ROBUST CONDITIONAL VALUE–AT–RISK UNDER PARALLELPIPE UNCERTAINTY: AN APPLICATION TO PORTFOLIO OPTIMIZATION

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ABSTRACT

ROBUST CONDITIONAL VALUE–AT–RISK UNDER PARALLELPIPE UNCERTAINTY: AN APPLICATION TO PORTFOLIO OPTIMIZATION

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In markets with high uncertainties, the trade–off between maximizing expected return and minimizing the risk is one of the main challenges in modeling and decision making. Since investors mostly shape their invested amounts towards certain assets and their risk version level according to their returns; scientists and practitioners has done studies on this subject since the beginning of the stock markets' establishment. Developments and inventions in the mathematical optimization provide a wide range of solutions to handle this problem. Mean–Variance Approach by Markowitz is one the oldest and best known approaches to the risk–return trade–off in the markets. However, it is a one time–step model and not very much prepared for highly volatile markets. After Markowitz, different optimization approaches have been invented for portfolio optimization, especially, in the tradition of Conditional Value–at–Risk. In this study, we modeled a Robust Optimization problem based on the data and used Robust Optimization approach to find a robust optimal solution to our portfolio optimization problem. This approach includes the use of Robust Conditional Value–at–Risk (RCVaR) under Parallelpipe Uncertainty sets, an evaluation and a numerical finding of the robust optimal portfolio allocation. We obtained and then traced back our robust linear programming model to the Standard Form of a Linear Programming model; then we solved it by a well–chosen algorithm and software package. The main idea is modeling a robust portfolio optimization problem that includes our development of RCVaR based on uncertainty–set–valued data. Our aim is, by considering the return–risk trade–off analysis under uncertain data, to obtain more robust, in fact, lower, risk–level under worst–case scenario by using RCVaR. Uncertainty in parameters, based on uncertainty in the prices, and a risk–return analysis are crucial parts of this study. Hence, the trade–off (antagonism) between accuracy and risk (variance), and robustness are our main issue. A numerical experiment is presented containing real–world data from stock markets. The thesis ends with a conclusion and an outlook to future studies.

Keywords : Robust Portfolio Optimization, Robust Optimization, Robust Conditional Value–at–Risk, Parallelpipe Uncertainty, Risk Management

ÖZ

PARALEL ŞERİT BELİRSİZLİĞİ ALTINDA SAĞLAM KOŞULLU RİSKE MARUZ DEĞER: PORTFÖY OPTİMİZASYONU UYGULAMASI

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Yüksek belirsizliğe sahip piyasalarda, beklenen getirinin maksimizasyonu ve riskin minimizasyonu arasındaki pazarlık, karar vermede ve modellemedeki başlıca uğraşılardan biridir. Yatırımcılar, çoğunlukla yatırımlarını belirli varlıklar üzerindeki yatırım miktalarını ve risk seviyelerini, varlıkların getirilerine göre düzenlediklerinden dolayı; bilim adamları ve uygulayıcılar, bu konu üzerinde borsaların kuruluşundan itibaren çalışmaktadırlar. Matematiksel Optimizasyon alanında olan gelişmeler, bu sorunlar ile ilgilenebilmek için çok sayıda çözüm önerisi sunmaktadırlar. Markowitz tarafından geliştirilen Ortalama–Varyans Yaklaşımı, risk–getiri pazarlığı ile ilgili en bilinen ve eski çalışmalardan biridir. Ancak bu yaklaşımın, tek dönemlik model olması ve yüksek volatileteye uygun olmaması durumları söz konusudur. Markowitz'den sonra, portföy optimizasyonu için farklı optimizasyon yaklaşımları geliştirilmiştir, özellikle, Koşullu Riske Maruz Değer geleneğine göre gelişmeler yaşanmıştır. Bu çalışmada, portföy optimizasyon modelimiz için, sağlam optimal çözümü bulmak adına, veri temelli Sağlam Optimizasyon Modeli geliştirilmiş ve Sağlam Optimizasyon yaklaşımı kullanılmıştır. Bu yaklaşım Paralel Şerit Belirsizlik kümesi altında, Sağlam Riske Maruz Değer kullanımını içermekte ve sağlam optimal portföy dağılımının bulunmasını içermektedir. Sağlam doğrusal programlama modelini elde ettikten sonra, standart formadaki doğrusal programlama haline çevrilmiş; iyi bilinen bir algoritma ve yazılım paketi ile çözülmüştür. Çalışmamızdaki ana düşüncemiz, belirsizlik kümesine dayalı veriye göre geliştirilmiş Sağlam Koşullu Riske Maruz Değer'i içeren sağlam portföy optimizasyonu problemini modellemektir. Amacımız, belirsizlik içeren veriyi

Sağlam Koşullu Riske Maruz Değer'de, en kötü senaryo durumunda kullanarak, risk–getiri değişimini göz önüne alıp, daha sağlam, ve aslında daha az risk seviyesi elde etmektir. Fiyatlardaki belirsizlik ve risk–getiri değişimi analizi temelinde, parametrelerdeki belirsizlik, bu çalışmanın önemli parçaları olmaktadır. Bu nedenle, doğruluk ve risk (varyans) arasındaki değişim (antagonizm) ve sağlamlık öncelikli meselelerimizdendir. Gerçek zamanlı piyasa verisinin kullanıldığı sayısal bir deney sunulmuştur. Tez, sonuç ve gelecekte yapılabilecek çalışmaların söz edilmesi ile son bulmaktadır.

Anahtar Kelimeler: Sağlam Portföy Optimizasyonu, Sağlam Optimizasyon, Sağlam Koşullu Riske Maruz Değer, Paralel Şerit Belirsizliği, Risk Yönetimi

For everyone who devoted their lives to intelligence, logic, and science

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LIST OF ABBREVIATIONS

CVaR	Conditional Value–at–Risk
CoV	Coefficient of Variation
E	Expected Value
JB test	Jarque–Bera Test
LP	Linear Programming
MPT	Modern Portfolio Theory
MVA	Mean–Variance Approach
MC	Monte–Carlo
$\mathbb{R}$	Set of Real Numbers
RCVaR	Robust Conditional Value–at–Risk
RO	Robust Optimization
Std	Standard Deviation
VaR	Value–at–Risk
Var	Variance
WCVaR	Worst–Case Conditional Value–at–Risk
w.r.t.	with respect to

CHAPTER 1

INTRODUCTION

Usage of the optimization models in finance has increased after Markowitz's great contribution, *Mean–Variance Approach*, to the portfolio selection theory [30]. Modern Portfolio Theory showed us the importance of the relation between risk taking and revenue (return) from the portfolio. While the risk–averse investors are focused on their risk levels, less risk–averse investors have improved their revenues by taking more risk. However, both of these investor types are affected by a common phenomenon: The main uncertainty comes from the random fluctuation of the prices of risky assets. Since returns from assets are calculated by asset prices, this type of uncertainty shows itself in returns also. Hence, investors also face uncertainty in their returns.

Uncertainty in the returns is one of the major effects for all kind of investments. Investors mostly shape their invested amounts towards certain assets and their risk version level according to the returns. Since the term risk aversion is related with psychology, any kind of random fluctuation on asset returns could affect the investors' behaviors, and vice versa.

Investors prefer to hedge their investment risks by using financial instruments or measuring their risk by risk management tools; such tools have been developed in recent decades. All of these approaches are summarized under one big term which is named as Risk Management. There are numerous risk management techniques in the literature. For instance, using financial derivatives like options, futures, swaps, etc., are risk management or risk hedging techniques. In a financial system, there is no possibility of terminating the risk of an investment, we can only avoid from risk as much as we can. Moreover, to define the risk level of the investment into specific asset(s), various risk measurement methodologies have been developed. However, from the very beginning, addressing the term "risk", the important fundamental is the phenomenon of "uncertainty".

Uncertainty about asset returns and the wider uncertainty about markets, especially, of the underlying prices, are sonorous principals behind the risk management literature. The investor fears from the uncertainty because it affects the market decisions, structure, and future. Since the total prediction of the future is almost impossible due to various randomness in the markets, by the help of uncertainty quantification and related risk management methodologies, the investors try to avoid the financial risk to their very best.

The classical Modern Portfolio Theory (MPT) or MVA "measures" risk by standard deviation and variance. In MPT, a trade–off is considered between risk and expected return; the objective function is to be minimized variance (risk) and also wants a certain level of expected return, i.e., to satisfy a target return constraint. Since MVA has drawbacks like irrationality of investors, and problem with calculation of expected values; researchers commenced to investigate new approaches to portfolio optimization. Because of the weak specifications of the MPT, Black and Litterman developed a new approach to portfolio optimization. In their approach, the investors can combine their views about the global look of the equities, bonds and currencies with a risk premium. Their results are intuitive and allow for diversified portfolios [14]. Recent researches, like Value–at–Risk (VaR) or Conditional Value–at–Risk (CVaR), prefer to consider probability distributions. Since all returns from the market are creating a probability distribution, VaR or CVaR are used to obtain a risk threshold through a pre–specified confidence level.

In MVA, the covariance matrix of the returns is used to define a risk measure as a quadratic function, called the Variance, for the portfolio vector, and its minimization is conducted. Nevertheless, portfolio optimization with CVaR, instead of the aforementioned Variance, is employed as a risk measure and its minimization is the issue. Also, these kinds of quantitative risk management techniques have two types of decision–making contexts: the return–risk trade–off and the utility maximization. These kinds of portfolio optimization approaches have three forms: (i) Minimizing of the portfolio risk, (ii) Maximizing of the expected return, or (iii) Minimize (or maximize) a combined goal with a penalty parameter. In some parametric sense, the three forms are equivalent to each other.

However, those risk measurement techniques still have uncertainty before they become applied. To deal with this uncertainty, according to the literature, various studies which consider CVaR, Robust Optimization (RO), and Robust CVaR are conducted. These researches are shown in different theoretical and applied approaches related to Robust Optimization. For instance in [28], CVaR is studied under different constraints, while other works like [15, 23, 24, 31, 39] emphasized risk management under the Worst–Case scenario CVaR and contributed to the RO and the risk management literature. During recent years, RO under different uncertainty sets is presented by [13, 26, 27, 48, 51]. As another contribution, Parallelpipe Uncertainty is considered in [33, 34, 35, 36]. In the next section, we will present all these researches in the literature with closer details.

In our research, we applied a Robust Optimization approach under *Parallelpipe Uncertainty* (given by particular polytopes, in fact, straight parallelpipes, in our thesis) into the portfolio optimization with CVaR. The main idea is modeling a robust portfolio optimization problem that includes our development of RCVaR based on uncertainty-set–valued data. Then we have a Robust Counterpart of the portfolio optimization problem with CVaR. Consequently, we evaluate this optimization problem further and arrive at a Standard Form of an LP problem.

Our aim is, by considering the return–risk trade–off analysis under uncertain data, to obtain more robust, in fact, lower, risk–level under worst–case scenario by using

Robust CVaR. Uncertainty in parameters, based on uncertainty in the prices, and a risk–return analysis are crucial parts of this study. Hence, the trade–off (antagonism) between accuracy and risk (variance), and robustness are our main issue.

1.1 State of the Research

After other new risk approaches which are classical now, CVaR was introduced into literature; and portfolio optimization with CVaR spread with various applications. Since CVaR is proposed as an extended version of VaR, CVaR is closely related to VaR. In this study, CVaR is used for another percentile risk measure. Krokhmal et al. (2002) used rate of return, different constraints and objective functions for calculating VaR and optimizing CVaR simultaneously. According to aims of studying return–optimization problems with convex constraints, one can use different optimization formulations. Using multiple CVaR constraints for different time zones and at different confidence intervals allows for shaping distributions according to the decision maker's preferences. The authors developed a new model for the optimization of the portfolio returns with CVaR constraints, and they used historical prices and conducted a case study on optimizing the portfolio of S&P100 stocks. As a result, that optimization approach is productive and efficient [28].

In the study of Quarante and Zaffaroni (2008), the weak points of the classical portfolio selection problems are discussed and illustrated. Since the optimization process leads to solutions which are likely to depend heavily on the parameters' perturbations, it is important to focus on "the often data" (data with high frequency). The term "parameter" might be considered as a random variable or a return in a portfolio optimization problem. When the data frequency is high, dependence on the parameter perturbations makes the theoretical and numerical results highly unreliable for practical studies. To deal with this situation, they used a RO methodology. In their study, to define the uncertainty set, Quaranta and Zaffaroni proposed "Soyster's approach". In their computational part, the methodology of RO is implemented to minimize the CVaR of a portfolio of shares and for finding a strategy of portfolio selection. In the conclusion, a robust counterpart of the model which is proposed in that study, decides on qualitatively better portfolios that are also more profitable when comparing it with the other approaches [39].

Cho (2008) proposed a worst–case robust multi–period portfolio optimization model using CVaR. Using scenarios trees, Cho suggested a min–max algorithm and optimization framework. The provided min–max algorithm is used for the determination of a worst case for a portfolio. Cho implemented the Worst–Case Robust Mean–Conditional Value–at–Risk portfolio optimization model, and analyses the properties of the robust portfolio. The accuracy of the model is measured by historical data. The performance of the Mean–Conditional Value–at–Risk model against the classical Mean–Variance Model (Markowitz's Modern Portfolio Theory) is evaluated. Another contribution of the paper is the performance of the Worst–Case robust portfolio optimization model against the non-robust model. Moreover, the complex solvers' performance is compared against the one with interior point solvers. In conclusion, back–testing results are that the Robust Mean–Conditional Value–at–Risk optimization model is minimizing the CVaR while giving better returns from the assets [15].

Huang et al. (2010) introduced the relative RCVaR where the underlying probability distribution of portfolio return is only known to belong to a certain set. In that study, the major criterion is given by worst–case scenarios that may rarely be realized in practice. Also, Huang et al. tracked most probable scenarios to find best optimistic portfolio weights. Benefits of this study are that the optimal portfolio results are less conservative than classical absolute robust approaches, and the fact that the portfolio selection problem could be formulated by linear programming. As an application, RCVaR algorithm is applied to a case which includes multiple experts [24].

Hasuike and Katagiri (2013) studied a robust portfolio selection problem with an uncertainty set of future returns and satisfaction of certain levels with total returns. An ellipsoidal set of future returns is proposed as an uncertainty set; then a robustness-based (worst–case objective function) selection problem is formulated as a bi–objective programming problem. The proposed interactive model in that study is a new robustness–based portfolio selection problem with bi–objective functions that maximizes the total profit and value of the robustness parameter (the diameter of an ellipsoidal set of future, returns representing the uncertainty set here), is the robustness parameter. Since the model is a bi–objective programming model, Hasuike and Katagiri introduced an interactive fuzzy satisfying method in this study and transformed the main problems into deterministic equivalent problems. During their study, fuzzy goals were inserted for both objective functions and an interactive fuzzy–satisfying method. After several steps, the given program is transformed into a deterministic equivalent problem. To apply the fuzzy–satisfying method, minimax problems should be solved obtaining M–Pareto optimal solutions. Hence, in that study, the exact solution algorithm is developed for explicit M–Pareto optimal solutions. The discoveries of Hasuike and Katagiri are applied on real–time data from the Tokyo stock exchange market and proved that the proposed model's solutions are more useful those by earlier approaches [23].

Natarajan et al. (2009) presented a unified theory that relates portfolio risk measures to robust optimization uncertainty sets. In their study, the most important contribution to the literature is adding together different risk measures and important results and adding the authors' perspective on computational tractability and robust optimization definitions of risk measures. They identified how risk measures such as Standard Deviation, Worst–Case VaR, and CVaR can be traced back to robust optimization uncertainty sets. They also emphasized that including worst–case outcomes in robust optimization can be used to generalize the concepts of these risk measures. Furthermore, they showed how an incoherent risk measure can be made becoming a coherent risk measure based upon information. Duality Theory is used to construct specific uncertainty sets that lead to coherent risk measures and address the computational tractability of the resulting problems. Moreover, the validity of the constraints' probability bounds is requested for practical purposes. As a computational example, the authors compared Worst–Case VaR and other risk measures proposed in that article. According to the authors' conclusion, practical benefits are provided by using new coherent risk measures [31].

Bertsimas and Brown (2009) proposed a new methodology to construct uncertainty sets in a RO framework for Linear Optimization models with uncertain parameters. In this approach, they emphasized the decision maker's risk preferences. Since a coherent risk measure addresses uncertainty in its data, the authors employed a convex uncertainty set in a robust optimization network. This is important for their study, because, according to their explanation, the uncertainty set becomes a consequence of the risk measure by the decision maker's selection. Also, they considered distortion risk measures. These kind of risk measures satisfy some additional risk hedging and distribution–invariance properties. Additionally, the authors emphasized that they did not want to make a contribution to risk theory, but they aim to make a contribution to the robust the optimization area [9].

