PERFECT DISCRETE MORSE FUNCTIONS ON CONNECTED SUMS

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

HANİFE VARLI

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY
IN
MATHEMATICS

OCTOBER 2017

Approval of the thesis:

PERFECT DISCRETE MORSE FUNCTIONS ON CONNECTED SUMS

submitted by **HANİFE VARLI** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy in Mathematics Department, Middle East Technical University** by,

Prof. Dr. Gülbin Dural Ünver
Dean, Graduate School of **Natural and Applied Sciences** ____________

Prof. Dr. Mustafa Korkmaz
Head of Department, **Mathematics** ____________

Assoc. Prof. Dr. Mehmetcik Pamuk
Supervisor, **Mathematics Department, METU** ____________

Prof. Dr. Neža Mramor Kosta
Co-supervisor, **Faculty of Computer and Information Science and Institute of Math., Phys. and Mech., Univ. of Ljubljana** ____________

Examining Committee Members:

Prof. Dr. Ali Sinan Sertöz
Mathematics Department, Bilkent University ____________

Assoc. Prof. Dr. Mehmetcik Pamuk
Mathematics Department, METU ____________

Prof. Dr. Hurşit Önsiper
Mathematics Department, METU ____________

Assoc. Prof. Dr. Ali Ulaş Özgür Kişisel
Mathematics Department, METU ____________

Assoc. Prof. Dr. Mesut Şahin
Mathematics Department, Hacettepe University ____________

Date: ____________

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last Name: HANİFE VARLI

Signature :

ABSTRACT

PERFECT DISCRETE MORSE FUNCTIONS ON CONNECTED SUMS

Varlı, Hanife
Ph.D., Department of Mathematics
Supervisor : Assoc. Prof. Dr. Mehmetcik Pamuk
Co-Supervisor : Prof. Dr. Neža Mramor Kosta

October 2017, 61 pages

Let K be a finite, regular cell complex and f be a real valued function on K. Then f is called a *discrete Morse function* if for all p-cell $\sigma \in K$, the following conditions hold:

$$n_1 = \#\{\tau > \sigma \mid f(\tau) \leq f(\sigma)\} \leq 1,$$
$$n_2 = \#\{\nu < \sigma \mid f(\nu) \geq f(\sigma)\} \leq 1.$$

A p-cell σ is called a *critical p-cell* if $n_1 = n_2 = 0$. A discrete Morse function f is called a *perfect discrete Morse function* if the number of critical p-cells of f equals to the p-th Betti number of K with reference to the coefficient group.

The main purpose of this thesis is to compose and decompose perfect discrete Morse functions on connected sums of closed, connected manifolds. We will first discuss the existence of perfect discrete Morse functions on finite complexes and closed, connected, triangulated n-manifolds.

Secondly, we will show that if the components of a connected sum M of closed, connected, triangulated n-manifolds admit a perfect discrete Morse function, then M admits a perfect discrete Morse function that coincides with the perfect discrete Morse functions on the components.

Next, we will find a separating sphere on a connected sum M of closed, connected, triangulated surfaces and 3-manifolds if M admits a perfect discrete Morse function

f. Finally, we will prove that f can be decomposed as perfect discrete Morse functions on each component of M after some local modifications of it.

Keywords: Perfect Discrete Morse Functions, Gradient Vector Fields, Connected Sums.

ÖZ

BAĞLANTILI TOPLAMLARDA MÜKEMMEL AYRIK MORSE FONKSİYONLARI

Varlı, Hanife
Doktora, Matematik Bölümü
Tez Yöneticisi : Doç. Dr. Mehmetcik Pamuk
Ortak Tez Yöneticisi : Prof. Dr. Neža Mramor Kosta

Ekim 2017 , 61 sayfa

K bir sonlu, düzgün hücre kompleksi ve f, K üzerinde tanımlı reel değerli bir fonksiyon olsun. Eğer bütün p-hücre $\sigma \in K$ için aşağıdaki şartlar sağlanırsa, f fonksiyonuna ayrık Morse fonksiyonu denir:

$$n_1 = \#\{\tau > \sigma \mid f(\tau) \leq f(\sigma)\} \leq 1,$$
$$n_2 = \#\{\nu < \sigma \mid f(\nu) \geq f(\sigma)\} \leq 1.$$

Eğer $n_1 = n_2 = 0$, bu durumda σ p-hücresine kritik p-hücre denir. Eğer bir f ayrık Morse fonksiyonunun kritik p-hücrelerinin sayısı K kompleksinin katsayı grubuna göre hesaplanan p'inci Betti sayısına eşitse, f fonksiyonuna mükemmel ayrık Morse fonksiyonu denir.

Bu tezin asıl amacı kapalı ve bağlantılı manifoldların bağlantılı toplamları üzerinde mükemmel ayrık Morse fonksiyonlarının nasıl oluşturulduğunu ve ayrıştırıldığını göstermektir. İlk olarak sonlu kompleksler ve kapalı, bağlantılı, üçgenleştirilmiş n-manifoldlar üzerinde mükemmel ayrık Morse fonksiyonlarının varlığını tartışacağız.

İkinci olarak, eğer kapalı, bağlantılı ve üçgenleştirilmiş n-manifoldların bir bağlantılı toplamı M içindeki bileşenler bir mükemmel ayrık Morse fonksiyonu içerirse, M'nin, bileşenleri üzerindeki mükemmel ayrık Morse fonksiyonları ile örtüşen, bir mükemmel ayrık Morse fonksiyonu içerdiğini göstereceğiz.

Daha sonra eğer kapalı, bağlantılı, üçgenleştirilmiş yüzeylerin ve 3-manifoldların bağlantılı toplamı bir f mükemmel ayrık Morse fonksiyonu içerirse, M üzerinde, bağlantılı toplamı ayrıştıran bir küre bulacağız. Son olarak f fonksiyonu üzerinde bazı lokal değişikliler yaptıktan sonra, bu fonksiyonun M bağlantılı toplamının herbir bileşeni üzerinde mükemmel ayrık Morse fonksiyonu verecek şekilde ayrıştırılabildiğini göstereceğiz.

Anahtar Kelimeler: Mükemmel Ayrık Morse Fonksiyonları, Yöntürevi Vectör Alanları, Bağlantılı Toplamlar.

To the memory of my mother
&
To my son Enes

ACKNOWLEDGMENTS

First and foremost, I would like to express my deepest thank to my advisor Assoc. Prof. Dr. Mehmetcik Pamuk who accepted me as his PhD student without any hesitation. I would like to thank him for his continuous support, invaluable guidance and patience throughout the PhD process. His support was not limited to the academic issues but extended to my personal life. I am very thankful to him for believing in me and making me courageous about being self-confident. It has been an honor to be his first PhD student.

I would like to thank Neža Mramor Kosta for being my co-advisor. I am very thankful to her not only for face to face studies but also Skype conversations. Her comments were always a source of encouragement. Her influence is felt throughout this thesis. I also want to thank for her hospitality during my visits to the University of Ljubljana and coming to my dissertation defense. Dear Neza, I am really happy to meet you.

I would like to thank the examining committee members. I shall also thank Prof. Dr. Mustafa Korkmaz, Prof. Dr. Turgut Önder, Prof. Dr. Sergey Finashin and Prof. Dr. Cem Tezer for teaching me topology and geometry via several courses. I also want to thank Prof. Dr. Hurşit Önsiper and Prof. Dr. Yıldıray Ozan for answering all my questions whenever I asked. I should not forget to thank Assoc. Prof. Dr. Semra Pamuk for her guidance, support and friendly conversation.

I am thankful to all the members of METU Mathematics family, academic and administrative, who provide a friendly working atmosphere.

Thanks to all the great friends I have made in METU, in particular, to Tülin Altunöz, Elçin Çalışkan, Hatice Ünlü Eroğlu, Dr. Adalet Çengel, Dr. Hatice Çoban, Dr. Sabahattin Ilbıra, Dr. Neslihan Nesliye Pelen and Bengi Ruken Yavuz for many fruitful discussions and their moral support.

I would like to thank Suzan Çınar and my dearest teachers Abdurrahman Hasançebi and Ayşegül Ergin for their support and encouragement throughout my education life.

I must include a special thanks to Prof. Dr. Yusuf Kaya and Ece Gülşah Çolak who guided me to the academic life.

I am also thankful to my brothers Ali İşal, Yusuf İşal and my sister Fatma Asal for their endless love and support through all my life.

This work is financially supported by TÜBİTAK-BİDEB National Graduate Scholarship Programme for PhD (2211)

Last but not least, I wish to express my deepest thanks to my husband, Sehmus for being very supportive at times when I had loads of work to do, giving me hope when I thought I cannot do this any more. I am forever indebted. I cannot forget my son, Enes, who is the love of my life. Enes, I am grateful for your existence.

TABLE OF CONTENTS

LIST OF FIGURES

FIGURES

CHAPTER 1

INTRODUCTION

Marston Morse [29] originally introduced Morse theory for differentiable manifolds. It helps one to examine the topological properties of manifolds via differentiable functions, called Morse functions. Since Morse theory does not enable one to investigate topological properties of discrete objects such as simplicial complexes and cellular complexes, computer-based applications need the discrete version of Morse theory.

In the 1990s, Robin Forman [9] proposed a discrete version of Morse theory which today is called the discrete Morse theory. This theory gives a way of studying the topology of discrete objects via critical cells of discrete Morse functions. A real valued function $f : K \to \mathbb{R}$ on a finite regular CW-complex K is called a **discrete Morse function** if for any p-cell $\alpha \in K$, there exists at most one $(p+1)$-cell $\beta \in K$ containing α such that $f(\beta) \leq f(\alpha)$, and there exists at most one $(p-1)$-cell $\nu \in K$ contained in α such that $f(\alpha) \leq f(\nu)$. A p-cell α is called a **critical** cell of f if $f(\nu) < f(\alpha)$ and $f(\alpha) < f(\beta)$ for all $(p-1)$-cell ν contained in α and $(p+1)$-cell β containing α. A fundamental result of discrete Morse theory is the construction of a CW-complex which is homotopy equivalent to the original complex with a smaller number of cells corresponding to the number of critical cells of a discrete Morse function on a given finite CW-complex and thus a more efficient way to examine the topological properties of the original complex. Thus, an important problem in this theory is to obtain a discrete Morse function on a given complex with the minimal number of critical cells. Such a discrete Morse function is called an **optimal** discrete Morse function.

Optimality of discrete Morse functions are extensively studied in literature [13], [19], [22], [23], [2], [4], [3]. Optimal discrete Morse functions have many applications in topology and geometry [1], [6], [7], [15]. These functions can be used in homology computations [10], [11], and they give effective algorithms for determining persistent homology information, which is important in computational topology and has applications in topological data analysis, image analysis, computer vision and material science [16], [18], [27]. For instance, optimal discrete Morse function are used to obtain topological information of a given combinatorial structure visually [24].

In this thesis, we study optimal discrete Morse functions whose number of critical p-cells equals to the p-th Betti number of the complex. A discrete Morse function that is optimal in this sense is called a **perfect** discrete Morse function.

Organization of this thesis is as follows:

In Chapter 3, we restate and prove several existence results of $\mathbb{F}$-perfect discrete Morse functions on 3-manifolds given in [4] for any dimension n. The proofs for dimension 3 carry over directly to dimension n.

In Chapter 4, we study the existence of $\mathbb{Z}$-perfect discrete Morse functions on a connected sum of closed, connected, oriented, triangulated n-manifolds. The main result of this chapter is as follows:

Theorem 1.0.1. *Let M_1 and M_2 be two n-dimensional closed, connected, oriented, triangulated manifolds, and f_1 and f_2 be the $\mathbb{Z}$-perfect discrete Morse functions on them, respectively. Then there exists a perfect discrete Morse function f on $M = M_1 \# M_2$ which coincides with f_1 and f_2, up to a constant on each summand, except on a neighbourhood of the two n-cells whose interiors are removed to form the connected sum.*

This theorem implies that a connected sum of n-manifolds has a cell decomposition with a minimal number of cells if each component has a cell decomposition with a minimal number of cells.

In Chapter 5, we show that a $\mathbb{Z}$-perfect discrete Morse function on a connected sum of closed, connected, oriented, triangulated surfaces or 3-manifolds can be decomposed into $\mathbb{Z}$-perfect discrete Morse functions on each summand. The main results of this chapter are as follows:

Let $M = M_1 \# M_2$ be a connected sum of two closed, connected, oriented, triangulated surfaces M_1 and M_2 of genera g_1 and g_2, respectively. Let f be a $\mathbb{Z}$-perfect discrete Morse function on M with the discrete gradient vector field V such that $V|_{M-M_2}$ has one critical 0-cell and $2g_1$ many critical 1-cells and $V|_{M-M_1}$ has one critical 2-cell and $2g_2$ many critical 1-cells.

Theorem 1.0.2. *There exist a separating* 1*-sphere* C *on* M *such that* $M = M_1 \#_C M_2$*, and none of the cells on* C *are paired with the cells in* $M - M_1$*.*

This theorem also works for non-orientable surfaces admitting $\mathbb{Z}_2$-perfect discrete Morse function. The proof is constructive and gives an algorithm for finding a separating 1-sphere on a connected sum of surfaces.

Theorem 1.0.3. *Let* C *be a separating* 1*-sphere on* M *such that* $M = M_1 \#_C M_2$ *and* $C \approx \partial(M - M_1) \approx \partial(M - M_2)$*. Assume that none of the cells on* C *are paired with the cells in* $M - M_1$*, and none of the critical cells of* f *in* M *lie on* C*. Then* $V|_{M-M_1}$ *and* $V|_{M-M_2}$ *can be extended to* M_2 *and* M_1*, respectively, as discrete gradient vector fields of perfect discrete Morse functions which agree with* f *on* $M - M_1$ *and* $M - M_2$ *except on a neighbourhood of* C *in* $M - M_1$ *and* $M - M_2$*.*

In this chapter, we also indicate that a $\mathbb{Z}$-perfect discrete Morse function on a connected sum of closed, connected, triangulated, oriented 3-manifolds can be decomposed as a $\mathbb{Z}$-perfect discrete Morse function on each component. Moreover, we describe a way to obtain a separating sphere on a connected sum which decompose M as M_1 and M_2 if M admits a $\mathbb{Z}$-perfect discrete Morse function.

Let $M = M_1 \# M_2$ be a connected sum of closed, connected, oriented, triangulated 3-manifolds, and f be a $\mathbb{Z}$-perfect discrete Morse function on it.

Theorem 1.0.4. *If the spine of* M *induced by the discrete gradient vector field of* f *is a wedge of spines of* M_1 *and* M_2*, then* M_1 *and* M_2 *have* $\mathbb{Z}$*-perfect discrete Morse functions* f_1 *and* f_2*, respectively, such that* f_1 *and* f_2 *agree with* f *on the spines of* M_1 *and* M_2*.*

In general, similar to the case in dimension 2, we have the following theorem on a separating sphere for the connected sum:

Theorem 1.0.5. *We can find a separating sphere S on M such that $M = M_1 \#_S M_2$, and the cells on S are never paired with the cells on $M - M_1$.*

Let $M = M_1 \#_S M_2$ such that $S \approx \partial(M - M_1) \approx \partial(M - M_2)$. Let f be a $\mathbb{Z}$-perfect discrete Morse function on M, and V be the gradient vector field induced by f. Assume that $V|_{M-M_2}$ has one critical 0-cell, $b_1(M_1)$ many critical 1-cells, $b_2(M_1)$ many critical 2-cells, and $V|_{M-M_1}$ has one critical 3-cell, $b_1(M_2)$ many critical 1-cells, $b_2(M_2)$ many critical 2-cells.

Once we have such a separating sphere as in Theorem (1.0.5), we can extend the gradient vector field to each summand:

Theorem 1.0.6. *In addition to the given conditions above, if there are no arrows on the cells of S pointing into $M - M_1$, then we can extend $V|_{M-M_2}$ to M_1 and $V|_{M-M_1}$ to M_2 as discrete gradient vector fields of $\mathbb{Z}$-perfect discrete Morse functions which coincide with V except on a neighbourhood of S in $M - M_1$ and $M - M_2$.*

Remark 1.0.7. *In the theorems given above, we study with $\mathbb{Z}$-perfect discrete Morse functions. But they also work for any suitable field coefficient $\mathbb{F}$.*

CHAPTER 2

BACKGROUND

2.1 Preliminaries

In this thesis, we work with closed, connected, triangulated manifolds. In this section we present some basic definitions and facts on CW-complexes and manifolds. This section is based on [7], [12] and [14].

Definition 2.1.1. *An n-cell σ is a topological space which is homeomorphic to the closed unit n-ball $D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$. The boundary of σ, $\partial\sigma$, is homeomorphic to $\partial D^n = \{x \in \mathbb{R}^n \mid |x| = 1\}$.*

Definition 2.1.2. *A finite CW-complex K is a topological space that can be written as*

$$K = \bigcup_n K^n$$

where K^0 is a finite set of points and K^n, the n-skeleton of K, is obtained by attaching finitely many n-cells σ_α via continuous attaching maps $f_\alpha : \partial\sigma_\alpha \approx S^{n-1} \to K^{n-1}$. That is,

$$K^n = K^{n-1} \coprod_\alpha \sigma_\alpha \Big/ x \sim f_\alpha(x), x \in \partial\sigma_\alpha.$$

A CW-complex is regular if attaching maps are homeomorphisms onto their images.

