SECOND-ORDER SCALAR-TENSOR FIELD THEORIES

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ABSTRACT

SECOND-ORDER SCALAR-TENSOR FIELD THEORIES

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We review Horndeski's scalar-tensor theory in this thesis. Partial differential equations that are satisfied by the Lagrangian limit its most general form. Demanding second-order field equations both for the metric and the scalar field, and choosing a four-dimensional spacetime also put restrictions on the most general form of the Lagrangian. Besides, by using similar techniques, in a four-dimensional spacetime, we find the most general form of the second-order Euler-Lagrange equations that are obtained from the Lagrangian through a variation of the metric. Finally, making use of relations between field equations and the Lagrangian, the most general form of the Lagrangian is obtained. Thus, one establishes the most general scalar-tensor theory in a four-dimensional spacetime.

Keywords: General Relativity, Modified Theories of Gravity, Scalar-Tensor Theory

ÖZ

İKİNCİ DERECEDEN SKALER-TENSÖR ALAN KURAMLARI

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Bu tezde Horndeski'nin skaler-tensör kuramı incelenmiştir. Lagrangian tarafından sağlanan kısmı diferansiyel denklemler onun en genel halini kısıtlar. Metrik ve skaler alan denklemlerinin ikinci dereceden olmasını istemek ve dört boyutlu bir uzayzamanı seçmek de Lagrangian'ın en genel halini kısıtlar. Ayrıca benzer teknikleri kullanarak, dört boyutlu bir uzayzamanda Lagrangian'ın metrik varyasyonundan elde edilen ikinci dereceden Euler-Lagrange denklemlerinin en genel halini buluruz. Son olarak alan denklemleri ve Lagrangian arasındaki ilişkilerden faydalanarak Lagrangian'ın en genel hali elde edilir. Böylece, dört boyutlu uzayzamanda en genel skaler-tensör kuramı elde edilmiş olur.

Anahtar Kelimeler: Genel Görelilik, Değiştirilmiş Kütleçekim Kuramları, Skaler-Tensör Kuramı

To the beautiful lady with whom we look at the same sky

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LIST OF ABBREVIATIONS

GR	General Relativity
FLRW	Friedmann–Lemaître–Robertson–Walker
LIGO	Laser Interferometer Gravitational-Wave Observatory

NOTATION:

Every index belongs to n dimensional spacetime and runs from 1 to n. There is no special meaning of any index. Einstein's summation convention , e.g. $u_\mu v^\mu = \sum_{\mu=1}^{n} u_\mu v^\mu$, is used throughout the text.

A comma denotes partial differentiation and a vertical bar denotes covariant differentiation.

The Levi-Civita symbol ϵ^{abcd} is defined as the permutation symbol which takes the values 0, 1 and -1.

Generalized Kronecker delta symbol is

$$\delta^{i_1\cdots i_k}_{j_1\cdots j_k} = \det \begin{pmatrix} \delta^{i_1}_{j_1} & \cdots & \delta^{i_k}_{j_1} \\ \vdots & \ddots & \vdots \\ \delta^{i_1}_{j_k} & \cdots & \delta^{i_k}_{j_k} \end{pmatrix}.$$

The determinant of the matrix formed by the components of the metric tensor is $g \equiv |\det(g_{ab})|$.

The Christoffel symbols are

$$\Gamma^a{}_{bc} = \frac{1}{2} g^{ha}(g_{bh,c} + g_{ch,b} - g_{bc,h}).$$

Components of the Riemann curvature tensor are

$$R^a{}_{bcd} = \Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d} + \Gamma^a{}_{ch}\Gamma^h{}_{bd} - \Gamma^a{}_{dh}\Gamma^h{}_{bc}$$

The Ricci tensor, the scalar curvature and the Einstein tensor are

$$R_{ab} \equiv R^h{}_{ahb}, \quad R \equiv R^h{}_h \quad \text{and} \quad G_{ab} \equiv R_{ab} - \frac{1}{2} g_{ab} R$$

respectively.

CHAPTER 1

INTRODUCTION

Newton's theory of gravity was remarkably successful in explaining closed elliptical orbits of planets around the Sun. This theory is still accurate enough if one uses it in astronomy and astrophysics, where the effect of the gravity is not strong enough. However, Newton's theory could not fully explain the precession of the perihelion of the planet Mercury. Due to Mercury's orbit's deviation from the theory, another planet named Vulcan is hypothesized by Urbain Le Verrier in the 19th century. However, such a planet has never been observed. Therefore, it was assumed that Newton's theory was incomplete. Actually, Newton was the first to consider modified theories to explain Lunar precession by including an inverse-cube term in his inverse-square law of the gravitational force.

After the theory of special relativity proposed by Einstein in 1905, he aimed generalizing his theory with the gravitation theory. Instead of the idea of action at a distance in Newton's theory, he considered gravitation as a curvature of spacetime. In 1915, Einstein published the geometric theory of gravitation, General Relativity (GR).

GR is considered as the best explanation of how spacetime behaves on macroscopic scales. One of the early successes of the theory was the correct prediction of the advance in the perihelion of Mercury. As a consequence of the theory, original ideas and their solutions are emerged. The idea of black holes, gravitational deflection of light, gravitational redshift and Shapiro delay are all predictions of GR.

The field equations found by Einstein have passed many experimental tests. These equations are

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \,, \tag{1.1}$$

where $G_{\mu\nu}$ is the Einstein tensor, $T_{\mu\nu}$ is the energy–momentum tensor, G is Newton's gravitational constant and c is the speed of light. These equations have been successful in explaining many phenomena which could not be explained otherwise. For instance, gravitational waves are natural consequence of Einstein's field equations. For the first time, the existence of gravitational waves was indirectly verified by measuring the period decay of binary neutron stars in 1974 [1]. On September 14, 2015, the first direct observation of gravitational waves was made by the LIGO team which will certainly lead to a new way of observing the universe [2].

Despite its successes, as in Newton's theory, GR has shortcomings in explaining some of the observed phenomena. Although it passes experiments performed in solar system scales, it has shortcomings on larger scales such as cosmological and galactic scales. Therefore, gravitation may differ in such scales. For example, if one tries to understand the galaxy rotation curve, which remains flat far away from the galaxy center, by using GR, significant amount of dark matter is needed. In other words, in a galactic scale limit, GR reduces to Newton's theory of gravity. As it happens in the Solar System, the theory suggests a decrease in the velocities of the orbits when radial distance from the center of mass increases. However, observations have shown that outer stars of a galaxy move faster than expected. Due to this fact, dark matter is hypothesized, which may be an explanation of these observations. Similarly, the accelerating expansion of the universe can only be explained in GR with the help of dark energy which seems to dominate the distribution of ordinary (baryonic) matter and energy in the universe. This need for the existence of considerable amount of dark energy and dark matter in the universe leads to speculations that GR may not be the complete theory of gravitation.

Although GR has had great success especially in the solar system experiments, alternative theories emerged before observational tests were made. Soon after its first publication in 1915, there were a flurry of theories in order to find a more unified version. Eddington's theory of connections [3], Weyl's scale independent theory [4], and the higher dimensional theories of Kaluza and Klein [5], [6] are some examples that can be presented. Eddington's studies influenced Dirac and thus, he discussed the idea of varying Newton's constant in time [7]. Later, this idea motivated Brans and Dicke [8]. In addition to the metric, in the Brans-Dicke theory, a scalar field ϕ is introduced which can be viewed as the varying gravitational constant G. They developed the prototypical form of scalar-tensor theories of gravity, which are still actively studied in the literature [9].

GR, together with a cosmological constant term in its field equations, is a unique second-order metric theory in a four-dimensional spacetime. The most general form of the action is

$$S = \int \left[\frac{1}{2\kappa}(R - 2\Lambda) + \mathcal{L}_M \right] \sqrt{g} d^4x,$$

where $\kappa = \frac{8\pi G}{c^4}$, Λ is a constant and $\mathcal{L}_M$ describes any matter fields. The field equations that are obtained from this action through a variation of the metric are

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \,. \tag{1.2}$$

Note that these equations are at most of second-order in the derivatives of the metric. As we mentioned, due to its constant cosmological term, GR has shortcomings in explaining different phases of the universe's expansion. As a consequence of this fact, one may make use of an additional degree of freedom in the Lagrangian such as a scalar field. Similarly, one may also consider higher dimensions than four in order to have different field equations rather than (1.2) which may explain the cosmic expansion.

In the seventies, Lovelock and Horndeski studied scalar-tensor theories in a more mathematical way to obtain the most general forms of both the Lagrangian and the Euler-Lagrange equations [10, 11, 12]. They applied and improved the methods developed by Rund on variational problems involving combined tensor fields [13].

Lovelock had shown that GR is the unique metric field theory of four-dimensional spacetime [14]. Moreover, Lovelock obtained the most general form of a metric theory in any dimension [15]. The Lovelock densities are

$$L_{(h/2)} = \frac{1}{2^h} \delta^{A_1 A_2 \cdots A_h}_{B_1 B_2 \cdots B_h} R_{A_1 A_2}{}^{B_1 B_2} \cdots R_{A_{h-1} A_h}{}^{B_{h-1} B_h}$$

and the Lovelock Lagrangian is

$$L = \sum_{h=0}^{k} c_h L_{(h)}$$

where $k = [(D-1)/2]$ and D is the dimension of spacetime. Here, $L_{(1)}$ is the Einstein-Hilbert term and $L_{(2)}$ is the Gauss-Bonnet term. However, for the case $h = D$, the Lovelock density becomes topological. Due to the Gauss-Bonnet theorem, the Einstein-Hilbert term is a topological invariant when $D = 2$. Therefore, the Einstein tensor is zero in a two-dimensional spacetime. Similarly, when $D = 4$, the variation of the Gauss-Bonnet term vanishes which keeps the Einstein field equations unique.

Due to this uniqueness, if we limit ourselves to four-dimensional spacetime, we have to consider some additional fields. A scalar could be the simplest way of adding extra degrees of freedom. Theories involving a scalar field together with a tensor field are called as scalar-tensor theories. The most general form of these theories in four-dimensional spacetime was constructed by Horndeski [16].

This study can be considered as a review of works done by Rund, Lovelock, and Horndeski on scalar-tensor field theories. In this thesis, we are seeking for the most general form of the Lagrangian which yields the most general form of the second-order scalar-tensor field equations in a four-dimensional spacetime.

In Chapter 2, we begin with choosing our Lagrangian of the form

$$L(g_{ij}, g_{ij,k}, g_{ij,kh}, \phi, \phi_{,i}). \tag{1.3}$$

Since the derivatives higher than second order create instabilities that lead to a theory with ghosts, we would like to obtain the second-order field equations from this Lagrangian. Therefore, the Lagrangian should be also at most of second-order. Moreover, for simplicity, we do not let our Lagrangian to have dependency on the second-order derivative of the scalar. By doing so, we can not construct the most general form of the field equations which is done in Chapter 3. Starting from the Lagrangian given in (1.3), we then work out the symmetries that are required for generating generalized tensor densities. These symmetric tensor densities are related to the partial derivatives of the Lagrangian with respect to the metric, the scalar and their derivatives. In addition to these symmetries, we obtain some important identities which put restrictions on these tensor densities. These identities are partial differential equations which are satisfied by the Lagrangian. Moreover, due to symmetries of these tensor densities, having a four-dimensional spacetime also restricts the general form of the Lagrangian. By virtue of these equations, relevant symmetries and dimensional restrictions, one can construct the most general form of the Lagrangian. To study the dynamics, we need to obtain the second-order field equations and to do so we derive the Euler-Lagrange equations corresponding to variations of the Lagrangian with respect to the metric and scalar field. Then we find the necessary conditions which guarantee that the field equations are functions of the metric, a scalar and their first two derivatives. After imposing these conditions on the Lagrangian the most general form of it is generated.

In Chapter 3, we look for the Lagrangian which has the Euler-Lagrange equations involving the metric, a scalar and their first two derivatives. Since there is no restriction in the beginning,

$$L = L(g_{ij}, g_{ij,i_1}, \ldots, g_{ij,i_1 \ldots i_p}, \phi, \phi_{,i_1}, \ldots, \phi_{,i_1 \ldots i_q}) \tag{1.4}$$

is the Lagrangian in this case. Approaching this problem in reverse order proves to be useful. In order to generate the most general form of the Lagrangian, we start by constructing the most general form of the Euler-Lagrange equations. Since the field equations are tensor densities, we may obtain new tensor densities out of field equations by taking their derivatives.

Generalized Bianchi identities for this Lagrangian defined as

$$E^{ij}{}_{|j}(L) = \frac{1}{2} g^{ij} \phi_{|j} E(L),$$

where E^{ij} and $E(L)$ are Euler-Lagrange equations through a variation of the Lagrangian with respect to the metric and the scalar, respectively. Having at most of second-order $E(L)$ means also having at most of second-order $E^{ij}{}_{|j}(L)$ at the same time. Obviously, this fact puts severe restrictions on the most general form of the field equations. Again by considering dimensional restrictions, we generate the most general form of the field equations obtained from the Lagrangian given in (1.4). Afterwards, by using the relations between the Lagrangian and its field equations, we will obtain the desired Lagrangian in a four-dimensional spacetime.

The implications of the results that we have obtained in chapters — will be discussed in Chapter 4. We will briefly examine the applicability of the methods that we have used to other field theories such as the bi-scalar-tensor theory and we will give a short summary of recent research on scalar-tensor theories.

In Appendix A, Riemann normal coordinates and their properties are explained. These coordinates are frequently used throughout this thesis. General forms of various tensor densities that possess certain symmetries are derived in Appendix B. Detailed calculations are relegated to Appendix C in order to provide fluency in reading.

CHAPTER 2

SCALAR-TENSOR FIELD THEORIES

In this chapter, our goal is to generate the most general Lagrangian of the form $L(g_{ij}, g_{ij,k}, g_{ij,kh}, \phi, \phi_{,i})$ which yields second-order field equations in a four dimensional spacetime. Therefore, we want to obtain

$$\begin{aligned} E^{ab} &= E^{ab}(g_{ij}, g_{ij,k}, g_{ij,kh}, \phi, \phi_{,i}, \phi_{,ij}), \\ E &= E(g_{ij}, g_{ij,k}, g_{ij,kh}, \phi, \phi_{,i}, \phi_{,ij}). \end{aligned}$$

In this chapter, we consider the first derivative of the scalar field ϕ only. By doing so, we study an easier but specific example of general scalar-tensor field theories. However, while we are studying this example, we will be developing some general methods which will be useful in the next chapter. Moreover, these general methods can be applied to various field theories.

To begin with, we are looking for an action invariant under arbitrary coordinate transformations of the form

$$\overline{x}^i = \overline{x}^i(x^\mu),$$

where the action is

$$I = \int L(g_{ij}, g_{ij,k}, g_{ij,kh}, \phi, \phi_{,i}) d^n x.$$

Therefore, we study the Lagrangians of the form $L(g_{ij}, g_{ij,k}, g_{ij,kh}, \phi, \phi_{,i})$ which satisfies

$$BL(g_{\mu\nu}, g_{\mu\nu,\rho}, g_{\mu\nu,\rho\sigma}, \phi, \phi_{,\mu}) = \overline{L}(\overline{g}_{ij}, \overline{g}_{ij,k}, \overline{g}_{ij,kh}, \overline{\phi}, \overline{\phi}_{,i}), \tag{2.1}$$

where g_{ij} is the metric, $g_{ij,k}$ is the first derivative of the metric, $g_{ij,kh}$ is the second derivative of the metric and

$$B \equiv \det\left(\frac{\partial x^\mu}{\partial \overline{x}^i}\right) > 0,$$

is the Jacobian. Since our Lagrangian is a scalar density, we may call it as *Lagrange scalar density*.

We can define the transformation matrix and its derivatives, which will be useful throughout the chapter, as

$$\begin{aligned} B^{\mu}{}_{i} &= \frac{\partial x^{\mu}}{\partial \overline{x}^{j}} \\ B^{\mu}{}_{ij} &= \frac{\partial B^{\mu}{}_{i}}{\partial \overline{x}^{j}} = \frac{\partial^2 x^{\mu}}{\partial \overline{x}^{j} \partial \overline{x}^{i}}, \\ B^{\mu}{}_{ijk} &= \frac{\partial^2 B^{\mu}{}_{i}}{\partial \overline{x}^{k} \partial \overline{x}^{j}} = \frac{\partial^3 x^{\mu}}{\partial \overline{x}^{k} \partial \overline{x}^{j} \partial \overline{x}^{i}}. \end{aligned}$$

We calculate the transformation of the scalar ϕ, the metric, and their derivatives as

$$\begin{aligned} \overline{\phi} =& \phi, \\ \overline{\phi}_{,i} =& B^{\mu}{}_{i} \phi_{,\mu}, \\ \overline{\phi}_{,ij} =& \phi_{,\mu\nu} B^{\mu}{}_{i} B^{\nu}{}_{j} + \phi_{,\mu} B^{\mu}{}_{ij}, \\ \overline{g}_{ij} =& g_{\mu\nu} B^{\mu}{}_{i} B^{\nu}{}_{j}, \\ \overline{g}_{ij,k} =& g_{\mu\nu,\rho} B^{\mu}{}_{i} B^{\nu}{}_{j} B^{\rho}{}_{k} + g_{\mu\nu} B^{\mu}{}_{ik} B^{\nu}{}_{j} + g_{\mu\nu} B^{\mu}{}_{i} B^{\nu}{}_{jk}, \\ \overline{g}_{ij,kh} =& g_{\mu\nu,\rho\sigma} B^{\mu}{}_{i} B^{\nu}{}_{j} B^{\rho}{}_{k} B^{\sigma}{}_{h} + g_{\mu\nu,\rho} \big(B^{\mu}{}_{ih} B^{\nu}{}_{j} B^{\rho}{}_{k} + B^{\mu}{}_{i} B^{\nu}{}_{jh} B^{\rho}{}_{k} \\ & + B^{\mu}{}_{i} B^{\nu}{}_{j} B^{\rho}{}_{kh} + B^{\mu}{}_{ik} B^{\nu}{}_{j} B^{\rho}{}_{h} + B^{\mu}{}_{i} B^{\nu}{}_{jk} B^{\rho}{}_{h} \big) \\ & + g_{\mu\nu} \big(B^{\mu}{}_{ikh} B^{\nu}{}_{j} + B^{\mu}{}_{i} B^{\nu}{}_{jkh} + B^{\mu}{}_{ik} B^{\nu}{}_{jh} + B^{\mu}{}_{ih} B^{\nu}{}_{jk} \big). \end{aligned} \tag{2.2}$$

Before going further, we define the following partial derivatives which will frequently appear

$$\begin{aligned} \Lambda^{ij} &\equiv \frac{\partial L}{\partial g_{ij}}, \\ \Lambda^{ij,k} &\equiv \frac{\partial L}{\partial g_{ij,k}}, \\ \Lambda^{ij,kh} &\equiv \frac{\partial L}{\partial g_{ij,kh}}, \\ \Phi &\equiv \frac{\partial L}{\partial \phi}, \\ \Phi^{i} &\equiv \frac{\partial L}{\partial \phi_{,i}}. \end{aligned}$$

We emphasize that the first three terms are symmetric in (i, j) and that the third term is also symmetric in (k, h). In the next section, we will show that $\Lambda^{ij,kh}$, Φ, and Φ^{i} are tensor densities that are of the fourth, zeroth, and the first rank, respectively.

2.1 The Construction of Certain Symmetric Tensor Densities

In this section, we will construct symmetric tensor densities. In order to do this, we will apply similar techniques developed originally by Rund [13]. Applying these methods reveals symmetries of tensor densities. Moreover, we can put some restrictions on tensor densities to obtain their most general form.

In Section 2.1.1, we will show that $\Lambda^{ij,kh}$ is a tensor density. In Section 2.1.2, we will obtain tensor densities Π^{ij} and $\Pi^{ij,k}$ which are very useful for obtaining the Euler-Lagrange equations in clear tensorial form. Then we will obtain important invariance identities for $\Lambda^{ij,kh}$, $\Pi^{ij,k}$ and Π^{ij} and in Section 2.1.3 to put restrictions on the Lagrangian L. These important identities can be obtained by taking the derivative of (2.1) with respect to $B^{\mu}{}_{abc}$, $B^{\mu}{}_{ab}$ and $B^{r}{}_{s}$, respectively.

This method can be used for different Lagrange scalar densities. There are various examples of application of this method on diverse Lagrange scalar densities which are conducted by Rund [13].

2.1.1 Proving that $\Lambda^{ij,kh}$s a Tensor Density

Taking the derivative of (2.1) with respect to $g_{\mu\nu,\rho\sigma}$ yields

$$\begin{aligned} B\frac{\partial L}{\partial g_{\mu\nu,\rho\sigma}} =& \frac{\partial \overline{L}}{\partial \overline{g}_{ij,kh}}\frac{\partial \overline{g}_{ij,kh}}{\partial g_{\mu\nu,\rho\sigma}} + \frac{\partial \overline{L}}{\partial \overline{g}_{ij,k}}\frac{\partial \overline{g}_{ij,k}}{\partial g_{\mu\nu,\rho\sigma}} + \frac{\partial \overline{L}}{\partial \overline{g}_{ij}}\frac{\partial \overline{g}_{ij}}{\partial g_{\mu\nu,\rho\sigma}} \\ &+ \frac{\partial \overline{L}}{\partial \overline{\phi}}\frac{\partial \overline{\phi}}{\partial g_{\mu\nu,\rho\sigma}} + \frac{\partial \overline{L}}{\partial \overline{\phi}_{,i}}\frac{\partial \overline{\phi}_{,i}}{\partial g_{\mu\nu,\rho\sigma}}. \end{aligned}$$

The only surviving term on the right hand side is the first one due to transformations that we have given in (2.2). Therefore, we have

$$B\Lambda^{\mu\nu,\rho\sigma} = \overline{\Lambda}^{ij,kh} B^{\mu}{}_{i} B^{\nu}{}_{j} B^{\rho}{}_{k} B^{\sigma}{}_{h}.$$

Note that $\Lambda^{\mu\nu,\rho\sigma}$ is a tensor density, as can be easily seen. Similarly, Φ^{μ} and Φ are also tensor densities. One can easily check this by taking the derivative of (2.1) with respect to $\phi_{,\mu}$ and ϕ, respectively. However, $\Lambda^{\mu\nu,\rho}$ and $\Lambda^{\mu\nu}$ are not tensor densities which can be proven by taking the derivative of equation (2.1) with respect to $g_{\mu\nu,\rho}$ and $g_{\mu\nu}$, respectively. These calculations can be found in the beginning of Section

2.1.2. Applying repeated partial differentiation of L with respect to $g_{\mu\nu,\rho\sigma}$, $\phi_{,\mu}$ and ϕ will produce a tensorial quantity.

2.1.2 Deriving the Tensor Densities Π^{ij} and $\Pi^{ij,k}$

We can differentiate equation (2.1) with respect to $g_{ab,cd}$, $g_{ab,c}$ and g_{ab} respectively. In light of (2.2), by considering nonvanishing terms, one has

$$B\Lambda^{ab,cd} = \overline{\Lambda}^{ij,kh}\frac{\partial \overline{g}_{ij,kh}}{\partial g_{ab,cd}}, \tag{2.3}$$

$$B\Lambda^{ab,c} = \overline{\Lambda}^{ij,kh}\frac{\partial \overline{g}_{ij,kh}}{\partial g_{ab,c}} + \overline{\Lambda}^{ij,k}\frac{\partial \overline{g}_{ij,k}}{\partial g_{ab,c}}, \tag{2.4}$$

$$B\Lambda^{ab} = \overline{\Lambda}^{ij,kh}\frac{\partial \overline{g}_{ij,kh}}{\partial g_{ab}} + \overline{\Lambda}^{ij,k}\frac{\partial \overline{g}_{ij,k}}{\partial g_{ab}} + \overline{\Lambda}^{ij}\frac{\partial \overline{g}_{ij}}{\partial g_{ab}}. \tag{2.5}$$

Note that only the first one is a tensor density but the rest are not. However, we can seek for tensor densities which include them.

Let us define a symmetric tensor h_{ij} which will be transforming similar to the metric tensor components as shown in (2.2). Now, we define a new quantity F which was introduced by du Plessis [17] as

$$F \equiv \Lambda^{ab,cd}h_{ab,cd} + \Lambda^{ab,c}h_{ab,c} + \Lambda^{ab}h_{ab}. \tag{2.6}$$

After multiplying each side of this equation with B, one can substitute equations (2.3), (2.4) and (2.5) into this to obtain

$$\begin{aligned} BF =& \overline{\Lambda}^{ij,kh}\left(\frac{\partial \overline{g}_{ij,kh}}{\partial g_{ab,cd}}h_{ab,cd} + \frac{\partial \overline{g}_{ij,kh}}{\partial g_{ab,c}}h_{ab,c} + \frac{\partial \overline{g}_{ij,kh}}{\partial g_{ab}}h_{ab}\right) \\ &+ \overline{\Lambda}^{ij,k}\left(\frac{\partial \overline{g}_{ij,k}}{\partial g_{ab,c}}h_{ab,c} + \frac{\partial \overline{g}_{ij,k}}{\partial g_{ab}}h_{ab}\right) + \overline{\Lambda}^{ij}\left(\frac{\partial \overline{g}_{ij}}{\partial g_{ab}}h_{ab}\right). \end{aligned}$$

Note that the quantities in the parentheses are equal to $\overline{h}_{ij,kh}$, $\overline{h}_{ij,k}$ and $\overline{h}_{ij}$, respectively. This is obvious from (2.2). Therefore, we have

$$BF = \overline{\Lambda}^{ij,kh}\overline{h}_{ij,kh} + \overline{\Lambda}^{ij,k}\overline{h}_{ij,k} + \overline{\Lambda}^{ij}\overline{h}_{ij}.$$

If we compare this with (2.6), we conclude that F is a scalar density. Therefore, we may define F as

$$F = \Lambda^{ij,kh}h_{ij|kh} + \Pi^{ij,k}h_{ij|k} + \Pi^{ij}h_{ij}. \tag{2.7}$$

It is remarkable that every term is a tensor density in this equation. Finally, we can obtain $\mathit{\Pi}^{ij,k}$ and $\mathit{\Pi}^{ij}$ after calculating $h_{ij|kh}$ and $h_{ij|k}$.

$$
\begin{aligned}
h_{ij|k} =& h_{ij,k} - \mathit{\Gamma}^a{}_{ik} h_{aj} - \mathit{\Gamma}^a{}_{jk} h_{ia}, \\
h_{ij|kh} =& h_{ij,kh} - \mathit{\Gamma}^a{}_{ik} h_{aj,h} - \mathit{\Gamma}^a{}_{jk} h_{ia,h} - \mathit{\Gamma}^a{}_{ik,h} h_{aj} - \mathit{\Gamma}^a{}_{jk,h} h_{ia} \\
& - \mathit{\Gamma}^b{}_{ih} h_{bj,k} - \mathit{\Gamma}^b{}_{jh} h_{ib,k} - \mathit{\Gamma}^b{}_{kh} h_{ij,b} + \mathit{\Gamma}^b{}_{ih} \left(\mathit{\Gamma}^c{}_{bk} h_{cj} + \mathit{\Gamma}^c{}_{jk} h_{bc}\right) \\
& + \mathit{\Gamma}^b{}_{jh} \left(\mathit{\Gamma}^c{}_{ik} h_{cb} + \mathit{\Gamma}^c{}_{bk} h_{ic}\right) + \mathit{\Gamma}^b{}_{kh} \left(\mathit{\Gamma}^c{}_{ib} h_{cj} + \mathit{\Gamma}^c{}_{jb} h_{ic}\right).
\end{aligned}
$$

If we substitute these two equations into (2.7), we obtain

$$
\begin{aligned}
F =& \mathit{\Lambda}^{ij,kh}\Bigg(h_{ij,kh} - \mathit{\Gamma}^a{}_{ik} h_{aj,h} - \mathit{\Gamma}^a{}_{jk} h_{ia,h} - \mathit{\Gamma}^a{}_{ik,h} h_{aj} - \mathit{\Gamma}^a{}_{jk,h} h_{ia} \\
& - \mathit{\Gamma}^b{}_{ih} h_{bj,k} - \mathit{\Gamma}^b{}_{jh} h_{ib,k} - \mathit{\Gamma}^b{}_{kh} h_{ij,b} + \mathit{\Gamma}^b{}_{ih} \left(\mathit{\Gamma}^c{}_{bk} h_{cj} + \mathit{\Gamma}^c{}_{jk} h_{bc}\right) \\
& + \mathit{\Gamma}^b{}_{jh} \left(\mathit{\Gamma}^c{}_{ik} h_{cb} + \mathit{\Gamma}^c{}_{bk} h_{ic}\right) + \mathit{\Gamma}^b{}_{kh} \left(\mathit{\Gamma}^c{}_{ib} h_{cj} + \mathit{\Gamma}^c{}_{jb} h_{ic}\right)\Bigg) \\
& + \mathit{\Pi}^{ij,k}\left(h_{ij,k} - \mathit{\Gamma}^a{}_{ik} h_{aj} - \mathit{\Gamma}^a{}_{jk} h_{ia}\right) + \mathit{\Pi}^{ij} h_{ij} \\
=& \mathit{\Lambda}^{ij,kh} h_{ij,kh} + \mathit{\Lambda}^{ij,kh}\big(- \mathit{\Gamma}^a{}_{ik} h_{aj,h} - \mathit{\Gamma}^a{}_{jk} h_{ia,h} - \mathit{\Gamma}^b{}_{ih} h_{bj,k} - \mathit{\Gamma}^b{}_{jh} h_{ib,k} \\
& - \mathit{\Gamma}^b{}_{kh} h_{ij,b}\big) + \mathit{\Pi}^{ij,k} h_{ij,k} + \mathit{\Lambda}^{ij,kh}\Bigg(- \mathit{\Gamma}^a{}_{ik,h} h_{aj} - \mathit{\Gamma}^a{}_{jk,h} h_{ia} \\
& + \mathit{\Gamma}^b{}_{ih} \left(\mathit{\Gamma}^c{}_{bk} h_{cj} + \mathit{\Gamma}^c{}_{jk} h_{bc}\right) + \mathit{\Gamma}^b{}_{jh} \left(\mathit{\Gamma}^c{}_{ik} h_{cb} + \mathit{\Gamma}^c{}_{bk} h_{ic}\right) \\
& + \mathit{\Gamma}^b{}_{kh} \left(\mathit{\Gamma}^c{}_{ib} h_{cj} + \mathit{\Gamma}^c{}_{jb} h_{ic}\right)\Bigg) + \mathit{\Pi}^{ij,k}\big(- \mathit{\Gamma}^a{}_{ik} h_{aj} - \mathit{\Gamma}^a{}_{jk} h_{ia}\big) + \mathit{\Pi}^{ij} h_{ij}.
\end{aligned}
$$

After renaming the indices, we should obtain

$$
F = \mathit{\Lambda}^{ij,kh} h_{ij,kh} + \mathit{\Lambda}^{ij,k} h_{ij,k} + \mathit{\Lambda}^{ij} h_{ij}.
$$

Therefore, we can write down the tensor densities that we need as

$$
\mathit{\Pi}^{ij,k} = \mathit{\Lambda}^{ij,k} + \mathit{\Gamma}^k{}_{ab} \mathit{\Lambda}^{ij,ab} + 2\mathit{\Gamma}^i{}_{ab} \mathit{\Lambda}^{aj,kb} + 2\mathit{\Gamma}^j{}_{ab} \mathit{\Lambda}^{ai,kb} \tag{2.8}
$$

and

$$
\begin{aligned}
\mathit{\Pi}^{ij} =& \mathit{\Lambda}^{ij} + \mathit{\Gamma}^i{}_{ab,c} \mathit{\Lambda}^{aj,bc} + \mathit{\Gamma}^j{}_{ab,c} \mathit{\Lambda}^{ai,bc} \\
& + \mathit{\Gamma}^i{}_{ab} \left(\mathit{\Pi}^{aj,b} - \mathit{\Gamma}^b{}_{cd} \mathit{\Lambda}^{aj,cd} - \mathit{\Gamma}^a{}_{cd} \mathit{\Lambda}^{cj,bd} - \mathit{\Gamma}^j{}_{cd} \mathit{\Lambda}^{ca,bd}\right) \\
& + \mathit{\Gamma}^j{}_{ab} \left(\mathit{\Pi}^{ai,b} - \mathit{\Gamma}^b{}_{cd} \mathit{\Lambda}^{ai,cd} - \mathit{\Gamma}^a{}_{cd} \mathit{\Lambda}^{ci,bd} - \mathit{\Gamma}^i{}_{cd} \mathit{\Lambda}^{ca,bd}\right).
\end{aligned} \tag{2.9}
$$

We remark obvious symmetries satisfied by these tensors as

$$
\mathit{\Pi}^{ij} = \mathit{\Pi}^{ji}, \qquad \mathit{\Pi}^{ij,k} = \mathit{\Pi}^{ji,k}. \tag{2.10}
$$

One can easily see that the relations, that we have in (2.8) and (2.9), are valid for the metric field theories that contain at most second-order derivative of the metric. Furthermore, these equations are still valid, even though the Lagrangian involves scalar fields and their derivatives, i.e., ϕ, $\phi_{,i_1\dots i_n}$.

2.1.3 Putting Restrictions on the Lagrangian

In this section, we will derive three identities that put restrictions on L. Therefore, we will differentiate (2.1) with respect to ${B^\mu}_{abc}$, ${B^\mu}_{ab}$ and ${B^r}_s$, respectively. We will divide this section into three parts, in order to make it easier to follow.

Part 1: Differentiation of (2.1) with respect to ${B^\mu}_{abc}$

Taking the derivative of (2.1) with respect to ${B^\mu}_{abc}$ yields

$$0 = \overline{\Lambda}^{ij,kh}\frac{\partial \overline{g}_{ij,kh}}{\partial {B^\mu}_{abc}} + \overline{\Lambda}^{ij,k}\frac{\partial \overline{g}_{ij,k}}{\partial {B^\mu}_{abc}} + \overline{\Lambda}^{ij}\frac{\partial \overline{g}_{ij}}{\partial {B^\mu}_{abc}} + \Phi\frac{\partial \phi}{\partial {B^\mu}_{abc}} + \overline{\Phi}^i\frac{\partial \overline{\phi}_{,i}}{\partial {B^\mu}_{abc}}.$$

The only non-zero term is the first one on the right hand side. Therefore, we have

$$\begin{aligned}0 = \overline{\Lambda}^{ij,kh}\frac{\partial \overline{g}_{ij,kh}}{\partial {B^\mu}_{abc}} =& \frac{1}{6}\overline{\Lambda}^{ij,kh} g_{\mu\nu}{B^\nu}_j\big(\delta^a_i\delta^b_k\delta^c_h + \delta^a_i\delta^b_h\delta^c_k + \delta^a_k\delta^b_i\delta^c_h + \delta^a_h\delta^b_k\delta^c_i + \delta^a_k\delta^b_h\delta^c_i \\ & + \delta^a_h\delta^b_i\delta^c_k\big) + \frac{1}{6}\overline{\Lambda}^{ij,kh} g_{\mu\nu}{B^\nu}_i\big(\delta^a_j\delta^b_k\delta^c_h + \delta^a_j\delta^b_h\delta^c_k + \delta^a_k\delta^b_j\delta^c_h \\ & + \delta^a_h\delta^b_k\delta^c_j + \delta^a_k\delta^b_h\delta^c_j + \delta^a_h\delta^b_j\delta^c_k\big).\end{aligned}$$

Now, with (i, j) symmetry, we have

$$\overline{\Lambda}^{ij,kh} g_{\mu\nu}{B^\nu}_j\big(\delta^a_i\delta^b_k\delta^c_h + \delta^a_i\delta^b_h\delta^c_k + \delta^a_k\delta^b_i\delta^c_h + \delta^a_h\delta^b_k\delta^c_i + \delta^a_k\delta^b_h\delta^c_i + \delta^a_h\delta^b_i\delta^c_k\big) = 0.$$

We have this equation for an arbitrary transformation. Moreover, in particular, this is also valid for the identity transformation

$$\overline{x}^i = x^i \qquad \text{with} \qquad {B^\mu}_i = \delta^\mu_i \qquad \text{and} \qquad {B^\mu}_{ij} = {B^\mu}_{ijk} = 0. \qquad (2.11)$$

As a result, we have

$$g_{\mu\nu}\big(\Lambda^{a\nu,bc} + \Lambda^{a\nu,cb} + \Lambda^{b\nu,ac} + \Lambda^{c\nu,ba} + \Lambda^{c\nu,ab} + \Lambda^{b\nu,ca}\big) = 0.$$

Using the symmetry properties that we have, we can rewrite this as

$$2g_{\mu\nu}\big(\Lambda^{\nu a,bc} + \Lambda^{\nu b,ac} + \Lambda^{\nu c,ba}\big) = 0.$$

Consequently, we obtain a remarkable identity for $\Lambda^{ij,kh}$ as

$$\Lambda^{ij,kh} + \Lambda^{ik,jh} + \Lambda^{ih,kj} = 0. \tag{2.12}$$

We emphasize that the identity above puts severe restrictions on L. Therefore, we will be frequently using this identity throughout this thesis.

