COMPACT-LIKE OPERATORS IN LATTICE-NORMED SPACES

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

ABDULLAH AYDIN

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY
IN
MATHEMATICS

MAY 2017

Approval of the thesis:

COMPACT-LIKE OPERATORS IN LATTICE-NORMED SPACES

submitted by **ABDULLAH AYDIN** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy in Mathematics Department, Middle East Technical University** by,

Prof. Dr. Gülbin Dural Ünver
Dean, Graduate School of **Natural and Applied Sciences** ________________

Prof. Dr. Mustafa Korkmaz
Head of Department, **Mathematics** ________________

Prof. Dr. Eduard Emel'yanov
Supervisor, **Department of Mathematics, METU** ________________

Examining Committee Members:

Prof. Dr. Süleyman Önal
Department of Mathematics, METU ________________

Prof. Dr. Eduard Emel'yanov
Department of Mathematics, METU ________________

Prof. Dr. Bahri Turan
Department of Mathematics, Gazi University ________________

Prof. Dr. Birol Altın
Department of Mathematics, Gazi University ________________

Assist. Prof. Dr. Kostyantyn Zhelturkhin
Department of Mathematics, METU ________________

Date: ________________

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last Name: ABDULLAH AYDIN

Signature :

ABSTRACT

COMPACT-LIKE OPERATORS IN LATTICE-NORMED SPACES

Aydın, Abdullah
Ph.D., Department of Mathematics
Supervisor : Prof. Dr. Eduard Emel'yanov

May 2017, 64 pages

Let (X,p,E) and (Y,m,F) be two lattice-normed spaces. A linear operator $T : X \to Y$ is said to be p-compact if, for any p-bounded net x_α in X, the net Tx_α has a p-convergent subnet in Y. That is, if x_α is a net in X such that there is a $e \in E_+$ satisfying $p(x_\alpha) \leq e$ for all α, then there exists a subnet x_{α_β} and $y \in Y$ such that $m(Tx_{\alpha_\beta} - y) \xrightarrow{o} 0$ in F. A linear operator $T : X \to Y$ is called p-continuous if $p(x_\alpha) \xrightarrow{o} 0$ in E implies $m(Tx_\alpha) \xrightarrow{o} 0$ in F, where x_α is a net in X. p-compact operators generalize several known classes of operators such as compact, weakly compact, order weakly compact, AM-compact operators, etc. Also, p-continuous operators generalize many classes of operators such as order continuous, norm continuous, Dunford-Pettis, etc. Similar to M-weakly and L-weakly compact operators, we define p-M-weakly and p-L-weakly compact operators and study some of their properties. We also study up-continuous and up-compact operators between lattice-normed vector lattices. We give some results about acting mixed-normed spaces on lattice normed spaces.

Keywords: Compact Operator, Vector Lattice, Lattice-Normed Space, Lattice-Normed Vector Lattice, up-Convergence, Mixed-Normed Space

ÖZ

KAFES(LATTİCE)-NORMLU UZAYLARDA KOMPACT-GİBİ OPERATÖRLER

Aydın, Abdullah
Doktora, Matematik Bölümü
Tez Yöneticisi : Prof. Dr. Eduard Emel'yanov

Mayıs 2017 , 64 sayfa

(X, p, E) ve (Y, m, F) iki kafes-normlu uzay olsun. X uzayında herhangi bir p-sınırlı x_α neti için eğer Y'deki Tx_α netinin p-yakınsak bir alt neti varsa, $T : X \to Y$ doğrusal operatörüne p-kompackt operatör denir. Yani, eğer x_α bir $e \in E_+$ elemanı için $p(x_\alpha) \leq e$ şartını şağlayan bir net ise; x_α'nın öyle bir x_{α_β} alt neti vardır ve $m(Tx_{\alpha_\beta} - y) \xrightarrow{o} 0$ şartını sağlar. Eğer $T : X \to Y$ doğrusal operatörü $p(x_\alpha) \xrightarrow{o} 0$ şartını sağlayan her x_α neti için $m(Tx_\alpha) \xrightarrow{o} 0$ şartını sağlarsa, T operatörüne p-sürekli denir. p-kompakt operatörler iyi bilenen zayıf kompakt, sıralı zayıf kompakt, AM-kompakt operatörler v.b, gibi kompakt operatörleri geneller. Aynı zamanda, p-sürekli operatörler sınırlı sürekli, norm sürekli, Dunford-Pettis v.b. bir çok operatörü genellediğini gösterdik. M-zayıf ve L-zayıf kompakt operatörlere benzer olarak p-M-zayıf ve p-L-zayıf kompakt operatörleri tanımladık ve onların bazı özellikleri uzerinde çalışmalar yapıtık. Bunlarla birlikte, kafes-normlu vektör kafesler üzerinde up-sürekli ve up-kompakt çalışmalar yaptık.

Anahtar Kelimeler: Kompakt Operatör, Vektör Kafes(Lattice), Kafes-Normlu Uzaylar, Kafes-Normlu Vektör kafes, Karışık-Normlu Uzaylar

To my beloved family

ACKNOWLEDGMENTS

First of all, I would like to record my deepest gratitude to my supervisor Prof. Dr. Eduard Emelyanov, not only for accepting me as his student but also for his academic support. I also would like to thank to him for his valuable guidance and constant support throughout the preparation of this thesis, and also for everything that he taught me. Also I would like to record my deepest gratitude to Prof. Dr. Şafak Alpay who accepted me as a doctoral student and helped me until he retired.

I offer my sincere thanks to the members of the examining committee for their valuable comments and suggestions. I extend my gratitude to all members of the Mathematics Department of Middle East Technical University for their continuous help during this long process. Many thanks go to all my colleagues and office mates at METU.

My greatest appreciation and respect to my mother Fatma Aydın and my kids Erva and Muhammed Tevfik Aydın for their unconditional love, generous support, trust and everything they have done for me throughout my whole life. I would also like to give my deepest thanks to my friend Mohammad A. A. Marabeh and Yrd. Doç. Dr Nazife Özcan Erkurşun.

Last, but not least, words fail me to express my appreciation to my wonderful and beautiful wife Gülsüm whose dedication, love and persistent confidence in me, has taken the load off my shoulder.

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

Unbounded order convergence (or uo-convergence, for short) was first defined for sequences on σ-Dedekind complete Riesz spaces by H. Nakano (1948) in [27], under the name "individual convergence. Nakano extended the individual ergodic theorem (known as Birkhoff's Ergodic Theorem) to particular Banach lattices "KB spaces. The name "unbounded order convergence" was first proposed by R. De Marr (1964) in [11]. The relation between weak and uo-convergence in Banach lattices were studied by A. W. Wickstead (1977) in [31]. The unbounded norm convergence was introduced by V. G. Troitsky (2004) in [29], under the name *d-convergence*. The name "unbounded norm convergence" was first proposed in [12] (2016). The unbounded p-convergence (or up-convergence, for short) was introduced in [7] (2016). We refer the reader for an exposition on uo-convergence to [16, 17], on un-convergence to [12] (see also recent paper [18]) and on up-convergence to [7]. For applications of uo-convergence, we refer to [14, 16, 17].

It is known that order convergence in vector lattices is not topological in general. Nevertheless, via order convergence, continuous-like operators (namely, order continuous operators) can be defined in vector lattices without using any topological structure. On the other hand, compact operators play an important role in functional analysis. Our aim in this thesis is to introduce and study compact-like operators in lattice-normed spaces and in lattice-normed vector lattices by developing topology-free techniques.

Let X be a vector space, E be a vector lattice, and $p : X \to E_+$ be a vector norm, then the triple (X, p, E) is called a *lattice-normed space*, abbreviated as LNS. If X is a vector lattice, and the vector norm p is monotone (i.e. $|x| \leq |y| \Rightarrow p(x) \leq p(y)$) then the triple (X, p, E) is called a *lattice-normed vector lattice*, abbreviated as LNVL. The lattice norm p in an LNS (X, p, E) is said to be *decomposable* if for all $x \in X$ and $e_1, e_2 \in E_+$, it follows from $p(x) = e_1 + e_2$, that there exist $x_1, x_2 \in X$ such that $x = x_1 + x_2$ and $p(x_k) = e_k$ for $k = 1, 2$. A net x_α in a lattice-normed vector lattice (X, p, E) is unbounded p-convergent to $x \in X$ if $p(|x_\alpha - x| \wedge u) \xrightarrow{o} 0$ for every $u \in X_+$.

It should be noticed that the theory of lattice-normed spaces is well- developed in the case of decomposable lattice norms (cf. [21, 20]). In [10] and [28] the authors studied some classes of operators in LNSs under the assumption that the lattice norms are decomposable. In this thesis, we usually do not assume lattice norms to be decomposable.

The structure of this thesis is as follows. In chapter 2, we give basic properties of vector lattices, normed vector lattices, lattice-normed spaces, and lattice-normed vector lattices.

In section 3.1, we recall definitions of p-continuous and p-bounded operators between lattice-normed spaces. We study the relation between p-continuous operators and norm continuous operators acting in mixed-normed spaces; see Proposition 3 and Theorem 5. We show that every p-continuous operator is p-bounded. We give a generalization of the fact that *any positive operator from a Banach lattice into a normed lattice is norm bounded*, in Theorem 6. Also, we show that p-continuity of adjoint of a p-bounded, sequentially p-continuous and positive operator is p-continuous in Theorem 7 and also under the some conditions we end this section by giving sequentially p-continuous and p-boundedness of adjoint of a positive operator in Theorem 8.

In section 3.2, we introduce the notion of p-compact and sequentially p-compact operators between lattice-normed spaces. These operators generalize several known classes of operators such as compact, weakly compact, order weakly compact, and AM-compact operators; see Example 4. A sequence T_n of dominated and sequentially p-compact operators is p-convergent to a dominated operator T, then it is also sequentially p-compact; see Theorem 9. Also the relations between sequentially p-compact operators and compact operators acting in mixed-normed spaces are investigated; see Propositions 7 and 8. Under some conditions, an order bounded operator is p-compact; see Proposition 9. We show that p-bounded finite rank operator is p-compact; see Proposition 11. Finally, We introduce the notion of a p-semicompact operator and study some of its properties.

In section 3.3, we define p-M-weakly and p-L-weakly compact operators which correspond respectively to M-weakly and L-weakly compact operators. We show that a p-bounded and sequentially p-continuous operator on some spacial space is p-M-weakly compact; see Proposition 16. Similar result obtained in the case p-L-weakly compact operators; see Proposition 17. Also we show that order bounded σ-order continuous operators is both p-M-weakly and p-L-weakly compact; see Proposition 18. In Theorem 10, Proposition 20 and 21 we obtain approximation results related to p-M-weakly and p-L-weakly compact operators. In Theorem 11, Under some conditions, p-M-weakly compact implies p-L-weakly compact. We give the relation of p-M-weakly compact with M-weakly compact and p-L-weakly compact with L-weakly compact, respectively see Proposition 22 and proposition 24. Also, several properties of these operators are investigated.

In section 3.4, the notion of (sequentially) up-continuous and (sequentially) up-compact operators are introduced. We show that an up-compact and p-semicompact operator is p-compact; see Proposition 27. Composition of a sequentially up-compact operator with a dominated lattice homomorphism has some results; see Theorem 13, Corollary 4, and Corollary 5. Finally, we give a relation between sequentially up-compact operators and GAM-compact operators; see Proposition 30.

CHAPTER 2

PRELIMINARIES

For the convenience of the reader, we present in this chapter the general background needed in the thesis, and we give some basic structural properties. We give basic definitions and properties of vector lattices. We refer the reader for more information about vector lattice to [2, 4, 5, 9, 22, 23, 32].

We give basic definitions and properties of functional analysis, normed vector lattices and Banach lattice. Moreover, we give definition and properties "atom" in vector lattice. More information can be found in [2, 3, 5, 26, 25]

Lastly, we give the definition of lattice-normed spaces (LNS), and lattice-normed vector lattices(LNVL). We refer the reader for more information in [7, 9, 20, 21].

2.1 Vector Lattice

Definition 1. *Let "$\leq$" be an order relation on a real vector space E. Then E is called an* ordered vector space *if, for any $x, y, z \in E$, the following conditions hold:*

(i) $x \leq y$ *implies* $x + z \leq y + z$.

(ii) $x \leq y$ *implies* $\alpha x \leq \alpha y$ *for all* $\alpha > 0$.

A vector x in an ordered vector space E is called *positive* whenever $x \geq 0$ holds. The set of all positive vectors of E will be denoted by $E_+ = \{x \in E : x \geq 0\}$. If $x \in E_+$ and $x \neq 0$, then we write $x > 0$. An ordered vector space E is called a *vector lattice (or Riesz space)* if, for each pair of vectors $x, y \in E$, the supremum and the infimum of the set $\{x, y\}$ both exist in E. The following classical notation will be used: $x \vee y = \sup\{x, y\}$ and $x \wedge y = \inf\{x, y\}$, and they are called *lattice operations*.

A subset A of E has a supremum $z \in E$, whenever $a \leq z$ for all $a \in A$ and, if there exists another $w \in E$ such that $a \leq w$, then $z \leq w$. Let $x \in E$, then $x^+ = x \vee 0$, $x^- = (-x) \vee 0$, and $|x| = (-x) \vee x$ are *the positive part, the negative part*, and *the modulus* of x, respectively.

In a vector lattice E, two elements x and y are said to be *disjoint* whenever $|x| \wedge |y| = 0$ holds, denoted by $x \perp y$. For a nonempty set A of E, the set $A^d := \{x \in E : x \perp a \text{ for all } a \in A\}$ is called the *disjoint complement* of A.

Some more useful lattice identities are in the next theorem; see details in [5].

Theorem 1. *Let E be a vector lattice. If $x, y, z \in E$, then we have:*

(i) $x + y = x \vee y + x \wedge y$.

(ii) $x = x^+ - x^-$, $|x| = x^+ + x^-$, *and* $x^+ \vee x^- = 0$.

(iii) $\big||x| - |y|\big| \leq |x - y| \leq |x| + |y|$.

(iv) $|x \vee z - z \vee y| \leq |x - y|$ *and* $|x \wedge z - z \wedge y| \leq |x - y|$. *Thus, it can be seen that* $|x^+ - y^+| \leq |x - y|$ *and* $|x^- - y^-| \leq |x - y|$.

(v) If $x, y, z \in E_+$ *then* $x \wedge (y + z) \leq x \wedge y + x \wedge z$.

Definition 2. *If x and y are two vectors in a vector lattice E with $x \leq y$, then the subset $[x, y]$ is called an* order interval *in E, and it is defined by $[x, y] := \{z \in E : x \leq z \leq y\}$.*

A subset A of a vector lattice E is *bounded from above* (respectively, *bounded from below*) if there is $x \in E$ with $a \leq x$ (respectively, $x \leq a$) for all $a \in A$. A subset A of E is called *order bounded* if it is bounded above and below (or, equivalently, if it is included in an order interval) or, equivalently, if there is $u \in E_+$ such that $A \subseteq [-u, u]$.

A vector subspace F of a vector lattice E is said to be a $sublattice$ of E if, for each $x_1, x_2 \in F$, we have $x_1 \vee x_2 \in F$.

Definition 3. *A subset A of a vector lattice is called* solid *whenever $|x| \leq |y|$ for any $y \in A$ and $x \in E$ implies $x \in A$. A solid vector subspace of a vector lattice is referred to as an* ideal. *Thus it can be seen, because of $x_1 \leq y$ and $x_2 \leq y$, $x_1 \vee x_2 \leq y$ for any $x_1, x_2, y \in E$. Thus, every ideal is a vector sublattice.*

Let A be a nonempty subset of E, then the *ideal generated by A* is the smallest ideal in E that contains A. This ideal is given by the formula, see [5, p.33]:

$$I_A := \{x \in E : \exists a_1, \ldots, a_n \in A \text{ and } \lambda \in \mathbb{R}_+ \text{with } |x| \leq \lambda \sum_{j=1}^{n} |a_j|\}.$$

For $x_0 \in E$, I_{x_0} generated by x_0 is referred as a *principal ideal*. This ideal has the form

$$I_{x_0} := \{x \in E : \exists \lambda \in \mathbb{R}_+ \text{with } |x| \leq \lambda |x_0|\}.$$

Also, the *solid hull of A* is the smallest solid set including A; that is $sol(A) = \{x \in E : \exists y \in A \text{ with } |x| \leq |y|\}$.

A sublattice F of E is called *order dense* in E if, for each $x > 0$, there is $0 < y \in F$ with $0 < y \leq x$; and F is said to be *majorizing* in E if, for each $x \in E_+$, there exists $y \in F$ such that $x \leq y$.

Definition 4. *A net x_α in a vector lattice is said to be* increasing *(in symbols $x_\alpha \uparrow$) whenever $\alpha \leq \beta$ implies $x_\alpha \leq x_\beta$. The notation $x_\alpha \uparrow x$ means that $x_\alpha \uparrow$ and* $\sup x_\alpha = x$ *both hold. The meaning of $x_\alpha \downarrow$ and $x_\alpha \downarrow x$ are analogous.*

A vector lattice E is said to be *Archimedean* if $\frac{1}{n}x \downarrow 0$ holds for each $x \in E_+$ and $n \in \mathbb{N}$. Throughout this thesis, all vector lattices are assumed to be Archimedean.

A vector lattice is called *order complete* or *Dedekind complete* (respectively, *σ-order complete* or *Dedekind σ-complete*) if every order bounded above subset (respectively, countable subset) has the supremum or, equivalently, whenever every nonempty bounded below subset (respectively, countable subset) has the infimum or, equivalently, if $0 \leq x_\alpha \uparrow \leq u$ then there is $x \in E$ such that $x_\alpha \uparrow x$.

Let E be a vector lattice. A subset D of E is called *directed upward* (respectively, *directed downward*), in symbols $D \uparrow$ (respectively, $D \downarrow$) whenever, for each pair

$x, y \in D$, there exists some $z \in D$ with $x \leq z$ and $y \leq z$ (respectively, $z \leq x$ and $z \leq y$).

Definition 5. *Let $(x_\alpha)_{\alpha \in A}$ be a net in a vector lattice E.*

(1) x_α *is* order convergent *(or o-convergent, for short) to $x \in E$, if there exists another net $(y_\beta)_{\beta \in B}$ satisfying $y_\beta \downarrow 0$ and, for any $\beta \in B$, there exists $\alpha_\beta \in A$ such that $|x_\alpha - x| \leq y_\beta$ for all $\alpha \geq \alpha_\beta$. In this case, we write $x_\alpha \xrightarrow{o} x$. It follows that an order convergent net has an order bounded tail, whereas an order convergent sequence is order bounded. For a net (x_α) in a vector lattice E and $x \in E$, we have $|x_\alpha - x| \xrightarrow{o} 0$ iff $x_\alpha \xrightarrow{o} x$ iff $|x_\alpha| \xrightarrow{o} |x|$. Thus, without loss of generality, we can only deal with order null nets in E_+. For an order bounded net (x_α) in an order complete vector lattice, we have $x_\alpha \xrightarrow{o} x$ iff $\inf_\alpha \sup_{\beta \geq \alpha} |x_\beta - x| = 0$; cf. [19].*

(2) $(x_\alpha)_{\alpha \in A}$ is said to be order Cauchy (or o-Cauchy) if the double net $(x_\alpha - x_{\alpha'})_{(\alpha, \alpha') \in A \times A}$ is order convergent to 0.

(3) x_α *is* unbounded order convergent *(or uo-convergent, for short) to $x \in E$ if $|x_\alpha - x| \wedge u \xrightarrow{o} 0$ for every $u \in E_+$; cf. [17]. In this case, we write $x_\alpha \xrightarrow{uo} x$. For more information on unbounded order convergence we refer to [7, 11, 12, 15, 16, 17, 18, 19]*

(4) A sequence x_n in E is said to be relatively uniformly convergent *to $x \in E$, whenever there exist some $u > 0$ and a sequence $\varepsilon_n \downarrow 0$ of real numbers such that $|x_n - x| \leq \varepsilon_n u$ holds for all n; cf. [5, p.100] . It can be seen that relatively uniformly convergence implies order convergence.*

Definition 6. *Let $T : E \to F$ be a mapping between two vector spaces which is linear, then it is called an* operator. *An operator $T : E \to F$ between two ordered vector spaces is said to be* positive *(in symbols $T \geq 0$) if $T(x) \geq 0$ for all $x \geq 0$.*

For positive operators, we have the following useful lemma; see [5, p.12].

Lemma 1. *If $T : E \to F$ is a positive operator between two vector lattices then, for each $x \in E$, we have $|Tx| \leq T|x|$.*

When we study the modulus, we need the next useful property.