Bertsimas and Sim (2002) proposed a new robust linear programming approach. This approach presents a new parameter to mediate robustness of the presented method against the conservativeness status. By using a pre–specified number, they protected their approach against the violation of constraints. This new method provided a feasible solution for every time less than this value. Unlike other studies, their model is a linear programming model, hence, it is efficient to solve. This approach easily generalizes to discrete linear optimization problems [13].

Zhu and Fukushima (2009) established the Worst–Case CVaR in a situation with uncertain data. They considered the minimization of the Worst–Case CVaR under mixture distribution uncertainty, Box Uncertainty and Ellipsoidal Uncertainty. In their study, Worst–Case CVaR is still a coherent risk measure. However, since they reflected under ellipsoidal, box and mixture distribution uncertainty, the optimization software is not conducted through an LP model. Moreover, the authors applied this extended risk measure on portfolio optimization [51].

Tütüncü and Koenig (2004) referred to a robust portfolio allocation problems with different assets, under uncertain data (unreliable portfolio asset returns). They described uncertainty sets that contain all possible realizations by using moments of returns instead of making a point estimation like in MVA. Their approach is a conservative one and also covers the Worst–Case situation. Their ultimate goal is finding the set of values for decision variables which solves the worst–case optimization problem. Additionally, they introduced a certain methodology to the determination of uncertainty intervals for robust optimization data. The application part of their study is performed on some historical data [48].

Jalilvand–Nejad et al. (2015) investigated different types of uncertainty sets in their study. They considered a LP model with uncertain data (coefficients), and they aimed to obtain a correlation matrix between uncertain coefficients. Additionally, they proposed a new polyhedral uncertainty set which its domain–bounded to the values of the correlation matrix. This matrix is obtained by historical data. Finally, the performance of this model is assessed by an application, and the investigation contributed to earlier studies [26].

Kirilyuk (2008) investigated polyhedral coherent risk measures and he applied to risk–return optimization problems. However, the data of this study reveal partial uncertainty. Since the author used Polyhedral Uncertainty, the portfolio optimization model in the study becomes reduced to a LP. In the application part of the work, an optimal investment allocation problem is discussed with some continuous problems under risk of catastrophic events [27].

Özmen et al. (2011) used recently developed Conic Multivariate Adaptive Regression Splines (CMARS) methodology under the existence of data with uncertainty and represented a new Robust CMARS (RCMARS) algorithm. The authors faced with uncertainty in input data and also output data. This kind of uncertainty affected input and output variables of the objective function about the model. Hence, they employed a robust optimization technique to cope with data uncertainty in inputs and outputs. They introduced, as they say, polyhedral, in the terminology of this thesis, in fact, parallelpipe and ellipsoidal uncertainties in order to get a robust optimization algorithm for CMARS. Since there can easily be a lack of computational power, the authors also implemented a so–called Weak Robustification into their approach and applications [35].

Özmen et al. (2012) studied on Generalized Partial Linear Model (GPLM) by using robust optimization. In their study, GPLM is used according to two different parts of the data. They included uncertainty to future scenarios into the non–linear part by RCMARS method and to an added linear/logit regression part, arriving at so-called CGPLM, but now under uncertainty. After this, the authors made a robustification of this method by dealing with the data uncertainty. As a result, they obtained a Robust Generalized Partial Linear Model (RCGPLM) and applied it to data from financial sector. They also used a weak robustification method due to computational difficulties. In a subsequent research, they developed Robust Conic GPLM method (RCGPLM) to forecast the default probabilities in 45 emerging markets. The linear part of the RCGPLM method consists of a discrete regression model, while the non–linear part consists of a continuous regression model. The objective of obtaining RCGPLM is to reduce the complexity of RCMARS by decreasing the number of variables or their non–linear involvement through the insertion of a robust linear model. In their study, RCGPLM is specified by involving, as the authors say: polyhedral, in fact, Parallelpipe Uncertainty. The reason for the usage of polyhedral uncertainty is to decrease the complexity of the model and to be able to continue the study with Conic Quadratic Programming. Additionally, the authors applied the model to a number of 1019 historical data from 45 emerging markets by considering the macroeconomic indicators. They also used a Weak Robustification because of the capacity of the problem. They named it WRCGPLM [33, 36].

Özmen and Weber (2014) gained from robust optimization when refining the MARS algorithm. To obtain this, they introduced data uncertainty in input and output variables to the MARS algorithm. They presented a new RMARS algorithm as a regression and classification tool. Since parameter estimation has a big impact on their study, RMARS models have much less variability on the parameter estimation and the accuracy. Since the trade-off is between model accuracy and model regularity, even model-robustness, RMARS models have less variability (i.e., less variance) while still having very good accuracy measures. As an application, the authors used financial datasets created by Monte–Carlo simulation [34].

1.2 Scope of the Thesis

The structure of this thesis is as follows: In the introduction given by Chapter 1, we provide a literature research about our study and the main contributions of this thesis. Chapter 2 presents the methodology behind our research; this chapter contains risk measures, uncertainty sets and robust optimization frameworks. Chapter 3 gives our contribution and our robust portfolio optimization approach. Chapter 4 presents a numerical example of applying Robust CVaR in a robust form of portfolio optimization. The last Chapter 5 concludes the thesis and indicates further studies for the future.

CHAPTER 2

METHODOLOGY OF THE RESEARCH

2.1 Risk Measures in Finance

Portfolio optimization and risk management with an emphasis on the need of uncertainty handling were in finance initiated by Markowitz's theory in the 20th century, where he introduced the Modern Portfolio Theory [30]. Since then, variance and standard deviation are fundamental and traditional risk measurement tools in financial sector. However, taking into account the so–called fat tails and other recent findings in finance decreased the accuracy of these risk measures; thus, this provided better risk management solutions. The MVA is often failing to address the Diversification Effect of a portfolio in a satisfying way. Hence, new risk measurement techniques have become and integrated into the financial sector by several studies.

Furthermore, in the literature, risk measures for finance are divided into two main categories: moment–based and quantile–based risk measures. The moment–based risk measures can be traced back to classical economic utility theory. On the other hand, quantile–based risk measures have arisen from recent developments in the theory of stochastic dominance. In this chapter, we will investigate two quantile–based risk measures: VaR and CVaR [31].

2.1.1 Value–at–Risk (VaR)

In finance industry, the most popular and well–known downside risk measure methodology is Value–at–Risk. It has been first developed by JP Morgan and made available through the RiskMetrics™ software in October 1994 [40]. This methodology basically explains the possible worst return loss (or portfolio loss) that can be expected with a certain confidence level (typically 95% or 99% and represented with $1-\alpha$) in a certain time period. Mathematically, the definition of the VaR (α–VaR) is as follows:

$$\text{VaR} = \zeta_\alpha(\xi) := \inf\left\{\zeta \in \mathbb{R} | P(\xi \leq \zeta) \geq \alpha\right\}, \tag{2.1}$$

where ξ denotes the random variable (asset return), α is the percentile of the distribution of the random variable and ζ is the threshold value [49].

VaR is a simple representation of the risk level (just one number); this is one of the main reasons of its popularity. It measures the downside risk of a portfolio or an asset. VaR is useful for the risk measurement process of non–linear instruments such as options, etc. Moreover, it is applicable under non–normal loss (return) distributions [49].

Furthermore, VaR does not take into account risks exceeding value at risk level. Also, it is non–sub–additive and non–convex.

From a practical and computational perspective, optimization of VaR is difficult to handle, unless the distribution of returns is assumed to be normal or log–normal [17]. It is difficult to obtain reliable optimization results if the portfolio consists of both long and short term. VaR also has several undesirable properties which are explained in the work [21].

2.1.2 Coherent Risk Measures

To obtain a coherent risk measure, we might desire additional properties such as structural, monotonicity, translational invariance and sub–additivity. Artzner et al. (1999) presented an axiomatic definition of risk measures which satisfy these properties [3].

Definition 2.1. *(Measure of Risk)* Let $\mathcal{G}$ represents the set of all risks in $\mathbb{R}^n$. A risk measure is a mapping from $\mathcal{G}$ into $\mathbb{R}$, e.g., $\rho : \mathcal{G} \to \mathbb{R}$.

Definition 2.2. *(Acceptance set associated with a risk measure).* An acceptance set associated with a risk measure ρ is the set denoted by $\mathcal{A}_\rho$, defined by

$$\mathcal{A}_\rho := \{\boldsymbol{\xi} \in \mathcal{G} | \rho(\boldsymbol{\xi}) \leq 0\} .$$

If a risk measure ρ satisfies the following properties, it is called a *coherent measure of risk*:

1. *Monotonicity*: If $\boldsymbol{\xi} \geq \mathbf{0}$, then $\rho \leq 0$. Since there are only positive returns, the risk should be non–positive.

2. *Sub–additivity*: $\rho(\boldsymbol{\xi}_1 + \boldsymbol{\xi}_2) \leq \rho(\boldsymbol{\xi}_1) + \rho(\boldsymbol{\xi}_2)$, where $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ are vectors of random variables. The risk of a portfolio of two assets should be less than or equal to the sum of their individual risks (Markowitz's MVA).

3. *Positive homogeneity*: Suppose c is any positive real number. Then, $\rho(c\boldsymbol{\xi}) = c\rho(\boldsymbol{\xi})$. When the portfolio value increases c times (i.e., by a factor of c), then the portfolio risk should be increased c times either.

4. *Translational invariance*: Suppose some $c \in \mathbb{R}$, then $\rho(\boldsymbol{\xi} + c) \leq \rho(\boldsymbol{\xi}) - c$. In fact c comes from a risk–free asset (e.g., government bond price, etc.) here, which means that it is not involved into the total portfolio risk (correctly expressed, one can indeed add $\boldsymbol{c}$ to the vector $\boldsymbol{\xi}$ as another Cartesian coordinate).

The mathematical determination those inequalities between vectors derived from, if $a \leq b \Leftrightarrow a_i \leq b_i \quad (i = 1, 2, \ldots, n)$, where $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)^T$ and $\boldsymbol{b} = (b_1, b_2, \ldots, b_n)^T$. Since VaR is not proper for these conditions, a new risk measurement technique has been founded. This coherent risk measure with its properties is satisfied and explored as *Conditional Value–at–Risk*.

2.1.3 Conditional Value–at–Risk (CVaR)

Since VaR has undesirable properties, a new risk measurement technique has been developed by Rockafellar and Uryasev (2000). CVaR (α–CVaR) is also called *Mean Excess Loss* and *Expected Shortfall* [1]. As a definition, CVaR is a risk measure with a certain probability level α; the α–VaR of an asset (portfolio) is the lowest amount ζ such that, with a probability α, the loss will not exceed ζ, whereas the α–CVaR is the conditional expectation of losses above the amount ζ. In the literature, for the parameter α three values are taken mostly: 90%, 95% and 99%. The definitions provided cause that the value of α–VaR cannot exceed the value of α–CVaR. Figure 2.1 shows the graphical look of CVaR and VaR properties [43].

As an alternative risk measurement methodology, CVaR has better properties than VaR. First, CVaR is a coherent risk measure according to [3], as it satisfies conditions of monotonicity, sub–additivity, positive homogeneity and translational invariance. Also, CVaR has a convex structure.

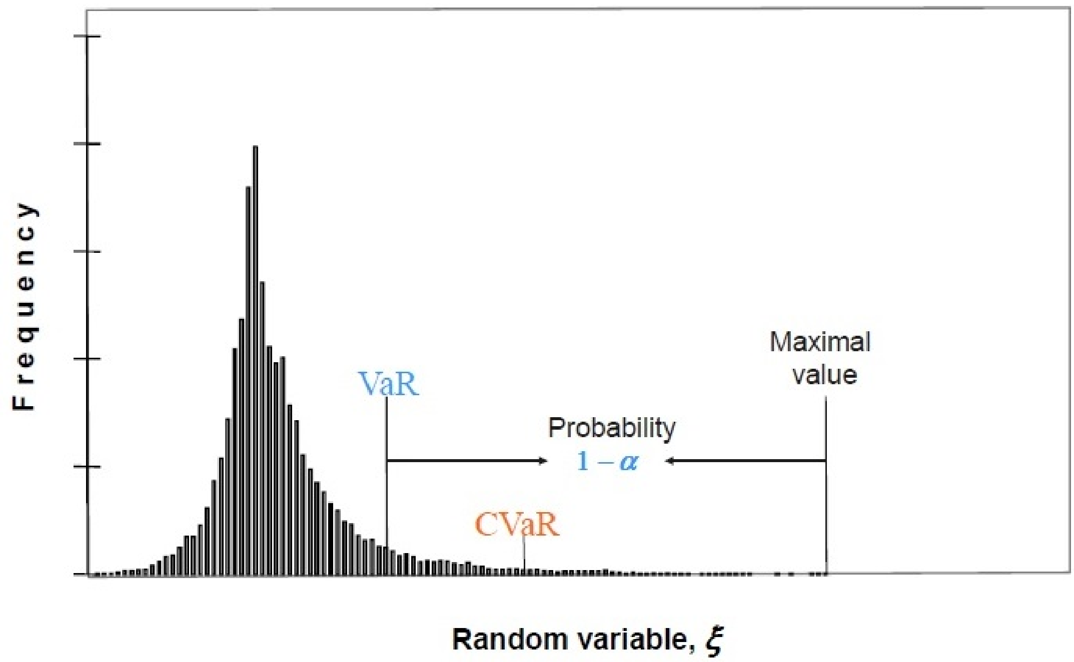


Figure 2.1: Representation of VaR and CVaR [49].

CVaR allows for an simple convenient representation of the risk as one number. It measures the downside risk just like VaR. Unlike VaR, CVaR can be used to non–symmetric loss distributions, too. CVaR takes into account the risks which exceed VaR, i.e., it is more conservative than VaR.

Minimizing the CVaR of an asset (or a portfolio) gives almost the same results as minimizing the VaR of an asset (portfolio). If the distribution of random variables (e.g., returns) is normal distributed, minimization of VaR and of CVaR give the same results. CVaR is applicable even for different financial instruments like linear and non–linear derivatives (options, futures, etc.), measuring market and credit risks or other circumstances in any corporation that is exposed to financial risks. CVaR could be used by hedge funds, banks, energy companies, insurance companies and elsewhere [43].

There are various benefits of CVaR applications. CVaR has steady statistical estimations unlike VaR. CVaR is continuous w.r.t. the confidence level α. CVaR is easy to control and easy to optimize for non–normal distributions. For optimization of CVaR, LP approaches are very useful –and helpful, even for big–size problems. Furthermore, the loss distribution of a portfolio can be shaped via CVaR constraints. Mostly, for continuous distributions, CVaR synchronizes with the conditional expected loss exceeding VaR. However, CVaR could be different from conditional expected loss exceeding VaR if the underlying random variable distribution is non–continuous [49].

2.1.3.1 Mathematical Definitions

Since CVaR is founded on return distributions, first we need to define the return function and then we could define the rest of the theory. Let $\boldsymbol{\xi}$ be the vector of returns (regarded as random variables) of assets (in a portfolio) and $\boldsymbol{x}$ be the vector of the assets' (portfolio) weights; then, $\boldsymbol{x}$ is the decision vector or vector of control variables of CVaR theory. Finally, $\boldsymbol{x}$ has to be chosen from a certain subset X of $\mathbb{R}^n$, where X is a convex set of feasible decisions, and based on a given random vector $\boldsymbol{\xi}$ in $\mathbb{R}^n$ [43].

Before starting the formal definition of the return functions, the backbone of our study, the definition of a portfolio should be presented.

Definition 2.3. The term return, *portfolio* in finance and economics represents a combination of different financial assets such as stocks, derivatives, bonds, and etc. The mathematical representation of a portfolio is:

$$\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T, \tag{2.2}$$

$$\sum_{j=1}^{n} x_j = 1, \tag{2.3}$$

$$x_j \geq 0 \quad (j = 1, 2, \ldots, n). \tag{2.4}$$

Here, n is number of financial assets, $\boldsymbol{x}$ is a portfolio, i.e., vector of assets, where $x_j \in \mathbb{R}$ is the relative amount (corresponding to a percentage) of the financial asset j.

Another alternative way to define a portfolio of the weights x_j is to replace its total sum to the total sum x which is the capital instead of 1, as follows:

$$\sum_{j=1}^{n} x_j = x.$$

This alternative is preferred in multi–period contexts with emphasis on the development of the capital (wealth) and on its re–balancing decisions.

In our research, we prefer to specify convex set X as follows.

Definition 2.4. The convex set X of portfolios is defined by:

$$X = \left\{ \boldsymbol{x} \in \mathbb{R}^n \,\middle|\, \sum_{i=1}^{n} x_i = 1;\; x_i \geq 0 \; (i = 1, 2, \ldots, n) \right\}.$$

Our research comfortably permits the inclusion of further linear constraints in the definition of convex set X.

