RUNGE-KUTTA SCHEME FOR
STOCHASTIC OPTIMAL CONTROL PROBLEMS

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF APPLIED MATHEMATICS
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

HACER ÖZ BAKAN

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY
IN
FINANCIAL MATHEMATICS

SEPTEMBER 2017

Approval of the thesis:

**RUNGE-KUTTA SCHEME FOR
STOCHASTIC OPTIMAL CONTROL PROBLEMS**

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ABSTRACT

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STOCHASTIC OPTIMAL CONTROL PROBLEMS

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September 2017, 93 pages

In this thesis, we analyze Runge-Kutta scheme for the numerical solutions of stochastic optimal control problems by using *discretize-then-optimize* approach. Firstly, we discretize the cost functional and the state equation with the help of Runge-Kutta schemes. Then, we state the discrete Lagrangian and take the partial derivative of it with respect to its variables to get the discrete optimality system. By comparing the continuous and discrete optimality conditions, we find a relationship between the Runge-Kutta coefficients of the state and adjoint equation, so that we present Runge-Kutta scheme for the adjoint pair $(p(t), q(t))$. Similar to the deterministic setting, the issue of convergence is important when dealing with a numerical scheme. In stochastic case, this can be achieved either by using the strong-order convergence or weak-order convergence criteria. We match the stochastic Taylor expansion on the exact solution of continuous optimality system with the stochastic Taylor expansion of approximate solution of our discrete optimality system, term by term, in order to get both strong and weak-order conditions. The thesis ends with a conclusion and a future outlook to forthcoming research and application.

Keywords : Stochastic optimal control, Runge-Kutta discretization, Stochastic differential equations, Stochastic-Taylor expansion, Hamiltonian, Optimization, Stochastic partial differential equation

ÖZ

STOKASTİK OPTİMAL KONTROL PROBLEMLERİ İÇİN RUNGE-KUTTA YÖNTEMİ

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Eylül 2017, 93 sayfa

Bu tezde, önce ayrıklaştırma sonra optimize etme yaklaşımı kullanarak, stokastik optimal kontrol problemlerinin numerik çözümleri için Runge-Kutta yöntemini inceledik. İlk önce maliyet fonksiyonu ve durum denklemi Runge-Kutta yöntemini ile ayrıklaştırdık. Sonra, ayrıklaştırılmış optimallik koşullarını elde etmek için, Lagrange fonksiyonunun ayrıklaştırılmış halini verdik ve onun kısmi türevlerini aldık. Sürekli ve ayrıklaştırlmış optimallik koşullarını karşılaştırarak durum denkleminin ve adjoint denkleminin Runge-Kutta katsayıları bir bağlantı bularak, adjoint denklemi için Runge-Kutta yöntemini elde ettik. Deterministik durumda olduğu gibi, stokastik durumunda da numerik metodun yakınsama konusu önemlidir. Stokastik durum için, güçlü ve zayıf yakınsama olmak üzere iki çeşit yakınsama vardır. Her iki durum için de yakınsama koşullarını elde edebilmek amacıyla, sürekli optimallik koşullarının gerçek çözümünü ve ayrıklaştırılmış optimallik koşullarının yaklaşık çözümünü karşılaştırdık. Bu tez bir değerlendirme ve gelecek çalışmalara bir bakış ile sonuçlandırılmıştır.

Anahtar Kelimeler : Stokastik optimal kontrol, Runge-Kutta ayrıklaştırması, Stokastik diferansiyel denklemler, Stokastik-Taylor açılımı, Hamiltonian, Optimizasyon, Stokastik kısmi diferansiyel denklemler

To My Family

ACKNOWLEDGMENTS

I would like to express my very great appreciation to my thesis supervisor Prof. Dr. Gerhard-Wilhelm Weber for his patient guidance, enthusiastic encouragement, and valuable advices during the development and preparation of this thesis. His willingness to give his time and to share his experiences have brightened my path.

I am very grateful to Assist. Prof. Dr. Fikriye Yılmaz for patiently guiding, motivating and encouraging me throughout this work.

I acknowledge Atılım University for having a chance to work as a research assistance during my PhD education. I wish to extend my warmest thanks to all members of the department mathematics.

Special thanks to my committee members, Assoc. Prof. Dr. Ali Devin Sezer, Prof. Dr. Gerhard-Wilhelm Weber, Assoc. Prof. Dr. Ümit Aksoy, Assoc. Prof. Dr. Mehmet Turan and Assist. Prof. Dr. Hamdullah Yücel for their insightful comments and suggestions. Likewise, I convey my gratitude to Prof. Dr. Mustafa Çelebi Pınar.

Furthermore, I would like to thank all members of the Institute for Applied Mathematics of METU for their endless friendship and kindness.

And, I would like to express my thanks to TUBİTAK for its financial support during my education.

I am grateful to my friend Sinem Kozpınar for her love, unfailing support and patience.

Finally, I would like to thank my family for their support and help. My special gratitude goes to my husband Gökhan for providing me with unfailing support and continuous encouragement through the process of researching and writing this thesis.

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a.s.	Almost surely
$\|\|.\|\|$	Euclidean norm
d	Infinitesimal differential or increment
$\mathbb{E}$	Expectation of a stochastic process
$\mathcal{F}$	σ-algebra
$\mathcal{F}(t)$	Filtration, depend on time t
HJB	Hamilton-Jacobi-Bellman
J	Stratonovich Integral
ODE	Ordinary Differential Equation
$\mathbb{Z}^+$	Set of Positive Integers
$\mathbb{R}$	Set of Real Numbers
SDE	Stochastic Differential Equation
SPDE	Stochastic Partial Differential Equation
$\circ$	Stratonovich calculus integration symbol
tr	Trace of a matrix
W, $W(t)$	1-dimensional Brownian motion, depend on time t

CHAPTER 1

INTRODUCTION

Stochastic optimal control problems play a crucial role in financial mathematics and economics. For instance, Merton reduced portfolio problem, how to allocate safe and risky assets while maximizing the expected utility, to a control problem and then he solved it by using stochastic control theory [40, 41]. Another example in finance is optimal production planning problem; here, a company has to adjust or to control its production rate in order to meet the demand while minimizing the expected total cost [66]. The number of these examples can be increased. At the same time, solving such stochastic optimal control problems is also an important issue and there exist different approaches to tackle stochastic optimal control problems. The first one is called as the *duality methods*; here, the problem is reduced to one of finite dimensions and solved by using the martingale representation theorem and Girsanov transformation [3, 4]. The second approach is *dynamic programming*; the method is characterized by means of the *Hamilton-Jacobi-Bellman (HJB) equation*, leading to a partial differential equation, whose solution gives the value function [66]. The last approach is *Pontryagin's Maximum Principle* [3, 48], which is developed separately and independently from the HJB equation, consists of the original state equation and the so-called *adjoint process* $(p(t), q(t))$, defined by a stochastic differential equation (SDE) combined with a final condition. So, the resulting system introduces a *forward-backward stochastic differential equation (FBSDE)*. It is an interesting question which of these methods should be used to solve the stochastic optimal control problems. Moreover, it is sometimes difficult to find the analytical solutions of stochastic optimal control programs, or the problem does not even admit a global solution at all with the help of one of the mentioned methods. In this case, numerical methods gain importance.

The numerical solution methods of SDEs are similar to techniques developed for ordinary differential equations (ODEs), but they are extended to satisfy the stochastic dynamics. The most efficient and widely used approach to obtain an approximation process is given by discrete-time approximations which are essentially based on the Itô-Taylor expansions [10, 28]. Euler and Milstein schemes can both be regarded as simple methods and they are widely used. Platen and Kloeden [28] provided a deep investigations of the Itô-Taylor expansions that leads to many numerical schemes. The Itô-Taylor schemes use the derivative of the drift and diffusion coefficients, and this increases the computational cost considerably. At this point, it is reasonable to employ derivative-free schemes such as Runge-Kutta type methods [6, 8, 7, 9, 13, 28, 42, 60]. Burrage and Burrage presented a general class of stochastic Runge-Kutta methods in [6]. In [59], Tian and Burrage discussed two-stage diagonally implicit stochastic Runge-Kutta methods with strong order-1 for strong solutions of the Stratonovich SDEs. Finally, Rößler developed many Runge-Kutta schemes for both the Itô and Stratonovich SDEs [14, 54, 55].

In stochastic calculus, there are two ways to measure the convergence of a numerical method. When sample paths, trajectories, of the solution are needed, strong convergence criteria are used. So, strong approximations are practical for problems requiring direct simulations of dynamical sytems such as filtering or testing estimators of Itô proceses, stochastic flows. Recent developments have showed that these approximations are also important for Multi-level Monte-Carlo method for SDEs [19]. However, if one deals with only the probability distribution or some moments of the solution process, weak convergence criteria are employed. The most typical application of weak approximations is Monte-Carlo simulation of option prices [33].

Stochastic Runge-Kutta schemes of strong-order were well studied in [6, 7, 8, 55]. Because of Jensen's inequality, mean-square convergence implies strong convergence of the same order [13]. Thus, mean-square convergence is used to measure the strong convergence. Burrage and Rößler [6, 7, 8, 55] made use of Rooted Tree analysis invented by Butcher [9] and stochastic Taylor approximations to obtain strong convergence. Burrage [6, 7, 8] derived strong order-1 conditions of the Runge-Kutta scheme for SDEs. However, Burrage could not exceed strong order-1 for any number of stages for the same Runge-Kutta scheme. By introducing an additional random variable to the classical Runge-Kutta method, Burrage got strong order-1.5 conditions with the help of the stochastic Taylor series. Rößler [52, 55] studied on a different kind of Runge-Kutta scheme from that of Burrage to obtain strong-order conditions by using the Rooted Tree Theory and stochastic Taylor series.

Many stochastic Runge-Kutta methods converging in weak sense were proposed in recent years [14, 29, 37, 42, 60, 61]. It is worth noting that these stochastic Runge-Kutta methods are similar, but they were expressed differently. Moreover, the way used to measure the convergence was different. For example, Komori [29] and Rößler [54] derived weak-order conditions by the aid of Rooted-Tree analysis invented by Butcher [9]. Tocino and Ardanuy [60] got weak-order conditions for stochastic Runge-Kutta method by comparing the truncated Itô-Taylor expansions of the exact solution and the solution from the Runge-Kutta method. They studied on the Itô SDEs and obtained a remainder term as well. For this reason, they had to choose the diffusion coefficient that minimizes the remainder term. On the other hand, Mackevicius [37] and later Rößler [14] investigated Runge-Kutta schemes of the Stratonovich SDEs to avoid the remainder term which Tocino and Ardanuy [60] experienced. They made use of Itô Formula to expand the expectation of Taylor expansions.

Runge-Kutta schemes were also well studied for optimal control problems of ODEs [5, 15, 21, 27, 58]. Hager [21] derived the Runge-Kutta scheme for the optimal control problems of ODEs. He discretized the state equation by a Runge-Kutta scheme and observed that the resulting optimality system, after transforming some variables, is a partitioned Runge-Kutta scheme. Then, by stating the discrete Lagrangian of the problem, he found a relationship between the Runge-Kutta coefficients of the state variable and adjoint variable, which leads to a symplectic scheme. Afterwards, he compared the Taylor expansion of the discrete and continuous problem to measure the convergence of the Runge-Kutta scheme and computed the order conditions up to 4 for the optimal control problems of ODEs.

Motivation of this thesis is the desire to derive a Runge-Kutta scheme for stochastic optimal control problems. We examine the studies of Hager [15, 21], Runge-Kutta scheme for optimal control problems of ODEs, and stochastic Runge-Kutta schemes especially, investigations of Burrage [6, 7, 8] and Rößler [14, 52, 54, 55]. Then, we aim to extend the results of Hager [15, 21] to stochastic optimal control problems of SDEs with the help of stochastic Runge-Kutta schemes.

The main objective of this thesis is to develop a Runge-Kutta scheme for the numerical solution of stochastic optimal control problems described by SDEs through Pontryagin's Maximum Principle. Stochastic optimal control problems can be solved numerically with the help of either the *optimize-then-discretize* or the *discretize-then-optimize* approach [22, 26, 62, 65]. In this study, we prefer the *discretize-then-optimize* approach to gain the advantage of standard optimization techniques. Firstly, the cost functional and the state equation are discretized, by using Runge-Kutta scheme. Then, we formulate the discrete Lagrangian function and take the partial derivatives with respect to its variables and equate them to zero to obtain the discrete optimality conditions. In the resulting optimality system, we get the Runge-Kutta discretization of the coupled adjoint process $(p(t), q(t))$, whose coefficients can be stated in terms of the Runge-Kutta coefficients of the state equation that is our main contribution. In order to compute the expectation, we use the Monte-Carlo method which, firstly, draws independent simulations, then, approximates the cost functional by using Runge-Kutta scheme. Finally, it averages the independent samples of the resulting cost functional to get an estimation of the expectation. We apply our Runge-Kutta scheme to some problems selected from the financial sector, and present a comparison of the numerical results with the exact solutions. We also employ Euler method to test the efficiency of our Runge-Kutta scheme.

We also aim to extend our Runge-Kutta method to stochastic optimal control problems governed by some SPDEs. We choose a special and an important problem in economics, finance and biology that is called as *optimal harvesting* problem. This problem is closely associated of daily life, e.g., agriculture, fisheries, forestry, gardening, tourism, city planing and water management. We first discretize the problem with respect to the space variable with the help of using the finite difference scheme and convert the given problem to an optimal control problem of system of SDEs. Then, by following the same methodology as done in the SDE case, we are able to derive a Runge-Kutta method on the numerical solution of stochastic control problems subject to system of SDEs.

In the second part of the thesis, our aim is to address strong convergence criteria in order to measure the convergence of our Runge-Kutta method for optimal control problems of SDEs. By assuming exact initial values, the Stratonovich-Taylor expansions of the exact solution and the solution from our Runge-Kutta scheme are compared to find the order of accuracy. In our Runge-Kutta scheme for stochastic optimal control problems, Runge-Kutta coefficients of the adjoint process have been obtained in terms of the Runge-Kutta coefficients of the state process. This yields additional order conditions to classical Runge-Kutta method of SDEs [6, 8] for the order of accuracy. In this work, such order conditions are derived explicitly.

In the third part of the thesis, we purpose to follow the idea of Mackevicius [37] and Rößler [14] in order to derive weak-order conditions of our Runge-Kutta scheme. By assuming exact initial values, expectations of the Taylor expansions of the exact solution and the solution from our Runge-Kutta scheme are compared to find the order of accuracy. In our Runge-Kutta scheme for stochastic optimal control problems, we show that Runge-Kutta discretization of the adjoint process is often different from the Runge-Kutta discretization of the state process. Herewith, there occur additional weak-order conditions to classical Runge-Kutta conditions of SDEs for the order of accuracy.

The outline of this thesis is as follows: In the preliminaries, presented in Chapter 2, we give the problem formulation and fundamental derivations of the Stochastic Pontryagin's Maximum Principle. In Chapter 3, the stochastic optimal control problem is discretized by Runge-Kutta schemes and then, by using *discretize-then-optimize* approach, the discrete optimality conditions for the stochastic optimal control programs described by SDEs are derived. In Chapter 4, our Runge-Kutta method is applied to stochastic optimal control problems of stochastic partial differential equations (SPDEs), as a special case of the optimal harvesting problem. In Chapter 5, we obtain strong order-1 and 1.5 conditions of our Runge-Kutta scheme for stochastic optimal control programs. In Chapter 6, we provide weak order-1 and 2 conditions of our Runge-Kutta scheme for stochastic optimal control problems. In Chapter 7, we conclude and give an outlook to future studies and applications.

CHAPTER 2

PRELIMINARIES

In this chapter, we introduce a stochastic optimal control problem which we use in this thesis. We derive of the stochastic Hamilton-Jacobi-Bellman (HJB) equation with the help of Itô Formula [24, 44, 57]. Then, we relate it with stochastic Pontryagin's Maximum Principle, an extension of deterministic Pontryagin's Maximum Principle [51], to present a continuous optimality system. More details about stochastic optimal control theory can be found in [32, 66].

2.1 Stochastic Optimal Control Problem

We let $(W(t))_{t_0 \leq t \leq T}$ (with $W(0) = 0$ a.s.) be a 1-dimensional Brownian motion on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}(t))_{t\in[t_0,T]}, \mathbb{P})$, where $t_0 > 0$ and $\Omega = [t_0, T]$ is a fixed finite horizon. On this probability space, the space of real-valued square integrable $(\mathcal{F}(t))$-adapted processes is defined in $L^2(t_0, T)$. While first we address scalar-valued processes, in later chapters, also vector-valued processes will be permitted.

We consider a controlled SDE

$$dy(t) = f(t, y(t), u(t))dt + h(t, y(t), u(t))dW(t) \quad (t \in [0, T]), \quad y(t_0) = y^0, \tag{2.1}$$

where $f(t, y(t), u(t))$ and $h(t, y(t), u(t))$ are continuously differentiable functions with respect to $(t, y(t), u(t))$ and their derivatives are uniformly bounded. Under these assumptions, we assure that Eqn. (2.1) has a unique solution [23]. Also, we let $u = (u(t))_{t\in[t_0,T]}$ is a control process in $\mathcal{A}$ which is a closed convex set in the control space $L^2(t_0, T)$.

The objective of the optimal control problem is:

$$(\mathcal{P}) \begin{cases} \underset{u\in\mathcal{A}}{\text{minimize}} & \mathbb{E}\left[\phi(T, y(T)) + \int_{t_0}^{T} g(s, y(s), u(s))ds\right] \\ \text{subject to} & dy(t) = f(t, y(t), u(t))dt + h(t, y(t), u(t))dW(t) \quad (t \in [t_0, T]), \\ & y(t_0) = y^0, \end{cases}$$

where $\Phi(T, y(T))$ and $g(t, y(t), u(t))$ are smooth functions with the continuous first-order derivatives. A control process $u^*(t)$ that solves this problem is called an *optimal control*.

2.2 Hamilton-Jacobi-Bellman (HJB) Equation

We define cost functional $J(t, y)$ by

$$J(t,y) = \min_{u \in \mathcal{A}} \mathbb{E}\left[\phi(T, y(T)) + \int_t^T g(s,y,u)ds\right];$$

we divide the cost functional in two parts, for any sufficiently small $\Delta t > 0$:

$$J(t,y) = \min_{u \in \mathcal{A}} \mathbb{E}\Bigg[\int_t^{t+\Delta t} g(s,y,u)ds + \underbrace{\Phi(T, y(T)) + \int_{t+\Delta t}^T g(s,y,u)ds}_{J(t+\Delta t, y)}\Bigg]. \quad (2.2)$$

Here, we note that the variables y and u depend on s; we use y and u instead of $y(s)$ and $u(s)$, respectively. From now on, for simplicity, we will use this abbreviation for variables and Brownian motion W, and for the increment dW instead of $dW(s)$ as well.

It is also important to note that we can write "min" in the aforementioned representations of the cost functions rather than "inf", as in our research, the infimum will be attained as a value.

By using the Itô Formula, we have

$$\begin{aligned} J(t+\Delta t, y) = J(t,y) + \int_t^{t+\Delta t} \Bigg(&\frac{\partial J(s,y)}{\partial s} + f^T(s,y,u)\frac{\partial J(s,y)}{\partial y} \\ &+ \frac{1}{2} tr\left\{ \frac{\partial^2 J(s,y)}{\partial y^2} h^T(s,y,u)h(s,y,u)\right\}\Bigg) ds \\ &+ \int_t^{t+\Delta t} \frac{\partial J(s,y)}{\partial y} h(s,y,u)dW, \end{aligned} \quad (2.3)$$

where $tr\left\{\frac{\partial^2 J(s,y)}{\partial y^2} h^T(s,y,u)h(s,y,u)\right\}$ stands for the trace of the matrix $\frac{\partial^2 J(s,y)}{\partial y^2} h^T(s,y,u)h(s,y,u)$. In the 1-dimensional case,

$$\frac{\partial^2 J(s,y)}{\partial y^2} h^T(s,y,u)h(s,y,u) = \frac{\partial^2 J(s,y)}{\partial y^2} h^2(s,y,u).$$

Now, by inserting Eqn. (2.3) into Eqn. (2.2), we get

$$
\begin{aligned}
J(t,y) &= \min_{u\in\mathcal{A}} \mathbb{E}\left[\int_t^{t+\Delta t} g(s,y,u)ds + J(t+\Delta t, y)\right] \\
&= \min_{u\in\mathcal{A}} \mathbb{E}\left[\int_t^{t+\Delta t} g(s,y,u)ds + J(t,y) + \int_t^{t+\Delta t}\left(\frac{\partial J(s,y)}{\partial s}\right.\right. \\
&\quad \left. + \ f^T(s,y,u)\frac{\partial J(s,y)}{\partial y} + \frac{1}{2}tr\left\{\frac{\partial^2 J(s,y)}{\partial y^2}h^T(s,y,u)h(s,y,u)\right\}\right) ds \\
&\quad \left. + \int_t^{t+\Delta t}\frac{\partial J(s,y)}{\partial y}h(s,y,u)dW\right] \\
&= \min_{u\in\mathcal{A}} \mathbb{E}\left[\int_t^{t+\Delta t} g(s,y,u)ds + J(t,y) + \int_t^{t+\Delta t}\left(\frac{\partial J(s,y)}{\partial s}\right.\right. \\
&\quad \left.\left. + \ f^T(s,y,u)\frac{\partial J(s,y)}{\partial y} + \frac{1}{2}tr\left\{\frac{\partial^2 J(s,y)}{\partial y^2}h^T(s,y,u)h(s,y,u)\right\}\right) ds\right].
\end{aligned}
$$

This gives for any sufficiently small $\Delta t > 0$:

$$
\begin{aligned}
0 = \min_{u\in\mathcal{A}} \mathbb{E}&\left[\int_t^{t+\Delta t}\left(g(s,y,u) + \frac{\partial J(s,y)}{\partial s} + f^T(s,y,u)\frac{\partial J(s,y)}{\partial y}\right.\right. \\
&\left.\left. + \ \frac{1}{2}tr\left\{\frac{\partial^2 J(s,y)}{\partial y^2}h^T(s,y,u)h(s,y,u)\right\}\right) ds\right],
\end{aligned}
$$

so that we obtain the following *Hamilton-Jacobi-Bellman (HJB) equation*

$$
\begin{aligned}
0 = \min_{u\in\mathcal{A}} &\left(g(t,y,u) + \frac{\partial J(t,y)}{\partial t} + f^T(t,y,u)\frac{\partial J(t,y)}{\partial y}\right. \\
&\left. + \ \frac{1}{2}tr\left\{\frac{\partial^2 J(t,y)}{\partial y^2}h^T(t,y,u)h(t,y,u)\right\}\right) \text{ a.s.}
\end{aligned} \tag{2.4}
$$

For the ease of exposition, the addition a.e. or a.s. is not always made in this thesis.

2.3 Stochastic Pontryagin's Maximum Principle

We define *Hamiltonian function* of the optimal control problem $(\mathcal{P})$:

$$
\mathcal{H}(t,y,u,p,q) := g(t,y,u) + f^T(t,y,u)p + \frac{1}{2}tr\left\{qh(t,y,u)h^T(t,y,u)\right\},
$$

where the coupled process, $(p(t), q(t))$, is adapted with respect to $(\mathcal{F}(t))_{t\in[t_0,T]}$ and

$$
\begin{aligned}
p(t) &= J_y(t,y), \\
q(t) &= \frac{\partial p}{\partial y} = J_{yy}(t,y).
\end{aligned} \tag{2.5}
$$

Here and in the entire thesis, we note that the subscripts, e.g., t, y, yy, denote the partial derivatives of $J(t, y)$ with respect to these variables. For notational convenience, in the rest of the whole work, we will use the variables, written as subscripts, as the partial derivatives.

Now, by using the Hamiltonian function, the HJB Eqn. (2.4) can be rewritten as:

$$-J_t(t, y) = \min_{u \in \mathcal{A}} \mathcal{H}(t, y(t), u(t), p(t), q(t)). \tag{2.6}$$

We assume that there exists a known optimal control $u^*(t, y(t), p(t), q(t))$ that solves the optimal control problem such that

$$\begin{aligned}
\mathcal{H}^*(t, y, p, q) &= \mathcal{H}(t, y, u^*(t, y, p, q), p, q) \\
&= g(t, y, p, q) + f^T(t, y, p, q)p + \frac{1}{2} tr \left\{ q h(t, y, p, q) h^T(t, y, p, q) \right\} \\
&= -J_t(t, y).
\end{aligned} \tag{2.7}$$

Herewith, we write the SDEs for the state and adjoint differentials on dy and dp, respectively:

$$\begin{aligned}
dy &= f(t, y, u^*)dt + h(t, y, u^*)dW \\
&= \mathcal{H}_p^*(t, y, p, q)dt + h(t, y, p, q)dW,
\end{aligned}$$

and applying the Itô Formula on the definition of $p(t)$ in Eqn. (2.5) leads to

$$\begin{aligned}
dp &= J_{yt}(t, y)dt + J_{yy}(t, y)dy + \frac{1}{2} J_{yyy}(t, y)dydy \\
&= \left(J_{yt}(t, y) + J_{yy}(t, y)f(t, y, u) + \frac{1}{2} tr \left\{ J_{yyy}(t, y)h(t, y, u)h^T(t, y, u) \right\} \right) dt \\
&\qquad + J_{yy}(t, y)h(t, y, u)dW.
\end{aligned} \tag{2.8}$$

Hence, we take the partial derivative of Eqn. (2.7) with respect to y in order to get $J_{yt}(t, y)$:

$$\begin{aligned}
-J_{yt}(t, y) &= \mathcal{H}_y^*(t, y, p, q) + \mathcal{H}_p^*(t, y, p, q)\frac{\partial p}{\partial y} + \mathcal{H}_q^*(t, y, p, q)\frac{\partial q}{\partial y} \\
&= \mathcal{H}_y^*(t, y, p, q) + J_{yy}(t, y)f(t, y, u) \\
&\qquad + \frac{1}{2} tr \left\{ J_{yyy}(t, y)h(t, y, u)h^T(t, y, u) \right\}.
\end{aligned} \tag{2.9}$$

Inserting Eqn. (2.9) into Eqn. (2.8) yields the adjoint equation:

$$\begin{aligned} dp &= -\mathcal{H}_y^*(t,y,p,q)dt + J_{yy}(t,y)h(t,y,p,q)dW \\ &= -\mathcal{H}_y^*(t,y,p,q)dt + qh(t,y,p,q)dW. \end{aligned}$$

Finally, we can state the system of forward-backward stochastic differential equation (FBSDE) of the problem $(\mathcal{P})$, the Stochastic Maximum Principle, as:

$$\begin{cases} \quad dy = \mathcal{H}_p(t,y,u,p,q)dt + h(t,y,u,p,q)dW \quad (t \in [t_0,T]), \\ \quad dp = -\mathcal{H}_y(t,y,u,p,q)dt + qh(t,y,u,p,q)dW \quad (t \in [t_0,T]), \\ \quad 0 = \mathcal{H}_u(t,y,u,p,q) \quad (t \in [t_0,T]), \\ y(t_0) = y^0, \\ p(T) = \phi_y(T,y(T)), \\ \mathcal{H}^*(t,y,p,q) = \min\limits_{u \in \mathcal{A}} \mathcal{H}(t,y,u,p,q). \end{cases}$$

In this thesis, we restrict our investigations to autonomous stochastic optimal control problems, where diffusion terms do not contain the control process in the form:

$$(\mathcal{P}_c) \begin{cases} \underset{u \in L^2(t_0,T)}{\text{minimize}} & \mathbb{E}\left[\Phi(y(T)) + \int_{t_0}^{T} g(y,u)dt\right] \\ \text{subject to} & dy = f(y,u)dt + h(y)dW \quad (t \in [t_0,T]), \\ & y(t_0) = y^0. \end{cases}$$

We note that every nonautonomous stochastic optimal control problem can be canonically transformed into an autonomous system with one additional equation.

In this case, the first-order optimality conditions of problem $(\mathcal{P}_c)$:

$$(\mathcal{OC}_c) \begin{cases} \quad dy = \mathcal{H}_p(y,u,p,q)dt + h(y)dW \quad (t \in [t_0,T]), \\ \quad dp = -\mathcal{H}_y(y,u,p,q)dt + h(y)qdW \quad (t \in [t_0,T]), \\ \quad 0 = \mathcal{H}_u(y,u,p,q) \quad (t \in [t_0,T]), \\ y(t_0) = y^0, \\ p(T) = \phi'(y(T)), \end{cases}$$

with

$$\mathcal{H}(y,u,p,q) = g(y,u) + f^T(y,u)p + \frac{1}{2}tr\left\{qh(y)h^T(y)\right\}, \tag{2.10}$$

where $\phi'(y(T))$ denotes the derivative of ϕ with respect to its variable y. We will use $'$ for the derivative of differentiable functions which depend on one variable only.

2.4 Runge-Kutta Scheme for SDEs

In this section, we recall the Runge-Kutta scheme for SDEs. We consider the following SDE

$$dy = f(y)dt + h(y)dW, \quad y_0 = y^0. \tag{2.11}$$

We introduce an equispaced discretization $0 = t_0 < t_1 < \ldots < t_k < \ldots < t_N = T$ of the time interval $[0, T]$. Let $\Delta := T/N$ denote the increments (step-size) and $\Delta W := W_{t_{k+1}} - W_{t_k}$ be $\mathcal{N}(0, \Delta)$-distributed Gaussian increment of the Brownian motion W.

We address an s-stage Runge-Kutta scheme [8] of Eqn. (2.11), for some $s \in \mathbb{Z}^+$:

$$\begin{cases} y_{k+1} = y_k + \Delta \sum_{i=1}^{s} \alpha_i f(y_{ki}) + \Delta W \sum_{i=1}^{s} \beta_i h(y_{ki}) \\ y_{ki} = y_k + \Delta \sum_{j=1}^{s} a_{ij} f(y_{kj}) + \Delta W \sum_{j=1}^{s} b_{ij} h(y_{kj}) \\ y_0 = y^0, \end{cases}$$

for $k = 0, 1, \ldots, N-1$, and $i = 1, 2, \ldots, s$, and the constants α_i, β_i, a_{ij}, b_{ij} are the Runge-Kutta coefficients. The Butcher array of the Runge-Kutta discretization of Eqn. (2.11) is given by

$$\begin{array}{c|ccc} c_1 & a_{11} & \ldots & a_{1s} \\ \vdots & \vdots & \ddots & \vdots \\ c_s & a_{s1} & \ldots & a_{ss} \\ \hline & \alpha_1 & \ldots & \alpha_s \end{array} \qquad \begin{array}{c|ccc} d_1 & b_{11} & \ldots & b_{1s} \\ \vdots & \vdots & \ddots & \vdots \\ d_s & b_{s1} & \ldots & b_{ss} \\ \hline & \beta_1 & \ldots & \beta_s \end{array}$$

where

$$c_i = \sum_{j=1}^{s} a_{ij} \qquad \text{and} \qquad d_i = \sum_{j=1}^{s} b_{ij},$$

for $i = 1, 2, \ldots, s$.

Here, we note that the Runge-Kutta coefficients α_i, β_i, a_{ij}, b_{ij}, constant real numbers, could be chosen arbitrarily or in a way such that some convergence properties are satisfied.

CHAPTER 3

RUNGE-KUTTA SCHEME FOR STOCHASTIC OPTIMAL CONTROL PROBLEMS OF SDEs

3.1 Introduction

In order to solve an stochastic optimal control problem, we need a discretization technique together with an optimization method. There are two possible options often referred to *discretize-then-optimize* and *optimize-then-discretize*. In this chapter, we construct a Runge-Kutta scheme for a class of optimal control problems of SDEs by following the *discretize-then-optimize* approach. Firstly, we discretize the cost functional and the state equation with the help of Runge-Kutta schemes. After we state the discrete Lagrangian, we take partial derivatives of it with respect to its variables to receive the discrete optimality system. Our main contribution is to get an implicit Runge-Kutta scheme for the adjoint pair $(p(t), q(t))$, whose Runge-Kutta coefficients can be written in terms of the Runge-Kutta coefficients of the state equation. Finally, we confirm our results with some numerical examples from the financial sector. We compare our numerical results with Euler method and exact solution to demonstrate the efficiency of our Runge-Kutta method.

Now, we recall our optimal control problem $(\mathcal{P}_c)$ as:

$$(\mathcal{P}_c)\begin{cases} \underset{u\in L^2(t_0,T)}{\text{minimize}} & \mathbb{E}\left[\Phi(y(T))] + \int_{t_0}^{T} g(y,u)dt\right] \\ \text{subject to} & dy = f(y,u)dt + h(y)dW \quad (t \in [t_0,T]), \\ & y(t_0) = y^0, \end{cases}$$

with the first-order optimality conditions of problem $(\mathcal{P}_c)$:

$$(\mathcal{OC}_c)\begin{cases} dy = \mathcal{H}_p(y,u,p,q)dt + h(y)dW \quad (t \in [t_0,T]), \\ dp = -\mathcal{H}_y(y,u,p,q)dt + h(y)qdW \quad (t \in [t_0,T]), \\ \quad 0 = \mathcal{H}_u(y,u,p,q) \quad (t \in [t_0,T]), \\ y(t_0) = y^0, \\ p(T) = \phi'(y(T)), \end{cases}$$

where

$$\mathcal{H}(y,u,p,q) = g(y,u) + f(y,u)p + \frac{1}{2}qh^2(y).$$

3.2 Runge-Kutta Scheme for Stochastic Optimal Control Problems of SDEs

Runge-Kutta schemes are applied to optimal control problems in [5, 15, 21, 27]. In [21], Hager showed that the resulting optimality system is a partitioned Runge-Kutta scheme, after some change of variables. In this chapter, we employ a Runge-Kutta scheme for stochastic optimal control problems of SDEs.

We introduce an equispaced discretization $0 = t_0 < t_1 < \ldots < t_k < \ldots < t_N = T$ of the time interval $[0,T]$. Let $\Delta := T/N$ denote the increments (step-size) and $\Delta W := W_{t_{k+1}} - W_{t_k}$ be $\mathcal{N}(0,\Delta)$-distributed Gaussian increment of the Brownian motion W.

Now, we state the s-stage Runge-Kutta discretization [21], for some $s \in \mathbb{Z}^+$, of the optimal control problem $(\mathcal{P}_c)$ as

$$(\mathcal{P}_d)\begin{cases} \text{minimize} & \mathbb{E}\left[\Phi(y_N) + \Delta \sum_{k=0}^{N-1}\sum_{i=1}^{s} \alpha_i g(y_{ki},u_{ki})\right] \\ \text{subject to} & y_{k+1} = y_k + \Delta \sum_{i=1}^{s} \alpha_i f(y_{ki},u_{ki}) + \Delta W \sum_{i=1}^{s} \beta_i h(y_{ki}), \\ & y_{ki} = y_k + \Delta \sum_{j=1}^{s} a_{ij} f(y_{kj},u_{kj}) + \Delta W \sum_{j=1}^{s} b_{ij} h(y_{kj}), \\ & y_0 = y^0, \end{cases}$$

for $k = 0,1,\ldots,N-1$, and $i = 1,2,\ldots,s$, and the constants α_i, β_i, a_{ij}, b_{ij} are the Runge-Kutta coefficients. The Butcher array of the Runge-Kutta discretization of problem $(\mathcal{P}_d)$ is given by

$$\begin{array}{c|ccc} c_1 & a_{11} & \ldots & a_{1s} \\ \vdots & \vdots & \ddots & \vdots \\ c_s & a_{s1} & \ldots & a_{ss} \\ \hline & \alpha_1 & \ldots & \alpha_s \end{array} \qquad \begin{array}{c|ccc} d_1 & b_{11} & \ldots & b_{1s} \\ \vdots & \vdots & \ddots & \vdots \\ d_s & b_{s1} & \ldots & b_{ss} \\ \hline & \beta_1 & \ldots & \beta_s \end{array}$$

where

$$c_i = \sum_{j=1}^{s} a_{ij} \qquad \text{and} \qquad d_i = \sum_{j=1}^{s} b_{ij},$$

for $i = 1,2,\ldots,s$.

