* and *1/k* were the constant of proportionality. It can be concluded that by the end of the instructional module, she understood the relationship between the slope and the constant of proportionality and learned that a constant multiplicative relationship existed between two quantities and it could be expressed in two ways. Yet, she still did not exactly understand the *key understandings 4.* To illustrate, in the second student teaching, although she made connection between the slope and the constant of proportionality, she did not mention the unit rate that was equal to both of them. However, she could make connections among different representations of a proportional situation later the instructional module.

Determining Proportional Relationships

Before the instructional module, Ela could not exactly explain the presence of proportionality by using the *key understandings*. She only mentioned that the ratio pairs in a proportional situation had to be equal (*key understanding 3*) and she called the relationship “division” instead of “ratio”. That is to say, she did not utilize proportional reasoning language in her explanation. However, after the instructional module, she explained the presence of proportionality by using both *key understanding 1* and *key understanding 3*.

Prior to instructional module, Ela mostly justified rationales of her classifications about proportional relationships by using *key understanding 3*. In addition, she inadequately utilized *key understanding 4* to justify a classification. That is to say, she mentioned that a proportional relationship could be represented symbolically as

$y = mx$, where m was the constant of proportionality, but she did not mention that m also the unit rate and slope. However, after the instructional module, Ela understood the relationship between the slope and the constant of proportionality although she did not mention the unit rate that was equal to both of them. Additionally, she utilized all of the *key understandings* to justify her classifications later the instructional module. Furthermore, her foremost justification was the *key understanding 1* whereas she did not use it in any situation before the instructional module. Moreover, unlike the results prior to instructional module, she did not use any inaccurate statements based on additive approach to decide proportionality. To illustrate, prior to the instructional module, Ela sometimes tried to find some additive patterns between quantities in the tables. She supposed that additive patterns between variables in a table representation could be evidence of a proportional relationship although she did not sure. On the other hand, by the end of the instructional module, it was seen that her doubts were removed, and she utilized *key understandings* to decide and explain proportional relationships. For example, unlike the pretest and pre-interview results, Ela could utilize *key understanding 1* and *key understanding 2* to justify her classifications. In addition, Ela could make connections among table, graph and algebraic expression of the given proportional situations after the instructional module.

4.4 Summary of the Results

In order to summarize participants' proportional reasoning, the findings about their approaches to different problem types, distinguishing proportional from nonproportional situations, and understanding mathematical relationships embedded in proportional situations before and after the instructional module based on proportional reasoning, are presented in the Table 4.10. The results obtained from the first and second administration of PRET, the first and second student teaching, and pre and post-interview. Additional information was gathered from the lesson plans and revision reports of the lesson plans in the student teachings.

Table 4.10 Pre-service teachers' proportional reasoning before and after the proportional reasoning instructional module

		Before			**After**		
		Gaye	Mine	Ela	Gaye	Mine	Ela
Approaches to different problem types	Solving problems accurately	±	±	±	+	±	+
	Using inaccurate additive strategy in proportional situations (making additive comparisons)	±	±	±	-	±	-
	Making qualitative comparisons not depending on numerical values	-	+	±	-	+	+
	Utilizing a broad range of strategies	-	-	±	+	+	+
	Using strategies highlighting multiplicative relationships	±	±	±	+	+	+
	Using efficient strategies	-	±	-	±	+	±
	Providing a meaningful explanation for solution	-	±	-	+	+	+
	Using proportional reasoning language in explanations	-	±	-	+	+	+
	Recognizing both within and between ratios	-	-	-	±	+	±
	Recognizing both part-to-part and part-to-whole ratios	-	+	+	-	+	+
	Flexibility in unitizing and reunitizing of quantities	-	-	-	-	-	-
	Building ratio tables and determining the rule for relating the number pairs in the table	-	-	-	+	+	+
Distinguishing proportional from nonproportional situations	Classifying relationships as proportional or nonproportional accurately	±	±	+	+	+	+
	Providing evidence to support claims about proportional relationships	-	±	±	+	+	+
	Misconception that all linear relationships are proportional	+	±	-	-	-	-
	Knowing that all proportional relationships are linear	-	-	+	+	+	+
	Identifying ratio as measure	±	±	±	+	±	+
	Using proportional reasoning language in explanations	±	±	±	+	+	+
	Providing example of proportional relationship and explain why	-	±	+	±	+	+
	Providing example of nonproportional relationship and explain why	±	+	+	+	+	+
Understanding mathematical relationships	Defining proportionality concepts	±	±	±	+	+	+
	Utilizing key understandings in definitions	-	±	±	+	+	+
	Using proportional reasoning language in definitions	-	-	-	+	+	+
	Utilizing key understandings to determine proportional relationship	±	±	±	+	+	+
	Using inaccurate additive approach	+	±	+	-	-	-
	Making connections among table, graph and algebraic expression of a proportional situation	-	-	-	±	+	+
	Understanding proportional relationships are multiplicative in nature	±	+	-	+	+	+

Table 4.10 (continued)

	Understanding proportional relationships are presented graphically by a line through the origin	-	-	-	+	+	+
	Understanding the rate pairs are equivalent in proportional relationships	±	+	+	+	+	+
	Understanding proportional relationships can be represented symbolically by the equation y = *m*x, where the *m* is the slope, unit rate, and constant of proportionality	±	±	-	±	±	±

Note: "-" indicates that the statement was not found,
"+" indicates that the statement was found,
"±" indicates that the statement was found only in some situations.

4.4.1 Summary of Findings about Approaches to Different Problem Types

Gaye, Mine and Ela generally accurately solved the proportion problems in the PRET, student teachings and interviews. Yet, before the proportional reasoning instructional module, all of them inaccurately solved a numerical comparison problem in the PRET (question 24) since they used an inaccurate additive strategy instead of a multiplicative one. Similarly, in the pretest, Gaye and Mine made an additive comparison to solve a qualitative problem (question 8) instead of a multiplicative comparison. However, Ela solved the same problem by making multiplicative comparisons. After the instructional module, Gaye and Ela could solve all of the proportion problems by utilizing multiplicative strategies rather than inaccurate additive strategies. On the other hand, Mine could not accurately solve a qualitative problem (question 8) and still made an additive comparison. In both pretest and posttest, while Gaye could not make qualitative comparisons that did not depend on numerical values, Mine could do. In the pretest, although Ela made multiplicative comparisons to solve qualitative problems, she could not make qualitative comparisons that did not depend on numerical values, whereas in the posttest, she could do.

After the proportional reasoning instructional module, all of the pre-service teachers utilized a broader range of strategies to solve proportion problems although they used limited number of strategies before the instructional module. In addition, after the instructional module, they mostly preferred to use strategies highlighting multiplicative relationships, whereas before the instructional module, they generally applied algebraic procedures without associating meaning. Moreover, all of them mostly used efficient strategies which facilitated the computations required to find missing values after the instructional module.

Prior to the instructional module, while Gaye and Mine could not express the meanings of quantities they calculated, Ela explained them in some problems at a limited level. Additionally, early on instructional module, their explanations and rationales for their solutions were generally inadequate that did not go beyond a description of the steps they had taken to determine the solution though Mine could provide meaningful explanations for some of her solutions. However, later the instructional module, they explicitly explained the meaning of the quantities, provided meaningful explanations for their solutions and used proportional reasoning language in their explanations.

In missing value problems, while Gaye, Mine and Ela recognized and used *between-ratios* before the instructional module, they had difficulty in recognizing *within-ratios*. In fact, they did not use *within-ratio* even in situations where it was integer. On the other hand, after the instructional module, Mine could recognize and used both *within* and *between-ratios* in the missing value problems, but Gaye and Ela only recognized *between-ratios*. However, since they sometimes utilized *unit rate strategy*, which was a *within-ratio strategy*, it can be said that they could realize *within-ratio* in only some situations.

Mine and Ela recognized and utilized both *part-to-part* and *part-to-whole ratios* before and after the instructional module. However, Gaye did not use *part-to-whole* *rations* in any solution of her, and in the pre-interview, she struggled to make sense of the quantities in a student's solution in which *part-to-whole ratio* was utilized. It can be concluded that she had difficulties in realizing *part-to-whole ratios* even after the instructional module.

None of the pre-service teacher had flexibility in unitizing and reunitizing quantities before and after the instructional module. In other words, they had difficulties in using composite units when the quantities suggest that it was more convenient than using singleton units even after the instructional module. However, by the end of the instructional module, Gaye, Mine and Ela built ratio tables and determined the rule for relating number pairs in the table by utilizing *factor of change* or *building-up* strategies. Furthermore, in some situations, they could view the ratio as reducible and increasable units and by this way; they could solve the missing value problems that had noninteger *between-ratios*. In addition, all of the pre-service teachers realized the constant ratio between elements of the same measure space and used the constant of proportionality to solve missing value problems after the instructional module.

4.4.2 Summary of Findings about Distinguishing Proportional from Nonproportional Situations

After the proportional reasoning instructional module, Gaye, Mine and Ela accurately classified all of the relationships in the PRET (questions 11-22) as proportional or nonproportional. However, prior to instructional module, Gaye and Mine made some inaccurate classifications; moreover, all of the pre-service teachers had some difficulties to provide evidence to support their claims. On the other hand, by the end of the instructional module, they could clearly explain why the relationships were proportional or not.

Before the instructional module, Gaye and Mine had a misconception that all linear relationships were proportional and did not know that all proportional relationships were linear, whereas Ela knew that linearity alone was not an indicator of proportional relationship, but it was a requirement for the variables to be proportional. After the instructional module, the misconception that all linear relationships were proportional appears to have been eliminated by the pre-service teachers. Prior to the instructional module, while Mine and Ela had some difficulties in understanding that proportional relationship increased or decreased with the same ratio, Gaye definitely did not know the statement. To illustrate, in the first student teaching, Gaye taught a linear and nonproportional relationship as if it was a proportional relationship. In contrast, in the second student teaching, she accurately classified the relationship between the same variables as nonproportional. Similarly, results after the instructional module revealed that all of the pre-service teachers learned that all proportional relationships were linear and proportional relationship increased or decreased with the same ratio.

Pre-service teachers had some difficulties in identifying ratio as a measure in the pretest. On the other hand, in the posttest, Gaye and Ela could realize that ratio was a proper method to measure concentration of a mixture, shade of paint, steepness of a ski ramp and "squareness" of a rectangle. However, Mine could not identify ratio as a measure of shade of paint even after the instructional module. Pretest results showed that the pre-service teachers sometimes did not use proportional reasoning language in their explanations; specifically, they sometimes did not use the word "ratio" while they were utilizing ratio as a measure. But, in the posttest, they used proportional reasoning language, especially the word "ratio", in their explanations.

Before the instructional module, Gaye and Mine sometimes could not provide valid examples of proportional relationships, while Ela could do. For example, in the pretest (question 5), Gaye and Mine was not able to create a missing value word problem in which the quantities related multiplicatively, whereas Ela could create a proportional problem. Additionally, in the pre-interview, only Gaye could not provide a valid example for a proportional real life situation.

After the instructional module, Mine and Ela could create valid examples of proportional relationships, but Gaye still had some difficulties. To illustrate, in the posttest (question 5), while Mine and Ela could create a proportional problem, Gaye could not do. In conclusion, Gaye could not write a word problem in which the quantities had proportional relationship even after the instructional module. On the other hand, in the post-interview, all of them were able to give examples of a proportional real life situation; moreover, they could justify why the examples were proportional.

According to results both before and after the instructional module, the pre-service teachers could provide valid examples of a nonproportional relationship and explain why their examples were nonproportional. Gaye's nonproportional example in the pre-interview was an exception because she could not clearly explain why her example was not proportional. The results after the instructional module indicated that pre-service teachers identified that additive and constant relationships were presented at nonproportional situations.

4.4.3 Summary of Findings about Understanding the Mathematical Relationships Embedded in Proportional Situations

After the proportional reasoning instructional module, Gaye, Mine and Ela could provide valid definitions of ratio and proportion concepts. However, prior to instructional module, they had difficulties to define the concepts. To illustrate, all of them asserted that ratio was the same thing with "division"; additionally, Mine argued that ratio and fraction were the same concepts. While Gaye could not make use of any *key understanding* to define ratio and proportion, Mine used *key understanding 1* and Ela used *key understanding 3* in some definitions before the instructional module. However, by the end of the module, all of them could utilize *key understanding 1* and *key understanding 3* to define ratio and proportion concepts. Although after the instructional module, all pre-service teachers could explain the difference between ratio and proportion, Gaye and Mine had difficulties before the instructional module. Unlike the results prior to instructional module, all of them could use proportional reasoning language in their definitions. Moreover, they could adequately utilize *key understandings* to determine proportional relationship contrary to the results prior to instructional module.

The results before the instructional module revealed that pre-service teachers probably did not know that a constant multiplicative relationship existed between two quantities and it could be expressed in two ways. Similarly, Gaye and Mine struggled to define the constant of proportionality in the pre-interview. On the other hand, after the instructional module, they appear to have learned the meaning of constant of proportionality and expressing it in two ways. However, Mine had difficulties to determine the constant of proportionality even after the instructional module. To illustrate, in the second student teaching, she stated that it was equal to both *within* and *between-ratios*.

Prior to instructional module, pre-service teachers sometimes used inaccurate statements based on additive approach to decide proportionality instead of *key understandings*. For example, Gaye and Ela tried to find some additive patterns between quantities to decide proportionality. Furthermore, Mine and Ela only mentioned that the direction of change of the related quantities had to be the same, but did not emphasize that the change between the quantities had to be in the same ratio in some situations. In contrast, by the end of the instructional module, they did not use any inaccurate statement and utilized the *key understandings* to decide proportionality of the situations.

Pre-service teachers had some difficulties in making connections among table, graph and algebraic expression of the given proportional situations before the instructional module. On the other hand, they mostly eliminated their difficulties after the module. However, in the second student teaching, Gaye had difficulty in finding the algebraic equation of a proportional relationship presented as a table although she accurately drew the graph of the relationship.

By the end of the instructional module, results indicated that pre-service teachers utilized the *key understandings* and mostly understood them whereas they had some difficulties earlier on the instructional module. To illustrate, before the instructional module, Gaye and Ela did not exactly understand the *key understanding 1* and all of them had some difficulties in the *key understanding 2* and *key understanding 4*. However, after the instructional module, they exactly understood the *key understandings*. The *key understanding 4* was an exception because the pre-service teachers knew that a proportional relationship could be represented symbolically as $y=kx$, where the k was the slope and the constant of proportionality, but they did not mention the unit rate that was equal to both of them.

CHAPTER V

CONCLUSION AND DISCUSSION

5.1 Pre-service Teachers' Proportional Reasoning

This study was designed to investigate pre-service middle school mathematics teachers' proportional reasoning before and after receiving a practice-based instructional module based on proportional reasoning. For this purpose, pre-service teachers' approaches to different problem types, distinguishing proportional from nonproportional situations, and understanding mathematical relationships embedded in proportional situations were examined. In this chapter, first, conclusions of the study are summarized respectively, and these conclusions are discussed under the related headings. Then, educational implications, limitations and recommendations for future research are presented.

5.1.1 Approaches to Different Problem Types

As it was expected pre-service teachers generally accurately solved the proportion problems which consisted of missing value, numerical comparison and qualitative reasoning problems before and after participation in the practice-based instructional module. However, giving correct answers does not ensure that proportional reasoning is taking place because proportions may be solved by using mechanical knowledge about equivalent fractions or about numerical relationships, or by applying algorithmic procedures (e.g., *cross-multiplication*) without the understanding of proportional relationships (Cramer et al., 1993; Lamon, 2007). That is to say, proportional reasoning goes beyond setting up a proportion and blindly applying rules and mechanical operations (Hoffer, 1988; Lamon, 2007).

Furthermore, Lamon (2007, 2011) argues that proportional reasoning is a mental process that involves argumentation and conscious analysis of the multiplicative relationships between variables. Besides, proportional reasoning requires expressing the meanings of quantities and variables in the context in which they are used, and using the language of proportionality (Lamon, 2007). Yet, the results prior to instructional module revealed that the pre-service teachers mostly solved proportion problems procedurally, and they did not have conceptual knowledge required to understand and explain proportional relationships in the problem situations. For instance, prior to the instructional module, although the pre-service teachers were extremely comfortable with numerical operations, they had difficulty in expressing the meanings of quantities they calculated. Moreover, the participants generally applied algebraic procedures such as *cross-multiplication* without associating meaning to solve proportion problems before the instructional module. Additionally, early on instructional module, their explanations and rationales for their solutions were mostly inadequate that did not go beyond a description of the steps they had taken to determine the solution, and lack of proportional reasoning language. On the other hand, by the end of the instructional module, the participants explicitly explained the meaning of the quantities, provided meaningful explanations for their solutions and used proportional reasoning language in their explanations. Furthermore, the results after the instructional module revealed that the pre-service teachers mostly preferred to use informal strategies (e.g., *factor of change*) highlighting multiplicative relationships, and relied less on the *cross-multiplication* and other formal strategies. In parallel with the results prior to the instructional module, research findings showed that students and teachers frequently use formal strategies, which are algebraic strategies in which rules and properties of algebra are used, to set up and solve proportion problems (Akkuş-Çıkla & Duatepe, 2002; Avcu & Avcu, 2010; Cramer & Post, 1993; Duatepe et al., 2005; Person et al., 2004). Additionally, similar to the results before the instructional module, in the studies by Akkuş-Çıkla and Duatepe (2002), and Person et al. (2004), the researchers found that pre-service teachers did not have conceptual knowledge required to solve and understand proportion problems. To illustrate, in the study by Person et al. (2004), when the interviewer asked a pre-service teacher how to introduce the concept of ratio to middle grade students, the participant mentioned techniques and number operations without giving a sense of proportions.

In the practice-based instructional module, pre-service teachers examined cases of students' works, which includes errors due to overreliance on *cross-multiplication* and other algebraic strategies, and they read an assigned article (Doğan & Çetin, 2009) about students' misconceptions of direct and inverse proportion, which mentioned that students had difficulties in solving proportion problems since they used formal strategies in a rote manner. These tasks may help them to notice that memorized algorithmic procedures might lead confusion due to the fact that they did not highlight multiplicative relationships between variables. Moreover, the mathematical tasks they solved by using informal strategies in the instructional module might help them to see how these strategies were meaningful and useful. The important point here is that as a result of the instructional module, the pre-service teachers questioned their own solution strategies and chose to change them with strategies highlighting multiplicative relationships. The claim was confirmed by the pre-service teachers during the posttest and post-interview because it was seen that they used factor of change strategy instead of cross-multiplication even in the scratch papers although it was not required. Moreover, as the participants shared and discussed their solutions with their peers in the instructional module, they might come to understand the meanings of quantities calculated and the relationships between variables, and this might be a reason that they provided meaningful and conceptual explanations for their solutions. Furthermore, the narrative case, in which a teacher used language of proportionality in her explanations, may have a role in the improvement of the participants' use of proportional reasoning language.