By using (2.12) and known symmetry properties of $\Lambda^{ij,kh}$, we can find another symmetry for $\Lambda^{ij,kh}$:

$$\begin{aligned} \Lambda^{ij,kh} + \Lambda^{ik,jh} + \Lambda^{ih,kj} &= 0, \\ \Lambda^{ji,kh} + \Lambda^{jk,ih} + \Lambda^{jh,ik} &= 0, \\ \Lambda^{ki,jh} + \Lambda^{kj,ih} + \Lambda^{kh,ij} &= 0. \end{aligned}$$

Summing these three equations yields zero. The sum of third terms of each line is $\Lambda^{ih,kj} + \Lambda^{jh,ik} + \Lambda^{kh,ij} = 0$ from equation (2.12). Therefore, the sum of the rest is also zero.

$$2(\Lambda^{ij,kh} + \Lambda^{ki,jh} + \Lambda^{jk,ih}) = 0.$$

When we compare this result with equation (2.12), the first and the second terms of both equations are the same, respectively. As a result, the third terms in both equations will be equal to each other. Therefore, we will have a new symmetry property of $\Lambda^{ij,kh}$ as

$$\Lambda^{ij,kh} = \Lambda^{kh,ij}.$$

Consequently, $\Lambda^{ij,kh}$ enjoys the following symmetry properties

$$\Lambda^{ij,kh} = \Lambda^{ji,kh} = \Lambda^{ij,hk} = \Lambda^{kh,ij}.$$

Part 2: Differentiation of (2.1) with respect to $B^{\mu}{}_{ab}$

In order to obtain an equation which is satisfied by $\Pi^{ij,k}$, we take the derivative of the equation (2.1) with respect to $B^{\mu}{}_{ab}$ which will yield

$$0 = \overline{\Lambda}^{ij,kh}\frac{\partial \overline{g}_{ij,kh}}{\partial B^{\mu}{}_{ab}} + \overline{\Lambda}^{ij,k}\frac{\partial \overline{g}_{ij,k}}{\partial B^{\mu}{}_{ab}} + \overline{\Lambda}^{ij}\frac{\partial \overline{g}_{ij}}{\partial B^{\mu}{}_{ab}} + \overline{\Phi}\frac{\partial \phi}{\partial B^{\mu}{}_{ab}} + \overline{\Phi}^{i}\frac{\partial \overline{\phi}_{,i}}{\partial B^{\mu}{}_{ab}}. \tag{2.13}$$

The only contribution is coming from the first two terms due to (2.2)

$$\begin{aligned} 0 =& \frac{1}{2}\overline{\Lambda}^{ij,kh}\Big(g_{\mu\nu,\sigma}\big(4B^{\nu}{}_{j}B^{\sigma}{}_{k}(\delta^a_i\delta^b_h + \delta^a_h\delta^b_i)\big) + g_{\nu\sigma,\mu}\big(B^{\nu}{}_{i}B^{\sigma}{}_{j}(\delta^a_k\delta^b_h + \delta^a_h\delta^b_k)\big)\Big) \\ &+ \frac{1}{2}\overline{\Lambda}^{ij,k}g_{\mu\nu}\big(2B^{\nu}{}_{j}(\delta^a_i\delta^b_k + \delta^a_k\delta^b_i)\big) + \Omega^{ab}{}_{\mu}, \end{aligned}$$

where the $\Omega^{ab}{}_{\mu}$ term contains terms which are linear in $B^{\nu}{}_{cd}$. Due to (2.11), there will be no contribution coming from this term. As previously applied, using the identity transformation of the form in (2.11), we have

$$
\begin{aligned}
0 =&\Lambda^{ij,kh}\bigg(g_{\mu\nu,\sigma}\big(4\delta^{\nu}_{j}\delta^{\sigma}_{k}(\delta^{a}_{i}\delta^{b}_{h}+\delta^{a}_{h}\delta^{b}_{i})\big)+g_{\nu\sigma,\mu}\big(\delta^{\nu}_{i}\delta^{\sigma}_{j}(\delta^{a}_{k}\delta^{b}_{h}+\delta^{a}_{h}\delta^{b}_{k})\big)\bigg)\\
&+\Lambda^{ij,k}g_{\mu\nu}\big(2\delta^{\nu}_{j}(\delta^{a}_{i}\delta^{b}_{k}+\delta^{a}_{k}\delta^{b}_{i})\big)\\
=&4\Lambda^{a\nu,\sigma b}g_{\mu\nu,\sigma}+4\Lambda^{b\nu,\sigma a}g_{\mu\nu,\sigma}+2\Lambda^{\nu\sigma,ab}g_{\nu\sigma,\mu}+2\Lambda^{a\nu,b}g_{\mu\nu}+2\Lambda^{b\nu,a}g_{\mu\nu}.
\end{aligned}
$$

As a result, we have found

$$
2\Lambda^{a\nu,\sigma b}g_{\mu\nu,\sigma}+2\Lambda^{b\nu,\sigma a}g_{\mu\nu,\sigma}+\Lambda^{\nu\sigma,ab}g_{\nu\sigma,\mu}+\Lambda^{a\nu,b}g_{\mu\nu}+\Lambda^{b\nu,a}g_{\mu\nu}=0. \tag{2.14}
$$

As stated in Appendix A, at the pole P of the Riemann normal coordinate system, we have

$$
g_{ab}\neq 0;\qquad \text{however,}\qquad g_{ab,c}=0,\qquad \text{therefore,}\qquad \Gamma^{a}{}_{bc}=0. \tag{2.15}
$$

Therefore, at the pole P, we can rewrite equation (2.14) together with equation (2.8) as

$$
\Pi^{a\nu,b}g_{\mu\nu}+\Pi^{b\nu,a}g_{\mu\nu}=0,
$$

as a result, we have

$$
\Pi^{ij,k}+\Pi^{kj,i}=0 \qquad\text{or}\qquad \Pi^{ij,k}=-\Pi^{kj,i}. \tag{2.16}
$$

By virtue of (2.10), it is possible to write

$$
\Pi^{ij,k}=-\Pi^{kj,i}=-\Pi^{jk,i}=\Pi^{ik,j}=\Pi^{ki,j}=-\Pi^{ji,k}.
$$

Therefore, we have a remarkable equation here

$$
\Pi^{ij,k}=0. \tag{2.17}
$$

Since this is a tensorial equation, once we obtain this equation at a particular point of our coordinate system, we can generalize it to be valid at any point of an arbitrary coordinate system.

Lemma 2.1 *If we have a scalar density of the form* $L(g_{ij},g_{ij,k},g_{ij,kh},\phi,\phi_{,i})$ *and* $\Lambda^{ij,kh}=0$ *then* $\Lambda^{ij,k}=0$.

This is an obvious result of equations (2.8) and (2.17) when we have the condition $\Lambda^{ij,kh} = 0$.

Part 3: Differentiation of (2.1) with respect to $B^r{}_s$

Note that $\frac{\partial B}{\partial B^r{}_s} = BA^s{}_r$, where $A^s{}_r$ is the inverse of $B^r{}_k$, that is $A^s{}_r B^r{}_k = \delta^s_k$. Taking the derivative of (2.1) with respect to $B^r{}_s$ yields

$$BA^s{}_r L = \overline{\Lambda}^{ij,kh} \frac{\partial \overline{g}_{ij,kh}}{\partial B^r{}_s} + \overline{\Lambda}^{ij,k} \frac{\partial \overline{g}_{ij,k}}{\partial B^r{}_s} + \overline{\Lambda}^{ij} \frac{\partial \overline{g}_{ij}}{\partial B^r{}_s} + \overline{\Phi} \frac{\partial \phi}{\partial B^r{}_s} + \overline{\Phi}^i \frac{\partial \overline{\phi}_{,i}}{\partial B^r{}_s}. \tag{2.18}$$

There is no contribution to the right hand side of this equation from the fourth term when equation (2.2) is used. Therefore, we have

$$\begin{aligned} BA^s{}_r L =& \overline{\Lambda}^{ij,kh} g_{\mu\nu,\rho\sigma} \big(B^\nu{}_j B^\rho{}_k B^\sigma{}_h \delta^\mu_r \delta^s_i + B^\mu{}_i B^\rho{}_k B^\sigma{}_h \delta^\nu_r \delta^s_j \\ & + B^\mu{}_i B^\nu{}_j B^\sigma{}_h \delta^\rho_r \delta^s_k + B^\mu{}_i B^\nu{}_j B^\rho{}_k \delta^\sigma_r \delta^s_h \big) \\ & + \overline{\Lambda}^{ij,k} g_{\mu\nu,\rho} \big(B^\nu{}_j B^\rho{}_k \delta^\mu_r \delta^s_i + B^\mu{}_i B^\rho{}_k \delta^\nu_r \delta^s_j + B^\mu{}_i B^\nu{}_j \delta^\rho_r \delta^s_k \big) \\ & + \overline{\Lambda}^{ij} g_{\mu\nu} \big(B^\nu{}_j \delta^\mu_r \delta^s_i + B^\mu{}_i \delta^\nu_r \delta^s_j \big) + \overline{\Phi}^i \delta^\mu_r \delta^s_i \phi_{,\mu} + \Omega^r{}_s, \end{aligned}$$

where term $\Omega^s{}_r$ contains terms which are linear in $B^\mu{}_{ab}$ and $B^\mu{}_{abc}$. Due to equation (2.11), there will be no contribution coming from this term. As applied before from the identity transformation given in (2.11), we have

$$\begin{aligned} \delta^s_r L =& \Lambda^{sj,kh} g_{rj,kh} + \Lambda^{is,kh} g_{ir,kh} + \Lambda^{ij,sh} g_{ij,rh} + \Lambda^{ij,ks} g_{ij,kr} \\ & + \Lambda^{sj,k} g_{rj,k} + \Lambda^{is,k} g_{ir,k} + \Lambda^{ij,s} g_{ij,r} + 2\Lambda^{sj} g_{rj} + \Phi^s \phi_{,r}. \end{aligned}$$

After renaming dummy indices, we have

$$\begin{aligned} \delta^s_r L =& \Lambda^{sj,ki} g_{rj,ki} + \Lambda^{js,ki} g_{jr,ki} + \Lambda^{ij,sk} g_{ij,rk} + \Lambda^{ij,ks} g_{ij,kr} \\ & + \Lambda^{sj,k} g_{rj,k} + \Lambda^{js,k} g_{jr,k} + \Lambda^{ij,s} g_{ij,r} + 2\Lambda^{sj} g_{rj} + \Phi^s \phi_{,r}. \end{aligned}$$

Using the symmetries, we have

$$\delta^s_r L = 2\Lambda^{sj,ki} g_{rj,ki} + 2\Lambda^{ij,sk} g_{ij,rk} + 2\Lambda^{sj,k} g_{rj,k} + \Lambda^{ij,s} g_{ij,r} + 2\Lambda^{sj} g_{rj} + \Phi^s \phi_{,r}.$$

after renaming dummy indices once more, we obtain

$$\delta^s_r L = 2\Lambda^{sj,ki} \big(g_{rj,ki} + g_{ik,rj} \big) + 2\Lambda^{sj,k} g_{rj,k} + \Lambda^{ij,s} g_{ij,r} + 2\Lambda^{sj} g_{rj} + \Phi^s \phi_{,r}. \tag{2.19}$$

At the pole P of the Riemann normal coordinate system, we have

$$R_{irkj} = \frac{1}{2} \big(g_{ij,rk} + g_{rk,ij} - g_{rj,ik} - g_{ik,rj} \big).$$

Upon multiplying this with $\Lambda^{ki,sj}$, we have

$$\Lambda^{ki,sj} R_{irkj} = \frac{1}{2}\big(g_{ij,rk}\Lambda^{ki,sj} + g_{rk,ij}\Lambda^{ki,sj} - g_{rj,ik}\Lambda^{ki,sj} - g_{ik,rj}\Lambda^{ki,sj}\big).$$

If α^{ij} is a quantity which is symmetric in (i,j), and $\phi_{ijkh} = \phi_{jikh} = \phi_{ijhk}$ together with $\phi_{ijkh} + \phi_{ikjh} + \phi_{ihkj} = 0$, then

$$\alpha^{ij}\phi_{aibj} = -\frac{1}{2}\alpha^{ij}\phi_{abij}. \tag{2.20}$$

Making use of (2.20), we find

$$\begin{aligned}\Lambda^{ki,sj} R_{irkj} =& -\frac{1}{4}g_{ij,rk}\Lambda^{ij,sk} - \frac{1}{4}g_{rk,ij}\Lambda^{ij,sk} - \frac{1}{2}g_{rj,ik}\Lambda^{ki,sj} - \frac{1}{2}g_{ik,rj}\Lambda^{ki,sj}\\ =& -\frac{1}{4}g_{ij,rk}\Lambda^{ij,sk} - \frac{1}{4}g_{rk,ij}\Lambda^{ij,sk} - \frac{1}{2}g_{rk,ij}\Lambda^{ji,sk} - \frac{1}{2}g_{ij,rk}\Lambda^{ji,sk}\\ =& -\frac{3}{4}g_{ij,rk}\Lambda^{ij,sk} - \frac{3}{4}g_{rk,ij}\Lambda^{ij,sk}\\ =& -\frac{3}{4}\Lambda^{ki,sj}\big(g_{ik,rj} + g_{rj,ik}\big).\end{aligned} \tag{2.21}$$

At the pole P of the Riemann normal coordinates, we have

$$\Lambda^{rs} = \Pi^{rs} - \Gamma^{r}{}_{ij,k}\Lambda^{is,jk} - \Gamma^{s}{}_{ij,k}\Lambda^{it,jk}. \tag{2.22}$$

Again at the pole P, we find

$$\Gamma^{r}{}_{ij,k} = \frac{1}{2}g^{rh}\big(g_{ih,jk} + g_{jh,ik} - g_{ij,hk}\big).$$

Inserting this into (2.22) yields

$$\begin{aligned}\Lambda^{rs} =& \Pi^{rs} - \frac{1}{2}\Lambda^{is,jk}g^{rh}\big(g_{ih,jk} + g_{jh,ik} - g_{ij,hk}\big)\\ &- \frac{1}{2}\Lambda^{it,jk}g^{sh}\big(g_{ih,jk} + g_{jh,ik} - g_{ij,hk}\big).\end{aligned}$$

In light of (2.20), we obtain

$$\Lambda^{rs} = \Pi^{rs} - \frac{1}{4}\Lambda^{is,jk}g^{rh}\big(g_{ih,jk} - g_{jk,ih}\big) - \frac{1}{4}\Lambda^{it,jk}g^{sh}\big(g_{ih,jk} - g_{jk,ih}\big).$$

Equation (2.21) reduces this to

$$\Lambda^{rs} = \Pi^{rs} + \frac{1}{3}g^{rh}\Lambda^{ik,sj}R_{khij} + \frac{1}{3}g^{sh}\Lambda^{ik,rj}R_{khij}.$$

After multiplying this with the metric, we have

$$g_{rt}\Lambda^{rs} = g_{rt}\Pi^{rs} + \frac{1}{3}\Lambda^{ik,sj}R_{ktij} + \frac{1}{3}g^{sh}g_{rt}\Lambda^{ik,rj}R_{khij},$$

after renaming indices, we find

$$g_{jr}\Lambda^{js} = g_{jr}\Pi^{js} + \frac{1}{3}\Lambda^{ik,sj}R_{krij} + \frac{1}{3}g^{sh}g_{jr}\Lambda^{ik,jt}R_{khit}.$$

By inserting this and equation (2.21) into equation (2.19) at the pole P of the Riemann normal coordinates, we obtain

$$\begin{aligned}\delta^s_r L =& -\frac{8}{3}\Lambda^{sj,ki}R_{irkj} + 2g_{jr}\Pi^{js} + \frac{2}{3}\Lambda^{ik,sj}R_{krij} + \frac{2}{3}g^{sh}g_{jr}\Lambda^{ik,jt}R_{khit} + \Phi^s\phi_{,r} \\ =& -2\Lambda^{sj,ki}R_{irkj} + 2g_{jr}\Pi^{js} + \frac{2}{3}g^{sh}g_{jr}\Lambda^{ik,jt}R_{khit} + \Phi^s\phi_{,r}.\end{aligned}$$

Since this is a tensorial equation, this identity is valid for any point of any coordinate system. Multiplying this with g^{ru} while renaming dummy indices yields

$$\begin{aligned}g^{su}L =& -2g^{ru}\Lambda^{sj,ki}R_{irkj} + 2\Pi^{us} + \frac{2}{3}g^{sh}\Lambda^{ik,ut}R_{khit} + g^{ru}\Phi^s\phi_{,r} \\ =& -2\Lambda^{sj,ki}R_i{}^u{}_{kj} + 2\Pi^{us} + \frac{2}{3}\Lambda^{ik,uj}R_i{}^s{}_{kj} + g^{ru}\Phi^s\phi_{,r}.\end{aligned} \tag{2.23}$$

The left hand side of this equation is symmetric in (s,u), thus the right hand side should also be symmetric in (s,u). The second term on the right hand side of this equation is symmetric in (s,u). Therefore, we have

$$\begin{aligned}&-2\Lambda^{sj,ki}R_i{}^u{}_{kj} + \frac{2}{3}\Lambda^{ik,uj}R_i{}^s{}_{kj} + g^{ru}\Phi^s\phi_{,r} \\ &\quad = -2\Lambda^{uj,ki}R_i{}^s{}_{kj} + \frac{2}{3}\Lambda^{ik,sj}R_i{}^u{}_{kj} + g^{rs}\Lambda^u\phi_{,r}.\end{aligned}$$

After rearranging, we find

$$-\frac{8}{3}\Lambda^{sj,ki}R_i{}^u{}_{kj} + g^{ru}\Phi^s\phi_{,r} = -\frac{8}{3}\Lambda^{uj,ki}R_i{}^s{}_{kj} + g^{rs}\Lambda^u\phi_{,r}.$$

Consequently, we obtain

$$\Lambda^{uj,ki}R_i{}^s{}_{kj} = \Lambda^{sj,ki}R_i{}^u{}_{kj} + \frac{3}{8}\left(g^{rs}\Lambda^u\phi_{,r} - g^{ru}\Phi^s\phi_{,r}\right).$$

If we substitute this into equation (2.23), we have

$$\begin{aligned}g^{su}L =& -2\Lambda^{sj,ki}R_i{}^u{}_{kj} + 2\Pi^{us} + \frac{2}{3}\Lambda^{sj,ki}R_i{}^u{}_{kj} \\ &+ \frac{1}{4}\left(g^{rs}\Lambda^u\phi_{,r} - g^{ru}\Phi^s\phi_{,r}\right) + g^{ru}\Phi^s\phi_{,r}.\end{aligned}$$

Following the long calculations, we end up with a very important identity

$$\frac{1}{2}g^{ij}L = \Pi^{ij} - \frac{2}{3}\Lambda^{im,kh}R_h{}^j{}_{km} + \frac{3}{8}g^{jh}\Phi^i\phi_{,h} + \frac{1}{8}g^{ih}\Phi^j\phi_{,h}. \tag{2.24}$$

Finally, we have very important identities (2.12), (2.17) and (2.24). These are very useful since they put severe restrictions on the Lagrangian. We will be using these results to obtain the Euler-Lagrange equations in the next section.

2.2 Properties of the Euler-Lagrange Equations

In this section, we will obtain some identities for the Euler-Lagrange equations. These identities are useful for finding field equations.

2.2.1 Finding the Tensorial Form of E^{ij}

We can write the Euler-Lagrange equations of our Lagrangian as

$$E^{ij}(L) = \frac{\partial}{\partial x^k}\big(\Lambda^{ij,k} - \frac{\partial}{\partial x^h}\Lambda^{ij,kh}\big) - \Lambda^{ij}, \tag{2.25}$$

$$E(L) = \frac{\partial}{\partial x^i}\Phi^i - \Phi. \tag{2.26}$$

Consider a symmetric tensor h_{ij}, we can write its first and second-order partial derivatives as

$$\begin{aligned} h_{ij,k}\Lambda^{ij,k} =& \big(h_{ij}\Lambda^{ij,k}\big)_{,k} - h_{ij}\Lambda^{ij,k}{}_{,k}, \\ h_{ij,kh}\Lambda^{ij,kh} =& \big(h_{ij,k}\Lambda^{ij,kh}\big)_{,h} - h_{ij,k}\Lambda^{ij,kh}{}_{,h} \\ =& \big(h_{ij}\Lambda^{ij,kh}\big)_{,h} - \big(h_{ij}\Lambda^{ij,kh}{}_{,h}\big)_{,k} + h_{ij}\Lambda^{ij,kh}{}_{,hk}. \end{aligned}$$

From (2.6), we can write

$$\begin{aligned} F =& - h_{ij}\big(\Lambda^{ij,k}{}_{,k} - \Lambda^{ij,kh}{}_{,kh} - \Lambda^{ij}\big) \\ &+ \big(h_{ij}\Lambda^{ij,k} + h_{ij,h}\Lambda^{ij,kh} - h_{ij}\Lambda^{ij,kh}{}_{,h}\big)_{,k}. \end{aligned} \tag{2.27}$$

Similarly, by using (2.7) for the first term in the parenthesis on the right hand side, we find

$$\begin{aligned} F =& - h_{ij}\big(\Pi^{ij,k}{}_{|k} - \Lambda^{ij,kh}{}_{|kh} - \Pi^{ij}\big) \\ &+ \big(h_{ij}\Pi^{ij,k} + h_{ij|h}\Lambda^{ij,kh} - h_{ij}\Lambda^{ij,kh}{}_{|h}\big)_{|k}. \end{aligned} \tag{2.28}$$

It is clear that the term inside the second parenthesis on the right hand side is a component of a $(1,0)$ type tensor density. For a tensor density of this type, we can write

$$A^j{}_{|k} = A^j{}_{,k} + \Gamma^j{}_{hk}A^h - w\Gamma^h{}_{kh}A^j,$$

where w is the weight of the tensor density. Since $w = 1$ in our case, we can write

$$A^k{}_{|k} = A^k{}_{,k} + \Gamma^k{}_{hk}A^h - \Gamma^h{}_{kh}A^k = A^k{}_{,k}.$$

Therefore, the covariant derivative of the second parenthesis on the right hand side of (2.28) reduces to partial derivative. Before taking the derivative, we have to calculate $h_{ij}\Pi^{ij,k} + h_{ij|h}\Lambda^{ij,kh} - h_{ij}\Lambda^{ij,kh}{}_{|h}$. We find easily that

$$h_{ij|h} = h_{ij,h} - \Gamma^a{}_{ih}\, h_{aj} - \Gamma^a{}_{jh}\, h_{ia}$$

and

$$\begin{aligned}\Lambda^{ij,kh}{}_{|h} =&\Lambda^{ij,kh}{}_{,h} + \Gamma^i{}_{ah}\,\Lambda^{aj,kh} + \Gamma^j{}_{ah}\,\Lambda^{ia,kh} + \Gamma^k{}_{ah}\,\Lambda^{ij,ah}\\ &+ \Gamma^h{}_{ah}\,\Lambda^{ij,ka} - \Gamma^h{}_{ah}\,\Lambda^{ij,ka}\\ =&\Lambda^{ij,kh}{}_{,h} + \Gamma^i{}_{ah}\,\Lambda^{aj,kh} + \Gamma^j{}_{ah}\,\Lambda^{ia,kh} + \Gamma^k{}_{ah}\,\Lambda^{ij,ah}.\end{aligned}$$

Using these two equations in conjunction with (2.8), we find

$$\begin{aligned}h_{ij}\Pi^{ij,k} + h_{ij|h}\Lambda^{ij,kh} - h_{ij}\Lambda^{ij,kh}{}_{|h} =&h_{ij}\big(\Lambda^{ij,k} + \Gamma^k{}_{ab}\,\Lambda^{ij,ab} + 4\Gamma^i{}_{ab}\,\Lambda^{aj,kb}\big)\\ &- h_{ij}\big(\Lambda^{ij,kh}{}_{,h} + 2\Gamma^i{}_{ah}\,\Lambda^{aj,kh} + \Gamma^k{}_{ah}\,\Lambda^{ij,ah}\big)\\ &- \Lambda^{ij,kh}\big(h_{ij,h} - 2\Gamma^a{}_{ih}\, h_{aj}\big)\\ =&h_{ij}\Lambda^{ij,k} + h_{ij,h}\Lambda^{ij,kh} - h_{ij}\Lambda^{ij,kh}{}_{,h}.\end{aligned}$$

Now, by using the fact that the covariant derivative of the second parenthesis on the right hand side of (2.28) reduces to ordinary partial derivative and from equations (2.27) and (2.28), we can easily obtain

$$h_{ij}\bigg(E^{ij}(L) - \big(\Pi^{ij,k}{}_{|k} - \Lambda^{ij,kh}{}_{|kh} - \Pi^{ij}\big)\bigg) = 0$$

for arbitrary symmetric h_{ij}. Therefore, we have

$$E^{ij}(L) = \Pi^{ij,k}{}_{|k} - \Lambda^{ij,kh}{}_{|kh} - \Pi^{ij}$$

which is a tensorial condition as expected. If we put equations (2.17) and (2.24) into the equation above, we find

$$E^{ij}(L) = -\Lambda^{ij,kh}{}_{|kh} - \frac{1}{2}g^{ij}L - \frac{2}{3}\Lambda^{im,kh}R_h{}^j{}_{km} + \frac{3}{8}g^{jh}\Phi^i\phi_{,h} + \frac{1}{8}g^{ih}\Phi^j\phi_{,h}. \quad (2.29)$$

This is a very useful equation since we only need to calculate $\Lambda^{ij,kh}$ and Φ^i to obtain an expression for $E^{ij}(L)$.

2.2.2 A Relation Between $E^{ij}{}_{|j}(L)$ and $E(L)$

At the pole P of the Riemann normal coordinate system, due to (2.25), a covariant derivative reduces to

$$E^{ij}{}_{|j}(L) = \Lambda^{ij,k}{}_{,kj} - \Lambda^{ij,kh}{}_{,khj} - \Lambda^{ij}{}_{,j}.$$

The second term on the right hand side is zero due to the symmetry that we have in equation (2.12). Making use of (2.8) and (2.17), we have

$$\Lambda^{ij,k} = -\Gamma^{k}_{\ ab}\Lambda^{ij,ab} - 2\Gamma^{i}_{\ ab}\Lambda^{aj,kb} - 2\Gamma^{j}_{\ ab}\Lambda^{ai,kb}.$$

If we use (2.20), we find

$$\Lambda^{ij,k} = -\Gamma^{k}_{\ ab}\Lambda^{ij,ab} + \Gamma^{i}_{\ ab}\Lambda^{kj,ab} + \Gamma^{j}_{\ ab}\Lambda^{ki,ab}.$$

From this, we can easily obtain

$$\Lambda^{ij,k}{}_{,kj} = \left(\Gamma^{i}_{\ ab}\Lambda^{kj,ab}\right)_{,kj}$$

and at the pole P of the Riemann normal coordinate system, we have

$$\Lambda^{ij,k}{}_{,kj} = \Gamma^{i}_{\ ab,kj}\Lambda^{kj,ab} + 2\Gamma^{i}_{\ ab,k}\Lambda^{kj,ab}{}_{,j}, \tag{2.30}$$

where at the pole P, we have

$$\begin{aligned}\Gamma^{i}_{\ ab,k} &= \frac{1}{2}g^{ih}(g_{ah,bk} + g_{bh,ak} - g_{ab,hk}),\\ \Gamma^{i}_{\ ab,kj} &= \frac{1}{2}g^{ih}(g_{ah,bkj} + g_{bh,akj} - g_{ab,hkj}).\end{aligned}$$

Using these in (2.30), we find

$$\begin{aligned}\Lambda^{ij,k}{}_{,kj} = &\frac{1}{2}g^{ih}(g_{ah,bkj} + g_{bh,akj} - g_{ab,hkj})\Lambda^{kj,ab}\\ &+g^{ih}(g_{ah,bk} + g_{bh,ak} - g_{ab,hk})\Lambda^{kj,ab}{}_{,j}.\end{aligned}$$

From (2.12) and (2.20), we obtain

$$\Lambda^{ij,k}{}_{,kj} = -\frac{1}{2}g^{ih}g_{ab,hkj}\Lambda^{kj,ab} - g^{ih}(g_{hk,ab} + g_{ab,hk})\Lambda^{kj,ab}{}_{,j}.$$

Multiplying (2.19) with g^{ri} and renaming the dummy indices yields

$$\begin{aligned}g^{ij}L =&2g^{ai}\Lambda^{jb,kh}\left(g_{ab,hk} + g_{hk,ab}\right) + 2g^{ai}\Lambda^{jb,k}g_{ab,k}\\ &+ g^{ai}\Lambda^{hb,j}g_{hb,a} + 2\Lambda^{ij} + g^{ai}\Phi^{j}\phi_{,a}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\Lambda^{ij} =&\frac{1}{2}g^{ij}L - g^{ai}\Lambda^{jb,kh}\left(g_{ab,hk} + g_{hk,ab}\right) - g^{ai}\Lambda^{jb,k}g_{ab,k}\\ &- \frac{1}{2}g^{ai}\Lambda^{hb,j}g_{hb,a} - \frac{1}{2}g^{ai}\Phi^{j}\phi_{,a}.\end{aligned}$$

Resulting from this, at the pole P of the Riemann normal coordinate system, we find

$$\begin{aligned} \Lambda^{ij}{}_{,j} =& \frac{1}{2} g^{ij} L_{,j} - g^{ai} \Lambda^{jb,kh} \big(g_{ab,hkj} + g_{hk,abj} \big) - g^{ai} \Lambda^{jb,kh}{}_{,j} \big(g_{ab,hk} + g_{hk,ab} \big) \\ & - g^{ai} \Lambda^{jb,k} g_{ab,kj} - \frac{1}{2} g^{ai} \Lambda^{hb,j} g_{hb,aj} - \frac{1}{2} g^{ai} \Lambda^{j}{}_{,j} \phi_{,a} - \frac{1}{2} g^{ai} \Phi^{j} \phi_{,aj}. \end{aligned} \tag{2.31}$$

If we calculate $L_{,j}$, we obtain

$$\frac{\partial L}{\partial x^j} = \frac{\partial g_{ab}}{\partial x^j} \Lambda^{ab} + \frac{\partial g_{ab,c}}{\partial x^j} \Lambda^{ab,c} + \frac{\partial g_{ab,cd}}{\partial x^j} \Lambda^{ab,cd} + \frac{\partial \phi}{\partial x^j} \Phi + \frac{\partial \phi_{,a}}{\partial x^j} \Lambda^{a}.$$

The first term on the right hand side is zero at P; therefore, we have

$$g^{ij} \frac{\partial L}{\partial x^j} = g^{ij} g_{ab,cj} \Lambda^{ab,c} + g^{ij} g_{ab,cdj} \Lambda^{ab,cd} + g^{ij} \phi_{,j} \Phi + g^{ij} \phi_{,aj} \Lambda^{a}.$$

If we insert this into (2.31), we find

$$\begin{aligned} \Lambda^{ij}{}_{,j} =& \frac{1}{2} g^{ij} g_{ab,cj} \Lambda^{ab,c} + \frac{1}{2} g^{ij} g_{ab,cdj} \Lambda^{ab,cd} + \frac{1}{2} g^{ij} \phi_{,j} \Phi + \frac{1}{2} g^{ij} \phi_{,aj} \Lambda^{a} \\ & - g^{ai} \Lambda^{jb,kh} \big(g_{ab,hkj} + g_{hk,abj} \big) - g^{ai} \Lambda^{jb,kh}{}_{,j} \big(g_{ab,hk} + g_{hk,ab} \big) \\ & - g^{ai} \Lambda^{jb,k} g_{ab,kj} - \frac{1}{2} g^{ai} \Lambda^{hb,j} g_{hb,aj} - \frac{1}{2} g^{ai} \Lambda^{j}{}_{,j} \phi_{,a} - \frac{1}{2} g^{ai} \Phi^{j} \phi_{,aj}. \end{aligned}$$

Simplifying this yields

$$\begin{aligned} \Lambda^{ij}{}_{,j} =& \frac{1}{2} g^{ij} g_{ab,cdj} \Lambda^{ab,cd} + \frac{1}{2} g^{ij} \phi_{,j} \Phi - g^{ai} \Lambda^{jb,kh} g_{hk,abj} \\ & - g^{ai} \Lambda^{jb,kh}{}_{,j} \big(g_{ab,hk} + g_{hk,ab} \big) - g^{ai} \Lambda^{jb,k} g_{ab,kj} - \frac{1}{2} g^{ai} \Lambda^{j}{}_{,j} \phi_{,a}. \end{aligned}$$

Note that $g^{ai} \Lambda^{jb,k} g_{ab,kj} = 0$, due to (2.16) at P. Therefore, we have

$$\begin{aligned} \Lambda^{ij}{}_{,j} =& \frac{1}{2} g^{ij} g_{ab,cdj} \Lambda^{ab,cd} + \frac{1}{2} g^{ij} \phi_{,j} \Phi - g^{ai} \Lambda^{jb,kh} g_{hk,abj} \\ & - g^{ai} \Lambda^{jb,kh}{}_{,j} \big(g_{ab,hk} + g_{hk,ab} \big) - \frac{1}{2} g^{ai} \Lambda^{j}{}_{,j} \phi_{,a}. \end{aligned}$$

Using all of these results to calculate $E^{ij}{}_{|j}(L)$, we finally obtain

$$E^{ij}{}_{|j}(L) = \Lambda^{ij,k}{}_{,kj} - \Lambda^{ij}{}_{,j}$$

which is also equal to

$$
\begin{aligned}
E^{ij}{}_{|j}(L) =& -\frac{1}{2}g^{ih}g_{ab,hkj}\Lambda^{kj,ab} - g^{ih}(g_{hk,ab}+g_{ab,hk})\Lambda^{kj,ab}{}_{,j} \\
&-\frac{1}{2}g^{ij}g_{ab,cdj}\Lambda^{ab,cd} - \frac{1}{2}g^{ij}\phi_{,j}\Phi + g^{ai}\Lambda^{jb,kh}g_{hk,abj} \\
&+g^{ai}(g_{ab,hk}+g_{hk,ab})\Lambda^{jb,kh}{}_{,j} + \frac{1}{2}g^{ai}\Lambda^{j}{}_{,j}\phi_{,a} \\
=& -\frac{1}{2}g^{ih}g_{ab,hkj}\Lambda^{kj,ab} - \frac{1}{2}g^{ij}g_{ab,cdj}\Lambda^{ab,cd} + g^{ai}g_{hk,abj}\Lambda^{jb,kh} \\
&-\frac{1}{2}g^{ij}\phi_{,j}\Phi + \frac{1}{2}g^{ij}\Lambda^{a}{}_{,a}\phi_{,j} \\
=& -\frac{1}{2}g^{ij}g_{ab,jcd}\Lambda^{cd,ab} - \frac{1}{2}g^{ij}g_{ab,cdj}\Lambda^{ab,cd} + g^{ij}g_{ab,dcj}\Lambda^{dc,ab} \\
&-\frac{1}{2}g^{ij}\phi_{,j}\Phi + \frac{1}{2}g^{ij}\Lambda^{a}{}_{,a}\phi_{,j} \\
=&\frac{1}{2}g^{ij}\phi_{,j}\big(\Lambda^{a}{}_{,a} - \Phi\big).
\end{aligned}
$$

Note that the term in the parenthesis is equal to (2.26). We state this important result as a theorem.

Theorem 2.1 *For a scalar density of the type*

$$L(g_{ij}, g_{ij,k}, g_{ij,kh}, \phi, \phi_{,i})$$

and if the corresponding Euler-Lagrange equations are $E^{ij}(L)$ and $E(L)$ are given by equations (2.25) and (2.26), then

$$E^{ij}{}_{|j}(L) = \frac{1}{2}g^{ij}\phi_{,j}E(L). \tag{2.32}$$

Note that even though we have found this relation at the pole P of the Riemann normal coordinate system, being a tensorial equation, it is guaranteed to be valid everywhere. As a consequence of this, if the Euler-Lagrange equations for the metric are satisfied, i.e., $E^{ij}(L) = 0$, then $E(L) = 0$. It is clear that the converse does not generally hold true.

The equation given in (2.32) is the generalization of the Bianchi identity. This equation reduces to divergence-free field equations in a metric field theory.

2.3 Degenerate Lagrange Densities in n Dimensions

We start by defining the following useful quantities that will be relevant in the discussion that follows

$$
\begin{aligned}
\Lambda^{ij,kh;rs,tu} &\equiv \frac{\partial}{\partial g_{rs,tu}} \frac{\partial L}{\partial g_{ij,kh}}, \\
\chi^{ij,kh;rs,tu} &\equiv \Lambda^{ij,kh;rs,tu} + \Lambda^{ij,ku;rs,th} + \Lambda^{ij,kt;rs,hu}, \\
\Lambda^{ij,kh;rs,tu;ab,cd} &\equiv \frac{\partial}{\partial g_{ab,cd}} \frac{\partial}{\partial g_{rs,tu}} \frac{\partial L}{\partial g_{ij,kh}}, \\
\Lambda^{ij,kh;a} &\equiv \frac{\partial}{\partial \phi_{,a}} \frac{\partial L}{\partial g_{ij,kh}}, \\
\Lambda^{ij,kh;ab} &\equiv \frac{\partial}{\partial g_{ab}} \frac{\partial L}{\partial g_{ij,kh}}, \\
\Lambda^{ij,kh;ab,c} &\equiv \frac{\partial}{\partial g_{ab,c}} \frac{\partial L}{\partial g_{ij,kh}}.
\end{aligned}
$$

Note that apart from the last term, all these terms are tensor densities.