Here, we note that the Runge-Kutta coefficients, α_i, β_i, a_{ij}, b_{ij}, are real constants. In this chapter, they are chosen arbitrarily to obtain numerical results. However, in Chapter 5 and Chapter 6, they are determined in a way such that strong and weak convergence properties are satisfied.

Now, we have a discrete state equation and a discrete cost functional. In the following theorem, we get discrete optimality conditions by defining the discrete Lagrangian.

Theorem 3.1. *If $\alpha_i,\ \beta_i,\ a_{ij},\ b_{ij}$ $(i, j = 1, 2, \ldots, s)$ are the Runge-Kutta coefficients in problem $(\mathcal{P}_d)$, then discrete first-order optimality conditions of problem $(\mathcal{P}_d)$ are obtained as*

$$(\mathcal{OC}_d)\left\{\begin{aligned} y_{k+1} &= y_k + \Delta\sum_{i=1}^{s}\alpha_i f(y_{ki}, u_{ki}) + \Delta W\sum_{i=1}^{s}\beta_i h(y_{ki}),\\ y_{ki} &= y_k + \Delta\sum_{j=1}^{s}a_{ij} f(y_{kj}, u_{kj}) + \Delta W\sum_{j=1}^{s}b_{ij} h(y_{kj}),\\ p_{k+1} &= p_k - \Delta\sum_{i=1}^{s}\tilde{\alpha}_i \mathcal{H}_y(y_{ki}, u_{ki}, p_{ki}, q_{ki}) + \Delta W\sum_{i=1}^{s}\tilde{\beta}_i h(y_{ki})q_{ki},\\ p_{ki} &= p_k - \Delta\sum_{j=1}^{s}\tilde{a}_{ij} \mathcal{H}_y(y_{kj}, u_{kj}, p_{kj}, q_{kj}) + \Delta W\sum_{j=1}^{s}\tilde{b}_{ij} h(y_{kj})q_{kj},\\ q_{ki}\psi_{ki} &= p_k - \Delta\sum_{j=1}^{s}\hat{a}_{ij} \mathcal{H}_y(y_{kj}, u_{kj}, p_{kj}, q_{kj}) + \Delta W\sum_{j=1}^{s}\hat{b}_{ij} h(y_{kj})q_{kj},\\ p_N &= \phi'(y_N),\\ y_0 &= y^0,\\ 0 &= \Delta\sum_{i=1}^{s}\alpha_i \mathcal{H}_u(y_{ki}, u_{ki}, p_{ki}, q_{ki}), \end{aligned}\right.$$

for $k = 0, 1, \ldots, N-1$, where the coefficients satisfy the following relations:

$$\begin{aligned} \tilde{\alpha}_i &:= \alpha_i, & \tilde{\beta}_i &:= \beta_i,\\ \tilde{a}_{ij} &:= \alpha_j - \frac{\alpha_j}{\alpha_i}a_{ji}, & \tilde{b}_{ij} &:= \beta_j - \frac{\beta_j}{\alpha_i}a_{ji},\\ \hat{a}_{ij} &:= \alpha_j - \frac{\alpha_j}{\beta_i}b_{ji}, & \hat{b}_{ij} &:= \beta_j - \frac{\beta_j}{\beta_i}b_{ji}, \end{aligned} \tag{3.1}$$

with

$$\psi_{ki} := \frac{\Delta\alpha_i h(y_{ki})}{\Delta W\beta_i} - \frac{h(y_{ki})}{h_y(y_{ki})}.$$

Proof. In order to prove this theorem, we follow the proof of Hager [21], in the deterministic case and we extend his proof to the stochastic settings. Let

$$k_{ki} := f(y_{ki}, u_{ki})$$

and

$$m_{ki} := h(y_{ki}),$$

so that we have

$$\begin{aligned}
g(y_{ki}, u_{ki}) &= g\left(y_k + \Delta \sum_{j=1}^{s} a_{ij} k_{kj} + \Delta W \sum_{j=1}^{s} b_{ij} m_{kj}, u_{ki}\right), \\
f(y_{ki}, u_{ki}) &= f\left(y_k + \Delta \sum_{j=1}^{s} a_{ij} k_{kj} + \Delta W \sum_{j=1}^{s} b_{ij} m_{kj}, u_{ki}\right), \\
h(y_{ki}) &= h\left(y_k + \Delta t \sum_{j=1}^{s} a_{ij} k_{kj} + \Delta W \sum_{j=1}^{s} b_{ij} m_{kj}\right).
\end{aligned}$$

Then, we can write the discrete Lagrangian as [11]:

$$\begin{aligned}
&\mathbb{E}\Big[\Phi(y_N) + p^0(y^0 - y_0) \\
&+ \sum_{k=0}^{N-1} \Bigg\{ \Delta \sum_{i=1}^{s} \alpha_i g\left(y_k + \Delta \sum_{j=1}^{s} a_{ij} k_{kj} + \Delta W \sum_{j=1}^{s} b_{ij} m_{kj}, u_{ki}\right) \\
&+ p_{k+1}\left(y_k - y_{k+1} + \Delta \sum_{i=1}^{s} \alpha_i k_{ki} + \Delta W \sum_{i=1}^{s} \beta_i m_{ki}\right) \\
&+ \sum_{i=1}^{s} \xi_{ki}\left(f\left(y_k + \Delta \sum_{j=1}^{s} a_{ij} k_{kj} + \Delta W \sum_{j=1}^{s} b_{ij} m_{kj}, u_{ki}\right) - k_{ki}\right) \\
&+ \sum_{i=1}^{s} \zeta_{ki}\left(h\left(y_k + \Delta t \sum_{j=1}^{s} a_{ij} k_{kj} + \Delta W \sum_{j=1}^{s} b_{ij} m_{kj}\right) - m_{ki}\right)\Bigg\}\Bigg],
\end{aligned}$$

where p^0, p_{k+1}, ξ_{ki}, ζ_{ki} are the Lagrange multipliers. Setting to 0 the partial derivatives of this Lagrangian function with respect to y_N, y_0, y_k, for $k = 0, 1, 2, \ldots, N-1$, and

k_{ki}, m_{ki}, u_{ki}, for $k = 0, 1, \ldots, N-1$, we get:

$$\begin{aligned}
p_N &= \Phi'(y_N), \\
p_1 &= p^0, \\
p_k - p_{k+1} &= \Delta \sum_{i=1}^{s} \alpha_i g_y(y_{ki}, u_{ki}) + \sum_{i=1}^{s} \xi_{ki} f_y(y_{ki}, u_{ki}) + \sum_{i=1}^{s} \zeta_{ki} h_y(y_{ki}), \\
\xi_{ki} &= \Delta \sum_{j=1}^{s} \Delta \alpha_j a_{ji} g_y(y_{ki}, u_{ki}) + \Delta \alpha_i p_{k+1} \\
&\quad + \Delta \sum_{j=1}^{s} a_{ji} f_y(y_{kj}, u_{kj}) \xi_{kj} + \Delta \sum_{j=1}^{s} a_{ji} h_y(y_{kj}) \zeta_{kj}, \\
\zeta_{ki} &= \Delta \sum_{j=1}^{s} \Delta W \alpha_j b_{ji} g_y(y_{ki}, u_{ki}) + \Delta W \beta_i p_{k+1} \\
&\quad + \Delta W \sum_{j=1}^{s} b_{ji} f_y(y_{kj}, u_{kj}) \xi_{kj} + \Delta W \sum_{j=1}^{s} b_{ji} h_y(y_{kj}) \zeta_{kj}, \\
0 &= \Delta \alpha_i g_u(y_{ki}, u_{ki}) + f_u(y_{ki}, u_{ki}) \xi_{ki}.
\end{aligned}$$

In order to compare the continuous optimality system, $(\mathcal{OC}_c)$, with the above system of equations, we set

$$\xi_{ki} := \Delta \alpha_i p_{ki}$$

and

$$\zeta_{ki} := \Delta W \beta_i q_{ki} \psi_{ki}$$

with

$$\psi_{ki} := \frac{\Delta \alpha_i h(y_{ki})}{\Delta W \beta_i} - \frac{h(y_{ki})}{h_y(y_{ki})},$$

where $\alpha_i \neq 0$ and $\beta_i \neq 0$ $(i = 1, 2, \ldots, s)$. If these values vanish, then the solution of the discrete problem may not converge to the solution of the continuous problem.

By eliminating ξ_{ki} and ζ_{ki} in the above equations, we obtain the desired result. □

It is worth noting that although our Runge-Kutta method in $(\mathcal{OC}_d)$ seems to be a symplectic Runge-Kutta, it is not symplectic. If we add one more condition, $\beta_i a_{ji} = \alpha_i b_{ji}$ ($i, j = 1, 2, \ldots, s$), to Eqns. (3.1), then the resulting Runge-Kutta scheme becomes a symplectic method [36, 63]. However, in the rest of the thesis, we will continue to study on our Runge-Kutta scheme, a more general scheme than a symplectic one.

3.3 Monte-Carlo Simulation and Implementation Details

We use Monte-Carlo method to approximate the conditional expectations. Monte-Carlo simulation of $\mathbb{E}[g(y,u)]$ is based on an approximation of the form

$$\bar{g}(y,u) = \frac{1}{M}\sum_{i=1}^{M} g_i(y_i,u_i), \tag{3.2}$$

where y_i and u_i are approximations of $y(t)$ and $u(t)$, respectively, at the ith orbit of the Monte-Carlo method [20, 30, 31].

Now, we give details of our computational efforts. Before summarizing the algorithm, we elaborate on the approximate cost functional and the gradient. The discrete cost functional coming from the *discretize-then-optimize* approach is

$$\mathbb{E}\left[\Phi(y_N) + \Delta\sum_{k=0}^{N-1}\sum_{i=1}^{s}\alpha_i g(y_{ki},u_{ki})\right], \tag{3.3}$$

where

$$y_{ki} = y_k + \Delta\sum_{j=1}^{s} a_{ij} f(y_{kj},u_{kj}) + \Delta W\sum_{j=1}^{s} b_{ij} h(y_{kj}).$$

Moreover, the discrete gradient is

$$\sum_{k=0}^{N}\sum_{i=1}^{s}\Delta\alpha_i\left(g_u(y_{ki},u_{ki}) + f_u(y_{ki},u_{ki})\right)p_{ki}. \tag{3.4}$$

Algorithm (*Discretize-then-Optimize* Approach with Gradient Descent)

1. Initialize the control and a tolerance $\epsilon > 0$. Choose the number of orbits M to be used in Monte-Carlo simulation.

2. $For\ i = 1\ to\ M$:

    i. Use Runge-Kutta scheme to discretize the state equation:

$$\begin{cases} y_{k+1} = y_k + \Delta\sum_{i=1}^{s}\alpha_i f(y_{ki},u_{ki}) + \Delta W\sum_{i=1}^{s}\beta_i h(y_{ki}), \\ y_{ki} = y_k + \Delta\sum_{j=1}^{s} a_{ij} f(y_{kj},u_{kj}) + \Delta W\sum_{j=1}^{s} b_{ij} h(y_{kj}), \\ y_0 = y^0, \end{cases}$$

    for $k = 0,1,\ldots,N-1$, and $i = 1,2,\ldots,s$.

ii. Use Runge-Kutta scheme to discretize the adjoint equation:

$$\begin{cases} p_{k+1} = p_k - \Delta \sum_{i=1}^{s} \tilde{\alpha}_i \mathcal{H}_y(y_{ki}, u_{ki}, p_{ki}, q_{ki}) + \Delta W \sum_{i=1}^{s} \tilde{\beta}_i h(y_{ki}) q_{ki}, \\ p_{ki} = p_k - \Delta \sum_{j=1}^{s} \tilde{a}_{ij} \mathcal{H}_y(y_{kj}, u_{kj}, p_{kj}, q_{kj}) + \Delta W \sum_{j=1}^{s} \tilde{b}_{ij} h(y_{kj}) q_{kj}, \\ q_{ki}\psi_{ki} = p_k - \Delta \sum_{j=1}^{s} \hat{a}_{ij} \mathcal{H}_y(y_{kj}, u_{kj}, p_{kj}, q_{kj}) + \Delta W \sum_{j=1}^{s} \hat{b}_{ij} h(y_{kj}) q_{kj}, \end{cases}$$

for $k = 0, 1, \dots, N-1$, and $i = 1, 2, \dots, s$.

iii. Compute the expected cost functional and the gradient from Eqns. (3.3)-(3.4) (Monte-Carlo).

iv. End *for loop.*

3. By using a line search algorithm, compute a descent direction.

4. Update the control u_k.

5. Compute $\epsilon_k = ||u_{k+1} - u_k||_2$. If $\epsilon_k < \epsilon$, then go to step 2.

In computations, we use the following discretization schemes for the state varible, $y(t)$, to obtain the 2-stage of stochastic Runge-Kutta scheme [17] given by

$$\begin{array}{c|cc} 0 & 0 & \\ 1 & 1 & 0 \\ \hline & 1/2 & 1/2 \end{array} \qquad \begin{array}{c|cc} 0 & 0 & \\ 2/3 & 2/3 & 0 \\ \hline & 1/4 & 3/4 \end{array}$$

By addressing the relations in Eqn. (3.1), we can obtain the corresponding Butcher array for the adjoint pair, $(p(t), q(t))$, as follows:

$$\begin{array}{c|cc} 0 & 1/2 & -1/2 \\ 1 & 1/2 & 1/2 \\ \hline & 1/2 & 1/2 \end{array} \qquad \begin{array}{c|cc} -1/2 & 1/4 & -3/4 \\ 1 & 1/4 & 3/4 \\ \hline & 1/4 & 3/4 \end{array}$$

and

$$\begin{array}{c|cc} -1/3 & 1/2 & -5/6 \\ 1 & 1/2 & 1/2 \\ \hline & 1/2 & 1/2 \end{array} \qquad \begin{array}{c|cc} -1 & 1/4 & -5/4 \\ 1 & 1/4 & 3/4 \\ \hline & 1/4 & 3/4 \end{array}$$

respectively.

3.4 Financial Applications

In many financial applications, the desired task can be achieved in an optimal or nearly an optimal manner. Merton used stochastic optimal control to study optimal portfolios of safe and risky assets for utility maximization [40, 41]. The criterion for portfolio selection, how to allocate stocks and bonds, is the maximization of the survival probability (or minimizing the ruin probability) or the minimization of the risk of firm. In this case, how much money is invested in stocks over the net wealth can be considered as the control variable. Such problems may be modeled by a stochastic optimal control problems.

Example 3.1. We consider the Black-Scholes type of an optimal control problem [16]:

$$\begin{cases} \underset{u\in L^2(0,T)}{\text{minimize}} & \frac{1}{2}\mathbb{E}\left[\int_0^T (y^* - y)^2 dt + \int_0^T u^2 dt\right] \\ \text{subject to} & dy = uydt + \sigma y dW, \quad y(0) = y^0, \end{cases} \tag{3.5}$$

where σ is a positive constant. We easily construct an exact solution as:

$$y(t) = y^0 e^{\int_0^t u(s)ds - \frac{\sigma^2}{2}t + \sigma W(t)}, \qquad u(t) = \frac{T-t}{\frac{1}{y^0} - Tt + \frac{t^2}{2}},$$

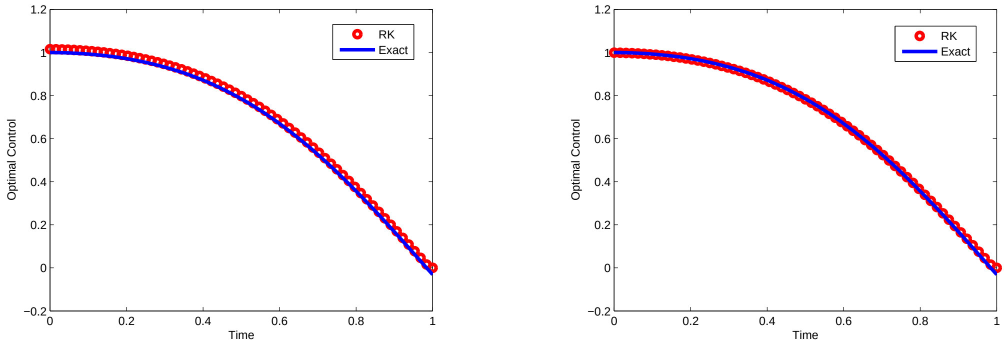
where

$$y^*(t) = \frac{e^{\sigma^2 t} - (T-t)^2}{\frac{1}{y^0} - Tt + \frac{t^2}{2}} + 1.$$

If we apply 2-stage stochastic Runge-Kutta schemes to Eqn. (3.5), then we get

$$\begin{cases} y_{k1} = y_k, \\ y_{k2} = y_k + \Delta u_{k1} y_{k1} + \frac{2}{3}\Delta W \sigma y_{k1}, \\ y_{k+1} = y_k + \frac{\Delta}{2}(u_{k1}y_{k1} + u_{k2}y_{k2}) + \frac{\Delta W}{4}\sigma(y_{k1} + 3y_{k2}), \\ p_{k1} = p_{k+1} + \Delta(y_{k2} - y^*_{k2} + p_{k2}u_{k2} + \sigma^2 q_{k2}y_{k2}) - \frac{3}{2}\Delta W \sigma q_{k2} y_{k2}, \\ p_{k2} = p_{k+1}, \\ q_{k1}\psi_{k1} = p_{k+1} + \frac{4}{3}\Delta(y_{k2} - y^*_{k2} + p_{k2}u_{k2} + \sigma^2 q_{k2}y_{k2}) - 2\Delta W \sigma q_{k2} y_{k2}, \\ q_{k2}\psi_{k2} = p_{k+1}, \\ p_k = p_{k+1} + \frac{\Delta}{2}\big(y_{k1} - y^*_{k1} + p_{k1}u_{k1} + \sigma^2 q_{k1}y_{k1} + y_{k2} - y^*_{k2} \\ \qquad + p_{k2}u_{k2} + \sigma^2 q_{k2}y_{k2}\big) - \frac{\Delta W}{4}\sigma(y_{k1}q_{k1} + 3y_{k2}q_{k2}). \end{cases}$$

We let $y^*(t) = y(t)$, and we choose $T = 1$ and $y_0 = 1$ in our numerical computation. Furthermore, we use 1000 paths in Monte-Carlo simulation. In Figures 3.1a and 3.1b, we compare the exact solution of control with the numerical control obtained from our Runge-Kutta scheme. We choose $\sigma = 0.1$ on the left and $\sigma = 0.3$ on the right side. It is easy to see that the graph of the optimal control with our Runge-Kutta scheme almost fits the graph of the optimal control of the exact solution.



(a) Optimal control for $\sigma = 0.1$.

(b) Optimal control for $\sigma = 0.3$.

Figure 3.1: Optimal control in Example 3.1.

In Table 3.1, we compare the Euler scheme with our Runge-Kutta method. As a stopping tolerance in gradient descent algorithm, $1e - 8$ is taken.

In order to show the efficiency of our Runge-Kutta scheme, we also obtain the results with Euler discretization. In this case, trapezoidal rule is applied to compute the approximate cost functional and its gradient. The state equation is discretized by means of forward Euler scheme, whereas backward Euler scheme is used for adjoint computation. Moreover, the same problem settings are employed as in our Runge-Kutta method. It can be understood from Table 3.1 that our Runge-Kutta scheme solves the problem faster and with less number of iterations, when compared to Euler scheme.

Table 3.1: Comparison of Runge-Kutta and Euler method with $\sigma = 0.1$ in Example 3.1.

	CPU Time (sec)		# of Iterations	
Δ	Euler	Runge-Kutta	Euler	Runge-Kutta
2^3	8.07	0.55	30	25
2^4	15.06	0.77	29	22
2^5	30.25	0.84	30	21
2^6	55.34	2.45	28	22
2^7	125.67	4.19	32	21
2^8	364.50	6.50	34	31

Example 3.2. We choose the following control problem as a second example:

$$\begin{cases} \underset{u\in L^2(0,T)}{\text{minimize}} & \frac{1}{2}\mathbb{E}\left[\int_0^T (y^* - y)^2 dt + \int_0^T (u-u^*)^2 dt\right] \\ \text{subject to} & dy = \frac{1}{2}u(u-u^*)ydt + \sigma y dW, \quad y(0) = y^0, \end{cases} \tag{3.6}$$

where σ is a positive scalar. We have the following continuous optimality system:

$$\begin{cases} dy = \frac{1}{2}u(u-u^*)ydt + \sigma y dW, \quad y(0) = y^0, \\ dp = (y^* - y + \sigma^2 p)dt + \sigma yq dW, \quad p(T) = 0, \\ u - u^* = -E\left[p\left(u - \frac{1}{2}u^*\right)\right]. \end{cases}$$

The exact solution is of the form

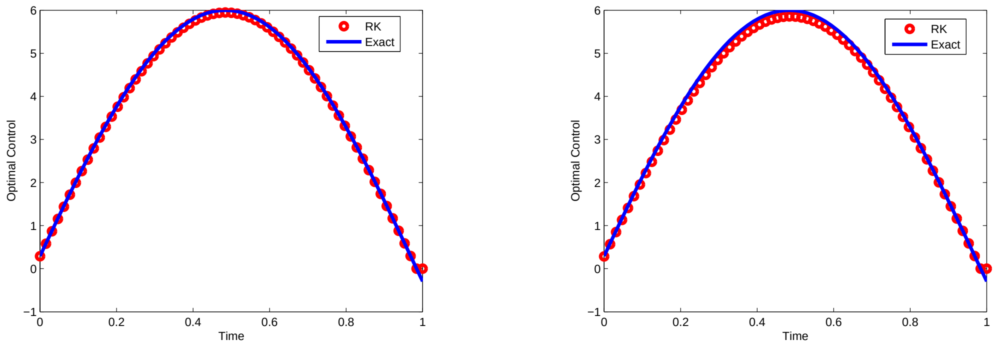
$$y(t) = y^*(t) = y^0 e^{-\frac{\sigma^2}{2}t + \sigma W(t)}, \qquad u(t) = u^*(t) = 6\sin(\pi t).$$

We used the same discretization scheme as done in the previous example. If we apply 2-stage stochastic Runge-Kutta schemes to Eqn. (3.6), we receive

$$\begin{cases} y_{k1} = y_k, \\ y_{k2} = y_k + \frac{\Delta}{2}u_{k1}(u_{k1} - u^*_{k1})y_{k1} + \frac{2}{3}\Delta W \sigma y_{k1}, \\ y_{k+1} = y_k + \frac{\Delta}{4}(u_{k1}(u_{k1} - u^*_{k1})y_{k1} + u_{k2}(u_{k2} - u^*_{k2})y_{k2}) + \frac{\Delta W}{4}\sigma(y_{k1} + 3y_{k2}), \\ p_{k1} = p_{k+1} + \Delta(y_{k2} - y^*_{k2} + \frac{1}{2}p_{k2}u_{k2}(u_{k2} - u^*_{k2}) + \sigma^2 q_{k2}y_{k2}) - \frac{3}{2}\Delta W \sigma q_{k2} y_{k2}, \\ p_{k2} = p_{k+1}, \\ q_{k1}\psi_{k1} = p_{k+1} + \frac{4}{3}\Delta(y_{k2} - y^*_{k2} + \frac{1}{2}p_{k2}u_{k2}(u_{k2} - u^*_{k2}) + \sigma^2 q_{k2}y_{k2}) - 2\Delta W \sigma q_{k2}y_{k2}, \\ q_{k2}\psi_{k2} = p_{k+1}, \\ p_k = p_{k+1} + \frac{\Delta}{2}\Big(y_{k1} - y^*_{k1} + \frac{1}{2}p_{k1}u_{k1}(u_{k1} - u^*_{k1}) + \sigma^2 q_{k1}y_{k1} + y_{k2} - y^*_{k2} \\ \qquad + \frac{1}{2}p_{k2}u_{k2}(u_{k2} - u^*_{k2}) + \sigma^2 q_{k2}y_{k2}\Big) - \frac{\Delta W}{4}\sigma(y_{k1}q_{k1} + 3y_{k2}q_{k2}). \end{cases}$$

We choose $T = 1$ and $y_0 = 1$ in the numerical computation. Moreover, we use 1000 paths in Monte-Carlo simulation. In Figures 3.2a and 3.2b, we compare the exact solution of optimal control with the numerical optimal control, obtained from our Runge-Kutta scheme.

In Table 3.2, one can see the efficiency of our Runge-Kutta method, when compared to Euler scheme with regard to time consumption.



(a) Optimal control for $\sigma = 0.3$. (b) Optimal control for $\sigma = 0.5$.

Figure 3.2: Optimal control in Example 3.2.

Table 3.2: Comparison of Runge-Kutta and Euler method with $\sigma = 0.3$ in Example 3.2.

	CPU Time (sec)		# of Iterations	
Δ	Euler	Runge-Kutta	Euler	Runge-Kutta
2^3	12.31	1.58	35	27
2^4	22.81	2.58	39	23
2^5	35.45	6.29	33	36
2^6	74.10	10.06	36	32
2^7	186.01	19.83	47	34
2^8	266.00	48.78	34	44

3.5 Summary

In this chapter, we mainly focused on a Runge-Kutta scheme for the optimal control problem of SDEs by following the *discretize-then-optimize* approach. Firstly, we discretized the cost functional and the state equation with the help of Runge-Kutta schemes. Then, by addressing the discrete Lagrangian, we got the Runge-Kutta discretizations of the adjoint pair $(p(t), q(t))$ and we derived the Runge-Kutta coefficients of the adjoint pair in terms of the Runge-Kutta coefficients of the state equation. We compared the numerical results with the exact solutions and Euler method. The numerical results agree with the exact solutions. The efficiency of our Runge-Kutta scheme comes from its time consumption. The Euler scheme consumes more CPU time than our Runge-Kutta method does.

CHAPTER 4

RUNGE-KUTTA SCHEME FOR STOCHASTIC OPTIMAL CONTROL PROBLEMS OF SOME SPDEs

4.1 Introduction

Optimal harvesting problem is an important model and tool in mathematical bioeconomics [1, 34, 56]. For instance, it is used in forestry, agriculture or a marine resource such as fish harvesting. This is an important problem in providing food and other organic resources for the people of the world, while caring for the world in a sustainable way. Early developments of this problem were studied in a deterministic environment [12]. However, environmental and human factors cause that the populations are in various states with some probabilities. So, this situation needs modeling in a stochastic environment. Optimal harvesting problem was investigated for the first time in a stochastic environment at the end of 1990's. Alvarez and Sheep [1] and Lungu and Øksendal [34] handled the optimal harvesting on different population models, which were modelled by stochastic optimal control theory. Later, Lungu and Øksendal [35], Øksendal [43] and Pinheiro [50] continued to study on that topic. More studies about optimal harvesting problems, e.g., by the example of the fisheries can be found in [2, 18, 25, 39, 47, 49, 64].

We obtained our Runge-Kutta scheme for optimal control problems of SDEs in Chapter 3. In this chapter, our aim is to solve an stochastic optimal control program of SPDEs by Runge-Kutta method. We choose Øksendal's optimal harvesting problem [35, 43]. In this problem, the density of the population is given by an SPDE and the problem is to maximize in a balanced way the total expected utility of the consumption and the terminal size of the population while controlling the harvesting rate. By using the finite difference scheme, we discretize the problem with respect to the space variable and convert the given program to optimal control problems of system of SDEs. Then, we employ our Runge-Kutta scheme for the resulting optimal control problem.

4.2 Formulation of Optimal Harvesting Problem

Let $(W(t))_{0\leq t\leq T}$ be a 1-dimensional Brownian motion on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}(t))_{t\in[0,T]}, \mathbb{P})$, where $T > 0$ is a time maturity and $\Omega \subset \mathbb{R}$ is a given set.

On this probability space, the space of real-valued square integrable $(\mathcal{F}(t))$-adapted processes is defined over $L^2(0,T)$. Defining $\mathcal{A} := [0,T] \times [0,1]$, we consider the objective of our optimal control problem to maximize an overall expected utility of the consumption

$$\mathbb{E}\left[\int_0^T \int_0^1 \frac{u^\gamma(t,x)}{\gamma} dxdt + \theta \int_0^1 y(T,x)dx\right],$$

with risk-aversion coefficient $\gamma \in (0,1)$ and regularization parameter $\theta > 0$. Here, $u \in L^2(\mathcal{A})$ is the stochastic control variable which stands for the harvesting rate. In this work, we address the following stochastic reaction-diffusion equation [43]:

$$\begin{cases} dy(t,x) = \left(\frac{1}{2}\Delta y(t,x) + \mu y(t,x) - u(t,x)\right) dt + \lambda y(t,x) dW \\ \qquad\qquad\qquad\qquad\qquad\qquad (x \in [0,1],\ t \in [0,T]), \\ y(0,x) = y_0(x) \quad (x \in [0,1]), \\ y(t,0) = y(t,1) = 0 \quad (t \in [0,T]), \end{cases} \tag{4.1}$$

where $y \in L^2(\mathcal{A})$ is the state variable which represents the density of the population living in an environment with a limited carrying capacity $K(t)$. Furthermore, $\mu > 0$ and λ are given constants in the stochastic carrying capacity $K(t)$, defined as $dK = \mu dt + \lambda dW$, and Δ is the Laplacian on $\mathbb{R}$:

$$\Delta y(t,x) = \frac{\partial^2 y(t,x)}{\partial x^2}.$$

See [45, 46] for more information on reaction-diffusion equations.

4.3 Discretization with Finite Difference Scheme

Our optimal control problem governed by SPDEs can be stated as:

$$(\mathcal{P}_1) \begin{cases} \underset{u \in L^2(\mathcal{A})}{\text{maximize}} & \mathbb{E}\left[\int_0^T \int_0^1 \frac{u^\gamma(t,x)}{\gamma} dxdt + \theta \int_0^1 y(T,x)dx\right] \\ \text{subject to} & \text{Eqn. (4.1)}. \end{cases}$$

Now, we use a finite difference scheme to approximate the space variable. We let the spatial length scale be $h := x_m - x_{m-1} = 1/M$ $(m = 1, 2, \ldots, M)$. For the space variable, $0 \leq x_0 < x_1 < \ldots < x_m < \ldots < x_M = 1$ denotes the equispaced discretization of space interval $[0,1]$. We let y^m and u^m correspond to $y(t,hm)$ and $u(t,hm)$, respectively, in the continuous case. Now, by applying the second-order central difference scheme for the space variable, we get

$$dy^m = \left(\frac{y^{m-1} - 2y^m + y^{m+1}}{2h^2} + \mu y^m - u^m\right) dt + \lambda y^m dW \quad (m = 1, 2, \ldots, M-1).$$

Then, we obtain a system of SDEs:

$$\begin{aligned}
dy^1 &= \left(\frac{y^0+y^2}{2h^2} - \frac{1}{2h^2}y^1 + \mu y^1 - u^1\right)dt + \lambda y^1 dW,\\
dy^2 &= \left(\frac{y^1+y^3}{2h^2} - \frac{1}{2h^2}y^2 + \mu y^2 - u^2\right)dt + \lambda y^2 dW,\\
&\vdots\\
dy^m &= \left(\frac{y^{m-1}+y^{m+1}}{2h^2} - \frac{1}{2h^2}y^m + \mu y^m - u^m\right)dt + \lambda y^m dW,\\
&\vdots\\
dy^{M-1} &= \left(\frac{y^{M-2}+y^M}{2h^2} - \frac{1}{2h^2}y^{M-1} + \mu y^{M-1} - u^{M-1}\right)dt + \lambda y^{M-1} dW,
\end{aligned}$$

which we rewrite in the matrix-vector notation:

$$d\mathbf{Y} = (\frac{1}{2}\mathbf{A}\mathbf{Y} + \mu\mathbf{Y} - \mathbf{U})dt + \lambda\mathbf{Y}dW,$$

where

$$\mathbf{A} := \begin{pmatrix} -1/h^2 & 1/2h^2 & 0 & \cdots & 0 & 0 \\ 1/2h^2 & -1/h^2 & 1/2h^2 & \cdots & 0 & 0 \\ 0 & 1/2h^2 & -1/h^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1/h^2 & 1/2h^2 \\ 0 & 0 & 0 & \cdots & 1/2h^2 & -1/h^2 \end{pmatrix},$$

$\mathbf{Y} := (y^1, y^2, \ldots, y^{M-1})^T$, $\mathbf{U} := (u^1, u^2, \ldots, u^{M-1})^T$, $\overline{\mathbf{U}} := (u^1, u^2, \ldots, u^M)^T$ and $\overline{\mathbf{Y}}(0) = \overline{\mathbf{Y}}^0 := (y_0^1, y_0^2, \ldots, y_0^M)^T$.

Now, we can rewrite our optimal control problem, governed by a system of SDEs, as follows:

$$\begin{cases} \underset{u \in L^2(\mathcal{A})}{\text{maximize}} & h\mathbb{E}\left[\int_0^T \sum_{m=1}^M \frac{(u^m)^\gamma}{\gamma} dt + \sum_{m=1}^M \theta(y_T^m)\right] \\ \text{subject to} & d\mathbf{Y} = (\frac{1}{2}\mathbf{A}\mathbf{Y} + \mu\mathbf{Y} - \mathbf{U})dt + \lambda\mathbf{Y}dW \quad (t \in [0,T]), \\ & \overline{\mathbf{Y}}(0) = \overline{\mathbf{Y}}^0. \end{cases}$$

For simplicity, to write the cost functional in a quadratic form, we restate the above problem in full matrix-vector form as

$$
(\mathcal{P}'')\begin{cases}\underset{\overline{\mathbf{U}}\in L^2(\mathcal{A})}{\text{maximize}} & h\mathbb{E}\left[\int_0^T \dfrac{\mathbf{V}(\overline{\mathbf{U}})^T\mathbf{V}(\overline{\mathbf{U}})}{\gamma}dt + \theta(\mathbf{Y}_T)^T\mathbf{1}\right],\\ \text{subject to} & d\mathbf{Y} = (\frac{1}{2}\mathbf{A}\mathbf{Y} + \mu\mathbf{Y} - \mathbf{U})dt + \lambda\mathbf{Y}dW \quad (t\in[0,T])\\ & \overline{\mathbf{Y}}(0) = \overline{\mathbf{Y}}^0,\end{cases}
$$

where $\mathbf{V} = \left((u^1)^{\gamma/2}, (u^2)^{\gamma/2}, \ldots, (u^M)^{\gamma/2}\right)^T$ and $\mathbf{1} = (1,1,\ldots,1)^T$. We note that the problem $(\mathcal{P}'')$ is continuous in time. In this work, our strategy is to follow the approach *discretize-then-optimize*. But, we need to write the continuous optimality conditions explicitly when choosing some parameters in the discrete optimality system. Herewith, we first derive the continuous optimality conditions in the following section.