Let α be an n-cell in a CW-complex K. If ν is a k-cell in $\partial\alpha$ for $k < n$, then we say ν is a face of α, and we use the notation $\nu < \alpha$. We also use the notation $\alpha^{(n)}$ to denote an n-cell α.

Definition 2.1.3. *An n-simplex is a convex hull of a set of $(n+1)$ affinely independent points, which are called the vertices of an n-simplex. A face of a simplex is the convex* *hull of a subset of its vertices.*

A 0-simplex is a vertex, 1-simplex is an edge, 2-simplex is a triangle, 3-simplex is a tetrahedron and so on.

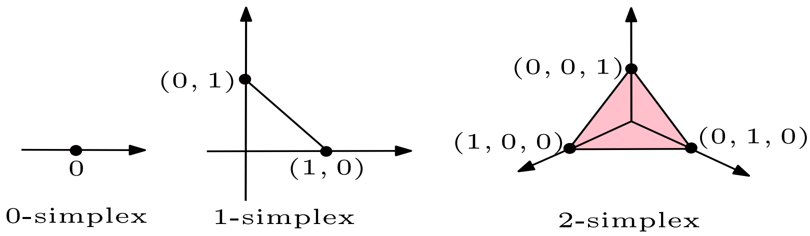

Figure 2.1: From left to right: a vertex, an edge, a triangle.

Definition 2.1.4. *A finite simplicial complex K in $\mathbb{R}^n$ is a finite collection of simplices such that*

1. *Every face of a simplex in K is also in K.*
2. *The intersection of any two simplices in K is a face of each simplex.*

Definition 2.1.5 ([32])**.** *Let K be a regular CW-complex, $\nu^{(n-1)} < \alpha^{(n)}$ be two cells of K and let ν be a free face of α, that is, it is not a face of any other cell. Then we say that K **collapses** to $L = K - \{\nu \cup \alpha\}$ and write $K \searrow L$. A complex K is called **collapsible** if it collapses to a point. The inverse of the collapse operation is called an **expansion**.*

Figure (2.2) is an example for a collapsible complex.

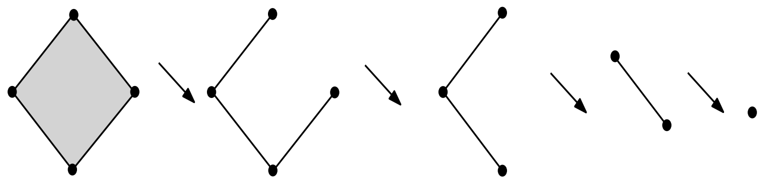

Figure 2.2: Collapsible 2-complex.

Definition 2.1.6. *Two regular CW-complexes are simple homotopy equivalent if there is a map between them which is homotopic to the composition of collapses and expansions.*

For example, a Möbius band and a cylinder are simple homotopy equivalent.

Remark 2.1.7. *Note that there are finite CW-complexes that are homotopy equivalent but not simple homotopy equivalent. As an example, consider the Dunce hat which is contractible but not collapsible ([34]) and a point.*

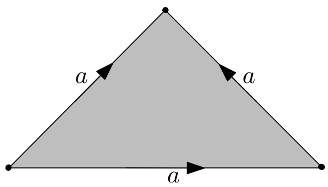

Figure 2.3: Dunce hat.

Next, we give the definition of a bisection which we use in the proofs of main theorems.

Definition 2.1.8 ([9])**.** *A bisection is a subdivision of a single cell in a regular CW-complex into two cells resulting in a new CW-complex with precisely one more cell than the original.*

Figure (2.4) is an example of a sequence of bisections on a 2-cell.

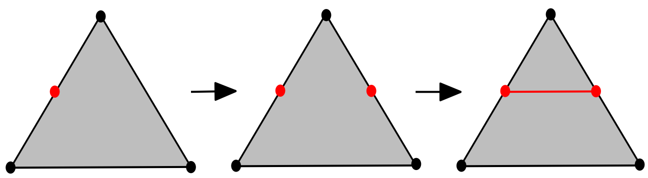
Figure 2.4: A sequence of bisections.

Definition 2.1.9. *Let M be an n-manifold. A triangulation of an n-manifold M is a simplicial complex K such that the union $\mid K \mid$ of its simplices is homeomorphic to M.*

Figure (2.5) is an example of a triangulation on the planer diagram of closed, oriented, genus 2 surface.

Definition 2.1.10. *Let K be a triangulated CW-complex, and $\alpha^{(k)} \in K$. The star of α in K is $St_K(\alpha) = \{\beta \in K | \alpha < \beta\}$, that is, it is a set of the cells in K that contain α. The link of α in K is the set of the cells which are in the star of α and do not consist of α on their boundary, that is, $Lk_K(\alpha) = \{\beta \in St_K(\alpha) | \alpha \cap \beta = \emptyset\}$*

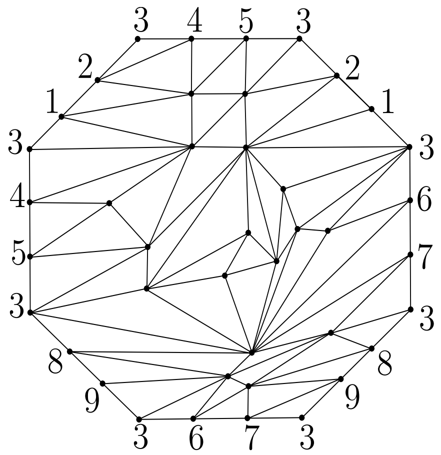

Figure 2.5: A triangulation on the genus 2 surface.

In the Figure (2.6), the gray dashed regions represent the star of the vertex ν and the star of the edge e. The blue colored circle, which is the boundary of $St(\nu)$, denotes the link of ν, and the two blue vertices form the link of e.

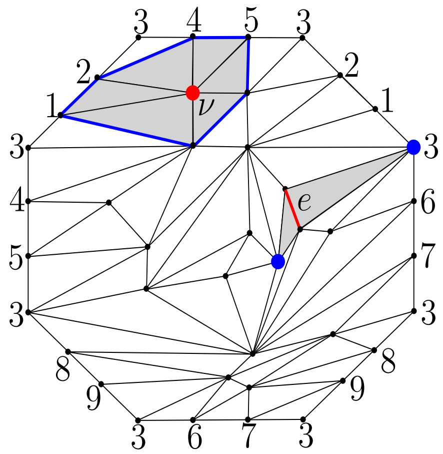

Figure 2.6: $St(\nu)$, $St(e)$ and $Lk(\nu)$, $Lk(e)$.

Definition 2.1.11. *A piecewise-linear n-ball (respectively a piecewise-linear n-sphere) is an n-dimensional CW-complex which is piecewise-linear homeomorphic to an n-simplex (respectively to the boundary of an $(n-1)$-simplex).*

Definition 2.1.12. *A triangulation of a manifold is piecewise-linear if the link of every vertex is a piecewise-linear sphere.*

Theorem 2.1.13 ([8], [33])**.** *Every smooth manifold has a piecewise-linear triangulation.*

Note that the converse of the Theorem (2.1.13) is not always true. For instance, Kervaire's 10-dimensional manifold admits a triangulation which is piecewise-linear but not homeomorphic to any smooth manifold ([17]).

Definition 2.1.14. *Let M be a compact, connected n-manifold with boundary and N be a subcomplex of M of dimension $k \leq n-1$. If M collapses to N and there is no further collapses on N, then N is called a spine of M. A spine of a closed, connected n-manifold is a spine of $M - int(D^n)$.*

Example 2.1.15. *A spine of the torus $\mathbb{T}^2$ is $S^1 \vee S^1$. A spine of the solid torus and solid Klein bottle is S^1.*

Definition 2.1.16. *Let M_1 and M_2 be n-dimensional manifolds. The connected sum $M_1 \# M_2$ is an n-manifold that is obtained by removing the interior of an n-ball from M_1 and M_2, and then gluing the resulting boundaries using an orientation reversing homeomorphism.*

Lemma 2.1.17 ([21])**.** *Let M_1 and M_2 be two closed, connected, oriented, triangulated 3-manifolds, and N_1 and N_2 be their spines, respectively. Then $N_1 \vee N_2$ is a spine of $M_1 \# M_2$.*

Definition 2.1.18. *A connected n-manifold M is called prime if it cannot be written as a connected sum of two n-manifolds neither of which is an n-sphere.*

The following theorem states an important property of 3-manifolds.

Theorem 2.1.19 ([20], [26])**.** *Every compact, orientable 3-manifold can be expressed as a connected sum of prime 3-manifolds, and this prime decomposition is unique up to insertion or deletion of S^3 summands.*

Note that Theorem (2.1.19) is not valid for an arbitrary n-dimensional manifold. For example, 2- and 4-dimensional manifolds do not have the unique prime decomposition property. Consider $\mathbb{R}P^2 \# \mathbb{T}^2$ as an example in dimension 2. It is homeomorphic to $\mathbb{R}P^2 \# \mathbb{K}^2$, but $\mathbb{T}^2$ is not homeomorphic to $\mathbb{K}^2$. Also consider $M = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \# \overline{\mathbb{C}P^2}$ in dimension 4. It is diffeomorphic to $(S^2 \times S^2) \# \overline{\mathbb{C}P^2}$, but $S^2 \times S^2$ and $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ are not even homotopy equivalent [25].

The following lemma gives a relation between the homology groups of a connected sum and the homology groups of each factor on the connected sum.

Lemma 2.1.20 ([12]). *Let M_1 and M_2 be closed, connected n-manifolds. Then*

$$H_0(M_1 \# M_2; \mathbb{Z}) = \mathbb{Z}, \text{ and } H_i(M_1 \# M_2; \mathbb{Z}) \cong H_i(M_1; \mathbb{Z}) \oplus H_i(M_2; \mathbb{Z})$$

for $0 < i < n$. If M_1 and M_2 are orientable, then $H_n(M_1 \# M_2; \mathbb{Z}) = \mathbb{Z}$. If either M_1 or M_2 is non-orientable, then $H_n(M_1 \# M_2; \mathbb{Z}) = 0$.

Remark 2.1.21. *Let K be a finite CW-complex. Throughout this thesis, b_i will denote the Betti number of K, that is, the rank of $H_i(K; \mathbb{Z})$ which is the number of $\mathbb{Z}$ summands in $H_i(K; \mathbb{Z})$, and $\chi(K)$ will denote the Euler characteristic of K.*

2.2 Discrete Morse Theory

In this section, we present some basic definitions and results on discrete Morse theory. Throughout this section, K will represent a finite regular CW-complex. This section is based on [9] and [10].

Definition 2.2.1 ([9]). *Let $f \colon K \to \mathbb{R}$ be a real valued function on K. We say that f is a discrete Morse function if for any p-cell $\alpha \in K$, it satisfies the following conditions:*

$$n_1 = \#\{\tau > \sigma \mid f(\tau) \leq f(\sigma)\} \leq 1,$$
$$n_2 = \#\{\nu < \sigma \mid f(\nu) \geq f(\sigma)\} \leq 1.$$

In other words, Definition (2.2.1) states that for any p-cell α, there can be at most one $(p+1)$-cell τ containing α such that $f(\tau) \leq f(\alpha)$. Similarly, there can be at most one $(p-1)$-cell ν contained in α such that $f(\nu) \geq f(\alpha)$. See Figure 2.7 for an example and a counterexample of a discrete Morse function on a 1-complex, where the numbers represents the function values on each cell.

Lemma 2.2.2 ([9]). *The numbers n_1 and n_2 above cannot be both one.*

Proof. By the way of contradiction, assume that $n_1 = n_2 = 1$, that is, there exist a $(p-1)$-cell ν and a $(p+1)$-cell β for a p-cell $\alpha \in K$ such that $f(\nu) \geq f(\alpha)$ and $f(\beta) \leq f(\alpha)$. Hence $f(\beta) \leq f(\alpha) \leq f(\nu)$. Since K is a regular CW-complex, there is at least one p-cell $\widetilde{\alpha}^{(p)} \neq \alpha^{(p)}$ such that $\nu^{(p-1)} < \widetilde{\alpha}^{(p)} < \beta^{(p+1)}$. Since f is a discrete Morse function, this implies that $f(\nu) < f(\widetilde{\alpha})$ and $f(\widetilde{\alpha}) < f(\beta)$, and this is a contradiction. □

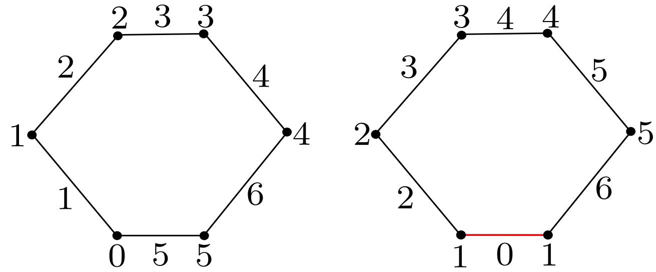

Figure 2.7: An example (left) and a counterexample (right) of a discrete Morse function on a 1-complex.

A p-cell $\alpha \in K$ is called a **critical** cell of f if $n_1 = n_2 = 0$. The value of f on a critical cell is called a critical value.

Example 2.2.3. *Every CW-complex K admits a discrete Morse function $f\colon K \to \mathbb{R}$ which can be defined as*

$$f(\alpha) = dim(\alpha)$$

for each $\alpha \in K$. For this particular discrete Morse function, we note that each cell in K is critical.

Observe that every p-cell, for $p \geq 1$, in K contains at least two $(p-1)$-cells in its boundary. Hence the minimum value of a discrete Morse function f on K is obtained at a vertex, which will be critical 0-cell of f by the definition of the discrete Morse function. Observe also that if K is a closed, connected, triangulated n-manifold, then every $(p-1)$-cell, for $p \leq n$, in K is contained in the boundary of at least two p-cells. Thus the maximum value of f is attained at an n-cell, which will be a critical cell of f.

Definition 2.2.4 ([9])**.** *Let f be a discrete Morse function on K and $c \in R$. The sublevel complex $K(c)$ is defined as follows:*

$$K(c) = \bigcup_{f(\beta)\leq c} \bigcup_{\alpha \leq \beta} \alpha.$$

In other words, $K(c)$ contains all cells β where $f(\beta) \leq c$ and all of their faces.

The following two lemmas show that restrictions and extensions of discrete Morse functions are also discrete Morse functions.

Lemma 2.2.5 ([9])**.** *Let K be a CW complex, L be a subcomplex of K and f be a discrete Morse function on M. Then the restriction of f is a discrete Morse function on L such that if $\sigma \in L$ is a critical cell of f, then it is a critical cell of the restriction function.*

Lemma 2.2.6 ([9])**.** *Let K be a CW complex, L be a subcomplex of K such that K collapses to L. Let f be a discrete Morse function on L and $c = max_{\sigma \subseteq L} f(\sigma)$. Then there exists an extension of f to K that is a discrete Morse function on K with*

$$L = K(c)$$

and there are no critical points in $K - L$.

The following theorems describe how critical cells affect the topology of the complex.

Lemma 2.2.7 ([9])**.** *Let $a < b \in \mathbb{R}$. If $f^{-1}((a, b])$ does not contain any critical cell, then $K(b)$ collapses to $K(a)$. See Figure (2.9) for an example.*

Lemma 2.2.8 ([9])**.** *Let $a < b \in \mathbb{R}$ and $\sigma^{(p)}$ be a unique critical cell in $f^{-1}((a, b])$. Then $K(b)$ is homotopy equivalent to $K(a)$ with a p-cell $\sigma^{(p)}$ attached along $\partial\sigma^{(p)}$. That is, $K(b)$ is homotopy equivalent to*

$$K(a) \bigcup_{\partial\sigma^{(p)}} \sigma^{(p)}.$$

Figure (2.8) is an example of a 2-complex K with a discrete Morse function, given by the numbers on each cell. The 0-cell with the value 0 and the 1-cell with the value 8 are the critical cells of the function.

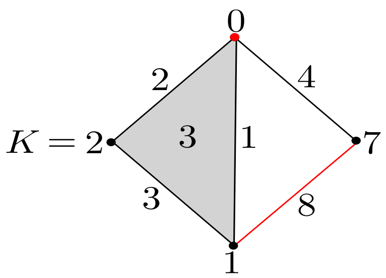

Figure 2.8: A discrete Morse function on a 2-complex.

On Figure (2.9), $K(4)$ collapses to $K(0)$ as it is stated in Lemma (2.2.7). Moreover, $K(8)$ is homotopy equivalent to $K(4)$ with one 1-cell attached along the boundary vertices as in Lemma (2.2.8).