The Euler-Lagrange equations associated with $L = L(g_{ij}, g_{ij,k}, g_{ij,kh}, \phi, \phi_{,i})$ are (2.25) and (2.26). Note that these equations are of the form

$$
\begin{aligned}
E^{tu} &= E^{tu}(g_{ij}, g_{ij,k}, g_{ij,kh}, g_{ij,khr}, g_{ij,khrs}, \phi, \phi_{,i}, \phi_{,ij}, \phi_{,ijk}), \\
E &= E(g_{ij}, g_{ij,k}, g_{ij,kh}, g_{ij,khr}, \phi, \phi_{,i}, \phi_{,ij}).
\end{aligned}
$$

We can calculate (2.25) and (2.26) as

$$
\begin{aligned}
E^{ij}(L) =& \frac{\partial g_{ab}}{\partial x^k} \frac{\partial}{\partial g_{ab}} \Lambda^{ij,k} + \frac{\partial g_{ab,c}}{\partial x^k} \frac{\partial}{\partial g_{ab,c}} \Lambda^{ij,k} + \frac{\partial g_{ab,cd}}{\partial x^k} \frac{\partial}{\partial g_{ab,cd}} \Lambda^{ij,k} + \frac{\partial \phi}{\partial x^k} \frac{\partial}{\partial \phi} \Lambda^{ij,k} \\
&+ \frac{\partial \phi_{,a}}{\partial x^k} \frac{\partial}{\partial \phi_{,a}} \Lambda^{ij,k} - \frac{\partial}{\partial x^h} \left(\frac{\partial g_{ab}}{\partial x^k} \frac{\partial}{\partial g_{ab}} \Lambda^{ij,kh} + \frac{\partial g_{ab,c}}{\partial x^k} \frac{\partial}{\partial g_{ab,c}} \Lambda^{ij,kh} \right. \\
&\left. + \frac{\partial g_{ab,cd}}{\partial x^k} \frac{\partial}{\partial g_{ab,cd}} \Lambda^{ij,kh} + \frac{\partial \phi}{\partial x^k} \frac{\partial}{\partial \phi} \Lambda^{ij,kh} + \frac{\partial \phi_{,a}}{\partial x^k} \frac{\partial}{\partial \phi_{,a}} \Lambda^{ij,kh} \right) - \Lambda^{ij}, \\
E(L) =& \frac{\partial g_{ab}}{\partial x^i} \frac{\partial}{\partial g_{ab}} \Phi^i + \frac{\partial g_{ab,c}}{\partial x^i} \frac{\partial}{\partial g_{ab,c}} \Phi^i + \frac{\partial g_{ab,cd}}{\partial x^i} \frac{\partial}{\partial g_{ab,cd}} \Phi^i \\
&+ \frac{\partial \phi}{\partial x^i} \frac{\partial}{\partial \phi} \Phi^i + \frac{\partial \phi_{,a}}{\partial x^i} \frac{\partial}{\partial \phi_{,a}} \Phi^i - \Phi.
\end{aligned}
$$

Here, we find the terms involving the fourth and the third-order derivatives of g_{ij} and

ϕ such that

$$
\begin{aligned}
E^{ij}(L) = &- g_{ab,cdkh}\Lambda^{ij,kh;ab,cd} - \phi_{,akh}\Lambda^{ij,kh;a} + g_{ab,cdk}\Bigg(\Lambda^{ij,k;ab,cd} \\
&- \Lambda^{ij,kd;ab,c} - \frac{\partial}{\partial x^h}\Lambda^{ij,kh;ab,cd}\Bigg) - g_{ab,ck}\frac{\partial}{\partial x^h}\Lambda^{ij,hk;ab,c} \qquad (2.33)\\
&- g_{ab,k}\frac{\partial}{\partial x^h}\Lambda^{ij,hk;ab} - \phi_{,ak}\frac{\partial}{\partial x^h}\Lambda^{ij,hk;a} - \phi_{,k}\frac{\partial}{\partial x^h}\Lambda^{ij,hk} + P^{ij},\\
E(L) = &g_{ab,cdi}\Lambda^{i;ab,cd} + P, \qquad (2.34)
\end{aligned}
$$

where we have defined

$$
\begin{aligned}
P^{ij} &= P^{ij}(g_{ab}, g_{ab,c}, g_{ab,cd}, \phi, \phi_{,a}, \phi_{,ab}),\\
P &= P(g_{ab}, g_{ab,c}, g_{ab,cd}, \phi, \phi_{,a}, \phi_{,ab})
\end{aligned}
$$

for convenience.

Lemma 2.2 *(i) $E(L)$ is at most of second-order in g_{ij} if and only if*

$$\Lambda^{i;ab,cd} + \Lambda^{d;ab,ci} + \Lambda^{c;ab,id} = 0; \qquad (2.35)$$

(ii) $E^{ij}(L)$ is at most of second-order in ϕ if and only if

$$\Lambda^{ij,kh;a} + \Lambda^{ij,ah;k} + \Lambda^{ij,ka;h} = 0; \qquad (2.36)$$

(iii) $E^{ij}(L)$ is at most of third-order in g_{ij} if and only if

$$
\begin{aligned}
\Lambda^{ij,kh;ab,cd} &+ \Lambda^{ij,kd;ab,ch} + \Lambda^{ij,kc;ab,hd}\\
&+ \Lambda^{ij,cd;ab,kh} + \Lambda^{ij,ch;ab,kd} + \Lambda^{ij,hd;ab,kc} = 0.
\end{aligned} \qquad (2.37)
$$

These three relations are direct consequences of symmetry relations associated with the derivatives of the metric. It is obvious that (2.35) and (2.36) are in fact identical conditions. Note that equation (2.37) can be written as

$$\chi^{ij,kh;ab,cd} = -\chi^{ab,kh;ij,cd}. \qquad (2.38)$$

Applying conditions of Lemma 2.2 leaves $E^{ij}(L)$ at most of third-order in g_{ij}. Therefore, we need to eliminate all third-order dependency of $E^{ij}(L)$. All third-order terms of the Euler-Lagrange equations of the metric are obtained from (2.33) as

$$\begin{aligned} &g_{ab,cdk}\left(\Lambda^{ij,k;ab,cd}-\Lambda^{ij,kd;ab,c}-\frac{\partial}{\partial x^h}\Lambda^{ij,kh;ab,cd}\right)-g_{ab,ck}g_{rs,tuh}\Lambda^{ij,hk;ab,c;rs,tu}\\ &-g_{ab,k}g_{rs,tuh}\Lambda^{ij,hk;ab;rs,tu}-\phi_{,ak}g_{rs,tuh}\Lambda^{ij,hk;a;rs,tu}-\phi_{,k}g_{rs,tuh}\Phi^{ij,hk;rs,tu}. \end{aligned} \tag{2.39}$$

The third term in the parenthesis can be written as

$$\begin{aligned} g_{ab,cdk}\frac{\partial}{\partial x^h}\Lambda^{ij,kh;ab,cd}=&g_{ab,cdk}g_{rs,tuh}\Lambda^{ij,hk;ab,cd;rs,tu}\\ &+g_{ab,cdk}g_{rs,th}\Lambda^{ij,hk;ab,cd;rs,t}+g_{ab,cdk}g_{rs,h}\Lambda^{ij,hk;ab,cd;rs}\\ &+g_{ab,cdk}\phi_{,h}\Lambda^{ij,hk;rs,tu}+g_{ab,cdk}\phi_{,rh}\Lambda^{ij,hk;rs,tu;r}. \end{aligned}$$

We can rewrite (2.39) using the equation above. The last four terms of (2.39) can be written as the third term in the parenthesis as shown above by index renaming. Therefore, we can write all third-order terms as

$$\begin{aligned} &-2g_{ab,cdk}\frac{\partial}{\partial x^h}\Lambda^{ij,kh;ab,cd}+g_{ab,cdk}g_{rs,tuh}\Lambda^{ij,hk;ab,cd;rs,tu}\\ &+g_{ab,cdk}\Lambda^{ij,k;ab,cd}-g_{ab,cdk}\Lambda^{ij,kd;ab,c}. \end{aligned} \tag{2.40}$$

By taking the derivative of (2.8) with respect to $g_{ab,cd}$ and $g_{ij,kd}$, respectively, we may write the last two terms in terms of $\Lambda^{ij,hk;rs,tu}$. Now, we can calculate them.

From the third term, we find

$$\begin{aligned} g_{ab,cdk}\Lambda^{ij,k;ab,cd}=g_{ab,cdk}\Big(&\Pi^{ij,k;ab,cd}-\Gamma^k_{\ rs}\Lambda^{ij,rs;ab,cd}\\ &-2\Gamma^i_{\ rs}\Lambda^{rj,ks;ab,cd}-2\Gamma^j_{\ rs}\Lambda^{ir,ks;ab,cd}\Big). \end{aligned}$$

From the fourth term, we have

$$\begin{aligned} -g_{ab,cdk}\Lambda^{ij,kd;ab,c}=-g_{ab,cdk}\Big(&\Pi^{ij,kd;ab,c}-\Gamma^c_{\ rs}\Lambda^{ij,kd;ab,rs}\\ &-2\Gamma^a_{\ rs}\Lambda^{ij,kd;rb,cs}-2\Gamma^b_{\ rs}\Lambda^{ij,kd;ar,cs}\Big). \end{aligned}$$

Note that according to equation (2.17), the first terms on the right hand sides of both equations above are zero. Using

$$g_{ab,cdk}\Lambda^{ij,kh;ab,cd}=\frac{1}{3}g_{ab,cdk}\chi^{ij,hk;ab,cd},$$

we can write the expression in (2.40) as

$$
\begin{aligned}
&-\frac{2}{3}g_{ab,cdk}\frac{\partial}{\partial x^h}\chi^{ij,hk;ab,cd}+g_{ab,cdk}g_{rs,tuh}\Lambda^{ij,hk;ab,cd;rs,tu}\\
&+g_{ab,cdk}\bigg(-\Gamma^{k}_{\ rs}\Lambda^{ij,rs;ab,cd}-\frac{2}{3}\Gamma^{i}_{\ rs}\chi^{rj,sk;ab,cd}-\frac{2}{3}\Gamma^{j}_{\ rs}\chi^{ir,sk;ab,cd}\bigg)\\
&-g_{ab,cdk}\bigg(-\Gamma^{c}_{\ rs}\Lambda^{ij,dk;ab,rs}-\frac{2}{3}\Gamma^{a}_{\ rs}\chi^{rb,sc;ij,kd}-\frac{2}{3}\Gamma^{b}_{\ rs}\chi^{ar,sc;ij,kd}\bigg).
\end{aligned}
\tag{2.41}
$$

We calculate the covariant derivative of $\chi^{ij,hk;ab,cd}$ as

$$
\begin{aligned}
\chi^{ij,hk;ab,cd}{}_{|h}=&\chi^{ij,hk;ab,cd}{}_{,h}+\Gamma^{i}_{\ rh}\chi^{rj,hk;ab,cd}+\Gamma^{j}_{\ rh}\chi^{ir,hk;ab,cd}+\Gamma^{h}_{\ rh}\chi^{ij,rk;ab,cd}\\
&+\Gamma^{k}_{\ rh}\chi^{ij,hr;ab,cd}+\Gamma^{a}_{\ rh}\chi^{ij,hk;rb,cd}+\Gamma^{b}_{\ rh}\chi^{ij,hk;ar,cd}\\
&+\Gamma^{c}_{\ rh}\chi^{ij,hk;ab,rd}+\Gamma^{d}_{\ rh}\chi^{ij,hk;ab,cr}-\Gamma^{r}_{\ hr}\chi^{ij,hk;ab,cd}.
\end{aligned}
$$

Now, we use the equation above to write all third-order terms in a compact form. By inserting the relation above into (2.41), we obtain

$$
\begin{aligned}
&-\frac{2}{3}g_{ab,cdk}\chi^{ij,hk;ab,cd}{}_{|h}+\frac{2}{3}g_{ab,cdk}\big(\Gamma^{i}_{\ rh}\chi^{rj,hk;ab,cd}+\Gamma^{j}_{\ rh}\chi^{ir,hk;ab,cd}\\
&+\Gamma^{h}_{\ rh}\chi^{ij,rk;ab,cd}+\Gamma^{k}_{\ rh}\chi^{ij,hr;ab,cd}+\Gamma^{a}_{\ rh}\chi^{ij,hk;rb,cd}+\Gamma^{b}_{\ rh}\chi^{ij,hk;ar,cd}\\
&+\Gamma^{c}_{\ rh}\chi^{ij,hk;ab,rd}+\Gamma^{d}_{\ rh}\chi^{ij,hk;ab,cr}-\Gamma^{r}_{\ hr}\chi^{ij,hk;ab,cd}-\frac{3}{2}\Gamma^{k}_{\ rh}\Lambda^{ij,rh;ab,cd}\\
&-\Gamma^{i}_{\ rh}\chi^{rj,hk;ab,cd}-\Gamma^{j}_{\ rh}\chi^{ir,hk;ab,cd}+\frac{3}{2}\Gamma^{c}_{\ rh}\Lambda^{ij,kd;ab,rh}+\Gamma^{a}_{\ rh}\chi^{rb,hc;ij,kd}\\
&+\Gamma^{b}_{\ rh}\chi^{ar,hc;ij,kd}\big)+g_{ab,cdk}g_{rs,tuh}\Lambda^{ij,hk;ab,cd;rs,tu}.
\end{aligned}
$$

After rearranging the terms and renaming indices, we find

$$
\begin{aligned}
&-\frac{2}{3}g_{ab,cdk}\chi^{ij,kh;ab,cd}{}_{|h}+\frac{2}{3}g_{ab,cdk}\bigg(\Gamma^{a}_{\ rh}\big(\chi^{ij,hk;rb,cd}+\chi^{rb,hk;ij,cd}\big)\\
&+\Gamma^{b}_{\ rh}\big(\chi^{ij,hk;ar,cd}+\chi^{ar,hk;ij,cd}\big)+\Gamma^{k}_{\ rh}\chi^{ij,hc;ab,rd}+\Gamma^{k}_{\ rh}\chi^{ij,hd;ab,cr}\\
&+\Gamma^{k}_{\ rh}\chi^{ij,hr;ab,cd}+\frac{3}{2}\Gamma^{k}_{\ rh}\Lambda^{ij,cd;ab,rh}-\frac{3}{2}\Gamma^{k}_{\ rh}\Lambda^{ij,rh;ab,cd}\bigg)\\
&+g_{ab,cdk}g_{rs,tuh}\Lambda^{ij,hk;ab,cd;rs,tu}.
\end{aligned}
\tag{2.42}
$$

One can write

$$
\begin{aligned}
\Lambda^{ij,cd;ab,rh}-\Lambda^{ij,rh;ab,cd}=&2\Lambda^{ij,cd;ab,rh}+\Lambda^{ij,rc;ab,hd}+\Lambda^{ij,ch;ab,rd}\\
&+\Lambda^{ij,rd;ab,ch}+\Lambda^{ij,hd;ab,rc}-\chi^{ij,cd;ab,rh}-\chi^{ab,cd;ij,rh}\\
=&\chi^{ij,cd;ab,rh}+\chi^{ij,dc;ab,rh}-\chi^{ij,cd;ab,rh}-\chi^{ab,cd;ij,rh}\\
=&\chi^{ij,dc;ab,rh}-\chi^{ab,cd;ij,rh}.
\end{aligned}
$$

Therefore, all third-order terms are

$$
\begin{aligned}
&-\frac{2}{3}g_{ab,cdk}\chi^{ij,kh;ab,cd}{}_{|h}+\frac{2}{3}g_{ab,cdk}\Big[\Gamma^{a}{}_{rh}\left(\chi^{ij,hk;rb,cd}+\chi^{rb,hk;ij,cd}\right)\\
&+\Gamma^{b}{}_{rh}\left(\chi^{ij,hk;ar,cd}+\chi^{ar,hk;ij,cd}\right)+\frac{3}{2}\Gamma^{k}{}_{rh}\left(\chi^{ij,hr;ab,cd}+\chi^{ab,hr;ij,cd}\right)\Big]\\
&+g_{ab,cdk}g_{rs,tuh}\Lambda^{ij,hk;ab,cd;rs,tu}.
\end{aligned}
\tag{2.43}
$$

Thus we have established

Lemma 2.3 *A necessary and sufficient condition to have no third-order dependency*

$$\frac{\partial E^{ab}}{\partial g_{ij,khr}}=0$$

is that

$$
\begin{aligned}
\chi^{ij,kh;ab,cd}{}_{|h}-\Big[&\Gamma^{a}{}_{rh}\left(\chi^{ij,hk;rb,cd}+\chi^{rb,hk;ij,cd}\right)+\Gamma^{b}{}_{rh}\left(\chi^{ij,hk;ar,cd}+\chi^{ar,hk;ij,cd}\right)\\
&+\frac{3}{2}\Gamma^{k}{}_{rh}\left(\chi^{ij,hr;ab,cd}+\chi^{ab,hr;ij,cd}\right)\Big]=0.
\end{aligned}
$$

We note that this is not a tensorial condition due to the bracketed term. In order to satisfy this equation for all coordinate transformations, the bracketed term must be equal to zero. The terms in the parentheses are zero if (2.38) is satisfied. However, the vanishing of the terms in the brackets does not imply that (2.38) will be satisfied.

Lemma 2.4 *In order not to have any third-order derivatives of the metric in E^{ab} under arbitrary transformations, the Lagrangian must satisfy*

$$\chi^{ij,kh;ab,cd}{}_{|h}=0$$

and

$$
\begin{aligned}
&\Gamma^{a}{}_{rh}\left(\chi^{ij,hk;rb,cd}+\chi^{rb,hk;ij,cd}\right)+\Gamma^{b}{}_{rh}\left(\chi^{ij,hk;ar,cd}+\chi^{ar,hk;ij,cd}\right)\\
&\quad+\frac{3}{2}\Gamma^{k}{}_{rh}\left(\chi^{ij,hr;ab,cd}+\chi^{ab,hr;ij,cd}\right)=0.
\end{aligned}
\tag{2.44}
$$

The condition (2.38) guarantees that there will be no fourth-order derivatives of the metric in $E^{ij}(L)$. Besides, equation (2.44) is satisfied due to (2.38). Therefore, we immediately have the following lemma.

Lemma 2.5 *E^{ab} is at most of second-order in derivatives of the metric i.e.,*

$$\frac{\partial E^{ab}}{\partial g_{ij,khrs}} = 0, \qquad \frac{\partial E^{ab}}{\partial g_{ij,khr}} = 0$$

if and only if

$$\chi^{ij,kh;ab,cd} = -\chi^{ab,kh;ij,cd}$$

and

$$\chi^{ij,kh;ab,cd}{}_{|h} = 0 \tag{2.45}$$

are satisfied.

As a result, (2.35), (2.38) and (2.45) are necessary and sufficient conditions to have the Euler-Lagrange equations that are at most of second-order both in the metric and the scalar field.

2.4 The Most General Lagrange Scalar Density in Four Dimensions

In this section, we start with deriving the most general form of the Lagrangian in a four-dimensional spacetime. After completing this task, we will impose the conditions that we have in Section 2.3 to have the Euler-Lagrange equations of the form

$$\begin{aligned} E^{ab} &= E^{ab}(g_{ij}, g_{ij,k}, g_{ij,kh}, \phi, \phi_{,i}, \phi_{,ij}), \\ E &= E(g_{ij}, g_{ij,k}, g_{ij,kh}, \phi, \phi_{,i}, \phi_{,ij}). \end{aligned}$$

As a result, at the end of this section, we will obtain the most general form of the Lagrangian which yields field equations above.

The detailed calculations in this section relegated to Appendices B and C.

Due to dimensional restrictions on our Lagrangian, in a spacetime of four dimensions, we have the equation below

$$\Lambda^{ij,kh;ab,cd;rs,tu;pq,lm} = 0. \tag{2.46}$$

The detailed proof of this can be found in Lemmas (B.10) and (B.11) in Appendix B. Integrating this equation yields

$$\Lambda^{ij,kh;ab,cd;rs,tu} = A\epsilon^{ij,kh;ab,cd;rs,tu}, \tag{2.47}$$

where

$$A = A(g_{ij}, g_{ij,k}, \phi, \phi_{,i})$$

and

$$\epsilon^{ij,kh;ab,cd;rs,tu} \equiv \sum_{tu}\sum_{rs}\sum_{cd}\sum_{ab}\sum_{kh}\sum_{ij} \epsilon^{ikac}\epsilon^{jhrt}\epsilon^{bdsu}/g,$$

where ϵ^{ikac} is the four-dimensional permutation symbol. The summation symbol is defined as

$$\sum_{ij} A^{ij\cdots}_{\cdots} \equiv A^{ij\cdots}_{\cdots} + A^{ji\cdots}_{\cdots}.$$

Making use of Lemmas (2.1) and (B.7), we obtain

$$A = A(\phi, \rho),$$

where $\rho \equiv g^{ij}\phi_{,i}\phi_{,j}$. As a result of Lemma B.12, we have $A^{i;j} = 0$ which implies $\frac{\partial A}{\partial \rho} = 0$. Therefore, we have $A = A(\phi)$. Due to (2.21), we can easily find

$$\Lambda^{ij,kh} g_{ij,kh} = \frac{2}{3}\Lambda^{ij,kh} R_{kijh}. \tag{2.48}$$

Upon integrating (2.46) together with the relation above, we obtain

$$\Lambda^{ij,kh;ab,cd} = \frac{2}{3} A \epsilon^{ij,kh;ab,cd;rs,tu} R_{trsu} + \psi^{ij,kh;ab,cd}, \tag{2.49}$$

where $\psi^{ij,kh;ab,cd} = \psi^{ij,kh;ab,cd}(g_{ab}, \phi, \phi_{,a})$ is a tensor density which has the same symmetry properties as $\Lambda^{ij,kh;ab,cd}$. As a result of Lemma B.12, we have

$$\psi^{ij,kh;ab,cd} = \alpha^{ijkhabcdr}\phi_{,r} + \beta^{ij,kh;ab,cd},$$

where

$$\alpha^{ijkhabcdr} = \alpha^{ijkhabcdr}(g_{ab}, \phi) \quad \text{and} \quad \beta^{ij,kh;ab,cd} = \beta^{ij,kh;ab,cd}(g_{ab}, \phi). \tag{2.50}$$

Here, since $\alpha^{ijkhabcdr}(g_{ab}, \phi)$ has a tensorial character and nine indices which is an odd number, it is very obvious that $\alpha^{ijkhabcdr} = 0$ [18].

Now, we can write equation (2.49) as

$$\Lambda^{ij,kh;ab,cd} = \frac{2}{3} A \epsilon^{ij,kh;ab,cd;rs,tu} R_{trsu} + \psi^{ij,kh;ab,cd}(g_{ab}, \phi).$$

If we integrate this equation again, we obtain

$$\Lambda^{ij,kh} = \frac{2}{9} A \epsilon^{ij,kh;ab,cd;rs,tu} R_{trsu} R_{cabd} + \frac{2}{3}\psi^{ij,kh;ab,cd} R_{cabd} + \mu^{ij,kh},$$

where $\mu^{ij,kh} = \mu^{ij,kh}(g_{ab}, \phi, \phi_{,a})$ is a tensor density. Note that $\Lambda^{ij,kh}$ and $\mu^{ij,kh}$ have the similar symmetry properties. Finally, the integration of this equation gives

$$L = \frac{4}{81} A\epsilon^{ij,kh;ab,cd;rs,tu} R_{trsu} R_{cabd} R_{kijh} + \frac{2}{9}\psi^{ij,kh;ab,cd} R_{cabd} R_{kijh} + \mu^{ij,kh} R_{kijh} + \lambda, \tag{2.51}$$

where $\lambda = \lambda(g_{ab}, \phi, \phi_{,a})$ is a scalar density. Therefore, from Lemma B.7, we have $\lambda = \lambda(\phi, \rho)$.

Now, for the first term, we adopt the results that we get from Appendix C.1. For the second term, we make use of Lemmas (B.5) and (B.6) in conjunction with the calculations in Appendix C.2. For the third term, we benefit from Lemma B.9. Finally, for the last term, by virtue of Lemma B.8, we find

$$\begin{aligned} L =& \alpha(*R^{ij}{}_{kh})(*R^{kh}{}_{rs})(*R^{rs}{}_{ij})/g + \beta\sqrt{g}(R^2 - 4R_{ij}R^{ij} + R_{ijkh}R^{ijkh}) \\ &+ \gamma * R^{ij}{}_{kh} R^{kh}{}_{ij} + \sigma\sqrt{g} R^{ij}\phi_{,i}\phi_{,j} + \mu\sqrt{g} R + \eta\sqrt{g}, \end{aligned} \tag{2.52}$$

where

$$*R^{ij}{}_{kh} = \epsilon^{ijrs} R_{rskh}, \qquad \alpha = \alpha(\phi), \qquad \beta = \beta(\phi), \qquad \gamma = \gamma(\phi),$$
$$\sigma = \sigma(\phi, \rho), \qquad \mu = \mu(\phi, \rho), \qquad \eta = \eta(\phi, \rho).$$

In order to complete our work, we have to impose the conditions that we have in equations (2.45) and (2.35) on our Lagrangian. However, for the former one, this process will undoubtedly lead to long calculations. Instead of imposing the condition (2.45), we can calculate the Euler-Lagrange equations and then decide on the possibilities to deal with the terms which include the third-order derivative of the metric. Since we have already imposed the condition that we have in (2.37) during this derivation, we will not have any terms which has the fourth-order derivative of the metric in the Euler-Lagrange equations.

For $L = \alpha(*R^{ij}{}_{kh})(*R^{kh}{}_{rs})(*R^{rs}{}_{ij})/g$, due to (2.29), we find the Euler-Lagrange equations as

$$\begin{aligned}E^{ij}(L) =& -3\alpha\epsilon^{efrs}\epsilon^{khab}\epsilon^{cdtu}\left(R_{rskh}R_{tuef}\frac{\partial R_{abcd}}{\partial g_{ij,kh}}\right)_{|kh} \\ & -6\alpha'\phi_{;k}\epsilon^{efrs}\epsilon^{khab}\epsilon^{cdtu}\left(R_{rskh}R_{tuef}\frac{\partial R_{abcd}}{\partial g_{ij,kh}}\right)_{|h} \\ & -3(\alpha''\phi_{;k}\phi_{;h}+\alpha'\phi_{;kh})\epsilon^{efrs}\epsilon^{khab}\epsilon^{cdtu}\left(R_{rskh}R_{tuef}\frac{\partial R_{abcd}}{\partial g_{ij,kh}}\right) \\ & -\frac{1}{2}\alpha g^{ij}\epsilon^{efrs}\epsilon^{khab}\epsilon^{cdtu}R_{rskh}R_{tuef}R_{abcd} \\ & -2\alpha\epsilon^{efrs}\epsilon^{khab}\epsilon^{cdtu}\left(R_{rskh}R_{tuef}\frac{\partial R_{abcd}}{\partial g_{im,kh}}\right)R_{h\ km}^{\ j},\end{aligned}$$

where a prime denotes a partial derivative with respect to ϕ.

The first term of this leads to terms that involve the third-order derivatives of the metric i.e., $R_{abcd;h}$. Therefore, $\alpha = 0$. Note that sum of all the terms of the form $R_{abcd;kh}$ are zero. It can be easily shown by calculating them. We note that this is a direct consequence of equation (2.37).

For the $\gamma(*R^{ij}_{\ \ kh})R^{kh}_{\ \ ij}$ term, by using equation (2.29), we calculate the Euler-Lagrange equations as

$$\begin{aligned}E^{ij}(L) =& -2\gamma\epsilon^{abrs}g^{tc}g^{ud}\left(R_{rstu}\frac{\partial R_{abcd}}{\partial g_{ij,kh}}\right)_{|kh} -4\gamma'\phi_{;k}\epsilon^{abrs}g^{tc}g^{ud}\left(R_{rstu}\frac{\partial R_{abcd}}{\partial g_{ij,kh}}\right)_{|h} \\ & -2(\gamma''\phi_{;k}\phi_{;h}+\gamma'\phi_{;kh})\epsilon^{abrs}g^{tc}g^{ud}\left(R_{rstu}\frac{\partial R_{abcd}}{\partial g_{ij,kh}}\right) \\ & -\frac{1}{2}\gamma g^{ij}\epsilon^{abrs}g^{tc}g^{ud}R_{rstu}R_{abcd} - \frac{4}{3}\gamma\epsilon^{abrs}g^{tc}g^{ud}\left(R_{rstu}\frac{\partial R_{abcd}}{\partial g_{im,kh}}\right)R_{h\ km}^{\ j}.\end{aligned}$$

Here, the first term vanishes. The second term yields the third-order derivatives of the metric. Therefore, we have $\gamma' = 0$ which means $\gamma = c$, where c is a constant.

A Lagrangian of the form $L = \sigma\sqrt{g}R^{ij}\phi_{,i}\phi_{,j} + \mu\sqrt{g}R$ should satisfy equation (2.35). Otherwise, we will have third-order derivatives of the scalar in the Euler-Lagrange equations. Consequently, we can calculate

$$\begin{aligned}\Lambda^{a;ij,kh} =& \frac{1}{4}\sqrt{g}\frac{\partial}{\partial\phi_{,a}}\Big(\sigma\big(\phi^{,i}\phi^{,h}g^{kj}+\phi^{,i}\phi^{,k}g^{hj}+\phi^{,j}\phi^{,h}g^{ki}+\phi^{,j}\phi^{,k}g^{hi} \\ & -2\phi^{,i}\phi^{,j}g^{kh}-2\phi^{,k}\phi^{,h}g^{ij}\big)\Big)+2g^{su}g^{rt}\frac{\partial R_{rstu}}{\partial g_{ij,kh}}g^{ab}\phi_{;b}\frac{\partial\mu}{\partial\rho}.\end{aligned}$$

Applying the chain rule in partial derivative yields

$$\begin{aligned}\Lambda^{a;ij,kh} =&\frac{1}{4}\sqrt{g}\bigg(\frac{\partial\sigma}{\partial\phi_{,a}}\big(\phi^{,i}\phi^{,h}g^{kj}+\phi^{,i}\phi^{,k}g^{hj}+\phi^{,j}\phi^{,h}g^{ki}+\phi^{,j}\phi^{,k}g^{hi}-2\phi^{,i}\phi^{,j}g^{kh}\\ &-2\phi^{,k}\phi^{,h}g^{ij}\big)\bigg)+\frac{1}{4}\sqrt{g}\sigma\frac{\partial}{\partial\phi_{,a}}\bigg(\big(\phi^{,i}\phi^{,h}g^{kj}+\phi^{,i}\phi^{,k}g^{hj}+\phi^{,j}\phi^{,h}g^{ki}\\ &+\phi^{,j}\phi^{,k}g^{hi}-2\phi^{,i}\phi^{,j}g^{kh}-2\phi^{,k}\phi^{,h}g^{ij}\big)\bigg)\\ &+\frac{\partial\mu}{\partial\rho}\phi^{;a}\big(g^{kj}g^{ih}+g^{hj}g^{ik}-2g^{kh}g^{ij}\big).\end{aligned}$$

By taking the derivative, we have

$$\begin{aligned}\Lambda^{a;ij,kh} =&\frac{1}{2}\sqrt{g}\frac{\partial\sigma}{\partial\rho}\bigg(\phi^{,a}\big(\phi^{,i}\phi^{,h}g^{kj}+\phi^{,i}\phi^{,k}g^{hj}+\phi^{,j}\phi^{,h}g^{ki}+\phi^{,j}\phi^{,k}g^{hi}-2\phi^{,i}\phi^{,j}g^{kh}\\ &-2\phi^{,k}\phi^{,h}g^{ij}\big)\bigg)+\frac{1}{4}\sqrt{g}\sigma\bigg(\big(g^{ai}\phi^{,h}g^{kj}+\phi^{,i}g^{ah}g^{kj}+g^{ai}\phi^{,k}g^{hj}\\ &+\phi^{,i}g^{ak}g^{hj}+g^{aj}\phi^{,h}g^{ki}+\phi^{,j}g^{ah}g^{ki}+g^{aj}\phi^{,k}g^{hi}+\phi^{,j}g^{ak}g^{hi}\\ &-2g^{ai}\phi^{,j}g^{kh}-2\phi^{,i}g^{aj}g^{kh}-2g^{ak}\phi^{,h}g^{ij}-2\phi^{,k}g^{ah}g^{ij}\big)\bigg)\\ &+\frac{\partial\mu}{\partial\rho}\phi^{;a}\big(g^{kj}g^{ih}+g^{hj}g^{ik}-2g^{kh}g^{ij}\big).\end{aligned}$$

Now, using the equation above, we can calculate the expression in equation (2.35) as

$$\begin{aligned}0 =&\Lambda^{a;ij,kh}+\Lambda^{k;ij,ah}+\Lambda^{h;ij,ka}\\ =&\sqrt{g}\frac{\partial\sigma}{\partial\rho}\big(\phi^{,a}\phi^{,i}\phi^{,h}g^{kj}+\phi^{,a}\phi^{,i}\phi^{,k}g^{hj}+\phi^{,a}\phi^{,j}\phi^{,h}g^{ki}+\phi^{,a}\phi^{,j}\phi^{,k}g^{hi}\\ &+\phi^{,a}\phi^{,i}\phi^{,h}g^{aj}+\phi^{,a}\phi^{,j}\phi^{,h}g^{ai}-\phi^{,a}\phi^{,i}\phi^{,j}g^{kh}-\phi^{,k}\phi^{,i}\phi^{,j}g^{ah}-\phi^{,h}\phi^{,i}\phi^{,j}g^{ka}\\ &-3\phi^{,k}\phi^{,a}\phi^{,h}g^{ij}\big)+\frac{1}{2}\sqrt{g}\sigma\big(g^{ai}\phi^{,h}g^{kj}+g^{ai}\phi^{,k}g^{hj}+g^{aj}\phi^{,h}g^{ki}+g^{aj}\phi^{,k}g^{hi}\\ &+g^{ki}\phi^{,a}g^{hj}+g^{hi}\phi^{,a}g^{kj}-2g^{ak}\phi^{,h}g^{ij}-2\phi^{,k}g^{ah}g^{ij}-2\phi^{,a}g^{kh}g^{ij}\big)\\ &+\sqrt{g}\frac{\partial\mu}{\partial\rho}\bigg(\phi^{;a}\big(g^{kj}g^{ih}+g^{hj}g^{ik}-2g^{kh}g^{ij}\big)+\phi^{;k}\big(g^{aj}g^{ih}+g^{hj}g^{ia}-2g^{ah}g^{ij}\big)\\ &+\phi^{;h}\big(g^{kj}g^{ia}+g^{aj}g^{ik}-2g^{ka}g^{ij}\big)\bigg).\end{aligned}$$

In order to satisfy this equation, we have $\frac{\partial\sigma}{\partial\rho}=0$ and $\frac{\partial\mu}{\partial\rho}=-\frac{1}{2}\sigma$. Therefore, $\mu=-\frac{1}{2}\sigma\rho+\theta(\phi)$.