4.4 First-Order Necessary Optimality Conditions

We recall the *Hamiltonian function* of the optimal control problem as:

$$
\mathcal{H}(t,\mathbf{Y},\overline{\mathbf{U}},\mathbf{P},\mathbf{Q}) = \frac{1}{\gamma}h\mathbf{V}(\overline{\mathbf{U}})^T\mathbf{V}(\overline{\mathbf{U}}) + \left(\frac{1}{2}\mathbf{A}\mathbf{Y} + \mu\mathbf{Y} - \mathbf{U}\right)^T\mathbf{P} + \frac{1}{2}tr\left\{\lambda^2\mathbf{Q}\mathbf{Y}\mathbf{Y}^T\right\},
$$

for a coupled process $(\mathbf{P}(t),\mathbf{Q}(t))$ that is adapted with respect to $(\mathcal{F}(t))_{t\in[0,T]}$, where $\mathbf{P}(t)$ is a vector having dimension $(M-1)$ and $\mathbf{Q}(t)$ is an $(M-1)\times(M-1)$-dimensional matrix. This pair satisfies the following continuous first-order necessary optimality system:

$$
\begin{cases} d\mathbf{Y} = \mathcal{H}_{\mathbf{P}}(t,\mathbf{Y},\overline{\mathbf{U}},\mathbf{P},\mathbf{Q})dt + \lambda\mathbf{Y}dW \quad (t\in[0,T]),\\ d\mathbf{P} = -\mathcal{H}_{\mathbf{Y}}(t,\mathbf{Y},\overline{\mathbf{U}},\mathbf{P},\mathbf{Q})dt + \lambda\mathbf{Q}\mathbf{Y}dW \quad (t\in[0,T]),\\ \mathbf{0} = \mathcal{H}_{\overline{\mathbf{U}}}(t,\mathbf{Y},\overline{\mathbf{U}},\mathbf{P},\mathbf{Q}) \quad (t\in[0,T]),\\ \overline{\mathbf{Y}}(0) = \overline{\mathbf{Y}}^0,\\ \mathbf{P}(T) = \theta\mathbf{1}.\end{cases}
$$

Let us consider the term $tr\left\{\lambda^2\mathbf{Q}\mathbf{Y}\mathbf{Y}^T\right\}$. After we perform the matrix multiplication, we evaluate the trace of the resulting matrix to get

$$
\begin{aligned}
tr\left\{\lambda^2\mathbf{Q}\mathbf{Y}\mathbf{Y}^T\right\} =& \lambda^2 y^1(q_{1,1}y^1 + q_{1,2}y^2 + \ldots + q_{1,M-1}y_t^{M-1})\\
&+ \lambda^2 y^2(q_{2,1}y^1 + q_{2,2}y^2 + \ldots + q_{2,M-1}y^{M-1})\\
&\vdots\\
&+ \lambda^2 y^{M-1}(q_{M-1,1}y^1 + q_{M-1,2}y^2 + \ldots + q_{M-1,M-1}y^{M-1}).
\end{aligned}
$$

We take the partial derivatives of the Hamiltonian function with respect to the coordinates of $\mathbf{Y}$. Then, we obtain

$$\frac{\partial}{\partial \mathbf{Y}}\left\{\frac{1}{2}tr\left\{\lambda^2 \mathbf{Q}\mathbf{Y}\mathbf{Y}^T\right\}\right\} = \lambda^2(\mathbf{Q}+\mathbf{Q}^T)\mathbf{Y}.$$

So, it is easy to see that

$$\mathcal{H}_{\mathbf{Y}}(t,\mathbf{Y},\overline{\mathbf{U}},\mathbf{P},\mathbf{Q}) = \frac{1}{2}\mathbf{A}^T\mathbf{P} + \mu\mathbf{P} + \lambda^2(\mathbf{Q}+\mathbf{Q}^T)\mathbf{Y}.$$

Now, we arrive at the so-called *adjoint equation*

$$d\mathbf{P} = -(\frac{1}{2}\mathbf{A}^T\mathbf{P} + \mu\mathbf{P} + \lambda^2(\mathbf{Q}+\mathbf{Q}^T)\mathbf{Y})dt + \lambda\mathbf{Q}\mathbf{Y}dW.$$

If we take the partial derivatives of the Hamiltonian function with respect to the coordinates of $\mathbf{U}$, then we obtain the gradient equation as follows:

$$\mathbf{0} = \mathcal{H}_{\overline{\mathbf{U}}}(t,\mathbf{Y},\overline{\mathbf{U}},\mathbf{P},\mathbf{Q}) = h\bar{\mathbf{V}}(\overline{\mathbf{U}}) - (\mathbf{P}^T,0)^T,$$

where $\bar{\mathbf{V}}(\overline{\mathbf{U}}) = \left((\mathbf{U}^1)^{\gamma/2},(\mathbf{U}^2)^{\gamma/2},\ldots,(\mathbf{U}^M)^{\gamma/2}\right)^T$. Then,

$$\mathbf{U}^i = \frac{1}{h}(\mathbf{P}^i)^{2/\gamma} \qquad (i=1,2,\ldots,M-1), \quad \mathbf{U}^M = 0.$$

Since $\mathbf{U}^M = 0$, it is enough if we just refer to $\mathbf{U}$ rather than $\overline{\mathbf{U}}$ from now on.

In the following section, we use Runge-Kutta method to formulate the discrete optimal control problem. At the end, we will obtain a Runge-Kutta scheme for the discrete adjoint variable.

4.5 Runge-Kutta Schemes for Optimal Harvesting Problem

We introduce a discretization $0 = t_0 < t_1 < \ldots < t_n < \ldots < t_N = T$ of the time interval $[0,T]$. Let $\Delta = T/N$ denote the increments (step-size) and ΔW be an $\mathcal{N}(0,\Delta)$-distributed Gaussian increment of the Brownian motion W.

Now, we state the s-stage Runge-Kutta discretization, for some $s \in \mathbb{Z}^+$, of the problem $(\mathcal{P}'')$ as

$$(\mathcal{P}')\begin{cases}\underset{\mathbf{U}\in L^2(\mathcal{A})}{\text{maximize}} & \dfrac{h}{\gamma}\mathbb{E}\left[\Delta\sum_{k=0}^{N-1}\sum_{i=1}^{s}\alpha_i\mathbf{V}(\mathbf{U}_{ki})^T\mathbf{V}(\mathbf{U}_{ki})+\theta(\mathbf{Y}_N)^T\mathbf{1}\right]\\ \text{subject to} & \mathbf{Y}_{k+1}=\mathbf{Y}_k+\Delta\sum_{i=1}^{s}\alpha_i(\frac{1}{2}\mathbf{A}\mathbf{Y}_{ki}+\mu\mathbf{Y}_{ki}-\mathbf{U}_{ki})+\Delta W\sum_{i=1}^{s}\beta_i\lambda\mathbf{Y}_{ki},\\ & \mathbf{Y}_{ki}=\mathbf{Y}_k+\Delta\sum_{j=1}^{s}a_{ij}(\frac{1}{2}\mathbf{A}\mathbf{Y}_{kj}+\mu\mathbf{Y}_{kj}-\mathbf{U}_{kj})+\Delta W\sum_{j=1}^{s}b_{ij}\lambda\mathbf{Y}_{kj},\\ & \overline{\mathbf{Y}}(0)=\overline{\mathbf{Y}}^0,\end{cases}$$

where $\mathbf{Y}_{ki}$ and $\mathbf{U}_{ki}$ have the dimension $(M-1)$ and the constants $\alpha_i,\ \beta_i,\ a_{ij},\ b_{ij}$, are the Runge-Kutta coefficients for $k=0,1,\dots,N-1$, and $i,j=1,2,\dots,s$. The Butcher array of the Runge-Kutta discretization of the system of problem $(\mathcal{P}')$ is given by

$$\begin{array}{c|ccc} c_1 & a_{11} & \dots & a_{1s}\\ \vdots & \vdots & \ddots & \vdots\\ c_s & a_{s1} & \dots & a_{ss}\\ \hline & \alpha_1 & \dots & \alpha_s\end{array}\qquad\qquad\begin{array}{c|ccc} d_1 & b_{11} & \dots & b_{1s}\\ \vdots & \vdots & \ddots & \vdots\\ d_s & b_{s1} & \dots & b_{ss}\\ \hline & \beta_1 & \dots & \beta_s\end{array}$$

In the following proposition, we achieve our discrete optimality conditions by defining the discrete Lagrangian.

Proposition 4.1. *If $\alpha_i\neq 0$ and $\beta_i\neq 0$ ($i=1,2,\dots,s$, and $k=0,1,\dots,N-1$) are the Runge-Kutta coefficients of the problem $(\mathcal{P}')$, then the discrete first-order necessary optimality conditions of the problem $(\mathcal{P}')$ are obtained as*

$$\begin{cases}\mathbf{Y}_{k+1}=\mathbf{Y}_k+\Delta\sum_{i=1}^{s}\alpha_i\left(\frac{1}{2}\mathbf{A}\mathbf{Y}_{ki}+\mu\mathbf{Y}_{ki}-\mathbf{U}_{ki}\right)+\Delta W\sum_{i=1}^{s}\beta_i\lambda\mathbf{Y}_{ki},\\ \mathbf{Y}_{ki}=\mathbf{Y}_k+\Delta\sum_{j=1}^{s}a_{ij}\left(\frac{1}{2}\mathbf{A}\mathbf{Y}_{kj}+\mu\mathbf{Y}_{kj}-\mathbf{U}_{kj}\right)+\Delta W\sum_{j=1}^{s}b_{ij}\lambda\mathbf{Y}_{kj},\\ \mathbf{P}_{k+1}=\mathbf{P}_k-\Delta\sum_{i=1}^{s}\tilde{\alpha}_i\mathcal{H}_{\mathbf{Y}}(\mathbf{Y}_{ki},\mathbf{U}_{ki},\mathbf{P}_{ki},\mathbf{Q}_{ki})+\Delta W\sum_{i=1}^{s}\tilde{\beta}_i\lambda\mathbf{Q}_{ki}\mathbf{Y}_{ki},\\ \mathbf{P}_{ki}=\mathbf{P}_k-\Delta\sum_{j=1}^{s}\tilde{a}_{ij}\mathcal{H}_{\mathbf{Y}}(\mathbf{Y}_{kj},\mathbf{U}_{kj},\mathbf{P}_{kj},\mathbf{Q}_{kj})+\Delta W\sum_{j=1}^{s}\tilde{b}_{ij}\lambda\mathbf{Q}_{kj}\mathbf{Y}_{kj},\\ \mathbf{Q}_{ki}\mathbf{\Psi}_{ki}=\mathbf{P}_k-\Delta\sum_{j=1}^{s}\hat{a}_{ij}\mathcal{H}_{\mathbf{Y}}(\mathbf{Y}_{kj},\mathbf{U}_{kj},\mathbf{P}_{kj},\mathbf{Q}_{kj})+\Delta W\sum_{j=1}^{s}\hat{b}_{ij}\lambda\mathbf{Q}_{kj}\mathbf{Y}_{kj},\\ \mathbf{P}_N=\mathbf{Y}_N,\\ \overline{\mathbf{Y}}(0)=\overline{\mathbf{Y}}^0,\end{cases}$$

where $\mathbf{P}_{ki}$ *and* $\mathbf{Q}_{ki}$ *have the dimensions* $(M-1)$ *and* $(M-1)\times(M-1)$*, respectively, for* $k = 0,\dots,N-1$*, and* $i = 1,2,\dots,s$*. The coefficients satisfy the following relations:*

$$\begin{aligned} \tilde{\alpha}_i &:= \alpha_i, & \tilde{\beta}_i &:= \beta_i, \\ \tilde{a}_{ij} &:= \alpha_j - \frac{\alpha_j}{\alpha_i} a_{ji}, & \tilde{b}_{ij} &:= \beta_j - \frac{\beta_j}{\alpha_i} a_{ji}, \\ \hat{a}_{ij} &:= \alpha_j - \frac{\alpha_j}{\beta_i} b_{ji}, & \hat{b}_{ij} &:= \beta_j - \frac{\beta_j}{\beta_i} b_{ji}, \end{aligned}$$

with

$$\mathbf{\Psi}_{ki} := \lambda \frac{\Delta \alpha_i}{\Delta W \beta_i} \mathbf{Q}_{ki}^{-1} (\mathbf{Q}_{ki} + \mathbf{Q}_{ki}^T) \mathbf{Y}_{ki} - \mathbf{Y}_{ki} \qquad (k = 0,\dots,N-1).$$

Proof. Let be

$$\mathbf{K}_{ki} := \frac{1}{2} \mathbf{A} \mathbf{Y}_{ki} + \mu \mathbf{Y}_{ki} - \mathbf{U}_{ki}$$

and

$$\mathbf{M}_{ki} := \lambda \mathbf{Y}_{ki},$$

so that we have

$$\begin{aligned} &\frac{1}{2}\mathbf{A}\mathbf{Y}_{ki} + \mu \mathbf{Y}_{ki} - \mathbf{U}_{ki} \\ &\quad = \frac{1}{2}\mathbf{A}\left(\mathbf{Y}_k + \Delta \sum_{j=1}^{s} a_{ij}\left(\frac{1}{2}\mathbf{A}\mathbf{Y}_{kj} + \mu\mathbf{Y}_{kj} - \mathbf{U}_{kj}\right) + \Delta W \sum_{j=1}^{s} b_{ij}\lambda \mathbf{Y}_{kj}\right) \\ &\qquad + \mu\left(\mathbf{Y}_k + \Delta \sum_{j=1}^{s} a_{ij}\left(\frac{1}{2}\mathbf{A}\mathbf{Y}_{kj} + \mu\mathbf{Y}_{kj} - \mathbf{U}_{kj}\right) + \Delta W \sum_{j=1}^{s} b_{ij}\lambda \mathbf{Y}_{kj}\right) \\ &\qquad - \mathbf{U}_{ki}, \end{aligned}$$

and

$$\lambda \mathbf{Y}_{ki} = \lambda\left(\mathbf{Y}_k + \Delta \sum_{j=1}^{s} a_{ij}\left(\frac{1}{2}\mathbf{A}\mathbf{Y}_{kj} + \mu\mathbf{Y}_{kj} - \mathbf{U}_{kj}\right) + \Delta W \sum_{j=1}^{s} b_{ij}\lambda \mathbf{Y}_{kj}\right).$$

Then, by using $\mathbf{K}_{ki}$ and $\mathbf{M}_{ki}$ introduced above, we can write the discretized Lagrangian as:

$$
\begin{aligned}
&\mathcal{L}(\mathbf{Y}_{ki}, \mathbf{U}_{ki}, \mathbf{P}_{ki}, \mathbf{Q}_{ki}, \boldsymbol{\Xi}_{ki}, \boldsymbol{Z}_{ki}) := \mathbb{E}\Big[\mathbf{P}^0(\mathbf{Y}^0 - \mathbf{Y}_0) \\
&\quad + \frac{h}{\gamma}\sum_{k=0}^{N-1} \Delta t \sum_{i=1}^{s} \alpha_i \mathbf{V}(\mathbf{U}_{ki})^T \mathbf{V}(\mathbf{U}_{ki}) + (\mathbf{Y}_N)^T \mathbf{1} \\
&\quad + \sum_{k=0}^{N-1} \Bigg\{ \sum_{i=1}^{s} \mathbf{P}_{k+1}^T \left(\mathbf{Y}_k - \mathbf{Y}_{k+1} + \Delta \sum_{i=1}^{s} \alpha_i \mathbf{K}_{ki} + \Delta W \sum_{i=1}^{s} \beta_i \mathbf{M}_{ki} \right) \\
&\quad + \sum_{i=1}^{s} \boldsymbol{\Xi}_{ki}^T \left(\frac{1}{2}\mathbf{A} \left(\mathbf{Y}_k + \Delta \sum_{j=1}^{s} a_{ij} \mathbf{K}_{kj} + \Delta W \sum_{j=1}^{s} b_{ij} \mathbf{M}_{kj} \right) \right. \\
&\quad \left. + \mu \left(\mathbf{Y}_k + \Delta \sum_{j=1}^{s} a_{ij} \mathbf{K}_{kj} + \Delta W \sum_{j=1}^{s} b_{ij} \mathbf{M}_{kj} \right) - \mathbf{U}_{ki} - \mathbf{K}_{ki} \right) \\
&\quad + \sum_{i=1}^{s} \boldsymbol{Z}_{ki}^T \left(\lambda \left(\mathbf{Y}_k + \Delta t \sum_{j=1}^{s} a_{ij} \mathbf{K}_{kj} + \Delta W \sum_{j=1}^{s} b_{ij} \mathbf{M}_{kj} \right) - \mathbf{M}_{ki} \right) \Bigg\} \Bigg],
\end{aligned}
$$

where $\mathbf{P}^0$, $\mathbf{P}_{k+1}$, $\boldsymbol{\Xi}_{ki}$, $\boldsymbol{Z}_{ki}$, are the vectors of Lagrange multipliers. Equating to zero the derivatives of this Lagrangian function with respect to all the coordinates of $\mathbf{Y}_N$, $\mathbf{Y}_0$, $\mathbf{Y}_k$, for $k = 1, 2, \ldots, N-1$, and of $\boldsymbol{\Xi}_{ki}$, $\boldsymbol{Z}_{ki}$, $\mathbf{U}_{ki}$, for $k = 0, 1, \ldots, N-1$, we obtain:

$$
\begin{aligned}
\mathbf{P}_N &= \theta \mathbf{1}, \\
\mathbf{P}_1 &= \mathbf{P}_0, \\
\mathbf{P}_k - \mathbf{P}_{k+1} &= \sum_{i=1}^{s} \frac{1}{2} \mathbf{A}^T \boldsymbol{\Xi}_{ki} + \mu \boldsymbol{\Xi}_{ki} + \sum_{i=1}^{s} \lambda \boldsymbol{Z}_{ki}, \\
\boldsymbol{\Xi}_{ki} &= \Delta \alpha_i \mathbf{P}_{k+1} + \Delta \sum_{j=1}^{s} \frac{1}{2} a_{ji} \mathbf{A}^T \boldsymbol{\Xi}_{kj} + \Delta \sum_{j=1}^{s} \frac{1}{2} a_{ji} \mu \boldsymbol{\Xi}_{kj} + \Delta \sum_{j=1}^{s} a_{ji} \boldsymbol{Z}_{kj}, \\
\boldsymbol{Z}_{ki} &= \Delta W \beta_i \mathbf{P}_{k+1} + \Delta W \sum_{j=1}^{s} \frac{1}{2} b_{ji} \mathbf{A}^T \boldsymbol{\Xi}_{kj} + \Delta W \sum_{j=1}^{s} \frac{1}{2} b_{ji} \mu \boldsymbol{\Xi}_{kj} \\
&\qquad + \Delta W \sum_{j=1}^{s} b_{ji} \lambda \boldsymbol{Z}_{kj}, \\
\mathbf{0} &= h \Delta \alpha_i \bar{\mathbf{V}}(\mathbf{U}_{ki}) - \boldsymbol{\Xi}_{ki}.
\end{aligned}
$$

Let us set

$$\boldsymbol{\Xi}_{ki} := \Delta \alpha_i \mathbf{P}_{ki}$$

and

$$\boldsymbol{Z}_{ki} := \Delta W \beta_i \mathbf{Q}_{ki} \boldsymbol{\Psi}_{ki}$$

with

$$\mathbf{\Psi}_{ki} := \lambda \frac{\Delta \alpha_i}{\Delta W \beta_i} \mathbf{Q}_{ki}^{-1} (\mathbf{Q}_{ki} + \mathbf{Q}_{ki}^T) \mathbf{Y}_{ki} - \mathbf{Y}_{ki},$$

where $\alpha_i \neq 0$ and $\beta_i \neq 0$ ($i = 1, 2, \ldots, s$). If these values vanish, then the solution of the discrete problem may not converge to the solution of the continuous problem.

By eliminating $\mathbf{\Xi}_{ki}$ and $\boldsymbol{Z}_{ki}$ in the above equations, and comparing with the first-order continuous optimality conditions given in Section 4, we obtain the desired result.

□

4.6 Numerical Application

We employ Monte-Carlo method to approximate conditional expectations [20, 30, 31]. To solve our stochastic optimal control problem, we use a gradient-descent type algorithm. We apply a line-search method to accelerate the implementation. In computations, we use the following discretization schemes for the state variable, y, to obtain the 2-stage of stochastic Runge-Kutta scheme [17] given by

$$\begin{array}{c|cc} 0 & 0 & \\ 1 & 1 & 0 \\ \hline & 1/2 & 1/2 \end{array} \qquad \begin{array}{c|cc} 0 & 0 & \\ 2/3 & 2/3 & 0 \\ \hline & 1/4 & 3/4 \end{array}$$

Now, we let be $\theta = 1$, $\gamma = 1/2$ and $\lambda = 1$ in problem $(\mathcal{P}_1)$. We perform the matrix-vector formulation as stated in Proposition 4.1. Furthermore, we employ a 2-step Runge-Kutta method. Since the adjoint equation is backward in time, by Proposition 4.1, we can write:

$$\begin{cases} \mathbf{Q}_{k2}\mathbf{\Psi}_{k2} = \mathbf{P}_{k+1}, \\ \mathbf{Q}_{k1}\mathbf{\Psi}_{k1} = \mathbf{P}_{k+1} + \dfrac{4\Delta}{3}\left(\dfrac{1}{2}\mathbf{A}^T\mathbf{P}_{k2} + \lambda^2(\mathbf{Q}_{k2} + \mathbf{Q}_{k2}^T)\mathbf{Y}_{k2}\right) - 2\Delta W \mathbf{Q}_{k2}\mathbf{Y}_{k2}, \\ \qquad \mathbf{\Psi}_{k1} = \lambda \mathbf{P}_{k+1}\dfrac{2\Delta}{\Delta W}\mathbf{Q}_{k1}^{-1}(\mathbf{Q}_{k1} + \mathbf{Q}_{k1}^T)\mathbf{Y}_{k1} - \mathbf{Y}_{k1}, \\ \qquad \mathbf{\Psi}_{k2} = \lambda \mathbf{P}_{k+1}\dfrac{2\Delta}{3\Delta W}\mathbf{Q}_{k2}^{-1}(\mathbf{Q}_{k2} + \mathbf{Q}_{k2}^T)\mathbf{Y}_{k2} - \mathbf{Y}_{k2}. \end{cases}$$

We rewrite these equations to get

$$
\begin{cases}
\mathbf{Y}(0) = \mathbf{Y}^0, \\
\mathbf{Y}_{k1} = \mathbf{Y}_k, \\
\mathbf{Y}_{k2} = \mathbf{Y}_k + \Delta\left(\frac{1}{2}\mathbf{A}\mathbf{Y}_{k1} + \mu\mathbf{Y}_{k1} - \mathbf{U}_{k1}\right) + \lambda\frac{2\Delta W}{3}\mathbf{Y}_{k1}, \\
\mathbf{Y}_{k+1} = \mathbf{Y}_k + \frac{\Delta}{2}\left(\frac{1}{2}\mathbf{A}\mathbf{Y}_{k1} + \mu\mathbf{Y}_{k1} - \mathbf{U}_{k1} + \frac{1}{2}\mathbf{A}\mathbf{Y}_{k2} + \mu\mathbf{Y}_{k2} - \mathbf{U}_{k2}\right) \\
\qquad + \lambda\frac{\Delta W}{4}(\mathbf{Y}_{k1} + 3\mathbf{Y}_{k2}), \\
\mathbf{P}_{k1} = \mathbf{P}_{k+1} + \Delta\left(\frac{1}{2}\mathbf{A}^T\mathbf{P}_{k2} + \lambda^2(\mathbf{Q}_{k2} + \mathbf{Q}_{k2}^T)\mathbf{Y}_{k2}\right) - \lambda\frac{3\Delta W}{2}\mathbf{Q}_{k2}\mathbf{Y}_{k2}, \\
\mathbf{P}_{k2} = \mathbf{P}_{k+1}, \\
\mathbf{P}_k = \mathbf{P}_{k+1} + \frac{\Delta}{2}\left(\frac{1}{2}\mathbf{A}^T\mathbf{P}_{k1} + \lambda^2(\mathbf{Q}_{k1} + \mathbf{Q}_{k1}^T)\mathbf{Y}_{k1} + \frac{1}{2}\mathbf{A}^T\mathbf{P}_{k2}\right. \\
\qquad \left. + \lambda^2(\mathbf{Q}_{k2} + \mathbf{Q}_{k2}^T)\mathbf{Y}_{k2}\right) - \lambda\frac{\Delta W}{4}(\mathbf{Q}_{k1}\mathbf{Y}_{k1} + 3\mathbf{Q}_{k2}\mathbf{Y}_{k2}), \\
\mathbf{P}_N = \mathbf{Y}_N,
\end{cases}
$$

where $k = 0, 1, \ldots, N-1$.

We use $h = 0.1$ and $\Delta = 2^{-7}$. In our Monte-Carlo simulation, the number of orbits is chosen as 100. In Figures 4.1-4.3, we present the numerical solutions of the optimal state and control variables with varying parameter μ.

Let us note that one of the advantages of our Runge-Kutta scheme is that it gives the Lagrange multiplier pair $(\mathbf{P}, \mathbf{Q})$ explicitly. However, if the *discretize-then-optimize* approach is used with the Euler method, the multiplier vector $\mathbf{Q}$ disappears. Thus, the solution with Euler method is far away from the solution obtained with our Runge-Kutta scheme.

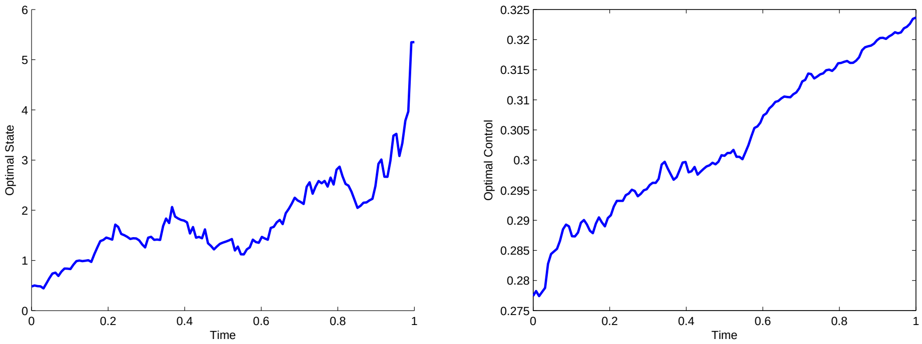


Figure 4.1: Density of the population (left), harvesting rate (right) for $\mu = 8$ at $x = 0.5$.

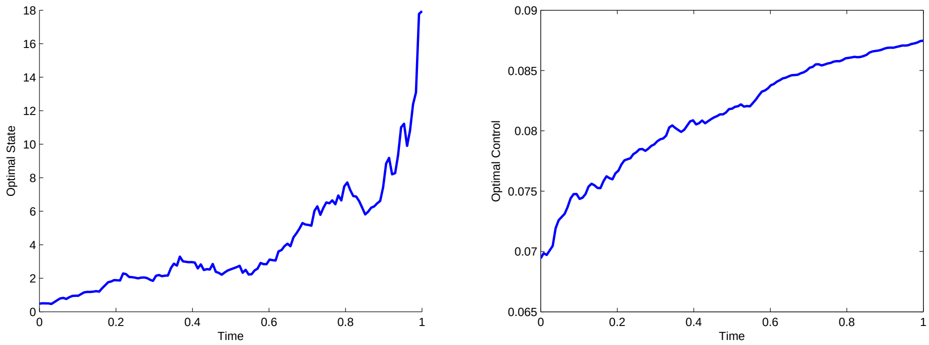


Figure 4.2: Density of the population (left), harvesting rate (right) for $\mu = 9$ at $x = 0.5$.

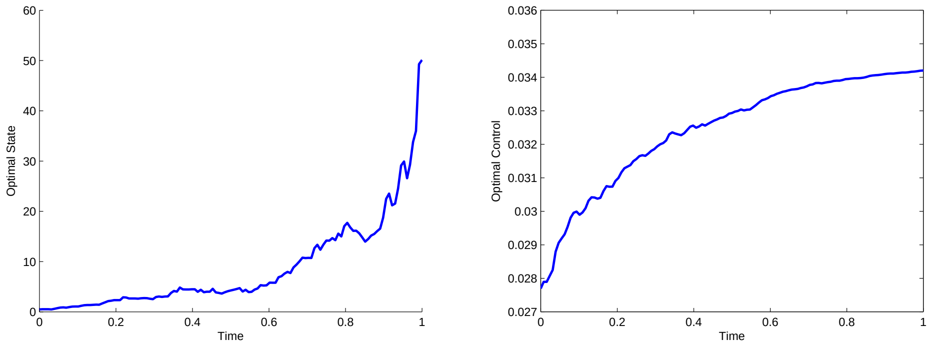


Figure 4.3: Density of the population (left), harvesting rate (right) for $\mu = 10$ at $x = 0.5$.

4.7 Summary

In this chapter, we studied the Runge-Kutta methods for the optimal control of a stochastic harvesting problem. A stochastic reaction-diffusion type of problem was chosen for modeling of the optimal harvesting. We formulated the control problem of SPDEs in terms of SDEs with the help of matrices and vectors. Furthermore, we showed that if a Runge-Kutta type method is applied to the constraint equation, then a similar scheme is obtained for the corresponding adjoint equation.

CHAPTER 5

STRONG-ORDER CONDITIONS OF THE RUNGE-KUTTA SCHEME FOR STOCHASTIC OPTIMAL CONTROL PROBLEMS

5.1 Introduction

In Chapter 3, we derived our Runge-Kutta scheme for stochastic optimal control problems of SDEs by using *discretize-then-optimize* approach. At this point, it is important to measure the accuracy of our Runge-Kutta approximation by using either the strong-order convergence or the weak-order convergence criteria. Strong approximations involve direct simulation of stochastic paths and this provides useful information about the qualitative behavior of the investigated model. Actually, strong convergence criteria of Runge-Kutta scheme for SDEs is investigated by Burrage [8] and strong-order conditions are derived. In this chapter, our aim is to get strong-order conditions of our Runge-Kutta scheme for stochastic optimal control problems.

We let $\boldsymbol{\zeta}(T)$ be a numerical approximation to $\mathbf{X}(t_N)$ after N steps with constant step size $\Delta := (t_N - t_0)/N$. Then $\boldsymbol{\zeta}(T)$ is said to converge strongly to $\mathbf{X}$ with order $r > 0$ if there exists a constant $C > 0$, which does not depend on Δ, and a $\Delta_0 > 0$ such that

$$\mathbb{E}\left[\|\boldsymbol{\zeta}(T) - \mathbf{X}(t_N)\|_2\right] \leq C\Delta^r, \qquad \Delta \in (0, \Delta_0),$$

where $\mathbf{X} = (y, p)^T$ are $\boldsymbol{\zeta} = (\hat{y}, \hat{p})^T$ the solution of continuous $(\mathcal{OC}_c)$ and discrete optimality system $(\mathcal{OC}_d)$ in Chapter 3, respectively. We notice that these optimality systems are stated in Itô forms. In this chapter, we address these optimality sytems in related Stratonovich forms.

Here, by assuming exact initial values, the Stratonovich-Taylor expansions of the exact solution and of the solution based on our Runge-Kutta scheme are compared to find the order of accuracy. Firstly, we obtain strong order-1 conditions of our Runge-Kutta method for the optimal control of SDEs. Then, we show why it is not possible to exceed the strong order-1 with our Runge-Kutta scheme, and we present the minimal truncation-error constants of our Runge-Kutta method for the optimal control of SDEs. By employing a more general Runge-Kutta scheme, we get strong order-1.5 conditions of the Runge-Kutta method on the optimal control of SDEs. In this chapter, such order conditions are derived explicitly. We confirm our results with numerical examples.

Now, we restate our optimal control problem as:

$$(\mathcal{P}_c)\begin{cases} \underset{u\in L^2(t_0,T)}{\text{minimize}} & \mathbb{E}\left[\Phi(y(T)) + \int_{t_0}^{T} \mathbb{E}[g(y,u)]dt\right] \\ \text{subject to} & dy = f(y,u)dt + h(y)dW \quad (t\in[t_0,T]), \\ & y(0) = y^0. \end{cases}$$

We recall the *Hamilton function* of the optimal control problem:

$$\mathcal{H}(y,u,p,q) = g(y,u) + f(y,u)p + \frac{1}{2}h^2(y)q,$$

with the following continuous first-order optimality system:

$$(\mathcal{OC}_c)\begin{cases} \quad dy = \mathcal{H}_p(y,u,p,q)dt + h(y)dW \quad (t\in[t_0,T]), \\ \quad dp = -\mathcal{H}_y(y,u,p,q)dt + h(y)qdW \quad (t\in[t_0,T]), \\ \quad\;\; 0 = \mathcal{H}_u(y,u,p,q) \quad (t\in[t_0,T]), \\ y(t_0) = y^0, \\ p(T) = \phi'(y(T)). \end{cases}$$

5.2 Problem Formulation and Discretization

Let $\mathbf{X}$, $\mathbf{F}$ and $\mathbf{H}$ denote the following pairs:

$$\mathbf{X} = \begin{pmatrix} y \\ p \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} f \\ -\mathcal{H}_y \end{pmatrix}, \quad \mathbf{H} = \begin{pmatrix} h \\ hq \end{pmatrix}.$$

With this notation, we can write constraint of problem $(\mathcal{P}_c)$ in the form

$$d\mathbf{X} = \mathbf{F}(y,u,p,q)dt + \mathbf{H}(y,q)dW$$

as an Itô SDE, or

$$d\mathbf{X} = \underline{\mathbf{F}}(y,u,p,q)dt + \mathbf{H}(y,q)\circ dW \tag{5.1}$$

as its related Stratonovich SDE with a modified drift coefficient which is defined by [28] with the vector

$$\underline{\mathbf{F}} = \mathbf{F} - \frac{1}{2}\mathbf{H}'\mathbf{H},$$

where $\underline{\mathbf{F}} = (\underline{f}, \underline{\mathcal{H}_y})^T$. For simplicity, we rewrite Eqn. (5.1) as

$$d\mathbf{X} = \underline{\mathbf{F}}(\mathbf{X})dt + \mathbf{H}(\mathbf{X})\circ dW, \tag{5.2}$$

where $\circ dW$ represents Stratonovich integral with respect to the Brownian motion W.

We note that any SDE in Itô form can be easily converted to its Stratonovich version, and vice versa [28]. While both SDEs have the same solution, the choice about which one is more appropriate to use, depends on the specific problem. Since Stratonovich calculus follows the same rules as Riemann-Stieltjes calculus, in this chapter, it is more advantageous to employ the Stratonovich representation of an SDE.

By following [8, 28], we get the Stratonovich-Taylor approximation of Eqn. (5.2) in the subsequent way.