The results of the study revealed that pre-service teachers utilized a broader range of strategies to solve proportion problems and made sense of these strategies as a result of their participation in the proportional reasoning instructional module. Furthermore, they preferred to use efficient strategies which facilitated the computations required to find missing values after the instructional module. The result indicated that the participants made progress in becoming a proportional reasoner. Correspondingly, solving multiple types of problems, missing value, numerical comparison, qualitative prediction and comparison problems with considerable flexibility, which meant to choose a strategy that is best suited to the problem, is an expected ability from a proportional reasoner (Cramer & Post, 1993; Cramer et al., 1993; Singh, 2000). Additionally, Lamon (2012) claims that a proportional reasoner exhibits greater efficiency in problem solving and utilizes a range of strategies, sometimes unique strategies, for dealing with problems. The reason for the development in the participants' repertoire of strategies may be due to the instruction and an assigned article (Duatepe, et al., 2005), which was about students' solution strategies, enabled them to have knowledge about different solution strategies. Moreover, mathematical tasks they solved in the instruction encouraged them to solve different types of proportion problems by using a range of strategies. In addition, discussions on different solution strategies used in the cases of students' works and a teacher's instruction, and problems solved in the instructional module might help them to develop a flexible set of strategies and to realize the connections between them. Similarly, Steele (2006) and Hillen (2005) found that the participants in their studies improved their ability to use multiple solution methods by the end of a practice-based professional development course in which they engaged in mathematical tasks, discussed mathematical ideas and examined cases of learning and teaching.

Some strategies might yield an incorrect answer to a proportion problem. In literature, the most commonly used incorrect approach was additive strategy (Ben-Chaim et al., 2012; Karplus et al., 1983). In fact, many studies indicate that both students and teachers frequently use additive strategies where multiplicative comparisons are required (Hart, 1988; Lesh et al., 1988; Simon & Blume, 1994b;

Singh, 2000; Sowder, Philipp et al., 1998). Further, inaccurate additive strategies do not seem to disappear with maturation (Hart, 1988). In a similar way, the results of the current study showed that pre-service teachers made use of inaccurate additive strategies to solve some proportion problems before the instructional module. For example, pre-service teachers used an inaccurate additive strategy instead of a multiplicative one to solve a numerical comparison problem in the PRET. According to several research studies, students often used inaccurate additive strategies when the proportion problem had noninteger ratios (Karplus et al., 1983, Cramer et al., 1993; Post, Cramer, Behr, Lesh, & Harel, 1993; Singh, 2000; Sowder, Philipp et al., 1998). Similarly, the numerical comparison problem in the PRET had noninteger ratios. Yet, we cannot conclude that the pre-service teachers used inaccurate additive strategies due to noninteger ratios because they could accurately utilize multiplicative strategies in the other problems which included noninteger ratios. The reason of their inaccurate additive strategy might be that the numerical comparison problem with a context of "squareness" was an unfamiliar problem type for pre-service teachers since middle school textbooks usually presented the traditional missing value problems (Sowder, Armstrong, et al., 1998). Another evidence for the claim was that their other inaccurate additive strategies were in the qualitative problems which were also less familiar problems. In a similar manner, Heller et al. (1989) found that when students encountered with a less familiar problem context, the difficulty of the problem increased. Correspondingly, Sowder, Philipp et al. (1998) argued that since additive reasoning had an important role in the early grades, students returned to additive reasoning in multiplicative situations when they encountered a difficulty. Conversely, after the instructional module, pre-service teachers did not utilize any inaccurate additive strategy to solve numerical comparison problems. In the practice-based instructional module, pre-service teachers examined students' works in which students used inaccurate additive strategies, and solved and discussed mathematical tasks which included numerical comparison problems. These tasks may help them to notice their own inaccurate additive strategies and to solve the problems with considerable flexibility.

The results also revealed that pre-service teachers had some difficulties in solving qualitative reasoning problems even after the instructional module. For instance, a pre-service teacher still made use of an additive strategy to solve a qualitative reasoning problem and another pre-service teacher could not make qualitative comparisons that did not depend on numerical values after the instructional module. However, qualitative problems require qualitative comparisons that do not depend on numerical values (Ben-Chaim et al., 2012). Additionally, proportional reasoning is a way of reasoning about multiplicative relationships, which includes both quantitative and qualitative process (Lesh et al., 1988). The reason for the difficulties in qualitative reasoning of pre-service teachers may be that the proportional reasoning instructional module did not have enough tasks about qualitative problems. Therefore, it can be recommended that in the future revisions of the instructional module, qualitative reasoning problems should be given more attention.

The results indicated that pre-service teachers did not have flexibility in unitizing and reunitizing quantities before and after the instructional module. In other words, they had difficulties in using composite units when the quantities suggest that it was more convenient than using singleton units although they could use single *unit rate strategy* (i.e., finding for one). Similarly, in Singh's (2000) study, a sixth grade student was unable to unitize the composite units to find a ratio unit although she accurately solved problems by using single *unit rate strategy*. Correspondingly, Lamon (1989) reported that most of the children in her study knew and preferred the single *unit rate strategy* before receiving any instruction. According to Lamon (1993a, 1993b, 1996), *unitizing* and *norming* are two important processes in the development of proportional reasoning. Moreover, she points out the importance of flexibility in *unitizing* that means conceptualizing a quantity with regard to many pieces with different sizes (Lamon, 2012). In a similar way, Lamon (1993a, 1993b) argues that one of the most important differences between those who reason proportionally and those who do not, is using composite units when the context suggests that using them is more effective than using singleton units. However, pre-service teachers in the current study could not adequately achieve the process. The reason may be that it is a long term process, which began in elementary school (Lamon, 1993a). Therefore, it is difficult to improve it through five-week instruction although the instruction had mathematical tasks that aimed to improve pre-service teachers' flexibility in unitizing. Moreover, it is an ability that requires a more advanced level of proportional reasoning, and is critical to the development of increasingly sophisticated mathematical ideas (Lamon, 1993a, 1994).

By the end of the instructional module, the participants could build ratio tables and determine the rule for relating number pairs in the table and could view the ratio as reducible and increasable units, which they could not do before the instructional module. According to Lamon (1999), this conceptualization allows to solve a broad range of problems including ones that have noninteger *between-ratios*. In the instructional module, there were considerable number of mathematical tasks which required building ratio tables and determining the rule in the tables. The tasks might enable pre-service teachers to see the ratio as reducible and increasable units, and practice on using ratio tables to solve proportion problems.

5.1.2 Distinguishing Proportional from Nonproportional Situations

Lamon (2007) argues that a proportional reasoner is able to distinguish situations in which proportionality is an appropriate mathematical model from situations in which proportionality is not appropriate. In other words, a proportional reasoner can determine whether the quantities in a problem situation are related additively, multiplicatively, or in some other way. However, the results indicated that pre-service teachers sometimes could not determine whether a situation was proportional or not and had difficulties in providing evidence to support their claims about proportionality before receiving the instructional module. Similar findings were reported in the studies by Atabaş (2014), Cramer et al. (1993) and Van Dooren et al. (2005). In the current study, the reason for pre-service teachers' difficulties in classifying relationships as proportional or not, may be due to their misconception that all linear relationships were proportional. Because, Gaye and Mine who had the misconception, sometimes could not determine whether a situation was proportional. Yet, Ela who did not have the misconception always could do. It can be concluded that before the instructional module, the pre-service teachers over generalized proportionality; in other words, they inaccurately applied proportional reasoning in situations that had nonproportional relationships. Van Dooren et al. (2003) refer to the overreliance on proportionality as "illusion of linearity" (p.113). As it was reviewed in the literature, there was a strong tendency to over generalize proportionality between many students and teachers (Atabaş, 2014; Cramer et al., 1993; Van Dooren et al., 2005; Van Dooren et al., 2003). For example, in a study by Cramer et al. (1993) 32 out of 33 pre-service elementary education teachers solved a nonproportional problem by setting up and solving a proportion. That is, the pre-service teachers did not recognize that the quantities in the problem were related additively instead of multiplicatively, and thus, they used a proportional strategy that was not work. On the other hand, by the end of the instructional module, the misconception that all linear relationships were proportional was eliminated by the pre-service teachers. Moreover, they could determine whether the quantities in a problem situation were related additively, multiplicatively, or in some other way. In the instructional module, pre-service teachers examined students' works in which students over generalized proportionality; moreover, they solved and discussed mathematical tasks that included not only proportional situations but also nonproportional situations in which variables had additive and constant relationships. According to Ball and Cohen (1999), in a practice-based professional development program, the documents of practice could be drawn from teachers' own teaching. Furthermore, they argued that teachers need opportunities to review their current practices and to examine others' practices. In a similar way, a video clip from Gaye's teaching, in which her linearity misconception was revealed, was watched and analyzed by pre-service teachers in the instructional module. These tasks mentioned above may enable pre-service teachers to eliminate their misconceptions, to improve their capacity to discriminate proportional situations and to provide evidence to support their claims about proportionality. Similarly, Hillen (2005) and Sowder et al. (1998) reported that pre-service teachers constructed a deeper understanding on proportional and nonproportional situations, and their difference with the aid of professional development opportunities.

Lamon (2007) asserts that without knowledge of intensive quantities, which are the quantities that cannot be measured directly, a student cannot be a proportional reasoner. An intensive quantity relates two extensive quantities (Simon, & Blume, 1994a). Multiplicative comparisons are utilized to indicate the intensive quantities (Sowder, Sowder, & Nickerson, 2012). Correspondingly, recognizing a ratio, which is a multiplicative comparison, as the proper measure of a given attribute is an indicator of the ability to reason multiplicatively (Simon & Blume, 1994b; Sowder et al., 2012). Yet, the results of the study showed that the pre-service teachers had some difficulties in recognizing ratio as a proper measure of a given attribute (e.g., concentration of a mixture) before receiving instruction. Similarly, in a study conducted by Simon and Blume (1994b) to investigate pre-service teachers' ability to identify ratio as a measure, pre-service teachers had difficulty in recognizing ratio as a proper measure for steepness of ski ramps. However, the results of the current study indicated that pre-service teachers mostly eliminated their difficulties in identifying ratio as measure after their participation in the instructional module. The reason may be that in the instructional module, pre-service teachers solved mathematical tasks which included intensive quantities such as squareness of a rectangle, and discussed how to measure the given attribute. Likewise, Lamon (2007) claims that students need opportunities to analyze intensive quantities and to engage in argumentation and justification about how to measure these quantities.

5.1.3 Understanding the Mathematical Relationships Embedded in Proportional Situations

The results prior to instructional module revealed that pre-service teachers had difficulties in defining ratio and proportion concepts and explaining the difference between them. In addition, they thought that ratio was the same thing with “division” or “fraction. Similarly, in the study by Akkuş-Çıkla and Duatepe (2002), the pre-service teachers had difficulty in defining ratio and proportion concepts and explaining the difference between them. However, by the end of the instructional module, pre-service teachers in the current study could define and distinguish ratio and proportion concepts. In the instructional module, pre-service teachers discussed the ratio and proportion concepts, and their connections and differences with other concepts such as fractions and rational numbers. These tasks may help them to understand these concepts conceptually.

The results of the study made salient that proportional reasoning instructional module in which pre-service teachers participated enhanced their understandings of ratio and proportion concepts, and proportional relationships as evidenced by their use of more *key understandings* than before the instructional module in defining proportionality concepts and determining proportional relationships. According to Sowder et al. (1998) understanding the concept of ratio is crucial in making the transition from additive to multiplicative thinking. In fact, prior to instructional module, pre-service teachers sometimes used inaccurate statements based on additive approach to decide proportionality instead of *key understandings*. However, a proportional reasoner should be able to understand mathematical relationships embedded in proportional situations such as proportional relationships’ multiplicative nature (*key understanding 1*) (Cramer et al., 1993). Similarly, Ölmez (2016) proposed that multiplicative relationships had a critical role in ensuring a powerful understanding of ratios and proportional relationships. For this reason, in the instructional module, some mathematical tasks aimed to highlight multiplicative nature of proportional relationships were solved and discussed. As a result, pre-service teachers understood the *key understanding 1* and utilized it in definitions and justifications. In addition, results indicated that pre-service teachers adequately utilized the other *key understandings* as well. Therefore, it can be concluded that they had a deeper understanding of proportional relationships and the mathematical ideas embedded in the relationships after the practice-based instructional module. Similarly, in the study by Silver et al. (2007), the results indicated that practice-based professional learning tasks provided many opportunities for teachers to learn mathematics such as building connections among related mathematical ideas and to rethink and reorganize the mathematics that they would encounter in their practice.

Ben-Chaim (2012) argues that corresponding elements of two sets are in a proportional relationship if there is a constant ratio (either direct or indirect) between them. In other words, the multiplicative relationship between two quantities has to be constant, either in the same, or opposite direction. However, results prior to the instructional module revealed that pre-service teachers probably did not know that a constant multiplicative relationship existed between two quantities and it could be expressed in two ways. On the other hand, they appear to learn the meaning of constant of proportionality after their participation in the instructional module. Yet, Mine had some difficulties to determine the constant of proportionality in the second student teaching although she accurately determined the concept in the post-interview. Underlying reason for her difficulty in the student teaching might be that she went back her old inaccurate statement about constant of proportionality while she was teaching since she did not have a deep understanding of the concept. It can be concluded that pre-service teachers might make mistakes in teaching if they did not have a deep understanding of the concept. Similarly, researchers claim that teachers need a broad and deep knowledge of concepts, principles and strategies they teach (Ball & Cohen, 1999; Ma, 1999; NBPTS, 2010). Another evidence for the claim was that while Gaye could make connections between the constant of proportionality, the slope and the algebraic expression of a proportional situation in the post-interview, she could not do in the second student teaching. Unfortunately, she had similar difficulties in making connections between these concepts prior to the instructional module. Further, she could not accurately write the algebraic expression of a proportional situation although she could do in the posttest. In the post-interview, when the researcher asked her to tell what happened in the student teaching, she said that she was nervous while she was teaching finding the equation of variables in a proportional situation, and she added, “It was very difficult and disappointing. I knew it, but I could not teach it.” It seems that she went back her old mistake while she was teaching. Correspondingly, Sowder, Philipp et al. (1998) claimed that students might return to their old mistakes when they encounter a difficulty.

5.2 Implications

The aim of the current study was to investigate pre-service middle school mathematics teachers’ proportional reasoning before and after receiving a practice-based instructional module based on proportional reasoning. This study provides an instructional module of how proportional reasoning can be improved in a practice-based professional development program. The instructional module also can be an example of the practice-based professional development program that Ball and Cohen (1999) proposed to use practice as a site for professional learning. Additionally, the study provides insights into pre-service teachers’ misconceptions and misunderstandings in proportionality concepts that can be useful for teacher educators. According to the conclusions and literature review, some practical implications and suggestions can be presented as follows.

The results of the study suggest that pre-service teachers can improve their proportional reasoning by completing a practice-based instructional module in which they engage in mathematical tasks, discuss mathematical ideas, examine students’ work and analyze narrative and video cases of teaching. Therefore, mathematics teacher educators should be informed about the practice-based professional development program that aims to deepen and sharpen teachers' knowledge through a practice-based curriculum and to improve teachers' capacity for innovative practice (Ball & Cohen, 1999; Smith, 2001; Silver, 2009). Similarly, Silver et al. (2007) states that practice-based professional learning tasks provide many opportunities for teachers to learn mathematics such as building connections among related mathematical ideas and to rethink and reorganize the mathematics that they would encounter in their practice. The results of the study also point to the importance of field-based opportunities, in which pre-service teachers taught ratio and proportion concepts to students in a real classroom in order to improve their own knowledge and understanding. The study also suggests that pre-service teachers need opportunities to review and discuss their own teaching and other's teaching. Thus, the documents of practice drawn from teachers' own teaching and others' teaching should be included in professional development courses. In a similar manner, Ball and Cohen (1999) argue that teachers need opportunities to review their current practices and to examine others' practices; moreover, they need to learn more about mathematics contents and students they teach. Furthermore, researchers suggest that in order to have an impact on pre-service teachers' knowledge and instructional practices, professional learning experiences should be closely tied to real classroom practices (Ball & Cohen, 1999; Smith 2001).

One of the important findings of the study was that pre-service teachers generally applied algebraic procedures such as *cross-multiplication* without associating meaning and used limited number of strategies to solve proportion problems before receiving any instruction. Similar findings were found in the studies with students (Avcu & Avcu, 2010; Cramer & Post, 1993; Duatepe, et al., 2005). However, proportional reasoning is a mental process that involves argumentation and conscious analysis of the multiplicative relationships between variables (Lamon, 2007, 2011). Therefore, both teachers and students should be encouraged to use different solution strategies highlighting multiplicative relationships to solve proportion problems.

Further, results indicated that familiarity with problem type is important in solving proportion problems. However, textbooks and teachers generally present traditional missing value problems and do not prefer to ask numerical and qualitative reasoning problems. It can be suggested to writers of mathematics textbooks, teachers and curriculum developers that they should include all types of proportional problems (i.e., missing value, numerical comparison, qualitative reasoning problems) into the mathematics lessons, books and curriculum. Additionally, teacher educators should also put emphasis on solving different problem types by using a broad range of strategies when designing their courses.

The results of the study made salient that there is a strong tendency to over generalize proportionality; that is, pre-service teachers inaccurately apply proportional reasoning in some situations that have nonproportional relationships before receiving instruction. The fact that both students and teachers frequently use additive strategies where multiplicative comparisons are required (Hart, 1988; Lesh et al., 1988). However, making the transition from additive to multiplicative thinking is crucial in understanding proportional relationships. Thus, teachers and teacher educators should be given more attention to the transition from additive to multiplicative thinking than it has traditionally received. That is, mathematical tasks that included not only proportional situations but also nonproportional situations in which variables had additive and constant relationships should be solved in mathematics classrooms and professional development programs. Similarly, Hillen (2005) and Sowder et al. (1998) suggest that pre-service teachers can construct a deeper understanding on proportional and nonproportional situations and their difference with the aid of professional development opportunities.

Lamon (1993a, 1993b) argues that one of the most important differences between those who think proportionally and those who do not is, using composite units when the context suggests that using them is more effective than using singleton units. However, pre-service teachers in the current study could not adequately achieve the process. In fact, it is a long term process, which began in elementary school (Lamon, 1993a). Therefore, mathematics teachers in elementary levels should give importance to develop their students' flexibility in unitizing and reunitizing quantities. In other words, mathematical tasks that aim to improve students' flexibility in unitizing and reunitizing quantities must begin in early elementary grades. Lamon (1993a) recommends teachers to present problems that could be solved using either one-units or composite units, which allowed flexibility in the choice of units. In addition, Lamon (1993a) suggests, "Situations allowing flexibility in the choice of units should encourage monitoring and regulation of the problem solving process and the adaptation of more complex units when they serve to increase the accuracy or efficiency of the solution." (p. 153).

The results of the study revealed that pre-service teachers might make mistakes in teaching proportionality concepts if they did not have a deep understanding of the concepts although they did not make the same mistakes in the posttest or post-interview. In order for teachers to present a conceptually based ratio and proportion curriculum to their students, they themselves must have a conceptually based understanding of the topic (Cramer et al., 1993; Lamon, 2007). Thus, teacher educators must put more emphasis on pre-service teachers' content knowledge when designing their courses so as to deepen and sharpen their knowledge of proportional reasoning. Moreover, teachers need a broad and deep knowledge of concepts, principles and strategies they teach (Ball & Cohen, 1999; Ma, 1999; NBPTS, 2010). Thus, pre-service teachers should be given opportunities to explore proportionality concepts in a deep way.