Theorem 2.2 *If $n=4$ then the most general $L=L(g_{ij},g_{ij,k},g_{ij,kh};\phi,\phi_{,i})$ for which the corresponding Euler-Lagrange equations are at most of second-order in g_{ij} and*

ϕ *is* $L = \beta_1 L_1 + \beta_2 L_2 + \beta_3 L_3 + \eta L_4 + c L_5$, *where*

$$\begin{aligned}
L_1 &= \sqrt{g}(R^2 - 4R_{ij}R^{ij} + R_{ijkh}R^{ijkh}),\\
L_2 &= \sqrt{g}G^{ij}\phi_{,i}\phi_{,j},\\
L_3 &= \sqrt{g}R,\\
L_4 &= \sqrt{g},\\
L_5 &= *R^{ij}{}_{kh}R^{kh}{}_{ij},
\end{aligned}$$

and $\beta_1 = \beta_1(\phi)$, $\beta_2 = \beta_2(\phi)$, $\beta_3 = \beta_3(\phi)$, $\eta = \eta(\phi, \rho)$ *and* c *is a constant.*

2.5 The Euler-Lagrange Equations of L

By virtue of (2.29) and (2.26) (or (2.32), we can calculate the Euler-Lagrange equations for the Lagrangian given in Theorem 2.2. These long calculations can be found in Appendix C.3.

$$\begin{aligned}
E^{ij}(\beta_1 L_1) =& 4\sqrt{g}\beta_1'\big(\phi^{|ia}R_a{}^j + \phi^{|ja}R_a{}^i + \frac{1}{2}R(g^{ab}\phi_{|ab}g^{ij} - \phi^{|ij})\\
&- g^{ij}\phi_{|ab}R^{ab} - g^{ab}\phi_{|ab}R^{ij} - \phi_{|ab}R^{aijb}\big)\\
&+ 4\sqrt{g}\beta_1''\big(\phi^{|i}\phi_{|a}R^{aj} + \phi^{|j}\phi_{|a}R^{ai} + \frac{1}{2}R(g^{ij}\phi_{|a}\phi^{|a} - \phi^{|i}\phi^{|j})\\
&- g^{ij}\phi_{|a}\phi_{|b}R^{ab} - R^{ij}\phi_{|a}\phi^{|a} - \phi_{|a}\phi_{|b}R^{aijb}\big),\\
E(\beta_1 L_1) =& -\beta_1' L_1,\\
E^{ij}(\beta_2 L_2) =& \sqrt{g}\beta_2\bigg(\frac{1}{2}g^{ij}\big((g^{ab}\phi_{|ab})^2 - \phi^{|ab}\phi_{|ab} - 2\phi_{|a}\phi_{|b}R^{ab} + \frac{1}{2}\phi_{|a}\phi^{|a}R\big)\\
&+ \phi^{|ia}\phi_{|a}{}^j - g^{ab}\phi_{|ab}\phi^{|ij} + \phi_{|a}\big(\phi^{|i}R^{aj} + \phi^{|j}R^{ai}\big)\\
&- \frac{1}{2}\phi^{|i}\phi^{|j}R - \frac{1}{2}\phi_{|a}\phi^{|a}R^{ij} - \phi_{|a}\phi_{|b}R^{aijb}\bigg)\\
&+ \frac{1}{2}\sqrt{g}\beta_2'\bigg(g^{ij}\big(\phi^{|a}\phi_{|a}g^{kh}\phi_{|kh} - \phi^{|a}\phi^{|b}\phi_{|ab}\big)\\
&- \phi^{|i}\phi^{|j}g^{|ab}\phi_{|ab} + \phi^{|a}\big(\phi^{|i}\phi_{|a}{}^j + \phi^{|j}\phi_{|a}{}^i\big) - \phi^{|ij}\phi_{|a}\phi^{|a}\bigg),\\
E(\beta_2 L_2) =& \sqrt{g}G^{ab}\big(\beta_2\phi_{|ab} + \beta_2'\phi_{|a}\phi_{|b}\big),\\
E^{ij}(\beta_3 L_3) =& \sqrt{g}\big(-\beta_3''\phi^{|i}\phi^{|j} + \beta_3''g^{ij}\phi_{|a}\phi^{|a} - \beta_3'\phi^{|ij} + \beta_3'g^{ij}g^{ab}\phi_{|ab} + \beta_3 G^{ij}\big),\\
E(\beta_3 L_3) =& -\sqrt{g}\beta_3' R,
\end{aligned}$$

$$
\begin{aligned}
E^{ij}(\eta L_4) =& \sqrt{g}\left(\frac{\partial \eta}{\partial \rho}\phi^{|i}\phi^{|j} - \frac{1}{2}g^{ij}\eta\right),\\
E(\eta L_4) =& 2\sqrt{g}\left(\frac{\partial^2 \eta}{\partial \phi \partial \rho}\phi_{|a}\phi^{|a} + 2\frac{\partial^2 \eta}{\partial^2 \rho}\phi_{|ab}\phi^{|a}\phi^{|b} + \frac{\partial \eta}{\partial \rho}g^{ab}\phi_{|ab} - \frac{1}{2}\frac{\partial \eta}{\partial \phi}\right),\\
E^{ij}(cL_5) =& 0,\\
E(cL_5) =& 0.
\end{aligned}
$$

Finally, we have found the field equations of the Lagrangian that is given in Theorem 2.2. Note that these field equations are second-order in derivatives of the metric and the scalar field.

CHAPTER 3

SECOND-ORDER SCALAR-TENSOR FIELD EQUATIONS IN A FOUR-DIMENSIONAL SPACETIME

In this chapter, we review the paper of Horndeski [16].

The field equations that we obtained in Section 2.5 are not the most general second-order field expressions due to fact that we have chosen our Lagrangian to be at most of second-order in the metric and at most of first order in ϕ. Now, we can look for the Lagrangian which yields second-order Euler-Lagrange equations. Therefore, we will not put any restriction on our Lagrangian in the beginning. Our Lagrangian can be of the form

$$L = L(g_{ij}, g_{ij,i_1}, \ldots, g_{ij,i_1 \ldots i_p}, \phi, \phi_{,i_1}, \ldots, \phi_{,i_1 \ldots i_q}), \tag{3.1}$$

where $p, q \geq 2$ in a four-dimensional spacetime.

The field equations of (3.1) are given by

$$E^{ij}(L) = \sum_{h=0}^{p} (-1)^{h+1} \frac{d}{dx^{i_1}} \cdots \frac{d}{dx^{i_h}} \frac{\partial L}{\partial g_{ij,i_1 \ldots i_h}} \tag{3.2}$$

and

$$E(L) = \sum_{h=0}^{q} (-1)^{h+1} \frac{d}{dx^{i_1}} \cdots \frac{d}{dx^{i_h}} \frac{\partial L}{\partial \phi_{,i_1 \ldots i_h}}, \tag{3.3}$$

where equations (3.2) and (3.3) follow from the variation of L with respect to g_{ij} and ϕ, respectively.

By employing similar techniques as in Section 2.2 on the new L, we can show easily that these two equations are related by

$$E^{ij}{}_{|j}(L) = \frac{1}{2} g^{ij} \phi_{|j} E(L). \tag{3.4}$$

We demand both $E^{ij}(L)$ and $E(L)$ to involve at most second-order in the derivatives of the metric g_{ij} and the scalar ϕ. Therefore, if $E^{ij}(L)$ is of second-order then $E^{ij}{}_{|j}(L)$ will be of third-order. However, by (3.4), we see that $E^{ij}{}_{|j}(L)$ has to be of second-order.

3.1 The Construction of the Most General Form of A^{ij}

From now on, we will be using the following notation: Given $A^{\cdots}_{\cdots}$ we define

$$A^{\cdots;ab,cd}_{\cdots} \equiv \frac{\partial A^{\cdots}_{\cdots}}{\partial g_{ab,cd}}, \qquad \text{and} \qquad A^{\cdots;ab}_{\cdots} \equiv \frac{\partial A^{\cdots}_{\cdots}}{\partial \phi_{,ab}}.$$

For example, we have

$$A^{ij;ab;cd,ef;rs} = \frac{\partial}{\partial \phi_{,rs}} \frac{\partial}{\partial g_{cd,ef}} \frac{\partial}{\partial \phi_{,ab}} A^{ij}. \tag{3.5}$$

Since (3.4) puts strict restrictions on $E^{ij}{}_{|j}(L)$, in a four-dimensional spacetime, we can seek for the most general symmetric tensor density of the form

$$A^{ij} = A^{ij}(g_{ab}, g_{ab,c}, g_{ab,cd}, \phi, \phi_{,c}, \phi_{,cd}) \tag{3.6}$$

which is such that

$$A^{ij}{}_{|j} = A^{ij}(g_{ab}, g_{ab,c}, g_{ab,cd}, \phi, \phi_{,c}, \phi_{,cd}). \tag{3.7}$$

Once we find the most general form of A^{ij}, we know that $E^{ij}(L)$ that we seek will be contained in A^{ij}.

As a result of (3.4), we should be able to express $A^{ij}{}_{|j}$ as follows

$$A^{ij}{}_{|j} = \phi^{|i} A, \tag{3.8}$$

where A is a scalar density of the form

$$A = A(g_{ab}, g_{ab,c}, g_{ab,cd}, \phi, \phi_{,c}, \phi_{,cd}).$$

As a result of equation (3.7), we can easily write

$$\frac{\partial A^{ij}{}_{|j}}{\partial g_{rs,tvu}} = 0 \tag{3.9}$$

and

$$\frac{\partial A^{ij}{}_{|j}}{\partial \phi_{,rst}} = 0. \tag{3.10}$$

Note that A^{ij} is a tensor density and repeated partial differentiation of it with respect to $g_{ab,cd}$ and $\phi_{,ab}$ will also yield tensor densities. In order to prove this, we may apply the same method that we have used in Section 2.1.1.

Since A^{ij} is a tensor density, we may write

$$BA^{tu} = B^t{}_r B^u{}_s \overline{A}^{rs}, \tag{3.11}$$

and taking the derivative with respect to $g_{\mu\nu,\rho\sigma}$ yields

$$\begin{aligned}
B\frac{\partial A^{tu}}{\partial g_{\mu\nu,\rho\sigma}} =& \frac{\partial \overline{A}^{rs}}{\partial \overline{g}_{ij,kh}} \frac{\partial \overline{g}_{ij,kh}}{\partial g_{\mu\nu,\rho\sigma}} B^t{}_r B^u{}_s + \frac{\partial \overline{A}^{rs}}{\partial \overline{g}_{ij,k}} \frac{\partial \overline{g}_{ij,k}}{\partial g_{\mu\nu,\rho\sigma}} B^t{}_r B^u{}_s \\
&+ \frac{\partial \overline{A}^{rs}}{\partial \overline{g}_{ij}} \frac{\partial \overline{g}_{ij}}{\partial g_{\mu\nu,\rho\sigma}} B^t{}_r B^u{}_s + \frac{\partial \overline{A}^{rs}}{\partial \overline{\phi}} \frac{\partial \overline{\phi}}{\partial g_{\mu\nu,\rho\sigma}} B^t{}_r B^u{}_s \\
&+ \frac{\partial \overline{A}^{rs}}{\partial \overline{\phi}_{,i}} \frac{\partial \overline{\phi}_{,i}}{\partial g_{\mu\nu,\rho\sigma}} B^t{}_r B^u{}_s + \frac{\partial \overline{A}^{rs}}{\partial \overline{\phi}_{,ij}} \frac{\partial \overline{\phi}_{,ij}}{\partial g_{\mu\nu,\rho\sigma}} B^t{}_r B^u{}_s .
\end{aligned}$$

Except for the first term all the terms vanish on the right hand side. Therefore, we have

$$BA^{tu;\mu\nu,\rho\sigma} = B^t{}_r B^u{}_s B^\mu{}_a B^\nu{}_b B^\rho{}_c B^\sigma{}_d \overline{A}^{rs;ab,cd}$$

which proves that $A^{ij;ab,cd}$ is a tensor density. Analogously, taking the partial derivative of (3.11) with respect to $\phi_{,\mu\nu}$ yields

$$BA^{tu;\mu\nu} = B^t{}_r B^u{}_s B^\mu{}_a B^\nu{}_b \overline{A}^{rs;ab},$$

which proves that $A^{ij;ab}$ is a tensor density.

Since $A^{ij}{}_{|j}$ is expected to be of third-order, applying similar methods above shows that equations (3.9) and (3.10) are tensorial conditions. Analogously to the derivation of (2.12), we can easily obtain

$$A^{ij;ab,cd} + A^{ij;ac,bd} + A^{ij;ad,cb} = 0. \tag{3.12}$$

Therefore, one has the following symmetries

$$A^{ij;ab,cd} = A^{ij;ba,cd} = A^{ij;cd,ab}.$$

Since A^{ij} is a tensor density of the form (3.6), we can calculate ${A^{ij}}_{|j}$ as

$$\begin{aligned} {A^{ij}}_{|j} =& \frac{\partial A^{ij}}{\partial g_{ab}} g_{ab,j} + \frac{\partial A^{ij}}{\partial g_{ab,c}} g_{ab,cj} + A^{ij;ab,cd} g_{ab,cdj} \\ &+ \frac{\partial A^{ij}}{\partial \phi} \phi_{,j} + \frac{\partial A^{ij}}{\partial \phi_{,a}} \phi_{,aj} + A^{ij;ab} \phi_{,abj} + A^{kj} \mathit{\Gamma}^{i}_{kj} \,. \end{aligned}$$

Equations (3.9) and (3.10) will hold if and only if

$$A^{ij;ab,cd} \frac{\partial g_{ab,cdj}}{\partial g_{rs,tvu}} = 0$$

and

$$A^{ij;ab} \frac{\partial \phi_{,abj}}{\partial \phi_{,rst}} = 0.$$

Therefore one finds that

$$A^{iu;rs,tv} + A^{it;rs,uv} + A^{iv;rs,tu} = 0 \tag{3.13}$$

and

$$A^{it;rs} + A^{ir;ts} + A^{is;rt} = 0, \tag{3.14}$$

respectively. Hence, one arrives at

$$A^{ij;ab,cd} = A^{ab;ij,cd} = A^{cd;ab,ij} \tag{3.15}$$

and

$$A^{ij;ab} = A^{ab;ij}, \tag{3.16}$$

respectively.

Now, we can use the definition which is used by Lovelock [19]:

Definition 3.1 *A quantity $B^{i_1 i_2 \ldots i_{2h-1} i_{2h} \ldots i_{2p}}$, where $p > 1$, is said to enjoy property $\mathcal{S}$ if it satisfies the following conditions:*

(i) it is symmetric in (i_{2h-1}, i_{2h}) for $h = 1, \ldots, p$;

(ii) it is symmetric under the interchange of the pair $i_1 i_2$ with the pair $i_{2h-1} i_{2h}$ for $h = 2, \ldots, p$;

(iii) it satisfies the cyclic identity involving any three of the four indices $(i_1 i_2)(i_{2h-1} i_{2h})$; i.e., when $h = 2$

$$B^{i_1 i_2 i_3 i_4 \ldots i_{2p}} + B^{i_2 i_3 i_1 i_4 \ldots i_{2p}} + B^{i_3 i_1 i_2 i_4 \ldots i_{2p}} = 0.$$

A quantity B^{ab} has property $\mathcal{S}$ if $B^{ab} = B^{ba}$.

By considering the symmetry relations above, we can say that when $n = 4$, then we have

$$A^{ij;i_1 i_2, i_3 i_4; i_5 i_6, i_7 i_8} = 0. \tag{3.17}$$

The proof of this is straightforward. Considering the symmetries (3.15) and (3.16), we are capable of interchanging any two groups of two indices. If three of any four indices in a group of four indices are the same, then from equation (3.12) (or (3.14), we get zero. Since there are 10 indices here, when $n = 4$, at least three indices must be the same. Making use of the symmetries introduced and (2.20), we can easily put these three indices in the same group of four indices. Therefore, we will have nothing but zeroes in a four-dimensional spacetime. Integrating equation (3.17) two times with respect to $g_{cd,ef}$ yields

$$A^{ab} = \beta^{abcdef} g_{cd,ef} + \beta^{ab}, \tag{3.18}$$

where $\beta^{abcdef} = \beta^{abcdef}(g_{ij}, g_{ij,k}, \phi, \phi_{,i}, \phi_{,ij})$, $\beta^{ab} = \beta^{ab}(g_{ij}, g_{ij,k}, \phi, \phi_{,i}, \phi_{,ij})$ and both enjoy property $\mathcal{S}$. By considering our symmetry relations, we can use the relation in (2.48) to find

$$\beta^{abcdef} g_{cd,ef} = \frac{2}{3} \beta^{abcdef} R_{ecdf} + J^{ab},$$

where $J^{ab} = J^{ab}(g_{ij}, g_{ij,k}, \phi, \phi_{,i}, \phi_{,ij})$ and J^{ab} is symmetric in (a, b). Now, we can rewrite (3.18) as

$$A^{ab} = \hat{\beta}^{abcdef} R_{ecdf} + \hat{\beta}^{ab}, \tag{3.19}$$

where $\hat{\beta}^{abcdef} = \hat{\beta}^{abcdef}(g_{ij}, g_{ij,k}, \phi, \phi_{,i}, \phi_{,ij})$, $\hat{\beta}^{ab} = \hat{\beta}^{ab}(g_{ij}, g_{ij,k}, \phi, \phi_{,i}, \phi_{,ij})$ and both enjoy property $\mathcal{S}$.

Employing the same technique used for (3.17), it can be easily shown that

$$\hat{\beta}^{abcdef;ij;kh} = 0. \tag{3.20}$$

As a result of this we can conclude that

$$\hat{\beta}^{abcdef;ij} = \hat{\beta}^{abcdef;ij}(g_{ij}, g_{ij,k}, \phi, \phi_{,i}).$$

As a consequence of Lemma 2.1, we have

$$\hat{\beta}^{abcdef;ij} = \hat{\beta}^{abcdef;ij}(g_{ij}, \phi, \phi_{,i}).$$

Consequently, we find

$$\hat{\beta}^{abcdef} = \xi^{abcdefgh}\phi_{,gh} + \alpha^{abcdef},$$

where $\xi^{abcdefij} = \xi^{abcdefij}(g_{ij}, \phi, \phi_{,i})$ with property $\mathcal{S}$ and

$$\alpha^{abcdef} = \alpha^{abcdef}(g_{ij}, g_{ij,k}, \phi, \phi_{,i}),$$

which also possesses property $\mathcal{S}$. Here we can replace $\phi_{,gh}$ by $\phi_{|gh} + \phi_{|r}\Gamma^{r}{}_{gh}$ to construct a tensorial equation of the form

$$\hat{\beta}^{abcdef} = \xi^{abcdefgh}\phi_{|gh} + \xi^{abcdef}, \tag{3.21}$$

where $\xi^{abcdef} = \xi^{abcdef}(g_{ij}, \phi, \phi_{,i})$ with property $\mathcal{S}$.

Now, applying the same techniques that we have used for equations (3.17) and (3.20), one finds when $n = 4$

$$\beta^{ab;cd;ef;gh;jk} = 0.$$

By integrating this several times, we obtain

$$\beta^{ab} = \psi^{abcdefgh}\phi_{|cd}\phi_{|ef}\phi_{|gh} + \psi^{abcdef}\phi_{|cd}\phi_{|ef} + \psi^{abcd}\phi_{|cd} + \psi^{ab}, \tag{3.22}$$

where $\psi^{abcdefgh}$, ψ^{abcdef}, ψ^{abcd} and ψ^{ab} are arbitrary tensor densities possessing property $\mathcal{S}$ and a function of g_{ij}, ϕ and $\phi_{,i}$. Combining our results that we get from equations (3.19), (3.21) and (3.22), we find

$$\begin{aligned} A^{ab} =& \xi^{abcdefgh}R^{ecdf}\,\phi_{|gh} + \xi^{abcdef}R^{ecdf} + \psi^{abcdefgh}\phi_{|cd}\phi_{|ef}\phi_{|gh} \\ &+ \psi^{abcdef}\phi_{|cd}\phi_{|ef} + \psi^{abcd}\phi_{|cd} + \psi^{ab}. \end{aligned} \tag{3.23}$$

Now, we need to find the most general form of these arbitrary tensor densities. To this end, we need to use similar techniques that we have used in Lemmas (B.2) and (B.3). The most general tensor densities which possess property $\mathcal{S}$ and a function of

g_{ij}, ϕ and $\phi_{,i}$ in a four-dimensional spacetime are

$$\begin{aligned}
\psi^{ab} =& \sqrt{g}C_1 g^{ab} + \sqrt{g}C_2\phi^{,a}\phi^{,b},\\
\psi^{abcd} =& \sqrt{g}C_3(g^{ac}g^{bd} + g^{ad}g^{bc} - 2g^{ab}g^{cd}) + \sqrt{g}C_4\big(\phi^{,a}\phi^{,c}g^{bd} + \phi^{,b}\phi^{,d}g^{ac}\\
&+ \phi^{,a}\phi^{,d}g^{bc} + \phi^{,b}\phi^{,c}g^{ad} - 2(\phi^{,a}\phi^{,b}g^{cd} + \phi^{,c}\phi^{,d}g^{ab})\big),\\
\psi^{abcdef} =& \frac{1}{\sqrt{g}}(C_5\phi_{,r}\phi_{,s} + C_6 g_{rs})\big(\epsilon^{acer}\epsilon^{bdfs} + \epsilon^{acfr}\epsilon^{bdes} + \epsilon^{ader}\epsilon^{bcfs} + \epsilon^{adfr}\epsilon^{bces}\big),\\
\psi^{abcdefgh} =& \frac{C_7}{\sqrt{g}}\big(\epsilon^{aceg}\epsilon^{bdfh} + \epsilon^{aceh}\epsilon^{bdfg} + \epsilon^{acfg}\epsilon^{bdeh} + \epsilon^{acfh}\epsilon^{bdeg}\\
&+ \epsilon^{adfh}\epsilon^{bceg} + \epsilon^{adfg}\epsilon^{bceh} + \epsilon^{adeh}\epsilon^{bcfg} + \epsilon^{adeg}\epsilon^{bcfh}\big),
\end{aligned}$$

where $C_1, \ldots, C_7$ are arbitrary functions of g_{ij}, ϕ and $\phi_{,i}$. However, as we have shown in Lemma B.7, $C_1, \ldots, C_7$ can be written as arbitrary functions of ϕ and ρ.

Using these together with the symmetry properties of the Riemann curvature tensor, (3.23) becomes

$$\begin{aligned}
A^{ab} =& \sqrt{g}\big(K_1\delta^{acde}_{fhjk}g^{fb}\phi_{|c}{}^{|h}R_{de}{}^{jk} + K_2\delta^{acd}_{efh}g^{eb}R_{cd}{}^{fh}\\
&+ K_3\delta^{acde}_{fhjk}g^{fb}\phi_{|c}\phi^{|h}R_{de}{}^{jk} + K_4\delta^{acde}_{fhjk}g^{fb}\phi_{|c}{}^{|h}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k}\\
&+ K_5\delta^{acd}_{efh}g^{eb}\phi_{|c}{}^{|f}\phi_{|d}{}^{|h} + K_6\delta^{acde}_{fhjk}g^{fb}\phi_{|c}\phi^{|h}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k}\\
&+ K_7\delta^{ac}_{de}g^{db}\phi_{|c}{}^{|e} + K_8\delta^{acd}_{efh}g^{eb}\phi_{|c}\phi^{|f}\phi_{|d}{}^{|h} + K_9 g^{ab} + K_{10}\phi^{|a}\phi^{|b}\big),
\end{aligned} \tag{3.24}$$

where $K_1, \ldots, K_{10}$ are arbitrary functions of ϕ and ρ.

3.2 The Consequences of Demanding that $A^{ij}{}_{|j} = \phi^{|i}A$

Equation (3.24) represents the most general form of the tensor density of the form (3.6) together with (3.7). We remark that both A^{ab} and $A^{ab}{}_{|b}$ are at most of second-order.

We can calculate $A^{ab}{}_{|b}$ by using the Ricci and Bianchi identities as

$$\begin{aligned}A^{ab}{}_{|b} =&\sqrt{g}\big(K_1'\delta^{acde}_{fhjk}\phi^{|f}\phi_{|c}{}^{|h}R_{de}{}^{jk} + 2\dot{K}_2\delta^{acd}_{efh}\phi_{|p}\phi^{|pe}R_{cd}{}^{fh}\\&+ K_3\delta^{acde}_{fhjk}\phi^{|h}\phi_{|c}{}^{|f}R_{de}{}^{jk} + K_5\delta^{acd}_{efh}\phi^{|m}\phi_{|d}{}^{|h}R_{mc}{}^{fe}\\&+ 2\dot{K}_1\delta^{acde}_{fhjk}\phi_{|p}\phi^{|pf}\phi_{|c}{}^{|h}R_{de}{}^{jk} + \frac{3}{2}K_4\delta^{acde}_{fhjk}\phi^{|m}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k}R_{mc}{}^{hf}\\&+ \frac{1}{2}K_1\delta^{acde}_{fhjk}\phi^{|m}R_{mc}{}^{hf}R_{de}{}^{jk} + K_2'\delta^{acd}_{efh}\phi^{|e}R_{cd}{}^{fh}\\&+ \frac{1}{2}K_7\delta^{ac}_{de}\phi^{|m}R_{mc}{}^{ed} + \frac{1}{2}K_8\delta^{acd}_{efh}\phi_{|c}\phi^{|f}\phi^{|m}R_{md}{}^{he}\\&+ 2\dot{K}_3\delta^{acde}_{fhjk}\phi_{|p}\phi^{|pf}\phi_{|c}\phi^{|h}R_{de}{}^{jk} + K_6\delta^{acde}_{fhjk}\phi_{|c}\phi^{|h}\phi^{|m}\phi_{|e}{}^{|k}R_{md}{}^{jf}\\&+ K_4'\delta^{acde}_{fhjk}\phi^{|f}\phi_{|c}{}^{|h}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k} + 2\dot{K}_5\delta^{acd}_{efh}\phi_{|p}\phi^{|pe}\phi_{|c}{}^{|f}\phi_{|d}{}^{|h}\\&+ K_6\delta^{acde}_{fhjk}\phi^{|h}\phi_{|c}{}^{|f}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k} + 2\dot{K}_8\delta^{acd}_{efh}\phi_{|p}\phi^{|pe}\phi_{|c}\phi^{|f}\phi_{|d}{}^{|h}\\&+ 2\dot{K}_6\delta^{acde}_{fhjk}\phi_{|p}\phi^{|pf}\phi_{|c}\phi^{|h}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k} + 2\dot{K}_4\delta^{acde}_{fhjk}\phi_{|p}\phi^{|pf}\phi_{|c}{}^{|h}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k}\\&+ K_5'\delta^{acd}_{efh}\phi^{|e}\phi_{|c}{}^{|f}\phi_{|d}{}^{|h} + 2\dot{K}_7\delta^{ac}_{de}\phi_{|p}\phi^{|pd}\phi_{|c}{}^{|e}\\&+ K_8\delta^{acd}_{efh}\phi^{|f}\phi_{|c}{}^{|e}\phi_{|d}{}^{|h} + (2\dot{K}_9 + K_{10})\phi_{|b}\phi^{|ab} + K_7'\delta^{ac}_{de}\phi^{|d}\phi_{|c}{}^{|e}\\&+ \phi^{|a}(K_9' + \rho K_{10}' + 2\dot{K}_{10}\phi^{|b}\phi^{|c}\phi_{|bc} + K_{10}\phi_{|c}{}^{|c})\big),\end{aligned}$$

where a prime denotes a partial derivative with respect to ϕ and a dot denotes a partial derivative with respect to ρ. This equation can be further reduced to

$$\begin{aligned}A^{ab}{}_{|b} =&\sqrt{g}\phi^{|a}Q + \sqrt{g}\big(\alpha\delta^{ace}_{hjk}\phi^{|d}\phi_{|c}{}^{|h}R_{de}{}^{jk} + \beta\delta^{ade}_{hjk}\phi^{|c}\phi_{|c}{}^{|h}R_{de}{}^{jk}\\&- \gamma\delta^{ace}_{fjk}\phi_{|p}\phi^{|pf}\phi_{|c}\phi^{|d}R_{de}{}^{jk} + \epsilon\delta^{abce}_{hjkm}\phi^{|p}\phi_{|c}{}^{|k}\phi_{|e}{}^{|m}R_{pb}{}^{jh}\\&+ \mu\delta^{ac}_{de}\phi^{|m}R_{mc}{}^{ed} + \nu\delta^{ade}_{fjk}\phi^{|p}\phi_{|p}{}^{|f}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k}\\&+ 2\omega\delta^{ac}_{de}\phi^{|p}\phi_{|p}{}^{|d}\phi_{|c}{}^{|e} + \xi\phi_{|p}\phi^{|ap}\big),\end{aligned} \tag{3.25}$$

where

$$\begin{aligned}Q =&\dot{K}_1\delta^{bcde}_{hjkm}\phi_{|b}{}^{|h}\phi_{|c}{}^{|j}R_{de}{}^{km} + (K_1' - K_3)\delta^{cde}_{fjk}\phi_{|c}{}^{|f}R_{de}{}^{jk}\\&- 2\dot{K}_3\delta^{cde}_{fjk}\phi^{|p}\phi_{|p}{}^{|f}\phi_{|c}R_{de}{}^{jk} - K_6\delta^{cde}_{fjk}\phi_{|c}\phi^{|m}\phi_{|e}{}^{|k}R_{md}{}^{jf}\\&- \frac{1}{8}K_1\delta^{bcde}_{hjkm}R_{bc}{}^{hj}R_{de}{}^{km} + K_2'\delta^{cd}_{fh}R_{cd}{}^{fh} - \frac{1}{2}K_8\delta^{cd}_{eh}\phi_{|c}\phi^{|m}R_{md}{}^{he}\\&- 2\dot{K}_8\delta^{cd}_{eh}\phi_{|p}\phi^{|pe}\phi_{|c}\phi_{|d}{}^{|h} + \frac{1}{2}\dot{K}_4\delta^{bcde}_{hjkm}\phi_{|b}{}^{|h}\phi_{|c}{}^{|j}\phi_{|d}{}^{|k}\phi_{|e}{}^{|m}\\&- 2\dot{K}_6\delta^{cde}_{fjk}\phi^{|p}\phi_{|p}{}^{|f}\phi_{|c}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k} + (K_4' - K_6)\delta^{cde}_{hjk}\phi_{|c}{}^{|h}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k}\\&+ (K_5' - K_8)\delta^{cd}_{fh}\phi_{|c}{}^{|f}\phi_{|d}{}^{|h} + K_9' + \rho K_{10}'\\&+ 2\dot{K}_{10}\phi^{|b}\phi^{|c}\phi_{|bc} + (K_{10} + K_7')\phi_{|c}{}^{|c}\end{aligned} \tag{3.26}$$

and

$$
\begin{aligned}
&\alpha = 2K_1' - 2K_3 + K_5 + \rho K_6, &&\beta = 2\dot{K}_2 - K_1' + K_3 + 2\rho\dot{K}_3,\\
&\gamma = 4\dot{K}_3 + K_6, &&\epsilon = 2\dot{K}_1 + \frac{3}{2}K_4,\\
&\mu = 2K_2' + \frac{1}{2}K_7 + \frac{1}{2}\rho K_8, &&\nu = 2\dot{K}_5 + 3K_6 - 3K_4' + 2\rho\dot{K}_6,\\
&\omega = \dot{K}_7 - K_5' + K_8 + \rho\dot{K}_8, &&\xi = 2\dot{K}_9 + K_{10} - K_7'.
\end{aligned}
\tag{3.27}
$$

Due to the fact that (3.25) must satisfy (3.8), there must be a scalar density T of the form

$$
T = T(g_{ab}, g_{ab,c}, g_{ab,cd}, \phi, \phi_{,c}, \phi_{,cd}),
$$

which is such that

$$
\begin{aligned}
\phi^{|a}T =&\sqrt{g}\big(\alpha\delta^{ace}_{hjk}\phi^{|d}\phi_{|c}{}^{|h}R_{de}{}^{jk} + \beta\delta^{ade}_{hjk}\phi^{|c}\phi_{|c}{}^{|h}R_{de}{}^{jk}\\
&- \gamma\delta^{ace}_{fjk}\phi_{|p}\phi^{|pf}\phi_{|c}\phi^{|d}R_{de}{}^{jk} + \epsilon\delta^{abce}_{hjkm}\phi^{|p}\phi_{|c}{}^{|k}\phi_{|e}{}^{|m}R_{pb}{}^{jh}\\
&+ \mu\delta^{ac}_{de}\phi^{|m}R_{mc}{}^{ed} + \nu\delta^{ade}_{fjk}\phi^{|p}\phi_{|p}{}^{|f}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k}\\
&+ 2\omega\delta^{ac}_{de}\phi^{|p}\phi_{|p}{}^{|d}\phi_{|c}{}^{|e} + \xi\phi_{|p}\phi^{|ap}\big).
\end{aligned}
\tag{3.28}
$$

Therefore, we need to find a solution for T to find the most general form of $A^{ab}{}_{|b}$. We can differentiate this equation once with respect to $g_{rs,tu}$ and twice with respect to $\phi_{,uw}$ to obtain

$$
\phi^{|a}T^{iq;uw;rs,tu} = \epsilon\sqrt{g}\delta^{abce}_{hjkm}\phi^{|p}\frac{\partial^2(\phi_{|c}{}^{|k}\phi_{|e}{}^{|m})}{\partial\phi_{,iq}\partial\phi_{,uw}}g^{jd}g^{hf}\frac{\partial R_{pbdf}}{\partial g_{rs,tu}}.
$$

Multiplying this with $g_{iq}g_{uw}g_{rs}$ yields

$$
\phi^{|a}g_{iq}g_{uw}g_{rs}T^{iq;uw;rs,tu} = 4\sqrt{g}\epsilon(\phi^{|a}g^{vt} + \phi^{|t}g^{va} + \phi^{|v}g^{ta}).
\tag{3.29}
$$

We can always construct a non-null vector field X^a which is orthogonal to $\phi_{,a}$:

$$
X_aX^a \neq 0, \quad \phi_{,a}X^a = 0.
$$

Multiplying (3.29) by X_aX_t yields

$$
0 = 4\epsilon\sqrt{g}\phi_{,v}
$$

and therefore, $\epsilon = 0$. Applying similar methods on (3.28) shows that

$$
\alpha = \beta = \gamma = \epsilon = \mu = \nu = \omega = \xi = 0.
$$

As a result, we have $B = 0$. Thus, we have eight partial differential equations following from (3.27). Two of these can be expressed in terms of the others

$$\nu = 2\dot{\alpha} + \gamma - 2\epsilon' \qquad \text{and} \qquad \omega = 2\dot{\mu} - \alpha' + \rho\gamma' - 2\beta'. \tag{3.30}$$

Consequently, the remaining six equations are as follows

$$\begin{aligned} K_4 &= -\frac{4}{3}\dot{K}_1, & K_5 &= 2K_3 - 2K_1' + 4\rho\dot{K}_3, \\ K_6 &= -4\dot{K}_3, & K_2 &= \frac{1}{2}F + W, \\ K_7 &= -2F' - 2W' - \rho K_8, & K_{10} &= -2F'' - 4W'' - \rho K_8' - 2\dot{K}_9, \end{aligned} \tag{3.31}$$

where K_1, K_3, K_8 and K_9 are arbitrary functions of ϕ and ρ, W is an arbitrary function of ϕ only and F is given by

$$F = F(\phi, \rho) = \int \big(K_1' - K_3 - 2\rho\dot{K}_3\big) d\rho. \tag{3.32}$$

By inserting (3.31) into (3.24), we obtain

$$\begin{aligned} A^{ab} =& \sqrt{g}\big(K_1\delta^{acde}_{fhjk}g^{fb}\phi_{|c}{}^{|h}R_{de}{}^{jk} + (\frac{1}{2}F + W)\delta^{acd}_{efh}g^{eb}R_{cd}{}^{fh} \\ &+ K_3\delta^{acde}_{fhjk}g^{fb}\phi_{|c}\phi^{|h}R_{de}{}^{jk} - \frac{4}{3}\dot{K}_1\delta^{acde}_{fhjk}g^{fb}\phi_{|c}{}^{|h}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k} \\ &+ (2K_3 - 2K_1' + 4\rho\dot{K}_3)\delta^{acd}_{efh}g^{eb}\phi_{|c}{}^{|f}\phi_{|d}{}^{|h} \\ &- 4\dot{K}_3\delta^{acde}_{fhjk}g^{fb}\phi_{|c}\phi^{|h}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k} - (2F' + 4W' + \rho K_8)\delta^{ac}_{de}g^{db}\phi_{|c}{}^{|e} \\ &+ K_8\delta^{acd}_{efh}g^{eb}\phi_{|c}\phi^{|f}\phi_{|d}{}^{|h} + K_9 g^{ab} \\ &- (2F'' + 4W'' + \rho K_8' + 2\dot{K}_9)\phi^{|a}\phi^{|b}\big). \end{aligned} \tag{3.33}$$

Using equations (3.25) and (3.26) together with (3.31) yields

$$
\begin{aligned}
A^{ab}{}_{|b} =& \sqrt{g}\phi^{|a}\big(\dot{K}_1\delta^{bcde}_{hjkm}\phi_{|b}{}^{|h}\phi_{|c}{}^{|j}R_{de}{}^{km} + (K_1' - K_3)\delta^{cde}_{fjk}\phi_{|c}{}^{|f}R_{de}{}^{jk} \\
&- 2\dot{K}_3\delta^{cde}_{fjk}\phi^{|p}\phi_{|p}{}^{|f}\phi_{|c}R_{de}{}^{jk} + 4\dot{K}_3\delta^{cde}_{fjk}\phi_{|c}\phi^{|m}\phi_{|e}{}^{|k}R_{md}{}^{jf} \\
&- \frac{1}{8}K_1\delta^{bcde}_{hjkm}R_{bc}{}^{hj}R_{de}{}^{km} + (\frac{1}{2}F' + W')\delta^{cd}_{fh}R_{cd}{}^{fh} \\
&- \frac{1}{2}K_8\delta^{cd}_{eh}\phi_{|c}\phi^{|m}R_{md}{}^{he} - 2\dot{K}_8\delta^{cd}_{eh}\phi_{|p}\phi^{|pe}\phi_{|c}\phi_{|d}{}^{|h} \\
&- \frac{2}{3}\ddot{K}_1\delta^{bcde}_{hjkm}\phi_{|b}{}^{|h}\phi_{|c}{}^{|j}\phi_{|d}{}^{|k}\phi_{|e}{}^{|m} + 8\ddot{K}_3\delta^{cde}_{fjk}\phi^{|p}\phi_{|p}{}^{|f}\phi_{|c}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k} \\
&+ (4\dot{K}_3 - \frac{4}{3}\dot{K}_1')\delta^{cde}_{hjk}\phi_{|c}{}^{|h}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k} \\
&+ (2K_3' - 2K_1'' + 4\rho\dot{K}_3' - K_8)\delta^{cd}_{fh}\phi_{|c}{}^{|f}\phi_{|d}{}^{|h} \\
&+ K_9' - \rho(2F''' + 4W''' + \rho K_8'' + 2\dot{K}_9') \\
&- 2(2\dot{F}'' + K_8' + \rho\dot{K}_8' + 2\ddot{K}_9)\phi^{|b}\phi^{|c}\phi_{|bc} \\
&- (4F'' + 8W'' + 2\rho K_8' + 2\dot{K}_9)\phi_{|c}{}^{|c}\big).
\end{aligned}
\tag{3.34}
$$

Now, the relation $A^{ab}{}_{|b} = \phi^{|a}A(g_{ij}, g_{ij,k}, g_{ij,kh}, \phi, \phi_{,h}, \phi_{,hk})$ between (3.33) and (3.34) holds.