Definition 2.5. The term *return* of an asset or a portfolio return represents a benefit or an interest of an asset of the portfolio holder after a certain period of time. In this time interval we assume the prices of each asset to be random and address those assets as *risky assets*. The return definition has similarities with *rate of return* definition. Hence, in this research we refer rate of return as return. The mathematical definition of an asset (ξ_j) or portfolio ($\boldsymbol{\xi}$) return is:

$$\xi := \left(\frac{p_T - p_0}{p_0} \right) = \left(\frac{p_T}{p_0} - 1 \right), \tag{2.5}$$

where p_T is the terminal price of the asset at the end of a period and p_0 is the initial price of the asset. The constant 1 does not lead to any change in our decision making, neither in total expected return nor in risk, as one can easily see. This definition can be applied for any discrete time points like t_1 and t_2, where $0 \leq t_1 < t_2 \leq T$, instead of 0 and T themselves (T is the end or terminal time of a given maturity).

In our context, let X denote the set of all the portfolios with different constraints, especially Equations (2.3, 2.4). We might add more constraints to this definition; however, we do not prefer to do that in this work. The vector $\boldsymbol{\xi}$ represents the n returns but the most important matter is that it also contains uncertainty; in fact, information of $\boldsymbol{\xi}$ comes as a market variable. Here, with all these different variables, we can develop a loss (or alternatively, reward) function for this risk measurement technique. Let $\boldsymbol{\xi}$ represent the vector of asset returns from different market scenarios and $\boldsymbol{x}$ represent the set of portfolio weights; hence, we have [49]

$$f(\boldsymbol{x}, \boldsymbol{\xi})$$

as the loss (or reward) function. Naturally, if this function's numerical result (value) is negative, this means that we have a loss function, and if it is positive, we have a reward function associated with our returns ($\boldsymbol{\xi}$). In this thesis, we address by f the loss; whenever we refer to the reward or total return, we use the notation $-f$.

For each $\boldsymbol{x}$, the function $f(\boldsymbol{x}, \boldsymbol{\xi})$ is a random variable having a distribution in $\mathbb{R}$ obtained by $\boldsymbol{\xi}$. The random vector $\boldsymbol{\xi}$'s density function in $\mathbb{R}$ is represented as $p(\boldsymbol{\xi})$.

The probability of $f(\boldsymbol{x}, \boldsymbol{\xi})$ not exceeding a threshold ζ is then given by

$$\Psi(\boldsymbol{x}, \zeta) = \int_{f(\boldsymbol{x}, \boldsymbol{\xi}) \leq \zeta} p(\boldsymbol{\xi}) d\boldsymbol{\xi}. \tag{2.6}$$

According to Equation (2.6), Ψ is the cumulative distribution function for the loss associated with the decision vector (the portfolio weights) $\boldsymbol{x}$. The distribution function at ζ is associated with a fixed Ψ value. This equation completely explains the behavior of the random variables and their fundamentals in VaR and CVaR definitions. The function Ψ is continuous from the right, however, it is not continuous from the left necessarily, because of the possibility of jumps in random variable values. But the authors of [43] assume that there are no jumps in the underlying process of random variables. With this assumption, the utilization and determination of $p(\boldsymbol{\xi})$ becomes more easy.

The α–VaR and α–CVaR values for the loss random variable associated with the decision vector $\boldsymbol{x}$ and a certain confidence interval α will be denoted by $\zeta_\alpha(\boldsymbol{x})$ and $\phi_\alpha(\boldsymbol{x})$. So, in the setting of [43],

$$\zeta_\alpha(\boldsymbol{x}) := \min \{\zeta \in \mathbb{R} : \Psi(\boldsymbol{x}, \zeta) \geq \alpha\} \tag{2.7}$$

and

$$\phi_\alpha(\boldsymbol{x}) := (1-\alpha)^{-1} \int_{f(\boldsymbol{x}, \boldsymbol{\xi}) \leq \zeta} f(\boldsymbol{x}, \boldsymbol{\xi}) p(\boldsymbol{\xi}) d\boldsymbol{\xi}. \tag{2.8}$$

As a consequence, a combination of these two functions $\zeta_\alpha(\boldsymbol{x})$ and $\phi_\alpha(\boldsymbol{x})$ in terms of some function F_α on $X \times \mathbb{R}$ gives us the main equation of CVaR:

$$F_\alpha(\boldsymbol{x}, \zeta) = \zeta + (1-\alpha)^{-1} \int_{\boldsymbol{\xi} \in \mathbb{R}^n} [f(\boldsymbol{x}, \boldsymbol{\xi}) - \zeta]^+ p(\boldsymbol{\xi}) d\boldsymbol{\xi}, \tag{2.9}$$

where we use the positive part as follows:

$$v^+ = \max \{v, 0\}.$$

Unlike the common acceptance, α–CVaR is not equal to an average of outcomes which are greater than α–VaR. To show this situation when the distribution is modeled by scenarios, CVaR could be determined by averaging a fractional number of scenarios [45].

Theorem 2.1. *(Characterization of VaR and CVaR [49]):*

a) α–VaR is a minimizer of the function F_α with respect to ζ:

$$\mathrm{VaR}_\alpha\left(f(\boldsymbol{x},\boldsymbol{\xi})\right) = \zeta_\alpha\left(\boldsymbol{x}\right) = \arg\min_\zeta F_\alpha(\boldsymbol{x},\zeta);$$

b) α–CVaR equals the minimal value (w.r.t. ζ) of the function F_α:

$$\mathrm{CVaR}_\alpha\left(f(\boldsymbol{x},\boldsymbol{\xi})\right) = \min_\zeta F_\alpha(\boldsymbol{x},\zeta).$$

Theorem 2.2. *(Minimization of CVaR [49]):*

a) α–VaR is a minimizer of F_α with respect to ζ:

$$\min_{\boldsymbol{x}}\mathrm{CVaR}_\alpha\left(f(\boldsymbol{x},\boldsymbol{\xi})\right) = \min_{\boldsymbol{x},\zeta} F_\alpha(\boldsymbol{x},\zeta).$$

According to these two theorems, minimization of $F_\alpha(\boldsymbol{x},\boldsymbol{\zeta})$ simultaneously calculates VaR $= \zeta_\alpha(\boldsymbol{x})$, the optimal portfolio weight (decision) $\boldsymbol{x}$ and the optimal CVaR. Moreover, minimization of CVaR can be traced back to approximately an LP model by discretizing the integral. The optimization approach of CVaR supported by Theorem 2.2 can be applied on Equation (2.9) of $F_\alpha(\boldsymbol{x},\boldsymbol{\zeta})$. The minimization of F_α over $X \times \mathbb{R}$ is located in the scientific field of Stochastic Programming or Stochastic Optimization. Since there is an expectation (the integral) in the definition of $F_\alpha(\boldsymbol{x},\boldsymbol{\zeta})$, stochastic programming approaches can give us the results. Eventually, a stochastic programming approach of α–CVaR minimization is possible after using Theorem 2.2 [43].

When a decision maker chooses a risk measure for assessing the exposed risk, another important matter is that the risk measurement model should be tractable. The choice of the risk measure can affect the tractability of the risk counterpart model. A model which satisfies the certain risk level in its constraints is proper for this aim. VaR has a non–convex and intractable risk counterpart. However, CVaR is generally more easy to optimize than VaR. Apart from the convexity, another important issue is whether a risk measure can be computed with any arbitrary accuracy. This is very important during an optimization process under high reliability, such as for providing a structural design [8, 31].

2.2 Robust Optimization

Data uncertainty is a big challenge for optimization problems. In the real world, the data are mostly incomplete or reveal a high uncertainty. Data uncertainty can be caused by several reasons [5]:

1. Measurement or estimation errors.

2. Implementation errors.

When a person performs an optimization on any topic, if some case of the aforementioned two kinds of errors is given –the reasons which are stated above stated are taken into account–, he or she cannot ignore that an even small amount of uncertainty can causes big changes in view of the result. Optimization algorithms are affected by data uncertainty for more than these two reasons even; those further reasons can increase the error propagation. That is why the determination and assessment of uncertainty (or perturbation) of the model's underlying data is important. Hence, the study field of optimization has a real need of a powerful methodology which is strong enough for the detection of optimization cases (optimal solution–finding processes) when data include uncertainty [5].

Nowadays, the technology of computation tools is sufficiently powerful to obtain meaningful and feasible results from optimization algorithms and software while it allows us to deal with a wide range of complex optimization problems, especially, given real–life data under uncertainty. A number of optimization methods have been used to handle this uncertainty. These methods take into account any uncertainty during the modeling or the computing or in the result.

Among these methods, the oldest one is *Sensitivity Analysis*. It deals with data uncertainty after an optimal solution has been obtained by an algorithm. This methods allows the optimizer to change the results between predefined intervals while the current solution remains optimal, assuming that only one parameter at a time deviates from its nominal value. Sensitivity analysis is easy to perform upon an LP while it uses duality theory; however, since it considers the results, it is a "post–mortem" (a–posteriori) tool [21, 38].

The second method is *Stochastic Programming*. It incorporates and intervenes directly into the computation of an optimal solution. According to the generated scenarios, a traditional stochastic LP model finds an optimal solution that produces the best average objective function value for all scenarios. Under uncertainty parameters with a pre–specified probability distribution, the objective function will includes a collection of random variables ($\boldsymbol{\xi}$) (e.g., in our work, the components of a random vector). After this, the LP method chooses the optimal one among the random variable collection by considering its constraints. Hence, it is a pro–active tool unlike sensitivity analysis. However, in very big optimization problems with a huge amount of data, it is ineffective to use it [21, 38].

The third methodology is named as *Dynamical Programming*. This technique is useful to deal with stochastic uncertain systems over multiple stages. The optimization problem is solved recursively by starting, e.g., in cases of backward dynamics, from the last state, returning to the to first state, and by computing each state individually. Unfortunately, this optimization approach may suffer from the curse of dimensionality. This means that the problem size increases exponentially as new scenarios or states are added to the problem [21].

Robust Optimization has evolved differently from the three other methodologies. It is more computationally attractive than others. It eventually treats uncertainty as deterministic, but does not limit parameter values to be point estimates. In the next subsection, robust optimization and its application areas are explained in closer detail.

2.2.1 Robust Optimization Framework

Now, we are going to describe some main RO algorithms, their pioneering works, application areas and a new areas of practical utilization through robustification. To understand RO, principally, LP –the best known and widely used approach– should become very well understood.

Scientists focused on RO heavily from both theoretical and practical perspectives, since it has a modeling framework for *immunizing* against parametric uncertainties in mathematical optimization. This optimization technique is useful when some parameter(s) include uncertainty ($\mathcal{U}$) and are only known to belong to some uncertainty set. RO finds an optimal solution that is feasible, e.g., according to *confidence intervals* in due to uncertain data [5].

The history of RO started in the early 1970s with Soyster [47]. He was one of the first researchers to investigate explicit approaches to robustness. In his study, he explained robust linear optimization in the case where the column vectors of the constraint matrix were constrained to belong to Ellipsoidal Uncertainty sets. According to some pre–specified intervals, the algorithm proposes a feasible solution to all input data points such that each element of the data has its own uncertainty; but, his solution proposal has been found to be overly conservative by some scientific authorities. In the 1990s and 2000s, Ben-Tal and Nemirovski [5, 6, 7] and El Ghaoui et al. [19, 20] proposed new approaches in RO. They bypassed the over conservatism by using ellipsoidal uncertainty sets for the data; the way how ellipsoids are chosen and evaluated is what they achieved. In the late 2000s, Bertsimas, Sim and Pachamanova presented new studies on RO by considering the *Price of Robustness* and *Polyhedral Uncertainty*. Furthermore, Ben-Tal and Nemirovski discussed Polyhedral Uncertainty in their research; also, Bertsimas et al. further contributed to this topic by many more details and insights [9, 10, 11, 12, 13].

RO is a complementary methodology to Stochastic Programming and Sensitivity Analysis. It seeks a robust feasible solution that will have an “acceptable” performance under most realizations of the uncertain inputs. Normally, for a random variable or a vector of random variables ($\boldsymbol{\xi}$) of uncertain parameters, a special distribution assumption is not needed. However, if the model includes such kind of information, it might be beneficial [1]. RO is a worst–case oriented methodology [5, 38].

This optimization technique is useful if some parameters come from an estimation process, which means that they have a certain "contamination" caused by estimation errors that we mentioned at the beginning of this chapter. Finally, there are hard constraints which must be satisfied under any conditions [38].

There are two main stages to constructing a robust model. The first one (i) is the *modeling* of the robust counterpart of the nominal model. The model has become robust when the robust counterpart of the model is solved within a pre–specified uncertainty sets for the uncertain parameters [21]. Second (ii) this robust counterpart is to be *solved* for the *worst–case* realization of the uncertain parameter associated with underlying data, uncertainty on input (or output) data based and expressed by a pre–determined uncertainty set. This means, we are minimizing the maximum, namely, the worst (or a disaster) situation for our *objective* (or objective function). Another definition for this situation is: an optimal solution is robust if it minimizes the *maximum relative regret* [38].

A visual representation of a robust solution approximation is shown in Figure 2.2. This figure represents the case of an ellipsoidal set (since our aim is to address a particular kind of uncertainty set with parallelpipe sets, we will have a different figure).

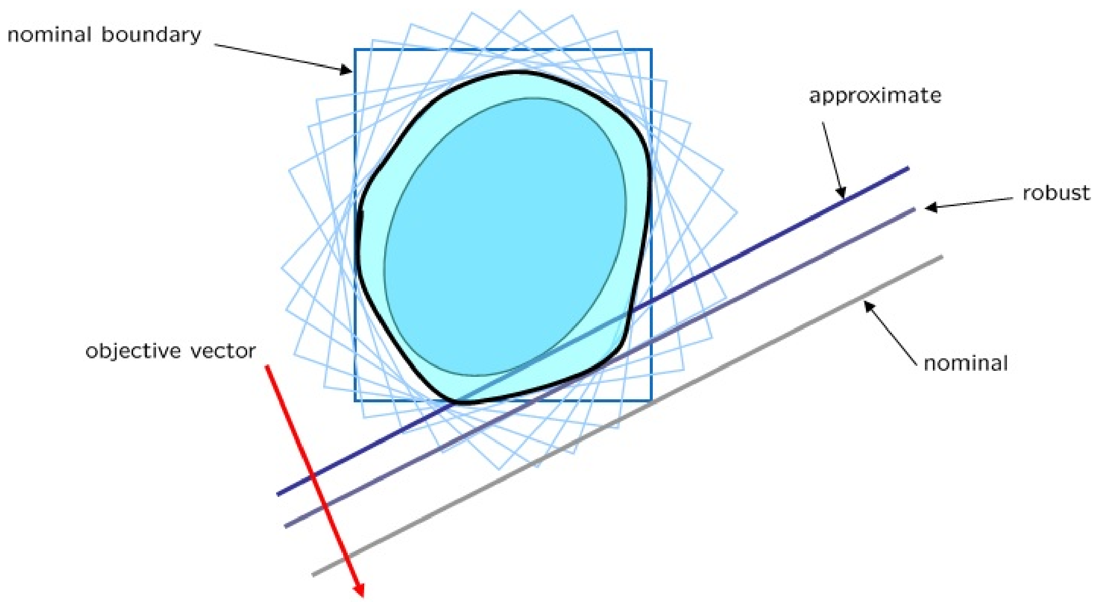


Figure 2.2: Approximating a robust solution (basic idea from [18]).

Although it has its difficulties, RO gives us another perspective to deal with data uncertainty. During the application of this technique, two criteria are strongly important. The first one is *tractability*. This criterion is about preserving the computational tractability of the nominal (actual) problem both theoretically and practically. This

[1] However, in the case of ellipsoidal uncertainty, usually a normal distribution is supposed.

means, the problem should be solvable easily and rapidly from a theoretical perspective. The second criterion is given by *probability bounds*. If the uncertain coefficients within uncertainty sets are under some certain probability distribution, the probability of a feasible robust solution should be accurate. By this guarantee, the trade–off between robustness of the solution and optimal (crisp) parameters can be specified [46].

Regarding this, first, we should consider a basic LP model which contains uncertainty:

$$\begin{aligned} &\text{minimize} \quad \boldsymbol{c}^T\boldsymbol{x} \\ &\text{subject to} \\ &\quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ &\quad \boldsymbol{x} \in X. \end{aligned} \tag{2.10}$$

Without loss of generality, our assumption is that only the matrix $\boldsymbol{A}$ implies the uncertainty and in Subsection 2.2.2, we shall consider uncertain matrix $\boldsymbol{A}$ in different uncertainty sets. Herewith, our robust counterpart of Equation (2.10) takes the following form:

$$\begin{aligned} &\text{minimize} \quad \boldsymbol{c}^T\boldsymbol{x} \\ &\text{subject to} \\ &\quad \breve{\boldsymbol{A}}\boldsymbol{x} = \boldsymbol{b}, \quad \forall \breve{\boldsymbol{A}} \in \mathcal{U}, \\ &\quad \boldsymbol{x} \in X. \end{aligned} \tag{2.11}$$

Subsequently, we will reflect on and evaluate this robustified problem of Equation (2.11) in every step which we are going to show next.