5.3 Limitations and Recommendations for Future Research

The present study was an effort to investigate pre-service middle school mathematics teachers' proportional reasoning before and after a practice-based instructional module based on proportional reasoning. Findings of this qualitative multiple case study relied on the data gathered form three pre-service mathematics teachers. Quantitative research studies, which provide a bigger picture of pre-service teachers' proportional reasoning, can be performed with pre-service teachers in large numbers. Additionally, in order to achieve the intended developments in pre-service teachers' proportional reasoning, their knowledge and understanding in the proportionality concepts should continue to be analyzed in different settings. Additionally, further research should be conducted not only with pre-service mathematics teachers, but also with in-service mathematics teachers. Moreover, further studies need to be done to explore how in-service mathematics teachers' proportional reasoning affect students' learning in ratio and proportion concepts.

This study only focused on one aspect of the pre-service mathematics teachers' mathematical knowledge for teaching proportional reasoning, which is common and specialized content knowledge. The studies that will investigate the other aspects of mathematical knowledge for teaching that are knowledge of students and content, and knowledge of content and teaching will also be beneficial for understanding the nature of pre-service mathematics teachers' mathematical knowledge for teaching proportional reasoning.

Finally, it is important to note that the practice-based professional development is an effective program that aims to deepen and sharpen teachers' knowledge through a practice-based curriculum and to improve teachers' capacity for innovative practice (Ball & Cohen, 1999; Smith, 2001; Silver, 2009). Therefore, further studies should be conducted to investigate the effect of the practice-based professional development program in improving pre-service teachers' knowledge and conceptions of other mathematics topics.

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APPENDICES

APPENDIX A

Turkish Version of the Proportional Reasoning Test

ORANTISAL AKIL YÜRÜTME TESTİ

Genel Açıklama: Aşağıdaki problemleri yaptığınız bütün işlemler açık olacak şekilde çözünüz. Yanlış yaptığınızı veya çözüme dâhil etmeyeceğinizi düşündüğünüz işlemleri silmeyiniz, sadece üzerini çiziniz. Gerekirse hesap makinesi kullanabilirsiniz.

Problem 1-4'de *x*'i iki farklı yolla bulunuz.

1. $\frac{4}{20} = \frac{x}{35}$

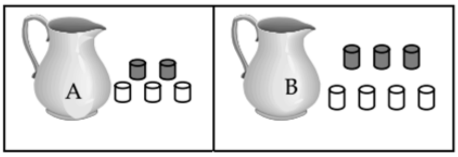
2. $\frac{2}{7} = \frac{6}{x}$

3. $\frac{3}{8} = \frac{x}{20}$

4. $\frac{9}{15} = \frac{12}{x}$

5. $\frac{3}{8} = \frac{x}{20}$ eşitliğinin oluşturulmasını ve çözülmesini gerektirecek bir sözel problem yazınız.

6. Aşağıdaki problemi iki farklı yoldan çözünüz.



Yukarıdaki şekilde görülen A ve B sürahilerinde portakal suyu yapılmaktadır. Koyu renkli bardaklarda portakal suyu konsantresi, açık renkli bardaklarda ise su vardır. Şekilde görüldüğü gibi A sürahisine 2 bardak portakal suyu konsantresi ve 3 bardak su, B sürahisine ise 3 bardak portakal suyu konsantresi ve 4 bardak su konulmuştur. Buna göre hangi sürahideki portakal tadı daha fazladır? Açıklayınız.

Adapted from Noelting, G. (1980). The development of proportional reasoning and the ratio concept: Part 1-Differentiation of stages. *Educational Studies in Mathematics, 11*, 217-253.

Problem 7 ve 8'de doğru şıkkı işaretleyiniz ve seçiminizi neye göre yaptığınızı açıklayınız.

7. Bir koşu parkurunda Esra, Gonca'dan daha kısa zamanda daha çok tur koşmuştur. Hangisi daha hızlı koşucudur?

   a) Esra
   b) Gonca
   c) Eşittirler.
   d) Verilen bilgiler yetersizdir.

Açıklamanız:

Taken from Cramer, K., Post, T., & Currier, S. (1993). Learning and teaching ratio and proportion: Research implications. In D. T. Owens (Ed.), *Research ideas for the classroom*. New York: Macmillan. p. 166.

8. Duygu'nun annesi her sabah, Duygu ve kardeşi için taze portakal ve elmaları kullanarak meyve suyu karışımı hazırlamaktadır. Eğer Duygu'nun annesi bugün, dün kullandığından daha az portakal ve elma kullandıysa hazırladığı meyve suyunun tadı nasıldır? Neden?

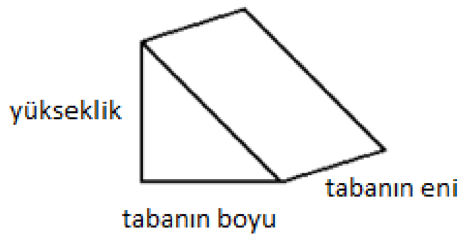
   a) Dünkünden daha çok portakallıdır.
   b) Dünkünden daha çok elmalıdır.
   c) Dünküyle aynıdır.
   d) Verilen bilgiler yetersizdir.

Açıklamanız:

9. Murat odasını boyamak istiyor. İstediği renge ulaşıncaya kadar mavi ve beyaz renk boyaları karıştırıyor. Odasını boyamaya başlıyor ama elindeki boyanın duvarlara yetmesi için çeyrek kutu boyaya daha ihtiyacı olduğunu fark ediyor. Murat, elindeki boyanın rengini değiştirmeden miktarını artırmak istiyor; bu amaçla bir bardak mavi, bir bardak beyaz olmak üzere eşit miktarlarda boyaları elindeki boya kabına ekliyor (2 bardak = 1 çeyrek kutu). Sizce, Murat'ın elindeki boyanın rengini değiştirmeden miktarını artırmak için kullandığı yöntem her zaman işe yarar mı? Neden?

   Taken from Heinz, K. R. (2000). *Conceptions of ratio in a class of preservice and practicing teachers*. Unpublished doctoral dissertation, The Pennsylvania State University, p. 150.

10. Termal kaplıcaları ile ünlü Afyon'da kayak yapmaya elverişli dağ sayısı azdır. Termal su ve kayağın aynı yerde olduğu bir merkezin çok ilgi çekeceğini düşünen bir grup girişimci, Afyon'da kayak yapılabilecek yamaçlar inşa etmeyi ve kolay kayılabilsin diye plastik fiber ile bu yamaçları kaplamayı planlamışlardır. Farklı eğimlerde toplam 3 yamaç inşa etmek isteyen girişimciler bu yamaçları eğimlerine göre sıralamak istemektedir. Her yamacın tabanının boyu, eni ve yamacın yüksekliği şekilde görüldüğü gibi ölçülebilmektedir. Bu bilgileri kullanarak yamaçların birbirlerine göre eğim sıralamasını hangi bilgileri kullanarak nasıl yaparsınız?



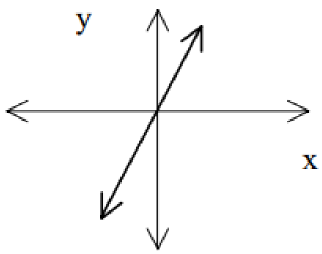
   Taken from Simon, M. A., & Blume, G. W. (1994b). Mathematical modeling as a component of understanding ratio-as-measure: A study of prospective elementary teachers. *Journal of Mathematical Behavior*, *13*, p. 187.

Problem 11-22'de değişkenler arasındaki ilişkilerin birbirleriyle orantılı olup olmadığını verilen boşluklara yazınız ve kararınızı nasıl verdiğinizi açıklayınız.

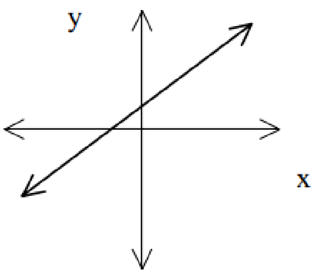
11. Ankara'da bir taksinin açılış fiyatı 2.2 TL ve kilometre başına artış fiyatı 1.9 TL ise bir yolcunun taksiyle gittiği kilometre ile verdiği ücret arasındaki ilişki __________________. Çünkü,

12. Giriş biletinin 12 TL olduğu bir sinemada, alınan bilet sayısı ile biletlere ödenen toplam para arasındaki ilişki __________________. Çünkü,

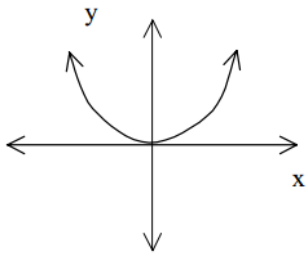
13. Bir maratonda koşan Sevil ve Nehir aynı hızda koşmaktadırlar. Eğer Sevil, Nehir koşmaya başlamadan önce başlayıp 2 km koştuysa, maratonda Sevil ve Nehir'in konumları arasındaki ilişki __________________. Çünkü,

14. 

x ile *y* __________________. Çünkü,

15. 

x ile *y* __________________. Çünkü,

16. 

x ile *y* __________________. Çünkü,

17. $y = 3x + 4{,}5$ *x* ile *y* __________________. Çünkü,

18. $y = 3x^2$ *x* ile *y* __________________. Çünkü,

19. $y = 2,5x$ x ile y ___________________. Çünkü,

20. x ile y ___________________. Çünkü,

x	*y*
4	6
6	9
8	12
10	15
12	18

21. x ile y ___________________. Çünkü,

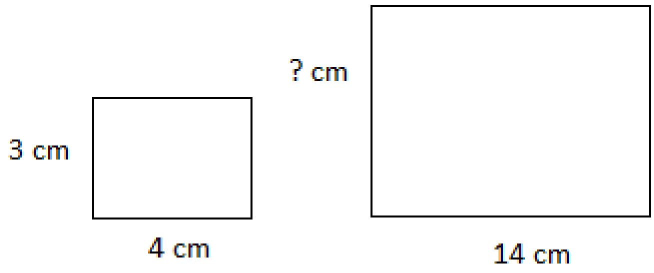
x	*y*
4	10
6	14
8	18
10	22
12	26

22. x ile y ___________________. Çünkü,

x	*y*
0	0
4	8
6	18
8	32
10	50

Adapted from Smith, M. S., Silver, E. A., Leinhardt, G., & Hillen, A. F. (2003).*Tracing the development of teachers' understanding of proportionality in a practice-based course*. Paper presented at the annual meeting of the American Educational Research Association, Chicago, IL, p. 54.

23. Volkan ve Emre, mezuniyet yıllığı için arkadaşlarının fotoğraflarını bozmadan büyütmek istiyorlar. Büyütmek istedikleri fotoğraflardan birinin eni 3 cm, boyu ise 4 cm'dir. Bu fotoğraf büyütüldükten sonra boyu 14 cm oluyorsa eni kaç cm olmuştur? Cevabınızı nasıl bulduğunuzu açıklayınız.



24. Bir terzi farklı boyutlarda örtüler dikmek için dikdörtgen şeklinde üç farklı kumaş kesiyor. Bu kumaşların boyutları 23 cm'e 35 cm, 139 cm'e 155 cm ve 56 cm'e 75 cm'dir. Terzinin kestiği kumaşlardan hangisi daha çok kareye benzer? Hangisi daha az kareye benzer? Cevabınızı nasıl bulduğunuzu açıklayınız.

Taken from Heinz, K. R. (2000). *Conceptions of ratio in a class of preservice and practicing teachers*. Unpublished doctoral dissertation, The Pennsylvania State University, p. 150.

APPENDIX B

Turkish Version of the Interview Schedules

BİRİNCİ GÖRÜŞME PLANI

Giriş

Merhaba. Sizinle, oran ve orantı konusuna yönelik bilgileriniz, görüşleriniz ve deneyimleriniz hakkında konuşmak istiyorum. Bu araştırmanın amacı, matematik öğretmen adaylarının orantısal akıl yürütme becerilerini incelemektir.

Görüşme yaklaşık bir saat sürecek. Bu zaman diliminde soracağım sorulara cevap vermek için uygun musunuz?

Bana söylediğiniz her şey kesinlikle gizli tutulacaktır. Kişilerin isimleri ve üniversitelerine ait bilgiler hiçbir yerde kullanılmayacaktır.

Sormak istediğiniz başka soru var mı?

Görüşmemizin ses kaydını yapmak istiyorum sizin için bir sakıncası var mı?

Kişisel Bilgiler

1. Kaç yaşındasınız?
2. Hangi liseden mezun oldunuz?
3. Lisans eğitimiz boyunca matematik öğretimiyle ilgili hangi dersleri aldınız? Bu derslerdeki notlarınız nelerdi?
4. Genel not ortalamanız nedir?
5. Matematik öğretimiyle ilgili deneyimleriniz nelerdir?

İçerik ve Süreçle İlgili Sorular

1. Kendi cümlelerinizle oranın ne demek olduğunu açıklar mısınız?
2. Kendi cümlelerinizle orantının ne demek olduğunu açıklar mısınız?
3. Oran ve orantı aynı kavramlar mıdır? Neden?
   Alt S: Oran ve orantı kavramlarının arasındaki farklar nelerdir?
4. Oran ve orantının günlük hayatta kullanımına örnekler verir misiniz?
5. İki çokluğun birbiriyle orantılı olup olmadığını nasıl anlarsın?

   - Ölçekteki 11,12 ve 13. sorulardaki değişkenlerin orantılı olup olmadığına nasıl karar verdiniz? Teker teker açıklar mısınız? Neden böyle düşünüyorsunuz?

- Ölçekteki 14,15 ve 16. sorulardaki değişkenlerin orantılı olup olmadığına nasıl karar verdiniz? Teker teker açıklar mısınız? Neden böyle düşünüyorsunuz?
  ALT S: Orantılı olan iki değişkenin grafiği nasıl olmalıdır? Neden?
- Ölçekteki 17,18 ve 19. sorulardaki değişkenlerin orantılı olup olmadığına nasıl karar verdiniz? Teker teker açıklar mısınız? Neden böyle düşünüyorsunuz?
  ALT S: Orantılı olan iki değişkenin denklemi nasıl olmalıdır? Neden?
  ALT S: y= mx denkleminde m neyi ifade eder?

- Ölçekteki 20,21 ve 22. sorulardaki değişkenlerin orantılı olup olmadığına nasıl karar verdiniz? Teker teker açıklar mısınız? Neden böyle düşünüyorsunuz?
  ALT S: (Eğer iki oranın eşitliğine bakmak gerekir derse) Hangi iki oranın eşitliğine bakmak gerekir? Neden?

6. Çoklukların aralarında orantılı ilişki olduğu bir günlük hayat duruma örnek verir misin? Neden bu durumdaki çokluklar orantılıdır?
   ALT S: (Eğer örnek veremezse) Ölçekteki 11,12 ve 13. sorularda değişkenler arasında orantılı ilişki vardır dediğiniz duruma benzer bir örnek verebilirsiniz.

7. Çoklukların aralarında orantısız ilişki olduğu bir günlük hayat duruma örnek verir misin? Neden bu durumdaki çokluklar orantısızdır?
   ALT S: (Eğer örnek veremezse) Ölçekteki 11,12 ve 13. sorularda değişkenler arasında orantısız ilişki vardır dediğiniz duruma benzer bir örnek verebilirsiniz.

8. Turgay öğretmen iki çokluğun orantılı olup olmadığını öğrencilerine anlatırken, öğrencilerinden Atacan: "O zaman öğretmenim, doğrusal ilişkilerin hepsi orantılıdır. Yani iki değişken arasında doğrusal bir ilişki varsa bu iki değişken aynı zamanda birbiriyle orantılıdır." demiştir. Sizce Atacan haklı mıdır? Neden, açıklar mısınız?

9. Orantı sabiti nedir? Örnek vererek açıklar mısınız?
   ALT S: Ölçekte orantı sabitini kullanmanızı gerektiren sorular var mıydı? Neden kullandınız? Hangi sorulardı? Bu sorular için orantı sabitini nedir söyler misiniz?

10. Birimli oran ile birimsiz oran nedir? Aralarındaki fark nedir? Örnek vererek açıklar mısınız?
    ALT S: "Üç ölçek şeker ile iki ölçek saf su karıştırıldığında, çözeltinin yoğunluğunu ya da ne kadar tatlı olduğunu matematiksel olarak ifade eden

değer nedir? ($\frac{3}{2}$ = 1,5 şeker/su). Bu oran birimli midir, birimsiz midir? Neden?
ALT S: “8 tane balonun fiyatı 12 TL ise 6 tane balonun fiyatı nedir?” sorusu için nasıl bir orantı oluşturursunuz? Bu orantıdaki oranlar birimli midir veya birimsiz midir? Neden?

11. Orantı çeşitleri nelerdir? Bu çeşitlere günlük hayattan örnekler verir misiniz?
ALT S: Doğru orantı nedir? Günlük hayattan doğru orantılı iki değişkene örnek verir misiniz? Bu iki değişkenin neden doğru orantılı olduğunu düşünüyorsunuz? Doğru orantının denklemi ve grafiği nasıldır?

ALT S: Ters orantı nedir? Günlük hayattan ters orantılı iki değişkene örnek verir misiniz? Bu iki değişkenin neden ters orantılı olduğunu düşünüyorsunuz? Ters orantının denklemi ve grafiği nasıldır?

12. Ölçekteki 1, 2, 3 ve 4. soruları hangi iki farkı yolla çözmüştünüz? Açıklar mısınız? (Kâğıdına bakıp hatırlaması sağlanabilir)
- Bu iki yolun birbirinden farkı nedir? Neden?

ALT S: Bir orantıda verilmeyen terimi bulurken hangi stratejileri kullanırsınız? Örnek vererek açıklar mısınız?
- Öğrencilerinize, bir orantıda verilmeyen terimi bulurken en etkili stratejinin ne olduğunu söylersiniz? Neden?
- Eğer öğrencileriniz bu stratejiyle anlamazsa başka nasıl (hangi stratejilerle) anlatırsınız?

13. Aşağıda bir problem ve bu probleme 5 öğrencinin verdiği cevapları görmektesiniz. Sizden öğrencilerin verdiği cevapları değerlendirmenizi isteyeceğim. Öncelikle size problemi anlamanız için birkaç dakika veriyorum.

- Aşağıda gördünüz gibi 5 öğrencinin hepsi Selin’in çikolatalı sütünün daha yoğun çikolata tadına sahip olduğu sonucuna varmışlardır. Buna göre, her bir cevabı tek tek göz önünde bulundurarak öğrencilerin Selin’in çikolatalı sütünün daha yoğun çikolata tadına sahip olduğunu nasıl bulduğunu açıklar mısınız?

ALT S: Öğrencilerin bulduğu her bir değerin anlamı nedir? (Öğrencilerin bulduğu değerler problemde neyi ifade etmektedir?)