Theorem 2. *(The Decomposition Property) Let E be a vector lattice, and let $x, y_1, y_2, \ldots, y_n \in E$ such that $|x| \leq |y_1 + \ldots + y_n|$ holds. Then there are $x_1, \ldots, x_n \in E$ satisfying $x = x_1 + x_2 + \ldots + x_n$ and $|x_i| \leq |y_i|$ for each $i = 1, \ldots, n$. Moreover, if x is positive then all x_i also can be chosen to be positive.*

The importance of the result lies in the fact that in order for a mapping $T : E_+ \to F_+$ to be the restriction of a (unique) positive operator from E to F it is necessary and sufficient to be additive on E_+; see [5, Thm. 1.10]. In next theorem, you can see details.

Theorem 3. *(Kantorovich) Let E and F be two vector lattices. If a mapping $T : E_+ \to F_+$ is additive, that is $T(x + y) = Tx + Ty$ for all $x, y \in E_+$, then it has a unique extension to a positive operator from E to F. Moreover, the extension is given by $\hat{T}x = Tx^+ - Tx^-$ for all $x \in E$.*

Throughout this thesis, $L(E, F)$ denotes the space of all operators between vector spaces E and F. We write $L(E)$ for $L(E, E)$. Thus, $L(E, F)$, under the ordering $T \leq S$ whenever $S - T$ is a positive operator (i.e., whenever $Tx \leq Sx$ holds for all $x \in E_+$), is an ordered vector space. For an operator $T \in L(E, F)$, we shall say that its *modulus* $|T|$ exists, whenever $|T| := (-T) \vee T$ exists in the sense that $|T|$ is the supremum of the set $\{-T, T\}$ in $L(E, F)$. If the modulus of an operator T exists, then next easy but important inequality holds: $|Tx| \leq |T|(|x|)$ for all $x \in E$; see [5, p.12].

The vector space of all order bounded operators between two vector lattices E and F will be denoted $L^\sim(E, F)$. The vector space $E^\sim$ of all order bounded linear functionals on E is called the *order dual* of E, i.e., $E^\sim = L^\sim(E, \mathbb{R})$. It can be seen easily that every positive operator is order bounded, and so the set of positive operators from E to F is a subset of $L^\sim(E, F)$. It follows that if F is order complete then each $T \in L^\sim(E, F)$ satisfies

$$T^+(x) = \sup\{Ty : 0 \leq y \leq x\} \text{ and } T^-(x) = \sup\{-Ty : 0 \leq y \leq x\}$$

for each $x \in E_+$. Moreover, it satisfies $T = T^+ - T^-$.

Let an operator $T \in L^\sim(E, F)$, then the *adjoint* of T is the operator $T' : F^\sim \to E^\sim$ defined by $\langle T'f, x\rangle = \langle f, Tx\rangle$ for all $f \in F^\sim$ and $x \in E$. An operator $T : X \to Y$

is said to be weakly continuous whenever $T : \big(X, \sigma(X, X')\big) \to \big(Y, \sigma(Y, Y')\big)$ is continuous.

Definition 7. *An operator $T \in L(E, F)$ is said to be:*

(1) order continuous *if $x_\alpha \xrightarrow{o} 0$ in E implies $Tx_\alpha \xrightarrow{o} 0$ in F,*

(2) σ-order continuous, *if $x_n \xrightarrow{o} 0$ in E implies $Tx_n \xrightarrow{o} 0$ in F,*

(3) an order bounded operator, *if it maps order bounded subsets of E to order bounded subsets of F.*

It is useful to note that a positive operator $T \in L(E, F)$ is order continuous iff $x_\alpha \downarrow 0$ in E implies $Tx_\alpha \downarrow 0$ in F (and also iff $0 \leq x_\alpha \uparrow x$ in E implies $Tx_\alpha \uparrow x$ in F.) We have the following useful property: every order continuous operator is order bounded.

A linear operator $T : E \to F$ between vector lattices is called a *lattice homomorphism* if $|Tx| = T|x|$ for all $x \in E$; cf. [4, Thm 1.17]. A one-to-one lattice homomorphism is referred as a *lattice isomorphism.* Two vector lattices E and F are said to be *lattice isomorphic* when there is a lattice isomorphism from E onto F. It is known that every lattice homomorphism $T \in L^{\sim}(E, F)$ is a positive operator; cf. [3, p.307]. Also note that the range of a lattice homomorphism is a vector sublattice.

If E is a vector lattice, then there is a (unique up to lattice isomorphism) order complete vector lattice E^δ that contains E as a majorizing order dense sublattice. We refer to E^δ as the *order completion* of E.

Definition 8. *A subset B of a vector lattice is said to be* order closed *whenever a net $(x)_\alpha \subseteq B$ and $x_\alpha \xrightarrow{o} x$ imply $x \in B$.*

An order closed ideal is referred to as a *band*; see [5, p.33]. The band generated by a subset A is the smallest (with respect to inclusion) band that includes A. A band B in a vector lattice E that satisfies $E = B \oplus B^d$ is referred to as a *projection band*, where $B^d = \{x \in E : x \perp b, \forall b \in B\}$ is the disjoint complement of B.
For $x_0 \in E$, the *principal band* generated by x_0 is the smallest band that includes x_0. We denote this band by B_{x_0}, and it is described as $B_{x_0} := \{x \in E : |x| \wedge n|x_0| \uparrow |x|\}$.

As usual, an operator $P : E \to E$ on a vector space is called a projection if $P^2 =$

P. Let B be a projection band in a vector lattice E. Thus, every $x \in E$ has the unique decomposition $x = x_1 + x_2$, where $x_1 \in B$ and $x_2 \in B^d$. Then a projection $P_B : E \to E$ is defined by the formula $P_B(x) := x_1$. Hence, P_B is called an *order projection* (or a *band projection*).
If P is a band projection then it is a lattice homomorphism and $0 \leq P \leq I$; i.e., $0 \leq Px \leq x$ for all $x \in E_+$. So band projections are order continuous.

A vector $e > 0$ is said to be a *weak order unit* in vector lattice E whenever the band generated by e satisfies $B_e = E$ (or, equivalently, whenever, for each $x \in E_+$, we have $x \wedge ne \uparrow x$). Also, note that $e > 0$ in a vector lattice space is a weak order unit iff $x \perp e$ implies $x = 0$. The element $e > 0$ is called a *strong unit* in E if, for every $x \in E$, there exists a positive number λ, depending on x, such that $|x| \leq \lambda e$.

The following theorem shows that the space of order bounded operators is vector lattice; see [5, Thm. 1.18].

Theorem 4. *(F. Riesz–Kantorovich). If E and F are vector lattices with F is order complete, then the ordered vector space $L^{\sim}(E, F)$ is an order complete vector lattice.*

Now, we give some basic definitions from the functional analysis.

Definition 9. *A collection $\mathcal{B}$ of subsets of a set X, which contains $\emptyset$ and X, is called a* Boolean algebra *or an* algebra*, if the following hold:*

(1) $A, B \in \mathcal{B}$ implies $A \cup B \in \mathcal{B}$;

(2) $A \in \mathcal{B}$ implies $A^c \in \mathcal{B}$.;

(3) $A, B \in \mathcal{B}$ implies $A \cap B \in \mathcal{B}$.

An algebra $\mathcal{B}$ of sets is a σ-algebra if every union of countable collection of sets in $\mathcal{B}$ belongs to $\mathcal{B}$. A collection $\mathcal{B}$ of Borel sets is the smallest σ-algebra that contains all open sets of real numbers.

Definition 10. *A* filter *$\mathcal{F}$ is a non-empty family of subsets of a set X satisfying the following properties:*

(1) $A_1, A_2, \ldots, A_n \in \mathcal{F}$ implies $A_1 \cap A_2 \cap \ldots \cap A_n \in \mathcal{F}$.

(2) $A \subseteq B$ and $A \in \mathcal{F}$ implies $B \in \mathcal{F}$.

A filter $\mathcal{U}$ is called an *ultrafilter* over a set X if, for all $A \subseteq X$, either $A \in \mathcal{U}$ or $A^c \in \mathcal{U}$.

2.2 Normed Vector Lattice

Let X be a normed vector space. If any Cauchy net in X is convergent some element in X, then X is called a *Banach space*. A subset A of a normed space X is called *norm bounded* if there is $x \in X$ such that $\|a\| \leq \|x\|$ for all $a \in A$. If X and Y are normed spaces, then we can define the notion of a bounded mapping. The boundedness of a linear mapping is equivalent to its continuity.

Definition 11. *A linear mapping $T : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ between two normed spaces is called* norm bounded *if there is a real number $k > 0$ such that $\|Tx\|_Y \leq k\|x\|_X$ for all $x \in X$.*

For normed spaces X and Y we use $B(X,Y)$ for the space of all norm bounded linear operators from X into Y. We write $B(X)$ for $B(X,X)$. If X is a normed space then $X^* = \{f : X \to \mathbb{R} : f \text{ is continuous}\}$ denotes the topological dual of X, and $B_X = \{x \in X : \|x\| \leq 1\}$ denotes the closed unit ball of X. Also we define $\mathbb{R}^{X^*}$ to be the set of all functions from X^* to $\mathbb{R}$. A subset A of normed space X is called *norm relatively compact* if its closure $\overline{A}$ is compact in X.

Definition 12. *An operator $T : X \to Y$ between two normed spaces is said to be a* compact operator *whenever T maps the closed unit ball B_X of X onto a norm relatively compact subset of Y.*

In other words, T is a compact operator iff, for every norm bounded sequence x_n in X, the sequence Tx_n has a norm convergent subsequence in Y. It can be seen that every compact operator is norm bounded (and, hence, is continuous).

Definition 13. *An operator $T : X \to Y$ between Banach spaces is called* weakly compact *if T carries norm bounded subsets of X to relatively weakly compact subsets of Y. Equivalently, if $T(B_X)$ is relatively weakly compact.*

A vector lattice X equipped with a norm $\|\cdot\|$ is said to be a *normed lattice* if $|x| \leq |y|$

in X implies $\|x\| \leq \|y\|$. If a normed lattice is norm complete, then it is called a *Banach lattice*. The spaces ℓ_p $(1 \leq p \leq \infty)$ and c_0 are order complete Banach lattices, where ordering is defined pointwise.

A lattice norm $(X, \|\cdot\|_X)$ is said to be *order continuous* whenever $x_\alpha \downarrow 0$ implies $\|x_\alpha\|_X \downarrow 0$ or, equivalently, $x_\alpha \xrightarrow{o} 0$ in X implies $\|x_\alpha\|_X \to 0$. If this condition holds for sequences, i.e., $x_n \downarrow 0$ implies $\|x_n\|_X \downarrow 0$, then $\|\cdot\|_X$ is said to be *σ-order continuous*. A normed lattice $(X, \|\cdot\|)$ is called a ***KB-space*** if, for $0 \leq x_\alpha \uparrow$ and $\sup_\alpha \|x_\alpha\| < \infty$, we get that the net (x_α) is norm convergent.

Now, assume X to be a normed lattice. Then a vector $0 < e \in X$ is called a *quasi-interior point* if $\overline{I_e} = X$, where I_e denotes the ideal generated by e. It can be shown that e is a quasi-interior point iff, for every $x \in X_+$, we have $\|x - x \wedge ne\| \to 0$, as $n \to \infty$.
Clearly, strong unit ⇒ quasi-interior point ⇒ weak unit.

Definition 14. *A positive vector $a \neq 0$ in a vector lattice X is called* atom *if, for any $x \in [0, a]$, there is $\lambda \in \mathbb{R}$ such that $x = \lambda a$; cf. [2, Def. 2.29].*

Let a be an atom in a vector lattice X. The principal band B_a generated by a is a projection band, and $B_a = I_a = span\{a\} = \{\lambda a : \lambda \in \mathbb{R}\}$, where I_a is the ideal generated by a. A vector lattice X is called *atomic* if the band generated by its atoms is X. If a vector lattice X is atomic, then it has a maximal orthogonal system of atoms a_γ, that is for $\gamma \neq \gamma'$, $a_\gamma \perp a_{\gamma'}$, and if $x \perp a_\gamma = 0$ for all γ then $x = 0$. Also, for any $x > 0$, there is an atom a such that $a \leq x$.

A lattice X is called *not atomic* if it has not atoms, but they do not form a maximal orthogonal system. X is also called *atomless* or *non-atomic* if it has no atom.

For any atom a corresponding to B_a, we have a band projection $p_a : X \to B_a$ denoted by $p_a(x) = f_a(x)a$, where the *biorthogonal function* $f_a : X \to \mathbb{R}$ defined by $f_a(x) = \lambda$, where λ is the real number such that $x = \lambda a$. Since band projections are lattice homomorphisms and are order continuous then so f_a for any atom a. Also it is known that f_a is norm continuous; cf. [2, p.31].

Last we characterize order convergence in atomic order complete vector lattices, but first we provide the following technical lemma.

Lemma 2. *Let X and Y be vector lattices. If $T : X \to Y$ is a order continuous lattice homomorphism and A is a subset of X such that $\sup A$ exists in X, then $T(\sup A) = \sup T(A)$.*

Proof. Note that T is an order continuous operator and $\{a_1 \vee \cdots \vee a_n : n \in \mathbb{N}, a_1, \ldots, a_n \in A\} \uparrow \sup A$. So $T\big(\{a_1 \vee \cdots \vee a_n : n \in \mathbb{N}, a_1, \ldots, a_n \in A\}\big) \uparrow T(\sup A)$. Furthermore, $T\big(\{a_1 \vee \cdots \vee a_n : n \in \mathbb{N}, a_1, \ldots, a_n \in A\}\big) = \{T(a_1 \vee \cdots \vee a_n) : n \in \mathbb{N}, a_1, \ldots, a_n \in A\} = \{Ta_1 \vee \cdots \vee Ta_n : n \in \mathbb{N}, a_1, \ldots, a_n \in A\} \uparrow \sup T(A)$. Hence, $T(\sup A) = \sup T(A)$. $\square$

The following useful lemma is used in some proofs of theorems in this thesis.

Lemma 3. *If X is an atomic order complete vector lattice and (x_α) is an order bounded net such that $f_a(x_\alpha) \to 0$ for any atom a, then $x_\alpha \xrightarrow{o} 0$.*

Proof. Suppose the contrary, then $\inf_\alpha \sup_{\beta \geq \alpha} |x_\beta| > 0$, so there is an atom a such that $a \leq \inf_\alpha \sup_{\beta \geq \alpha} |x_\beta|$. Hence, $a \leq \sup_{\beta \geq \alpha} |x_\beta|$ for any α. Let f_a be the biorthogonal functional corresponding to a, then $1 = f_a(a) \leq f_a\big(\sup_{\beta \geq \alpha} |x_\beta|\big) = \sup_{\beta \geq \alpha} |f_a(x_\beta)|$ for each α. Thus, $\limsup_\alpha |f_a(x_\alpha)| \geq 1$, which is a contradiction. $\square$

In a normed space X, a net x_α is *weakly convergent* to $x \in X$ (shortly, $x_\alpha \xrightarrow{w} x$) if $f(x_\alpha) \to f(x)$ holds in $\mathbb{R}$ for all $f \in X^*$. Since $|f(x_\alpha - x)| \leq |f||x_\alpha - x|$ holds for all α, then we have the following implication: x_α absolutely weakly convergent to x implies $x_\alpha \xrightarrow{w} x$.

In a normed lattice $(X, \|\cdot\|)$, a net x_α is *unbounded norm convergent* to $x \in X$; see [12], written as $x_\alpha \xrightarrow{un} x$, if $\||x_\alpha - x| \wedge u\| \to 0$ for every $u \in X_+$; see [12]. Clearly, if the norm is order continuous then uo-convergence implies un-convergence.

Let E be an Archimedean vector lattice with a strong unit e. Then we can define in a natural manner a norm $\|x\|_e = \inf\{k : k \geq 0 \text{ and } |x| \leq ke\}$. This is a lattice norm on E and also, for any sequence in E, the notions of norm convergence, e-uniform convergence and relatively uniform convergence are equivalent; see [23, Thm. 62.4].

Definition 15. *An operator $T \in B(X,Y)$ from a normed lattice X into a normed space Y is called M-weakly compact, whenever* $\lim \|Tx_n\| = 0$ *holds for every norm bounded disjoint sequence x_n in X, and $T \in B(X,Y)$ from a normed space X into a normed lattice Y is called L-weakly compact, whenever* $\lim \|y_n\| = 0$ *holds for every disjoint sequence y_n in $sol(T(B_X))$; see [5, Def. 5.59].*

The following standard fact will be used throughout this thesis.

Lemma 4. *Let $(X, \|\cdot\|)$ be a normed space. Then $x_n \xrightarrow{\|\cdot\|} x$ iff, for any subsequence x_{n_k}, there is a further subsequence $x_{n_{k_j}}$ such that $x_{n_{k_j}} \xrightarrow{\|\cdot\|} x$.*

Definition 16. *A Banach lattice E is said to be:*

*(1) An AL-*space*, whenever its norm is additive on E_+ in the sense that $\|x+y\| = \|x\| + \|y\|$ holds for all $x, y \in E_+$.*

*(2) An AM-*space *if $x \wedge y = 0$ in E implies $\|x \vee y\| = \max\{\|x\|, \|y\|\}$.*

It is know that, in an AM-space with strong unit, every norm bounded set is order bounded; cf. [32, Exr. 122.8].

Let X be a normed space. The norm on the topological dual X^* is defined by the formula

$$\|f\|_{X^*} = \sup_{\|x\| \leq 1} |f(x)|.$$

For each $x \in X$, there is an $f \in X^*$ with $f(x) = \|x\|$ and $\|f\| \leq 1$. In particular, for each $x \in X$,

$$\|x\| = \sup\{|f(x)| : f \in X^*, \|f\| \leq 1\}$$

is called the norm formula.

Definition 17. *Let X be a vector lattice and Y be a normed space. An operator $T : X \to Y$ is called:*

(1) order weakly compact *if $T[-x, x]$ is relatively weakly compact for all $x \in X_+$; see [26, Def. 3.4.1].*

*(2) AM-*compact *if $T[-x, x]$ is relatively compact for every x in X_+; see [32, p.496].*

2.3 Lattice Normed Spaces

Definition 18. *Let X be a vector space and let E be a vector lattice. Then $p : X \to E_+$ is called the* vector norm (*see [22, 1.8.1]*) *if the following conditions hold*

(i) $p(x) = 0 \Leftrightarrow x = 0$,

(ii) $p(\lambda x) = |\lambda| p(x)$ *for all* $\lambda \in \mathbb{R}$ *and* $x \in X$,

(iii) $p(x + y) \leq p(x) + p(y)$ *for all* $x, y \in X$.

Thus, the triple (X, p, E) is called a *lattice-normed space*, abbreviated as LNS; c.f [20]. The lattice norm p in an LNS (X, p, E) is said to be *decomposable* if, for all $x \in X$ and $e_1, e_2 \in E_+$, it follows from $p(x) = e_1 + e_2$, that there exist $x_1, x_2 \in X$ such that $x = x_1 + x_2$ and $p(x_k) = e_k$ for $k = 1, 2$. If X is a vector lattice and the vector norm p is monotone (i.e. $|x| \leq |y| \Rightarrow p(x) \leq p(y)$) then the triple (X, p, E) is called a *lattice-normed vector lattice*, abbreviated as LNVL; see [7]. In this thesis, we usually use the pair (X, E) or just X to refer to an LNS (X, p, E), if there is no confusion.

We abbreviate the convergence $p(x_\alpha - x) \xrightarrow{o} 0$ as $x_\alpha \xrightarrow{p} x$ and say in this case that x_α p-converges to x. A net $(x_\alpha)_{\alpha \in A}$ in an LNS (X, p, E) is said to be *p-Cauchy* if the net $(x_\alpha - x_{\alpha'})_{(\alpha, \alpha') \in A \times A}$ p-converges to 0. An LNS (X, p, E) is called (*sequentially*) *p-complete* if every p-Cauchy (sequence) net in X is p-convergent. In an LNS (X, p, E), a subset A of X is called *p-bounded* if there exists $e \in E$ such that $p(a) \leq e$ for all $a \in A$. An LNVL (X, p, E) is called *op-continuous* if $x_\alpha \xrightarrow{o} 0$ implies $p(x_\alpha) \xrightarrow{o} 0$. A net x_α in an LNVL (X, p, E) is said to be *unbounded p-convergent* to $x \in X$ (shortly, x_α *up*-converges to x or $x_\alpha \xrightarrow{up} x$), if $p(|x_\alpha - x| \wedge u) \xrightarrow{o} 0$ for all $u \in X_+$; see [7, Def.6].

Let (X, p, E) be an LNS and $(E, \|\cdot\|_E)$ be a normed lattice. The *mixed norm* on X is defined by $p\text{-}\|x\|_E = \|p(x)\|_E$ for all $x \in X$ (see, for example, [20, 7.1.1, p.292]). In this case the normed space $(X, p\text{-}\|\cdot\|_E)$ is called a *mixed-normed space*.