Let $\boldsymbol{\mathcal{L}}^0$ and $\boldsymbol{\mathcal{L}}^1$ be vector-valued operators of 2 variables defined as

$$\boldsymbol{\mathcal{L}}^0\boldsymbol{\Phi} := \frac{\partial \boldsymbol{\Phi}}{\partial \mathbf{X}}\underline{\mathbf{F}} \qquad \text{and} \qquad \boldsymbol{\mathcal{L}}^1\boldsymbol{\Phi} := \frac{\partial \boldsymbol{\Phi}}{\partial \mathbf{X}}\mathbf{H},$$

where $\boldsymbol{\Phi}$ is any twice continuously differentiable vector-valued function of 2 variables. We note that $\partial\boldsymbol{\Phi}/\partial\mathbf{X}$ stands for Jacobian matrix of $\boldsymbol{\Phi}$ so that $\boldsymbol{\mathcal{L}}^0\boldsymbol{\Phi}$ and $\boldsymbol{\mathcal{L}}^1\boldsymbol{\Phi}$ are vectors. Now, application of deterministic Chain Rule gives:

$$\begin{aligned}
\boldsymbol{\Phi}(\mathbf{X}(t)) &= \boldsymbol{\Phi}(\mathbf{X}(t_0)) + \int_{t_0}^{t} \frac{\partial \boldsymbol{\Phi}(\mathbf{X}(s))}{\partial \mathbf{X}} \underline{\mathbf{F}}(\mathbf{X}(s))ds \\
&\qquad + \int_{t_0}^{t} \frac{\partial \boldsymbol{\Phi}(\mathbf{X}(s))}{\partial \mathbf{X}} \mathbf{H}(\mathbf{X}(s)) \circ dW(s) \\
&= \boldsymbol{\Phi}(\mathbf{X}(t_0)) + \int_{t_0}^{t} \boldsymbol{\mathcal{L}}^0\boldsymbol{\Phi}(\mathbf{X}(s))ds + \int_{t_0}^{t} \boldsymbol{\mathcal{L}}^1\boldsymbol{\Phi}(\mathbf{X}(s)) \circ dW(s). \qquad (5.3)
\end{aligned}$$

If we choose $\boldsymbol{\Phi}(\mathbf{X}(t)) = \mathbf{X}(t)$, we obtain the original Stratonovich SDE (5.1) in integral form:

$$\mathbf{X}(t) = \mathbf{X}(t_0) + \int_{t_0}^{t} \underline{\mathbf{F}}(\mathbf{X}(s))ds + \int_{t_0}^{t} \mathbf{H}(\mathbf{X}(s)) \circ dW(s). \qquad (5.4)$$

Similarly, for $\boldsymbol{\Phi}(\mathbf{X}(t)) = \underline{\mathbf{F}}(\mathbf{X}(t))$ and $\boldsymbol{\Phi}(\mathbf{X}(t)) = \mathbf{H}(\mathbf{X}(t))$, Eqn. (5.3) reduces to

$$\underline{\mathbf{F}}(\mathbf{X}(t)) = \underline{\mathbf{F}}(\mathbf{X}(t_0)) + \int_{t_0}^{t} \boldsymbol{\mathcal{L}}^0\underline{\mathbf{F}}(\mathbf{X}(s))ds + \int_{t_0}^{t} \boldsymbol{\mathcal{L}}^1\underline{\mathbf{F}}(\mathbf{X}(s)) \circ dW(s), \qquad (5.5)$$

$$\mathbf{H}(\mathbf{X}(t)) = \mathbf{H}(\mathbf{X}(t_0)) + \int_{t_0}^{t} \boldsymbol{\mathcal{L}}^0\mathbf{H}(\mathbf{X}(s))ds + \int_{t_0}^{t} \boldsymbol{\mathcal{L}}^1\mathbf{H}(\mathbf{X}(s)) \circ dW(s). \qquad (5.6)$$

Substituting Eqn. (5.5) and Eqn. (5.6) into Eqn. (5.4) implies that

$$\begin{aligned}
\mathbf{X}(t) &= \mathbf{X}(t_0) + \int_{t_0}^{t} \left(\underline{\mathbf{F}}(\mathbf{X}(t_0)) + \int_{t_0}^{s} \boldsymbol{\mathcal{L}}^0\underline{\mathbf{F}}(\mathbf{X}(z))dz + \int_{t_0}^{s} \boldsymbol{\mathcal{L}}^1\underline{\mathbf{F}}(\mathbf{X}(z)) \circ dW(z) \right) ds \\
&\qquad + \int_{t_0}^{t} \Bigg(\mathbf{H}(\mathbf{X}(t_0)) + \int_{t_0}^{s} \boldsymbol{\mathcal{L}}^0\mathbf{H}(\mathbf{X}(z))dz \\
&\qquad\qquad + \int_{t_0}^{s} \boldsymbol{\mathcal{L}}^1\mathbf{H}(\mathbf{X}(z)) \circ dW(z) \Bigg) \circ dW(s) \\
&= \mathbf{X}(t_0) + \underline{\mathbf{F}}(\mathbf{X}(t_0))J_0 + \mathbf{H}(\mathbf{X}(t_0))J_1 \\
&\qquad + \int_{t_0}^{t}\int_{t_0}^{s} \boldsymbol{\mathcal{L}}^0\underline{\mathbf{F}}(\mathbf{X}(z))dzds + \int_{t_0}^{t}\int_{t_0}^{s} \boldsymbol{\mathcal{L}}^1\underline{\mathbf{F}}(\mathbf{X}(z)) \circ dW(z)ds \\
&\qquad + \int_{t_0}^{t}\int_{t_0}^{s} \boldsymbol{\mathcal{L}}^0\mathbf{H}(\mathbf{X}(z))dz \circ dW(s) + \boldsymbol{\mathcal{L}}^1\mathbf{H}(\mathbf{X}(z)) \circ dW(z) \circ dW(s).
\end{aligned}$$

If we continue to apply Eqn. (5.3) to the integrand functions appearing in the above equation, we reach the Stratonovich-Taylor approximation of Eqn. (5.2):

$$\begin{aligned}\mathbf{X}(t) = \mathbf{X}(t_0) + (&\underline{\mathbf{F}}J_0 + \mathbf{H}J_1 + \underline{\mathbf{F}}'\underline{\mathbf{F}}J_{00} + \underline{\mathbf{F}}'\mathbf{H}J_{10} + \mathbf{H}'\underline{\mathbf{F}}J_{01} + \mathbf{H}'\mathbf{H}J_{11} \\ &+ \underline{\mathbf{F}}'\underline{\mathbf{F}}'\underline{\mathbf{F}}J_{000} + \underline{\mathbf{F}}''\underline{\mathbf{F}}\underline{\mathbf{F}}J_{000} + \underline{\mathbf{F}}'\underline{\mathbf{F}}'\mathbf{H}J_{100} + \underline{\mathbf{F}}''\underline{\mathbf{F}}\mathbf{H}J_{100} \\ &+ \underline{\mathbf{F}}'\mathbf{H}'\underline{\mathbf{F}}J_{010} + \underline{\mathbf{F}}''\mathbf{H}\underline{\mathbf{F}}J_{010} + \underline{\mathbf{F}}'\mathbf{H}'\mathbf{H}J_{110} + \underline{\mathbf{F}}''\mathbf{H}\mathbf{H}J_{110} \\ &+ \underline{\mathbf{F}}'\mathbf{H}'\underline{\mathbf{F}}J_{001} + \mathbf{H}''\underline{\mathbf{F}}\underline{\mathbf{F}}J_{001} + \mathbf{H}'\underline{\mathbf{F}}'\mathbf{H}J_{101} + \mathbf{H}''\underline{\mathbf{F}}\mathbf{H}J_{101} \\ &+ \mathbf{H}'\mathbf{H}'\underline{\mathbf{F}}J_{011} + \mathbf{H}''\mathbf{H}\underline{\mathbf{F}}J_{011} + \mathbf{H}'\mathbf{H}'\mathbf{H}J_{111} + \mathbf{H}''\mathbf{H}\mathbf{H}J_{111})(\mathbf{X}(t_0)) + \mathbf{R}.\end{aligned} \tag{5.7}$$

Here, $\mathbf{R}$ represents the remainder term and $J_{j_1 j_2 \ldots j_k}$ stands for a Stratonovich multiple integral, where integration is with respect to ds if $j_i = 0$, or $\circ dW(s)$ if $j_i = 1$. For example, in one dimension,

$$J_{110} = \int_{t_0}^{t}\int_{t_0}^{s_3}\int_{t_0}^{s_2} \circ dW(s_1) \circ dW(s_2) ds_3.$$

Let us note that the derivatives should be viewed in an operator context. For instance, the first derivative of a vector-valued function $\underline{\mathbf{F}}$ is the Jacobian matrix, so that $\underline{\mathbf{F}}'\underline{\mathbf{F}}$ corresponds to multiplying the Jacobian matrix by the vector $\underline{\mathbf{F}}$ to give a vector. The second derivative $\underline{\mathbf{F}}''$ operates on a pair of vectors $(\underline{\mathbf{F}}, \underline{\mathbf{F}})$ to give a vector $\underline{\mathbf{F}}''\underline{\mathbf{F}}\underline{\mathbf{F}}$.

We introduce an equispaced discretization $0 = t_0 < t_1 < \ldots < t_k < \ldots < t_N = T$ of the time interval $[0, T]$. Let $\Delta := T/N$ denote the time increments (step-size).

Now, we state the s-stage Runge-Kutta discretization, for some $s \in \mathbb{Z}^+$, of the optimal control problem $(\mathcal{P})$ in the Stratonovich form as:

$$(\mathcal{P}_d')\begin{cases} \text{minimize} & \mathbb{E}\left[\phi(y_N) + J_0 \sum_{k=0}^{N-1}\sum_{i=1}^{s} \alpha_i g(y_{ki}, u_{ki})\right] \\ \text{subject to} & y_{k+1} = y_k + J_0 \sum_{i=1}^{s} \alpha_i \underline{f}(y_{ki}, u_{ki}) + J_1 \sum_{i=1}^{s} \beta_i h(y_{ki}), \\ & y_{ki} = y_k + J_0 \sum_{j=1}^{s} a_{ij} \underline{f}(y_{kj}, u_{kj}) + J_1 \sum_{j=1}^{s} b_{ij} h(y_{kj}), \\ & y_0 = y^0, \end{cases}$$

for $k = 0, 1, \ldots, N-1$, where the constants $\alpha_i,\ \beta_i,\ a_{ij},\ b_{ij}$ $(i = 1, 2, \ldots, s)$ are the Runge-Kutta coefficients. The Butcher array of the Runge-Kutta discretization of system of problem $(\mathcal{P}_d')$ is given by

$$\begin{array}{c|ccc} c_1 & a_{11} & \ldots & a_{1s} \\ \vdots & \vdots & \ddots & \vdots \\ c_s & a_{s1} & \ldots & a_{ss} \\ \hline & \alpha_1 & \ldots & \alpha_s \end{array} \qquad \begin{array}{c|ccc} d_1 & b_{11} & \ldots & b_{1s} \\ \vdots & \vdots & \ddots & \vdots \\ d_s & b_{s1} & \ldots & b_{ss} \\ \hline & \beta_1 & \ldots & \beta_s \end{array}$$

Theorem 5.1. [6, 8] *Let $\alpha_i,\ \beta_i,\ a_{ij},\ b_{ij}$ ($i, j = 1, 2, \ldots, s$) be the Runge-Kutta coefficients. If the coefficients of stochastic Runge-Kutta method for SDEs (2.11) fulfill the following conditions:*

$$\text{A1.}\ \sum_{i=1}^{s}\alpha_i = 1, \qquad \text{A2.}\ \sum_{i=1}^{s}\beta_i = 1, \qquad \text{A3.}\ \sum_{i,j=1}^{s}\beta_i b_{ij} = \frac{1}{2},$$

then the stochastic Runge-Kutta method converges of order-1 in the strong sense.

5.3 Strong Order-1 Conditions of Runge-Kutta Method with Minimal Truncation Error Constants for Stochastic Optimal Control Problems

In Chapter 3, we have derived the discrete optimality conditions, $(\mathcal{OC}_d)$, for stochastic optimal control problems of SDEs in Itô form. Similar discrete optimality conditions can be also derived for stochastic optimal control problems of SDEs in the Stratonovich form.

Theorem 5.2. *If $\alpha_i,\ \beta_i,\ a_{ij},\ b_{ij}$ ($i, j = 1, 2, \ldots, s$) are the Runge-Kutta coefficients in the system of problem $(\mathcal{P}'_d)$, then the discrete first-order optimality conditions associated to the system of problem $(\mathcal{P}'_d)$ are obtained as:*

$$(\mathcal{OC}'_d)\begin{cases} y_{k+1} = y_k + J_0\sum_{i=1}^{s}\alpha_i \underline{f}(y_{ki}, u_{ki}) + J_1\sum_{i=1}^{s}\beta_i h(y_{ki}), \\ y_{ki} = y_k + J_0\sum_{j=1}^{s}a_{ij}\underline{f}(y_{kj}, u_{kj}) + J_1\sum_{j=1}^{s}b_{ij}h(y_{kj}), \\ p_{k+1} = p_k - J_0\sum_{i=1}^{s}\tilde{\alpha}_i\underline{\mathcal{H}}_y(y_{ki}, u_{ki}, p_{ki}, q_{ki}) + J_1\sum_{i=1}^{s}\tilde{\beta}_i h(y_{ki})q_{ki}, \\ p_{ki} = p_k - J_0\sum_{j=1}^{s}\tilde{a}_{ij}\underline{\mathcal{H}}_y(y_{kj}, u_{kj}, p_{kj}, q_{kj}) + J_1\sum_{j=1}^{s}\tilde{b}_{ij}h(y_{kj})q_{kj}, \\ q_{ki}\psi_{ki} = p_k - J_0\sum_{j=1}^{s}\hat{a}_{ij}\underline{\mathcal{H}}_y(y_{kj}, u_{kj}, p_{kj}, q_{kj}) + J_1\sum_{j=1}^{s}\hat{b}_{ij}h(y_{kj})q_{kj}, \\ p_N = \phi'(y_N), \\ y_0 = y^0, \\ 0 = \Delta\sum_{i=1}^{s}\alpha_i\underline{\mathcal{H}}_u(y_{ki}, u_{ki}, p_{ki}, q_{ki}), \end{cases}$$

for $k = 0, 1, \ldots, N-1$, where the coefficients satisfy the subsequent relations:

$$\tilde{\alpha}_i := \alpha_i, \qquad \tilde{\beta}_i := \beta_i,$$
$$\tilde{a}_{ij} := \alpha_j - \frac{\alpha_j}{\alpha_i} a_{ji}, \qquad \tilde{b}_{ij} := \beta_j - \frac{\beta_j}{\alpha_i} a_{ji},$$
$$\hat{a}_{ij} := \alpha_j - \frac{\alpha_j}{\beta_i} b_{ji}, \qquad \hat{b}_{ij} := \beta_j - \frac{\beta_j}{\beta_i} b_{ji},$$

with

$$\psi_{ki} := \frac{J_0 t \alpha_i h(y_{ki})}{J_1 \beta_i} - \frac{h(y_{ki})}{h'(y_{ki})}.$$

We shall not give the proof here, since it is quite similar to the Itô case, Theorem 3.1, which is given in Chapter 3.

Now, we will obtain strong order-1 conditions of our Runge-Kutta scheme by matching the Stratonovich-Taylor series expansion of the exact solution and the approximation defined by the Runge-Kutta method for SDEs over one step, assuming exact initial values. For this reason, we use a similar notation to Hager (see page 261 in [21]) and we first write the Stratonovich-Taylor series expansion of the approximation defined via the Runge-Kutta method, by benefiting from the approach of Butcher [6, 8, 9].

In order to study order conditions of discrete optimality conditions $(\mathcal{OC}'_d)$, the $(\mathcal{OC}'_d)$ will be written as a function of t. By using Butcher approach, we write $t_n = t_0$, and for a given $t = t_0 + \Delta$. For a given initial iteration values y_k and p_{k+1}, the solutions y_{ki} and p_{ki} are functions of t, denoted by $y_{ki}(t)$ and $p_{ki}(t)$, respectively. Let the values $p_k(t)$ and $y_{k+1}(t)$ stand for the iterates p_k and y_{k+1}, respectively, which can be calculated as $y(t)$ and $p(t)$ with intermediate values $y_{ki} = y_{ki}(t)$ and $p_{ki} = p_{ki}(t)$. For this reason, let $\boldsymbol{\zeta}(t) = (y(t), p(t))^T$ be the vector of length $2N(s+1)$ and let $\boldsymbol{\zeta}_{ki}(t)$, $\boldsymbol{\zeta}_{s+1}(t)$, $\underline{\tilde{\mathbf{F}}}(\boldsymbol{\zeta}(t))$ and $\tilde{\mathbf{H}}(\boldsymbol{\zeta}(t))$ denote the following pairs:

$$\boldsymbol{\zeta}_{ki}(t) = \begin{pmatrix} y_{ki}(t) \\ p_{ki}(t) \end{pmatrix} \quad (1 \le i \le s), \quad \boldsymbol{\zeta}_{s+1}(t) = \begin{pmatrix} y_{k+1}(t) \\ p_k(t) \end{pmatrix} \quad (i = s+1),$$

$$\underline{\tilde{\mathbf{F}}}(\boldsymbol{\zeta}(t)) = \underline{\tilde{\mathbf{F}}}_i(\boldsymbol{\zeta}(t)) = \begin{pmatrix} \sum_{j=1}^s a_{ij} \underline{f}(\boldsymbol{\zeta}_{kj}(t)) \\ \sum_{j=1}^s \tilde{a}_{ij} \underline{\mathcal{H}}_y(\boldsymbol{\zeta}_{kj}(t)) \end{pmatrix} \quad (1 \le i \le s+1),$$

and

$$\tilde{\mathbf{H}}(\boldsymbol{\zeta}(t)) = \tilde{\mathbf{H}}_i(\boldsymbol{\zeta}(t)) = \begin{pmatrix} \sum_{j=1}^s b_{ij} h(\boldsymbol{\zeta}_{kj}(t)) \\ \sum_{j=1}^s \tilde{b}_{ij} h(\boldsymbol{\zeta}_{kj}(t)) q(\boldsymbol{\zeta}_{kj}(t)) \end{pmatrix} \quad (1 \le i \le s+1),$$

where $k = 0, 1, \ldots, N-1$, is the index of the Runge-Kutta scheme in the discrete optimality conditions $(\mathcal{OC}'_d)$, with

$$a_{s+1,j} = \tilde{a}_{s+1,j} = \alpha_j \quad (1 \le j \le s),$$
$$b_{s+1,j} = \tilde{b}_{s+1,j} = \beta_j \quad (1 \le j \le s).$$

By using the above notation, we can state the discrete optimality conditions $(\mathcal{OC}'_d)$ in the form

$$\boldsymbol{\zeta}(t) = \boldsymbol{\zeta}(t_0) + (t - t_0)\underline{\tilde{\mathbf{F}}}(\boldsymbol{\zeta}(t)) + J_1\tilde{\mathbf{H}}(\boldsymbol{\zeta}(t)) \quad (1 \le i \le s+1), \tag{5.8}$$

where $J_1 = \Delta W = W(t) - W(t_0)$ so that $J_1(t_0) = 0$.

The term $\underline{\tilde{\mathbf{F}}}(\boldsymbol{\zeta}(t))$ (and $\tilde{\mathbf{H}}(\boldsymbol{\zeta}(t))$, analogously) can be represented by using a Taylor-series expansion:

$$\underline{\tilde{\mathbf{F}}}(\boldsymbol{\zeta}(t)) = \underline{\tilde{\mathbf{F}}}(\boldsymbol{\zeta}(t_0)) + \sum_{n=1}^{\infty} \frac{\boldsymbol{\mathcal{L}}_{\Delta}^{n}\underline{\tilde{\mathbf{F}}}(\boldsymbol{\zeta}(t_0))}{n!}, \tag{5.9}$$

where $\boldsymbol{\mathcal{L}}_{\Delta}$ is the vector-valued differential operator of 2 variables, given by

$$\boldsymbol{\mathcal{L}}_{\Delta}\boldsymbol{\Phi} := \Delta\frac{\partial\boldsymbol{\Phi}}{\partial t} + J_1\frac{\partial\boldsymbol{\Phi}}{\partial J_1},$$

since $\boldsymbol{\Phi}$ is any twice continuously differentiable vector-valued function of 2 variables. Thus, it is seen that

$$\begin{aligned}\boldsymbol{\mathcal{L}}_{\Delta}\underline{\tilde{\mathbf{F}}}(\boldsymbol{\zeta}(t)) &= \Delta\underline{\tilde{\mathbf{F}}}'(\boldsymbol{\zeta}(t))\left(\underline{\tilde{\mathbf{F}}}(\boldsymbol{\zeta}(t)) + (t-t_0)\underline{\tilde{\mathbf{F}}}'(\boldsymbol{\zeta}(t))\frac{\partial\boldsymbol{\zeta}(t)}{\partial t} + J_1\tilde{\mathbf{H}}'(\boldsymbol{\zeta}(t))\frac{\partial\boldsymbol{\zeta}(t)}{\partial t}\right)\\ &\quad + J_1\underline{\tilde{\mathbf{F}}}'(\boldsymbol{\zeta}(t))\left((t-t_0)\underline{\tilde{\mathbf{F}}}'(\boldsymbol{\zeta}(t))\frac{\partial\boldsymbol{\zeta}(t)}{\partial J_1} + \tilde{\mathbf{H}}(\boldsymbol{\zeta}(t)) + J_1\tilde{\mathbf{H}}'(\boldsymbol{\zeta}(t))\frac{\partial\boldsymbol{\zeta}(t)}{\partial J_1}\right),\end{aligned}$$

so that

$$\boldsymbol{\mathcal{L}}_{\Delta}\underline{\tilde{\mathbf{F}}}(\boldsymbol{\zeta}(t_0)) = J_0\underline{\tilde{\mathbf{F}}}'(\boldsymbol{\zeta}(t_0))\underline{\tilde{\mathbf{F}}}(\boldsymbol{\zeta}(t_0)) + J_1\underline{\tilde{\mathbf{F}}}'(\boldsymbol{\zeta}(t_0))\tilde{\mathbf{H}}(\boldsymbol{\zeta}(t_0)).$$

Similarly,

$$\boldsymbol{\mathcal{L}}_{\Delta}\tilde{\mathbf{H}}(\boldsymbol{\zeta}(t_0)) = J_0\tilde{\mathbf{H}}'(\boldsymbol{\zeta}(t_0))\underline{\tilde{\mathbf{F}}}(\boldsymbol{\zeta}(t_0)) + J_1\tilde{\mathbf{H}}'(\boldsymbol{\zeta}(t_0))\tilde{\mathbf{H}}(\boldsymbol{\zeta}(t_0)).$$

Then, the Stratonovich-Taylor approximation of Eqn. (5.8) looks as follows:

$$\begin{aligned}\boldsymbol{\zeta}(t) &= \boldsymbol{\zeta}(t_0) + J_0\underline{\tilde{\mathbf{F}}}(\boldsymbol{\zeta}(t)) + J_1\left(\tilde{\mathbf{H}}(\boldsymbol{\zeta}(t))\right)\\ &= \boldsymbol{\zeta}(t_0) + J_0\left(\underline{\tilde{\mathbf{F}}}(\boldsymbol{\zeta}(t_0)) + \boldsymbol{\mathcal{L}}\underline{\tilde{\mathbf{F}}}(\boldsymbol{\zeta}(t_0)) + \frac{1}{2!}\boldsymbol{\mathcal{L}}^2\underline{\tilde{\mathbf{F}}}(\boldsymbol{\zeta}(t_0)) + \ldots\right)\\ &\qquad + J_1\left(\tilde{\mathbf{H}}(\boldsymbol{\zeta}(t_0)) + \boldsymbol{\mathcal{L}}\tilde{\mathbf{H}}(\boldsymbol{\zeta}(t_0)) + \frac{1}{2!}\boldsymbol{\mathcal{L}}^2\tilde{\mathbf{H}}(\boldsymbol{\zeta}(t_0)) + \ldots\right),\end{aligned}$$

$$
\begin{aligned}
\boldsymbol{\zeta}(t) = \boldsymbol{\zeta}(t_0) + \Bigg(& J_0\underline{\tilde{\mathbf{F}}} + J_0^2\underline{\tilde{\mathbf{F}}}'\underline{\tilde{\mathbf{F}}} + J_0J_1\underline{\tilde{\mathbf{F}}}'\tilde{\mathbf{H}} + \frac{1}{2}J_0^3\underline{\tilde{\mathbf{F}}}''\underline{\tilde{\mathbf{F}}}\underline{\tilde{\mathbf{F}}} + J_0^3\underline{\mathbf{F}}'\underline{\tilde{\mathbf{F}}}'\underline{\tilde{\mathbf{F}}} \\
& + J_0^2J_1\underline{\tilde{\mathbf{F}}}'\underline{\tilde{\mathbf{F}}}'\tilde{\mathbf{H}} + \frac{1}{2}J_0^2J_1\underline{\tilde{\mathbf{F}}}''\underline{\tilde{\mathbf{F}}}\tilde{\mathbf{H}} + J_0^2J_1\underline{\tilde{\mathbf{F}}}'\tilde{\mathbf{H}}'\underline{\tilde{\mathbf{F}}} \\
& + \frac{1}{2}J_0^2J_1\underline{\tilde{\mathbf{F}}}''\tilde{\mathbf{H}}\underline{\tilde{\mathbf{F}}} + \frac{1}{2}J_0J_1^2\underline{\tilde{\mathbf{F}}}''\tilde{\mathbf{H}}\tilde{\mathbf{H}} + J_0J_1^2\underline{\tilde{\mathbf{F}}}'\tilde{\mathbf{H}}'\tilde{\mathbf{H}}\Bigg)(\boldsymbol{\zeta}(t_0)) + \ldots \quad (5.10) \\
& + \Bigg(J_1\tilde{\mathbf{H}} + J_0J_1\tilde{\mathbf{H}}'\underline{\tilde{\mathbf{F}}} + J_1^2\tilde{\mathbf{H}}'\tilde{\mathbf{H}} + \frac{1}{2}J_0^2J_1\tilde{\mathbf{H}}''\underline{\tilde{\mathbf{F}}}\underline{\tilde{\mathbf{F}}} + J_0^2J_1\tilde{\mathbf{H}}'\underline{\tilde{\mathbf{F}}}'\underline{\tilde{\mathbf{F}}} \\
& + J_0J_1^2\tilde{\mathbf{H}}'\underline{\tilde{\mathbf{F}}}'\tilde{\mathbf{H}} + \frac{1}{2}J_0J_1^2\tilde{\mathbf{H}}''\tilde{\mathbf{H}}\underline{\tilde{\mathbf{F}}} + J_0J_1^2\tilde{\mathbf{H}}'\tilde{\mathbf{H}}'\underline{\tilde{\mathbf{F}}} + \frac{1}{2}J_0J_1^2\tilde{\mathbf{H}}''\tilde{\mathbf{H}}\underline{\tilde{\mathbf{F}}} \\
& + \frac{1}{2}J_1^3\tilde{\mathbf{H}}''\tilde{\mathbf{H}}\tilde{\mathbf{H}} + J_1^3\tilde{\mathbf{H}}'\tilde{\mathbf{H}}'\tilde{\mathbf{H}}\Bigg)(\boldsymbol{\zeta}(t_0)) + \ldots
\end{aligned}
$$

As we mentioned, in the case of the Stratonovich-Taylor expansion of the exact solution, the derivatives should be considered in an operator context.

For simplicity,

$$
\underline{\tilde{\mathbf{F}}}_i = \begin{pmatrix} \sum_{j=1}^s a_{ij}\underline{f} \\ -\sum_{j=1}^s \tilde{a}_{ij}\underline{\mathcal{H}}_y \end{pmatrix} = \sum_{j=1}^s a_{ij}\underline{\mathbf{f}}^0 + \sum_{j=1}^s \tilde{a}_{ij}(-\underline{\boldsymbol{\mathcal{H}}}_y^0) \quad (1 \le i \le s+1),
$$

where

$$
\underline{\mathbf{f}}^0 = \begin{pmatrix} \underline{f} \\ 0 \end{pmatrix}, \quad \underline{\boldsymbol{\mathcal{H}}}_y^0 = \begin{pmatrix} 0 \\ -\underline{\mathcal{H}}_y \end{pmatrix},
$$

and

$$
\tilde{\mathbf{H}}_i = \begin{pmatrix} \sum_{j=1}^s b_{ij}h \\ \sum_{j=1}^s \tilde{b}_{ij}hq \end{pmatrix} = \sum_{j=1}^s b_{ij}\mathbf{h}^0 + \sum_{j=1}^s \tilde{b}_{ij}(\mathbf{hq})^0 \quad (1 \le i \le s+1),
$$

with

$$
\mathbf{h}^0 = \begin{pmatrix} h \\ 0 \end{pmatrix}, \quad (\mathbf{hq})^0 = \begin{pmatrix} 0 \\ hq \end{pmatrix},
$$

and

$$
(\underline{\tilde{\mathbf{F}}}_i')_j = a_{ij}(\underline{\mathbf{f}}^0)' + \tilde{a}_{ij}((\underline{\boldsymbol{\mathcal{H}}}_y)^0)'.
$$

The J-integrals J_0, J_1, J_{11}, in Eqns. (5.7) and (5.10) are of order Δ, $\Delta^{0.5}$ and Δ, respectively, so that strong order-1 conditions of our Runge-Kutta scheme can be analyzed as stated below.

Since $J_1 \sim \mathcal{N}(0, \Delta)$, we have [28]:

$$
\mathbb{E}[J_1^{2k+1}] = 0, \ \ \mathbb{E}[J_1^{2k}] = \frac{2k!}{k!2^k}\Delta^k, \ \ \mathbb{E}[J_{10}J_1] = \frac{1}{2}\Delta^2, \ \ \mathbb{E}[J_{10}^2] = \frac{1}{3}\Delta^3,
$$

and

$$J_{1\dots 1} = \frac{J_1^p}{p!}.$$

i) Since

$$\underline{\tilde{\mathbf{F}}}_{s+1} = \begin{pmatrix} \sum_{i=1}^s \alpha_i \underline{f} \\ -\sum_{i=1}^s \alpha_i \underline{\mathcal{H}}_y \end{pmatrix} = \sum_{i=1}^s \alpha_i \underline{\mathbf{F}} \quad (1 \leq i \leq s+1),$$

it holds

$$\mathbb{E}\left[J_0 - J_0 \sum_{i=1}^s \alpha_i\right]^2 = \mathbb{E}\left[\Delta\left(1 - \sum_{i=1}^s \alpha_i\right)\right]^2,$$

$$\Rightarrow \sum_{i=1}^s \alpha_i = 1 \quad \text{(Condition A1. in Theorem 5.1).}$$

ii) The second strong order-1 term is J_1:

$$J_1 \tilde{\mathbf{H}}_{s+1} = J_1 \sum_{i=1}^s \beta_i \mathbf{H},$$

which implies that

$$\mathbb{E}\left[J_1 - J_1 \sum_{i=1}^s \beta_i\right]^2 = \mathbb{E}\left[J_1\left(1 - \sum_{i=1}^s \beta_i\right)\right]^2,$$

$$\Rightarrow \sum_{i=1}^s \beta_i = 1 \quad \text{(Condition A2. in Theorem 5.1).}$$

iii) The final strong order-1 term comes from the integral J_{11}:

$$J_1^2 \tilde{\mathbf{H}}' \tilde{\mathbf{H}} = J_1^2 \sum_{i=1}^s (\tilde{\mathbf{H}}'_{s+1})_i (\tilde{\mathbf{H}})_i = J_1^2 \sum_{i,j=1}^s \beta_i \mathbf{H}'(b_{ij}\mathbf{h}^0 + \tilde{b}_{ij}(\mathbf{hq})^0).$$

If

$$\sum_{i,j=1}^s \beta_i b_{ij} = \sum_{i,j=1}^s \beta_i \tilde{b}_{ij},$$

then

$$\mathbb{E}\left[\frac{J_1^2}{2} - J_1^2\sum_{i,j=1}^{s}\beta_i b_{ij}\right]^2 = \mathbb{E}\left[J_1^2\left(\frac{1}{2} - \sum_{i,j=1}^{s}\beta_i b_{ij}\right)\right]^2$$
$$= 3\Delta^2\left(\frac{1}{2} - \sum_{i,j=1}^{s}\beta_i b_{ij}\right)^2,$$
$$\Rightarrow \sum_{i,j=1}^{s}\beta_i b_{ij} = \frac{1}{2} \quad \text{(Condition A3. in Theorem 5.1).}$$

Similarly, we can obtain

$$\sum_{i,j=1}^{s}\beta_i\tilde{b}_{ij} = \sum_{i,j=1}^{s}\beta_i\beta_j\left(1 - \frac{a_{ji}}{\alpha_i}\right) = \frac{1}{2} \quad \text{(New condition to Theorem 5.1).} \tag{5.11}$$

At this point, the fundamental issue is to construct a family of methods satisfying strong order-1 conditions. In the case of $s = 2$ with the explicit scheme for the state equation and related implicit scheme for the adjoint equation, we have 4 conditions for strong order-1, and there are 6 unknowns. Thus, free parameters guarantee the existence of a solution. Therefore, one can find different methods (different coefficients) which satisfy the strong order-1 conditions for $s \geq 2$. When constructing these methods, one needs to be careful about the associated truncation-error constants, since larger truncation-error constants can cause a reduction in the effectiveness of the method. For this reason, we aim to construct a Runge-Kutta method of strong order-1 with minimum local truncation-error constants.

The terms corresponding to the $\Delta^{1.5}$ in Eqns. (5.7) and (5.10) arise from the following J-integrals: J_{01}, J_{10} and J_{111}. Herewith, the minimum local truncation-error constants of our Runge-Kutta scheme to have strong order-1 are analyzed below.

iv) The first strong order-1.5 term looks as follows:

$$J_0J_1\tilde{\mathbf{H}}'\underline{\tilde{\mathbf{F}}} = J_0J_1\sum_{i=1}^{s}(\tilde{\mathbf{H}}'_{s+1})_i(\underline{\tilde{\mathbf{F}}})_i = J_0J_1\sum_{i,j=1}^{s}\beta_i\mathbf{H}'(a_{ij}\mathbf{f}^0 + \tilde{a}_{ij}(-\boldsymbol{\mathcal{H}}_y)^0).$$

If

$$\sum_{i,j=1}^{s}\beta_i a_{ij} = \sum_{i,j=1}^{s}\beta_i\tilde{a}_{ij},$$

then

$$\mathbb{E}\left[J_{01} - J_0J_1\sum_{i=1}^{s}\beta_i a_{ij}\right]^2 = \left(\frac{1}{3} - \sum_{i,j=1}^{s}\beta_i a_{ij} + \left(\sum_{i,j=1}^{s}\beta_i a_{ij}\right)^2\right)\Delta^3. \tag{5.12}$$

Here, note that the quadratic equation in Eqn. (5.12) does not have any real root, so that this term cannot be zero. Hence, this result prevents us from getting strong order-1.5. However, the minimal value of the function in Eqn. (5.12) is $\Delta^3/12$.

v) The second strong order-1.5 term:

$$J_1 J_0 \underline{\tilde{\mathbf{F}}}' \tilde{\mathbf{H}} = J_1 J_0 \sum_{i=1}^{s} (\underline{\tilde{\mathbf{F}}}'_{s+1})_i (\tilde{\mathbf{H}})_i = J_1 J_0 \sum_{i,j=1}^{s} \alpha_i \underline{\mathbf{F}}' (b_{ij} \mathbf{h}^0 + \tilde{b}_{ij} (\mathbf{hq})^0).$$

If

$$\sum_{i,j=1}^{s} \alpha_i b_{ij} = \sum_{i,j=1}^{s} \alpha_i \tilde{b}_{ij},$$

then

$$\mathbb{E}\left[J_{10} - J_1 J_0 \sum_{i,j=1}^{s} \alpha_i b_{ij}\right]^2 = \left(\frac{1}{3} - \sum_{i,j=1}^{s} \alpha_i b_{ij} + \left(\sum_{i,j=1}^{s} \alpha_i b_{ij}\right)^2\right) \Delta^3, \tag{5.13}$$

and the minimal value of the function in Eqn. (5.13) is also $\Delta^3/12$.

vi) The third strong order-1.5 term:

$$\frac{J_1^3}{2} \tilde{\mathbf{H}}'' \tilde{\mathbf{H}} \tilde{\mathbf{H}} = \frac{J_1^3}{2} \sum_{i=1}^{s} (\tilde{\mathbf{H}}''_{s+1})_i (\tilde{\mathbf{H}})_i^2 = \frac{J_1^3}{2} \sum_{i,j=1}^{s} \beta_i \mathbf{H}'' \left(b_{ij} \mathbf{h}^0 + \tilde{b}_{ij} (\mathbf{hq})^0\right)^2.$$

If

$$\sum_{i,j=1}^{s} \beta_i b_{ij}^2 = \sum_{i,j=1}^{s} \beta_i b_{ij} \tilde{b}_{ij} = \sum_{i,j=1}^{s} \beta_i \tilde{b}_{ij}^2,$$

then

$$\mathbb{E}\left[J_{111} - \frac{J_1^3}{2} \sum_{i,j=1}^{s} \beta_i b_{ij}^2\right]^2 = \mathbb{E}\left[\frac{1}{9} - \frac{2}{3} \sum_{i,j=1}^{s} \beta_i b_{ij}^2 + \left(\sum_{i,j=1} \beta_i b_{ij}^2\right)^2\right] \frac{15}{4} \Delta^3; \tag{5.14}$$

the minimal value of the function in Eqn. (5.14) is attained as 0.

The last strong order-1.5 term is

$$\begin{aligned} J_1^3 \tilde{\mathbf{H}}' \tilde{\mathbf{H}}' \tilde{\mathbf{H}} &= J_1^3 \sum_{i,j,k=1}^{s} (\tilde{\mathbf{H}}'_{s+1})_i (\tilde{\mathbf{H}}')_i (\tilde{\mathbf{H}})_j \\ &= J_1^3 \sum_{i,j,k=1}^{s} \beta_i \mathbf{H}' \left(b_{ij} (\mathbf{h}^0)' + \tilde{b}_{ij} ((\mathbf{hq})^0)'\right) \left(b_{jk} \mathbf{h}^0 + \tilde{b}_{jk} (\mathbf{hq})^0\right). \end{aligned}$$

If

$$\sum_{i,j,k=1}^{s} \beta_i b_{jk} b_{ij} = \sum_{i,j,k=1}^{s} \beta_i b_{jk} \tilde{b}_{ij} = \sum_{i,j,k=1}^{s} \beta_i \tilde{b}_{jk} b_{ij} = \sum_{i,j,k=1}^{s} \beta_i \tilde{b}_{jk} \tilde{b}_{ij},$$

then we can obtain

$$\begin{aligned} &\mathbb{E}\left[\frac{J_1^3}{6} - J_1^3 \sum_{i,j,k=1}^{s} \beta_i b_{jk} b_{ij}\right]^2 \\ &= \mathbb{E}\left[\frac{1}{36} - \frac{1}{3}\sum_{i,j,k=1}^{s} \beta_i b_{jk} b_{ij} + \left(\sum_{i,j,k=1}^{s} \beta_i b_{jk} b_{ij}\right)^2\right] 15\Delta^3, \end{aligned} \tag{5.15}$$

and the minimal value of the function in Eqn. (5.14) is also 0.