Problem: Emre ve Selin çikolata sosu ve süt kullanarak çikolatalı süt hazırlamak istemektedirler. Emre 3 fincan çikolata sosunu 5 fincan süt ile karıştırırken, Selin 5 fincan çikolata sosunu 8 fincan süt ile karıştırarak çikolatalı süt hazırlamışlardır. Hangisinin hazırladığı çikolatalı süt daha yoğun çikolata tadına sahiptir? Nasıl bulduğunuzu açıklayınız.
<u>Öğrenci 1</u> Emre: $\frac{3}{5} = 0{,}6$ Selin: $\frac{5}{8} = 0{,}625$ Bu yüzden, Selin'in çikolatalı sütü daha yoğun çikolata tadına sahiptir.
<u>Öğrenci 2</u> Emre: $\frac{3}{5} = \frac{24}{40}$ Selin: $\frac{5}{8} = \frac{25}{40}$ Bu yüzden, Selin'in çikolatalı sütü daha yoğun çikolata tadına sahiptir.
<u>Öğrenci 3</u> Emre: $\frac{5}{3} = 1{,}67$ Selin: $\frac{8}{5} = 1{,}6$ Bu yüzden, Selin'in çikolatalı sütü daha yoğun çikolata tadına sahiptir.
<u>Öğrenci 4</u> Emre: $\frac{3}{8} = 0{,}375$ Selin: $\frac{5}{13} = 0{,}385$ Bu yüzden, Selin'in çikolatalı sütü daha yoğun çikolata tadına sahiptir.

Öğrenci 5

24 25

$\frac{3}{5}$ $\frac{5}{8}$

Bu yüzden, Selin'in çikolatalı sütü daha yoğun çikolata tadına sahiptir.

14. Son olarak oran ve orantı öğretimi ve öğrenimi ile ilgili olarak söylemek istediğiniz bir şey var mı?

İKİNCİ GÖRÜŞME PLANI

Giriş

Merhaba. Sizinle, oran ve orantı konusuna yönelik bilgileriniz, görüşleriniz ve deneyimleriniz hakkında konuşmak istiyorum. Bu araştırmanın amacı, matematik öğretmen adaylarının orantısal akıl yürütme becerilerini incelemektir.

Görüşme yaklaşık bir saat sürecek. Bu zaman diliminde soracağım sorulara cevap vermek için uygun musunuz?

Bana söylediğiniz her şey kesinlikle gizli tutulacaktır. Kişilerin isimleri ve üniversitelerine ait bilgiler hiçbir yerde kullanılmayacaktır.

Sormak istediğiniz başka soru var mı?

Görüşmemizin ses kaydını yapmak istiyorum sizin için bir sakıncası var mı?

Kişisel Bilgiler

1. Adınız soyadınız?

İçerik ve Süreçle İlgili Sorular

1. Kendi cümlelerinizle oranın ne demek olduğunu açıklar mısınız?
2. Kendi cümlelerinizle orantının ne demek olduğunu açıklar mısınız?
3. Oran ve orantı aynı kavramlar mıdır? Neden?
   Alt S: Oran ve orantı kavramlarının arasındaki farklar nelerdir?
4. Oran ve orantının günlük hayatta kullanımına örnekler verir misiniz?
5. İki çokluğun birbiriyle orantılı olup olmadığını nasıl anlarsın?
   - Ölçekteki 11,12 ve 13. sorulardaki değişkenlerin orantılı olup olmadığına nasıl karar verdiniz? Teker teker açıklar mısınız? Neden böyle düşünüyorsunuz?
   - Ölçekteki 14,15 ve 16. sorulardaki değişkenlerin orantılı olup olmadığına nasıl karar verdiniz? Teker teker açıklar mısınız? Neden böyle düşünüyorsunuz?
     ALT S: Orantılı olan iki değişkenin grafiği nasıl olmalıdır? Neden?
   - Ölçekteki 17,18 ve 19. sorulardaki değişkenlerin orantılı olup olmadığına nasıl karar verdiniz? Teker teker açıklar mısınız? Neden böyle düşünüyorsunuz?
     ALT S: Orantılı olan iki değişkenin denklemi nasıl olmalıdır? Neden?
     ALT S: $y= mx$ denkleminde m neyi ifade eder?
   - Ölçekteki 20,21 ve 22. sorulardaki değişkenlerin orantılı olup olmadığına nasıl karar verdiniz? Teker teker açıklar mısınız? Neden böyle düşünüyorsunuz?
     ALT S: (Eğer iki oranın eşitliğine bakmak gerekir derse) Hangi iki oranın eşitliğine bakmak gerekir? Neden?

6. Çoklukların aralarında orantılı ilişki olduğu bir günlük hayat duruma örnek verir misin? Neden bu durumdaki çokluklar orantılıdır?
ALT S: (Eğer örnek veremezse) Ölçekteki 11,12 ve 13. sorularda değişkenler arasında orantılı ilişki vardır dediğiniz duruma benzer bir örnek verebilirsiniz.

7. Çoklukların aralarında orantısız ilişki olduğu bir günlük hayat duruma örnek verir misin? Neden bu durumdaki çokluklar orantısızdır?
ALT S: (Eğer örnek veremezse) Ölçekteki 11,12 ve 13. sorularda değişkenler arasında orantısız ilişki vardır dediğiniz duruma benzer bir örnek verebilirsiniz.

8. Turgay öğretmen iki çokluğun orantılı olup olmadığını öğrencilerine anlatırken, öğrencilerinden Atacan: "O zaman öğretmenim, doğrusal ilişkilerin hepsi orantılıdır. Yani iki değişken arasında doğrusal bir ilişki varsa bu iki değişken aynı zamanda birbiriyle orantılıdır." demiştir. Sizce Atacan haklı mıdır? Neden, açıklar mısınız? Başka bir öğrenci Semih "İki değişken birbiriyle orantılı ise bu iki değişken arasında doğrusal bir ilişki vardır." demiştir. Sizce Semih haklı mıdır? Neden, açıklar mısınız?

9. Orantı sabiti nedir? Örnek vererek açıklar mısınız?
ALT S: Ölçekte orantı sabitini kullanmanızı gerektiren sorular var mıydı? Neden kullandınız? Hangi sorulardı? Bu sorular için orantı sabitini söyler misiniz?

10. Birimli oran ile birimsiz oran nedir? Aralarındaki fark nedir? Örnek vererek açıklar mısınız?
ALT S: "Üç ölçek şeker ile iki ölçek saf su karıştırıldığında, çözeltinin yoğunluğunu ya da ne kadar tatlı olduğunu matematiksel olarak ifade eden değer nedir? ($\frac{3}{2}$ = 1,5 şeker/su). Bu oran birimli midir, birimsiz midir? Neden?
ALT S: "8 tane balonun fiyatı 12 TL ise 6 tane balonun fiyatı nedir?" sorusu için nasıl bir orantı oluşturursunuz? Bu orantıdaki oranlar birimli midir veya birimsiz midir? Neden?

11. Orantı çeşitleri nelerdir? Bu çeşitlere günlük hayattan örnekler verir misiniz?
ALT S: Doğru orantı nedir? Günlük hayattan doğru orantılı iki değişkene örnek verir misiniz? Bu iki değişkenin neden doğru orantılı olduğunu düşünüyorsunuz? Doğru orantının denklemi ve grafiği nasıldır?
ALT S: Ters orantı nedir? Günlük hayattan ters orantılı iki değişkene örnek verir misiniz? Bu iki değişkenin neden ters orantılı olduğunu düşünüyorsunuz? Ters orantının denklemi ve grafiği nasıldır?

12. Ölçekteki 1, 2, 3 ve 4. soruları hangi iki farkı yolla çözmüştünüz? Açıklar mısınız? (Kâğıdına bakıp hatırlaması sağlanabilir)
    - Bu iki yolun birbirinden farkı nedir? Neden?

    ALT S: Bir orantıda verilmeyen terimi bulurken hangi stratejileri kullanırsınız? Örnek vererek açıklar mısınız?

    - Öğrencilerinize, bir orantıda verilmeyen terimi bulurken en etkili stratejinin ne olduğunu söylersiniz? Neden?
    - Öğrencilerinize, sayısal karşılaştırma problemlerinde en etkili stratejinin ne olduğunu söylersiniz? Neden?
    - Eğer öğrencileriniz bu stratejiyle anlamazsa başka nasıl (hangi stratejilerle) anlatırsınız?
    -

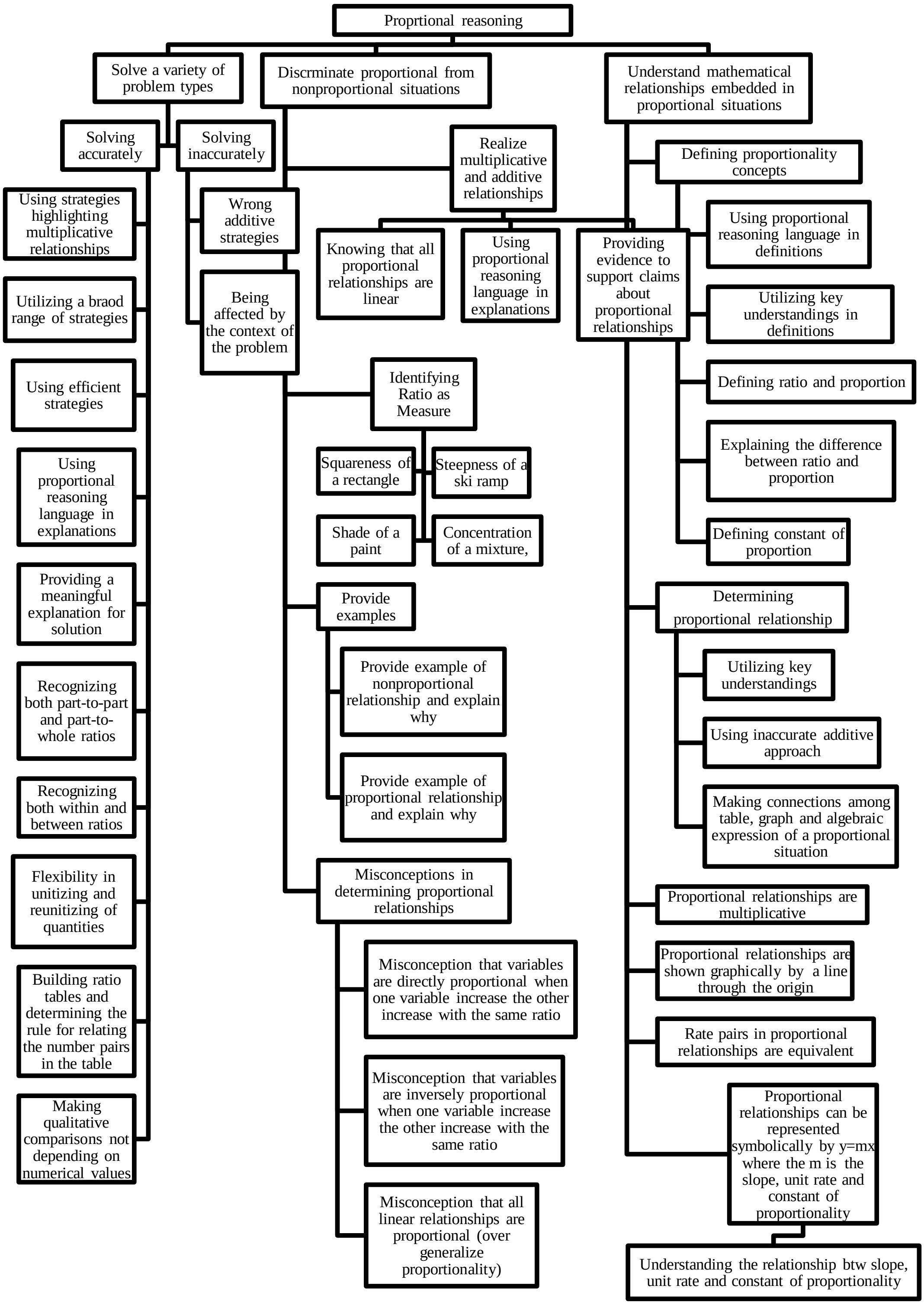
13. Ortaokul öğrencilerine anlattığınız ilk derste eksikleriniz nelerdi? İkinci derste bu eksikleri ne ölçüde kapattığınıza inanıyorsunuz? Eksiklerinizi kapatmakta aldığınız eğitimin ne kadar katkısı oldu? Nasıl?

14. Orantısal akıl yürütme odaklı aldığınız eğitimin size neler kattığını düşünüyorsunuz? Bu eğitime başlamadan önce bilmediğiniz ve anlamadığınız neleri şu anda bildiğinizi ve anladığınızı düşünüyorsunuz?

15. Son olarak, oran ve orantı öğretimi ve öğrenimi ile ilgili olarak söylemek istediğiniz bir şey var mı?

APPENDIX C

List of Coding Categories



APPENDIX D

The Objectives That Each Participant Taught In the Student Teachings

<table>
<tr><th>Participants</th><th>Objectives In the Turkish Mathematics Curriculum</th></tr>
<tr><td>Gaye</td><td>• Doğru orantılı iki çokluk arasındaki ilişkiyi tablo veya denklem olarak ifade eder.
✓ Doğru orantılı çokluklar arasında çarpmaya dayalı bir ilişki olduğu dikkate alınır.
• Doğrusal ilişki içeren gerçek yaşam durumlarına ait tablo, grafik ve denklemi oluşturur ve yorumlar.
✓ Doğrunun eksenleri hangi noktalarda kestiği, eksenlere paralelliği, orijinden geçip geçmediği ve benzeri durumların gerçek yaşamla ilişkisi kurulur.
✓ Doğrunun grafiği yorumlanırken doğru üzerindeki noktaların x ve y koordinatları arasındaki ilişki, eksenleri hangi noktalarda kestiği, orijinden geçip geçmediği, eksenlere paralelliği ve benzeri durumlar ele alınır.</td></tr>
<tr><td>Mine</td><td>• Gerçek yaşam durumlarını, tabloları veya doğru grafiklerini inceleyerek iki çokluğun orantılı olup olmadığına karar verir.
✓ İki oran eşitliğinin orantı olarak adlandırıldığı vurgulanır. Doğru orantılı çokluklar ele alınır. Doğru orantılı çokluklara ait grafiklerin orijinden geçtiği dikkate alınır.
• Doğru orantılı iki çokluk arasındaki ilişkiyi tablo veya denklem olarak ifade eder.
✓ Doğru orantılı çokluklar arasında çarpmaya dayalı bir ilişki olduğu dikkate alınır.
• Doğru orantılı iki çokluğa ait orantı sabitini belirler ve yorumlar.
✓ Verilen gerçek yaşam durumları, bunlara ilişkin tablolar veya doğru grafikleri incelenerek orantı sabitini belirlemeye yönelik çalışmalar yapılır.</td></tr>
<tr><td>Ela</td><td>• Doğru orantılı iki çokluğa ait orantı sabitini belirler ve yorumlar.
✓ Verilen gerçek yaşam durumları, bunlara ilişkin tablolar veya doğru grafikleri incelenerek orantı sabitini belirlemeye yönelik çalışmalar yapılır.
• Gerçek yaşam durumlarını ve tabloları inceleyerek iki çokluğun ters orantılı olup olmadığına karar verir.
✓ Ters orantılı çoklukların çarpımının sabit olduğunu keşfetmeye yönelik çalışmalara yer verilir.
• Doğru ve ters orantıyla ilgili problemleri çözer.
✓ Ölçek, karışım, indirim ve artış durumlarına ilişkin problemlere yer verilir.</td></tr>
</table>

APPENDIX E

Turkish Version of Informed Consent Form

Gönüllü Katılım Formu

İlköğretim Matematik Bölümü Öğretmen adaylarının matematik öğretimi bilgileri ile ilgili bir araştırma yapmaktayız. Sizin velisi olduğunuz …………………………………… adlı öğrencinin de bu araştırmaya katılmasını öneriyoruz; çünkü araştırma süresince çocuğunuz matematik dersinde öğrenmekte zorluk çektiği konularla ilgili dersler alacaktır. Çalışmaya katılım gönüllülük esasına dayalıdır. Kararınızdan önce araştırma hakkında sizi bilgilendirmek istiyoruz. Bu bilgileri okuyup anladıktan sonra çocuğunuzun araştırmaya katılmasını isterseniz formu imzalayınız.

Araştırma sonuçlarının eğitim ve bilimsel amaçlarla kullanımı sırasında çocuğunuzun kişisel bilgileriniz ihtimamla korunacaktır. Araştırmaya yönelik oluşabilecek sorularla ilgili olarak Öğr. Gör. Mutlu PİŞKİN TUNÇ'a … numaralı telefondan ulaşabilirsiniz.

Katılımcının Beyanı

Bana yapılan tüm açıklamaları ayrıntılarıyla anlamış bulunmaktayım. Kendi başıma belli bir düşünme süresi sonunda adı geçen bu araştırma projesinde çocuğumun "katılımcı" olarak yer alması kararını aldım. Bu konuda yapılan daveti büyük bir memnuniyet ve gönüllülük içerisinde kabul ediyorum.