A net x_α in an LNS (X, p, E) is said to *relatively uniformly p-convergent* to $x \in X$ (written as $x_\alpha \xrightarrow{rp} x$) if there is $e \in E_+$ such that, for any $\varepsilon > 0$, there is α_ε satisfying $p(x_\alpha - x) \leq \varepsilon e$ for all $\alpha \geq \alpha_\varepsilon$. In this case we say that x_α *rp*-converges to x. A net x_α

in an LNS (X, p, E) is called *rp-Cauchy* if the net $(x_\alpha - x_{\alpha'})_{(\alpha,\alpha')\in A\times A}$ rp-converges to 0. It is easy to see that, for a sequence x_n in an LNS (X, p, E), $x_n \xrightarrow{\text{rp}} x$ iff there exist $e \in E_+$ and a numerical sequence $\varepsilon_k \downarrow 0$ such that, for all $k \in \mathbb{N}$, there is $n_k \in \mathbb{N}$ satisfying $p(x_n - x) \leq \varepsilon_k e$ for all $n \geq n_k$. An LNS (X, p, E) is said to be *rp-complete* if every rp-Cauchy sequence in X is rp-convergent. It should be noticed that in rp-complete LNSs every rp-Cauchy net is rp-convergent. Indeed, assume x_α to be an rp-Cauchy net in an rp-complete LNS (X, p, E). Then, there exists an element $e \in E_+$ such that, for all $n \in \mathbb{N}$, there is α_n such that $p(x_{\alpha'} - x_\alpha) \leq \frac{1}{n}e$ for all $\alpha, \alpha' \geq \alpha_n$. We select a strictly increasing sequence α_n. Then it is clear that x_{α_n} is an rp-Cauchy sequence, and so there is $x \in X$ such that $x_{\alpha_n} \xrightarrow{\text{rp}} x$. Let $n_0 \in \mathbb{N}$. Hence, there is α_{n_0} such that, for all $\alpha \geq \alpha_{n_0}$, we have $p(x_\alpha - x_{\alpha_{n_0}}) \leq \frac{1}{n_0}e$ and, for all $n \geq n_0$, $p(x - x_{\alpha_{n_0}}) \leq \frac{1}{n_0}e$, that implies $x_\alpha \xrightarrow{\text{rp}} x$.

We recall the following result (see for example [20, 7.1.2,p.293]). If (X, p, E) is an LNS such that $(E, \|\cdot\|_E)$ is a Banach space, then $(X, p\text{-}\|\cdot\|_E)$ is norm complete iff the LNS (X, p, E) is rp-complete. On the other hand, it is not difficult to see that if an LNS is sequentially p-complete then it is rp-complete. Thus, the following result follows readily.

Lemma 5. *Let (X, p, E) be an LNS such that $(E, \|\cdot\|_E)$ is a Banach space. If (X, p, E) is sequentially p-complete then $(X, p\text{-}\|\cdot\|_E)$ is a Banach space.*

Consider LNSs (X, p, E) and (Y, m, F). An operator $T : X \to Y$ is said to be *dominated* if there is a positive operator $S : E \to F$ satisfying $m(Tx) \leq S(p(x))$ for all $x \in X$. In this case, S is called a *dominant* for T. The set of all dominated operators from X to Y is denoted by $M(X, Y)$. In the ordered vector space $L^\sim(E, F)$ of all order bounded operators from E into F, if the least element of all dominants of an operator T exists, then such element is called the *exact dominant* of T and denoted by $[\![T]\!]$; see [20, 4.1.1,p.142].

By considering [20, 4.1.3(2), p.143] and Kaplan's example [5, Ex.1.17], we see that not every dominated operator possesses the exact dominant. On the other hand, if X is decomposable and F is order complete then every dominated operator $T : X \to Y$ has the exact dominant $[\![T]\!]$; see [20, 4.1.2,p.142].

We refer the reader for more information on LNSs to [9, 13, 21, 20] and [7]. It should be noticed that the theory of lattice-normed spaces is well- developed in the case of decomposable lattice norms (cf. [21, 20]). In [10] and [28], the authors studied some classes of operators in LNSs under the assumption that the lattice norms are decomposable. In this thesis, we usually do not assume lattice norms to be decomposable.

CHAPTER 3

MAIN RESULTS

In this chapter we use basic definitions and properties of [8]. We recall definitions of p-continuous and p-bounded operators between lattice-normed spaces. We introduce the notion of p-compact and sequentially p-compact operators between lattice-normed spaces. These operators generalize several known classes of operators. Also, we define p-M-weakly and p-L-weakly compact operators which correspond respectively to M-weakly and L-weakly compact operators. In addition, the notions of (sequentially) up-continuous and (sequentially) up-compact operators acting between lattice-normed vector lattices, are introduced.

3.1 p-Continuous and p-Bounded Operators

Firstly, we recall the notion of p-continuous operator between LNSs which generalizes the notion of order continuous operator in a vector lattice.

Definition 19. *Let X, Y be two LNSs and $T \in L(X,Y)$. Then T is called:*

(1) p-continuous *if $x_\alpha \xrightarrow{\text{p}} 0$ in X implies $Tx_\alpha \xrightarrow{\text{p}} 0$ in Y and, if the condition holds only for sequences, then T is called* sequentially p-continuous*;*

(2) p-bounded *if it maps p-bounded sets in X to p-bounded sets in Y.*

The following lemma gives the linearity of p-continuous operators.

Lemma 6. *Let (X,p,E) and (Y,m,F) be two LNSs.*

(i) If $T,S : (X,p,E) \to (Y,m,F)$ are p-continuous operators, then $\lambda S + \mu T$ is

p-continuous for real numbers λ and μ. In particular, if $H = T - S$ then H is order continuous.

(ii) If $-T_1 \leq T \leq T_2$, with T_1 and T_2 are positive and p-continuous operators, then T is p-continuous.

Proof. (i) Let λ and μ be any two real numbers, then we have

$$\begin{aligned} m(\lambda S x_\alpha + \mu T x_\alpha) &\leq m(\lambda S x_\alpha) + m(\mu T x_\alpha) \\ &= |\lambda| m(S x_\alpha) + |\mu| m(T x_\alpha) \\ &\leq (|\lambda| + |\mu|)(m(S x_\alpha) + m(T x_\alpha)) \xrightarrow{o} 0. \end{aligned}$$

Therefore, $\lambda S + \mu T$ is p-continuous.

(ii) If $-T_1 \leq T \leq T_2$, with T_1 and T_2 are positive and p-continuous, then

$$0 \leq T + T_1 \leq T_2 + T_1$$

and so $T + T_1$ is a positive and p-continuous operator. Therefore, $T = (T + T_1) - T_1$ is p-continuous. □

Following remark give relations about p-continuous and p-bounded operators with rp-continuous, bo-continuous and dominated operators.

Remark 1.

(i) The collection of all p-continuous operators between LNSs is a vector space.

(ii) Using rp-convergence, one can introduce the following notion:
A linear operator T from an LNS (X, E) into another LNS (Y, F) is called rp-continuous if $x_\alpha \xrightarrow{\text{rp}} 0$ in X implies $Tx_\alpha \xrightarrow{\text{rp}} 0$ in Y. But this notion is not so interesting, because it coincides with p-boundedness of an operator (see [9, Thm. 5.3.3 (a)]).

(iii) A p-continuous (respectively, sequentially p-continuous) operator between two LNSs is also known as bo-continuous (respectively, sequentially bo-continuous); see e.g. [20, 4.3.1, p.156].

(iv) Let (X, E) be a decomposable LNS and let F be an order complete vector lattice. Then $T \in M_n(X, Y)$ iff its exact dominant $|T|$ is order continuous [20, Thm.4.3.2], where $M_n(X, Y)$ denotes the set of all dominated bo-continuous operators from X to Y.

(v) Every dominated operator is p-bounded. Indeed, let $T \in M(X, Y)$, then there is a dominant operator $S : E \to F$ of T. Let A be a p-bounded set in X, then there is $e \in E$ such that $p(a) \le e$ for all $a \in A$. Thus, $m(Ta) \le S\big(p(a)\big) \le Se$ for all $a \in A$, since S is positive. Therefore $T(A)$ is p-bounded in Y.

The converse does not need be true, for example, consider the identity operator $I : (\ell_\infty, |\cdot|, \ell_\infty) \to (\ell_\infty, \|\cdot\|, \mathbb{R})$. It is p-bounded. Indeed, let a subset A be p-bounded in ℓ_∞, then there is $e \in \ell_\infty$ such that $|a| \le e$ holds for all $a \in A$. Since $(\ell_\infty, \|\cdot\|)$ is a Banach lattice, then we have $\|a\| \le \|e\|$ for all $a \in A$. Hence, I is p-bounded.
However, the identity operator $I : (\ell_\infty, |\cdot|, \ell_\infty) \to (\ell_\infty, \|\cdot\|, \mathbb{R})$ is not dominated (see [9, Rem., p.388]).

Next, we give some examples of p-continuity and p-boundedness of operators in a particular case of LNSs.

Example 1.

(i) Let X and Y be vector lattices, then $T \in L(X, Y)$ is (σ-) order continuous iff $T : (X, |\cdot|, X) \to (Y, |\cdot|, Y)$ is (sequentially) p-continuous.

(ii) Let X and Y be vector lattices, then $T \in L^\sim(X, Y)$ iff $T : (X, |\cdot|, X) \to (Y, |\cdot|, Y)$ is p-bounded.

(iii) Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces. Then $T \in B(X, Y)$ iff $T : (X, \|\cdot\|_X, \mathbb{R}) \to (Y, \|\cdot\|_Y, \mathbb{R})$ is p-continuous iff $T : (X, \|\cdot\|_X, \mathbb{R}) \to (Y, \|\cdot\|_Y, \mathbb{R})$ is p-bounded.

(iv) Let X be a vector lattice and $(Y, \|\cdot\|_Y)$ be a normed space. Then an operator T from X to Y is called order-to-norm continuous *if $x_\alpha \xrightarrow{o} 0$ in X implies $Tx_\alpha \xrightarrow{\|\cdot\|_Y} 0$ (see [24, Sect.4, p.468]). Therefore, $T : X \to Y$ is order-to-norm continuous iff $T : (X, |\cdot|, X) \to (Y, \|\cdot\|_Y, \mathbb{R})$ is p-continuous.*

In the next lemma, we have a relation between order continuous operator and p-continuous operator. Also, it gives a relation between order bounded operator and p-bounded operator.

Lemma 7. *Given an op-continuous LNVL (Y, m, F) and a vector lattice X.*

(i) If $T : X \to Y$ is (σ-) order continuous, then $T : (X, |\cdot|, X) \to (Y, m, F)$ is (sequentially) p-continuous.

(ii) If $T : X \to Y$ is order bounded, then $T : (X, |\cdot|, X) \to (Y, m, F)$ is p-bounded.

Proof. (i) Assume a net x_α in X to be p-convergent to zero in $(X, |\cdot|, X)$. Then we have $x_\alpha \xrightarrow{o} 0$ in X. Thus, $Tx_\alpha \xrightarrow{o} 0$ in Y, as T is order continuous. Since (Y, m, F) is *op*-continuous, then $m(Tx_\alpha) \xrightarrow{o} 0$ in F. Therefore, $Tx_\alpha \xrightarrow{p} 0$ in Y, and so T is p-continuous.
The sequential case is similar.

(ii) Let A be p-bounded set in X. That is, A is order bounded in X, and so $T(A)$ is order bounded in Y, as T is order bounded. Thus, there exists $y \in Y_+$ such that $|Ta| \leq y$ for all $a \in A$. Hence, $m(|Ta|) \leq m(y)$ for all $a \in A$. Therefore, $T : (X, |\cdot|, X) \to (Y, m, F)$ is p-bounded. □

In the following proposition, we have the partial answer of the converse of Lemma 7.

Proposition 1. *Let (X, p, E) be an op-continuous LNVL, let (Y, m, F) be an LNVL, and let $T : (X, p, E) \to (Y, m, F)$ be a (sequentially) p-continuous positive operator. Then $T : X \to Y$ is (σ-) order continuous.*

Proof. We show only the order continuity of T, the sequential case is analogous. Assume $x_\alpha \downarrow 0$ in X. Since X is *op*-continuous, then $p(x_\alpha) \xrightarrow{o} 0$. Hence, we get $T(x_\alpha) \xrightarrow{p} 0$ in Y, as T is p-continuous. Since $0 \leq T$, then $Tx_\alpha \downarrow$. Also we have $m(Tx_\alpha) \xrightarrow{o} 0$, so it follows from [7, Prop.1] that $Tx_\alpha \downarrow 0$. Thus, T is order continuous. □

The next proposition give relation about norm bounded operator and sequentially p-continuous operator.

Proposition 2. *Let $(X, \|\cdot\|_X)$ be a σ-order continuous Banach lattice. Then $T \in B(X)$ iff $T : (X, |\cdot|, X) \to (X, \|\cdot\|_X, \mathbb{R})$ is sequentially p-continuous.*

Proof. ($\Rightarrow$) Assume $T \in B(X)$, and let $x_n \xrightarrow{\mathrm{p}} 0$ in $(X, |\cdot|, X)$. Then $x_n \xrightarrow{\mathrm{o}} 0$ in X. Since $(X, \|\cdot\|_X)$ is a σ-order continuous Banach lattice, then $x_n \xrightarrow{\|\cdot\|_X} 0$, and hence $Tx_n \xrightarrow{\|\cdot\|_X} 0$, as T is norm continuous. Therefore, $T : (X, |\cdot|, X) \to (X, \|\cdot\|_X, \mathbb{R})$ is sequentially p-continuous.

($\Leftarrow$) Assume $T : (X, |\cdot|, X) \to (X, \|\cdot\|_X, \mathbb{R})$ to be sequentially p-continuous. Suppose $x_n \xrightarrow{\|\cdot\|_X} 0$, and let x_{n_k} be a subsequence of x_n. Then, clearly, $x_{n_k} \xrightarrow{\|\cdot\|_X} 0$. Since $(X, \|\cdot\|_X)$ is a Banach lattice, there is a subsequence $x_{n_{k_j}}$ such that $x_{n_{k_j}} \xrightarrow{\mathrm{o}} 0$ in X (cf. [30, Thm.VII.2.1]), and so $x_{n_{k_j}} \xrightarrow{\mathrm{p}} 0$ in $(X, |\cdot|, X)$. Since T is sequentially p-continuous, then $Tx_{n_{k_j}} \xrightarrow{\|\cdot\|_X} 0$. Thus, it follows from Lemma 4, that $Tx_n \xrightarrow{\|\cdot\|_X} 0$. $\square$

In following proposition, we get when sequentially p-continuous operator is norm continuous.

Proposition 3. *Let (X, p, E) be an LNVL with a Banach lattice $(E, \|\cdot\|_E)$ and (Y, m, F) be an LNS with a σ-order continuous normed lattice $(F, \|\cdot\|_F)$. If $T : (X, p, E) \to (Y, m, F)$ is sequentially p-continuous, then $T : (X, p\text{-}\|\cdot\|_E) \to (Y, m\text{-}\|\cdot\|_F)$ is norm continuous.*

Proof. Let x_n be a sequence in X such that $x_n \xrightarrow{p\text{-}\|\cdot\|_E} 0$ (i.e. $\|p(x_n)\|_E \to 0$). Given a subsequence x_{n_k} of x_n, then $\|p(x_{n_k})\|_E \to 0$. Since $(E, \|\cdot\|_E)$ is a Banach lattice, there is a further subsequence $x_{n_{k_j}}$ such that $p(x_{n_{k_j}}) \xrightarrow{\mathrm{o}} 0$ in E (cf. [30, Thm.VII.2.1]). Hence, $x_{n_{k_j}} \xrightarrow{\mathrm{p}} 0$ in (X, p, E). Now, the p-continuity of T implies $m(Tx_{n_{k_j}}) \xrightarrow{\mathrm{o}} 0$ in F. But $(F, \|\cdot\|_F)$ is σ-order continuous, and so $\|m(Tx_{n_{k_j}})\|_F \to 0$ or $m\text{-}\|Tx_{n_{k_j}}\|_F \to 0$. Hence, Lemma 4 implies $m\text{-}\|Tx_n\|_F \to 0$. So T is norm continuous. $\square$

The next theorem is a partial converse of Proposition 3.

Theorem 5. *Suppose (X, p, E) to be an LNS with an order continuous (respectively, σ-order continuous) normed lattice $(E, \|\cdot\|_E)$ and (Y, m, F) to be an LNS with an atomic Banach lattice $(F, \|\cdot\|_F)$. Assume further that:*

(i) $T : (X, p\text{-}\|\cdot\|_E) \to (Y, m\text{-}\|\cdot\|_F)$ *is norm continuous,*

(ii) $T : (X, p, E) \to (Y, m, F)$ *is* p*-bounded.*

Then $T : (X, p, E) \to (Y, m, F)$ *is* p*-continuous* (*respectively, sequentially* p*-continuous*).

Proof. We assume that $(E, \|\cdot\|_E)$ is an order continuous normed lattice and show the p-continuity of T, the other case is similar. Suppose $x_\alpha \xrightarrow{p} 0$ in (X, p, E), then $p(x_\alpha) \xrightarrow{o} 0$ in E, and so there is α_0 such that $p(x_\alpha) \leq e$ for all $\alpha \geq \alpha_0$. Thus, $(x_\alpha)_{\alpha \geq \alpha_0}$ is p-bounded and, since T is p-bounded, then $(Tx_\alpha)_{\alpha \geq \alpha_0}$ is p-bounded in (Y, m, F).

Since $(E, \|\cdot\|_E)$ is order continuous and $p(x_\alpha) \xrightarrow{o} 0$ in E, then $\|p(x_\alpha)\|_E \to 0$ or $p\text{-}\|x_\alpha\|_E \to 0$. The norm continuity of $T : (X, p\text{-}\|\cdot\|_E) \to (Y, m\text{-}\|\cdot\|_F)$ ensures that $\|m(Tx_\alpha)\|_F \to 0$ or $m\text{-}\|Tx_\alpha\|_F \to 0$. In particular, $\|m(Tx_\alpha)\|_F \to 0$ for $\alpha \geq \alpha_0$.

Let $a \in F$ be an atom, and f_a be the biorthogonal functional corresponding to a. Then $f_a\big(m(Tx_\alpha)\big) \to 0$. Since $m(Tx_\alpha)$ is order bounded for all $\alpha \geq \alpha_0$ and $f_a\big(m(Tx_\alpha)\big) \to 0$ for any atom $a \in F$, the atomicity of F implies that $m(Tx_\alpha) \xrightarrow{o} 0$ in F, as $\alpha_0 \leq \alpha \to \infty$. Thus, $T : (X, p, E) \to (Y, m, F)$ is p-continuous. □

The next result extends the well-known fact that every order continuous operator between vector lattices is order bounded, and its proof is similar to [1, Thm.2.1].

Proposition 4. *Let* T *be a* p*-continuous operator from a LNS* (X, p, E) *to another LNS* (Y, m, F)*, then* T *is* p*-bounded.*

Proof. Assume that $T : X \to Y$ is p-continuous. Let $A \subset X$ be p-bounded (i.e. there is $e \in E$ such that $p(a) \leq e$ for all $a \in A$). Let $I = \mathbb{N} \times A$ be an index set with the lexicographic order. That is: $(m, a') \leq (n, a)$ iff $m < n$ or else $m = n$ and $p(a') \leq p(a)$. Clearly, I is directed upward. Define the following net as $x_{(n,a)} = \frac{1}{n}a$. Then $p(x_{(n,a)}) = \frac{1}{n}p(a) \leq \frac{1}{n}e$. So $p(x_{(n,a)}) \xrightarrow{o} 0$ in E or $x_{(n,a)} \xrightarrow{p} 0$. By p-continuity of T, we get $m(Tx_{(n,a)}) \xrightarrow{o} 0$. So there is a net $(z_\beta)_{\beta \in B}$ such that $z_\beta \downarrow 0$ in F and, for any $\beta \in B$, there exists $(n', a') \in I$ satisfying $m(Tx_{(n,a)}) \leq z_\beta$ for all $(n, a) \geq (n', a')$. Fix $\beta_0 \in B$. Then there is $(n_0, a_0) \in I$ satisfying $m(Tx_{(n,a)}) \leq z_{\beta_0}$ for all $(n, a) \geq (n_0, a_0)$. In particular, $(n_0 + 1, a) \geq (n_0, a_0)$ for all $a \in A$.

Thus, $m(Tx_{(n_0+1,a)}) = m(\frac{1}{n_0+1}Ta) \leq z_{\beta_0}$ or $m(Ta) \leq (n_0+1)z_{\beta_0}$ for all $a \in A$. Therefore, T is p-bounded. □

The following remark give two examples: one of examples about the converse of Proposition 4 is not true and another example about a p-continuous operator does not need to be order bounded.

Remark 2.