3.3 Construction of Useful Tensor Densities

In this section, we will apply methods similar to those employed in Section 2.1.3. We remark that we have $\phi_{,ij}$ dependency in the Lagrangian which produces additional terms to the calculations carried out in Chapter 2.

We have the Lagrangian of the form

$$L = L(g_{ij}, g_{ij,k}, g_{ij,kh}, \phi, \phi_{,i}, \phi_{,ij}), \tag{3.35}$$

which satisfies

$$BL(g_{\mu\nu}, g_{\mu\nu,\rho}, g_{\mu\nu,\rho\sigma}, \phi, \phi_{,\mu}, \phi_{,\mu\nu}) = \overline{L}(\overline{g}_{ij}, \overline{g}_{ij,k}, \overline{g}_{ij,kh}, \overline{\phi}, \overline{\phi}_{,\mu}, \overline{\phi}_{,\mu\nu}). \tag{3.36}$$

We need to define a new quantity since we now have second-order derivatives of the scalar ϕ, i.e., $\phi_{,ij}$ in the Lagrangian,

$$\varPhi^{ij} \equiv \frac{\partial L}{\partial \phi_{,ij}}.$$

Taking the derivative of (3.36) with respect to $\phi_{,ij}$ yields

$$\begin{aligned} B\frac{\partial L}{\partial \phi_{,\mu\nu}} =& \frac{\partial \overline{L}}{\partial \overline{g}_{ij,kh}}\frac{\partial \overline{g}_{ij,kh}}{\partial \phi_{,\mu\nu}} + \frac{\partial \overline{L}}{\partial \overline{g}_{ij,k}}\frac{\partial \overline{g}_{ij,k}}{\partial \phi_{,\mu\nu}} + \frac{\partial \overline{L}}{\partial \overline{g}_{ij}}\frac{\partial \overline{g}_{ij}}{\partial \phi_{,\mu\nu}} \\ &+ \frac{\partial \overline{L}}{\partial \overline{\phi}}\frac{\partial \overline{\phi}}{\partial \phi_{,\mu\nu}} + \frac{\partial \overline{L}}{\partial \overline{\phi}_{,i}}\frac{\partial \overline{\phi}_{,i}}{\partial \phi_{,\mu\nu}} + \frac{\partial \overline{L}}{\partial \overline{\phi}_{,ij}}\frac{\partial \overline{\phi}_{,ij}}{\partial \phi_{,\mu\nu}}. \end{aligned} \tag{3.37}$$

From (2.2), only the last term contributes. Therefore, we have

$$B\Phi^{\mu\nu} = \overline{\Phi}^{ij} B^{\mu}{}_{i} B^{\nu}{}_{j}.$$

Consequently, Φ^{ij} is a tensor density. It can be easily seen that the representations of $\Pi^{ij,k}$ and Π^{ij} defined in Section 2.1.2 are still valid. It can also be easily seen that although $\Lambda^{ij,kh}$ and Φ are tensor densities, Φ^{i} is not.

By taking the derivative of equation (3.36) with respect to $\phi_{,i}$, we obtain

$$B\frac{\partial L}{\partial \phi_{,\mu}} = \frac{\partial \overline{L}}{\partial \overline{\phi}_{,i}}\frac{\partial \overline{\phi}_{,i}}{\partial \phi_{,\mu}} + \frac{\partial \overline{L}}{\partial \overline{\phi}_{,ij}}\frac{\partial \overline{\phi}_{,ij}}{\partial \phi_{,\mu}}. \tag{3.38}$$

Taking the derivatives yields

$$B\Phi^{\mu} = \overline{\Phi}^{i} B^{\mu}{}_{i} + \overline{\Phi}^{ij} B^{\mu}{}_{ij}.$$

Since this is not a tensor density, we can seek for its tensorial form. As we did before in Section 2.1.2, we can define a new quantity K as

$$K \equiv \Phi^{ab} h_{,ab} + \Phi^{a} h_{,a} + \Phi h, \tag{3.39}$$

where h is a scalar. After multiplying each side of this equation with B, we can substitute equations (3.37) and (3.38) into this to obtain

$$BK = \overline{\Phi}^{ij}\left(\frac{\partial \overline{\phi}_{,ij}}{\partial \phi_{,ab}} h_{,ab} + \frac{\partial \overline{\phi}_{,ij}}{\partial \phi_{,a}} h_{,a}\right) + \overline{\Phi}^{i}\frac{\partial \overline{\phi}_{,i}}{\partial \phi_{,a}} h_{,a} + \overline{\Phi} h.$$

Due to (2.2), we can rewrite this

$$BK = \overline{\Phi}^{ab}\overline{h}_{,ab} + \overline{\Phi}^{a}\overline{h}_{,a} + \overline{\Phi}\,\overline{h}.$$

We conclude that K is a scalar density. Therefore, we may define K as

$$K = \zeta^{ab} h_{|ab} + \zeta^{a} h_{|a} + \zeta h,$$

where ζ^{ab}, ζ^a and ζ are tensor densities. We can easily calculate the following covariant derivatives

$$h_{|ab} = h_{,ab} - \Gamma^{\mu}{}_{ab} h_{,\mu}, \qquad\qquad h_{|a} = h_{,a}.$$

Inserting these into the equation above yields

$$\begin{aligned} K =& \zeta^{ab}(h_{,ab} - \Gamma^{\mu}{}_{ab} h_{,\mu}) + \zeta^a h_{,a} + \zeta h \\ =& \zeta^{ab} h_{,ab} + (\zeta^a - \Gamma^{a}{}_{rs} \zeta^{rs}) h_{,a} + \zeta h. \end{aligned}$$

If we compare this with (3.39), we conclude that $\zeta^{ab} = \Phi^{ab}$ and $\zeta = \Phi$ as expected. Moreover, we have also obtained the tensor density $\zeta^a = \Phi^a + \Gamma^{a}{}_{rs} \zeta^{rs}$.

In order to construct an equation for $\Pi^{ij,k}$ of the Lagrangian of the form (3.35), we take the derivative of the equation (3.36) with respect to $B^{\mu}{}_{ab}$ to obtain

$$0 = \overline{\Lambda}^{ij,kh} \frac{\partial \overline{g}_{ij,kh}}{\partial B^{\mu}{}_{ab}} + \overline{\Lambda}^{ij,k} \frac{\partial \overline{g}_{ij,k}}{\partial B^{\mu}{}_{ab}} + \overline{\Lambda}^{ij} \frac{\partial \overline{g}_{ij}}{\partial B^{\mu}{}_{ab}} + \overline{\Phi} \frac{\partial \phi}{\partial B^{\mu}{}_{ab}} + \overline{\Phi}^{i} \frac{\partial \overline{\phi}_{,i}}{\partial B^{\mu}{}_{ab}} + \overline{\Phi}^{ij} \frac{\partial \overline{\phi}_{,ij}}{\partial B^{\mu}{}_{ab}}.$$

Due to last term on the right hand side, instead of (2.14), we now have

$$0 = 2\Lambda^{a\nu,\sigma b} g_{\mu\nu,\sigma} + 2\Lambda^{b\nu,\sigma a} g_{\mu\nu,\sigma} + \Lambda^{\nu\sigma,ab} g_{\nu\sigma,\mu} + \Lambda^{a\nu,b} g_{\mu\nu} + \Lambda^{b\nu,a} g_{\mu\nu} + \Phi^{ab} \phi_{,\mu}.$$

At the pole P of the Riemann normal coordinate system, we have

$$\Pi^{a\nu,b} g_{\mu\nu} + \Pi^{b\nu,a} g_{\mu\nu} + \Phi^{ab} \phi_{|\mu} = 0.$$

Since this is a tensorial condition, it can be generalized to any point of any coordinate system. Multiplying this with $g_{\mu\rho}$ yields

$$\Pi^{a\rho,b} + \Pi^{b\rho,a} + \Phi^{ab} \phi^{|\rho} = 0.$$

In order to solve this equation for $\Pi^{ab,\rho}$, we can use symmetries. Therefore, we write

$$\Pi^{a\rho,b} = -\Pi^{b\rho,a} - \Phi^{ab} \phi^{|\rho}.$$

Due to fact that there is a symmetry in (a, ρ) on the left hand side, we can write

$$\Pi^{a\rho,b} = -\Pi^{ba,\rho} - \Phi^{\rho b} \phi^{|a}.$$

Similarly, we have

$$\Pi^{b\rho,a} = -\Pi^{ab,\rho} - \Phi^{\rho a} \phi^{|b}.$$

By combining these two, we immediately find

$$\Pi^{ab,h} = \frac{1}{2}(\Phi^{ab}\phi^{|h} - \Phi^{hb}\phi^{|a} - \Phi^{ha}\phi^{|b}). \tag{3.40}$$

Taking the derivative of (3.36) with respect to $B^r{}_s$, one can construct an equation for Π^{ij}. Therefore, we obtain

$$\begin{aligned} BA^s{}_r L =& \overline{\Lambda}^{ij,kh}\frac{\partial \overline{g}_{ij,kh}}{\partial B^\mu{}_a} + \overline{\Lambda}^{ij,k}\frac{\partial \overline{g}_{ij,k}}{\partial B^\mu{}_a} + \overline{\Lambda}^{ij}\frac{\partial \overline{g}_{ij}}{\partial B^\mu{}_a} \\ &+ \overline{\Phi}\frac{\partial \phi}{\partial B^\mu{}_a} + \overline{\Phi}^i\frac{\partial \overline{\phi}_{,i}}{\partial B^\mu{}_a} + \overline{\Phi}^{ij}\frac{\partial \overline{\phi}_{,ij}}{\partial B^\mu{}_a}. \end{aligned}$$

Note that, the only difference between this and (2.18) is the last term. Therefore, instead of (2.19), we have

$$\begin{aligned} \delta^s_r L =& 2\Lambda^{sj,ki}\big(g_{rj,ki} + g_{ik,rj}\big) + 2\Lambda^{sj,k} g_{rj,k} \\ &+ \Lambda^{ij,s} g_{ij,r} + 2\Lambda^{sj} g_{rj} + \Phi^s\phi_{,r} + 2\Phi^{sj}\phi_{,rj}. \end{aligned} \tag{3.41}$$

At the pole P of the Riemann normal coordinate system, we have $\Phi^a = \zeta^a$, similar to the result obtained in (2.23), we have

$$g^{su}L = -2\Lambda^{sj,ki}R_i{}^u{}_{kj} + 2\Pi^{su} + \frac{2}{3}\Lambda^{ik,uj}R_i{}^s{}_{kj} + \zeta^s\phi^{|u} + 2\Phi^{sj}\phi_{|j}{}^{|u}. \tag{3.42}$$

Since this is a tensorial equation, it can be generalized to any point of any coordinate system. After renaming indices and some rearranging, we obtain

$$\Pi^{ab} = \frac{1}{3}R_k{}^b{}_{mh}\Lambda^{hk,am} - R_k{}^a{}_{mh}\Lambda^{hk,bm} - \frac{1}{2}\phi^{|a}\zeta^{|b} - \Phi^{bs}\phi_{|s}{}^{|a} + \frac{1}{2}g^{ab}L. \tag{3.43}$$

Analogous to what was done in Section 2.2, we obtain field equations

$$E^{ab}(L) = -\Lambda^{ab,kh}{}_{|kh} + \Pi^{ab,k}{}_{|k} - \Pi^{ab}, \tag{3.44}$$

and

$$E(L) = -\Phi^{kh}{}_{|kh} + \zeta^k{}_{|k} - \Phi.$$

In order to calculate these, we need the following identities that we derived in this section

$$\zeta^a = \frac{\partial L}{\partial \phi_{,a}} + \Gamma^a_{rs}\Phi^{rs}, \tag{3.45}$$

$$\Pi^{ab,h} = \frac{1}{2}(\Phi^{ab}\phi^{|h} - \Phi^{hb}\phi^{|a} - \Phi^{ha}\phi^{|b}), \tag{3.46}$$

$$\begin{aligned} \Pi^{ab} =& \frac{1}{3}R_k{}^b{}_{mh}\Lambda^{hk,am} - R_k{}^a{}_{mh}\Lambda^{hk,bm} - \frac{1}{2}\phi^{|a}\zeta^{|b} \\ & -\Phi^{bs}\phi_{|s}{}^{|a} + \frac{1}{2}g^{ab}L. \end{aligned} \tag{3.47}$$

3.4 Lagrange Scalar Densities

According to Lovelock [20], the Lagrangian L defined in (3.1) may be obtained by examining $g_{ij}A^{ij}$. Therefore, in order to obtain the Lagrangian, we calculate

$$\begin{aligned} g_{ab}A^{ab} =& \sqrt{g}\big(K_1\delta^{cde}_{hjk}\phi_{|c}{}^{|h}R_{de}{}^{jk} - \frac{4}{3}\dot{K}_1\delta^{cde}_{hjk}\phi_{|c}{}^{|h}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k} \\ &+ K_3\delta^{cde}_{hjk}\phi_{|c}\phi^{|h}R_{de}{}^{jk} - 4\dot{K}_3\delta^{cde}_{hjk}\phi_{|c}\phi^{|h}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k} \\ &+ (F+2W)\delta^{cd}_{fh}R_{cd}{}^{fh} + 2(2K_3 - 2K_1' + 4\rho\dot{K}_3)\delta^{cd}_{fh}\phi_{|c}{}^{|f}\phi_{|d}{}^{|h} \\ &- 3(2F' + 4W' + \rho K_8)\phi_{|c}{}^{|c} + 2K_8\delta^{cd}_{fh}\phi_{|c}\phi^{|f}\phi_{|d}{}^{|h} + 4K_9 \\ &- \rho(2F'' + 4W'' + \rho K_8' + 2\dot{K}_9)\big). \end{aligned} \tag{3.48}$$

As a result of the equation above, we can write L_1, L_2, L_3, L_4, L_5 and L_6 as

$$\begin{aligned} L_1 &= \sqrt{g}M_1\phi_{|c}{}^{|c}, \\ L_2 &= \sqrt{g}M_2\delta^{cd}_{ef}R_{cd}{}^{ef} - 4\sqrt{g}\dot{M}_2\delta^{cd}_{fk}\phi_{|c}{}^{|f}\phi_{|d}{}^{|k}, \\ L_3 &= \sqrt{g}M_3\delta^{cd}_{ef}\phi_{|c}\phi^{|e}\phi_{|d}{}^{|f}, \\ L_4 &= \sqrt{g}M_4\delta^{cde}_{fjk}\phi_{|c}{}^{|f}R_{de}{}^{jk} - \frac{4}{3}\sqrt{g}\dot{M}_4\delta^{cde}_{fjk}\phi_{|c}{}^{|f}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k}, \\ L_5 &= \sqrt{g}M_5\delta^{cde}_{fjk}\phi_{|c}\phi^{|f}R_{de}{}^{jk} - 4\sqrt{g}\dot{M}_5\delta^{cde}_{fjk}\phi_{|c}\phi^{|f}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k}, \\ L_6 &= \sqrt{g}M_6, \end{aligned}$$

where $M_1, \ldots, M_6$ are arbitrary functions of ϕ and ρ.

We can calculate field equations by using (3.44), (3.45), (3.46) and (3.47) together. Therefore, we have

$$\begin{aligned} E^{ab}(L_1) =& \sqrt{g}\rho\dot{M}_1\delta^{ac}_{de}g^{db}\phi_{|c}{}^{|e} - \sqrt{g}\dot{M}_1\delta^{acd}_{efh}g^{eb}\phi_{|c}\phi^{|f}\phi_{|d}{}^{|h} \\ &+ \sqrt{g}M_1'(\frac{1}{2}g^{ab}\rho - \phi^{|a}\phi^{|b}), \end{aligned} \tag{3.49}$$

$$\begin{aligned} E^{ab}(L_2) =& -\sqrt{g}\dot{M}_2\delta^{acde}_{fhjk}g^{fb}\phi_{|c}\phi^{|h}R_{de}{}^{jk} + \sqrt{g}(\rho\dot{M}_2 - \frac{1}{2}M_2)\delta^{acd}_{efh}g^{eb}R_{cd}{}^{fh} \\ &- 2\sqrt{g}(2\rho\ddot{M}_2 + \dot{M}_2)\delta^{acd}_{efh}g^{eb}\phi_{|c}{}^{|f}\phi_{|d}{}^{|h} \\ &+ 4\sqrt{g}\ddot{M}_2\delta^{acde}_{fhjk}g^{fb}\phi_{|c}\phi^{|h}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k} + 2\sqrt{g}(M_2' + 2\rho\dot{M}_2')\delta^{ac}_{de}g^{db}\phi_{|c}{}^{|e} \\ &- 8\sqrt{g}\dot{M}_2'\delta^{acd}_{efh}g^{eb}\phi_{|c}\phi^{|f}\phi_{|d}{}^{|h} + 2\sqrt{g}\rho M_2''g^{ab} - 2\sqrt{g}M_2''\phi^{|a}\phi^{|b}, \end{aligned} \tag{3.50}$$

$$\begin{aligned} E^{ab}(L_3) =& \sqrt{g}(\rho^2\dot{M}_3 + \frac{3}{2}\rho M_3)\delta^{ac}_{de}g^{db}\phi_{|c}{}^{|e} - \sqrt{g}(\rho\dot{M}_3 \\ &+ \frac{3}{2}M_3)\delta^{acd}_{efh}g^{eb}\phi_{|c}\phi^{|f}\phi_{|d}{}^{|h} + \frac{1}{2}\sqrt{g}M_3'\rho^2g^{ab} - \frac{1}{2}\sqrt{g}\rho M_3'\phi^{|a}\phi^{|b}, \end{aligned} \tag{3.51}$$

$$
\begin{aligned}
E^{ab}(L_4) =& \sqrt{g}\dot{M}_4\delta^{acdef}_{hjkpq}g^{hb}\phi_{|c}\phi^{|j}\phi_{|d}{}^{|k}R_{ef}{}^{pq} \\
&+ \sqrt{g}\dot{M}_4\rho\delta^{adef}_{hkpq}g^{hb}\phi_{|d}{}^{|k}R_{ef}{}^{pq} - \frac{1}{2}\sqrt{g}\rho M_4'\delta^{acd}_{efh}g^{eb}R_{cd}{}^{fh} \\
&+ \sqrt{g}M_4'\delta^{acde}_{fhjk}g^{fb}\phi_{|c}\phi^{|h}R_{de}{}^{jk} + \frac{4}{3}\sqrt{g}\ddot{M}_4\delta^{acdef}_{hjkpq}g^{hb}\phi_{|c}\phi^{|j}\phi_{|d}{}^{|k}\phi_{|e}{}^{|p}\phi_{|f}{}^{|q} \\
&- \frac{4}{3}\sqrt{g}(\rho\ddot{M}_4 + \dot{M}_4)\delta^{acde}_{fhjk}g^{fb}\phi_{|c}{}^{|h}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k} \\
&+ \sqrt{g}(2M_4' + 2\rho\dot{M}_4')\delta^{ade}_{fjk}g^{fb}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k} - 4\sqrt{g}\dot{M}_4'\delta^{acde}_{fhjk}g^{fb}\phi_{|c}\phi^{|h}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k} \\
&+ 2\sqrt{g}M_4''\delta^{acd}_{efh}g^{eb}\phi_{|c}\phi^{|f}\phi_{|d}{}^{|h},
\end{aligned} \tag{3.52}
$$

$$
\begin{aligned}
E^{ab}(L_5) =& \sqrt{g}(\rho^2\dot{M}_5 + \frac{1}{2}\rho M_5)\delta^{acd}_{efh}g^{eb}R_{cd}{}^{fh} - \sqrt{g}(\rho\dot{M}_5 + M_5)\delta^{acde}_{fhjk}g^{fb}\phi_{|c}\phi^{|h}R_{de}{}^{jk} \\
&- \sqrt{g}(2M_5 + 10\rho\dot{M}_5 + 4\rho^2\ddot{M}_5)\delta^{acd}_{efh}g^{eb}\phi_{|c}{}^{|f}\phi_{|d}{}^{|h} \\
&+ 4\sqrt{g}(2\dot{M}_5 + \rho\ddot{M}_5)\delta^{acde}_{fhjk}g^{fb}\phi_{|c}\phi^{|h}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k} \\
&- \sqrt{g}(2M_5' + 4\rho\dot{M}_5')\delta^{ace}_{jfr}g^{jb}\phi_{|c}\phi^{|f}\phi_{|e}{}^{|r},
\end{aligned} \tag{3.53}
$$

$$
E^{ab}(L_6) = \sqrt{g}\dot{M}_6\phi^{|a}\phi^{|b} - \frac{1}{2}\sqrt{g}M_6 g^{ab}. \tag{3.54}
$$

Equations (3.48), and (3.49)-(3.54) now permit us to conclude that

$$
\begin{aligned}
E^{ab}(g_{ij}A^{ij}) =& \sqrt{g}\big(\rho\dot{K}_1\delta^{adef}_{hkpq}g^{hb}\phi_{|d}{}^{|k}R_{ef}{}^{pq} + \rho\dot{K}_3\delta^{acde}_{fhjk}g^{fb}\phi_{|c}\phi^{|h}R_{de}{}^{jk} \\
&+ (\frac{1}{2}J - W)\delta^{acd}_{efh}g^{eb}R_{cd}{}^{fh} - \frac{4}{3}\frac{\partial}{\partial\rho}(\rho\dot{K}_1)\delta^{acde}_{fhjk}g^{fb}\phi_{|c}{}^{|h}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k} \\
&- 4\frac{\partial}{\partial\rho}(\rho\dot{K}_3)\delta^{acde}_{fhjk}g^{fb}\phi_{|c}\phi^{|h}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k} \\
&+ (-2\rho\dot{K}_1' + 6\rho\dot{K}_3 + 4\rho^2\ddot{K}_3)\delta^{ade}_{fjk}g^{fb}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k} \\
&+ \rho\dot{K}_8\delta^{acd}_{efh}g^{eb}\phi_{|c}\phi^{|f}\phi_{|d}{}^{|h} + (-2J' + 4W' - \rho^2\dot{K}_8)\delta^{ac}_{de}g^{db}\phi_{|c}{}^{|e} \\
&+ (\rho\dot{K}_9 - 2K_9)g^{ab} + (-J'' + 4W'' + 2\dot{K}_9 - 2\rho\ddot{K}_9 - \rho^2\dot{K}_8')\phi^{|a}\phi^{|b}\big),
\end{aligned} \tag{3.55}
$$

where

$$
J = \int \big(\frac{\partial}{\partial\phi}(\rho\dot{K}_1) - \rho\dot{K}_3 - 2\rho\frac{\partial}{\partial\rho}(\rho\dot{K}_3)\big)d\rho.
$$

Integration by parts together with equation (3.32) yields

$$
J = -F + \rho\dot{F}.
$$

If we compare (3.55) with equation (3.33), then we deduce that

$$\begin{aligned}\mathcal{L} =&\sqrt{g}\big(\mathcal{K}_1\delta^{cde}_{hjk}\phi_{|c}{}^{|h}R_{de}{}^{jk} - \frac{4}{3}\dot{\mathcal{K}}_1\delta^{cde}_{hjk}\phi_{|c}{}^{|h}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k}\\ &+ \mathcal{K}_3\delta^{cde}_{hjk}\phi_{|c}\phi^{|h}R_{de}{}^{jk} - 4\dot{\mathcal{K}}_3\delta^{cde}_{hjk}\phi_{|c}\phi^{|h}\phi_{|d}{}^{|j}\phi_{|e}{}^{|k}\\ &+ (\mathcal{F}+2\mathcal{W})\delta^{cd}_{fh}R_{cd}{}^{fh} + 2(2\mathcal{K}_3 - 2\mathcal{K}_1' + 4\rho\dot{\mathcal{K}}_3)\delta^{cd}_{fh}\phi_{|c}{}^{|f}\phi_{|d}{}^{|h}\\ &- 3(2\mathcal{F}' + 4\mathcal{W}' + \rho\mathcal{K}_8)\phi_{|c}{}^{|c} + 2\mathcal{K}_8\delta^{cd}_{fh}\phi_{|c}\phi^{|f}\phi_{|d}{}^{|h} + 4\mathcal{K}_9\\ &- \rho(2\mathcal{F}'' + 4\mathcal{W}'' + \rho\mathcal{K}_8' + 2\dot{\mathcal{K}}_9)\big),\end{aligned} \tag{3.56}$$

where

$$\begin{aligned}&\mathcal{K}_1 = \int \frac{1}{\rho}K_1 d\rho, \quad \mathcal{K}_3 = \int \frac{1}{\rho}K_3 d\rho,\\ &\mathcal{K}_8 = \int \frac{1}{\rho}K_8 d\rho, \quad \mathcal{K}_9 = \rho^2\int \frac{1}{\rho^3}K_9 d\rho,\\ &\mathcal{W} = -W, \qquad\quad \mathcal{F} = \int \big(\mathcal{K}_1' - \mathcal{K}_3 - 2\rho\dot{\mathcal{K}}_3\big)d\rho.\end{aligned}$$

Note that during this derivation, we have excluded the possibility of the additional terms in $\mathcal{L}$ that yields vanishing $E^{ij}(\mathcal{L})$. One can consider to include some additional terms that do not change the field equations. Therefore, in a four-dimensional space-time, the most general second-order Euler-Lagrange equations are derivable from $\mathcal{L}$.

CHAPTER 4

DISCUSSION

In this thesis, we have studied scalar-tensor field theories and we have obtained some important results in a four-dimensional spacetime. We have also derived useful identities for an n dimensional spacetime. One may easily apply the methods here on many other field theories. The results that we have obtained in this thesis can be helpful for those who study similar field theories.

In Chapter 2, we have started with a specific Lagrangian which is of the form

$$L(g_{ij}, g_{ij,k}, g_{ij,kh}, \phi, \phi_{,i}). \tag{4.1}$$

Choosing the Lagrangian to be a scalar density puts severe restrictions on it. In an n dimensional spacetime, we have the following identities

$$\Lambda^{ij,kh} + \Lambda^{ik,jh} + \Lambda^{ih,kj} = 0, \tag{4.2}$$

$$2\Lambda^{a\nu,\sigma b} g_{\mu\nu,\sigma} + 2\Lambda^{b\nu,\sigma a} g_{\mu\nu,\sigma} + \Lambda^{\nu\sigma,ab} g_{\nu\sigma,\mu} + \Lambda^{a\nu,b} g_{\mu\nu} + \Lambda^{b\nu,a} g_{\mu\nu} = 0, \tag{4.3}$$
$$\frac{1}{2} g^{ij} L = \Pi^{ij} - \frac{2}{3}\Lambda^{im,kh} R_{h\ \ km}^{\ j} + \frac{3}{8} g^{jh}\Phi^i \phi_{,h} + \frac{1}{8} g^{ih}\Phi^j \phi_{,h},$$

which should be satisfied by the Lagrangian. These partial differential equations are called *invariance identities*. We have obtained the field equations by using these equations and their symmetry relations. We have also used similar techniques in Section 3.3 for the Lagrangian of the form $L = L(g_{ij}, g_{ij,k}, g_{ij,kh}, \phi, \phi_{,i}, \phi_{,ij})$ to obtain its invariance identities.

Variation of the Lagrangian in (4.1) with respect to the metric yields

$$E^{ij}(L) = -\Lambda^{ij,kh}{}_{|kh} - \frac{1}{2} g^{ij} L - \frac{2}{3}\Lambda^{im,kh} R_{h\ \ km}^{\ j} + \frac{3}{8} g^{jh}\Phi^i \phi_{,h} + \frac{1}{8} g^{ih}\Phi^j \phi_{,h}.$$

Note that every term in this equation is a tensor density and the equation is valid for a n dimensional spacetime.

(4.2) and (4.3) are not only valid for L but also for any tensor densitiy which is a function of g_{ij}, $g_{ij,k}$, $g_{ij,kh}$, ϕ and $\phi_{,i}$. For example, a tensor density

$$A^{\cdots}_{\cdots} = A^{\cdots}_{\cdots}(g_{ij}, g_{ij,k}, g_{ij,kh}, \phi, \phi_{,i})$$

satisfies

$$\frac{\partial A^{\cdots}_{\cdots}}{\partial g_{ij,kh}} + \frac{\partial A^{\cdots}_{\cdots}}{\partial g_{ik,jh}} + \frac{\partial A^{\cdots}_{\cdots}}{\partial g_{ih,kj}} = 0. \tag{4.4}$$

Due to Lemma 2.1, for a tensor density $A^{\cdots}_{\cdots}(g_{ij}, g_{ij,k}, g_{ij,kh}, \phi, \phi_{,i})$, if $\frac{\partial A^{\cdots}_{\cdots}}{\partial g_{ij,kh}} = 0$ then $\frac{\partial A^{\cdots}_{\cdots}}{\partial g_{ij,k}} = 0$.

By virtue of the following relation which can be considered as a generalization of the Bianchi identity

$$E^{ij}{}_{|j}(L) = \frac{1}{2} g^{ij} \phi_{,j} E(L), \tag{4.5}$$

whenever one has $E^{ij}(L) = 0$ then $E(L) = 0$ is automatically satisfied. Furthermore, this important relation implies that both $E(L)$ and $E^{ij}{}_{|j}(L)$ are functions of the metric, a scalar, and their first two derivatives. This result puts severe restrictions on the most general form of $E^{ij}(L)$.

We should note that we have put everything into tensorial form in order to make use of the Riemann normal coordinates. If we find an expression which is in a tensorial form at a pole P of the Riemann normal coordinates, we can generalize it to any point of any coordinate system.

Although we do not work with vector-metric field theories here, techniques used in this thesis can be applied to those as well. An interested reader can easily adopt the techniques used here not only to a different Lagrangian but also to a different number of dimensions.

If we choose the Lagrangian to be of the form

$$L(g_{ij}, g_{ij,k}, g_{ij,kh}), \tag{4.6}$$

in a four-dimensional spacetime, the most general form of the Lagrangian which yields the field equations involving the metric, a scalar and their first two derivatives is the following

$$L = \beta\sqrt{g}(R^2 - 4R_{ij}R^{ij} + R_{ijkh}R^{ijkh}) + \gamma(*R^{ij}{}_{kh}R^{kh}{}_{ij}) + \mu\sqrt{g}R + \eta\sqrt{g}, \quad (4.7)$$

where β, γ, μ and η are arbitrary constants and $*R^{ij}{}_{kh} = \epsilon^{ijrs}R_{rskh}$. Since β, γ, μ and η are just constants, by applying techniques similar to those developed in Section 2.5, the Euler-Lagrange equations are found to be

$$R_{ij} - \frac{1}{2}g_{ij}R + \Lambda g_{ij} = 0, \tag{4.8}$$

where Λ is a constant. This is nothing but Einstein's vacuum field equations with a cosmological constant. Therefore, if the Lagrangian is of the form (4.6), then Einstein's field equations are unique in a four-dimensional spacetime. Note that the first term on the right hand side of (4.7), called the Gauss-Bonnet term, can be written as a total derivative in a four-dimensional spacetime. In addition to this, one can rewrite the second term on the right hand side of (4.7) as a total derivative without any dimensional restrictions.

The Gauss-Bonnet term contributes to the field equations in dimensions higher than four. For the Lagrangian $L = \sqrt{g}(R^2 - 4R_{ij}R^{ij} + R_{ijkh}R^{ijkh})$, we find the field equations

$$\begin{aligned} E^{ij}(L) = &-\frac{1}{2}\sqrt{g}g^{ij}(R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}) \\ &+ 2\sqrt{g}\big(RR^{ij} - 2R^j{}_kR^{ik} - 2R^{kh}R_h{}^j{}_k{}^i + R^{ihkm}R^j{}_{hkm}\big), \end{aligned}$$

which are identically zero in a four-dimensional spacetime due to the Lanczos identity. Therefore, one may consider dimensions $n \geq 5$ to obtain field equations other than Einstein's in a metric field theory.

For the Lagrangian of the form (4.6), from (4.5) we immediately have the Bianchi identity that automatically follows from diffeomorphism invariance, $E^{ij}{}_{|j}(L) = 0$. Note that this result is obviously in accordance with the field equations (4.8). As a result, if one wants to construct field equations different than (4.8) in a four-dimensional spacetime, one has to use a Lagrangian different than (4.6). In that case, there will be no divergence-free field equations, i.e. $E^{ij}{}_{|j}(L) \neq 0$.

In Chapter 3, we have started by constructing the most general form of the Euler-Lagrange equations that are functions of the metric, a scalar and their first two derivatives in a four-dimensional spacetime. Therefore, we have generated the most general form of the tensor density $A^{ij} = A^{ij}(g_{ab}, g_{ab,c}, g_{ab,cd}, \phi, \phi_{,c}, \phi_{,cd})$, which includes the most general form of the Euler-Lagrange equations E^{ij}. Making use of (4.5), we have concluded that ${E^{ij}}_{|j}$ must also be at most of second-order. Due to this fact, one is able to eliminate some terms from the most general form of A^{ij}. Then using the relation between the Lagrangian and its field equations, we have found the Lagrangian which yields the most general form of the Euler-Lagrange equations in a space of four dimensions. As a result, the Lagrangian

$$\begin{aligned}\mathcal{L} = \sqrt{g}\big(&\mathcal{K}_1 \delta^{cde}_{hjk} {\phi_{|c}}^{|h} {R_{de}}^{jk} - \frac{4}{3}\dot{\mathcal{K}}_1 \delta^{cde}_{hjk} {\phi_{|c}}^{|h} {\phi_{|d}}^{|j} {\phi_{|e}}^{|k} + \mathcal{K}_3 \delta^{cde}_{hjk} \phi_{|c} \phi^{|h} {R_{de}}^{jk} \\ &- 4\dot{\mathcal{K}}_3 \delta^{cde}_{hjk} \phi_{|c} \phi^{|h} {\phi_{|d}}^{|j} {\phi_{|e}}^{|k} + (\mathcal{F} + 2\mathcal{W}) \delta^{cd}_{fh} {R_{cd}}^{fh} \\ &+ 2(2\mathcal{K}_3 - 2\mathcal{K}_1' + 4\rho\dot{\mathcal{K}}_3) \delta^{cd}_{fh} {\phi_{|c}}^{|f} {\phi_{|d}}^{|h} - 3(2\mathcal{F}' + 4\mathcal{W}' + \rho\mathcal{K}_8) {\phi_{|c}}^{|c} \\ &+ 2\mathcal{K}_8 \delta^{cd}_{fh} \phi_{|c} \phi^{|f} {\phi_{|d}}^{|h} + 4\mathcal{K}_9 - \rho(2\mathcal{F}'' + 4\mathcal{W}'' + \rho\mathcal{K}_8' + 2\dot{\mathcal{K}}_9)\big),\end{aligned} \tag{4.9}$$

yields field equations which are functions of the metric, a scalar and their first two derivatives in a four-dimensional spacetime. Obviously, one may include some additional terms in this Lagrangian if their contributions to the Euler-Lagrange equations vanish.

The Lagrangian given in (4.9) is a motivation for those who study and construct new modified gravity theories. For example, the theory called *the Fab Four* is a subset of Horndeski's theory [21]. This theory is established in order to find a solution for the cosmological constant problem on FLRW backgrounds. This theory possesses self-tuning properties that may provide a partial solution to the cosmological constant problem. The Fab Four consist of four pieces

$$\begin{aligned}\mathcal{L}_{john} &= \sqrt{g} V_{john}(\phi) G^{\mu\nu} \phi_{|\mu} \phi_{|\nu}, \\ \mathcal{L}_{paul} &= \sqrt{g} V_{paul}(\phi) P^{\mu\nu\alpha\beta} \phi_{|\mu} \phi_{|\alpha} \phi_{|\nu\beta}, \\ \mathcal{L}_{george} &= \sqrt{g} V_{george}(\phi) R, \\ \mathcal{L}_{ringo} &= \sqrt{g} V_{ringo}(\phi) \hat{G},\end{aligned}$$

where $\hat{G} = R^2 - 4R_{ij}R^{ij} + R_{ijkh}R^{ijkh}$ is the Gauss-Bonnet term, $\epsilon_{\mu\nu\alpha\beta}$ is the Levi-Civita tensor and $P^{\mu\nu\alpha\beta} = -\frac{1}{4}\epsilon^{\mu\nu\lambda\sigma} R_{\lambda\sigma\gamma\delta} \epsilon^{\alpha\beta\gamma\delta}$ is the double dual of the Riemann tensor. Note that, these Lagrange scalar densities are included in (4.9). The Fab Four Lagrangian terms have potential to explain the matter-dominated phase of the universe expansion and late-time acceleration phase at least at the classical level [22].

By following similar techniques that we use in Chapter 3, Ohashi et al. constructed the most general second-order field equations of bi-scalar-tensor theory in four dimensions [23]. They have chosen the Lagrangian of the form

$$L = L(g_{ij}, g_{ij,i_1}, \ldots, g_{ij,i_1 \ldots i_p}, \phi^I, \phi^I{}_{,i_1}, \ldots, \phi^I{}_{,i_1 \ldots i_q}),$$

where $I = 1, 2$. By considering more than one scalars in the Lagrangian, Ohashi et al. have shown that one can obtain new terms in the field equations that are not included in the Horndeski theory.

In this thesis, we have chosen torsion-free metric connection. However, one may consider torsion to have results different than Horndeski's. Valdivia et al. recently have shown that effects of torsion can be critical in the very early universe [24]. Although they did not generate the most general Lagrangian for a spacetime with torsion, they generalized Horndeski's Lagrangian for the case when the torsion is nonvanishing.