Katılımcının Velisi:
Adı, soyadı:
İmza:

APPENDIX F

Scores of the Participants on Each Item of the PRET

		Pretest			**Posttest**		
Qs	Objectives	Gaye	Mine	Ela	Gaye	Mine	Ela
1	Approaches to different problem types	3	2	3	3	4	3
2	Approaches to different problem types	2	2	3	4	4	4
3	Approaches to different problem types	3	3	3	4	4	4
4	Approaches to different problem types	3	3	3	4	4	4
5	Distinguishing proportional situations	1	1	4	1	4	4
6	Approaches to different problem types	2-3	3-0	3-4	4-4	4-4	4-4
7	Approaches to different problem types	2	3	3	3	3	4
8	Approaches to different problem types	2	1	4	3	1	4
9	Distinguishing proportional situations & Understanding mathematical relationships	1	2	1	4	3	4
10	Distinguishing proportional situations & Understanding mathematical relationships	4	2	2	4	2	4
11	Distinguishing proportional situations & Understanding mathematical relationships	1	1	3	3	3	3
12	Distinguishing proportional situations & Understanding mathematical relationships	2	2	3	2	3	3
13	Distinguishing proportional situations & Understanding mathematical relationships	1	3	2	3	3	3
14	Distinguishing proportional situations & Understanding mathematical relationships	3	3	3	4	4	4
15	Distinguishing proportional situations & Understanding mathematical relationships	1	2	2	4	4	4
16	Distinguishing proportional situations & Understanding mathematical relationships	2	2	3	3	4	4
17	Distinguishing proportional situations & Understanding mathematical relationships	3	3	3	3	3	4
18	Distinguishing proportional situations & Understanding mathematical relationships	2	3	3	3	3	3
19	Distinguishing proportional situations & Understanding mathematical relationships	3	3	3	3	3	4
20	Distinguishing proportional situations & Understanding mathematical relationships	2	3	3	3	4	3
21	Distinguishing proportional situations & Understanding mathematical relationships	1	4	3	3	3	4
22	Distinguishing proportional situations & Understanding mathematical relationships	1	3	3	2	3	4
23	Approaches to different problem types	3	2	2	4	4	4
24	Approaches to different problem types & Distinguishing proportional situations	1	1	1	4	4	4
Total		**52**	**57**	**71**	**82**	**85**	**94**

APPENDIX G

Courses offered by the Elementary Mathematics Education Program

UNDERGRADUATE CURRICULUM	
FIRST YEAR	
First Semester	**Second Semester**
General Mathematics	Discrete Mathematics
Turkish I: Written Expression	Geometry
Atatürk's Principles and Revolutionary History I	Atatürk's Principles and Revolutionary History II
Computer I	Turkish II: Oral Expression
Foreign Language I	Computer II
Introduction to Education	Foreign Language II
	Educational Psychology
SECOND YEAR	
Third Semester	**Fourth Semester**
Calculus I	Calculus II
Linear Algebra I	Linear Algebra II
Physics I	Physics II
Scientific Research Methods	Instructional Technologies and Material Design
Instructional Principles and Methods	Elective Course
Elective Course	
THIRD YEAR	
Fifth Semester	**Sixth Semester**
Calculus III	Differential Equations
Analytical Geometry I	Analytical Geometry II
Statistics and Probability I	Statistics and Probability II
Introduction to Algebra	Methods of Teaching Mathematics II
History of Science	History of Turkish Education
Methods of Teaching Mathematics I	Community Service
Elective Course	Measurement and Evaluation
FOURTH YEAR	
Seventh Semester	**Eighth Semester**
Elementary Number Theory	Philosophy of Mathematics
History of Mathematics	Turkish Educational System and School Management
Guidance	Teaching Practice
School Experience	Elective Course
Classroom Management	Elective Course
Special Education	
Elective Course	

APPENDIX H

Sample Activities in the Practice-Based Instructional Module

I. Hafta-Çiçek Problemi (Flower Problem)

Problem: İki hafta önce, iki çiçeğin boyları 8 cm ve 12 cm olarak ölçülmüştür. Bugün ise çiçekler sırasıyla 11 cm ve 15 cm boyundadır. 8 cm'lik çiçek mi yoksa 12 cm'lik çiçek mi daha fazla uzamıştır?
Bu problem için geçerli olabilecek iki farklı cevap bulunuz ve cevaplarınızı açıklayınız.

Çözüm: I. Yol : Her iki çiçek de aynı miktarda uzamıştır (3 cm).
Bu çözüm, *Toplamsal Akıl Yürütmeye* dayanmaktadır.
İki yeni ölçmedeki sonuç için, her bir ölçüme bir tek nicelik eklenmiştir.
II. Yol: Çiçeklerin uzama miktarlarını başlangıç boylarıyla karşılaştırmaktır.
Çiçeklerden birincisi kendi boyunun $\frac{3}{8}$ ' i, ikincisi ise $\frac{3}{12}$ 'si kadar uzamıştır. Bu *çarpımsal bakış açısına* göre, ilk çiçek daha çok uzamıştır.
Bu çözüm, değişim durumuna **orantısal olarak** bakmaktadır.

- Burada, hem toplamsal hem de çarpımsal akıl yürütme, farklı cevaplar olsa da geçerli cevaplar üretmiştir.

I. Hafta- Tarla Problemi (Field Problem)

Bir çiftçi, 3 tarlaya sahiptir. İlki 185x245 m, ikincisi 75x114 m, üçüncüsü ise 455x508 m ebatlarındadır. Bu üç tarlaya gökyüzünden bakarsanız hangi tarla size en çok karemsi görünür? Hangisi en az karemsi görünür? Cevaplarınızı açıklayınız.

Öğrenciler Tarla Problemi'nde hangi yanılgılara düşebilir?

- Boy ve en arasındaki farklara bakarak yanlış bir akıl yürütmeyle (toplamsal) en çok fark olanın veya en az fark olanın daha karemsi olduğunu söyleyebilirler.
- En büyük veya en küçük dikdörtgenin sırf büyük veya sırf küçük olduğu için daha karemsi görüneceğini düşünerek yanılabilirler (görsel karşılaştırma).

I. Hafta -Tartışalım

- ❖ Bütün oranlar rasyonel sayı mıdır?

π sayısı aslında bir orandır; yani dairenin çevresinin çapına oranıdır. Ama π rasyonel bir sayı değildir.

$1:\sqrt{2}$ bir karenin köşegenine oranıdır. Bu oran da rasyonel bir sayı değildir.

- ❖ Bütün rasyonel sayılar aynı zamanda oran mıdır?

Evet! Bütün rasyonel sayılar $\frac{a}{b}$ şeklinde yazılabilir.

$$\mathbb{Q} = \{\frac{a}{b} | a, b \in \mathbb{Z} \wedge b \neq 0\}$$

- ❖ Bütün oranlar aynı zamanda parça-bütün karşılaştırması mıdır?

Hayır, sadece bazıları! Parça-bütün ve parça-parça karşılaştırmaları aynı çoklukları karşılaştıran oranlardır.

Örnek: Ege'nin 4'ü mavi, 5'i kırmızı olmak üzere 12 tane kalemi var. (parça-bütün, parça-parça)

4:5, 5:12, 4:12

Örnek değil: Bir araba 2 saatte 110 kilometre yol alıyor.

110 km : 2 sa

- ❖ Bütün parça-bütün karşılaştırmaları oran mıdır?

Evet orandır.

Örnek: Ege'nin 4'ü mavi, 5'i kırmızı olmak üzere 12 tane kalemi var. (parça-bütün, parça-parça)

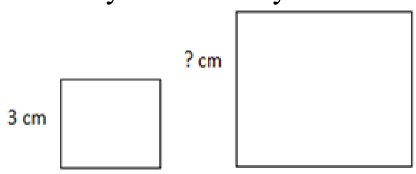
4:5, 5:12, 4:12

II. Hafta - Fotoğraf Büyütme Sorusu (Photo Enlargement Problem)

Öğrenci ne yaptı?

Aşağıdaki Fotoğraf Büyütme Problemi'ne bir öğrencinin verdiği cevabı inceleyiniz. Sizce öğrenci haklı mıdır? Neden? Siz o öğrencinin öğretmeni olsanız, öğrenciye nasıl bir açıklama yaparsınız.

Problem: Bir fotoğrafın eni 3 cm, boyu ise 4 cm'dir. Bu fotoğraf büyütüldükten sonra boyu 20 cm oluyorsa eni kaç cm olmuştur?



Öğrenci Cevabı: Bence cevap 19 cm'dir. Çünkü 4 cm'de 20 cm'e çıktıysa 16 cm büyümüş demektir. Bu yüzden diğer kenarında aynı miktarda büyüyüp 3+16=19 cm'e çıkması gerekir.

- ➢ Öğrenci çarpımsal durumda toplamsal durum gibi düşünmüştür.

II. Hafta – Yaş Problemi (Age Problem)

Öğrenci ne yaptı?

Aşağıdaki problemi çözerken öğrencileriniz hangi yanılgılara düşebilir? Siz, öğrencilerinizi bu yanılgılarından vazgeçirmek için nasıl bir açıklama yaparsınız?

Problem: Su 12 yaşında, abisi 15 yaşındadır. Su şimdiki yaşının iki katı yaşına gelince abisi kaç yaşında olur?

Öğrenci Yanılgısı: Öğrenciler toplamsal durumda çarpımsal durum gibi düşünebilir. "Kendisinin yaşı iki katına çıktıysa abisinin yaşı de iki katına çıkar. Bu yüzden abisi 30 yaşındadır" diye cevap verebilirler.

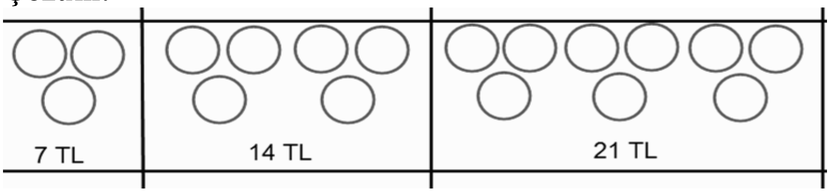
III. Hafta- Dikdörgen ve Top Problemleri (Rectangle and Ball Problems)

Problem: Eni 3cm, boyu 4 cm olan dikdörtgenin kenarlarını tekrarlı toplama yaparak en-boy oranını değişmeden aşağıdaki gibi büyütelim.

- Öğrenciler öncelikle, oranı ifade eden çoklukları tekrarlı toplama yaparak, nitel anlamda yapıyı bozmadan yeni durumlar oluşturabileceklerini kavrarlar.
- Boyu ilk boyu kadar artınca, eni de ilk eni kadar artıyor. Burada çarpımsal bir ilişkilendirme yapmış oluyoruz. Toplamsal ilişkilendirme yapan bir öğrenci boyu 4 cm artıyorsa eni de 4 cm artar diye yanlış bir çıkarımda bulunurdu.

Problem: "3 top 7 TL ederse, 9 top kaç TL eder?" şeklinde verilen problemi tekrarlı toplama yaparak çözünüz.

Çözüm:



- Öğrenciler "her 3 top 7 TL eder" ilişkisini ve bu ilişkinin değişmezliğini kullanarak çözüme ulaşırlar.

III. Hafta- İş Problemi (Labor Problem)

Problem: İki arkadaşın üzerinde tek başlarına çalıştıklarında 6 saat ve 4 saatte bitirdikleri bir işi, ikisi beraber çalıştığında ne kadar sürede bitireceklerdir?
Bu problemi bilinen veya tanıdık işlemsel çözümlere başvurmadan yapılan iş miktarı ile geçen süre arasında bir ilişkilendirme yaparak çözünüz.

Çözüm: Bir saatlik süre ile işin bitimi için gereken toplam süre arasındaki ilişki, bir saat içerisinde yapılan iş miktarı ve toplam iş miktarı arasındaki ilişki ile aynıdır.
1. kişi------1 saatte işin 1/6 sını bitirir.
2.kişi------- 1 saatte işin 1/4 ünü bitirir.
Birlikte-----1 saatte işin 5/12 sini bitirirler.
Birlikte----- Kaç saatte işin 12/12 sini bitirirler.
Tüm iş miktarı 12/12, bir saat içinde yapılan iş miktarı 5/12 nin 12/5 katıdır. İşin bitmesi için gerekli olan toplam süre de aynı şekilde 1 saatlik sürenin 12/5 katı olmak zorundadır. İşi bitirmek için geçen toplam süre 12/5 saattir.

Öğrenci ne yaptı?

Aşağıda aynı problem için bir öğrencinin çözümü vardır. Öğrenci kesin bir cevap bulamamıştır. Öğrenci nasıl düşünmüştür? Ona çözüme ulaşması için nasıl yardım edebilirsiniz?

Öğrencinin Çözümü:
1. kişi------1 saatte işin 1/6 sını bitirir.

2.kişi------- 1 saatte işin 1/4 ünü bitirir.

Birlikte-----1 saatte işin 5/12 sini bitirirler.

1 saatte ikisi birlikte işin 1/6+1/4 = 5/12 sini yaparlar. Diğer 1 saat içerisinde işin 5/12 si daha biter. Böylece 2 saat içerisinde işin 10/12 sini bitirirler. Geriye işin 2/12 si kaldı, o zaman bu iki arkadaş işi 2 saatten daha fazla bir sürede bitirebilir diyebiliriz ama tam olarak kaç saatte bitireceklerini söyleyemeyiz.

- Burada açığa çıkan sonuç; problem ile oran kavramı tam olarak bağdaşlaştırılamamış olduğundan, öğrencilerin sadece tekrarlı toplamayı kullanarak oran kavramını gerektiren durumlarda en azından belirli bir noktadan öteye gidemeyecekleridir.

III. Hafta - Sınıfta Neler Oluyor?

İzleyeceğiniz video Gaye öğretmenin sınıfında çekilmiştir. Sizce Gaye öğretmenin doğru orantı tanımını verirken yaptığı hata nedir? Bu küçük hata öğrencilerin hangi yanlış anlamalara sahip olmalarına sebep olabilir? Bu hata nasıl düzeltilebilir?

- "İki çokluktan biri artarken diğeri de artarsa doğru orantılı olurlar. İki çokluktan biri azalırken diğeri artıyorsa ters orantılı olurlar." şeklindeki gelenekleşmiş yanlış ifadeler öğrencilerin zihninde kavramların yanlış algılanmasına ve öğrenme güçlüklerine sebep olmaktadır.

- Bunun yanında, doğru orantılı çokluklarda miktarların birbirine bölümlerinin; ters orantılı çokluklarda ise miktarların çarpımlarının sabit olduğu bilgisinin ezberci bir yaklaşımla öğrenilmesi de yanlıştır. Yanlış anlamalara ve yanlış kavram imajları gelişimine sebep olacak şekildeki gelenekleşmiş anlatımdan uzak durulmalıdır.
- Hem ders kitaplarında, hem de dersin işlenişinde kavramların doğru olarak ifade edilmesi; öğrenmenin kolay ve kalıcı oluşunda önemlidir. Öğrencilerin, oranı verilen çokluklara sayısal değerler vererek çoklukların arasındaki doğru ya da ters orantı ilişkisini görmeleri sağlanmalıdır.
- Artışın veya azalışın aynı oranda olduğuna dikkat çekilmelidir.

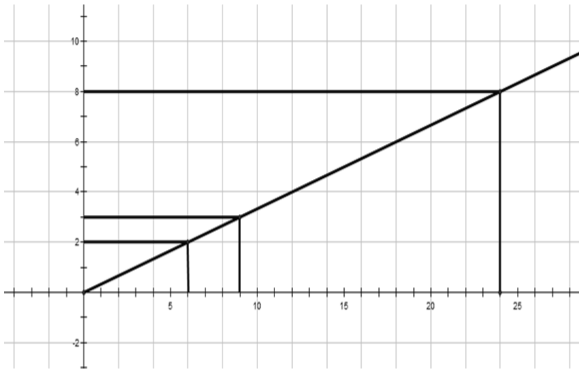
IV. Hafta- Benzer Dikdörtgenler Problemi (Similar Rectangles Problem)

Problem: A, B ve C ölçüleri verilen üç dikdörtgendir. A'nın ölçüleri, 2cm'e 6 cm; B'nin ölçüleri 3cm'e 9 cm ve C'nin ölçüleri 8 cm'e 24 cm'dir. Her dikdörtgenin kendi içindeki oranını bulunuz. Elde ettiğiniz sonuç, bu üç dikdörtgenin benzer olduğuna yönelik sizi ikna eder mi? Bir de A ile B ve A ile C dikdörtgenleri için aralarındaki oranı inceleyiniz. Bu oranlar niçin farklıdır?

Çözüm:
Her dikdörtgenin <u>kendi içindeki oranı:</u>
A için; 2:6=1:3 B için; 3:9=1:3 C için; 8:24=1:3
Kenarları oranı aynı olduğu için dikdörtgenler benzerdir.

A ile B ve A ile C dikdörtgenleri için <u>aralarındaki oranı:</u>
A ile B için; 2:3=6:9 A ile C için; 2:8=6:24
İki dikdörtgenin aralarındaki oran bu iki dikdörtgenin "arasındadır."
Bir köşede sıralanarak yerleştirilen orantısal dikdörtgenlerin köşelerinden gelen doğrunun eğimi, iki kenarın oranına eşittir.

Eğim= $\frac{dikey}{yatay}$



IV. Hafta- Pasta Problemi (Cake Problem)

Aşağıda verilen problemleri cevaplamak için oran tablolarını kullanınız. Bu tablolarda verilen değişkenlerin ilişkilerini gösteren denklem ve grafikleri oluşturunuz.

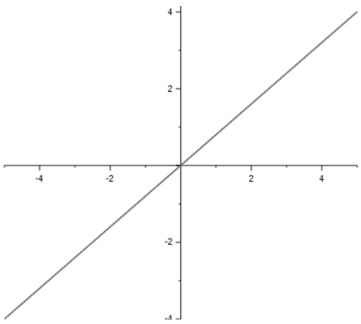
Problem: Kardelen öğretmen bir matematik öğretmenidir ve iki tane sınıfın dersine girmektedir. Dersine girdiği sınıflardaki öğrencilerin hepsine eşit şekilde paylaştırmak üzere küçük pastalar alıp sürpriz yapmak istemektedir. Eğer bir sınıfındaki 16 öğrenci için 20 tane pasta almışsa diğer sınıfındaki 36 öğrenci için kaç tane pasta alması gerekir?

Çözüm:

Öğrenci Sayısı	16	32	4	36
Pasta Sayısı	20	40	5	45

Denklemi: ö = 4/5 p veya
p=5/4 ö

Grafiği:



V. Hafta – Mısır Gevreği Problemi (Cornflakes Problem)

Aşağıda verilen problemi “Birim Oran” stratejisini kullanarak çözünüz.

Problem: Markete giden Gülsün Hanım, çocuklarına kahvaltıda yemeleri için mısır gevreği almak istiyor. Süper Mısır Gevreğinin 160 gramı 3,36 TL ve Harika Mısır Gevreğinin 120 gramı 2,64 TL’dir. Buna göre, Gülsün Hanım hangi mısır gevreğini alırsa daha karlı olur?

Hatırlatma!

- Neye birim diyeceğinize sizin karar verebileceğinizi unutmayın.

Çözüm: Bu problemde, birimi 40 gram almak yani; her iki mısır gevreğinin 40 gramı için olan fiyatları hesaplayarak karşılaştırma yapmak daha uygundur. Böylece, problem, birimi 1 alınca yani; mısır gevreklerinin 1 gramlarının fiyatlarının hesaplanması için yapılan işlemlerden daha kolay işlemlerle, daha kısa zamanda çözülür.

Süper Mısır Gevreği	160g	3,36TL
	40g	0,84 TL
Harika Mısır Gevreği	120g	2,64 TL
	40 g	0,88 TL

O halde, “Süper Mısır Gevreği” daha hesaplıdır.

APPENDIX I

Turkish Summary

ORTAOKUL MATEMATİK ÖĞRETMEN ADAYLARININ UYGULAMAYA DAYALI BİR ÖĞRETİM MODÜLÜNÜN ÖNCESİNDE VE SONRASINDA ORANTISAL AKIL YÜRÜTMELERİ

GİRİŞ

Öğretmenler, öğrettikleri kavramlarla ilgili kavramsal bilgiye sahip olmalıdır (Ball & Cohen, 1999; Ma, 1999; National Board for Professional Teaching Standards [NBPTS], 2010). Dahası, öğretmenlerden kavramlar arasındaki matematiksel ilişkileri, bu kavramların problem çözmedeki, diğer disiplinlerdeki ve günlük hayattaki uygulamaları hakkında da bilgi sahibi olmaları beklenmektedir (NBPTS, 2010). Amerika Ulusal Matematik Öğretmenleri Konseyi (NCTM, 2000) sağlam ve kapsamlı bir kavramsal bilginin oluşturulmasına büyük önem vermektedir. Benzer şekilde, Türkiye'deki matematik müfredatı da kavramların anlamlarının derinlemesine öğrenilmesini amaçlayan kavramsal bilgiye önemini vurgulamaktadır (Milli Eğitim Bakanlığı, 2013). Yapılan araştırmalar, ilk ve orta dereceli okullarda öğretmenlik yapan matematik öğretmenlerinin, kavramlarla ilgili bilgilerinin çoğunlukla işlemsel bilgiye dayandığını ve birçok kavramla ilgili kavram yanılgılarının olduğunu göstermiştir (Ball, 1990; Ma, 1999; Tirosh, 2000). Bu durumun, matematik öğretmenlerinin, orantısal ilişkilerle ve rasyonel sayılarla ilgili bilgi ve anlamalarında da geçerli olduğu görülmüştür (Ball, 1990; Chick, 2003; Cramer & Lesh, 1988; Harel & Behr, 1995; Lacampagne, Post, Harel, & Behr, 1988). Fakat bilindiği üzere, öğretmenlerin bilgi ve anlamaları öğretim kalitesinde önemli bir role sahiptir (Ball, Bass, Sleep, & Thames, 2005). Bu bağlamda, öğretmenlerin ve öğretmen adaylarının kavramlarla ilgili sahip olduğu bilgiler, araştırılması gereken önemli bir konudur.