(i) *It is known that the converse of Proposition 4 is not true. For example, let $X = C[0,1]$, then $X^* = X^\sim$. So, for any $0 \neq \varphi \in X^*$, we have $\varphi : (X, |\cdot|, X) \to (\mathbb{R}, |\cdot|, \mathbb{R})$ is p-bounded. Indeed, let A be a p-bounded subset in X. Since φ is order bounded, $\varphi(A)$ is order bounded in $\mathbb{R}$. Thus, φ is p-bounded.*

On the other hand, let us define a functional $\varphi : (X, |\cdot|, X) \to (\mathbb{R}, |\cdot|, \mathbb{R})$ denoted by $\varphi(f) := f(0)$. We show that φ is not a σ-order continuous functional. Take a sequence (f_n) in $C[0,1]$ defined by $f_n(t) = -nt + 1$, whenever $0 \leq t \leq \frac{1}{n}$, and by $f_n(t) = 0$, whenever $\frac{1}{n} \leq t \leq 1$. Then, clearly, $f_n \downarrow 0$. However, $\varphi(f_n) = f(0) = 1 \not\to 0$. Thus, φ is not σ-order continuous or $\varphi \notin X_c^\sim$ and so $\varphi \notin X_n^\sim$, as $X_n^\sim \subseteq X_c^\sim$, (here $X_c^\sim$ denotes the σ-order continuous dual of X, and $X_n^\sim$ denotes the order continuous dual of X). Thus, φ is not order continuous.

(ii) *If $T : (X, E) \to (Y, F)$ between two LNVLs is p-continuous, then an operator between two vector lattices $T : X \to Y$ does not need to be order bounded. Let us consider Lozanovsky's example. If $T : L_1[0,1] \to c_0$ is defined by*

$$T(f) = \Big(\int_0^1 f(x) sinx\, dx, \int_0^1 f(x) sin2x\, dx, \cdots \Big),$$

then it can be shown that T is norm bounded, but it is not order bounded $\big($cf. [5, Exer.10, p.289]$\big)$. So $T : (L_1[0,1], \|\cdot\|_{L_1}, \mathbb{R}) \to (c_0, \|\cdot\|_\infty, \mathbb{R})$ is p-continuous, and $T : L_1[0,1] \to c_0$ is not order bounded.

The following example is an example of LNVL.

Example 2. *Let $(X, \|\cdot\|_X)$ be a normed lattice. Put $E := \mathbb{R}^{X^*}$, and define $p : X \to E_+$ by $p(x)[f] = |f|(|x|)$ for $f \in X^*$. Then (X, p, E) is an LNVL. Indeed:*

(1) *If* $x=0$ *then* $p(0)[f]=|f|(|0|)=0$. *Thus,* $p(0)=0$.
Now, suppose $p(x)=0$. *We show that* $x=0$. *For each* $f\in X^*$, $p(x)[f]=|f|(|x|)=0$. *So, by the norm formula in [25, Prop. 6.10], we have*

$$\|x\|=\sup\{\big|f(|x|)\big|:f\in B_{X^*}\}=0,$$

so $x=0$.

(2) *For each* $\alpha\in\mathbb{R}$ *and* $f\in X^*$,

$$p(\alpha x)[f]=|f|(|\alpha x|)=|\alpha||f|(|x|)=|\alpha|p(x)[f].$$

So, $p(\alpha x)=|\alpha|p(x)$.

(3) *For any* $x,y\in X$,

$$\begin{aligned} p(x+y)[f]=|f|(|x+y|) &\leq |f|(|x|+|y|) \\ &\leq |f|(|x|)+|f|(|y|) \\ &= p(x)[f]+p(y)[f] \\ &= \big(p(x)+p(y)\big)[f]. \end{aligned}$$

Thus, $p(x+y)\leq p(x)+p(y)$.

Finally, we should show that p *is monotone. Take two elements* $x,y\in X$ *such that* $x\leq y$. *Thus,* $|f|(|x|)\leq|f|(|y|)$ *for all* $f\in X^*$. *Hence,* $p(x)[f]\leq p(y)[f]$ *for all* $f\in X^*$, *and so* p *is monotone.*

Recall that $T\in L(X,Y)$, where X and Y are normed spaces, is called a *Dunford-Pettis operator* if $x_n\xrightarrow{\text{w}}0$ in X implies $Tx_n\xrightarrow{\|\cdot\|}0$ in Y. Following proposition gives relation between Dunford-Pettis operator and sequentially p-continuous operator.

Proposition 5. *Let* $(X,\|\cdot\|_X)$ *be a normed lattice and* $(Y,\|\cdot\|_Y)$ *be a normed space. Put* $E:=\mathbb{R}^{X^*}$, *and define* $p:X\to E_+$ *by* $p(x)[f]=|f|(|x|)$ *for* $f\in X^*$. *It follows from Example 2, that* (X,p,E) *is an LNVL.*

(i) If $T\in L(X,Y)$ *is a Dunford-Pettis operator, then* $T:(X,p,E)\to(Y,\|\cdot\|_Y,\mathbb{R})$ *is sequentially* p*-continuous.*

(ii) The converse holds true if the lattice operations of X *are weakly sequentially continuous.*

Proof. (i) Assume that $x_n \xrightarrow{\text{p}} 0$ in X. Then $p(x_n) \xrightarrow{\text{o}} 0$ in E, and hence $p(x_n)[f] \to 0$ or $|f|(|x_n|) \to 0$ for all $f \in X^*$. From which, it follows that $|x_n| \xrightarrow{\text{w}} 0$ and so, by inequality $|f(x_n)| \leq |f|(|x_n|)$, we have $x_n \xrightarrow{\text{w}} 0$ in X. Since T is a Dunford-Pettis operator, then $Tx_n \xrightarrow{\|\cdot\|_Y} 0$.

(ii) Assume that $x_n \xrightarrow{\text{w}} 0$. Since the lattice operations of X are weakly sequentially continuous, then we get $|x_n| \xrightarrow{\text{w}} 0$. So, for all $f \in X^*$, we have $|f|(|x_n|) \to 0$ or $p(x_n)[f] \to 0$. Thus, $x_n \xrightarrow{\text{p}} 0$ and, since T is sequentially p-continuous, we get $Tx_n \xrightarrow{\|\cdot\|_Y} 0$. Therefore, T is a Dunford-Pettis operator. □

Remark 3. *It should be noticed that there are many classes of Banach lattices that satisfy the condition* (ii) *of Proposition 5. For example, the lattice operations of atomic order continuous Banach lattices, AM-spaces, and Banach lattices with atomic topological dual are all weakly sequentially continuous* (*see, respectively, [26, Prop. 2.5.23], [5, Thm. 4.31], and [6, Cor. 2.2]*)

It is known that any positive operator from a Banach lattice into a normed lattice is norm continuous or, equivalently, is norm bounded (see e.g., [5, Thm.4.3]). Similarly, we have the following result.

Theorem 6. *Let (X, p, E) be a sequentially p-complete LNVL such that $(E, \|\cdot\|_E)$ is a Banach lattice, and let $(Y, \|\cdot\|_Y)$ be a normed lattice. If $T : X \to Y$ is a positive operator, then T is p-bounded as an operator from (X, p, E) into $(Y, \|\cdot\|_Y, \mathbb{R})$.*

Proof. Assume that $T : (X, p, E) \to (Y, \|\cdot\|_Y, \mathbb{R})$ is not p-bounded. Then there is a p-bounded subset A of X such that $T(A)$ is not norm bounded in Y. Thus, there is $e \in E_+$ such that $p(a) \leq e$ for all $a \in A$, but $T(A)$ is not norm bounded in Y. Hence, for any $n \in \mathbb{N}$, there is $x_n \in A$ such that $\|Tx_n\|_Y \geq n^3$. Since $|Tx_n| \leq T|x_n|$, we may assume, without loss of generality, that $x_n \geq 0$. Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2} x_n$ in the mixed-norm space $(X, p\text{-}\|\cdot\|_E)$, which is a Banach lattice, due to Lemma 5. Then

$$\sum_{n=1}^{\infty} p\text{-}\|\frac{1}{n^2} x_n\|_E = \sum_{n=1}^{\infty} \frac{1}{n^2} \|p(x_n)\|_E \leq \|e\|_E \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

Since the series $\sum_{n=1}^{\infty} \frac{1}{n^2} x_n$ is absolutely convergent, it converges to some element, say x, i.e. $x = \sum_{n=1}^{\infty} \frac{1}{n^2} x_n \in X$. Clearly, $x \geq \frac{1}{n^2} x_n$ for every $n \in \mathbb{N}$ and, since $T \geq 0$, then $T(x) \geq \frac{1}{n^2}Tx_n$, which implies $\|Tx\|_Y \geq \frac{1}{n^2}\|Tx_n\|_Y \geq n$ for all $n \in \mathbb{N}$; a contradiction. $\square$

Recall that, for an order bounded net x_α, we have $x_\alpha \xrightarrow{o} 0$ iff $x_\alpha \to 0$ coordinatewise (c.f [30, Thm. III.9.2]). If the net is not order bounded then, in general, we have not such characterization. For instance, consider the sequence e_n in c_0. It is not order bounded in c_0 and $e_n \to 0$ coordinatewise.

In next example, we can see that the condition of sequential p-completeness in Theorem 6 can not be removed.

Example 3. *Let $T : (c_{00}, |\cdot|, \ell_\infty) \to (\mathbb{R}, |\cdot|, \mathbb{R})$ be defined by $T(x_n) = \sum_{n=1}^{\infty} nx_n$. Then $T \geq 0$ and the LNVL $(c_{00}, |\cdot|, \ell_\infty)$ is not sequentially p-complete. Indeed, take a sequence $x_n = (1, 1, \cdots, 1, 0, 0, \cdots) = \sum_{i=1}^{n} e_n \in c_{00}$. We show that x_n is p-Cauchy in c_{00}, or we have to show that x_n is order Cauchy in ℓ_∞. Let $y_n = (0, 0, \cdots, 1, 1, 1, \cdots) = \sum_{i=n}^{\infty} e_n$ in ℓ_∞. Then clearly $y_n \downarrow 0$ in ℓ_∞.*
For $n_0 \in \mathbb{N}$, if $n > m \geq n_0$ then

$$p(x_n - x_m) = |x_n - x_m| = \left| \sum_{i=1}^{n} e_i - \sum_{i=1}^{m} e_i \right| = \sum_{i=m+1}^{n} e_i \leq \sum_{i=n_0}^{\infty} e_i = y_{n_0}.$$

Thus, x_n is p-Cauchy in $(c_{00}, |\cdot|, \ell_\infty)$.
The sequence x_n is order bounded in ℓ_∞ and coordinatewise converges to $(1, 1, 1, \cdots)$. But, since $(1, 1, 1, \cdots) \notin c_{00}$, then x_n is not p-convergent in $(c_{00}, |\cdot|, \ell_\infty)$. Thus $(c_{00}, |\cdot|, \ell_\infty)$ is not sequentially p-complete.

Consider the p-bounded sequence e_n in $(c_{00}, |\cdot|, \ell_\infty)$. Since $Te_n = n$ for all $n \in \mathbb{N}$, the sequence Te_n is not norm bounded in $\mathbb{R}$. Hence, T is not p-bounded.

It is well-known that the adjoint of an order bounded operator between two vector lattices is always order bounded and order continuous (see, for example, [5, Thm.1.73]). The following two results deal with a similar situation.

Theorem 7. *Let $(X, \|\cdot\|_X)$ be a normed lattice and Y be a vector lattice. Let $Y_c^\sim$ denote the σ-order continuous dual of Y. If $0 \leq T : (X, \|\cdot\|_X, \mathbb{R}) \to (Y, |\cdot|, Y)$ is sequentially p-continuous and p-bounded, then the operator $T^\sim : (Y_c^\sim, |\cdot|, Y_c^\sim) \to (X^*, \|\cdot\|_{X^*}, \mathbb{R})$ defined by $T^\sim(f) := f \circ T$ is p-continuous.*

Proof. First, we prove that $T^{\sim}(f) \in X^*$ for each $f \in Y_c^{\sim}$. Assume $x_n \xrightarrow{\|\cdot\|} 0$. Since T is sequentially p-continuous, then $Tx_n \xrightarrow{o} 0$ in Y. Since f is σ-order continuous, then $f(Tx_n) \to 0$ or $(f \circ T)(x_n) \to 0$. Hence, we have $f \circ T \in X^*$.

Next, we show that $T^{\sim}$ is p-continuous. Assume $0 \leq f_\alpha \xrightarrow{o} 0$ in $Y_c^{\sim}$. Show that $\|T^{\sim} f_\alpha\|_{X^*} \to 0$ or $\|f_\alpha \circ T\|_{X^*} \to 0$.

Now, $\|f_\alpha \circ T\|_{X^*} = \sup\limits_{x \in B_X} |(f_\alpha \circ T)x|$. Since B_X is p-bounded in $(X, \|\cdot\|_X, \mathbb{R})$ and T is a p-bounded operator, then $T(B_X)$ is order bounded in Y. So there exists $y \in Y_+$ such that $-y \leq Tx \leq y$ for all $x \in B_X$. Hence $-f_\alpha y \leq (f_\alpha \circ T)x \leq f_\alpha y$ for all $x \in B_X$ and for all α. So $\|f_\alpha \circ T\|_{X^*} \subseteq [-f_\alpha y, f_\alpha y]$ for all α. It follows from [30, Thm.VIII.2.3] that $\lim\limits_{\alpha} f_\alpha y = 0$ as $f_\alpha \xrightarrow{o} 0$. Thus, $\lim\limits_{\alpha} \|f_\alpha \circ T\|_{X^*} = 0$. Therefore, $T^{\sim}$ is p-continuous. □

Theorem 8. *Let X be a vector lattice and Y be an AL-space. Assume that $0 \leq T : (X, |\cdot|, X) \to (Y, \|\cdot\|_Y, \mathbb{R})$ is sequentially p-continuous. Define $T^{\sim} : (Y^*, \|\cdot\|_{Y^*}, \mathbb{R}) \to (X^{\sim}, |\cdot|, X^{\sim})$ by $T^{\sim}(f) = f \circ T$. Then $T^{\sim}$ is sequentially p-continuous and p-bounded.*

Proof. Clearly, if $f \in Y^*$ then $f \circ T$ is order bounded as T is positive, and so $T^{\sim}(f) \in X^{\sim}$.

We prove that $T^{\sim}$ is p-bounded. Let $A \subseteq Y^*$ be a p-bounded set in $(Y^*, \|\cdot\|_{Y^*}, \mathbb{R})$, then there is $0 < c < \infty$ such that $\|f\|_{Y^*} \leq c$ for all $f \in A$. Since Y^* is an AM-space with a strong unit, then A is order bounded in Y^*; i.e., there is a $g \in Y_+^*$ such that $-g \leq f \leq g$ for all $f \in A$. That is $-g(y) \leq f(y) \leq g(y)$ for any $y \in Y_+$, which implies $-g(Tx) \leq f(Tx) \leq g(Tx)$ for all $x \in X_+$. Thus, $-g \circ T \leq f \circ T \leq g \circ T$ or $-g \circ T \leq T^{\sim} f \leq g \circ T$ for every $f \in A$. Therefore, $T^{\sim}(A)$ is p-bounded in $(X^{\sim}, |\cdot|, X^{\sim})$.

Next, we show that $T^{\sim}$ is sequentially p-continuous. Assume $0 \leq f_n \xrightarrow{\|\cdot\|_{Y^*}} 0$ in $(Y^*, \|\cdot\|_{Y^*})$. Since Y^* is an AM-space with a strong unit, say e, then $f_n \xrightarrow{\|\cdot\|_e} 0$. It follows from [23, Thm. 62.4] that f_n e-converges to zero in Y^*. Thus, there is a sequence $\varepsilon_k \downarrow 0$ in $\mathbb{R}$ such that, for all $k \in \mathbb{N}$, there is $n_k \in \mathbb{N}$ satisfying $f_n \leq \varepsilon_k e$ for all $n \geq n_k$. In particular, $f_n(Tx) \leq \varepsilon_k e(Tx)$ for all $x \in X_+$ and for all $n \geq n_k$.

From which it follows that $f_n \circ T$ e-converges to zero in $X^\sim$, and so $f_n \circ T \xrightarrow{o} 0$ in $X^\sim$. Hence, $T^\sim(f_n) \xrightarrow{o} 0$ in $X^\sim$, and $T^\sim$ is sequentially p-continuous. □

3.2 p-Compact Operators

Given normed spaces X and Y. Recall that an operator T from X to Y is said to be compact if $T(B_X)$ is relatively compact in Y. Equivalently, T is compact iff, for any norm bounded sequence x_n in X, there is a subsequence x_{n_k} such that the sequence Tx_{n_k} is convergent in Y. Motivated by this, we introduce the notion of p-compact and sequentially p-compact operators between lattice-normed spaces. notions.

Definition 20. *Let X, Y be two LNSs and $T \in L(X,Y)$. Then:*

(1) T is called p-compact *if, for any p-bounded net x_α in X, there is a subnet x_{α_β} such that $Tx_{\alpha_\beta} \xrightarrow{p} y$ in Y for some $y \in Y$.*

(2) T is called sequentially p-compact *if, for any p-bounded sequence x_n in X, there is a subsequence x_{n_k} such that $Tx_{n_k} \xrightarrow{p} y$ in Y for some $y \in Y$.*

Whew one use the relatively uniformly p-convergence, then the following definitions also can be defined.

Definition 21. *Let X, Y be two LNSs and $T \in L(X,Y)$. Then:*

(1) T is called rp-compact *if, for any p-bounded net x_α in X, there is a subnet x_{α_β} such that $Tx_{\alpha_\beta} \xrightarrow{rp} y$ in Y for some $y \in Y$.*

(2) T is called sequentially rp-compact *if, for any p-bounded sequence x_n in X, there is a subsequence x_{n_k} such that $Tx_{n_k} \xrightarrow{rp} y$ in Y for some $y \in Y$.*

The next remark have relations between rp-compact operators and p-compact operators.

Remark 4.

(i) Every (sequentially) rp-compact is (sequentially) p-compact since every rp-convergence implies order convergence.

(ii) The converse of (i) in the sequential case does not need to be true. To see this, consider the identity operator $I : (\ell_\infty, |\cdot|, \ell_\infty) \to (\ell_\infty, |\cdot|, \ell_\infty)$. Then, by the standard diagonal argument, I is sequentially p-compact. Indeed, let $x^n = (x_1^n, x_2^n, x_3^n, \cdots) = (x_j^n)_{j\in\mathbb{N}}$ be a p-bounded sequence in ℓ_∞. So, there is a sequence $(y_j)_{j\in\mathbb{N}}$ in ℓ_∞ such that, for each $n \in \mathbb{N}$, $|x_j^n| \leq y_j$ for all $j \in \mathbb{N}$. Since $(x_1^n)_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}$, then there is a subsequence $x_1^{n_k}$ such that $x_1^{n_k} \to x_1$ in $\mathbb{R}$, as $n \to \infty$.

Now, the sequence $(x_2^{n_k})_{k\in\mathbb{N}}$ is bounded (by y_2) in $\mathbb{R}$, and so there is a subsequence $x_2^{n_{k_j}}$ such that $x_2^{n_{k_j}} \to x_2$, as $j \to \infty$ in $\mathbb{R}$. Going in this way, we can find a common subsequence, say x^{n_k}, of x^n such that $x^{n_k} \to (x_1, x_2, x_3, \cdots) \in \ell_\infty$.

On the other hand, I is not sequentially pr-compact. Indeed, consider the sequence x_n in ℓ_∞, given by $x_n = \sum\limits_{i=n}^{\infty} e_i$, where e_i's are the standard unit vectors in ℓ_∞. It is clear that, for all $n \neq m$, $|x_n - x_m| \not\leq \frac{1}{2}\mathbb{1}$, where $\mathbb{1} = (1, 1, 1, \cdots) \in \ell_\infty$. Let x_{n_k} be a subsequence of x_n. If x_{n_k} is pr-Cauchy, then there exists $0 \leq u \in \ell_\infty$ such that, for all $\varepsilon > 0$, there is $k_\varepsilon \in \mathbb{N}$ satisfying $|x_{n_k} - x_{n_j}| \leq \varepsilon u$ for all $j, k \geq k_\varepsilon$. Since $\mathbb{1}$ is a strong unit in ℓ_∞, then there is $\lambda > 0$ such that $u \leq \lambda\mathbb{1}$. Take $\varepsilon_0 = \frac{1}{2\lambda} > 0$, then there is $k_0 \in \mathbb{N}$ such that, for all $j > k \geq k_0$, $|x_{n_k} - x_{n_j}| \leq \varepsilon_0 u \leq \varepsilon_0\lambda\mathbb{1} = \frac{1}{2}\mathbb{1}$, a contradiction. Therefore, no subsequence of x_n can be pr-convergent.

(iii) We do not know whether or not every rp-compact operator is sequentially rp-compact, and whether or not the vice versa is true.

In the following example, we show that p-compact operators generalize many well-known classes of operators.

Example 4.

(i) Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces. Then $T : (X, \|\cdot\|_X, \mathbb{R}) \to (Y, \|\cdot\|_Y, \mathbb{R})$ is (sequentially) p-compact iff $T : X \to Y$ is compact.