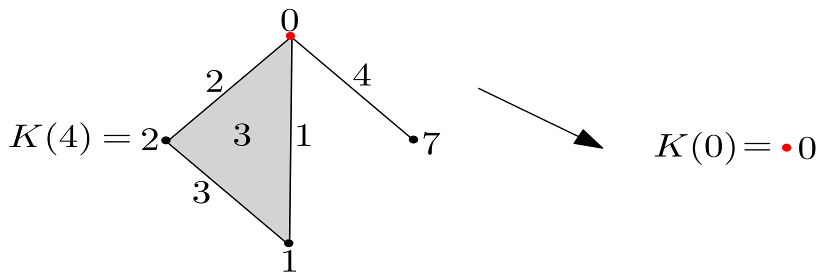

Figure 2.9: $K(4) \searrow K(0)$

Remark 2.2.9. *Throughout this thesis, $m_p(f)$ will denote the number of critical p-cells of a discrete Morse function f.*

The following theorem determines the homotopy type of a given complex with a discrete Morse function.

Theorem 2.2.10 ([9])**.** *Let f be a discrete Morse function on K. Then K is homotopy equivalent to a CW-complex with one p-cell for each critical p-cell.*

It is obvious that the complex given in Figure (2.8) is homotopy equivalent to a CW-decomposition of a 1-sphere with one 0-cell and one 1-cell.

The following corollary gives the discrete version of the well known Morse inequalities.

Corollary 2.2.11 ([9])**.** *Let $f\colon K \to \mathbb{R}$ be a discrete Morse function on an n-dimensional complex K, $\mathbb{F}$ be a field and $b_i(M;\mathbb{F})$ be the i^{th} Betti number of K with respect to $\mathbb{F}$. Then, for $0 \leq i \leq n$,*

1. $m_i(f) - m_{i-1}(f) + \ldots \pm m_0(f) \geq b_i(M;\mathbb{F}) - b_{i-1}(M;\mathbb{F}) + \ldots \pm b_0(M;\mathbb{F})$,
2. $m_i(f) \geq b_i(M;\mathbb{F})$,
3. $\chi(K) = \Sigma_i (-1)^i m_i(f)$

Definition 2.2.12 ([9])**.** *A discrete vector field V on K is a collection of pairs of cells $\{\alpha^{(p)} < \beta^{(p+1)}\}$ such that each cell is in at most one pair. If $\{\alpha^{(p)}, \beta^{(p+1)}\} \in V$, then we write $V(\alpha) = \beta$*

For a graphic representation of a discrete vector field we draw arrows pointing from $\alpha^{(p)}$ to $\beta^{(p+1)}$ as in Figure (2.10).

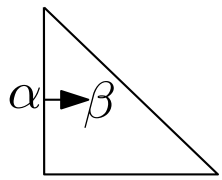

Figure 2.10: $f(\alpha) \geq f(\beta)$.

Definition 2.2.13 ([9])**.** *A V-path of dimension $(p+1)$ is a sequence of cells*

$$\alpha_0^{(p)} < \beta_0^{(p+1)} > \alpha_1^{(p)} < \beta_1^{(p+1)} > \ldots < \beta_k^{(p+1)} > \alpha_{k+1}^{(p)}$$

such that $\{\alpha_i^{(p)}, \beta_i^{(p+1)}\} \in V$ and $\alpha_i^{(p)} \neq \alpha_{i+1}^{(p)}$ for each $i = 0, 1, \ldots, k$. A V-path is a non-trivial closed path if for some $k > 0$ we have $\alpha_0 = \alpha_{k+1}$.

Definition 2.2.14 ([9])**.** *The discrete gradient vector field of a discrete Morse function f on K consists of pairs $\{\alpha^{(p)}, \beta^{(p+1)}\}$ such that $\alpha < \beta$ and $f(\alpha) \geq f(\beta)$.*

A V-path on a discrete gradient vector field is called a gradient path.

We note that a cell is critical if and only if it is neither the tail nor the head of an arrow. See Figure (2.11) for several gradient paths and critical cells on a planar diagram of the torus.

Remark 2.2.15. *If there are no arrows pointing from L to $K - L$ in Lemma (2.2.5), then the restricted function has only the critical cells of f and no other critical cells.*

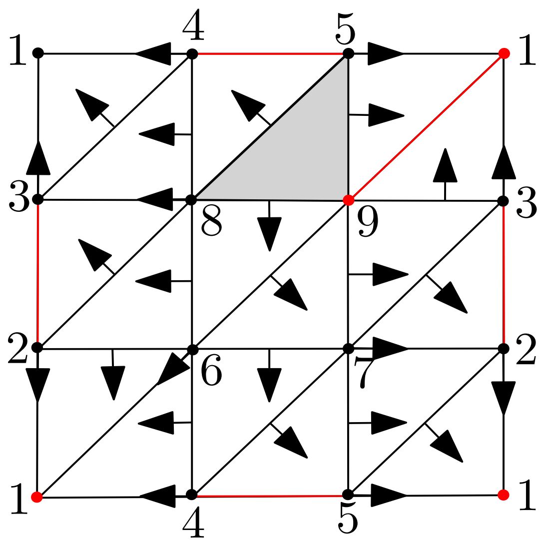

Figure 2.11: A collection of discrete gradient paths on the torus.

The gradient paths denote the direction in which function values descend. Thus a discrete gradient vector field does not admit any non-trivial closed gradient path. This fact is stated in the following theorem.

Theorem 2.2.16 ([10]). *A discrete vector field V is the discrete gradient vector field of a discrete Morse function if and only if none of the V-paths forms a non-trivial cycle.*

In discrete Morse theory, discrete gradient vector fields are often more useful than the underlying discrete Morse functions for the combinatorial purposes. We give the following definition concerning equivalence relation between discrete Morse functions.

Definition 2.2.17 ([2]). *Let f and g be two discrete Morse functions on K. Then f and g are called equivalent if for every pair of cells $(\alpha^{(p)} < \beta^{(p+1)}) \in K$*

$$f(\alpha) < f(\beta) \text{ if and only if } g(\alpha) < g(\beta).$$

The following theorem gives us an opportunity to work with the discrete gradient vector fields instead of the underlying discrete Morse functions.

Theorem 2.2.18 ([2]). *Let f and g be two discrete Morse functions on K. Then f and g are equivalent if and only if f and g have the same critical cells and induce the same discrete gradient vector field.*

The following theorem shows that bisection does not affect the existence of a discrete Morse function on a given complex. That is, an extension of a discrete Morse function f to the subdivided complex obtained via a sequence of bisections is also a discrete Morse function.

Theorem 2.2.19 ([9, Theorem 12.1]). *Let K be a polyhedron, g be a discrete Morse function on K and α be a p-cell of K. Assume that $\widetilde{K}$ is a subdivision of K obtained from a bisection*

$$\alpha^{(p)} = {\alpha_1}^{(p)} \cup \nu^{(p-1)} \cup {\alpha_2}^{(p)}.$$

Then $\widetilde{K}$ admits a discrete Morse function $\widetilde{g}$ which satisfies the following properties:

1. *$\beta \neq \alpha$ is a critical cell of g if and only if β is a critical cell of $\widetilde{g}$.*

2. *α is a critical cell of g if and only if α_1 is a critical cell of $\widetilde{g}$, and α_2 is not a critical cell of $\widetilde{g}$.*

3. *Let V_g and $V_{\widetilde{g}}$ be the discrete gradient vector fields induced by g and $\widetilde{g}$, respectively. Then, except the cell α, $V_g = V_{\widetilde{g}}$. That is, if $\beta_1 \neq \alpha \neq \beta_2$, then*

$$V_g(\beta_1) = \beta_2 \quad \textit{if and only if} \quad V_{\widetilde{g}}(\beta_1) = \beta_2$$

and

$$V_g(\alpha) = \beta_1 \quad \textit{if and only if} \quad V_{\widetilde{g}}(\alpha_1) = \beta_2 \quad \textit{or} \quad V_{\widetilde{g}}(\alpha_2) = \beta_2$$
$$V_g(\beta_1) = \alpha \quad \textit{if and only if} \quad V_{\widetilde{g}}(\beta_1) = \alpha_1 \quad \textit{or} \quad V_{\widetilde{g}}(\beta_1) = \alpha_2$$

In ([9]), Forman proved that homology groups of a complex can be computed via discrete Morse functions. Let f be a discrete Morse function on K, and C_i be the free groups generated by critical i-cells of f. There are boundary maps $\partial_i : C_i \to C_{i-1}$ such that $\partial_{i-1} \circ \partial_i = 0$ for each i, and thus the following complex:

$$C_* : 0 \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_{i+1}} C_i \xrightarrow{\partial_i} C_{i-1} \xrightarrow{\partial_{i-1}} \dots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

is called Morse complex arising from f. The boundary maps are computed from gradient V-paths which start in the boundary of a critical i-cell and end in a critical $(i-1)$-cell as follows:

We choose an orientation for each critical cell. Then for any critical i-cell β, we define

$$\partial(\beta) = \Sigma_\alpha c_{(\alpha,\beta)} \alpha$$

where

$$c_{(\alpha,\beta)} = \Sigma_\gamma m(\gamma)$$

and γ is a discrete gradient path which starts in the boundary of the critical i-cell β and ends in a critical $(i-1)$-cell α. Note that γ induces an orientation on α given by the orientation of β. If this orientation coincides with the orientation of α, then we say $m(\gamma) = 1$, otherwise, we say $m(\gamma) = -1$.

Theorem 2.2.20 ([9]). *$H_*(C_*) \cong H_*(K, \mathbb{Z})$.*

Here is an example for homology computation on torus via Morse complex.

Example 2.2.21. *On Figure (2.12) from the previous example of a discrete gradient vector field on a torus, blue arrows represent the orientation of the critical* 2*-cell and critical* 1*-cells. The orientation of the vertices* 1 *and* 9 *are* $-$ *and* $+$*, respectively. The green arrows and signs denote the orientation on each critical cell induced by the discrete gradient paths.*

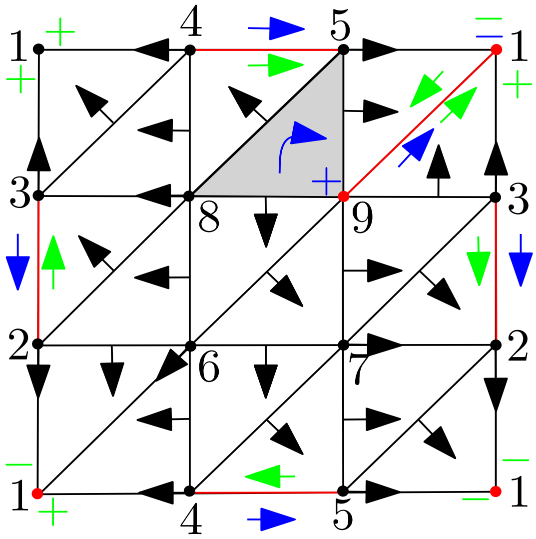

Figure 2.12: Orientation induced by the discrete gradient paths on the torus.

The Morse complex on the torus is as follows:

$$C_* : 0 \xrightarrow{\partial_3} C_2([5,8,9]) \xrightarrow{\partial_2} C_1([1,9],[2,3],[4,5]) \xrightarrow{\partial_1} C_0([1],[9]) \xrightarrow{\partial_0} 0.$$

Note that $c_{([1,9],[5,8,9])} = 0$*,* $c_{([2,3],[5,8,9])} = 0$*,* $c_{([4,5],[5,8,9])} = 0$*,* $c_{([1],[1,9])} = 1$*,* $c_{([9],[1,9])} = 1$*,* $c_{([1],[2,3])} = 0$*,* $c_{([9],[2,3])} = 0$ *and* $c_{([1],[4,5])} = 0$*,* $c_{([9],[4,5])} = 0$*. Thus* $\partial_2([5,8,9]) = 0$*,* $\partial_1([1,9]) = [1] + [9]$*,* $\partial_1([2,3]) = 0$ *and* $\partial_1([4,5]) = 0$*. Then the Morse homology groups of the torus are as follows:* $H_2(\mathbb{T}^2,\mathbb{Z}) = \mathbb{Z}$ *is generated by the critical* 2*-cell* $[5,8,9]$*,* $H_1(\mathbb{T}^2,\mathbb{Z}) = \mathbb{Z}\oplus\mathbb{Z}$ *is generated by the critical* 1*-cells* $[2,3],[4,5]$*, and* $H_0(\mathbb{T}^2,\mathbb{Z}) = \mathbb{Z} \cong \langle[1],[9]\rangle/\langle[1]+[9]\rangle$*.*

CHAPTER 3

EXISTENCE OF PERFECT DISCRETE MORSE FUNCTIONS

In this chapter, we discuss existence of $\mathbb{F}$-perfect discrete Morse functions, where $\mathbb{F}$ is a field or $\mathbb{Z}$, on finite complexes and closed, connected, triangulated n-manifolds. We add this chapter for the integrity of this thesis and we do not claim the originality of this chapter. The results of this chapter are based on previous results from [3], [4], [19], [23].

3.1 Existence of perfect discrete Morse functions

We start this section with the definition of an $\mathbb{F}$-perfect discrete Morse function where $\mathbb{F}$ is a field or $\mathbb{Z}$.

Definition 3.1.1. *A discrete Morse function $f : K \to \mathbb{R}$ on a finite complex K is called an $\mathbb{F}$-perfect discrete Morse function if $m_i(f) = b_i(K;\mathbb{F})$ where $b_i(K;\mathbb{F}) =$ rank$H_i(K;\mathbb{F})$ with the coefficient group $\mathbb{F}$ and $i = 1, 2, \ldots,$ dim(K).*

Let L be a module over a principal ideal domain G. Observe that L is a group with operators from G. In smooth Morse theory, Pitcher [30] stated the Morse inequalities in terms of Betti numbers of a smooth manifold M with reference to the coefficient group L and torsion coefficients of $H_i(M, L)$ in the following theorem. Let T_i be the submodule of elements of finite order in $H_i(M, L)$ and t_i be the number of torsion coefficients which is the smallest number of the cyclic submodules of which T_i is a direct sum.

Theorem 3.1.2 ([30, Theorem 14.2])**.** *Let f be a smooth Morse function on a compact n-dimensional Riemannian manifold M. Then*

1. $m_0(f) \geq b_0$, $m_i(f) \geq b_i + t_i + t_{i-1}$ *for* $i = 1, 2, \ldots, n$,

2. $m_i(f) - m_{i-1}(f) + \ldots + (-1)^i m_0(f) \geq b_i - b_{i-1} + \ldots + (-1)^i b_0 + t_i$,

3. $m_n(f) - m_{n-1}(f) + \ldots + (-1)^n m_0(f) = b_n - b_{n-1} + \ldots + (-1)^n b_0$.

The following lemma is a discrete version of the inequalities given in Theorem (3.1.2) under the coefficient group $\mathbb{Z}$.

Lemma 3.1.3. *Let f be a discrete Morse function on a finite CW-complex K of dimension n. Then,*

$$m_i(f) \geq b_i + t_i + t_{i-1} \text{ for } i = 1, 2, \ldots, n.$$

where $b_i := rank H_i(K, \mathbb{Z})$ and t_i is the number of torsion coefficients of $H_i(K; \mathbb{Z})$.

Proof. Let $C_i(K, \mathbb{Z})$ be the free $\mathbb{Z}$-module generated by the critical i-cells of f, and

$$C_* : 0 \to C_n(K, \mathbb{Z}) \xrightarrow{\partial_n} C_{n-1}(K, \mathbb{Z}) \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_1} C_0(K, \mathbb{Z}) \xrightarrow{\partial_0} 0$$

be the resulting Morse chain complex. Then, by Theorem (2.2.20),

$$H_i(C_*) \cong H_i(K, \mathbb{Z}).$$

Observe that rank$(C_i(K, \mathbb{Z}))$ = rank$(Ker\partial_i)$ + rank$(Im\partial_i)$ and rank$(Ker\partial_i) \geq b_i + t_i$, rank$(Im\partial_i) \geq t_{i-1}$. Thus rank$(C_i(K, \mathbb{Z})) \geq b_i + t_i + t_{i-1}$ where rank$(C_i(K, \mathbb{Z})) = m_i(f)$. □

Proposition 3.1.4 ([23]). *Every closed, triangulated surface admits a $\mathbb{Z}$-perfect discrete Morse function if and only if it is orientable.*

Figure (3.1) is an example of a discrete gradient vector field on the torus induced by a perfect discrete Morse function.

Theorem 3.1.5 ([23]). *Every closed, triangulated, non-orientable surface admits a $\mathbb{Z}_2$-perfect discrete Morse function.*

Corollary 3.1.6 ([23]). *Every closed, connected surface admits a $\mathbb{Z}_2$-perfect discrete Morse function.*

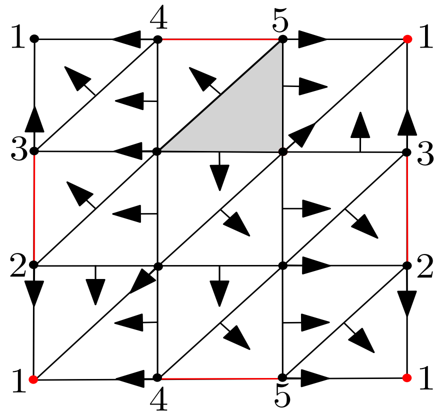

Figure 3.1: A discrete gradient vector field on the torus.