Öğretmen adaylarının bilgi ve öğretimlerinde bir etki yaratmak isteniyorsa, öğretmen eğitimi, gerçek sınıf ortamlarındaki öğrenme ve öğretme süreciyle yakından ilişkili olmalıdır (Ball & Cohen, 1999; Smith 2001). Buna karşın yapılan araştırmalar, geleneksel öğretmen eğitimi programlarının bu ilişkiyi kurmakta yeterli olmadığını göstermektedir (Smith, 2001). Ball ve Cohen (1999), öğretmen ve öğretmen adayları için, uygulamanın araç olarak kullanıldığı bir mesleki gelişim modeli önermektedir. Bu öğretmen eğitimi modeli, genellikle uygulamaya dayalı mesleki gelişim modeli (practice-based professional development) olarak adlandırılmaktadır; amacı ise uygulamaya dayalı bir müfredat yardımıyla öğretmenlerin bilgilerini artırmak ve uygulamaya yönelik deneyim ve kapasitelerini geliştirmektir (Ball & Cohen, 1999; Smith, 2001; Silver, 2009). Benzer bir şekilde bu çalışmada da, uygulamaya dayalı bir öğretim modülü, öğretmen adaylarının orantısal akıl yürütme becerilerini geliştirmek için kullanılmıştır.

Orantısal akıl yürütme matematiksel akıl yürütmenin bir türüdür ve günlük hayattaki pek çok durum orantısal kurallara göre işler ve çalışır (Cramer & Post, 1993). Benzer şekilde, Baykul (2002)' a göre günlük hayatta sıkça karşılaşılan faiz, yüzde, indirim, komisyon hesaplamalarında ve yol problemlerinin çözümünde orantısal akıl yürütme becerisinden sıkça yararlanılır. Bunun yanında, oran ve orantı kavramlarını derinlemesine anlamak için orantısal akıl yürütme becerisine sahip olmak gerekmektedir (Lesh, Post ve Behr, 1988). Bu bağlamda, bu çalışmanın amacı ortaokul matematik öğretmen adaylarının uygulamaya dayalı bir öğretim modülünün öncesinde ve sonrasında orantısal akıl yürütmelerini incelemektir.

Cramer, Post ve Currier (1993) orantısal akıl yürütme becerisini, orantı yoluyla matematiksel olarak şekillendirilen bir durumu tanıyabilme, orantılı olmayan bir durumdan ayırt edebilme, bu durumu sembolik olarak ifade edebilme ve orantı problemlerini çözebilme becerisi olarak tanımlamaktadır. Orantısal akıl yürütme becerisi, ilköğretim düzeyindeki birçok matematiksel kavramın öğrenilmesinde mihenk taşı iken; lise matematik müfredatındaki ileri düzey matematik kavramlarının öğrenilmesi için gerekli alt yapıyı oluşturan bir köşe taşıdır (Lamon, 2012; Lesh et al., 1988).

Literatür incelendiğinde, orantısal düşünme yeteneğini değerlendirmek için üç farklı problem tipinin tanımlandığı görülmüştür (Cramer ve diğ., 1993; Heller, Post, Behr ve Lesh, 1990; Post, Behr ve Lesh 1988). Bu problem tipleri; bilinmeyen değeri bulma, sayısal karşılaştırma ve niteliksel akıl yürütme problemleridir. Bilinmeyen değeri bulma problem tipinde amaç; a/b = c/d gibi bir orantıda üç çokluk verilmişken dördüncü çokluğun bulunmasıdır (Lamon, 2007). Tipik bir bilinmeyen değeri bulma problemi şöyledir; "300 km yolu 4 saatte alan bir otomobil, aynı hızla giderse 750 km'lik yolu kaç saatte alır?" (Kayhan, Duatepe ve Akkuş-Çıkla, 2004). Bu problemde, orantıdaki üç çokluk verilmiştir ve bilinmeyen çokluk sorulmaktadır. Verilenler; gidilen yol (300 km) ve seyahat süresi (4 saat) ile bilinmeyen bir sürede gidilen yoldur (750 km), istenen ise bilinmeyen süredir. Sayısal karşılaştırma probleminde amaç; iki tane oran verilmişken, sayısal bir cevaba ihtiyaç duymadan oranların karşılaştırılmasıdır. Noelting (1980)'in "Portakal Suyu Problemi" tipik bir sayısal karşılaştırma sorusudur. Bu problemde, portakal suyu konsantresi ve suyla yapılan iki karışımın içindeki portakal suyu konsantresi ve su miktarları verilir ve karışımların tatlarının karşılaştırılması istenir. Diğer problem tipi ise niteliksel akıl yürütme problemleridir. Bu problem tipinde sayısal değerler verilmez ve amaç; sayısal değerlere bağlı olmaksızın karşılaştırmalar yapmaktır. Bu problem tipinin niteliksel tahmin ve niteliksel karşılaştırma olmak üzere iki çeşidi vardır (Cramer et al., 1993).

Araştırmalar orantı problemlerinin çözümünde kullanılan pek çok çözüm stratejisinin olduğunu göstermiştir (Baroody ve Coslick, 1998; Ben-Chaim, Keret ve Ilany, 2012; Cramer ve Post, 1993; Cramer ve diğ., 1993; Kaput ve West, 1994; Lamon, 2007, 2012). Bazı araştırmacılar (Baroody ve Coslick, 1998; Kaput ve West, 1994) bu stratejileri formal ve informal stratejiler olarak ikiye ayırmıştır. Bu araştırmacılara göre, formal stratejiler cebir kurallarının kullanıldığı cebirsel stratejiler (içler-dışlar çarpımı gibi) iken informal stratejiler (birim oran, değişim çarpanı gibi) çoğunlukla orantısal ilişkilerin kullanıldığı stratejilerdir. Cramer ve Post (1993), öğrencilerin orantı problemlerini çözmede informal stratejileri kullanmaya yönlendirilmesini önermişlerdir; hatta formal stratejilerin, öğrencilerin informal stratejileri tam olarak kullanıp içselleştirdiğine emin olununcaya kadar öğretilmemesi gerektiğine vurgu yapmışlardır. Fakat, pek çok çalışmada, öğrencilerin ve hatta öğretmenlerin orantı problemlerini çözerken anlamdan yoksun ve ezbere işlemlerden ibaret olan içler-dışlar çarpımı stratejisini kullandıkları görülmüştür (Ben-Chaim ve diğ., 2012; Cramer ve Post, 1993). Hiç şüphesiz ki en çok kullanılan formal strateji içler-dışlar çarpımı stratejisidir. Bu stratejide, içler-dışlar çarpımı algoritmasıyla orantı kurulur ve eşitlik çözülür (Van de Walle, 2010). En çok kullanılan informal stratejiler ise değişim çarpanı, birim oran, arttırma ve kesir stratejisidir (Cramer ve Post, 1993). Bazı stratejiler orantı problemlerinde yanlış cevap bulunmasına sebep olabilir. En çok kullanılan yanlış çözüm stratejisi ise çarpımsal ilişkiler yerine toplamsal ilişkilerin kullanıldığı toplamsal ilişki stratejisidir (Ben-Chaim ve diğ., 2012; Karplus, Pulos ve Stage, 1983).

Orantısal akıl yürütme becerisiyle ilgili yürütülen pek çok çalışmada, öğrencilerin ve hatta öğretmenlerin oran, orantı kavramlarını anlamlandırmada ve özellikle bu kavramların yer aldığı problemleri çözmede zorluk çektiği görülmüştür (Heller, Ahlegren, Post, Behr ve Lesh, 1989; Ben-Chaim, Fey, Fitzgerald, Benedetto ve Miller, 1998; Singh, 2000). Orantı problemlerinde doğru sonuca ulaşmak orantısal düşünme yeteneğine sahip olunduğunu göstermez, çünkü orantısal ilişkiler fark edilmeden, ezbere algoritmik işlemler yapılarak da (içler-dışlar çarpımı gibi) doğru sonuca ulaşılabilir (Lamon, 2007).

Araştırmanın Amacı

Bu araştırmanın amacı, ortaokul matematik öğretmen adaylarının uygulamaya dayalı bir öğretim modülüne katılımlarından önceki ve sonraki orantısal akıl yürütmelerini incelemektir. Daha ayrıntılı olarak, öğretmen adaylarının, farklı orantısal problem tiplerini yani; bilinmeyen değeri bulma, sayısal karşılaştırma ve niteliksel akıl yürütme problemlerini çözerken yaklaşımlarını, orantısal durumları orantısal olmayan durumlardan ayırt edebilmelerini ve orantısal durumların altında yatan matematiksel ilişkileri anlayabilmelerini ortaya çıkarmaktır.

Araştırmanın Problem Cümlesi ve Alt Problemler

2. Matematik öğretmen adaylarının orantısal akıl yürütmeyle ilgili uygulamaya dayalı bir öğretim modülüne katılımlarından önceki orantısal akıl yürütmelerinin doğası nedir?

    3.4. Matematik öğretmen adaylarının orantısal akıl yürütmeyle ilgili uygulamaya dayalı bir öğretim modülüne katılmadan önce farklı problem tiplerine yaklaşımları nedir?
    3.5. Matematik öğretmen adayları orantısal akıl yürütmeyle ilgili uygulamaya dayalı bir öğretim modülüne katılmadan önce orantısal durumları orantısal olmayan durumlardan nasıl ayırt ederler?
    3.6. Matematik öğretmen adayları orantısal akıl yürütmeyle ilgili uygulamaya dayalı bir öğretim modülüne katılmadan önce orantısal durumların altında yatan matematiksel ilişkileri nasıl anlarlar?

3. Matematik öğretmen adaylarının orantısal akıl yürütmeyle ilgili uygulamaya dayalı bir öğretim modülüne katılımlarından sonraki orantısal akıl yürütmelerinin doğası nedir?

    3.1 Matematik öğretmen adaylarının orantısal akıl yürütmeyle ilgili uygulamaya dayalı bir öğretim modülüne katıldıktan sonra farklı problem tiplerine yaklaşımları nedir?

3.2 Matematik öğretmen adayları orantısal akıl yürütmeyle ilgili uygulamaya dayalı bir öğretim modülüne katıldıktan sonra orantısal durumları orantısal olmayan durumlardan nasıl ayırt ederler?
3.3 Matematik öğretmen adayları orantısal akıl yürütmeyle ilgili uygulamaya dayalı bir öğretim modülüne katıldıktan sonra orantısal durumların altında yatan matematiksel ilişkileri nasıl anlarlar?

4. Matematik öğretmen adaylarının orantısal akıl yürütmeyle ilgili uygulamaya dayalı bir öğretim modülüne katılımlarından önceki ve sonraki orantısal akıl yürütmeleri arasındaki farklar nelerdir?

   4.1 Matematik öğretmen adaylarının orantısal akıl yürütmeyle ilgili uygulamaya dayalı bir öğretim modülüne katılımlarından önce ve sonra farklı problem tiplerine yaklaşımları arasındaki farklar nelerdir?
   4.2 Matematik öğretmen adaylarının orantısal akıl yürütmeyle ilgili uygulamaya dayalı bir öğretim modülüne katılımlarından önce ve sonra orantısal durumları orantısal olmayan durumlardan ayırt etmeleri arasındaki farklar nelerdir?
   4.3 Matematik öğretmen adaylarının orantısal akıl yürütmeyle ilgili uygulamaya dayalı bir öğretim modülüne katılımlarından önce ve sonra orantısal durumların altında yatan matematiksel ilişkileri anlamaları arasındaki farklar nelerdir?

YÖNTEM

Araştırma Deseni

Bu çalışmanın amacı ortaokul matematik öğretmen adaylarının orantısal akıl yürütmeyle ilgili uygulamaya dayalı bir öğretim modülüne katılımlarından önceki ve sonraki orantısal akıl yürütmelerini incelemektir. Bu çalışmada, öğretmen adaylarının orantısal akıl yürütmeleri birçok veri toplama aracından elde edilen veriler kullanılarak detaylı olarak incelendiği için durum çalışması tekniği kullanılmıştır (Yin, 2009). "Ortaokul matematik öğretmen adaylarının orantısal akıl yürütmeleri" olgusunu daha iyi anlayabilmek için bu çalışma, birden fazla katılımcıyla yürütülmüştür; bu yüzden çalışmanın deseni çoklu durum çalışmasıdır (Creswell, 2007; Yin, 2009).

Katılımcılar

Araştırmanın katılımcıları iki aşamada seçilmiştir. Birinci aşamada "kolay ulaşılabilir durum örneklemesi" yöntemi kullanılmıştır. Bu aşamada, Özel Öğretim Yöntemleri II ve Topluma Hizmet Uygulamaları derslerine kayıtlı, 28'i kadın, 12'si erkek olmak üzere 40 üçüncü sınıf ortaokul matematik öğretmen adayı seçilmiştir. Bu öğretmen adayları, Batı Karadeniz'de bir devlet üniversitesinde, 2012-2013 öğretim yılı bahar döneminde ilköğretim matematik öğretmenliği programına kayıtlı öğrencilerdir. İlk aşamadaki öğretmen adaylarının hepsi Orantısal Akıl Yürütme Testi'ni (OAYT) çözmüştür.

Örneklem seçiminin ikinci aşamasında, amaçlı örneklem yöntemlerinden biri olan "maksimum çeşitlilik örneklemesi" kullanılmıştır (Patton, 2002). Patton'a (2002) göre amaçlı örneklemenin gücü zengin bilgiye sahip olduğu düşünülen durumların derinlemesine çalışılmasına olanak vermesindedir. Üç öğretmen adayı, Orantısal Akıl Yürütme Testi'nden aldıkları puanlara göre seçilmiştir. Buna göre, en yüksek puanlı öğretmen adaylarının arasından Ela, orta düzeydeki puanlar arasından Mine ve en düşük puanlı öğretmen adayları arasından da Gaye çalışmaya katılmaları için davet edilmiştir. Çalışmaya katılımın tamamen gönüllülük esasına dayandığı ve elde edilen bulguların not vermek amacıyla kullanılmayacağı belirtilmiştir. Bunun yanında, öğretmen adayları Topluma Hizmet Uygulamaları dersinde yürütülecek olan sosyal sorumluluk projesi hakkında da bilgilendirilmiştir. Sonuç olarak üç öğretmen adayı da çalışmaya katılmayı kabul etmiştir.

Veri Toplama Süreci

Bu çalışmanın verileri 2013 yılının Mart ayından Mayıs ayına kadar yaklaşık iki aylık bir sürede toplanmıştır. Öncelikle Orantısal Akıl Yürütme Testi, Özel Öğretim Yöntemleri II dersini alan bütün öğretmen adaylarına uygulanmıştır. Öğretmen adaylarına bu testi çözmeleri için 60 dakikada verilmiştir. Daha önce bahsedildiği gibi testi çözen öğretmen adaylarının içinden üç öğretmen adayı, Orantısal Akıl Yürütme Testi'nin ilk uygulamasındaki puanlarına göre seçilmiştir. Sonrasında seçilen öğretmen adayları Topluma Hizmet Uygulamaları dersi kapsamında bir sosyal sorumluluk projesini yürütmeye başlamışlardır. Bu projenin amacı, ailesinin ekonomik durumu çocuklarını dershaneye veya özel derse göndermek için uygun olmayan ortaokul öğrencilerine anlamadıkları matematik konularında yardımcı olmaktır. Ortaokul öğrencileri matematik öğretmenleri tarafından istekli olmalarına ve ailelerinin sosyo-ekonomik durumlarına bakılarak seçilmiştir. Öğrencilerle yapılan dersler, 80 dakika (40+40) sürmüştür. Dersler, haftanın bir günü, okulun boş bir sınıfında, öğrencilerin okul saatlerinin dışında bir zamanda, altı hafta boyunca yürütülmüştür. Gaye ve Ela dörder 8.sınıf öğrencisiyle, Mine ise sekiz 7. Sınıf öğrencisiyle olmak üzere, dersler toplam 16 öğrenciyle yapılmıştır. Öğretmen adaylarının öğrencilere öğrettiği matematik konularından biri oran ve orantı konusudur. Her bir öğretmen adayı, oran ve orantı konusundaki anlattıkları kazanımları, uygulamaya dayalı öğretim modülünün öncesinde ve sonrasında olmak üzere, farklı öğrenci gruplarına ikişer kez anlatmışlardır. Bu esnada, öğretmen adayları araştırmacı tarafından gözlemlenmiştir ve her ders videoya kaydedilmiştir. Öğretim deneyimlerinden önce ve sonra öğretmen adaylarıyla yarı-yapılandırılmış görüşmeler yapılmıştır. Uygulamaya dayalı öğretim modülü, öğretmen adaylarının mödülden önceki orantısal akıl yürütmeleri ile ilgi veriler toplandıktan sonra başlatılmıştır. Uygulamaya dayalı öğretim modülü, beş hafta boyunca Özel Öğretim Yöntemleri II dersinin bir parçası olarak yürütülmüştür. Modülden hemen sonra Orantısal Akıl Yürütme Testi araştırmanın katılımcıları olan üç öğretmen adayına son test olarak uygulanmıştır. Daha sonrasında, öğretmen adayları ikinci kez oran ve orantı konusunu anlatmışlar ve ikinci görüşmeler yapılmıştır. Her öğretim deneyiminden önce öğretmen adayları ders planları hazırlamışlardır. Bununla birlikte, ikinci öğretim deneyimlerinden önce, öğretmen adaylarından, ikinci ders planlarında değişiklik yaptılarsa bu değişikliklere neden gerek duyduklarını açıkladıkları bir düzeltme raporu yazmaları istenmiştir.

Veri Toplama Araçları

Öğretmen adaylarının orantısal akıl yürütmelerini derinlemesine ve tüm yönleriyle anlamak için veri toplama sürecinde çok sayıda veri toplama aracı kullanılmıştır. Orantısal Akıl Yürütme Testi, görüşmeler ve öğretim deneyimlerinin gözlemleri ana veri toplama kaynakları olarak kullanılmıştır. Bunların yanında, ders planları ve düzeltme raporları da öğretmen adaylarının orantısal akıl yürütmelerini anlamak için yardımcı veri toplama araçları olarak kullanılmıştır. Veri toplama kaynaklarıyla ilgili detaylı bilgi aşağıda verilmiştir.