(ii) Let X be a vector lattice and Y be a normed space. An operator $T : X \to Y$ is AM-compact operator iff $T : (X, |\cdot|, X) \to (Y, \|\cdot\|_Y, \mathbb{R})$ is p-compact.

(iii) Let X be a normed space and $(Y, \|\cdot\|_Y)$ be a normed lattice. Let $E := \mathbb{R}^{Y^*}$, and consider the LNVL (Y, p, E), where $p(y)[f] = |f|(|y|)$ for all $f \in Y^*$. Then $T \in L(X, Y)$ is weakly compact iff $T : (X, \|\cdot\|_X, \mathbb{R}) \to (Y, p, E)$ is sequentially p-compact.

(iv) Let X be a vector lattice and $(Y, \|\cdot\|_Y)$ be a normed lattice. Let $E := \mathbb{R}^{Y^*}$, and consider the LNVL (Y, p, E), where $p(y)[f] = |f|(|y|)$ for all $f \in Y^*$. Then $T \in L(X, Y)$ is order weakly compact iff $T : (X, |\cdot|, X) \to (Y, p, E)$ is sequentially p-compact.

Remark 5. *It is known that any compact operator is norm continuous, but in general we may have a p-compact operator which is not p-continuous. Indeed, consider the following example taken from [24]. Denote by $\mathcal{B}$ the Boolean algebra of the Borel subsets of $[0,1]$ equal up to measure null sets. Let $\mathcal{U}$ be any ultrafilter on $\mathcal{B}$. It can be shown that the linear operator $\varphi_{\mathcal{U}} : L_\infty[0,1] \to \mathbb{R}$ defined by*

$$\varphi_{\mathcal{U}}(f) := \lim_{A \in \mathcal{U}} \frac{1}{\mu(A)} \int_A f d\mu$$

is AM-compact, but it is not order-to-norm continuous. Indeed, consider a sequence A_n of members of $\mathcal{U}$ with $\mu(A_n) \to 0$. Then $\chi_{A_n} \downarrow 0$, however $f(\chi_{A_n}) = 1$ for each $n \in \mathbb{N}$; see [24, Ex.4.2].
That is, the operator $\varphi_{\mathcal{U}} : (L_\infty[0,1], |\cdot|, L_\infty[0,1]) \to (\mathbb{R}, |\cdot|, \mathbb{R})$ is p-compact, but it is not p-continuous.

In following example, we can see that a sequentially p-compact operator does not need to be p-bounded.

Example 5. *Let's consider again Lozanovsky's example. Take $T : L_1[0,1] \to c_0$ defined by*

$$T(f) = \Big(\int_0^1 f(x) sinx \, dx, \int_0^1 f(x) sin2x \, dx, \cdots \Big).$$

Then, it can be shown that T is not order bounded $\big($cf. [5, Exer.10, p.289]$\big)$. Indeed, if $\mathbb{1}$ denotes the constant function one on $[0,1]$, we can show that $T([-1,1])$ is not an order bounded subset of c_0. To see this, assume, by way of contradiction, that there is some $0 \leq \alpha = (\alpha_1, \alpha_2, \ldots) \in c_0$ such that $|Tu| \leq \alpha$ holds for all $u \in [-1,1]$. Let $\delta = \frac{1}{\pi} - \frac{1}{4} > 0$. Now, for each n, let s_n denote the sign function of $\sin(nx)$. That

is $s_n(x) = 1$ if $\sin(nx) \geq 0$, and $s_n(x) = -1$ if $\sin(nx) < 0$. Clearly, each s_n is measurable and $|s_n(x)| = 1$ holds for all x, and so $s_n \in [-1, 1]$.
Fix any $n \in \mathbb{N}$ with $n \geq 4$ and let k be the unique natural number satisfying $k\pi < n < (k+1)\pi$. In particular, note that $\frac{k}{n} > \frac{1}{\pi} - \frac{1}{n} \geq \delta > 0$. Hence, we have

$$\begin{aligned} \alpha_n \geq \int_0^1 s_n(x)\sin(nx)dx &= \int_0^1 |\sin(nx)|dx \\ &= \frac{1}{n}\int_0^n |\sin(t)|dt \\ &\geq \frac{1}{n}\int_0^{k\pi} |\sin(t)|dt = \frac{1}{n}(2k) > 2\delta, \end{aligned}$$

and so $\alpha_n \not\to 0$, which is impossible. This contradiction shows that T is not order bounded; that is can be seen in [4, Exer. 30.(b)].

So T is not p-bounded as an operator from the LNS $\big(L_1[0,1], |\cdot|, L_1[0,1]\big)$ into the LNS $\big(c_0, |\cdot|, c_0\big)$.

On the other hand, let f_n be a p-bounded sequence in $\big(L_1[0,1], |\cdot|, L_1[0,1]\big)$, then f_n is order bounded in $L_1[0,1]$. By the standard diagonal argument, there are a subsequence f_{n_k} and a sequence $a = (a_k)_{k\in\mathbb{N}} \in c_0$ such that $Tf_{n_k} \xrightarrow{o} a$ in c_0. Therefore, $T : \big(L_1[0,1], |\cdot|, L_1[0,1]\big) \to (c_0, |\cdot|, c_0)$ is sequentially p-compact.

The following lemma gives the linearity of p-compact operators.

Lemma 8. *Let (X, p, E) and (Y, m, F) be two LNSs. If $S, T : (X, p, E) \to (Y, m, F)$ are (sequentially) p-compact operators, then $S + T$ and λT are also (sequentially) p-compact for every scalar λ.*

Proof. Let x_α be a p-bounded net in (X, p, E). Since T is p-compact, then there is a subnet x_{α_β} of x_α such that $m(Tx_{\alpha_\beta} - y) \xrightarrow{o} 0$ in F for some $y \in Y$. As x_{α_β} is also p-bounded in X and S is p-compact, then there is a subnet $x_{\alpha_{\beta_\gamma}}$ of x_{α_β} such that $m(Sx_{\alpha_{\beta_\gamma}} - z) \xrightarrow{o} 0$ in F for some $z \in Y$, and so $m(Tx_{\alpha_{\beta_\gamma}} - y) \xrightarrow{o} 0$ in F. Thus, $m\big((S+T)(x_{\alpha_{\beta_\gamma}}) - y - z\big) \xrightarrow{o} 0$. Hence, $S + T$ is p-compact. □

Let (X, E) be a decomposable LNS and (Y, F) be an LNS such that F is order complete. Then, by [20, 4.1.2, p.142], each dominated operator $T : X \to Y$ has the exact dominant $[\![T]\!]$. Therefore, the triple $\big(M(X, Y), p, L^{\sim}(E, F)\big)$ is an LNS, where $p : M(X,Y) \to L_+^\sim(E,F)$ is defined by $p(T) = |T|$ (see, for example, [20, 4.2.1, p.150]). Thus, if T_α is a net in $M(X,Y)$, then $T_\alpha \xrightarrow{\text{p}} T$ in $M(X,Y)$, whenever $|T_\alpha - T| \xrightarrow{\text{o}} 0$ in $L^\sim(E,F)$.

In the next theorem, we show that if a net of sequentially p-compact dominated operators p-convergent to a dominated operator, then it is also sequentially p-compact.

Theorem 9. *Let (X,p,E) be a decomposable LNS and (Y,q,F) be a sequentially p-complete LNS such that F is order complete. If T_m is a sequence in $M(X,Y)$ and each T_m is sequentially p-compact with $T_m \xrightarrow{\text{p}} T$ in $M(X,Y)$, then T is sequentially p-compact.*

Proof. Let x_n be a p-bounded sequence in X, then there is $e \in E_+$ such that $p(x_n) \le e$ for all $n \in \mathbb{N}$. By the standard diagonal argument, there exists a subsequence x_{n_k} such that, for any $m \in \mathbb{N}$, $T_m x_{n_k} \xrightarrow{\text{p}} y_m$ for some $y_m \in Y$.

We show that y_m is a p-Cauchy sequence in Y.

$$\begin{aligned} q(y_m - y_j) &= q(y_m - T_m x_{n_k} + T_m x_{n_k} - T_j x_{n_k} + T_j x_{n_k} - y_j) \\ &\le q(y_m - T_m x_{n_k}) + q(T_m x_{n_k} - T_j x_{n_k}) + q(T_j x_{n_k} - y_j). \end{aligned}$$

The first and the third terms in the last inequality both order converge to zero as $m \to \infty$ and $j \to \infty$, respectively. Since $T_m \in M(X,Y)$ for all $m \in \mathbb{N}$,

$$q(T_m x_{n_k} - T_j x_{n_k}) \le |T_m - T_j|(p(x_{n_k})) \le |T_m - T_j|(e).$$

Since $T_m \xrightarrow{\text{p}} T$ in $M(X,Y)$, by [30, Thm.VIII.2.3], that $|T_m - T_j|(e) \xrightarrow{\text{o}} 0$ in F as $m, j \to \infty$. Thus, $q(y_m - y_j) \xrightarrow{\text{o}} 0$ in F, as $m, j \to \infty$. Therefore, y_m is p-Cauchy. Since Y is sequentially p-complete, there is $y \in Y$ such that $q(y_m - y) \xrightarrow{\text{o}} 0$ in F, as $m \to \infty$. Hence,

$$\begin{aligned} q(Tx_{n_k} - y) &\le q(Tx_{n_k} - T_m x_{n_k}) + q(T_m x_{n_k} - y_m) + q(y_m - y) \\ &\le |T_m - T|(p(x_{n_k})) + q(T_m x_{n_k} - y_m) + q(y_m - y) \\ &\le |T_m - T|(e) + q(T_m x_{n_k} - y_m) + q(y_m - y). \end{aligned}$$

Fix $m \in \mathbb{N}$ and let $k \to \infty$, then

$$\limsup_{k\to\infty} q(Tx_{n_k} - y) \le |T_m - T|(e) + q(y_m - y).$$

But $m \in \mathbb{N}$ is arbitrary, so $\limsup\limits_{k\to\infty} q(Tx_{n_k} - y) = 0$. Hence, $q(Tx_{n_k} - y) \xrightarrow{o} 0$. Therefore, T is sequentially p-compact. □

The following proposition say that the union of p-compact operator with left hand side with p-continuous operator is also p-compact. Similarly, the union of p-compact operator with right hand side with p-bounded operator is also p-compact.

Proposition 6. *Let (X, p, E) be an LNS and $R, T, S \in L(X)$.*

(i) If T is (sequentially) p-compact and S is (sequentially) p-continuous, then $S \circ T$ is (sequentially) p-compact.

(ii) If T is (sequentially) p-compact and R is p-bounded, then $T \circ R$ is (sequentially) p-compact.

Proof. (i) Assume x_α to be a p-bounded net in X. Since T is p-compact, there are a subnet x_{α_β} and $x \in X$ such that $p(Tx_{\alpha_\beta} - x) \xrightarrow{o} 0$. It follows from the p-continuity of S, that $p\big(S(Tx_{\alpha_\beta}) - Sx\big) \xrightarrow{o} 0$. Therefore, $S \circ T$ is p-compact.

(ii) Assume x_α to be a p-bounded net in X. Since R is p-bounded, Rx_α is p-bounded. Now, the p-compactness of T implies that there are a subnet x_{α_β} and $z \in X$ such that $p\big(T(Rx_{\alpha_\beta}) - z\big) \xrightarrow{o} 0$. Therefore, $T \circ R$ is p-compact.
The sequential case is analogous. □

In following two propositions, we have relations between compact operator on acting mixed norm and sequentially p-compact operators.

Proposition 7. *Let (X, p, E) be an LNS, where $(E, \|\cdot\|_E)$ is a normed lattice, and let (Y, m, F) be an LNS, where $(F, \|\cdot\|_F)$ is a Banach lattice. If $T : (X, p\text{-}\|\cdot\|_E) \to (Y, m\text{-}\|\cdot\|_F)$ is compact, then $T : (X, p, E) \to (Y, m, F)$ is sequentially p-compact.*

Proof. Let x_n be a p-bounded sequence in (X, p, E). Then there is $e \in E$ such that $p(x_n) \leq e$ for all $n \in \mathbb{N}$. So $\|p(x_n)\|_E \leq \|e\|_E < \infty$, since $(E, \|\cdot\|_E)$ is a normed lattice. Hence, x_n is norm bounded in $(X, p\text{-}\|\cdot\|_E)$. Since T is compact, there are a subsequence x_{n_k} and $y \in Y$ such that $m\text{-}\|Tx_{n_k} - y\|_F \to 0$ or $\|m(Tx_{n_k} - y)\|_F \to 0$. Since $(F, \|\cdot\|_F)$ is a Banach lattice, then, by [30, Thm.VII.2.1], there is a further subsequence $x_{n_{k_j}}$ such that $m(Tx_{n_{k_j}} - y) \xrightarrow{o} 0$. Therefore, $T : (X, p, E) \to (Y, m, F)$ is sequentially p-compact. $\square$

Proposition 8. *Let (X, p, E) be an LNS, where $(E, \|\cdot\|_E)$ is an AM-space with a strong unit. Let (Y, m, F) be an LNS, where $(F, \|\cdot\|_F)$ is a σ-order continuous normed lattice. If $T : (X, p, E) \to (Y, m, F)$ is sequentially p-compact, then $T : (X, p\text{-}\|\cdot\|_E) \to (Y, m\text{-}\|\cdot\|_F)$ is compact.*

Proof. Let x_n be a normed bounded sequence in $(X, p\text{-}\|\cdot\|_E)$. That is $p\text{-}\|x_n\|_E = \|p(x_n)\|_E \leq k < \infty$ for all $n \in \mathbb{N}$. Since $(E, \|\cdot\|_E)$ is an AM-space with a strong unit, then $p(x_n)$ is order bounded in E. Thus, x_n is a p-bounded sequence in (X, p, E). Since T is sequentially p-compact, there are a subsequence x_{n_k} and $y \in Y$ such that $m(Tx_{n_k} - y) \xrightarrow{o} 0$ in F. Since $(F, \|\cdot\|_F)$ is σ-order continuous, then $\|m(Tx_{n_k} - y)\|_F \to 0$ or $m\text{-}\|Tx_{n_k} - y\|_F \to 0$. Thus, the operator $T : (X, p\text{-}\|\cdot\|_E) \to (Y, m\text{-}\|\cdot\|_F)$ is compact. $\square$

The following result could be known but, since we do not have a reference for it, we include a proof for the sake of completeness.

Lemma 9. *Let X be an atomic vector lattice, and let x_α be a net in X. Then $x_\alpha \xrightarrow{uo} 0$ iff, for any atom $a \in X$, $p_a(x_\alpha) \xrightarrow{o} 0$ "coordinatewise".*

Proof. $(\Rightarrow)$ Assume $x_\alpha \xrightarrow{uo} 0$ and, without loss generality, we may assume $x_\alpha \geq 0$ for all α. Then, $x_\alpha \wedge u \xrightarrow{o} 0$ for all $u \in X_+$.

Let $a \in X_+$ be an atom, then $x_\alpha \wedge a \xrightarrow{o} 0$. Since p_a, which is the band projection onto $span\{a\}$, is order continuous, then $p_a(x_\alpha \wedge a) \xrightarrow{o} 0$. But, p_a is a lattice homomorphism, and so $p_a x_\alpha \wedge p_a a = p_a(x_\alpha \wedge a) \xrightarrow{o} 0$, but $p_a x_\alpha \in span\{a\}$ for all a i.e. $p_a x_\alpha = \lambda_\alpha a$, where $\lambda_\alpha \in \mathbb{R}$. Thus, $p_a x_\alpha \wedge a \xrightarrow{o} 0$ and so $\lambda_\alpha \wedge a \xrightarrow{o} 0$ or $\min\{\lambda_\alpha, 1\} a \xrightarrow{o} 0$. So, $\lambda_\alpha \to 0$ in $\mathbb{R}$. Hence, $p_a x_\alpha = \lambda_\alpha a \xrightarrow{o} 0$.

$(\Leftarrow)$ Conversely, suppose that for each atom $a \in X_+$, $p_a x_\alpha \xrightarrow{o} 0$. We show $x_\alpha \wedge u \xrightarrow{o} 0$. Again, assume, without loss of generality, that $x_\alpha \geq 0$ for all α. Let $u \in X_+$. Let Ω be the collection of all atoms in X. Let $F(\Omega)$ be the collection of all finite subset of

X. Let $\Delta = F(\Omega) \times \mathbb{N}$. For each $\delta = (F, n) \in \Delta$, put $y_\delta = \frac{1}{n} \sum\limits_{a \in F} p_a u + \sum\limits_{a \in \Omega \setminus F} P_a u$, where P_a denotes the band projection onto $span\{a\}$.

We show $y_\delta \downarrow 0$. Let $\delta_1, \delta_2 \in \Delta$ such that $\delta_1 \leq \delta_2$ or $(F_1, n_1) \leq (F_2, n_2)$, which is equivalent to $F_1 \subseteq F_2$ and $n_1 \leq n_2$. Then

$$n_1 \leq n_2 \Leftrightarrow \frac{1}{n_1} \geq \frac{1}{n_2} \Rightarrow \frac{1}{n_1} \sum_{a \in F_1} p_a u \geq \frac{1}{n_2} \sum_{a \in F_2} p_a u. \tag{3.1}$$

Note that:

$$\frac{1}{n_2} \sum_{a \in F_2} p_a u = \frac{1}{n_2} \sum_{a \in F_1} p_a u + \frac{1}{n_2} \sum_{a \in F_2 \setminus F_1} p_a u. \tag{3.2}$$

Note: Note that:

$$\sum_{a \in \Omega \setminus F_1} p_a u = \sum_{a \in F_2 \setminus F_1} p_a u + \sum_{a \in \Omega \setminus F_2} p_a u. \tag{3.3}$$

Also we have:

$$\sum_{a \in F_2 \setminus F_1} p_a u \geq \frac{1}{n_2} \sum_{a \in F_2 \setminus F_1} p_a u. \tag{3.4}$$

It follows from (3.3) and (3.4) that

$$\sum_{a \in \Omega \setminus F_1} p_a u \geq \sum_{a \in F_2 \setminus F_1} p_a u + \sum_{a \in \Omega \setminus F_2} p_a u. \tag{3.5}$$

By (3.1) and (3.5), we get:

$$\frac{1}{n_1} \sum_{a \in F_1} p_a u + \sum_{a \in \Omega \setminus F_1} p_a u \geq \frac{1}{n_2} \sum_{a \in F_1} p_a u + \frac{1}{n_2} \sum_{a \in F_2 \setminus F_1} p_a u + \sum_{a \in \Omega \setminus F_2} p_a u.$$

Thus, by (3.2), we have:

$$\frac{1}{n_1} \sum_{a \in F_1} p_a u + \sum_{a \in \Omega \setminus F_1} p_a u \geq \frac{1}{n_2} \sum_{a \in F_2} p_a u + \sum_{a \in \Omega \setminus F_2} p_a u.$$

Hence, $y_{\delta_1} \geq y_{\delta_2}$. So, $y_\delta \downarrow 0$.

Next, we show that $\inf\limits_{\delta \in \Delta} y_\delta = 0$. Let $0 \leq x \leq y_\delta$ for all $\delta \in \Delta$. Let $a \in \Omega$ be arbitrary atom, let $F = \{a\}$, and let $n \in \mathbb{N}$. Then, $0 \leq x \leq \frac{1}{n} p_a u + \sum\limits_{b \in \Omega \setminus \{a\}} p_b u$. We apply p_a for both sides of the above inequality and we get

$$\begin{aligned} 0 \leq p_a x \leq \frac{1}{n} p_a u + 0 \quad &\Rightarrow \quad 0 \leq p_a x \leq \frac{1}{n} p_a u + 0 \qquad (\forall n \in \mathbb{N}) \\ &\Rightarrow \quad 0 \leq p_a x \leq 0 \\ &\Rightarrow \quad p_a x = 0. \end{aligned}$$

But $a \in \Omega$ was arbitrary, so $p_a x = 0$ for all $a \in \Omega$. Since X is atomic, we get that $x = 0$. Therefore, $y_\delta \downarrow 0$.

Finally, we show $x_\alpha \wedge u \xrightarrow{\text{o}} 0$. Let $F \in F(\Omega)$ and $n \in \mathbb{N}$.

$$\begin{aligned} x_\alpha \wedge u = \sum_{a\in\Omega} p_a(x_\alpha \wedge u) &= \sum_{a\in F} p_a(x_\alpha \wedge u) + \sum_{a\in\Omega\setminus F} p_a(x_\alpha \wedge u) \\ &\leq \sum_{a\in F} p_a(x_\alpha) + \sum_{a\in\Omega\setminus F} p_a(u) \\ &\leq \sum_{a\in F} p_a(x_\alpha) + \sum_{a\in F} p_a(x_\alpha) + \frac{1}{n}\sum_{a\in F} p_a(u) + \sum_{a\in\Omega\setminus F} p_a(u) \\ &= \sum_{a\in F} p_a(x_\alpha) + y_\delta \xrightarrow{\text{o}} 0. \end{aligned}$$

Therefore, $x_\alpha \wedge u \xrightarrow{\text{o}} 0$ or $x_\alpha \xrightarrow{\text{uo}} 0$. □

The next remark is very useful and we use it many times.