2.2.2 Uncertainty Sets

Uncertainty sets are important elements of RO. When the relevant data reveal uncertainty, the robust counterpart of the mathematical problem is a worst–case formulation of the mathematical model with uncertainty. In this context, the mathematical parameters of the model include uncertainty over a bounded interval (scale). Instead of nominal values, the uncertainty sets can take any values within a bounded and usually (but not necessarily) symmetric region. This set is known as an uncertainty set $\mathcal{U}$. The scale and structure of such a set $\mathcal{U}$ is defined by the decision modeler There are different structural shapes which refer to the geometry: Box, Polyhedral, Ellipsoidal, etc.

Soyster's approach in RO is one of the earliest approaches in this area of research and application [47]. Fundamentally, in his study, an uncertain optimization problem for each coefficient may include disturbances (or perturbations) of any coefficient, e.g., within a specified interval. However, this approach is regarded as very conservative by recent studies, although it gives feasible solutions. As shown in Subsections 2.2.2.1, if the perturbation level of a box uncertainty set is put to a certain level, the model is identical to Soyster's approach and it is over conservative.

Hence, new uncertainty set structures for RO are defined in different works by [6, 12, 13, 19, 20] to avoid the conservatism. Ben–Tal and Nemirovski (1998), El–Ghaoui and Lebret (1997), El–Ghaoui, Oustry, and Lebret (1998) proposed Ellipsoidal Uncertainty for the robust counterpart in their independent studies. Bertsimas and Sim (2003, 2004) introduced another geometrical structure, named as Polyhedral Uncertainty set, for their research's robust counterpart to avoid conservatism. Their polyhedral uncertainty has an advantage over the ellipsoidal set. Calculation and modeling under Polyhedral Uncertainty is much easier than under Ellipsoidal Uncertainty.

However, Ellipsoidal Uncertainty sets can give more robust solutions than Polyhedral Uncertainty sets. The reasons behind of this robustness difference is based on the probabilistic and statistical approach of having confidence regions. When addressing the case that Parallelpipe Uncertainty sets are products of intervals, one looks at products of intervals (see Subsection 3.2). The main reasons of the robustness difference between Ellipsoidal and Parallelpipe Uncertainty (under Parallelpipe Uncertainty) are: (i) Confidence ellipsoids are bigger than confidence intervals in each dimension. (ii) Ellipsoids (confidence ellipsoids) take into account the existing covariances between the coordinates where random variables (e.g., the components of $\boldsymbol{\xi}$) take their values. For further information, one might refer to [2, 4].

According to the general optimization theory with uncertain random variables, in the RO literature, the general form of $\mathcal{U}$, i.e., the common form of an uncertainty set, is as follows [32]:

$$\mathcal{U} = \left\{ \breve{\boldsymbol{\xi}} = \boldsymbol{\xi} + \sum_{i=1}^{K} \rho_i \boldsymbol{\xi}^i \in \mathbb{R}^K \middle| \boldsymbol{\rho} \in Z \right\}, \tag{2.12}$$

where $\boldsymbol{\xi}$ is the nominal value of the uncertain vector $\breve{\boldsymbol{\xi}}$, the vectors $\boldsymbol{\xi}^i$ are so–called possible scenarios of it, and $\boldsymbol{\rho} = (\rho_1, \rho_2, \ldots, \rho_K)^T$ is a so–called perturbation vector. The set Z represents the structure of the uncertainty set. These sets might be box (i.e., box–shaped), ellipsoidal or polyhedral [32]. We emphasize that in many applications the elements of $\mathcal{U}$ are represented in the form of matrices. In fact, we, in our work, shall regard K as the format $M \times N$.

Detailed information about Box, Polyhedral and Ellipsoidal uncertainty sets are shown in Subsections 2.2.2.1, 2.2.2.2 and 2.2.2.3.

2.2.2.1 Box Uncertainty

When an optimization problem has uncertainty in its parameters, it is assumed that terms ξ_{ij} are random variables of the problem and also they are independent. Their absolute values are occurring in intervals $[0, \Psi_i]$. Each perturbation has a connection with any other one, in terms of shape, size and position. This relationship creates a box which constitutes called Box Uncertainty in the literature. Such a Box Uncertainty set can be expressed as follows [26, 29]:

$$\mathcal{U}^{\boldsymbol{A}} := \left\{ (\breve{a}_{ij})_{i=1,\ldots,M;\ j=1,\ldots,N} \middle| \breve{a}_{ij} = a_{ij} + \xi_{ij}\Delta_{ij};\ |\xi_{ij}| \leq \Psi_i, \forall i,j \right\}. \tag{2.13}$$

In Equation (2.13), a_{ij}, $\breve{a}_{ij}$, Δ_{ij} denote the nominal, actual and maximum positive perturbation of corresponding uncertain coefficient values, respectively. Furthermore, Ψ_i is the perturbation bound for all of the uncertain coefficients related to the ith constraint of the optimization model. Put in a more clear way, the result of the RO modeling associated with the uncertainty set as stated in Equation (2.13) is acceptable if and only if, the absolute value of all perturbations within uncertain coefficients is less than Ψ_i, respectively. The robust counterpart formulation of Equation (2.23) for the case of Equation (2.13) can be represented as follows [26, 29]:

$$\begin{aligned}
&\text{minimize} \quad \boldsymbol{c}^T\boldsymbol{x} \\
&\text{subject to} \\
&\quad \sum_{j=1}^{N} a_{ij}x_j + \Psi_i \sum_{j=1}^{N} \Delta_{ij}y_j \leq b_i \quad (i = 1, 2, \ldots, M), \\
&\quad -y_j \leq x_j \leq y_j \quad (j = 1, 2, \ldots, N), \\
&\quad \boldsymbol{l} \leq \boldsymbol{x} \leq \boldsymbol{u}, \\
&\quad \boldsymbol{y} \geq \boldsymbol{0}.
\end{aligned} \tag{2.14}$$

In this problem, if $\Psi_i = 1$, the model will be identical the one in [47]. Figure 2.3 is associated with several values of Ψ_i. It shows that if the value of Ψ_i increases, the conservation of the model increases also [26].

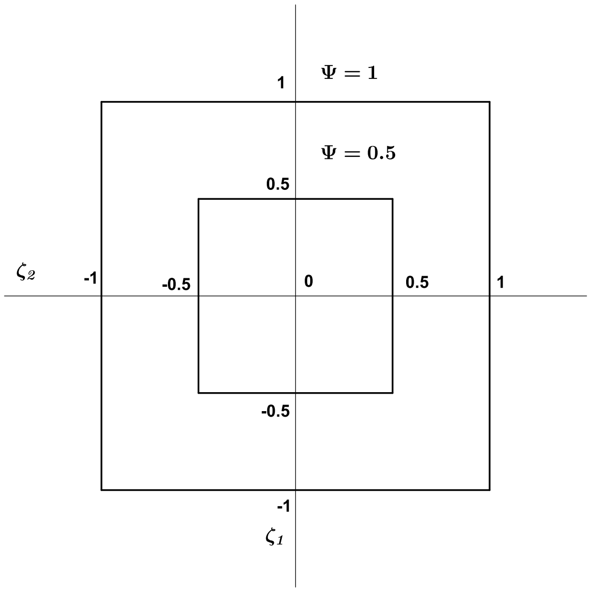


Figure 2.3: Box Uncertainty sets [26].

2.2.2.2 Ellipsoidal Uncertainty

After years of Soyster's approach, Ben–Tal and Nemirovski (1998, 2000) proposed an alternative and new methodology to avoid from too high conservativeness in RO [6, 7]. They implemented the idea of ellipsoidal uncertainty in this field of research. The following uncertainty set is an ellipsoidal set [26, 29]:

$$\mathcal{U}^{\boldsymbol{A}} := \left\{ (\breve{a}_{ij})_{i=1,\ldots,M;\ j=1,\ldots,N} \middle| \breve{a}_{ij} = a_{ij} + \xi_{ij}\Delta_{ij}, \forall i,j;\ \sum_{j=1}^{N} \xi_{ij}^2 \leq \Omega_i^2, \forall i \right\}, \tag{2.15}$$

where $\mathcal{U}^{\boldsymbol{A}}$ expresses the ellipsoidal uncertainty set and Ω_i^2 defines the so–called borders of the set. Eventually, this set has an ellipsoidal structure. Big values of such parameters, like Ω_i^2, imply big sizes of the ellipsoid. The robust counterpart of an LP model can be represented as follows [26, 29]:

$$\begin{aligned}
&\text{minimize} \quad \boldsymbol{c}^T\boldsymbol{x} \\
&\text{subject to} \\
&\quad \sum_{j=1}^{N} a_{ij}x_j + \sum_{j=1}^{N} \Delta_{ij}y_{ij} + \Omega_i\sqrt{\sum_{j=1}^{N} \Delta_{ij}^2 z_{ij}^2} \leq b_i, \quad \forall i, \\
&\quad -y_{ij} \leq x_j - z_{ij} \leq y_{ij}, \quad \forall i, j, \\
&\quad \boldsymbol{l} \leq \boldsymbol{x} \leq \boldsymbol{u}, \\
&\quad \boldsymbol{y} \geq \boldsymbol{0},
\end{aligned} \tag{2.16}$$

where y_{ij} are integer variables and z_{ij} are dual variables of the primer model. The robustness level of this approach can be determined by changing the values of the Ω_i parameter. The disadvantage of this model is that it leads to an optimization problem with a higher computational and complex structure, compared to Box and Polyhedral uncertainty. Hence, it gives the most robust solution to us. Figure 2.4 illustrates the ellipsoidal uncertainty set with two coefficients [26].

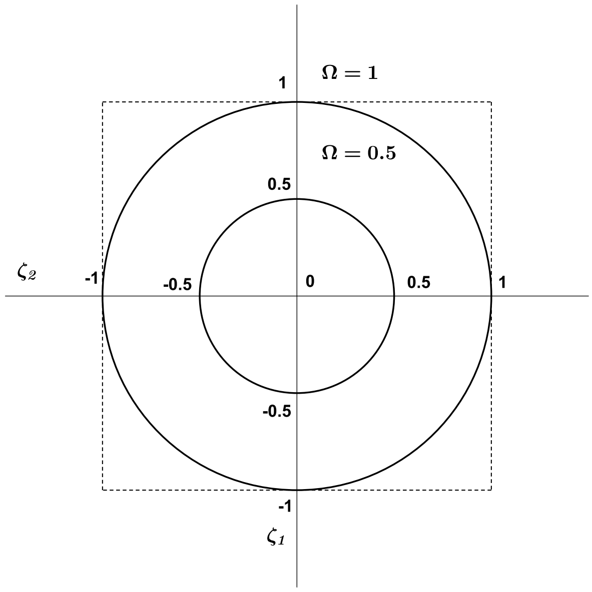


Figure 2.4: Ellipsoidal Uncertainty sets [26].

2.2.2.3 Polyhedral Uncertainty

Since the ellipsoidal uncertainty is hard to solve, Bertsimas and Sim (2004) proposed new kind of uncertainty structure [13]. Polyhedral Uncertainty reduces the level of complexity; however, the solution becomes less robust, compared with the ellipsoidal case. Their model's solution is a robust solution that is protected against all scenarios. To confirm this, they added a boundary Γ_i for the coefficients which limits the perturbation number. At most Γ_i coefficients of the ith constraint are allowed to be perturbed. The polyhedral uncertainty set for the coefficient matrix $\boldsymbol{A}$ can be formulated as follows [26, 29]:

$$\mathcal{U}^{\boldsymbol{A}} := \left\{ (\breve{a}_{ij})_{i=1,\ldots,M;\ j=1,\ldots,N} \middle| \breve{a}_{ij} = a_{ij} + \xi_{ij}\Delta_{ij}, \forall i,j;\ \sum_{j=1}^{N} |\xi_{ij}| \leq \Gamma_i, \forall i \right\}. \tag{2.17}$$

For an LP optimization problem with two coefficients, such uncertainty sets are described in Figure 2.5 [26].

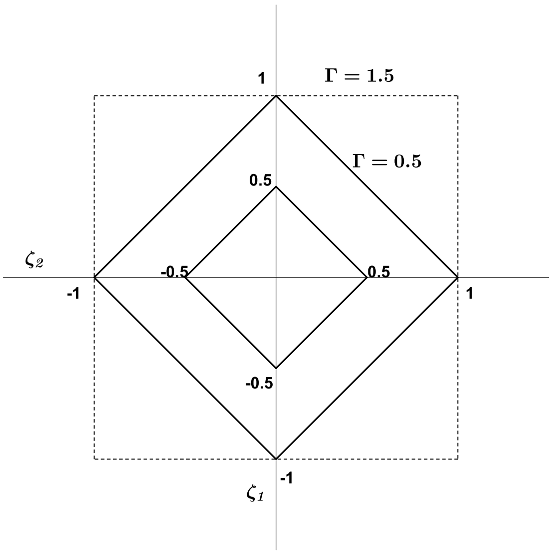


Figure 2.5: Polyhedral Uncertainty sets [26].

The robust counterpart of an LP problem with Polyhedral Uncertainty can be stated as follows [26, 29]:

$$
\begin{aligned}
&\text{minimize} \quad \boldsymbol{c}^T\boldsymbol{x}\\
&\text{subject to}\\
&\sum_{j=1}^{N} a_{ij}x_j +\\
&\max_{\{S_i\cup\{t_i\}|S_i\subseteq J_i,|S_i|=\lfloor\Gamma_i\rfloor,t_i\in J_i\setminus S_i\}} \quad \left\{\sum_{j\in S_i}\Delta_{ij}y_j + (\Gamma_i - \lfloor\Gamma_i\rfloor)\Delta_{it}y_t\right\} \le b_i, \quad \forall \boldsymbol{i},\\
&-y_j \le x_j \le y_j, \quad \forall \boldsymbol{j},\\
&\boldsymbol{l} \le \boldsymbol{x} \le \boldsymbol{u},\\
&\boldsymbol{y} \ge \boldsymbol{0},
\end{aligned}
\tag{2.18}
$$

where J_i is the set of column indexes j with coefficients a_{ij} of the ith constraint that are subject to uncertainty. Here, Γ_i is a parameter which takes its value within the interval $[0, |J_i|]$ for any i. In all cases, the model's solution is protected so that, up to $\lfloor\Gamma_i\rfloor$[2] many coefficients are allowed to change and one coefficient, a_{it}, changes by $(\Gamma_i - \lfloor\Gamma_i\rfloor)a_{it}$, as explained in [46]. The robust counterpart of an ordinary LP model

[2] This lower bracket term represents the greatest integer which does not exceed Γ_i.

like in Equation (2.23) associated with Equation (2.18) is as follows:

$$
\begin{aligned}
&\text{mininimize} \quad \boldsymbol{c}^T\boldsymbol{x}\\
&\text{subject to}\\
&\quad \sum_{j=1}^{N} a_{ij}x_j + z_i\mathit{\Gamma}_i + \sum_{j\in J_i} p_{ij} \le b_i, \quad \forall i,\\
&\quad z_i + p_{ij} \ge \Delta_{ij}y_j, \quad \forall j \in J_i,\ \forall i\\
&-y_j \le x_j \le y_j, \quad \forall j,\\
&\quad l_j \le x_j \le u_j, \quad \forall j,\\
&\quad p_{ij} \ge 0, \quad \forall j \in J_i,\ \forall i,\\
&\quad y_j \ge 0, \quad \forall j,\\
&\quad z_i \ge 0, \quad \forall i.
\end{aligned}
\tag{2.19}
$$

Furthermore, when our LP model includes an invertible covariance matrix $\boldsymbol{\Sigma} \in \mathbb{R}^{(M\cdot N)(M\cdot N)}$ which represents uncertainty and dependences in matrix coefficients, another approach should be deployed to the model which can be found in [37] for closer details. According to Pachamanova's study, the following polyhedral uncertainty set should be defined:

$$
\mathcal{U}^{\boldsymbol{A}} := \left\{ \breve{\boldsymbol{A}} \middle| \left|\left| \boldsymbol{\Sigma}^{-1/2} \left(vec\left(\breve{\boldsymbol{A}}\right) - vec\left(\breve{\boldsymbol{A}}\right)\right) \right|\right|_1 \le \mathit{\Gamma} \right\}, \tag{2.20}
$$

where all the appearing letters represent matrices. In this equation, $\breve{\boldsymbol{A}} \in \mathbb{R}^{M\cdot N}$ and $\breve{\boldsymbol{A}} \in \mathbb{R}^{M\cdot N}$ denotes the matrices of actual values and expected values of the uncertain coefficient matrix $\boldsymbol{A}$, respectively. In Equation (2.20), the vectors $vec(\breve{\boldsymbol{A}}) \in \mathbb{R}^{(M\cdot N)\times 1}$ and $vec(\breve{\boldsymbol{A}}) \in \mathbb{R}^{(M\cdot N)\times 1}$ are generated by consequently aligning the rows of the matrix one and another. These two matrices precisely represent the vectors which are equal to $\breve{\boldsymbol{A}}$ and $\breve{\boldsymbol{A}}$, respectively, through a standard reshape. Furthermore, $||\boldsymbol{x}||_1$ represents the L_1 norm of a vector $\boldsymbol{x}$ and equals $\sum_j |x_j|$ [26].