Theorem 3.1.7 ([3])**.** *Let $\mathbb{F}$ be a field. A 2-complex K that is $\mathbb{F}$-acyclic and non-collapsible does not admit any $\mathbb{F}$-perfect discrete Morse functions.*

For example, the Dunce hat does not admit a perfect discrete Morse function for any coefficient field.

The following proposition is mentioned in ([4]) for 3-dimensional manifolds without any proof. For the integrity of this thesis, we will give a proof.

Proposition 3.1.8. *Let M be a closed, connected, triangulated n-manifold with a $\mathbb{Z}$-perfect discrete Morse function defined on it. Then M is orientable and $H_i(M;\mathbb{Z})$ is free for $i = 0, 1, 2, \ldots, n$.*

Proof. Let f be a $\mathbb{Z}$-perfect discrete Morse function on M. By Lemma (3.1.3), we have

$$b_0 = m_0(f) \geq b_0,$$
$$b_i = m_i(f) \geq b_i + t_i + t_{i-1} \text{ for } i = 1, 2, \ldots, n-1,$$
$$b_n = m_n \geq b_n + t_{n-1}.$$

Thus $t_i = 0$ for all $i = 1, 2, \ldots, n-1$ which implies that $H_i(M;\mathbb{Z})$ is free. Since $t_{n-1} = 0$, then M is orientable by [12, Corollary 3.28]. □

Corollary 3.1.9 ([4, Corollary 1])**.** *Let M be a closed, connected, triangulated n-manifold such that $\pi_1(M)$ is finite and non-trivial. Then M cannot admit any $\mathbb{Z}$-perfect discrete Morse functions.*

Proof. Assume that $\pi_1(M)$ is finite and non-trivial, then

1. if $H_1(M;\mathbb{Z}) \neq 0$, then $H_1(M;\mathbb{Z}) = \pi_1(M)/\pi_1'(M)$ will have a torsion subgroup where $\pi_1'(M)$ is the commutator subgroup of $\pi_1(M)$. Thus, by Proposition (3.1.8), M does not admit a $\mathbb{Z}$-perfect discrete Morse function.

2. if $H_1(M;\mathbb{Z}) = 0$, then the abelianization of $\pi_1(M)$, which is $\pi_1(M)/\pi_1'(M)$, will be trivial. Since $\pi_1(M)$ is non-trivial, it is not abelian, and thus it should have at least two generators. So, 1-skeleton of any CW-complex structure on M will contain a wedge of at least two circles. That is, it will contain at least two 1-cells. Therefore, by Theorem (2.2.10), M cannot admit a $\mathbb{Z}$-perfect discrete Morse function since for a $\mathbb{Z}$-perfect discrete Morse function f on M, $m_1(f) = 0$.

□

For instance, the lens space $L(p,q)$ with $\pi_1(L(p,q)) = \mathbb{Z}_p$, and the Homology sphere M with $\pi_1(M) \neq 0$ and $H_1(M;\mathbb{Z}) = 0$ cannot admit any $\mathbb{Z}$-perfect discrete Morse functions.

Corollary 3.1.10 ([4, Corollary 2]). *Let M be a closed, connected, triangulated n-manifold such that $\pi_1(M)$ contains a torsion subgroup. Then M does not admit a $\mathbb{Z}$-perfect discrete Morse function.*

The following Theorem indicates that the existence of an $\mathbb{F}$-perfect discrete Morse function on a closed manifold is strongly related with its subcomplexes.

Theorem 3.1.11 ([4, Theorem 5]). *Let M be a closed, connected, orientable triangulated n-manifold, and $\mathbb{F}$ be either $\mathbb{Z}$ or a field. Then there exists an $\mathbb{F}$-perfect discrete Morse function on M if and only if M has a spine that admits an $\mathbb{F}$-perfect discrete Morse function.*

Proof. Let f be an $\mathbb{F}$-perfect discrete Morse function on M. Since M is a closed, connected, orientable manifold, $b_0(M;\mathbb{F}) = b_n(M;\mathbb{F}) = 1$ and $b_i(M;\mathbb{F}) = b_j(M;\mathbb{F})$ for $i+j=n$, by Poincare duality theorem. Thus $m_0(f) = m_n(f) = 1$ and $m_i(f) = b_i(M;\mathbb{F})$ for $i = 1, 2, \ldots, n-1$. Let β be the unique critical n-cell in M, and $N = M - \text{int}(\beta)$. Clearly N can be collapsed along all n-paths and some i-paths for

$i = 1, 2, \ldots, n-1$ to a subcomplex K of dimension $\leq (n-1)$ of M such that there is no more collapse on K. That is, K is a spine of M and $f|_K$ is a discrete Morse function by Lemma (2.2.5) such that $m_i(f) = m_i(f|_K)$ because of the construction of K. Since K and N are simple homotopy equivalent, $b_i(K;\mathbb{F}) = b_i(N;\mathbb{F})$ and $b_i(N;\mathbb{F}) = b_i(M;\mathbb{F})$ by Mayer Vietoris for $i = 1, 2, \ldots, n-1$. Therefore, $b_i(K;\mathbb{F}) = m_i(f) = m_i(f|_K)$ and $f|_K$ is an $\mathbb{F}$-perfect discrete Morse function on K.

Conversely, let K be a spine of M such that $M - \text{int}(\gamma) \searrow K$ where γ is an n-cell of M and h be an $\mathbb{F}$-perfect discrete Morse function on K. By Lemma (2.2.6), we can extend h to an $\mathbb{F}$-perfect discrete Morse function f on $M - \text{Int}(\gamma)$. Then, we can also extend f to M as an $\mathbb{F}$-perfect discrete Morse function by defining $f(\gamma) = 1 + \max(\alpha)$ where $\alpha < \gamma$. $\square$

Example 3.1.12. *A* 3*-sphere* S^3 *with a triangulation on it admits a* $\mathbb{Z}$*-perfect discrete Morse function since it has a spine which is a point. And* $S^1 \times S^2$ *is another example which admits a* $\mathbb{Z}$*-perfect discrete Morse function. Note that* $S^1 \vee S^2$ *is a spine of* $S^1 \times S^2$*. Let* f_1 *and* f_2 *be* $\mathbb{Z}$*-perfect discrete Morse functions on* S^1 *and* S^2*, respectively. Let us identify the critical* 0*-cells* ν *and* ω *on* S^1 *and* S^2 *to obtain* $S^1 \vee S^2$*. Then*

$$f(\alpha) = \begin{cases} \mathit{min}\{f_1(\alpha), f_2(\alpha)\} & \mathit{if} \quad \alpha = \nu = \omega \\ f_1(\alpha) & \mathit{if} \quad \alpha \in S^1, \alpha \neq \nu \\ f_2(\alpha) & \mathit{if} \quad \alpha \in S^2, \alpha \neq \omega \end{cases}$$

is a $\mathbb{Z}$*-perfect discrete Morse function on* $S^1 \vee S^2$*. Thus, by Theorem (3.1.11),* $S^1 \times S^2$ *admits a* $\mathbb{Z}$*-perfect discrete Morse function. We can say that, by universal coefficient theorem for homology,* S^3 *and* $S^1 \times S^2$ *also admit* $\mathbb{F}$*-perfect discrete Morse function for any field coefficients* $\mathbb{F}$*.*

CHAPTER 4

COMPOSING PERFECT DISCRETE MORSE FUNCTIONS ON A CONNECTED SUM

In this chapter, we present our main result on composing perfect discrete Morse functions on a connected sum of manifolds. We give an explicit algorithm for the perfect discrete Morse function on the connected sum that coincides almost everywhere with the perfect discrete Morse functions on the two components of the connected sum.

4.1 Main Result

We start this section with stating a lemma regarding uniqueness of critical 0-cell and n-cell of a perfect discrete Morse function.

Lemma 4.1.1. *If f is a perfect discrete Morse function on a connected complex K, then it has precisely one critical vertex. If K is a triangulation of a closed triangulated manifold of dimension n, then f has only one critical n-cell.*

Proof. Since K is a connected complex, then $b_0(K, \mathbb{Z}_2) = 1$. If K is a triangulation of a closed, triangulated manifold of dimension n, then $b_n(K, \mathbb{Z}_2) = 1$. Since f is a perfect discrete Morse function on K, and the number of the critical cells of a discrete Morse function is independent of the coefficients, then, by Corollary (2.2.11), $m_0(f) = b_0(K, \mathbb{Z}_2)$ and $m_n(f) = b_n(K, \mathbb{Z}_2)$. Thus f has exactly one critical 0-cell and one critical n-cell. □

Now we give the main result of this chapter and its proof. Throughout the section, we work with $\mathbb{Z}$-perfect discrete Morse functions but the proof works for any field coefficients.

Theorem 4.1.2. *Let M_1 and M_2 be two n-dimensional closed, connected, oriented, triangulated manifolds, and f_1 and f_2 be the perfect discrete Morse functions on them, respectively. Then there exists a perfect discrete Morse function f on $M = M_1 \# M_2$ which coincides with f_1 and f_2, up to a constant on each summand, except on a neighbourhood of the two n-cells whose interiors are removed to form the connected sum.*

Proof. We work with the discrete gradient vector fields V_1 and V_2 induced by the perfect discrete Morse functions f_1 and f_2, respectively. Now, let us show that V_1 and V_2 give a discrete gradient vector field V induced by a perfect discrete Morse function, called f, on M which agrees with V_1 and V_2 on each summand. By Lemma (4.1.1), V_1 and V_2 have only one critical 0-cell and one critical n-cell.

Let α be the unique critical n-cell of M_1 and β be a non-critical n-cell of M_2 with the unique critical 0-cell ν in its boundary. We form M with a discrete gradient vector field V defined on it as follows:

First, we attach a tube $L = \partial\alpha \times [0, 1]$ to $M_1 - \mathring{\alpha}$, where $\mathring{\alpha}$ represents the interior of the cell α, along $\partial\alpha \times \{0\}$ with the natural product cell decomposition. Then we extend the discrete vector field $V_1|_{M_1-\alpha}$,which is a discrete gradient vector field with no additional critical cells by Lemma (2.2.5) and Remark (2.2.15), to $(M_1 - \alpha) \cup L$ such that we pair each cell σ in $\partial\alpha \times \{1\}$ with its co-face $\sigma \times (0, 1)$ in L. For an example in dimension 2, see Figure (4.1).

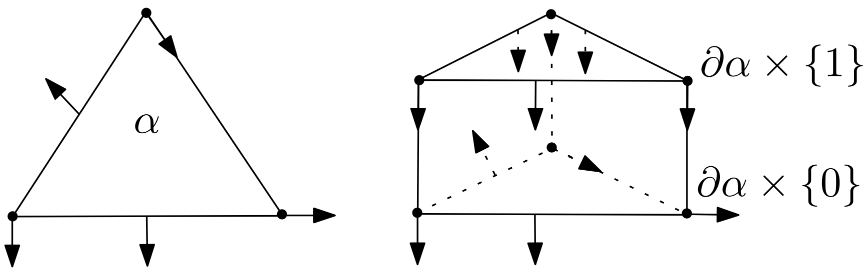

Figure 4.1: The discrete gradient vector field on $\partial\alpha \times \{1\}$.

We subdivide β to obtain an n-cell β' whose boundary and itself will be critical in M_2 after the subdivision as in the following way: Let $H : \beta \times [0, 1] \to \beta$ be a deformation retraction of β onto the critical 0-cell ν such that $H(x, t) = h_t(x)$ where $h_t : \beta \to \beta$

is given by

$$h_t(x) = (1-t)x + t\nu$$

for all $x \in \beta$. Then $\beta' = H_{1/2}(\beta)$ gives a smaller copy of β, and thus the map $H(x,t)$ provides a subdivision of β in M_2. Let $\beta'' = \beta - \beta'$ be the complement of β' in β. In other words, $\beta'' = \mathrm{Lk}_\beta(\nu) \times [0, 1/2]$ where $\mathrm{Lk}_\beta(\nu)$ denotes the link of ν in β. Then there exists one to one correspondence between the cells of $\beta - \nu$ and $\beta'' - \beta'$. See Figure (4.2) for an example in dimension 2.

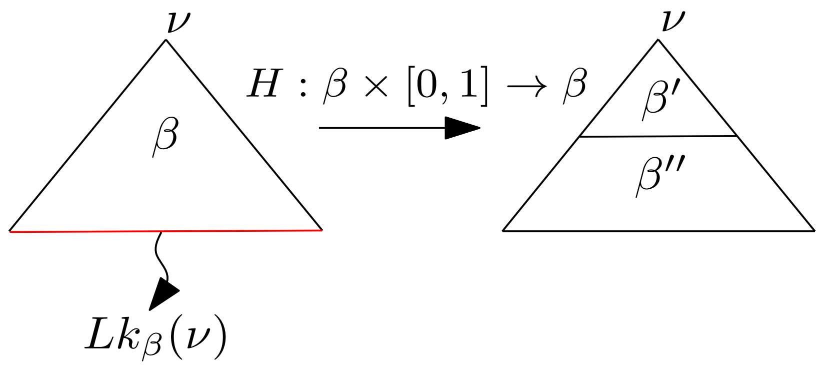

Figure 4.2: The subdivison on β.

We extend the discrete gradient vector field $V_2|_{M_2-\mathring{\beta}}$ to $(M_2 - \mathring{\beta}) \cup (\beta' \cup \beta'')$ after the subdivision on β as follows: For any pair $(\sigma, \tau) \in V_2$ with $\sigma \in \beta$ and either $\tau \in \beta$ or $\tau \in (M_2 - \beta)$, there exists a corresponding pair (σ'', τ'') with $\sigma'' \in \beta''$ and either $\tau'' \in \beta''$ or and $\tau'' = \tau \in (M_2 - \beta)$. Hence the cells on β' will be unpaired, that is, they will be critical cells for the extension of $V_2|_{M_2-\mathring{\beta}}$ let us call this vector field as $\widetilde{V}_2$. See Figure (4.3) for an example in dimension 2.

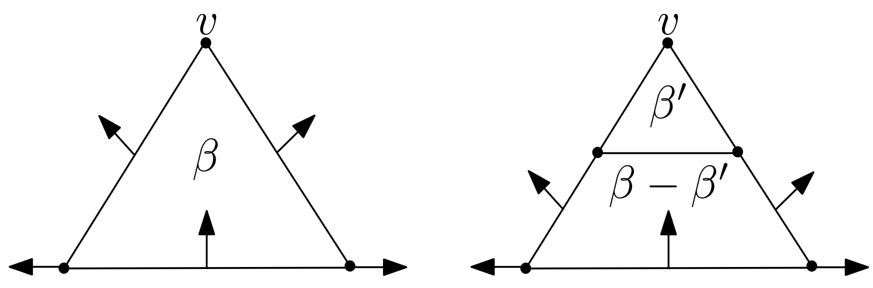

Figure 4.3: The discrete gradient vector field on β and its subdivision.

Finally, we form the connected sum $M = M_1 \# \widetilde{M_2}$ by attaching $\widetilde{M_2} - \mathring{\beta}'$ to $(M_1 - \alpha) \cup L$ by identifying the resulting boundary of $\widetilde{M_2} - \mathring{\beta}' \cong \partial\beta'$ with $\partial\alpha \times \{1\}$. The discrete vector field V on M is given by

$$V(\gamma)=\begin{cases} V_1(\gamma) & ;\ \gamma\in M_1-\mathring{\alpha} \\ \gamma\times(0,1) & ;\ \gamma\in\partial\alpha\times\{1\}\cong\partial\beta' \\ V_2(\sigma) & ;\ \gamma=\sigma''\in(\beta''-\beta') \\ V_2(\gamma) & ;\ \gamma\in M_2-\beta \end{cases}$$

The discrete vector field V is a discrete gradient vector field on M since neither V_1 nor V_2 admit any non-trivial loops and all the arrows on $\partial\alpha\times\{1\}$ point towards M_1. We have removed the critical n-cell $\alpha\in M_1$ and we have paired the critical 0-cell $\nu\in\widetilde{M_2}$. The number of the critical cells in V are

$$\begin{aligned} m_0(V) &= 1=b_0(M), \\ m_i(V) &= m_i(V_1)+m_i(V_2)=b_i(M_1)+b_i(M_2)=b_i(M), i=1,2,\ldots,n-1, \\ m_n(V) &= 1=b_n(M). \end{aligned}$$

Therefore, V is induced by a perfect discrete Morse function f that satisfies the conditions of the theorem.