Remark 6. *If X is an atomic KB-space then every order bounded net has an order convergent subnet. Indeed, let x_α be an order bounded net in X. Then clearly x_α is norm bounded and so, by [18, Thm.7.5], there is a subnet x_{α_β} such that $x_{\alpha_\beta} \xrightarrow{\text{un}} x$ for some $x \in X$. But, in atomic order continuous Banach lattices, un-convergence coincides with pointwise convergence (see [18, Cor. 4.14]). Therefore, by Lemma 9, $x_{\alpha_\beta} \xrightarrow{\text{uo}} x$. Thus, $x_{\alpha_\beta} \xrightarrow{\text{o}} x$ since x_α is order bounded.*

The following proposition give information about when a order bounded operator is p-compact.

Proposition 9. *Let X be a vector lattice and (Y, m, F) be an op-continuous LNVL such that Y is atomic KB-space. If $T \in L^\sim(X, Y)$ then $T : (X, |\cdot|, X) \to (Y, m, F)$ is p-compact.*

Proof. Let x_α be a p-bounded net in $(X, |\cdot|, X)$ then x_α is order bounded in X. Since T is order bounded, then Tx_α is order bounded in Y. Since Y is an atomic KB-space, then, by Remark 6, there are a subnet x_{α_β} and $y \in Y$ such that $Tx_{\alpha_\beta} \xrightarrow{\text{o}} y$. Since (Y, m, F) is op-continuous, $m(Tx_{\alpha_\beta} - y) \xrightarrow{\text{o}} 0$. Thus, T is p-compact. □

In the next proposition, under some conditions, we can see that p-bounded operator is p-compact.

Proposition 10. *Let (X, p, E) and $(Y, |\cdot|, Y)$ be two LNVLs such that Y is an atomic KB-space. If $T : (X, p, E) \to (Y, |\cdot|, Y)$ is p-bounded then T is p-compact.*

Proof. Let x_α be a p-bounded net in X. Since T is p-bounded, Tx_α is order bounded in Y. Since $(Y, \|\cdot\|_Y)$ is atomic KB-space, then, by Remark 6, there is a subnet x_{α_β} such that $Tx_{\alpha_\beta} \xrightarrow{o} y$ for some $y \in Y$. Therefore, T is p-compact. □

It is known that every dominated operator is p-bounded. Thus, in Proposition 10, if we take T as a dominated operator instead of p-boundednees of T, that is also holds.

Recall that, for given sequence of measurable functionals f_n we have $f_n \to 0$ a.e. iff $f_n \xrightarrow{uo} 0$. If the given sequence f_n is order bounded, then $f_n \to 0$ a.e. iff $f_n \xrightarrow{o} 0$. We use this characterization in the following remark.

Remark 7.

(i) We can not omit the atomicity in Propositions 9 and 10; consider the identity operator $I : (L_1[0,1], |\cdot|, L_1[0,1]) \to (L_1[0,1], |\cdot|, L_1[0,1])$. The sequence of Rademacher functions, that is the function $r_n : [0,1] \to \mathbb{R}$ defined by $r_n(t) = sgn \sin(2^n \pi t)$ for $t \in [0,1]$, is order bounded by $\mathbb{1}$ and has no order convergent subsequence. Indeed, let r_{n_k} be a subsequence of r_n such that $r_{n_k} \to f$. Then $r_{n_k}(x) \to f(x)$ a.e. for each $x \in [0,1]$. But, for each $x \in [0,1]$, there are infinitely many n's such that $r_{n_k}(x) = 1$ and infinitely many n's such that $r_{n_k}(x) = -1$. So, I is not p-compact.

(ii) The identity operator $I : (\ell_1, |\cdot|, \ell_1) \to (\ell_1, |\cdot|, \ell_1)$ satisfies the conditions of Proposition 9, where $X = \ell_1$, $(Y, m, F) = (\ell_1, |\cdot|, \ell_1)$, $Y = \ell_1$, which is atomic KB-space, because ℓ_1 has no copy of c_0 (note that $x = (\frac{1}{n}) \in c_0$ but $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$, so $(\frac{1}{n}) \notin \ell_1$); and also $(\ell_1, |\cdot|, \ell_1)$ is op-continuous, so I is p-compact. This shows that the identity operator on an infinite dimensional space can be p-compact.

(iii) We do not know whether or not the identity operator on the LNS $I : (L_\infty[0,1], |\cdot|, L_\infty[0,1]) \to (L_\infty[0,1], |\cdot|, L_\infty[0,1])$ could be p-compact or sequentially p-compact.

Recall that, for vector lattices E and F, if $f : E \to \mathbb{R}$ is order bounded and $u \in F$, then the symbol $f \otimes u$ stands for *a rank one operator* (i.e. $f \otimes u \in L^{\sim}(E, F)$):

$$(f \otimes u)(x) = f(x)u \qquad (x \in E).$$

Any operator $T : E \to F$ of the form $T = \sum_{i=1}^{n} f_i \otimes u_i$, where $f_i \in E^{\sim}$ and $u_i \in F$ for any $i = 1, 2, \cdots, n$, is called a *finite rank operator*; see [5, p.64]. It is known that a finite rank operator is compact. Similarly, we have the following result.

Proposition 11. *Let (X, p, E) and (Y, m, F) be LNSs. Let $T : (X, p, E) \to (Y, m, F)$ be a p-bounded finite rank operator. Then T is p-compact.*

Proof. Without lost of generality, we may suppose that T is given by $Tx = f(x)y_0$ for some p-bounded functional $f : (X, p, E) \to (\mathbb{R}, |\cdot|, \mathbb{R})$ and $y_0 \in Y$.

Let x_α be a p-bounded net in X, then $f(x_\alpha)$ is bounded in $\mathbb{R}$. So there is a subnet x_{α_β} such that $f(x_{\alpha_\beta}) \to \lambda$ for some $\lambda \in \mathbb{R}$, since every bounded net has convergent subnet in $\mathbb{R}$. Now, $m(Tx_{\alpha_\beta} - \lambda y_0) = m\big((fx_{\alpha_\beta} - \lambda)y_0\big) = |f(x_{\alpha_\beta}) - \lambda| m(y_0) \xrightarrow{o} 0$ in F. Thus, T is p-compact. □

In the next example, we can see that the p-boundedness of T in Proposition 11 can not be removed.

Example 6. *Let (X, p, E) be an LNS and $f : (X, p, E) \to (\mathbb{R}, |\cdot|, \mathbb{R})$ be a linear functional which is not p-bounded. Then there is a p-bounded sequence x_n such that $|f(x_n)| \geq n$ for all $n \in \mathbb{N}$. Therefore, any rank one operator $T : (X, p, E) \to (Y, m, F)$ given by the rule $Tx = f(x)y_0$, where $0 \neq y_0 \in Y$, is not p-compact.*

Recall that:

(1) A subset A of a normed lattice $(X, \|\cdot\|)$ is called *almost order bounded* if, for any $\varepsilon > 0$, there exists an order interval $[x, y]$ in X such that $A \subseteq [x, y] + \varepsilon B_X$. In this case, there exists, of course, a symmetric order interval $[-u, u]$ such that $A \subseteq [-u, u] + \varepsilon B_X$, where $u \geq 0$. It can be seen that subset A is almost order bounded if, for any $\varepsilon > 0$, there is $u_\varepsilon \in X_+$ such that

$$\|(|x| - u_\varepsilon)^+\| = \||x| - u_\varepsilon \wedge |x|\| \leq \varepsilon \quad (\forall x \in A).$$

For a given Banach lattice X, by ([32, Thm. 111.2]), we have the following:

$$order\ boundedness \Rightarrow almost\ order\ boundedness \Rightarrow norm\ boundedness.$$

(2) Given an LNVL (X, p, E). A subset A of X is said to be p-*almost order bounded* if, for any $w \in E_+$, there is $x_w \in X$ such that

$$p\big((|x| - x_w)^+\big) = p(|x| - x_w \wedge |x|) \le w \quad (\forall x \in A),$$

see [7, Def.7]. If $(X, \|\cdot\|)$ is a normed lattice, then a subset A of X is p-almost order bounded in $(X, \|\cdot\|, \mathbb{R})$ iff A is almost order bounded in X. On the other hand, if X is a vector lattice, a subset in $(X, |\cdot|, X)$ is p-almost order bounded iff it is order bounded in X.

(3) An operator $T \in L(X, Y)$, where X is a normed space and Y is a normed lattice, is called *semicompact* if $T(B_X)$ is almost order bounded in Y.

Definition 22. *Let (X, E) be an LNS and (Y, F) be an LNVL. A linear operator $T : X \to Y$ is called* p-semicompact *if, for any p-bounded set A in X, we have that $T(A)$ is p-almost order bounded in Y.*

The following remark has relation between p-semicompact operator, and p-bounded and order bounded operators.

Remark 8.

(i) Any p-semicompact operator is a p-bounded operator.

(ii) Take $T \in L(X, Y)$, where X is a normed space and Y is a normed lattice. Then T is semicompact iff $T : (X, \|\cdot\|_X, \mathbb{R}) \to (Y, \|\cdot\|_Y, \mathbb{R})$ is p-semicompact.

(iii) For vector lattices X and Y, we have $T \in L^{\sim}(X, Y)$ iff $T : (X, |\cdot|, X) \to (Y, |\cdot|, Y)$ is p-semicompact.

In the next proposition, we have a relation between p-semicompact operator and semicompact operator.

Proposition 12. *Let (X, p, E) be an LNS with an AM-space $(E, \|\cdot\|_E)$ possessing a strong unit, and (Y, m, F) be an LNVL with a normed lattice $(F, \|\cdot\|_F)$. If $T : (X, p, E) \to (Y, m, F)$ is p-semicompact, then $T : (X, p\text{-}\|\cdot\|_E) \to (Y, m\text{-}\|\cdot\|_F)$ is semicompact.*

Proof. Consider the closed unit ball B_X of $(X, p\text{-}\|\cdot\|_E)$. Then $p\text{-}\|x\|_E \leq 1$ or $\|p(x)\|_E \leq 1$ for all $x \in B_X$. We show that $T(B_X)$ is almost order bounded in $(Y, m\text{-}\|\cdot\|_F)$. Given $\varepsilon > 0$. Let a $w \in F_+$ such that

$$\|w\|_F = \varepsilon. \tag{3.6}$$

Since $\|p(x)\|_E \leq 1$ for all $x \in B_X$ and $(E, \|\cdot\|_E)$ is an AM-space with a strong unit, there exists $e \in E_+$ such that $p(x) \leq e$ for all $x \in B_X$. Thus, B_X is p-bounded in (X, p, E) and, since T is p-semicompact, we get $T(B_X)$ to be p-almost order bounded in (Y, m, F). So, for $w \in F_+$ in (3.6), there is $y_w \in Y_+$ such that $m\big((|Tx| - y_w)^+\big) \leq w$ for all $x \in B_X$, which implies $\|m\big((|Tx| - y_w)^+\big)\|_F \leq \|w\|_F$ for all $x \in B_X$. Hence, $m\text{-}\|(|Tx| - y_w)^+\|_F \leq \varepsilon$ for all $x \in B_X$. Therefore, T is semicompact. □

In the following proposition, we see that every positive operator, which is bounded by p-semicompact operator, is also p-semicompact.

Proposition 13. *Let (X, p, E) and (Y, m, F) be two LNVLs. Suppose a positive linear operator $T : X \to Y$ to be p-semicompact. If $0 \leq S \leq T$, then S is p-semicompact.*

Proof. Let A be a p-bounded set in X. Put $|A| := \{|a| : a \in A\}$. Clearly $|A|$ is p-bounded. Since T is p-semicompact, then $T(|A|)$ is p-almost order bounded. Given $w \in F_+$, there is $y_w \in Y_+$ such that

$$m\big((T|a| - y_w)^+\big) \leq w \qquad (a \in A).$$

Thus, for any $a \in A$,

$$S|a| \leq T|a| \Rightarrow (S|a| - y_w)^+ \leq (T|a| - y_w)^+ \Rightarrow m\big((S|a| - y_w)^+\big) \leq w.$$

Since $(|Sa| - y_w)^+ \leq (S|a| - y_w)^+$, we have

$$m\big((|Sa| - y_w)^+\big) \leq m\big((S|a| - y_w)^+\big) \leq w \quad (\forall a \in A).$$

Therefore, $S(A)$ is p-almost order bounded, and S is p-semicompact. □

The following proposition shows that union of p-compact operator from right hand with p-semicompact operator is p-compact.

Proposition 14. *Let (X, p, E) be an LNS, (Y, m, F) be an LNVL, and (Z, q, G) be an LNS. If $T : (X, p, E) \to (Y, m, F)$ is a p-semicompact operator and $S : (Y, m, F) \to (Z, q, G)$ is a p-compact operator, then $S \circ T : (X, p, E) \to (Z, q, G)$ is a p-compact operator.*

Proof. We know that any p-semicompact operator is p-bounded, and so the result follows directly from Proposition 6 *(ii)*. $\square$

A linear operator T from an LNS (X, E) to a Banach space $(Y, \|\cdot\|_Y)$ is called *generalized AM-compact* or *GAM-compact* if, for any p-bounded set A in X, $T(A)$ is relatively compact in $(Y, \|\cdot\|_Y)$; see [28, p.1281]. Clearly, $T : (X, p, E) \to (Y, \|\cdot\|_Y, \mathbb{R})$ is GAM-compact iff it is (sequentially) p-compact.

In following proposition, we have a relation between GAM-compact operator and sequentially p-compact operator.

Proposition 15. *Let (X, p, E) be an LNS and (Y, m, F) be an op-continuous LNVL with a norming Banach lattice $(Y, \|\cdot\|_Y)$. If $T : (X, p, E) \to (Y, \|\cdot\|_Y)$ is GAM-compact, then $T : (X, p, E) \to (Y, m, F)$ is sequentially p-compact.*

Proof. Let x_n be a p-bounded sequence in X. Since T is GAM-compact, then there are a subsequence x_{n_k} and for $y \in Y$ such that $\|Tx_{n_k} - y\|_Y \to 0$. As $(Y, \|\cdot\|_Y)$ is Banach lattice, then, by [30, Thm.VII.2.1], there is a subsequence $x_{n_{k_j}}$ such that $Tx_{n_{k_j}} \xrightarrow{o} y$ in Y. Then, by op-continuity of (Y, m, F), we get $Tx_{n_{k_j}} \xrightarrow{p} y$ in Y. Hence, T is sequentially p-compact. $\square$

In particular, if (X, p, E) is an LNS, $(Y, \|\cdot\|_Y)$ is a Banach lattice and $T : (X, p, E) \to (Y, \|\cdot\|_Y)$ is GAM-compact operator then we get that $T : (X, p, E) \to (Y, |\cdot|, Y)$ is sequentially p-compact since $(Y, |\cdot|, Y)$ is always, op-continuous LNVL.

3.3 p-M-Weakly and p-L-Weakly Compact Operators

Recall that an operator $T \in B(X, Y)$ from a normed lattice X into a normed space Y is called *M-weakly compact*, whenever $\lim \|Tx_n\| = 0$ holds for every norm bounded disjoint sequence x_n in X, and $T \in B(X,Y)$ from a normed space X into a normed lattice Y is called *L-weakly compact*, whenever $\lim \|y_n\| = 0$ holds for every disjoint sequence y_n in $sol(T(B_X))$ (see, for example, [26, Def.3.6.9]). Similarly we have:

Definition 23. *Let $T : (X,p,E) \to (Y,m,F)$ be a p-bounded and sequentially p-continuous operator between LNSs.*

(1) If X is an LNVL and $m(Tx_n) \xrightarrow{o} 0$ for every p-bounded disjoint sequence x_n in X, then T is said to be p-M-weakly compact.

(2) If Y is an LNVL and $m(y_n) \xrightarrow{o} 0$ for every disjoint sequence y_n in $sol(T(A))$, where A is a p-bounded subset of X, then T is said to be p-L-weakly compact.

In next remark, we have some spacial cases of p-M-weakly compact and p-L-weakly compact operators.

Remark 9.

(1) Let $(X,\|\cdot\|_X)$ be a normed lattice and $(Y,\|\cdot\|_Y)$ be a normed space. Assume $T \in B(X,Y)$. Then $T : (X,\|\cdot\|_X,\mathbb{R}) \to (Y,\|\cdot\|_Y,\mathbb{R})$ is p-M-weakly compact iff $T : X \to Y$ is M-weakly compact.

(2) Let $(X,\|\cdot\|_X)$ be a normed space and $(Y,\|\cdot\|_Y)$ be a normed lattice. Assume $T \in B(X,Y)$. Then $T : (X,\|\cdot\|_X,\mathbb{R}) \to (Y,\|\cdot\|_Y,\mathbb{R})$ is p-L-weakly compact iff $T : X \to Y$ is L-weakly compact.

In the sequel, the following fact will be used frequently.

Remark 10. *If x_n is a disjoint sequence in a vector lattice X, then $x_n \xrightarrow{uo} 0$ (see [16, Cor.3.6]). If, in addition, x_n is order bounded in X, then clearly $x_n \xrightarrow{o} 0$.*

It is shown below two propositions and corollaries that, in some cases, the collection of p-M and p-L-weakly compact operators can be very large.

Proposition 16. *Assume X to be a vector lattice and $(Y,\|\cdot\|_Y)$ to be a normed space. If $T : (X,|\cdot|,X) \to (Y,\|\cdot\|_Y,\mathbb{R})$ is p-bounded and sequentially p-continuous, then T is p-M-weakly compact.*

Proof. Let x_n be a p-bounded disjoint sequence in $(X, |\cdot|, X)$. Then x_n is order bounded in X and, by Remark 10, we get $x_n \xrightarrow{o} 0$. That is $x_n \xrightarrow{p} 0$ in $(X, |\cdot|, X)$. Since T is sequentially p-continuous, then $Tx_n \xrightarrow{\|\cdot\|_Y} 0$. Therefore, $T : (X, |\cdot|, X) \to (Y, \|\cdot\|_Y, \mathbb{R})$ is p-M-weakly compact. □

Corollary 1. *Let $(X, \|\cdot\|_X)$ be a normed lattice and Y be a vector lattice. Let $Y_c^\sim$ denote the σ-order continuous dual of Y. If $0 \leq T : (X, \|\cdot\|_X, \mathbb{R}) \to (Y, |\cdot|, Y)$ is sequentially p-continuous and p-bounded, then the operator $T^\sim : (Y_c^\sim, |\cdot|, Y_c^\sim) \to (X^*, \|\cdot\|_{X^*}, \mathbb{R})$ defined by $T^\sim(f) := f \circ T$ is p-M-weakly compact.*

Proof. Theorem 7 implies that $T^\sim$ is p-continuous, and so it is p-bounded, by Proposition 4. Thus, we get from Proposition 16, that $T^\sim$ is p-M-weakly compact. □

Proposition 17. *Assume $(X, \|\cdot\|_X)$ to be a normed lattice and Y to be a vector lattice. If $T : (X, \|\cdot\|_X, \mathbb{R}) \to (Y, |\cdot|, Y)$ is a p-bounded and sequentially p-continuous operator, then T is p-L-weakly compact.*

Proof. Let A be a p-bounded set in $(X, \|\cdot\|_X, \mathbb{R})$. Since T is a p-bounded operator, then $T(A)$ is p-bounded in $(Y, |\cdot|, Y)$, i.e. $T(A)$ is order bounded and hence $sol(T(A))$ is order bounded. Let y_n be a disjoint sequence in $sol(T(A))$. Then, by Remark 10, we have $y_n \xrightarrow{o} 0$ in Y, i.e. $y_n \xrightarrow{p} 0$ in $(Y, |\cdot|, Y)$. Thus, T is p-L-weakly compact. □

Corollary 2. *Let X be a vector lattice and Y be an AL-space. Assume $0 \leq T : (X, |\cdot|, X) \to (Y, \|\cdot\|_Y, \mathbb{R})$ to be sequentially p-continuous. Define $T^\sim : (Y^*, \|\cdot\|_{Y^*}, \mathbb{R}) \to (X^\sim, |\cdot|, X^\sim)$ by $T^\sim(f) = f \circ T$. Then $T^\sim$ is p-L-weakly compact.*

Proof. Theorem 8 implies that $T^\sim$ is sequentially p-continuous and p-bounded, and so we get, by Proposition 17, that $T^\sim$ is p-L-weakly compact. □

It is known that any order continuous operator is order bounded, but this fails for σ-order continuous operators; see [5, Exer.10, p.289]. Therefore, we need the order boundedness condition in the following proposition.