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APPENDIX A

RIEMANN NORMAL COORDINATES

We frequently make use of Riemann normal coordinates in this thesis in order to simplify long calculations. Although these coordinates are specific coordinates, if we find a tensorial condition in these coordinates we can generalize it to any arbitrary coordinate system. Note that tensors are invariant under coordinate transformations. Therefore, in order to utilize Riemann normal coordinates, we look for the tensorial form of the quantities.

In a curved spacetime, one can locally construct a coordinate system where free particles move along straight lines. These straight lines can be found by using geodesic equations. The general form of the geodesic equations in an arbitrary coordinate system can be written as

$$\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha{}_{\beta\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} = 0,$$

where λ is an affine parameter. Since free particles are moving on straight lines at the pole P of this coordinate system, we immediately find $\frac{d^2x^\alpha}{d\lambda^2} = 0$. Thus

$$\Gamma^\alpha{}_{\beta\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} = 0$$

and $\Gamma^\alpha{}_{\beta\gamma} = 0$. By considering the spacetime metric, one can always construct an orthonormal coordinate system to have $\boldsymbol{e}_\alpha \cdot \boldsymbol{e}_\beta = \eta_{\alpha\beta}$, where $\boldsymbol{e}_\alpha$ is a base vector of the coordinate α and $\eta_{\alpha\beta}$ is the metric of the flat spacetime. At the pole P of this coordinate system, we have

$$g_{\alpha\beta}(P) = \eta_{\alpha\beta},$$
$$g_{\alpha\beta,\mu}(P) = 0,$$
$$g_{\alpha\beta,\mu\nu}(P) \neq 0.$$

Therefore, as an example, components of the Riemann curvature tensor at the pole of this coordinate system can be calculated as

$$R_{\alpha\beta\mu\nu}(P) = g_{\alpha\nu,\beta\mu}(P) - g_{\alpha\mu,\beta\nu}(P).$$

APPENDIX B

THE CONSTRUCTION OF GENERAL TENSOR DENSITIES

Lemma B.1 *If $\xi = \xi(g_{rs}, \phi)$ is a scalar density then one can always express it as*

$$\xi = \mu(\phi)\sqrt{g}. \tag{B.1}$$

Proof. Taking the derivative of the metric $g_{rs} = B^{\mu}_{r} B^{\nu}_{s} g_{\mu\nu}$ with respect to B^{a}_{b} yields

$$\frac{\partial g_{rs}}{\partial B^{a}_{b}} = \delta^{\mu}_{a} \delta^{b}_{r} B^{\nu}_{s} g_{\mu\nu} + \delta^{\nu}_{a} \delta^{b}_{s} B^{\mu}_{r} g_{\mu\nu}. \tag{B.2}$$

Since we have a scalar density ξ, we may define a scalar ψ by

$$\psi = \xi/\sqrt{g}.$$

By applying $\frac{\partial}{\partial B^{a}_{b}}$ to ψ, we obtain

$$\frac{\partial \psi}{\partial B^{a}_{b}} = \frac{\partial g_{\mu\nu}}{\partial B^{a}_{b}} \frac{\partial \psi}{\partial g_{\mu\nu}} = \frac{\partial \psi}{\partial g_{b\mu}} g_{a\mu} + \frac{\partial \psi}{\partial g_{b\nu}} g_{a\nu} = 2\frac{\partial \psi}{\partial g_{b\nu}} g_{a\nu} = 0.$$

Therefore, ψ is a function of the scalar ϕ. We have completed the proof.

Lemma B.2 *If $\xi_{ij} = \xi_{ij}(g_{rs}, \phi)$ is a tensor then*

$$\xi_{ij} = \mu g_{ij} + \delta^{n}_{2} \sigma \sqrt{g} \epsilon_{ij},$$

where μ and σ are arbitrary functions of the scalar ϕ, ϵ_{ij} is the two dimensional Levi-Civita symbol and δ^{a}_{b} is the Kronecker delta.

Proof. Taking the derivative of $\xi_{rs} = B^{\mu}_{r} B^{\nu}_{s} \xi_{\mu\nu}$ with respect to B^{a}_{b} yields

$$\frac{\partial g_{\mu\nu}}{\partial B^{a}_{b}} \frac{\partial \xi_{rs}}{\partial g_{\mu\nu}} = \xi_{ra} \delta^{b}_{s} + \xi_{as} \delta^{b}_{r}.$$

Therefore, we have the following equation

$$2\frac{\partial \xi_{rs}}{\partial g_{b\nu}} g_{a\nu} = \xi_{ra}\delta_s^b + \xi_{as}\delta_r^b.$$

Multiplying each side by g^{ac} yields

$$2\frac{\partial \xi_{rs}}{\partial g_{bc}} = g^{ac}\xi_{ra}\delta_s^b + g^{ac}\xi_{as}\delta_r^b. \tag{B.3}$$

Since the left hand side is symmetric in (b, c), so must the right hand side. Therefore, we have

$$2\frac{\partial \xi_{rs}}{\partial g_{bc}} = g^{ac}\xi_{ra}\delta_s^b + g^{ac}\xi_{as}\delta_r^b = g^{ab}\xi_{ra}\delta_s^c + g^{ab}\xi_{as}\delta_r^c.$$

With $b = s$, we have

$$g^{ac}\xi_{ra}\delta_b^b + g^{ac}\xi_{ab}\delta_r^b = g^{ab}\xi_{ra}\delta_b^c + g^{ab}\xi_{ab}\delta_r^c.$$

In an n dimensional spacetime, we have

$$ng^{ac}\xi_{ra} + g^{ac}\xi_{ar} = g^{ac}\xi_{ra} + g^{ab}\xi_{ab}\delta_r^c.$$

After rearranging this, we obtain

$$(n-1)g^{ac}\xi_{ra} + g^{ac}\xi_{ar} = g^{ab}\xi_{ab}\delta_r^c.$$

Multiplying each side by g_{cd} yields

$$(n-1)\xi_{rd} + \xi_{dr} = g^{ab}\xi_{ab}g_{dr}. \tag{B.4}$$

Due to the symmetry in (d, r) of the right hand side, for $n \neq 2$, we have $\xi_{rd} = \frac{1}{n}\lambda g_{rd}$, where $\lambda = g^{ab}\xi_{ab}$. Substituting this result into equation (B.3) yields

$$2\frac{\partial}{\partial g_{bc}}(\frac{1}{n}\lambda g_{rs}) = g^{ac}\frac{1}{n}\lambda g_{ra}\delta_s^b + g^{ac}\frac{1}{n}\lambda g_{as}\delta_r^b.$$

After straightforward calculations, we have

$$2\frac{\partial g_{rs}}{\partial g_{bc}}\frac{1}{n}\lambda + 2\frac{\partial \lambda}{\partial g_{bc}}\frac{1}{n}g_{rs} = \frac{1}{n}\lambda\delta_r^c\delta_s^b + \frac{1}{n}\lambda\delta_s^c\delta_r^b,$$

$$2\frac{1}{n}\lambda\delta_r^b\delta_s^c + 2\frac{\partial \lambda}{\partial g_{bc}}\frac{1}{n}g_{rs} = \frac{1}{n}\lambda\delta_r^c\delta_s^b + \frac{1}{n}\lambda\delta_s^c\delta_r^b.$$

Due to the (b, c) symmetry of the last equation, we get $\frac{\partial\lambda}{\partial g_{bc}} = 0$. Therefore, λ is a function of the scalar ϕ. For $n = 2$ (B.4) implies

$$\xi_{rd} = \frac{1}{2}\lambda g_{rd} + \frac{1}{2}(\xi_{rd} - \xi_{dr}).$$

Since the second term on the right is proportional to ϵ_{rd}, we have

$$\xi_{rd} = \frac{1}{2}\lambda g_{rd} + \sigma\sqrt{g}\epsilon_{dr},$$

where λ and σ are arbitrary functions of the scalar ϕ, which completes the proof.

Lemma B.3 *If $\xi_{ijkh} = \xi_{ijkh}(g_{rs}, \phi)$ is a tensor then for $n > 2$,*

$$\xi_{ijkh} = \alpha g_{ij} g_{kh} + \beta g_{ik} g_{jh} + \gamma g_{ih} g_{jk} + \lambda \delta_4^n \nu \sqrt{g}\epsilon_{ijkh},$$

where α, β, γ and λ are arbitrary functions of the scalar ϕ and ϵ_{ijkh} is the four-dimensional Levi-Civita symbol.

Proof. Taking the derivative of $\xi_{ijkm} = B_i^\mu B_j^\nu B_k^\gamma B_m^\delta \xi_{\mu\nu\gamma\delta}$ with respect to B_b^a yields

$$\begin{aligned}\frac{\partial g_{rs}}{\partial B_b^a}\frac{\partial \xi_{ijkm}}{\partial g_{rs}} =& \delta_a^\mu \delta_i^b B_j^\nu B_k^\gamma B_m^\delta \xi_{\mu\nu\gamma\delta} + \delta_a^\nu \delta_j^b B_i^\mu B_k^\gamma B_m^\delta \xi_{\mu\nu\gamma\delta} \\ &+ \delta_a^\gamma \delta_k^b B_i^\mu B_j^\nu B_m^\delta \xi_{\mu\nu\gamma\delta} + \delta_i^\mu \delta_m^b B_i^\mu B_j^\nu B_k^\gamma \xi_{\mu\nu\gamma\delta}.\end{aligned}$$

Therefore, we have the following equation

$$\delta_i^b \xi_{ajkm} + \delta_j^b \xi_{iakm} + \delta_k^b \xi_{ijam} + \delta_m^b \xi_{ijka} = 2\frac{\partial \xi_{ijkm}}{\partial g_{b\nu}} g_{a\nu}.$$

Multiplying this with g^{ah} yields

$$2\frac{\partial \xi_{ijkm}}{\partial g_{bh}} = g^{ah}\delta_i^b \xi_{ajkm} + g^{ah}\delta_j^b \xi_{iakm} + g^{ah}\delta_k^b \xi_{ijam} + g^{ah}\delta_m^b \xi_{ijka}.$$

After renaming indices, we obtain

$$2\frac{\partial \xi_{ijkm}}{\partial g_{rs}} = g^{hs}\delta_i^r \xi_{hjkm} + g^{hs}\delta_j^r \xi_{ihkm} + g^{hs}\delta_k^r \xi_{ijhm} + g^{hs}\delta_m^r \xi_{ijkh}.$$

Due to the symmetry in (r, s), we have

$$\begin{aligned}2\frac{\partial \xi_{ijkm}}{\partial g_{rs}} &= g^{hs}\delta_i^r \xi_{hjkm} + g^{hs}\delta_j^r \xi_{ihkm} + g^{hs}\delta_k^r \xi_{ijhm} + g^{hs}\delta_m^r \xi_{ijkh} \\ &= g^{hr}\delta_i^s \xi_{hjkm} + g^{hr}\delta_j^s \xi_{ihkm} + g^{hr}\delta_k^s \xi_{ijhm} + g^{hr}\delta_m^s \xi_{ijkh}.\end{aligned}$$

Multiplying this with g_{st} yields

$$\delta_i^r \xi_{tjkm} + \delta_j^r \xi_{itkm} + \delta_k^r \xi_{ijtm} + \delta_m^r \xi_{ijkt} \tag{B.5}$$

$$= g_{it} g^{rs} \xi_{sjkm} + g_{jt} g^{rs} \xi_{iskm} + g_{kt} g^{rs} \xi_{ijsm} + g_{mt} g^{rs} \xi_{ijks}. \tag{B.6}$$

We sum over (r, i) and replace t by i and m by h to get

$$(n-1)\xi_{ijkh} + \xi_{jikh} + \xi_{kjih} + \xi_{hjki} = g_{ij}g^{rs}\xi_{rskh} + g_{ik}g^{rs}\xi_{rjsh} + g_{ih}g^{rs}\xi_{rjks}. \quad \text{(B.7)}$$

Now, we can define the following quantities

$$g^{rs}\xi_{rskh} = \lambda g_{kh}, \qquad g^{rs}\xi_{rjsh} = \mu g_{jh}, \qquad g^{rs}\xi_{rjhs} = \rho g_{jh}, \quad \text{(B.8)}$$

where λ, μ and ρ are arbitrary functions of the scalar ϕ. Let us define $g^{rs}\xi_{khrs} = \sigma g_{kh}$, where σ is any function of the scalar ϕ, then

$$g^{rs}\xi_{rskh} - g^{rs}\xi_{khrs} = (\lambda - \sigma)g_{kh}.$$

We multiply each side with g^{kh} to get

$$g^{kh}g^{rs}\xi_{rskh} - g^{kh}g^{rs}\xi_{khrs} = (\lambda - \sigma)g^{kh}g_{kh} = 0.$$

Therefore, $\sigma = \lambda$. It is easily shown in similar way that

$$g^{rs}\xi_{khrs} = \lambda g_{kh}, \qquad g^{rs}\xi_{jrhs} = \mu g_{jh}, \qquad g^{rs}\xi_{jrsh} = \rho g_{jh}. \quad \text{(B.9)}$$

We may rewrite equation (B.7) as

$$(n-1)\xi_{ijkh} + \xi_{jikh} + \xi_{kjih} + \xi_{hjki} = \lambda g_{ij}g_{kh} + \mu g_{ik}g_{jh} + \rho g_{ih}g_{jk}. \quad \text{(B.10)}$$

We may obtain three equations from equation (B.6) by summing over (r, j), (r, h) and (r, h) pairs and replacing t by j, k and h, respectively.

$$(n-1)\xi_{ijkh} + \xi_{jikh} + \xi_{ikjh} + \xi_{ihkj} = \lambda g_{ij}g_{kh} + \mu g_{ik}g_{jh} + \rho g_{ih}g_{jk}, \quad \text{(B.11)}$$

$$(n-1)\xi_{ijkh} + \xi_{kjih} + \xi_{ikjh} + \xi_{ijhk} = \lambda g_{ij}g_{kh} + \mu g_{ik}g_{jh} + \rho g_{ih}g_{jk}, \quad \text{(B.12)}$$

$$(n-1)\xi_{ijkh} + \xi_{hjki} + \xi_{ihkj} + \xi_{ijhk} = \lambda g_{ij}g_{kh} + \mu g_{ik}g_{jh} + \rho g_{ih}g_{jk}. \quad \text{(B.13)}$$

We calculate $(B.10) + (B.11) - (B.12) - (B.13)$ to obtain

$$2(\xi_{jikh} - \xi_{ijhk}) = 0, \qquad \xi_{jikh} = \xi_{ijhk}. \quad \text{(B.14)}$$

If we substitute this into $(B.11) - (B.12)$, then one finds

$$\xi_{ihkj} - \xi_{kjih} = 0, \qquad \xi_{ihkj} = \xi_{kjih} = \xi_{hijh}. \quad \text{(B.15)}$$

From (B.10), we obtain three similar equations by interchanging i with j, k and h, respectively.

$$(n-1)\xi_{jikh}+\xi_{ijkh}+\xi_{kijh}+\xi_{hikj}=\lambda g_{ij}g_{kh}+\rho g_{ik}g_{jh}+\mu g_{ih}g_{jk}, \tag{B.16}$$

$$(n-1)\xi_{kjih}+\xi_{jkih}+\xi_{ijkh}+\xi_{hjik}=\rho g_{ij}g_{kh}+\mu g_{ik}g_{jh}+\lambda g_{ih}g_{jk}, \tag{B.17}$$

$$(n-1)\xi_{hjki}+\xi_{jhki}+\xi_{kjhi}+\xi_{ijkh}=\mu g_{ij}g_{kh}+\lambda g_{ik}g_{jh}+\rho g_{ih}g_{jk}. \tag{B.18}$$

We calculate $[(n-1)(B.10)+2(B.16)-(B.17)-(B.18)]$, due to symmetries, we have

$$\begin{aligned}(n-1)^2\xi_{ijkh}+3(n-1)\xi_{jikh}=&(n-1)\lambda g_{ij}g_{kh}+(n-1)\mu g_{ik}g_{jh}\\&+(n-1)\rho g_{ih}g_{jk}+2\lambda g_{ij}g_{kh}+2\rho g_{ik}g_{jh}\\&+2\mu g_{ih}g_{jk}-\rho g_{ij}g_{kh}-\mu g_{ik}g_{jh}\\&-\lambda g_{ih}g_{jk}-\mu g_{ij}g_{kh}-\lambda g_{ik}g_{jh}-\rho g_{ih}g_{jk}.\end{aligned}$$

We can put this result into a compact form as

$$(n-1)^2\xi_{ijkh}+3(n-1)\xi_{jikh}=\alpha g_{ij}g_{kh}+\beta g_{ik}g_{jh}+\gamma g_{ih}g_{jk}, \tag{B.19}$$

where

$$\begin{aligned}\alpha&=(n+1)\lambda-\rho-\mu,\\ \beta&=(n-2)\mu-\lambda+2\rho,\\ \gamma&=(n-2)\rho-\lambda+2\mu.\end{aligned}$$

Interchanging i with j in equation (B.19) yields

$$(n-1)^2\xi_{jikh}+3(n-1)\xi_{ijkh}=\alpha g_{ij}g_{kh}+\beta g_{jk}g_{ih}+\gamma g_{jh}g_{ik}. \tag{B.20}$$

In order to eliminate ξ_{jikh}, we calculate $[(n-1)(B.19)-3(B.20)]$

$$\begin{aligned}\big((n-1)^2-9\big)(n-1)\xi_{ijkh}=&(n-4)(n+2)(n-1)\xi_{ijkh}\\=&a g_{ij}g_{kh}+b g_{ik}g_{jh}+c g_{ih}g_{jk}.\end{aligned}$$

For $n>2$ and $n\neq 4$, we have proved the lemma. When $n=4$, we need to go back to equation (B.19)

$$9(\xi_{ijkh}+\xi_{jikh})=\alpha g_{ij}g_{kh}+\beta(g_{ik}g_{jh}+g_{ih}g_{jk}).$$

Let us define a tensor A_{ijkh} such that

$$\begin{aligned} A_{ijkh} \equiv & \xi_{ijkh} - \xi_{ijhk} - \xi_{ikjh} + \xi_{ikhj} - \xi_{ihkj} + \xi_{ihkj} - \xi_{jikh} + \xi_{jihk} \\ & + \xi_{jkih} - \xi_{jkhi} + \xi_{jhki} - \xi_{jhik} - \xi_{kjih} + \xi_{kjhi} + \xi_{kijh} - \xi_{kihj} \\ & + \xi_{khij} - \xi_{khji} - \xi_{hjki} + \xi_{hjik} + \xi_{hikj} - \xi_{hijk} + \xi_{hkji} - \xi_{hkij}. \end{aligned}$$

Due to the fact that A_{ijkh} is anti-symmetric in every pair of indices, it has only one independent component in a four-dimensional spacetime. Clearly, it is proportional to the Levi-Civita symbol such that

$$A_{ijkh} = \psi(g_{rs}, \phi)\epsilon_{ijkh}.$$

By virtue of the symmetries that we have in (B.15), we obtain

$$A_{ijkh} = 4(\xi_{ijkh} - \xi_{jikh} + \xi_{jhki} - \xi_{hjki} + \xi_{hikj} - \xi_{ihkj}). \tag{B.21}$$

For $n = 4$, from (B.10) and (B.16), we have

$$2\xi_{ijkh} = -(\xi_{jikh} + \xi_{kjih} + \xi_{hjki}) + \lambda g_{ij} g_{kh} + \beta(g_{ik} g_{jh} + g_{ih} g_{jk}), \tag{B.22}$$

$$2\xi_{jikh} = -(\xi_{ijkh} + \xi_{kijh} + \xi_{hikj}) + \lambda g_{ij} g_{kh} + \beta(g_{ik} g_{jh} + g_{ih} g_{jk}). \tag{B.23}$$

We calculate $(B.22) - (B.23)$ to obtain

$$4(\xi_{ijkh} - \xi_{jikh}) = \xi_{hikj} - \xi_{ihkj} + \xi_{jhki} - \xi_{hjki}.$$

The right hand side of equation (B.21) can be written in terms of the first two terms. As a result, we obtain

$$\xi_{ijkh} = a g_{ij} g_{kh} + b g_{ik} g_{jh} + a g_{ih} g_{jk} + \frac{1}{24}\psi(g_{rs}, \rho)\epsilon_{ijkh},$$

where a, b and c are arbitrary functions of the scalar ϕ. From Lemma B.1, we have $\psi(g_{rs}, \phi) = \beta\sqrt{g}$, where β is an arbitrary function of the scalar ϕ.

Lemma B.4 *If $\xi_{ijkh} = \xi_{ijkh}(g_{ab}, \phi)$ is a tensor and*

$$\xi_{ijkh} = \xi_{jikh} = \xi_{ijhk}$$

together with

$$\xi_{ijkh} + \xi_{ikjh} + \xi_{ihkj} = 0$$

then for $n \geq 2$

$$\xi_{ijkh} = \alpha\big(g_{ij} g_{kh} - \frac{1}{2}(g_{ik} g_{jh} + g_{ih} g_{jk})\big),$$

where α arbitrary function of the scalar ϕ.

Proof. The result is obvious by applying the given symmetries to Lemma B.3.

Lemma B.5 *If* $\xi_{ij,kh;rs,tu} = \xi_{ij,kh;rs,tu}(g_{ab}, \phi)$ *is a tensor and*

$$\xi_{ij,kh;rs,tu} = \xi_{rs,tu;ij,kh} = \xi_{ji,kh;rs,tu} = \xi_{ij,hk;rs,tu} \tag{B.24}$$

together with

$$\xi_{ij,kh;rs,tu} + \xi_{ih,jk;rs,tu} + \xi_{ik,hj;rs,tu} = 0 \tag{B.25}$$

and

$$\xi_{ij,kh;rs,tu} + \xi_{ij,ku;rs,th} + \xi_{ij,kt;rs,hu} + \xi_{rs,kh;ij,tu} + \xi_{rs,ku;ij,th} + \xi_{rs,kt;ij,hu} = 0, \tag{B.26}$$

then for $n > 3$*,*

$$\xi_{ij,kh;rs,tu} R^{kijh} R^{trsu} = \frac{[(2n-5)\alpha_{ij,kh;rs,tu} + 2\alpha_{ij,rs;kh,tu}] R^{kijh} R^{trsu}}{(2n-3)(n-3)},$$

where

$$\begin{aligned} \alpha_{ij,kh;rs,tu} = g^{lm}[&g_{ij}\xi_{ml,kh;rs,tu} + g_{ik}\xi_{mj,lh;rs,tu} + g_{ih}\xi_{mj,kl;rs,tu} \\ &+ g_{ir}\xi_{mj,kh;ls,tu} + g_{is}\xi_{mj,kh;rl,tu} + g_{it}\xi_{mj,kh;rs,lu} + g_{iu}\xi_{mj,kh;rs,tl}] \end{aligned} \tag{B.27}$$

Note that commas and semicolons are used for identifying the only symmetries. They have no other meanings whatsoever here.

Proof. We can write

$$\xi_{ij,kh;rs,tu} = B_i^\alpha B_j^\beta B_k^\gamma B_h^\delta B_r^\mu B_s^\nu B_t^\rho B_u^\lambda \xi_{\alpha\beta,\gamma\delta;\mu\nu,\rho\lambda}.$$

Taking the derivative of both sides with respect to $\frac{\partial}{\partial B_b^a}$ yields,

$$\begin{aligned} \frac{\partial \xi_{ij,kh;rs,tu}}{\partial B_b^a} =& \frac{\partial g_{\theta\varphi}}{\partial B_b^a} \frac{\partial \xi_{ij,kh;rs,tu}}{g_{\theta\varphi}} = 2g_{a\theta} \frac{\partial \xi_{ij,kh;rs,tu}}{g_{\theta b}} \\ =& \delta_i^b \xi_{aj,kh;rs,tu} + \delta_j^b \xi_{ia,kh;rs,tu} + \delta_k^b \xi_{ij,ah;rs,tu} + \delta_h^b \xi_{ij,ka;rs,tu} \\ &+ \delta_r^b \xi_{ij,kh;as,tu} + \delta_s^b \xi_{ij,kh;ra,tu} + \delta_t^b \xi_{ij,kh;rs,au} + \delta_u^b \xi_{ij,kh;rs,ta}. \end{aligned}$$

We multiply each side with $g^{a\mu}$ and replace b with ν to obtain

$$\begin{aligned} 2\frac{\partial \xi_{ij,kh;rs,tu}}{g_{\mu\nu}} =& \delta_i^\nu g^{a\mu} \xi_{aj,kh;rs,tu} + \delta_j^\nu g^{a\mu} \xi_{ia,kh;rs,tu} + \delta_k^\nu g^{a\mu} \xi_{ij,ah;rs,tu} \\ &+ \delta_h^\nu g^{a\mu} \xi_{ij,ka;rs,tu} + \delta_r^\nu g^{a\mu} \xi_{ij,kh;as,tu} + \delta_s^\nu g^{a\mu} \xi_{ij,kh;ra,tu} \\ &+ \delta_t^\nu g^{a\mu} \xi_{ij,kh;rs,au} + \delta_u^\nu g^{a\mu} \xi_{ij,kh;rs,ta}. \end{aligned}$$

Due to the symmetry in (μ, ν), we arrive at

$$
\begin{aligned}
&\delta_i^\nu g^{a\mu}\xi_{aj,kh;rs,tu} + \delta_j^\nu g^{a\mu}\xi_{ia,kh;rs,tu} + \delta_k^\nu g^{a\mu}\xi_{ij,ah;rs,tu} + \delta_h^\nu g^{a\mu}\xi_{ij,ka;rs,tu} \\
&\quad + \delta_r^\nu g^{a\mu}\xi_{ij,kh;as,tu} + \delta_s^\nu g^{a\mu}\xi_{ij,kh;ra,tu} + \delta_t^\nu g^{a\mu}\xi_{ij,kh;rs,au} + \delta_u^\nu g^{a\mu}\xi_{ij,kh;rs,ta} \\
&= \delta_i^\mu g^{a\nu}\xi_{aj,kh;rs,tu} + \delta_j^\mu g^{a\nu}\xi_{ia,kh;rs,tu} + \delta_k^\mu g^{a\nu}\xi_{ij,ah;rs,tu} + \delta_h^\mu g^{a\nu}\xi_{ij,ka;rs,tu} \\
&\quad + \delta_r^\mu g^{a\nu}\xi_{ij,kh;as,tu} + \delta_s^\mu g^{a\nu}\xi_{ij,kh;ra,tu} + \delta_t^\mu g^{a\nu}\xi_{ij,kh;rs,au} + \delta_u^\mu g^{a\nu}\xi_{ij,kh;rs,ta}.
\end{aligned}
$$

We multiply each side with $g_{\mu\sigma}$ to obtain

$$
\begin{aligned}
&\delta_i^\nu \xi_{\sigma j,kh;rs,tu} + \delta_j^\nu \xi_{i\sigma,kh;rs,tu} + \delta_k^\nu \xi_{ij,\sigma h;rs,tu} + \delta_h^\nu \xi_{ij,k\sigma;rs,tu} \\
&\quad + \delta_r^\nu \xi_{ij,kh;\sigma s,tu} + \delta_s^\nu \xi_{ij,kh;r\sigma,tu} + \delta_t^\nu \xi_{ij,kh;rs,\sigma u} + \delta_u^\nu \xi_{ij,kh;rs,t\sigma} \\
&= g_{i\sigma} g^{a\nu}\xi_{aj,kh;rs,tu} + g_{j\sigma} g^{a\nu}\xi_{ia,kh;rs,tu} + g_{k\sigma} g^{a\nu}\xi_{ij,ah;rs,tu} + g_{h\sigma} g^{a\nu}\xi_{ij,ka;rs,tu} \\
&\quad + g_{r\sigma} g^{a\nu}\xi_{ij,kh;as,tu} + g_{s\sigma} g^{a\nu}\xi_{ij,kh;ra,tu} + g_{t\sigma} g^{a\nu}\xi_{ij,kh;rs,au} + g_{u\sigma} g^{a\nu}\xi_{ij,kh;rs,ta}.
\end{aligned}
$$

Summing over (ν, i) and replacing σ by i yields

$$
\begin{aligned}
&(n-1)\xi_{ij,kh;rs,tu} + \xi_{ji,kh;rs,tu} + \xi_{kj,ih;rs,tu} + \xi_{hj,ki;rs,tu} \\
&\quad + \xi_{rj,kh;is,tu} + \xi_{sj,kh;ri,tu} + \xi_{tj,kh;rs,iu} + \xi_{uj,kh;rs,ti} \\
&= g_{ij} g^{a\nu}\xi_{\nu a,kh;rs,tu} + g_{ik} g^{a\nu}\xi_{\nu j,ah;rs,tu} + g_{ih} g^{a\nu}\xi_{\nu j,ka;rs,tu} + g_{ir} g^{a\nu}\xi_{\nu j,kh;as,tu} \\
&\quad + g_{is} g^{a\nu}\xi_{\nu j,kh;ra,tu} + g_{it} g^{a\nu}\xi_{\nu j,kh;rs,au} + g_{iu} g^{a\nu}\xi_{\nu j,kh;rs,ta}.
\end{aligned}
$$

Upon using (B.25) for the second, the third and the fourth terms and using the equation (B.27) as well, we have

$$
\begin{aligned}
&(n-1)\xi_{ij,kh;rs,tu} + \xi_{rj,kh;is,tu} + \xi_{sj,kh;ri,tu} + \xi_{tj,kh;rs,iu} + \xi_{uj,kh;rs,ti} \\
&\qquad = \alpha_{ij,kh;rs,tu}.
\end{aligned}
\tag{B.28}
$$

As a result of (B.26), we may write

$$
\begin{aligned}
&(\xi_{rj,kh;is,tu} + \xi_{ij,kh;rs,tu} + \xi_{sj,kh;ri,tu} + \xi_{rj,tu;is,kh} \\
&\qquad + \xi_{ij,tu;rs,kh} + \xi_{sj,tu;ri,kh})R^{kijh}R^{trsu} = 0.
\end{aligned}
$$

Due to the symmetries, we have

$$
\begin{aligned}
&(2\xi_{rj,kh;is,tu} + \xi_{ij,kh;rs,tu} + 2\xi_{rj,tu;is,kh} + \xi_{ij,rs;kh,tu})R^{kijh}R^{trsu} \\
&\qquad = (4\xi_{rj,kh;is,tu} + \xi_{ij,kh;rs,tu} + \xi_{ij,rs;kh,tu})R^{kijh}R^{trsu} = 0.
\end{aligned}
$$

Considering the Riemann curvature tensor's symmetries, we calculate

$$\begin{aligned}
&\alpha_{ij,kh;rs,tu}R^{kijh}R^{trsu}\\
&=[(n-1)\xi_{ij,kh;rs,tu}+\xi_{rj,kh;is,tu}+\xi_{sj,kh;ri,tu}+\xi_{tj,kh;rs,iu}+\xi_{uj,kh;rs,ti}]R^{kijh}R^{trsu}\\
&=[(n-1)\xi_{ij,kh;rs,tu}+\xi_{rj,kh;is,tu}+\xi_{sj,kh;ri,tu}+\xi_{sj,kh;ut,ir}+\xi_{rj,kh;ut,si}]R^{kijh}R^{trsu}\\
&=[(n-1)\xi_{ij,kh;rs,tu}+2\xi_{rj,kh;is,tu}+2\xi_{sj,kh;ri,tu}]R^{kijh}R^{trsu}\\
&=[(n-1)\xi_{ij,kh;rs,tu}+4\xi_{rj,kh;is,tu}]R^{kijh}R^{trsu}\\
&=[(n-2)\xi_{ij,kh;rs,tu}-\xi_{ij,rs;kh,tu}]R^{kijh}R^{trsu}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\alpha_{ij,rs;kh,tu}R^{kijh}R^{trsu}\\
&=[(n-1)\xi_{ij,rs;kh,tu}+\xi_{kj,rs;ih,tu}+\xi_{hj,rs;ki,tu}+\xi_{tj,rs;kh,iu}+\xi_{uj,rs;kh,ti}]R^{kijh}R^{trsu}\\
&=[(n-1)\xi_{ij,rs;kh,tu}-\xi_{ij,rs;kh,tu}+\xi_{rj,tu;kh,is}+\xi_{rj,ut;kh,si}]R^{kijh}R^{trsu}\\
&=[(n-2)\xi_{ij,rs;kh,tu}+2\xi_{rj,tu;kh,is}]R^{kijh}R^{trsu}\\
&=[(n-\frac{5}{2})\xi_{ij,rs;kh,tu}-\xi_{ij,kh;rs,tu}]R^{kijh}R^{trsu}.
\end{aligned}$$

We may eliminate the $\xi_{ij,rs;kh,tu}$ term by using the last two equations to obtain

$$\begin{aligned}
\big((2n-5)\alpha_{ij,kh;rs,tu}+2\alpha_{ij,rs;kh,tu}\big)R^{kijh}R^{trsu}&\\
=[(2n-5)(n-2)-1]\,\xi_{ij,kh;rs,tu}R^{kijh}R^{trsu}&\\
=[(2n-3)(n-3)]\,\xi_{ij,kh;rs,tu}R^{kijh}R^{trsu},&
\end{aligned}$$

which completes the proof.

Lemma B.6 *Under the conditions of Lemma B.5 and for $n>2$*

$$g^{tu}\xi_{ij,kh;rs,tu}=\frac{1}{n-2}\beta_{ij,kh;rs} \tag{B.29}$$

and

$$\begin{aligned}
g^{jt}\xi_{ij,kh;rs,tu}=&-\frac{1}{2}g^{jt}\xi_{iu,kh;rs,tj}\\
&+\mu\delta_4^n\sqrt{g}(g_{kr}\epsilon_{iuhs}+g_{ks}\epsilon_{iuhr}+g_{hr}\epsilon_{iuks}+g_{hs}\epsilon_{iukr}),
\end{aligned} \tag{B.30}$$

where

$$\beta_{ij,kh;rs}=\lambda(g_{ij}g_{khrs}-\frac{1}{2}g_{ik}g_{jhrs}-\frac{1}{2}g_{ih}g_{jkrs}-\frac{1}{2}g_{ir}g_{khjs}-\frac{1}{2}g_{is}g_{khjr}),$$

$$g_{khrs}=g_{kh}g_{rs}-\frac{1}{2}(g_{kr}g_{hs}+g_{ks}g_{hr})$$

and λ, μ are arbitrary functions of the scalar ϕ.

Once again, note that commas and semicolons are used for identifying the only symmetries. They have no other meanings whatsoever here.