Eqns. (5.12)-(5.15) constitute the truncation-error constants. These equations are minimized if

$$(\mathcal{S}) \begin{cases} \displaystyle\sum_{i,j=1}^{s} \beta_i a_{ij} = \sum_{i,j=1}^{s} \beta_i \tilde{a}_{ij} = \frac{1}{2}, \\ \displaystyle\sum_{i,j=1}^{s} \alpha_i b_{ij} = \sum_{i,j=1}^{s} \alpha_i \tilde{b}_{ij} = \frac{1}{2}, \\ \displaystyle\sum_{i,j=1}^{s} \beta_i b_{ij}^2 = \sum_{i,j=1}^{s} \beta_i b_{ij} \tilde{b}_{ij} = \sum_{i,j=1}^{s} \beta_i \tilde{b}_{ij}^2 = \frac{1}{3}, \\ \displaystyle\sum_{i,j,k=1}^{s} \beta_i b_{jk} b_{ij} = \sum_{i,j,k=1}^{s} \beta_i b_{jk} \tilde{b}_{ij} = \sum_{i,j,k=1}^{s} \beta_i \tilde{b}_{jk} b_{ij} = \sum_{i,j,k=1}^{s} \beta_i \tilde{b}_{jk} \tilde{b}_{ij} = \frac{1}{6}, \end{cases}$$

in which case the minima of the functions in the system $(\mathcal{S})$ are, respectively,

$$\frac{\Delta^3}{12},\ \frac{\Delta^3}{12},\ 0,\ 0.$$

Let us note that to get strong order-1.5, all of the coefficients of terms containing Δ^3 must be zero. Since the coefficients of Eqns. (5.12) and (5.13) can not be zero, it is impossible to exceed strong order-1 for any number of stages without introducing another type of random variable in the method formulation of the Runge-Kutta scheme in $(\mathcal{P}_d')$.

For a 2-stage explicit method on the state equation and implicit method on the adjoint equation, 4 conditions of strong order-1 must be satisfied, and 11 conditions of the local truncation error-constants in $(\mathcal{S})$ have to be fulfilled. By inserting the 4 conditions into 11 conditions, one can set up and solve an unconstrained multi-objective minimization problem with one variable, namely, β_1. The range of optimal solution is obtained by finding the maximum and the minimum values for β_1 such that $\beta_1 \in [-0.4584, 0.7500]$. We notice that each line in ($\mathcal{S}$) corresponds to a truncation-error constant term if these inequalities are satisfied, so that we have 4 truncation error constants. Otherwise, we address the maximum norm of the equations for each line in ($\mathcal{S}$) to find the truncation-error constants. For simplicity, we choose $\beta_1 = 0.5$. Thus, we have found that the principal truncation-error constants are

$$\frac{\Delta^3}{12}, \frac{\Delta^3}{12}, \frac{5\Delta^3}{48}, \frac{5\Delta^3}{12}, \tag{5.16}$$

and the solution is represented by the following tableaus:

$$\begin{array}{c|cc} & 0 & 0 \\ & 1 & 0 \\ \hline & 0.5 & 0.5 \end{array} \qquad \begin{array}{c|cc} & 0 & 0 \\ & 1 & 0 \\ \hline & 0.5 & 0.5 \end{array} \qquad \text{(for the state equation)},$$

$$\begin{array}{c|cc} & 0.5 & -0.5 \\ & 0.5 & 0.5 \\ \hline & 0.5 & 0.5 \end{array} \qquad \begin{array}{c|cc} & 0.5 & -0.5 \\ & 0.5 & 0.5 \\ \hline & 0.5 & 0.5 \end{array} \qquad \text{(for the adjoint equation)}.$$

Therefore, in the above analysis, it seems that without introducing an additional random variable to the classical Runge-Kutta method, it is not possible to exceed the strong order-1 for any number of stages. We state this result in the following theorem.

Theorem 5.3. *Let $\alpha_i, \beta_i, a_{ij}, b_{ij}$ $(i, j = 1, 2, \ldots, s)$ be Runge-Kutta coefficients in problem $(\mathcal{P}'_d)$. If the coefficients of our Runge-Kutta method for the stochastic optimal control problem satisfy equations A1-A3 in Theorem 5.1 and Eqn. (5.11), then the stochastic Runge-Kutta method has maximum strong order-1 for any number of stages. Moreover, as a special case, $s = 2$, the optimal principal truncation-error coefficients of explicit method for the state equation and related implicit method for the adjoint equation are given in Eqn. (5.16).*

5.4 Strong Order-1.5 Conditions of Runge-Kutta Method for Stochastic Optimal Control Problems

In the previous section, we have obtained strong order-1 conditions of our Runge-Kutta scheme for stochastic optimal control problems. Likewise, Burrage and Burrage (1996), we are not able to exceed strong order-1 by using the Runge-Kutta method in the system of problem $(\mathcal{P}'_d)$. For this reason, they assumed that every random variables, $J_1 \sum_{j=1}^{s} b_{ij}$ and $J_1 \sum_{i=1}^{s} \beta_i$, can be written as a linear combination of p different random variables, $\theta_1, \theta_2, \ldots, \theta_p$, in order to study strong-order properties of the Runge-Kutta method, especially, order-1.5 and higher strong-order conditions. We follow their assumption to receive strong order-1.5 conditions of our Runge-Kutta scheme for stochastic optimal control problems. Herewith, problem $(\mathcal{P}'_d)$ is a specific case of the

Runge-Kutta method:

$$(\mathcal{P}_p)\begin{cases} \text{minimize} & \mathbb{E}\left[\phi(y_N) + \Delta \sum_{k=0}^{N-1}\sum_{i=1}^{s} \alpha_i g(y_{ki}, u_{ki})\right] \\ \text{subject to} & y_{k+1} = y_k + \Delta \sum_{i=1}^{s} \alpha_i \underline{f}(y_{ki}, u_{ki}) + \sum_{l=1}^{p}\left(\sum_{i=1}^{s} \beta_i^{(l)} h(y_{ki})\right)\theta_l, \\ & y_{ki} = y_k + \Delta \sum_{j=1}^{s} a_{ij}\underline{f}(y_{kj}, u_{kj}) + \sum_{l=1}^{p}\left(\sum_{j=1}^{s} b_{ij}^{(l)} h(y_{kj})\right)\theta_l, \\ & y_0 = y^0, \end{cases}$$

for $k = 0, 1, \ldots, N-1$; $i = 1, 2, \ldots, s$, where $\theta_1, \theta_2, \ldots, \theta_p$, are random variables that can be written in terms of multiple Stratonovich integral and have the same strong order as with J_1. It was taken as $p = 2$ with $\theta_1 = J_1$ and $\theta_2 = J_{10}/\Delta$ to obtain strong-order conditions, such that the problem $(\mathcal{P}_p)$ can be rewritten as:

$$(\mathcal{P}_2)\begin{cases} \text{minimize} & \mathbb{E}\left[\phi(y_N) + \Delta \sum_{k=0}^{N-1}\sum_{i=1}^{s} \alpha_i g(y_{ki}, u_{ki})\right] \\ \text{subject to} & y_{k+1} = y_k + \Delta \sum_{i=1}^{s} \alpha_i \underline{f}(y_{ki}, u_{ki}) + \sum_{i=1}^{s}\left(\beta_i^{(1)} J_1 + \beta_i^{(2)} \frac{J_{10}}{\Delta}\right) h(y_{ki}), \\ & y_{ki} = y_k + \Delta \sum_{j=1}^{s} a_{ij}\underline{f}(y_{kj}, u_{kj}) + \sum_{j=1}^{s}\left(b_{ij}^{(1)} J_1 + b_{ij}^{(2)} \frac{J_{10}}{\Delta}\right) h(y_{kj}), \\ & y_0 = y^0. \end{cases}$$

for $k = 0, 1, \ldots, N-1$, and $i = 1, 2, \ldots, s$.

By comparing the Stratonovich-Taylor series expansions of the exact solution and of the approximation method defined by the Runge-Kutta method for SDEs, respectively, Burrage and Burrage (1996) obtained strong-order conditions presented in the following theorem.

Theorem 5.4. [6, 8] *Let $\alpha_i,\ \beta_i^{(1)},\ \beta_i^{(2)},\ a_{ij},\ b_{ij}^{(1)},\ b_{ij}^{(2)}$ $(i, j = 1, 2, \ldots, s)$ be the Runge-Kutta coefficients. If the coefficients of stochastic Runge-Kutta method for SDEs (2.11), the constraint equations in the problem $(\mathcal{P}_2)$, fulfill the subsequent conditions:*

A1. $\sum_{i=1}^{s} \alpha_i = 1,$

A2. $\sum_{i=1}^{s} \beta_i^{(1)} = 1,$

A3. $\sum_{i=1}^{s} \beta_i^{(2)} = 0,$

A4. $\sum_{i,j=1}^{s} \beta_i^{(1)} b_{ij}^{(1)} = \frac{1}{2},$

A5. $\sum_{i,j=1}^{s} \beta_i^{(1)} b_{ij}^{(2)} = -\sum_{i,j=1}^{s} \beta_i^{(2)} b_{ij}^{(1)},$

A6. $\sum_{i,j=1}^{s} \beta_i^{(2)} b_{ij}^{(2)} = 0,$

then the stochastic Runge-Kutta method of Eqn. (2.11) converges to order-1 in the strong sense. In addition, if the conditions

$$\text{A7.} \quad \sum_{i,j=1}^{s} \beta_i^{(1)} a_{ij} = 1, \qquad \text{A8.} \quad \sum_{i,j=1}^{s} \beta_i^{(2)} a_{ij} = -1,$$

$$\text{A9.} \quad \sum_{i,j=1}^{s} \alpha_i b_{ij}^{(1)} = 0, \qquad \text{A10.} \quad \sum_{i,j=1}^{s} \alpha_i b_{ij}^{(2)} = 1,$$

$$\text{A11.} \quad \sum_{i,j=1}^{s} \beta_i^{(2)} (b_{ij}^{(1)})^2 = -2 \sum_{i,j=1}^{s} \beta_i^{(1)} b_{ij}^{(1)} b_{ij}^{(2)}, \qquad \text{A12.} \quad \sum_{i,j=1}^{s} \beta_i^{(1)} (b_{ij}^{(1)})^2 = \frac{1}{3},$$

$$\text{A13.} \quad \sum_{i,j=1}^{s} \beta_i^{(1)} (b_{ij}^{(2)})^2 = -2 \sum_{i,j=1}^{s} \beta_i^{(2)} b_{ij}^{(1)} b_{ij}^{(2)}, \qquad \text{A14.} \quad \sum_{i,j=1}^{s} \beta_i^{(2)} (b_{ij}^{(2)})^2 = 0,$$

$$\text{A15.} \quad \sum_{i,j,k=1}^{s} \beta_i^{(1)} b_{ij}^{(1)} b_{jk}^{(1)} = \frac{1}{6}, \qquad \text{A16.} \quad \sum_{i,j,k=1}^{s} \beta_i^{(2)} b_{ij}^{(2)} b_{jk}^{(2)} = 0,$$

$$\text{A17.} \quad \sum_{i,j,k=1}^{s} \beta_i^{(2)} b_{ij}^{(1)} b_{jk}^{(1)} + \beta_i^{(1)} b_{ij}^{(2)} b_{jk}^{(1)} + \beta_i^{(1)} b_{ij}^{(1)} b_{jk}^{(2)} = 0,$$

$$\text{A18.} \quad \sum_{i,j,k=1}^{s} \beta_i^{(1)} b_{ij}^{(2)} b_{jk}^{(2)} + \beta_i^{(2)} b_{ij}^{(2)} b_{jk}^{(1)} + \beta_i^{(2)} b_{ij}^{(1)} b_{jk}^{(2)} = 0,$$

are fulfilled, then the stochastic Runge-Kutta method converges to order-1.5 in the strong sense.

We remark that the discrete optimality conditions ($\mathcal{OC}_d$) is derived by using the specific case of the Runge-Kutta method, problem ($\mathcal{P}_d$). However, by following the same procedure in Chapter 3, a similar system of equations can also be derived for the problem $(\mathcal{P}_2)$ with the conditions:

$$\tilde{\alpha}_i := \alpha_i, \qquad \tilde{\beta}_i^{(1)} := \beta_i^{(1)}, \qquad \tilde{\beta}_i^{(2)} := \beta_i^{(2)},$$

$$\tilde{a}_{ij} := \alpha_j - \frac{\alpha_j}{\alpha_i} a_{ji}, \qquad \tilde{b}_{ij}^{(1)} := \beta_j^{(1)} - \frac{\beta_j^{(1)}}{\alpha_i} a_{ji}, \qquad \tilde{b}_{ij}^{(2)} := \beta_j^{(2)} - \frac{\beta_j^{(2)}}{\alpha_i} a_{ji}.$$

In order to study order conditions of discrete optimality conditions $(\mathcal{OC}'_d)$, the $(\mathcal{OC}'_d)$ will be written as a function of t. By using Butcher approach, we write $t_n = t_0$, and for a given $t = t_0 + \Delta$. For a given initial iteration values y_k and p_{k+1}, the solutions y_{ki} and p_{ki} are functions of t, denoted by $y_{ki}(t)$ and $p_{ki}(t)$, respectively. Let the values $p_k(t)$ and $y_{k+1}(t)$ stand for the iterates p_k and y_{k+1}, respectively, which can be calculated as $y(t)$ and $p(t)$ with intermediate values $y_{ki} = y_{ki}(t)$ and $p_{ki} = p_{ki}(t)$. For this reason, let $\boldsymbol{\zeta}(t) = (y(t), p(t))^T$ be the vector of length $2N(s+1)$ and let $\boldsymbol{\zeta}_{ki}(t)$, $\boldsymbol{\zeta}_{s+1}(t)$, $\underline{\tilde{\mathbf{F}}}(\boldsymbol{\zeta}(t))$

and $\tilde{\mathbf{H}}^l(\boldsymbol{\zeta}(t))$ denote the following pairs:

$$\boldsymbol{\zeta}_{ki}(t) = \begin{pmatrix} y_{ki}(t) \\ p_{ki}(t) \end{pmatrix} \quad (1 \leq i \leq s), \qquad \boldsymbol{\zeta}_{s+1}(t) = \begin{pmatrix} y_{k+1}(t) \\ p_{k+1}(t) \end{pmatrix} \quad (i = s+1),$$

$$\underline{\tilde{\mathbf{F}}}(\boldsymbol{\zeta}(t)) = \underline{\tilde{\mathbf{F}}}_i(\boldsymbol{\zeta}(t)) = \begin{pmatrix} \sum_{j=1}^{s} a_{ij} \underline{f}(\boldsymbol{\zeta}_{kj}(t)) \\ \sum_{j=1}^{s} \tilde{a}_{ij} \underline{\mathcal{H}}_y(\boldsymbol{\zeta}_{kj}(t)) \end{pmatrix} \quad (1 \leq i \leq s+1)$$

and

$$\tilde{\mathbf{H}}^{(l)}(\boldsymbol{\zeta}(t)) = \tilde{\mathbf{H}}_i^{(l)}(\boldsymbol{\zeta}(t)) = \begin{pmatrix} \sum_{j=1}^{s} b_{ij}^{(l)} h(\boldsymbol{\zeta}_{kj}(t)) \\ \sum_{j=1}^{s} \tilde{b}_{ij}^{(l)} h(\boldsymbol{\zeta}_{kj}(t)) q(\boldsymbol{\zeta}_{kj}(t)) \end{pmatrix} \quad (1 \leq i \leq s+1 \text{ and } 1 \leq l \leq p),$$

with the convention that

$$\begin{aligned} a_{s+1,j} &= \tilde{a}_{s+1,j} = \alpha_j \qquad (1 \leq i \leq s), \\ b_{s+1,j} &= \tilde{b}_{s+1,j} = \beta_j \qquad (1 \leq i \leq s). \end{aligned}$$

By using the above notation, we can state the discrete optimality conditions of problem $(\mathcal{P}_p)$ in the form

$$\boldsymbol{\zeta}(t) = \boldsymbol{\zeta}(t_0) + (t - t_0)\underline{\tilde{\mathbf{F}}}(\boldsymbol{\zeta}(t)) + \sum_{l=1}^{p} \tilde{\mathbf{H}}^l \theta_l(\boldsymbol{\zeta}(t)) \quad (1 \leq i \leq s+1), \tag{5.17}$$

where $\theta_l(t_0) = 0$, $l = 1, 2, \ldots, p$.

The term $\underline{\tilde{\mathbf{F}}}(\boldsymbol{\zeta}(t))$ (and $\tilde{\mathbf{H}}(\boldsymbol{\zeta}(t))$, analogously) can be elaborated by applying the Taylor-series expansion:

$$\underline{\tilde{\mathbf{F}}}(\boldsymbol{\zeta}(t)) = \underline{\tilde{\mathbf{F}}}(\boldsymbol{\zeta}(t_0)) + \sum_{n=1}^{\infty} \frac{\boldsymbol{\mathcal{L}}_{\Delta}^{n} \underline{\tilde{\mathbf{F}}}(\boldsymbol{\zeta}(t_0))}{n!}, \tag{5.18}$$

where $\boldsymbol{\mathcal{L}}$ is the vector-valued differential operator of 2 variables, given by

$$\boldsymbol{\mathcal{L}}_{\Delta}\boldsymbol{\Phi} := \Delta \frac{\partial \boldsymbol{\Phi}}{\partial t} + \sum_{l=1}^{p} \theta_l \frac{\partial \boldsymbol{\Phi}}{\partial \theta_l},$$

for $\boldsymbol{\Phi}$ is any twice continuously differentiable vector-valued function of 2 variables.

Thus, it is seen that

$$\boldsymbol{\mathcal{L}}_\Delta \underline{\tilde{\mathbf{F}}}(\boldsymbol{\zeta}(t)) = \Delta \underline{\tilde{\mathbf{F}}}'(\boldsymbol{\zeta}(t)) \left(\underline{\tilde{\mathbf{F}}}(\boldsymbol{\zeta}(t)) + (t-t_0)\underline{\tilde{\mathbf{F}}}'(\boldsymbol{\zeta}(t))\frac{\partial \boldsymbol{\zeta}}{\partial t} + \sum_{l=1}^{p} \theta_l (\tilde{\mathbf{H}}^{(l)})'(\boldsymbol{\zeta}(t))\frac{\partial \boldsymbol{\zeta}}{\partial t} \right)$$
$$+ \sum_{l=1}^{p} \theta_l \underline{\tilde{\mathbf{F}}}'(\boldsymbol{\zeta}(t)) \Bigg((t-t_0)\underline{\tilde{\mathbf{F}}}'(\boldsymbol{\zeta}(t))\frac{\partial \boldsymbol{\zeta}}{\partial \theta_l} + \tilde{\mathbf{H}}^{(l)}(\boldsymbol{\zeta}(t))$$
$$+ \sum_{l=1}^{p} \theta_l (\tilde{\mathbf{H}}^{(l)})'(\boldsymbol{\zeta}(t))\frac{\partial \boldsymbol{\zeta}}{\partial \theta_l} \Bigg),$$

so that

$$\boldsymbol{\mathcal{L}}_\Delta \underline{\tilde{\mathbf{F}}}(\boldsymbol{\zeta}(t_0)) = \Delta \underline{\tilde{\mathbf{F}}}'(\boldsymbol{\zeta}(t_0))\underline{\tilde{\mathbf{F}}}(\boldsymbol{\zeta}(t_0)) + \underline{\tilde{\mathbf{F}}}'(\boldsymbol{\zeta}(t_0)) \sum_{l=1}^{p} \theta_l \tilde{\mathbf{H}}^{(l)}(\boldsymbol{\zeta}(t_0)).$$

Similarly,

$$\boldsymbol{\mathcal{L}}_\Delta \tilde{\mathbf{H}}^{(l)}(\boldsymbol{\zeta}(t_0)) = \Delta (\tilde{\mathbf{H}}^{(l)})'(\boldsymbol{\zeta}(t_0))\underline{\tilde{\mathbf{F}}}(\boldsymbol{\zeta}(t_0)) + (\tilde{\mathbf{H}}^{(l)})'(\boldsymbol{\zeta}(t_0)) \sum_{l=1}^{p} \theta_l (\tilde{\mathbf{H}}^{(l)})'(\boldsymbol{\zeta}(t_0))$$

for $l = 1, 2, \ldots, p$.

Then, the Stratonovich-Taylor approximation of discrete optimality system of problem $(\mathcal{P}_p)$ looks as follows:

$$\boldsymbol{\zeta}(t) = \boldsymbol{\zeta}(t_0) + \Delta \underline{\tilde{\mathbf{F}}}(\boldsymbol{\zeta}(t)) + \sum_{l=1}^{p} \theta_l \left(\tilde{\mathbf{H}}^{(l)}(\boldsymbol{\zeta}(t)) \right)$$
$$= \boldsymbol{\zeta}(t_0) + \Delta \left(\underline{\tilde{\mathbf{F}}}(\boldsymbol{\zeta}(t_0)) + \boldsymbol{\mathcal{L}}_\Delta \underline{\tilde{\mathbf{F}}}(\boldsymbol{\zeta}(t_0)) + \frac{1}{2!}\boldsymbol{\mathcal{L}}_\Delta^2 \underline{\tilde{\mathbf{F}}}(\boldsymbol{\zeta}(t_0)) + \ldots \right)$$
$$+ \sum_{l=1}^{p} \theta_l \left(\tilde{\mathbf{H}}^{(l)}(\boldsymbol{\zeta}(t_0)) + \boldsymbol{\mathcal{L}}_\Delta \tilde{\mathbf{H}}^{(l)}(\boldsymbol{\zeta}(t_0)) + \frac{1}{2!}\boldsymbol{\mathcal{L}}_\Delta^2 \tilde{\mathbf{H}}^{(l)}(\boldsymbol{\zeta}(t_0)) + \ldots \right).$$

Hence,

$$\boldsymbol{\zeta}(t) = \boldsymbol{\zeta}(t_0) + \Bigg(\Delta \underline{\tilde{\mathbf{F}}} + \Delta^2 \underline{\tilde{\mathbf{F}}}'\underline{\tilde{\mathbf{F}}} + \Delta \sum_{l=1}^{p} \theta_l \underline{\tilde{\mathbf{F}}}'\tilde{\mathbf{H}}^{(l)} + \frac{1}{2}\Delta^3 \underline{\tilde{\mathbf{F}}}''\underline{\tilde{\mathbf{F}}}\underline{\tilde{\mathbf{F}}} + \Delta^3 \underline{\tilde{\mathbf{F}}}'\underline{\tilde{\mathbf{F}}}'\underline{\tilde{\mathbf{F}}}$$
$$+ \Delta^2 \sum_{l=1}^{p} \theta_l \underline{\tilde{\mathbf{F}}}'\underline{\tilde{\mathbf{F}}}'\tilde{\mathbf{H}}^{(l)} + \frac{1}{2}\Delta^2 \sum_{l=1}^{p} \theta_l \underline{\tilde{\mathbf{F}}}''\underline{\tilde{\mathbf{F}}}\tilde{\mathbf{H}}^{(l)} + \Delta^2 \sum_{l=1}^{p} \theta_l \underline{\tilde{\mathbf{F}}}'(\tilde{\mathbf{H}}^{(l)})'\underline{\tilde{\mathbf{F}}}$$
$$+ \frac{1}{2}\Delta^2 \sum_{l=1}^{p} \theta_l \underline{\tilde{\mathbf{F}}}''\tilde{\mathbf{H}}^{(l)}\underline{\tilde{\mathbf{F}}} + \frac{1}{2}\Delta \sum_{l=1}^{p} \theta_l \underline{\tilde{\mathbf{F}}}''\tilde{\mathbf{H}}^{(l)} \sum_{l=1}^{p} \theta_l \tilde{\mathbf{H}}^{(l)}$$
$$+ \Delta \sum_{l=1}^{p} \theta_l \underline{\tilde{\mathbf{F}}}'(\tilde{\mathbf{H}}^{(l)})' \sum_{l=1}^{p} \theta_l \tilde{\mathbf{H}}^{(l)} \Bigg) (\boldsymbol{\zeta}(t_0)) + \ldots$$

$$
\begin{aligned}
&+ \Bigg(\sum_{l=1}^{p} \theta_l \tilde{\mathbf{H}}^{(l)} + \Delta \sum_{l=1}^{p} \theta_l (\tilde{\mathbf{H}}^{(l)})' \underline{\tilde{\mathbf{F}}} + \sum_{l=1}^{p} \theta_l (\tilde{\mathbf{H}}^{(l)})' \sum_{l=1}^{p} \theta_l \tilde{\mathbf{H}}^{(l)} \\
&+ \frac{1}{2} \Delta^2 \sum_{l=1}^{p} \theta_l (\tilde{\mathbf{H}}^{(l)})'' \underline{\tilde{\mathbf{F}}\tilde{\mathbf{F}}} + \Delta^2 \sum_{l=1}^{p} \theta_l (\tilde{\mathbf{H}}^{(l)})' \underline{\tilde{\mathbf{F}}}' \underline{\tilde{\mathbf{F}}} \\
&+ \Delta \sum_{l=1}^{p} \theta_l (\tilde{\mathbf{H}}^{(l)})' \underline{\tilde{\mathbf{F}}}' \sum_{l=1}^{p} \theta_l \tilde{\mathbf{H}}^{(l)} + \frac{1}{2} \Delta \theta_l (\tilde{\mathbf{H}}^{(l)})'' \sum_{l=1}^{p} \theta_l \tilde{\mathbf{H}}^{(l)} \sum_{l=1}^{p} \underline{\tilde{\mathbf{F}}} \\
&+ \Delta \sum_{l=1}^{p} \theta_l (\tilde{\mathbf{H}}^{(l)})' \sum_{l=1}^{p} \theta_l (\tilde{\mathbf{H}}^{(l)})' \underline{\tilde{\mathbf{F}}} + \frac{1}{2} \Delta \sum_{l=1}^{p} \theta_l (\tilde{\mathbf{H}}^{(l)})'' \sum_{l=1}^{p} \theta_l \tilde{\mathbf{H}}^{(l)} \underline{\tilde{\mathbf{F}}} \\
&+ \frac{1}{2} \sum_{l=1}^{p} \theta_l (\tilde{\mathbf{H}}^{(l)})'' \sum_{l=1}^{p} \theta_l \tilde{\mathbf{H}}^{(l)} \sum_{l=1}^{p} \theta_l \tilde{\mathbf{H}}^{(l)} \\
&+ \sum_{l=1}^{p} \theta_l (\tilde{\mathbf{H}}^{(l)})' \sum_{l=1}^{p} \theta_l (\tilde{\mathbf{H}}^{(l)})' \sum_{l=1}^{p} \theta_l \tilde{\mathbf{H}}^{(l)} \Bigg) (\boldsymbol{\zeta}(t_0)) + \ldots
\end{aligned}
$$

As we mentioned, in the case of the Stratonovich-Taylor expansion of the exact solution, various derivatives should be considered in an operator context. Here, let us note that we write the above Stratonovich-Taylor expansion for an arbitrary positive p. However, strong-order conditions of our Runge-Kutta scheme for stochastic optimal control problem are particularly obtained for $p = 2$. In the subsequent theorem, we state such order conditions.

Theorem 5.5. *Let* $\alpha_i,\ \beta_i^{(1)},\ \beta_i^{(2)},\ a_{ij},\ \tilde{a}_{ij},\ b_{ij}^{(1)},\ \tilde{b}_{ij}^{(1)},\ b_{ij}^{(2)},\ \tilde{b}_{ij}^{(2)}$ *(*$i, j = 1, 2, \ldots, s$*) be the Runge-Kutta coefficients. If the coefficients of our Runge-Kutta method for the stochastic optimal control problem satisfy both conditions A1-A6 in Theorem 5.4 and*

B1. $\displaystyle\sum_{i,j=1}^{s} \beta_i^{(1)} \tilde{b}_{ij}^{(1)} = \frac{1}{2},$ B3. $\displaystyle\sum_{i,j=1}^{s} \beta_i^{(2)} \tilde{b}_{ij}^{(2)} = 0,$

B2. $\displaystyle\sum_{i,j=1}^{s} \beta_i^{(1)} \tilde{b}_{ij}^{(2)} = -\sum_{i,j=1}^{s} \beta_i^{(2)} \tilde{b}_{ij}^{(1)},$

then our Runge-Kutta method for stochastic optimal control problems converges to order-1 in the strong sense. In addition, if both conditions A7-A18 in Theorem 5.4 and

B4. $\displaystyle\sum_{i,j=1}^{s} \beta_i^{(1)} \tilde{a}_{ij} = 1,$ B5. $\displaystyle\sum_{i,j=1}^{s} \beta_i^{(2)} \tilde{a}_{ij} = -1,$

B6. $\displaystyle\sum_{i,j=1}^{s} \alpha_i \tilde{b}_{ij}^{(2)} = 1,$ B7. $\displaystyle\sum_{i,j=1}^{s} \alpha_i \tilde{b}_{ij}^{(1)} = 0,$

B8. $\sum_{i,j=1}^{s} \beta_i^{(1)} b_{ij}^{(1)} \tilde{b}_{ij}^{(1)} = \frac{1}{6}$, B9. $\sum_{i,j=1}^{s} \beta_i^{(2)} b_{ij}^{(2)} \tilde{b}_{ij}^{(2)} = 0$,

B10. $\sum_{i,j=1}^{s} \left(\beta_i^{(1)} b_{ij}^{(2)} + \beta_i^{(2)} b_{ij}^{(1)} \right) \tilde{b}_{ij}^{(2)} = -\sum_{i,j=1}^{s} \beta_i^{(2)} b_{ij}^{(2)} \tilde{b}_{ij}^{(1)}$,

B11. $\sum_{i,j=1}^{s} \left(\beta_i^{(1)} b_{ij}^{(2)} + \beta_i^{(2)} b_{ij}^{(1)} \right) \tilde{b}_{ij}^{(1)} = -\sum_{i,j=1}^{s} \beta_i^{(1)} b_{ij}^{(1)} \tilde{b}_{ij}^{(2)}$,

B12. $\sum_{i,j=1}^{s} \beta_i^{(1)} (\tilde{b}_{ij}^{(1)})^2 = \frac{1}{3}$, B13. $\sum_{i,j=1}^{s} \beta_i^{(2)} (\tilde{b}_{ij}^{(2)})^2 = 0$,

B14. $\sum_{i,j=1}^{s} \beta_i^{(2)} (\tilde{b}_{ij}^{(1)})^2 = -2 \sum_{i,j=1}^{s} \beta_i^{(1)} \tilde{b}_{ij}^{(1)} \tilde{b}_{ij}^{(2)}$,

B15. $\sum_{i,j=1}^{s} \beta_i^{(1)} (\tilde{b}_{ij}^{(2)})^2 = -2 \sum_{i,j=1}^{s} \beta_i^{(2)} \tilde{b}_{ij}^{(1)} \tilde{b}_{ij}^{(2)}$,

B16. $\sum_{i,j,k=1}^{s} \beta_i^{(1)} \tilde{b}_{ij}^{(1)} b_{jk}^{(1)} = \frac{1}{6}$, B17. $\sum_{i,j,k=1}^{s} \beta_i^{(2)} \tilde{b}_{ij}^{(2)} b_{jk}^{(2)} = 0$,

B18. $\sum_{i,j,k=1}^{s} \left(\beta_i^{(2)} \tilde{b}_{ij}^{(1)} b_{jk}^{(1)} + \beta_i^{(1)} \left(\tilde{b}_{ij}^{(2)} b_{jk}^{(1)} + \tilde{b}_{ij}^{(1)} b_{jk}^{(2)} \right) \right) = 0$,

B19. $\sum_{i,j,k=1}^{s} \left(\beta_i^{(1)} \tilde{b}_{ij}^{(2)} b_{jk}^{(2)} + \beta_i^{(2)} \left(\tilde{b}_{ij}^{(2)} b_{jk}^{(1)} + \tilde{b}_{ij}^{(1)} b_{jk}^{(2)} \right) \right) = 0$,

B20. $\sum_{i,j,k=1}^{s} \beta_i^{(1)} b_{ij}^{(1)} \tilde{b}_{jk}^{(1)} = \frac{1}{6}$, B21. $\sum_{i,j,k=1}^{s} \beta_i^{(2)} b_{ij}^{(2)} \tilde{b}_{jk}^{(2)} = 0$,

B22. $\sum_{i,j,k=1}^{s} \left(\beta_i^{(2)} b_{ij}^{(1)} \tilde{b}_{jk}^{(1)} + \beta_i^{(1)} \left(b_{ij}^{(2)} \tilde{b}_{jk}^{(1)} + b_{ij}^{(1)} \tilde{b}_{jk}^{(2)} \right) \right) = 0$,

B23. $\sum_{i,j,k=1}^{s} \left(\beta_i^{(1)} b_{ij}^{(2)} \tilde{b}_{jk}^{(2)} + \beta_i^{(2)} \left(b_{ij}^{(2)} \tilde{b}_{jk}^{(1)} + b_{ij}^{(1)} \tilde{b}_{jk}^{(2)} \right) \right) = 0$,

B24. $\sum_{i,j,k=1}^{s} \beta_i^{(1)} \tilde{b}_{ij}^{(1)} \tilde{b}_{jk}^{(1)} = \frac{1}{6}$, B25. $\sum_{i,j,k=1}^{s} \beta_i^{(2)} \tilde{b}_{ij}^{(2)} \tilde{b}_{jk}^{(2)} = 0$,

B26. $\sum_{i,j,k=1}^{s} \left(\beta_i^{(2)} \tilde{b}_{ij}^{(1)} \tilde{b}_{jk}^{(1)} + \beta_i^{(1)} \left(\tilde{b}_{ij}^{(2)} \tilde{b}_{jk}^{(1)} + \tilde{b}_{ij}^{(1)} \tilde{b}_{jk}^{(2)} \right) \right) = 0$,

B27. $\sum_{i,j,k=1}^{s} \left(\beta_i^{(1)} \tilde{b}_{ij}^{(2)} \tilde{b}_{jk}^{(2)} + \beta_i^{(2)} \left(\tilde{b}_{ij}^{(2)} \tilde{b}_{jk}^{(1)} + \tilde{b}_{ij}^{(1)} \tilde{b}_{jk}^{(2)} \right) \right) = 0$,

are fulfilled, our Runge-Kutta method for stochastic optimal control problems converges to order-1.5 in the strong sense.