For example, assuming there is no pairing between the faces of the critical n-cell α of f_1, and the values of f_2 is greater than or equal to the values of f_1, f can be defined as

$$f(\gamma)=\begin{cases} f_1(\gamma) & ;\ \gamma\in M_1-\mathring{\alpha} \\ f_1(\tau)+C/2 & ;\ \gamma=\tau\times(0,1), \gamma=\tau\times\{1\}\in L, \tau\in\partial\alpha \\ f_2(\gamma)+C & ;\ \gamma\in M_2-\beta \\ f_2(\tau)+C & ;\ \gamma=\tau''\in(\beta''-\beta') \end{cases}$$

where C is a constant bigger than $f_1(\alpha)+1$. □

CHAPTER 5

DECOMPOSING PERFECT DISCRETE MORSE FUNCTIONS

In this chapter, we give our main results on decomposing perfect discrete Morse functions on connected sums of closed, connected, oriented, triangulated surfaces and 3-dimensional manifolds. Before stating the main results of this chapter, we give some useful lemmas and theorems which are used in the proofs of the main theorems. Throughout this chapter, manifold will always refer to a closed, connected, oriented, triangulated manifold.

5.1 Preliminaries

We start this section with an observation for a discrete Morse function on an oriented surface.

Definition 5.1.1 ([5]). *Let f be a discrete Morse function on a manifold with boundary. If a cell on the boundary is a critical cell of f, then it is called a boundary critical cell.*

Figure (5.1) is an example of a discrete Morse function on a rectangle with three boundary critical cells given as v and e.

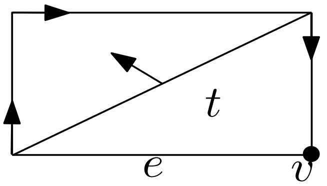

Figure 5.1: A discrete gradient vector field with boundary critical cells.

Lemma 5.1.2. *Let f be a discrete Morse function defined on a 2-manifold M. Let D be a triangulated open disk in M whose closure contains exactly one critical 0-cell or one critical 2-cell in its interior. Then the restriction function $f|_{M-D}$ has the same number of boundary critical 0-cells as boundary critical 1-cells.*

Proof. A 2-manifold M of genus g has Euler characteristic $\chi(M) = 2 - 2g$. When we remove a 2-disk D from M, Euler characteristic decreases by one. That is,

$$\begin{aligned}\chi(M - D) &= 2 - 2g - 1 \\ &= 1 - 2g. \end{aligned} \tag{5.1}$$

Let us assume that D includes only a single critical 0-cell, and there exist m_0 many critical 0-cells, m_1 many critical 1-cells and m_2 many critical 2-cells of f in M. Assume also that $f|_{M-D}$ has n_0 many boundary critical 0-cells and n_1 many boundary critical 1-cells. Thus, $f|_{M-D}$ has (m_0+n_0-1) many critical 0-cells, (m_1+n_1) many critical 1-cells and m_2 many critical 2-cells. Then, by Corollary (2.2.11), we have

$$\begin{aligned}\chi(M) &= m_0 - m_1 + m_2 = 2 - 2g, \text{ and} \\ \chi(M - D) &= (m_0 + n_0 - 1) - (m_1 + n_1) + m_2 \\ &= (m_0 - m_1 + m_2) + n_0 - n_1 - 1 \\ &= (2 - 2g) + n_0 - n_1 - 1 \\ &= 1 - 2g + (n_0 - n_1). \end{aligned} \tag{5.2}$$

The equations (5.1) and (5.2) together imply that $n_0 = n_1$. □

Moreover, the restriction of f to the resulting boundary 1-sphere $C = \partial(M - D)$, $f|_C$, is a discrete Morse function by Lemma 2.2.5 and

$$\chi(C) = 0 = m_0(f|_C) - m_1(f|_C)$$

implies that $m_0(f|_C) = m_1(f|_C)$ where $m_0(f|_C)$ and $m_1(f|_C)$ represent the number of critical 0-cells and the critical 1-cells of $f|_C$, respectively.

Now, let us give two preliminary Lemmas about discrete Morse functions which we use in the following proofs. We include their proofs for the completeness of the text.

Lemma 5.1.3. *Let K be a finite cell complex with a discrete Morse function f defined on it and let V be the discrete gradient vector field induced by f. Then*

1. *The 1-paths in V can merge but they cannot split.*
2. *All 1-paths in V form a tree at the critical 0-cell(minimal vertex) if f is a perfect discrete Morse function and K is a connected complex.*

Proof.

1. Suppose, on the contrary, that a 1-path splits at a vertex. Then this implies that the vertex is paired with two different edges, that is, it will be at least in two pairs in V, which is a contradiction.
2. Since K is connected, $b_0(K) = 1$ and since f is a perfect discrete Morse function, it has exactly one critical 0-cell in K. We note that every 1-path ends at the critical 0-cell, and thus all 1-paths form a tree at the unique critical vertex.

□

Lemma 5.1.4. *Let M be a compact triangulated n-manifold, and V be the discrete gradient vector field induced by a discrete Morse function f on M. Then*

1. *The n-paths in V can split but they cannot merge.*
2. *If M is closed, connected and orientable, then every regular n-cell τ is connected to the critical n-cell by a unique gradient path starting in the boundary of the critical cell and ending in τ.*

Proof.

1. Every $(n-1)$-cell is a common face of at most two n-cells (which depends on whether M is a manifold with or without boundary). If two n-paths would merge at an $(n-1)$-cell σ, then σ would be the common face of its n-dimensional pair in V as well as at least two other n-cells, but this is not possible.
2. If τ is an n-cell on an n-path in V, then its pair is an $(n-1)$-cell which has precisely one other coface since M is closed. If τ is not the critical n-cell, then

this is the only possible previous n-cell on the path. By repeating this process up to the unique critical n-cell, we eventually obtain a unique gradient n-path which starts in the boundary of the critical n-cell.

□

5.2 Decomposing Perfect Discrete Morse Functions on 2-Manifolds

Let M be a 2-manifold given as a connected sum of two manifolds. Also, let f be a perfect discrete Morse function defined on M. In this section, we show how to decompose f as perfect discrete Morse function on each summand using the discrete gradient vector field induced by f. To achieve this, we choose a separating circle satisfying certain properties, given in terms of the vector field, and then attach suitable disks to both sides.

The first thing one needs to do is to decide which critical cells belong to the same component. (We use Poincare duality and cohomology ring structure of M to understand the placement of critical cells). Note that M is a genus g surface for some g. We may take $\{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$ as a basis for $H_1(M)$, where α_i's and β_i's are given by the critical 1-cells of f. These homology classes are obtained by our perfect discrete Morse function following the 1-paths emanating from the critical 1-cells. Note that in any basis for the first homology group of a closed, connected, orientable surface of genus g, homology generators come as pairs. That is for any α in a basis for $H_1(M)$, there should be a class β such that the number of transverse intersections is odd between any representatives of α and β. This follows from the fact that $H^2(M) \cong \mathbb{Z}$ and a generator is given by the cup product of two 1-dimensional cohomology generators, say a and b, such that $(a \cup b)[M] = 1$. In other words, $a([M] \cap b) = 1$. In terms of homology, this can be explained as the cohomology classes a and b have Poincare duals intersecting transversally at an odd number of points (i.e. these classes should have an odd geometric intersection and their algebraic intersection should be 1). One can not count these intersection numbers by considering the 1-paths from the critical cells. Because these paths do not intersect transversally but instead they may merge together. To get the correct pairing we are going to work with the dual homology generators obtained from the 2-paths from the critical 2-cell to the critical 1-cells. To clarify the situation, let us try to explain these homology generators. There are exactly two 2-paths from the critical 2-cell to a critical 1-cell. The core of these 2-paths is a non-trivial homology class which is dual to the homology generator produced by the critical 1-cell. This duality can be explained in this way: when one considers the cohomology class that is the Poincare dual of the homology class from the critical cell, its value is 1 on this homology class coming from the core. Actually, this dual class must be homologous to one of the elements in the basis, the one we started with and the class homologous to the dual class, must belong to the same component of M. But instead of looking for this homologous class in the basis, which might be too cumbersome, we opt for counting the number of transverse intersections of these dual elements. Now we count the number of transverse intersections of these dual classes, for which one might need to perturb these cycles, and the classes with an odd number of intersections and hence their duals must belong to the same component.

Theorem 5.2.1. *Let $M = M_1 \# M_2$ be a connected sum of two triangulated 2-manifolds M_1 and M_2 of genera g_1 and g_2, respectively, and f be a perfect discrete Morse function defined on M. Then there exists a separating 1-sphere C on M so that $M = M_1 \#_C M_2$ and none of the cells on C are paired with the cells in $M - (M_1 - \mathit{int}(D_1))$ where D_1 is a 2-disk.*

The proof of the theorem requires some minor subdivisions of the triangulation, but these are localised in the neighbourhood of C. By abuse of notation the subdivided manifold and its regions will be denoted by the same symbols.

Lemma 5.2.2. *Let M and f be given as in Theorem (5.2.1). The union of the critical 2-cell and all the gradient paths from the critical 2-cell to any pair of critical 1-cells forms a subcomplex $N \subset M$ with boundary consisting of a union of 1-spheres meeting at finitely many vertices or connected by arcs which are formed by the interior 1-paths of N.*

Proof. By construction, the region N is a connected subcomplex of M with boundary. The boundary of N is a union of 1-spheres which are connected by either 1-paths (these are the interior arcs where two 2-paths from N touch) or connected by the critical 1-cells in N. Otherwise, the 2-paths forming N would lie on a 2-cycle which is not possible for a discrete Morse function. □

The following two lemmas deal with these two situations about the resulting boundary of N.

Lemma 5.2.3. *Let P be the star of the cells on a 1-path γ in N along which different 2-paths forming N as in Lemma (5.2.2) meet, and let $P' = P \cap N$. After some necessary subdivisions on the cells in $P' - \gamma$, that are cofaces of the cells on γ, we can separate these 2-paths such that they do not meet along γ anymore.*

Proof. In order to separate the 2-paths containing γ on their common boundary, we apply the following steps:

1. We bisect those 1-cells in P' that are mentioned above. Let ν be the terminal point of γ in N.
2. In order to extend the vector field, we pair all the new vertices, but the vertices in the star of ν, with their cofaces in the star of the 0-cells on γ.
3. We pair all the newly introduced vertices in the star of ν with their cofaces if ν is either critical or it is not critical but it is not paired with a 1-cell on the boundary of N.
4. If ν is paired with a 1-cell that is on the boundary of N, then we pair the new vertices, except the one on the edge that is paired with ν, with their unpaired cofaces in the star of ν and we pair the exceptional vertex with its unpaired coface (see Figure (5.2)).
5. Next, we bisect all the 2-cells in P'. Let P'' be the star of the cells on γ in N obtained after the bisections in the above steps.
6. We have already paired the 0-cells and now we pair the 1-cells in P'' with their cofaces in P''.

At the end of the steps above, we extend the vector field to the subdivided cells without creating any cycle by Theorem (2.2.19). □

Lemma 5.2.4. *Let S be the star of a wedge point ω on the boundary of N obtained as in Lemma (5.2.2), and let $S' = S \cap N$. After subdividing the cells on S' that contain ω on their boundary, we can separate the 2-paths in N that meet at the wedge point ω.*

Proof. In order to separate the 2-paths, we construct a smaller copy $\widetilde{S}$ of S' as in the following way:

1. We bisect all the 1-cells in the open star of ω in S' and pair all the new vertices with their cofaces in the star of ω if ω is the critical 0-cell or ω is not critical but it is paired with a 1-cell in $M - N$.

2. If ω is paired with a 1-cell in S', then we pair the vertex that is on this 1-cell with its unpaired coface, and we pair the remaining vertices with their cofaces containing ω.

3. We bisect all the 2-cells in S'. Let S'' be the new star of ω in N obtained after bisecting the 1-cells and 2-cells as in the above steps.

4. We have already paired all the 0-cells and now we pair all the unpaired 1-cells in S'' with their cofaces in S''.

As a consequence of the above steps, we separate the 2-paths which meet at ω, and we can extend the vector field to the subdivided cells without creating any cycle by Theorem (2.2.19) (Figure (5.3) is an example of subdivision and extension of a vector field). □

Recall that our aim is to obtain a single circle as the boundary after following the 2-paths from the critical 2-cell to the critical 1-cells that belong to the $M - (M_1 - \text{int}(D_1))$ part. The next lemma shows that for a pair of critical 1-cells on the same component the resulting boundary components are always connected by 1-paths. We use these connecting 1-paths to reduce the number of boundary components to one.

Lemma 5.2.5. *Let α and β be two of the critical 1-cells in M that give a pair of non-trivial intersecting first homology generators. Suppose that the stars of α, β and the* *unique critical 2-cell do not contain any other critical cells. If the resulting boundary of the subcomplex N obtained by tracing the 2-paths that end in α and β is a disjoint union of 1-spheres, then there must be 0-cells on the boundary components that are paired with interior 1-cells on 1-paths in N that connects each disjoint boundary components.*

Proof. Suppose on the contrary that there is no such 1-path i.e., for such two critical 1-cells α and β assume that the boundary of the region N we obtain tracing the 2-paths from the critical 2-cell has $n > 1$ disconnected boundary components but there is no 1-path connecting the disjoint boundary components. Then by cutting M along the boundary of N we get a triangulated surface of genus 1 containing α and β, and a genus $g_1 + g_2 - 1$ surface with a discrete Morse function defined on them by the restriction of f. This is the case since α and β give a pair of non-trivial intersecting first homology generators. Observe that the genus $g_1 + g_2 - 1$ surface does not have any boundary critical cells by construction, so by Euler characteristic calculations we should have

$$1 - 2(g_1 + g_2 - 1) = 2 - 2(g_1 + g_2 - 1) - n.$$

The above equation yields that $n = 1$, which contradicts with our assumption on the number of disjoint boundary components. □

Next, we consider the case where non-trivial intersecting critical 1-cells produce a connected boundary.

Lemma 5.2.6. *If the resulting boundary of N obtained in Lemma (5.2.5) is a single 1-sphere C, then there can not be any 0-cells or 1-cells on C that are paired with the interior cells of N.*

Proof. By the construction of N, it is easy to see that there can not be any 1-cell on C that is paired with the interior 2-cells of N. Suppose that there are $n \geq 1$, 0-cells on C that are paired with the interior cells of N. Since the unique critical 0-cell of f belongs to $M - \text{int}(N)$, there must be 1-paths in N that begin in these 0-cells and end at some of the 0-cells on C (considering only the parts of these paths in N). Note that these 1-paths may occur as the common boundary of different 2-paths. If this is the case, then we separate the 2-paths such that they do not contain these 1-paths in their boundary anymore and extend the vector field by using Lemma (5.2.3). Next, we form a new subcomplex N'' of the subdivided M by tracing the 2-paths ending in α and β. The boundary of N'' is a union of at least two disjoint 1-spheres. But, this would imply disjoint boundary components with no 0-cells on the boundary that are paired with interior cells of N'' which contradicts Lemma 5.2.5. □

Proof of Theorem 5.2.1. By Lemma (2.1.20) and f being a perfect discrete Morse function we have

$$\begin{aligned} b_0(M) = m_0(f) &= 1, \\ b_1(M) = m_1(f) &= b_1(M_1) + b_1(M_2) = 2g_1 + 2g_2, \\ b_2(M) = m_2(f) &= 1. \end{aligned}$$

Let A be a set of g_2 many pairs of homologically transversal critical 1-cells. In order to find a separating 1-sphere C which satisfies the condition given in the theorem concerning how the cells are paired on it, we apply the following steps in the given order whenever they are necessary:

1. First of all, if the star of a critical cell consists of an other critical cell, we bisect the star to separate the critical cells from each other.
2. Then we construct a subcomplex M_2' of M by following the 2-paths that start in the boundary of the critical 2-cell and end at the critical 1-cells in A by using Lemma (5.2.2).
3. Next, we separate the 2-paths that meet along 1-paths, thus forming disconnected boundary components, using Lemma (5.2.3). (see Figure (5.2) as an example of the separation of 2-paths where the gray region represents M_2' and the red edges represent the resulting boundary).
4. Now, we construct a new subcomplex M_2'' of M(after these modifications) by tracing the 2-paths ending at the cells in A. Since we separate the 2-paths meeting along 1-paths, the resulting boundary of M_2'' is connected. But it might be a union of wedges of several 1-spheres since different 2-paths can meet at a vertex, that is, the 2-cells on these 2-paths can share only a vertex on their common boundary.

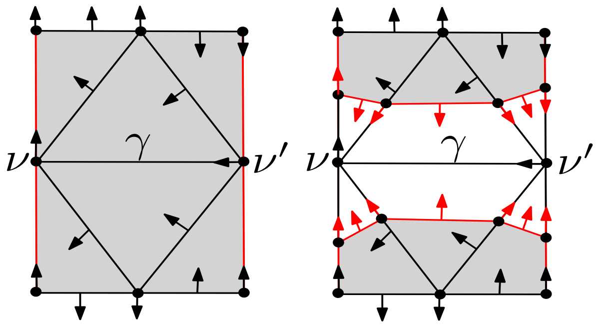

Figure 5.2: A separation of the 2-paths meeting along a 1-path in M_2'.