Proof. We may define a new tensor $\beta_{ij,kh;rs}$ such that

$$g^{tu}\xi_{ij,kh;rs,tu} \equiv \kappa_1 \beta_{ij,kh;rs}, \tag{B.31}$$

$$g^{tu}\xi_{ij,rs;kh,tu} \equiv \kappa_2 \beta_{ij,rs;kh}, \tag{B.32}$$

where κ_1 and κ_2 are arbitrary functions of the scalar ϕ. Therefore, after multiplying each line with the inverse metric, we have

$$g^{ij}g^{kh}g^{rs}g^{tu}\xi_{ij,kh;rs,tu} = \kappa_1 g^{ij}g^{kh}g^{rs}\beta_{ij,kh;rs},$$
$$g^{ij}g^{kh}g^{rs}g^{tu}\xi_{ij,rs;kh,tu} = \kappa_2 g^{ij}g^{kh}g^{rs}\beta_{ij,rs;kh}.$$

By comparing these two, we have

$$\begin{aligned} &g^{ij}g^{kh}g^{rs}g^{tu}\xi_{ij,kh;rs,tu} - g^{ij}g^{kh}g^{rs}g^{tu}\xi_{ij,rs;kh,tu} \\ &= \kappa_1 g^{ij}g^{kh}g^{rs}\beta_{ij,kh;rs} - \kappa_2 g^{ij}g^{kh}g^{rs}\beta_{ij,rs;kh} = 0. \end{aligned}$$

Therefore, $\kappa_1 = \kappa_2$. Together with this, we may also use (B.25) to obtain the symmetries such that

$$\beta_{ij,kh;rs} = \beta_{ij,rs;kh}, \tag{B.33}$$

$$\beta_{ij,kh;rs} + \beta_{ik,jh;rs} + \beta_{ih,kj;rs} = 0. \tag{B.34}$$

This tensor is a function of the the metric and the scalar, i.e.,

$$\beta_{ij,kh;rs} = \beta_{ij,kh;rs}(g_{ab}, \phi).$$

Since we have a tensor, we can write

$$\beta_{ij,kh;rs} = B^{\alpha}_{i}B^{\beta}_{j}B^{\gamma}_{k}B^{\delta}_{h}B^{\mu}_{r}B^{\nu}_{s}\beta_{\alpha\beta,\gamma\delta;\mu\nu}.$$

Taking the derivative of both sides with respect to $\frac{\partial}{\partial B^a_b}$ yields

$$\begin{aligned} \frac{\partial \beta_{ij,kh;rs}}{\partial B^a_b} =& \frac{\partial g_{\theta\varphi}}{\partial B^a_b}\frac{\partial \beta_{ij,kh;rs}}{g_{\theta\varphi}} = 2g_{a\theta}\frac{\partial \beta_{ij,kh;rs}}{g_{\theta b}} \\ =& \delta^b_i \beta_{aj,kh;rs} + \delta^b_j \beta_{ia,kh;rs} + \delta^b_k \beta_{ij,ah;rs} \\ &+ \delta^b_h \beta_{ij,ka;rs} + \delta^b_r \beta_{ij,kh;as} + \delta^b_s \beta_{ij,kh;ra}. \end{aligned}$$

We multiply each side with $g^{a\mu}$ and replace b with ν to obtain

$$
\begin{aligned}
2\frac{\partial \beta_{ij,kh;rs}}{g_{\mu\nu}} =& \delta_i^\nu g^{a\mu}\beta_{aj,kh;rs} + \delta_j^\nu g^{a\mu}\beta_{ia,kh;rs} + \delta_k^\nu g^{a\mu}\beta_{ij,ah;rs} \\
&+ \delta_h^\nu g^{a\mu}\beta_{ij,ka;rs} + \delta_r^\nu g^{a\mu}\xi_{ij,kh;as} + \delta_s^\nu g^{a\mu}\beta_{ij,kh;ra}.
\end{aligned}
$$

Due to the symmetry in (μ, ν), we find

$$
\begin{aligned}
&\delta_i^\nu g^{a\mu}\beta_{aj,kh;rs} + \delta_j^\nu g^{a\mu}\beta_{ia,kh;rs} + \delta_k^\nu g^{a\mu}\beta_{ij,ah;rs} \\
&\quad + \delta_h^\nu g^{a\mu}\beta_{ij,ka;rs} + \delta_r^\nu g^{a\mu}\beta_{ij,kh;as} + \delta_s^\nu g^{a\mu}\beta_{ij,kh;ra} \\
&= \delta_i^\mu g^{a\nu}\beta_{aj,kh;rs} + \delta_j^\mu g^{a\nu}\beta_{ia,kh;rs} + \delta_k^\mu g^{a\nu}\beta_{ij,ah;rs} \\
&\quad + \delta_h^\mu g^{a\nu}\beta_{ij,ka;rs} + \delta_r^\mu g^{a\nu}\beta_{ij,kh;as} + \delta_s^\mu g^{a\nu}\beta_{ij,kh;ra}.
\end{aligned}
$$

We multiply each side with $g_{\mu\sigma}$ to obtain

$$
\begin{aligned}
&\delta_i^\nu \beta_{\sigma j,kh;rs} + \delta_j^\nu \beta_{i\sigma,kh;rs} + \delta_k^\nu \beta_{ij,\sigma h;rs} \\
&\quad + \delta_h^\nu \beta_{ij,k\sigma;rs} + \delta_r^\nu \beta_{ij,kh;\sigma s} + \delta_s^\nu \beta_{ij,kh;r\sigma} \\
&= g_{i\sigma} g^{a\nu}\beta_{aj,kh;rs} + g_{j\sigma} g^{a\nu}\beta_{ia,kh;rs} + g_{k\sigma} g^{a\nu}\beta_{ij,ah;rs} \\
&\quad + g_{h\sigma} g^{a\nu}\beta_{ij,ka;rs} + g_{r\sigma} g^{a\nu}\beta_{ij,kh;as} + g_{s\sigma} g^{a\nu}\beta_{ij,kh;ra}.
\end{aligned}
$$

Summing over (ν, i) and replacing σ by i yields

$$
\begin{aligned}
&(n-1)\beta_{ij,kh;rs} + \beta_{ji,kh;rs} + \beta_{kj,ih;rs} + \beta_{hj,ki;rs} + \beta_{rj,kh;is} + \beta_{sj,kh;ri} \\
&= g_{ij} g^{a\nu}\beta_{\nu a,kh;rs} + g_{ik} g^{a\nu}\beta_{\nu j,ah;rs} + g_{ih} g^{a\nu}\beta_{\nu j,ka;rs} \\
&\quad + g_{ir} g^{a\nu}\beta_{\nu j,kh;as} + g_{is} g^{a\nu}\beta_{\nu j,kh;ra}.
\end{aligned}
$$

According to equation (B.25), the sum of the second, the third and the fourth terms vanishes. Similarly, we can calculate the first, the fifth, and the sixth terms in the following way

$$
\begin{aligned}
&(n-1)\beta_{ij,kh;rs} + \beta_{rj,kh;is} + \beta_{sj,kh;ri} \\
&\quad = (n-2)\beta_{ij,kh;rs} \\
&\quad = g_{ij} g^{a\nu}\beta_{\nu a,kh;rs} + g_{ik} g^{a\nu}\beta_{\nu j,ah;rs} + g_{ih} g^{a\nu}\beta_{\nu j,ka;rs} \\
&\qquad + g_{ir} g^{a\nu}\beta_{\nu j,kh;as} + g_{is} g^{a\nu}\beta_{\nu j,kh;ra}.
\end{aligned}
$$

By virtue of (2.20), we may rewrite the latter equation as

$$
\begin{aligned}
(n-2)\beta_{ij,kh;rs} =& g_{ij} g^{a\nu}\beta_{a\nu,kh;rs} - \frac{1}{2} g_{ik} g^{a\nu}\beta_{a\nu,jh;rs} - \frac{1}{2} g_{ih} g^{a\nu}\beta_{a\nu,jk;rs} \\
&- \frac{1}{2} g_{ir} g^{a\nu}\beta_{a\nu,kh;js} - \frac{1}{2} g_{is} g^{a\nu}\beta_{a\nu,kh;jr}
\end{aligned}
\tag{B.35}
$$

We may show all symmetries of $g^{ij}\beta_{ij,kh;rs}$ by using the symmetries of $\beta_{ij,kh;rs}$ as

$$g^{ij}\beta_{ij,kh;rs} = \sigma\xi_{khrs}, \tag{B.36}$$

$$\xi_{khrs} = \xi_{hkrs} = \xi_{rskh}.$$

Therefore, we calculate

$$g^{rs}(\beta_{ij,kh;rs} + \beta_{ik,jh;rs} + \beta_{ih,kj;rs}) = \sigma(\xi_{ijkh} + \xi_{ikjh} + \xi_{ihkj}) = 0,$$

where σ is any function of the scalar ϕ. Making use of Lemma B.4, we find

$$\xi_{ijkh} = a[g^{ij}g^{hk} - \frac{1}{2}(g^{ik}g^{jh} + g^{ih}g^{jk})]$$

Then, we may define

$$g_{ijkh} \equiv [g^{ij}g^{hk} - \frac{1}{2}(g^{ik}g^{jh} + g^{ih}g^{jk})]. \tag{B.37}$$

By using (B.31), (B.35), (B.36) and (B.37), we may write

$$\begin{aligned}(n-2)g^{tu}\xi_{ij,kh;rs,tu} = \lambda\Bigg(& g_{ij}g_{khrs} - \frac{1}{2}g_{ik}g_{jhrs} - \frac{1}{2}g_{ih}g_{jkrs} \\ & - \frac{1}{2}g_{ir}g_{khjs} - \frac{1}{2}g_{is}g_{khjr}\Bigg)\end{aligned}$$

and

$$\beta_{ij,kh;rs} = \lambda\Bigg(g_{ij}g_{khrs} - \frac{1}{2}g_{ik}g_{jhrs} - \frac{1}{2}g_{ih}g_{jkrs} - \frac{1}{2}g_{ir}g_{khjs} - \frac{1}{2}g_{is}g_{khjr}\Bigg),$$

where λ is arbitrary function of the scalar ϕ. Consequently, we have

$$g^{tu}\xi_{ij,kh;rs,tu} = \frac{1}{n-2}\beta_{ij,kh;rs}$$

We may also define a tensor $\psi_{i,kh;rs,u}$ as

$$\psi_{i,kh;rs,u} \equiv g^{jt}\xi_{ij,kh;rs,tu}. \tag{B.38}$$

We may obtain a relation related to this tensor by multiplying (B.26) by g^{kr}

$$\begin{aligned}&\psi_{h,ij;tu,s} + \psi_{u,ij;th,s} + \psi_{t,ij;hu,s} \\ &\qquad - \frac{1}{2(n-2)}\left[\beta_{sh,ij;tu} + \beta_{su,ij;th} + \beta_{st,ij;hu}\right] = 0.\end{aligned}$$

From (B.34), we reduce the latter equation to

$$\psi_{h,ij;tu,s} + \psi_{u,ij;th,s} + \psi_{t,ij;hu,s} = 0. \tag{B.39}$$

We also have symmetries from (B.24) as

$$\psi_{i,kh;rs,u} = \psi_{i,hk;rs,u} = \psi_{u,rs;kh,i}. \tag{B.40}$$

We multiply (B.28) with g^{jt} to obtain

$$\begin{aligned}
(n-1)\psi_{i,kh;rs,u} &+ \psi_{r,kh;is,u} + \psi_{s,kh;ri,u} + \frac{1}{n-2}\beta_{kh;rs,iu} + \psi_{u,kh;rs,i} \\
=&\frac{1}{n-2}\beta_{kh;rs,iu} - \frac{1}{2(n-2)}g_{ik}g^{jt}\beta_{jh,rs;tu} \\
&- \frac{1}{2(n-2)}g_{ih}g^{jt}\beta_{jk,rs;tu} + g_{ir}g^{a\nu}\psi_{\nu,kh;as,u} \\
&+ g_{is}g^{a\nu}\psi_{\nu,kh;ra,u} + \psi_{i,kh;rs,u} + g_{iu}g^{a\nu}g^{jt}\xi_{\nu j,kh;rs,ta}.
\end{aligned} \tag{B.41}$$

By using (2.20), (B.34), and (B.39), we calculate

$$\begin{aligned}
g_{ir}g^{a\nu}\psi_{\nu,kh;as,u} &= -\frac{1}{2}g_{ir}g^{a\nu}\psi_{s,kh;a\nu,u} = -\frac{1}{2}g_{ir}g^{a\nu}g^{\beta\gamma}\xi_{\beta s,kh;a\nu,u\gamma} \\
&= -\frac{1}{2(n-2)}g_{ir}g^{\beta\gamma}\beta_{\beta s,kh;u\gamma} = \frac{1}{4(n-2)}g_{ir}\xi_{uskh}.
\end{aligned}$$

Similarly, we have

$$g_{is}g^{a\nu}\psi_{\nu,kh;ra,u} = \frac{1}{4(n-2)}g_{is}\xi_{urkh}.$$

The last term of (B.41) can be calculated by using (B.26) as follows

$$\begin{aligned}
g_{iu}g^{a\nu}g^{jt}\big(&\xi_{\nu j,kh;rs,ta} + \xi_{\nu a,kh;rs,tj} + \xi_{\nu t,kh;rs,ja} \\
&+ \xi_{\nu j,rs;kh,ta} + \xi_{\nu a,rs;kh,tj} + \xi_{\nu t,rs;kh,ja}\big) \\
=&\frac{2}{(n-2)}g_{iu}\xi_{khrs} + 4g_{iu}g^{a\nu}g^{jt}\xi_{\nu j,kh;rs,ta} = 0.
\end{aligned}$$

Therefore, we find

$$g_{iu}g^{a\nu}g^{jt}\xi_{\nu j,kh;rs,ta} = -\frac{1}{2(n-2)}g_{iu}\xi_{khrs}.$$

We may further reduce (B.41) as

$$\begin{aligned}
&(n-3)\psi_{i,kh;rs,u} + \psi_{u,kh;rs,i} \\
=&\frac{1}{4(n-2)}g_{ik}\xi_{hurs} + \frac{1}{4(n-2)}g_{ih}\xi_{kurs} + \frac{1}{4(n-2)}g_{ir}\xi_{uskh} \\
&+ \frac{1}{4(n-2)}g_{is}\xi_{urkh} - \frac{1}{2(n-2)}g_{iu}\xi_{khrs} \\
=&-\frac{1}{2}\beta_{iu,kh;rs}.
\end{aligned} \tag{B.42}$$

Interchanging i and u yields

$$(n-3)\psi_{u,kh;rs,i} + \psi_{i,kh;rs,u} = -\frac{1}{2}\beta_{iu,kh;rs}. \tag{B.43}$$

We calculate $(n-3)(B.42) - (B.43)$ to obtain

$$\begin{aligned}\big((n-3)(n-3)-1\big)\psi_{i,kh;rs,u} &= -\frac{n-4}{2}\beta_{iu,kh;rs}\\ = \big((n-2)(n-4)\big)\psi_{i,kh;rs,u} &= -\frac{n-4}{2}\beta_{iu,kh;rs}.\end{aligned}$$

For $n > 2$ and $n \neq 4$, we have

$$\psi_{i,kh;rs,u} = -\frac{1}{2(n-2)}\beta_{iu,kh;rs} \tag{B.44}$$

or

$$g^{jt}\xi_{ij,kh;rs,tu} = -\frac{1}{2}g^{jt}\xi_{iu,kh;rs,tj}.$$

For $n > 2$ and $n = 4$, we have

$$\psi_{i,kh;rs,u} + \psi_{u,kh;rs,i} = -\frac{1}{2}\beta_{iu,kh;rs}.$$

Therefore, the most general form of $\psi_{i,kh;rs,u}$ can be written by considering the anti-symmetry in (i, u) in the following way

$$\begin{aligned}\psi_{i,kh;rs,u} = &-\frac{1}{4}\beta_{iu,kh;rs} + \Lambda_1 g_{rs}\epsilon_{iukh} + \Lambda_2 g_{kh}\epsilon_{iurs}\\ &+ \Lambda_3 g_{kr}\epsilon_{iuhs} + \Lambda_4 g_{ks}\epsilon_{iuhr} + \Lambda_5 g_{hr}\epsilon_{iuks} + \Lambda_6 g_{hs}\epsilon_{iukr},\end{aligned}$$

where Λ_1, Λ_2, Λ_3, Λ_4, Λ_5 and Λ_6 are arbitrary functions of the scalar ϕ. $\Lambda_1 = \Lambda_2 = 0$ due to (r, s) and (k, h) symmetry. Using these two symmetries, we also have $\Lambda_3 = \Lambda_4 = \Lambda_5 = \Lambda_6$. Thus, we have successfully completed the proof.

$$\begin{aligned}g^{jt}\xi_{ij,kh;rs,tu} = &-\frac{1}{2}g^{jt}\xi_{iu,kh;rs,tj}\\ &+ \mu\delta^n_4\sqrt{g}(g_{kr}\epsilon_{iuhs} + g_{ks}\epsilon_{iuhr} + g_{hr}\epsilon_{iuks} + g_{hs}\epsilon_{iukr}),\end{aligned}$$

where μ is any function of the scalar ϕ.

Lemma B.7 *If ψ is a scalar which depends on the variables g_{ij}, ϕ and $\phi_{,i}$, i.e., $\psi = \psi(g_{ij}, \phi, \phi_{,i})$, then $\psi = \psi(\phi, \rho)$, where $\rho \equiv g^{ij}\phi_{,i}\phi_{,j}$.*

Proof. If we have a scalar $\psi = \psi(g_{ij}, \phi, \phi_{,i})$, and we have

$$\psi(g_{ij}, \phi, \phi_{,i}) = \overline{\psi}(\overline{g}_{ij}, \overline{\phi}, \overline{\phi}_{,i}).$$

Taking the derivative of this with respect to $B^a{}_b$, together with (2.2), yields

$$\frac{\partial\psi}{\partial B^a{}_b} = \frac{\partial g_{\alpha\beta}}{\partial B^a{}_b}\frac{\partial\psi}{\partial g_{\alpha\beta}} + \frac{\partial\phi_{,i}}{\partial B^a{}_b}\frac{\partial\psi}{\partial\phi_{,i}} = 2\frac{\partial\psi}{\partial g_{b\beta}}g_{a\beta} + \frac{\partial\psi}{\phi_{,b}}\phi_{,a} = 0.$$

If we multiply this with g^{ar}, we obtain

$$2\frac{\partial\psi}{\partial g_{br}} = -\frac{\partial\psi}{\phi_{,b}}\phi^{,r}, \tag{B.45}$$

where $\phi^{,r} = g^{ar}\phi_{,a}$. Due to symmetry in b and r on the right hand side, the left hand side must be symmetric in (b, c). Therefore, we have

$$\frac{\partial\psi}{\phi_{,b}}\phi^{,r} = \frac{\partial\psi}{\phi_{,r}}\phi^{,b}.$$

Multiplying this equation by $\phi_{,r}$ and defining $\rho \equiv g^{ij}\phi_{,i}\phi_{,j}$ and $\lambda \equiv \rho^{-1}\phi_{,r}\frac{\partial\psi}{\partial\phi_{,r}}$, we obtain

$$\frac{\partial\psi}{\partial\phi_{,r}} = \lambda\phi^{,r}. \tag{B.46}$$

Here, we can define $\psi(g_{ij}, \phi, \phi_{,i}) = F(g_{ij}, \phi, \phi_{,i}, \rho)$. By taking the derivative of this with respect to $\phi_{,r}$ yields

$$\frac{\partial\psi}{\partial\phi_{,r}} = \frac{\partial g_{\alpha\beta}}{\partial\phi_{,r}}\frac{\partial F}{\partial g_{\alpha\beta}} + \frac{\partial\phi}{\partial\phi_{,r}}\frac{\partial F}{\partial\phi} + \frac{\partial\phi_{,i}}{\partial\phi_{,r}}\frac{\partial F}{\partial\phi_{,i}} + \frac{\partial\rho}{\partial\phi_{,r}}\frac{\partial F}{\partial\rho},$$

$$\frac{\partial\psi}{\partial\phi_{,r}} = \frac{\partial F}{\partial\phi_{,r}} + \frac{\partial\rho}{\partial\phi_{,r}}\frac{\partial F}{\partial\rho}. \tag{B.47}$$

We can calculate the second term on the right hand side in the following way

$$\frac{\partial\rho}{\partial\phi_{,r}}\frac{\partial F}{\partial\rho} = 2g^{ij}\phi_{,i}\delta^r_j\frac{\partial F}{\partial\rho} = 2\phi^{,r}\frac{\partial F}{\partial\rho}.$$

Consequently, we can rewrite (B.47) as

$$\frac{\partial\psi}{\partial\phi_{,r}} = \frac{\partial F}{\partial\phi_{,r}} + 2\phi^{,r}\frac{\partial F}{\partial\rho}$$

By comparing this result with (B.46), we find

$$2\frac{\partial F}{\partial\rho} = \lambda \qquad \text{and} \qquad \frac{\partial F}{\partial\phi_{,r}} = 0. \tag{B.48}$$

Therefore, we find that $\psi = F(g_{ij}, \phi, \rho)$. Considering the obtained result, the right hand side of (B.45) vanishes. Therefore, the left hand side must also vanish. As a result, we have

$$\psi = \psi(\phi, \rho). \tag{B.49}$$

Lemma B.8 *If ψ is a scalar density which depends on the variables g_{ij}, ϕ and $\phi_{,i}$, i.e., $\psi = \psi(g_{ij}, \phi, \phi_{,i})$, then $\psi = \sqrt{g}\kappa(\phi, \rho)$, where κ is a constant.*

Proof. Using the methods and results employed in Lemmas B.1 and B.7, this can be easily established.

Lemma B.9 *If $\psi_{ijkh} = \psi_{ijkh}(g_{ab}, \phi, \phi_{,a})$ is a tensor and*

$$\psi_{ijkh} = \psi_{jikh} = \psi_{ijhk}$$

together with

$$\psi_{ijkh} + \psi_{ikjh} + \psi_{ihkj} = 0,$$

then for $n \geq 3$

$$\psi_{ijkh} R^{kijh} = \alpha R^{ij} \phi_{,i} \phi_{,j} + \beta R, \tag{B.50}$$

where $\alpha = \alpha(\phi, \rho)$ and $\beta = \beta(\phi, \rho)$.

The detailed proof of this identity is long. However, this result is immediate when one considers the most general form of the tensor $\psi_{ijkh} = \psi_{ijkh}(g_{ab}, \phi, \phi_{,a})$ with the given symmetry properties. A rigorous proof of this can be obtained by following the same techniques that we used in Lemmas B.3 and B.7.

Lemma B.10 *If $n = 4$ and if $\chi^{ij,kh;ab,cd}$ satisfies equation (2.38), then*

$$\Lambda^{ij,kh;ab,cd;rs,tu} = A\epsilon^{ij,kh;ab,cd;rs,tu},$$

where

$$A = A(g_{ij}, g_{ij,k}, g_{ij,kh}, \phi, \phi_{,i})$$

is a scalar and

$$\epsilon^{ij,kh;ab,cd;rs,tu} \equiv \sum_{tu} \sum_{rs} \sum_{cd} \sum_{ab} \sum_{kh} \sum_{ij} \epsilon^{ikac} \epsilon^{jhrt} \epsilon^{bdsu} / g,$$

where ϵ^{ikac} is the four-dimensional permutation symbol which has the values 0, 1 and -1. The summation symbol is defined as

$$\sum_{ij} A^{ij\cdots}_{\cdots} \equiv A^{ij\cdots}_{\cdots} + A^{ji\cdots}_{\cdots}.$$

Proof. Here, we can use the following notation

$$\Lambda^{ij,kh;ab,cd;rs,tu} \equiv ij, kh; ab, cd; rs, tu$$

without using the summation convention. Since we have $n = 4$, some of the indices must be equal to each other.

If five or more indices are equal, then we will have at least two of these indices in the two of three groups (ij, kh), (ab, cd) and (rs, tu). In light of (2.20), we can put them in the form $\cdot\cdot, ii; \cdot\cdot, ii; \cdot\cdot, \cdot i$. By using (2.37), this term immediately vanishes. If three or four are equal in the same group of indices, then due to (2.12), we will have zero.

If four of the indices are equal to each other, we can calculate every possible combination. However, considering the results above, there is no need to check every possible combination. Let us say we have four i, three j and k, two h.

If our indices are in the form $jk, ii; jk, hi; jk, hi$, from (2.37), we have

$$2jk, ii; jk, hi; jk, hi + 2jk, ii; jk, hh; jk, ii + 2jk, ii; jk, hi; jk, hi = 0.$$

Note that the second term vanishes by (2.37). The first and the third terms are the same. Therefore, this term is equal to zero.

If our indices are in the form of $jj, ii; jk, hi; kk, hi$, from (2.37), we will get the same result as the one above.

If our indices are in the form $hh, ii; jk, ji; jk, ki$, again, we will get the same result.

If three of the indices are equal, then there must be three i, j, k and h. Let us say we have $ij, kh; ij, kh; ij, kh$. We may use (2.37) to conclude that

$$4(ij, kh; ij, kh; ij, kh) + 2(ij, kh; ij, kk; ij, hh) = 0.$$

From the second term, making use of (2.37) for k in the first two group of indices, we also find

$$6(ij, kh; ij, kk; ij, hh) = 0.$$

Therefore, we conclude that $ij, kh; ij, kh; ij, kh = 0$.

We infer that if we do not have any equal indices in any group, the result is zero as shown above, unless we have $jk, ih; jk, ih; ij, kh$.

If we have three equal indices in any two groups, then the other two pair of indices should not be equal in order not to obtain a zero. Therefore, any combination such as $kh, ii; kh, \cdot i; \cdot\cdot, \cdot\cdot = 0$, as shown above. Therefore, only the following combinations will survive

$$
\begin{aligned}
&ii, hh; ik, jh; kk, jj, \qquad ii, hh; ik, jj; hj, kk, \\
&ii, jk; ij, hh; jh, kk, \qquad ii, jk; ij, kh; jk, hh, \\
&jk, ih; jk, ih; ij, kh.
\end{aligned}
$$

However, each term can be written in terms of the first one due to (2.37). Therefore, we may write

$$
\begin{aligned}
ii, hh; ik, jj; hj, kk &= -ii, hh; ik, jh; kk, jj, \\
ii, jk; ij, hh; jh, kk &= -\frac{1}{2} ii, hh; ik, jh; kk, jj, \\
ii, jk; ij, kh; jk, hh &= -\frac{1}{4} ii, hh; ik, jh; kk, jj, \\
jk, ih; jk, ih; ij, kh &= \frac{1}{8} ii, hh; ik, jh; kk, jj.
\end{aligned}
$$

Therefore, $\Lambda^{ij,kh;ab,cd;rs,tu}$ has only one independent component when $n = 4$. Since we can obtain every non-zero permutation of indices from the first one, examining only this term will suffice. It can be easily shown that $ii, hh; ik, jh; kk, jj$ is anti-symmetric under the interchange of any two groups of three equal indices as a consequence of (2.37).

Note that every equal indices in any group should be together as two, before or after the comma. Otherwise, the term vanishes, namely, $i\cdot, i\cdot; \cdot\cdot, \cdot\cdot; \cdot\cdot, \cdot\cdot = 0$. Now, we can define a new quantity which has the same symmetry properties as $\Lambda^{ij,kh;ab,cd;rs,tu}$,

$$
\epsilon^{ij,kh;ab,cd;rs,tu} \equiv \sum_{tu} \sum_{rs} \sum_{cd} \sum_{ab} \sum_{kh} \sum_{ij} \epsilon^{ikac} \epsilon^{jhrt} \epsilon^{bdsu} / g.
$$

Therefore, if $\chi^{ij,kh;ab,cd}$ satisfies (2.38), we have

$$
\Lambda^{ij,kh;ab,cd;rs,tu} = A\epsilon^{ij,kh;ab,cd;rs,tu}, \tag{B.51}
$$

where $A = A(g_{ij}, g_{ij,k}, g_{ij,kh}, \phi, \phi_{,i})$.

Lemma B.11 *If $n = 4$ and if $\chi^{ij,kh;ab,cd}$ satisfies equation (2.38), then*

$$
\Lambda^{ij,kh;ab,cd;rs,tu;pq,lm} = 0.
$$

Proof. Note that we use the notation that we employed in Lemma B.10, where there is no summation on repeated indices.

We have two options to obtain non-zero terms from the results of Lemma B.10.

$$ii, hh; ik, jh; kk, jj; ij, kh, \qquad ii, hh; ik, jh; kk, jj; ih, kj$$

Other combinations of indices that we have in the last group of four indices are directly zero as shown before. If we apply (2.38) to $(ik, jh; ij, kh)$, we calculate

$$\begin{aligned} ik, jh; ij, kh + ik, jk; ij, hh + ik, jh; ij, kh \\ + ij, jh; ik, kh + ij, jk; ik, hh + ij, jh; ik, kh = 0. \end{aligned}$$

This can be written as

$$2(ik, jh; ij, kh) + ik, jk; ij, hh + 2(ik, kh; ij, jh) + ik, hh; ij, jk = 0$$

After using (2.20), we obtain

$$2(ik, jh; ij, kh) - \frac{1}{2}(ij, kk; ij, hh) - (ih, kk; ih, jj) - \frac{1}{2}(ik, hh; ik, jj) = 0$$

The last three terms are zero. Therefore, the first term also is zero. The same procedure shows that $(ik, jh; ih, kj)$ is also zero. Thus

$$\Lambda^{ij,kh;ab,cd;rs,tu;pq,lm} = 0 \tag{B.52}$$

Due to (B.51) and (B.52), we can conclude that $A = A(g_{ij}, g_{ij,k}, \phi, \phi_{,i})$. As a result of Lemmas (2.1) and (B.7), we have $A = A(\phi, \rho)$.

Lemma B.12 *If $n = 4$, then (2.35) implies that $\Lambda^{a;b;ij,kh;rs,tu} = 0$.*

Proof. We again employ our notation without summation. For the last two groups of four indices, we can use the results that we have in Lemma B.10. Therefore, the only possible non-zero terms are of the form

$$i; j; ii, hh; kk, jj, \qquad k; j; ii, hh; ik, jh.$$

Obviously the first term is zero due to (2.35). By virtue of the symmetry relation that we have in Appendix C.5, the second term can be written as

$$k; j; ii, hh; ik, jh = j; h; ii, hh; ik, kj,$$

which vanishes again due to equation (2.35). Therefore, if $n = 4$, then

$$\Lambda^{a;b;ij,kh;rs,tu} = 0. \tag{B.53}$$

APPENDIX C

EXTENDED CALCULATIONS

In this appendix, some long calculations arising from the derivation of the most general form of the Lagrangian and the derivation of the field equations are made.

C.1 Calculation of the First Term of (2.51)

Starting from the definition of $\epsilon^{ij,kh;ab,cd;rs,tu}$, we have

$$\epsilon^{ij,kh;ab,cd;rs,tu} \equiv \sum_{tu}\sum_{rs}\sum_{cd}\sum_{ab}\sum_{kh}\sum_{ij} \epsilon^{ikac}\epsilon^{jhrt}\epsilon^{bdsu}/g$$

We calculate

$$\begin{aligned} &A\epsilon^{ij,kh;ab,cd;rs,tu}R_{trsu}R_{cabd}R_{kijh} \\ &= A\sum_{tu}\sum_{rs}\sum_{cd}\sum_{ab}\sum_{kh}\sum_{ij} \epsilon^{ikac}\epsilon^{jhrt}\epsilon^{bdsu}R_{trsu}R_{cabd}R_{kijh}/g \\ &= A\lambda(*R^{jh}{}_{su})(*R^{su}{}_{ca})(*R^{ac}{}_{jh})/g, \end{aligned}$$

where λ is a constant.

C.2 Calculation of the Second Term of (2.51)

Calculation of the second term on the right hand side of equation (2.51) can be completed here. From Lemma B.5, we have

$$\xi_{ij,kh;rs,tu}R^{kijh}R^{trsu} = \frac{[(2n-5)\alpha_{ij,kh;rs,tu} + 2\alpha_{ij,rs;kh,tu}]R^{kijh}R^{trsu}}{(2n-3)(n-3)}, \qquad \text{(C.1)}$$

where from equation (B.27), we have

$$\alpha_{ij,kh;rs,tu} = g^{lm}[g_{ij}\xi_{ml,kh;rs,tu} + g_{ik}\xi_{mj,lh;rs,tu} + g_{ih}\xi_{mj,kl;rs,tu} + g_{ir}\xi_{mj,kh;ls,tu} + g_{is}\xi_{mj,kh;rl,tu} + g_{it}\xi_{mj,kh;rs,lu} + g_{iu}\xi_{mj,kh;rs,tl}].$$

Using equations (2.20), (B.29) and (B.38), we calculate

$$\begin{aligned}\alpha_{ij,kh;rs,tu} =& \frac{1}{n-2}[g_{ij}\beta_{kh,rs;tu} - \frac{1}{2}g_{ik}\beta_{jh,rs;tu} - \frac{1}{2}g_{ih}\beta_{jk,rs;tu}] \\ &+ g_{ir}\psi_{j,kh;tu,s} + g_{is}\psi_{j,kh;tu,r} + g_{it}\psi_{j,kh;rs,u} + g_{iu}\psi_{j,kh;rs,t}, \\ \alpha_{ij,rs;kh,tu} =& \frac{1}{n-2}[g_{ij}\beta_{rs,kh;tu} - \frac{1}{2}g_{ir}\beta_{js,kh;tu} - \frac{1}{2}g_{is}\beta_{jr,kh;tu}] \\ &+ g_{ik}\psi_{j,rs;tu,h} + g_{ih}\psi_{j,rs;tu,k} + g_{it}\psi_{j,rs;kh,u} + g_{iu}\psi_{j,rs;kh,t}.\end{aligned}$$

Consequently, we find

$$\begin{aligned}&(2n-5)\alpha_{ij,kh;rs,tu} + 2\alpha_{ij,rs;kh,tu} \\ &= \frac{2n-3}{n-2}g_{ij}\beta_{kh,rs;tu} + g_{ik}[2\psi_{j,rs;tu,h} + \frac{2n-5}{2(n-2)}\beta_{jh,rs;tu}] \\ &+ g_{ih}[2\psi_{j,rs;tu,k} + \frac{2n-5}{2(n-2)}\beta_{jk,rs;tu}] \\ &+ g_{ir}[(2n-5)\psi_{j,kh;tu,s} + \frac{1}{n-2}\beta_{js,kh;tu}] \\ &+ g_{is}[(2n-5)\psi_{j,kh;tu,r} + \frac{1}{n-2}\beta_{jr,kh;tu}] \\ &+ g_{it}[(2n-5)\psi_{j,kh;rs,u} + 2\psi_{j,rs;kh,u}] \\ &+ g_{iu}[(2n-5)\psi_{j,kh;rs,t} + 2\psi_{j,rs;kh,t}].\end{aligned} \tag{C.2}$$

We can use equation (B.30) together with equation (B.44) in order to rewrite the latter as

$$\begin{aligned}&(2n-5)\alpha_{ij,kh;rs,tu} + 2\alpha_{ij,rs;kh,tu} \\ &= \frac{2n-3}{n-2}g_{ij}\beta_{kh,rs;tu} - \frac{2n-3}{2(n-2)}g_{ik}\beta_{jh,rs;tu} - \frac{2n-3}{2(n-2)}g_{ih}\beta_{jk,rs;tu} \\ &- \frac{2n-3}{2(n-2)}g_{ir}\beta_{js,kh;tu} - \frac{2n-3}{2(n-2)}g_{is}\beta_{jr,kh;tu} - \frac{2n-3}{2(n-2)}g_{it}\beta_{ju,kh;rs} \\ &- \frac{2n-3}{2(n-2)}g_{iu}\beta_{jt,kh;rs} + \kappa_{ijkhrstu},\end{aligned}$$

where $\kappa_{ijkhrstu}$ represents all the terms containing the metric and ϵ_{ijkh} together, arising from equation (B.30). Multiplying this with $R^{kijh}R^{trsu}$ yields

$$
\begin{aligned}
&\big((2n-5)\alpha_{ij,kh;rs,tu} + 2\alpha_{ij,rs;kh,tu}\big)R^{kijh}R^{trsu} \\
&\quad= \bigg(\frac{2n-3}{n-2}g_{ij}\beta_{kh,rs;tu} - \frac{2n-3}{2(n-2)}g_{ik}\beta_{jh,rs;tu} - \frac{2n-3}{2(n-2)}g_{ih}\beta_{jk,rs;tu} \\
&\qquad - \frac{2n-3}{2(n-2)}g_{ir}\beta_{js,kh;tu} - \frac{2n-3}{2(n-2)}g_{is}\beta_{jr,kh;tu} - \frac{2n-3}{2(n-2)}g_{it}\beta_{ju,kh;rs} \\
&\qquad - \frac{2n-3}{2(n-2)}g_{iu}\beta_{jt,kh;rs} + \kappa_{ijkhrstu}\bigg)R^{kijh}R^{trsu}.
\end{aligned}
\tag{C.3}
$$

We are ready to calculate the right hand side of equation (C.3) term by term. From the first term, we have

$$
\begin{aligned}
&\frac{2n-3}{n-2}g_{ij}\beta_{kh,rs;tu}R^{kijh}R^{trsu} \\
&\quad=\lambda\frac{2n-3}{n-2}g_{ij}\bigg(g_{kh}g_{rstu} - \frac{1}{2}g_{kr}g_{hstu} - \frac{1}{2}g_{ks}g_{hrtu} \\
&\qquad - \frac{1}{2}g_{kt}g_{rshu} - \frac{1}{2}g_{ku}g_{rsht}\bigg)R^{kijh}R^{trsu} \\
&\quad=-\lambda\frac{2n-3}{n-2}R^{kh}R^{trsu}\bigg(g_{kh}\big(g_{rs}g_{tu} - \frac{1}{2}g_{rt}g_{su} - \frac{1}{2}g_{ru}g_{st}\big) \\
&\qquad - \frac{1}{2}g_{kr}\big(g_{hs}g_{tu} - \frac{1}{2}g_{ht}g_{su} - \frac{1}{2}g_{hu}g_{st}\big) \\
&\qquad - \frac{1}{2}g_{ks}\big(g_{hr}g_{tu} - \frac{1}{2}g_{ht}g_{ru} - \frac{1}{2}g_{hu}g_{rt}\big) \\
&\qquad - \frac{1}{2}g_{kt}\big(g_{rs}g_{hu} - \frac{1}{2}g_{rh}g_{su} - \frac{1}{2}g_{ru}g_{sh}\big) \\
&\qquad - \frac{1}{2}g_{ku}\big(g_{rs}g_{ht} - \frac{1}{2}g_{rh}g_{st} - \frac{1}{2}g_{rt}g_{sh}\big)\bigg).
\end{aligned}
$$

Applying contractions reduces the latter

$$
\begin{aligned}
&\lambda\frac{2n-3}{n-2}\bigg(R^2 - 0 + \frac{1}{2}R^2 - \frac{1}{2}R_{rs}R^{rs} + 0 - \frac{1}{4}R_{ru}R^{ru} - \frac{1}{2}R_{sr}R^{sr} - \frac{1}{4}R_{st}R^{st} \\
&\qquad + 0 - \frac{1}{2}R_{tu}R^{tu} + 0 - \frac{1}{4}R_{ts}R^{ts} - \frac{1}{2}R_{ut}R^{ut} - \frac{1}{4}R_{ru}R^{ru} + 0\bigg) \\
&\quad=\lambda\frac{2n-3}{n-2}\left(\frac{3}{2}R^2 - 3R_{ij}R^{ij}\right).
\end{aligned}
$$

The second term on the right hand side of equation (C.3) vanishes due to symmetry relations of the metric and the Riemann curvature tensor.