Proof. We use the same logic as we did in Section 5.3, to find strong-order conditions of our Runge-Kutta scheme for stochastic control problems of SDEs. So, for simplicity,

$$\tilde{\underline{\mathbf{F}}}_i = \begin{pmatrix} \sum_{j=1}^{s} a_{ij} \underline{f} \\ -\sum_{j=1}^{s} \tilde{a}_{ij} \underline{\mathcal{H}}_y \end{pmatrix} = \sum_{j=1}^{s} a_{ij} \underline{\mathbf{f}}^0 + \sum_{j=1}^{s} \tilde{a}_{ij} (-\underline{\boldsymbol{\mathcal{H}}}_y^0) \quad (1 \leq i \leq s+1),$$

where

$$\underline{\mathbf{f}}^0 = \begin{pmatrix} \underline{f} \\ 0 \end{pmatrix}, \quad \underline{\boldsymbol{\mathcal{H}}}_y^0 = \begin{pmatrix} 0 \\ -\underline{\mathcal{H}}_y \end{pmatrix},$$

and

$$\tilde{\mathbf{H}}_i^{(l)} = \begin{pmatrix} \sum_{j=1}^{s} b_{ij} h \\ \sum_{j=1}^{s} \tilde{b}_{ij}^{(l)} hq \end{pmatrix} = \sum_{j=1}^{s} b_{ij}^{(l)} \mathbf{h}^0 + \sum_{j=1}^{s} \tilde{b}_{ij}^{(l)} (\mathbf{hq})^0 \quad (1 \leq i \leq s+1; \quad l = 1, 2)$$

with

$$\mathbf{h}^0 = \begin{pmatrix} h \\ 0 \end{pmatrix}, \quad (\mathbf{hq})^0 = \begin{pmatrix} 0 \\ hq \end{pmatrix},$$

and

$$(\tilde{\underline{\mathbf{F}}'}_i)_j = a_{ij} (\underline{\mathbf{f}}^0)' + \tilde{a}_{ij} ((\underline{\boldsymbol{\mathcal{H}}}_y)^0)'.$$

The J-integrals J_0, J_1, J_{11}, are of order Δ, $\Delta^{0.5}$ and Δ, respectively, so that order conditions of our Runge-Kutta scheme for stochastic control problems to have strong order-1 are analyzed below, subsequently.

i) Since

$$\tilde{\underline{\mathbf{F}}}_{s+1} = \begin{pmatrix} \sum_{i=1}^{s} \alpha_i \underline{f} \\ -\sum_{i=1}^{s} \alpha_i \underline{\mathcal{H}}_y \end{pmatrix} = \sum_{i=1}^{s} \alpha_i \underline{\mathbf{F}} \quad (1 \leq i \leq s+1),$$

it holds

$$\mathbb{E} \left[J_0 - \Delta \sum_{i=1}^{s} \alpha_i \right]^2 = \mathbb{E} \left[\Delta \left(1 - \sum_{i=1}^{s} \alpha_i \right) \right]^2,$$

$$\Rightarrow \sum_{i=1}^{s} \alpha_i = 1 \quad \text{(Condition A1. in Theorem 5.4).}$$

ii) The second strong order-1 term is J_1:

$$\sum_{l=1}^{p} \theta_l \tilde{\mathbf{H}}_{s+1}^{(l)} = \left(\sum_{i=1}^{s} \beta_i^{(1)} J_1 + \sum_{i=1}^{s} \beta_i^{(2)} \frac{J_{10}}{\Delta} \right) \mathbf{H},$$

which leads us to the implication

$$
\begin{aligned}
&\mathbb{E}\left[J_1 - \left(\sum_{i=1}^{s} \beta_i^{(1)} J_1 + \sum_{i=1}^{s} \beta_i^{(2)} \frac{J_{10}}{\Delta}\right)\right]^2 \\
&= \mathbb{E}\left[\left(1 - \sum_{i=1}^{s} \beta_i^{(1)}\right) J_1 - \sum_{i=1}^{s} \beta_i^{(2)} \frac{J_{10}}{\Delta}\right]^2 \\
&= \mathbf{z}^T \begin{pmatrix} \mathbb{E}[J_1^2] & -\mathbb{E}[J_1 J_{10}/\Delta] \\ -\mathbb{E}[J_1 J_{10}/\Delta] & \mathbb{E}[J_{10}^2/\Delta^2] \end{pmatrix} \mathbf{z} \\
&= \mathbf{z}^T \begin{pmatrix} 1/3 & -1/2 \\ -1/2 & 1 \end{pmatrix} \mathbf{z}\Delta^2 \geq 0 \\
&\Rightarrow \mathbf{z}^T = (z_1, z_2) = (0, 0) \\
&\Rightarrow \sum_{i=1}^{s} \beta_i^{(1)} = 1,\ \sum_{i=1}^{s} \beta_i^{(2)} = 0
\end{aligned}
$$

(Conditions A2. and A3. in Theorem 5.4).

iii) The final strong order-1 term comes from the integral J_{11}:

$$
\begin{aligned}
\sum_{l=1}^{p} \theta_l (\tilde{\mathbf{H}}^{(l)})' \sum_{l=1}^{p} \theta_l \tilde{\mathbf{H}}^{(l)} &= \sum_{l=1}^{p} \theta_l \sum_{i=1}^{s} ((\tilde{\mathbf{H}}^{(l)})'_{s+1})_i \sum_{l=1}^{p} \theta_l \sum_{i=1}^{s} (\tilde{\mathbf{H}})_i \\
&= \sum_{i,j=1}^{s} \left(\beta_i^{(1)} J_1 + \beta_i^{(2)} \frac{J_{10}}{\Delta}\right) \mathbf{H}' \\
&\quad \times \left(\left(b_{ij}^{(1)} J_1 + b_{ij}^{(2)} \frac{J_{10}}{\Delta}\right) \mathbf{h}^0 + \left(\tilde{b}_{ij}^{(1)} J_1 + \tilde{b}_{ij}^{(2)} \frac{J_{10}}{\Delta}\right) (\mathbf{hq})^0\right).
\end{aligned}
$$

If

$$
\begin{aligned}
&\sum_{i,j=1}^{s} \left(\beta_i^{(1)} J_1 + \beta_i^{(2)} \frac{J_{10}}{\Delta}\right) \left(b_{ij}^{(1)} J_1 + b_{ij}^{(2)} \frac{J_{10}}{\Delta}\right) \\
&\quad = \sum_{i,j=1}^{s} \left(\beta_i^{(1)} J_1 + \beta_i^{(2)} \frac{J_{10}}{\Delta}\right) \left(\tilde{b}_{ij}^{(1)} J_1 + \tilde{b}_{ij}^{(2)} \frac{J_{10}}{\Delta}\right),
\end{aligned}
$$

then, we get the implication

$$
\begin{aligned}
&\mathbb{E}\left[\frac{J_1^2}{2} - \sum_{i,j=1}^{s} \left(\beta_i^{(1)} J_1 + \beta_i^{(2)} \frac{J_{10}}{\Delta}\right) \left(b_{ij}^{(1)} J_1 + b_{ij}^{(2)} \frac{J_{10}}{\Delta}\right)\right]^2 \\
&= \mathbb{E}\left[\frac{J_1^2}{2} - \sum_{i,j=1}^{s} \beta_i^{(1)} b_{ij}^{(1)} J_1^2 - \sum_{i,j=1}^{s} \left(\beta_i^{(1)} b_{ij}^{(2)} + \beta_i^{(2)} b_{ij}^{(1)}\right) J_1 \frac{J_{10}}{\Delta} - \beta_i^{(2)} b_{ij}^{(2)} \frac{J_{10}^2}{\Delta^2}\right]^2
\end{aligned}
$$

$= \mathbf{z}^T \mathbf{C} \mathbf{z}$ (with $\mathbf{C}$ defined by the following line)

$$= \mathbf{z}^T \begin{pmatrix} \mathbb{E}[J_1^4] & \mathbb{E}[J_1^3 J_{10}/\Delta] & \mathbb{E}[J_1^2 J_{10}^2/\Delta^2] \\ \mathbb{E}[J_1^3 J_{10}/\Delta] & \mathbb{E}[J_1^2 J_{10}^2/\Delta^2] & \mathbb{E}[J_1 J_{10}^3/\Delta^3] \\ \mathbb{E}[J_1^2 J_{10}^2/\Delta^2] & \mathbb{E}[J_1 J_{10}^3/\Delta^3] & \mathbb{E}[J_{10}^4] \end{pmatrix} \mathbf{z}$$

$$= \Delta^2 \mathbf{z}^T \begin{pmatrix} 3 & 3/2 & 5/6 \\ 3/2 & 5/6 & 1/2 \\ 5/6 & 1/2 & 1/3 \end{pmatrix} \mathbf{z} \geq 0$$

$$\Rightarrow \mathbf{z}^T = (z_1, z_2, z_3) = (0, 0, 0)$$

$$\Rightarrow \sum_{i,j=1}^{s} \beta_i^{(1)} b_{ij}^{(1)} = \frac{1}{2}, \quad \sum_{i,j=1}^{s} \beta_i^{(1)} b_{ij}^{(2)} = -\sum_{i,j=1}^{s} \beta_i^{(2)} b_{ij}^{(1)}, \quad \sum_{i,j=1}^{s} \beta_i^{(2)} b_{ij}^{(2)} = 0$$

(Conditions A4-A6. in Theorem 5.4).

Similarly, we can obtain 3 extra conditions:

B1. $$\sum_{i,j=1}^{s} \beta_i^{(1)} \tilde{b}_{ij}^{(1)} = \frac{1}{2}$$

$$\Rightarrow \sum_{i,j=1}^{s} \beta_i^{(1)} \left(\beta_j^{(1)} - \beta_j^{(1)} \frac{a_{ji}}{\alpha_i} \right) = \sum_{i,j=1}^{s} \beta_i^{(1)} \beta_j^{(1)} f_i = \frac{1}{2},$$

B2. $$\sum_{i,j=1}^{s} \beta_i^{(1)} \tilde{b}_{ij}^{(2)} = -\sum_{i,j=1}^{s} \beta_i^{(2)} \tilde{b}_{ij}^{(1)}$$

$$\Rightarrow \sum_{i,j=1}^{s} \beta_i^{(1)} \left(\beta_j^{(2)} - \frac{\beta_j^{(2)}}{\alpha_i} a_{ji} \right) = -\sum_{i,j=1}^{s} \beta_i^{(2)} \left(\beta_j^{(1)} - \frac{\beta_j^{(1)}}{\alpha_i} a_{ji} \right),$$

$$\Rightarrow \sum_{i,j=1}^{s} \beta_i^{(1)} \beta_j^{(2)} f_i = -\sum_{i,j=1}^{s} \beta_i^{(2)} \beta_j^{(1)} f_i,$$

B3. $$\sum_{i,j=1}^{s} \beta_i^{(2)} \tilde{b}_{ij}^{(2)} = 0$$

$$\Rightarrow \sum_{i,j=1}^{s} \beta_i^{(2)} \left(\beta_j^{(2)} - \frac{\beta_j^{(2)}}{\alpha_i} a_{ji} \right) = \sum_{i,j=1}^{s} \beta_i^{(2)} \beta_j^{(2)} f_i = 0,$$

where

$$f_i = \sum_{j=1}^{s} \left(1 - \frac{a_{ij}}{\alpha_i} \right).$$

The expressions corresponding to the $\Delta^{1.5}$ terms arise from the J-integrals: J_{01}, J_{10} and J_{111}, accordingly, so that order conditions of our Runge-Kutta scheme to have strong order-1.5 are analyzed below.

iv) The first strong order-1.5 term turns out to be

$$\Delta \sum_{l=1}^{p} \theta_l (\tilde{\mathbf{H}}^{(l)})' \underline{\tilde{\mathbf{F}}} = \Delta \sum_{l=1}^{p} \theta_l \sum_{i=1}^{s} ((\tilde{\mathbf{H}}^{(l)})'_{s+1})_i (\underline{\tilde{\mathbf{F}}})_i$$
$$= \Delta \sum_{i,j=1}^{s} \left(\beta_i^{(1)} J_1 + \beta_i^{(2)} \frac{J_{10}}{\Delta} \right) \mathbf{H}' (a_{ij} \underline{\mathbf{f}}^0 + \tilde{a}_{ij} (-\underline{\boldsymbol{\mathcal{H}}}_y)^0).$$

If

$$\sum_{i,j=1}^{s} a_{ij} \left(\beta_i^{(1)} J_1 + \beta_i^{(2)} \frac{J_{10}}{\Delta} \right) = \sum_{i,j=1}^{s} \tilde{a}_{ij} \left(\beta_i^{(1)} J_1 + \beta_i^{(2)} \frac{J_{10}}{\Delta} \right),$$

then

$$\mathbb{E} \left[J_{01} - \sum_{i,j=1}^{s} a_{ij} \left(\beta_i^{(1)} J_1 + \beta_i^{(2)} \frac{J_{10}}{\Delta} \right) \right]^2$$
$$= \mathbb{E} \left[J_1 - \frac{J_{10}}{\Delta} - \sum_{i,j=1}^{s} a_{ij} \left(\beta_i^{(1)} J_1 + \beta_i^{(2)} \frac{J_{10}}{\Delta} \right) \right]^2$$
$$= \mathbb{E} \left[\left(1 - \sum_{i,j=1}^{s} \beta_i^{(1)} a_{ij} \right) J_1 - \left(1 + \sum_{i,j=1}^{s} \beta_i^{(2)} c_i \right) \frac{J_{10}}{\Delta} \right]^2,$$

(using the analysis of quadratic forms)

$$\Rightarrow \sum_{i,j=1}^{s} \beta_i^{(1)} a_{ij} = 1, \; \sum_{i,j=1}^{s} \beta_i^{(2)} a_{ij} = -1$$

(Conditions A7. and A8. in Theorem 5.4).

Similarly, we can obtain 2 additional conditions:

B4. $\sum_{i,j=1}^{s} \beta_i^{(1)} \tilde{a}_{ij} = \sum_{i,j=1}^{s} \beta_i^{(1)} \alpha_j f_i = 1,$

B5. $\sum_{i,j=1}^{s} \beta_i^{(2)} \tilde{a}_{ij} = \sum_{i,j=1}^{s} \beta_i^{(2)} \alpha_j f_i = -1.$

v) The second strong order-1.5 term:

$$\Delta \sum_{l=1}^{p} \theta_l \underline{\tilde{\mathbf{F}}}' \tilde{\mathbf{H}}^{(l)}$$
$$= \Delta \sum_{l=1}^{p} \theta_l \sum_{i=1}^{s} (\underline{\tilde{\mathbf{F}}}'_{s+1})_i (\tilde{\mathbf{H}}^{(l)})_i$$
$$= \Delta \sum_{i,j=1}^{s} \alpha_i \underline{\mathbf{F}}' \left(\left(b_{ij}^{(1)} J_1 + b_{ij}^{(2)} \frac{J_{10}}{\Delta} \right) \mathbf{h}^0 + \left(\tilde{b}_{ij}^{(1)} J_1 + \tilde{b}_{ij}^{(2)} \frac{J_{10}}{\Delta} \right) (\mathbf{hq})^0 \right).$$

If

$$\sum_{i,j=1}^{s} \alpha_i \left(b_{ij}^{(1)} J_1 + b_{ij}^{(2)} \frac{J_{10}}{\Delta} \right) = \sum_{i,j=1}^{s} \alpha_i \left(\tilde{b}_{ij}^{(1)} J_1 + \tilde{b}_{ij}^{(2)} \frac{J_{10}}{\Delta} \right),$$

then we have the implication

$$\begin{aligned}
&\mathbb{E}\left[J_{10} - \Delta \sum_{i,j=1}^{s} \alpha_i \left(b_{ij}^{(1)} J_1 + b_{ij}^{(2)} \frac{J_{10}}{\Delta} \right) \right]^2 \\
&\quad = \mathbb{E}\left[J_{10} \left(1 - \sum_{i,j=1}^{s} \alpha_i b_{ij}^{(2)} \right) - \sum_{i,j=1}^{s} \alpha_i b_{ij}^{(1)} J_0 J_1 \right]^2 \\
&\quad = \mathbf{z}^T \begin{pmatrix} \mathbb{E}[J_{10}^2] & -\mathbb{E}[J_0 J_1 J_{10}] \\ -\mathbb{E}[J_0 J_1 J_{10}] & \mathbb{E}[J_0^2 J_1^2] \end{pmatrix} \mathbf{z} \\
&\quad = \Delta^3 \mathbf{z}^T \begin{pmatrix} 1/3 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{z} \geq 0, \\
&\quad \Rightarrow \mathbf{z}^T = (z_1, z_2) = (0, 0) \\
&\quad \Rightarrow \sum_{i,j=1}^{s} \alpha_i b_{ij}^{(2)} = 1, \ \sum_{i,j=1}^{s} \alpha_i b_{ij}^{(1)} = 0
\end{aligned}$$

(Conditions A9. and A10. in Theorem 5.4).

Similarly, we can get 2 more conditions:

$$\text{B6.} \quad \sum_{i,j=1}^{s} \alpha_i \tilde{b}_{ij}^{(2)} = 1 \quad \Rightarrow \quad \sum_{i,j=1}^{s} \alpha_i \beta_j^{(2)} f_i = 1,$$

$$\text{B7.} \quad \sum_{i,j=1}^{s} \alpha_i \tilde{b}_{ij}^{(1)} = 0 \quad \Rightarrow \quad \sum_{i,j=1}^{s} \alpha_i \beta_j^{(1)} f = 0.$$

vi) The third strong order-1.5 term:

$$\begin{aligned}
&\sum_{l=1}^{p} \theta_l (\tilde{\mathbf{H}}^{(l)})'' \sum_{l=1}^{p} \theta_l \tilde{\mathbf{H}}^{(l)} \sum_{l=1}^{p} \theta_l \tilde{\mathbf{H}}^{(l)} \\
&\quad = \sum_{l=1}^{p} \theta_l \sum_{i=1}^{s} (\tilde{\mathbf{H}}''_{s+1})_i \sum_{l=1}^{p} \theta_l \sum_{i=1}^{s} (\tilde{\mathbf{H}}^{(l)})_i (\tilde{\mathbf{H}}^{(l)})_i \\
&\quad = \sum_{i,j=1}^{s} \left(\beta_i^{(1)} J_1 + \beta_i^{(2)} \frac{J_{10}}{\Delta} \right) \mathbf{H}'' \\
&\qquad \times \left(\left(b_{ij}^{(1)} J_1 + b_{ij}^{(2)} \frac{J_{10}}{\Delta} \right) \mathbf{h}^0 + \left(\tilde{b}_{ij}^{(1)} J_1 + \tilde{b}_{ij}^{(2)} \frac{J_{10}}{\Delta} \right) (\mathbf{hq})^0 \right)^2.
\end{aligned}$$

If

$$\sum_{i,j=1}^{s}\left(\beta_i^{(1)}J_1+\beta_i^{(2)}\frac{J_{10}}{\Delta}\right)\left(b_{ij}^{(1)}J_1+b_{ij}^{(2)}\frac{J_{10}}{\Delta}\right)^2$$
$$=2\sum_{i,j=1}^{s}\left(\beta_i^{(1)}J_1+\beta_i^{(2)}\frac{J_{10}}{\Delta}\right)\left(b_{ij}^{(1)}J_1+b_{ij}^{(2)}\frac{J_{10}}{\Delta}\right)\left(\tilde{b}_{ij}^{(1)}J_1+\tilde{b}_{ij}^{(2)}\frac{J_{10}}{\Delta}\right)$$
$$=\sum_{i,j=1}^{s}\left(\beta_i^{(1)}J_1+\beta_i^{(2)}\frac{J_{10}}{\Delta}\right)\left(\tilde{b}_{ij}^{(1)}J_1+\tilde{b}_{ij}^{(2)}\frac{J_{10}}{\Delta}\right)^2,$$

then

$$\mathbb{E}\left[J_{111}-\frac{1}{2}\sum_{i,j=1}^{s}\left(\beta_i^{(1)}J_1+\beta_i^{(2)}\frac{J_{10}}{\Delta}\right)\left(b_{ij}^{(1)}J_1+b_{ij}^{(2)}\frac{J_{10}}{\Delta}\right)^2\right]^2$$
$$=\mathbb{E}\left[\frac{J_1^3}{6}-\frac{1}{2}\sum_{i,j=1}^{s}\left(\beta_i^{(1)}J_1+\beta_i^{(2)}\frac{J_{10}}{\Delta}\right)\left(b_{ij}^{(1)}J_1+b_{ij}^{(2)}\frac{J_{10}}{\Delta}\right)^2\right]^2$$
$$=\mathbf{z}^T\mathbf{C}\mathbf{z},$$

where

$$z_1=\frac{1}{6}-\frac{1}{2}\sum_{i,j=1}^{s}\beta_i^{(1)}(b_{ij}^{(1)})^2,\quad z_2=\frac{1}{2}\sum_{i,j=1}^{s}\beta_i^{(2)}(b_{ij}^{(1)})^2+2\sum_{i,j=1}^{s}\beta_i^{(1)}b_{ij}^{(1)}b_{ij}^{(2)},$$
$$z_3=\frac{1}{2}\sum_{i,j=1}^{s}\beta_i^{(1)}(b_{ij}^{(2)})^2+2\sum_{i,j=1}^{s}\beta_i^{(2)}b_{ij}^{(1)}b_{ij}^{(2)},\quad z_4=\frac{1}{2}\sum_{i,j=1}^{s}\beta_i^{(2)}(b_{ij}^{(2)})^2=0,$$

and

$$c_{ij}=\mathbb{E}\left[J_1^{8-(i+j)}-\left(\frac{J_{10}}{\Delta}\right)^{i+j-2}\right],$$

$$\Delta^3\mathbf{z}^T\begin{pmatrix}15 & 15/2 & 4 & 9/4\\ 15/2 & 4 & 9/4 & 4/3\\ 4 & 9/4 & 4/3 & 5/6\\ 9/4 & 4/3 & 5/6 & 5/9\end{pmatrix}\mathbf{z}\geq 0,$$

where 0 is attained (i.e., holds as an equality) if and only if

$$\sum_{i,j=1}^{s}\beta_i^{(1)}(b_{ij}^{(1)})^2=\frac{1}{3},\qquad \sum_{i,j=1}^{s}\beta_i^{(2)}(b_{ij}^{(1)})^2=-2\sum_{i,j=1}^{s}\beta_i^{(1)}b_{ij}^{(1)}b_{ij}^{(2)},$$
$$\sum_{i,j=1}^{s}\beta_i^{(2)}(b_{ij}^{(2)})^2=0,\qquad \sum_{i,j=1}^{s}\beta_i^{(1)}(b_{ij}^{(2)})^2=-2\sum_{i,j=1}^{s}\beta_i^{(2)}b_{ij}^{(1)}b_{ij}^{(2)},$$

(Conditions A11.-A14. in Theorem 5.4).

Similarly, we can receive 8 additional conditions:

$$\text{B8.} \quad \sum_{i,j=1}^{s} \beta_i^{(1)} b_{ij}^{(1)} \tilde{b}_{ij}^{(1)} = \frac{1}{6} \qquad \Rightarrow \qquad \sum_{i,j=1}^{s} \beta_i^{(1)} \beta_j^{(1)} b_{ij}^{(1)} f_i = \frac{1}{6},$$

$$\text{B10.} \quad 2\sum_{i,j=1}^{s} (\beta_i^{(1)} b_{ij}^{(2)} + \beta_i^{(2)} b_{ij}^{(1)}) \tilde{b}_{ij}^{(1)} = -2\sum_{i,j=1}^{s} \beta_i^{(1)} b_{ij}^{(1)} \tilde{b}_{ij}^{(2)}$$

$$\Rightarrow \sum_{i,j=1}^{s} (\beta_i^{(1)} b_{ij}^{(2)} + \beta_i^{(2)} b_{ij}^{(1)}) \beta_j^{(1)} f_i = -\sum_{i,j=1}^{s} \beta_i^{(1)} b_{ij}^{(1)} \beta_j^{(2)} f_i,$$

$$\text{B11.} \quad 2\sum_{i,j=1}^{s} \beta_i^{(2)} b_{ij}^{(2)} \tilde{b}_{ij}^{(2)} = 0 \qquad \Rightarrow \qquad \sum_{i,j=1}^{s} \beta_i^{(2)} \beta_j^{(2)} b_{ij}^{(2)} f_i = 0,$$

$$\text{B12.} \quad \sum_{i,j=1}^{s} \beta_i^{(1)} (\tilde{b}_{ij}^{(1)})^2 = \frac{1}{3} \qquad \Rightarrow \qquad \sum_{i,j=1}^{s} \beta_i^{(1)} (\beta_j^{(1)})^2 f_i^2 = \frac{1}{3},$$

$$\text{B13.} \quad \sum_{i,j=1}^{s} \beta_i^{(2)} (\tilde{b}_{ij}^{(1)})^2 = -2\sum_{i,j=1}^{s} \beta_i^{(1)} \tilde{b}_{ij}^{(1)} \tilde{b}_{ij}^{(2)}$$

$$\Rightarrow \sum_{i,j=1}^{s} \beta_i^{(2)} (\beta_j^{(1)})^2 f_i^2 = -2\sum_{i,j=1}^{s} \beta_i^{(1)} \beta_j^{(1)} \beta_j^{(2)} f_i^2,$$

$$\text{B14.} \quad \sum_{i,j=1}^{s} \beta_i^{(1)} (\tilde{b}_{ij}^{(2)})^2 = -2\sum_{i,j=1}^{s} \beta_i^{(2)} \tilde{b}_{ij}^{(1)} \tilde{b}_{ij}^{(2)}$$

$$\Rightarrow \sum_{i,j=1}^{s} \beta_i^{(1)} (\beta_j^{(2)})^2 f_i^2 = -2\sum_{i,j=1}^{s} \beta_i^{(2)} \beta_j^{(1)} \beta_j^{(2)} f_i^2,$$

$$\text{B15.} \quad \sum_{i,j=1}^{s} \beta_i^{(2)} (\tilde{b}_{ij}^{(2)})^2 = 0 \qquad \Rightarrow \qquad \sum_{i,j=1}^{s} \beta_i^{(2)} (\beta_j^{(2)})^2 f_i^2 = 0.$$

The last strong order-1.5 term is

$$\sum_{l=1}^{p} \theta_l (\tilde{\mathbf{H}}^{(l)})' \sum_{l=1}^{p} \theta_l (\tilde{\mathbf{H}}^{(l)})' \sum_{l=1}^{p} \theta_l \tilde{\mathbf{H}}^{(l)} = \sum_{l=1}^{p} \theta_l \sum_{i,j=1}^{s} (\tilde{\mathbf{H}}'_{s+1})_i (\tilde{\mathbf{H}}')_i (\tilde{\mathbf{H}})_j$$
$$= \sum_{i,j,k=1}^{s} \left(\beta_i^{(1)} J_1 + \beta_i^{(2)} \frac{J_{10}}{\Delta} \right) \mathbf{H}'$$
$$\times \left(\left(b_{ij}^{(1)} J_1 + b_{ij}^{(2)} \frac{J_{10}}{\Delta} \right) (\mathbf{h}^0)' + \left(\tilde{b}_{ij}^{(1)} J_1 + \tilde{b}_{ij}^{(2)} \frac{J_{10}}{\Delta} \right) ((\mathbf{hq})^0)' \right)$$
$$\times \left(\left(b_{jk}^{(1)} J_1 + b_{jk}^{(2)} \frac{J_{10}}{\Delta} \right) \mathbf{h}^0 + \left(\tilde{b}_{jk}^{(1)} J_1 + \tilde{b}_{jk}^{(2)} \frac{J_{10}}{\Delta} \right) (\mathbf{hq})^0 \right).$$

If

$$\sum_{i,j,k=1}^{s}\left(\beta_i^{(1)}J_1+\beta_i^{(2)}\frac{J_{10}}{\Delta}\right)\left(b_{jk}^{(1)}J_1+b_{jk}^{(2)}\frac{J_{10}}{\Delta}\right)\left(b_{ij}^{(1)}J_1+b_{ij}^{(2)}\frac{J_{10}}{\Delta}\right)$$
$$=\sum_{i,j,k=1}^{s}\left(\beta_i^{(1)}J_1+\beta_i^{(2)}\frac{J_{10}}{\Delta}\right)\left(b_{jk}^{(1)}J_1+b_{jk}^{(2)}\frac{J_{10}}{\Delta}\right)\left(\tilde{b}_{ij}^{(1)}J_1+\tilde{b}_{ij}^{(2)}\frac{J_{10}}{\Delta}\right)$$
$$=\sum_{i,j,k=1}^{s}\left(\beta_i^{(1)}J_1+\beta_i^{(2)}\frac{J_{10}}{\Delta}\right)\left(\tilde{b}_{jk}^{(1)}J_1+\tilde{b}_{jk}^{(2)}\frac{J_{10}}{\Delta}\right)\left(b_{ij}^{(1)}J_1+b_{ij}^{(2)}\frac{J_{10}}{\Delta}\right)$$
$$=\sum_{i,j,k=1}^{s}\left(\beta_i^{(1)}J_1+\beta_i^{(2)}\frac{J_{10}}{\Delta}\right)\left(\tilde{b}_{jk}^{(1)}J_1+\tilde{b}_{jk}^{(2)}\frac{J_{10}}{\Delta}\right)\left(\tilde{b}_{ij}^{(1)}J_1+\tilde{b}_{ij}^{(2)}\frac{J_{10}}{\Delta}\right),$$

then we can obtain

$$\mathbb{E}\left[\frac{J_1^3}{6}-\sum_{i,j,k=1}^{s}\left(\beta_i^{(1)}J_1+\beta_i^{(2)}\frac{J_{10}}{\Delta}\right)\left(b_{jk}^{(1)}J_1+b_{jk}^{(2)}\frac{J_{10}}{\Delta}\right)\left(b_{ij}^{(1)}J_1+b_{ij}^{(2)}\frac{J_{10}}{\Delta}\right)\right]^2.$$

The above analysis implies

$$\sum_{i,j,k=1}^{s}\beta_i^{(1)}b_{ij}^{(1)}b_{jk}^{(1)}=\frac{1}{6},\quad \sum_{i,j,k=1}^{s}\beta_i^{(2)}b_{ij}^{(2)}b_{jk}^{(2)}=0,$$
$$\sum_{i,j,k=1}^{s}\beta_i^{(2)}b_{ij}^{(1)}b_{jk}^{(1)}+\sum_{i,j,k=1}^{s}\beta_i^{(1)}\left(b_{ij}^{(2)}b_{jk}^{(1)}+b_{ij}^{(1)}b_{jk}^{(2)}\right)=0,$$
$$\sum_{i,j,k=1}^{s}\beta_i^{(1)}b_{ij}^{(2)}b_{jk}^{(2)}+\sum_{i,j,k=1}^{s}\beta_i^{(2)}\left(b_{ij}^{(2)}b_{jk}^{(1)}+b_{ij}^{(1)}b_{jk}^{(2)}\right)=0$$

(Conditions A15.-A18. in Theorem 5.4).

Similarly, we obtain 12 further conditions:

B16. $$\sum_{i,j,k=1}^{s}\beta_i^{(1)}\tilde{b}_{ij}^{(1)}b_{jk}^{(1)}=\frac{1}{6}\quad\Rightarrow\quad\sum_{i,j,k=1}^{s}\beta_i^{(1)}\beta_j^{(1)}b_{jk}^{(1)}f_i=\frac{1}{6},$$

B17. $$\sum_{i,j,k=1}^{s}\beta_i^{(2)}\tilde{b}_{ij}^{(2)}b_{jk}^{(2)}=0,\quad\Rightarrow\quad\sum_{i,j,k=1}^{s}\beta_i^{(2)}\beta_j^{(2)}f_ib_{jk}^{(2)}=0,$$

B18. $$\sum_{i,j,k=1}^{s}\beta_i^{(2)}\tilde{b}_{ij}^{(1)}b_{jk}^{(1)}+\sum_{i,j,k=1}^{s}\beta_i^{(1)}\left(\tilde{b}_{ij}^{(2)}b_{jk}^{(1)}+\tilde{b}_{ij}^{(1)}b_{jk}^{(2)}\right)=0$$
$$\Rightarrow\sum_{i,j,k=1}^{s}\beta_i^{(2)}\beta_j^{(1)}f_ib_{jk}^{(1)}+\beta_i^{(1)}\beta_j^{(2)}f_ib_{jk}^{(1)}+\beta_i^{(1)}\beta_j^{(1)}f_ib_{jk}^{(2)}=0,$$

B19. $\sum_{i,j,k=1}^{s} \beta_i^{(1)}\tilde{b}_{ij}^{(2)}b_{jk}^{(2)} + \beta_i^{(2)}\tilde{b}_{ij}^{(2)}b_{jk}^{(1)} + \beta_i^{(2)}\tilde{b}_{ij}^{(1)}b_{jk}^{(2)} = 0$

$$\Rightarrow \sum_{i,j,k=1}^{s} \beta_i^{(1)}\beta_j^{(2)}f_i b_{jk}^{(2)} + \beta_i^{(2)}\beta_j^{(2)}f_i b_{jk}^{(1)} + \beta_i^{(2)}\beta_j^{(1)}f_i b_{jk}^{(2)} = 0,$$

B20. $\sum_{i,j,k=1}^{s} \beta_i^{(1)}b_{ij}^{(1)}\tilde{b}_{jk}^{(1)} = \frac{1}{6} \quad \Rightarrow \quad \sum_{i,j,k=1}^{s} \beta_i^{(1)}b_{ij}^{(1)}\beta_k^{(1)}f_j = \frac{1}{6},$

B21. $\sum_{i,j,k=1}^{s} \beta_i^{(2)}b_{ij}^{(2)}\tilde{b}_{jk}^{(2)} = 0 \quad \Rightarrow \quad \sum_{i,j,k=1}^{s} \beta_i^{(2)}b_{ij}^{(2)}\beta_k^{(2)}f_j = 0,$

B22. $\sum_{i,j,k=1}^{s} \beta_i^{(2)}b_{ij}^{(1)}\tilde{b}_{jk}^{(1)} + \beta_i^{(1)}b_{ij}^{(2)}\tilde{b}_{jk}^{(1)} + \beta_i^{(1)}b_{ij}^{(1)}\tilde{b}_{jk}^{(2)} = 0$

$$\Rightarrow \sum_{i,j,k=1}^{s} \beta_i^{(2)}b_{ij}^{(1)}\beta_k^{(1)}f_j + \beta_i^{(1)}b_{ij}^{(2)}\beta_k^{(1)}f_j + \beta_i^{(1)}b_{ij}^{(1)}\beta_k^{(2)}f_j = 0,$$

B23. $\sum_{i,j,k=1}^{s} \beta_i^{(1)}b_{ij}^{(2)}\tilde{b}_{jk}^{(2)} + \beta_i^{(2)}b_{ij}^{(2)}\tilde{b}_{jk}^{(1)} + \beta_i^{(2)}b_{ij}^{(1)}\tilde{b}_{jk}^{(2)} = 0$

$$\Rightarrow \sum_{i,j,k=1}^{s} \beta_i^{(1)}b_{ij}^{(2)}\beta_k^{(2)}f_j + \beta_i^{(2)}b_{ij}^{(2)}\beta_k^{(1)}f_j + \beta_i^{(2)}b_{ij}^{(1)}\beta_k^{(2)}f_j = 0,$$

B24. $\sum_{i,j,k=1}^{s} \beta_i^{(1)}\tilde{b}_{ij}^{(1)}\tilde{b}_{jk}^{(1)} = \frac{1}{6} \quad \Rightarrow \quad \sum_{i,j,k=1}^{s} \beta_i^{(1)}\beta_j^{(1)}f_i\beta_k^{(1)}f_j = \frac{1}{6},$

B25. $\sum_{i,j,k=1}^{s} \beta_i^{(2)}\tilde{b}_{ij}^{(2)}\tilde{b}_{jk}^{(2)} = 0 \quad \Rightarrow \quad \sum_{i,j,k=1}^{s} \beta_i^{(2)}\beta_j^{(2)}f_i\beta_k^{(2)}f_j = 0,$

B26. $\sum_{i,j,k=1}^{s} \beta_i^{(2)}\tilde{b}_{ij}^{(1)}\tilde{b}_{jk}^{(1)} + \beta_i^{(1)}\tilde{b}_{ij}^{(2)}\tilde{b}_{jk}^{(1)} + \beta_i^{(1)}\tilde{b}_{ij}^{(1)}\tilde{b}_{jk}^{(2)} = 0$

$$\Rightarrow \sum_{i,j,k=1}^{s} \beta_i^{(2)}\beta_j^{(1)}f_i\beta_k^{(1)}f_j + \beta_i^{(1)}\beta_j^{(2)}f_i\beta_k^{(1)}f_j + \beta_i^{(1)}\beta_j^{(1)}f_i\beta_k^{(2)}f_j = 0,$$

B27. $\sum_{i,j,k=1}^{s} \beta_i^{(1)}\tilde{b}_{ij}^{(2)}\tilde{b}_{jk}^{(2)} + \beta_i^{(2)}\tilde{b}_{ij}^{(2)}\tilde{b}_{jk}^{(1)} + \beta_i^{(2)}\tilde{b}_{ij}^{(1)}\tilde{b}_{jk}^{(2)} = 0$

$$\Rightarrow \sum_{i,j,k=1}^{s} \beta_i^{(1)}\beta_j^{(2)}f_i\beta_k^{(2)}f_j + \beta_i^{(2)}\beta_j^{(2)}f_i\beta_k^{(1)}f_j + \beta_i^{(2)}\beta_j^{(1)}f_i\beta_k^{(2)}f_j = 0.$$

□

In the previous section, we showed that it is possible to find different schemes for our Runge-Kutta method on the regarded stochastic optimal control problem, which converges to strong order-1. For strong order-1.5 we remark that we have 18 conditions which come from Runge-Kutta methods for SDEs, and 27 further conditions due to our stochastic optimal control problem that result in 45 conditions. Although the number of conditions does not depend on the number of stages, s, of the method, the number of variables is controlled by s. Moreover, these equations are nonlinear. So, even in the Runge-Kutta methods for SDEs case, it is not easy to solve these equations. Therefore, mostly, explicit schemes become our methods of choice with the aim of simplifying these equations to be solved. For instance, Burrage [8] considered an explicit scheme for these 18 conditions, but had to increase the number of stage. For $s = 4$, Burrage solved these conditions and obtained 4-stage explicit scheme in terms of free parameters.