5. Next, we separate the 2-paths in M_2'' meeting at wedge points on the resulting boundary by using Lemma (5.2.4). Observe that, after this step, none of the 2-paths ending at the critical 1-cells meet neither along 1-paths nor at vertices in this refined M (see Figure (5.3) for an illustration of the separation).

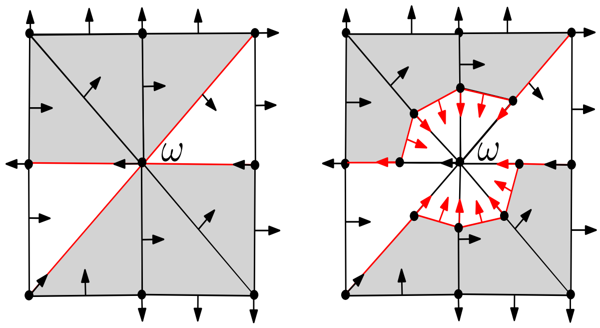

Figure 5.3: A separation of the 2-paths meeting at a vertex ω in M_2''.

6. In the next step, we trace all the 2-paths that begin in the boundary of critical 2-cell and end at the critical 1-cells to form a subcomplex $\widetilde{M_2}$ of M with a boundary 1-sphere. Since we pair the cells obtained after each refinement of M as in Lemma (5.2.3) and Lemma (5.2.4), none of the cells on the boundary 1-sphere, say C, is paired with the interior cells of $\widetilde{M_2}$.

7. For the last thing to check: if C admits the unique critical 0-cell of f, then we bisect the cells in the star of this critical 0-cell in $\widetilde{M_2}$ by using Lemma (5.2.4) to push it into $M - \widetilde{M_2}$.

Finally, we can consider $M_1 = (M - \widetilde{M_2}) \cup_C D_1$ and $M_2 = \widetilde{M_2} \cup_C D_2$, where D_1 and D_2 are 2 dimensional disks with $\partial D_1 \approx \partial D_2 \approx C$. Note that $\widetilde{M_2}$ represents $M - (M_1 - \text{int}(D_1))$ part of the connected sum M. $\square$

Note that if A contains at least one pair of critical 1-cells that does not give non-trivial intersecting first homology generators, we can not find a separating 1-sphere on M that satisfy the conditions given in the statement of the Theorem (5.2.1) due to Lemma (5.2.5) and Lemma (5.2.6).

Now, after these preliminaries we are ready to decompose a perfect discrete Morse function defined on a connected sum of surfaces.

Let $M = M_1 \#_C M_2$ be a connected sum of two 2-manifolds M_1 and M_2 of genera g_1 and g_2, respectively. Let f be a perfect discrete Morse function on M with the induced discrete gradient vector field V so that $V|_{M-M_2}$ has one critical 0-cell and $2g_1$ many critical 1-cells and $V|_{M-M_1}$ has one critical 2-cell and $2g_2$ many critical 1-cells where the notations $M - M_1$ and $M - M_2$ represent $M - (M_1 - \text{int}(D_1))$ and $M - (M_2 - \text{int}(D_2))$, respectively, and D_i for $i = 1, 2$ denote a 2-disk. Assume also that $C \approx \partial(M - M_1) \approx \partial(M - M_2)$ is a separating 1-sphere in M and none of the cells on C are paired with the cells in $M - M_1$ and none of the critical cells of f in M lie on C. By Theorem 5.2.1, we know that such a separating 1-sphere always exists.

Theorem 5.2.7. *Let M be a connected sum given as above. Then $V|_{M-M_1}$ and $V|_{M-M_2}$ can be extended to M_2 and M_1, respectively, as a discrete gradient vector field of perfect discrete Morse functions that agree with f on $M - M_1$ and $M - M_2$ except on the cells in the star of the cells on C.*

Proof. We can extend $V|_{M-M_1}$ to a discrete gradient vector field on $M_2 = (M - M_1) \cup_C D_2$, where D_2 is a triangulated disk with boundary C and an interior vertex ν (as in Figure (5.4)), as in the following way:

1. For each pair (α, β) of $V|_{M-M_1}$ on C, we form a corresponding pair (α', β') where $\alpha', \beta' \in D_2$ are cofaces of α and β, respectively.

2. For each boundary critical cell σ of $V|_{M-M_1}$, we form a pair (σ, σ') where $\sigma' \in D_2$ is a coface of σ.

The vertex ν remains unpaired and is the unique critical 0-cell of the extension V_2 of $V|_{M-M_1}$ to M_2. Note that V_2 is a discrete gradient vector field since $V|_{M-M_1}$ is a discrete gradient vector field by Lemma (2.2.5), there is no cell on C paired with an interior cell of $M - M_1$, and all 1-paths on C end at ν. Let f_2 be a discrete Morse function which induces V_2, then it has the following numbers of critical cells:

$$\begin{aligned} m_0(f_2) &= 1 = b_0(M_2), \\ m_1(f_2) &= b_1(M_2), \\ m_2(f_2) &= 1 = b_2(M_2), \end{aligned}$$

which implies that f_2 is a perfect discrete Morse function. Specifically, we can define f_2 such that it is equal to f on the interior cells of $M_2 - D_2$ and it attains values on the cells in D_2 so that they descend along the gradient paths in $V_2|_{D_2}$.

The following figure shows how one can extend $V|_{M-M_1}$ to D_2.

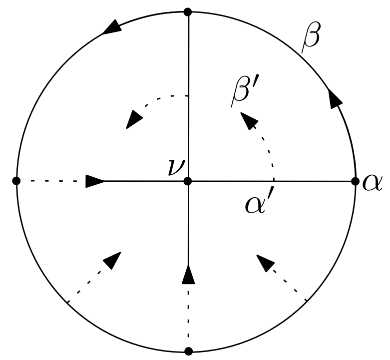

Figure 5.4: The discrete gradient vector field on the disk with the critical 0-cell in the center.

We extend $V|_{M-M_2}$ to a discrete gradient vector field on $M_1 = (M - M_2) \cup_C D_1$, where D_1 is a triangulated disk with boundary C and an interior vertex ω (as in Figure (5.5)), as in the following way:

1. We pair the interior vertex ω with one of the interior edges of D_1.
2. For each remaining interior 1-cell α_i, we form a pair (α_i, β_i), where $\beta_i \in D_1$ is a coface of α_i in counter-clockwise direction.

There must be exactly one 2-cell, say τ, in D_1 that remains unpaired since the number of interior 1-cells and 2-cells are the same. This cell is the unique critical 2-cell of the extension V_1 of $V|_{M-M_2}$ to M_1. Since none of the cells on C are paired with the cells on $M - M_1$, the vector field V_1 is a discrete gradient vector field induced by a discrete Morse function f_1 with the following numbers of critical cells:

$$\begin{aligned} m_0(f_1) &= 1 = b_0(M_1), \\ m_1(f_1) &= b_1(M_1), \\ m_2(f_1) &= 1 = b_2(M_1). \end{aligned}$$

Thus f_1 is a perfect discrete Morse function. Note that f_1 can be defined so that it has the same value with f on the cells in $M_1 - D_1$, and it has a value on the critical 2-cell τ which is big enough, and it has values on the remaining cells in D_1 which descends along the gradient paths in D_1 and which are all greater than the values on C.

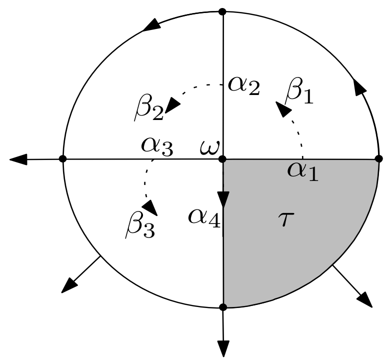

Figure 5.5: The discrete gradient vector field on the disk with a critical 2-cell.

□

Remark 5.2.8. *We can give an alternative proof for the extension of the discrete Morse function $f|_{M-M_2}$ to M_1 as a perfect discrete Morse function as follows:*

We triangulate the disc D_1 with a unique interior vertex v. Then we choose one of the triangles in D_1, say τ. Obviously, $(D_1 - int(\tau)) \searrow C$. We form M_1 so that $(M - M_2) \cup_C D_1 \approx M_1$, and we note that $(M_1 - int(\tau)) \searrow (M - M_2)$. Hence, $f|_{M-M_2}$ can be extended to $M_1 - \tau$ as a discrete Morse function without any new critical cells by Lemma (2.2.6). Let g be this extension of $f|_{M-M_2}$ to $(M_1 - \tau)$. We define a discrete Morse function g' on M_1 that is an extension of g to M_1 as in the following way:

$$g'(\sigma) = \begin{cases} g(\sigma) & \text{if } \; \sigma \in M_1 - \tau, \\ max\{g(\partial\sigma)\} + c & \text{if } \; \sigma = \tau, \end{cases}$$

where c is an arbitrary constant bigger than $g(\partial\sigma)$. Therefore, g' is a perfect discrete Morse function with a unique critical 2-cell T.

The following is an immediate corollary of Theorems (5.2.1) and (5.2.7).

Corollary 5.2.9. *Let $M = M_1 \# M_2$ be a connected sum of two closed, connected, oriented surfaces of genera g_1 and g_2, respectively, and let f be a perfect discrete Morse function on M. We can extend $f_{|_{M-M_2}}$ to M_1 and $f_{|_{M-M_1}}$ to M_2 as perfect discrete Morse functions after some modifications in the star of some necessary cells*

.

To clear up the process described in Theorem (5.2.1), we work it out in the following example.

Example 5.2.10. *Let $M = M_1 \# M_2$ be a connected sum of two tori with a perfect discrete Morse function which induces the gradient vector field on a triangulation of M depicted in Figure (5.6).*

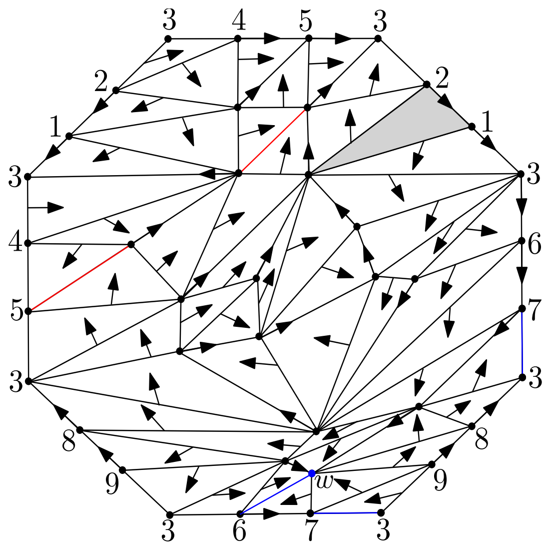

Figure 5.6: A discrete gradient vector field on the genus 2 orientable surface.

The gray triangle is the critical 2-cell, the red and blue edges are the critical 1-cells, and the vertex w is the critical 0-cell for the given discrete gradient vector field. We will form a region $\widetilde{M_2}$ as in Theorem (5.2.1) such that $M_2 = \widetilde{M_2} \cup_C D_2$ is a torus, and C is a separating circle in M.

In Figure (5.7), the blue region denotes the subcomplex M_2' obtained by following the 2-paths beginning in the boundary of the critical 2-cell and ending at the red critical

1-*cells. Different* 2-*paths in* M_2' *meet along the* 1-*path* γ*. Thus the resulting boundary depicted with orange in the figure is disconnected.*

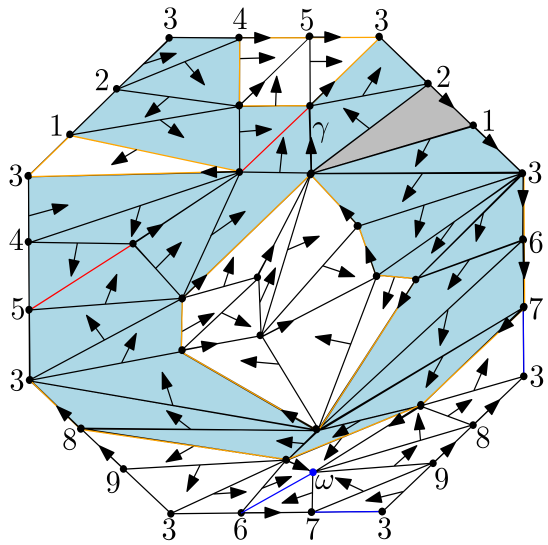

Figure 5.7: The 2-paths ending at the critical 1-cells.

Using Theorem (5.2.1), we separate these 2-*paths by refining* M *and extend the vector field as in Figure (5.8).*

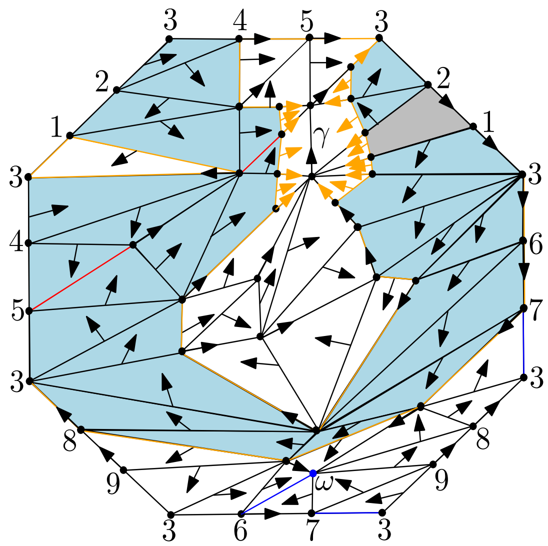

Figure 5.8: A separation of the 2-paths meeting along the 1-path γ.

Now, we trace the 2-*paths up to the red critical* 1-*cells in the refined* M *to obtain a subcomplex* M_2'' *of* M*. Observe that the resulting boundary is connected but a wedge of three circles at the vertex* 3 *which is obtained due to the different* 2-*paths in* M_2'' *that meet at the vertex numbered* 3*. Figure (5.9) denotes the separation of the* 2-*paths meeting at the wedge vertex.*

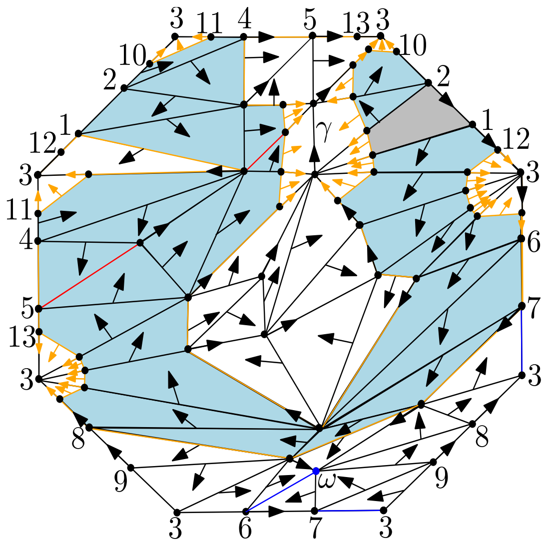

Figure 5.9: A separation of the 2-paths meeting at the wedge vertex 3.

Finally we trace all the 2-paths ending at the red critical 1-cells in the refined M in Figure (5.9) to form $\widetilde{M_2}$ with a boundary which is a 1-sphere. Since we separate the 2-paths in M which give rise to disconnected boundary components or wedge points on the resulting boundary of the traced 2-paths, the boundary of $\widetilde{M_2}$ is a 1-sphere. We can easily see the boundary 1-sphere by starting from an arbitrary vertex, for instance, the vertex numbered 4 and following consecutive vertices on the orange curve in Figure (5.9).

5.3 Decomposing Perfect Discrete Morse Functions on 3-Manifolds

In this section, we prove that one can decompose a $\mathbb{Z}$-perfect discrete Morse function f on a connected sum M of 3-manifolds under a condition on f, which we are going to explain now: The vector field induced by f might produce 2-paths from the critical 2-cells to the critical 1-cells. Throughout our method for decomposition, we first decide how to group the critical cells i.e., which critical cells belong to which component. Then we are going to find a separating sphere as in the 2-dimensional case with a certain arrow configuration on it. After grouping the critical cells, for our purposes, the vector field may produce paths only from one component to the other not from both components to the other. Otherwise we do not know how to extend the vector field completely to each summand in the decomposition. Although we are not going to keep repeating this condition, let us assume that we are working with such a discrete Morse function in this chapter. Moreover, we construct a separating 2-sphere for the connected sum if it admits a $\mathbb{Z}$-perfect discrete Morse function.

But first, let us prove a particular and simpler version of the main theorem of this chapter whose proof uses the ideas introduced in [4, Theorem 6].