Proposition 18. *If $T : X \to Y$ is an order bounded σ-order continuous operator between vector lattices, then $T : (X, |\cdot|, X) \to (Y, |\cdot|, Y)$ is both p-M-weakly and p-L-weakly compact.*

Proof. Clearly, $T : (X, |\cdot|, X) \to (Y, |\cdot|, Y)$ is both sequentially p-continuous and p-bounded.

First, we show that T is p-M-weakly compact. Let x_n be a p-bounded disjoint sequence of X. Then, by Remark 10, we get $x_n \xrightarrow{o} 0$ in X, and so $Tx_n \xrightarrow{o} 0$ in Y, as T is order continuous. Therefore, T is p-M-weakly compact.

Next, we show that T is p-L-weakly compact. Let A be a p-bounded set in $(X, |\cdot|, X)$, then A is order bounded in X. Thus, $T(A)$ is order bounded, and so $sol(T(A))$ is order bounded in Y. If y_n is a disjoint sequence in $sol(T(A))$, then again, by Remark 10, $y_n \xrightarrow{o} 0$ or $y_n \xrightarrow{p} 0$ in $(Y, |\cdot|, Y)$. Therefore, T is p-L-weakly compact. □

Next, we show that p-M-weakly and p-L-weakly compact operators satisfy the domination property.

Proposition 19. *Let (X, p, E) and (Y, m, F) be LNVLs, and let $S, T : X \to Y$ be two linear operators such that $0 \leq S \leq T$.*

(i) If T is p-M-weakly compact, then S is p-M-weakly compact.

(ii) If T is p-L-weakly compact, then S is p-L-weakly compact.

Proof. *(i)* Since T is sequentially p-continuous and p-bounded, then it is easily seen that S is sequentially p-continuous and p-bounded. Let x_n be a p-bounded disjoint sequence in X. Then $|x_n|$ is also p-bounded and disjoint. Since T is p-M-weakly compact, then $m(T|x_n|) \xrightarrow{o} 0$ in F. Now, $0 \leq S|x_n| \leq T|x_n|$ for all $n \in \mathbb{N}$ and, since the lattice norm is monotone, we get $m(S|x_n|) \xrightarrow{o} 0$ in F. Now, $|Sx_n| \leq S|x_n|$ for all $n \in \mathbb{N}$, and so $m(Sx_n) = m(|Sx_n|) \leq m(S|x_n|) \xrightarrow{o} 0$ in F. Thus, S is p-M-weakly compact.

(ii) It is easy to see that S is sequentially p-continuous and p-bounded. Let A be a p-bounded subset of X. Put $|A| = \{|a| : a \in A\}$. Clearly, $sol(S(A)) \subseteq sol(S(|A|))$ and, since $0 \leq S \leq T$, we have $sol(S(|A|)) \subseteq sol(T(|A|))$. Let y_n be a disjoint

sequence in $sol(S(A))$, then $y_n \in sol(T(|A|))$ and, since T is p-L-weakly compact, then $m(S|x_n|) \xrightarrow{o} 0$ in F. Therefore, S is p-L-weakly compact. □

Corollary 3. *Let (X, p, E) and (Y, m, F) be two LNVLs, and let $-T_1 \le T \le T_2$, where both operators $T_1, T_2 : X \to Y$ are positive.*

(i) If T_1 and T_2 are p-M-weakly compact, then T is p-M-weakly compact.

(ii) If T_1 and T_2 are p-L-weakly compact, then T is p-L-weakly compact.

Let us give the following useful lemma [5, Lem. 4.35].

Lemma 10. *Let E be a vector lattice, and let x_n be a sequence of E_+. If some $x \in E_+$ satisfies $2^{-n}x_n \le x$ for all n, then the sequence u_n defined by*

$$u_n = \left[x_{n+1} - 4\sum_{i=1}^{n} x_i - 2^{-n}x\right]^+$$

is a disjoint sequence.

The following result is a variant of [5, Thm.4.36].

Theorem 10. *Let (X, p, E) be a sequentially p-complete LNVL such that $(E, \|\cdot\|_E)$ is a Banach lattice, and let (Y, m, F) be an LNS. Assume $T : (X, p, E) \to (Y, m, F)$ to be sequentially p-continuous, and let A be a p-bounded solid subset of X.*

If $m(Tx_n) \xrightarrow{o} 0$ holds for each disjoint sequence x_n in A then, for each atom a in F and each $\varepsilon > 0$, there exists $0 \le u \in I_A$ satisfying

$$f_a\big(m(T(|x| - u)^+)\big) < \varepsilon$$

for all $x \in A$, where I_A denotes the ideal generated by A in X.

Proof. Suppose the claim is false. Then there are an atom $a_0 \in F$ and $\varepsilon_0 > 0$ such that, for each $u \ge 0$ in I_A, we have $f_{a_0}\big(m(T(|x| - u)^+)\big) \ge \varepsilon_0$ for some $x \in A$. In particular, there exists a sequence x_n in A such that

$$f_{a_0}\big(m(T(|x_{n+1}| - 4^n\sum_{i=1}^{n}|x_i|)^+)\big) \ge \varepsilon_0 \quad (\forall n \in \mathbb{N}). \tag{3.7}$$

Now, put $y = \sum\limits_{n=1}^{\infty} 2^{-n}|x_n|$. Lemma 5 implies $y \in X$. Also let $w_n = (|x_{n+1}| - 4^n \sum\limits_{i=1}^{n}|x_i|)^+$ and $v_n = (|x_{n+1}| - 4^n \sum\limits_{i=1}^{n}|x_i| - 2^{-n}y)^+$. By Lemma 10, the sequence v_n is disjoint. Also, since A is solid and $0 \leq v_n < |x_{n+1}|$ holds, we see that v_n is in A and so, by the hypothesis, $m(Tx_n) \xrightarrow{o} 0$.

On the other hand, $0 \leq w_n - v_n \leq 2^{-n}y$ and so $p(w_n - v_n) \leq 2^{-n}p(y)$. Thus, $p(w_n - v_n) \xrightarrow{o} 0$ in F. Since T is sequentially p-continuous, then $m\big(T(w_n - v_n)\big) \xrightarrow{o} 0$ in F. Now, $m(Tw_n) \leq m(T(w_n - v_n)) + m(Tv_n)$ implies $m(Tw_n) \xrightarrow{o} 0$ in F. In particular, $f_{a_0}\big(m(Tw_n)\big) \to 0$ as $n \to \infty$, which contradicts (3.7). □

In [5, Thm.5.60], the approximation properties were provided for M-weakly and L-weakly compact operators. The following two propositions are similar to [5, Thm.5.60] in the case of p-M-weakly and p-L-weakly compact operators.

Proposition 20. *Let (X, p, E) be a sequentially p-complete LNVL with a Banach lattice $(E, \|\cdot\|_E)$, (Y, m, F) be an LNS, $T : (X, p, E) \to (Y, m, F)$ be a p-M-weakly compact operator, and A be a p-bounded solid subset of X. Then, for each atom a in F and each $\varepsilon > 0$, there exists some $u \in X_+$ such that*

$$f_a\big(m(T(|x| - u)^+)\big) < \varepsilon$$

holds for all $x \in A$.

Proof. Let A be a p-bounded solid subset of X. Since T is p-M-weakly compact, then $m(Tx_n) \xrightarrow{o} 0$ for every disjoint sequence in A. By Theorem 10, for any atom $a \in F$ and for $\varepsilon > 0$, there exists $u \in X_+$ such that $f_a\big(m(T(|x| - u)^+)\big) < \varepsilon$ for all $x \in A$. □

Proposition 21. *Let (X, p, E) be an LNS and (Y, m, F) be a sequentially p-complete LNVL with a Banach lattice F. Assume $T : (X, p, E) \to (Y, m, F)$ to be p-L-weakly compact and A to be p-bounded in X. Then, for atom a in F and $\varepsilon > 0$, there exists some $u \in Y_+$ in the ideal generated by $T(X)$ satisfying*

$$f_a\big(m(|Tx| - u)^+)\big) < \varepsilon$$

for all $x \in A$.

Proof. Let A be a p-bounded subset of X. Since T is p-L-weakly compact, $m(y_n) \xrightarrow{o} 0$ for any disjoint sequence y_n in $sol(T(A))$. Consider the identity operator $I : (Y, m, F) \to (Y, m, F)$. By Theorem 10, for any atom $a \in F$ and $\varepsilon > 0$, there exists $u \in Y_+$ in the ideal generated by $sol(T(A))$ (and so in the ideal generated by $T(X)$) such that

$$f_a\big(m((|y| - u)^+)\big) < \varepsilon$$

for all $y \in sol(T(A))$. In particular,

$$f_a\big(m((|Tx| - u)^+)\big) < \varepsilon$$

for all $x \in A$. □

Recall that, for any atomic vector lattice X, $f_a : X \to \mathbb{R}$ is biorthogonal functional for all a. Then, if $x_\alpha \xrightarrow{o} 0$ in X, we have $f_a(x_\alpha) \to 0$ for all a. The converse holds true if the net x_α is order bounded.

The next two theorems provide relations between p-M-weakly and p-L-weakly compact operators, which are known for M-weakly and L-weakly compact operators; e.g. [5, Thm.5.67 and Exer.4(a), p:337]

Theorem 11. *Let (X, p, E) be a sequentially p-complete LNVL with a norming Banach lattice $(E, \|\cdot\|_E)$, (Y, m, F) be an op-continuous LNVL with an atomic norming lattice F, and $T \in L^\sim(X, Y)$. If $T : (X, p, E) \to (Y, m, F)$ is p-M-weakly compact, then T is p-L-weakly compact.*

Proof. Let A be a p-bounded subset of X, and let y_n be a disjoint sequence in $sol(T(A))$. Then there is a sequence x_n in A such that $|y_n| \le |Tx_n|$ for all $n \in \mathbb{N}$. Let $a \in F$ be an atom. Given $\varepsilon > 0$, then, by Proposition 20, there is $u \in X_+$ such that

$$f_a\big(m(T(|x| - u)^+)\big) < \varepsilon$$

for all $x \in sol(A)$. In particular, for all $n \in \mathbb{N}$, we have

$$f_a\big(m(T(x_n^+ - u)^+)\big) < \varepsilon \quad \text{and} \quad f_a\big(m(T(x_n^- - u)^+)\big) < \varepsilon.$$

Thus, for each $n \in \mathbb{N}$,

$$\begin{aligned} |y_n| &\leq |Tx_n| \leq |Tx_n^+| + |Tx_n^-| \\ &= |T(x_n^+ - u)^+ + T(x_n^+ \wedge u)| + |T(x_n^- - u)^+ + T(x_n^- \wedge u)| \\ &\leq |T(x_n^+ - u)^+| + |T(x_n^+ \wedge u)| + |T(x_n^- - u)^+| + |T(x_n^- \wedge u)| \\ &\leq |T(x_n^+ - u)^+| + |T(x_n^- - u)^+| + |T|(x_n^+ \wedge u) + |T|(x_n^- \wedge u) \\ &\leq |T(x_n^+ - u)^+| + |T(x_n^- - u)^+| + 2|T|u. \end{aligned}$$

By Riesz decomposition property, for all $n \in \mathbb{N}$, there exist $u_n, v_n \geq 0$ such that $y_n = u_n + v_n$ and $0 \leq u_n \leq |T(x_n^+ - u)^+| + |T(x_n^- - u)^+|$, $0 \leq v_n \leq 2|T|u$. Since y_n is disjoint sequence and $v_n \leq |y_n|$ for all $n \in \mathbb{N}$, the sequence v_n is disjoint. Moreover, it is order bounded. Hence, by Remark 10, we get $v_n \xrightarrow{o} 0$. Since (Y, m, F) is *op*-continuous, then $m(v_n) \xrightarrow{o} 0$. In particular, $f_a\big(m(v_n)\big) \to 0$, as $n \to \infty$, since f_a is order continuous for every atom a. So, for given $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $f_a(m(v_n)) < \varepsilon$ for all $n \geq n_0$. Thus, for any $n \geq n_0$, we have

$$\begin{aligned} f_a\big(m(y_n)\big) &\leq f_a\big(m(u_n)\big) + f_a\big(m(v_n)\big) \\ &\leq f_a\big(m(T(x_n^+ - u)^+)\big) + f_a\big(m(T(x_n^- - u)^+)\big) + \varepsilon \leq 3\varepsilon. \end{aligned}$$

Hence, $f_a\big(m(y_n)\big) \to 0$ as $n \to \infty$.
Since T is p-bounded, then $m\big(Tx_n\big)$ is order bounded in F and so, by inequality $m(y_n) \leq m\big(Tx_n\big)$, $m(y_n)$ is order bounded in F. By Lemma 3, the atomicity of F implies $m(y_n) \xrightarrow{o} 0$ in F. Therefore, T is p-L-weakly compact. □

Theorem 12. *Let (X, p, E) and (Y, m, F) be LNVLs. If $T : (X, p, E) \to (Y, m, F)$ is a p-L-weakly compact lattice homomorphism, then T is p-M-weakly compact.*

Proof. Let x_n be a p-bounded disjoint sequence in X. Since T is lattice homomorphism, then Tx_n is disjoint in Y. Clearly $Tx_n \in sol\big(\{Tx_n : n \in \mathbb{N}\}\big)$. Since T is a p-L-weakly compact, then $m\big(T(x_n)\big) \xrightarrow{o} 0$ in F. Therefore, T is p-M-weakly compact. □

We end up this section by an investigation of the relation between p-M-weakly (respectively, p-L-weakly) compact operators and M-weakly (respectively, L-weakly) compact operators acting in mixed-normed spaces.

Proposition 22. *Given an LNVL (X, p, E) with $(E, \|\cdot\|_E)$, which is an AM-space with a strong unit. Let an LNS (Y, m, F) be such that $(F, \|\cdot\|_F)$ is a σ-order continuous normed lattice. If $T : (X, p, E) \to (Y, m, F)$ is p-M-weakly compact, then $T : (X, p\text{-}\|\cdot\|_E) \to (Y, m\text{-}\|\cdot\|_F)$ is M-weakly compact.*

Proof. By Proposition 3, it follows that $T : (X, p\text{-}\|\cdot\|_E) \to (Y, m\text{-}\|\cdot\|_F)$ is norm continuous. Let x_n be a norm bounded disjoint sequence in $(X, p\text{-}\|\cdot\|_E)$. Then $p\text{-}\|x_n\|_E \leq k < \infty$ or $\|p(x_n)\|_E \leq k < \infty$ for all $n \in \mathbb{N}$. Since $(E, \|\cdot\|_E)$ is an AM-space with a strong unit, then there is $e \in E_+$ such that $p(x_n) \leq e$ for all $n \in \mathbb{N}$. Thus, x_n is a p-bounded disjoint sequence in (X, p, E). Since $T : (X, p, E) \to (Y, m, F)$ is p-M-weakly compact, then $m(Tx_n) \xrightarrow{o} 0$ in F. It follows from the σ-order continuity of $(F, \|\cdot\|_F)$, that $\|m(Tx_n)\|_F \to 0$ or $\lim\limits_{n\to\infty} m\text{-}\|Tx_n\|_F = 0$. Therefore, $T : (X, p\text{-}\|\cdot\|_E) \to (Y, m\text{-}\|\cdot\|_F)$ is M-weakly compact. □

Proposition 23. *Suppose (X, p, E) to be an LNVL with a σ-order continuous normed lattice $(E, \|\cdot\|_E)$, and (Y, m, F) to be an LNS with an atomic normed lattice $(F, \|\cdot\|_F)$. Assume further that*:

(i) $T : (X, p, E) \to (Y, m, F)$ is p-bounded;

(ii) $T : (X, p\text{-}\|\cdot\|_E) \to (Y, m\text{-}\|\cdot\|_F)$ is M-weakly compact.

Then $T : (X, p, E) \to (Y, m, F)$ is p-M-weakly compact.

Proof. The assumptions, together with Theorem 5, imply that $T : (X, p, E) \to (Y, m, F)$ is sequentially p-continuous.

Assume x_n to be a p-bounded disjoint sequence in (X, p, E). Then x_n is disjoint and norm bounded in $(E, p\text{-}\|\cdot\|_E)$. Since $T : (X, p\text{-}\|\cdot\|_E) \to (Y, m\text{-}\|\cdot\|_F)$ is M-weakly compact, then $\lim\limits_{n\to\infty} m\text{-}\|Tx_n\|_F = 0$ or $\lim\limits_{n\to\infty} \|m(Tx_n)\|_F = 0$. Since x_n is p-bounded and $T : (X, p, E) \to (Y, m, F)$ is p-bounded, then $m(Tx_n)$ is order bounded in F. Let $a \in F$ be an atom, then

$$\left|f_a\big(m(Tx_n)\big)\right| \leq \|f_a\| \|m(Tx_n)\|_F \to, 0 \quad \text{as } n \to \infty.$$

Since F is atomic and $m(Tx_n)$ is order bounded, then, by Lemma 3, $m(Tx_n) \xrightarrow{o} 0$. Therefore, $T : (X, p, E) \to (Y, m, F)$ is p-M-weakly compact. □

Proposition 24. *Assume (X, p, E) to be an LNS with an AM-space $(E, \|\cdot\|_E)$ possessing a strong unit, and (Y, m, F) to be an LNVL with a σ-order continuous normed lattice $(F, \|\cdot\|_F)$. If $T : (X, p, E) \to (Y, m, F)$ is p-L-weakly compact, then $T : (X, p\text{-}\|\cdot\|_E) \to (Y, m\text{-}\|\cdot\|_F)$ is L-weakly compact.*

Proof. Proposition 3 implies that $T : (X, p\text{-}\|\cdot\|_E) \to (Y, m\text{-}\|\cdot\|_F)$ is norm continuous. Let B_X be the closed unit ball of $(X, p\text{-}\|\cdot\|_E)$. Then $p\text{-}\|x\|_E \leq 1$ or $\|p(x)\|_E \leq 1$ for all $x \in B_X$. Since $(E, \|\cdot\|_E)$ is an AM-space with a strong unit, then there is an element $e \in E_+$ such that $p(x) \leq e$ for each $x \in B_X$. So B_X is p-bounded. Let y_n be a disjoint sequence in $sol(T(B_X))$. Since $T : (X, p, E) \to (Y, m, F)$ is p-L-weakly compact, then $m(y_n) \xrightarrow{o} 0$ in F. Since $(F, \|\cdot\|_F)$ is σ-order continuous normed lattice, then $\|m(y_n)\|_F \to 0$ or $\lim\limits_{n\to\infty} m\text{-}\|y_n\|_F = 0$. So $T : (X, p\text{-}\|\cdot\|_E) \to (Y, m\text{-}\|\cdot\|_F)$ is L-weakly compact. □

Proposition 25. *Let (X, p, E) be an LNS with a σ-order continuous normed lattice, (Y, m, F) be an LNVL with an atomic normed lattice $(F, \|\cdot\|_F)$. Assume that*:

(i) $T : (X, p, E) \to (Y, m, F)$ *is p-bounded;*

(ii) $T : (X, p\text{-}\|\cdot\|_E) \to (Y, m\text{-}\|\cdot\|_F)$ *is L-weakly compact.*

Then $T : (X, p, E) \to (Y, m, F)$ is p-L-weakly compact.

Proof. Theorem 5 implies that $T : (X, p, E) \to (Y, m, F)$ is sequentially p-continuous. Let A be a p-bounded set. Then there is $e \in E_+$ such that $p(a) \leq e$ for all $a \in A$. Hence, $\|p(a)\|_E \leq \|e\|_E$ for all $a \in A$ or $p\text{-}\|a\|_E \leq \|e\|_E$ for each $a \in A$. Thus, A is norm bounded in $(X, p\text{-}\|\cdot\|_E)$. Let y_n be a disjoint sequence in $sol(T(A))$. Since $T : (X, p\text{-}\|\cdot\|_E) \to (Y, m\text{-}\|\cdot\|_F)$ is L-weakly compact, then $\lim\limits_{n\to\infty} m\text{-}\|y_n\|_F = 0$ or $\lim\limits_{n\to\infty} \|m(y_n)\|_F = 0$.
Since $T : (X, p, E) \to (Y, m, F)$ is p-bounded and A is p-bounded, then $T(A)$ is p-bounded in Y, and so $sol(T(A))$ is p-bounded in Y. Hence, y_n is a p-bounded sequence in (Y, m, F); i.e. $m(y_n)$ is order bounded in F. Let $a \in F$ be an atom and consider its biorthogonal functional f_a. Then

$$|f_a\big(m(y_n)\big)| \leq \|f_a\|\|m(y_n)\|_F \to 0 \ \text{ as } n \to \infty.$$

So, for any atom $a \in F$, $\lim\limits_{n\to\infty} f_a\big(m(y_n)\big) = 0$ and, since $m(y_n)$ is order bounded in an atomic vector lattice F, by Lemma 3, we have $m(y_n) \xrightarrow{\text{o}} 0$ in F. Thus, T is p-L-weakly compact. $\square$

3.4 up-Continuous and up-Compact Operators

In this section, we define the notions of (sequentially) up-continuous and (sequentially) up-compact operators acting between lattice-normed vector lattices. Also, using the up-convergence in LNVLs, we introduce the following notions.