The robust counterpart with uncertain covariance matrix of an LP model as Equation (2.23) is described by [37] and it is equivalently represented as follows:

$$
\begin{aligned}
&\text{minimize} \quad \boldsymbol{c}^T\boldsymbol{x}\\
&\text{subject to}\\
&\quad (\boldsymbol{x}_i^T)vec(\breve{\boldsymbol{A}}) + u^i\mathit{\Gamma} \le b_i \quad (i = 1, 2, \ldots, M),\\
&\quad u^i\boldsymbol{e}^T \ge \boldsymbol{\Sigma}^{1/2}\boldsymbol{x}_i \quad (i = 1, 2, \ldots, M),\\
&\quad u^i\boldsymbol{e}^T \ge -\boldsymbol{\Sigma}^{1/2}\boldsymbol{x}_i \quad (i = 1, 2, \ldots, M),\\
&\quad u^i \ge 0 \quad (i = 1, 2, \ldots, M),
\end{aligned}
\tag{2.21}
$$

where the vector $\boldsymbol{x}_i \in \mathbb{R}^{(M\cdot N)\times 1}$ includes $\boldsymbol{x}$ in the entries $(i-1)\cdot N + 1$ through $i \cdot N$ of the vector $\boldsymbol{x}$, and zero everywhere else $(i = 1, 2, \ldots, M)$. Moreover, $\boldsymbol{e} \in \mathbb{R}^M$ is the vector of entries 1 and u^i are added variables to the deterministic linear programming model of polyhedral uncertainty [26].

2.2.3 Robust Optimization under Parallelpipe Uncertainty

Consider a standard optimization problem and then a Linear Optimization or LP model with uncertain data. We minimize (or maximize) the objective function. Our general optimization problem is:

$$\begin{aligned} &\text{minimize} \quad f(\boldsymbol{x}, \boldsymbol{\xi}) \\ &\text{subject to} \\ &\quad g_i(\boldsymbol{x}, \boldsymbol{\xi}) \leq 0 \quad (i = 1, 2, \ldots, M), \\ &\quad \boldsymbol{x} \in X, \end{aligned} \tag{2.22}$$

where $\boldsymbol{x}$ is the vector of decision variables, $\boldsymbol{\xi}$ is the vector of data (uncertain), f [3] and g_i $(i = 1, 2, ..., M)$ given convex functions and X is a possibly (usually) convex set (it is well–known in optimization theory [22]).

Regarding to this model, our LP model (which is very practical for us often) and a *Standard Form* of an LP model are as follows, respectively:

$$\begin{aligned} &\text{minimize} \quad \boldsymbol{c}^T\boldsymbol{x} \\ &\text{subject to} \\ &\quad \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}, \\ &\quad \boldsymbol{l} \leq \boldsymbol{x} \leq \boldsymbol{u}; \end{aligned} \tag{2.23}$$

$$\begin{aligned} &\text{minimize} \quad \boldsymbol{c}^T\boldsymbol{x} \\ &\text{subject to} \\ &\quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \\ &\quad \boldsymbol{x} \in X. \end{aligned} \tag{2.24}$$

In our study, we assume that data uncertainty affects only the matrix $\boldsymbol{A}$ in the above Equations (2.23) and (2.24). The parameters $\boldsymbol{l}$ and $\boldsymbol{u}$ comprise lower and upper bounds of decision variables. We note that in our research, we allow components of the vector $\boldsymbol{u}$ also to attain ∞ as a value, in order to serve as "upper bounds" of the real-valued variables x_j in the model; this, in fact, allows for unboundedness from above in those random and decision variables. Herewith and by other standard arguments that we shall apply, we could move from one representation or form of an LP model to another. Now, we imply uncertainty into our matrix $\boldsymbol{A}$. We assume without loss of generality that in our optimization model, the vectors $\boldsymbol{c}$, $\boldsymbol{b}$, $\boldsymbol{l}$ and $\boldsymbol{u}$ are not subject to uncertainty. Our objective function is to be minimized: hence, we could use a *height variables*, $\boldsymbol{z}$ (by an epigraph argument) and add $z - \boldsymbol{c}^T\boldsymbol{x} \geq 0$ as an inequality constraint; and we include this constraint into the vector–matrix notation of the constraints $\boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}$. Soon we will employ such kinds of standard arguments comfortably.

Hence, there is a model uncertainty in our study. Let us consider a row i in matrix $\boldsymbol{A}$, and referring to the entries, we denote by each entry of $\boldsymbol{A}$, denoted by a_{ij}, $j \in J_i$, the set of columns met by row i which are subject to uncertainty. Actually, every entry a_{ij}

[3] There might be a possible confusion of the reader since this f is just a general objective function. However, in further sections, the function f is a loss function within CVaR and RCVaR.

(nominal value of the parameter coefficient) in row i, $j \in J_i$, is modeled as a symmetric and bounded random variable $\breve{a}_{ij}$, $j \in J_i$ [7].

In this investigation, we want to solve an LP model by turning it to a robust LP model. Since Ellipsoidal Uncertainty turns our optimization problem into a Conic Quadratic Programming model, it is more robust but less tractable. Therefore, we prefer to imply Parallelpipe Uncertainty (as we name it now) in the sense of Özmen, et al. [34, 35]. This new type of uncertainty set has similarities with Box Uncertainty in Subsection 2.2.2.1, but generalizes the Box type by fully variable, indeed independent side lengths of our Parallelpipes. In fact, our uncertainty set $\mathcal{U}$ will have different structural properties than others.

In Parallelpipe Uncertainty, the definition and construction of an uncertainty set is different than from a one with Polyhedral Uncertainty. Despite of similarities with the other uncertainty sets, differences of the parallelpipes are revealed regarding their structural occurrence. Parallelpipe uncertainty sets are products of the entry–wise intervals in matrix $\breve{\boldsymbol{A}}$; we shall look at such intervals in Equation (2.26) soon. The benefit of parallelpipes to our research is that instead of a single price we consider several and flexibly varying prices of an asset. The specific definition as a product of intervals is explained in further sections with all the necessary details.

So, any of our regarded Parallelpipe Uncertainty sets $\mathcal{U}$ will be built up by entries in our uncertain matrix $\boldsymbol{A}$ where the corresponding uncertainty may, to some degree (or budget), differ from one entry to another. According to the models which are described above, the following criteria are important in RO modeling [37]:

Formulation flexibility: How to allow for expressing dependencies among the uncertain coefficients along a set of constraints;

Conservativeness of optimal solution: Associated with probabilistic guarantees, formulating the RO problem by a pre–decided level of conservativeness;

Tractability: Preserving the RO model's computational tractability, e.g., computationally being easy to solve.

Consequently, according to our Parallelpipe Uncertainty sets, the new closed–form general Robust LP model with uncertainty implied is

$$
\begin{aligned}
&\text{minimize} \quad \boldsymbol{c}^T\boldsymbol{x} \\
&\text{subject to} \\
&\quad \breve{\boldsymbol{A}}\boldsymbol{x} = \boldsymbol{b}, \quad \forall \breve{\boldsymbol{A}} \in \mathcal{U}, \\
&\quad \boldsymbol{x} \in X.
\end{aligned}
\tag{2.25}
$$

In the literature, Polyhedral Uncertainty based upon uncertain matrix entries intervals. The major property of intervals involved towards our set $\mathcal{U}$ in the sense of Polyhedral Uncertainty is that uncertain coefficients $\breve{a}_{ij}$ $(j \in J_i)$, lie in intervals [9]

$$[\hat{a}_{ij} - \Delta_{ij}, \hat{a}_{ij} + \Delta_{ij}], \tag{2.26}$$

where $\hat{a}_{ij}$ is the arithmetic mean of the $\breve{a}_j$ and Δ_{ij} is the perturbation term for representing the margin of uncertainty in the corresponding random variable $(i = 1, \ldots, M;\ j = 1, 2, \ldots, N)$. Change in the value of perturbation term Δ_{ij} could be determined by considering the standard deviations (σ_{ij}) of random variables (components ξ_j of $\boldsymbol{\xi}$) or in general, the uncertain matrix entries $(\breve{a}_{ij})$.

From Section 2.2.2.3 we know that the value of perturbation is restricted by a value $\mathit{\Gamma}_i$ at each level which; this value is the semi–length of an interval on parametric uncertainties which is called a *budget of uncertainty*. Such a restriction of perturbation with a parameter, $\mathit{\Gamma}_i$, could be applied to our intervals when considering a Polyhedral Uncertainty matrix $\breve{\boldsymbol{A}}$ [46].

In this research, we refer to Equation (2.26) differently, namely, from the perspective of Parallelpipe Uncertainty. However, with our entries $\breve{a}_{ij}$ we address return values based on uncertain prices. Since Parallelpipe Uncertainty is based on intervals of the uncertain parameters, definition and size of price and return intervals are important for the set $\mathcal{U}$ of matrices $\breve{\boldsymbol{A}}$. For prices and returns we employed interval calculus and suitable definitions. In fact, Equation (2.26) is benefited by us with regard to prices and returns.

CHAPTER 3

ROBUST PORTFOLIO OPTIMIZATION WITH ROBUST CVaR UNDER PARALLELPIPE UNCERTAINTY

In this chapter, there are two major sections which describe our main contribution. Since, our research differs from the involvement of box, ellipsoidal and polyhedral uncertainty in the literature, we call our RO approach Parallelpipe Uncertainty. However, before using this new uncertainty set, CVaR should be prepared carefully for this application. Section 3.1 explains the discretization and worst–case approximation of CVaR according to the existing literature and stated for our scientific purposes. Section 3.2 offers our contribution to literature by considering Parallelpipe Uncertainty. In this section, we present our modeling approach for numerical applications, explain the logic behind, Parallelpipe Uncertainty, and conclude the theoretical and methodological part of our research.

3.1 Robust Conditional Value–at–Risk (RCVaR)

Robustification of CVaR starts with the uncertainty on the distribution of portfolio returns, based on the underlying prices that include uncertainty. In the markets, the distribution of the asset returns is partially unknown. This situation is supporting our claim to involve uncertainty in the design or our model matrix $\boldsymbol{A}$. Instead of assuming one certain probability distribution of the random variable vector ($\boldsymbol{\xi}$), our assumption is that some density function of the portfolio returns only belongs to a certain set P of distributions of various types, i.e., $p(\cdot) \in P$, which covers all the possible (or reasonable) return scenarios (including the worst–case return scenario).

Since our data (including returns) reveal uncertainty, to prepare a RO under a *Parallelpipe Uncertainty* set, we prefer to use the WCVaR optimization model with underlying probability distribution, being an element of some given set P. From that viewpoint, we prefer to use [25, 51]. The function $f(\boldsymbol{x}, \boldsymbol{\xi}) := -(x_1\xi_1 + x_2\xi_2 + \ldots + x_n\xi_n)$ is the loss (if the multiplication has positive sign, then it is a reward) of the portfolio with decision vector $\boldsymbol{x} \in X \subseteq \mathbb{R}^n$ and random vector $\boldsymbol{\xi} \in \mathbb{R}^n$ that presents the portfolio returns at the maturity time T.

Suppose $E(|f(\boldsymbol{x},\boldsymbol{\xi})|) < +\infty$ in view of all $\boldsymbol{x} \in X$. For the simplicity and tractability of the optimization model, we assume $\boldsymbol{\xi} \in \mathbb{R}^n$ has a continuous density function $p(\boldsymbol{\xi})$. By the theory of [44], all the results can also be applied to discontinuous distributions. Now, Equations (2.1), (2.6), (2.7), (2.8), and (2.9) will lead us to a new approximation. If X is a convex set in $\mathbb{R}^n$, and the function $f(\boldsymbol{x},\boldsymbol{\xi})$ is convex with respect to $\boldsymbol{x}$, then the optimization problem is called *Convex Programming* problem.

Our further task in the construct on an optimization model by using the CVaR approach, consisting in the determination of the density of the random vector $\boldsymbol{\xi}$ with a given maturity time T. Stated in another way, we need to know the probability distribution of the random vector $\boldsymbol{\xi}$ at time T. However, in the real life and in most of the practical cases, the distribution of the random vector (returns) $\boldsymbol{\xi}$ is partially unknown. This means, our random variable vector (along the different data underlying the matrix in the LP model) contains uncertainty from the assets' prices. Hence, [51] assumed that the density function is only known to belong to a certain set P of distributions, i.e., $p(\cdot) \in P$.

Therefore, the precise and yet general definition of the Robust Conditional Value–at–Risk (or Worst–Case Conditional Value–at–Risk) in the boundaries of coherent risk measures definition of [3] and with the given confidence level α, in short RCVaR$_\alpha$, for a given portfolio $\boldsymbol{x} \in X$ with respect the set of distributions, P, is defined as [16, 25, 51]:

$$\text{RCVaR}_\alpha(\boldsymbol{x}) := \sup_{p(\cdot)\in P} \text{CVaR}_\alpha^p(\boldsymbol{x}). \tag{3.1}$$

In CVaR$_\alpha^p$, the index p serves as reminder about the reference (dependence) of CVaR on the return distribution p. Likewise we add an index p at the function F_α to express the same reference and dependence. Then, by the definition from Equation (2.9), RCVaR$_\alpha$ is

$$\text{RCVaR}_\alpha(\boldsymbol{x}) := \sup_{p(\cdot)\in P} \min_{\zeta\in\mathbb{R}} F_\alpha^p(\boldsymbol{x},\zeta). \tag{3.2}$$

Hence, to minimize the RCVaR over $\boldsymbol{x} \in X$ is equivalent to the following min–sup–min problem:

$$\text{minimize}_{\boldsymbol{x}\in X} \sup_{p(\cdot)\in P} \min_{\zeta\in\mathbb{R}} F_\alpha^p(\boldsymbol{x},\zeta). \tag{3.3}$$

The definition above give us the related connection between RCVaR and CVaR. Now, let us suppose that we have a discrete probability which, in fact, we shall later–on achieve by a discretization. In fact, this assumption provides us an easy path for the tractability purposes of our forthcoming main LP model.

Let the sample space of random vector $\boldsymbol{\xi}$ be given by $\{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots, \boldsymbol{\xi}_m\}$ with discrete probability $\Pr \boldsymbol{\xi}_i = \pi_i$ and $\sum_{i=1}^m \pi_i = 1$, $\pi_j \geq 0$ $(i = 1, 2, \ldots, m)$. Moreover, we denote the discrete probability as $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_m)^T$ and define [16, 43, 51]:

$$M_\alpha(\boldsymbol{x},\zeta,\boldsymbol{\pi}) := \zeta + \frac{1}{(1-\alpha)} \sum_{i=1}^m \pi_i [f(\boldsymbol{x},\boldsymbol{\xi}_i) - \zeta]^+. \tag{3.4}$$

For a given $\boldsymbol{x}$ and $\boldsymbol{\pi}$ the related CVaR is defined as [44]

$$\mathrm{CVaR}_\alpha(\boldsymbol{x},\boldsymbol{\pi}) := \min_{\zeta\in\mathbb{R}} M_\alpha(\boldsymbol{x},\zeta,\boldsymbol{\pi}). \tag{3.5}$$

As a particular case, P is presented now as $P_{\boldsymbol{\pi}} \subset \mathbb{R}^m$ and the RCVaR for fixed $\boldsymbol{x} \in X$ with respect to $P_{\boldsymbol{\pi}}$ is defined as

$$\mathrm{RCVaR}_\alpha(\boldsymbol{x}) := \sup_{\tilde{\boldsymbol{\pi}}\in P_{\boldsymbol{\pi}}} \mathrm{CVaR}_\alpha(\boldsymbol{x},\tilde{\boldsymbol{\pi}}) \tag{3.6}$$

or, in our case

$$\mathrm{RCVaR}_\alpha(\boldsymbol{x}) := \sup_{\tilde{\boldsymbol{\pi}}\in P_{\boldsymbol{\pi}}} \min_{\zeta\in\mathbb{R}} M_\alpha(\boldsymbol{x},\zeta,\tilde{\boldsymbol{\pi}}). \tag{3.7}$$

So far, we developed the theoretical part of the RCVaR model. Now, we need the computational aspects of RCVaR minimization over $\boldsymbol{x} \in X$ and with X being the convex set of portfolio weight vectors. Let us consider the following objective function and problem:

$$\begin{aligned}&\text{minimize}_{\zeta,\boldsymbol{x},\boldsymbol{\pi}} \quad \zeta + \frac{1}{1-\alpha}\sum_{i=1}^{m} \pi_i[f(\boldsymbol{x},\boldsymbol{\xi}_i)-\zeta]^+\\ &\text{subject to all constraints.}\end{aligned} \tag{3.8}$$