Theorem 5.3.1. *Let $M = M_1 \# M_2$ be a connected sum of 3-manifolds M_1 and M_2 with a $\mathbb{Z}$-perfect discrete Morse function f defined on it such that the spine of M induced by f is a wedge of spines of M_1 and M_2. Then M_1 and M_2 have $\mathbb{Z}$-perfect discrete Morse functions f_1 and f_2 defined on them separately such that they agree with f on the spines of M_1 and M_2, respectively.*

Proof. Let K_1 and K_2 be spines of M_1 and M_2, respectively, such that $K = K_1 \vee K_2$ is the spine of M induced by f. Since K is a spine of M, we have $b_i(K) = b_i(M)$ for $i = 0, 1, 2$. Let $g := f|_K$ be the restriction of f to K. By Lemma (2.2.5), g is a discrete Morse function on K. Indeed, it is a $\mathbb{Z}$-perfect discrete Morse function on K by Theorem (3.1.11). Then, we have

$$m_0(g) = b_0(K) = 1,$$
$$m_i(g) = b_i(K) \text{ for } i = 1, 2.$$

Since K_1 and K_2 are subcomplexes of K, the restriction functions $g|_{K_1}$ and $g|_{K_2}$ are discrete Morse functions on K_1 and K_2, respectively, by Lemma (2.2.5). In general, the number of critical i-cells, for $i = 0, 1, 2$, of $g|_{K_1}$ and $g|_{K_2}$ might be bigger than number of critical i-cells of g. Note that, $K = K_1 \vee K_2$ is the spine obtained by collapsing along the discrete gradient paths induced by f, and g does not have any extra critical cells. Thus, $g|_{K_1}$ and $g|_{K_2}$ can not have any extra critical i-cells,for $i = 1, 2$, which are not critical for g. This is the main difference between working with $K_1 \# K_2$ and $K = K_1 \vee K_2$. While working with the connected sum, restrictions to components might have extra critical cells. The function g is a $\mathbb{Z}$-perfect discrete Morse function on K, and so it has a unique critical 0-cell. Let v be the wedge point of K. If the vertex v is the critical 0-cell of g, then it is critical for both $g|_{K_1}$ and $g|_{K_2}$. Thus $m_0(g|_{K_1}) = 1 = b_0(K_1)$ and $m_0(g|_{K_2}) = 1 = b_0(K_2)$. If the vertex v is not a critical 0-cell of g, then the critical 0-cell of g is either in K_1 or in K_2. Assume that the critical 0-cell of g is in K_1. Then the vertex v has to be paired with a 1-cell in K_1 since all 1-paths form a tree rooted at the unique critical 0-cell by Lemma (5.1.3). Hence, v will be a critical 0-cell for $g|_{K_2}$. Then we have

$$\begin{aligned}
&m_0(g) = m_0(g|_{K_1}) = 1 = b_0(K_1),\\
&m_0(g) = m_0(g|_{K_2}) = 1 = b_0(K_2),\\
&m_i(g) = m_i(g|_{K_1}) + m_i(g|_{K_2}) = b_i(M_1) + b_i(M_2) = b_i(M) \text{ for } i = 1, 2.
\end{aligned}$$

As a result, we have the following equalities:

$$m_i(g) = m_i(g|_{K_1}) + m_i(g|_{K_2}) = b_i(K_1) + b_i(K_2) = b_i(M) \text{ for } i = 1, 2.$$

We know that $m_i(g|_{K_j}) \geq b_i(K_j)$, for $i = 1, 2$ and $j = 1, 2$ by Corollary (2.2.11). Then we have

$$\begin{aligned}
m_i(g|_{K_1}) = b_i(K_1),\\
m_i(g|_{K_2}) = b_i(K_2).
\end{aligned}$$

which means that $g|_{K_1}$ and $g|_{K_2}$ are $\mathbb{Z}$-perfect discrete Morse functions on K_1 and K_2, respectively. By Theorem (3.1.11), M_1 and M_2 admit $\mathbb{Z}$-perfect discrete Morse functions that agree with f on the spines K_1 and K_2. □

For an arbitrary perfect discrete Morse function f on M, the spine induced by f is not necessarily a wedge. For such a function, we first show that one can always find a suitable separating 2-sphere in the sense of the following theorem. But first, let us recall the definition and a theorem on the double of a manifold.

Definition 5.3.2 ([31])**.** *Let M be an n-manifold with boundary. The double $2M$ of M is a closed n-manifold obtained by gluing the two copies of M along the boundaries by the identity map.*

Theorem 5.3.3 ([28])**.** *Let M be an orientable 3-manifold with boundary. Then the double $2M$ is an orientable 3-manifold.*

Theorem 5.3.4. *Let $M = M_1 \# M_2$ be a connected sum of 3-manifolds M_1 and M_2 with a $\mathbb{Z}$-perfect discrete Morse function f defined on it. Then we can find a separating 2-sphere S on M such that $M = M_1 \#_S M_2$ and the cells on S are never paired with the cells on $M - (M_1 - int(D_1))$ where D_1 is a 3-disk.*

Before we give the proof of Theorem (5.3.4), we give two lemmas stating that, after some necessary subdivisions, following certain paths from the critical 3-cell we can separate 3-paths that meet along some 1-paths or 2-paths and get a 3-manifold with connected boundary.

Let M and f be given as in Theorem (5.3.4), and C be the set of $b_1(M_2)$ many critical 1-cells and $b_2(M_2)$ many critical 2-cells of f in M. Let P_2 be the set of the gradient 2-paths that end at the critical 1-cells in C. Then the union P_2, the unique critical 3-cell, the gradient 3-paths that start in the boundary of the critical 3-cell and end at the critical 2-cells in C and the gradient 3-paths that end at the 2-cell in P_2 form a subcomplex N of M with boundary.

Lemma 5.3.5. *Let R be the star of the 1-cells and 2-cells on a 2-path γ in N along which different 3-paths meet, that is, γ is on the common boundary of different 3-paths in N. Let $R' = R \cap N$. After some necessary subdivisions on the cells in $R' - \gamma$, that are cofaces of the cells on γ, we separate these 3-paths such that in the subdivision γ is not in the common boundary of the 3-paths.*

Proof. To separate these 3-paths, we follow the steps below in the given order (see also Figure (5.10)):

1. We bisect all the 1-cells in R' that intersects with γ at a single vertex.
2. To extend the vector field, we pair these new vertices with their cofaces that are either in the star of the intersection vertex or with their cofaces outside the star depending on whether the intersection point is not paired with this coface or is paired with this coface.
3. We bisect the 2-cells in R' by connecting the vertices that we introduced after bisections.
4. We pair the remaining 1-cells with their cofaces that intersects with γ.
5. We pair the remaining 2-cells with their cofaces in the star of γ in N.

Finally, we extend the vector field given by f without creating any non-trivial cycle by Theorem (2.2.19). Observe that all the 3-paths in the star of γ end at the regular

2-cells on γ, and thus the 3-paths that end at the cells in C or in P_2 do not contain γ on their boundary. Figure (5.10) is an example of this separation and extension where the blue arrows denote the 3-paths that end at a critical 2-cell in C and contain the 2-path γ on their boundary. The front and back faces of these 3-paths are contained the boundary of N. The figure on the right denotes the separation of these 3-paths where the blue arrows denote the 3-paths that end at the critical 2-cell in C, red arrows show the extension of the vector field. The front and back faces on the boundary of these 3-paths that are colored in gray denote the resulting boundary of a subcomplex N' of M, which is obtained by tracing all 3-paths that end at the critical 2-cells in C and at the 2-cells in P_2 after we separate the 3-paths in N on the left figure.

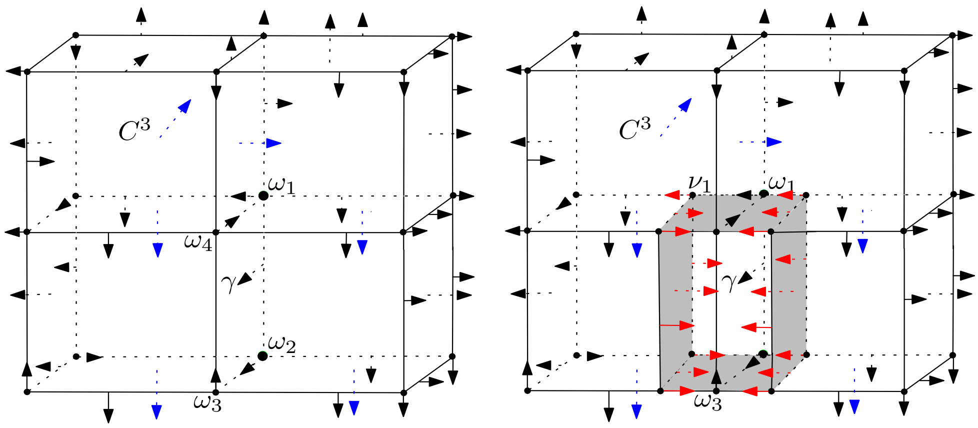

Figure 5.10: A separation of the 3-paths that contain a 2-path on their common boundary.

□

Lemma 5.3.6. *Let S be the star of the cells on an interior 1-path μ in N that is on the common boundary of different 3-paths in N and let $S' = S \cap N$. After some necessary subdivisions on the cells in $S' - \mu$ that are cofaces of the cells on μ, we separate these 3-paths such that μ is not contained in the boundary of these paths anymore.*

Proof. Let $S'' \subset S'$ be the open star of the cells on μ, and σ and σ' be the initial and terminal vertices of μ in N, respectively. In order to separate the 3-paths, we subdivide S' as in the following way (see also Figure (5.11)):

1. We bisect all the 1-cells in S' that intersect μ at a single vertex.and pair these

new vertices with their cofaces in the star of the 0-cells on μ if σ' is not paired a with 1-cell in S'.

2. If σ' is paired with a 1-cell in S', then we pair the new vertex with its unpaired coface and pair the remaining new vertices with their cofaces containing the 0-cells on μ.

3. We bisect the 2-cells in S' by connecting the vertices that we introduced after bisections.

4. We extend the vector field by pairing the remaining 1-cells with their cofaces that contain either the 0-cells or 1-cells on μ.

5. Finally, we pair the remaining 2-cells with their cofaces in the star of μ in N.

Consequently, we extend the vector field without creating any non-trivial cycle using Theorem (2.2.19). Note that all the 3-paths in the star of γ end at regular 2-cells. Thus, we separate the 3-paths such that μ is not contained in their boundary. Figure (5.11) is an illustration of this extension and separation where the orange colored arrows represent the 2-paths in P_2 that end at the orange colored critical 1-cell in C, and blue colored arrows denote the 3-paths that end at the 2-cells in P_2 and admit the 1-path μ on their common boundary. The front and back faces on the boundary of these 3-paths on the left figure form the boundary of N. The figure on the right denote the separation of these 3-paths such that the boundary of these paths does not contain μ anymore. The front and back faces on the boundary of the blue 3-paths with the gray colored cells in the left figure form the boundary of a subcomplex N' of M obtained after we trace these blue 3-paths.

□

Proof of Theorem 5.3.4. Before we start decomposing M in accordance with the given perfect discrete Morse function f on it, let us make an observation about which cells should belong to the same part. Since M is an orientable 3-manifold $H_3(M;\mathbb{Z}) \cong \mathbb{Z}$, and f is perfect, there is a unique critical 3-cell. We think M as the union of $M - M_1$ and $M - M_2$ where the notations $M - M_1$ and $M - M_2$ represent $M - (M_1 - \text{int}(D_1))$ and $M - (M_2 - \text{int}(D_2))$, respectively, and D_i, for $i = 1, 2$, denotes a 3-disk, and we

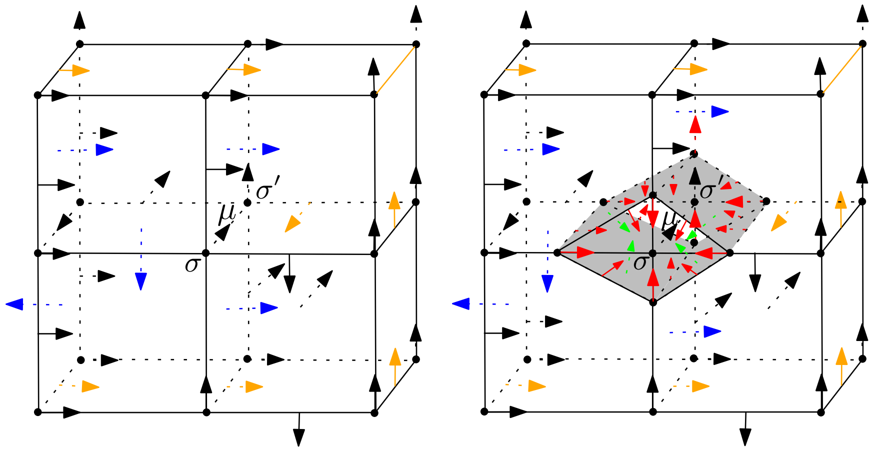

Figure 5.11: A separation of the 3-paths that contain a 1-path on their common boundary.

are going to put this critical 3-cell to the $M - M_1$ part. When we trace back a 3-path from a critical 2-cell to the unique critical 3-cell, we get a solid torus whose core is a 1-dimensional homology generator obtained from the dual cell decomposition. From the cup product structure of M, this homology generator must be homologous to exactly one of the homology generators that we obtain by considering the 1-paths from a critical 1-cell to the unique critical 0-cell of f. We will put these 2- and 1-cells on the same part of M. Then we will form $M - M_1$ part of M with a boundary sphere S which then will serve as a separating sphere on M, and the remaining part will be $M - M_2$.

Let C be the set of the critical 1-cells and 2-cells that belong to the same part of M and C_2 be the set of the 2-paths that end at the critical 1-cells in C.

1. To begin with, if the star of a critical cell consists of another critical cell, we bisect the star to separate these cells.

   Let M_2' be a subcomplex of M that contains the unique critical 3-cell, the 3-paths that end at the critical 2-cells in C and at the regular 2-cells in C_2. Since different 3-paths that form M_2' may meet along some 1-paths and 2-paths which are on their common boundary, there might be some boundary 0-cells and 1-cells that are paired with interior 1-cells and 2-cells of M_2'.

2. Let μ and γ be such 1- and 2-paths that lie in the interior of M_2', respectively, such that initial point of these paths are on the boundary of M_2'. Since the

unique critical 0-cell of f belongs to $M - M_2'$, terminal point of μ in M_2' is on the boundary of M_2'.

3. We first separate the 3-paths such that their boundaries do not contain γ anymore as in Lemma (5.3.5) (see Figure (5.10)). We repeat this process for all boundary 1-cells that are paired with interior 2-cells of M_2'.

4. Next we separate the 3-paths such that μ is not on their boundary anymore by using Lemma (5.3.6)(see Figure (5.11)). We repeat this operation for all boundary 0-cells that form a pair with the interior 1-cells in the locally refined M_2' in step 4.

   After these modifications, let M_2'' be the subcomplex M obtained by tracing all the 3-paths that end at the critical 2-cells in C and end at the regular 2-cells in C_2. Since we separate all the 3-paths which give the existence of the boundary cells in M_2' that are paired with some of the interior cells of M_2', none of the cells on the resulting boundary of M_2'' are paired with the interior cells of M_2''. But the boundary of M_2'' may not be a manifold, that is, there may be some non-manifold edges and vertices on the boundary since two different 3-paths in M_2' might have only edges or vertices on their common boundary.

5. To get rid of these non-manifold cells, we bisect the star of these cells in M_2'' to separate these 3-paths that share non-manifold cells on their boundary and extend the vector field by using Lemma (5.3.5) and Lemma (5.3.6)(see Figure (5.12 for the separation of the 3-paths, which are given with the blue arrows, that cause the existence of a non-manifold edge).

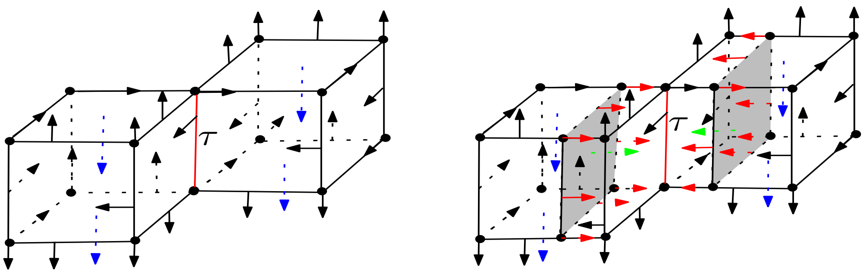

Figure 5.12: A non-manifold edge τ on $\partial(M_2'')$ and a separation of the 3-paths meeting at τ.

6. We construct a new subcomplex $\widetilde{M_2}$ of the modified M by tracing all 3-paths

beginning in the boundary of the unique critical 3-cell and ending at the critical 2-cells in C and at the regular 2-cells in C_2. The resulting boundary of $\widetilde{M_2}$ is a 2-manifold such that none of the cells on it are paired with the interior cells of $\widetilde{M_2}$.

7. If the boundary of $\widetilde{M_2}$ admits the unique critical 0-cell of f, then we may push it into $M - \widetilde{M_2}$ by reversing the a 1-path in $M - \widetilde{M_2}$ that end at this 0-cell.