Definition 24. *Let X, Y be two LNVLs and $T \in L(X,Y)$. Then:*

(1) T is called up-continuous, *if $x_\alpha \xrightarrow{\text{up}} 0$ in X implies $Tx_\alpha \xrightarrow{\text{up}} 0$ in Y. If the condition holds for sequences, then T is called* sequentially up-continuous;

(2) T is called up-compact *if, for any p-bounded net x_α in X, there is a subnet x_{α_β} such that $Tx_{\alpha_\beta} \xrightarrow{\text{up}} y$ in Y for some $y \in Y$;*

(3) T is called sequentially-up-compact *if, for any p-bounded sequence x_n in X, there is a subsequence x_{n_k} such that $Tx_{n_k} \xrightarrow{\text{up}} y$ in Y for some $y \in Y$.*

For any two positive operators $S, T : (X,p,E) \to (Y,m,F)$, we have S is up-continuous operator, whenever T is up-continuous and $0 \leq S \leq T$. Indeed, let $0 \leq x_\alpha \xrightarrow{\text{up}} 0$ in X. Thus, by up-continuity of T, $m(|Tx_\alpha| \wedge w) = m(Tx_\alpha \wedge w) \xrightarrow{\text{o}} 0$ in F for all $w \in Y_+$. Then, by the inequality $0 \leq Sx_\alpha \leq Tx_\alpha$ for all α, we have $0 \leq m(Sx_\alpha \wedge w) \leq m(Tx_\alpha \wedge w)$. Thus, $m(Sx_\alpha \wedge w) \xrightarrow{\text{o}} 0$ in F. If we take an arbitrary net $x_\alpha \xrightarrow{\text{up}} 0$, then, by the fact that $x_\alpha = (x_\alpha^+ - x_\alpha^-) \xrightarrow{\text{up}} 0$ iff $x_\alpha^+ \xrightarrow{\text{up}} 0$ and $x_\alpha^- \xrightarrow{\text{up}} 0$, we get the up-continuity of S.

In following remark, we have some special case of up-continuous and up-compact operators.

Remark 11.

(i) The notion of up-continuous operators is motivated by two recent notions, namely: σ-unbounded order continuous (σuo-continuous) mappings between vector lattices (see [14, p.23]), and un-continuous functionals on Banach lattices (see [18, p.269]).

(ii) Let $(X, \|\cdot\|_X)$ be a normed space and $(Y, \|\cdot\|_Y)$ be a normed lattice. An operator $T \in B(X,Y)$ is called (sequentially) un-compact *if, for every norm bounded net x_α (respectively, every norm bounded sequence x_n), its image has a un-convergent subnet (respectively, subsequence); see [18, Sec.9, p.275]. Therefore, $T \in B(X,Y)$ is (sequentially) un-compact iff $T : (X, \|\cdot\|_X, \mathbb{R}) \to (Y, \|\cdot\|_Y, \mathbb{R})$ is (sequentially) up-compact.*

The next proposition gives relations between p-continuous (respectively, p-compact) and up-continuous (respectively, up-compact). Since the proof is clear, we do not give details.

Proposition 26. *Let (X,E) and (Y,F) be two LNVLs and $T : (X,E) \to (Y,F)$ be a linear operator. Then the following assertions are true.*

(i) If T is (sequentially) p-continuous then T is (sequentially) up-continuous.

(ii) If T is (sequentially) p-compact then T is (sequentially) up-compact.

Proof. Since every p-convergent net is also up-convergent, the statements follow directly. □

The next proposition is partial converse of Proposition 26, (ii).

Proposition 27. *Let (X,E) and (Y,F) be two LNVLs and $T \in L(X,Y)$. If T is a up-compact and p-semicompact operator, then T is p-compact.*

Proof. Let x_α be a p-bounded net in X. Then Tx_α is a p-almost order bounded net in Y, as T is p-semicompact operator. Moreover, since T is up-compact, then there is a subnet x_{α_β} such that $Tx_{\alpha_\beta} \xrightarrow{\text{up}} y$ for some $y \in Y$. It follows by [7, Prop.9], that $Tx_{\alpha_\beta} \xrightarrow{\text{p}} y$. Therefore, T is p-compact. □

Similar to Proposition 6, we have the following result.

Proposition 28. *Let (X,E) be an LNVL, and let $S, T, G : (X,E) \to (X,E)$ be linear operators. Then the following assertions are true.*

(i) If S is p-bounded and T is up-compact, then $T \circ S : (X,E) \to (X,E)$ is up-compact.

(ii) If T is up-compact and G is up-continuous, then $G \circ T : (X, E) \to (X, E)$ is up-compact.

Proof. The proof of (i) is obvious by definition of up-compactness of T.
We show (ii). Let x_α be a p-bounded net in X. Since T is up-compact, then there is a subnet x_{α_β} of x_α such that Tx_{α_β} is up-convergent in (X, E). By the up-continuity of G, we have $G(Tx_{\alpha_\beta})$ to be up-convergent in (X, E). Therefore, $G \circ T$ is up-compact. □

The following proposition has a relation between p-bounded operator and up-compact operator.

Proposition 29. *Let X be a vector lattice, (Y, m, F) be an op-continuous LNVL such that $(Y, \|\cdot\|_Y)$ is an atomic KB-space. If $T : (X, |\cdot|, X) \to (Y, \|\cdot\|_Y, \mathbb{R})$ is p-bounded, then $T : (X, |\cdot|, X) \to (Y, m, F)$ is up-compact.*

Proof. Let x_α be a p-bounded net in $(X, |\cdot|, X)$, then Tx_α is norm bounded in Y. Since Y is atomic KB-space, then, by [18, Theorem 7.5], there are a subnet x_{α_β} of x_α and $y \in Y$ such that $Tx_{\alpha_\beta} \xrightarrow{\text{un}} y$. On atomic order Banach lattices, un-convergence and uo-convergence agree. Hence, $Tx_{\alpha_\beta} \xrightarrow{\text{uo}} y$. Since (Y, m, F) is op-continuous, then $Tx_{\alpha_\beta} \xrightarrow{\text{up}} y$. Thus, T is up-compact. □

Now we investigate a composition of a sequentially up-compact operator with a dominated lattice homomorphism.

Theorem 13. *Let (X, p, E) be an LNVL, (Y, m, F) be an LNVL with an order continuous Banach lattice $(F, \|\cdot\|_F)$, and (Z, q, G) be an LNVL with a Banach lattice $(G, \|\cdot\|_G)$. If $T \in L(X, Y)$ is a sequentially up-compact operator and $S \in L(Y, Z)$ is a dominated surjective lattice homomorphism, then $S \circ T$ is sequentially up-compact.*

Proof. Let x_n be a p-bounded sequence in X. Since T is sequentially up-compact, then there is a subsequence x_{n_k} such that $Tx_{n_k} \xrightarrow{\text{up}} y$ in Y for some $y \in Y$. Let $u \in Z_+$. Since S is surjective lattice homomorphism, we have some $v \in Y_+$ such that $Sv = u$. Since $Tx_{n_k} \xrightarrow{\text{up}} y$, then $m(|Tx_{n_k} - y| \wedge v) \xrightarrow{\text{o}} 0$ in F. Clearly, F is order complete and so, by [1, Prop.1.5], there are $f_k \downarrow 0$ and $k_0 \in \mathbb{N}$ such that

$$m(|Tx_{n_k} - y| \wedge v) \leq f_k \qquad (k \geq k_0). \tag{3.8}$$

Note also $\|f_k\|_F \downarrow 0$ in F, as $(F, \|\cdot\|_F)$ is an order continuous Banach lattice. Since S is dominated, then there is a positive operator $R : F \to G$ such that

$$q\big(S(|Tx_{n_k} - y| \wedge v)\big) \leq R\big(m(|Tx_{n_k} - y| \wedge v)\big).$$

Taking into account that S is a lattice homomorphism and $Sv = u$, we get, by (3.8),

$$q(|S \circ Tx_{n_k} - Sy| \wedge u) \leq Rf_k \qquad (k \geq k_0). \tag{3.9}$$

Since R is positive, then, by [5, Thm.4.3], it is norm continuous. Hence, $\|Rf_k\|_G \downarrow 0$. Also, by [30, Thm.VII.2.1], there is a subsequence f_{k_j} of $(f_k)_{k \geq k_0}$ such that $Rf_{k_j} \xrightarrow{\text{o}} 0$ in G, and so $Rf_{k_j} \downarrow 0$ in G. So (3.9) becomes

$$q(|S \circ Tx_{n_{k_j}} - Sy| \wedge u) \leq Rf_{k_j} \qquad (j \in \mathbb{N}).$$

Since $u \in Z_+$ is arbitrary, $S \circ T(x_{n_{k_j}}) \xrightarrow{\text{up}} Sy$. Therefore, $S \circ T$ is sequentially up-compact. □

Remark 12. *In connection with the proof of Theorem 13, it should be mentioned that, if the operator T is up-compact and S is a surjective lattice homomorphism with an order continuous dominant, then it can be easily seen that $S \circ T$ is up-compact.*

Recall that up-convergence of a net x_α to zero in a sublattice Y of vector lattice X defined as $|x_\alpha| \wedge u \xrightarrow{\text{o}} 0$ for all $u \in Y_+$. For an LNVL (X, p, E), a sublattice Y of X is called *up-regular* if, for any net y_α in Y, the convergence $y_\alpha \xrightarrow{\text{up}} 0$ in Y implies $y_\alpha \xrightarrow{\text{up}} 0$ in X; see [7, Def.10 and Sec.3.4].

Corollary 4. *Let (X, p, E) be an LNVL, (Y, m, F) an LNVL with an order continuous Banach lattice $(F, \|\cdot\|_F)$, and (Z, q, G) an LNVL with a Banach lattice $(G, \|\cdot\|_G)$. If $T \in L(X, Y)$ is a sequentially up-compact operator, $S \in L(Y, Z)$ is a dominated lattice homomorphism, and $S(Y)$ is up-regular in Z, then $S \circ T$ is sequentially up-compact.*

Proof. Since S is a lattice homomorphism, then $S(Y)$ is a vector sublattice of Z. So $(S(Y), q, G)$ is an LNVL. Thus, by Theorem 13, we have $S \circ T : (X, p, E) \to (S(Y), q, G)$ is sequentially up-compact.

Next, we show that $S \circ T : (X, p, E) \to (Z, q, G)$ is sequentially up-compact. Let x_n be a p-bounded sequence in X. Then there is a subsequence x_{n_k} such that $S \circ T(x_{n_k}) \xrightarrow{\text{up}} z$ in $S(Y)$ for some $z \in S(Y)$. Since $S(Y)$ is up-regular in Z, we have $S \circ T(x_{n_k}) \xrightarrow{\text{up}} z$ in Z. Therefore, $S \circ T : X \to Z$ is sequentially up-compact. $\square$

The next result is a p-version of Proposition 9.4 in [18].

Corollary 5. *Let (X, p, E) be an LNVL, (Y, m, F) an LNVL with an order continuous Banach lattice $(F, \|\cdot\|_F)$, and (Z, q, G) an LNVL with a Banach lattice $(G, \|\cdot\|_G)$. If $T \in L(X, Y)$ is a sequentially up-compact operator, $S \in L(Y, Z)$ is a dominated lattice homomorphism, and $I_{S(Y)}$ (the ideal generated by $S(Y)$) is up-regular in Z, then $S \circ T$ is sequentially up-compact.*

Proof. Let x_n be a p-bounded sequence in X. Since T is sequentially up-compact, there exist a subsequence x_{n_k} and $y_0 \in Y$ such that $Tx_{n_k} \xrightarrow{\text{up}} y_0$ in Y. Let $0 \le u \in I_{S(Y)}$. Then there is $y \in Y_+$ such that $0 \le u \le Sy$. Therefore, we have for a dominant R:

$$q\big(S(|Tx_{n_k} - y_0| \wedge y)\big) \le R\big(m(|Tx_{n_k} - y_0| \wedge y)\big),$$

and so

$$q\big((|STx_{n_k} - Sy_0| \wedge Sy)\big) \le R\big(m(|Tx_{n_k} - y_0| \wedge y)\big).$$

It follows from $0 \le u \le Sy$, that

$$q\big((|STx_{n_k} - Sy_0| \wedge u)\big) \le R\big(m(|Tx_{n_k} - y_0| \wedge u)\big).$$

Now, the argument given in the proof of Theorem 13 can be repeated here as well. Thus, we have that $S \circ T : (X, p, E) \to (I_{S(Y)}, q, G)$ is sequentially up-compact. Since $I_{S(Y)}$ is up-regular in Z, then it can be easily seen that $S \circ T : X \to Z$ is sequentially up-compact. $\square$

We conclude this section by a result which might be compared with Proposition 9.9 in [18].

Proposition 30. *Let (X, p, E) be an LNS and let $(Y, \|\cdot\|_Y)$ be a σ-order continuous normed lattice. If $T : (X, p, E) \to (Y, |\cdot|, Y)$ is sequentially up-compact and p-bounded, then $T : (X, p, E) \to (Y, \|\cdot\|_Y)$ is GAM-compact.*

Proof. Let x_n be a p-bounded sequence in X. Since T is up-compact, there exist a subsequence x_{n_k} and for $y \in Y$ such that $Tx_{n_k} \xrightarrow{\text{up}} y$ in $(Y, |\cdot|, Y)$. Then, by σ-order continuity of $(Y, \|\cdot\|_Y)$, we have $Tx_{n_k} \xrightarrow{\text{un}} y$ in Y. Moreover, since T is p-bounded, then Tx_n is p-bounded in $(Y, |\cdot|, Y)$ or order bounded in Y, and so we get $Tx_{n_k} \xrightarrow{\|\cdot\|_Y} y$. Therefore, T is GAM-compact. $\square$

REFERENCES

[1] Y. Abramovich and G. Sirotkin. On order convergence of nets. *Positivity*, 9(3):287–292, 2005.

[2] Y. A. Abramovich and C. D. Aliprantis. *An invitation to operator theory*, volume 50 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2002.

[3] C. D. Aliprantis and K. C. Border. *Infinite dimensional analysis*. Springer, Berlin, third edition, 2006. A hitchhiker's guide.

[4] C. D. Aliprantis and O. Burkinshaw. *Locally solid Riesz spaces with applications to economics*, volume 105 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2003.

[5] C. D. Aliprantis and O. Burkinshaw. *Positive operators*. Springer, Dordrecht, 2006. Reprint of the 1985 original.

[6] B. Aqzzouz and A. Elbour. Some results on discrete Banach lattices. *Creat. Math. Inform.*, 19(2):110–115, 2010.

[7] A. Aydın, E. Yu. Emelyanov, N. Erkurşun Özcan, and M. A. A. Marabeh. Unbounded p-convergence in lattice-normed vector lattices. *arXiv:1609.05301v2*, 2016.

[8] A. Aydın, E. Yu. Emelyanov, N. Erkurşun Özcan, and M. A. A. Marabeh. Compact-like operators in lattice-normed spaces, 2017.

[9] A. V. Bukhvalov, A. E. Gutman, V. B. Korotkov, A. G. Kusraev, S. S. Kutateladze, and B. M. Makarov. *Vector lattices and integral operators*, volume 358 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht, 1996.

[10] V. B. Cherdak. On the order spectrum of r-compact operators in lattice-normed spaces. *Sibirsk. Mat. Zh.*, 32(1):148–152, 221, 1991.

[11] R. De Marr. Partially ordered linear spaces and locally convex linear topological spaces. *Illinois J. Math.*, 8:601–606, 1964.

[12] Y. Deng, M. O'Brien, and V.G. Troitsky. Unbounded norm convergence in banach lattices. *To appear in Positivity, DOI:10.1007/s11117-016-0446-9*, 2016.

[13] E. Yu. Emel'yanov. Infinitesimal analysis and vector lattices. *Siberian Adv. Math.*, 6(1):19–70, 1996. Siberian Advances in Mathematics.

[14] E. Yu. Emel'yanov and M. A. A. Marabeh. Two measure-free versions of the brezis-lieb lemma. *Vladikavkaz. Mat. Zh.*, 18(1):21–25, 2106.

[15] N. Gao. Unbounded order convergence in dual spaces. *J. Math. Anal. Appl.*, 419(1):347–354, 2014.

[16] N. Gao, V. G. Troitsky, and F. Xanthos. Uo-convergence and its applications to cesáro means in banach lattices. *To appear in Israel Journal of Math.*, 2016.

[17] N. Gao and F. Xanthos. Unbounded order convergence and application to martingales without probability. *J. Math. Anal. Appl.*, 415(2):931–947, 2014.

[18] M. Kandić, M. A. A. Marabeh, and V. G. Troitsky. Unbounded norm topology in Banach lattices. *J. Math. Anal. Appl.*, 451(1):259–279, 2017.

[19] S. Kaplan. On unbounded order convergence. *Real Anal. Exchange*, 23(1):175–184, 1997/98.

[20] A. G. Kusraev. *Dominated operators*, volume 519 of *Mathematics and its Applications*. Kluwer Academic Publishers, Dordrecht, 2000.

[21] A. G. Kusraev and S. S. Kutateladze. *Boolean valued analysis*, volume 494 of *Mathematics and its Applications*. Kluwer Academic Publishers, Dordrecht, 1999.

[22] S. S. Kutateladze, editor. *Nonstandard analysis and vector lattices*, volume 525 of *Mathematics and its Applications*. Kluwer Academic Publishers, Dordrecht, 2000.

[23] W. A. J. Luxemburg and A. C. Zaanen. *Riesz spaces. Vol. I.* North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., New York, 1971. North-Holland Mathematical Library.

[24] O. V. Maslyuchenko, V. V. Mykhaylyuk, and M. M. Popov. A lattice approach to narrow operators. *Positivity*, 13(3):459–495, 2009.

[25] R. Meise and D. Vogt. *Introduction to functional analysis*, volume 2 of *Oxford Graduate Texts in Mathematics*. The Clarendon Press, Oxford University Press, New York, 1997. Translated from the German by M. S. Ramanujan and revised by the authors.

[26] P. Meyer-Nieberg. *Banach lattices*. Universitext. Springer-Verlag, Berlin, 1991.

[27] H. Nakano. Ergodic theorems in semi-ordered linear spaces. *Ann. of Math. (2)*, 49:538–556, 1948.

[28] M. Pliev. Narrow operators on lattice-normed spaces. *Cent. Eur. J. Math.*, 9(6):1276–1287, 2011.

[29] V. G. Troitsky. Measures on non-compactness of operators on Banach lattices. *Positivity*, 8(2):165–178, 2004.

[30] B. Z. Vulikh. *Introduction to the theory of partially ordered spaces*. Translated from the Russian by Leo F. Boron, with the editorial collaboration of Adriaan C. Zaanen and Kiyoshi Iséki. Wolters-Noordhoff Scientific Publications, Ltd., Groningen, 1967.

[31] A. W. Wickstead. Weak and unbounded order convergence in Banach lattices. *J. Austral. Math. Soc. Ser. A*, 24(3):312–319, 1977.

[32] A. C. Zaanen. *Riesz spaces. II*, volume 30 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, 1983.

CURRICULUM VITAE

PERSONAL INFORMATION

Surname, Name: Aydın, Abdullah
Nationality: Turkish (TC)
Date and Place of Birth: 31.12.1986, Adıyaman
Marital Status: Married
Phone: 0 436 249 4949

EDUCATION

Degree	Institution	Year of Graduation
M.S.	Gazi University, Mathematics	2009
B.S.	Adıyaman University, Mathematics	2007
High School	Kahta Lisesi	2003

PROFESSIONAL EXPERIENCE

Year	Place	Enrollment
2009-2010	Muş Alparslan University-Mathematic	Research Assistant
2010-2016	METU- Mathematic	Research Assistant
2016-Present	Muş Alparslan University-Mathematic	Research Assistant

PUBLICATIONS

1. A. Aydın, E. Yu. Emelyanov, N. Erkurşun Özcan and M. A. A. Marabeh, Unbounded p-convergence in Lattice-Normed Vector Lattices, 2016, arXiv:1609.05301.
2. A. Aydın, S. G. Gorokhova and H. Gül, Nonstandard hulls of lattice-normed ordered vector spaces, to appear in Turkish Journal of Mathematics, DOI: 10.3906/mat-1612-59.
3. A. Aydın, E. Yu. Emelyanov, N. Erkurşun Özcan and M. A. A. Marabeh, Compact-like operators in lattice-normed spaces 2017, arXiv:1701.03073.

GIVEN TALKS

(1) Unbounded p-convergence in Lattice-Normed Vector Lattices, POSITIVITY Conference, December 3-4, 2016, Department of Mathematics, Abant Izzet Baysal University, Bolu, Turkey.

(2) Unbounded p-convergence in Lattice-Normed Vector Lattices, Graduate Seminar, April 4, 2017, Department of Mathematics, METU, Turkey.