Additionally, a lower bound for returns in the model can be implemented in order to satisfy an investor's risk attitude; however, this extension will be used in later parts of our research. In Equation (3.8), the term $[f(\boldsymbol{x},\boldsymbol{\xi}_i)-\zeta]^+$ should be expressed in a simplified manner within our optimization problem. Hence, we use a vector of $\boldsymbol{z}$, *height variables* beyond the positive part terms in Equation (3.8), and case–wise evaluate the $[f(\boldsymbol{x},\boldsymbol{\xi}_i)-\zeta]^+$–term. As we say, we minimize the hight variables in the epigraphs of the aforementioned positive–part functions. So,

$$\begin{aligned}&z_i \geq f(\boldsymbol{x},\boldsymbol{\xi}_i)-\zeta,\\ &z_i \geq 0 \quad (i=1,2,\ldots,m).\end{aligned} \tag{3.9}$$

We insert this new type of variables to the $\min_{\zeta,\boldsymbol{x}}$–kind of problem formulation with the discretized goal function; then we obtain the problem representation

$$\begin{aligned}&\text{minimize}_{\zeta,\boldsymbol{x},\boldsymbol{\pi},\boldsymbol{z}} \quad \zeta + \frac{1}{(1-\alpha)}\sum_{i=1}^{m} \pi_i z_i\\ &\text{subject to}\\ &z_i + \boldsymbol{x}^T\boldsymbol{\xi}_i + \zeta \geq 0,\\ &z_i \geq 0 \quad (i=1,2,\ldots,m),\\ &\boldsymbol{x} \in X.\end{aligned} \tag{3.10}$$

By using this model, we could implement all possible, but piecewise–linear convex optimization terms into linear constraints, continuing with a linear objective function also [42]. However, before finishing this determination, we need to reformulate the objective function to a tractable one for computational purposes. Above, we are involved with Stochastic Programming since CVaR is based upon it. Since our main aim is a Robustification of CVaR, we are going to gradually imply RO rules after this point. Hence, according to the steps above, the RCVaR minimization is equivalent to

$$\begin{aligned}&\text{minimize}_{\zeta,\boldsymbol{x},\Lambda,p(\cdot)}\ \Lambda\\&\text{subject to}\\&\zeta+\frac{1}{(1-\alpha)}\int_{\boldsymbol{\xi}\in\mathbb{R}^n}[f(\boldsymbol{x},\boldsymbol{\xi})-\zeta]^+p(\boldsymbol{\xi})d\boldsymbol{\xi}\le\Lambda\quad\forall p(\cdot)\in P,\\&\text{and all the other constraints.}\end{aligned}\tag{3.11}$$

Again to numerally overcome the difficulty of integration, we immediately approximate the integral by a Riemann–kind of sum. Then, we have the constraint

$$\begin{aligned}&\text{minimize}_{\zeta,\boldsymbol{x},\Lambda,\boldsymbol{z}}\ \Lambda\\&\text{subject to}\\&\zeta+\frac{1}{(1-\alpha)}\sum_{i=1}^{m}\tilde{\pi}_i z_i\le\Lambda\quad\forall\tilde{\boldsymbol{\pi}}\in P_\pi,\\&\text{and all the other constraints.}\end{aligned}\tag{3.12}$$

At last we arrive are the following equivalent LP model [42, 51]:

$$\begin{aligned}&\text{minimize}_{(\zeta,\boldsymbol{x},\boldsymbol{z},\Lambda,\boldsymbol{\pi})\in\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}\times\mathbb{R}^m}\ \Lambda\\&\text{subject to}\\&\zeta+\frac{1}{(1-\alpha)}(\tilde{\boldsymbol{\pi}}_{ij})^T\boldsymbol{z}_j\le\Lambda\quad\forall\tilde{\boldsymbol{\pi}}\in P_\pi,\\&z_i+\boldsymbol{x}^T\boldsymbol{\xi}_i+\zeta\ge0\quad(i=1,2,\ldots,m),\\&\boldsymbol{z}\ge\boldsymbol{0},\\&\boldsymbol{x}\in X.\end{aligned}\tag{3.13}$$

In fact, according to Theorem 2 of [51], if $P_{\boldsymbol{\pi}}\subset\mathbb{R}^m$ is a compact convex set, then for each $\boldsymbol{x}$, we obtain the following result:

Theorem 3.1 (Theorem 2 from [51])**.** *The closed form RCVaR model is:*

$$\text{RCVaR}_\alpha(\boldsymbol{x})=\min_{\zeta\in\mathbb{R}}\max_{\tilde{\boldsymbol{\pi}}\in P_{\boldsymbol{\pi}}}M_\alpha(\boldsymbol{x},\zeta,\tilde{\boldsymbol{\pi}}).$$

By Theorem 3.1, our RCVaR minimization model turns into the subsequent form:

$$
\begin{aligned}
&\text{minimize}_{(\zeta,\boldsymbol{x},\boldsymbol{z},\Lambda)\in\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}} \quad \Lambda \\
&\text{subject to} \\
&\max_{\tilde{\boldsymbol{\pi}}\in P_{\boldsymbol{\pi}}} \quad \zeta + \frac{1}{(1-\alpha)}(\tilde{\boldsymbol{\pi}})^T\boldsymbol{z}_j \leq \Lambda, \\
&z_i + \boldsymbol{x}^T\boldsymbol{\xi}_j + \zeta \geq 0 \quad (i = 1, 2, \ldots, m), \\
&\boldsymbol{z} \geq \mathbf{0}, \\
&\boldsymbol{x} \in X,
\end{aligned}
\tag{3.14}
$$

from where we obtain the problem representation of Equation (3.13) equivalently.

Furthermore, the decision maker could add a new constraint as mentioned in Subsection 3.1. This additional constraint may denote the minimum required return amount for the model:

$$\boldsymbol{\mu}_\xi^T\boldsymbol{x} \geq R,$$

where $\boldsymbol{\mu}_\xi$ stands for the vector of mean–asset returns and R denotes the targeted expected portfolio return (reward or return). According to [43], the optimal portfolio weights ($\boldsymbol{x}$) from CVaR model are the same as the weights derived from MVA if the returns of the portfolio are normal distributed and $\alpha \geq 0.5$. In our research, this constraint could be used in the nominal model, however, in our robust model, it is not always necessary to use. In the next section, we will reflect about this question in further detail, addressing also cases of need for such an additional constraint and, then, its robustification within the context of our RO approach.

Here, either addressing original objective function or, equivalently, minimizing a height variable of that function is a question of usefulness and practicability in the given context. Finally, this minimization problem's form is a matter of focus to some aspects, such as objective function versus constraints and optimality versus feasibility.

We recall that hight variables are often used in order to overcome non–differentiability, such as as resulting from max–type functions, e.g., our positive–part functions.

Any focus on the representation of all the constraints allows for an overall problem representation and programming in a matrix–vector form, i.e., by arrays that permit deeper structural investigations in terms of sub–matrices, stability, and parametric control.

3.2 Robust Portfolio Optimization with Robust CVaR under Parallelpipe Uncertainty

The multiple steps presented in the last section prepared us a general form for an organized program of our RO problem. However, when the optimization modeling is an issue, vectors and the matrices are to be included into our model of the problem. Furthermore, we are going to imply the given but uncertain data at the place of our samples now, and we shall do this in the multi–valued way of intervals, according to each coordinate of the random return vector $\boldsymbol{\xi}$. In the uniform kind of linear constraints, we shall address all elements of those intervals. Hence, we will present and discuss the LP model and the Robust LP model of our research using a matrix–vector form in this section.

Herewith, we changed our equation style. In this section, since we explained the WC-VaR above, we use a discretized version of CVaR. We do not need all possible convex optimization terms as we did in Equations (3.13) and (3.14), as we shall discuss. To make it clear, the representation of CVaR in a discretized form as an LP model under various constraints is as follows:

$$\begin{aligned}&\text{minimize} \quad \zeta + \frac{1}{(1-\alpha)} \sum_{i=1}^{m} [f(\boldsymbol{x}, \boldsymbol{\xi}_i) - \zeta]^{+} p(\boldsymbol{\xi}_i),\\ &\text{subject to all constraints,}\end{aligned} \tag{3.15}$$

where m is number of periods (data). However, this type of model is not convenient for the purpose of this study, so, by expanding the problem dimension, Equation (3.15) is reduced to LP form again by using positive parts $v^{+} = \max\{v, 0\}$ and $z_j \geq f(\boldsymbol{x}, \boldsymbol{\xi}_j) - \zeta$. For Standard Form purposes we write $\boldsymbol{\zeta} = (\zeta_1, \zeta_2)^T$ as a new decision vector in our forthcoming models; here, we will substitute ζ by $(\zeta_1 - \zeta_2)$ with $\zeta_1, \zeta_2 \geq 0$. Consequently, the discretized version of $\min_{\boldsymbol{x} \in X}$ (CVaR) is equivalently expressed by

$$\begin{aligned}&\text{minimize}_{(\boldsymbol{\zeta}, \boldsymbol{x}, \boldsymbol{z}) \in \mathbb{R}^2 \times \mathbb{R}^n \times \mathbb{R}^m} \quad \zeta_1 - \zeta_2 + \frac{1}{(1-\alpha) \cdot m} \sum_{i=1}^{m} \boldsymbol{z}_i\\ &\text{subject to}\\ &\boldsymbol{z}_i \geq f(\boldsymbol{x}, \boldsymbol{\xi}_i) - (\zeta_1 - \zeta_2) \quad (i = 1, 2, \ldots, m),\\ &\sum_{j=1}^{n} x_j = 1,\\ &\boldsymbol{x} \geq \boldsymbol{0}, \quad \boldsymbol{z} \geq \boldsymbol{0}, \quad \boldsymbol{\zeta} \geq \boldsymbol{0}.\end{aligned} \tag{3.16}$$

In this work, our move $p(\xi_i) = \pi_i$, from Stochastic Programming to a handy RO will be complete when we concentrate on particular values for each of the discrete probabilities whose sum amounts to one. Instead, we reflect uncertainty in the form of the random vectors $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots, \boldsymbol{\xi}_m$ and their coordinate–wise treatment in the form of intervals. During our research, we confine us to the case of equal (uniform) weights $p(\xi_i) = \pi_i = 1/m$, but we underline that: (i) our approach works likewise well with any other (i.e., non-uniform) discrete distribution, too, and (ii) we could also permit generalized set–valued, e.g., interval–valued discrete probabilities, but we do not prefer this additional modeling complexity.

Let us note that the function $-f$, which we will canonically have at the right–hand side of the first constraint, signifies the reward or (total) return. Most of the other studies, especially in the tradition of Markowitz's MVA, added an expected return lower bound (or target return value) constraint into their models. At the end of Section 3.1, we mentioned and discussed about it. In our research, since we are following an RO approach, we do not necessitate that constraint. Our algorithm works with min–max operators which means we are maximizing our ζ (as a discussion of our original objective functions in Equation (2.23) and (2.24) shows, together with an immediate case study), while we are minimizing our height variables $\boldsymbol{z}$, herewith, in tendency, pushing–up the total return in the second constraint of Equation (3.13). Hence, we are in no need to use an additional target expected return constraint, unless our results lead to unsatisfactory and too small total returns. In such a case, we would implement the additional constraint –in a robustified way, however. Actually, we will replace the mean returns of assets by interval–valued counterparts, implying uncertainty.

Now, we can present the model matrices and vectors of Equation (3.16) in the general LP form of Equation (2.23) as follows [1]:

$$\boldsymbol{A} := \left[\begin{array}{cccc|cccc|cc} \xi_1^1 & \xi_2^1 & \ldots & \xi_n^1 & 1 & 0 & \ldots & 0 & 1 & -1 \\ \xi_1^2 & \xi_2^2 & \ldots & \xi_n^2 & 0 & 1 & \ldots & 0 & 1 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \xi_1^m & \xi_2^m & \ldots & \xi_n^m & 0 & 0 & \ldots & 1 & 1 & -1 \\ 1 & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 & 0 & 0 \end{array}\right],$$

$$\boldsymbol{x} := \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ z_1 \\ z_2 \\ \vdots \\ z_m \\ \zeta_1 \\ \zeta_2 \end{bmatrix},$$

[1] For the sake of convenience and better reading, we do not include the various dimensions at the matrix and vector notations explicitly.

$$\boldsymbol{c} := \begin{bmatrix} 0 & \dots & 0 & \dfrac{1}{m \cdot (1-\alpha)} & \dots & \dfrac{1}{m \cdot (1-\alpha)} & 1 & -1 \end{bmatrix}^T,$$

$$\boldsymbol{b} := \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

In the Standard Form of an LP model, as presented in Equation (2.24), the matrix representation of Equation (3.16) turns to become

$$\begin{aligned}
&\text{minimize}_{(\boldsymbol{\zeta},\boldsymbol{x},\boldsymbol{z},\boldsymbol{e}) \in \mathbb{R}^2 \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m} \quad \zeta_1 - \zeta_2 + \frac{1}{(1-\alpha) \cdot m} \sum_{i=1}^{m} z_j \\
&\text{subject to} \\
&z_1 + \xi_1^1 x_1 + \xi_2^1 x_2 + \ldots + \xi_n^1 x_n - (\zeta_1 - \zeta_2) - e_1 = 0, \\
&z_2 + \xi_1^2 x_1 + \xi_2^2 x_2 + \ldots + \xi_n^2 x_n - (\zeta_1 - \zeta_2) - e_2 = 0, \\
&\vdots \\
&z_m + \xi_1^m x_1 + \xi_2^m x_2 + \ldots + \xi_n^m x_n - (\zeta_1 - \zeta_2) - e_m = 0, \\
&x_1 + x_2 + x_3 + \ldots + x_n = 1, \\
&\boldsymbol{x} \geq \boldsymbol{0}, \quad \boldsymbol{z} \geq \boldsymbol{0}, \quad \boldsymbol{e} \geq \boldsymbol{0}, \quad \boldsymbol{\zeta} \geq \boldsymbol{0}.
\end{aligned} \tag{3.17}$$

To obtain the form of Equation (2.24), we implied a vector of surplus variables ($\boldsymbol{e}$) to relevant constraints in Equation (3.16). This vector $\boldsymbol{e}$, such to say, subtracts the artificial amount of portfolio return to satisfy the equality component–wise in our model. The matrix–vector representation of this Standard Form model will be given terms of $\boldsymbol{A}$, $\boldsymbol{c}$, $\boldsymbol{b}$, and $\boldsymbol{x}$. First we will give their updated definitions. Then, we will present the interval–, in fact, parallelpipe– valued of the set of matrices $\boldsymbol{A}$. Now the coefficient matrix of the model in Equation (3.17) is found to be

$$\boldsymbol{A} := \left[\begin{array}{cccc|cccc|cc|cccc}
\xi_1^1 & \xi_2^1 & \dots & \xi_n^1 & 1 & 0 & \dots & 0 & 1 & -1 & -1 & 0 & \dots & 0 \\
\xi_1^2 & \xi_2^2 & \dots & \xi_n^2 & 0 & 1 & \dots & 0 & 1 & -1 & 0 & -1 & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\xi_1^m & \xi_2^m & \dots & \xi_n^m & 0 & 0 & \dots & 1 & 1 & -1 & 0 & \dots & 0 & -1 \\
1 & 1 & \dots & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0
\end{array}\right],$$

where ξ_i^j $(i = 1, \ldots, m;\ j = 1, \ldots, n)$ represents the random variables, namely, the percentage returns as introduced in Equation (2.5).

The right–hand side of Equation (3.17) contains the vector $\boldsymbol{b}$ which is defined as:

$$\boldsymbol{b} := \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix};$$

in our context, the vector of decision variables matrix $\boldsymbol{x}$ (portfolio weights and various auxiliary variables) is:

$$\boldsymbol{x} := \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ z_1 \\ z_2 \\ \vdots \\ z_m \\ \zeta_1 \\ \zeta_2 \\ e_1 \\ e_2 \\ \vdots \\ e_m \end{bmatrix},$$

and the vector $\boldsymbol{c}$ is

$$\boldsymbol{c} := \begin{bmatrix} 0 & \dots & 0 & \frac{1}{m \cdot (1-\alpha)} & \dots & \frac{1}{m \cdot (1-\alpha)} & 1 & -1 & 0 & \dots & 0 \end{bmatrix}^T.$$

Let us underline that this modeling is applicable for any set of our dimensions, especially, for any given number of risky assets. Here, optimization with respect to $\boldsymbol{x}$ and in view of the matrices and vectors introduced above should provide the solution which must satisfy all constraints by considering the data–based (return) uncertainty in the coefficient matrix $\boldsymbol{A}$. This type of optimization problem, i.e., where uncertain entries, here: the returns (random variables $\boldsymbol{\xi}$) are expressed through uncertainty sets, here: by intervals, and where we aim at an optimal solution, is called the *Robust Counterpart* of the original LP problem of CVaR (which revealed certain parameters only). Robust counterparts represent a worst–case situation. More specifically, RO addresses some optimization problem in a parametric worst–case consideration. Of course, this worst–case situation in our problem will be specified in constraints and in further sections it will be closely evaluated.