The third term on the right hand side of equation (C.3) gives the same result as the first term with a factor of $\frac{1}{2}$ by interchanging j and h. We calculate the fourth term on the right hand side of equation (C.3) as

$$
\begin{aligned}
&-\frac{2n-3}{n-2} g_{ir}\beta_{js,kh;tu} R^{kijh} R^{trsu} \\
&\quad =\lambda \frac{2n-3}{n-2} R^{kijh} R_i^{\ tsu} \bigg(g_{kh}\big(g_{js}g_{tu} - \frac{1}{2} g_{jt}g_{su} - \frac{1}{2} g_{ju}g_{st}\big) \\
&\quad - \frac{1}{2} g_{kj}\big(g_{hs}g_{tu} - \frac{1}{2} g_{ht}g_{su} - \frac{1}{2} g_{hu}g_{st}\big) \\
&\quad - \frac{1}{2} g_{ks}\big(g_{hj}g_{tu} - \frac{1}{2} g_{ht}g_{ju} - \frac{1}{2} g_{hu}g_{jt}\big) \\
&\quad - \frac{1}{2} g_{kt}\big(g_{js}g_{hu} - \frac{1}{2} g_{jh}g_{su} - \frac{1}{2} g_{ju}g_{sh}\big) \\
&\quad - \frac{1}{2} g_{ku}\big(g_{js}g_{ht} - \frac{1}{2} g_{jh}g_{st} - \frac{1}{2} g_{jt}g_{sh}\big)\bigg).
\end{aligned}
\tag{C.4}
$$

Applying contractions reduces the latter

$$
\begin{aligned}
&\lambda \frac{2n-3}{n-2} \bigg(- R_{ij}R^{ij} + 0 - \frac{1}{2} R_{ij}R^{ij} - \frac{1}{2} R_{ih}R^{ih} + 0 - \frac{1}{4} R_{ih}R^{ih} \\
&\quad + 0 + \frac{1}{4} R_{ihkj}R^{kijh} + \frac{1}{4} R_{ijkh}R^{kijh} - \frac{1}{2} R_{ikjh}R^{kijh} + 0 \\
&\quad + \frac{1}{4} R_{ikjh}R^{kijh} - \frac{1}{2} R_{ihjk}R^{kijh} + 0 + \frac{1}{4} R_{ijhk}R^{kijh}\bigg) \\
&= - \lambda \frac{2n-3}{n-2} \bigg(\frac{9}{4} R_{ij}R^{ij} + \frac{1}{4} R^{ijkh}\big(- R_{jhik} - R_{jkih} \\
&\quad + 2R_{jikh} - R_{jihk} + 2R_{jhki} - R_{jkhi}\big)\bigg) \\
&= - \lambda \frac{2n-3}{n-2} \bigg(\frac{9}{4} R_{ij}R^{ij} - \frac{3}{4} R^{ijkh}\big(R_{ikjh} + R_{ijkh}\big)\bigg).
\end{aligned}
\tag{C.5}
$$

Due to the first Bianchi identity

$$
R^{ijkh}\big(R_{ikjh} + R_{ijhk} + R_{ihkj}\big) = 0,
$$

we have

$$
R^{ijkh}R_{ijkh} = R^{ijkh}\big(R_{ikjh} - R_{ihjk}\big) = 2R^{ijkh}R_{ikjh}. \tag{C.6}
$$

This can be used in the last term of equation (C.5). As a result, the fourth term on the right hand side of equation (C.3) can be written as

$$
-\frac{2n-3}{n-2} g_{ir}\beta_{js,kh;tu} R^{kijh} R^{trsu} = -\lambda \frac{2n-3}{n-2} \bigg(\frac{9}{4} R_{ij}R^{ij} - \frac{9}{8} R_{ijkh}R^{ijkh}\bigg).
$$

It can be easily shown that the fourth and the fifth terms on the right hand side of equation (C.3) are equal. Interchanging r and s together with interchanging t and

u leads to the equality due to the symmetry relations of the tensor $\beta_{ij,kh;rs}$ and the Riemann curvature tensor.

Similarly, it is also easy to show that the fourth, the sixth, and the seventh terms on the right hand side of equation (C.3) are all equal. Therefore, we calculate equation (C.3) such that

$$\begin{aligned}
&\big((2n-5)\alpha_{ij,kh;rs,tu} + 2\alpha_{ij,rs;kh,tu}\big)R^{kijh}R^{trsu} \\
&\quad = \lambda\frac{2n-3}{2(n-2)}\left(\frac{9}{2}R^2 - 18R_{ij}R^{ij} + \frac{9}{2}R_{ijkh}R^{ijkh}\right) + \kappa_{ijkhrstu}R^{kijh}R^{trsu} \\
&\quad = \lambda\frac{9(2n-3)}{4(n-2)}\left(R^2 - 4R_{ij}R^{ij} + R_{ijkh}R^{ijkh}\right) + \kappa_{ijkhrstu}R^{kijh}R^{trsu}.
\end{aligned}$$

Using (B.30) and for $n=4$, we can calculate the last term of the latter as

$$\begin{aligned}
&\kappa_{ijkhrstu}R^{kijh}R^{trsu} \\
&\quad = \mu n\sqrt{g}\Big((2n-5)g_{ir}\big(g_{kt}\epsilon_{jshu} + g_{ku}\epsilon_{jsht} + g_{ht}\epsilon_{jsku} + g_{hu}\epsilon_{jskt}\big) \\
&\qquad + (2n-5)g_{is}\big(g_{kt}\epsilon_{jrhu} + g_{ku}\epsilon_{jrht} + g_{ht}\epsilon_{jrku} + g_{hu}\epsilon_{jrkt}\big) \\
&\qquad + (2n-7)g_{it}\big(g_{kr}\epsilon_{juhs} + g_{ks}\epsilon_{juhr} + g_{hr}\epsilon_{juks} + g_{hs}\epsilon_{jukr}\big) \\
&\qquad + (2n-7)g_{iu}\big(g_{kr}\epsilon_{jths} + g_{ks}\epsilon_{jthr} + g_{hr}\epsilon_{jtks} + g_{hs}\epsilon_{jtkr}\big)\Big)R^{kijh}R^{trsu},
\end{aligned}$$

where μ is an arbitrary function of the scalar ϕ. It can be easily shown that every term of the latter is proportional to $(*R^{ij}{}_{kh})R^{kh}{}_{ij}$ and the sum is non-zero. Therefore, we have

$$\kappa_{ijkhrstu}R^{kijh}R^{trsu} = \gamma(*R^{ij}{}_{kh})R^{kh}{}_{ij},$$

where γ is an arbitrary function of the scalar ϕ.

C.3 Calculations of Field Equations

We use equations (2.29) and (2.26) (or (2.32) to obtain the Euler-Lagrange equations. Therefore, in this appendix, we will frequently use the following equations to calculate field equations.

By considering the second-order terms, we can easily calculate

$$\frac{\partial R_{abcd}}{\partial g_{ij,kh}} = \frac{1}{4}\big(D^{ijkh}_{abcd} + D^{ijkh}_{badc} - D^{ijkh}_{abdc} - D^{ijkh}_{bacd}\big),$$

where

$$D^{ijkh}_{abcd} \equiv \frac{1}{2}(\delta^i_a\delta^j_d + \delta^i_d\delta^j_a)(\delta^k_b\delta^h_c + \delta^k_c\delta^h_b).$$

In order to obtain the Euler-Lagrange equations that follow from L_1, we first note the following

$$\frac{\partial R^2}{\partial g_{ij,kh}} = 2Rg^{ac}g^{bd}\frac{\partial R_{abcd}}{\partial g_{ij,kh}},$$

$$\begin{aligned}\frac{\partial(R_{ac}R^{ac})}{\partial g_{ij,kh}} =& g^{bd}g^{rt}g^{sa}g^{uc}\frac{\partial(R_{abcd}R_{rstu})}{\partial g_{ij,kh}}\\ =& g^{bd}R^{ac}\frac{\partial R_{abcd}}{\partial g_{ij,kh}} + g^{rt}R^{su}\frac{\partial R_{rstu}}{\partial g_{ij,kh}}\\ =& 2g^{bd}R^{ac}\frac{\partial R_{abcd}}{\partial g_{ij,kh}},\end{aligned}$$

$$\frac{\partial(R_{abcd}R^{abcd})}{\partial g_{ij,kh}} = 2R^{abcd}\frac{\partial R_{abcd}}{\partial g_{ij,kh}}.$$

Finally, we can calculate the following

$$\begin{aligned}&\frac{\partial}{\partial g_{ij,kh}}\big(R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}\big)\\ &\quad= 2\frac{\partial R_{abcd}}{\partial g_{ij,kh}}\big(g^{ac}g^{bd}R - 4g^{bd}R^{ac} + R^{abcd}\big)\\ &\quad= \frac{1}{4}\big((\delta^i_a\delta^j_d + \delta^i_d\delta^j_a)(\delta^k_b\delta^h_c + \delta^k_c\delta^h_b) + (\delta^i_b\delta^j_c + \delta^i_c\delta^j_b)(\delta^k_a\delta^h_d + \delta^k_d\delta^h_a)\\ &\qquad - (\delta^i_a\delta^j_c + \delta^i_c\delta^j_a)(\delta^k_b\delta^h_d + \delta^k_d\delta^h_b) - (\delta^i_b\delta^j_d + \delta^i_d\delta^j_b)(\delta^k_a\delta^h_c + \delta^k_c\delta^h_a)\big)\\ &\qquad \times \big(g^{ac}g^{bd}R - 4g^{bd}R^{ac} + R^{abcd}\big)\\ &\quad= \frac{1}{4}\bigg(g^{ih}g^{kj}R - 4g^{kj}R^{ih} + R^{ikhj} + g^{ik}g^{hj}R - 4g^{hj}R^{ik} + R^{ihkj}\\ &\qquad + g^{jh}g^{ki}R - 4g^{ki}R^{jh} + R^{jkhi} + g^{jk}g^{hi}R - 4g^{hi}R^{jk} + R^{jhki}\\ &\qquad + g^{kj}g^{ih}R - 4g^{ih}R^{kj} + R^{kijh} + g^{hj}g^{ik}R - 4g^{ik}R^{hj} + R^{hijk}\\ &\qquad + g^{ki}g^{jh}R - 4g^{jh}R^{ki} + R^{kjih} + g^{hi}g^{jk}R - 4g^{jk}R^{hi} + R^{hjik}\\ &\qquad - \big(g^{ij}g^{kh}R - 4g^{kh}R^{ij} + R^{ikjh} + g^{ij}g^{hk}R - 4g^{hk}R^{ij} + R^{ihjk}\\ &\qquad + g^{ji}g^{kh}R - 4g^{kh}R^{ji} + R^{jkih} + g^{ji}g^{hk}R - 4g^{hk}R^{ji} + R^{jhik}\big)\\ &\qquad - \big(g^{kh}g^{ij}R - 4g^{ij}R^{kh} + R^{kihj} + g^{hk}g^{ij}R - 4g^{ij}R^{hk} + R^{hikj}\\ &\qquad + g^{kh}g^{ji}R - 4g^{ji}R^{kh} + R^{kjhi} + g^{hk}g^{ji}R - 4g^{ji}R^{hk} + R^{hjki}\big)\bigg).\end{aligned}$$

After rearranging this, we obtain

$$\frac{\partial}{\partial g_{ij,kh}}\left(R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}\right)$$
$$= g^{ih}g^{kj}R + g^{ik}g^{jh}R - 2g^{ij}g^{kh}R - 2g^{kj}R^{ih} - 2g^{jh}R^{ik} - 2g^{ki}R^{jh}$$
$$- 2g^{hi}R^{jk} + 4g^{kh}R^{ij} + 4g^{ij}R^{kh} + 2R^{ikhj} + 2R^{ihkj}.$$

C.3.1 The Euler-Lagrange Equations of L_1 of Theorem 2.2

If we use the identity $R_{|h} = 2R^{a}{}_{h|a}$, we can calculate the covariant derivative of the equation above as

$$\frac{\partial R_{abcd}}{\partial g_{ij,kh}}\left(2g^{ac}g^{bd}R_{|h} - 8g^{bd}R^{ac}{}_{|h} + 2R^{abcd}{}_{|h}\right)$$
$$= 2g^{kj}R^{ia}{}_{|a} + 2g^{ik}R^{ja}{}_{|a} - 4g^{ij}R^{ka}{}_{|a} - 2g^{kj}R^{ia}{}_{|a} - 2g^{jh}R^{ik}{}_{|h} - 2g^{ik}R^{ja}{}_{|a}$$
$$- 2g^{hi}R^{jk}{}_{|h} + 4g^{kh}R^{ij}{}_{|h} + 4g^{ij}R^{ka}{}_{|a} + 2R^{ikhj}{}_{|h} + 2R^{ihkj}{}_{|h}. \tag{C.7}$$

In light of second Bianchi identity, one can easily find the following identities which helps simplifying (C.7)

$$R^{ikhj}{}_{|h} + R^{k}{}_{h}{}^{hj|i} + R_{h}{}^{ihj}{}_{|k} = 0,$$
$$R^{ikhj}{}_{|h} = R^{kj|i} - R^{ij|k},$$

and

$$R^{ihkj}{}_{|h} + R^{ihj}{}_{h}{}^{|k} + R^{ih}{}_{h}{}^{k|j} = 0,$$
$$R^{ikhj}{}_{|h} = -R^{ij|k} + R^{ik|j}.$$

After substituting these in (C.7), we obtain

$$\frac{\partial R_{abcd}}{\partial g_{ij,kh}}\left(2g^{ac}g^{bd}R_{|h} - 8g^{bd}R^{ac}{}_{|h} + 2R^{abcd}{}_{|h}\right) = 0.$$

Therefore, we have

$$\frac{\partial R_{abcd}}{\partial g_{ij,kh}}\left(2g^{ac}g^{bd}R_{|hk} - 8g^{bd}R^{ac}{}_{|hk} + 2R^{abcd}{}_{|hk}\right) = 0.$$

By inserting the relations above into (2.29), we can calculate the Euler-Lagrange equations derived from L_1 as

$$
\begin{aligned}
E^{ij}(\beta_1 L_1) = & -\beta_1\sqrt{g}\frac{\partial R_{abcd}}{\partial g_{ij,kh}}\big(2g^{ac}g^{bd}R_{|kh} - 8g^{bd}R^{ac}{}_{|kh} + 2R^{abcd}{}_{|kh}\big) \\
& - 2\beta_1'\sqrt{g}\phi_{|k}\frac{\partial R_{abcd}}{\partial g_{ij,kh}}\big(2g^{ac}g^{bd}R_{|h} - 8g^{bd}R^{ac}{}_{|h} + 2R^{abcd}{}_{|h}\big) \\
& - \sqrt{g}(\beta_1''\phi_{|k}\phi_{|h} + \beta_1'\phi_{|kh})\frac{\partial R_{abcd}}{\partial g_{ij,kh}}\big(2g^{ac}g^{bd}R - 8g^{bd}R^{ac} + 2R^{abcd}\big) \\
& - \frac{1}{2}\sqrt{g}\beta_1 g^{ij}\big(R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}\big) \\
& - \frac{2}{3}\sqrt{g}\beta_1\frac{\partial R_{abcd}}{\partial g_{im,kh}}\big(2g^{ac}g^{bd}R - 8g^{bd}R^{ac} + 2R^{abcd}\big)R_h{}^j{}_{km}.
\end{aligned}
$$

After taking the relevant derivatives by using the methods described in the beginning of this appendix, we finally obtain

$$
\begin{aligned}
E^{ij}(\beta_1 L_1) = & 4\sqrt{g}\beta_1'\big(\phi^{|ia}R_a{}^j + \phi^{|ja}R_a{}^i + \frac{1}{2}R(g^{ab}\phi_{|ab}g^{ij} - \phi^{|ij}) \\
& - g^{ij}\phi_{|ab}R^{ab} - g^{ab}\phi_{|ab}R^{ij} - \phi_{|ab}R^{aijb}\big) \\
& + 4\sqrt{g}\beta_1''\big(\phi^{|i}\phi_{|a}R^{aj} + \phi^{|j}\phi_{|a}R^{ai} + \frac{1}{2}R(g^{ij}\phi_{|a}\phi^{|a} - \phi^{|i}\phi^{|j}) \\
& - g^{ij}\phi_{|a}\phi_{|b}R^{ab} - R^{ij}\phi_{|a}\phi^{|a} - \phi_{|a}\phi_{|b}R^{aijb}\big) \\
& - \frac{1}{2}\sqrt{g}\beta_1 g^{ij}(R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}) \\
& - \frac{2}{3}\sqrt{g}\beta_1\big(-3RR^{ij} + 6R^j{}_kR^{ik} + 6R^{kh}R_h{}^j{}_k{}^i - 3R^{ihkm}R^j{}_{hkm}\big).
\end{aligned}
$$

The field equation above is valid for n dimensional spacetime. However, in a four-dimensional spacetime, one can further reduce this equation. Since in our case, we have a four-dimensional spacetime, we are able to make use of the Lanczos identity [25]. The detailed derivation of the Lanczos identity can be found in Appendix C.4. Lanczos has shown that if $n = 4$ then

$$
\frac{1}{4}\delta^e_f\big(R^{abcd}R_{abcd} - 4R^{ab}R_{ab} + R^2\big) = R^{ebcd}R_{fbcd} - 2R_{fb}{}^e{}_dR^{bd} - 2R_{fb}R^{eb} + RR^e{}_f.
$$

Therefore, in a four-dimensional spacetime, we have

$$
\begin{aligned}
E^{ij}(\beta_1 L_1) = & 4\sqrt{g}\beta_1'\big(\phi^{|ia}R_a{}^j + \phi^{|ja}R_a{}^i + \frac{1}{2}R(g^{ab}\phi_{|ab}g^{ij} - \phi^{|ij}) \\
& - g^{ij}\phi_{|ab}R^{ab} - g^{ab}\phi_{|ab}R^{ij} - \phi_{|ab}R^{aijb}\big) \\
& + 4\sqrt{g}\beta_1''\big(\phi^{|i}\phi_{|a}R^{aj} + \phi^{|j}\phi_{|a}R^{ai} + \frac{1}{2}R(g^{ij}\phi_{|a}\phi^{|a} - \phi^{|i}\phi^{|j}) \\
& - g^{ij}\phi_{|a}\phi_{|b}R^{ab} - R^{ij}\phi_{|a}\phi^{|a} - \phi_{|a}\phi_{|b}R^{aijb}\big).
\end{aligned}
$$

By virtue of (2.26), we can easily calculate

$$E(\beta_1 L_1) = -\beta_1' L_1.$$

C.3.2 The Euler-Lagrange Equations of L_2 of Theorem 2.2

Due to (2.29), we calculate the Euler-Lagrange equations derived from L_2 as

$$\begin{aligned} E^{ij}(\beta_2 L_2) = &- \left(\sqrt{g}\beta_2\phi_{|r}\phi_{|s}\big(g^{ar}g^{cs}g^{bd}\frac{\partial R_{abcd}}{\partial g_{ij,kh}} - \frac{1}{2}g^{rs}g^{bd}g^{ac}\frac{\partial R_{abcd}}{\partial g_{ij,kh}}\big)\right)_{|kh} \\ &- \frac{1}{2}\sqrt{g}\beta_2 g^{ij}G^{ab}\phi_{|a}\phi_{|b} + \frac{3}{4}\sqrt{g}\beta_2 G^{ia}\phi_{|a}\phi^{|j} + \frac{1}{4}\sqrt{g}\beta_2 G^{ja}\phi_{|a}\phi^{|i} \\ &- \frac{2}{3}\left(\sqrt{g}\beta_2\phi_{|r}\phi_{|s}\big(g^{ar}g^{cs}g^{bd}\frac{\partial R_{abcd}}{\partial g_{im,kh}} - \frac{1}{2}g^{rs}g^{bd}g^{ac}\frac{\partial R_{abcd}}{\partial g_{im,kh}}\big)\right)R_{h\ km}^{\ j}. \end{aligned}$$

Upon taking the relevant derivatives, we find

$$\begin{aligned} E^{ij}(\beta_2 L_2) = &- \frac{1}{4}\sqrt{g}\beta_2\Bigg(\big(\phi^{|i}\phi^{|h}g^{kj} + \phi^{|i}\phi^{|k}g^{hj} + \phi^{|j}\phi^{|h}g^{ki} + \phi^{|j}\phi^{|k}g^{hi} \\ &- 2\phi^{|i}\phi^{|j}g^{kh} - 2\phi^{|k}\phi^{|h}g^{ij}\big) - \phi_{|a}\phi^{|a}\big(g^{kj}g^{ih} + g^{hj}g^{ik} - 2g^{kh}g^{ij}\big)\Bigg)_{|kh} \\ &- \frac{1}{2}\sqrt{g}\beta_2'\phi_{|k}\Bigg(\big(\phi^{|i}\phi^{|h}g^{kj} + \phi^{|i}\phi^{|k}g^{hj} + \phi^{|j}\phi^{|h}g^{ki} + \phi^{|j}\phi^{|k}g^{hi} \\ &- 2\phi^{|i}\phi^{|j}g^{kh} - 2\phi^{|k}\phi^{|h}g^{ij}\big) - \phi_{|a}\phi^{|a}\big(g^{kj}g^{ih} + g^{hj}g^{ik} - 2g^{kh}g^{ij}\big)\Bigg)_{|h} \\ &- \frac{1}{4}\sqrt{g}(\beta_2''\phi_{|k}\phi_{|h} + \beta_2'\phi_{|kh})\Bigg(\big(\phi^{|i}\phi^{|h}g^{kj} + \phi^{|i}\phi^{|k}g^{hj} + \phi^{|j}\phi^{|h}g^{ki} \\ &+ \phi^{|j}\phi^{|k}g^{hi} - 2\phi^{|i}\phi^{|j}g^{kh} - 2\phi^{|k}\phi^{|h}g^{ij}\big) \\ &- \phi_{|a}\phi^{|a}\big(g^{kj}g^{ih} + g^{hj}g^{ik} - 2g^{kh}g^{ij}\big)\Bigg) \\ &- \frac{1}{2}\sqrt{g}\beta_2\big(g^{ij}R^{ab}\phi_{|a}\phi_{|b} - \frac{1}{2}g^{ij}R\phi_{|a}\phi^{|a}\big) \\ &- \frac{1}{6}\sqrt{g}\beta_2\Bigg(\big(\phi^{|i}\phi^{|h}g^{km} + \phi^{|i}\phi^{|k}g^{hm} + \phi^{|m}\phi^{|h}g^{ki} + \phi^{|m}\phi^{|k}g^{hi} \\ &- 2\phi^{|i}\phi^{|m}g^{kh} - 2\phi^{|k}\phi^{|h}g^{im}\big) \\ &- \phi_{|a}\phi^{|a}\big(g^{km}g^{ih} + g^{hm}g^{ik} - 2g^{kh}g^{im}\big)\Bigg)R_{h\ km}^{\ j} \\ &+ \sqrt{g}\beta_2\big(\frac{3}{4}R^{ia}\phi_{|a}\phi^{|j} + \frac{1}{4}R^{ja}\phi_{|a}\phi^{|i} - \frac{1}{2}R\phi^{|i}\phi^{|j}\big). \end{aligned}$$

After rearranging this, we obtain

$$
\begin{aligned}
E^{ij}(\beta_2 L_2) =& \sqrt{g}\beta_2\bigg(\frac{1}{2}g^{ij}\big((g^{ab}\phi_{|ab})^2 - \phi^{|ab}\phi_{|ab} - 2\phi_{|a}\phi_{|b}R^{ab} + \frac{1}{2}\phi_{|a}\phi^{|a}R\big) \\
& + \phi^{|ia}\phi_{|a}{}^{j} - g^{ab}\phi_{|ab}\phi^{|ij} + \phi_{|a}\big(\phi^{|i}R^{aj} + \phi^{|j}R^{ai}\big) \\
& - \frac{1}{2}\phi^{|i}\phi^{|j}R - \frac{1}{2}\phi_{|a}\phi^{|a}R^{ij} - \phi_{|a}\phi_{|b}R^{aijb}\bigg) \\
& + \frac{1}{2}\sqrt{g}\beta_2'\bigg(g^{ij}\big(\phi^{|a}\phi_{|a}g^{kh}\phi_{|kh} - \phi^{|a}\phi^{|b}\phi_{|ab}\big) \\
& - \phi^{|i}\phi^{|j}g^{|ab}\phi_{|ab} + \phi^{|a}\big(\phi^{|i}\phi_{|a}{}^{j} + \phi^{|j}\phi_{|a}{}^{i}\big) - \phi^{|ij}\phi_{|a}\phi^{|a}\bigg).
\end{aligned}
$$

Using equation (2.26), we can easily calculate

$$
\begin{aligned}
E(\beta_2 L_2) &= 2\sqrt{g}\big(\beta_2 G^{ai}\phi_{|a}\big)_{|i} - \sqrt{g}\beta_2' G^{ab}\phi_{|a}\phi_{|b} \\
&= \sqrt{g}G^{ab}\big(\beta_2\phi_{|ab} + \beta_2'\phi_{|a}\phi_{|b}\big).
\end{aligned}
$$

C.3.3 The Euler-Lagrange Equations of L_3 of Theorem 2.2

From (2.29), we can calculate the Euler-Lagrange equations obtained from L_3 as

$$
\begin{aligned}
E^{ij}(\beta_3 L_3) =& \sqrt{g}\bigg(- (\beta_3''\phi_{|k}\phi_{|h} + \beta_3'\phi_{|kh})\big(\frac{1}{2}g^{ih}g^{kj} + \frac{1}{2}g^{ik}g^{hj} - g^{ij}g^{kh}\big) \\
& - \frac{1}{2}\beta_3 g^{ij}R - \frac{2}{3}\beta_3\big(\frac{1}{2}g^{ik}g^{hm} - g^{im}g^{kh}\big)R_h{}^{j}{}_{km}\bigg).
\end{aligned}
$$

After rearranging this, we obtain

$$
E^{ij}(\beta_3 L_3) = \sqrt{g}\bigg(- \beta_3''\phi^{|i}\phi^{|j} + \beta_3'' g^{ij}\phi_{|a}\phi^{|a} - \beta_3'\phi^{|ij} + \beta_3' g^{ij}g^{ab}\phi_{|ab} + \beta_3 G^{ij}\bigg).
$$

By virtue of (2.26), we can easily calculate

$$
E(\beta_3 L_3) = -\sqrt{g}\beta_3' R.
$$

C.3.4 The Euler-Lagrange Equations of L_4 of Theorem 2.2

Due to (2.29), we calculate the Euler-Lagrange equations derived from L_4 as

$$
\begin{aligned}
E^{ij}(\eta L_4) =& -\frac{1}{2}g^{ij}\eta\sqrt{g} + \frac{3}{8}g^{jh}\phi_{,h}\frac{\partial\rho}{\partial\phi_{,i}}\sqrt{g}\frac{\partial\eta}{\partial\rho} + \frac{1}{8}g^{ih}\phi_{,h}\frac{\partial\rho}{\partial\phi_{,j}}\sqrt{g}\frac{\partial\eta}{\partial\rho} \\
=& -\frac{1}{2}g^{ij}\eta\sqrt{g} + \frac{3}{4}g^{jh}\phi_{,h}g^{ib}\phi_{,b}\sqrt{g}\frac{\partial\eta}{\partial\rho} + \frac{1}{4}g^{ih}\phi_{,h}g^{jb}\phi_{,b}\sqrt{g}\frac{\partial\eta}{\partial\rho}.
\end{aligned}
$$

After rearranging this, we obtain

$$E^{ij}(\eta L_4) = \sqrt{g}\bigg(\frac{\partial \eta}{\partial \rho}\phi^{|i}\phi^{|j} - \frac{1}{2}g^{ij}\eta\bigg).$$

Making use of (2.32), we can calculate $E(L)$ from $E^{ij}{}_{|j}(L)$. After taking the covariant derivative, we find

$$\begin{aligned}
E^{ij}{}_{|j}(\eta L_4) =& \sqrt{g}\bigg(\frac{\partial^2 \eta}{\partial\phi\partial\rho}\phi_{|j}\phi^{|i}\phi^{|j} + 2\frac{\partial^2\eta}{\partial^2\rho}g^{ab}\phi_{|aj}\phi_{|b}\phi^{|i}\phi^{|j} + \frac{\partial\eta}{\partial\rho}\phi^{|ij}\phi_{|j} \\
&+ \frac{\partial\eta}{\partial\rho}\phi^{|i}g^{ab}\phi_{|ab} - \frac{1}{2}\frac{\partial\eta}{\partial\phi}g^{ij}\phi_{|j} - \frac{\partial\eta}{\partial\rho}\phi^{|ij}\phi_{|j}\bigg) \\
=& \sqrt{g}\bigg(\frac{\partial^2 \eta}{\partial\phi\partial\rho}\phi_{|j}\phi^{|i}\phi^{|j} + 2\frac{\partial^2\eta}{\partial^2\rho}g^{ab}\phi_{|aj}\phi_{|b}\phi^{|i}\phi^{|j} \\
&+ \frac{\partial\eta}{\partial\rho}\phi^{|i}g^{ab}\phi_{|ab} - \frac{1}{2}\frac{\partial\eta}{\partial\phi}g^{ij}\phi_{|j}\bigg).
\end{aligned}$$

As a result, we have

$$E(\eta L_4) = 2\sqrt{g}\bigg(\frac{\partial^2\eta}{\partial\phi\partial\rho}\phi_{|a}\phi^{|a} + 2\frac{\partial^2\eta}{\partial^2\rho}\phi_{|ab}\phi^{|a}\phi^{|b} + \frac{\partial\eta}{\partial\rho}g^{ab}\phi_{|ab} - \frac{1}{2}\frac{\partial\eta}{\partial\phi}\bigg).$$

C.3.5 The Euler-Lagrange Equations of L_5 of Theorem 2.2

From (2.29), we calculate the Euler-Lagrange equations derived from L_5 as

$$\begin{aligned}
E^{ij}(cL_5) =& -2c\epsilon^{abrs}g^{tc}g^{ud}\big(R_{rstu}\frac{\partial R_{abcd}}{\partial g_{ij,kh}}\big)_{|kh} - \frac{1}{2}cg^{ij}\epsilon^{abrs}g^{tc}g^{ud}R_{rstu}R_{abcd} \\
&- \frac{4}{3}c\epsilon^{abrs}g^{tc}g^{ud}\big(R_{rstu}\frac{\partial R_{abcd}}{\partial g_{im,kh}}\big)R_{h\ \,km}^{\ \,j}.
\end{aligned}$$

To assist in our calculation of $E^{ij}(cL_5)$, we calculate

$$\begin{aligned}
\epsilon^{abrs}g^{tc}g^{ud}\big(R_{rstu}\frac{\partial R_{abcd}}{\partial g_{ij,kh}}\big) =& \frac{1}{8}\big(\epsilon^{ikrs}g^{th}g^{uj}R_{rstu} + \epsilon^{ihrs}g^{tk}g^{uj}R_{rstu} \\
&+ \epsilon^{jkrs}g^{th}g^{ui}R_{rstu} + \epsilon^{jhrs}g^{tk}g^{ui}R_{rstu} \\
&+ \epsilon^{kirs}g^{tj}g^{uh}R_{rstu} + \epsilon^{hirs}g^{tj}g^{uk}R_{rstu} \\
&+ \epsilon^{kjrs}g^{ti}g^{uh}R_{rstu} + \epsilon^{hjrs}g^{ti}g^{uk}R_{rstu} \\
&- \epsilon^{ikrs}g^{tj}g^{uh}R_{rstu} - \epsilon^{ihrs}g^{tj}g^{uk}R_{rstu} \\
&- \epsilon^{jkrs}g^{ti}g^{uh}R_{rstu} - \epsilon^{jhrs}g^{ti}g^{uk}R_{rstu} \\
&- \epsilon^{kirs}g^{th}g^{uj}R_{rstu} - \epsilon^{hirs}g^{tk}g^{uj}R_{rstu} \\
&- \epsilon^{kjrs}g^{th}g^{ui}R_{rstu} - \epsilon^{hjrs}g^{tk}g^{ui}R_{rstu}\big),
\end{aligned}$$

which reduces to

$$\begin{aligned}
&\epsilon^{abrs} g^{tc} g^{ud} \big(R_{rstu} \frac{\partial R_{abcd}}{\partial g_{ij,kh}} \big) \\
&= \frac{1}{4} R_{rstu} \big(\epsilon^{ikrs} (g^{th} g^{uj} - g^{tj} g^{uh}) + \epsilon^{ihrs} (g^{tk} g^{uj} - g^{tj} g^{uk}) \\
&\quad + \epsilon^{jkrs} (g^{th} g^{ui} - g^{ti} g^{uh}) + \epsilon^{jhrs} (g^{tk} g^{ui} - g^{ti} g^{uk}) \big).
\end{aligned}$$

After long calculations where one uses the first and the second Bianchi, and the Ricci identities, we see that all the terms vanish. Consequently, we obtain

$$E^{ij}(cL_5) = 0.$$

Therefore, from equation (2.32), we immediately have

$$E(cL_5) = 0.$$

C.4 Derivation of the Lanczos Identity

In a n dimensional spacetime, the Weyl tensor is given by

$$\begin{aligned}
C_{abcd} =& R_{abcd} - \frac{1}{n-2} (g_{ac} R_{bd} - g_{ad} R_{cb} - g_{bc} R_{ad} + g_{bd} R_{ac}) \\
&+ \frac{1}{(n-1)(n-2)} R (g_{ac} g_{db} - g_{ad} g_{cb}),
\end{aligned} \tag{C.8}$$

together with obvious symmetries

$$C_{abcd} = -C_{bacd}, \qquad C_{abcd} = C_{cdab}.$$

The trace of the Weyl tensor is zero, $C^{a}{}_{bad} = 0$. By considering this and the symmetry relations, we write

$$\delta^{\alpha\beta ab}_{\mu\nu cd} C^{cd}{}_{ab} = 4 C^{\alpha\beta}{}_{\mu\nu}. \tag{C.9}$$

Since $\delta^{\alpha\beta abe}_{\mu\nu cdf} = 0$ when $n = 4$, we have $\delta^{\alpha\beta abe}_{\mu\nu cdf} C^{cd}{}_{ab} = 0$. By virtue of this and (C.9), we conclude

$$C^{[cd}{}_{[ab} \delta^{e]}_{f]} = 0.$$

Consequently, we have

$$\begin{aligned}
&\frac{1}{9} \big(C^{cd}{}_{ab} \delta^{e}_{f} + C^{de}{}_{ab} \delta^{c}_{f} + C^{ec}{}_{ab} \delta^{d}_{f} + C^{cd}{}_{bf} \delta^{e}_{a} + C^{de}{}_{bf} \delta^{c}_{a} + \\
&\qquad C^{ec}{}_{ba} \delta^{d}_{a} + C^{cd}{}_{fa} \delta^{e}_{b} + C^{de}{}_{fa} \delta^{c}_{b} + C^{ec}{}_{fa} \delta^{d}_{b} \big) = 0.
\end{aligned}$$

Upon multiplying this with $C^{ab}{}_{cd}$, we obtain

$$C^{abcd}C_{abcd}\delta^e_f = 4C^{ebcd}C_{fbcd}.$$

Calculating both sides by using (C.8) when $n = 4$ yields

$$\begin{aligned}
&R^{abcd}R_{abcd}\delta^e_f - 2R^{ab}R_{ab}\delta^e_f + \frac{1}{3}R^2\delta^e_f \\
&\qquad = 4\big(R^{ebcd}R_{fbcd} - 2R_{fb}{}^e{}_dR^{bd} - 2R_{fb}R^{eb} + RR^e{}_f + \frac{1}{2}R^{ab}R_{ab}\delta^e_f - \frac{1}{6}R^2\delta^e_f\big).
\end{aligned}$$

After rearranging the terms, we find

$$\frac{1}{4}\delta^e_f\big(R^{abcd}R_{abcd} - 4R^{ab}R_{ab} + R^2\big) = R^{ebcd}R_{fbcd} - 2R_{fb}{}^e{}_dR^{bd} - 2R_{fb}R^{eb} + RR^e{}_f.$$

This is known as the Lanczos identity.

C.5 Symmetries of $\Lambda^{a;b;ij,kh}$

By the compatibility of the partial derivatives, it is obvious that

$$\Lambda^{a;b;ij,kh} = \Lambda^{b;a;ij,kh}.$$

If we use this together with (2.35), then we have

$$\Lambda^{a;b;ij,kh} + \Lambda^{a;i;bj,kh} + \Lambda^{a;j;ib,kh} = 0. \tag{C.10}$$

As a result of (C.10), the second and the third terms can be calculated as

$$\begin{aligned}
\Lambda^{a;i;bj,kh} &= -\Lambda^{j;i;ba,kh} - \Lambda^{b;i;aj,kh}, \\
\Lambda^{a;j;ib,kh} &= -\Lambda^{i;j;ab,kh} - \Lambda^{b;j;ia,kh}.
\end{aligned}$$

By inserting these into equation (C.10), we obtain

$$\Lambda^{a;b;ij,kh} - \Lambda^{j;i;ba,kh} - \Lambda^{b;i;aj,kh} - \Lambda^{i;j;ab,kh} - \Lambda^{b;j;ia,kh} = 0. \tag{C.11}$$

Again, by using (C.10), the sum of the third and the fifth terms on the left hand side are

$$-\Lambda^{b;i;aj,kh} - \Lambda^{b;j;ia,kh} = \Lambda^{b;a;ij,kh}.$$

Therefore, (C.11) can be written as

$$\begin{aligned}
\Lambda^{a;b;ij,kh} - \Lambda^{j;i;ba,kh} - \Lambda^{i;j;ab,kh} + \Lambda^{b;a;ij,kh} &= 0 \\
2\Lambda^{a;b;ij,kh} - 2\Lambda^{i;j;ab,kh} &= 0.
\end{aligned}$$

Consequently, we have the following symmetry

$$\Lambda^{a;b;ij,kh} = \Lambda^{i;j;ab,kh}. \tag{C.12}$$