In our Runge-Kutta method for stochastic optimal control programs, we have 45 conditions. Because of additional more complicated 27 conditions, it is even harder to solve them. Herewith, MAPLE is a good choice to deal with these conditions. By letting $s \geq 4$, these conditions may be solved in terms of the free parameters. Or, by imposing conditions such as an explicit method on the state equation and an implicit method on the adjoint equation, it may be easier to find a solution. However, in this study our aim is to derive a Runge-Kutta method for stochastic optimal control problems and to investigate the convergence of the solution, i.e., to show how such conditions can be obtained.

5.5 Numerical Application

In this section, we choose two numerical examples whose exact solutions we know. Herewith, we can compute the convergence rates explicitly. To solve the optimization problem, we employ a gradient-descent method with a stopping criterion by the error margin of $1e-8$. We also use 1000 paths of Monte-Carlo simulation in each example.

Example 5.1. As a first numerical example, we consider the following optimal control problem [16]:

$$\begin{cases} \underset{u \in L^2(0,T)}{\text{minimize}} & \dfrac{1}{2}\mathbb{E}\left[\displaystyle\int_0^T (y^* - y)^2 dt + \int_0^T (u - u^*)^2 dt\right] \\ \text{subject to} & dy = \dfrac{1}{2}u(u - u^*)ydt + \sigma y dW, \quad y(0) = y^0, \end{cases}$$

where σ is merely a positive scalar, often called as volatility. We note that this example is from the financial sector and that it can be regarded as a continuous modeling task under so-called regularization. We have the following continuous optimality system:

$$\begin{cases} dy = \dfrac{1}{2}u(u - u^*)ydt + \sigma y dW, \quad y(0) = y_0, \\ dp = (y^* - y + \sigma^2 p)dt + \sigma y q dW, \quad p(T) = 0, \\ u - u^* = -\mathbb{E}\left[p\left(u - \dfrac{1}{2}u^*\right)\right]. \end{cases}$$

The exact solution (y, u) is of the form:

$$y(t) = y^*(t) = y^0 e^{-\frac{\sigma^2}{2}t+\sigma W(t)}, \qquad u(t) = u^*(t) = 6\sin(\pi t).$$

We employ the following Runge-Kutta scheme which satisfies the conditions in Theorem 5.5, strong order-1 conditions.

$$\begin{array}{c|cc} 1/2 & 1/4 & 1/4 \\ 1/2 & 1/4 & 1/4 \\ \hline & 1/2 & 1/2 \end{array} \qquad\qquad \begin{array}{c|cc} 1/2 & 1/4 & 1/4 \\ 1/2 & 1/4 & 1/4 \\ \hline & 1/2 & 1/2 \end{array}$$

We choose $T = 1$, $y_0 = 1$ and $\sigma = 0.1$ for our numerical computation. In Figure 5.1, we compare the exact solution of the optimal control with the numerical optimal control obtained from our Runge-Kutta scheme.

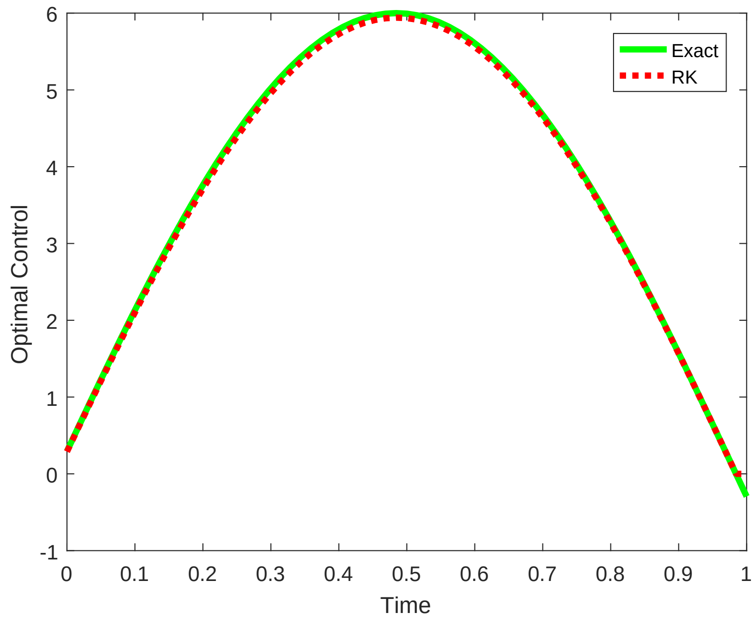


Figure 5.1: Optimal control with $\sigma = 0.1$ in Example 5.1.

If the absolute error is given by $E_i = |u(t_i) - \bar{u}(t_i)|$, where $u(t_i)$ is the exact value of u and $\bar{u}(t_i)$ denotes the the approximate value of u at t_i, then the order of convergence rate is computed by the following formula:

$$\text{Rate} = \frac{\log(E_i/E_{i+1})}{\log(\Delta t_i/\Delta t_{i+1})}.$$

In Table 5.1, we choose $\Delta t_i = 1/2^i$. We see that the absolute error E_{i+1} is the half of E_i, and $\Delta t_i/\Delta t_{i+1} = 2$. Thus, the calculated order of convergence is 1, as we expected.

Table 5.1: Convergence Rate of our Runge-Kutta method with $\sigma = 0.1$ in Example 5.1.

i	Δt_i	E_i (Absolute Error)	Order
6	2^{-6}	0.2944	1
7	2^{-7}	0.1472	1
8	2^{-8}	0.0736	1
9	2^{-9}	0.0368	1
10	2^{-10}	0.0184	1

Example 5.2. In this example, we investigate the subsequent Black-Scholes type of optimal control problem [16]:

$$\begin{cases} \underset{u \in L^2(0,T)}{\text{minimize}} & \frac{1}{2}\mathbb{E}\left[\int_0^T (y^* - y)^2 dt + \int_0^T u^2 dt\right] \\ \text{subject to} & dy = uydt + \sigma ydW, \quad y(0) = y^0, \end{cases}$$

where $\sigma > 0$ is a constant and $y^*(t)$ is given. Again, this example can be interpreted in terms of financial modeling under regularization. We can construct an exact solution of the form:

$$y(t) = y(0)e^{\int_0^t u(s)ds - \frac{\sigma^2}{2}t + \sigma W(t)}, \qquad u(t) = \frac{T - t}{\frac{1}{y_0} - Tt + \frac{t^2}{2}},$$

where

$$y^*(t) = \frac{e^{\sigma^2 t} - (T-t)^2}{\frac{1}{y^0} - Tt + \frac{t^2}{2}} + 1.$$

Table 5.2: Convergence Rate of our Runge-Kutta method with $\sigma = 0.1$ in Example 5.2.

i	Δt_i	E_i (Absolute Error)	Order
6	$1/2^{-6}$	0.0312	1
7	$1/2^{-7}$	0.0156	1
8	$1/2^{-8}$	0.0078	1
9	$1/2^{-9}$	0.0039	1
10	$1/2^{-10}$	0.0019	1

We apply the same 2-stage of stochastic Runge-Kutta scheme as in Example 5.1. Let $y^*(t) = y(t)$. We also choose $T = 1$, $y_0 = 1$ and $\sigma = 0.1$ in the numerical computation. In Figure 5.2, a comparison between the exact solution of the control and the numerical control obtained from our Runge-Kutta scheme is given.

In Table 5.2, we choose $\Delta t_i = 1/2^i$. We asses the ratio by $E_i/E_{i+1} = \Delta t_i/\Delta t_{i+1} = 2$, so that the computed order of convergence is 1.

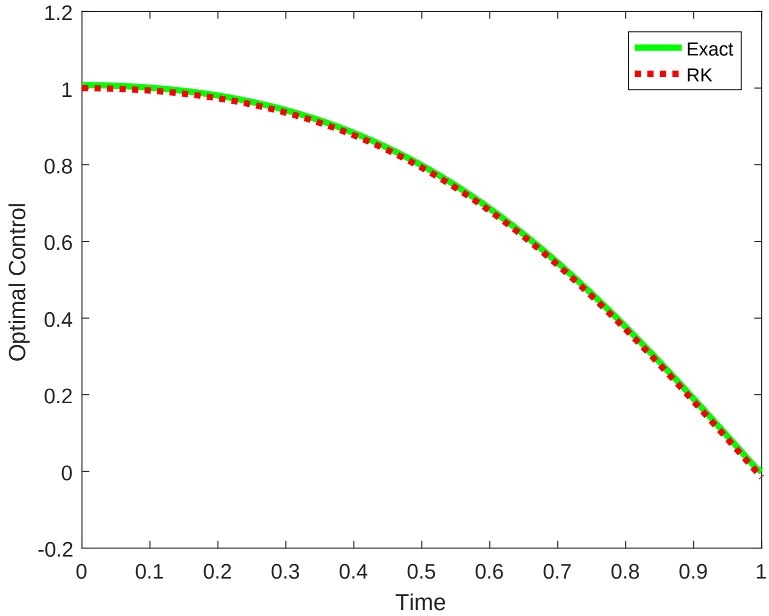


Figure 5.2: Optimal control with $\sigma = 0.1$ in Example 5.2.

5.6 Summary

In this chapter, firstly, we provided strong order-1 conditions of our Runge-Kutta scheme for the optimal control problems of SDEs for any number of s-stages. We showed that strong order-1.5 can not be obtained for any stages s. Moreover, minimal local truncation-error constants for strong order-1 were obtained for $s = 2$. To do this, we compared the Stratonovich-Taylor expansions of the exact solution and our Runge-Kutta scheme. By considering a general Runge-Kutta scheme, we were able to get strong order-1.5 conditions of our Runge-Kutta scheme for the optimal control problems of SDEs. We obtained additional order conditions on the classical Runge-Kutta schemes to SDEs for both order-1 and order-1.5. Finally, by choosing the step-size Δ small enough, the accuracy of the scheme was received. We confirmed our results in two numerical applications.

CHAPTER 6

WEAK-ORDER CONDITIONS OF THE RUNGE-KUTTA SCHEME FOR STOCHASTIC OPTIMAL CONTROL PROBLEMS

6.1 Introduction

In some cases, it is necessary to approximate certain moments of solution $\mathbf{X}$, e.g., $\mathbb{E}[\mathbf{X}]$, $\mathbb{E}[\mathbf{X}^2]$ or, more generally, $\mathbb{E}[\mathbf{\Phi}(\mathbf{X})]$ for some vector-valued of 2 variables function $\mathbf{\Phi}$, instead of simulating the sample paths which are close to the solution $\mathbf{X}$. Simulating of such moments gives information about the probability distribution of the solution $\mathbf{X}$ rather than a good approximation of sample paths. This results in a much weaker criteria, in so-called *weak convergence*. Actually, weak convergence criteria of Runge-Kutta scheme for SDEs is investigated by Mackevicius [37] and weak-order conditions are derived. In this chapter, our aim is to seek weak-order conditions of our Runge-Kutta scheme for stochastic optimal control problems on the addressed class of SDEs.

We let $\boldsymbol{\zeta}(T)$ be a numerical approximation to $\mathbf{X}(t_N)$ after N steps with constant step size $\Delta = (t_N - t_0)/N$. Then $\boldsymbol{\zeta}(T)$ is said to converge weakly to $\mathbf{X}$ with order $r > 0$, if for each function $\mathbf{\Phi}$ which is $2(r+1)$-times continuously differentiable vector-valued of 2 variables, there exists a constant $C > 0$ which does not depend on Δ, and a $\Delta_0 > 0$ such that

$$\|\mathbb{E}[\mathbf{\Phi}(\boldsymbol{\zeta}(T))] - \mathbb{E}[\mathbf{\Phi}(\mathbf{X}(t_N))]\|_2 \leq C\Delta^r, \qquad \Delta \in (0, \Delta_0),$$

where $\mathbf{X} = (y,p)^T$ and $\boldsymbol{\zeta} = (\hat{y},\hat{p})^T$ are the solutions of continuous $(\mathcal{OC}_c)$ and discrete optimality systems $(\mathcal{OC}_d)$ in Chapter 3, respectively. We notice that these optimality systems are stated in Itô forms. In this chapter, we address these optimality systems in related Stratonovich forms.

Mackevicius [37, 38] showed that there is no second-order weak Runge-Kutta approximations for Itô SDEs. If one wants to achieve weak-order Runge-Kutta approximations for some Itô SDE, it could firstly be rewritten in Stratonovich SDEs form. In this chapter, we follow the idea of Mackevicius [37] and Rößler [14, 52] to derive weak-order conditions of our Runge-Kutta scheme of the Stratonovich form of the stochastic optimal control problem. As done in the previous chapter, we use the Stratonovich form of both continuous and discrete optimality systems. Then, we make use of the Itô Formula to expand stochastic Taylor-series on the exact solution and the solution of our Runge-Kutta scheme to find the order of accuracy.

6.2 Problem Formulation

We use the same problem formulation given in Chapter 5. Let us first recall $\mathbf{X}$, $\mathbf{F}$ and $\mathbf{H}$ denoting the following pairs:

$$\mathbf{X} = \begin{pmatrix} y \\ p \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} f \\ -\mathcal{H}_y \end{pmatrix}, \quad \mathbf{H} = \begin{pmatrix} h \\ hq \end{pmatrix}.$$

With this notation, we can write problem $(\mathcal{P}_c)$ in the form

$$d\mathbf{X} = \mathbf{F}(y,u,p,q)dt + \mathbf{H}(y,q)dW$$

as an Itô SDE, or

$$d\mathbf{X} = \underline{\mathbf{F}}(y,u,p,q)dt + \mathbf{H}(y,q) \circ dW \tag{6.1}$$

as its related Stratonovich SDE with a modified drift coefficient which is defined by [28]

$$\underline{\mathbf{F}} = \mathbf{F} - \frac{1}{2}\mathbf{H}'\mathbf{H}.$$

For simplicity, we restate Eqn. (6.1) as

$$d\mathbf{X} = \underline{\mathbf{F}}(\mathbf{X})dt + \mathbf{H}(\mathbf{X}) \circ dW. \tag{6.2}$$

We recall the discrete optimality conditions of problem $(\mathcal{P}'_d)$ posed in Theorem 5.2:

$$(\mathcal{OC}'_d) \left\{ \begin{aligned} y_{k+1} &= y_k + J_0 \sum_{i=1}^{s} \alpha_i \underline{f}(y_{ki}, u_{ki}) + J_1 \sum_{i=1}^{s} \beta_i h(y_{ki}), \\ y_{ki} &= y_k + J_0 \sum_{j=1}^{s} a_{ij} \underline{f}(y_{kj}, u_{kj}) + J_1 \sum_{j=1}^{s} b_{ij} h(y_{kj}), \\ p_{k+1} &= p_k - J_0 \sum_{i=1}^{s} \tilde{\alpha}_i \underline{\mathcal{H}}_y(y_{ki}, u_{ki}, p_{ki}, q_{ki}) + J_1 \sum_{i=1}^{s} \tilde{\beta}_i h(y_{ki}) q_{ki}, \\ p_{ki} &= p_k - J_0 \sum_{j=1}^{s} \tilde{a}_{ij} \underline{\mathcal{H}}_y(y_{kj}, u_{kj}, p_{kj}, q_{kj}) + J_1 \sum_{j=1}^{s} \tilde{b}_{ij} h(y_{kj}) q_{kj}, \\ q_{ki}\psi_{ki} &= p_k - J_0 \sum_{j=1}^{s} \hat{a}_{ij} \underline{\mathcal{H}}_y(y_{kj}, u_{kj}, p_{kj}, q_{kj}) + J_1 \sum_{j=1}^{s} \hat{b}_{ij} h(y_{kj}) q_{kj}, \\ p_N &= \phi'(y_N), \\ y_0 &= y^0, \\ 0 &= \Delta \sum_{i=1}^{s} \alpha_i \underline{\mathcal{H}}_u(y_{ki}, u_{ki}, p_{ki}, q_{ki}), \end{aligned} \right.$$

for $k = 0, 1, \ldots, N-1$, where the coefficients satisfy the subsequent relations:

$$
\begin{aligned}
\tilde{\alpha}_i &:= \alpha_i, & \tilde{\beta}_i &:= \beta_i, \\
\tilde{a}_{ij} &:= \alpha_j - \frac{\alpha_j}{\alpha_i} a_{ji}, & \tilde{b}_{ij} &:= \beta_j - \frac{\beta_j}{\alpha_i} a_{ji}, \\
\hat{a}_{ij} &:= \alpha_j - \frac{\alpha_j}{\beta_i} b_{ji}, & \hat{b}_{ij} &:= \beta_j - \frac{\beta_j}{\beta_i} b_{ji},
\end{aligned}
$$

with

$$
\psi_{ki} := \frac{J_0 t \alpha_i h(y_{ki})}{J_1 \beta_i} - \frac{h(y_{ki})}{h'(y_{ki})}.
$$

6.3 Weak-Order Conditions of Runge-Kutta Method for Stochastic Optimal Control Problems

To obtain weak-order conditions of our Runge-Kutta method, we need to expand $\mathbb{E}[\mathbf{\Phi}(\mathbf{X}(t_0 + \Delta))]$ and $\mathbb{E}[\mathbf{\Phi}(\boldsymbol{\zeta}(t_0 + \Delta))]$ using the Itô Formula for some sufficiently smooth vector-valued of 2 variables function $\mathbf{\Phi}$. So, we first consider the vector-valued diffusion operator of 2 variables $\boldsymbol{\mathcal{L}}$ [28] for the solution $\mathbf{X}$ of the Stratonovich SDE from Eqn. (6.2):

$$
\boldsymbol{\mathcal{L}}\mathbf{\Phi} := \frac{\partial \mathbf{\Phi}}{\partial \mathbf{X}} \left(\underline{\mathbf{F}} + \frac{1}{2}\mathbf{H}'\mathbf{H} \right) + \frac{1}{2}\frac{\partial^2 \mathbf{\Phi}}{\partial \mathbf{X}^2}\mathbf{H}\mathbf{H}.
$$

Then, the Itô Formula yields

$$
\begin{aligned}
\mathbb{E}\left[\mathbf{\Phi}(\mathbf{X}(t_0+\Delta))\right] &= \mathbf{\Phi}(\mathbf{X}(t_0)) + \int_{t_0}^{t_0+\Delta} \mathbb{E}\left[\boldsymbol{\mathcal{L}}\mathbf{\Phi}(\mathbf{X}(s))ds\right] \\
&= \mathbf{\Phi}(\mathbf{X}(t_0)) + \int_{t_0}^{t_0+\Delta} \left(\boldsymbol{\mathcal{L}}\mathbf{\Phi}(\mathbf{X}(t_0)) + \int_{t_0}^{s} \mathbb{E}\left[\boldsymbol{\mathcal{L}}^2\mathbf{\Phi}(\mathbf{X}(u)\right] du \right) ds \\
&= \mathbf{\Phi}(\mathbf{X}(t_0)) + \int_{t_0}^{t_0+\Delta} \boldsymbol{\mathcal{L}}\mathbf{\Phi}(\mathbf{X}(t_0))ds + \int_{t_0}^{t_0+\Delta}\int_{t_0}^{s} \boldsymbol{\mathcal{L}}^2\mathbf{\Phi}(\mathbf{X}(t_0))duds \\
&\quad + \int_{t_0}^{t_0+\Delta}\int_{t_0}^{s}\int_{t_0}^{u} \mathbb{E}\left[\boldsymbol{\mathcal{L}}^3\mathbf{\Phi}(\mathbf{X}(v))\right] dvduds \\
&= \mathbf{\Phi}(\mathbf{X}(t_0)) + \boldsymbol{\mathcal{L}}\mathbf{\Phi}(\mathbf{X}(t_0))\Delta + \boldsymbol{\mathcal{L}}^2\mathbf{\Phi}(\mathbf{X}(t_0))\frac{1}{2}\Delta^2 \\
&\quad + \int_{t_0}^{t_0+\Delta}\int_{t_0}^{s}\int_{t_0}^{u} \mathbb{E}\left[\boldsymbol{\mathcal{L}}^3\mathbf{\Phi}(\mathbf{X}_v)\right] dvduds,
\end{aligned}
$$

and

$$\begin{aligned}\boldsymbol{\mathcal{L}}^2\mathbf{\Phi} =&\frac{\partial \mathbf{\Phi}}{\partial \mathbf{X}}\left(\underline{\mathbf{F}}'\underline{\mathbf{F}} + \frac{1}{2}\mathbf{H}'\mathbf{H}'\underline{\mathbf{F}} + \frac{1}{2}\mathbf{H}''\underline{\mathbf{F}}\mathbf{H} + \frac{1}{2}\mathbf{H}'\underline{\mathbf{F}}'\mathbf{H} + \frac{1}{4}\mathbf{H}'\mathbf{H}'\mathbf{H}'\mathbf{H}\right.\\ &\left. + \frac{1}{4}\mathbf{H}''\mathbf{H}\mathbf{H}'\mathbf{H} + \frac{1}{2}\underline{\mathbf{F}}''\mathbf{H}\mathbf{H} + \frac{3}{4}\mathbf{H}''\mathbf{H}'\mathbf{H}\mathbf{H} + \frac{1}{4}\mathbf{H}'''\mathbf{H}\mathbf{H}\mathbf{H}\right)\\ &+ \frac{\partial^2 \mathbf{\Phi}}{\partial \mathbf{X}^2}\left(\underline{\mathbf{F}}\underline{\mathbf{F}} + \mathbf{H}\mathbf{H}'\underline{\mathbf{F}} + \underline{\mathbf{F}}\mathbf{H}\mathbf{H}' + \frac{3}{4}\mathbf{H}\mathbf{H}'\mathbf{H}\mathbf{H}' + \mathbf{H}\mathbf{H}\mathbf{H}'\mathbf{H}' + \mathbf{H}''\mathbf{H}\mathbf{H}\mathbf{H}\right.\\ &\left. + \mathbf{H}\mathbf{H}\underline{\mathbf{F}}'\right) + \frac{\partial^3 \mathbf{\Phi}}{\partial \mathbf{X}^3}\left(\underline{\mathbf{F}}\mathbf{H}\mathbf{H} + \mathbf{H}'\mathbf{H}\mathbf{H}\mathbf{H} + \frac{1}{2}\mathbf{H}'\mathbf{H}\mathbf{H}\mathbf{H}\right) + \frac{\partial^4 \mathbf{\Phi}}{\partial \mathbf{X}^4}\left(\frac{1}{4}\mathbf{H}\mathbf{H}\mathbf{H}\mathbf{H}\right).\end{aligned}$$

We point out that the expectation of multiple Itô integral including at least one integration with respect to Brownian motion is zero [28, 53]. Moreover, we note that the derivatives should be viewed in an operator context. For example, the first derivative of a vector-valued function $\underline{\mathbf{F}}$ is a Jacobian matrix, so that $\underline{\mathbf{F}}'\underline{\mathbf{F}}$ corresponds to multiplying the Jacobian matrix by the vector $\underline{\mathbf{F}}$ to give a vector. The second derivative $\underline{\mathbf{F}}''$ operates on a pair of the vector $(\mathbf{H}, \mathbf{H})$ in order to give a vector $\underline{\mathbf{F}}''\mathbf{H}\mathbf{H}$.

As in Section 5.3, in order to study order conditions of discrete optimality conditions $(\mathcal{OC}'_d)$, the $(\mathcal{OC}'_d)$ will be written as a function of t. By using Butcher approach, we write $t_n = t_0$, and for a given $t = t_0 + \Delta$. Let us recall the notations Section 5.3 so that let $\boldsymbol{\zeta}(t) = (y(t), p(t))^T$ be the vector of length $2N(s+1)$ and let $\boldsymbol{\zeta}_{ki}(t)$, $\boldsymbol{\zeta}_{s+1}(t)$, $\tilde{\underline{\mathbf{F}}}(\boldsymbol{\zeta}(t))$ and $\tilde{\mathbf{H}}(\boldsymbol{\zeta}(t))$ denote the following pairs:

$$\boldsymbol{\zeta}_{ki}(t) = \begin{pmatrix} y_{ki}(t) \\ p_{ki}(t) \end{pmatrix} \quad (1 \leq i \leq s), \quad \boldsymbol{\zeta}_{s+1}(t) = \begin{pmatrix} y_{k+1}(t) \\ p_{k+1}(t) \end{pmatrix} \quad (i = s+1),$$

$$\tilde{\underline{\mathbf{F}}}(\boldsymbol{\zeta}(t)) = \tilde{\underline{\mathbf{F}}}_i(\boldsymbol{\zeta}(t)) = \begin{pmatrix} \sum_{j=1}^{s} a_{ij}\underline{f}(\boldsymbol{\zeta}_{kj}(t)) \\ \sum_{j=1}^{s} \tilde{a}_{ij}\underline{\mathcal{H}}_y(\boldsymbol{\zeta}_{kj}(t)) \end{pmatrix} \quad (1 \leq i \leq s+1),$$

and

$$\tilde{\mathbf{H}}(\boldsymbol{\zeta}(t)) = \tilde{\mathbf{H}}_i(\boldsymbol{\zeta}(t)) = \begin{pmatrix} \sum_{j=1}^{s} b_{ij}h(\boldsymbol{\zeta}_{kj}(t)) \\ \sum_{j=1}^{s} \tilde{b}_{ij}h(\boldsymbol{\zeta}_{kj}(t))q(\boldsymbol{\zeta}_{kj}(t)) \end{pmatrix} \quad (1 \leq i \leq s+1),$$

where $k = 0, 1, \ldots, N-1$, is the index of the Runge-Kutta scheme in the discrete optimality conditions $(\mathcal{OC}'_d)$, with

$$\begin{aligned} a_{s+1,j} &= \tilde{a}_{s+1,j} = \alpha_j \quad (1 \leq j \leq s), \\ b_{s+1,j} &= \tilde{b}_{s+1,j} = \beta_j \quad (1 \leq j \leq s). \end{aligned}$$

By using the above notation, we can state the discrete optimality conditions $(\mathcal{OC}'_d)$ in the form

$$\boldsymbol{\zeta}(t) = \boldsymbol{\zeta}(t_0) + (t - t_0)\tilde{\underline{\mathbf{F}}}(\boldsymbol{\zeta}(t)) + \Delta W\tilde{\mathbf{H}}(\boldsymbol{\zeta}(t)) \quad (1 \leq i \leq s+1), \tag{6.3}$$

which is Eqn. (5.8), where $\Delta = t - t_0$ and $\Delta W = W(t) - W(t_0)$.

Then, one-step Runge-Kutta approximation in Eqn. (6.3) can be stated as

$$\begin{cases} \boldsymbol{\zeta}(t_0) = \mathbf{X}(t_0), \\ \boldsymbol{\zeta}(t) = \mathbf{A}(\boldsymbol{\zeta}(t), \Delta, \Delta W). \end{cases} \tag{6.4}$$

It is clear that $\mathbf{A}(\boldsymbol{\zeta}(t_0), 0, 0) = \boldsymbol{\zeta}(t_0)$, but in the following expansion, we briefly write $\mathbf{A}(\mathbf{X}(t_0), 0, 0) := \mathbf{A}(\mathbf{X}(t_0))$.

The corresponding vector-valued diffusion operator of 2 variables, $\boldsymbol{\mathcal{L}}_\Delta$, for expansion of $\mathbb{E}[\boldsymbol{\Phi}(\boldsymbol{\zeta}(t_0 + \Delta))]$ is given by

$$\boldsymbol{\mathcal{L}}_\Delta \boldsymbol{\Phi} := \frac{\partial \boldsymbol{\Phi}}{\partial u} + \frac{1}{2}\frac{\partial^2 \boldsymbol{\Phi}}{\partial \tilde{W}^2},$$

where $u = \Delta$ and $\tilde{W}(u) = \Delta W$. For simplicity, we write $\tilde{W} := \tilde{W}(u)$.

Then, the Itô Formula gives

$$\begin{aligned}
\mathbb{E}\left[\boldsymbol{\Phi}(\boldsymbol{\zeta}(t_0 + \Delta))\right] &= \boldsymbol{\Phi}(\mathbf{X}(t_0)) + \int_{t_0}^{t_0+\Delta} \mathbb{E}\left[\boldsymbol{\mathcal{L}}_\Delta \boldsymbol{\Phi}(\mathbf{A}(\boldsymbol{\zeta}(s), s, \tilde{W}(s)))ds\right] \\
&= \boldsymbol{\Phi}(\mathbf{X}(t_0)) + \int_{t_0}^{t_0+\Delta} \left(\boldsymbol{\mathcal{L}}_\Delta \boldsymbol{\Phi}(\mathbf{A}(\mathbf{X}(t_0))) + \int_{t_0}^{s} \mathbb{E}\left[\boldsymbol{\mathcal{L}}_\Delta^2 \boldsymbol{\Phi}(\mathbf{A}(\boldsymbol{\zeta}(u), u, \tilde{W}(u)))\right] du\right) ds \\
&= \boldsymbol{\Phi}(\mathbf{X}(t_0)) + \int_{t_0}^{t_0+\Delta} \boldsymbol{\mathcal{L}}_\Delta \boldsymbol{\Phi}(\mathbf{A}(\mathbf{X}(t_0)))ds + \int_{t_0}^{t_0+\Delta}\int_{t_0}^{s} \boldsymbol{\mathcal{L}}_\Delta^2 \boldsymbol{\Phi}(\mathbf{A}(\mathbf{X}(t_0)))duds \\
&\quad + \int_{t_0}^{t_0+\Delta}\int_{t_0}^{s}\int_{t_0}^{u} \mathbb{E}\left[\boldsymbol{\mathcal{L}}_\Delta^3 \boldsymbol{\Phi}((\boldsymbol{\zeta}(v), v, \tilde{W}(v))))\right] dvduds \\
&= \boldsymbol{\Phi}(\mathbf{X}(t_0)) + \boldsymbol{\mathcal{L}}\boldsymbol{\Phi}(\mathbf{A}(\mathbf{X}(t_0)))\Delta + \boldsymbol{\mathcal{L}}_\Delta^2 \boldsymbol{\Phi}(\mathbf{A}(\mathbf{X}(t_0)))\frac{1}{2}\Delta^2 \\
&\quad + \int_{t_0}^{t_0+\Delta}\int_{t_0}^{s}\int_{t_0}^{u} \mathbb{E}\left[\boldsymbol{\mathcal{L}}_\Delta^3 \boldsymbol{\Phi}(\mathbf{X}(v))\right] dvduds,
\end{aligned}$$

with

$$\boldsymbol{\mathcal{L}}_\Delta \boldsymbol{\Phi}(\mathbf{A}) = \frac{\partial \boldsymbol{\Phi}}{\partial \mathbf{X}}\left(\frac{\partial \mathbf{A}}{\partial u} + \frac{1}{2}\frac{\partial^2 \mathbf{A}}{\partial \tilde{W}^2}\right) + \frac{\partial^2 \boldsymbol{\Phi}}{\partial \mathbf{X}^2}\frac{1}{2}\left(\frac{\partial \mathbf{A}}{\partial \tilde{W}}\right)^2$$

and

$$\begin{aligned}
\boldsymbol{\mathcal{L}}_\Delta^2 \boldsymbol{\Phi}(\mathbf{A}) =& \frac{\partial \boldsymbol{\Phi}}{\partial \mathbf{X}}\left(\frac{\partial^2 \mathbf{A}}{\partial s^2} + \frac{\partial^3 \mathbf{A}}{\partial \tilde{W}^2 \partial u} + \frac{1}{4}\frac{\partial^4 \mathbf{A}}{\partial \tilde{W}^4}\right) + \frac{\partial^2 \boldsymbol{\Phi}}{\partial \mathbf{X}^2}\left(\left(\frac{\partial \mathbf{A}}{\partial u}\right)^2\right. \\
&+ \left.\frac{\partial \mathbf{A}}{\partial \Delta}\frac{\partial^2 \mathbf{A}}{\partial \tilde{W}^2} + 2\frac{\partial \mathbf{A}}{\partial \tilde{W}}\frac{\partial^2 \mathbf{A}}{\partial u \partial \tilde{W}} + \frac{\partial \mathbf{A}}{\partial \tilde{W}}\frac{\partial^3 \mathbf{A}}{\partial \tilde{W}^3} + \frac{3}{4}\left(\frac{\partial^2 \mathbf{A}}{\partial \tilde{W}^2}\right)\right) \\
&+ \frac{\partial^3 \boldsymbol{\Phi}}{\partial \mathbf{X}^3}\left(\frac{\partial \mathbf{A}}{\partial u}\left(\frac{\partial \mathbf{A}}{\partial \tilde{W}}\right)^2 + \frac{3}{4}\left(\frac{\partial \mathbf{A}}{\partial \tilde{W}}\right)^2\frac{\partial^2 \mathbf{A}}{\partial \tilde{W}^2}\right) + \frac{\partial^4 \boldsymbol{\Phi}}{\partial \mathbf{X}^4}\left(\frac{1}{4}\left(\frac{\partial \mathbf{A}}{\partial \tilde{W}}\right)^4\right).
\end{aligned}$$

Now, we reach the intended weak order-1 and order-2 conditions of our Runge-Kutta scheme by solving the conditions:

$$\boldsymbol{\mathcal{L}}\mathbf{\Phi}(\mathbf{X}(t_0)) = \boldsymbol{\mathcal{L}}_\Delta\mathbf{\Phi}(\mathbf{A}(\mathbf{X}(t_0)))$$

and

$$\boldsymbol{\mathcal{L}}^2\mathbf{\Phi}(\mathbf{X}(t_0)) = \boldsymbol{\mathcal{L}}_\Delta^2\mathbf{\Phi}(\mathbf{A}(\mathbf{X}(t_0))),$$

respectively.