Those matrices and vectors, i.e., those "arrays", are backbones of this study. From now on, the entry contents of the coefficient matrix $\boldsymbol{A}$ will be widened through products of uncertainty intervals from a set $\mathcal{U}$. This *immunizes* our problem against parameter uncertainty, in fact, against underlying data uncertainty in the prices, eventually.

To obtain a matrix $\boldsymbol{A}$ with uncertainty intervals, Equation (2.5) from Definition 2.5 will be employed. This equation is rigid if we only address for nominal returns. In this work's context, the return formula should be denoted with Parallelpipe Uncertainty based on intervals. The return interval determination formula –now to be translated for Parallelpipe Uncertainty– is given in Equation (2.26). For the required parallelpipe setting of our returns with intervals, we shall employ the following definition, herewith generalizing Equation (2.5) where, for some non–negativity reasons, we address a return defined by the ratio of price at end time and price at beginning time.

Definition 3.1. Suppose we have an asset which has prices as intervals. In that sense, our return formula is based on

$$\frac{[a,b]}{[c,d]} = \left[\frac{a}{d}, \frac{b}{c}\right] \quad (b \geq a \geq 0;\ d \geq c > 0). \tag{3.18}$$

In our study, where a, b, c and d will be in the role of lower and upper bounds of asset prices, respectively. In this research, our goal is obtaining return intervals by using price intervals. These price intervals are calculated by Equation (2.26). Hence, we have

$$\left[\xi^l, \xi^u\right] := \frac{\left[p_T^l, p_T^u\right]}{\left[p_0^l, p_0^u\right]} - [1,1] = \left[\frac{p_T^l}{p_0^u} - 1, \frac{p_T^u}{p_0^l} - 1\right], \tag{3.19}$$

where ξ^l and ξ^u are lower and upper bounds of a return interval, respectively. Here l and u represent lower and upper bounds indexes of a price (or a return). The nominator refers to the end and the denominator stands for the beginning of a time interval or time subinterval.

For detailing the introduction of percentage return from Subsection 2.1.3, Equation (2.5), within our present interval–valued setting, each return in some entry of the matrix $\boldsymbol{A}$ is generalized, in fact, "randomized", with Equation (3.19). There is a fundamental issue here on how to obtain the new (perturbed, uncertain) matrix $\breve{\boldsymbol{A}}$. Since our matrix has 0, 1 and -1 values related to the various (also auxiliary) decision variables of the decision vector and the constraints of the model, we have to consider and represent them for the needed calculations. In fact, there are two certain options during the application of the model: either we treat all entries of $\breve{\boldsymbol{A}}$ as intervals, even if some entries are numbers (0, 1 or -1), i.e., degenerate intervals, or we could consider just the intervals which have positive lengths (i.e., which are not scalars), encountered for uncertainty representation by (non–degenerate) intervals. For practicability purposes, we consider just the intervals in the matrix $\breve{\boldsymbol{A}}$, in order to find the number of all vertex points by the combinations between all beginning and all end points of such intervals. According to this preference, the new matrix $\breve{\boldsymbol{A}}$ which located in the uncertainty set $\mathcal{U}$ is

$$
\breve{\boldsymbol{A}} \in \left[\begin{array}{ccc|cccc|cc|cccc}
\left[\xi_1^{l,1}, \xi_1^{u,1}\right] & \dots & \left[\xi_n^{l,1}, \xi_n^{u,1}\right] & 1 & 0 & \dots & 0 & 1 & -1 & -1 & 0 & \dots & 0 \\
\left[\xi_1^{l,2}, \xi_1^{u,2}\right] & \dots & \left[\xi_2^{l,2}, \xi_2^{u,2}\right] & 0 & 1 & \dots & 0 & 1 & -1 & 0 & -1 & \dots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\left[\xi_1^{l,m}, \xi_1^{u,m}\right] & \dots & \left[\xi_n^{l,m}, \xi_n^{u,m}\right] & 0 & 0 & \dots & 1 & 1 & -1 & 0 & \dots & 0 & -1 \\
1 & \dots & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0
\end{array} \right],
$$

i.e., $\breve{\boldsymbol{A}} \in \mathcal{U}$ and

$$
\mathcal{U} := \text{conv}(\breve{\boldsymbol{A}}^1, \breve{\boldsymbol{A}}^2, \dots, \breve{\boldsymbol{A}}^{2^{m \cdot n}}) \tag{3.20}
$$

is the convex hull of the canonical *vertices* $\breve{\boldsymbol{A}}^l$, where $l = 1, 2, \dots, 2^{m \cdot n}$. The matrix $\breve{\boldsymbol{A}}$ is situated in $\mathbb{R}^{M \times N}$, where $M = (m + 1)$ and $N = (n + 2m + 2)$. A set which is the convex hull of finitely many points is called a *polytope*, as a special case of a polyhedron (which might be unbounded). The convex hull $\mathcal{U}$ is an $m \cdot n$–dimensional (non–degenerate) polytope, placed in a higher dimensional Euclidean space. That higher dimension is $M \cdot N$. For closer information about the related convex analysis we refer to [41].

In fact, $\mathcal{U}$ is a polytope given by all convex combinations of its vertices $\breve{\boldsymbol{A}}^1, \breve{\boldsymbol{A}}^2, \dots, \breve{\boldsymbol{A}}^{2^{m \cdot n}}$. The elements of an uncertainty set $\mathcal{U}$ are founded by the Cartesian product of the uncertainty intervals, and also by the degenerate intervals which consist of scalar entries, namely, 0, 1 and -1; the products of uncertainty intervals are our parallelpipes. Herewith, these uncertainty sets turn out to be straight parallelpipes; they are lower dimensional because of the scalar entries.

To understand this calculation and geometrical shape more clearly, the Cartesian products of the coefficient matrix with uncertain contents is represented in the Figure 3.1. For the sake of simplicity, we do not include dimensions (entries) here where the intervals are degenerate.

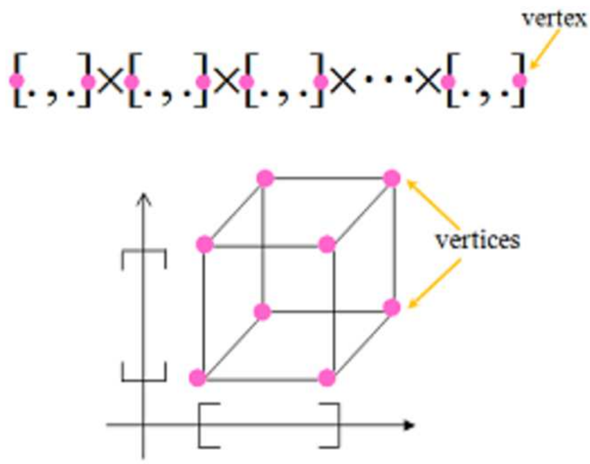


Figure 3.1: Parallelpipe Uncertainty Set: Cartesian product of intervals for 3 entries [32].

This figure is representing the Cartesian product of intervals or, equivalently, the convex hull of vertices as an element ($\breve{\boldsymbol{A}}$) of $\mathcal{U}$ in a simplified manner. Here, any element matrix $\breve{\boldsymbol{A}}$ is a vector with uncertainty; these elements altogether generate a parallelpipe and $\mathcal{U}$ is a polytope with maximum $2^{m \cdot n}$ vertices, since we have $m \cdot n$ (non–degenerate) interval–valued entries and take into account all combinatorial cases given by the boundary points of the intervals. The matrix $\breve{\boldsymbol{A}}$ can be represented as a vector with uncertainty that, on the other hand, is represented through a parallelpipe C. By taking into account all these parallelpipes we have a special type of uncertainty set named *Parallelpipe Uncertainty* set. At the first view, this new uncertainty set looks like a box, however, in Box Uncertainty sets, lengths of intervals are same for every row of the uncertainty matrix $\breve{\boldsymbol{A}}$. In a Parallelpipe Uncertainty set, the lengths of intervals may vary among each other, e.g., along the columns and the rows of the regarded uncertainty matrix $\breve{\boldsymbol{A}}$.

The coefficient matrices represented as $\breve{\boldsymbol{A}}$ generates parallelpipe C. Let C be a parallelpipe that encompasses entries of our uncertain returns. Then, C is

$$C = [\xi_1^{l,1}, \xi_1^{u,1}] \times [\xi_2^{l,1}, \xi_2^{u,1}] \times \ldots \times [\xi_n^{l,m}, \xi_n^{u,m}] \times \ldots =: \prod_{k=1}^{m \cdot n} C_k, \tag{3.21}$$

where $\xi^l \leq \xi \leq \xi^u$. Here, ξ^l is the lower bound and ξ^u stands for the upper bound of the intervals in the $m \cdot n$ return–value dimensions (non–degenerate uncertainty intervals). We recall that there are degenerate intervals like $[a, a] = \{a\}$ inserted into the set–valued coefficient matrix. They take 0, 1 and -1 values, but since they are single–valued and their upper and lower limits are the same, we may say that for representing and programming the set C, these trivial or degenerate intervals of single constant values mean no complexity and no coding problem.

CHAPTER 4

APPLICATION AND STATISTICAL COMPARISON OF RCVaR

As a numerical part of this study, we used RCVaR based on historical financial data to obtain an optimal portfolio allocation. By this way, we could illustrate the implementation of RCVaR to historical financial data. Here, we strongly refer to our discussions of earlier sections to optimize a portfolio by using RCVaR and satisfy a certain minimum portfolio return.

4.1 Data

Three historical price datasets are chosen for this numerical application. We created a portfolio which contains three different financial assets. The components of our portfolio are Intel, Aaon and Microsoft monthly stock prices from NASDAQ stock exchange. We used the historical monthly price data of these financial instruments from March 2000, to September 2016. Graphical representations of these datasets are shown in Figure 4.1.

Figure 4.1: Historical monthly price data of Intel, Aaon and Microsoft from NASDAQ stock market.

At the beginning of the numerical application, all price datasets are turned into a percentage return series by Equation (2.5). It is one of the contributions of our study, because mostly researches are eventually done with return calculation based on using logarithmic returns.

Many of the investigations take into particular account the return distribution. One of the fundamental properties of CVaR is that with Gaussian shaped data it gives same results when compared to VaR and MVA, as we mentioned before. Nevertheless, CVaR also works on non–normal datasets either; in fact, here it is not important whether our data are normal or not[1]. Because of this reason, the Q–Q plot of portfolio assets' data are shown in Figure 4.2 as an example.

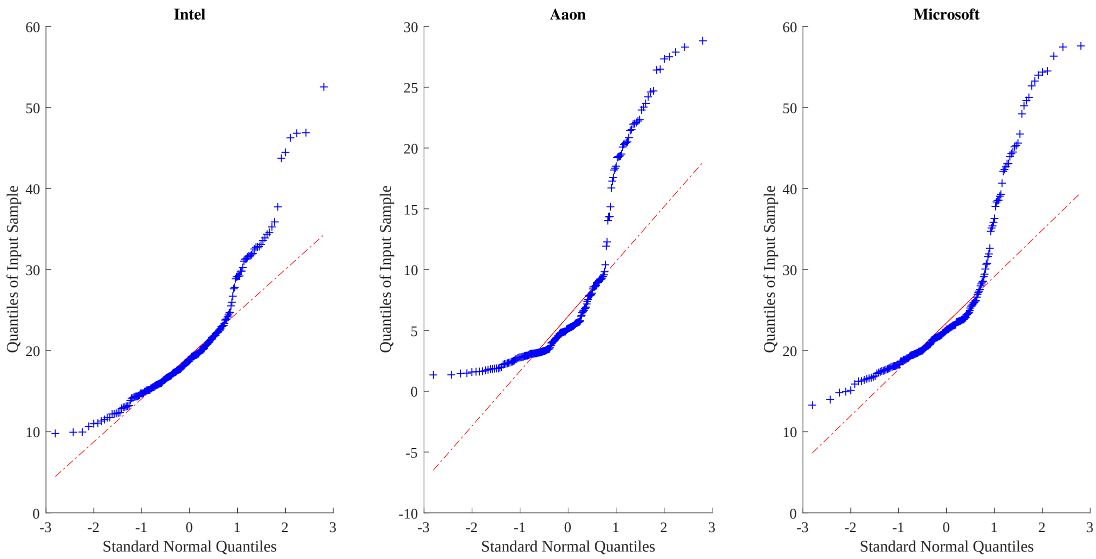


Figure 4.2: Normal Q–Q plot of portfolio assets.

Lastly, descriptive statistics of our asset price datasets are presented in Table 4.1. Besides the regular statistical results like maximum, minimum, etc., skewness, kurtosis, and JB test show important description about our data. The skewness illustrates symmetry of the data around mean. As a consequence, the data are not normally distributed since we have a positive skewness. Additionally, the data distribution has positive excess kurtosis (leptokurtic); in fact, the kurtosis values of Asset 1 and 3 are higher than the one of Asset 2. Therefore, our data have fat tails, which means that an investor will be risk averse. Another evidence for non–normally distributed data is given by JB–test as shown in Table 4.1. According to p–values, null hypothesis (H_0) should be rejected, since those values are lower than the significance level.

[1] According to the data distribution analysis on assets, our datasets reveal Burr (4P), Loglogistic and Burr distributions, respectively.

Table 4.1: Descriptive statistics of portfolio asset prices.

	Intel (Asset 1)	Aaon (Asset 2)	Microsoft (Asset 3)
Max	52.5396	28.8200	57.6000
Min	9.8004	1.3455	13.2876
Mean	20.7719	8.1515	25.9097
Median	18.8469	5.1746	22.6234
Std	7.6186	7.2733	10.0486
Variance	58.0426	52.9011	100.9752
Kurtosis	5.6145	3.5911	4.5977
Skewness	1.5109	1.3692	1.5718
CoV	0.3667	0.8922	0.3878
JB test (p–value)	2.2e-16	7.327e-15	2.2e-16

4.2 Algorithm

As a beginning, we used nominal CVaR algorithm to obtain asset allocations and risk levels of our portfolio. This application is conducted by using Equation (3.16). In this sense, we only employed CVaR algorithm on MATLAB and obtained our results.

For RCVaR application, our robust algorithm is founded on some basic LP algorithm; an Interior Point Method is chosen in this research. All optimization and CVaR codes are conducted in MATLAB Software according to the problem representation as given in Equations (3.16, 3.17); furthermore, both the nominal and the new robust optimization model are performed also. In this numerical part, we set $\alpha = 0.99$ to obtain a maximum conservatism.

To apply RCVaR, i.e., our RO technique upon CVaR optimization model, first, we created uncertainty intervals based on our price data by Equation (2.26). Second, we transformed the price intervals to the return data in the form of using Equation (3.19). We included these uncertainty intervals into the real–world data–based matrix $\boldsymbol{A}$ in each dimension and corresponding entry; then the uncertainty matrices with *Parallelpipe Uncertainty* are constructed. Consequently, we included those prepared matrices into the problem representation of Equation (3.16). The part or "block" of coefficient matrix $\breve{\boldsymbol{A}}$ with uncertainty entries shows the following form:

$$\begin{aligned} \breve{\boldsymbol{A}}_1 &:= \begin{bmatrix} \breve{a}_1^1 & \breve{a}_2^1 & \breve{a}_3^1 \\ \breve{a}_1^2 & \breve{a}_2^2 & \breve{a}_3^2 \\ \vdots & \vdots & \vdots \\ \breve{a}_1^{199} & \breve{a}_2^{199} & \breve{a}_3^{199} \end{bmatrix} \\ &\in \begin{bmatrix} [0.018, 1.318] & [-0.011, 0.069] & [-1.203, 0] \\ [0.008, 0.948] & [-0.012, 0.067] & [-0.228, 0.059] \\ \vdots & \vdots & \vdots \\ [-0.028, 0.914] & [0.002, 0.082] & [-0.001, 0.034] \end{bmatrix}. \end{aligned} \tag{4.1}$$

Naturally, the matrix $\breve{\boldsymbol{A}}_1$, in order to reach the full size of matrix $\breve{\boldsymbol{A}}$, should attain more row(s) and, especially, more columns. Herewith, the auxiliary variables of our linear optimization problem in Standard Form narrows down the degree of freedom as far as it came from a very small number n of risky assets. Of course, not all of the auxiliary decision variables and related columns would be needed for that purpose. Eventually, our whole example comfortably works with portfolios whose number n of risky assets is larger than three. We choose a number 3 of assets because of ease of notation and understanding.

Remark 4.1. If Parallelpipe Uncertainty sets are used, then there can be a computational drawback in the numerical experiment. The number of vertices might be too large and to handle them computationally causes a high complexity. Additionally, the matrix $\breve{\boldsymbol{A}}$ has a very big dimension in our numerical practice, and our computer capacity is not enough for such a size of the coefficient matrix. Hence, we employed a *Weak Parallelpipe Robustification* to solve that practical problem. Weak robustification means an entry–wise robustification with respect to the matrix $\breve{\boldsymbol{A}}$. This finite robustification process goes row by row, and it represents all the other data according to interval midpoints, as shown in Figure 3.1 (*ceteris paribus*). Eventually, our "weak" version of RO approach addresses the worst– (robust) case with respect to all entry–wise robustifications [32].