1. Orantısal Akıl Yürütme Testi

Bu çalışmada, oran ve orantı literatüründeki problemler derlenerek ve uyarlanarak Hillen (2005) tarafından geliştirilen Orantısal Akıl Yürütme Testi ön test ve son test olarak kullanılmıştır. Orantısal Akıl Yürütme Testi yirmi dört açık uçlu matematik sorusunu içermektedir. Ölçek araştırmacı tarafından Türkçe'ye çevrilmiştir. Ölçeğin geçerliliğini sağlamak için gerekli çalışmalar yapıldıktan sonra Orantısal Akıl Yürütme Testi'ne son hali verilmiştir. Bu testin amacı, öğretmen adaylarının farklı orantısal problem tiplerini yani; bilinmeyen değeri bulma, sayısal karşılaştırma ve niteliksel akıl yürütme problemlerini çözerken kullandıkları çözüm stratejilerini ve süreçlerini, orantısal durumları orantısal olmayan durumlardan ayırt edebilmelerini ve orantısal durumların altında yatan matematiksel ilişkileri anlayabilmelerini ölçmektir. Bu testte öğretmen adaylarının orantısal akıl yürütmelerini doğru bir şekilde ölçebilmek için farklı soru tiplerine ve farklı içerik ve sayısal ilişkilere sahip problemlere yer verilmiştir.

2. Görüşmeler

Öğretmen adaylarının orantısal akıl yürütmelerini daha detaylı incelemek için öğretmen adaylarıyla her bir öğretim deneyiminden sonra yarı-yapılandırılmış görüşmeler yapılmıştır. Görüşme sorularının ana amacı öğretmen adaylarının farklı orantısal problem tiplerine yani; bilinmeyen değeri bulma, sayısal karşılaştırma ve niteliksel akıl yürütme problemlerine yaklaşımlarını, orantısal durumları orantısal olmayan durumlardan ayırt edebilmelerini ve orantısal durumların altında yatan matematiksel ilişkileri anlayabilmelerini ölçmektir. Bunlara ek olarak bazı görüşme soruları öğretmen adaylarına Orantısal Akıl Yürütme Testi'ndeki sorulara verdikleri cevapları tekrar düşünme fırsatı vermiştir. Bu sorularda öğretmen adaylarına Orantısal Akıl Yürütme Testi'ndeki bazı cevaplarını ve cevaplarının gerekçelerini ayrıntılı bir şekilde açıklamaları istenmiştir. Ön görüşmede öğretmen adaylarına bazı kişisel bilgi sorularından sonra orantısal akıl yürütmelerini incelemek için 14 açık uçlu soru sorulmuştur. Son görüşmede de ön görüşmeye paralel sorular sorulmuştur. İlk ve son görüşmelere ait görüşme planları Ek B'de verilmiştir.

3. Öğretim Deneyimlerinin Gözlemleri

Bu çalışmada Orantısal Akıl Yürütme Testi'nden ve görüşmelerden elde edilen verilerin yanında öğretmen adaylarının öğretim deneyimlerinden el edilen gözlem sonuçları da veri toplama aracı olarak kullanılmıştır. Gözlemlerin asıl amacı öğretmen adaylarının farklı orantısal problem tiplerine yani; bilinmeyen değeri bulma, sayısal karşılaştırma ve niteliksel akıl yürütme problemlerine yaklaşımları, orantısal durumları orantısal olmayan durumlardan ayırt edebilmeleri ve orantısal durumların altında yatan matematiksel ilişkileri anlayabilmeleri ile ilgili bulguları geliştirmek ve kontrol etmektir. Diğer bir değişle, öğretim deneyimlerinin gözlemlerinin amacı öğretmen adaylarının oran ve orantı kavramlarına yönelik kavramsal anlamalarını ortaya çıkarmaktır. Gözlem verileri toplanırken yapılandırılmış bir gözlem formu kullanılmamıştır ama araştırmacı her ders anlatımında sınıfta bulunmuş ve saha notları almıştır.

4. Ders Planları ve Düzeltme Raporları

Çalışmanın katılımcıları uygulamaya dayalı öğretim modülünden önce ve sonra yaptıkları ders anlatımları için ders planları hazırlamışlardır. Önceden bahsedildiği gibi Gaye ve Ela 8.sınıf öğrencilerine, Mine ise 7. sınıf öğrencilerine oran ve orantı konularını anlatmışlardı. İlk ders anlatımından önce öğretmen adayları ve araştırmacı buluşup öğretmen adaylarının Milli Eğitim Bakanlığı Ortaokul Matematik Müfredatındaki hangi kazanımları anlatacaklarına karar vermişlerdir. Her bir öğretmen adayının anlattıkları kazanımlar Ek D'de verilmiştir. Öğretmen adayları aynı kazanımları anlattıkları ikişer ders planı hazırlamışlardır. İlk ders planı ilk öğretim deneyiminde uygulandıktan sonra öğretmen adaylarına ders planlarını isterlerse değiştirebilecekleri söylenmiştir. Buna ek olarak ders planlarında herhangi bir değişiklik veya düzeltme yaparlarsa yaptıkları değişikliğin gerekçelerini anlatacakları bir düzeltme raporu yazmaları istenmiştir.

Orantısal Akıl Yürütmeye Yönelik Uygulamaya Dayalı Öğretim Modülü

Orantısal akıl yürütmeye yönelik uygulamaya dayalı bir öğretim modülü Özel Öğretim Yöntemleri II dersinin bir parçası olarak üçüncü sınıf ortaokul matematik öğretmen adaylarına verilmiştir. Öğretim modülü, 2012-2013 öğretim yılı bahar döneminde beş haftalık bir süreç boyunca öğretmen adaylarıyla yürütülmüştür. Öğretim modülünün öğretmeni Özel Öğretim Yöntemleri II dersinin de öğretmeni olan bu çalışmanın araştırmacısıdır. Öğretim modülüne Özel Öğretim Yöntemleri II dersine kayıtlı olan bütün öğretmen adayları katılmıştır fakat bu araştırmaya orantısal akıl yürütmeyle ilgili derinlemesine inceleme yapmak için sadece üç öğretmen adayı dahil edilmiştir. Araştırmacı ve öğretmen adayları haftada bir gün üç veya dört seanslık dersler yapmak için bir araya gelmişlerdir. Her ders 50 dakika sürmüştür ve toplamda öğretmen adaylarıyla 16 ders yapılmıştır. Uygulamaya dayalı öğretim modülünün temel amacı öğretmen adaylarının orantısal akıl yürütmelerini geliştirmektir. Bu amaç doğrultusunda öğretmen adaylarıyla çeşitli etkinlikler yürütülmüştür. Bu etkinlikler içinde; farklı çözüm stratejileri kullanarak orantısal problemleri çözmek, bu problemlerin cevaplarını savunmak ve çözümlerini açıklamak, öğrenci yanılgılarını incelemek, katılımcıların öğretim deneyimlerinde çekilmiş video kesitlerini izleyip, öğretmen ve öğrencilerin yanlışları ve doğruları üzerine tartışmak ve bir ortaokul matematik öğretmeni olan Meral Öğretmen'in sınıfında oran ve orantı konusu işlenirken neler olduğunu anlatan örnek durumu analiz edip tartışmak vardır. Etkinlikler oran ve orantı öğretimi ve orantısal akıl yürütmenin gelişimiyle ilgili literatür derlenerek oluşturulmuştur. Meral Öğretmen'in sınıfında neler olduğuyla ilgili örnek olay ise Smith ve diğ. (2005) tarafından yazılan kitaptan alınmış ve araştırmacı tarafından Türkçe'ye çevrilmiştir. Uygulamaya dayalı öğretim modülünde uygulanan bazı örnek etkinlikler Ek H'de verilmiştir.

Veri Analizi

Bu çalışmada veri analizi öğretmen adaylarının uygulamaya dayalı bir öğretim modülünün öncesinde ve sonrasındaki orantısal akıl yürütmelerini ortaya çıkarmak için yapılmıştır. Araştırma sorularına cevap verebilmek için Orantısal Akıl Yürütme Testi'nin ilk ve ikinci uygulamasından, ön ve son görüşmelerden, öğretim deneyimlerinin gözlemlerinden, ders planlarından ve düzeltme raporundan elde edilen veriler analiz edilmiştir. İçerik analizi "toplanan verilerin önce kavramsallaştırılması, daha sonra da ortaya çıkan kavramlara göre mantıklı bir biçimde düzenlenmesi ve buna göre veriyi açıklayan temaların saptanması" (Yıldırım & Şimşek, 2013, s. 227) için kullanılmıştır. Bu amaçla öncelikle orantısal akıl yürütmeyle ilgili literatür taranmış ve araştırma sorularında görüldüğü gibi üç ana tema literatürden elde edilmiştir. Bunlar; farklı orantısal problem tiplerine yaklaşımlar, orantısal durumları orantısal olmayan durumlardan ayırt edebilmek ve orantısal durumların altında yatan matematiksel ilişkileri anlayabilmektir. Bir sonraki aşamada ses ve görüntü kayıtları alınan bütün görüşmeler ve öğretim deneyimleri yazıya aktarılmış ve diğer veri kaynaklarıyla birlikte veri yığını düzenlenmiştir. Sonrasında görüşmelerin ve öğretim deneyimlerinin yazıya aktarılan kayıtları, saha notları, ders planları, düzeltme raporları, ön ve son testler tekrar tekrar okunup incelenerek büyük boyutlardaki veri yığınını daha anlamlı hale getirmek için üç ana temanın altında kategoriler ve alt kategoriler oluşturulmuştur.

BULGULAR

Bu çalışmanın amacı; ortaokul matematik öğretmen adaylarının uygulamaya dayalı bir öğretim modülüne katılımlarından önceki ve sonraki orantısal akıl yürütmelerini incelemektir. Bu amaç doğrultusunda, çalışmanın bulguları üç ana başlık altında verilmiştir. Birinci başlık; öğretmen adaylarının farklı problem tiplerine yani; bilinmeyen değeri bulma, sayısal karşılaştırma ve niteliksel akıl yürütme problemlerine yaklaşımlarıyla ilgili bulguları içermektedir. İkinci başlık; öğretmen adaylarının orantısal durumları orantısal olmayan durumlardan ayırt edebilmeleriyle ilgili bulguları ve üçüncü başlık da; öğretmen adaylarının orantısal durumların altında yatan matematiksel ilişkileri anlayabilmeleriyle ilgili bulguları içermektedir. Çalışmanın bulguları, Orantısal Akıl Yürütme Testi'nin birinci ve ikinci uygulamasından, birinci ve ikinci görüşmelerden ve birinci ve ikinci öğretim deneyimlerinin gözlemlerinden elde edilmiştir. Bunların yanında, ders planları ve düzeltme raporları da öğretmen adaylarının orantısal akıl yürütmeleriyle ilgili bulgu elde etmek için kullanılmıştır.

Öğretmen Adaylarının Farklı Problem Tiplerine Yaklaşımlarıyla İlgili Bulgular

Gaye, Mine ve Ela, Orantısal Akıl Yürütme Testi'ndeki, görüşmelerdeki ve öğretim deneyimlerindeki orantısal problemleri genellikle doğru çözmüşlerdir. Fakat uygulamaya dayalı öğretim modülünden önce, öğretmen adaylarının hepsinin

Orantısal Akıl Yürütme Testi'ndeki bir sayısal karşılaştırma problemini (24. soru) çarpımsal bir strateji yerine toplamsal bir strateji kullandıkları için yanlış çözdükleri görülmüştür. Benzer şekilde, ön testte Gaye ve Mine bir niteliksel akıl yürütme probleminde (8. soru) çarpımsal karşılaştırma yapmak yerine toplamsal bir karşılaştırma yapıp problemi yanlış çözmüşlerdir. Bu bulgular, öğretmen adaylarının öğretim modülünden önce, bazı orantısal problemlerin çözümünde, çarpımsal ilişkiler yerine toplamsal ilişkilerin kullanıldığı yanlış bir çözüm stratejisi olan toplamsal ilişki stratejisini kullandıklarını göstermiştir. Oysa öğretim modülünden sonra, Gaye ve Ela hiçbir problemin çözümünde yanlış toplamsal ilişki stratejisini kullanmamış, Mine ise son testte sadece bir niteliksel akıl yürütme probleminde (8. soru) yanlış toplamsal ilişki stratejisini kullanmıştır. Ön test ve son testte, niteliksel problemlerin çözümünde, Gaye, sayısal değerlere dayanmayan niteliksel karşılaştırmalar yapamazken, Mine yapabilmiştir. Ela ise ön testte, niteliksel problemlerin çözümünde, çarpımsal karşılaştırmalar yapmasına rağmen sayısal değerlere dayanmayan niteliksel karşılaştırmalar yapamamıştır. Buna karşın, Ela son testte sayısal değerlere dayanmayan niteliksel karşılaştırmalar yapmayı başarmıştır.

Uygulamaya dayalı öğretim modülünden önce, öğretmen adayları orantı problemlerini çözmek için sınırlı sayıda strateji kullanırken, öğretim modülünden sonra, orantı problemlerini çözmek için farklı çözüm stratejileri kullanabilmişlerdir. Buna ek olarak, öğretmen adayları öğretim modülünden sonra, genellikle çarpımsal ilişkilerin kullanıldığı informal stratejileri tercih etmişlerdir. Oysa öğretmen adayları modülden önce, anlamdan yoksun ezbere kuralların kullanıldığı formal stratejileri (içler dışlar çarpımı gibi) tercih etmekteydiler. Dahası öğretmen adayları, öğretim modülünden sonra işlem kolaylığı sağlayan etkili stratejileri fark edip kullanmaya başlamışlardır.

Öğretim modülünden önce, Gaye ve Mine yaptıkları işlemlerden elde ettikleri çoklukların anlamlarını açıklayamazken, Ela sınırlı bir biçimde açıklayabiliyordu. Bunun yanında, modülden önce, öğretmen adaylarının problem çözümlerine yönelik açıklamaları ve kullandıkları stratejiye yönelik gerekçeleri çoğunlukla yetersizdi öyle ki; açıklamaları uyguladıkları işlem basamaklarını söylemekten öteye gitmiyordu (önce çarptım sonra böldüm gibi). Fakat öğretim modülünden sonra, öğretmen adayları, yaptıkları işlemlerden elde ettikleri çoklukların anlamlarını ve çözümlerini anlaşılır bir şekilde açıklayabiliyorlar ve seçtikleri stratejinin gerekçesini söyleyebiliyorlardı.

Öğretim modülünden önce, bilinmeyen değeri bulma problemlerinde, öğretmen adaylarının hepsi çoklukların aralarındaki oranları fark ediyor ve kullanıyorlardı. Ama öğretmen adayları çoklukların kendi içindeki oranları fark etmek de zorluk çekiyorlardı. Örneğin, öğretmen adaylarının kendi içindeki oranları bu oranların tam sayı olduğu durumlarda bile kullanmadığı görülmüştür. Diğer taraftan öğretim modülünden sonra, Mine çoklukların kendi içindeki ve aralarındaki oranları fark edebiliyor ve kullanıyorken, Gaye ve Ela sadece çoklukların aralarındaki oranları kullanıyorlardı. Bununla birlikte Mine ve Ela öğretim modülünün öncesinde ve sonrasında parça-parça ve parça-bütün oranlarını fark edebiliyor ve kullanıyorken, Gaye parça-bütün oranlarını öğretim modülüne katıldıktan sonra dahi fark edemiyordu.

Öğretmen adaylarından hiçbiri öğretim modülünden önce ve sonra çoklukları birden farklı birimlere ayırmak konusunda başarılı değildi. Daha doğrusu öğretmen adayları öğretim modülünden sonra bile birimi işlem kolaylığı sağlayan birden başka bir sayı seçmeyi düşünememişlerdir. Diğer taraftan, öğretim modülüne katıldıktan sonra Gaye, Mine ve Ela'nın orantı problemlerini çözerken oran tabloları oluşturdukları ve tablodaki kuralı bulmak için değişim çarpanı ve tekrarlı toplama stratejilerini kullandıkları görülmüştür. Dahası bazı durumlarda öğretmen adaylarının oranı azaltılabilen ve arttırılabilen bir birim olarak görebildikleri gözlemlenmiştir. Bunlara ek olarak, öğretmen adaylarının öğretim modülünden sonra aynı ölçüm uzayındaki çoklukların aralarında sabit bir oran olduğunu bildikleri ve orantı sabitini problemlerde kullanabildikleri ortaya çıkmıştır.

Öğretmen Adaylarının Orantısal Durumları Orantısal Olmayan Durumlardan Ayırt Edebilmeleriyle İlgili Bulgular

Uygulamaya dayalı öğretim modülünden sonra, Gaye, Mine ve Ela Orantısal Akıl Yürütme Testi'ndeki, verilen değişkenler arasındaki ilişkinin orantısal olup olmadığının sınıflandırılmasının istendiği sorulardaki (11-22. sorular) bütün ilişkileri doğru bir şekilde sınıflandırmışlardır. Oysa öğretim modülünden önce, Gaye ve Mine bazı değişkenler arasındaki ilişkilerin orantılı olup olmadığına doğru karar verememişlerdir. Bunun yanında öğretim modülünden önce öğretmen adayları, verilen ilişkilerin neden orantılı veya neden orantısız olduğunu açıklamada zorlanırken, modülden sonra yaptıkları sınıflandırmaların gerekçelerini açık bir şekilde anlatabilmişlerdir.

Öğretim modülünden önce, Gaye ve Mine'nin bütün doğrusal ilişkilerin orantılı olduğuna dair bir kavram yanılgısı vardı ve bütün orantılı ilişkilerin doğrusal olduğunu bilmiyorlardı. Buna karşın Ela, öğretim modülünden önce iki değişkenin arasında doğrusal bir ilişki olmasının tek başına bu iki değişkenin orantılı olmasına karar vermek için yeterli olmadığını ama bu değişkenlerin orantılı olabilmesi için doğrusal olmaları gerektiğini biliyordu. Orantısal akıl yürütmeyi geliştirmeyi amaçlayan öğretim modülünden sonra öğretmen adaylarının bütün doğrusal ilişkilerin orantılı olduğuna dair kavram yanılgılarının kaybolduğu görülmüştür. Öğretim modülünün öncesinde Mine ve Ela orantısal ilişkilerin aynı oranda artıp aynı oranda azalmasını anlamakta bazı zorluklar çekerken, Gaye orantısal ilişkilerin aynı oranda artıp aynı oranda azalacağını hiç bilmemekteydi. Örneğin, ilk ders anlatımında Gaye aralarında doğru orantılı ilişkili olan bir gerçek hayat durumuna şu problemi örnek olarak vermiştir; "Bir fidan 50cm iken her yıl 20 cm uzamaktadır. Bu fidanın ilk üç yıl için büyüme zaman grafiğini çiziniz." Gaye bu problemi sınıfta aşağıdaki gibi çözmüştür:

Gaye: Burası sıfıra sıfır noktasıydı dimi. Sıfırdan başlayabiliriz ama bizim elimizdeki fidan 0 boyda değil bunun bir uzunluğu var. Ne kadarmış 50 cm imiş o zaman boy kısmından 50 cm i gösterelim (50 cm'i y ekseninde gösterdi). Buradan başlayacağız yani. Hiç zamansız, sıfır zamanında, daha hiç zaman geçmemişken, bu fidanın boyu 50 cm'dir. Çünkü elimizdeki fidan 50 cm idi. (koordinat düzlemine sayıları yerleştirdi). Bir yılsonunda fidanın boyu ne kadar olur? 70 olur dimi? 2 yıl sonra; 90, 3 yıl sonra; 110 bu şekilde devam eder (noktaları birleştiriyor).
Öğrenci: Öğretmenim bu doğru orantı mı oluyor?
Gaye: Doğru orantı ıı oluyor (tahtaya bakıyor). Neden sizce?
Öğrenci: Demiştik ya doğru orantıda biri artarken diğeri de artıyor.
Gaye: Ne artarken ne artmış burada?
Öğrenci: Zaman artıkça boyda artıyordu.
Gaye: Evet, zaman artıkça boy da artmış o zaman doğru orantı vardır.