Let V be the discrete gradient vector field on M obtained after the necessary subdivisions in the above steps. Observe that $\widetilde{M_2}$ and the remaining subcomplex $M - \widetilde{M_2}$ are 3-manifolds with boundary, and the restriction of V to $M - \widetilde{M_2}$ has no boundary critical cells. So the number of critical cells of $V|_{M-\widetilde{M_2}}$ are

$$\begin{aligned} m_0(V|_{M-\widetilde{M_2}}) &= b_0(M_1) = 1, \\ m_1(V|_{M-\widetilde{M_2}}) &= b_1(M_1), \\ m_2(V|_{M-\widetilde{M_2}}) &= b_2(M_1). \end{aligned}$$

Let $2(M - \widetilde{M_2})$ be the double of $M - \widetilde{M_2}$. By Definition (5.3.2) and Theorem (5.3.3), $2(M - \widetilde{M_2})$ is a closed, connected, oriented 3-manifold. Thus, we have

$$\begin{aligned} 0 = \chi(2(M - \widetilde{M_2})) &= 2\chi(M - \widetilde{M_2}) - \chi(\partial(M - \widetilde{M_2})), \\ \chi(M - \widetilde{M_2}) &= 1 - b_1(M_1) + b_2(M_1). \end{aligned}$$

The above equations together imply that $\chi(\partial(M - \widetilde{M_2})) = 2$. Since $\partial(M - \widetilde{M_2})$ is a closed, oriented 2-manifold, by classification of surfaces, it is a 2-sphere S^2.

Finally, the sphere $S = S^2$ is a separating sphere for the two components $M - M_1$ and $M - M_2$ in the connected sum that are represented by $\widetilde{M_2}$ and $M - \widetilde{M_2}$, respectively, in the proof.

□

In the following theorem, we show how to decompose a $\mathbb{Z}$-perfect discrete Morse function defined on a manifold M, which is a connected sum, as $\mathbb{Z}$-perfect discrete Morse functions on each summand. As a consequence of this, we show how to obtain each prime factor of M.

Let $M = M_1 \# M_2$ be a connected sum of 3-manifolds M_1 and M_2 such that neither M_1 nor M_2 is a 3-sphere, and $D_i^3 \subset M_i$ be an embedded 3-disk for $i = 1, 2$. Let f be a $\mathbb{Z}$-perfect discrete Morse function on M and V be the discrete gradient vector field induced by f such that the critical 3-cell, $b_2(M_2)$ many critical 2-cells and $b_1(M_2)$ many critical 1-cells belong to M_2 part, and the remaining critical cells belong to M_1 part of the connected sum. Suppose that $M - M_1$ and $M - M_2$ part of the M is formed as in the proof of Theorem (5.3.4) with a boundary sphere S obtained by modifying V in a neighbourhood of some specific paths. Let W be this modification of V. Note that

$$\begin{aligned} m_0(W|_{M-M_2}) &= 1, \\ m_1(W|_{M-M_2}) &= b_1(M_1), \\ m_2(W|_{M-M_2}) &= b_2(M_2), \end{aligned}$$

and

$$\begin{aligned} m_0(W|_{M-M_1}) &= n_0, \\ m_1(W|_{M-M_1}) &= b_1(M_2) + n_1, \\ m_2(W|_{M-M_1}) &= b_2(M_2) + n_2, \\ m_3(W|_{M-M_1}) &= 1 \end{aligned}$$

where n_i is the number of boundary critical cells of $W|_{M-M_2}$ for $i = 0, 1, 2$.

Theorem 5.3.7. *Under the above conditions, we can extend $W|_{M-M_2}$ to M_1 and $W|_{M-M_1}$ to M_2 as perfect discrete gradient vector fields such that the extensions coincide with V everywhere on $M - M_2$ and $M - M_1$ except some cells around the separating sphere S given above.*

Proof. We will extend $W|_{M-M_2}$ to a discrete gradient vector field on $M_1 = (M - M_2) \cup_S D_1$ where D_1 is a triangulated 3-disk with boundary S and an interior vertex. Let Δ be a 3-cell in D_1. Obviously $(M_1 - \text{int}(\Delta)) \searrow (M - M_2)$. By the construction of S in Theorem (5.3.4), $W|_{M-M_2}$ has no boundary critical cells. Therefore, we can extend $W|_{M-M_2}$ to a discrete gradient vector field W' on $M_1 - \text{int}(\delta)$ without creating any extra critical cells by Lemma (2.2.6). Let g be a discrete Morse function corresponding to W' on $M_1 - \text{int}(\Delta)$. Then a function $\widetilde{g} : M_1 \to \mathbb{R}$ defined as

$$\widetilde{g}(\alpha) = \begin{cases} g(\alpha) & ; \quad \alpha \in M_1 - \text{int}(\Delta) \\ 1 + \max\{g(\partial\alpha)\} & ; \quad \alpha = \Delta \end{cases}$$

is a discrete Morse function on M_1 with the following number of critical cells:

$$\begin{aligned} m_0(\widetilde{g}) &= 1 = b_0(M_1), \\ m_1(\widetilde{g}) &= b_1(M_1), \\ m_2(\widetilde{g}) &= b_2(M_1), \\ m_3(\widetilde{g}) &= 1 = b_3(M_1). \end{aligned}$$

That is, $\widetilde{g}$ is a perfect discrete Morse function on M_1 corresponding the extension of the discrete gradient vector field $W|_{M-M_2}$ on M_1.

Now we will extend $W|_{M-M_1}$ to a discrete gradient vector field on $M_2 = (M - M_2) \cup_S D_2$ where D_2 is a triangulated disk with boundary S and with an interior vertex ω, in the following way:

1. For each boundary critical cell α on $M - M_1$, we form a pair (α, α') where α' is the coface of α in int(D_2).
2. For each $(\sigma, \beta) \in W|_{M-M_1}$, we form a corresponding pair (σ', β') where σ' and β' are the cofaces of σ and β in int(D_2), respectively.

The vertex ω remains unpaired, and it is the unique critical 0-cell of the obtained extension V of $W|_{M-M_1}$ to M_2. Observe that V is a discrete gradient vector field since the cells on S are never paired with interior cells of $M - M_1$. The number of the critical cells of V is as follows:

$$\begin{aligned} m_0(V) &= 1 = b_0(M_2), \\ m_1(V) &= b_1(M_2), \\ m_2(V) &= b_2(M_2), \\ m_3(V) &= 1 = b_3(M_2). \end{aligned}$$

Hence V is a discrete gradient vector field induced by a perfect discrete Morse function on M_2, that is, it is a perfect discrete gradient vector field.

The Figure 5.13 is an example of a discrete gradient vector field $(W|_{M-M_1})|_S$ where v_9 is a critical 0-cell, e_1, e_2 and e_3 are critical 1-cells, and σ_1, σ_2, σ_3 and σ_4 are critical 2-cells.

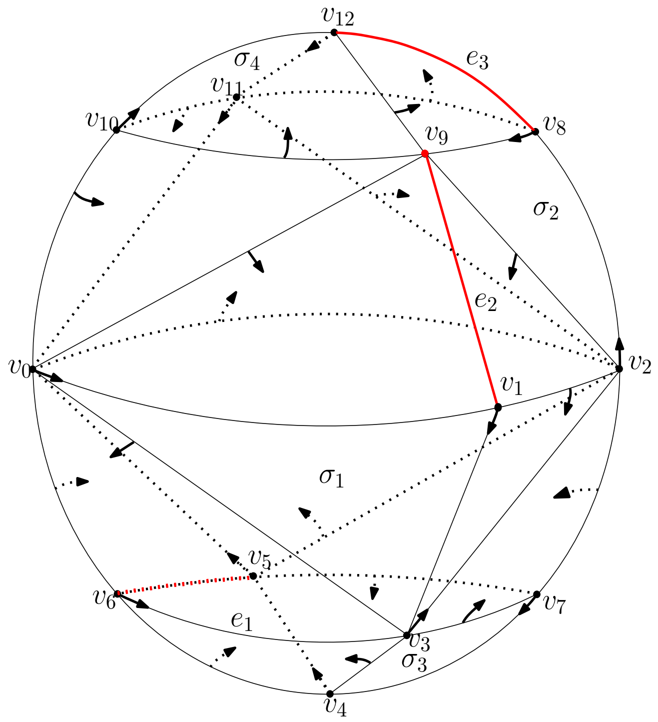

Figure 5.13: A discrete gradient vector field on a sphere S^2.

The vector field on Figure 5.14 denotes the extension of $(W|_{M-M_1})|_S$ to D_2^3 with only one critical 0-cell which is the vertex ω. On this extension, v_9 is paired with the 1-cell $[v_9, \omega]$, e_1, e_2 and e_3 are paired with the 2-cells $[v_5, v_6, \omega]$, $[v_1, v_9, \omega]$, $[v_8, v_{12}, \omega]$, respectively, and σ_1, σ_2, σ_3, and σ_4 are paired with the 3-cells $[v_0, v_1, v_3, \omega]$, $[v_2, v_8, v_9, \omega]$, $[v_3, v_4, v_7, \omega]$ and $[v_{10}, v_{11}, v_{12}, \omega]$, respectively. The pairs of the remaining interior cells depend on the pairs of the cells on their faces on S.

□

Remark 5.3.8. *One should note that we cannot extend the methods that we used in Theorems (5.3.4) and (5.3.7) to higher dimensional manifolds because higher dimensional manifolds cannot be classified only by using the homology groups and Euler characteristic. Moreover, these manifolds do not have a unique connected sum de-*

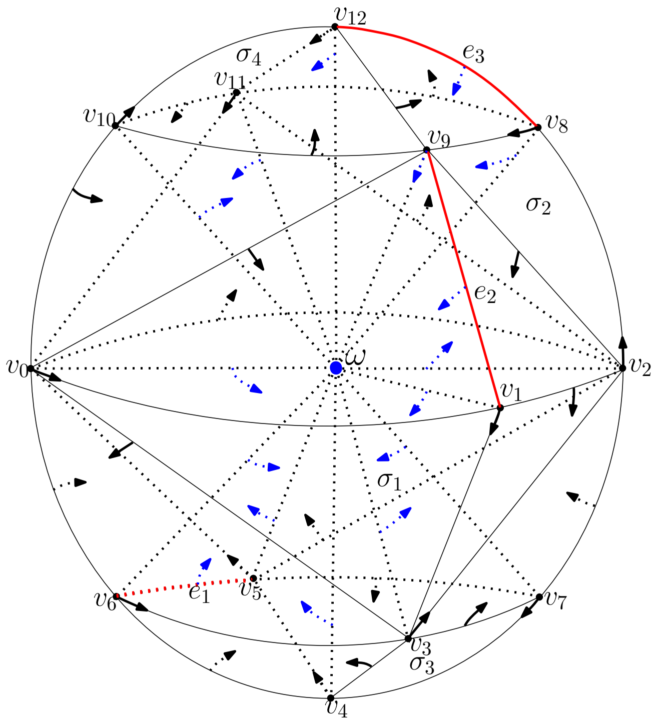

Figure 5.14: A perfect discrete gradient vector field on D^3.

composition as in dimension 3*, see Chapter (2) for an example.*

REFERENCES

[1] K. Adiprasito and B. Benedetti. Tight complexes in 3-space admit perfect discrete morse functions. *European J. Combin.*, 45:71–84, 2015.

[2] R. Ayala, L. M. Fernández, and J. A. Vilches. Characterizing equivalent discrete morse functions. *Bull. Braz. Math. soc.*, 40(2):225–235, 2009.

[3] R. Ayala, D. Fernández-Ternero, and J. A. Vilches. Perfect discrete morse functions on 2-complexes. *Pattern Recognition Letters*, 33(11):1495–1500, 2012.

[4] R. Ayala, D. Fernández-Ternero, and J. A. Vilches. Perfect discrete morse functions on triangulated 3- manifolds. *Ferri M., Frosini P., Landi C., Cerri A., Di Fabio B. (eds) Computational Topology in Image Context. Lecture Notes in Comput. Sci., Springer, Heidelberg*, 7309:11–19, 2012.

[5] B. Benedetti. Discrete morse theory for manifolds with boundary. *Trans. Amer. Math. Soc.*, 364(12):6631–6670, 2012.

[6] B. Benedetti. Discrete morse theory is at least as perfect as morse theory. *available at arXiv:1010.0548v5*, 2014.

[7] B. Benedetti. Smoothing discrete morse theory. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 16(2):335–368, 2016.

[8] S.S. Cairns. Triangulation of the manifold of class one. *Bull. Amer. Math. Soc.*, 41(8):549–552, 1935.

[9] R. Forman. Morse theory for cell complexes. *Adv. Math.*, 134:90–145, 1998.

[10] R. Forman. A user's guide to discrete morse theory. *Sem. Lothar. Combin.*, 48:35, 2002.

[11] S. Harker, K. Mischaikow, M. Mrozek, and V. Nanda. Discrete morse theoretic algorithms for computing homology of complexes and maps. *Found. Comput. Math.*, 14(1):151–184, 2014.

[12] A. Hatcher. *Algebraic Topology*. Cambridge University Press, Cambridge, 2002.

[13] P. Hersh. On optimizing discrete morse functions. *Adv. in Appl. Math.*, 35(3):294–322, 2005.

[14] C. Hog-Angeloni and W. Metzler. *Two-dimensional homotopy and combinatorial group theory*, volume 197 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1993.

[15] G. Jerše and N. M. Kosta. Ascending and descending regions of a discrete morse function. *Comput. Geom.*, 42(6-7):639–651, 2009.

[16] G. Jerše and N. M. Kosta. Tracking features in image sequences using discrete morse functions. *Image-A: Applicable Mathematics in Image Engineering*, 1(1):27–32, 2010.

[17] M. A. Kervaire. A manifold which does not admit any differentiable structure. *Commentarii mathematici Helvetici*, 34:257–270, 1960.

[18] H. King, K. Knudson, and N. M. Kosta. Birth and death in discrete morse theory. *J. Symbolic Comput.*, 78:41–60, 2017.

[19] H. King, K. Knudson, and N. Mramor. Generating discrete Morse functions from point data. *Experiment. Math.*, 14(4):435–444, 2005.

[20] H. Kneser. Geschlossene flchen in dreidimensionalen mannigfaltigkeiten. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 38:248–259, 1929.

[21] G.W. Knutson. A characterization of closed 3-manifolds with spines containing no wild arcs. *Proc. Amer. Math. Soc.*, 21:310–314, 1969.

[22] T. Lewiner, H. Lopes, and G. Tavares. Optimal discrete morse functions for 2-manifolds. *Comput. Geom.*, 26(3):221–233, 2003.

[23] T. Lewiner, H. Lopes, and G. Tavares. Toward optimality in discrete morse theory. *Experiment. Math.*, 12(3):271–285, 2003.

[24] T. Lewiner, H. Lopes, and G. Tavares. Applications of forman's discrete morse theory to topology visualization and mesh compression. *IEEE Transactions on Visualization and Computer Graphics*, 10(5):499–508, 2004.

[25] D. Mcduff. The structure of rational and ruled symplectic 4-manifolds. *J. Amer. Math. Soc.*, 3(3):679–712, 1990.

[26] J. Milnor. A unique decomposition theorem for 3-manifolds. *Amer. J. Math.*, 84, 1962.

[27] K. Mischaikow and V. Nanda. Morse theory for filtrations and efficient computation of persistent homology. *Discrete Comput. Geom.*, 50(2):330–353, 2013.

[28] E.E. Moise. *Geometric topology in dimensions* 2 *and* 3. Graduate Texts in Mathematics, 47. Springer-Verlag, New York-Heidelberg, 1977.

[29] M. Morse. The foundations of the calculus of variations in the large in m-space. *Trans. Amer. Math. Soc.*, 31(3):379–404, 1929.

[30] E. Pitcher. Inequalities of critical point theory. *Bull. Amer. Math. Soc.*, 64, 1958.

[31] D. Rolfsen. *Knots and Links*. Mathematics Lecture Series, 7. Publish or Perish, Inc., Houston,TX, 1990.

[32] J. H. C. Whitehead. Simplicial spaces, nuclei and m-groups. *Proc. London Math. Soc.*, 45(1):243–327, 1939.

[33] J. H. C. Whitehead. On c^1-complexes. *Ann. of Math.*, 41(2):809–824, 1940.

[34] E. C. Zeeman. On the dunce hat. *Topology*, 2:341–358, 1964.

CURRICULUM VITAE

PERSONAL INFORMATION

Surname, Name: Varlı, Hanife
Nationality: Turkish (TC)
Date and Place of Birth: 10.08.1984, Samsun
E-mail: hanife_isal@hotmail.com

EDUCATION

Degree	Institution	Year of Graduation
B.S.	Zonguldak Karaelmas University Mathematics	2009
High School	19 May Super High School	2004

RESEARCH INTERESTS

Algebraic Topology, Computational Topology, Discrete Morse Theory

PROFESSIONAL EXPERIENCE

Year	Place	Enrollment
2010 - 2011	Çankırı Karatekin University Mathematics	Research and Teaching Assistant
2011 - 2017	METU Mathematics	Research and Teaching Assistant