Remark 4.2. One major assumption during the calculation of returns series from prices series is that first period returns for each asset have zero value. The logic behind the assumption is that an investor does not have a return when he/she invested in a stock on the market. Then, the investor only invests money.

One criticism might come up to minds could be the curse of dimensionality or over–determination problem. The usage of medium amount data could be seen as huge amount however in finance, since there are tremendous uncertainty at everywhere, more data are needed than in some more classical engineering applications, even when compared to the number of decision variables

4.3 Nominal CVaR Application

This application provided a numerical result for portfolio optimization. The considered model is here the objective given in Equation (2.9) and all its constraints. Additionally, we employed constraint on minimum required return amount which is presented in Sections 3.1 and 3.2 ($R = 0.0062$). As we discussed in those Sections, such a constraint can be inserted into the model and its given constraints to improve the entire return and risk level.

Two different confidence levels are implied into the CVaR model. First, since $\alpha = 0.95$ is considered as a standard value, we employed it for a first experiment. Second, we optimized our portfolio regarding the conservation level $\alpha = 0.99$. During the RO process, our research is interested in worst–case situations. The second confidence–level value is employed for various difficult situations when the decision should be more conservative then a usually before.

Table 4.2: Nominal portfolio weights and CVaR results.

	Intel (Asset 1)	**Aaon (Asset 2)**	**Microsoft (Asset 3)**	**CVaR**
α=0.95	0.0840	0.4760	0.4400	0.1217
α=0.99	0.1991	0.5030	0.2979	0.1717

According to the nominal portfolio optimization results in Table 4.2, the decision vector put particular weight on Asset 2 (Aaon) and Asset 3 (Microsoft). Furthermore, the confidence interval changed the amount of risk (objective value) which our portfolio faced and the distribution of our asset allocation (the decision vector).

4.4 RCVaR Application

Since our main aim is robust portfolio allocation, we used our optimization model on the real–time data. Herewith, by using Equation (3.17), we obtained our robust solutions.

Table 4.3: Robust portfolio weights and CVaR results.

	Intel (Asset 1)	**Aaon (Asset 2)**	**Microsoft (Asset 3)**	**CVaR**
Nominal	0.1991	0.5030	0.2979	0.1717
Robust	0,1482	0,4650	0,3868	0,1683

From the results in Table 4.3, while our portfolio risk minimized as we aimed, asset weights are allocated differently than nominal case.

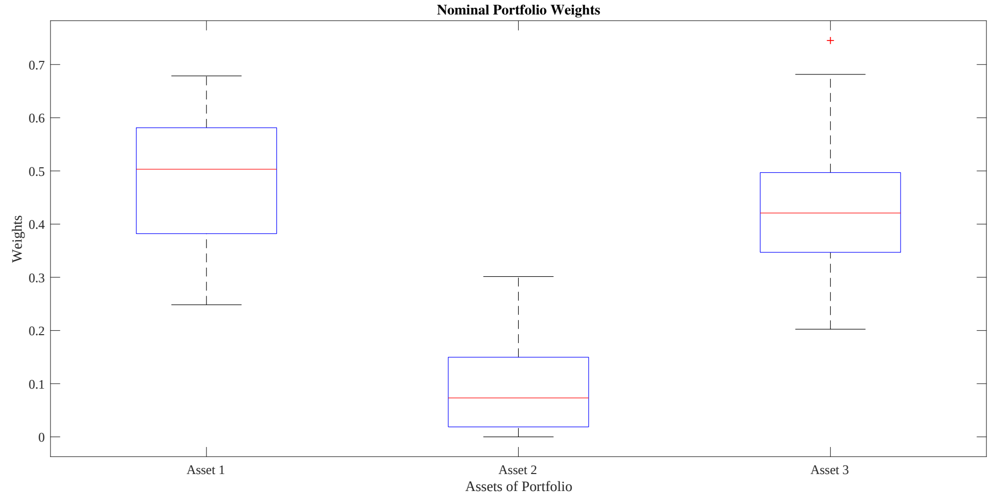
4.5 Stability Measuring by Monte–Carlo (MC) Simulation

After we achieved nominal and RO–supported portfolio optimization by using CVaR and RCVaR, in this section, we compared CVaR and RCVaR algorithms using different price datasets generated by MC simulation based on our real data's descriptive statistics. The comparison based on statistical properties (variance) of obtained results, i.e., risk values. Moreover, we generated 3 different asset prices for 199 months between March 2000, and September 2016. We generated our random asset prices under *uniform distribution*.

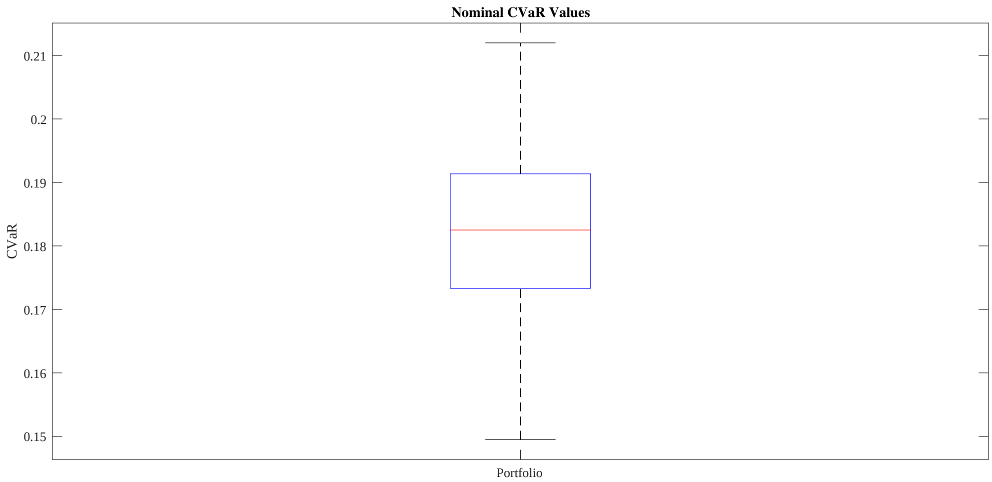
MC simulation provides uncertain model scenarios and allows to use them for our purpose. Since the perturbed prices by MC simulation are generated under specific probability distribution, they include uncertainty. This situation permits us to use our RO technique on new data sets.

In this context, 199 *uniformly distributed asset prices* were generated under 40 different price scenarios. We made a portfolio optimization with a Robust Portfolio Optimization to each price scenario. Optimal asset weights and CVaR results for each scenario are presented in Figure 4.3, respectively. The algorithm of this scenario application is based upon [50].

These results explain us the variations of weights among similar portfolio prices and their returns. Here, Robust Optimization aims to reduce these variations to obtain a more robust objective result.



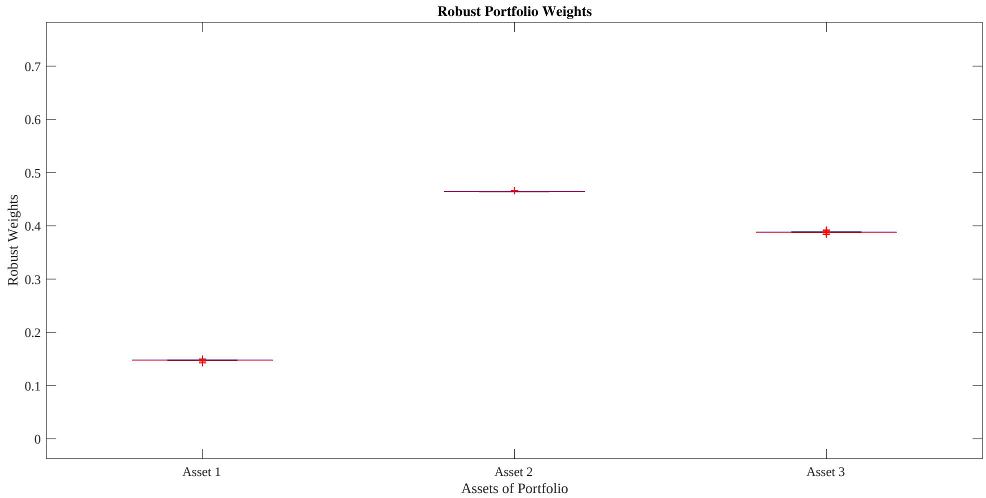
(a) Variety of portfolio weights per simulation.



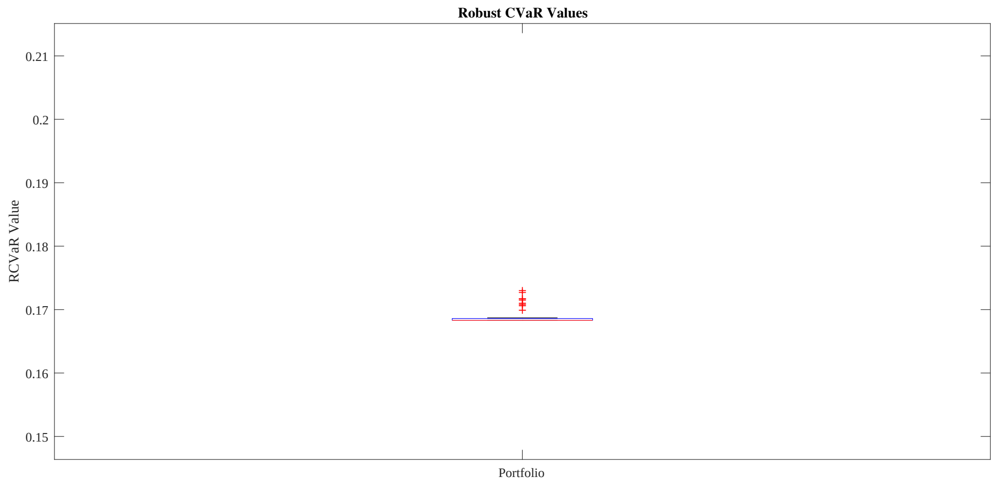
(b) Variety of CVaR values per simulation.

Figure 4.3: Nominal simulation results.

For our simulation study of RCVaR, we obtained 40 different interval values under Parallelpipe Uncertainty. According to those intervals, 40 different uncertainty scenarios were generated. Hereby, the weights of all the generated uniformly distributed portfolios are shown in Figure 4.4a. Furthermore, the RCVaR results are displayed in Figure 4.4b. All these calculations and optimization codes have been constructed by MATLAB Software.



(a) Variety of robust portfolio weights per simulation.



(b) Variety of RCVaR values per simulation.

Figure 4.4: Robust simulation results.

The model in Equation (3.17) aimed to reduce the risk in the portfolio and to obtained a robust portfolio allocation. From Figure 4.4b, we may claim that while our portfolio risk is minimizing, asset weights of the portfolio converged to their robust values based on our Parallelpipe Uncertainty set.

From Figures 4.3 and Figures 4.4 we observed that the variability of the portfolio weights has decreased. Moreover, sample of 40 nominal and robust simulated portfolio optimization results are given in Table 4.4.

Table 4.4: Sample of simulation results for nominal and robust models.

Portfolio	Nominal Weights			CVaR	Robust Weights			RCVaR
1	0.5554	0.1034	0.3411	0.1494	0.1479	0.4643	0.3878	0.1683
2	0.3691	0	0.6308	0.1885	0.1475	0.4645	0.3881	0.1683
3	0.5509	0.0097	0.4392	0.1988	0.1482	0.4650	0.3868	0.1683
4	0.4948	0.0096	0.4954	0.1890	0.1482	0.4650	0.3868	0.1683
5	0.5052	0.0266	0.4681	0.1876	0.1480	0.4644	0.3876	0.1715
6	0.5935	0	0.4064	0.1768	0.1477	0.4645	0.3878	0.1706
7	0.4170	0.1443	0.4386	0.1742	0.1477	0.4643	0.3879	0.1699
8	0.2681	0.3015	0.4304	0.1928	0.1476	0.4645	0.3879	0.1687
9	0.2484	0.0701	0.6815	0.1967	0.1477	0.4644	0.3879	0.1685
10	0.4341	0.0562	0.5097	0.2045	0.1477	0.4643	0.3879	0.1730

For more clear visualizations, nominal and robust portfolio optimization weights according to 40 simulations are presented in Figure 4.5, related to our numerical results, respectively.

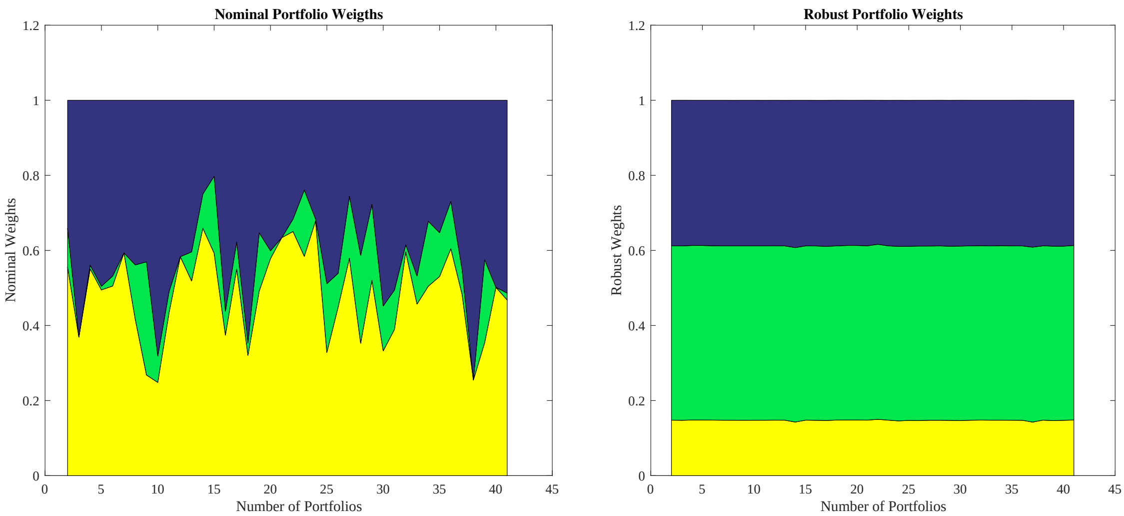


Figure 4.5: Area plots of nominal and robust portfolio weights.

One can understand that the portfolio optimization with RCVaR under Parallelpipe Uncertainty provides a stable portfolio asset allocation and reduced risk level. This result is a good explanation for a risk management and portfolio optimization without an initial wealth assumption as we discussed before.

CHAPTER 5

CONCLUSION AND FUTURE WORK

The trade–off between maximizing expected return and minimizing the risk under uncertainty is great challenge to the decision making process for all quantitative investors. In financial and energy markets, deciding on an investment is difficult without sufficient and proper information. Uncertainty in the given information as given, i.e., by the data, affects the amount, variety, risk, and return on investments. In this research, we prepared and conducted a robust decision making model algorithm and methodology applied on given real–time data. Our aim was to find the robust portfolio optimization results (the selected quantities of assets) by using a well–developed mathematical approach.

Our mathematical tools are RO, CVaR, and Parallelpipe Uncertainty sets. By considering the robust optimization framework at the literature, we invented RCVaR methodology under Parallelpipe Uncertainty sets. In that respect, by taking into account the amount of uncertainty in the real–world data from a stock market, we created uncertainty intervals with perturbation terms. These uncertainty intervals are employed to obtain Parallelpipe Uncertainty sets, and we implemented them into the model or design matrix of the optimization problem of CVaR. We aimed to minimize the CVaR value of our portfolio, and at the same time, we calculated meaningful value of portfolio weights. We used our new RCVaR methodology on real–data. Moreover, for comparison, we generated new portfolios by MC simulation on those data for using CVaR and compared them with the portfolio given the same number of different RCVaR scenarios. Both the variety of portfolio weights and the risk values were compared among two approaches.

As a result, our RCVaR methodology under Parallelpipe Uncertainty provided us more stable portfolio weights and reduced our portfolio risk. By considering Parallelpipe Uncertainty, we traced back our robustified LP problem to an ordinary LP problem, exploiting the linearity of the robust program and the interval foundation of its uncertainty set. We canonically referred to the vertex points of the uncertainty set; then, we obtained a lower risk level in simulation and real–world applications. Our advances on Robust Portfolio Optimization illustrated that during the optimization processes, if the data uncertainty is addressed carefully, robust optimal results could be achieved. Improvements in the asset allocation reduced the risk level of the portfolio. In a more clear way, RCVaR methodology illustrated us the importance of robust portfolio allocation.

For our future work on this new model and methodology, robust portfolio optimization in multi–periods with given initial and, then, gradually rebalanced and reallocated wealth (rather than equating the budget constraint to one) could become beneficial for the practice of risk management and for the optimization literature. Since there might be possible correlations between random variables, as an extension, Correlated Parallelpipe Uncertainty set could be possible to taking into account. Additionally, time consistency is a rising topic in the optimization field of mathematics. Herewith, time consistent robustly optimized portfolios should be an interesting research topic in the near future.

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