Theorem 6.1. *Let $\alpha_i,\ \beta_i,\ a_{ij},\ \tilde{a}_{ij},\ b_{ij}\ \tilde{b}_{ij}$, be the Runge-Kutta coefficients for $i,j = 1,2,\ldots,s$. If the coefficients of the Runge-Kutta method for the stochastic optimal control problems fulfill the following conditions:*

1. $\sum_{i=1}^{s}\alpha_i = 1,$ 2. $\sum_{i=1}^{s}\beta_i = 1,$ 3. $\sum_{i=1}^{s}\beta_i b_i = \sum_{i=1}^{s}\beta_i\tilde{b}_i = \frac{1}{2},$

then our Runge-Kutta method for the stochastic optimal control problems converges to order-1 in the weak sense. In addition, if the following conditions are also satisfied:

4. $\sum_{i=1}^{s}\alpha_i a_i = \sum_{i=1}^{s}\alpha_i\tilde{a}_i = \frac{1}{2},$
5. $\sum_{i=1}^{s}\alpha_i b_i^2 = \sum_{i=1}^{s}\alpha_i\tilde{b}_i^2 = \sum_{i=1}^{s}\alpha_i b_i\tilde{b}_i = \frac{1}{2},$
6. $\sum_{i=1}^{s}\alpha_i\sum_{j=1}^{s}b_{ij}b_j = \sum_{i=1}^{s}\alpha_i\sum_{j=1}^{s}b_{ij}\tilde{b}_j = \sum_{i=1}^{s}\alpha_i\sum_{j=1}^{s}\tilde{b}_{ij}b_j = \sum_{i=1}^{s}\alpha_i\sum_{j=1}^{s}\tilde{b}_{ij}\tilde{b}_j,$
7. $\sum_{i=1}^{s}\beta_i\sum_{j=1}^{s}a_{ij}b_j = \sum_{i=1}^{s}\beta_i\sum_{j=1}^{s}a_{ij}\tilde{b}_j = \sum_{i=1}^{s}\beta_i\sum_{j=1}^{s}\tilde{a}_{ij}b_j = \sum_{i=1}^{s}\beta_i\sum_{j=1}^{s}\tilde{a}_{ij}\tilde{b}_j,$
8. $\sum_{i=1}^{s}\alpha_i\sum_{j=1}^{s}b_{ij}b_j + \sum_{i=1}^{s}\beta_i\sum_{j=1}^{s}a_{ij}b_j = \frac{1}{4},$
9. $\sum_{i=1}^{s}\beta_i a_i b_i = \sum_{i=1}^{s}\beta_i a_i\tilde{b}_i = \sum_{i=1}^{s}\beta_i\tilde{a}_i b_i = \sum_{i=1}^{s}\beta_i\tilde{a}_i\tilde{b}_i = \frac{1}{4},$
10. $\sum_{i,j=1}^{s}\beta_i a_j b_{ij} = \sum_{i,j=1}^{s}\beta_i a_j\tilde{b}_{ij} = \sum_{i,j=1}^{s}\beta_i\tilde{a}_j b_{ij} = \sum_{i,j=1}^{s}\beta_i\tilde{a}_j\tilde{b}_{ij} = \frac{1}{4},$

11. $\sum_{i=1}^{s} \beta_i b_i^3 = \sum_{i=1}^{s} \beta_i b_i^2 \tilde{b}_i = \sum_{i=1}^{s} \beta_i b_i \tilde{b}_i^2 = \sum_{i=1}^{s} \beta_i \tilde{b}_i^3 = \frac{1}{4},$

12. $\sum_{i,j=1}^{s} \beta_i b_{ij} b_i b_j = \sum_{i,j=1}^{s} \beta_i b_{ij} \tilde{b}_i b_j = \sum_{i,j=1}^{s} \beta_i \tilde{b}_{ij} \tilde{b}_i b_j$

$= \sum_{i,j=1}^{s} \beta_i b_{ij} b_i \tilde{b}_j = \sum_{i,j=1}^{s} \beta_i b_{ij} \tilde{b}_i \tilde{b}_j = \sum_{i,j=1}^{s} \beta_i \tilde{b}_{ij} \tilde{b}_i \tilde{b}_j = \frac{1}{8},$

13. $\sum_{i,j=1}^{s} \beta_i b_{ij} b_j^2 = \sum_{i,j=1}^{s} \beta_i b_{ij} \tilde{b}_j b_j = \sum_{i,j=1}^{s} \beta_i \tilde{b}_{ij} \tilde{b}_j b_j = \sum_{i,j=1}^{s} \beta_i b_{ij} \tilde{b}_j^2 = \sum_{i,j=1}^{s} \beta_i \tilde{b}_{ij} \tilde{b}_j^2 = \frac{1}{12},$

14. $\sum_{i,j,k=1}^{s} \beta_i b_{ij} b_{jk} b_k = \sum_{i,j,k=1}^{s} \beta_i b_{ij} \tilde{b}_{jk} b_k = \sum_{i,j,k=1}^{s} \beta_i \tilde{b}_{ij} \tilde{b}_{jk} b_k$

$= \sum_{i,j,k=1}^{s} \beta_i b_{ij} b_{jk} \tilde{b}_k = \sum_{i,j,k=1}^{s} \beta_i b_{ij} \tilde{b}_{jk} \tilde{b}_k = \sum_{i,j,k=1}^{s} \beta_i \tilde{b}_{ij} \tilde{b}_{jk} \tilde{b}_k = \frac{1}{24},$

15. $\sum_{i=1}^{s} \alpha_i b_i = \sum_{i=1}^{s} \alpha_i \tilde{b}_i = \frac{1}{2},$

16. $\sum_{i=1}^{s} \beta_i a_i = \sum_{i=1}^{s} \beta_i \tilde{a}_i = \frac{1}{2},$

17. $\sum_{i=1}^{s} \beta_i b_i^2 = \sum_{i=1}^{s} \beta_i b_i \tilde{b}_i = \sum_{i=1}^{s} \beta_i \tilde{b}_i^2 = \frac{1}{3},$

18. $\sum_{i=1}^{s} \beta_i b_{ij} b_j = \sum_{i=1}^{s} \beta_i \tilde{b}_{ij} b_j = \sum_{i=1}^{s} \beta_i b_{ij} \tilde{b}_j = \sum_{i=1}^{s} \beta_i \tilde{b}_{ij} \tilde{b}_j = \frac{1}{6},$

then our Runge-Kutta method for the stochastic optimal control problems converges to order-2 in the weak sense, where $a_i := \sum_{j=1}^{s} a_{ij}$, $\tilde{a}_i := \sum_{j=1}^{s} \tilde{a}_{ij}$, $b_i := \sum_{j=1}^{s} b_{ij}$, $\tilde{b}_i := \sum_{j=1}^{s} \tilde{b}_{ij}$.

Proof. For simplicity,

$$\tilde{\underline{\mathbf{F}}}_i = \begin{pmatrix} \sum_{j=1}^{s} a_{ij} \underline{f} \\ -\sum_{j=1}^{s} \tilde{a}_{ij} \underline{\mathcal{H}}_y \end{pmatrix} = \sum_{j=1}^{s} a_{ij} \underline{\mathbf{f}}^0 + \sum_{j=1}^{s} \tilde{a}_{ij} (-\underline{\boldsymbol{\mathcal{H}}}_y^0) \quad (1 \le i \le s+1),$$

where

$$\underline{\mathbf{f}}^0 = \begin{pmatrix} \underline{f} \\ 0 \end{pmatrix}, \quad \underline{\boldsymbol{\mathcal{H}}}_y^0 = \begin{pmatrix} 0 \\ -\underline{\mathcal{H}}_y \end{pmatrix},$$

and

$$\tilde{\mathbf{H}}_i = \begin{pmatrix} \sum_{j=1}^{s} b_{ij} \mathbf{h} \\ \sum_{j=1}^{s} \tilde{b}_{ij} \mathbf{hq} \end{pmatrix} = \sum_{j=1}^{s} b_{ij} \mathbf{h}^0 + \sum_{j=1}^{s} \tilde{b}_{ij} (\mathbf{hq})^0 \quad (1 \le i \le s+1),$$

with

$$\mathbf{h}^0 = \begin{pmatrix} h \\ 0 \end{pmatrix}, \quad (\mathbf{hq})^0 = \begin{pmatrix} 0 \\ hq \end{pmatrix},$$

and

$$(\underline{\tilde{\mathbf{F}}}_i')_j = a_{ij}(\underline{\mathbf{f}}^0)' + \tilde{a}_{ij}((\underline{\boldsymbol{\mathcal{H}}}_y)^0)'.$$

First, let us equate the following equations in order to derive the proposed weak order-1 conditions of our Runge-Kutta scheme for stochastic optimal control problems:

$$\boldsymbol{\mathcal{L}}_\Delta \boldsymbol{\Phi}(\mathbf{A}) = \frac{\partial \boldsymbol{\Phi}}{\partial \mathbf{X}} \left(\frac{\partial \mathbf{A}}{\partial u} + \frac{1}{2} \frac{\partial^2 \mathbf{A}}{\partial \tilde{W}^2} \right) + \frac{\partial^2 \boldsymbol{\Phi}}{\partial \mathbf{X}^2} \frac{1}{2} \left(\frac{\partial \mathbf{A}}{\partial \tilde{W}} \right)^2$$

and

$$\boldsymbol{\mathcal{L}} \boldsymbol{\Phi} = \frac{\partial \boldsymbol{\Phi}}{\partial \mathbf{X}} \left(\underline{\mathbf{F}} + \frac{1}{2} \mathbf{H}' \mathbf{H} \right) + \frac{1}{2} \frac{\partial^2 \boldsymbol{\Phi}}{\partial \mathbf{X}^2} \mathbf{H} \mathbf{H}.$$

i) We have

$$\begin{aligned} &\frac{\partial \mathbf{A}}{\partial u} \left(\boldsymbol{\zeta}(t), u, \tilde{W} \right) = \underline{\tilde{\mathbf{F}}}(\boldsymbol{\zeta}(t)) + u \underline{\tilde{\mathbf{F}}}'(\boldsymbol{\zeta}(t)) \frac{\partial \boldsymbol{\zeta}(t)}{\partial u} + \tilde{W} \tilde{\mathbf{H}}'(\boldsymbol{\zeta}(t)) \frac{\partial \boldsymbol{\zeta}(t)}{\partial u}, \\ &\frac{\partial \mathbf{A}}{\partial u} \left(\mathbf{X}(t_0), 0, 0 \right) = \sum_{i=1}^{s} \alpha_i \underline{\mathbf{F}}(\mathbf{X}(t_0)). \end{aligned}$$

Therefore,

$$\sum_{i=1}^{s} \alpha_i \underline{\mathbf{F}}(\mathbf{X}(t_0)) = \underline{\mathbf{F}}(\mathbf{X}(t_0)) \quad \Rightarrow \quad \sum_{i=1}^{s} \alpha_i = 1 \quad (\text{Condition 1. in Theorem 6.1}).$$

ii) The partial derivative of $\mathbf{A}$ with respect to $\tilde{W}$ is found to be:

$$\begin{aligned} &\frac{\partial \mathbf{A}}{\partial \tilde{W}} \left(\boldsymbol{\zeta}(t), u, \tilde{W} \right) = u \underline{\tilde{\mathbf{F}}}'(\boldsymbol{\zeta}(t)) \frac{\partial \boldsymbol{\zeta}(t)}{\partial \tilde{W}} + \tilde{\mathbf{H}}(\boldsymbol{\zeta}(t)) + \tilde{W} \tilde{\mathbf{H}}'(\boldsymbol{\zeta}(t)) \frac{\partial \boldsymbol{\zeta}(t)}{\partial \tilde{W}} \\ &\Rightarrow \quad \frac{\partial \mathbf{A}}{\partial \tilde{W}} \left(\mathbf{X}(t_0) \right) = \sum_{i=1}^{s} \beta_i \mathbf{H}(\mathbf{X}(t_0)). \end{aligned}$$

Herewith,

$$\left(\sum_{i=1}^{s} \beta_i \mathbf{H}(\mathbf{X}(t_0)) \right)^2 = \mathbf{H}(\mathbf{X}(t_0)) \mathbf{H}(\mathbf{X}(t_0)) \quad \Rightarrow \quad \sum_{i=1}^{s} \beta_i = 1$$

$$(\text{Condition 2. in Theorem 6.1}).$$

iii) The second-order partial derivative of $\mathbf{A}$ with respect to $\tilde{W}$ is:

$$\begin{aligned}\frac{\partial^2 \mathbf{A}}{\partial \tilde{W}^2}\left(\boldsymbol{\zeta}(t), u, \tilde{W}\right) =& u\left(\tilde{\underline{\mathbf{F}}}''(\boldsymbol{\zeta}(t))\left(\frac{\partial \boldsymbol{\zeta}(t)}{\partial \tilde{W}}\right)^2 + \tilde{\underline{\mathbf{F}}}'(\boldsymbol{\zeta}(t))\frac{\partial^2 \boldsymbol{\zeta}(t)}{\partial \tilde{W}^2}\right) \\ &+ \tilde{\mathbf{H}}'(\boldsymbol{\zeta}(t))\frac{\partial \boldsymbol{\zeta}(t)}{\partial \tilde{W}} + \tilde{\mathbf{H}}'(\boldsymbol{\zeta}(t))\frac{\partial \boldsymbol{\zeta}(t)}{\partial \tilde{W}} \\ &+ \tilde{W}\left(\tilde{\mathbf{H}}''(\boldsymbol{\zeta}(t))\left(\frac{\partial \boldsymbol{\zeta}(t)}{\partial \tilde{W}}\right)^2 + \tilde{\mathbf{H}}'(\boldsymbol{\zeta}(t))\frac{\partial^2 \boldsymbol{\zeta}(t)}{\partial \tilde{W}^2}\right).\end{aligned} \tag{6.5}$$

Since

$$\frac{\partial \boldsymbol{\zeta}(t_0)}{\partial \tilde{W}} = \tilde{\mathbf{H}}(\boldsymbol{\zeta}(t_0)),$$

we have

$$\begin{aligned}\frac{\partial^2 \mathbf{A}}{\partial \tilde{W}^2}(\mathbf{X}(t_0)) &= 2\sum_{i,j=1}^{s} \beta_i \mathbf{H}'(\mathbf{X}(t_0))\tilde{\mathbf{H}}(\mathbf{X}(t_0)) \\ &= 2\sum_{i=1}^{s} \beta_i \mathbf{H}'(\mathbf{X}(t_0)) \sum_{j=1}^{s}\left(b_{ij}\mathbf{h}^0 + \tilde{b}_{ij}(\mathbf{hq})^0\right)(\mathbf{X}(t_0)).\end{aligned}$$

If

$$\sum_{i,j=1}^{s} \beta_i b_{ij} = \sum_{i,j=1}^{s} \beta_i \tilde{b}_{ij},$$

then

$$\begin{aligned}&2\sum_{i=1}^{s} \beta_i b_{ij}\mathbf{H}'(\mathbf{X}(t_0))\mathbf{H}(\mathbf{X}(t_0)) = \mathbf{H}'(\mathbf{X}(t_0))\mathbf{H}(\mathbf{X}(t_0)) \\ &\quad \Rightarrow \quad \sum_{i,j=1}^{s} \beta_i b_{ij} = \sum_{i,j=1}^{s} \beta_i \tilde{b}_{ij} = \frac{1}{2} \quad \text{(Condition 3. in Theorem 6.1)}.\end{aligned}$$

Now, by comparing the following equations, we can reach the proposed weak order-2 conditions of our Runge-Kutta scheme for stochastic optimal control problems:

$$\boldsymbol{\mathcal{L}}^2\boldsymbol{\Phi}(\mathbf{X}(t_0)) = \boldsymbol{\mathcal{L}}_{\Delta}^2\boldsymbol{\Phi}(\mathbf{A}(\mathbf{X}(t_0))).$$

i) Let us consider the terms including $\partial\boldsymbol{\Phi}/\partial\mathbf{X}$.

The first term is:

$$\begin{aligned}\frac{\partial^2 \mathbf{A}}{\partial u^2}\left(\boldsymbol{\zeta}(t), u, \tilde{W}\right) =& \underline{\mathbf{F}}'(\boldsymbol{\zeta}(t))\frac{\partial \boldsymbol{\zeta}(t)}{\partial u} + \underline{\mathbf{F}}'(\boldsymbol{\zeta}(t))\frac{\partial \boldsymbol{\zeta}(t)}{\partial u} \\ &+ u\left(\underline{\mathbf{F}}''(\boldsymbol{\zeta}(t))\left(\frac{\partial \boldsymbol{\zeta}(t)}{\partial u}\right)^2 + \underline{\mathbf{F}}'(\boldsymbol{\zeta}(t))\frac{\partial^2 \boldsymbol{\zeta}(t)}{\partial u^2}\right) \\ &+ \tilde{W}\left(\underline{\mathbf{H}}''(\boldsymbol{\zeta}(t))\left(\frac{\partial \boldsymbol{\zeta}(t)}{\partial u}\right)^2 + \underline{\mathbf{H}}'(\boldsymbol{\zeta}(t))\frac{\partial^2 \boldsymbol{\zeta}(t)}{\partial u^2}\right).\end{aligned}$$

Since

$$\frac{\partial \boldsymbol{\zeta}(t_0)}{\partial u} = \tilde{\underline{\mathbf{F}}}(\boldsymbol{\zeta}(t_0)),$$

we have

$$\begin{aligned}\frac{\partial^2 \mathbf{A}}{\partial u^2}\left(\mathbf{X}(t_0)\right) &= 2\sum_{i,j=1}^{s} \alpha_i \underline{\mathbf{F}}'(\mathbf{X}(t_0))\tilde{\underline{\mathbf{F}}}(\mathbf{X}(t_0)) \\ &= 2\sum_{i=1}^{s} \alpha_i \underline{\mathbf{F}}'(\mathbf{X}(t_0))\sum_{j=1}^{s}\left(a_{ij}\mathbf{f}^0 + \tilde{a}_{ij}\underline{\boldsymbol{\mathcal{H}}}_y^0\right)\mathbf{X}(t_0).\end{aligned}$$

If

$$\sum_{i,j=1}^{s} \alpha_i a_{ij} = \sum_{i,j=1}^{s} \alpha_i \tilde{a}_{ij},$$

then

$$\begin{aligned}&2\sum_{i=1}^{s} \alpha_i a_{ij}\underline{\mathbf{F}}'(\mathbf{X}(t_0))\underline{\mathbf{F}}(\mathbf{X}(t_0)) = \underline{\mathbf{F}}'(\mathbf{X}(t_0))\underline{\mathbf{F}}(\mathbf{X}(t_0)) \\ &\Rightarrow \quad \sum_{i,j=1}^{s} \alpha_i a_{ij} = \sum_{i,j=1}^{s} \alpha_i \tilde{a}_{ij} = \frac{1}{2} \quad \text{(Condition 4. in Theorem 6.1)}.\end{aligned}$$

The second term can be obtained by taking the partial derivative of Eqn. (6.5) with respect to u:

$$\begin{aligned}\frac{\partial^3 \mathbf{A}}{\partial \tilde{W}^2 \partial u}\left(\boldsymbol{\zeta}(t), u, \tilde{W}\right) =& \left(\tilde{\underline{\mathbf{F}}}''(\boldsymbol{\zeta}(t))\left(\frac{\partial \boldsymbol{\zeta}(t)}{\partial \tilde{W}}\right)^2 + \tilde{\underline{\mathbf{F}}}'(\boldsymbol{\zeta}(t))\frac{\partial^2 \boldsymbol{\zeta}(t)}{\partial \tilde{W}^2}\right) \\ &+ u\left(\tilde{\underline{\mathbf{F}}}'''(\boldsymbol{\zeta}(t))\frac{\partial \boldsymbol{\zeta}(t)}{\partial u}\left(\frac{\partial \boldsymbol{\zeta}(t)}{\partial \tilde{W}}\right)^2 + 2\tilde{\underline{\mathbf{F}}}''(\boldsymbol{\zeta}(t))\frac{\partial \boldsymbol{\zeta}(t)}{\partial \tilde{W}}\frac{\partial^2 \boldsymbol{\zeta}(t)}{\partial \tilde{W}\partial u}\right)\end{aligned}$$

$$+ u\left(\tilde{\underline{\mathbf{F}}}''(\boldsymbol{\zeta}(t))\frac{\partial\boldsymbol{\zeta}(t)}{\partial u}\frac{\partial^2\boldsymbol{\zeta}(t)}{\partial\tilde{W}^2} + \tilde{\underline{\mathbf{F}}}'(\boldsymbol{\zeta}(t))\frac{\partial^3\boldsymbol{\zeta}(t)}{\partial\tilde{W}^2\partial u}\right)$$
$$+ 2\left(\tilde{\mathbf{H}}''(\boldsymbol{\zeta}(t))\frac{\partial\boldsymbol{\zeta}(t)}{\partial\tilde{W}}\frac{\partial\boldsymbol{\zeta}(t)}{\partial u} + \tilde{\mathbf{H}}'(\boldsymbol{\zeta}_{ki})\frac{\partial^2\boldsymbol{\zeta}(t)}{\partial\tilde{W}\partial u}\right)$$
$$+ \tilde{W}\left(\tilde{\mathbf{H}}'''(\boldsymbol{\zeta}(t))\frac{\partial\boldsymbol{\zeta}(t)}{\partial u}\left(\frac{\partial\boldsymbol{\zeta}(t)}{\partial\tilde{W}}\right)^2 + 2\tilde{\mathbf{H}}''(\boldsymbol{\zeta}(t))\frac{\partial\boldsymbol{\zeta}(t)}{\partial\tilde{W}}\frac{\partial^2\boldsymbol{\zeta}(t)}{\partial u\partial\tilde{W}}\right)$$
$$+ \tilde{W}\left(\tilde{\mathbf{H}}''(\boldsymbol{\zeta}(t))\frac{\partial^2\boldsymbol{\zeta}(t)}{\partial\tilde{W}^2} + \mathbf{H}'(\boldsymbol{\zeta}(t))\frac{\partial^3\boldsymbol{\zeta}(t)}{\partial\tilde{W}^2\partial u}\right).$$

Since

$$\frac{\partial^2\boldsymbol{\zeta}(t_0)}{\partial u\partial\tilde{W}} = \tilde{\underline{\mathbf{F}}}'(\boldsymbol{\zeta}(t_0))\frac{\partial\boldsymbol{\zeta}(t_0)}{\partial\tilde{W}} + \tilde{\mathbf{H}}'(\boldsymbol{\zeta}(t_0))\frac{\partial\boldsymbol{\zeta}(t_0)}{\partial u},$$

and

$$\frac{\partial^2\boldsymbol{\zeta}(t_0)}{\partial\tilde{W}^2} = \tilde{\mathbf{H}}'(\boldsymbol{\zeta}(t_0))\frac{\partial\boldsymbol{\zeta}(t_0)}{\partial\tilde{W}} + \tilde{\mathbf{H}}'(\boldsymbol{\zeta}(t_0))\frac{\partial\boldsymbol{\zeta}(t_0)}{\partial\tilde{W}},$$

we have

$$\begin{aligned}\frac{\partial^3\mathbf{A}}{\partial\tilde{W}^2\partial u}(\mathbf{X}(t_0)) &= \sum_{i,j,l=1}^{s}\alpha_i\left(\underline{\mathbf{F}}''\tilde{\mathbf{H}}\tilde{\mathbf{H}} + 2\underline{\mathbf{F}}'\tilde{\mathbf{H}}'\tilde{\mathbf{H}}\right)\\ &\quad + 2\sum_{i=1}^{s}\beta_i\left(\mathbf{H}''\tilde{\mathbf{H}}\tilde{\underline{\mathbf{F}}} + \tilde{\underline{\mathbf{F}}}'\mathbf{H}'\tilde{\mathbf{H}} + \mathbf{H}'\tilde{\mathbf{H}}'\tilde{\underline{\mathbf{F}}}\right)(\mathbf{X}(t_0))\\ &= \left(\frac{1}{2}\underline{\mathbf{F}}''\mathbf{H}\mathbf{H} + \frac{1}{2}\underline{\mathbf{F}}'\mathbf{H}'\mathbf{H} + \frac{1}{2}\mathbf{H}\underline{\mathbf{F}}\mathbf{H}'' + \frac{1}{2}\mathbf{H}'\mathbf{H}'\underline{\mathbf{F}}\right)(\mathbf{X}(t_0)).\end{aligned}$$

So that, Conditions 5. and 10. in Theorem 6.1 can be easily deduced from the aforementioned equation.

Now, consider the third term $\partial^4\mathbf{A}/\partial\tilde{W}^4$:

$$\begin{aligned}\frac{\partial^3\mathbf{A}}{\partial\tilde{W}^3}\left(\boldsymbol{\zeta}(t),u,\tilde{W}\right) =& u\left(\underline{\mathbf{F}}'''(\boldsymbol{\zeta}(t))\left(\frac{\partial\boldsymbol{\zeta}(t)}{\partial\tilde{W}}\right)^3 + 2\underline{\mathbf{F}}''(\boldsymbol{\zeta}(t))\frac{\partial\boldsymbol{\zeta}(t)}{\partial\tilde{W}}\frac{\partial^2\boldsymbol{\zeta}(t)}{\partial\tilde{W}^2}\right)\\ &+ \left(\mathbf{H}''(\boldsymbol{\zeta}(t))\left(\frac{\partial\boldsymbol{\zeta}(t)}{\partial\tilde{W}}\right)^2 + \mathbf{H}'(\boldsymbol{\zeta}(t))\frac{\partial^2\boldsymbol{\zeta}(t)}{\partial\tilde{W}^2}\right)\\ &+ \tilde{W}\left(\mathbf{H}'''(\boldsymbol{\zeta}(t))\left(\frac{\partial\boldsymbol{\zeta}(t)}{\partial\tilde{W}}\right)^3 + \mathbf{H}''(\boldsymbol{\zeta}(t))\frac{\partial\boldsymbol{\zeta}(t)}{\partial\tilde{W}}\frac{\partial^2\boldsymbol{\zeta}(t)}{\partial\tilde{W}^2}\right)\\ &+ \tilde{W}\left(\mathbf{H}''(\boldsymbol{\zeta}(t))\frac{\partial\boldsymbol{\zeta}(t)}{\partial\tilde{W}}\frac{\partial^2\boldsymbol{\zeta}(t)}{\partial\tilde{W}^2} + \mathbf{H}'(\boldsymbol{\zeta}(t))\frac{\partial^3\boldsymbol{\zeta}(t)}{\partial\tilde{W}^3}\right),\end{aligned}$$

which implies

$$
\begin{aligned}
&\frac{\partial^4 \mathbf{A}}{\partial \tilde{W}^4}\left(\mathbf{X}(t_0)\right) \\
&\qquad = \sum_{i,j,l=1}^{s} \beta_i \left(\tilde{\mathbf{H}}\tilde{\mathbf{H}}\tilde{\mathbf{H}}\tilde{\mathbf{H}}''' + 6\tilde{\mathbf{H}}''\tilde{\mathbf{H}}\tilde{\mathbf{H}}'\tilde{\mathbf{H}} + 3\tilde{\mathbf{H}}'\tilde{\mathbf{H}}''\tilde{\mathbf{H}}\tilde{\mathbf{H}} + 6\tilde{\mathbf{H}}'\tilde{\mathbf{H}}'\tilde{\mathbf{H}}'\tilde{\mathbf{H}}\right)(\mathbf{X}(t_0)) \\
&\qquad = \frac{1}{4}\left(\mathbf{HHHH}''' + \mathbf{H}''\mathbf{HH}'\mathbf{H} + 3\mathbf{H}''\mathbf{H}'\mathbf{HH} + \mathbf{H}'\mathbf{H}'\mathbf{H}'\mathbf{H}\right)(\mathbf{X}(t_0)). \qquad (6.6)
\end{aligned}
$$

Hence, Conditions 11. and 14. in Theorem 6.1 can be obtained from Eqn. (6.6).

ii) Let us consider the terms including $\partial^2\mathbf{\Phi}/\partial\mathbf{X}^2$.

The partial derivative,

$$
\begin{aligned}
2\frac{\partial \mathbf{A}}{\partial \tilde{W}}\frac{\partial^2 \mathbf{A}}{\partial u \partial \tilde{W}}(\mathbf{X}(t_0)) &= 2\sum_{i,j=1}^{s} \beta_i (\mathbf{H}\tilde{\underline{\mathbf{F}}}'\tilde{\mathbf{H}} + \mathbf{H}\tilde{\mathbf{H}}'\tilde{\underline{\mathbf{F}}})(\mathbf{X}(t_0)) \\
&= (\mathbf{H}\underline{\mathbf{F}}'\mathbf{H} + \mathbf{H}\mathbf{H}'\underline{\mathbf{F}})(\mathbf{X}(t_0)),
\end{aligned}
$$

yields Conditions 15. and 16. in Theorem 6.1.

Moreover, from the following equation

$$
\begin{aligned}
\frac{\partial \mathbf{A}}{\partial \tilde{W}}\frac{\partial^3 \mathbf{A}}{\partial \tilde{W}^3}(\mathbf{X}(t_0)) &= 3\sum_{i,j=1}^{s} \beta_i (\mathbf{H}\mathbf{H}''\tilde{\mathbf{H}}\tilde{\mathbf{H}} + \mathbf{H}\mathbf{H}'\tilde{\mathbf{H}}'\tilde{\mathbf{H}})(\mathbf{X}(t_0)) \\
&= (\mathbf{H}\mathbf{H}''\mathbf{H}\mathbf{H} + \mathbf{H}\mathbf{H}'\mathbf{H}'\mathbf{H})(\mathbf{X}(t_0)),
\end{aligned}
$$

we obtain Conditions 17. and 18. in Theorem 6.1.

By performing a similar procedures for the other terms, one can obtain weak-order conditions, which are the same conditions that we have already obtained for our Runge-Kutta scheme for stochastic optimal control problems.

□

We note that the first equations in Theorem 6.1, which do not include Runge-Kutta coefficients of $\tilde{a}_{ij}$ and $\tilde{a}_{ij}$, constitute the weak order-1 and weak order-2 conditions of Runge-Kutta method for SDEs. These conditions are derived by [37, 52] for different Runge-Kutta scheme for Stratonovich SDEs. The out of first 17 conditions, which we find in Theorem 6.1 are the additional conditions because of the stochastic optimal control problem.

At this point, the fundamental issue is to construct a family of methods satisfying weak order-1 and order-2 conditions, respectively. It is not surprising that weak order-1 conditions and strong order-1 conditions are the same, since strong convergence implies weak convergence. In the case of $s = 2$ with an explicit scheme for the state equation and a related implicit scheme for the adjoint equation, we have 4 conditions for weak order-1, and there are 6 unknowns. Thus, free parameters guarantee the existence of a solution. Therefore, one can find different methods (i.e., particular coefficients) which satisfy weak order-1 conditions for $s \geq 2$. For example, the following tableaus fulfill weak order-1 conditions:

$$\begin{array}{c|cc} & 0 & 0 \\ & 1 & 0 \\ \hline & 0.5 & 0.5 \end{array} \qquad \begin{array}{c|cc} & 0 & 0 \\ & 1 & 0 \\ \hline & 0.5 & 0.5 \end{array} \qquad \text{(for the state equation)},$$

$$\begin{array}{c|cc} & 0.5 & -0.5 \\ & 0.5 & 0.5 \\ \hline & 0.5 & 0.5 \end{array} \qquad \begin{array}{c|cc} & 0.5 & -0.5 \\ & 0.5 & 0.5 \\ \hline & 0.5 & 0.5 \end{array} \qquad \text{(for the adjoint equation)}$$

and

$$\begin{array}{c|cc} & 1/4 & 1/4 \\ & 1/4 & 1/4 \\ \hline & 1/2 & 1/2 \end{array} \qquad \begin{array}{c|cc} & 1/4 & 1/4 \\ & 1/4 & 1/4 \\ \hline & 1/2 & 1/2 \end{array} \qquad \text{(for the state equation)},$$

$$\begin{array}{c|cc} & 1/4 & 1/4 \\ & 1/4 & 1/4 \\ \hline & 1/2 & 1/2 \end{array} \qquad \begin{array}{c|cc} & 1/4 & 1/4 \\ & 1/4 & 1/4 \\ \hline & 1/2 & 1/2 \end{array} \qquad \text{(for the adjoint equation)}.$$

As for the weak order-2, in [14, 37] it is shown that the classical Runge-Kutta method requires $s \geq 4$ in the explicit case. In our Runge-Kutta method, we have more than 50 equations which need to be fulfilled, such that $s \geq 5$ is needed. One can make use of MAPLE to solve these equations. However, we mainly focus on the deriavation of Runge-Kutta method for stochastic optimal control problems and investigating the convergence of the solution, herewith showing a way how such conditions can be achieved.

6.4 Summary

In this chapter, we again used the Stratonovich form of both continuous and discrete optimality systems. However, to obtain weak-order conditions of our Runge-Kutta scheme for stochastic optimal control problems, this time, we made use of the Itô Formula to expand stochastic Taylor-series for the exact solution of continuous optimality system and the approximate solution of our discrete optimality system. After taking the expectation of stochastic Taylor series, we compared these expansions. Hence, we succeeded to get weak order-1 and weak order-2 conditions of our Runge-Kutta scheme for stochastic optimal control problems.

CHAPTER 7

CONCLUSION AND OUTLOOK

In this thesis, we proposed a Runge-Kutta method for numerical solution of stochastic optimal control problems based on Pontryagin's Maximum Principle. In Chapter 3, we derived such a method for stochastic optimal control problems of SDEs. We followed *discretize-then-optimize* approach. After we presented a Runge-Kutta discretization for both cost functional and state equation, we introduced discrete Lagrangian for our discrete optimal control problem. By taking the partial derivatives of the discrete Lagrangian with respect to its variables, we achieved the discrete optimality system of our stochastic optimal control problem. The main advantage of our method is that a Runge-Kutta discretization of adjoint pair is derived and Runge-Kutta coefficients of adjoint pair are obtained in terms of Runge-Kutta coefficients of the state equation. In order to test our Runge-Kutta scheme, some examples were selected from the financial sector and a comparison with simulation made for the exact solution was illustrated. Numerical results also revealed that our Runge-Kutta scheme is more efficient in terms of time consumption when compared to Euler scheme.

We also derived a Runge-Kutta method for the numerical solution of stochastic control problems of some SPDEs in Chapter 4. We chose a special, emerging problem, that is an *optimal harvesting* problem. This is an important problem in ensuring food and other organic material for the people of the world, while caring for humankind in a sustainable manner. Such problems exist in agriculture, fisheries, forestry, gardening, tourism, city planing and water management, which are closely associated areas of daily life, modern industries and scientific research. By using the finite difference scheme, we discretized the problem with respect to the space variable and converted the given problem to an optimal control problem of system of SDEs. Then, by following the same methodology as in the SDE case, we were able to derive a Runge-Kutta method on the numerical solution of stochastic control problems subject to system of SDEs.

When dealing with a numerical scheme, the issue of convergence is important in order to judge the quality of the scheme. In stochastic calculus, the desired task can be achieved in two different ways. If sample paths of the solution are subject of interest, *strong convergence* criteria are used. Since it requires the sample paths to be close, the same Brownian motion is used in the simulation. For this reason, we first focused on strong convergence properties of our Runge-Kutta scheme for stochastic optimal control problems in Chapter 5. Because of the simplified nature of Stratonovich calculus, we preferred to use the related Stratonovich form for our stochastic optimal control problem to examine strong convergence properties of our Runge-Kutta method. By following the same methodology as in Itô SDE case, we obtained the discrete optimality system of our problem in the related Stratonovich form. Then, we expanded the exact solution from the continuous optimality system and the approximate solution from our discrete optimality system in Stratonovich-Taylor series. In order to find the strong-order of accuracy, we matched these two Stratonovich-Taylor series expansions by assuming exact initial values. We employed the mean-square convergence since mean-square convergence implies the strong convergence as a result of Jensen's inequality. We were able to obtain strong order-1 conditions of our Runge-Kutta scheme for stochastic optimal control problems. We also illustrated why we can not exceed strong order-1. Since it is not possible to get a 0 error from coefficients of order-1.5 terms, which constitute the principal truncation error constants, we minimized the error constants to obtain a good method which converges strongly to order-1. Thereafter, by using the idea that each random variable can be written as a linear combination of 2 or more random variables that have the same order with mentioned random variable and can be stated as in terms of multiple Stratonovich integrals, we reformulated our problem and we achieved strong order-1.5 conditions of our Runge-Kutta scheme for stochastic optimal control programs. In our Runge-Kutta scheme for stochastic optimal control problems, Runge-Kutta coefficients of the adjoint process were obtained in terms of the Runge-Kutta coefficients of the state process. This caused additional order conditions to the classical Runge-Kutta method of SDEs for the strong-order of accuracy. We derived such order conditions explicitly. Eventually, we verified our results in numerical examples.

If one deals with only the probabilistic aspects of the solution or some moments, it is more appropriate to employ a much weaker condition: *weak convergence* criteria. In this case, different Brownian motions or even random processes which have similar moment properties with Brownian motions can be used in each numerical solution. In Chapter 6, we paid attention to weak convergence properties of our Runge-Kutta scheme for stochastic optimal control problems. As in Chapter 5, we used the Stratonovich form of both continuous and discrete optimality systems. However, we made use of the Itô Formula to expand the expectation of stochastic Taylor-series for the exact solution of a continuous optimality system and the approximate solution of our discrete optimality system to find weak-order accuracy. In Chapter 3, with our Runge-Kutta scheme for stochastic optimal control problems, we show that Runge-Kutta discretization of the adjoint process is often different from the Runge-Kutta discretization of the state process. Herewith, there occur additional weak-order conditions to classical Runge-Kutta conditions of SDEs for the weak-order of accuracy.

As a further study, in the formulation of stochastic optimal control problem, we can consider control processes in diffusion process. Moreover, a stochastic optimal control problem of a coupled state equation can be investigated, too. Since we use Pontrayagin's Maximum Principle, we can also allow the existence of delays in the stochastic optimal control problem, to find excellent theoretical results in such a wider framework with delay and to use them for numerical solution procedures. Herewith, a Runge-Kutta method for stochastic optimal control with delay can be another research direction. Furthermore, jumps and regime switching dynamics may be introduced into stochastic optimal control of SDEs in order to propose a Runge-Kutta scheme. We can permit the existence of control processes in the jump term, too; then we may speak of a Runge-Kutta scheme of impulse control. Finally, strong and weak convergence of the proposed future research and application can be investigated.

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