Görüldüğü gibi Gaye, öğrencilerine doğrusal ama orantılı olmayan bir ilişkiyi orantılı bir ilişki gibi anlatmıştır. Ancak Gaye öğretim modülünden sonra öğrencilere anlattığı ikinci derste aynı problemi öğrencilere tekrar sormuştur ve bu sefer bu problemdeki değişkenler arasındaki ilişkinin orantısal olmadığını doğru bir şekilde açıklayabilmiştir. Benzer şekilde öğretim modülünden sonra elde edilen veriler, öğretmen adaylarının hepsinin bütün orantılı ilişkilerin doğrusal olduğunu ve aynı oranda artıp aynı oranda azaldığını bildiğini ortaya koymuştur.

Öğretmen adayları, ön testte oranı, herhangi bir özelliği (portakal suyu karışımının portakal tadı, boya karışımının tonu gibi) ölçmek için kullanılabilecek bir ölçüm olarak görmekte zorlanmaktadırlar. Buna karşın son testte Gaye ve Ela oranın, bir meyve suyu karışımın yoğunluğunu bulmakta, bir boya karışımının tonunu bulmakta, bir kayak rampasının eğimini bulmakta ve bir dikdörtgenin ne kadar kareye benzediğini bulmakta uygun bir ölçüm olduğuna karar vermişlerdir. Fakat Mine'nin öğretimin modülünün sonrasında bile oranın bir boya karışımının tonunu bulmakta uygun bir ölçüm olduğunu bilmediği ortaya çıkmıştır. Ön test sonuçları öğretmen adaylarının açıklamalarında orantısal akıl yürütme dilini kullanmadıklarını göstermiştir. Özellikle, öğretmen adayları bazı durumlarda oranı belli bir özelliği ölçmek için kullanmalarına rağmen "oran" kelimesini açıklamalarında kullanmamışlardır. Ancak son testte öğretmen adaylarının orantısal akıl yürütme dilini ve özellikle “oran” kelimesini açıklamalarında kullandıkları görülmüştür. Örneğin Ela Orantısal Akıl Yürütme Testi’nin ikinci uygulamasında bir problemi çözerken (9. soru) aşağıdaki açıklamayı yapmıştır:

> Murat’ın yöntemi her zaman işe yaramaz. Bu yöntemin işe yaraması için ilk karışımdaki beyaz boyanın mavi boyaya oranının ikinci karışımdaki beyaz boyanın mavi boyaya oranına eşit olması gerekir. Yani ilk karışımdaki boyaların birbirine oranını bilmemiz gerekir. Eğer boyaların oranı 1/1 ise, Murat’ın yöntemi işe yarar. Ama baştaki oranları bilmiyoruz; o yüzden bu yöntem her zaman işe yaramaz.

Gaye ve Mine, uygulamaya dayalı öğretim modülünden önce çoklukların aralarında orantısal ilişkili olduğu durumlara her zaman geçerli örnekler veremezken Ela verebiliyordu. Örneğin ön testte (5. soru), Gaye ve Mine çoklukların aralarında orantısal ilişkili olduğu bir sözel problem yazamazken Ela yazabildi. Ela’nın yazdığı problem: “Sekiz eş parçaya bölünen bir tarlanın 3 parçasına domates ekilmiştir ve hasat edilmiştir. Çiftçi çeşitli ürün elde etmek için aynı tarlayı 20 eş parçaya bölecektir, tarlanın kaç parçasına domates ekerse ilk elde ettiği domates oranını elde eder?” Görüldüğü gibi Ela geçerli bir orantı problemi yazmıştır. Öğretim modülünden sonra ise Mine ve Ela çoklukların aralarında orantısal ilişkili olduğu durumlara geçerli örnekler verebilirken, Gaye’nin hala bu konuda zorluk çektiği görülmüştür.

Öğretim modülünün öncesindeki ve sonraki bulgular öğretmen adaylarının çolukların aralarında orantısal ilişkilerin olmadığı durumlara örnekler verirken zorlanmadıklarını göstermiştir. Bunun yanında öğretmen adayları verdikleri örnek durumlardaki çoklukların neden orantısal olmadıklarını da açıklayabilmişlerdir. Fakat Gaye’nin ön görüşmedeki orantısal olmayan ilişkiye verdiği örnek bir istisnadır; çünkü Gaye bu örnekteki çoklukların aralarındaki ilişkinin neden orantısal olmadığını açıklayamamıştır. Ayrıca öğretim modülünden sonraki bulgular öğretmen adaylarının, çoklukların aralarında toplamsal ve sabit ilişkili olduğu durumların orantısal olmayan durumlar olduğunu bildiğini ortaya çıkarmıştır.

Öğretmen Adaylarının Orantısal Durumların Altında Yatan Matematiksel İlişkileri Anlayabilmeleriyle İlgili Bulgular

Uygulamaya dayalı öğretim modülünden önce Gaye, Mine ve Ela oran ve orantı kavramlarını tanımlamakta zorluk çekerlerken öğretim modülünden sonra bu kavramların doğru tanımlarını yapabildiler. Öğretim modülünden önce öğretmen adayları "oran" kavramının "bölme" kavramıyla aynı şey olduğunu düşünüyordu. Buna ek olarak Mine "oran" ın "bölme" ile aynı kavramlar olduklarını iddia ediyordu. Fakat öğretim modülünden sonra öğretmen adayları oran ve orantı kavramlarını doğru bir şekilde ve bu kavramların dayandığı matematiksel ilişkileri kullanarak tanımlamışlardır. Bu matematiksel ilişkiler; orantılı ilişkilerin çarpımsal bir doğası olduğu (*matematiksel ilişki 1*) ve orantılı ilişkilerde oran çiftlerinin birbirine eşit olduğuydu (*matematiksel ilişki 3*). Öğretim modülünden önce Gaye ve Mine oran ve orantı kavramlarının farkını açıklamakta zorlanıyordu; fakat öğretim modülünden sonraki bulgular öğretmen adaylarının hepsinin bu iki kavramın farklı kavram olduklarını bildiklerini ve farklarını açıklayabildiklerini göstermiştir. Öğretim modülünden önceki bulguların aksine öğretmen adayları orantısal akıl yürütme dilini tanımlarında kullanabilmişlerdir. Bununla birlikte öğretmen adayları tanımlarını yaparken orantısal durumların altında yatan matematiksel ilişkileri de kullanmışlardır.

Uygulamaya dayalı öğretim modülünden önceki bulgular öğretmen adaylarının, orantısal iki çokluğun arasında sabit çarpımsal bir ilişkinin olduğunu ve bu ilişkinin iki farklı yolla (k ve 1/k gibi) gösterilebileceğini bilmediklerini ortaya koymuştur. Örneğin, Gaye ve Mine ön görüşmede orantı sabitini tanımlamada zorluk çekmişlerdir. Buna karşın, öğretim modülünden sonra öğretmen adaylarının orantı sabitinin tanımını öğrendikleri ve iki farklı şekilde gösterebildikleri görülmüştür.

Fakat Mine öğretim modülünden sonrasında da orantı sabitini belirlemekte zorluk çekmiştir. Örneğin ikinci ders anlatımında orantı sabitinin bir orantıdaki çoklukların hem aralarındaki orana hem de içlerindeki orana eşit olduğunu söylemiştir.

Orantısal akıl yürütmeyi geliştirmeyi amaçlayan öğretim modülünden önce öğretmen adayları, bazı durumlarda çoklukların arasındaki orantılı ilişkilerin varlığına karar verebilmek için orantılı durumların altında yatan matematiksel ilişkileri kullanmak yerine toplamsal ilişkileri kullanmışlardır. Mesela; Gaye ve Ela bazı durumların orantılı olup olmadığına karar verebilmek için çokluklar arasında toplamsal örüntüler bulmaya çalışmıştır. Bunun yanında öğretmen adayları genellikle çoklukların orantılı olabilmesi için aynı zamanda artıp azalmalarının gerektiğine vurgu yapmışlar ama aynı oranda artıp azalmaları gerektiğine değinmemişlerdir. Fakat öğretim modülünden sonra öğretmen adayları, çoklukların aralarındaki orantılı ilişkilere karar verebilmek için aynı zamanda artıp azalma gibi yanlış ifadeler yerine orantılı durumların altında yatan matematiksel ilişkileri kullanmışlardır.

Öğretmen adayları, öğretim modülünden önce verilen orantısal durumların tablo, grafik ve cebirsel gösterimleri arasında ilişkiler kurmakta zorlanmaktaydılar. Buna karşın öğretim modülünden sonra bu ilişkileri çoğunlukla kurdukları görülmüştür. Fakat ikinci ders anlatımında Gaye, tabloyla verilen orantısal durumun grafiğini çizebilmesine rağmen cebirsel denklemini bulmakta zorlanmıştır.

Uygulamaya dayalı öğretim modülünün sonunda elde edilen bulgular öğretmen adaylarının orantısal durumların altında yatan matematiksel ilişkileri çoğunlukla anlayabildiklerini ve kullanabildiklerini göstermiştir. Oysa öğretim modülünün öncesinde öğretmen adayları bu konuda zorluk çekmekteydiler. Örneğin, öğretim modülünden önce Gaye ve Ela orantılı ilişkilerin çarpımsal bir doğası olduğunu (*matematiksel ilişki 1*) tam olarak bilmiyorlardı. Bunun yanında öğretim modülünden önceki bulgular öğretmen adaylarının, orantısal bir durumun grafiğinin orijinden geçen bir doğru grafiği olduğunu (*matematiksel ilişki 2*) ve orantısal bir durumun cebirsel olarak denkleminin *k* eğim, birim oran ve orantı sabiti olmak üzere *y=kx* denklemiyle gösterildiğini (*matematiksel ilişki 4*) tam olarak bilmediklerini ortaya çıkarmıştır. Buna karşın, uygulamaya dayalı öğretim modülünden sonraki bulgular öğretmen adaylarının orantısal durumların altında yatan matematiksel ilişkileri tam olarak anlayabildiklerini ortaya koymuştur. Fakat *matematiksel ilişki 4*'ün bir istisna olduğu görülmüştür; çünkü öğretmen adayları orantısal bir durumun cebirsel olarak denkleminin *y=kx* denklemiyle gösterildiğini ve *k'* nın eğim ve orantı sabiti olduğunu anlamışlar fakat *k*'nin aynı zamanda birim oran olduğundan hiç bahsetmemişlerdir.

SONUÇ

Bu araştırma kapsamında geliştirilen orantısal akıl yürütmeye yönelik uygulamaya dayalı öğretim modülünün öğretmen adaylarının orantısal akıl yürütmelerine önemli derecede katkıda bulunduğu görülmüştür. Öğretim modülünün öncesinde, öğretmen adayları, orantı problemlerini çözmek için genellikle içler-dışlar çarpımı gibi anlamdan yoksun cebirsel kuralları uygulamışlar ve sınırlı sayıda strateji kullanmışlardır. Bunun yanında, orantısal durumları orantısal olmayan durumlardan ayırt etmekte ve orantısal durumların içerdiği matematiksel ilişkileri anlamada zorluk çektikleri görülmüştür. Buna karşın, öğretim modülünün sonrasında, öğretmen adayları orantı problemlerini çözerken çoğunlukla, çarpımsal ilişkilerin kullanıldığı informal stratejileri (değişim çarpanı gibi) kullanmayı tercih ederken, içler-dışlar çarpımı ve diğer formal stratejileri kullanmayı pek tercih etmemişlerdir. Ayrıca, orantı problemlerini çözmek için farklı stratejiler kullanmışlar ve bu stratejileri anlamlandırabilmişlerdir. Buna ek olarak, verilen çoklukların aralarında toplamsal, çarpımsal veya başka ilişkilerin olup olmadığını belirleyebilmişlerdir. Bunun yanında, öğretmen adaylarının orantısal durumlardaki matematiksel ilişkileri anlayabildikleri görülmüştür.

APPENDIX J

Curriculum Vitae

PERSONAL INFORMATION

Surname, Name: Pişkin Tunç, Mutlu
Nationality: Turkish (TC)
Date and Place of Birth: 30 March 1984, Uşak
Marital Status: Married
Phone: +90 372 3233870
email: mutlupiskin@gmail.com

EDUCATION

Degree	**Institution**	**Year of Graduation**
MS	METU Elementary Science and Mathematics Education	2010
BS	Gazi University Mathematics Education	2007
High School	Mamak Anatolian High School, Ankara	2002

WORK EXPERIENCE

Year	**Place**	**Enrollment**
2011- Present	Bülent Ecevit University	Instructor
2008-2011	Abant Izzet Baysal University	Research Assistant
2008	Düzce, Aydınpınar İlköğretim Okulu	Teacher

FOREIGN LANGUAGES

Advanced English

PUBLICATIONS

1. Uçar, Z. T., Akdoğan, E., Pişkin, M., & Taşçı D. (2009), "İlköğretim Öğrencilerinin Matematiksel İnançları", *I. Uluslararası Türkiye Eğitim Araştırmaları Kongresi*, Çanakkale,1-3 Mayıs, 2009.

2. Pişkin M., Akkaş E., Uçar Z. T., Taşçı, D. (2009). "Çocukların Gözünde Matematik." *8. Matematik Sempozyumu Sergi ve Şenlikleri*, Ankara.

3. Pişkin, M., Çakıroğlu, E., & Bulut, S. (2010), "Preservice Mathematics Teachers' Views about Using Concrete Models in Mathematics Classrooms", *The European Conference on Educational Research 2010,* Helsinki, Finlandiya, 25-27 Ağustos, 2010.

4. Toluk Uçar, Z., Pişkin, M., Akdoğan, E. N. ve Taşçı, D. (2010). "İlköğretim Öğrencilerinin Matematik, Matematik Öğretmenleri ve Matematikçiler Hakkındaki İnançları", *Eğitim ve Bilim,* 35(155), 131-144.

5. Pişkin, M., & Durmuş, S. (2010). "Sınıf Öğretmeni Adaylarının Matematiğe Karşı Öz-Yeterlik Algıları", *E-Journal of New World Sciences Academy Educational Sciences,*5(3) 1189–1196.

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9. Pişkin-Tunç M., Durmuş S., & Akkaya R. (2012). "İlköğretim Matematik Öğretmen Adaylarının Matematik Öğretiminde Somut Materyalleri ve Sanal Öğrenme Nesnelerini Kullanma Yeterlikleri." *MATDER Matematik Eğitimi Dergisi, 1*(1), 13-20.

10. Pişkin-Tunç, M., & Durmuş, S. (2012). "Pre-service elementary school classroom and mathematics teachers' interpretations about the definition of angle concept." *Energy Education Science and Technology, Part B-Social and Educational Studies,* 4(1), 131-140.

11. Pişkin-Tunç, M. (2012). "Prospective mathematics teachers' efficacies regarding teaching mathematics." *The European Conference on Educational Research 2012,* Cadiz, İspanya.

12. Pişkin-Tunç, M. & Haser, Ç. (2012). "Sınıf Öğretmeni Adaylarının Matematik Öğretimine İlişkin İnanışlarının İncelenmesi." *X. Ulusal Fen Bilimleri ve Matematik Eğitimi Kongresi,* Niğde.

13. Akkaya, R., Durmuş, S., Pişkin-Tunç, M. (2012). "İlköğretim Matematik Öğretmen Adaylarının Somut Materyal ve Sanal Manipülatifleri Eğitim Süreçleri Boyunca Kullanabilme Durumlarının Belirlenmesi." *X. Ulusal Fen Bilimleri ve Matematik Eğitimi Kongresi,* Niğde.

14. Karataş, İ., Pişkin-Tunç, M., Akıncı, M., Demiray E., Yılmaz N. (2012). "İlköğretim matematik öğretmen adaylarının matematik öğretiminde teknolojik pedagojik alan bilgilerinin gelişimi." *I. Türk Bilgisayar ve Matematik Eğitimi Sempozyumu,* Trabzon.

15. Durmuş S., Akkaya R., Pişkin-Tunç, M. (2013). "Awareness and proficiency levels of pre-service primary school mathematics teachers related to the use of concrete materials and virtual manipulatives in mathematics education." *The International Journal of Arts & Science Conference.*

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17. Pişkin-Tunç, M., & Karacı, G. (2015). "Sınıf Öğretmeni Adaylarının Matematik Kavramına İlişkin Algılarının Zihinsel İmgeler Yardımıyla İncelenmesi." *II. Türk Bilgisayar ve Matematik Eğitimi Sempozyumu*, Adıyaman.

18. Pişkin-Tunç, M., Karataş, İ., Yılmaz, N., & Karacı, G. (2015). "İlköğretim Matematik Öğretmen adaylarının Teknolojik Pedagojik Alan Bilgisi." *II. International Dynamic, Explorative and Active Learning (IDEAL) Conference,* Amasya.

APPENDIX K

Tez Fotokopisi İzin Formu

ENSTİTÜ

Fen Bilimleri Enstitüsü ☐

Sosyal Bilimler Enstitüsü ☒ X

Uygulamalı Matematik Enstitüsü ☐

Enformatik Enstitüsü ☐

Deniz Bilimleri Enstitüsü ☐

YAZARIN

Soyadı : Pişkin Tunç
Adı : Mutlu
Bölümü : İlköğretim

TEZİN ADI (İngilizce) : Pre-Service Middle School Mathematics Teachers' Proportional Reasoning Before and After a Practice-Based Instructional Module

TEZİN TÜRÜ : Yüksek Lisans ☐ Doktora ☒ X

1. Tezimin tamamından kaynak gösterilmek şartıyla fotokopi alınabilir. ☐
2. Tezimin içindekiler sayfası, özet, indeks sayfalarından ve/veya bir bölümünden kaynak gösterilmek şartıyla fotokopi alınabilir. ☐
3. Tezimden bir bir (1) yıl süreyle fotokopi alınamaz. ☒ X

TEZİN KÜTÜPHANEYE TESLİM TARİHİ: