PROSPECTIVE MIDDLE SCHOOL MATHEMATICS TEACHERS' DEVELOPING UNDERSTANDING OF GEOMETRIC TRANSFORMATIONS

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I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

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ABSTRACT

PROSPECTIVE MIDDLE SCHOOL MATHEMATICS TEACHERS' DEVELOPING UNDERSTANDING OF GEOMETRIC TRANSFORMATIONS

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The purpose of this study was to explore prospective middle school mathematics teachers' developing understanding of geometric transformations during the implementation of a seven-week instructional sequence designed to have them gain a mapping understanding of geometric transformations. This instructional sequence involved exploring geometric transformations in both static and dynamic geometry environments.

Participants of the study were sixteen prospective middle school mathematics teachers. Design-based research was conducted to explore their developing understanding of geometric transformations during the implementation of instructional sequence and this development was evaluated by using APOS theory. Moreover, Transformation Geometry Questionnaire developed by the researcher was administered before and after the implementation of instructional sequence.

Findings of the study showed that the participants had difficulties in defining, identifying and performing geometric transformations prior to study. During and after the implementation, they showed development in defining, identifying, and performing geometric transformations.

This study also revealed that the participants were at or below the level of action conception prior to the implementation. Eight out of sixteen participants conceived geometric transformations as one to one and onto functions defined on the plane and thus they gained a complete object conception of geometric transformations at the end of the study. Seven participants reached an understanding that fell within a continuum between process and object conception at the end of the study.

This study contributed to the literature by uncovering several critical ideas required for understanding geometric transformations as functions. Recommendations for future research were also presented.

Keywords: Prospective Middle School Mathematics Teachers, Geometric Transformations, Function, APOS, Dynamic Geometry Environment, Design-Based Research

ÖZ

ORTAOKUL MATEMATİK ÖĞRETMEN ADAYLARININ GEOMETRİK DÖNÜŞÜMLER İLE İLGİLİ GELİŞEN ANLAYIŞLARI

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Bu çalışmada ortaokul matematik öğretmen adaylarına yedi haftalık bir öğretim dizisi uygulanarak onların geometrik dönüşümlerle ilgili gelişen anlayışlarını incelemek ve geometrik dönüşümleri düzlemden düzleme tanımlı fonksiyon olarak ele almalarını sağlamak amaçlanmıştır. Öğretim dizisi hem statik ortamda hem de dinamik geometri ortamında geometrik dönüşümlerin incelenmesini içermiştir.

Araştırmanın katılımcıları on altı ortaokul matematik öğretmen adayından oluşmuştur. Katılımcıların öğretim dizisi boyunca geometrik dönüşümlerle ilgili gelişen anlayışlarını inceleyebilmek için tasarım temelli araştırma yöntemi kullanılmıştır ve bu gelişimleri APOS teorisi kullanılarak değerlendirilmiştir. Ayrıca araştırmacı tarafından geliştirilen Dönüşüm Geometrisi Testi öğretim dizisinin öncesinde ve sonrasında uygulanmıştır.

Araştırmanın bulguları katılımcıların çalışma öncesinde geometrik dönüşümleri tanımlama, belirleme ve gerçekleştirmede güçlük yaşadıklarını ortaya koymuştur. Öğretim dizisinin uygulanması sırasında ve sonrasında katılımcıların geometrik dönüşümleri tanımlama, belirleme ve gerçekleştirmede gelişen bir anlayış göstermişlerdir.

Çalışma ayrıca katılımcıların çalışma öncesinde geometrik dönüşümleri anlamada eylem düzeyinde olduklarını göstermiştir. Çalışmanın sonunda, katılımcılardan sekizi geometrik dönüşümleri birebir ve örten fonksiyonlar olarak algılayabilmiş ve böylece tam olarak nesne düzeyine ulaşabilmişlerdir. Ayrıca, yedi katılımcı süreç ve nesne düzeyleri arasında bir yerde anlayış sergilemişlerdir.

Bu çalışma geometrik dönüşümleri fonksiyon olarak ele almayla ilgili kritik fikirleri ortaya çıkararak alan yazına katkıda bulunmuştur. İleride yürütülebilecek olası araştırmalara yönelik tavsiyelere de değinilmiştir.

Anahtar Kelimeler: Ortaokul Matematik Öğretmen Adayları, Geometrik Dönüşümler, Fonksiyon, APOS, Dinamik Geometri Ortamı, Tasarım Tabanlı Araştırma

To My Son

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LIST OF ABBREVIATIONS

MoNE: Ministry of National Education

CCSSM: Common Core State Standards for Mathematics

NCTM: National Council of Teachers of Mathematics

DGE: Dynamic Geometry Environment

HLT: Hypothetical Learning Trajectory

CHAPTER I

INTRODUCTION

Before the end of the 19^{th} century, Klein (1872) published an influential research program, Erlangen Program, in order to propose a new solution to the problem of how to classify and characterize geometries on the basis of projective geometry and group theory. Besides, Klein's discovery resulted in three effects:

> "First, geometries such as affine geometry, detailed in the Erlanger Program, became subjects to study in themselves. Second, transformation became a fundamental concept in geometry – the means by which geometries could be created and compared. Finally, because Klein lived for over fifty years after the first announcement of his program and was known as an outstanding teacher of teachers, transformations became widely used in Germany and a few other European countries" (Usiskin, 1974, p. 354).

Before the Erlangen Program, geometry was regarded as the exploration of points, lines, planes, and three dimensional space using deductive logic and formal proof (Edwards, 2003). Similarly, Lakoff and Nunes (2000) proposed that until the nineteenth century, space was naturally continuous and they emphasized the following points: the space does not include objects, instead it is regarded as a background setting in which objects are located and is independent of objects located in itself. Similarly, planes are absolutely continuous and are not formed by objects but there are locations on planes where objects can be situated. In a similar fashion, lines are continuous like the trace of a moving point. Finally, points are locations in space, on line, or on planes and they cannot exist independent of the space, line or plane in which they are located.

As a result of the Erlangen Program, the focal point of geometry has shifted from particular objects with concrete, visualizable referents such as points, lines and planes to the concepts of invariance, group theory, and mappings (Edwards, 2003). This made it possible to match discrete numbers to discrete points on lines and in space and thereafter Lakoff and Nunes (2000) coined the term 'discretization program' as a substitute for the Erlangen Program. The naturally continuous conception and the discrete conception of space are opposite of each other and a large proportion of the new discretized mathematics has been created by conceptual metaphors. The space-as-set-of-points metaphor central to the discretization program is presented below.

> "A space is just a set of elements with certain relations holding among the elements. There is nothing inherently spatial about a "space". What are called "points" are just elements of the set of any sort. They are discrete entities, distinct from one another. Like any member of sets, the points exist independently of any sets they are in. Spaces, planes, and lines – being sets – do not exist independently of the points that constitute them. A line is a set of points with certain relations holding among the points. A plane is a set of points with other relations holding among the points. A geometrical figure, like a circle or a triangle, is a subset of the points in a space, with certain relations among the points. There is thus nothing inherently spatial about a circle or triangle. For example, a circle is a subset of the elements of the space with certain relations to one another" (ibid, p. 263).

Naturally continuous conception of space and the set of points conception of space are different. The former is natural to our conceptualization of the real world we operate in and thus it is formed unconsciously and automatically. However, the latter is a reconceptualization of the former one and thus it has to be intentionally constructed by the individual (Lakoff & Nunes, 2000).

Inspired by the aforementioned two conceptualizations of space, Edwards (1991, 1992, 1997) investigated students' understanding of geometric transformations and found that students' understanding was quite different from that of contemporary mathematicians. According to Edwards (2003), the characteristics of learners' understanding of transformations refer to motion conception while the characteristics of mathematicians' understanding of transformations refer to mapping conception. Edwards' (2003) comparison of learners' and mathematicians' understanding of geometric transformations is presented in Table 1.1.

Table 1.1. The comparison of learners' and mathematicians' understanding of geometric transformations

Learner's Transformations	Mathematician's Transformations
The plane is an empty, invisible *background*	The plane is a set of points
Geometric figures sit *on* the plane	Geometric figures are subsets of points of the plane
Transformations are *physical motions* of geometric figures on top of the plane	Transformations are *mappings* of all the points of the plane
The "objects" of geometry are points, lines, circles, triangles.	The "objects" of geometry are groups of transformations

(Edwards, 2003, p.8, emphasis added)

In terms of the discretization program, there are no motions in geometric transformations of the plane. Instead, transformations are viewed as the mappings of the whole plane (Edwards, 2003). Russell (2009) made a similar point that the meanings attributed to the term "motion" in geometry and in daily life are completely different from each other. According to him, in mathematics, motion "is not something which changes, but is usually, on the contrary, something incapable of change" and he added that "a motion is a one-one relation, in which the referent and the relatum are both points, and in which every point may appear as referent and again as relatum" (p. 412). Coxeter and Greitzer (1968) defined transformation as "mapping of the whole plane onto itself so that every point P has a unique image P', and every point Q' has a unique prototype Q" (p. 80). They added that the idea of mapping played a prominent role in many areas of mathematics, to illustrate in the equation y = f(x), the set of values of x are mapped onto the set of corresponding values of y. In addition, Martin (1982) gave a more detailed definition of the concept of mapping. He stated that "mapping f is said to be *onto* if for every point P there is a point Q such that $f(Q) = P$; mapping f is said to be one-to-one if $f(R) = f(S)$ implies $R = S$. So, a transformation is just a mapping on the points that is both one-to-one and onto" (p. 2).

As implied by Martin's (1982) definition of mapping, geometric transformations and functions are interconnected concepts. This connection between the two concepts has gained some recognition and helping students to make this connection is advocated by several national and international curriculum documents (e.g., Common Core State Standards for Mathematics [CCSSM], 2010; Ministry of National Education [MoNE], 2013). For instance, CCSSM (2010) gives emphasis to students describing “transformations as functions that take points in the plane as inputs and give other points as outputs” (p. 29). Despite this, studies consistently report that learners’ understanding of transformations does not change significantly over the elementary to secondary and to undergraduate years (e.g., Edwards, 2003; Harper, 2003; Yanık, 2011, 2013). Thus, learners’ understanding of geometric transformations in general and the shift in their understanding from motion to mapping conception in particular was described briefly henceforth in the light of limited studies at hand.

1.1. Learners’ Understanding of Geometric Transformations

There are many studies examining learners’ pre-existing knowledge or understanding of geometric transformations (e.g., Hollebrands, 2004, Ramful, Ho & Lowrie, 2015, Thaqi, Gimenez, & Rosich, 2011, Yanık, 2011; Yanık, 2014). Research has shown that both students and prospective teachers had similar difficulties related to geometric transformations. Some of these difficulties were related to defining transformations with their parameters (Hollebrands, 2004; Thaqi et al., 2011; Yanık, 2011; Yanık, 2014). For instance, Thaqi and others (2011) found that when defining rotation, prospective elementary teachers mentioned only the angle of rotation but not the center of rotation. Similarly, high school students’ definition of rotation included the angle of rotation explicitly, while the center of the rotation seemed to be implicit (Hollebrands, 2004). Similarly, when defining translation, neither middle school students nor prospective middle school teachers mentioned the magnitude or the direction of the translation vector (Yanık, 2011, 2014).

Some of the other difficulties that the learners possessed were related to identifying transformations (e.g., Harper, 2003; Hollebrands, 2004; Thaqi et al., 2011). For instance, Harper (2003) stated that when a figure and its rotated image were presented, none of the four prospective elementary teachers could identify a single rotation. The prospective elementary teachers tried to explain this rotation by a combination of simple steps without identifying any specific center and angle of rotation.

Learners had difficulty in performing transformations as well. For instance, in performing translation middle or high school students (e.g., Hollebrands, 2004; Yanık, 2014) and prospective teachers (e.g., Harper, 2003; Yanık, 2011) failed to use the translation vector. Both middle school students and prospective teachers used the translation vector as a reflection line or as a direction indicator (Yanık, 2011, 2014) and some of the high school students used it as a location on which the figure should be placed (Hollebrands, 2004).

Learners also had difficulties in considering the relationship among a pre-image, a parameter(s), and an image in geometric transformations (e.g., Hollebrands, 2004). For instance, Hollebrands (2004) and Ramful and others (2015) reported that while performing reflection, learners were not able to carefully consider that reflection line must be the perpendicular bisector of the segments formed by joining corresponding pre-image and image points. They added that learners had more difficulty in performing reflection with inclined reflection lines than with horizontal and vertical reflection lines. Ramful and others (2015) ascribed leaners' erroneous conception of reflection related to inclined reflection lines to their intuitive understanding of perpendicularity as being opposite. They further explained that without formally knowing the criterion of perpendicularity, learners could reflect pre-images along vertical and horizontal lines by the help of the intuitive idea of being opposite and that this intuitive idea did not work in performing reflection with inclined reflection lines.

Students' performance in tasks involving inclined reflection lines might also be influenced by the nature of the object. For instance, Ramful and others (2015) found out that reflecting a line segment along an inclined reflection line was more challenging for students than reflecting geometric shapes. They indicated that the reflection of a geometric figure as a whole was easier for students to understand and attributed this to the fact that a geometric figure as a whole had more defining properties when compared to an individual line segment (Ramful et al., 2015). On the other hand, Kuchemann (1981) revealed that reflecting a point along an inclined reflection line was easier for students compared to that of a line segment.

Another difficulty that learners have is related to using appropriate mathematical terminology in describing geometric transformations. To give an example, Harper (2003) found out that prospective elementary school teachers did not incorporate mathematical terminology in describing geometric transformations and that they rather used informal, everyday language such as “flip, flop, fold, shift, move, slide, turn, swing, and pivot” in explaining those transformations (p. 2912).

Research studies revealed that learners from different ranges of ages mainly held motion conception of transformations (e.g., Edwards, 2003; Glass, 2001; Hollebrands, 2003, 2004; Portnoy, Grundmeier & Graham, 2006; Thaqi et al., 2011; Yanık; 2011, 2014). Edwards (2003) indicated that learners from elementary to undergraduate level “had the same initial expectations of how transformations would work, and they made the same kinds of errors” (p. 4). She explained that learners holding a motion conception of transformations viewed the plane as an invisible background, thought that geometric figures sit on this plane, and considered transformations as physical movements of geometric figures on the top of the plane. For instance, in Yanık’s (2011, 2014) studies middle school students and prospective middle school mathematics teachers had motion conception of translations. Actually, they misinterpreted the motion conception itself since nearly half of them misconceived that rolling was a key aspect of translation in addition to displacement. Middle school students gave “rolling a ball or a movement of a tire” as examples of translation (Yanık, 2014). Besides, prospective teachers argued that sliding of an object on a smooth plane without rolling would be a non-example of translation (Yanık, 2011). These studies confirmed Edwards’ (2003) argument that overall understandings of learners from elementary to undergraduate levels are similar to each other in geometric transformations. Edwards (2003) indicated that learners’ understanding of transformations originated from their “embodied experience” in the physical world. She further stated that “the embodied, natural understanding of motion that the learners brought to the experience is the source of their misconceptions. These ‘misconceptions’ are in actuality, conceptions that are adaptive and functional outside of the context of formal mathematics” (p. 9).

1.2. Purpose Statement and Research Questions

Research studies on learners’ developing understanding of geometric transformations show that it is challenging for learners to reach a complete mapping understanding of geometric transformations and that they need more time and experience to develop coherent understanding of transformations (Yanık, 2006). For instance, although Hollebrands (2003) studied with four high achieving students in geometry, only one of them reached an object conception of geometric transformations (i.e., the idea that transformations are one-to-one and onto functions), while two of them began to show an improvement towards mapping understanding of transformations.

Although, earlier studies provide some ideas about high school students' and prospective middle school mathematics teachers' developing understanding of geometric transformations (e.g., Hollebrands, 2003; Yanık & Flores, 2009; Yanık 2013), it is still unclear whether other learners might follow similar paths and how they progress from motion to mapping conception. There are several studies in which researchers developed instructional units in order to help a small number of learners' (i.e., three or four participants) progress from motion to mapping conception through individual teaching experiments (e.g., Hollebrands, 2003; Yanık & Flores, 2009). However, it needs to be further examined how learners show progress in a much more populated classroom environment. Moreover, further studies need to focus on the use of dynamic geometry environments in the teaching of geometric transformations (Hollebrands, 2003). Therefore, I went further in this direction and tried to fill these gaps by exploring the growth in a cohort of prospective middle school mathematics teachers' understanding of geometric transformations in static and dynamic environments. Therefore, the following research question was sought to answer in the current study:

1. How do prospective middle school mathematics teachers' understanding of geometric transformations develop during the instructional unit?
    a. How do prospective mathematics teachers' competencies to identify geometric transformations develop as they engage in the instructional unit?
    b. How do prospective mathematics teachers' competencies to perform geometric transformations develop as they engage in the instructional unit?
    c. How do prospective middle school mathematics teachers' understanding of geometric transformations develop in action-object continuum?

1.3. Definitions of Important Terms

The current study was delimited to exploration of prospective teachers' understanding of reflections, translation, and rotations. Hence, the definitions and properties of these transformation types are presented in more detail in the following parts.

Transformation

"A transformation on the plane is a one-to-one correspondence from the set of points in the plane onto itself. For a given transformation f, this means that for every point P there is a unique point Q such that $f(P)=Q$ and, conversely, for every point R there is unique point S such that $f(S)=R$" (Martin, 1982, p.1)

Reflection

"Reflection R_l is a transformation on the plane defined by $R_l\,(P)=P$ if point P is on the line l and $R_l(P)=Q$ if l is the perpendicular bisector of $\overline{PQ}$. Point Q is called reflection of point P along line l." (Martin, 1982, p.14)

This definition implies that a reflection is defined by a line (i.e., the parameter of a reflection) and this line is called a reflection line. Since all points in the plane are reflected along this line, no identity function (i.e., a specific reflection that maps all points in the plane onto themselves) exists in the family of reflection. However, points on the reflection line are fixed under any reflection. That is, these points map to themselves in all reflections.

The inverse of a reflection is equal to itself. That is, if the image of a point is reflected along the same line then the reflection of the image coincides with that point.

Translation

Translation $T_{\vec{v}}$ is a transformation on the plane defined as $Q=T_{\vec{v}}\,(P)=P+\vec{v}$ such that $\vec{v}=\overrightarrow{PQ}$. Point Q is called translation of Point P with $\vec{v}$ (Argün, Arıkan, Bulut & Halıcıoğlu, 2014).

As the definition implies, the parameter of a translation is a vector and it is called a translation vector. If the translation vector is $\vec{v}=(0,0)$, then all points in the plane map to themselves. That is, identity function of family of translations is $T_{\vec{v}}$

such that $\vec{v} = (0,0)$. However, for all members of translation family, none of the points in the plane remain fixed except for identity translation.

For each member of translation family, there exists a translation which is equal to the inverse of that translation. If the translation vector of a translation is $\vec{v}$, then the translation vector of the inverse of this translation is $-\vec{v}$.

Rotation

Rotation $R_{C,\alpha}$ is a transformation on the plane defined by $Q = R_{C,\alpha}(P) = (x.cos\alpha - y.sin\alpha\, , x.sin\alpha + y.cos\alpha)$. Point Q is called rotation of point P(x, y) around the origin (i.e., point C) by angle α (Argün et al., 2014).

This definition implies that the parameters of a rotation are a point and an angle which are called the center of rotation and the angle of rotation. If the angle of rotation is equal to 0° or to a multiple of 360° then all points in the plane map to themselves. That is, identity function of family of rotations is $R_{C,\alpha}$ such that C is any point on the plane and α is equal to 0° or to a multiple of 360°. However, for all members of rotation family, the center of rotation always remains fixed after rotation regardless of the angle of rotation.

For each member of rotation family, there exists a rotation which is equal to the inverse of that rotation. Regardless of the center of rotation, if the angle of rotation is α, then the angle of rotation of the inverse of this rotation is –α.

1.4. Significance of the Study

Transformation geometry includes construction and manipulation of mental images (Leong & Lim-Teo 2003). Therefore, working with transformation geometry would improve skills in spatial and geometric reasoning (Clements & Battista, 1992; Edwards, 1997). Moreover, it has the potential to foster students' reasoning and justification skills and it gives learners opportunities to describe patterns, make generalizations and develop spatial abilities (Portnoy et al., 2006). Transformation geometry emphasizes sensitivity, conjecturing, transformation, inquisitiveness and helps students to explore the abstract mathematical concepts of congruence, symmetry, similarity, and parallelism; enrich their geometrical experience, thought and imagination; and thus improve their spatial abilities (Peterson, 1973).

Transformation geometry can be used as an approach to teaching a variety of school geometry topics such as angles, congruent triangles, circles, areas and perimeters as well (e.g., Kort, 1971; Usiskin, 1972, 1974; Usiskin and Coxford, 1972). The use of transformations based approach proves useful in consolidating students' knowledge of geometry concepts (Usiskin, 1972) and in the in greater retention of topic such as congruence, similarity, and symmetry (Kort, 1971). Geometric transformations might be used in the teaching of other school mathematics topics as well. Usiskin (1974) argued that transformations might be used as a subject that integrates various school mathematics concepts such as complex numbers, trigonometric functions, logarithms, vectors and so forth. He advocated that the use of transformation approach in the teaching of mathematics in that it made proofs very simple to understand, it benefited much to slow pacing students and consequently appeared to provide a promising avenue for developing stronger and clearer understandings in school mathematics.

Transformation geometry can also be viewed as a basis for advanced geometry topics. Specifically, the mathematical properties used in several fields of geometry such as topology, projective geometry and affine geometry can be described by means of transformations (Williford, 1972). Hence, the concepts of transformation and invariance are regarded as fundamental notions that dominate all mathematics and science (Dienes & Golding, 1967; Kapur, 1970 as cited in Williford, 1972).

Geometric transformations are also used in the teaching of tertiary-level mathematics topics such as groups, matrix algebra and so forth (Hollebrands, 2003; Küchemann, 1991). For instance, Küchemann (1991) expressed the importance introducing transformations to learners as providing insights into mathematical structures of advanced level concepts, especially the structure of the concept of group, by having them discover generic rules about composition of transformations. In addition, Küchemann asserted that geometric transformations would provide an explicit reification of matrix algebra and emphasized that this would be acknowledged as an influential manifestation of the uniformity of mathematics.

Transformations make it available for students to apply the idea of function which is one of the most important concepts pervading all mathematical topics

(Jackson, 1975). More specifically, thinking transformations as “a one-to-one correspondence from the set of points in the plane onto itself” (Martin, 1982, p. 1) has been a central idea in approaching geometric transformations as functions. Hollebrands (2003) stressed that acquisition of this idea is important for students to understand the domain and range of transformation and to understand what is needed for a transformation to be one-to-one and onto. By focusing on geometric transformations as functions, several researchers proposed that learners' understanding of transformations proceeds from motion understanding to mapping understanding (e.g., Hollebrands, 2003; Yanık & Flores, 2009; Thaqi et al., 2011). In studies that implemented instructional units on prospective teachers, it was found out that their understanding of geometric transformations developed, they realized the role of parameters (i.e., vectors, line of symmetry, and center of rotation) in transformations. Moreover, they were able to understand that domain and plane are key concepts in understanding transformations as functions (e.g., Harper, 2002; Portnoy et al., 2006; Yanık 2006; Yanık & Flores, 2009).

In this sense, the ideas taught to children for learning transformation geometry in elementary or middle grades can be transferred to the ideas about relations and functions in high school. Conversely, ideas about relations and functions can also be used in the teaching of ideas about geometry transformation. Thus, it is possible to say that there is a reflexive relationship among functions and geometric transformations. Indeed, Usiskin (1969) and Kort (1971) have focused on this idea and developed a one year curriculum for the purpose of teaching tenth grade geometry topics with transformations based approach. Their results showed that transformation approach helped students learn better when compared to those who learnt by non-transformation geometry approach. Similarly, transformation geometry approach students had higher achievement in relations, functions, and inverse functions. Kort (1971) suggested that the continuity of the mathematics curriculum from primary to university level mathematics is desired, and mathematicians or mathematics educators should emphasize transformations in the school or college curriculum. If it is agreed upon that transformations do or can play a significant unifying role, the mathematics curriculum at all grade levels and areas, not just geometry, should integrate transformations and concepts that are strongly associated with each other. Hollebrands (2003) made the same point about geometric transformations and stated that the study of geometric transformations is vital since "it provides opportunities for students to think about important mathematical concepts (e.g., functions, symmetry); it provides a context within which students can view mathematics as an interconnected discipline and it provides opportunities for students to engage in higher-level reasoning activities using a variety of representations" (p. 55).

Due to their importance, Turkish school mathematics curricula also give considerable emphasis on geometric transformation concepts through elementary to high school. Transformation geometry has been integrated into middle school mathematics curriculum as a sub-learning domain of geometry after the implementation of reform movements in Turkey in 2005. Transformation geometry topics has maintained their importance thereafter. Before starting high school, students need to have essential knowledge about geometric transformations. That is, by the end of 8^{th} grade, students are expected to translate, reflect and rotate geometric objects and determine the images of those objects in a coordinate plane. However, research studies showed that prospective middle school teachers had difficulties in defining and identifying geometric transformations with their parameters and in performing geometric transformations (e.g., Harper, 2003; Yanık, 2011). Similarly, these concepts might be regarded as novel concepts for most of the prospective middle school mathematics teachers participated in this study since they have not been formally taught transformations in their elementary, middle or high school years. Particularly, in Turkey, since transformation geometry has been put into practice after 2005, it is not possible that the participating prospective teachers were taught geometric transformations in their middle or secondary school years at least in this content. Besides, there is no such compulsory and comprehensive courses related to transformations in Turkish teacher education programs. Therefore, prospective teachers need to remedy their difficulties related to geometric transformations before they start their teaching profession and teach these concepts to their students. By this way, they can have the knowledge of geometric transformations within the scope of the content they will teach in their teaching.

The quality of prospective teachers' teaching of geometric transformations is to a certain extent dependent upon the breadth and depth of their knowledge of this topic. More explicitly, since teachers' knowledge of mathematics is crucial for effective classroom instruction (Ball, Hill & Bass, 2005), prospective teachers must reconceptualize that geometric transformations are mappings on the plane as indicated by the formal definitions of geometric transformations. Briefly, prospective teachers should have mapping conception of geometric transformations rather than motion conceptions to provide a more quality instruction to their students (Edwards, 2003).

Gaining mapping conception of geometric transformations might influence prospective teachers' selection and generation of examples related to this topic during actual classroom practices. Example selection plays a critical role in the teaching and learning of mathematics since well-chosen examples may improve students' learning whereas the poor ones might impede their understanding (Zodik & Zaslavsky, 2008). Thus, it is challenging for teachers to generate carefully planned examples in that it requires them to take into consideration many aspects related to the specific content being taught (Zaslavsky & Zodik, 2007). When prospective teachers largely adopt the formal definition perspective, they may not experience some possible challenges in the course of selecting geometric transformation examples that can better convey the content to their students. Similarly, prospective teachers' understanding of geometric transformations might have some influence on careful selection and designing of activities implemented in the classroom.

Besides, teaching does not only involve delivering predetermined curriculum rather, it involves paying attention to students' questions, predicting some possible student difficulties and dealing with unexpected classroom events (Rowland & Zazkis, 2013). Thus, gaining knowledge of geometric transformations beyond the curriculum is crucial for prospective teachers in dealing with the unexpected and contingent events in the classroom.

Edwards (2003) indicated that motion conception originates from learners' "embodied experience" in the physical world, and in order to have a mapping conception one needs to reconceptualize his/her understanding of geometric transformations. Exposing prospective middle school mathematics teachers to the formal teaching of geometric transformations in teacher education programs is important in that such instruction might help them become aware and reconceptualize their understanding of geometric transformations before they start their teaching career. However, a large proportion of mathematics teacher education programs do not overtly speak to this issue and do not systematically train prospective teachers to cope with transformation geometry concepts in an educated way. As stated by Leikin and Zazkis (2010), systematic knowledge is obtained through studying mathematics and pedagogy at teacher education programs and universities. Therefore, in order to have prospective middle school mathematics teachers gain systematic knowledge of geometric transformations before they start their teaching profession, there is a need to design courses in teacher education programs that might improve their knowledge of geometric transformations and that help them shift from motion conception to mapping conception.

Due to the aforementioned reasons, designing an instructional unit that helps prospective teachers improve their knowledge of geometric transformations and that helps them shift from motion conception to mapping conception is significant. Therefore, in this study a seven-week instructional sequence was developed in order to explore the growth in a cohort of prospective middle school mathematics teachers' understanding of geometric transformations in the context of a semester long course. This instructional sequence includes seven activities and related pre-constructed, dynamic geometry sketches. Therefore, this study is significant in terms of attempting to prove a carefully designed, tested, and revised instructional sequence that can be used in other classrooms aiming at remedying learners' difficulties related to geometric transformations and focusing on teaching geometric transformations as mappings on the plane.

This study is significant in terms of adding and testing new critical ideas (i.e., ideas related to functional dependency of points in the plane) for having learners gain mapping conception of geometric transformations as well. In designing the instructional sequence, the ideas reported in the literature as critical for having learners gain mapping conception of geometric transformations were took into consideration. Although Yanık (2006) and Hollebrands (2003) explained several critical ideas required for understanding geometric transformations as mappings on the plane, these ideas did not suffice to reach a complete mapping conception of geometric transformations for their participants and the picture was still incomplete.

Namely, it is still unclear what would be a more complete mapping conception of geometric transformations and how might learners progress from motion to mapping conception of transformations in the context of a course. Therefore, the researcher of the current study conjectured that understanding functional dependency of points in the plane would help prospective teachers understand geometric transformations as functions and tested this idea through designing and implementing activities. Therefore, this study is significant in terms of its contribution to the research on critical ideas for understanding geometric transformations as mappings on the plane.

In previous studies, researchers developed instructional units in order to help a small number of learners (i.e., three or four participants) progress from motion conception to mapping conception through individual teaching experiments (e.g., Hollebrands, 2003; Yanık & Flores, 2009). These studies provide some ideas about learners' developing understanding of geometric transformations. However, learners' development in a classroom environment may differ from that in the individual teaching experiments. Therefore, it is still unclear whether learners might follow similar paths in a much more populated classroom environment and it needs to be explored how learners progress from motion to mapping conception in the classroom context. Thus, this study is significant since it attempted to fill this gap by investigating sixteen prospective middle school mathematics teachers' developing understanding of geometric transformations in a classroom environment that require participation in whole class discussions.

In this study, both static and dynamic geometry environments were used to enhance prospective teachers' understanding of geometric transformations. Most of the tasks were developed in a way that participants study first in a paper-pencil and then in a dynamic geometry environment. Thus, tasks involving use of a dynamic geometry environment and their implementations in the context of a classroom environment might provide some ideas about how a dynamic geometry environment can be used in improving prospective teachers' knowledge and understanding of geometric transformations.

1.5. Instructional Unit

In this study, the use of design research methodology was important in terms of achieving my purposes and goals. By the help of the instructional sequence I devised, prospective teachers explored geometric transformations in both dynamic geometry environment and static environment. The instructional sequence is explained in more detail in the methodology chapter. Briefly, it included five phases and in the first phase prospective teachers were presented finite figures and frieze patterns that were constructed by using geometric transformations. Participants had an opportunity to explore these geometric transformations in pair work and to share their ideas in whole class discussions. In the second phase of the instructional sequence, prospective teachers had a chance to perform single geometric transformations both in a paper and pencil environment with a compass and a ruler and in a dynamic geometry environment. By the help of the pre-constructed GeoGebra files, participants explored the specific properties of reflection, rotation and translation dynamically. Similarly, composition of two or more transformations were first studied in a paper and pencil environment with a compass and a ruler and then they were studied through pre-constructed GeoGebra files. By this way, they had an opportunity to see the relations within and across the three geometric transformation types. At the end of the second phase, prospective teachers were expected to have a full understanding of transformations from a geometric perspective. By the help of the first two phases, prospective teachers were expected to be ready for learning complex ideas about transformations in the following phases.

The third phase of the instructional sequence helped prospective teachers systematically learn functional thinking on its own. In a dynamic geometry environment, they explored the dependent variable, independent variable, and the notion of function as a relationship between these variables. This phase helped prospective teachers explore functional ideas in a dynamic geometry environment, and thus they had a deeper and more robust understanding of functions. The first three phases of the instructional sequence formed the basis of the subsequent phases.

The fourth and fifth phases of the instructional sequence which formed the main focus of the current study, provided prospective teachers opportunities to make connections among geometry and algebra. In other words, these phases of the instructional sequence focused on having prospective teachers connect geometric transformations with ideas of algebraic functions. Thus, they were expected to build a mapping understanding of transformations. In brief, after the implementation of the instructional sequence, prospective middle school mathematics teachers' knowledge of functions would become integrated with geometry and by this way, they would have deeper and more connected knowledge of mathematics. Consequently, their unified mathematics knowledge might influence their future students' learning of mathematics positively.

CHAPTER II

LITERATURE REVIEW

The aim of this study was to explore prospective middle school mathematics teachers' developing understanding of geometric transformations. In light of this purpose, this chapter was presented under four main parts. First, an overview of geometric transformations and two different conceptualizations of geometric transformations are presented. Second, more detailed information about understanding geometric transformations as functions (i.e., mapping conception) are presented. Third, the teaching of geometric transformations and functions through dynamic geometry environments is presented. Finally, related studies conducted with middle or high school students and prospective teachers regarding their pre-existing and developing understanding of geometric transformations are reviewed. The related studies are reviewed under the following main titles: students' understanding of geometric transformations; prospective teachers' understanding geometric transformations; and development in learners' knowledge and understanding of geometric transformations.

2.1. An Overview of Geometric Transformations and Motion and Mapping Conceptions

Battista (2007) defined geometry as "complex interconnected network of concepts, ways of reasoning, and representation systems that is used to conceptualize and analyze physical and imagined spatial environments" (p. 843). Geometry is a means of providing students with the ability to reason mathematically, make and validate conjectures, and classify and define geometric objects. Geometric ideas are worthwhile in solving and representing problems in real life situations and in other mathematical areas (NCTM, 2000).

Specific geometry content and spatial reasoning are two quite different yet related components of geometric reasoning ability. Spatial reasoning is an intuition about shapes and the relationships among them. It includes the ability to mentally imagine objects. That is, it requires the ability to turn things in the mind (Van de Walle, 2016). Spatial abilities can be regarded as a kind of cognitive activity that helps learners to produce spatial images and to use them in the solution of different problems (Hegarty & Waller, 2005). There is much in common between spatial reasoning and transformation geometry in that they both include construction and manipulation of mental images (Leong & Lim-Teo, 2003). Therefore, working with transformation geometry would improve skills in spatial and geometric reasoning (Clements & Battista, 1992; Edwards, 1997).

Apart from using in school mathematics, geometric transformations are also used in the teaching of tertiary-level mathematics topics such as groups, matrix algebra and so forth (Hollebrands, 2003; Kuchemann, 1981). For instance, Kuchemann (1981) expressed the importance introducing transformations to learners as providing insights into mathematical structures of advanced level concepts, especially the structure of the concept of group, by having them discover generic rules about composition of transformations. In addition, Kuchemann (1981) asserted that geometric transformations would provide an explicit reification of matrix algebra and emphasized that this would be acknowledged as an influential manifestation of the uniformity of mathematics. Kort (1971) made the same point that geometric transformations can play a significant unifying role in establishing continuity of mathematical topics spanning elementary through tertiary level education.

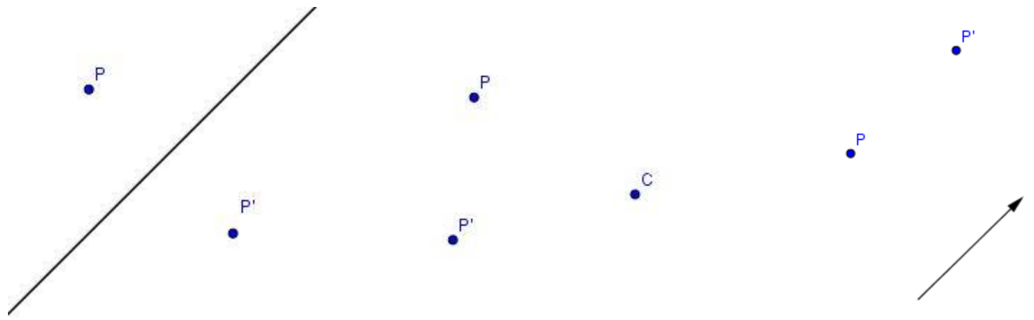
There are two different conceptualizations of geometric transformations as motion and mapping conceptualization (Edwards, 2003; Hollebrands, 2003; Yanik & Flores, 2009). By considering her prior studies (i.e., Edwards, 1991, 1992, 1997), Edwards (2003) suggested that motion conception refers to learners' initial conceptualization of geometric transformations and that mapping conception of geometric transformations are representative of contemporary mathematicians' conceptualization of transformation geometry. Edwards (2003) further indicated that motion conception originates from learners' "embodied experience" in the physical world, and in order to have a mapping conception one needs to reconceptualize his/her understanding of geometric transformations.

Moreover, Edwards (2003) explained that learners with motion conception of geometric transformations think that points on the plane are carried onto other points on the plane depending on a particular distance and direction. Learners with a motion conception of geometric transformations consider the plane as a background and manipulate the geometric figures on the top of it. On the other hand, motion is not a necessary part of geometric transformation in the mapping conception. Learners with mapping conception can consider geometric transformation as a special function which maps all points in the plane to other points based on a specific distance and direction (Hollebrands, 2003). They conceive that the plane comprises unlimited number of points and geometric figures are sets of points which are subsets of the plane and they do not consider geometric figures as objects sitting on the plane. Learners with such interpretation of geometric transformations consider the domain of function as all points in the plane (Hollebrands, 2003). Accordingly, geometric transformation must be applied to all points in the plane rather than to a single object.

National and international curriculum documents (e.g., MoNE, 2011; CCSSM, 2010) and several researchers (e.g., Hollebrands, 2003; Steketee & Scher, 2011, 2016; Steketee, 2012; Yanık & Flores, 2009; Yanık, 2013) advocated relating geometric transformations and functions in the teaching of mathematics. Relating these two concepts requires learners to have a mapping conception. For instance, Hollebrands (2003) implemented an instructional unit to have high school students connect geometric transformations with functions. In a similar fashion, Yanık and Flores (2009) aimed at getting prospective teachers to connect geometric transformations and functions. Steketee (2012) went further to claim that the formal teaching of functions should begin with geometric transformations that take points as inputs and outputs. He stressed that by this way students might develop stronger and clearer ideas related to functions (i.e., dependent and independent variables, domain, range, relative rate of change, inverse functions, compositions of functions, and function notation). He proposed four arguments that suggested the use of geometric transformations instead of numeric functions in introducing function concepts as cognitive, kinesthetic, visual, and structural. These arguments are explained in more detail in the further sections.

2.2. Mapping Conception: Geometric Transformations as Functions

A function is defined as “a set of ordered pairs, a correspondence, a graph, a dependent variable, a formula, an action, a process, or an object” (Selden & Selden, 1992, as cited in Kieran, 2007, p. 730). “A transformation is just a mapping on the points that is both one-to-one and onto” (Martin, 1982, p. 2). This definition indicates that transformations can be conceived as functions. However, these functions are not numeric functions. Namely, as stated by Hollebrands (2003) and Steketee and Scher (2011), these functions are not equivalent to functions that map $R \rightarrow R$, because they take two numbers, not one, to define the position of a point in the two-dimensional plane. Thus, transformations are mappings from R^2 to R^2. Actually, in their studies Steketee and his colleague used the term “geometric functions” to refer to geometric transformations of a point in the plane. Examples of geometric transformations that map a point on the plane to another point on the plane are given in Figure 2.1.



(a) (b) (c)

Figure 2.1. Reflection (a), rotation (b) and translation (c) examples on the plane

According to Hollebrands (2003), Martin’s (1982) definition of transformation highlights two crucial ideas: understanding the domain and range of a transformation and understanding what it means for a transformation to be one-to-one and onto. She explained that students’ understanding of the inputs and parameters for a transformation, relationships among pre-images, images and parameters, and of what properties are preserved by a transformation are other ideas included in understanding transformations as functions. Because of the importance of these ideas, each one was explained in more detail in the following paragraphs based on the studies of Hollebrands (2003) and Steketee and Scher (2011).

Domain and range of a transformation are different from domain and range of a function in an algebra class. That is, functions usually have the set of all real numbers as their domains and they map real numbers to real numbers. In other words, they map R to R. However, geometric transformations map all points in the plane to other points in the plane. That is, they map R^2 to R^2. When a transformation is applied to an object in the plane, one can misinterpret that an object is a particular input. However, actually, the transformation is not applied only to the object, but to the all other points in the plane as well. For instance, let's consider triangle ABC and reflect it along line *d.* When students are asked to state which points remain fixed, they may falsely think that only the points on the triangle ABC changes. However, the transformation is applied to all the points in the plane including both the triangle and the reflection line. Thus, only the points on the reflection line remains unchanged due to the fact that the points on it map to themselves. Shortly, the idea that "domain consists of all points in the plane, and it is not just a single object on which a transformation is applied" play an important role in understanding transformations as functions.

Another important idea is that transformations are special functions that are both one-to-one and onto. A transformation T(P)=P' is an onto function since every point in the range (every point P' in the plane) has a corresponding point in the domain (a point P in the plane). Besides, the transformation T is one-to-one since it satisfies the following conditional statement: "If A and B are two different points in the domain, then the output values A' and B' in the range will be different from each other."

Another key idea related to understanding transformations as functions involves understanding inputs as variables. An input varies means that an input can be any element (e.g., a number, a geometric figure, a set) from the domain on which the function is defined. For instance, when the transformation T(P)=P' is considered, the point P is interpreted as being any point in the plane.

Moreover, parameters and their effects on the transformations are important in understanding transformations as functions. They are parts of the definitions of functions. They determine the particular member of a function family. For instance, in the transformation given by $R_{C,\,\alpha}$ (P) = P', R indicates the family of rotations, P indicates the variable, C indicates the center of rotation as one of the parameters of rotation and α indicates the other parameter of the rotation. The parameters indicate the particular member from the family of rotations that will be applied to *P*. Parameters are a vector for a translation, a line of reflection for a reflection, a center of rotation and an angle of rotation for a rotation, and a center of a dilation and a scale factor for a dilation.

In order to understand that geometric transformations are functions, one should know certain characteristics of geometric transformations. Some of the characteristics of geometric transformations are the relations among the pre-images, parameters and images, invariant properties and the fixed points under the transformation (i.e., points that map to themselves). Preservation of length and preservation of collinearity are also among the characteristics of geometric transformations.

2.3. The Teaching of Geometric Transformations and Functions through Dynamic Geometry Environments

International studies such as TIMSS and PISA have shown that students' geometry achievement is rather low. To improve student performance in geometry, several researchers spoke to the use of computer based approaches (e.g., Battista, 2007; Clements & Battista, 1992). More specifically, researchers have advocated students' use of dynamic geometry environments in the teaching and learning of geometry (Marrades & Gutierrez, 2000). The dynamic characteristic of dynamic geometry environments comes from the ability to drag the points and lines on the screen. Depending on the geometric rules used in construction, some properties of the geometric figure changes or remains the same. This feature of dynamic geometry software help students recognize the relevant attributes of a given geometric concept (Jones, 2005).

Many educators state that dynamic geometry programs can promote higher order thinking skills such as generalization (e.g. Knuth & Hartman, 2005; Vincent, 2005). In his review of research on dynamic geometry environments, Jones (2002) stated that "interacting with DGS (dynamic geometry software) can help students to explore, conjecture, construct and explain geometrical relationships. It can even provide them with the basis from which to build deductive proofs." (p. 20). Similarly, Battista (2001) and Hativa (1984) mentioned that dynamic geometry environments promote the teaching and learning of geometry by establishing dynamic and fruitful interactions among the teacher and his/her students.

In particular, some researchers reported the superiority of dynamic geometry software in enhancing learners' knowledge and understanding of geometric transformations (Dixon, 1997; Edwards, 1991; Güven, 2012; Hollebrands, 2003; Harper, 2003; Tatar et al., 2014; Yanık, 2013). For instance, in her qualitative study, Harper (2003) focused on translations, reflections, and rotations and she found that after four prospective elementary school teachers were involved with transformation geometry in dynamic geometry environment, their use of mathematical vocabulary became more complex; they were able to reflect images and draw reflection lines by considering the properties of reflections; they were more informed about the role of a vector in representing the direction and magnitude of a translation; and finally they could determine the center and angle of a single rotation. In an experimental study focusing on reflections and rotations, Dixon (1997) found that eight grade students who used dynamic geometry were more successful than the traditional group in identifying and applying reflections and rotations. In another study focusing on geometric translations, Yanık (2013) found out that GeoGebra helped prospective middle school mathematics teachers explore the properties of translations, make conjectures, use several strategies, and build new understandings. Besides, participants became more confident in defining, identifying, and performing translations upon completion of the study.

Despite not collecting empirical data, several researchers advocated teaching geometric transformations as functions through dynamic geometry environments (e.g., Flores & Yanık, 2016; Steketee & Scher, 2011) and designed a series of activities for this purpose. For instance, Steketee and Scher (2011) used the Geometer's Sketchpad, to introduce activities that help middle school students explore transformations as geometric functions. The activities comprised of core mathematical ideas. The researchers emphasized nine insights as a result of these activities: functions need not be numeric; there is no one primary representation of function; variables really vary; variables can vary continuously; functions can be viewed automatically or collectively; in the collective view, functions map pre-image to image, domain to range; functions can be multidimensional; functions, transformations, loci and mappings are related ways of expressing the same deep mathematical concept; deep connections are found between algebra and geometry. The researchers concluded that providing students with the opportunity to explore a range of geometric representations of functions, help them extend their experiences and overcome their difficulties and misconceptions that may be a big problem later. The researchers added that the middle school students might not fully develop a complex understanding of functions, however, allowing them to use drag and trace tools in dynamic geometry environment to explore a variety of geometric functions might help them to form a basis for their future mathematical development.

Similarly, Flores and Yanık (2016) designed research based activities that would help middle school students in building knowledge of translations. These activities included several examples and non-examples of translations and the students are expected to study on them on GeoGebra. After explorations of these activities, the students were expected to shift from motion conception to mapping conception and thus improve their knowledge of translations. The researchers suggested increasing the number of geometric figures to be translated gradually in order to emphasize that domain was all points in the plane. For instance, in one of the activities, the researcher focused on translation of two shapes and stated that it was possible to translate three shapes at the same time. They argued that by increasing the number of shapes on the screen step by step, the students may extend the domain of translation from a number of shapes in the plane to the entire plane.

Although, Flores and Yanık's (2016) idea related to teaching domain of translation might be beneficial for students in extending the domain of translation, they never mentioned about the power of Trace tool in dynamic geometry software. However, Steketee and Scher (2011) spoke to the use of Drag and Trace Tools in helping learners understand the domain and range of geometric transformations as plane. They emphasized that by this way students could build a sense of continuous variation of the input and output. A translation example constructed by using Drag and Trace Tools and the traces of Points P and P' are presented in Figure 2.2.

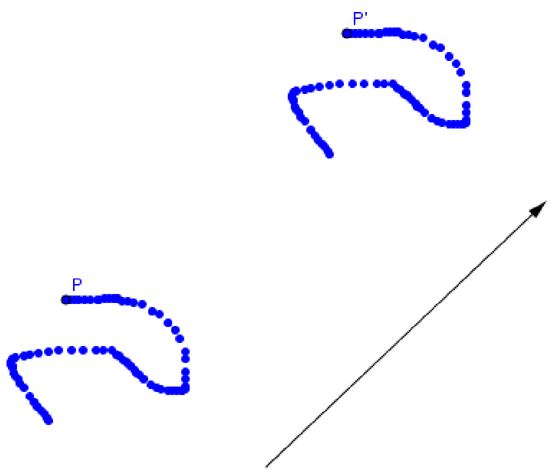


Figure 2.2. A translation example with activated Trace tool

The aforementioned studies show that dynamic geometry environments (DGEs) play a crucial role in understanding geometric transformations as functions, especially the domain and range. Apart from this, there are some other studies that point to the role of DGEs in grasping the notion of function as a relationship between variables (e.g., Falcade, Laborde & Mariotti, 2007; Hazzan and Goldenberg, 1997). For instance, Hazzan and Goldenberg (1997) introduced undergraduate mathematics majors the notion of functional dependency between geometrical variables through dynamic geometry environment. They indicated that students' learning of functions have been investigated in terms of symbols and graphs in a static context and therefore they attempted to model functional relationships that were not specified by symbols or represented by graphs through dynamic geometry environment. They devised a dynamic geometry context with no algebra, graphs or numbers. The nature of independent variable, constant function, undefined function, and continuous functions were the issues that formed the main concern of the researchers. They prepared geometric constructions in DGE and let students drag certain objects on the construction. As a response to dragging, the construction on the screen changed dynamically while obeying the construction rules. While dragging in a continuous manner, students observed how the entire construction responded dynamically. Therefore, in DGE, students had opportunity to explore functional relationship that were not specified by symbols or represented by graphs. The researchers indicated that the differences between algebraic and dynamic representations of functions provided students with experiences that can expand their ideas of functions. Hazzan and Goldenberg (1997) summarized two perspectives on interpretation of dynamic change as a function. In the first perspective, dragging a specific point in R^2 to another specific point in R^2 along a particular path (i.e., movement of the user's hand) was the function that transformed the pre-image of an object to its image. This function was defined on a construction space in which each element was a construction and the preimage was the construction before the dragging. In the second perspective, the construction rules determined a dependency and this dependency was defined as a function. That is, function was a construction that related the values in R^2 to values in R^n. In this perspective, the input of a function was defined on R^2 (a position of a point) while the output was defined on either R^2 (the position of another point), or R (a measurement performed on some feature), or even in R^n (e.g., the position, slope, and length of a segment). As the input was dragged, the output varied with respect to the dependencies determined by the construction rules.

Hazzan and Goldenberg (1997) used only the Dragging tool of dynamic geometry environment when introducing functional dependency. However, Falcade and others (2007) used both the Dragging tool and Trace tool to have tenth grade students explore the notion of function as covariation. The researchers emphasized that these two tools might play a potential role in introducing the notions of domain and range of a function. They conducted a teaching experiment aimed to introduce students the notion of function as covariation, starting within the geometrical domain in DGE. In their study, they focused on the use of Trace tool and its potentialities in for constructing the meaning of function. They showed how students related the idea of function to the idea of trajectory (i.e., the set of points representing a covariation). Falcade and others (2007) stated that Dragging and Trace tools contributed to the emergence of meanings related to the notion of independent and dependent variables. They added that students intentionally used the Trace tool to check their conjectures and this showed that Trace tool functioned as a potential semiotic mediator. Besides, this tool contributed to the emergence of the idea of function as a point to point correspondence. Finally, the researchers stated that they reached their goal and suggested further studies to introduce the notion of graph of a function as a trajectory.

The research studies reviewed above emphasize that geometric transformations should be taught in conjunction with the notion of functions in a dynamic geometry environment. Similarly, some other researchers emphasize that functions should be taught in conjunction with geometric transformations (e.g., Steketee, 2012). For instance, Steketee (2012) claimed that beginning the formal teaching of functions with geometric transformations in a dynamic geometry environment might help students develop stronger and clearer ideas related to functions (i.e., dependent and independent variables, domain, range, relative rate of change, inverse functions, compositions of functions, and function notation). He proposed four arguments to use geometric transformations instead of numeric functions in introducing the function concept: cognitive, kinesthetic, visual, and structural argument.

Cognitive argument was related to construction of an independent variable, construction of a function mechanism for operating on the dependent variable and for producing a dependent variable, and finally creating a dependent variable. This function mechanism could be a reflection line, translation vector and so forth and students could point to this mechanism as the embodiment of the function. However, when students began with numeric functions, they had no opportunity to construct visible numeric function, for instance a visible “add 5” object. The numeric function existed only as a mental construct and students had to create an abstraction. Therefore, introducing geometric functions by using dynamic geometry environment was found to be more concrete and compelling than using numeric functions.

Kinesthetic argument was related to the variability of variables. In a dynamic environment, students could create an independent variable and could vary it simply by dragging. While dragging the independent variable, students could explore a variety of values of independent variable easily and could observe the behavior of both variables. Continuous variation was a natural concept resulting from students’ direct experience since they perceived the action of dragging as continuous. However, with the numeric function, students’ direct experience with these variables was not continuous, and it was limited to the discrete variation represented by their choice of numeric values.

Visual argument was related to the visual nature of the function behavior. By the help of Trace tool in dynamic geometry environment, covariation became visually and dynamically accessible and students could explore the various features of a function's behavior. However, to identify the behavior of a numeric function, students had to analyze discrete pairs of values by performing calculations and by matching patterns. This procedure did not involve either the direct experience of continuity of the function or the relative rate of change of the variables.

Structural argument had to do with students' attainment of interconnected concepts such as domain, range, function notation, composition, and functions as mappings. Domain and range were difficult for students to learn in numerical function context since they seemed to be ambiguous and there were not any concrete objects linking these two concepts. On the other hand, by the help of dynamic geometry environment the domain could be restricted to a specific geometric path and thus these concepts became visible and concrete. Moreover, in a numeric function a label like $f(\mathrm{x})$ did not make much sense for students however with geometric functional notation was more meaningful. For instance, the notation $R_{\mathrm{C}, 90}(x)$ could easily be understood by the students as "the rotation around center C by 90 degrees of point x". Finally, when the domain was restricted, a traced visual image was generated for the range corresponding to this domain and by this way students came to understand functions as mappings from one point to another point.

The four arguments of Steketee (2012) suggest using geometric transformations in the teaching of functions and also other studies suggest teaching geometric transformations as special functions (i.e., Hollebrands, 2003; Steketee & Scher, 2011). All these studies indicate a reflexive relationship between geometric transformations and functions. Based on this relationship, the researcher attempted to design an instructional unit that integrates geometric transformations and functions. By this way the researcher expected that such an instructional unit might prove useful in enhancing learners' understanding of both geometric transformations and functions.

Thus far, an overview of research on the teaching of geometric transformations and functions through dynamic geometry environments was provided. In the following section, previous research findings related to geometric transformations are reviewed.

2.4. Studies Related to Geometric Transformations

Over the last few decades, a large number of studies have been carried out to explore learners' (elementary, middle and high school students, and prospective teachers) understanding of geometric transformations. Several studies focused on learners' pre-existing knowledge or understanding of geometric transformations (e.g., Hollebrands, 2004; Ramful et al., 2015, Thaqi et al., 2011, Yanık, 2011; Yanık, 2014). Moreover, several researchers focused on the positive effects and superiorities of DGEs on students' understanding of geometric translations (e.g., Edwards, 1991; Güven, 2012; Harper, 2003; Tatar et al., 2014). In some other studies, researchers developed instructional units aiming at having students understand geometric transformations as functions (e.g., Hollebrands, 2003; Portnoy et al., 2006; Sünker & Zenbat 2012; Yanık & Flores 2009; Yanık 2013). Thus, in this part, research studies regarding geometric transformations were reviewed in three main sections: (*i*) students' pre-existing knowledge and understanding of geometric transformations, (*ii*) prospective teachers' pre-existing knowledge and understanding of geometric transformations, and (*iii*) development in learners' knowledge and understanding of geometric transformations.

2.4.1. Students' Pre-existing Knowledge and Understanding of Geometric Transformations

In the transformation geometry literature, there are studies exploring middle school students' (e.g., Yanık, 2014) and high school students' (e.g., Hollebrands, 2004; Ramful et al., 2015) pre-existing knowledge and understanding of geometric transformations. For instance, Yanık (2014) explored the nature of middle-grade students' understanding of geometric translations and the possible sources of their conceptions. One hundred and ten 6^{th} grade students aged 11 and 12 were administered a written instrument related to geometric translations. To provide more detailed information about students' conceptions, the researcher selected a total of 24 students with different levels of understandings and conducted interviews both with these students and their teachers. Besides, the researcher examined the student textbook, student artifacts and teacher materials such as slides and handouts to see how these materials approached geometric translations and shape students' conceptions of translations. Analysis of written instrument data showed that students had two major categories of understandings in geometric translations: translational motion, and both translational and rotational motion. Specifically, the conceptions they had about these motion types varied in complexity. Namely, students' conceptions for both types of motions varied as undefined, partially defined and defined motions. Meanwhile, students had five different interpretations of a translation vector (*i*) reference line, (*ii*) vector as a symmetry line, (*iii*) vector as a direction indicator, (*iv*) vector as a parameter, and (*v*) vector as an abstract tool. Yanık (2014) also found out that mathematics and science textbooks, classroom instruction, real life examples and everyday language were the major sources of students' conceptions of geometric translations. He claimed that teachers' examples such as "rolling a ball" and "movement of a car" included both rotational motion and translational motion at the same time, and led students to hold misconceptions about geometric translations. He also warned that teachers must not use such careless explanations to avoid misconceptions about geometric translations. Yanık (2014) found that students' science textbook introduced the topic of force and motion through examples such as "pushing a desk, dragging objects, moving shopping cart". He stated that these examples might have caused students to hold the belief that an external force needed to be applied to objects in order to move them and that they might lead to conflicts in students' interpretation of geometric translations. Yanık also explained that the term "translation" in Turkish (ötele) referred implicitly to a physical action that involves force, motion, and displacement in Turkish dictionaries and that this conventional meaning might have been influential in students' execution of translation tasks and in their thinking of rotational motion of objects as translation examples.

Similarly, Ramful and others (2015) conducted a case study with two students (Brittany and Sara) aged 14 and 15. The researchers wanted to explore the reasons of conceptual difficulties related with inclined lines of symmetry, the enactment of visual strategies in bilateral symmetry and reflection tasks and the interaction of visual and analytic strategies in the generation of an image from the object. The participants were administered four sets of tasks. In the first set of tasks, the two students were asked to find symmetry lines of a polygon such as a parallelogram or a square and the symmetry lines of alphanumeric characters such as X and Z. In the second set of tasks, they were asked to reflect the given line segments and polygons on a grid paper along the given reflection lines. In the third set of tasks, they had to complete the given partial geometric figures in a way that fulfill the number of symmetry lines pre-determined by the researcher. Finally, in the fourth set of tasks, they were asked to reflect the given geometric objects along inclined symmetry lines and the objects were not given on a grid.

The researchers found that although Brittany did not have difficulty in tasks involving vertical and horizontal symmetry lines, tasks involving inclined symmetry lines revealed inadequacy of her conception of symmetry. In solving reflection tasks, she concentrated on equidistance, congruence of length and ‘exactly opposite’ as the intuitive counterpart of perpendicularity criterion for performing reflection. She was not formally aware of the perpendicularity criterion and was not aware that the points on the line of symmetry was invariant under reflection. The researcher attributed Brittany’s erroneous conception of inclined symmetry lines to her intuitive understanding of perpendicularity as being opposite. On the other hand, Sara’s positioning of the ruler to obtain right angles was an indicator of her formal conception of equidistance and perpendicularity.

The researchers speculated that the nature of the object influenced students’ performance in tasks involving inclined symmetric lines. Namely, reflection of line segments along inclined symmetry lines was more challenging than reflecting the ones who had geometric shapes. They also speculated that it could be easier to conceive or imagine the reflection of a geometric object as a whole since it had more defining properties than that of the singular line segment. The researchers recommended that it was necessary to give more emphasis on perpendicularity criterion in reflection.

Hollebrands (2004) conducted a study to gain insights into high school students’ prior understanding of geometric transformations. Namely, she conducted task-based interviews with six 10th grade students to determine their knowledge of translations, reflections and rotations. The participants took two algebra courses in 8th and 9th grades and did not study geometric transformations either in middle or elementary school.

In the reflection task, students were asked to define a reflection and next reflect a polygon along an inclined line which did not intersect the polygon. Students defined a reflection most commonly as “a reflection is like a mirror” and noted that after reflection the size of the polygon would remain unchanged while its orientation would change. The students seemed to have difficulty in understanding the perpendicularity criterion in reflection as described by Ramful et al. (2015). Besides, the researcher attributed their difficulty with the reflection to the position (i.e. orientation) of the reflection line given in the task. In general, none of the students took into account the relationships between the corresponding points on the polygon and its image and the reflection line.

When students were asked to define a rotation, most of them explained it as “turning an object”. All of them explicitly included the angle of rotation in their explanations while the center of rotation seemed to be implicit. Two tasks were administered to students to perform rotations of two triangles. In the first task, the center of rotation was outside the triangle. Although they seemed to relate rotation with turning, they used centers of rotations differently. Two of the students did not use the given point as the center of rotation. The researcher stated that majority of the remaining students seemed to envision turning the triangle about a point rather than considering the equal distances between the corresponding points and the center of rotation. In the second task, the center of rotation was on the hypotenuse of the triangle. In this task, three of the students rotated the triangle around different points rather than the given point as the center of rotation. As in reflection tasks, students did not consider the relationship between the corresponding points on the triangle and the center of rotation in the given rotation tasks. The researcher suggested that activities involving drawing a polygon on a paper, placing the pencil on the center of rotation and turning the paper a certain number of degrees may help students understand the role of the center of rotation.

Two tasks were administered to students to perform translations. Students seemed to have more difficulty in translation tasks than the reflection and rotation tasks and they were not familiar with the translation vector. In the first task, a line segment and a translation vector were given and students asked to perform translation. In the second task, image of a triangle and a translation vector were given and students were asked to draw the pre-image of the triangle on which the translation was applied. In both tasks, all students except one did not know how to use the translation vector to translate figures. Besides, some students used the translation vector as a location on which the figure should be placed and determined the figures in a way that they coincided with the translation vector. Hollebrands (2004) concluded that the students conceived transformations as motions or actions that were applied to a figure. Besides, they were not aware of the role of parameters in performing transformations. Finally, they did not pay attention to the relationship between pre-image, image and the parameters. To help students move from motion of figures conception to considering all points in the plane, she recommended the use of points rather than a single polygon as a pre-image. Besides, she suggested using DGEs to sample the entire plane by dragging a pre-image point.

2.4.2. Prospective Teachers' Pre-existing Knowledge and Understanding of Geometric Transformations

In the transformation geometry literature, there are studies exploring prospective elementary (e.g., Thaqi et al., 2011) and middle school teachers' (e.g., Yanık, 2011) knowledge and understanding of geometric transformations. For instance, Thaqi et al. (2011) sought to examine and compare how Kosovan and Spanish prospective elementary teachers' made sense of geometric transformations. The researcher stated that in a prior research, the first author conducted a curricular-cultural analysis of textbooks and teachers' training materials and found big differences between these two countries. In light of this idea, the researchers also found it necessary to explore whether participants' cultural background was related to their pre-existing knowledge of geometric transformations.

Thirteen and fifteen prospective elementary teachers were selected from Spain and Kosovo respectively and their ages ranged between 18 and 22. Initially, a questionnaire was used to identify participants' beliefs, meanings, and prototypes about transformations. Besides, a semi structured interview including 14 open ended questions was used to reveal participants' use of terminology and their ideas about transformations.

The results of the study showed that none of the prospective teachers had consolidated knowledge about transformations, and they were not aware of the idea that transformations could be regarded as specific functions. More specifically, Kosovan prospective teachers assumed a transformation perspective by using deep mathematical expressions while Spanish prospective teachers conceived transformations as simple relations between the objects and their images. A few Kosovan prospective teachers had knowledge about invariance of shape and size under transformations. Besides, two Kosovan prospective teachers defined transformation as defined motions of all points on the plane. Most of the Spanish prospective teachers identified transformations as repetitions however they did not accept the concept of symmetry under the set of isometric transformations. In general, in both countries prospective teachers could not state all parameters to define transformations. When defining rotation, prospective teachers mentioned only the angles but not the center of rotations or equal distances. The researchers concluded that Kosovan and Spanish prospective teachers' low transformation geometry performances revealed their inadequacies in their previous backgrounds. In Spain, natural images influenced prospective teachers' construction of the idea of projection however less attention was given to the role of properties in defining. Kosovan students had better understandings about these properties due to the influence of German-Russian tradition. However, in general, none of the prospective teachers in the two countries had a complete understanding of transformations and none of them could conceive transformations as functions.

Similarly, Yanık (2011) examined prospective middle school mathematics teachers' pre-conceptions of geometric translations. The participants of the study were 44 sophomore prospective middle school mathematics teachers enrolled in a large urban public university in the center of Turkey. The researcher conducted semi structured clinical interviews including 13 tasks to probe participants' understanding of translations. He asked participants to read tasks and think aloud to share their reasoning during the interviews without attempting to advance participants' understanding of translations or vectors. The results of this study showed that all prospective teachers except for two, considered the motion itself as translation. Furthermore, they exhibited two different motion conceptions: translation as rotational motion and translation as translational motion. Nearly half of the participants considered translation as rotational motion. They stated that disks or a wheel rolling down or along the horizontal line were the examples of translation.

They indicated that in addition to displacement rolling was a key aspect of translation. Other half of the participants indicated that linear motion and displacement were the major characteristics of translation. Their examples were sliding an object or carrying an object to another location without rotating. Participants did not give information about vectors or an indicator that specifies the direction and the distance between pre-image and image. Only one participant had mapping understanding. He explained translation as constructing the same shape in other place. That is, the shape is not moved instead, another shape is constructed in other place. Although prospective teachers had some ideas about translation, the majority of them were unsure about the role of vectors in translation.

The results regarding prospective teachers' understanding of vectors in translation showed that 12 out 44 participants had consistent ideas about the concept of vectors in geometric translations. That is, these twelve participants considered the direction and magnitude of the vector when performing translations. Other types of conceptions were vectors as force, vectors as a line of symmetry, and vectors as a direction indicator. In short, the results of the study showed that understanding plane and domain for translations were crucial factors that shaped prospective teachers' conceptions of translation. Participants who had motion and mapping conceptions had different conceptualization of plane and domain of translation. Besides, participants had difficulty in understanding the role of vectors in translations. For those reasons, the researcher stated that further studies should be conducted to explore pre-conceptions identified in this study and other possible pre-conceptions involved in conceiving translations. Moreover, he stated that it might be useful to investigate classroom teaching that emphasizes different conceptions of transformation. Besides, he suggested researchers to explore how students' progress from motion conceptions to mapping conceptions of transformations.

To summarize, studies related to learners' understanding of geometric transformations showed similar results. For instance, middle school students (Yanık, 2014), high school students (Hollebrands, 2004) prospective elementary and middle school teachers (Thaqi et al., 2011; Yanık, 2011) had motion conception of geometric transformations. In more detail, middle school students in Yanık's (2014) study and prospective teachers in Yanık's (2011) study had translational motion or both translational and rotational motion conceptions of translations. Actually, they misinterpreted motion conception itself since nearly half of them misconceived that rolling was the key aspect of translations.

Besides, learners were not aware of the role of parameters in transformations (Hollebrands, 2004; Thaqi et al., 2011; Yanık, 2011; Yanık, 2014). For example, since middle school students' and prospective teachers' definition of translation partially involved or even did not involve the idea of vector (the parameter of translation) their motion conceptions were either undefined, partially defined or defined motions (Yanık, 2011; Yanık, 2014). Besides, when defining rotation, students mentioned only the angles but not the centers of rotations (Thaqi et al., 2011).

To help students move from motion of figures conception to considering all points in the plane, Hollebrands (2004) recommended the use of points rather than a single polygon as a pre-image. Besides, she suggested using DGEs to sample the entire plane by dragging a pre-image point.

High school students in Hollebrands' (2004) study did not pay attention to the relationship between pre-image, image and the parameters and thus had difficulty in transformation tasks. Besides, Ramful et al.'s (2015) study showed that in solving reflection tasks, the equidistance, congruence of length and perpendicularity involved in this relationship played an important role. One of the two participants' formal conception of equidistance and perpendicularity helped her solve tasks involving vertical, horizontal, and inclined lines of reflections.

Studies described above aimed to explore middle or high school students' (Hollebrands, 2004; Ramful et al., 2015; Yanık, 2014) and prospective teachers' (Yanık, 2011; Thaqi et al., 2011) understanding of transformations without giving any instruction. However, some other studies developed learning environments in DGEs to enhance learners' understanding of transformations and explored the effects of DGEs in learners' understanding of geometric transformations. These studies are explained in the following section.

2.4.3. Development in Learners' Knowledge and Understanding of Geometric Transformations

In the transformation geometry literature, there are studies exploring the effect of dynamic geometry environments on students' (e.g. Edwards, 1991; Güven, 2012) and prospective teachers' (e.g., Harper, 2003; Tatar et al., 2014) knowledge of geometric transformations. In these studies, the researchers aimed to improve learners' knowledge of properties of geometric transformations and of identifying and performing geometric transformations by using dynamic geometry environments. Moreover, some other studies aimed to have middle school students (e.g., Sünker & Zembat, 2012), high school students (e.g., Flanagan, 2001; Hollebrands, 2003), and prospective teachers (e.g., Portnoy et al., 2006; Yanık, 2006; Yanık & Flores, 2009; Yanık, 2013) gain mapping conception of geometric transformations by using dynamic geometry environments. These studies are reviewed in the following parts.

2.4.3.1. Effect of DGEs in learners' knowledge of geometric transformations

In the transformation geometry literature, there are studies exploring effect of dynamic geometry environments in students' (e.g. Edwards, 1991; Güven, 2012) and prospective teachers' (e.g., Harper, 2003; Tatar et al., 2014) knowledge of geometric transformations. For instance, Edwards (1991) investigated the effect of using computer microworld on middle school students' learning of transformation geometry. Specifically, she aimed at building a model of middle school students' initial learning and describing students' use of computer microworld during the learning of transformation geometry. The computer microworld provided translation, rotation, reflection and change of scale examples to students. The researcher focused on the effectiveness of the microworld in helping students build a correct initial understanding of transformations, the changes that occurred in students' knowledge and strategies after studying with a curriculum of activities in the microworld and on the mechanisms of conceptual change that might lie behind the changes in their knowledge, strategy, and behavior. A total of 12 students aged between 11 and 14 (sixth, seventh and eighth grade students) participated in the study. All of them had studied Logo in their regular curriculum however none of them had received transformation geometry instruction before.

Before and after the treatment, students were administered a spatial reasoning test, a visual reasoning test (i.e., the card rotation and the paper folding test), a sorting shapes task, and a describing shapes task. Then, students were introduced transformation geometry in the classroom sessions. At the end of the treatment, a final test was administered in addition to these three measures. The final test required students to execute and to identify transformations. Six items were related to single transformations and six items were related to composition of transformations. Analysis of data showed that there was a significant change in students' transformation geometry knowledge. The results of the study suggested that the microworld and the designed activities were effective in assisting the students to construct a working knowledge of transformations. Besides, students' performance in identifying and executing rotations and reflections on paper improved during the study. Students correctly responded 70% of the 24 items included the final test. Before the treatment, students were considering transformations as a combination of local changes. At the end of the study, they started to reason transformations as whole-plane operations. Besides, they started to make use of both graphical and the symbolic representation of transformations.

Another dynamic geometry environment used by the researchers to improve middle school students' knowledge of geometric transformations was Cabri. For instance, Güven (2012) conducted a quasi-experimental research to examine the effect of Cabri on middle school students' understanding of translations, rotations and reflections. Sixty-eight eight grade students participated in the study and 36 of them were in the experimental group and the remaining students were in the control group. The experimental group students studied geometric transformations in the dynamic geometry environment and the control group students studied them in a paper-pencil environment (dotted and isometric worksheets). Eight lesson hours were allocated to the teaching of geometric transformations for both groups. Specifically, 2 lesson hours for translations and reflections, 2 lesson hours for rotation and finally 4 lesson hours for composite transformations were spent. A multiple choice Transformation Geometry Achievement Test consisting of 15 items and an open ended Learning Levels of Transformation Geometry Test comprising 15 items were used to gather data in pre-test and post-test. The results showed that Cabri based instruction method had a moderate effect on students' transformation geometry achievement and high effect on their levels of understanding of transformation geometry. The researcher concluded that dynamic observation of features of geometric transformation in a DGE improved students' understanding of these features and consequently their achievement in transformation geometry.

Dynamic geometry environments are used by researchers to improve prospective teachers' knowledge of geometric transformations as well. By using the Geometer's Sketchpad (GSP), Harper (2003) aimed to determine the knowledge of prospective elementary teachers about geometric transformations; the kinds of interactions they displayed when using this dynamic software; and the changes that occurred in their knowledge during or after instruction with this software. Four female prospective elementary teachers studying at a public university in the USA participated in the study. Each participant was interviewed before and after three instructional sessions with the Geometer's Sketchpad. The instructional sessions consisted of a series of problem-based Geometer's Sketchpad activities. The first, second and third sessions primarily involved reflection, translation, and rotation tasks respectively. Prospective teachers' interactions with Geometer's Sketchpad during the solution of these tasks were video recorded. The researcher audio- and video-recorded all interviews and instructional sessions to critically review each session.

The findings showed that prospective elementary school teachers lacked a complete understanding of geometric transformations prior to the implementation of instructional sessions. However, their understanding enhanced after presenting geometric transformation tasks with the help of GSP. Overall, prospective teachers' language and ability to predict and construct transformed images developed after the implementation of the instructional sessions. Specifically, participants' vocabulary became more sophisticated and they used formal mathematical terminology such as "translate, translation vector, reflect, line of reflection, rotate, equidistant, perpendicular center, and angle of rotation" while they used informal terms such as "flip, flop, fold, shift, move, slide, turn, spin, swing, and pivot" prior to instruction. In addition, after the instruction, all participants became able to construct a single reflection line by considering the given properties of a reflection. Besides, they were more knowledgeable with the use of a vector to represent a translation's direction and magnitude. Furthermore, they were able to identify the center and angle of a single rotation to map a figure with its rotated image. The findings of this study also revealed that GSP provided immediate visual feedback to prospective teachers and helped them conjecture, test and revise their solutions. Apart from this, they could use the non-transformation features of GSP to solve tasks that emphasized the properties of geometric transformations with which they were familiar and this encouraged them to use their own strategies and constructions to solve the given geometric transformations. Harper (2003) concluded that her study verified the advantages of integrating innovative technological tools into teacher education programs and recommended that this study could be useful for teacher educators in their instruction about geometric transformations with their prospective teachers, and as well as for mathematicians teaching mathematics content courses for elementary school teachers.

Another dynamic geometry environment used by the researchers to improve prospective teachers' knowledge of geometric transformations was GeoGebra. For instance, Tatar et al., (2014) examined the effects of using GeoGebra on prospective mathematics teachers' achievement in reflection symmetry. Besides, the researchers attempted to describe participants' views about using GeoGebra in the teaching of mathematics. 30 freshman prospective mathematics teachers participated in the study. A Reflection Knowledge Test comprising 16 tasks, developed by the researchers, was used to gather quantitative data. To provide qualitative data regarding participants' views about using GeoGebra, observations and a survey including three open-ended questions were conducted. The Reflection Knowledge Test was used as pre-test and post-test. After pre-test, reflection symmetry was taught to prospective teachers via GeoGebra for four lesson hours. The researchers finalized the study by conducting the survey after post-test.

The results showed that there was a significant difference between prospective teachers' pre and post test scores. This revealed that GeoGebra had a positive impact on prospective teachers' reflection symmetry achievement. Most of the participants stated that GeoGebra would contribute positively to students' learning of mathematics. In particular, they indicated that GeoGebra were mainly advantageous in terms of the following aspects: visualization; time saving; and interest in the lesson/attracting attention. On the other hand, they indicated the following disadvantages of using GeoGebra in mathematics lessons: learning difficulty arising from negative attitudes towards computers; discouraging students' inquiry due to readily given information; and distraction of students' attention and so forth. Finally, participants mainly expressed the following opinions when they were asked to whether they would use GeoGebra in professional lives: the learning environment should have sufficient equipment and the instruction would be more permanent if traditional instruction given first and GeoGebra instruction is given next. The researcher recommended that it was important to train teachers how to use dynamic and interactive learning environments such as Cabri, GeoGebra and GSP and on the kind of material they could create for teaching mathematics.

To summarize, research has shown that dynamic geometry environments had positive impact on middle school students' (e.g., Güven, 2012; Edwards, 1991) and prospective teachers' (e.g., Harper, 2003; Tatar et al., 2014) knowledge and understanding of geometric transformations. For instance, Güven (2012) examined middle school students' understanding of geometric transformations through Van-Hiele's Level of Geometric Thinking before and after the instruction and found that Cabri based instruction had high effect on students' level of understanding. Moreover, the results of the Edwards' (1991) study suggested that the microworld and the designed activities were effective in assisting the students to construct a working knowledge of transformations. Besides, students' performance in identifying and executing rotations and reflections on paper improved during the study. Before the treatment, students were considering transformations as a combination of local changes. At the end of the study, they started to reason transformations as whole-plane operations.

Similar results were reported in DGE studies conducted with prospective teachers. For instance, Tatar et al. (2014) stated that GeoGebra had a positive impact on prospective teachers' reflection symmetry performance. Besides, Harper (2003) found that prospective elementary school teachers' understanding of translations, reflections and rotations enhanced after studying a series of problem-based

Geometer's Sketchpad activities. After the instruction, all participants became able to construct a single reflection line by considering the given properties of a reflection. Besides, they were able to use a vector to represent a translation's direction and magnitude. Furthermore, they were able to identify the center and angle of a single rotation.

2.4.3.2 Developing instructional units for teaching geometric transformations as functions

In the transformation geometry literature, there are studies exploring middle school students' (e.g., Sünker & Zembat, 2012), high school students' (e.g., Flanagan, 2001; Hollebrands, 2003) and prospective teachers' (e.g., Portnoy et al., 2006; Yanık, 2006; Yanık & Flores, 2009; Yanık, 2013) developing understanding of geometric transformations. Among these, some researchers focused on a single type of transformation such as translation while others investigated all types of transformation geometry concepts simultaneously (i.e., translation, reflection, rotation, and dilation).

There are studies that developed an instructional unit using both dynamic and static environments for middle school students (e.g., Sünker & Zembat, 2012). For instance, Sünker and Zembat (2012) investigated the understanding required to develop the meaning of translations conceptually at the middle school level. Researchers developed a curriculum piece supported by the use of geometry software (i.e. Wingeom-tr) by conducting a teaching experiment with four sixth grade students. Data were collected through interviews and the results showed that understanding vectors namely the parameter of a translation was necessary in abstracting the meaning of translations. The researchers recommended that in order for students to make sense of translations mathematically, they should be provided the relationship between domain and range first in dynamic environments and alternately in static environments and by focusing on vectors.

There are teaching experiment studies that developed an instructional unit in dynamic geometry environments for high school students (e.g., Flanagan, 2001; Hollebrands, (2003). For instance, Flanagan (2001) focused on the nature of high school students' understanding of transformations namely translations, reflections, rotations, and dilations within a technological environment using Geometer's

Sketchpad and TI-92 calculator. The researcher implemented a seven-week instructional unit on six 10th grade students. Data were collected by in-depth clinical interviews, small group and whole group discussions, and by students' written materials. The researcher found that there were key understandings that seemed to reflect deeper understanding of transformations. These included "understanding the domain, a focus on theoretical objects rather than concrete objects, and understanding of how to interpret and make use of multiple representations" (p. 357). She suggested that the study proved useful not only in terms of exploring students' understanding of transformations in a technological environment but also in making sense of mathematics classroom teaching in general.

In another teaching experiment, Hollebrands (2003) investigated high school students' understanding of geometric transformations in the context of Geometer's Sketchpad. She implemented an instructional unit on translations, reflections, rotations, and dilations for seven weeks. The researcher conducted a teaching experiment on six students to investigate their understanding of geometric transformations. Four students whose data sets were almost complete formed the cases of this study. Major data sources were transcripts of whole-class discussions, of the participants working collaboratively in small groups and pairs, and of task-based interviews conducted with each participant. Students' conceptions of transformations as functions were analyzed by using Action-Process-Object-Schema (APOS) Theory. The findings showed that students initially had motion understanding of transformations and later they moved towards reasoning about transformations as functions which was regarded as mapping understanding. In more detail, Hollebrands (2003) found out that three out of four high school students started to consider all points in the plane as the domain towards the end of the instructional unit. Furthermore, these three students appeared to be able to predict the results of applying transformations on each point without actually performing them and this way of thinking suggested a process conception. On the other hand, none of her participants could reach an object conception of geometric transformations at the end of the instructional unit. She arrived at a conclusion that in order to have a mapping understanding of transformations, one needed to understand four fundamental concepts: the domain of transformations as all points in the plane; parameters (e.g., vector, reflection line, center of rotation, and angle of rotation) that define transformations; the relations and properties of transformations; and transformations as being one-to-one mappings of points in the plane onto points in the plane.

There are studies that developed an instructional unit in dynamic geometry environments for prospective teacher as well. For instance, Yanık (2006) monitored prospective elementary teachers' progress in understanding rigid transformations namely translation, reflection, and rotation. He employed APOS Theory as the theoretical framework of his study. He conducted clinical interviews with four prospective teachers and found out that these four participants had an action conception of transformations from the perspective of APOS theory. Besides, two of those participants showed some understanding of process conception of transformations as the teaching experiment went on. These two participants were selected for further analysis for their understanding of transformations. The results showed that these two prospective teachers constructed similar learning trajectories in conceptualizing transformations and the order of their development was first "transformations as undefined motions of single objects", then "transformations as defined motions of single objects", finally "transformations as defined motion of all points in the plane" (p.57). Furthermore, understanding parameters, their relationship with geometrical objects, the effect of transformations, and the domain of the transformations were important factors that affected the progression of participants through mentioned stages. Both participants followed the same phases and thus their cases were combined and presented as a case of "growing from undefined motion of single objects to defined motion of the plane" (p.57). Yanık (2006) suggested that parameters, domain, and plane were important factors in transformations therefore they should be emphasized in the teaching of transformations.

In another teaching experiment that focused on translations, Yanik and Flores (2009) explored the development of a master student's, Jeff's, understanding of translations. Data were collected through clinical interviews and teaching episodes. Towards the end of the teaching experiment, Jeff's growing understanding of translation became apparent. The model of Jeff's understanding of translations were presented in Figure 2.3.

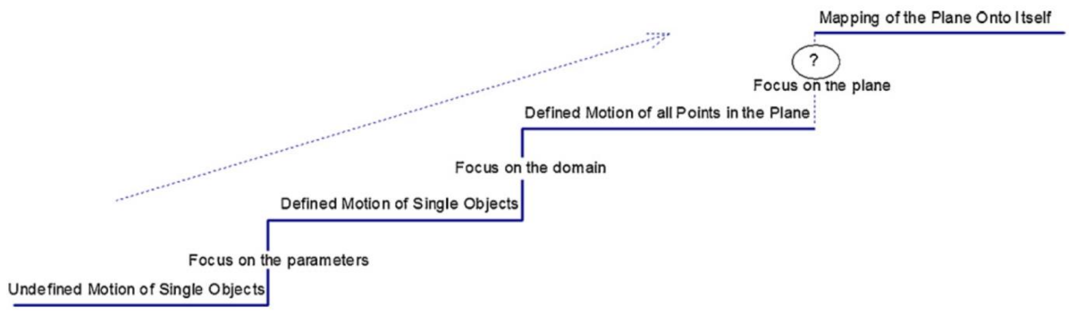


Figure 2.3. Jeff's hypothetical learning trajectory (Yanık & Flores, 2009, p.46).

This model described Jeff's developing ideas of translations as he moved from an understanding of transformations as undefined motion of single objects toward an understanding of transformations as the mapping of plane onto itself.

The findings of the teaching experiment suggested that understanding the vectors (parameter of translation) helped Jeff realize that transformations were actually determined by vectors. By the end of the study, Jeff understood how changing the vectors would affect the result of transformations. Besides, understanding domain as all points in the plane was a new and difficult concept for Jeff. Initially, Jeff was thinking the domain as a single geometrical point or a line rather than all points in the plane. By the end of the study, Jeff was able to consider the domain as all points on the plane for translations. However, his understanding of plane in relation to geometric figures was still the main difficulty for him to see the nature of mapping of plane. Since he was thinking that geometrical objects were not part of the plane, the mapping was like placing the objects in a different location on the plane. Furthermore, Jeff considered geometric figures separate from the plane in which they were located. This was different from the perspective of contemporary mathematics in which points, lines or figures were located at particular places on the plane but they were not separate from places where they were located and they were actually subsets of the plane (Smart, 1998). Since Jeff was thinking of objects as individual entities, which were independent from the plane and from each other, he applied the transformations only to discrete objects in the first two stages. Therefore, when he was provided multiple discrete geometric figures to be translated, he did not consider all of the initial geometric relationships among the figures. By the end of the study, although Jeff was thinking that the plane was static and consisted of infinite number of distinct points, he held the belief that everything was independent of the plane except for the coordinate system. According to Jeff, when a transformation of any type was performed, the objects themselves were mapped onto new coordinates of the same plane. Understanding the geometric figures as subsets of the plane, however, with certain relationships among points, seemed a difficult concept for Jeff throughout the study. Yanık and Flores(2009) pointed out that although the model described above might not be applicable for all prospective teachers, it was still a useful tool for understanding how participants perceived and made meaning of rigid transformations based on the tasks that were asked during the teaching experiment.

There are studies that developed an instructional unit in a non-technological context for prospective teacher (e.g., Portnoy et al., 2006). For instance, Portnoy et al. (2006) explored 19 prospective middle and high school teachers' understanding of geometric transformations in a non-technological context. They developed mathematical activities entitled Connections materials. The following series of activities were included in these materials: "isometries of the plane; rotations, reflections, translations and glides; compositions; proof with isometries; the human vertices; and isometries and linear algebra" (p. 198). The implemented curriculum module aimed to have participants explore the relationship between transformation geometry and linear algebra. Six activities were implemented for 6 weeks in a semester long geometry course. Data were collected through interviews, concept maps, and journal entries. Of all the participants who participated in the activities, eight agreed to be interviewed and four of them completed all three interviews. The first interview was conducted at the middle of the implementation and focused on participants' beliefs about mathematics. The second interview was conducted after the implementation and participants were asked to draw and explain their concept map of isometries, explain the relations between isometries and other mathematical domains such as linear and abstract algebra, reflect to and on two proofs of isosceles triangle theorem (i.e., the standard synthetic proof and the proof by using geometric transformations). The final interview focused on pedagogical component of the materials and hence on the relationship between undergraduate mathematics and high school mathematics.

The results of the study showed that participants had difficulties in understanding and constructing proofs using geometric transformations. Actually, participants' views about geometric transformations and geometric objects conflicted with views about the role of proof in mathematics. The results of the study showed that prospective teachers had operational conceptions of transformations rather than structural. They thought transformations primarily as processes or procedures that could be applied to geometric objects such as sliding, flipping, and rotating. Four participants' operational views were seen during the interviews. The researchers saw that participants' lack of object view of transformations also supported their operational views. Moreover, three of the four participants had a perceived view of geometric objects and did not completely develop conceived views. They also saw that the participants were working with perceived geometric objects (i.e. drawings) rather than working with conceived geometric objects (i.e. mind's eye, uniform, etc.). The researchers briefly stated that the participants' conceived transformations as procedures and viewed geometric objects (e.g., a triangle) as "perceived" and this affected their ability to write and develop proofs related to geometric transformations.

By focusing on translation, Yanık (2013) explored the nature of prospective middle school mathematics teachers' understanding of translation in the context of GeoGebra. Individual teaching experiment methods were used to explore participants' development of understanding of translations. Four prospective teachers who held either translational or rotational motion conceptions of translations participated in the study. The study was conducted in three phases as semi structured clinical interviews, teaching episodes, and retrospective analysis.

The findings of the study showed that GeoGebra improved participants' understanding of translations. Namely, they progressed from motion conception to mapping conception of translations. More specifically, the dragging and measurement features of GeoGebra helped them to explore the properties of translations, to make conjectures, use several strategies and build new understandings. When participants focused on the result of translation they continued to hold motion conception. However, when they focused on the coordinate system and points as locations they could shift from motion to mapping conception of translations. GeoGebra's immediate visual feedback helped them see points as locations rather than physical entities on the plane. Besides, the participants became more confident and were able to identify, explain, perform and represent translations at the end of the study. Yanık (2013) concluded that transition from motion to mapping conception was a difficult task for prospective teachers. Besides, he recommended researchers to focus on other prospective teachers' understanding of geometric translations to find out whether they follow the same path described in his study.

Studies related to students' understanding of transformations have revealed similar results (Flanagan, 2001; Hollebrands, 2003; Sünker & Zembat, 2012). For instance, understanding the domain of transformation played a critical role in understanding transformations as functions which was regarded as mapping understanding. Besides, parameters such as vectors, reflection lines, centers of rotations, and angles of rotations were fundamental concepts needed to have a mapping understanding. Besides, dynamic geometry environment helped students to develop their understanding.

In the aforementioned studies, none of the participants had mapping understanding of transformations at the beginning of the studies. However, the studies that implemented instructional units on prospective teachers found out that prospective teachers' lack of knowledge of transformations diminished and they realized the role of parameters (i.e., vectors, line of symmetry, and center of rotation) in transformations (e.g., Portnoy et al., 2006; Yanık 2006; Yanık & Flores, 2009; Yanık, 2013). More importantly, learners were able to understand that domain and plane were key concepts in understanding translations as functions. Therefore, prospective teachers' understanding of transformations evolved from motion to mapping as a result of the instructions.

Another point is that the findings obtained from studies using elementary and high school students as participants (e.g., Flanagan, 2001; Hollebrands, 2003; Sünker & Zembat, 2012) were similar to those who used prospective teachers as study participants (e.g., Portnoy et al., 2006; Yanik 2006; Yanık 2011; Yanık & Flores, 2009). In all the studies described in this review, the researchers emphasized that understanding the domain and parameter of a transformation played a critical role in students' understanding of transformations as functions that could be regarded as mapping understanding.

CHAPTER III

METHODOLOGY

The purpose of this study was to explore prospective middle school mathematics teachers' evolving understanding of geometric transformations in the context of a semester long course. Through this purpose, the following research question was formulated:

1. How do prospective middle school mathematics teachers' understanding of geometric transformations develop during the instructional unit?
    a. How do prospective mathematics teachers' competencies to identify geometric transformations develop as they engage in the instructional unit?
    b. How do prospective mathematics teachers' competencies to perform geometric transformations develop as they engage in the instructional unit?
    c. How do prospective middle school mathematics teachers' understanding of geometric transformations develop in action-object continuum?

In this chapter, the research design and design principles and implementation of the instructional sequence were described first. Next, participants of the study, data collection and data analysis procedures were described. Finally, the methods that were employed to ensure the trustworthiness of the current research and the researcher role and bias were explained.

3.1. Design Based Research

As educational research deals usually with issues and problems that are different from the ones included in everyday practice, there is a need for new approaches that address directly to the problems of practice (National Research Council [NRC], 2002). More specifically, mathematics education research is reconceptualized as a design science similar to engineering and other emergent interdisciplinary fields (Lesh & Sriraman, 2005). Design-based research can be defined as "an emerging paradigm for the study of learning in context through the systematic design and study of instructional strategies and tools” (The Design Based Research Collective, 2003, p. 5).

In traditional educational research, research is often conducted in contrived settings in order to investigative what strategies or interventions are effective in teaching. In contrast, design based research involves testing learning theories in actual settings through a cycle that includes testing, modifying, retesting, and remodifying (Gorard, Roberts & Taylor, 2004). In this study, the researcher developed a hypothetical learning trajectory related to prospective middle school mathematics teachers' developing understanding of geometric transformations. The researcher tried to test this hypothetical learning trajectory through testing, modifying, retesting, and remodifying the instructional sequence regarding geometric transformations.

Moreover, design-based research contrasts with laboratory studies of learning in terms of the following aspects of methodology:

> “conducting research in messy, non-laboratory settings, conducting research involving many dependent variables, characterizing the situations as opposed to controlling variables, using flexible design revision instead of staying with fixed procedures, valuing social interaction over social isolation in learning, generating profiles rather than testing hypothesis, valuing participants’ input for design and analysis, rather than relying solely on the judgements of the researcher” (Collins, 1999, as cited in Kelly, 2004, p. 119-120).

Similarly, Lesh and Sriraman (2005) explained the characteristics of design science that are particularly appropriate for mathematics education in this way: the participants being examined are in part the products of human creativity; the participants being examined are complex systems; rather than just testing, the researchers need to design for power, sharability and reusability; the participants and the conceptual systems are continually changing; participants being investigated are affected by social constraints and affordances; there is a need to more than one single theory for providing practical solutions to practically complicated problems; and the development frequently includes a series of iterative design cycles. Design studies are based strongly on prior research and theory (Shavelson, Phillips, Towne & Feuer, 2003). In this study, in order to develop an instruction unit for having prospective teachers gain an understanding of geometric transformations as functions, the researcher took into account the critical ideas reported in the literature.

Design studies are carried out in educational settings aiming to trace the evolution of learning in classrooms and schools, to test and build theories of teaching and learning, and to produce instructional tools that survive the challenges of everyday practice (Shavelson et al., 2003). In this study, the instructional unit developed by the researcher was implemented in a classroom environment and prospective teachers' developing understanding of geometric transformations, namely their progressions from motion to mapping conception of geometric transformations, were traced. The instructional unit comprising hypothetical learning trajectory, transformation geometry activities and the corresponding GeoGebra files are instructional tools that might help teachers, teacher educators and students in overcoming difficulties related to geometric transformations and gaining a mapping understanding of geometric transformations.

Kelly (2004) stated that "design studies should produce an artifact that outlasts the study and can be adopted, adapted, and used by others (e.g. either researchers or teachers)" (p.116). The artifact does not have to be concrete like computer programs and it might describe aspects of "activity structures, institutions, scaffolds, and curricula" (The Design-Based Research Collective, 2003, p. 6). Design studies give information "about a model of practice, about learning, about the design and use of a piece of software or learning environments" and teaching experiments are examples of design studies which aim to study the process of engagement between a teacher and a student. (Kelly, 2004, p.116). By this way, Kelly considers the artifact as the process of teaching experiment. In this study, the participants interacted with each other during pairwise and whole discussions of transformation geometry tasks. Thus, this design experiment (in particular classroom teaching experiment) gave information about the learning of geometric transformations in a social environment. Besides, this study gave information about the design and implementation of the instructional unit in which a dynamic geometry environment was used.

Design experiments might differ from each other in terms of the participants being explored. Namely, design experiments might be conducted with an individual student who interact one-on-one with a researcher in a series of teaching sessions or a group of students in a classroom or pre-service teachers in a university course, or practicing teachers who collaborate with researchers as members of a professional teaching community (Cobb & Gravemeijer, 2008). This study was carried out with 16 prospective middle school mathematics teachers. Thus, classroom teaching experiment methodology was used to explore participants' developing understanding of geometric transformations.

Regardless of their types, design experiments have the following crosscutting features: iterative, process focused, interventionist, collaborative, multileveled, utility oriented, and theory driven (Cobb, Confrey, diSessa, Lehrer & Schauble, 2003). Shavelson and others (2003) summarized the characteristics of design experiments as follows: *iterative* since they are strictly related to design-analysis-redesign cycles that lead to learning or to the development of an artifact; *process focused* since they aim keeping track of either individuals', groups', or school systems' learning by understanding consecutive patterns in their thinking and reasoning and the effect of instructional artifacts on these thinking and reasoning; *interventionist* since they test theory and instructional artifacts by design and modification of real world contexts; *collaborative* since team members share knowledge and work together; usually *multileveled* since they might include analysis of multiple settings such as classrooms, schools or districts; *utility oriented* since they intend to develop efficient instructional tools to foster learning; *theory driven* since they test and contribute to the theory by design-analysis-redesign of instructional activities and artifacts.

One type of design research is design experiments in classrooms in which a research team is responsible for the learnings of a group of students (Cobb & Gravemeijer, 2008). In a classroom teaching experiment (i.e., a design experiment in classroom), students are considered to contribute to the evolving classroom mathematical practices as they reorganize their individual mathematics activities. In this study, both pairwise discussions and whole class discussions were carried out by the participants. Participants shared their understandings with each other during pairwise and whole class discussions. By this way, they revised and refined their understandings and developed a shared understanding of geometric transformations.

Classroom teaching experiments can be conducted either in collaboration with teachers or a researcher may act as a teacher (Cobb, 2000). In this study, the researcher also acted as a classroom teacher.

3.2. Designing and Piloting the Instructional Sequence

In this study, classroom teaching experiment methodology was used to investigate prospective middle school mathematics teachers' evolving understanding of geometric transformations. A classroom teaching experiment has three phases: "preparing for the experiment, experimenting to support learning, and conducting retrospective analysis of the data generated during the course of the experiment" (Cobb & Gravemeijer, 2008, p.68). Besides, a classroom teaching experiment begins with the formulation of a hypothetical learning trajectory (HLT). A HLT is made up of three components: learning goals for students; planned learning or instructional activities; and a conjectured learning process in which teacher anticipates how students' thinking and understanding might evolve when the learning activities are enacted in the classroom (Cobb, 2000). In this study, in order to have prospective middle school mathematics teachers gain a mapping understanding of transformations and to remedy their difficulties experienced in geometric transformations a HLT was designed. In designing the HLT, ideas that were found crucial in understanding geometric transformations as functions and learners' difficulties related to these key ideas as reported in the previous studies were taken into consideration. The HLT designed in this study was implemented in the pilot study.

The pilot study was conducted with eleven volunteered prospective middle school mathematics teachers who attended an elective mathematics education course for fourteen weeks in the fall semester of 2014 – 2015 academic year. Two junior and nine senior prospective teachers participated in the pilot study. In the first week participants were informed about the study and in the second week they took the first version of the Transformation Geometry Questionnaire (TGQ) developed by the researcher. Then, in the following weeks I implemented the first version of seven activities of the instructional sequence. After the implementation of the instructional sequence, the TGQ was administered.

During the implementation of instructional sequence in the pilot study, a mathematics educator took notes related to the implementation. Each class session was videotaped and small group discussions were audio recorded. In the class sessions, prospective teachers worked on geometric transformation tasks in groups of

2 or 3. For each task included in the activity sheets, participants first provided written responses to them in small group discussions and then they discussed their responses in whole class discussions. At the end of the activities, activity sheets were collected.

The researcher, a mathematics educator, and two volunteered prospective teachers who participated in the activities met after each class session. In these meetings, challenging ideas or ideas that were not well-communicated during the class sessions, issues about tasks, and guidelines in the activities, the adequacy of time allocated for completion of each task, the issues related to implementation of the activities and the role of the researcher covering the lessons were discussed. In addition, the meetings helped in evaluating the effectiveness of the GeoGebra files prepared by the researcher in advance for some of the tasks. More specifically, the meetings helped determine whether the GeoGebra files served their purpose, whether they contributed to prospective teachers' generalization of ideas related to geometric transformations and to reaching the goal of each task. It is thought that prospective teachers' participation in these meetings proved very useful in terms of the soundness of the evaluations. The designed activities were revised during and after the pilot study based on the data gathered through weekly meeting notes, observer notes taken by the researcher and the mathematics educator, video and audio recordings collected during the implementation, and the activity sheets completed by the prospective teachers.

The HLT formed before the pilot study is presented in Table 3.1. It consisted of five phases and each phase is explained in the following parts. Besides, revisions made after the pilot study are explained. There was no need to revise the learning goals and the sub-learning goals after the pilot study. Only some of the tasks included in the first four phases needed some small revisions. Since these revisions did not affect the learning goals, the HLT implemented in the pilot study and the main study were the same and it is presented in Table 3.1.

Table 3.1. Hypothetical Learning Trajectory formed in this study

Phases	Learning Goals	Sub-Learning Goals	Supporting Tasks	Tools
1	Define transformations with their parameters	Identify transformations Identify parameters of transformations Use parameters to define transformations Perform transformations to construct frieze patterns and finite figures	Activity 1 and Activity 2	Ruler, protractor and GeoGebra files
2	Perform transformations and their compositions	Perform transformations by using a ruler, a compass and a protractor Perform compositions of transformations Identify a single transformation equal to composition of transformations	Activity 3 and Activity 4	Ruler, compass, protractor and GeoGebra files
3	Exploring functional dependency of points in the plane (R^2)	Classify points in the plane as dependent and independent points by exploring their behaviors in GeoGebra Express the coordinates of dependent and independent points by comparing the corresponding changes in their direction and speed in GeoGebra Distinguishing functional relations from non-functional relations Express mathematically the rule of the functional relation among dependent and independent points in GeoGebra	Activity 5	GeoGebra files
4	Exploring transformations as $R^2 \rightarrow R^2$ functions	Explore variables of transformations as points in the plane Explore the domain of transformations as plane (R^2) Define geometric transformations as $R^2 \rightarrow R^2$ functions	Activity 6	GeoGebra files
5	Exploring functional properties of transformations	Determine points that remain fixed after transformation Determine the identity function of a transformation family Determine inverse function of a transformation Explore transformations as one to one and onto functions	Activity 7	GeoGebra files

3.2.1. Phase One

Parameters and their effects on the transformations are important in understanding geometric transformations as functions. They are parts of the definitions of functions. They determine the particular member of a function family (Hollebrands, 2003; Yanık, 2009). However, related literature showed that learners' some of the difficulties about geometric transformations were related to defining transformations with their parameters (Hollebrands, 2004; Thaqi et al., 2011; Yanık, 2011; Yanık, 2014). For instance, when defining rotation, prospective elementary teachers mentioned only the angle but not the center (Thaqi et al., 2011). Besides, when defining translation, prospective middle school teachers mentioned either the magnitude or the direction of the translation vector (Yanık, 2011, 2014).

Learners' some other difficulties were related to identifying transformations (e.g., Harper, 2003; Hollebrands, 2004; Thaqi et al., 2011). Harper (2003) stated that when a figure and its rotated image were presented, prospective elementary teachers could not explain this transformation by using only one rotation and they could not identify the parameters of the rotation. They tried to explain this rotation by a combination of simple steps without identifying any specific center and angle of rotation. In short, research has shown that prospective teachers explained geometric transformations as either partially defined or undefined. Besides, they had difficulty in identifying the geometric transformations.

The results of these studies showed that it was critical to design tasks that would help the prospective teachers in identifying geometric transformations and understanding the role of parameters in geometric transformations. With this implication in mind, the first phase of the HLT was prepared to remedy prospective teachers' aforementioned difficulties in geometric transformations. Thus, the aims of the first phase were to present numerous finite figures and frieze patterns that were constructed by using geometric transformations and to create opportunities for learners to explore these geometric transformations in pair work and share their ideas in whole class discussions. In the first phase, there were two activities and these activities are explained in the following paragraphs.

3.2.1.1. Activity One

Activity One was related to reflection and rotation. In the first version of the Activity 1, there were two sub-sections related to identifying reflections and rotations. The first sub-section was related to identification of rotation and its parameters and the second sub-section was related to identification of reflection and its parameter. Each sub-section had four tasks and each task involved three finite figures. Thus, the participants of the pilot study examined twenty-four finite figures in total and identified the geometric transformations involved in these figures. After the pilot study, the researcher decided to avoid wasting time due to duplicate tasks involving identification of reflection or rotation. Besides, the researcher decided to avoid prospective teachers' identification of geometric transformations without fully understanding by mixing different types of geometric transformations. Therefore, after the pilot study, the sub-sections were combined and participants examined the geometric transformations in a mixed order. Besides, in order to save time, the number of tasks was decreased to five and the number of finite figures in each task was decreased to two. Therefore, the main study included ten finite figures in total and the types of transformations were mixed.

In the main study, the first six tasks involved identification of reflection and rotation in finite figures and identification of the parameters of these geometric transformations. In all of these tasks, the center of rotation was on the figure itself and the line of reflection was passing through the figure. The seventh task involved utilization of reflection and rotation in constructing finite figures. The details of these tasks are explained in the following paragraphs.

Tasks 1–5 consisted of two finite figures that were formed by the same geometric transformation(s). Task 1 and Task 3 had to with identification of both reflection and rotation. Task 2, Task 4 and Task 5 had to do with identification of only rotation. and it included seven tasks (see Appendix C).

Task 6 included eight finite figures and each pair of participants were expected to select one of these finite figures from a bag. The finite figures included in Task 6 were different from the ten finite figures included in Task 1 through 5. When eight figures were on the smart board, each pair was expected to explain their own finite figure by using related transformations and have other participants determine the selected finite figure. To clarify, the two finite figures included in Task 6 are presented in Figure 3.1.

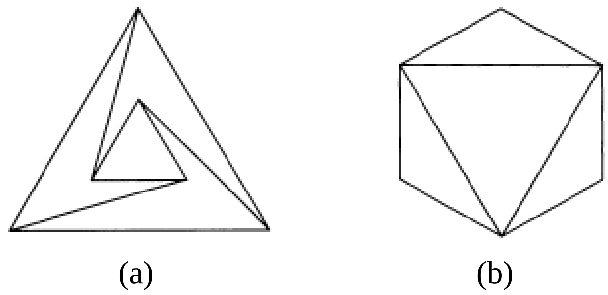


Figure 3.1 Two finite figures in Task 6 of Activity 1

While the first finite figure in Figure 3.1 involves only 120 degree rotations, the second finite figure involves both 120 degrees rotation and three reflections. The pair who selected one of the finite figures in Figure 3.1 has to explain their figure by using both reflection and rotation and their parameters.

In sum, in the first six tasks, the prospective teachers were expected to identify geometric transformations and define these transformations with their parameters. The aim of Task 7 was to have the prospective teachers perform geometric transformations to construct their own finite figures. Each prospective teacher was expected to decide on the unit, the angle of rotation and the reflection lines her/himself and to obtain two finite figures one of which had to be obtained only by performing rotation while the second one had to be obtained by performing reflection and rotation.

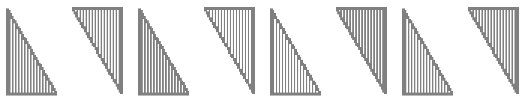
The first version of Task 7 of Activity 1 was “construct finite figures by using the geometric transformations that you examined and explain your finite figures mathematically”. Evaluation of the activity sheets of participants in the pilot study showed that this statements was too general and some of the participants constructed finite figures by using only rotation and some others by using rotation and reflection. Moreover, participants did not explain the unit of their finite figures and thus did not explain how they applied the geometric transformations on that unit. Therefore, the researcher decided to ask prospective teachers in the main study to construct two finite figures with the following conditions: the first one must involve only rotation and the second one must involve both rotation and reflection. Besides, this general statement in the pilot study was divided into eight sub-tasks in order to understand how participants perform geometric transformations. Therefore, main study participants constructed two finite figures and they were explicitly asked to explain their own unit and the transformation(s) applied to this unit in order to construct the corresponding finite figure.

3.2.1.2. Activity Two

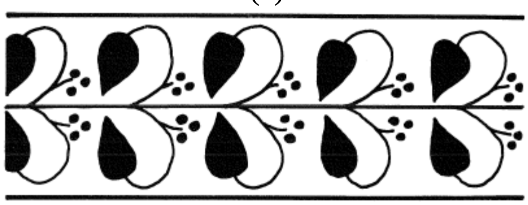
Activity Two was related to reflection, translation and rotation. There were eight tasks in the first version of Activity Two. After the pilot study a new task was designed as the sixth task. Nine tasks including the new one are explained in the following paragraphs.

The first seven tasks involved identification of reflection, translation and rotation in frieze patterns and identification of the parameters of these geometric transformations. Task 8 and Task 9 involved performing geometric transformations to construct frieze patterns.

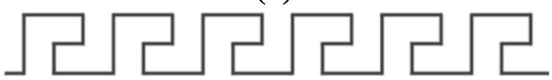
In Task 1 through 7, participants were asked to identify geometric transformations involved in the frieze patterns given in each task. In each of the first five tasks, participants were presented a frieze pattern in which the unit of patterns was a right triangle (see Figure 3.2a). Task 6 and Task 7 included five and eight different frieze patterns respectively and the units of these patterns were unobvious (see, for example, Figure 3.1b, c). In Figure 3.2, one of the frieze patterns included in the first five tasks, one of the frieze patterns included in Task 6 and one of the frieze patterns in Task 7 are presented.



(a)



(b)



(c)

Figure 3.2 Sample frieze patterns in three different tasks of Activity 2

Task 1 of Activity 2 had to do with identification of only translation. Tasks 2 to 4 had to with identification of vertical, horizontal and both horizontal and vertical reflection respectively and Task 5 had to do with identification of rotation (see Figure 3.2c). In these tasks, the center of rotation was outside the figure and the line of reflections were not passing through the figure.

In the pilot study, the unit of the frieze patterns in Tasks 1 – 5 in Activity 2 was the letter R. Besides, the boundaries of frieze patterns were framed with straight lines. After the implementation of the pilot study, the researcher decided to simplify these tasks by changing the unit to a right triangle and deleting the straight lines in the frieze patterns. Because, the letter R made the tasks more challenging. Besides, the participants had some confusions due to the straight lines.

As mentioned before, Task 6 was prepared after the pilot study as transition from the identification of geometric transformations in frieze patterns with obvious units to identification of geometric transformations in frieze patterns with unobvious units. Namely, in pilot study the researcher noticed that the participants experienced difficulties in identifying geometric transformations due to their inability to determine the unit of the frieze patterns. Thus, the researcher decided to prepare Task 6 in the main study to check and clarify the important points about deciding the unit of a frieze pattern before moving on to identifying transformations in the frieze patterns with unobvious units. Therefore, iIn Task 6, the participants were asked to identify the transformations included in these frieze patterns, identify the units of the frieze patterns and match these frieze patterns with the frieze patterns included in Tasks 1 through 5 by considering the type of transformations used to construct those patterns.

In Task 7 of Activity 2, prospective teachers examined frieze patterns different from the ones they examined in Task 1 thorough 6. In Task 7, each pair randomly selected one pattern from a bag of eight frieze patterns. The participants were expected to examine their own pattern and identify the geometric transformations included in this frieze pattern. As in Task 6, they were expected to match their own frieze pattern with the frieze patterns included in Task 1 through 5 by considering the type of transformations used.

Task 8 and Task 9 involved performing geometric transformations to construct their own frieze patterns. In Task 8, each prospective teacher was expected to choose one of the following letters as the unit of their pattern: Ç, F, G, J, L, P, and R and to construct a frieze pattern by performing geometric transformations on their unit. In Task 9, each prospective teacher was expected to decide on a unit and to construct a frieze pattern by performing geometric transformations on their unit.

3.2.2. Phase Two

In order to understand that transformations are functions, one should know the relations among the pre-images, parameters and images, invariant properties, preservation of length and collinearity (Hollebrands, 2003). However, research has shown that learners could not consider the relationship among a pre-image, a parameter(s), and an image in geometric transformations (e.g., Hollebrands, 2004). In particular, learners could not carefully consider that reflection line must be the perpendicular bisector of the segments formed by joining corresponding pre-image and image points in reflection. Therefore, reflection with inclined reflection lines were found to be especially challenging for learners to perform (Hollebrands, 2004; Ramful, et al., 2015). Meanwhile, it was reported that the intuitive idea of being opposite worked in performing vertical or horizontal lines whereas it did not work in inclined lines and it was concluded that it was mandatory for learners to know criterion of perpendicularity formally in order to perform reflection with inclined lines (Ramful et al., 2015).

Learners also have difficulties in using the translation vector in performing translations (e.g., Harper, 2003; Hollebrands, 2004; Yanık, 2011; Yanık, 2014). They use the translation vector as a reflection line or as a direction indicator (Yanık, 2011, 2014) or as a location on which the figure should be placed (Hollebrands, 2004).

To overcome the aforementioned difficulties, the second phase of the HLT focused on the relations among the pre-image, the parameter(s), and the image, invariant properties, preservation of length and collinearity under geometric transformations in the course of participants' execution of single translations and composition of transformations. In designing tasks related to geometric transformations, a rich variety of parameters were selected. For instance, in reflection tasks, vertical, horizontal and particularly inclined lines were used as parameters.

In this phase, there were two activities, and one or more GeoGebra files were prepared in advance for each task of these activities. These activities are explained in the following paragraphs.

3.2.2.1. Activity Three

Activity Three was related mainly to performing geometric transformations. There were four tasks in the first version of Activity Three. After the pilot study a new task was designed as first task. Therefore, there were five tasks in Activity Three implemented in the main study. All tasks including the new one are explained in the following paragraphs.

There was no identification task in the first version of the Activity 3 implemented in the pilot study. Namely, in pilot study, Activity 1 and Activity 2 were related to identification of geometric transformations and Activity 3 was related to performing geometric transformations. During the implementation of Activity 2 in the pilot study, the researcher noticed participants' difficulty in identifying geometric transformations. Thus, the researcher decided to prepare Task 1 in the main study to check and clarify the important points about identification of geometric transformations before moving on to performing tasks in Activity 3. Task 1 included three sub-tasks and each of these sub-tasks involved translation, reflection and rotation, respectively. In this task, letter F was given on a grid and the participants were asked to identify the geometric transformation and its parameter(s).

Task 2 (i.e., the first task of the Activity 3 in the pilot study) involved performing single transformations, and Task 3, Task 4 and Task 5 involved performing compositions of geometric transformations. In more detail, Task 2 was related to performing geometric transformations on a grid and it consisted of three sub-tasks. In Task 2A, prospective teachers were asked to translate the letter F by a given vector. In Task 2B, they were asked to reflect the letter F along an inclined reflection line. Finally, in Task 2C, they were asked to perform a rotation to the letter F around a given center and angle of rotation. After performing geometric transformations on paper, participants were expected to explore the same transformations by means of the corresponding pre-constructed GeoGebra file. Through this file, prospective teachers were expected to recognize the relations among the pre-images, parameters and images, invariant properties, and preservation of length.

Task 3, Task 4, and Task 5 were related to performing compositions of translations, reflections and rotations, respectively. Task 3 had six sub-tasks and as prospective teachers progressed through these sub-tasks, they were expected to recognize that the composition of two or more translations is equal to a single translation. In this task, the letter F was given on a coordinate plane. In Task 3A, two vectors were given and prospective teachers were asked to translate F initially with the first vector to obtain F' and retranslate F' with the second vector to obtain F''. Similarly, Task 3B and Task 3C required performing three and four successive translations respectively. Task 3D involved comparing the results of Task 3A, Task 3B, and Task 3C. Task 3E involved finding a single transformation alternative to the composition of translations given in Task 3A, 3B, and 3C. Finally, Task 3F had to do with deciding whether it was possible to find a single transformation that is equal to the composition of translations without actually performing two, three, and four translations. Namely, in Task 3F, the prospective teachers were expected to recognize that the sum of translation vectors yielded the translation vectors of the composition of translations. After performing composition of translations on paper, participants were expected to explore compositions of translations for other translation vectors by means of the corresponding pre-constructed GeoGebra file. By this way, prospective teachers were expected to generalize that compositions of translations were always equal to a single translation.

Task 4 involved recognizing that the composition of two reflections corresponds to a single translation when reflection lines are parallel to each other and to a single rotation when reflection lines intersect with each other. This task had three sub-tasks. The translation and rotation tasks included in Task 1A and 1C were re-presented in Task 4A and 4B respectively. However, Task 1A and 1C were about identifying single transformations while Task 4A and 4B asked how the image could be obtained by two performing two successive reflections on pre-image. In Task 4A, the prospective teachers were expected to find out that composition of two successive reflections in which two reflection lines are parallel to each other corresponds to a single translation. In Task 4B, the prospective teachers were expected to find out that composition of two successive reflections in which two reflection lines intersect with each other corresponds to a single rotation. Meanwhile, the prospective teachers were expected to recognize that the intersection point of the reflection lines yields the center of rotation and the angle measure between these two lines is half of the measurement of the angle of rotation. Finally, Task 4C involved comparing the responses given for Task 4A and 4B.

After performing compositions of reflections on paper in Task 4, participants were expected to explore compositions of reflections for different reflection lines by means of the corresponding pre-constructed GeoGebra file. By this way, prospective teachers were expected to generalize that composition of two reflections corresponds to a single translation when reflection lines are parallel to each other and to a single rotation when reflection lines intersect with each other.

Task 5 involved composition of three rotations. In this task, the prospective teachers were expected to recognize that the composition of two or more rotations corresponds to a single translation if the sum of measures of each angle of rotation is equal to 360° or to a multiple of 360° and corresponds to a single rotation in other cases (i.e., if the sum of measures of each angle of rotation is not equal to 360° or to a multiple of 360°). Task 5 included a flag and the points A, B, and C. Besides, this task included four different compositions of rotations tasks with different angle measures. For instance, the first composition had 90°, 220°, and 50° angle of rotations while the second composition had 90°, 150°, and 120° angle of rotations. By selecting different angle measures which all add up to 360°, the critical role of 360° in compositions of rotations was emphasized. The prospective teachers were expected to notice that each composition was equal to a single translation however each translation vector was different. After performing composition of three rotations on paper, participants were expected to explore composition of three rotations for other centers of rotations and angles of rotations by means of the corresponding pre-constructed GeoGebra file. By this way, prospective teachers were expected to generalize the ideas gained through paper and pencil environment.

3.2.2.2. Activity Four

Activity Four was mainly related to performing composition of geometric transformations and it included seven tasks. In general, Tasks 1 through 4 were related to composition of translation and reflection and each task included two sub-tasks. Since this composition was equal to a glide reflection in some cases, Task 1 and Task 2 were related to identification of glide reflection and to its properties. Task 3 and Task 4 involved two sub-tasks related to performing composition of a reflection and a translation. Task 5 and Task 6 involved sub-tasks related to performing composition of a reflection and a rotation. Finally, Task 7 was related to performing composition of a rotation and a translation.

Tasks 1A and 1B involved performing composition of a reflection and a translation in which the reflection line and the translation vector were parallel to each other. In both sub-tasks, participants were asked to find a single transformation alternative to the two transformations. They would not find a single reflection, a rotation or a translation since the composition in each sub-task was a glide reflection. The glide reflection was a new concept for participants and Task 2A and 2B involved identification of glide reflection. In these sub-tasks, participants were expected to identify a reflection line and a translation vector which are parallel to each other. Understanding the notion of glide reflection was important since the some of the compositions can only be explained by a glide reflection.

After the implementation of Activity 4 in the pilot study, new sections were added to each task except for Task 3 to make the ideas gained by the prospective teachers during explorations more understandable. As mentioned before, in some tasks participants were asked to identify or to perform geometric transformations and then to explore these transformations in GeoGebra files in order to make generalizations about them. The participants in the pilot study solved these types of tasks in paper and then they explored them in corresponding GeoGebra files. However, it was difficult for the researcher to understand prospective teachers' explorations by using only audio recordings of pair discussions. Thus, the researcher decided to ask participants to summarize their ideas in the activity sheets, in order to understand their ideas gained through the exploration of GeoGebra files. Therefore, in the main study, the researcher allocated some space in activity sheets to prospective teachers for writing down the summary of ideas they gained after exploring GeoGebra files.

Tasks 3A and 3B included composition of a reflection and a translation. In both sub-tasks, the reflection line was perpendicular to the translation vector. Thus, each composition was equal to a single reflection. Tasks 4A and 4B included composition of a reflection and a translation as well. However, in these two sub-tasks, the translation vector was not perpendicular to reflection line and thus the compositions were equal to glide reflections. In Task 4, the prospective teachers were expected to use the ideas they gained in Task 1 and Task 2 and to identify a translation vector and a reflection line of the glide reflection.

After performing composition of reflection and translation on paper in Task 4, participants were expected to explore the compositions for different reflection lines and translation vectors by means of the corresponding pre-constructed GeoGebra file. By this way, prospective teachers were expected to generalize that composition of reflection and translation corresponds to a single reflection when reflection line and the translation vector are parallel to each other and to a glide reflection when the reflection line and the translation vector are not parallel to each other.

Task 5A and 5B involved composition of a reflection and a rotation and in both sub-tasks the center of rotation was on the reflection line. Therefore, each composition was equal to a single reflection. Task 6A and 6B involved composition of a reflection and a rotation as well. However, in these two sub-tasks, the center of rotation was not on the reflection line and thus the compositions were equal to glide reflections. Therefore, in Task 6, the prospective teachers were expected to determine a reflection line and a translation vector which are parallel to each other. After performing the composition of reflection and rotation on paper, participants were expected to explore the compositions for different reflection lines, angles of rotations and centers of rotations by means of the corresponding pre-constructed GeoGebra file. By this way, prospective teachers were expected to generalize that composition of reflection and rotation corresponds to a single reflection when the center of rotation is on the reflection line and to a glide reflection when the center of rotation is not on the reflection line.

Task 7 involved composition of a rotation and a translation. In this task, the composition was equal to a single rotation. Task 1C of Activity 3 was related to finding the center of rotation. In this task, prospective teachers were expected to use the ideas related to finding the center of rotation that they learnt in Task 1C of Activity 3. After performing the composition of rotation and translation on paper, participants were expected to explore the compositions for different translation vectors, angles of rotations and centers of rotations by means of the corresponding GeoGebra file. By this way, prospective teachers were expected to generalize that the composition of rotation and translation corresponds to a single rotation.

3.2.3. Phase Three

Laborde (1995) expressed that “designing a problem situation in a computer based environment requires a new analysis of the mathematical objects, operations and didactical feedback involved in the situation" (p.37). Similarly, Hazzan and Goldenberg (1997) stated that the different behaviors of functions can be observed in different environments. More specifically, they argued that exploration, manipulation, and discussion of functions in a dynamic geometry environment is different from doing them in a static environment. They explained that in static environment the behavior of functions are verbal descriptions generated through conceptualizing symbols, tables or graphs in mind whereas in dynamic geometry environment the behavior of functions can be observed directly and thus it is open to experimentation. A great number of studies have reported that students have difficulty in understanding function concepts despite their centrality in mathematics (Falcade et al., 2007). The teaching of functions in DGE might help learners focus their mathematical thinking on bigger and more abstract mathematical ideas than is common in paper and pencil environment (Hazzan & Goldenberg, 1997) and thus, might help them overcome the difficulties they encounter when learning functions. Meanwhile, since this study aimed to have prospective teachers view geometric transformations as functions through the use of GeoGebra, it is mandatory for them to learn certain characteristics of functions from a geometric perspective. Namely, in this transition phase, prospective teachers are expected to have the pre-existing knowledge that enables them to understand geometric transformations as functions in DGE.

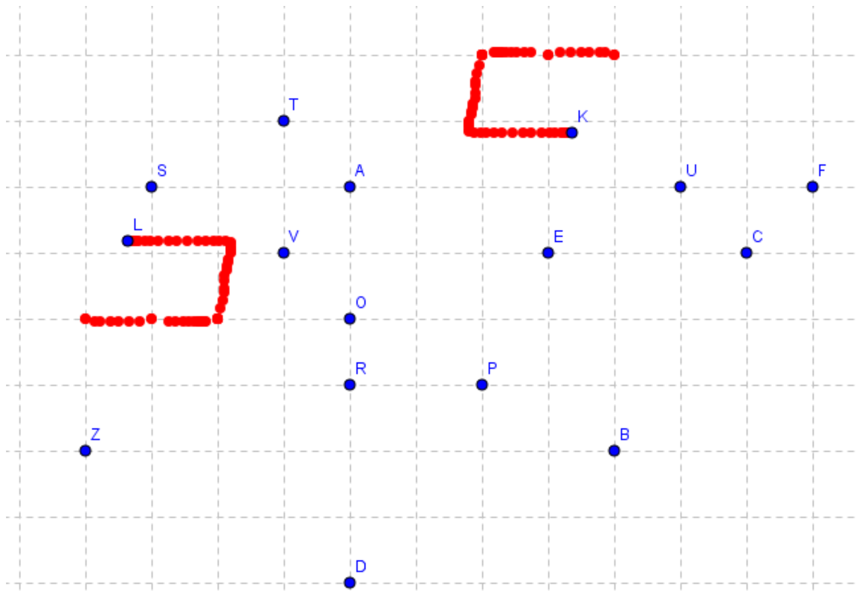
Since learners are most often exposed to functions in a numerical setting, they have difficulty in understanding that functions are relationships between variables

(Carlson, 1998). More precisely, learners are often introduced to functional relationships between variables merely in terms of discrete variables and consequently they have difficulty understanding the multifaceted idea of variable (Trigueros & Ursini, 1999). By considering the learner difficulties reported in the relevant literature, and by acknowledging that functions have a distinct face in DGE than static environment, I plan to implement Phase 3 as a transitional phase. Hence, in the third phase of the instructional sequence, functional thinking was taught to prospective teachers systematically. In this activity they are expected to use GeoGebra to explore the dependent variable, independent variable, and the notion of function as a relationship between these variables. Namely, these explorations are seen mandatory for prospective teachers in order to understand geometric transformations as functions. Prior to this study, they are also expected to be familiar with algebraic functions that include single real numbers as inputs and outputs. In this phase, participants are expected to extend their knowledge of algebraic functions to functions defined on the plane in which variables are ordered pairs of real numbers. Briefly, this phase aim at having prospective teachers explore functional ideas in a dynamic geometry environment, and thus at enabling them to have a deeper understanding and more abstract thinking of functions which would pave the way for a mapping conception of geometric transformations.

3.2.3.1. Activity Five

Activity Five involved exploration of functional dependency of points in the plane. It is important to note here that there was no need to revise tasks in Activity Five after the pilot study and thus the Activity Five in the pilot study and the main study were the same. This activity included four tasks and in these tasks participants were expected to classify points in the plane as dependent and independent points, to observe the relative speed and direction of dependent and independent points, to distinguish functional relations from non-functional relations and to express the rules of all relations in GeoGebra.

The aim of Task 1 was to have participants think about the idea of dependent and independent variables as points in the plane. The GeoGebra file for Task 1 included 16 points and these points were constructed in a way that some of the points were depended upon other points. When participants dragged some of the points, they observed a change in some other points. However, it was not be possible for other points to drag. For instance, in the corresponding GeoGebra file (see Figure 3.3), when Point K is dragged, the movement of Point L in relation to the change in the position of Point K is observed. However, when there is an attempt to drag Point L, it is seen that this point cannot be dragged.



Note: The traces of Point K and Point L were activated only for this figure for the purpose of explaining the nature of this task to the readers.

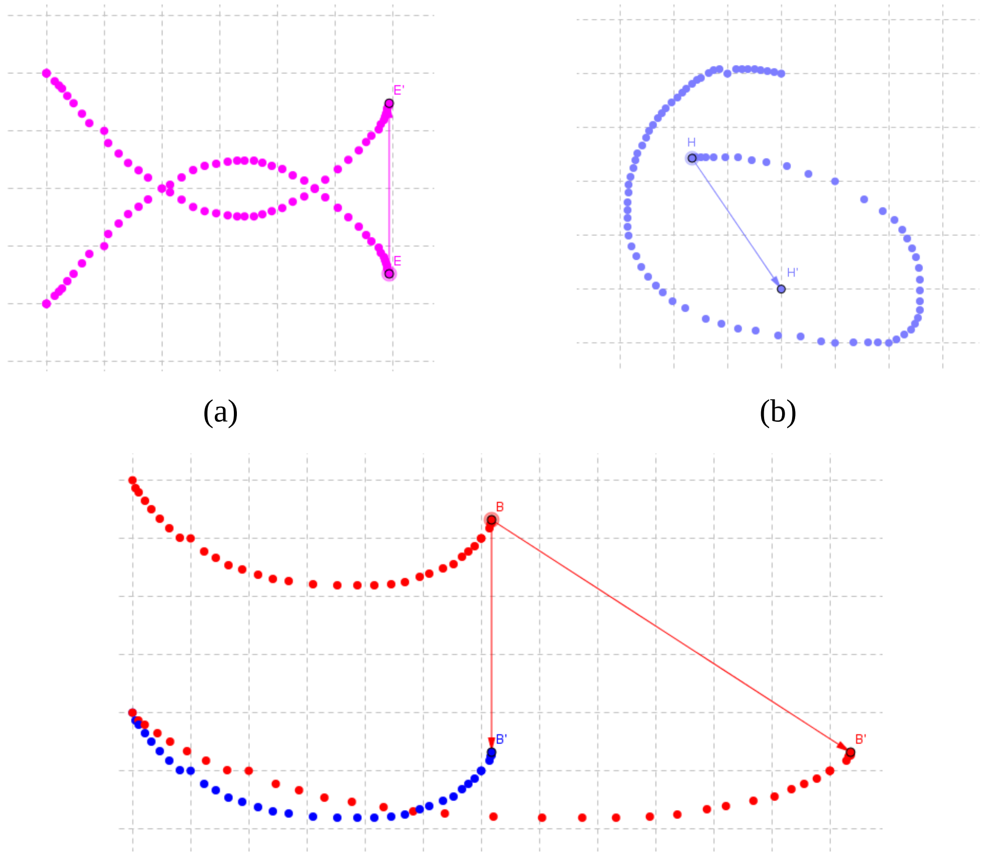
Figure 3.3. GeoGebra screenshot for Task 1 of Activity 5

Task 1A of Activity 5 involved observing the behaviors of the given points by dragging and determining the relationships among given points. Task 1B involved classifying the given 16 points into two by considering their behavioral characteristics. The participants were expected to consider movability of the points, and to give a name to these two groups of points based on their characteristics.

Tasks 2 involved observing relative directions and relative speeds of related points, namely independent and dependent points (points given in Task 1). The aim of this task was to have participants connect relative speeds and directions to the coordinates of independent and dependent points. By making these connections, participants were expected to understand the functional relationship among independent and dependent points.

Task 3 involved exploring eight relations given in four GeoGebra files and deciding whether these relations were functions or not. As an example, Relations EE', HH' and BB' are presented in Figure 3.4. Relations AA', DD', EE', GG' and HH' satisfied all properties of a function. In Relation AA', when Point A was dragged on the screen, the single point A' always moved correspondingly. In Relation EE', when Point E was dragged on the screen, the single point E' always moved correspondingly as well (see Figure 3.4a). The impacts of the independent variables in these two relations on the dependent variables were different from each other. Actually, the Relation AA' was an example of translation and Relation EE' was a reflection along a horizontal line. In this task, these relations were examined in terms of being a function and participants were not expected to relate them with the geometric transformations. The geometric transformations were only used to construct the dependent variable in Relations AA' and EE'. In the Relation GG', Point G' was constructed by projecting Point G on a horizontal line. When Point D was dragged on the screen, Point L appeared and corresponded to D' as well. In Relation DD', matching Point L and Point D with Point D' was not problem for being a function. The Relation HH' was an example of a constant function (see figure 3.4b). Namely, in Relation HH', the Point H' was constructed in a way that kept it remain invariant despite all the draggings of Point H in any direction.

Relations BB', CC', and FF' were not functions since they did not satisfy the properties of being a function (see Relation BB' in Figure 3.4c). In relation BB', when Point B was dragged on the screen, two Point B's moved correspondingly. In Relation FF', when Point F was dragged on the screen, two Point F's moved correspondingly as well. In relation CC', when Point C was dragged on the screen, Point C' appeared in some regions and disappeared in some other regions. The Relation CC' was not a function when the domain was the whole plane. However, it became a function if the domain was restricted to a certain region of the screen.



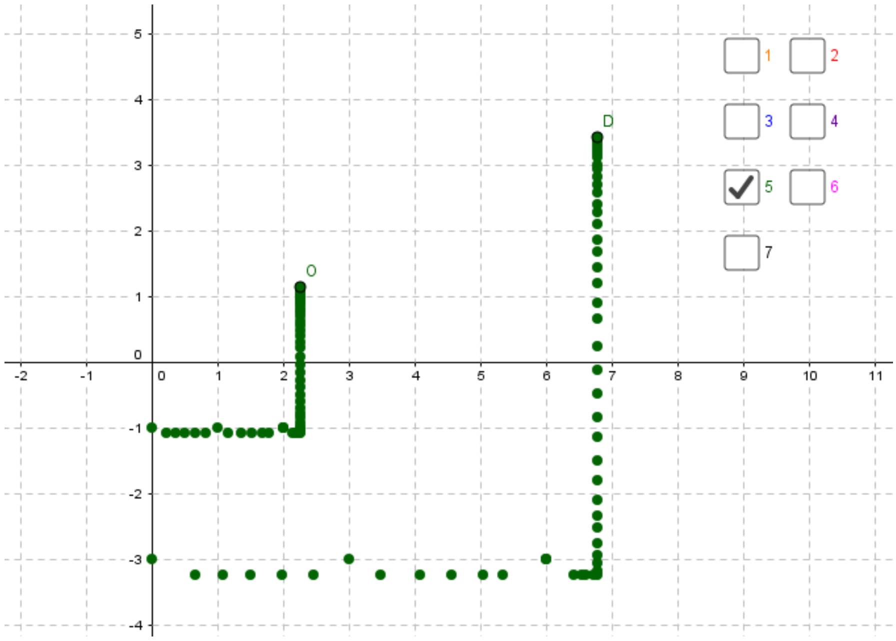
(a) (b)

(c)

Note: The traces of points were activated only for this figure for the purpose of explaining the nature of this task to the readers.

Figure 3.4. GeoGebra screenshots for Relation EE' (a), Relation HH' (b) and Relation BB' (C) in Task 3 of Activity 5

In Task 4, the points given in Task 1 were presented on a coordinate plane and participants were required to express the rules of the relations between the points. As an example Relation OD is presented in Figure 3.5.



Note: The traces of Point O and Point D were activated only for this figure for the purpose of explaining the nature of this task to the readers.

Figure 3.5. GeoGebra screenshot for Relation OD in Task 4 of Activity 5

The construction rules for the dependent points were presented as follows: A→Z, (x, y) → (x - 4, y - 4); E→C, (x, y) → (2x, y); S→R, (x, 0) → (0, x); O→D, (x, y) → (3x, 3y); P→U, (x, y) → (x+3, y+3); T →V, (x, y) → (-y, x); P→F, (x, y)→(x+5, y+5); K→L, (x, y) → (-x, -y).

3.2.4. Phase Four

In order to understand that transformations are functions, one should know that both the domain and range of a transformation is a plane. Namely, geometric transformations map all points in the plane to other points in the plane (Hollebrands, 2003). However, it was revealed that learners from different ranges of ages mainly viewed the plane as an invisible background, thought that geometric figures sit on this plane, and considered transformations as physical movements of geometric figures on the top of the plane (Edwards, 2003). This type of reasoning, motion conception of transformations, was mainly held by the learners (e.g., Edwards, 2003; Glass, 2001; Hollebrands, 2003, 2004; Portnoy et al., 2006; Thaqi et al., 2011; Yanık; 2011, 2014). For learners to move away from motion conception and to understand mapping conception, it is critical for them to conceptualize domain and range of transformations (Hollebrands, 2003; Flores & Yanık, 2009).

Steketee and Scher (2011) expressed that the domain and range of geometric transformations could be introduced to students by the help of Drag and Trace tools and added that by this way students might build a sense of continuous variation of the input and output. Besides, to help students move from motion of figures conception to considering all points in the plane, Hollebrands (2004) recommended the use of points rather than a single polygon as a pre-image. Besides, she suggested using the Drag tool in DGEs to sample the entire plane by dragging a pre-image point. The idea of sampling the plane by dragging an independent point (called pre-image point by Hollebrands (2004)) was considered in developing Activity 5 to emphasize the functional dependency between dependent and independent points. As mentioned before, in Activity 5, the participants explored the function concept from a geometric perspective. The same idea was considered in Activity 6 of Phase 4, and this time the focus was on the domain of functions.

By the help of Activity 6, the prospective teachers were expected to integrate their knowledge and understanding of geometric transformations obtained through the first four activities with the ones obtained through Activity 5. Consequently, by this integration, they are expected to consider geometric transformations as mappings from plane to plane.

3.2.4.1. Activity Six

Unlike the previous activities, there was no student activity sheet in Activity 6. This activity consisted of pair discussions and whole class discussions. It involved re-examining the functions in the Activity 5 by activating the Trace tool in GeoGebra and noticing that some of these functions correspond to specific geometric transformations. More specifically, this activity involved determining domain and range of functions and their elements, exploring functions from R to R through a dynagraph, pondering on the graph of the function from R^2 to R^2, exploring the domain of the relations defined on a specific region on the plane, and defining the domain and range of geometric transformations as a plane. At the end of the activity, participants were expected to come to reason geometric transformations as functions defined from R^2 to R^2.

At the beginning of the Activity 6, prospective teachers were expected to re-examine Relation AA' and Relation CC' which have been examined earlier in

Activity 5. These two relations had been examined in terms of being a function or not in Activity 5. In Activity 6, the participants were expected to determine the domain and range of these relations and their elements. As mentioned before, The Relation AA' was a function defined on R^2 and the Relation CC' was a function defined on a specific region in R^2.

Participants were expected to examine functions from R to R through a dynagraph example. By doing so, participants were expected to shift from R to R^2 when considering the domain of functions represented with $f(x, y)$. Next, participants were expected to think about the graph of Relation AA' which is a function from R^2 to R^2.

Prospective teachers were expected to examine Relation AA' and Relation BB' in Activity 6 (Note that these relations were not the same with the Relations AA' and BB' examined previously in Activity 5). Both of these two relations were defined on a specific region and participants were asked to determine the regions and express them mathematically. Besides, they were expected to find the rule of these relations. For instance, the rule of Relation AA' was $(x, y) \rightarrow (x-4, y-4)$ and it was defined as $[2, 4] \times [1, 3] \longrightarrow [-2, 0] \times [-1, -3]$. Exploring these relations while the Trace tool was activated might help understand that these relations were in fact translations and that all translations were functions.

By the help of the Trace tool, the participants were expected to focus on the idea that the functional relationship between points were also valid for geometric transformations. Briefly, the prospective teachers were expected to experience dynamically that if a pre-image point is dragged, the image point dependent upon this point moves as well and the traces of these two points form two shapes. If the construction rule between the pre-image and image points is a geometric transformation, then these two shapes will also exemplify a transformation which is applied to the first shape to obtain the second shape.

The participants were expected to examine other relations which exemplified translation, reflection, and rotation. These relations were actually the relations that were accepted as functions by the prospective teachers the previous week. Participants were expected to relate these relations with geometric transformations and then to express these transformations with their parameters.

3.2.5. Phase Five

In order to understand that transformations are functions, one should know certain characteristics of transformations (Hollebrands, 2003). Some of the characteristics of geometric transformations are the fixed points under the transformations (i.e., points that map to themselves), identity functions (i.e., a specific reflection that maps all points in the plane onto themselves) of each family of transformation, inverse function of each member of transformation families, and being a one-to-one and onto function. The last phase of the instructional sequence involved Activity 7 and this activity was devoted to the teaching of aforementioned characteristics of geometric transformations.

3.2.5.1. Activity Seven

Like the previous activity, there was no student activity sheet in Activity 7. This activity consisted of pair discussions and whole class discussions as well. The discussion questions were about certain characteristics of each type of geometric transformations. These characteristics of geometric transformations are the fixed points under the transformations, identity functions of each family of transformation, inverse function of each member of transformation families, and being a one-to-one and onto function.

3.2.6. Summary of Task Revisions during Pilot Study

The major revisions of the activities were about task guidelines included in the activity sheets. The task guidelines were revised in order to improve the clarity of the statements and to be able to ask what was intended by the tasks. In addition, pilot study results showed that some of the tasks were more challenging for prospective teachers than expected. Such issues were handled either by subdividing the tasks, by creating new simpler and clearer tasks that were to be implemented before the challenging ones or by omitting some tasks that were recognized to be repetitive. I also noticed that some tasks served the same purpose. Thus, such tasks were either deleted or refined and moved into different sections in order for them to be more functional. This helped the researcher save time in the implementation of the activities as well.

The time given to prospective teachers for completing each activity was also revised. For instance, it was observed that for tasks that involved performing a single transformation or composition of transformations, prospective teachers needed more time than anticipated. Issues related to time were handled either by revising the concerning task or changing the time allocated to the given task for completion.

Finally, GeoGebra files that were prepared for some tasks were also revised. The important points that the prospective teachers need to focus on while exploring a GeoGebra file were made more explicit by adding guiding statements to the file. For instance, The GeoGebra file for Task 5B of Activity 4 is presented in Figure 3.6.

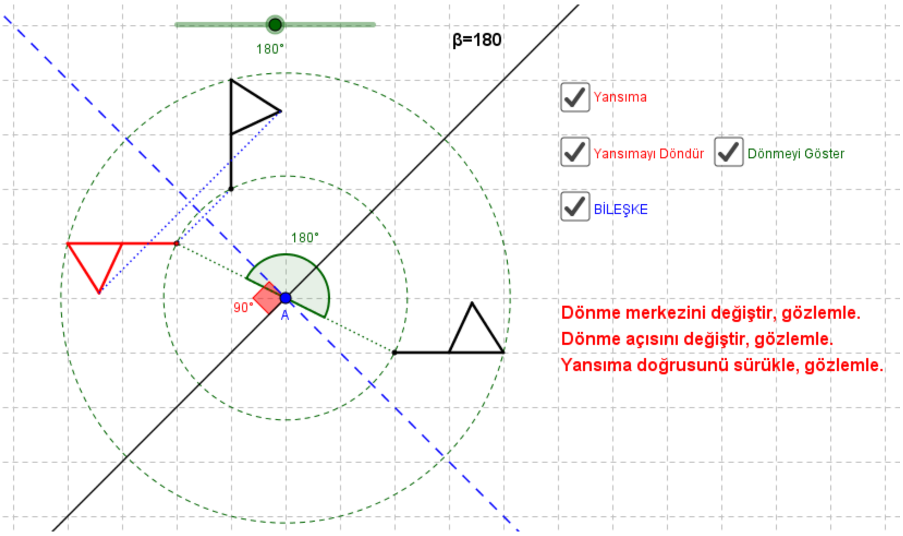


Figure 3.6. A GeoGebra screenshot of Task 5B of Activity 4

After the pilot study, the researcher decided to add the text "Dönme merkezini değiştir, gözlemle. Dönme açısını değiştir, gözlemle. Yansıma doğrusunu sürükle, gözlemle." to the GeoGebra file as presented in Figure 3.6 in order to have prospective teachers change the center of rotation, the angle of the rotation and the reflection line and observe the consequences and then to have participants focus on the relations in the composition of reflection and rotation.

3.3. Implementation of the Instructional Sequence

After the revisions made based on the pilot study, the HLT was implemented in the main study. As mentioned before, since the revisions were made only on tasks included in the activities and these revisions did not affect the learning goals, the HLT implemented in main study was the same as the HLT presented in Table 3.1.

The main study was conducted with sixteen volunteered prospective middle school mathematics teachers who attended a semester-long elective mathematics education course. The course lasted fourteen weeks in the spring semester of 2014 – 2015 academic year. Three junior and thirteen senior prospective middle school mathematics teachers participated in the main study. In the first week of the course, participants were informed about the study. In the second week TGQ developed by the researcher was administered (see Appendix A) and then, all participants were interviewed related to the TGQ during the second week. In the following seven weeks, the revised versions of seven activities were implemented (see Appendix C). The activities were entitled as Activity 1, Activity 2, Activity 3, Activity 4, Activity 5, Activity 6, and Activity 7 and were implemented in the same order. Each activity was completed in one week, and each week a three-hour lesson was held (see Appendix D for course schedule)

During the implementation of the instructional sequence, participants worked in pairs. Prospective teachers were allowed to choose their own partners. Namely, the researcher did not have any effect on the formation of the small groups. For each task included in the activity sheets, participants first discussed their ideas in the small groups and prepared their written responses, and then they presented and discussed their responses in the whole class discussions. Depending on the task being worked on, the time allocated to prospective teachers for small group discussions varied. Namely, it changed between five and fifteen minutes. The researcher did not interrupt the prospective teachers while they were discussing on the related task with their partners. The researcher paid attention to the small group discussions to understand prospective teachers' ways of thinking and their solution methods. The researcher guided the whole class discussions based on her observations and evaluations related to small group discussions. For instance, the researcher became aware of the solution methods used by the prospective teachers during small group discussions and she consequently took into account these methods in deciding on the optimum order for pairs to share their responses in whole class discussions.

After the implementation of the instructional sequence presented in Section 3.2, the TGQ was administered and then follow-up interviews were conducted. In the following part, the participants of the study are described.

3.4. Participants

This study was carried out at a state university in Central Anatolia in Turkey. Participants of the study were prospective middle school mathematics teachers enrolled in elementary mathematics education program. The main study was conducted with volunteered prospective middle school mathematics teachers who attended a spring semester elective mathematics education course for fourteen weeks. This elective course was developed for the implementation of the instructional sequence developed in this study. Therefore, the content of this elective course was only about transformation geometry concepts.

Sixteen prospective middle school mathematics teachers (4 male, 12 female) who were enrolled in this course were the participants of this study. While three of the prospective teachers were junior, thirteen of them were senior students. Participants' demographic characteristics are presented in Table 3.2.

Table 3.2. Participants' demographic characteristics

	Juniors	Seniors	Total
Male		4	4
Female	3	9	12
Total	3	13	16

The senior prospective elementary mathematics teachers were in their last semester and thus completed all mathematics courses in the EME program. Completing these courses was important for the present study since it would help prospective teachers in developing analytic thinking skills necessary for completing the activities in the instructional sequence related to geometric transformations. Meanwhile, the three junior prospective teachers completed all mathematics courses except for Analytic Geometry II and Elementary Number Theory courses in elementary mathematics education (EME) program. They were taking Analytic Geometry II course at the time of data collection. The instructor of the Analytic Geometry II course explicitly stated that the content of this course did not involve transformation geometry concepts. Finally, none of the participant took any course in general related to technology or related to dynamic geometry environment.

The topic of transformation geometry has been integrated into middle school mathematics curriculum as a sub-learning domain of geometry after the implementation of reform movements in Turkey in 2005. Since transformation geometry has been put into practice after 2005, it was unlikely that the prospective middle school mathematics teachers participated in this study had been formally taught geometric transformations before. To ensure this, participants were directly asked whether they had learnt transformation geometry topics in their middle and secondary school years. They all indicated that they had never learnt these topics before. Besides, there were no such elective or compulsory courses related to geometric transformations in the teacher education program in which the participants were enrolled. Thus, the participants did not also learn these topics in their undergraduate education.

3.5. Data Collection

The purpose of this study was to explore prospective middle school mathematics teachers' evolving understanding of geometric transformations. The qualitative and quantitative research methods were used in data collection process. The data collection tools used in this study were: (1) The Transformation Geometry Questionnaire as a pre-test and posttest for evaluating prospective middle school mathematics teachers' knowledge of geometric transformations; (2) follow-up interviews regarding the pre-test and post-test; (3) activity sheets and other written works completed by the participants; (4) video-recorded classroom observations; (5) audio-recorded pairwise discussions; (6) field notes taken by the researcher; and (7) audio-recorded research team meetings. Each data collection tool is explained in the following parts.

3.5.1. Transformation Geometry Questionnaire

In this study, the Transformation Geometry Questionnaire (TGQ) was developed by the researcher. The first version of the TGQ was revised after the pilot study and these revisions were explained in the previous parts.

The TGQ was administered before and after the implementation of the instructional sequence as a pre-test and post-test to measure participants' knowledge and understanding of transformations (see Appendix A). The tasks included in this questionnaire required participants to use a ruler, a compass and a protractor when identifying or performing transformations. The questionnaire included ten open-ended tasks. The table of specifications prepared for this questionnaire is presented in Table 3.3.

Table 3.3. Table of Specifications for the Transformation Geometry Questionnaire

Tasks	Sub-Tasks	Objective
1	A	Define reflection
	B	Define translation
	C	Define rotation
	D	Define glide reflection
2	A and F	Identify rotation
	B and E	Identify reflection
	C	Identify translation
	D	Identify glide reflection
3	A, B, and C	Perform reflection
4	A, B, and C	Perform translation
5	A, B, and C	Perform rotation
6	A	Perform reflection
	B	Perform translation
	C	Perform rotation
7		Perform rotation
8		Perform reflection
9	A and B	Relate geometric transformations and functions
10		Perform compositions of transformations

The first task asked participants to define transformations, give examples of transformations, and explain the properties of each transformation. This task included four sub-tasks and each sub-task was related to reflection, translation, rotation, and glide reflection, respectively.

In the second task, there were six sub-tasks and in each sub-task a flag and its transformed image were given and the participants were asked to determine the single transformation applied to the flag, to explain this transformation with its parameter, and construct the parameter(s). Sub-tasks 2B and 2E were related to identification of rotation. In Task 2B, the reflection line and the pre-image and its reflected image coincided at one point and thus the reflection line coincided with this point as well. In task 2E, the pre-image, image, and the reflection line did not coincide. In these two tasks, participants were expected to identify the reflection and to construct its reflection line.

Tasks 2A and 2F were related to identification of rotation and the participants were expected to identify the rotation and to determine the center of rotation and the angle of rotation which were the parameters of the rotation. The center of rotation in Task 2A was on the end point of the flag. However, in task 2F the center of rotation was outside the flag and identification of this point involved more steps than the previous one. Therefore, it can be said that identification of the center of rotation in Task 2F was more complex than Task 2A.

In Task 2C, participants were expected to identify the translation and construct the translation vector. In Task 2D, participants were asked to identify the glide reflection, namely the translation followed by the reflection. Participants were expected to construct the translation vector and the reflection line which were perpendicular to each other.

Tasks 3 to 8 were about performing transformations. Task 3, Task 4, and Task 5 were related to performing reflections, translations, and rotations respectively and each of them had three sub-tasks. In each sub-task, a point and the parameter(s) were given on a grid and participants were asked to perform transformation on that point with given parameter(s). More specifically, the reflection in Task 3A had a horizontal reflection line, while reflections in Tasks 3B and 3C had inclined reflection lines. In Tasks 4A, 4B, and 4C, the three translation vectors differed from each other in terms of their magnitude and direction and in terms of their locations with reference to the given points. In Tasks 5A, 5B and 5C, the angles of rotations and centers of rotations were different from each other. Namely, angles of rotations were 45°, 90°, and 120° and the centers of rotations differed from each other in terms of their locations with reference to the given points.

Tasks 6A, Task 6B and Task 6C were related to performing reflections, translations and rotations respectively. In each of them, the letter P was given on a grid and participants were asked to perform transformations with given parameters. In Tasks 3, 4, and 5, points were given to perform transformation on, while Task 6 required participants to perform transformations on a more complex geometric figure which required consideration of more defining properties.

In Task 7, the participants were asked to perform 140° rotation on a geometric figure which was formed by a triangle and a semicircle. In this task, the geometric figure was not given on a grid. In Task 8, the same geometric figure was given and participants were asked to reflect it along an inclined reflection line. Task 8 was not given on a grid as well.

Task 9 asked prospective teachers to explain whether geometric transformations were functions or not and to justify their answers. In addition, the participants who counted geometric transformations as functions were asked to determine the domain, range and the elements of these two sets.

Finally, Task 10 was about compositions of transformations. Namely, the participants were asked whether the composition of any two geometric transformations could be explained by a single geometric transformation. Participants were required to explain their answers through several examples.

3.5.1.1. Validity and reliability of the TGQ

Validity refers to appropriate, meaningful, correct, and useful interpretations of any measurement (Fraenkel & Wallen, 2012). Thus, it is about the goal of the test and what it measures. To establish the construct validity of the TGQ, two mathematics educators and one mathematician with a doctoral degree examined the first versions of the transformation geometry tasks with respect to the table of specifications. Moreover, the appropriateness of the tasks to the participants, the representativeness of the content by the tasks, the appropriateness of the test format such as clarity of directions and language, and quality of printing were checked. In the revision of items, suggestions given by the experts were taken into consideration. For instance, one of the experts suggested that there were too many tasks related to identification of geometric transformations in the TGQ and that some of them served the same purpose. By considering his suggestion, the researcher deleted some of the tasks. By doing these revisions, the researcher avoided wasting time with duplicate tasks involving identification of geometric transformations and the balance between the components of the TGQ (e.g., defining and performing geometric transformations) was established. Moreover, all experts suggested that more space should be allocated to tasks involving performing geometric transformations. Therefore, the researcher reorganized the pages by considering this suggestion. Finally, all experts also suggested some revisions related to wording of the tasks. For instance, they suggested revising the statement “construct the image of given figure after the reflection along line *d*” as “reflect the given figure along line *d*”.

Pilot testing is important in establishing the construct validity of the instrument, which means whether the items measure the construct they are intended to measure, and to ensure that the instructions, questions, format, and scale items are clear (Creswell, 2003). After revising the TGQ items (tasks) in line with the expert opinions, the TGQ was piloted. By the help of this pilot testing, necessary revisions were made. For instance, wordings of some tasks were revised in order to make them more understandable. Moreover, the pilot study showed that some of the tasks should be divided into sub-tasks. For instance, in tasks involving defining geometric transformations in the pilot study, participants were asked to define each geometric transformation and give an example for each of them. After the pilot study, these tasks were divided into three sub-tasks. To illustrate, reflection task was subdivided into three as (*i*) define the reflection, (*ii*) give an example for reflection, and (*iii*) explain your example and the properties of reflection.

Finally, the revised TGQ was again reviewed by two mathematics educators and one mathematician who were all holding a doctoral degree in their fields. They examined the TGQ by considering the table of specifications. As in the first examination, they evaluated the questionnaire in terms of the appropriateness of the tasks to the participants, representativeness of the content presented in each task, the appropriateness of the test format such as clarity of directions and language, and quality of printing. The feedbacks of the three aforementioned academicians helped in making the remaining minor revisions.

Reliability refers to the consistency of scores obtained from the instrument (Fraenkel & Wallen, 2012). In this study, inter-rater reliability was used as an evidence for reliability. To establish the inter-rater reliability, a detailed rubric was prepared by the researcher to score the responses of prospective middle school mathematics teachers for each transformation geometry task in the pilot study (see, Appendix B). The researcher and one mathematics educator with a doctoral degree scored participants' responses separately. The correlation between the two sets of scores, accounting for the inter-rater agreement between the researcher and the mathematics educator, was obtained as 0.92. The discrepancies between the two coders were resolved in a number of meetings and the TGQ was put into its final form. Revisions were done on the rubric during the resolution of these discrepancies. For instance, when the rubric was given to the mathematics educator to score participants' responses to Task 5A, there were statements "when given point is not rotated precisely, give 1 point". The mathematics educator suggested that the term 'precisely' was not understandable and it was difficult to decide which response was precise or not. Therefore, this statement was changed as follows "Given point is not rotated precisely. That is, either the angle of rotation or the distance from the center is not correctly determined".

3.5.2. Interviews

Semi-structured interviews were carried out with all participants before and after the implementation of the instructional sequence. More specifically, these interviews were follow-up interviews regarding the Transformation Geometry Questionnaire. The follow-up interviews focused on clarifying prospective teachers' responses to the tasks included in TGQ. Thus, these follow-up interviews helped the researcher to better understand and interpret participants' responses to the tasks in TGQ.

The first follow-up interviews were conducted before the implementation of the first activity and after the administration of the TGQ as a pre-test. The second follow-up interviews were conducted after the administration of TGQ as a post-test. These follow-up interviews were conducted with all participants, namely with 16 prospective teachers during a one-week period. Each interview lasted about 30 minutes. All interviews were audio- and video-recorded. The interviews were audio recorded as a precaution for unclear recordings of video camera. The video camera was adjusted in a way that displayed participants' actions and written work while they were explaining their responses. By using camera recordings I was able to capture how participants used a compass, a ruler and a protractor and for which purposes they used these tools in solving relevant tasks. I was also able to determine how participants carried out perpendicularity, equidistance, and angle measures while performing geometric transformations.

Before conducting interviews, the researcher examined participants’ written responses to the tasks included in the TGQ. Then, based on participants’ blank and incorrect responses, the researcher determined a particular set of questions to be asked to each prospective teacher in addition to general interview questions. Thus, the interview questions differed from one participant to another.

Through general interview questions, the participants were asked to explain how they identified or performed the geometric transformations. For instance, in identification of rotation tasks, the participants were requested to answer questions such as “Why did you think that it was a rotation? How did you measure the angle of rotation? How did you did you determine the center of rotation? What other points did you consider?” I used prompts in cases where the participants’ explanations were unclear or inadequate. For instance, when participants could not explain reflection transformation, I posed prompts such as “Did you consider any equidistance between some points? Did you consider perpendicularity?” The prompts were used until I adequately understood each participant’s way of thinking and solution process.

3.5.3. Written Responses to the Tasks Included in the Activities

In this study, an instructional sequence developed by the researcher was implemented. As explained before, this instructional sequence included seven activities. The first five activities included a large number of tasks to be responded in small group and whole class discussions. There were eight pairs and each pair completed one activity sheet in one week and altogether eight activity sheets were collected each week. Therefore, participants’ responses to tasks included in the activity sheets formed the crucial part of the data of this study and consequently activity sheets served as an important data collection tool.

3.5.4. Classroom Observations

Observation is "the systematic description of events, behaviors, and artifacts in the social setting chosen for study" (Marshall & Rossman, 1999, p.79). It provides first-hand information to the researchers (Creswell, 2003), and it enables to observe what happens directly, gain new standpoints and an in-depth understanding of events

(Merriam, 2009). Besides, it is critical for researchers in directly experiencing the process and becoming familiar with the context and consequently in drawing inferences (Yin, 2011). In this study, classroom observations were carried out by two researchers, namely by the instructor of the instructional sequence and another researcher who had doctoral degree in mathematics education. I, as an instructor observed pair-wise discussions while participants were completing their tasks. Meanwhile, I took field notes about pairs' way of thinking and considered these notes when guiding whole class discussions. The second researcher was a non-participant observer. He was responsible for technical issues and for observing general aspects such as behaviors of participants, the role of the researcher, and the events in the classroom. He also took notes about these issues when necessary. In brief, the classroom observations focused on prospective teachers' discussions surrounding geometric transformation concepts and their behaviors.

Each class session was audio- and video-taped. There were three video cameras recording different regions of the classroom. One camera was placed at the back of the classroom. This camera recorded participants' verbal and written explanations for tasks on the smart board. In addition, it recorded participants' exploration of geometric transformations on the smart board in GeoGebra environment. The second and the third camera were placed in front of the classroom and focused on the classroom interactions that took place between the instructor and the participants and among the participants. Moreover, all groups were audio-taped to gather pair-wise discussion data during the course of the instructional sequence.

3.5.5. Research Team Meetings

The research team consisted of me, a researcher who had a doctoral degree in mathematics education and two prospective middle school mathematics teachers who were also the participants of the study. The second researcher was a non-participant observer.

After each class session, the team members met together to discuss the implemented activity. The researcher facilitated these meetings and they lasted about thirty minutes. In these meetings, participants' ideas about the implemented activity played an important role in revising the succeeding activities. For instance, if the two prospective teachers expressed that there was a de-emphasized point, a challenging or an unclear point, I noted these issues down and took them into consideration in revising and implementing the following activities. All research team meetings were documented and audio-recorded.

Before the implementation of the subsequent activities, I met with the researcher to examine the field notes taken during the implementation and the ones taken in the prior team meeting. Later, we discussed and tried to handle all issues included in these notes by making necessary modifications and revisions on the instructional sequence.

3.6. Data Analysis

Design-based research involves ongoing and retrospective analyses. The ongoing analysis is usually related to the goal of supporting participants' learning and it is conducted while the experiment is in process (Cobb & Gravemeijer, 2008). In this study, during the ongoing analysis, the researcher tried to understand prospective middle school mathematics teachers' reasoning and accordingly made spontaneous decisions about the implementation and the learning environment during and between class sessions. The retrospective analysis is conducted for the entire data set collected during the implementation. In retrospective analysis, the data should be analyzed in a systematic way.

In following part, first quantitative data analysis related to the TGQ was explained. Next, qualitative analysis procedures of other data were presented. Finally, the APOS Theory framework used for analyzing participants' understanding of geometric transformations as functions was explained.

3.6.1. Analysis of Transformation Geometry Questionnaire Data

The TGT was administered before the implementation of instructional sequence. Then, by considering the responses and the table of specifications, the rubric developed in the pilot study were revised. For instance, after the pilot study, some of the tasks in the TGQ were deleted. Therefore, it was required to reconsider the maximum scores of each task in the TGQ. Similarly, some of the tasks (e.g., tasks involving defining geometric transformations) were divided into sub-tasks and thus the scores of each task were also divided. Next, the revised rubric was examined by two experts. Similar to the first examination, they focused on the appropriateness of the rubric for scoring the participants' responses and the comprehensiveness of the all possible responses to the tasks. By using the final version of rubric, participants' TGQ scores were determined (see Appendix B). To give an example, according to the rubric, the maximum score for identification of translation sub-task (i.e., Task 2C) was three points. According to the rubric, when a participant could identify the translation but s/he could not identify the translation vector, s/he got 1 point. When s/he identified the translation vector partially, namely the magnitude or direction, s/he got 2 points. When s/he could identify the translation and could explain the magnitude and the direction of the translation vector, s/he got 3 points. After the implementation of the instructional unit, the TGQ was administered as a post-test. Participants' post-test scores were calculated in a similar way.

As mentioned before, follow-up interviews were conducted related to pre- and post- administration of the TGQ. During the evaluation of participants' responses to the TGQ items, the follow-up interviews were also considered. For instance, the participants' unclear or very short written responses to TGQ were elaborated in the follow-up interviews. Hence, participants' written and verbal responses were considered together when determining their TGQ scores. Participants' pre- and post-test mean scores and standard deviations for each task of the TGQ were calculated. Besides, for each component in TGQ, participants' mean scores and standard deviations were calculated.

3.6.2. Qualitative Data Analysis

In this study, prospective middle school mathematics teachers' understanding of geometric transformations was explored. To analyze their understanding, the APOS Theory applied to transformation geometry concepts by Hollebrands (2003) was used (see Table 3.4). By using this framework, participants' initial and final understanding of geometric transformations before and after they participated in the study were evaluated. For this purpose, video and audio recordings of follow-up interviews were transcribed by the researcher. Transcripts of the follow-up interviews regarding the TGQ were used to analyze participants' understanding of geometric transformations through APOS Theory.

Participants' developing understanding of geometric transformations were evaluated during the course of the study by the help of APOS Theory. As mentioned before, activity sheets completed by the participants during the class sessions were collected. There were eight activity sheets for each of the first five class sessions and the researcher examined forty activity sheets in total.

The audio recordings of pairwise discussions were analyzed in order to evaluate prospective teachers' responses to tasks included in the activities and thereby to evaluate their understanding of geometric transformations. Namely, by the help of these recordings, their ways of thinking were tried to be understood. In short, these audio recordings were used to understand and analyze participants' thinking processes that reflect their understanding of geometric transformations during the implementation of the instructional unit.

In addition to the activity sheets and audio recordings of pairwise discussions, whole class discussions were used to evaluate participants' understanding of geometric transformations through APOS Theory. For this purpose, all classroom sessions were transcribed by the researcher. During the whole class discussions, participants shared their responses and their reasoning with other participants. The researcher tried to find clues related to their understanding of geometric transformations.

Finally, transcripts of follow-up interviews regarding the post implementation of the TGQ were used to evaluate participants' understanding of the geometric transformations upon implementation of the instructional sequence.

In the following sections, the use of APOS theory in mathematics education is summarized first. Next, examples illustrating the application of this theory to geometric transformations are presented.

3.6.2.1. APOS Theory

APOS Theory is principally concerned with the nature of mathematical knowledge and the development of a mathematical concept (Asiala et al., 1996). This theory can help researchers understand students' learning process by observing them while they are attempting to construct their own understandings related to a mathematical concept and suggest directions for pedagogy which might promote students' learning process (Dubinsky & McDonald, 2001). This theory posits that "the growth of understanding is highly non-linear with starts and stops; the student develops partial understandings, repeatedly returns to the same piece of knowledge, and periodically summarizes and ties ideas together (Asiala et al., 1996, p.13).

APOS theory has proved very useful in exploring learners' understanding of a wide range of advanced mathematical concepts related to calculus, abstract algebra, statistics, discrete mathematics, and so forth (Dubinsky & McDonald, 2001). Within this theoretical perspective, learners progress through the following four levels in the course of learning a mathematical concept: action, process, object, and schema (Asiala et al., 1996). These levels are the fundamental components of APOS Theory and they are briefly summarized in the following paragraphs.

In *action* level, when a learner experiences a novel mathematical concept, concept formation starts with transformation of existing mental or physical objects. Transformations in this level is essentially external and learners require external cues such as formulas, algorithms, and similar examples to carry out these transformations (Dubinsky, 1991). By the external clues, each step of transformations is explained explicitly and the learners can carry out these transformations only by following these steps. Namely, they cannot image the transformations yet in action level (Dubinsky & McDonald, 2001). For instance, students who are limited to an action conception of a function concept require "an explicit expression in order to think about the concept of function and can do little more than substitute for the variable in the expression and manipulate it" (Dubinsky, Weller, McDonald & Brown, 2005, p. 338). Thus, the explicit expression functions as an external cue that tells how the action must be carried out by substituting specific values (Arnon et al., 2013). To give another example, a learner with an action understanding of function concept needs to have explicit expressions of each of the two functions in order to think about their composition for specific values. However, the learner may probably not compose these two functions in more general situations such as in functions with split domains and in functions in which expressions are not given (Breidenbach, Dubinsky, Hawks & Nichols, 1992).

As a learner repeats and reflects on the actions, he/she internalizes them and the actions turns into *processes*. Namely, the learner shifts from relying on external cues to having internal control over them by internalization of actions. Learners with process understanding are able to carry out the steps without actually performing each of them explicitly and they start to skip some steps and reverse these steps as well (Arnon et al., 2013). Dubinsky and others (2005) describe and give an example of a process conception of function concept as follows:

> "As an individual repeats and reflects on an action, it may be interiorized into a mental process. A process is a mental structure that performs the same operation as the action being interiorized, but wholly in the mind of the individual, thus enabling her or him to imagine performing the transformation without having to execute each step explicitly. Thus, for example, an individual with a process understanding of function will construct a mental process for a given function and think in terms of inputs, possibly unspecified, and transformations of those inputs to produce outputs" (p. 339).

The construction of an *object* conception is achieved through the mechanism of encapsulation. An action (a static structure) turns into a process (dynamic structure) when it is internalized. Process is a transformation which is performed by the learner. Encapsulation takes place when a learner applies an action or a process to a process. In other words, when a learner starts to reason that a dynamic structure is a static structure to which actions or processes may be applied, the process becomes encapsulated. Briefly, while process is an internalized transformation, the object is a static structure to which action or process is applied (Arnon et al., 2013). Dubinsky and others (2005) explain object conception as follows:

> "If one becomes aware of the process as a totality, realizes that transformations can act on that totality and can actually construct such transformations (explicitly or in one's imagination), then we say the individual has encapsulated the process into a cognitive object. For the function concept, encapsulation allows one to apply transformations of functions such as forming a set of functions, defining arithmetic operations on such a set, equipping it with a topology, etc." (p. 339).

Finally, a learner's *schema* is the whole body of knowledge possessed by him or her for a specific mathematical concept. More precisely, a schema for a specific mathematical concept is a learner's collection of actions, processes, objects, and other schemas which are connected to each other by some principles to form a framework in that learner's mind. These collections of actions, processes, objects, and other schemas are brought into play when a learner confronts with a problem relating to that mathematical concept (Dubinsky & McDonald, 2001). For instance, a function schema involves different types of functions such as real-valued functions, multivariable functions, vector-valued functions, and/or proposition-valued functions. Some of these functions might have been constructed as processes or objects. To give an example, Arnon and others (2013) explain function schema as follows:

> "One example of a Schema is the function Schema. It can be composed of different types of functions such as real-valued functions, multivariable functions, vector-valued functions, and/or proposition-valued functions. These different types of functions may have been constructed as Processes or Objects, together with the operations that can be applied to them. For some students, different types of functions may be related by the common idea of a function Process: an operation applied to a set of inputs to obtain a set of outputs. Functions differ in the types of inputs involved, the nature of the operations applied to those inputs, and the results of the operations. Although individuals' Schemas may include the same types of functions, their components or the types relations constructed among them may differ." (p.111).

Traditionally, researchers mainly focused on learners' understanding of numeric functions by using APOS Theory (e.g., Dubinsky & Harel, 1992; Breidenbach et al., 1992). However, there are very few research using this theory to understand learners understanding of geometric transformations (e.g., Hollebrands, 2003; Yanık, 2006). In this study, we focused on prospective middle school teachers understanding of geometric transformations, and in the following part we provided some specific examples which correspond to the various levels of APOS Theory.

3.6.2.2. Illustrations of action, process, and object conception of geometric transformations

APOS Theory framework proposed by Flanagan (2001) and Hollebrands (2003) for analyzing learners' understanding of geometric transformations are summarized in Table 3.4. According to this framework, seeing geometric transformations as motions indicates an action conception of geometric transformations. For instance, in this study, defining translation as a movement of an object or defining rotation as turning of an object in defining transformation geometry tasks in the TGQ were regarded as indicators of an action conception of geometric transformations. Moreover, in the follow-up interviews regarding the TGQ, explaining geometric transformations by holding a physical object and sliding it on the table were regarded as indicators of an action conception of translations. Because, it contains only the physical movement of the concrete object.

According to the framework given in Table 3.4, seeing geometric transformations as motions that is applied to a single object indicates an action conception of geometric transformations. In other words, one who has an action conception of geometric transformations views the domain of a transformation as single object rather than all points in the plane. For instance, holding a physical object and sliding it on the table also indicates that the transformation is applied only to the single object rather than all points in the plane.

Understanding that transformations are applied to all points in the plane, not just to a single object is an indicator of a process conception of geometric transformations. This means that one who has a process conception of geometric transformations views the domain of geometric transformations as all points in the plane. For instance, while discussing the domain and range of the relations by dragging the independent points in the GeoGebra file, some of the participants explicitly stated that the domain is a plane (i.e., R^2) and the points (x, y) are elements of this plane. These explicit explanations of the participants regarding domain were regarded as an indicator of process conception of geometric transformations. Besides, when prospective teachers responded to tasks related to defining geometric transformations as functions defined in the plane and explained their reasoning explicitly in the TGQ, their understandings corresponded to a process conception of geometric transformations.

As presented in the framework given in Table 3.4, the ability to anticipate the results of an action through mental rather than physical activities and the ability to envision the results and discuss the invariant features is an indicator of the process conception of geometric transformations. For instance, while participants were discussing the points that remained unchanged (i.e., fixed points) after the performing of geometric transformations they showed some evidences that they performed geometric transformations mentally. Therefore, their determination of points that remained fixed after translation, reflection, and rotation were regarded as an indicator of a process conception. When a participant can differentiate points that remain fixed from points that remain unfixed, this indicates that s/he is aware of the fact that all points in the plane map to other points in the plane and that s/he mentally performed geometric transformations in her/his mind.

Thinking transformations as one-to-one and onto functions that map points in the plane to points in the plane is an indicator of object conception of geometric transformations. For instance, while discussing the properties of geometric transformations during the exploration of the GeoGebra file, some of the participants explicitly stated that they were one to one and onto. These explicit explanations of the participants related to geometric transformations were regarded as an indicator of object conception of geometric transformations. Besides, when prospective teachers responded to tasks involving defining geometric transformations as one to one and onto functions defined in the plane and explained their reasoning explicitly in TGQ, their understandings corresponded to an object conception of geometric transformations.

Arnon et al.'s (2013) notion of action conception in APOS Theory corresponds to Edwards (2003) notion of motion conception of geometric transformations. For instance, according to APOS Theory given in Table 3.4, one who has an action conception of geometric transformations views the domain of a transformation as single object rather than all points in the plane. Similarly, Edwards (2003) indicated the idea that transformations are applied only to the geometric figures rather than the whole plane. This indicates the motion conception of geometric transformations.

Some of the characteristics of learners with process conception correspond to mapping conception of geometric transformations. For instance, understanding that transformations are applied to all points in the plane, not just to a single object is one of the characteristics of process conception. Edwards (2003) indicated that conceiving transformations as mappings of all points of the plane corresponds to the mapping conception. However, holding the characteristics of process conception does not mean holding complete mapping conception. In order to have a complete mapping conception of geometric transformations one needs to reach an object conception of geometric transformations. Mapping conception of geometric transformations requires understanding geometric transformations as one-to-one and onto functions (Edwards, 2003). This understanding corresponds to object conception of geometric transformations in APOS theory.

Table 3.4. APOS Theory framework for analyzing learners' understanding of geometric transformations (Flanagan, 2001; Hollebrands, 2003)

Action Conception of Geometric Transformations
1. Involve an understanding that a transformation is a motion that is applied to a single object
2. Learner might view a transformation as a snapshot. That is, student may consider only the pre-image and image and not consider the relations that might hold if either object is varied
3. Learner might view as the domain of the transformation single objects rather than all points in the plane and may not view the input for the transformation as an object that can vary
4. Learner might not be aware of the characteristics and properties that will remain invariant for the transformation
5. Learner may not be aware that an inverse for the transformation exists
6. Learner may view the parameter as something that indicates an action rather than being a mathematical object
7. May be unaware of the effects that varying the parameters will have on the transformation
8. May view translations as nothing more than finding an output value for a function when given an input value
9. May need to be given a formula or description of a particular translation in order to perform the action
10. May perform the transformation by simply implementing a sequence of commands in the computer without knowing in advance what result to expect because he or she is unable to visualize the results of the action prior to performing it
11. May find the image of point P under a translation by vector v simply by substituting in the coordinates of P into an equation $x' = x + a$, $y' = y + b$ to find the coordinates of the image point P'
Process Conception of Geometric Transformations
1. Learner is able to mentally perform an action without necessarily running through all the specific steps
2. Learner is able to conceive of the mental transformation as a complete activity that includes an object, the performance of an action on that object, and a new object that results from that action
3. Learner is able to anticipate the results of an action through mental rather than physical activities and envision the results and discuss the invariant features (e.g., a translation preserves distances between points)
4. May be able to reason about properties that remain invariant under a transformation
5. May be able to reason about parameters and the effects of changing parameters on the transformation
6. May come to understand that inputs are variables because the transformation is viewed as repeatable
7. Understand that transformations are applied to all points in the plane, not just to a single object
8. May begin to develop understandings of transformations as one-to-one and onto functions that have inverses
Object Conception of Geometric Transformations
1. Reason about the composition of two or more transformations
2. Think about the properties that would be preserved by the composite transformation
3. Think transformation as one-to-one and onto function that maps points in the plane to points in the plane

3.7. Trustworthiness of the Study

Validity and reliability play a crucial role in analyzing the findings and in determining the quality of the study (Patton, 2002). Lincoln and Guba (1985) used the term ‘trustworthiness’ to refer to the validity and reliability of qualitative research studies. To establish the trustworthiness of a study, they used the terms credibility, transferability, dependability, and confirmability as equivalents for internal validity, external validity, reliability, and objectivity.

In order to address the issue of credibility, I used the following validation strategies: triangulation, member checking, using thick description, clarifying researcher bias, prolonged engagement, persistent observation, and peer debriefing (Creswell, 2007; Lincoln & Guba, 1985)

In this study, data triangulation, investigator triangulation, and methodological triangulation were used to increase the credibility of the study. Namely, there were sixteen prospective middle school mathematics teachers as data source (data triangulation), the second researcher was an observer of the implementation and the second coder of the data (investigator triangulation), and different types of data including observations, interviews, and participants’ written responses to TGQ were collected (methodological triangulation).

In this study, member checking was used by elaborating on participants’ written responses to TGQ in follow-up interviews. By this way, I was able to see whether there were any conflicts between me and participants in terms of their understanding of geometric transformations.

In addition to member checking, I made thick and rich descriptions about the design principles and development process of the instructional sequence, the implementation of the instructional sequence, and data analysis procedures. Besides, participants’ constructions, explanations and written responses were presented to reflect their particular ways of reasoning about transformation geometry concepts.

I tried to clarify researcher bias by acknowledging and describing my beliefs and biases about the current study in the researcher role and bias section. Besides, prolonged engagement helped me to build trust and establish rapport with the participants. Meanwhile, I conducted persistent observations through which I observed the group work and class discussions of participants. Peer debriefing is “the review of data and research process by someone who is familiar with the research of the phenomenon being explored” (Creswell & Miller, 2000, p.129). Thus, the second researcher who joined team meetings shared his ideas and provided feedbacks for the present study.

Transferability is concerned the generalizability of the findings of a research study. In quantitative research studies, external validity refers to making generalizations to the population. However, in qualitative research studies, generalizability does not serve for making inferences from a sample to a wider population. In more detail, transferability in qualitative studies serve for making analytic and argumentative generalizations “depending on the degree of similarity between sending and receiving contexts” (Linconln & Guba, 1985, p. 297). Besides, Miles and Huberman (1994) suggested researchers to provide “thick descriptions for the readers to assess the potential transferability and appropriateness for their own settings” (p. 279). Lincoln and Guba (1985) expressed that researchers need to provide adequate contextual information about the fieldwork site in order for the readers to transfer the findings to their own contexts. They further added that detailed descriptions of the research site, participants, the research process, and data collection and data analysis procedures needs to be provided to achieve transferability. Thus, in this study, I thickly described all elements of the research and therefore the readers would be able to determine to what extent they would apply the findings of this study and generalize them to similar contexts.

Dependability refers to the consistency and stability of the study over time and across researchers (Miles & Huberman, 1994). In order to establish dependability of the study, another researcher coded the randomly selected parts of the transcripts. This made it available to cross-check the findings of my study. The second coder was experienced in coding qualitative data. He was informed about the purpose and research questions of the study in detail. Besides, he was informed about the coding process before starting the actual coding and hence clarified the focal points of data analysis. The differences between the coders were discussed and the data coding process ended when two coders reached an almost full consensus with each other.

Confirmability is related to objectivity of the researcher and the power of scientific method comes from objectivity (Patton, 2002). Shenton (2004) recommended researchers to use triangulation to increase confirmability. In this study, confirmability is established by using triangulation and rich description of content. Besides, the analyses were supported by presenting participants' constructions, quoting from the interviews and discussions in order to ensure confirmability.

3.8. Researcher Role and Bias

Researchers are key instruments for collecting and analyzing data in qualitative studies (Merriam, 2009). Thus, a researcher's views and beliefs might affect his/her interpretations in a qualitative study. This brings along the issue of subjectivity as a main concern of qualitative research studies. To be aware of the possible biases I might have as a researcher, in this study I used the key strategy of reflexivity. Reflexivity can be defined as "the process of researchers' reflecting upon their actions and values during research (e.g., in producing accounts and writing accounts), and the effects that they may have" (Robson, 2011, p.531). With this definition in mind, I hereafter expressed my actions, values and beliefs about the current study.

I strictly believe that it is important for middle school prospective teachers to understand geometric transformations as functions and consequently have mapping conception of geometric transformations. This conception can help them connect algebra and geometry concepts and therefore provides opportunities for them to see mathematics as a unified and interconnected discipline. Besides, having such mathematical perspective is important for prospective teachers since they will be teachers in the future and convey this perspective to their students and shape these students' conceptions as well. During the study, I was aware about my personal feelings and thus I tried to put them aside while analyzing my data. Besides, I contacted with the participants and shared my interpretations with themselves and made sure that my interpretations matched with their understandings. Finally, after writing up my findings I checked whether I used more quotations from one participant than other. When I noticed such an issue, I went back to my analysis and made sure that there was no bias on the part of that participant.

Since researchers play a major role in data collection, interpretation and analysis, it is significant for them to explicitly state their roles in their own studies as well. In this study, I explained the purpose of my thesis to the participants at the beginning to clarify my role to them. I explained that it was not compulsory to participate in the study and made sure which participants were voluntary to participate in my study. I also informed them that the video recordings and the interview transcripts were going to be kept confidential. During data analysis, I did not pay attention to participants' names and used pseudonyms in place of their actual names.

CHAPTER 4

RESULTS

The purpose of this study was to explore prospective middle school mathematics teachers' evolving understanding of geometric transformations in the context of a semester long course. Through this purpose, the following research question was formulated:

1. How do prospective middle school mathematics teachers' understanding of geometric transformations develop during the instructional unit?
    a. How do prospective mathematics teachers' competencies to identify geometric transformations develop as they engage in the instructional unit?
    b. How do prospective mathematics teachers' competencies to perform geometric transformations develop as they engage in the instructional unit?
    c. How do prospective middle school mathematics teachers' understanding of geometric transformations develop in action-object continuum?

In this chapter, the results of this study are presented. First, the results related to prospective middle school mathematics teachers' pre-existing knowledge and understanding of geometric transformations are presented. Then, the findings related to their evolving understanding of geometric transformations during the instructional unit is presented in six parts. Prospective teachers' identification of geometric transformations and their performing of single and compositions of transformations are presented in the first three parts. In the next three parts their exploration of functional dependency of points in the plane, $R^2 \rightarrow R^2$ functions, and of functional properties of transformations are presented. Finally, prospective middle school mathematics teachers' knowledge and understanding of geometric transformations after the implementation of the instructional unit are presented.

4.1. Prospective Middle School Mathematics Teachers' Pre-Existing Knowledge and Understanding of Geometric Transformations

Before the implementation of the instructional unit, all prospective teachers were administered the Transformation Geometry Questionnaire (TGQ) as a pre-test. After pre-test, follow-up interviews were conducted with each participant. The results of the TGQ are presented in two sections. First, quantitative results regarding prospective teachers' pre-existing knowledge of geometric transformations are presented based on pre-test data. Next, qualitative results regarding interpretation of prospective teachers' understanding of geometric transformations through APOS theory are presented based on pre-test and follow-up interview data.

4.1.1. Prospective Middle School Mathematics Teachers' Pre-Existing Knowledge of Geometric Transformations

Descriptive results regarding prospective middle school teachers' pre-existing knowledge of geometric transformations are presented in Table 4.1.

Table 4.1. Descriptive results regarding participants' knowledge of geometric transformations

		Pre-Test Scores	
Task #	Max	*M (%)*	*SD*
1	24	8.38 (34.91)	4.06
2	18	9.63 (53.50)	3.98
3	3	2.00 (66.67)	0.73
4	3	1.69 (56.33)	1.54
5	6	4.56 (76.00)	1.63
6	9	2.19 (24,33)	2.04
7	4	0.25 (6.25)	0.58
8	4	1.25 (31.25)	1.69
9	18	2.5 (13.89)	3.83
10	11	0 (0)	0
Total	100	32.44 (32.44)	12.31

As seen in Table 4.1, TGQ includes ten tasks and maximum score of each task differs from each other. For that reason, the percentage of each task was calculated in addition to mean scores. As seen in Table 4.1, prospective teachers' transformation geometry performance for each task ranged between 0 and 76.00 out of 100 points in pre-test. Prospective teachers displayed the highest performance in Task 5, the lowest performance in Task 9 and no performance in Task 10. More detailed information related to each component of TGQ is presented in Table 4.2.

Table 4.2. Descriptive results for each component of TGQ

			Pre-Test Scores	
Components of TGT	Related Tasks	Points	*M %*	*SD*
Define geometric transformations	1	24	8.38(34.91)	4.06
Identify rotation	2A, 2F	6	2.88(48.00)	1.71
Identify reflection	2B, 2E	6	3.31(55.17)	1.74
Identify translation	2C	3	1.81(60.33)	1.10
Identify glide reflection	2D	3	1.63(54.33)	1.26
Perform reflection	3, 6A, 8	10	3.81(38.10)	2.14
Perform translation	4, 6B	6	2.94(49.00)	2.57
Perform rotation	5, 6C, 7	13	5.19(39.92)	2.43
Relate geometric transformations with functions	9	18	2.5(13.89)	3.83
Perform compositions of transformations	10	11	0(0)	0

As seen in Table 4.2, prospective teachers' transformation geometry performance for each component ranged between 0 and 60.33 out of 100 points in pre-test. While they displayed lower performance in tasks involving defining geometric transformations, performing reflection, and performing rotation (34.91%, 38.10%, 39.92% respectively), they had relatively higher performance in tasks involving identification of reflection, translation, and glide reflection (55.17%, 60.33%, and 54.33% respectively). However, prospective teachers displayed no performance in the composition of transformations task (0%) and almost no performance in relating geometric transformations and functions task (13.89%).

4.1.2. Interpretation of Prospective Middle School Mathematics Teachers' Initial Understanding of Geometric Transformations through the Lens of APOS Theory

In this part, results regarding analysis of prospective teachers' initial understanding of geometric transformations through APOS theory are presented. The pre-test and follow-up interview data showed that all participants except for Prospective Teacher 5 (PT5) were in Action level prior to the implementation of the instructional unit. PT5's understanding of geometric transformations was below the Action level. Detailed information about each prospective teacher's initial understanding of geometric transformations is presented in Table 4.3.

Table 4.3. Summary of prospective teachers' ideas related to geometric transformations based on pre-test and follow-up interview data

		Prospective Teachers															
	Ideas related to geometric transformations	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Reflection	An image in the mirror	x	x	x			x	x		x		x		x	x		x
	An image that appears in water or glass				x						x						
	An image of a figure with respect to a line								x			x	x			x	
	The object becomes reversed			x	x			x		x			x	x	x	x	x
	Size of the object remains same	x			x							x	x		x	x	x
	Distances to the line are equal	x		x				x				x	x			x	
Rotation	Spinning or turning the objects	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
	Requires an angle	x	x	x	x			x	x	x	x	x	x	x	x	x	
	Requires a center	x	x	x			x		x	x	x		x		x	x	
	Size of the object remains same	x			x							x		x		x	x
Translation	Moving an object	x				x			x								
	Sliding an object		x												x	x	
	Carrying an object							x		x	x	x	x				x
	Displacement			x	x		x							x			
	Requires a direction	x	x		x	x			x	x	x	x	x		x	x	x
	Requires an amount/unit	x	x		x	x	x	x	x	x	x	x	x		x	x	x
	Size of the object remains same	x	x	x	x			x		x	x	x		x	x	x	x

Pre-test and follow-up interview data showed that except PT5, all prospective teachers held an action conception of geometric transformations. They had a conception that a transformation is a motion that is applied to a single object. They viewed the domain of geometric transformations as single objects rather than all points in the plane (i.e., R^2). Prospective teachers' ideas related to geometric transformations are explained in the following paragraphs.

When asked to define what reflection is, 10 out of 16 prospective teachers expressed the following statement: "The image in the mirror" (PT1, PT2, PT3, PT6, PT7, PT9, PT11, PT13, PT14, and PT16). Three participants defined reflection as "an image that appears in water or glass" (PT4, PT9, and PT10). It can be seen that these reflection examples are selected from participants' daily life experiences. Shortly, they used everyday terms to define reflection. On the other hand, four participants (PT8, PT11, PT12, and PT15) used the coordinate plane to define reflection and stated that "reflection is the image of an object along x and y - axes." Thus, these four participants made use of the mathematical term 'coordinate plane' in their reflection examples.

When prospective teachers were asked to explain the properties of reflection, nine of them stated that "reflection makes the object become reversed" (PT3, PT4, PT7, PT9, PT12, PT13, PT14, PT15, and PT16) and seven of them stated that "after reflection the size of the object remains the same" (PT1, PT4, PT11, PT12, PT14, PT15, and PT16). Six participants expressed that "in reflection distances to the line are equal" (PT1, PT3, PT7, PT11, PT12, and PT15). Only PT1 focused on the critical attributes of the object itself when explaining properties of reflection. Namely, PT1 stressed that "the properties of a geometric figure, for instance of a square, such as side lengths and angle measures remain the same after reflection. Thus, a square is again a square after reflection".

When prospective teachers were asked to define rotation, all prospective teachers expressed that "rotations are spinning or turning the objects". For instance, PT14 defined rotation as "turning a figure by using a specific rule". The example she gave for rotation is presented in Figure 4.1.

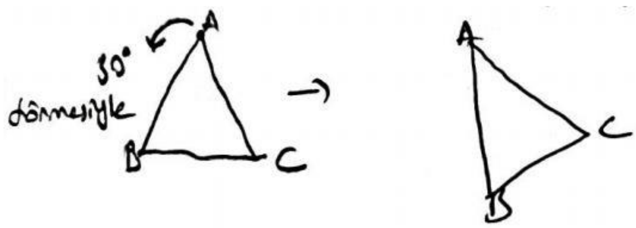


Figure 4.1. PT14's example in defining rotation.

When participants' use of parameters to define rotation was examined, based on the pre-test and follow-up interview data, it was seen that nine of them (PT1, PT2, PT3, PT8, PT9, PT10, PT12, PT14, and PT15) mentioned both a specific point as 'center of rotation' and angle measure as 'angle of rotation' in their definitions. Five participants (PT4, PT6, PT7, PT11, and PT13) mentioned either 'center of rotation' or 'angle of rotation' in their definitions. The participants who mentioned at least one of the parameters of rotation in their definitions also noted that the direction of rotation must be specified. PT16 did not mention the two parameters. However, she emphasized that the direction of rotation must be specified as well. Finally, six of the participants (PT1, PT4, PT11, PT13, PT15, and PT16) mentioned the size of the object as a property that remains unchanged after rotation.

When prospective teachers were asked to define translation, they utilized one of the following terms in their statements: moving (PT1, PT5 and PT8), sliding (PT2, PT14 and PT15) carrying (PT7, PT9, PT10, PT11, PT12 and PT16) and displacement (PT3, PT4, PT6 and PT13).

When participants' use of parameters to define translation was examined, based on the pre-test and follow-up interview data, it was seen that PT3 and PT13 did not mention any parameter of translation in their definitions or examples and they only stated that "translation is a displacement". For instance, PT13 gave "walking of a human" as an example of translation. As can be seen, this statement does not include the parameter of translation. PT6 and PT7 mentioned only the amount of translation in their definitions. The remaining twelve participants mentioned either the terms "direction" and "amount" or "direction" and "unit" in their examples. Moreover, excluding PT5, PT6, PT8, and PT12, all participants stated that the figure remained the same after translation and that its size remained unchanged.

PT5 had an understanding of geometric transformations below Action level. Namely, he could not provide any evidence that support that he held at least one of the ideas displayed in Action level. For instance, he defined reflection as “the bouncing off light with some angle measure after hitting on a transparent object”. This definition showed that PT5’s knowledge of reflections was limited to the reflection topic taught in science lessons. The following example he gave for reflection provided further evidence that he did not know that the concept of reflection is also a mathematical concept.

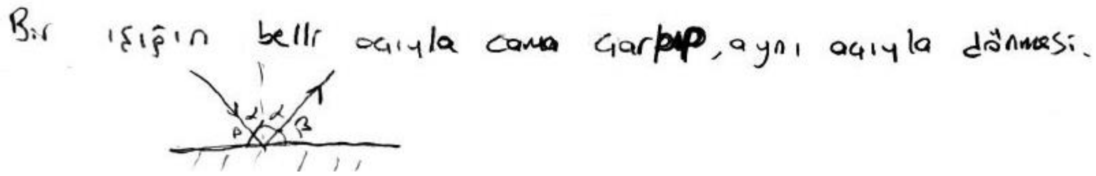


Figure 4.2. PT5’s example for definition of reflection.

Participants in Action level were able to provide reflection examples that included an object, a mirror, and the image of that object. However, it is important to note that PT5’s reflection example included neither an object nor an image equidistant from the mirror. Besides, according to PT5, reflection takes place only in one side of the mirror. He expressed that a reflection is “hitting and turning of the light with a specific angle”. Due to these differences, PT5 was interpreted to be below Action level.

Therefore, there was not any prospective teacher holding a process conception of geometric transformations before the implementation of the instructional unit. Although some of them were able to use parameters to define geometric transformations and to reason about properties that remain invariant under some of transformations, they defined them as a motion of single objects. They did not present any evidence to support the idea that transformations are applied to all points in the plane and consequently any evidence that shows that they held a process conception of geometric transformations.

4.2. Identifying Geometric Transformations

In this part, findings related to prospective teachers’ identification of transformations during the instructional unit are presented. The data came from identification tasks included in Activity 1, Activity 2, and Activity 3. In these activities, prospective teachers were asked to identify transformations presented in several contexts. They were either finite figures with unobvious units, frieze patterns with obvious and unobvious units, or 2D geometric figures on a grid paper such as letter F.

This part is presented under four headings as prospective teachers' identification of reflections, rotations, translations, and finally overall evaluation of their understanding of these three geometric transformations.

4.2.1. Identifying Reflections

Prospective teachers' identification of reflections was determined via tasks that ask them to identify geometric transformations in different contexts. Namely, findings related to prospective teachers' identification of reflections in the finite figures, reflections in the frieze patterns with obvious units, and reflections on a grid paper are presented respectively. Prospective teachers' identification of reflections in the finite figures are presented below.

4.2.1.1. Reflections in the finite figures

Prospective teachers were asked to identify the transformations included in the finite figures. Based on the data obtained through activity sheets and pairwise discussions regarding Task 1 and Task 3 of Activity 1, the prospective teachers' identification of reflections is presented Table 4.3.

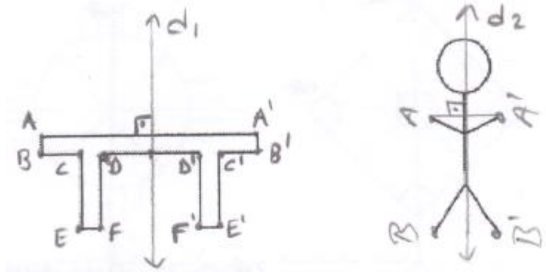
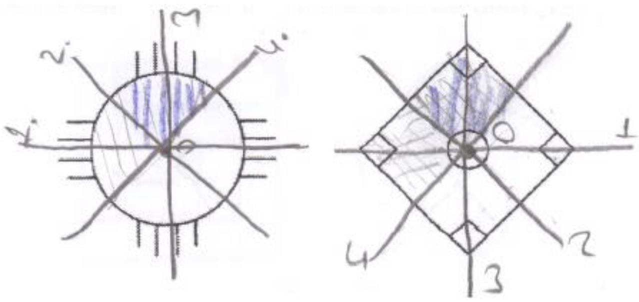
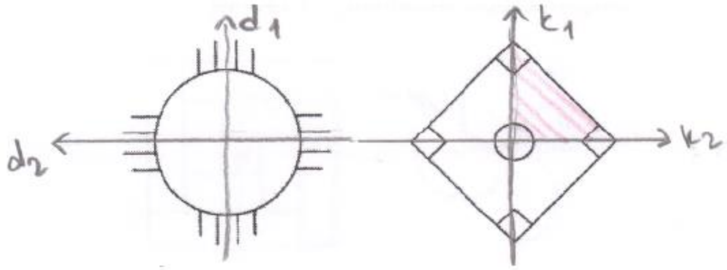
Table 4.3. Prospective teachers' identification of reflections in the finite figures in Task 1 and Task 3 of Activity 1

Pairs	Task 1		Task 3				
	Ref.	*l*	Ref.	l_1	l_2	l_3	l_4
PT1 & PT2	✓	✓	✓	✓	✓	✓	✓
PT3 & PT4	✓	✓	✓	✓	✓		
PT5 & PT6	✓	✓	✓	✓			
PT7 & PT8	✓	✓					
PT9 & PT10	✓	✓	✓	✓	✓		
PT11 & PT12	✓	✓	✓	✓			
PT13 & PT14	✓	✓	✓		✓		
PT15 & PT16	✓	✓	✓	✓	✓		

Note: PT: Prospective teacher, Ref: Reflection, *l*: line of reflection

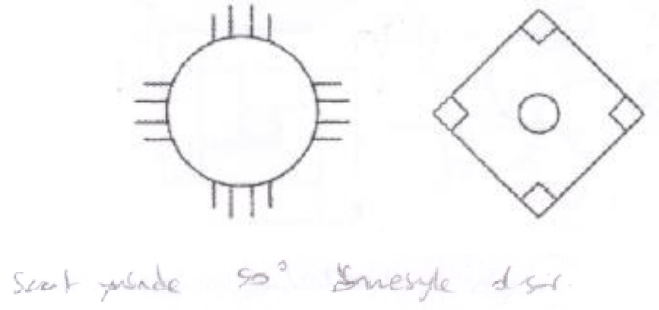
When Table 4.3 is examined it can be seen that except PT7 & PT8, all pairs were able to identify the reflections in finite figures given in Task 1 and Task 3. Moreover, all pairs were able to identify the reflection line in Task 1 as the parameter. However, only PT1 & PT2 were able to identify all four reflection lines for the finite figures given in Task 3. Three pairs (PT3 & PT4, PT9 & PT10 and PT15 & PT16) could identify two reflection lines, another three pairs (PT5 & PT6, PT11 & PT12 and PT13 & PT14) could identify only one reflection line, yet PT7 & PT8 could not identify any reflection line in Task 3. Examples of prospective teachers' responses to Task 1 and Task 3 of Activity 1 are presented in Table 4.4.

Table 4.4. Prospective teachers’ sample responses to Task 1 and Task 3 of Activity 1

PT11 & PT12’s response to Task 1	 A-A', B-B', C-C', D-D', E-E', F-F' d doğrusuna esit uzaklıktadır. d2 doğrusu ekseninde katlandığında cisimler çakışır. [Points A-A', B-B', C-C', D-D', E-E', F-F' are equidistant to the line d. Objects conincide when they are folded along line d_2]
PT1 & PT2’s response to Task 3	 belirlenen 1, 2, 3, 4. eksenlere göre yansıma [Reflection along lines 1, 2, 3, and 4]
PT15 & PT16’s response to Task 3	 d1 veya d2 doğrularına göre yansıması alınmıştır. k1 veya k2 doğrularına göre yansıması alınmıştır [It is reflected along line d_1 or d_2. It is reflected along line k_1 or k_2]

Four reflection lines identified by PT1 & PT2 and two reflection lines identified by PT15 & PT16 in Task 3 are seen in Table 4.4. As mentioned before, PT7 & PT8 could not identify the reflection in Task 3. Their response to Task 3 is presented in Table 4.5.

Table 4.5. PT7 & PT8's response to Task 3 of Activity 1

PT7 & PT8's response to Task 3	 [90° rotation in clockwise direction]

As seen in Table 4.5, they identified a rotation and its angle measure as 90 degrees. This response was not wrong but it was insufficient. Due to the nature of reflection and rotation concepts, all figures including reflection also include rotation. Here, PT7 & PT8's response includes only rotation. Since their response did not provide any information about a reflection and/or a line of reflection, their response was coded as "misidentification of reflection and its parameter". However, it is not possible to infer that they did not know how to identify the reflection and the reflection line.

As mentioned before, all pairs were able to identify the reflection line in Task 1 whereas only PT1 & PT2 were able to identify all four reflection lines in Task 3. This difference might have been due to the fact that prospective teachers could construct the finite figure by performing one or two reflections in Task 3. More precisely, when one half of the finite figure is considered as a unit in Task 3, the whole figure can be constructed after performing one reflection. Similarly, the quarter of the finite figure given in this task can be considered as a unit and the whole figure can be constructed by performing two reflections. Another reason for this might be related to the type of reflection line. Namely, diagonal reflections might be more difficult for prospective teachers than vertical and horizontal reflections. The prospective teachers who were able to identify only one or two reflection lines for Task 3 can be given as examples that support this idea. Note that these prospective teachers could identify either horizontal, vertical or both but not diagonal reflection lines.

After prospective teachers worked in pairs and identified reflections in their activity sheets, prospective teachers discussed their responses with other pairs. In whole class discussion, prospective teachers were asked to state which geometric transformations they identified for each task. Next, they were asked to explain in detail the transformations they identified. That is, the researcher wanted to find out whether all participants could explain transformations with their parameters.

The researcher asked each pair how they responded to Task 1A. All pairs stated that reflection was performed on the figures. When the researcher asked the class to explain the reflection they identified, PT13 responded as “a reflection which has a line of reflection passing through the middle of each figure”. Other participants stated that they found the same reflection line. As can be seen in Table 4.3, other participants could also identify the same reflection lines in Task 1. The following dialogue took place among the researcher and the prospective teachers while discussing Task 3 of Activity 1:

> Researcher: Which transformations did you identified in Task 3?
> PT3: Reflection
> PT10: Reflection
> PT16: Reflection
> Researcher: How did you identify the reflection?
> PT3: Vertical
> PT10: Horizontal
> PT16: There are vertical and horizontal reflection lines in those figures.
> Researcher: How many reflection lines did you find at most?
> PT1: There are two more reflection lines passing through the diagonals.

During the whole class discussion, prospective teachers were able to share and extend their ideas related to reflection lines. That is, PT3 referred to the vertical reflection line, PT10 referred to the horizontal reflection line, and PT1 referred to the two diagonal reflection lines. By this way, all prospective teachers had an opportunity to see all reflection lines belonging to the given figures.

In order for prospective teachers to ponder more on reflection, the researcher asked them to identify the units of the given figures in Task 1. These figures were reflected on the smart board and the researcher asked the class to find the units of them. Initially, researcher asked the class to find the unit of the ‘stick man’. PT15 responded as “each left or right part of the stick man”. Next, the researcher asked them to find the unit of the other figure in Task 1. PT1 responded as “the letter T” and all participants confirmed the answers of PT1 and PT15.

4.2.1.2. Reflections in the frieze patterns with obvious units

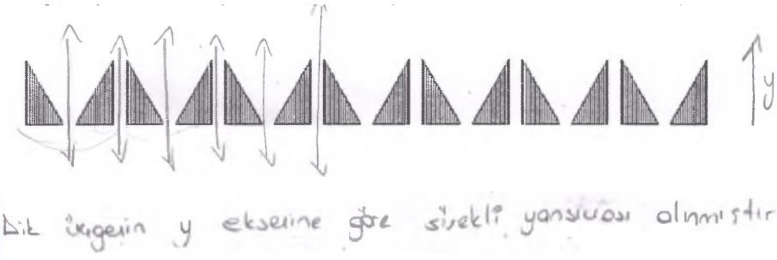
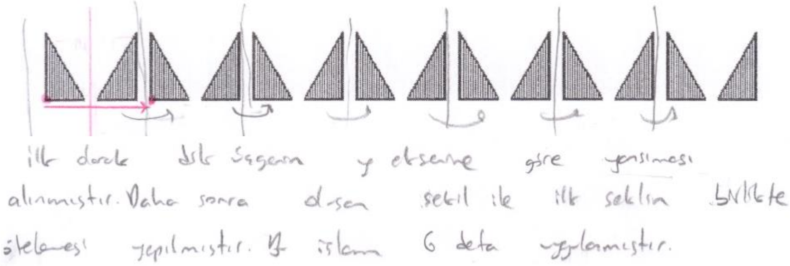
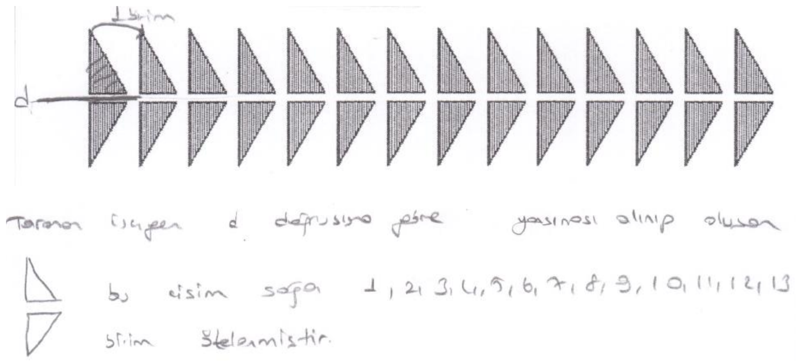
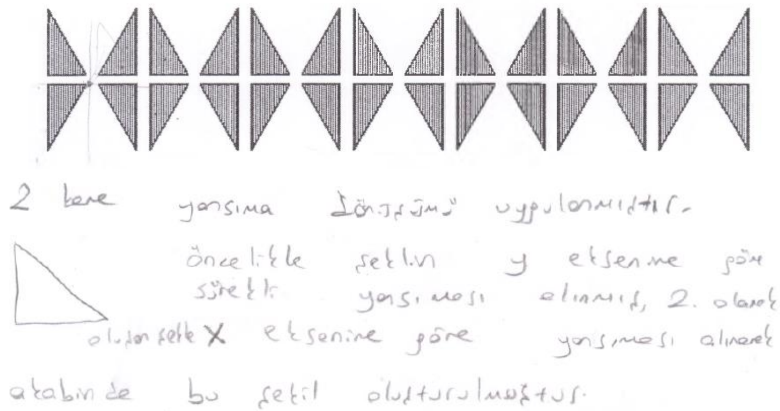
Prospective teachers were asked to identify the transformations included in the frieze patterns whose units are all right triangles. Based on the data obtained through activity sheets and pairwise discussions regarding Task 2, Task 3 and Task 4 of Activity 2, the prospective teachers’ identification of reflection is presented in Table 4.6.

Table 4.6. Prospective teachers’ identification of reflection in the frieze patterns in Task 2, Task 3, and Task 4 of Activity 2

Pairs	Task 2		Task 3		Task 4		
	Ref.	l_1	Ref.	l_1	Ref.	l_1	l_2
PT1 & PT2	✓	✓	✓	✓	✓	✓	✓
PT3 & PT4	✓	✓	✓	✓	✓	✓	✓
PT5 & PT6	✓		✓		✓	✓	✓
PT7 & PT8	✓	✓	✓	✓	✓	✓	✓
PT9&PT10	✓	✓	✓	✓	✓	✓	✓
PT11&PT12	✓	✓	✓	✓	✓	✓	✓
PT13&PT14	✓	✓	✓	✓	✓	✓	✓
PT15&PT16	✓	✓	✓	✓	✓	✓	✓

When Table 4.6 is examined it can be seen that all pairs were able to identify reflections in these three tasks. Moreover, except for PT5 & PT6, all pairs were able to identify all reflection lines. More specifically, PT5 & PT6 were not able to identify the reflection lines for Task 2 and Task 3. Examples of prospective teachers’ responses to Task 2, Task 3, and Task 4 of Activity 2 are presented in Table 4.7.

Table 4.7. Prospective teachers' sample responses to Task 2, Task 3 and, Task 4 of Activity 2

PT15 & PT16's response to Task 2	 [The right triangle was reflected along y-axis succesively]
PT7 & PT8's response to Task 2	 [First, the right triangle was reflected along y-axis, then the resultant triangle together with the first triangle were translated succesively. This process was repeated six times.]
PT1 & PT2's response to Task 3	 [The shaded triangle was reflected along the line d. Then, the resultant triangle and the first triangle were translated 1,2,3…13 units.]
PT5 & PT6's response to Task 4	 [Reflection was applied twice. First, the figure was reflected along y-axis successively. Then, the resultant figure were reflected along x-axis.]

Two different solution methods were used by the prospective teachers to obtain the frieze pattern given in Task 2. PT15 & PT16 and PT7 & PT8's sample responses to this task are presented in Table 4.7. The solution method used by PT15 & PT16 included recursive vertical reflections. They reflected the first triangle vertically and obtained the second triangle. Then, they reflected the second triangle vertically and obtained the third triangle. Their solution method continued like this. The solution method used by PT7 & PT8 included obtaining the second triangle by a vertical reflection and then applying recursive translations to the first two triangles. They translated the first two triangles six times to obtain the frieze pattern.

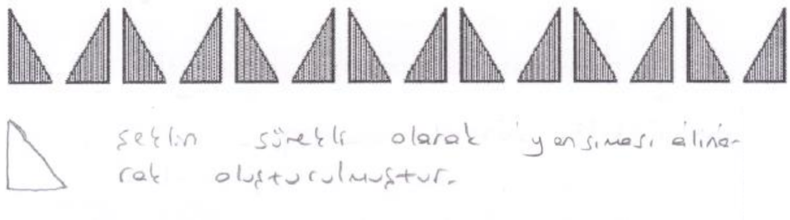
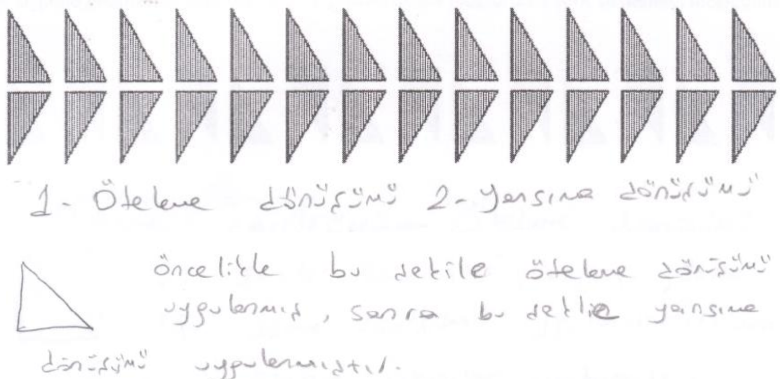
After prospective teachers worked in pairs and identified reflections in their activity sheets, they discussed their responses with other pairs. In Task 2, the researcher asked them to explain which transformations they identified in the given frieze pattern. All pairs responded as reflection. Actually, when Table 4.6 is examined, it can be seen that all pairs could identify reflection in this task. Moreover, the following dialogue occurred during the whole class discussion of Task 2 of Activity 2:

> Researcher: How might this frieze pattern be constructed?
> PT4: I reflected the initial right triangle first, then I obtained the second triangle. Next, I reflected the second triangle and obtained the third one. It goes on like this (see PT15 & PT16's response in Table 4.7)
> Researcher: (By turning to the class) What do you think?
> PT16: Actually, after obtaining the second triangle, the first two triangles might be treated as the unit of pattern and these two triangles might be reflected at once.
> PT11: I think that after obtaining the second triangle by reflection, we can translate the first two triangles concurrently and successively to obtain this frieze pattern (see PT7 & PT8's response in Table 4.7).
> Researcher: As you can see, this pattern can be obtained in several different ways. However, to be more economical, which way should we prefer?
> Students: We should reflect the first triangle for ones and translate both this triangle and the obtained triangle concurrently.

The possible ways for obtaining the frieze pattern given in Task 2 are articulated in the above given dialogue. The dialogue revealed prospective teachers' three different ways in the construction of the frieze pattern. Actually, two of them are exemplified in Table 4.7. The solution way suggested by PT4 was the same with the response of PT15 & PT16 presented in Table 4.7 and it included recursive vertical reflections. Similarly, the solution way suggested by PT11 was the same with the response of PT7 & PT8 presented in Table 4.7 and it included obtaining the second triangle by a vertical reflection and then applying recursive translations to the first two triangles. The third solution way was suggested by PT16. This way involved obtaining the second triangle by a vertical reflection and then applying recursive vertical reflections to the first two triangles.

As mentioned before only PT5 & PT6 were not able to identify the reflection lines for Task 2 and Task 3. Their responses to Task 2 and Task 3 were presented in Table 4.8.

Table 4.8. PT5 & PT6's responses to Task 2 and Task 3 of Activity 2

PT5 & PT6's response to Task 2	 [It was constructed by reflecting the figure succesively.]
PT5 & PT6's response to Task 3	 [1- Translation. 2- Reflection. Initialy, translation was applied to this figure. Next, reflection was applied to this figure.]

As seen in PT5 & PT6's responses to Task 2 and Task 3, they only identified the reflections but not their parameters (i.e., reflection lines). Since their responses did not include any explanation about the reflection line, their responses were coded as "misidentification of the parameter". However, as seen in Table 4.6, they identified both reflection lines in Task 4. Therefore, it can be said that PT5 & PT6's difficulty in identifying the reflection line in Task 2 and Task 3 was resolved in Task 4.

Finally, it is important to note here that PT7 & PT8, those who did not include reflection in their responses in Task 3 of Activity 1, were able to identify all reflections and reflection lines in Task 2, Task 3, and Task 4 in Activity 2. Their identification of reflection and reflection line in Task 2 is presented in Table 4.7. Therefore, PT7 & PT8's response showed that their misidentification of reflection in Activity 1 disappeared in tasks included in Activity 2.

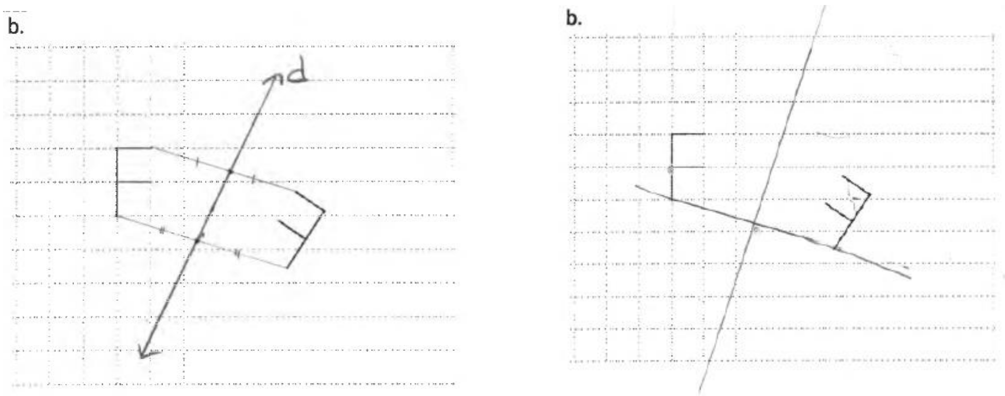
4.2.1.3. Reflections on a grid paper

In Task 1 of Activity 3, the prospective teachers were asked to identify the transformations given on a grid paper. In Task 1B, reflection was applied to the letter 'F' and the participants were asked to identify this reflection and explain it with its parameter. Prospective teachers were provided rulers and protractors. Based on the data obtained through activity sheets and pairwise discussions regarding Task 1B of Activity 3, the prospective teachers' identification of reflection is presented Table 4.9.

Table 4.9. Prospective teachers' identification of reflection in Task 1B of Activity 3

Pairs	Task 1B	
	Ref.	*l*
PT1 & PT2	✓	✓
PT3 & PT4	✓	✓
PT5 & PT6	✓	
PT7 & PT8		
PT9 & PT10	✓	✓
PT11 & PT12	✓	✓
PT13 & PT14	✓	✓
PT15 & PT16		

The reflection in Task 1B was identified by six pairs. However, among these pairs PT5 & PT6 could not identify the reflection line and they only stated that reflection was applied to the letter. PT7 & PT8 and PT15 & PT16 could not identify the reflection and the reflection line. Examples of prospective teachers' responses to Task 1B of Activity 3 are presented in Figure 4.3.



(a) (b)

Figure 4.3. PT13 & PT14's (a) and PT9 & PT10's (b) response to Task 1B

As seen in Figure 4.3, the responses provided by PT13 & PT14 and PT9 & PT10 include different solution methods for construction of the reflection line. PT13 & PT14 used the property that reflection lines are equidistant from the corresponding pre-image and image points. They constructed two mid-points between corresponding pre-image and image points and then constructed the reflection line passing through these two mid-points. PT9 & PT10 used the property that reflection lines are perpendicular bisectors of line segments joining the corresponding pre-image and image points. In solving Task 1B, PT11 & PT12 used another method and it is presented in Figure 4.4.

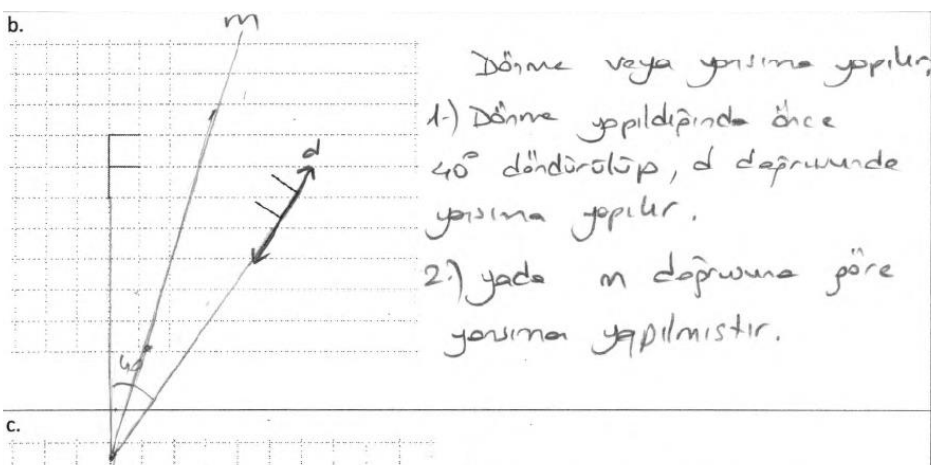


Figure 4.4. PT11 & PT12's response to Task 1B

As can be understood in Figure 4.4, PT11 & PT12 used two different methods for Task 1B and both of them were correct. In their first solution method, PT11 & PT12 used rotation and reflection together. In this method they stated that "F was rotated 40°" around the intersection point of the extended lines. Later, the resultant "F was reflected along line *d*" coinciding with F. In their second method, they stated that "a reflection along line m was applied". They constructed the angle bisector between the extended lines and then they considered this angle bisector as a reflection line. PT11 & PT12's second method was explained by PT11 in the whole class discussion as reported in the following paragraph.

After pairwise discussions, I asked all pairs to state which transformations they identified in Task 1B. Besides, prospective teachers' three different methods appeared in their activity sheets for finding the reflection line were also mentioned by them in whole class discussions. In whole class discussion, as a response to Task 1B, the pairs PT11 & PT12 and PT15 &PT16 stated that both reflection and rotation were used and the remaining pairs indicated that only reflection was used. Then, I requested P11 to come to the board and explain his answer to Task 1B. The explanations of P11 were as follows:

> PT11: I joined the two Fs in this way (He extended the lower parts of the two Fs and the two extended lines intersecting somewhere below them). The angle measure between these two lines is 40°. If I draw a line passing from the intersection point and if this angle measure is 20°, then this line (pointing to the angle bisector) becomes the reflection line (see Figure 4.4).
> Researcher: Well, do we always do the same thing to determine the reflection line?
> PT11: Yes, we need to determine the intersection point and this angle.

Just then, I turned back to the class and asked all prospective teachers to state their own ideas about PT11's solution.

> Researcher: Well, what does the class say about this?
> PT4: Teacher, it seems reasonable, however PT11's method is a little bit difficult.
> Researcher: Why is it difficult, PT4?
> PT4: For instance, if the reflection line had been perpendicular to the two Fs, the extension lines could not have such an intersection point.
> Research: Your friend is right. In that case, what would you do PT11?

PT11: In that case I… we would look at the distances (meanwhile, PT11 drew a line segment from F to the angle bisector). But, we had drawn by the method I explained previously (PT11 sat on his desk and PT13 came to the board)

As understood from the dialogue, PT11 & PT12's solution method was not accepted by other participants. Besides, PT11 could not justify that their solution method was valid for other cases in which such an intersection point did not exist. After discussing PT11 & PT12's solution method for finding the reflection line, I turned back to the class and asked all prospective teachers to state their own solution methods.

Researcher: Well, is there anybody who drew the reflection line by using a different method?

PT13: First, we drew these distances (PT13 selected two points on F and joined these points with the corresponding ones on F'). Next, we found the midpoints of these distances and drew a line passing through these midpoints. This line is our reflection line. Moreover, these are perpendicular to each other as well (the distances and the reflection line) (see Figure 4.3a).

Researcher: Well, is there any comment?

PT12: Teacher, you know we found the distance between a point on F and the corresponding point on F'. If we draw a perpendicular line to the midpoint of this distance, this line will be our reflection line. This reflection line is perpendicular to other distances between F and F' as well (see Figure 4.3b)

Researcher: PT12 states that the perpendicular bisector of the line segment joining Point A (a point on F) and A' (the corresponding point on F') is also the reflection line. What do you think? Is that line can be reflection line?

PT2: Yes. The line segments between F and F' are all parallel to each other. Thus, if one of the aforementioned line segments is perpendicular to reflection line, then other line segments will already be perpendicular as well.

In the above given dialogue, two different solution methods were explained by the prospective teachers. PT13 explained the solution method presented in Figure 4.3a and PT12 explained the solution method presented in Figure 4.3b. It is important to note that PT12's method was the same with PT9 & PT10's solution method. Besides, PT2 supported PT12's idea by providing the mathematical idea behind PT12's solution method.

As mentioned before, PT15 & PT16 could not identify the reflection. Similar to PT11 & PT12, PT15 & PT16 attempted to explain the reflection via composition of two geometric transformations in Task 1B. However, PT15 & PT16's solution method was not correct. To illustrate, PT15 & PT16's response to this sub-task is presented in Figure 4.5.

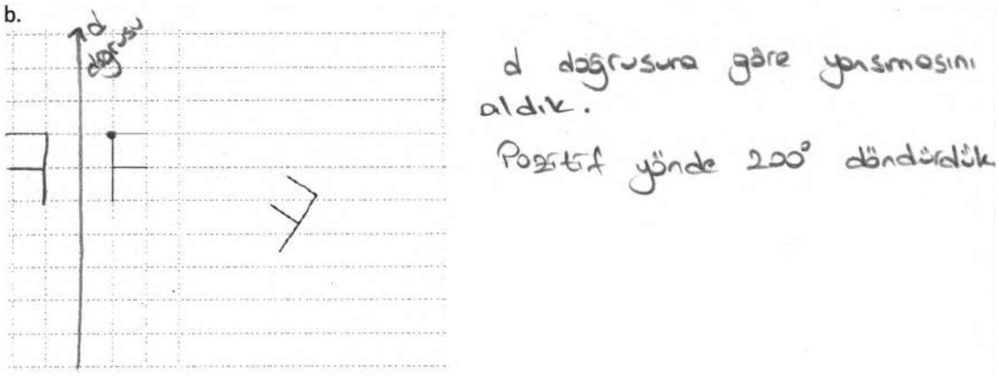


Figure 4.5. PT15 & PT16's response to Task 1B

In Figure 4.5, PT15 & PT16 stated that they "reflected letter F along line d and then rotated the resultant F 200 degrees". They did not mention the center of rotation. Actually, the angle of rotation seemed between 270 and 360 degrees, not 200 degrees. PT15 & PT16 could not identify the center of rotation and angle of rotation correctly and thus this was not regarded as a totally correct answer. Indeed, the aim of this sub-task was to have prospective teachers identify the single reflection and its parameter. In this respect, their response was coded as "misidentification of reflection and its parameter".

Another pair who could not identify the reflection and its parameter was PT7 & PT8. In their activity sheet there was no response to Task 1B. Another pair, PT5 & PT6, responded as there was a reflection but they could not construct or explain the reflection line in their response.

It is important to note that all prospective teachers, including PT5 & PT6, PT7 & PT8, and PT15 & PT16, were able to identify the reflection and the reflection line at the end of the Activity 2. However, these three pairs had difficulty in identification of the reflection and its parameter in Task 1B of Activity 3. There might be several reasons for this difficulty. First, as the instructional unit proceeded, the tasks included in the further activities became more complex. For instance, the reflections presented to the prospective teachers in the first activity had reflection lines that passed through the figure. Differently, the reflection lines in Activity 2 did not pass through the figure. In Task 1B of Activity 3, the reflection lines did not pass through the figure as well. Meanwhile, the reflection lines included in Activity 2 were either vertical or horizontal while the one included in Activity 3 was a diagonal reflection line. Besides, the reflection in this task was given on a grid and prospective teachers were asked to determine the location and the slope of the reflection line exactly. Thus, the nature of reflection line in Activity 3 (i.e., not passing through the figure, being a diagonal line, and being given on a grid) might have influenced its complexity. Although Table 4.9 shows that PT11 & PT12 could identify the reflection and its parameter in Task 1B of Activity 3, their response was valid for only this task and in other tasks they did not have any idea about finding the reflection line as revealed in whole class discussions. Thus, four pairs including PT11 & PT12 had difficulty in identification of a diagonal reflection line that did not pass through the figure and that was given on a grid.

4.2.2. Identifying Rotations

Prospective teachers' identification of rotations was determined via tasks that ask them to identify geometric transformations in different contexts. Namely, findings related to prospective teachers' identification of rotations in the finite figures, rotations in the frieze patterns with obvious units, and rotations on a grid paper are presented respectively. Prospective teachers' identification of rotations in the finite figures are presented below.

4.2.2.1. Rotations in the finite figures

Prospective teachers were asked to identify the transformations included in the finite figures. Based on the data obtained through activity sheets and pairwise discussions regarding Task 2, Task 4, and Task 5 of Activity 1, the prospective teachers' identification of rotations are presented in Table 4.10.

Table 4.10. Prospective teachers' identification of rotations in the finite figures in Task 2, Task 4 and Task 5 of Activity 1

Pairs	Task 2			Task 4			Task 5		
	Rot	C	A	Rot	C	A	Rot	C	A
PT1 & PT2	✓	✓	✓	✓	✓	✓	✓	✓	✓
PT3 & PT4	✓	✓	✓	✓	✓		✓	✓	
PT5 & PT6	✓	✓		✓	✓		✓	✓	
PT7 & PT8	✓		✓	✓			✓		✓
PT9 & PT10	✓	✓	✓	✓	✓	✓	✓	✓	
PT11 & PT12	✓	✓	✓	✓	✓	✓	✓	✓	✓
PT13 & PT14	✓	✓	✓	✓	✓		✓	✓	✓
PT15 & PT16	✓	✓	✓	✓		✓	✓		✓

Note: PT: Prospective teacher, Rot: Rotation, C: Center of rotation, A: Angle of rotation

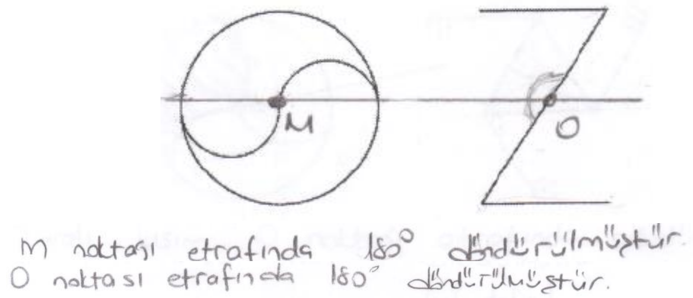
Table 4.10 shows that all pairs were able to identify rotations in these three tasks. However, not all pairs were able to identify centers and angles as parameters of the rotations. Only PT1 & PT2 and PT11 & PT12 were able to identify all parameters of the rotations included in these three tasks. In more detail, six, three and three pairs could identify both parameters of rotation in Task 2, Task 4 and Task 5 respectively. In Task 2, all pairs except for PT5 & PT6 and PT7 & PT8 could identify both parameters. PT5 & PT6 could identify only the center of rotation and PT7 & PT8 could identify only the angle of rotation.

In Task 4, PT1 & PT2, PT9 & PT10 and PT11 & PT12 could identify both parameters. PT3 & PT4, PT5 & PT6 and PT13 & PT14 could identify only the center of rotation, PT15 & PT16 could identify only the angle of rotation while PT7 & PT8 could identify none of the two parameters of the rotation in Task 4.

In Task 4, PT1 & PT2, PT11 & PT12 and PT13 & PT14 could identify both parameters. PT3 & PT4, PT5 & PT6 and PT9 & PT10 could identify only the center of rotation while PT7 & PT8 and PT15 & PT16 could identify only the angle of rotation in Task 5. Prospective teachers' sample responses to Task 2, Task 4, and Task 5 of Activity 1 are presented in Table 4.11.

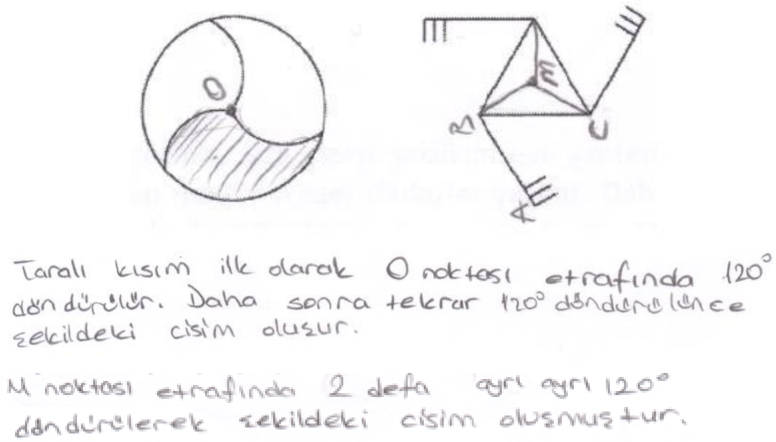
Table 4.11. Prospective teachers' sample responses to Task 2, Task 4 and, Task 5 of Activity 1

PT9 & PT10' s response to Task 2



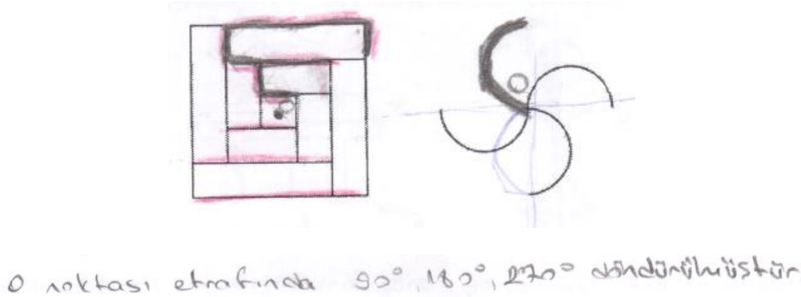
[It was rotated 180° around point M. It was rotated 180° around point O.]

PT11 & PT12' s response to Task 4



[The shaded area was rotated 120° around point O. Then, it was rotated 120° again. 120° rotations were applied twice.]

PT1 & PT2' s response to Task 5



[90°, 180°, and 270° rotation around point O.]

As mentioned before, only PT1 & PT2 and PT11 & PT12 were able to identify all parameters of rotations included in these three tasks. PT7 & PT8 did not mention about the center of rotation in Task 2 and Task 5. PT7 & PT8 could not identify the angle of rotation and the center of rotation in Task 4. Besides, in Task 4 they explained that the angles of rotations were 90 and 60 degrees for the first and the second figure respectively. The remaining pairs could not identify either the angle of rotation or the center of rotation in Task 2, Task 4, and Task 5. For instance, as a response to Task 2, PT5 & PT6 stated that "rotation was applied around this point" without including the angle of rotation. As a response to Task 4, PT15 & PT16 stated that "120 degrees rotation was applied" without including the center of rotation.

After prospective teachers worked in pairs and completed the activity sheets, they discussed their responses with other pairs. In Task 2, Task 4 and Task 5, similar dialogues took place during whole discussion. The dialogue that took place in the discussion of Task 2 is given below:

> Researcher: Which transformations did you identify in Task 2?
> PT9: Rotation
> PT13: Rotation
> PT11: Rotation
> Researcher: Can you explain the rotation?
> PT1: 180 degrees rotation.
> Researcher: Any other detail PT5?
> PT5: The center of rotation is the center of the circle (the first figure in Table 4.11)
> Researcher: Well, what is the center of the Z? (the second figure in Table 4.11)
> PT15: The midpoint of the line segment located at the middle of Z

As understood by the above given dialogue, prospective teachers shared their responses to Task 2 with other prospective teachers. Table 4.10 shows that most of the participants correctly identified both parameters of the rotation. Therefore, there was no conflict among participants in whole class discussion in terms of the rotations identified.

In order for prospective teachers to ponder more on rotation, the researcher asked them to identify the units of the given figures. The figures in Task 2 were reflected on the smart board and the researcher asked the class to find the unit of each of these two figures. The following dialogue took place among the researcher and the participants:

> Researcher: PT3, would you come to the board and draw the units of the two figures in Task 2 please?

PT3: Ok, I am coming. (For the first figure, PT3 shaded the inside of the upper part. For the second figure, she traced the upper half part, namely the upper line segment and the upper half of the middle line segment on the letter Z with a board marker)

Researcher: (By facing towards the class) Do you agree with your friend for both figures?

PT13: In the second figure, she traced the line segments. However, she shaded a region in the first one. She could have rotated only the lines of the first figure as she did in the second figure rather than shading the inner region.

Researcher: Well, let's examine your idea. (The researcher traced the upper part of the first figure that was shaded by PT13. Namely, the two smaller arcs inside the figure and the larger arc looking downwards) Let's rotate the figure I traced. (The researcher rotated the traced figure 180 degrees with a different colored pen. Therefore, the smaller two arcs inside the figure were traced with two different colors.) Is it correct now?

PT13: The smaller two arcs coincided with each other. Maybe, we can find a new unit which is smaller than this one.

PT12: Let's draw a horizontal line passing through the center of rotation and rotate one of the two congruent parts formed by the horizontal line.

Researcher: Ok, then, who wants to tell me the smallest unit?

PT14: One of the smaller arcs inside the figure…

PT13: …and the outer semi-circle.

Researcher: Let's see the rotations in GeoGebra file that I prepared before.

The dialogue that took place in the classroom during the discussion of the units of the figures given in Task 2 are presented above. PT3 explained one unit for each figure. PT3 referred to a region as a unit for the first figure while to line segments for the second figure. PT13 noticed this issue and argued that line segments should be used as a unit for the first figure as well. At that time, the researcher traced the border of the region mentioned by PT3. PT13 interrupted again and stated that the unit must be smaller than the traced region owing to the fact that the two smaller arcs inside the figure coincided when rotation was applied. This conflict was resolved when PT12 put forward the idea of dividing the figure into two by a horizontal line passing through its center of rotation. As indicated by PT13 and PT14, the unit of the first figure was a combination of the inner semi-circle and the outer semi-circle.

The researcher opened the GeoGebra file given in Figure 4.6. By this file, the researcher aimed at having prospective teachers see how rotations were applied to the units. The two figures given in Task 2 are presented in Figure 4.6.

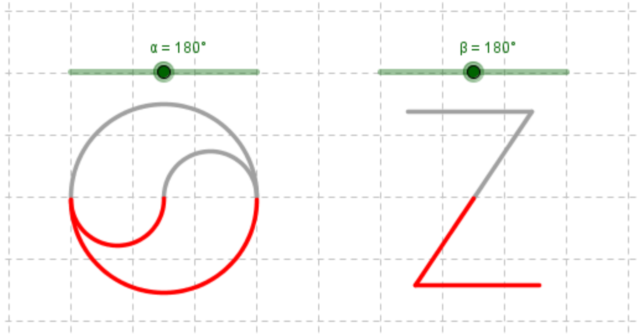


Figure 4.6. The geometric figures included in Task 2.

This GeoGebra file was prepared in advance by the researcher. The red lines were obtained by applying 180 degrees rotation to the grey lines. The researcher moved the sliders back and forth to have the class see the change in angle measure and subsequently have them see the change in the red lines in both dynamic figures. In short, the researcher tried to have the prospective teachers realize that they could use either the grey lines or the red lines as a unit to obtain the two figures by 180 degrees rotation. The two units are presented in Figure 4.7.

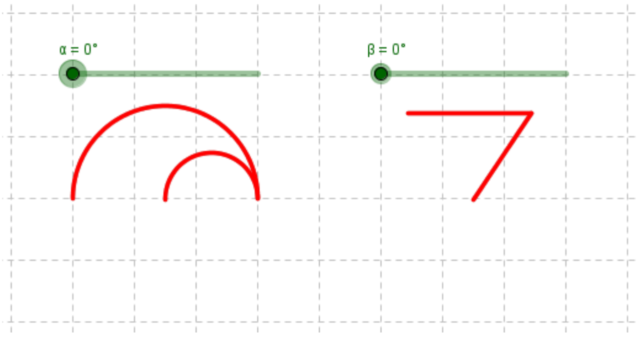


Figure 4.7. The units of the figures given in Task 2.

After discussing Task 2 thoroughly, the participants continued to identify the units of other geometric figures given in Task 4 and Task 5 by coming to the smart board and by showing their own units. PT6 identified the units of both figures in Task 4. Finally, PT13 and PT10 identified the rotated units of the first and second figures given in Task 5 respectively.

4.2.2.2. Rotations in the frieze patterns with obvious units

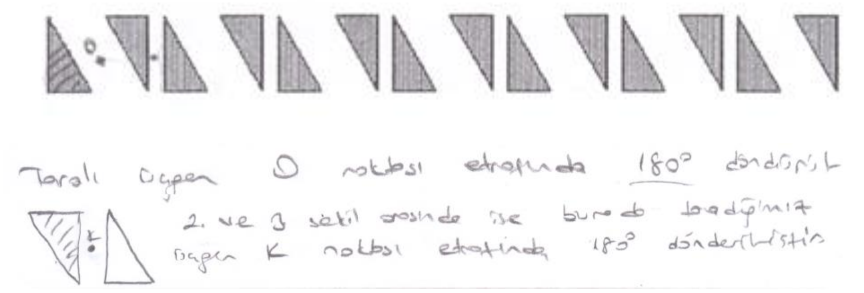
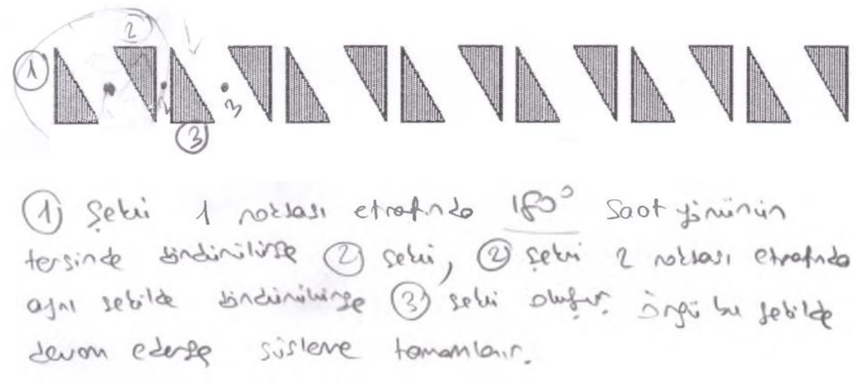
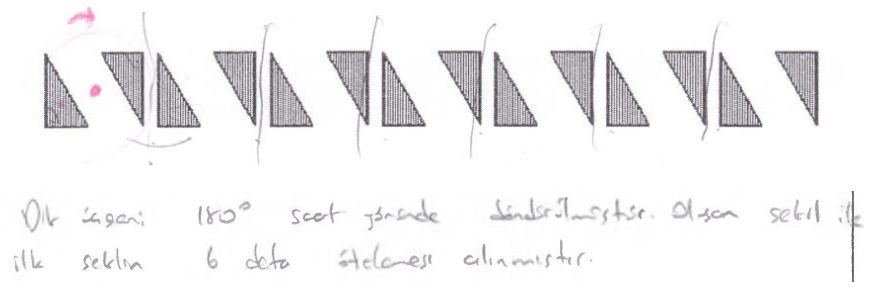
Prospective teachers were asked to identify the transformations included in the frieze patterns whose units were all right triangles. Based on the data obtained through pairwise discussions in Task 5 of Activity 2, the prospective teachers' identification of rotations are presented in Table 4.12.

Table 4.12. Prospective teachers' identification of rotations in the frieze patterns in Task 5 of Activity 2

Pairs	Task 5		
	Rot.	C	A
PT1 & PT2	✓	✓	✓
PT3 & PT4	✓	✓	✓
PT5 & PT6	✓		✓
PT7 & PT8	✓	✓	✓
PT9 & PT10	✓		✓
PT11 & PT12			
PT13 & PT14	✓		✓
PT15 & PT16	✓		✓

All pairs except for PT11 & PT12 were able to identify the rotation in pairwise discussions. However, only PT1 & PT2, PT3 & PT4, and PT7 & PT8 could identify both the center of rotation and the angle of rotation. While PT11 & PT12 could not identify any of the two parameters, the remaining pairs could identify only the angle of rotation. The responses of the pairs who could identify both parameters for Task 5 of Activity 2 are presented in Table 4.13.

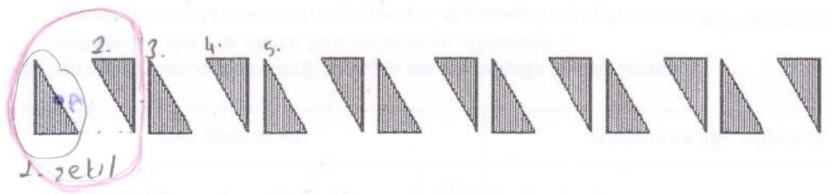
Table 4.13. Prospective teachers' sample responses to Task 5 of Activity 2

PT1 & PT2' s response to Task 5	 [The shaded triangle was rotated 180° around point O. The second triangle, shaded on the left, was rotated 180° around point K.]
PT3 & PT4' s response to Task 5	 [When Figure 1 is rotated 180° around the first point in counterclockwise direction, the second figure is obtained. Similarly, when Figure 2 is rotated around the second point, the third figure is obtained. If the cycle continues in this way, the pattern becomes completed.]
PT7 & PT8's response to Task 5	 [The rigt triangle was rotated 180° around the red point. The resultant and the first triangles were translated six times.]

The two solution methods used by the prospective teachers to obtain the frieze pattern are presented in Table 4.13. The first one was used by PT1 & PT2 and PT3 & PT4 and the second one was used by PT7 & PT8. More specifically, PT1 & PT2 rotated the first triangle 180 degrees around point O and obtained the second triangle. Then, they rotated the second triangle 180 degrees around point K and they obtained the third triangle. In other words, they performed 180 degrees rotations successively to obtain the whole frieze pattern. PT7 & PT8 used rotation and translation to obtain the frieze pattern. Similar to PT1 & PT2 and PT3 & PT4, they obtained the second triangle by performing 180 degrees rotation around the same point. However, they considered these first two triangles a unit and then translated them together six times and obtained the frieze pattern.

The incomplete solution method used by PT15 & PT16 as a response to Task 5 of Activity 2 is presented in Table 4.14.

Table 4.14. PT15 & PT16's response to Task 5 of Activity 2



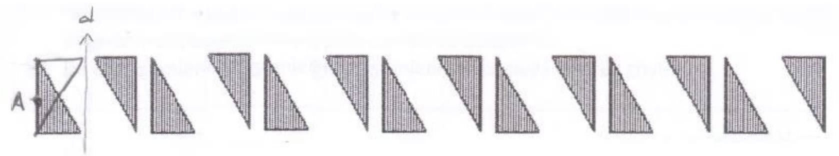
PT15 & PT16's response to Task 5	1. şekil önce 180° döndürülmüş ardından sağa doğru ötelenmiştir. ve 2. şekil oluşturulmuştur. 2. şekil sadece 180° döndürülerek 3. şekil oluşturulmuştur. 3. şekil önce 180° döndürülmüş ve ardından sağa doğru ötelenmiş ve 4. şekil oluşmuştur. 4. şekil sadece 180° döndürülerek 5. şekil oluşmuştur. A merkezinde veya kırmızı işaretli birim sadece sağa doğru ötelenme ile örüntü oluşturulmuştur. [The first figure was initially rotated 180° and then translated to the right and thus the second figure was obtained. The third figure was obtained by only 180° rotation of the second figure. The third figure was initially rotated 180° and then translated to the right and thus the fourth figure was obtained. The fifth figure was obtained by only 180° rotation of the fourth figure or the unit specified by the red circle was translated to the right.]

As seen in Table 4.14, PT15 & PT16 rotated the first triangle 180 degrees around point A (a point on the hypotenuse of the triangle). They obtained a triangle whose hypotenuse coincided with the hypotenuse of the first triangle. Since they rotated the first triangle around a point on this triangle, the obtained triangle did not coincide with the second triangle. In order to obtain the second triangle, they translated the obtained triangle to the right. Therefore, they had to translate the obtained triangle to the right. After this point, they proposed two different methods. In the first one, they continued with successive rotations. They stated that they rotated the second triangle 180 degrees (without indicating the center of rotation) and obtained the third triangle. Similar to the first one, the third triangle was rotated and then translated to obtain the fourth triangle. The interesting point in PT15 & PT16's response was that they could not identify a single rotation but they could provide a correct response by performing a rotation whose center was on the figure and a translation successively. In their second method, they considered the first two triangles as a unit and translated the unit successively to obtain the whole frieze pattern.

PT9 & PT10 and PT13 & PT14 stated that they obtained the frieze pattern by rotating the triangle 180 degrees successively. PT9 & PT10's center of rotation was at the lower-right corner of the first triangle and PT13 & PT14's center of rotation was at the gravity center of the first triangle. However, selecting these points as a center of rotation and rotating the first triangle 180 degrees does not yield the second triangle and these two pairs might have not been aware of this issue. Another pair, PT5 & PT6 stated that they obtained the frieze pattern by rotating the triangle 180 degrees but they could not indicate the center of rotation.

As mentioned before, PT11 & PT12 provided an incorrect response for Task 5 of Activity 2. This incorrect solution method is presented in Table 4.15.

Table 4.15. PT11 & PT12's response to Task 5 of Activity 2

	
PT11 & PT12's response to Task 5	1. Üçgene A noktasına göre 180° dönme dönüşümü uygulanmıştır. 2. Oluşan üçgene d doğrusuna göre yansıma dönüşümü uygulanmış 3. Son olarakta ortaya çıkan 2 üçgene her defasında 2 cm olmak üzere 6 defa öteleme dönüşümü uygulanmıştır. [1. The triangle was rotated 180° around point A. 2. The resultant triangle was reflected along line d. 3. Finally, the first and the resultant triangles were translated 2 cm six times.]

As seen in Table 4.15, PT11 & PT12 stated that they rotated the first triangle 180 degrees around point A and then reflected the obtained triangle and finally translated the two triangles six times. Their solution was regarded incorrect. Although they stated that there was a 180 degrees rotation, they could not perform the 180 degrees rotation correctly. Therefore, they obtained a different triangle and their solution method had different steps than the previous methods. In other words, they used reflection to obtain the second triangle in addition to rotation. Finally, they considered the first and the second triangles as a unit and translated the unit successively to obtain the whole frieze pattern.

In whole class discussion, the researcher asked the participants to explain which transformations were used to construct the given frieze pattern. PT9 answered as translation and rotation. Then, the researcher requested one of the participants to show the transformations on the board. The following dialogue appeared between the researcher and the participants during whole class discussion:

> PT15: 180 degrees rotation around this point (See in Table 4.14)
> Researcher: Do any of you have an idea about your friend's rotation?
> PT13: We rotated the first triangle around its gravity center. Both our solution and PT15's solution are insufficient in obtaining the second triangle in its true position. After rotation, we need a translation to obtain the second triangle given in the frieze pattern.
> Researcher: Well, could any of you obtain the second triangle by performing only one rotation on the first triangle? (PT1 & PT2 and PT3 & PT4 raised their fingers to answer this task) Come on PT3, show your answer on the board!
> PT3: We chose the point between the first and the second triangles. Then, we rotated the first triangle 180 degrees around this point. After obtaining the second figure, we can use rotation to obtain the third triangle (see in Table 4.13) or translate the first and the second triangle concurrently to obtain the third and the fourth triangle.

In the above given dialogue, PT15 explained their solution method presented in Table 4.14. Since PT15 selected the midpoint of the hypotenuse of the first triangle as the center of rotation, the hypotenuse of the resulting triangle coincided with the hypotenuse of the first triangle. Thus, PT15 could not generate the frieze pattern correctly due to determining the center of rotation incorrectly. Then PT13 stated that both their own and PT15's solution methods were insufficient in obtaining the second triangle. PT13's explanation was valid but there did not exist such an explanation in PT13 & PT14's activity sheet. It seems PT13 realized that their response (i.e., 180 degrees rotation around the gravity center of the first triangle) was insufficient to obtain the frieze pattern when he saw PT15's insufficient solution method during whole class discussion. After PT13 shared her idea, PT3 explained the single rotation that was applied to the first triangle to obtain the second triangle. Different from their activity sheet, PT3 also explained that the first two triangles could be translated together to generate the frieze pattern.

The critical issue in prospective teachers' responses to Task 5 of Activity 2 was their inability to identify the center of rotation. Although six pairs (i.e., all pairs except for PT7 & PT8 and PT15 & PT16) were able to identify the center of rotation in tasks included in Activity 1, only three pairs were able to identify the center of rotation in Task 5 of Activity 2. The decrease in identification of center of rotation might be related to the location of the center of rotation. That is, in Activity 1 all centers of rotations were on the figures but in Activity 2 the center of rotation was outside the figure. Identification of centers of rotations located outside the figure might be more difficult than the ones located on the figure. On the other hand, PT7 & PT8's inability in identification of the center of rotation in Activity 1 was resolved in Activity 2. However, PT15 & PT16 could not identify the center of rotation in Activity 2 as well.

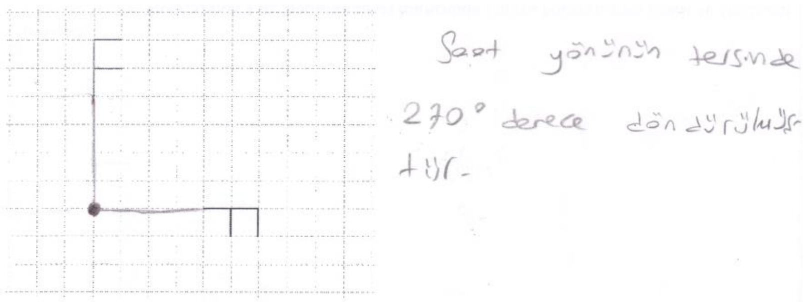
4.2.2.3. Rotations on a grid paper

In Task 1 of Activity 3, prospective teachers were asked to identify the transformations on a grid paper. In Task 1C, rotation was applied to the letter 'F' and the participants were asked to identify this rotation and explain it with its parameter. Prospective teachers were provided rulers and protractors. Based on the data obtained through activity sheets and pairwise discussions regarding Task 1C of Activity 3, the prospective teachers' identification of rotation is presented Table 4.16. All pairs could identify the rotation in Task 1C, whereas two pairs could not identify the parameters. Namely, PT9 & PT10 and PT15 & PT16 could not identify the center of rotation and angle of rotation.

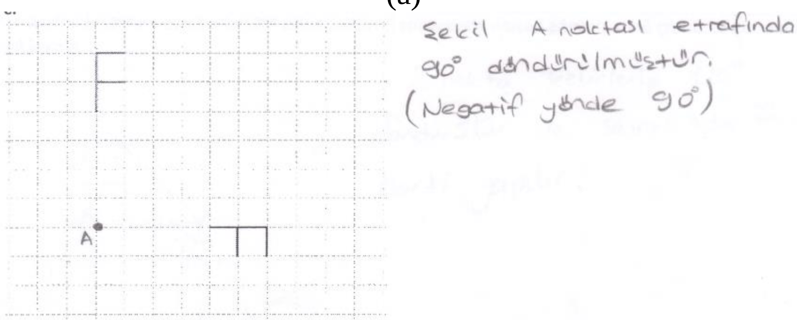
Table 4.16. Prospective teachers’ identification of rotation in Task 1C of Activity 3

Pairs	Task 1C		
	Rot.	C	A
PT1 & PT2	✓	✓	✓
PT3 & PT4	✓	✓	✓
PT5 & PT6	✓	✓	✓
PT7 & PT8	✓	✓	✓
PT9 & PT10	✓		
PT11 & PT12	✓	✓	✓
PT13 & PT14	✓	✓	✓
PT15 & PT16	✓		

Prospective teachers’ sample responses to Task 1C of Activity 3 are presented in Figure 4.8.



(a)

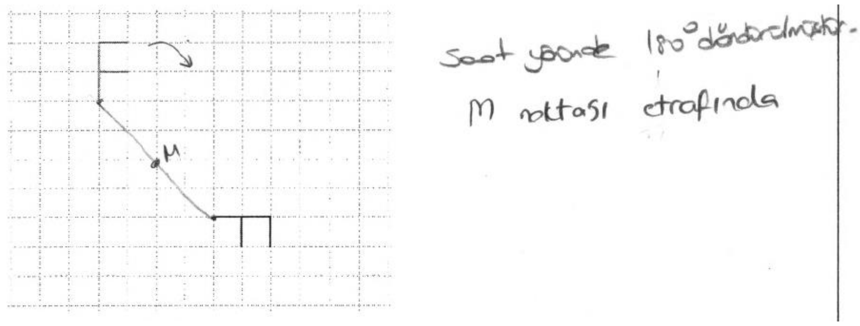


(b)

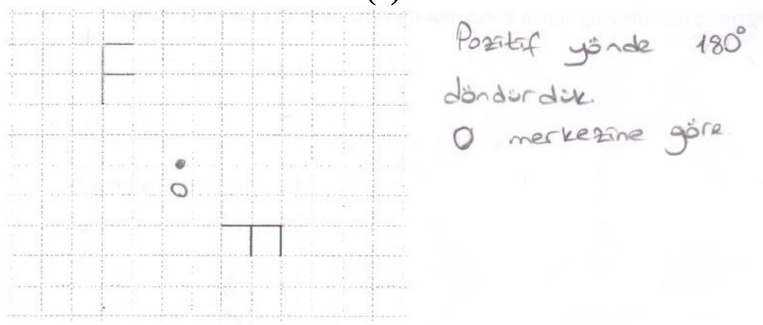
Figure 4.8. PT5 & PT6’s (a) and PT11 & PT12’s (b) responses to Task 1C of Activity 3

As seen in Figure 4.8, PT5 & PT6 stated that “there was a 270 degrees rotation in counterclockwise direction” and PT11 & PT12 stated that “there was a 90 degrees rotation around point A in negative direction”. The difference between these responses arises from considering different Fs as a pre-image. Although PT5 & PT6 did not include the center of rotation in their explanation, they constructed a huge point on the grid to point to the center of rotation. Therefore, both responses were accepted as correct.

As mentioned before, PT9 & PT10 and PT15 & PT16 could not identify the center of rotation and angle of rotation in Task 1C of Activity 3. Their incorrect solution methods are presented in Figure 4.9.



(a)



(b)

Figure 4.9. PT9 & PT10’s (a) and PT15 & PT16’s responses to Task 1C of Activity 3

As seen in Figure 4.9, PT9 & PT10 stated that “there was a 180 degrees rotation around point M in clockwise direction” and PT15 & PT16 stated that “there was 180 degrees rotation around point O in positive direction”. However, their responses were incorrect.

After pairwise discussion of Task 1C, the researcher asked all pairs to state which transformations they identified in this sub-task. All pairs stated that rotation was applied to F. Besides, researcher asked them to explain how they identified the rotation. PT3 was volunteered to come to the board and explained how she did it. She extended the lower parts of the two Fs and the extended lines intersected at a point which she named as the center of rotation (similar to PT5 & PT6’s responses in Figure 4.9). Meanwhile, she stated that the angle measure between the two extended lines was 90° and named it as the angle of rotation. PT3 correctly identified the center of rotation in Task 1C. However, her solution method was unique to this task and therefore it was not generalizable to the other rotation tasks involving identification of center of rotation. Namely, she could not have found the center of rotation by extending the lower parts of the two Fs, if it had been at a different point.

I asked the prospective teachers to articulate whether any of them solved Task 1C by using another method different from that of PT3. A different solution was proposed by PT9. She came to the board and demonstrated the center of rotation and angle of rotation which she identified together with PT10 during pair work (see Figure 4.10). PT9 stated that “the center of rotation should be equidistant from F to F' ”. Although PT9’s statement was correct, PT9 & PT10 solved Task 1C incorrectly in pairwise and whole class discussions. As soon as PT9 explained their response to the class, many of the prospective teachers argued that the center of rotation determined by them was wrong. Besides, PT3 explained that their center of rotation is located at a point which is the same distance from both F and F' for all points and it was correct according to PT9’s statement. Other pairs also stated that the point identified by PT3 satisfied this condition and that they did not agree with PT9’s response. It seemed that PT9 & PT10 considered only one point on F while finding the center of rotation and that they did not check the truth of their answer for other points on F.

It is important to note that although only three pairs were able to identify the center of rotation in Task 5 of Activity 2, six pairs were able to identify the center of rotation in Task 1B of Activity 3. The increase in prospective teachers’ identification of center of rotation might be related to whole class discussions carried out in Activity 2. That is, they might have learnt the critical points in identification of the center of rotation. However, PT15 & PT16 and PT9 & PT10 still could not identify the center of rotation in Activity 3. Thus, it can be said that these two pairs were still unable to identify rotation after the implementation of Activity 3.

4.2.3. Identifying Translations

Prospective teachers' identification of translation was determined via tasks that ask them to identify geometric transformations in two different contexts. Namely, findings related to prospective teachers' identification of translations in the frieze patterns with obvious units and translations on a grid paper are presented respectively. Prospective teachers' identification of translation in the frieze patterns with obvious units are presented below.

4.2.3.1. Translations in the frieze patterns with obvious units

Prospective teachers were asked to identify the transformations included in the frieze patterns whose units were all right triangles in Task 1 and Task 3 of Activity 2. Prospective teachers' identification of translations are presented in Table 4.17 based on the data obtained through activity sheets and pairwise discussions regarding these tasks.

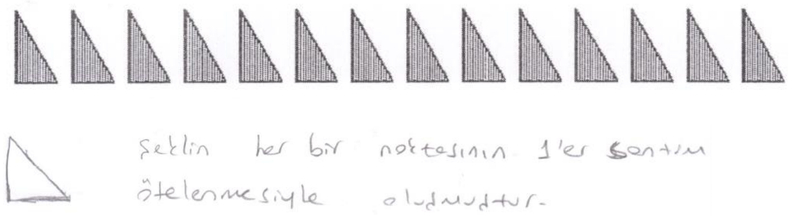
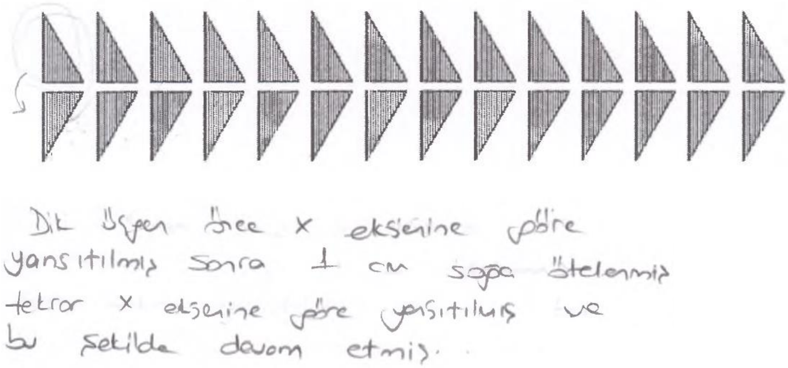
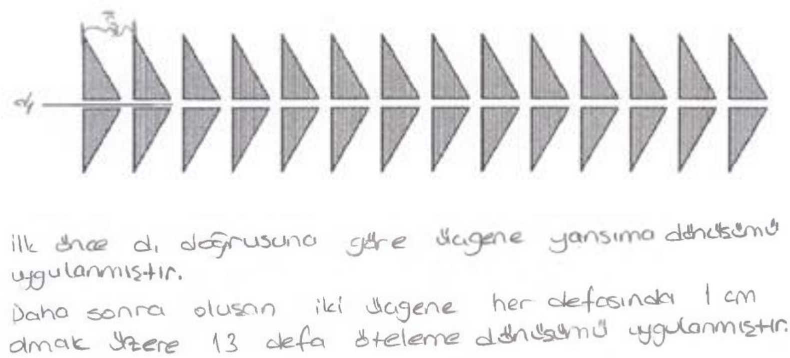
Table 4.17. Prospective teachers' identification of translations in the frieze patterns in Task 1 and Task 3 of Activity 2

Pairs	Task 1		Task 3	
	Trans.	$\vec{v}$	Trans.	$\vec{v}$
PT1 & PT2	✓		✓	
PT3 & PT4	✓		✓	
PT5 & PT6	✓		✓	
PT7 & PT8	✓		✓	
PT9 & PT10	✓		✓	✓
PT11 & PT12	✓		✓	
PT13 & PT14	✓	✓	✓	
PT15 & PT16	✓		✓	

Note: PT: Prospective teacher, Trans: Translation, v: Translation vector

As can be seen in Table 4.17, all pairs were able to identify translations in Task 1 and Task 3. Among all pairs, only PT9 & PT10 and PT13 & PT14 were able to identify the translation vector only in Task 3 and Task 1 respectively. Examples of prospective teachers' responses to Task 1 and Task 3 of Activity 2 are presented in Table 4.18.

Table 4.18. Prospective teachers' sample responses to Task 1 and Task 3 of Activity 2

PT5 & PT6's response to Task 1	 [Each point of the figure was translated 1 cm.]
PT9 & PT10's response to Task 3	 [The right triangle was initially reflected along x-asis and then traslated 1 cm to the right. Then reflected along x-axsis again. This continued in this way.]
PT11 & PT12's response to Task 3	 [First, the triangle was reflected along d_1. Then, the first and the resultant the triangles were translated 1 cm each time and this was repeated thirteen times.]

When participants' responses to Task 3 were examined, it was seen that there were two different solution methods. One of them was used by PT11 & PT12. As seen in Table 4.18, they explained that they reflected the first triangle horizontally, and later that they translated the first triangle and the resultant triangle concurrently and successively to obtain the frieze pattern. PT9 & PT10 reflected the first triangle along x-axis as well but they continued differently. They explained that they translated only the obtained triangle 1 cm to the right and then reflected the obtained triangle along x-axis. They stated that they obtained the first four triangles by this method and that their frieze pattern continued like this.

As mentioned before, only PT9 & PT10 and PT13 & PT14 were able to identify the translation vector only in Task 3 and Task 1 respectively. PT13 & PT14 wrote the following statement: "*a* units in positive direction of *x* axis" for Task 1. Since, this statement includes the meaning of the term 'vector', magnitude and direction, it was accepted as the translation vector. All pairs except for PT9 & PT10 and PT13 & PT14 wrote one of the following statements: "translation", "translation to the right", and "1 cm translation" for these identification of translation tasks. Since these statements were not sufficient for the meaning of the term vector, they were not accepted as correct identifications. The remaining identification of translation tasks were evaluated in the same manner.

After prospective teachers worked in pairs and completed their activity sheets, they discussed their responses with other pairs. The researcher asked each pair how they responded to Task 1. All pairs answered as translation. As the concept of translation was not covered in the first activity, the researcher aimed at drawing participants' attention to the definition and parameter of translation. During the classroom discussion of Task 1, the following dialogue took place:

Researcher: Well, how was this translation applied?
PT9: The right triangle was translated 1 centimeter. (In activity sheet, the distance between corresponding points on the two right triangles is 1 centimeter)
Researcher: Well, who wants to define translation?
PT13: Moving a shape or a point along a specified direction and distance.
Researcher: Then, what is needed to apply a translation?
PT4: Distance
PT13: Direction

Researcher: Is there a concept that encompasses distance and direction?
PT13: It must be a vector.
Researcher: Well, what is a vector?
PT6: A line segment with a magnitude and direction.
Researcher: Ok, then is it sufficient to determine a translation vector to apply translation?
Students: Yes!

As mentioned before PT13 & PT14 was the only pair who was able to identify the translation vector as the parameter in Task 1 (see Table 4.17). As understood from the dialogue, PT13's knowledge of translation vector had been influential in the emergence of the aforementioned dialogue.

4.2.3.2. Translations on a grid paper

In Task 1A of Activity 3, translation was applied to the letter 'F' and the participants were asked to identify this transformation and explain it with its parameter. Prospective teachers were provided rulers and protractors. Based on the data obtained through activity sheets and pairwise discussions regarding Task 1A of Activity 3, prospective teachers' identification of reflections is presented Table 4.19.

Table 4.19. Prospective teachers' identification of translation in Task 1A of Activity 3

Pairs	Task 1A	
	Trans.	$\vec{v}$
PT1 & PT2	✓	✓
PT3 & PT4	✓	✓
PT5 & PT6	✓	✓
PT7 & PT8	✓	✓
PT9 & PT10	✓	✓
PT11 & PT12	✓	✓
PT13 & PT14	✓	✓
PT15 & PT16	✓	✓

As can be seen in Table 4.19, all pairs could identify the translation and its vector correctly in Task 1A. As an example, PT15 & PT16's response to Task 1A is presented in Figure 4.10.

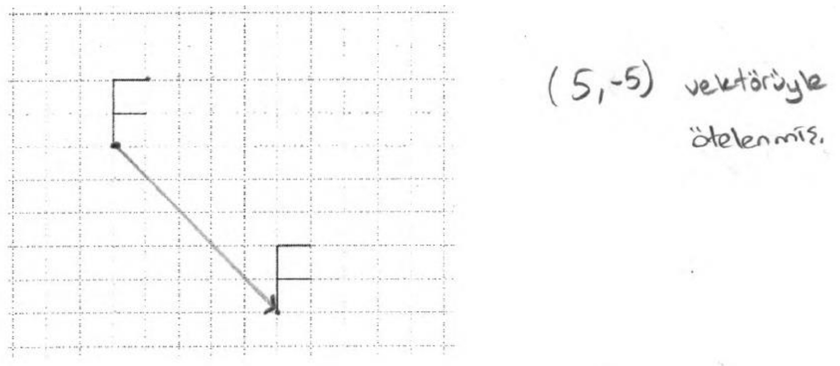


Figure 4.10. PT15 & PT16's response to Task 1A of Activity 3

As seen in Figure 4.10, PT15 & PT16 stated that "translation was applied with vector (5, -5)". After the discussion of Task 1A in pair work, the researcher asked all pairs to state which transformations they identified in this task. All pairs stated that translation was used in Task 1A. The researcher requested P15 to come to the board and explain her answer to Task 1A. PT15's explanation for Task 1A was as follows:

> PT15: We said translation because there is no change in the orientation and slope of the F. I selected a point in this way. I treated each grid as one unit. I am going to translate as a vector. I see that there are 5 grids in *x*-axis. Thus, *x* is equal to 5. By looking down from the same point, I see that there are also 5 grids. Since it is downwards, *y* is equal to -5. Hence, we said that the F shape was translated via the vector (5,-5).

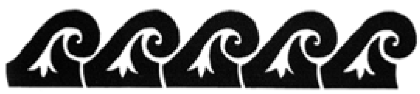
It is important to note that only PT9 & PT10 was able to identify the translation vector in Task 3 of Activity 2. Similarly, only PT13 & PT14 was able to identify the translation vector in Task 1 of Activity 2. Notably, all pairs could identify the translation and its vector correctly in Task 1A of Activity 3. Whole class discussions related to translation and translation vector in Activity 2 might have helped the prospective teachers in solving this task.

4.2.4. Identifying Reflections, Rotations, and Translations

To explore participants' understandings of transformations in greater depth, the researcher allocated some more time on identifying transformations. Namely, this part focused on identifying transformations in the frieze patterns whose units could not be seen easily at first glance. The data came from Task 6 and Task 7 of Activity 2.

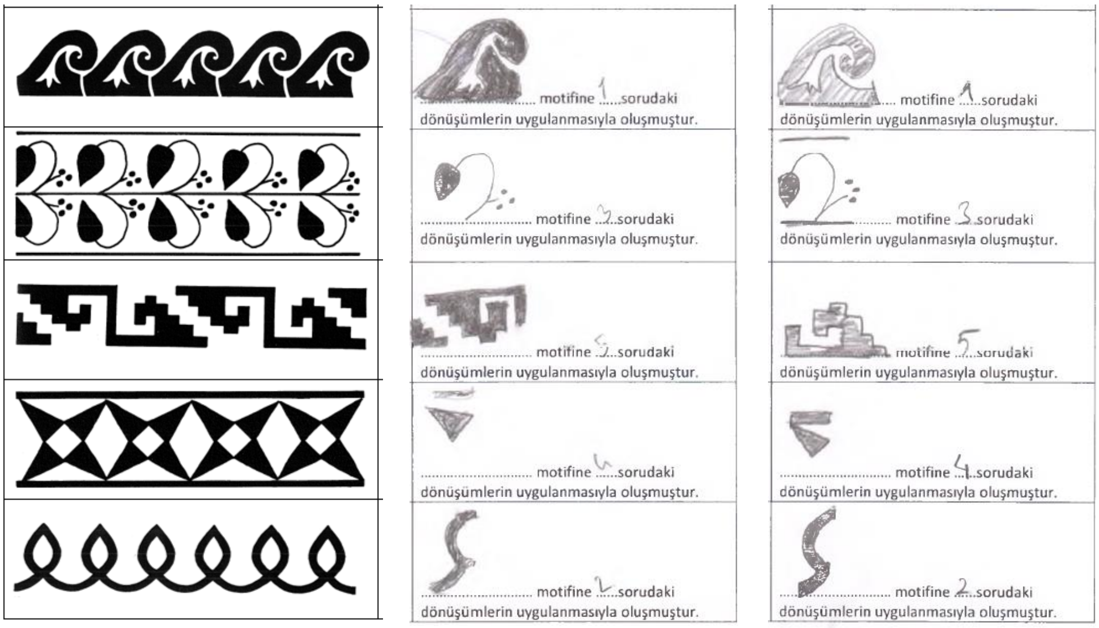
Task 6 included five different frieze patterns (see Appendix C). In this task, the participants were asked to identify the transformations included in the frieze patterns, identify the unit of the frieze patterns and match these frieze patterns with the frieze patterns included in Task 1 through 5 of Activity 2 by considering the type of transformations used to construct those patterns. A sample response that shows the matching between two frieze patterns and the transformation(s) used in these frieze patterns are presented in Table 4.20.

Table 4.20. Matching between two frieze patterns included in Task 6 of Activity 2

The frieze patterns examined before Task 6	The corresponding frieze patterns in Task 6	The transformation(s) used
		Translation

The two frieze patterns presented in Table 4.20 were constructed by using translations. Other frieze patterns examined in Task 1 through Task 5, the corresponding frieze patterns given in Task 6, and the transformations used to construct these frieze patterns are presented in Appendix C. When participants' responses to Task 6 were examined it was seen that all pairs were able to identify the transformations used and match frieze patterns correctly.

Task 6 asked prospective teachers to identify the units of the given frieze pattern in addition to having them identify the type of transformation used. All pairs were able to draw the units of the first, fourth and fifth frieze patterns correctly. However, when their drawings for the second and third frieze pattern were examined, it was seen that PT5 & PT6, PT7 & PT8, and PT15 & PT16 failed to draw some necessary parts of the unit of the second frieze pattern, while PT5 & PT6 and PT7 & PT8 included also unnecessary parts to the unit of the third frieze pattern. To illustrate, correct units drawn by PT11 & PT12, the missing unit drawn for the second frieze pattern and the unnecessary unit drawn for the third frieze pattern by PT7 & PT8 for Task 6 are presented in Figure 4.11.



(a) (b) (c)

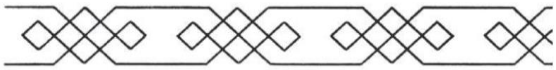
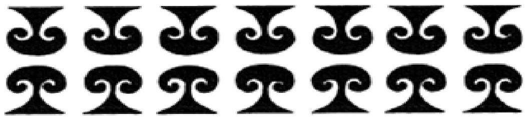
Figure 4.11. Frieze patterns given in Task 6 (a), PT7 & PT8’ s (b) and PT11 & PT12’ s responses (c) to Task 6

As seen in Figure 4.11, PT11 & PT12 identified all units correctly. When two pairs’ responses for the second frieze pattern were compared, it can be seen that PT7 & PT8 failed to include the lines over and under the unit which were necessary for obtaining a pattern identical to the given pattern. Meanwhile, PT5 & PT6 and PT15 & PT16 also constructed the same unit with PT7 & PT8.

As mentioned before, PT5 & PT6 and PT7 & PT8 were not able to identify the smallest unit of the third frieze pattern. That is, they drew some unnecessary parts in addition to the desired unit as in the drawing of PT7 & PT8 in Figure 4.11. This issue becomes more apparent when the drawings of PT7 & PT8 and PT11 & PT12 are compared (see Figure 4.11).

In Task 7 of Activity 2, prospective teachers examined frieze patterns different from the ones they examined before in Task 1 thorough 6. In Task 7, each pair randomly selected one pattern from a bag of eight frieze patterns. The participants were expected to examine their own pattern and identify the geometric transformations included in this frieze pattern. As in Task 6, they were expected to match their own frieze pattern with the frieze patterns included in Task 1 through 5 by considering the type of transformations used. The eight frieze patterns were projected onto the smart board by the researcher. After each pair examined their own frieze pattern, participants were requested to come to the board and write the type of the frieze pattern number next to selected frieze pattern. The frieze patterns given in Task 7 and pairs' responses to this task are presented in Table 4.21.

Table 4.21. Frieze patterns given in Task 7 and prospective teachers' responses

Pattern #	Randomly selected patterns	Pairs	Response to Task 7
1		PT1 & PT2	1
2		PT7 & PT8	2
3		PT3 & PT4	3
		PT11 & PT12	3
4		PT13 & PT14	4
		PT15 & PT16	3
5		PT9 & PT10	3
		PT5 & PT6	1

As seen in Table 4.21, the translation included in the first frieze pattern was identified correctly by PT1 & PT2. The vertical reflection and translation included in the second frieze pattern were identified correctly by PT7 & PT8. The horizontal reflections and translations included in the following two frieze patterns were identified both by PT3 & PT4 and PT11 & PT12 correctly. Finally, PT13 & PT14 identified the vertical and horizontal reflections and translation correctly. However, three out of eight pairs (i.e., PT5 & PT6, PT9 & PT10, and PT15 & PT16) were not able to match the frieze patterns correctly.

PT5 & PT6 and PT9 & PT10 could not correctly identify the 180 degrees rotation and translation included in their own frieze patterns. Namely, PT5 & PT6 claimed that their own frieze pattern included only translations and PT9 & PT10 claimed that their own frieze pattern included horizontal reflections and translations.

After prospective teachers matched all frieze patterns with the relevant geometric transformations on the board, the researcher requested the class to examine whether their classmates' matchings were true or false. Next, the researcher asked the participants to explain how these frieze patterns were obtained by the geometric transformations and to identify the unit of each frieze pattern. By this way, the participants had the chance to examine other frieze patterns in addition to their own frieze pattern.

During whole class discussions, prospective teachers discussed only the units which were not identified correctly by some pairs. The last two frieze patterns given in Table 4.21 can be obtained correctly by 180 degrees rotation and translation. PT5 & PT6 gave an incorrect answer due to not being able to identify the smallest unit of their frieze pattern. Namely, since they accepted the 180 degrees rotated form of the actual unit as their own unit, they could only identify translation and answered as 1. The correct unit was explained by other prospective teachers.

Similar to PT5 & PT6, PT15 & PT16 identified a larger unit than the actual one for the sixth frieze pattern. Actually, the half part of the single mushroom in the sixth frieze pattern is the correct unit. The sixth frieze pattern is obtained correctly if this half part is first reflected vertically (i.e., one mushroom is obtained) and the obtained figure is reflected horizontally and finally the resultant figure (i.e., two mushrooms) is translated repeatedly. In this frieze pattern, PT15 & PT16 incorrectly identified the unit due to ignoring vertical reflection by choosing the mushroom as the unit of the pattern. PT9 & PT10 misidentified the transformations and they did not suggest any justification for their answers during whole class discussion.

In general, it was found out that most of the prospective teachers could identify the transformations used in the frieze patterns with unobvious units given in Task 6 and Task 7. In more detail, all prospective teachers were able to identify the geometric transformations included in the frieze patterns given in Task 6. However, PT5 & PT6, PT7 & PT8, and PT15 & PT16 had difficulty in identifying the smallest units on which the geometric transformations were applied to construct the frieze patterns in this task. When prospective teachers' responses to Task 7 were examined, it was seen that three pairs (i.e., PT5 & PT6, PT9 & PT10, and PT15 & PT16) could not identify the geometric transformations. However, in whole class discussion it was understood that PT5 & PT6 and PT15 & PT16 identified a larger unit than the actual one for the sixth frieze pattern and that they could not identify the transformations completely. Meanwhile, PT9 & PT10 could not identify the rotation and they did not suggest any justification for their answers during whole class discussion. PT9 & PT10's inability to identify the rotation might be related to the complexity of their frieze pattern. That is, it can be said that the two frieze patterns including rotations might have been more difficult for them than the other frieze patterns presented in Task 7.

4.2.5. Summary of Prospective Teachers' Identification of Geometric Transformations

To date, findings related to prospective teachers' identification of transformations in Activity 1, 2, and 3 were presented. Prospective teachers identified geometric transformations in finite figures with unobvious units, frieze patterns with obvious and unobvious units, and on a grid paper. Since identification of geometric transformations involves identification of parameters, the prospective teachers identified them as well. In addition, prospective teachers were administered tasks that asked them to identify the units of the finite figures and frieze patterns and thereby they developed their understanding of geometric transformations and their parameters to a large extent.

For all identification tasks, the prospective teachers first worked in pairs and then they discussed each task in whole class discussions. During whole class discussions, all prospective teachers had the opportunity to share their ideas. Correct and incorrect responses were discussed thoroughly. Therefore, at the end of whole class discussions, all prospective teachers had an opportunity to see the correct answer of each task. Findings related to prospective teachers' identification of transformations in Activity 1, 2, and 3 are summarized in Table 4.22 and Table 4.23. In these tables, a defined transformation meant participants could identify the transformation and its all parameters. For instance, defined reflection meant participants could identify the reflection and its all reflection lines and undefined reflection meant reflection was identified but none of the parameters was identified. Partially defined meant that the reflection was identified and some of the parameters were identified but some of them were not.

Table 4.22. Summary of identification of geometric transformations

		A1 T1	A1 T3	A2 T2	A2 T3	A2 T4	A3 T1B	Total
Reflection	Defined Reflection	8	1	7	7	8	5	36
	Partially Defined Reflection	-	6	1	-	-	1	8
	Undefined Reflection	-	-	-	1	-	-	1
	Not identified	-	1	-	-	-	2	3

		A1 T2	A1 T4	A1 T5	A2 T5	A3 T1C	Total
Rotation	Defined Rotation	6	3	3	3	6	21
	Partially Defined Rotation	2	4	5	4	-	15
	Undefined Rotation	-	1	-	-	2	3
	Not identified	-	-	-	1	-	1

		A2 T1	A2 T3	A3 T1A	Total
Translation	Defined Translation	1	1	8	10
	Partially Defined Translation	-	-	-	-
	Undefined Translation	7	7	-	14
	Not identified	-	-	-	-

Table 4.23. Summary of prospective teachers' identification of geometric transformations

	Reflection						Rotation					Translation		
	A1 T1	A1 T3	A2 T2	A2 T3	A2 T4	A3 T1B	A1 T2	A1 T4	A1 T5	A2 T5	A3 T1C	A2 T1	A2 T3	A3 T1A
PT1 & PT2	DRe	DRe	DRe	DRe	DRe	DRe	DRo	DRo	DRo	DRo	DRo	UT	UT	DT
PT3 & PT4	DRe	pDRe	DRe	DRe	DRe	DRe	DRo	pDRo	pDRo	DRo	DRo	UT	UT	DT
PT5 & PT6	DRe	pDRe	pDRe	URe	DRe	URe	pDRo	pDRo	pDRo	pDRo	DRo	UT	UT	DT
PT7 & PT8	DRe	-	DRe	DRe	DRe	-	pDRo	URo	pDRo	DRo	DRo	UT	UT	DT
PT9 & PT10	DRe	pDRe	DRe	DRe	DRe	DRe	DRo	DRo	pDRo	pDRo	URo	UT	DT	DT
PT11 & PT12	DRe	pDRe	DRe	DRe	DRe	DRe	DRo	DRo	DRo	-	DRo	UT	UT	DT
PT13 & PT14	DRe	pDRe	DRe	DRe	DRe	DRe	DRo	pDRo	DRo	pDRo	DRo	DT	UT	DT
PT15 & PT16	DRe	pDRe	DRe	DRe	DRe	-	DRo	pDRo	pDRo	pDRo	URo	UT	UT	DT

Note: A1 T1: Task 1 of Activity 1, DRe: Defined reflection, pDRe: Partially defined reflection, URe: Undefined reflection, DRo: Defined rotation, pDRo: Partially defined rotation, URo: Undefined rotation, DT: Defined translation, UT: Undefined translation

4.2.5.1. Summary of prospective teachers' identification of reflections

In the identification of reflection part, each pair of prospective teachers were administered six tasks and were asked to identify six reflections and the parameters of these reflections. Namely, prospective teachers were expected to define these reflections with their reflection lines. Since there were eight pairs and six tasks, all pairs were expected to define 48 reflections in total. As seen in Table 4.22, total number of defined reflections was 36. This shows that most of the prospective teachers could define the reflections with their all parameters (i.e., all reflection lines that the given figure has). Besides, total number of partially defined reflections was 8. This shows that one sixth of the pairs could define the reflections with some of the reflection lines. Furthermore, 1 out of 48 reflections was identified but it was not defined with any reflection line. On the other hand, 3 out of 48 reflections were not identified.

As seen in Table 4.22, the reflections and all their reflection lines in four tasks (i.e., Task 1 of Activity 1, Task 2 of Activity 2, Task 3 of Activity 2, and Task 4 of Activity 2) were identified by either seven or eight pairs. However, prospective teachers had difficulty in identifying all parameters in Task 3 of Activity 1. More precisely, only one pair defined reflection with its all reflection lines in this task. To state differently, six pairs defined the reflection partially and one pair could even not identify the reflection itself in this task. Meanwhile, the reflection in Task 1B of Activity 3 was defined with all reflection lines by five pairs. One pair defined the reflection partially and two pairs could even not identify the reflection itself in Task 1B of Activity 3.

Pair by pair examination of prospective teachers' identification of reflections resulted in three categories. As mentioned before, there were altogether six reflection tasks. The first category comprised one pair, namely PT1 & PT2, and this pair could identify all reflections and all parameters included in these six tasks. Second category comprised PT3 & PT4, PT9 & PT10, PT11 & PT12, and PT13 & PT14. These four pairs defined five reflections with all reflection lines and one reflection with some of the reflection lines. It is important to note that these four pairs had difficulty in the same task, namely in Task 3 of Activity 1. Actually, except for PT1 & PT2, all pairs had difficulty in identifying the four reflection lines included in this task. The third category comprised three pairs (i.e., PT5 & PT6, PT7 & PT8, and PT15 & PT16). These pairs had difficulty in other tasks in addition to Task 3 of Activity 1. PT5 & PT6, PT7 & PT8, and PT15 & PT16 had difficulty in identifying the reflection in Task 1B of Activity 3.

When we turn back to the four pairs (i.e., PT3 & PT4, PT9 & PT10, PT11 & PT12, and PT13 & PT14) who had difficulty only in Task 3 of Activity 1, it can be said that they might have resolved this difficulty since they could solve all remaining identification of reflection tasks. In whole class discussion related to this task, the diagonal reflection lines and the idea of the smallest unit were mentioned. Therefore, the ideas they gained in whole class discussion might have helped them in other tasks.

Prospective teachers' difficulty in Task 3 of Activity 1 might be related to the type and number of the reflection lines. First, the two figures included in this task have a horizontal, a vertical and two diagonal reflection lines. The diagonal reflection lines might have been more difficult to identify for prospective teachers. The second reason might be related to the number of reflection lines. The prospective teachers could construct the finite figure by performing one reflection when they considered half of the figure as a unit. Similarly, they could construct the finite figure by performing two reflections when they considered the quarter of the finite figure as a unit. After identifying one or two of these reflection lines, prospective teachers might have not found it necessary to find other ways to construct the finite figure and consequently they might not have found it necessary to identify other reflection lines as well.

Finally, PT5 & PT6, PT7 & PT8, and PT15 & PT16' difficulty in Task 1B of Activity 3 might be related to the characteristics and difficulty level of tasks. It is important to note that six identification of reflection tasks have different characteristics and difficulty level. In other words, the first task in the first activity is simpler than the task in the third activity. Besides, there were vertical, horizontal, and diagonal reflections and some of these reflection lines passed inside the figure and some others passed outside the figure. The last task was presented on a grid and it included a diagonal reflection. The pairs in the third category were able to identify most of the reflections given in the first and second activities while they had trouble in identifying the reflection given in the third activity. This indicates that the reflection task given the third activity might have been more difficult for prospective teachers than the other two tasks given in the first and second activities. Therefore, pairs' failure to identify the reflection task given in the third activity does not mean that they could not make progress in identifying reflections during the first three weeks of the instructional unit.

4.2.5.2. Summary of prospective teachers' identification of rotation

In the identification of rotation part, each pair of prospective teachers were administered five rotation tasks and they were asked to identify these rotations and their parameters (i.e., the center of rotations and the angle of rotations). In other words, prospective teachers were expected to define these rotations with centers of rotations and angles of rotations. As seen in Table 4.22, total number of defined rotations was 21. This meant that half of the rotations were defined with both parameters since there were eight pairs and five tasks and there were 40 rotations in total. Besides, the total number of partially defined rotations was 15. This meant that nearly half of the rotations were defined either with an angle of rotation or a center of rotation. Therefore, it can be said that most of the prospective teachers could identify the rotations and could identify at least one of their parameter. On the other hand, three rotations were not defined with any of the parameters and one rotation was even not identified by the pairs.

As seen in Table 4.22, not even one rotation was defined with its all parameters by whole participants. In particular, the rotations included in Task 2 of Activity 1 and Task 1C of Activity 3 were defined with both parameters by six pairs and the rotations included in Task 4 and Task 5 of Activity 1 and Task 5 of Activity 2 were defined with both parameters by three pairs. Besides, the rotations in the first four tasks were partially defined by two, four, five and four pairs respectively. The rotations included in Task 4 of Activity 1 and Task 1C of Activity 3 were identified by one and two pairs respectively but none of the parameters were identified in these tasks. Finally, one pair could not identify the rotation included in Task 5 of Activity 2.

Pair by pair examination of prospective teachers' identification of rotations resulted in three categories. As mentioned before, there were altogether five rotation tasks. The first category comprised one pair, namely PT1 & PT2, and this pair could identify all rotations and all their parameters (i.e., center of rotations and angle of rotations) included in these five tasks. Second category comprised PT3 & PT4, PT11 & PT12, and PT13 & PT14. These three pairs defined at least three rotations with their all parameters. Third category comprised PT5 & PT6, PT7 & PT8, PT9 & PT10, and PT15 & PT16. These four pairs defined one or two rotations with their all parameters. It is important to note that the same pair of prospective teachers fell under reflection and rotation categorizations with one exception. Namely, PT9 & PT10 was in the second category in identification of reflection categorization and was in the third category in identification of rotation categorization.

PT11 & PT12 could define four rotations with both angle of rotation and center of rotation but they could not identify the rotation in Task 5 of Activity 2. Other two pairs in the second category (i.e., PT3 & PT4 and PT13 & PT14) partially defined two rotations. These three pairs could define the rotation in Task 1C of Activity 3. Thus, it can be said that they could develop themselves through the process of identification of rotations. This development might be related to the whole class discussions.

The four pairs in the third category (i.e., PT5 & PT6, PT7 & PT8, PT9 & PT10, and PT15 & PT16) could identify all rotations but they had difficulty in defining these rotations with their parameters. Similar to the pairs in the second category, PT5 & PT6, and PT7 & PT8 could define the rotation in Task 1C of Activity 3. Thus, it can be said that they could overcome their inability in identifying the center of rotation and angle of rotation. This development might be related to the whole class discussions. Therefore, excluding PT9 & PT10 and PT15 & PT16, all pairs were able to identify both parameters of rotation in Task 1C of Activity 3. Hence, they were able to develop their understanding of rotation and its parameters by the help of Activity 1, 2 and 3. However, PT9 & PT10, and PT15 & PT16 still could not identify the rotation in Task 1C of Activity 3. Therefore, it can be concluded that their inability in identifying rotation continued and this issue needs to be resolved in the latter activities.

4.2.5.3. Summary of prospective teachers' identification of translation

In the identification of translation part, each pair of prospective teachers were administered six translation tasks and they were asked to identify and define these translations with their translation vectors. Since there were eight pairs and three tasks, all pairs were expected to define 24 translations in total. As seen in Table 4.22, total number of defined translations was 10 and undefined translations was 14. This showed that all pairs could identify the translation. However, nearly half of these translations were defined with the translation vectors.

As seen in Table 4.22, all pairs were able to identify the translations in all tasks. However, prospective teachers had difficulty identifying the translation vectors in Task 1 and Task 3 of Activity 2. More precisely, only one pair defined the translation with its translation vector and other participants could only identify the translation itself. On the other hand, all pairs were able to define the translation with its translation vector in Task 1A of Activity 3. This showed that prospective teachers developed their understanding during the implementation of Activity 2 through Activity 3.

4.3. Performing Geometric Transformations

In this part, findings related to performing geometric transformations are presented. The data came from Task 7 of Activity 1, Task 8 and Task 9 of Activity 2, and Task 2 of Activity 3. Participants were expected to perform geometric transformations to construct their own finite figures and frieze patterns. Moreover, they performed geometric transformations on a grid paper. Based on the data obtained through individual and pairwise activity sheets, and pairwise discussions, prospective teachers' performing of geometric transformations are presented in Table 4.24.

Table 4.24. Prospective teachers' performing of geometric transformations

<table>
<tr><th rowspan="2"></th><th colspan="2">Performing transformations to construct finite figures</th><th colspan="2">Performing transformations to construct frieze patterns</th><th colspan="3">Performing geometric transformations on a grid</th></tr>
<tr><th>First finite figure</th><th>Second finite figure</th><th>First frieze pattern</th><th>Second frieze pattern</th><th>Translation</th><th>Reflection</th><th>Rotation</th></tr>
<tr><td>PT1</td><td>✓</td><td>✓</td><td>✓</td><td>✓</td><td rowspan="2">✓</td><td rowspan="2">✓</td><td rowspan="2">✓</td></tr>
<tr><td>PT2</td><td>✓</td><td>✓</td><td>✓</td><td>✓</td></tr>
<tr><td>PT3</td><td>✓</td><td>✓</td><td>✓</td><td>✓</td><td rowspan="2">✓</td><td rowspan="2">✓</td><td rowspan="2">✓</td></tr>
<tr><td>PT4</td><td>✓</td><td>✓</td><td>✓</td><td>✓</td></tr>
<tr><td>PT5</td><td>✓</td><td>✓</td><td>✓</td><td>✓</td><td rowspan="2">✓</td><td rowspan="2">✓</td><td rowspan="2">✓</td></tr>
<tr><td>PT6</td><td>✓</td><td>✓</td><td>✓</td><td>✓</td></tr>
<tr><td>PT7</td><td>✓</td><td>✓</td><td>✓</td><td>✓</td><td rowspan="2">✓</td><td rowspan="2">✓</td><td rowspan="2">✓</td></tr>
<tr><td>PT8</td><td>✓</td><td>✓</td><td>✓</td><td>✓</td></tr>
<tr><td>PT9</td><td>✓</td><td>✓</td><td>✓</td><td>✓</td><td rowspan="2">✓</td><td rowspan="2">✓</td><td rowspan="2">✓</td></tr>
<tr><td>PT10</td><td>✓</td><td>✓</td><td>✓</td><td>✓</td></tr>
<tr><td>PT11</td><td>✓</td><td>✓</td><td>✓</td><td></td><td rowspan="2">✓</td><td rowspan="2">✓</td><td rowspan="2">✓</td></tr>
<tr><td>PT12</td><td>✓</td><td>✓</td><td>✓</td><td>✓</td></tr>
<tr><td>PT13</td><td>✓</td><td>✓</td><td>✓</td><td>✓</td><td rowspan="2">✓</td><td rowspan="2">✓</td><td rowspan="2">✓</td></tr>
<tr><td>PT14</td><td>✓</td><td>✓</td><td>✓</td><td></td></tr>
<tr><td>PT15</td><td>✓</td><td>✓</td><td>✓</td><td>✓</td><td rowspan="2">✓</td><td rowspan="2">✓</td><td rowspan="2">✓</td></tr>
<tr><td>PT16</td><td>✓</td><td>✓</td><td>✓</td><td>✓</td></tr>
</table>

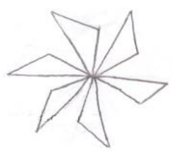
As seen in Table 4.24 all prospective teachers were able perform geometric transformations and construct finite figures and frieze patters except for PT11 and PT14. Actually, PT11 and PT14 were able to perform geometric transformations to construct frieze patterns but one of their frieze patterns did not comply with any type of frieze patterns. Moreover, all pairs were able to perform translation, reflection and rotation with given parameters on a grid. Findings related to performing geometric transformations were presented in more detail in the following sections.

4.3.1. Performing Geometric Transformations to Construct Finite Figures

In Task 7 of Activity 1, prospective teachers worked individually and they were expected to construct two finite figures. The first figure had to be obtained only by performing rotation while the second one had to be obtained by performing reflection and rotation. For both figures, they decided on the unit, the angle of rotation and the reflection lines themselves. Since they were expected to construct finite figures, the centers of the rotations were the centers of the figures themselves.

When prospective teachers' first finite figures were examined it was seen that almost all of them used 90° or 180° rotations in constructing finite figures. Besides, it was seen that one, seven, six and two of the participants used 60°, 90°, 180° and 120° rotations respectively in construction of their finite figures. In more detail, PT15 used 60° rotation, PT1, PT2, PT4, PT7, PT9, PT13, and PT14 used 90° rotation, PT6 and PT16 used 120° rotation, and PT3, PT5, PT8, PT10, PT11, and PT12 used 180° rotation to construct finite figures that were obtained merely by rotation. Examples of finite figures that were obtained merely by rotation are presented in Table 4.25.

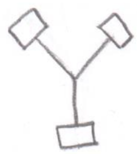
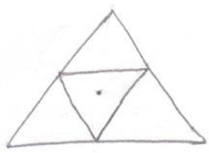
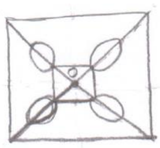
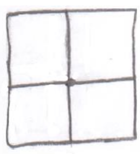
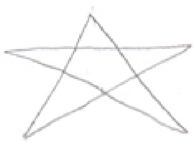
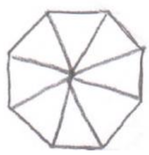
Table 4.25. Prospective teachers' sample finite figures obtained by performing only rotation

Prospective Teachers	Finite figures constucted	Transformation
PT15		60° Rotation
PT1, PT13, PT14	  	90° Rotation
PT3, PT10, PT12	  	180° Rotation

Each prospective teacher decided on a unit and an angle measure himself/herself and rotated the unit with that angle. Since the constructed figures were all finite figures, the centers of the rotations were the centers of figures themselves. For instance, see Table 4.18 for the finite figures obtained through 90° rotations (symmetry type C_4) by PT1, PT13 and PT14 and for the finite figures obtained through 180° rotations (symmetry type C_2) by PT3, PT10 and PT12. As can be seen, only PT15 constructed a figure that included 60° rotation (symmetry type C_6). In short, prospective teachers provided examples of C_2, C_4, and C_6 symmetry types.

Similar to the construction of their first finite figures, prospective teachers decided on their own units and rotated them in different angle measures or reflected them along the lines they constructed in the construction of their second finite figures. The centers of the rotations were still the centers of the figures themselves. Sample finite figures constructed by the prospective teachers through rotation and reflection are presented in Table 4.26.

Table 4.26. Prospective teachers' sample finite figures obtained by performing rotation and reflection

Prospective teachers	Sample finite figures constucted	Transformation
PT4, PT11	 	120° rotation and 3 reflections
PT1, PT2, PT9, PT13	   	90° rotation and 4 reflections
PT7		72° rotation and 5 reflections
PT10		45° rotation and 8 reflections

When prospective teachers' figures were examined it was seen that almost all of them used 90° rotations and four reflections or 120° rotations and three reflections to construct their finite figures. In more detail, PT4, PT5, PT8, PT11, and PT12 used 120° rotations and 3 reflections, PT1, PT2, PT3, PT6, PT9, PT13, PT14, PT15, and PT16 used 90° rotations and 4 reflections, PT7 used 72° rotations and 5 reflections and PT10 used 45° rotations and 8 reflections to construct a finite figure.

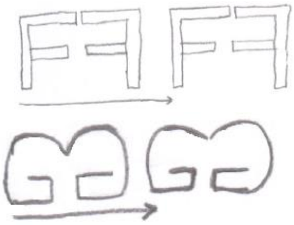
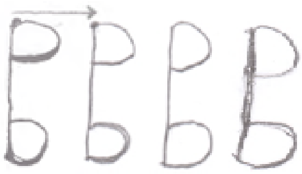
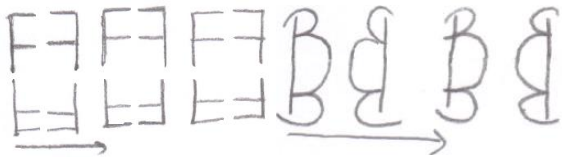
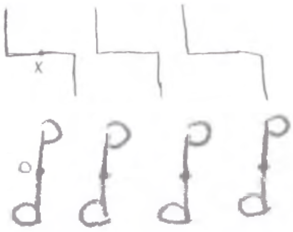
In Table 4.26, the finite figures obtained by PT4 and PT11 through 120° rotations and three reflections (symmetry type D_3); by PT1, PT2, PT9, and PT13 through 90° rotations and four reflections (symmetry type D_4); by PT7 through 72° rotations and five reflections (symmetry type D_5) and by PT10 through 45° rotations and eight reflections (symmetry type D_8) are presented. In short, prospective teachers provided examples of D_3, D_4, D_5 and D_8 symmetry types.

4.3.2. Performing Geometric Transformations to Construct Frieze Patterns

In Task 8 and Task 9 of Activity 2, prospective teachers worked individually and they constructed frieze patterns by using different types of geometric transformations. More specifically, in Task 8, the participants were asked to choose one of the following letters as the unit of their patterns: Ç, F, G, J, L, P, and R and construct frieze patterns by performing geometric transformations on their own units.

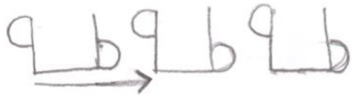
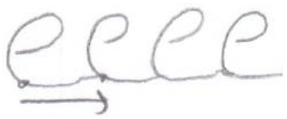
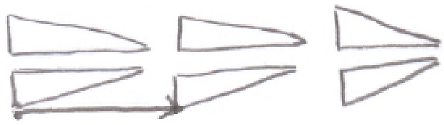
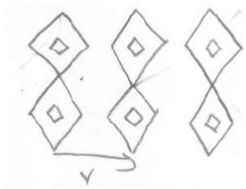
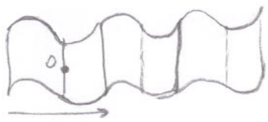
When participants' responses to Task 8 were examined it was seen that six prospective teachers (PT3, PT4, PT6, PT7, PT8, and PT9) used vertical reflection and translation, one prospective teacher (PT5) used horizontal reflection and translation, six prospective teachers (PT2, PT10, PT12, PT13, PT15, and PT16) used both horizontal and vertical reflections and translation, and finally three prospective teachers (PT1, PT11, and PT14) used 180 degrees rotation and translation to construct their frieze patterns. The types of geometric transformations applied to the letters by the participants and the sample frieze patterns constructed by them in Task 8 are presented in Table 4.27.

Table 4.27. Transformations used and sample frieze patterns constructed by the participants in Task 8 of Activity 2

Prospective teachers	Sample frieze patterns constucted	Transformations
PT8, PT9		Vertical reflection and translation
PT5		Horizontal reflection and translation
PT2, PT13		Horizontal and vertical reflections and translation
PT11, PT14		180 degrees rotation and translation

In Task 9, the participants were not limited to using the units determined by the researcher. Namely, they were asked to decide on a unit and then construct their own frieze patterns by performing geometric transformations. When participants' responses to Task 9 were examined it was seen that two prospective teachers (PT3 and PT7) used translation, four prospective teachers (PT5, PT8, PT15, and PT16) used vertical reflection and translation, one prospective teacher (PT4) used horizontal reflection and translation, two prospective teachers (PT12 and PT13) used both horizontal and vertical reflections and translation, and finally five prospective teachers (PT1, PT2, PT6, PT9, and PT10) used 180° rotation and translation to construct their frieze patterns. The remaining two participants (PT11 and PT14) constructed geometric figures that could not be accepted as frieze patterns. Thus, their figures were not included among the categories presented in Table 4.28. The types of geometric transformations used by the participants and the sample frieze patterns constructed by them in Task 9 are presented in Table 4.28.

Table 4.28. Transformations used and sample frieze patterns constructed by the participants in Task 9 of Activity 2

Prospective teachers	Sample frieze patterns constucted	Transformations
PT3, PT7	 	Translation
PT15, PT16	 	Vertical reflection and translation
PT4		Horizontal reflection and translation
PT12, PT13	 	Horizontal and vertical reflections and translation
PT2, PT6	 	180 degrees rotation and translation

4.3.3. Performing Geometric Transformations on a Grid Paper

Task 2 of Activity 3 is related to performing geometric transformations on a grid paper and it consists of three sub-tasks. In each sub-task, a transformation and parameters of the transformation were given and prospective teachers were asked to perform this transformation. Besides, they were expected to explain their constructions in detail. In Task 2A, prospective teachers were asked to translate the letter F by using the vector u = (3, 1). In Task 2B, they were asked to reflect the letter F along the given line. Finally, in Task 2C, they were asked to perform a 120° rotation for the letter F around a given point. The examination of prospective teachers' activity sheets and the audio recordings of their pairwise discussions showed that they could perform all of the requested transformations correctly.

After the completion of the transformations by the pairs, whole class discussion was carried out by having prospective teachers inspect the GeoGebra files I prepared in advance. The aim of these files was to elicit critical features that arose in pair work about transformations and to help prospective teachers make generalizations about those critical features. The corresponding GeoGebra screenshots are presented in Figure 4.13, Figure 4.15, and Figure 4.17.

As mentioned before, in Task 2A, prospective teachers were asked to translate the letter F by using the given vector $u = (3, 1)$. Examination of participants' activity sheets and audio recordings of pair discussions showed that all pairs were able to perform translation by using the vector u correctly. As an example, PT1 & PT2's response to Task 2A is presented in Figure 4.12. They expressed that "it was translated with vector u"

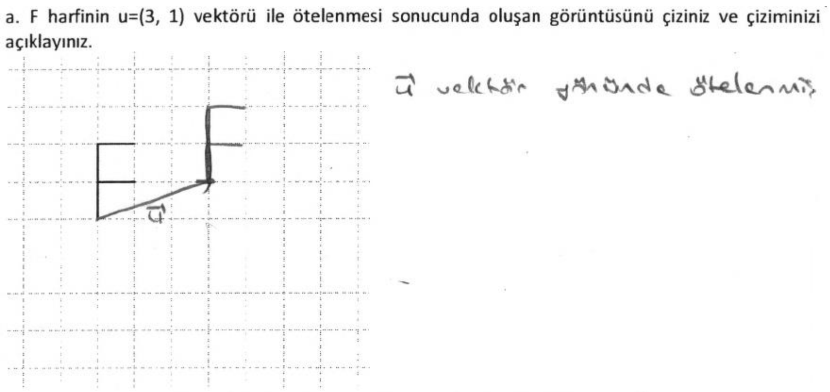


Figure 4.12. PT1 & PT2's response to Task 2A of Activity 3

In whole class discussion, to emphasize the important features of translation dynamically in Task 2A, I asked them to study on the GeoGebra file that I prepared in advance. The corresponding GeoGebra screenshot is presented in Figure 4.13.

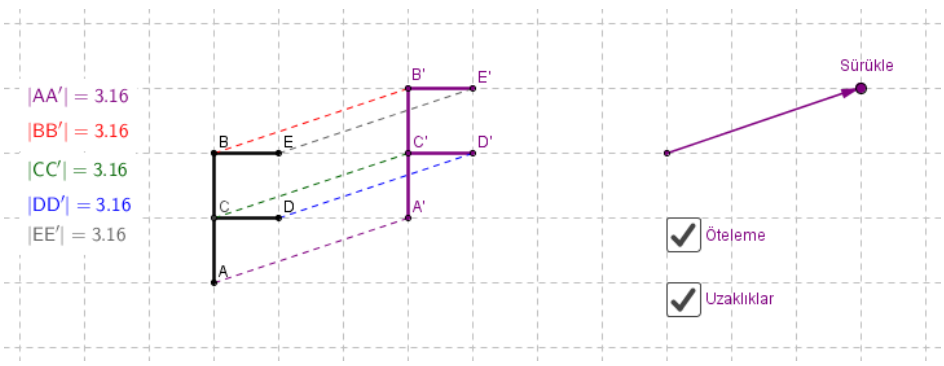


Figure 4.13. GeoGebra screenshot for Task 2A of Activity 3

Since I had witnessed that all prospective teachers performed transformations correctly during pair work, I decided to have one prospective teacher show the answer on the smart board for Tasks 2A, 2B, and 2C to save time. A prospective teacher came to the board, clicked the ‘öteleme’ button, and then the translated F, namely F', appeared on the board. Later, she clicked the ‘uzaklıklar’ button and it was seen that the distances between the selected points on F and the corresponding ones on F' were all the same ($|AA'| = |BB'| = |CC'| = |DD'| = |EE'| = 3.16$). Here, the most important point was to explore dynamically the critical features of reflection. Therefore, by changing the translation vector or the location of F and observing the consequences were important. The prospective teacher changed the magnitude and direction of the vector by dragging on the smart board, then it was seen on the screen that the equality $|AA'| = |BB'| = |CC'| = |DD'| = |EE'|$ was also satisfied for all other translations. Then, I let prospective teachers spend some time on GeoGebra file in pairs.

In Task 2B, prospective teachers were asked to reflect F along the given line. Examination of participants’ activity sheets and audio recordings of pair discussions showed that all pairs were able to perform reflection correctly. As an example, PT15 & PT16’s response to this task is presented in Figure 4.14.

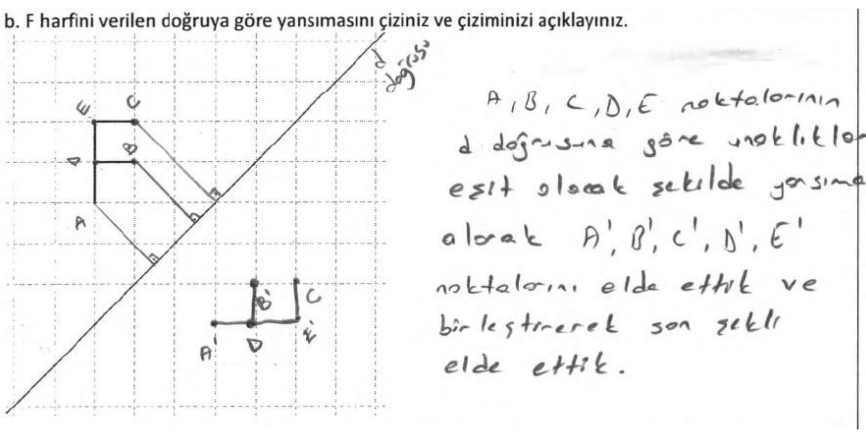


Figure 4.14. PT15 & PT16’s response to Task 2B of Activity 3

As seen in Figure 4.14, PT15 & PT16 stated that “we reflected points A, B, C, D, E along the line d in a way that the distances are equal, we obtained A', B', C', D', E' and then obtained the final figure by joining them”. In whole class discussion of Task 2B, I opened the GeoGebra file that I prepared in advance to emphasize the features of reflection. The corresponding GeoGebra screenshot is presented in Figure 4.15.

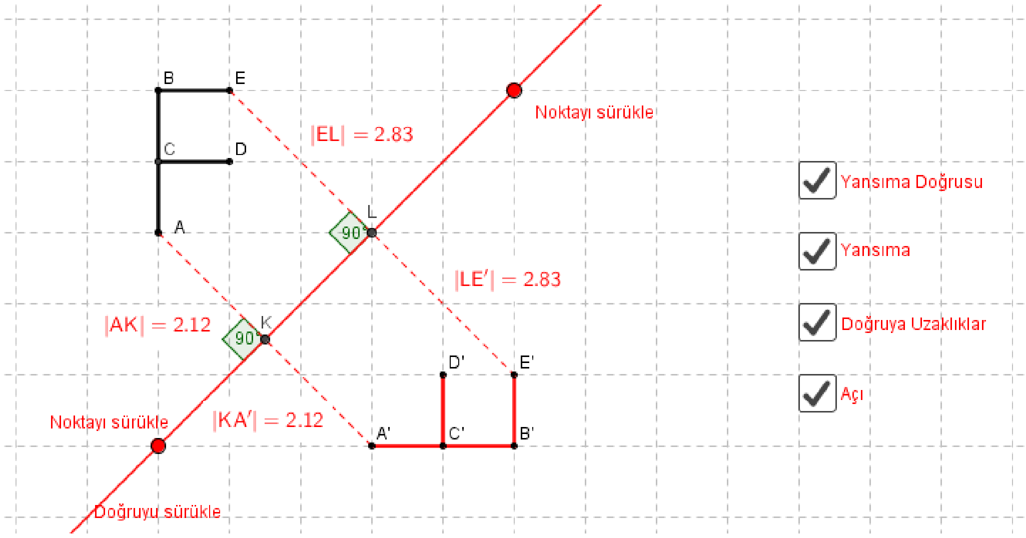


Figure 4.15. GeoGebra screenshot for Task 2B of Activity 3

A prospective teacher came to the board and clicked the 'yansıma' button and had his classmates see the reflection of F. Next, he clicked the other buttons one by one and showed that the distances between the reflection line and the corresponding points on F and F' were the same. Later, he changed the location of the reflection line without altering its slope to reexamine the aforementioned distances. Finally, he dragged the points on the reflection line (each dragging changed the slope of the reflection line) and demonstrated that the distances to the reflection line were also the same for different reflections. To have prospective teachers generalize critical features of reflection, I let them spend some time on GeoGebra file in pair work.

Finally, in Task 2C, prospective teachers were asked to perform a 120° rotation for the letter F around a given point. Examination of participants' activity sheets and audio recordings of pair discussions showed that all pairs were able to perform the rotation correctly. Different from Task 2A and 2B, I asked one of the volunteer prospective teachers to perform the rotation on the board before actually exploring it through GeoGebra. The task was projected on the smart board and PT15 came to the board voluntarily.

> PT15: Let's consider a point on F (she labeled it as point A), and rotate it 120°. I initially drew a straight line (from point A to the center of rotation). Later, I drew another straight line in a way that there is 120° angle measure between the two lines. Point A' is located somewhere on this line. Now, we will determine the location of point A' (she measured the distance

between center of rotation and point A by a ruler and then determined point A' by marking the point on the line having the same distance from center of rotation). I found the point A'. We will repeat this method for other points on F as well. Finally, we will join all the new points and get F'.

Researcher: In this case, you did not need to use a compass, right?

PT15: Yes, but actually we could have used it. We could have directly obtained A' by placing the point of the compass at the center of rotation and turning it without measuring distances.

Researcher: Your friend initially constructed the angle. Later, she expressed that there were two ways. She said that we could measure the distances on the arms of an angle or we could use a compass (I showed this via a huge compass that I brought to the classroom).

PT3 & PT4's solution to Task 2C exemplifies use of a compass in performing rotation. Their solution to this task is presented in Figure 4.16.

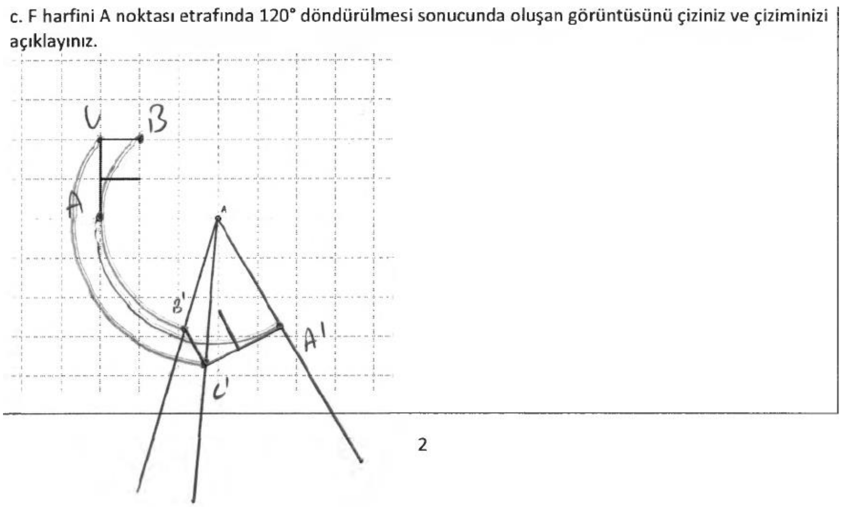


Figure 4.16. PT3 & PT4's solution to Task 2C of Activity 3

PT3 & PT4 determined three different points on F as seen in Figure 4.16. They constructed three line segments each of which forms a 120° angle measure with the line segments joining each point on F and the center of rotation. Later, they constructed three arcs starting from points A, B, and C by placing the point of the compass at the center of rotation. These three arcs and the corresponding line segments intersected at points A', B', and C'.

After performing rotation by using a ruler and a compass on the board, prospective teachers went on to examine the GeoGebra file which I prepared in advance for rotation. The corresponding GeoGebra screenshot is presented in Figure 4.17.

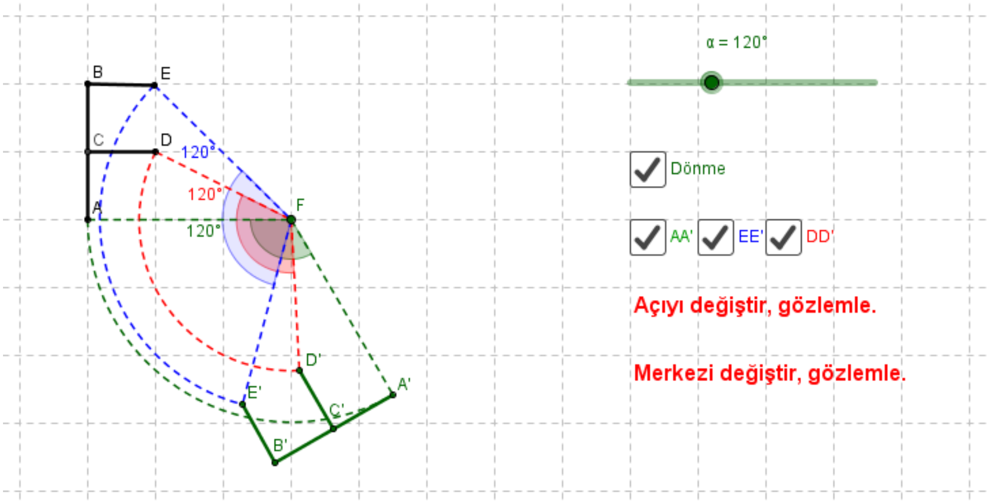


Figure 4.17. GeoGebra screenshot for Task 2C of Activity 3

In whole class discussion part, one of the prospective teachers changed the angle of rotation on smart board. It was observed that the angle measures of arc AA', DD' and EE' were all the same and were all equal to the angle of rotation. Next, the prospective teacher kept the angle measure constant and changed only the location of the center of rotation by dragging. During this time, the simultaneous change on F' was observed. Then, I let them spend some time on GeoGebra file in pair work.

To sum up, all pairs were able to perform translation, reflection, and rotation correctly in pair work. That is, they were able to use parameters of each transformation correctly. Moreover, they started to reason the effects of changing parameters on transformations and the relation among pre-image and image. More importantly, they explored the critical features of each transformation by using dynamic features of GeoGebra files that were prepared by the researcher before the class session in whole class discussions and pair discussions.

4.4. Performing Compositions of Geometric Transformations

According to Flanagan (2001), an object understanding of transformations might involve reasoning about the composition of two or more transformations and thinking about the properties that would be preserved by the composite transformations. For that reason, a considerable part of the results presented are related to composition of transformations. In this part, findings related to performing compositions of geometric transformations are presented. The data came from Task 3, Task 4 and Task 5 of Activity 3 and Task 1, Task 4, Task 5, Task 6, and Task 7 of Activity 4. The general aim of these tasks was to have prospective teachers perform geometric transformations successively and to identify to which single transformation (i.e., reflection, rotation, or translation) the composition of transformations correspond. That is, prospective teachers were expected to explain composition of geometric transformations by using only one geometric transformation. Therefore, in this part they were expected to combine and use the knowledge related to identification of geometric transformations and performing geometric transformations. Task 3, Task 4, and Task 5 of Activity 3 are related to performing compositions of translations, reflections and rotations respectively. They included one task with six, three, and two sub-tasks respectively. Task 1, Task 4, Task 5, Task 6, and Task 7 of Activity 4 are related to compositions of different types of transformations. Based on the data obtained through both pairwise and whole class discussions and activity sheets, prospective teachers' performing of composition of geometric transformations are presented in Table 4.29.

Table 4.29. Prospective teachers' performing of composition of geometric transformations

	Comp. of Trans	Comp. of Refs	Comp. of Rots	Comp. of Ref. and Tran	Comp. of Refl. and Rot	Comp. of rot. and tran
PT1 & PT2	✓	✓	✓	✓	✓	✓
PT3 & PT4	✓	✓	✓	✓	✓	
PT5 & PT6	✓	✓	✓	✓	✓	✓
PT7 & PT8	✓	✓	✓	✓	✓	
PT9 & PT10	✓	✓	✓			✓
PT11 & PT12	✓	✓	✓	✓	✓	✓
PT13 & PT14	✓	✓	✓	✓	✓	✓
PT15 & PT16	✓	✓	✓	✓	✓	✓

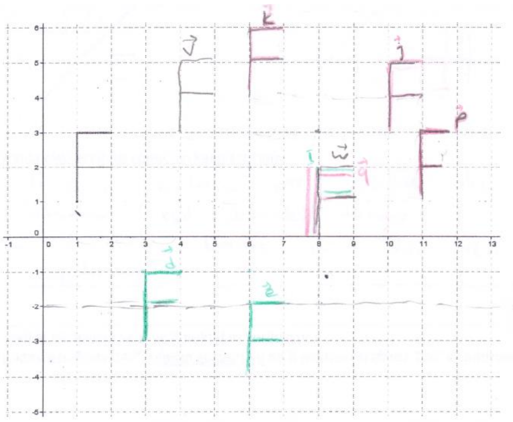
Note: Ref: Reflection, Rot: Rotation, Tran: Translation

As seen in Table 4.29, excluding PT3 & PT4, PT7 & PT8, and PT9 & PT10, all pairs were able to solve all tasks related to performing compositions of geometric transformations with given parameters on a grid. PT3 & PT4 and PT7 & PT8 were able to perform translation after rotation successively however they could not identify the center of the rotation that belongs to their composition. PT9 & PT10 could perform all compositions excluding the ones that involve glide reflections. Namely, this pair had difficulty in explaining glide reflections. The results related to the performing compositions of transformations are presented in the following parts.

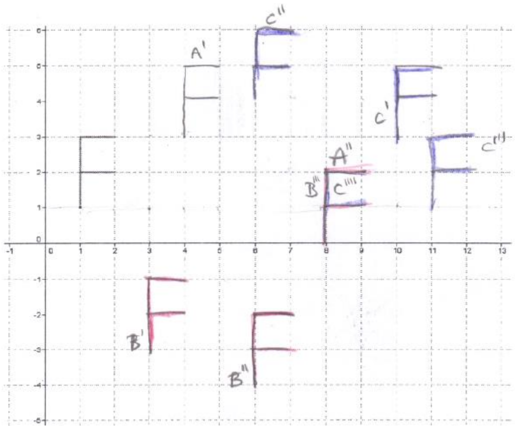
4.4.1. Performing Compositions of Translations

Task 3 of Activity 3 has six sub-tasks and as prospective teachers progress through these sub-tasks, they are expected to recognize that the composition of translations equals to a single translation. In this task, the letter F was given on a coordinate plane. In Task 3A, two vectors were given and prospective teachers were asked to translate F initially with the first vector to obtain F' and retranslate F' with the second vector to obtain F''. Similarly in Task 3B, prospective teachers were expected to perform three translations successively. They were expected to perform four translations successively in Task 3C. In Task 3D, prospective teachers were expected to compare the results of Task 3A, Task 3B, and Task 3C. In Task 3E, they were asked to find a single transformation alternative to the composition of translations given in Task 3A, Task 3B, and Task 3C. Finally, in Task 3F prospective teachers were asked whether it is possible to find the composition of translations without actually performing two, three, and four translations given in Task 3A, Task 3B, and Task 3C. Namely, in Task 3F the prospective teachers were expected to recognize that the sum of vectors yields the translation vector of the composition of translations.

The examination of prospective teachers’ activity sheets showed that all of them could perform translations in Task 3A, 3B, and 3C. To illustrate, PT7 & PT8’s and PT9 & PT10’s constructions of translations are presented in Figure 4.18.



(a)



(b)

Figure 4.18. PT7 & PT8's (a) and PT9 & PT10's (b) constructions of translations in Task 3 of Activity 3

When prospective teachers were asked to compare the translation vectors of Task 3A, Task 3B, and Task 3C in Task 3D, they all responded as "they coincide with each other" or "they are the same". This showed that they were able to perform composition of translations correctly and saw the coincidence.

Besides, in Task 3E all pairs were able to arrive at (7, -1). That is, they were able to find a single transformation alternative to the composition of translations and they indicated that "the translation vector of composition of translations is equal to (7, -1)". When prospective teachers were asked whether it is possible to find the composition of translations without actually performing two, three, and four translations in Task 3F they stated that they either "added the vectors without drawing" or "found the composition vector of the given vectors" to find composition of translations. Prospective teachers' responses showed that they were able to notice how to find the translation vector of the composition of two or more translations in pair work. Examples of their written responses to Task 3E and 3F are given in Table 4.30.

Table 4.30. Examples of prospective teachers’ responses to Task 3E and 3F of Activity 3

Pairs	Responses to Task 3E and 3F
PT9 & PT10	(7,-1) vektörü ile ötelemiştir. [It was translated by the vector (7, -1).] Verilen vektörleri toplayarak son durumu direkt elde edebiliriz. [We can directly obtain the final situation by adding the given vectors.]
PT11 & PT12	(7, -1) vektörü ile öteleme dönüşümü uygulanmış. [The translation was applied by the vector (7, -1).] Verilen vektörleri toplayarak tek bir seferde öteleme vektörü bulunur. [The translation vector can be obtanined by adding the given vectors.]

As seen in Table 4.30, PT9 & PT10 and PT11 & PT12 explained that the sum of vectors yields the translation vector of the composition of translations. It can be concluded that prospective teachers were able to perform composition of translations, find a single translation alternative to the composition of translations and understand how to find the translation vector of that composition in pair work.

After the completion of Task 3, whole class discussion was carried out by having prospective teachers inspect the GeoGebra file I prepared in advance. These file were used to help prospective teachers make generalizations about composition of translations. The corresponding GeoGebra screenshot is presented in Figure 4.19.

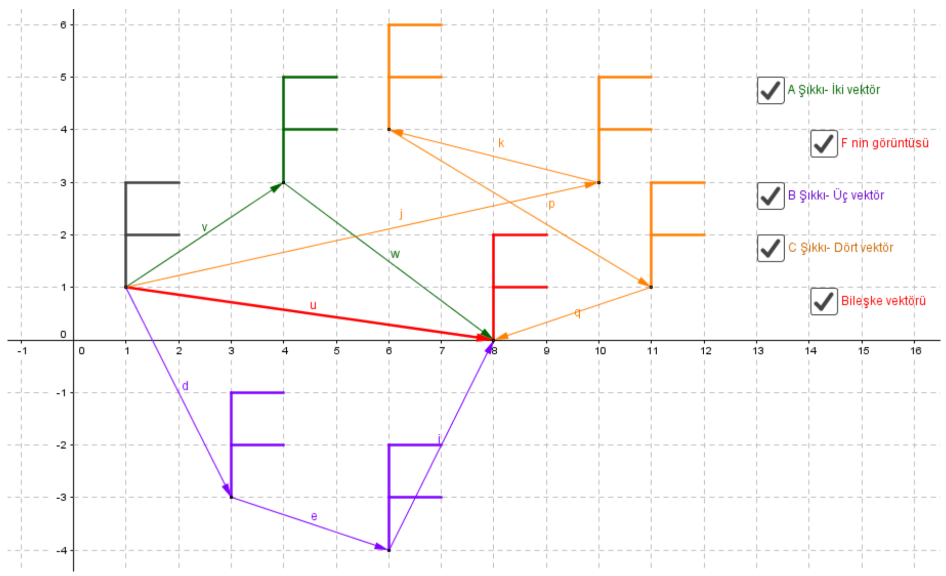


Figure 4.19. GeoGebra screenshot for Task 3 of Activity 3

During prospective teachers' exploration of composition of translations in GeoGebra, as a clue, I asked them to find any case (situation) in which composition of translations is not explained by a single translation. Besides, I asked them to summarize the ideas they developed during the exploration. Some of these ideas are presented in Table 4.31.

Table 4.31. Examples of prospective teachers' ideas developed during the exploration of GeoGebra in Task 3 of Activity 3

Pairs	Ideas developed about composition of translations
PT1 & PT2	vektörler toplamına göre ötelemeye eşittir. [Based on the addition of vectors, it is equal to a translation.]
PT3 & PT4	Öteleme Her koşulda. [It is a translation in all circumstances.]
PT7 & PT8	ötelemelerin bileşkesi her zaman ötelemeyi verir. [Composition of translations always yields a translation.]
PT15 & PT16	Öteleme vektörlerinin bileşkesi vektörlerin toplamına eşittir ve sonuçta öteleme yapılıyor. [The composition of translation vectors is equal to the sum of vectors and thus translation was applied.]

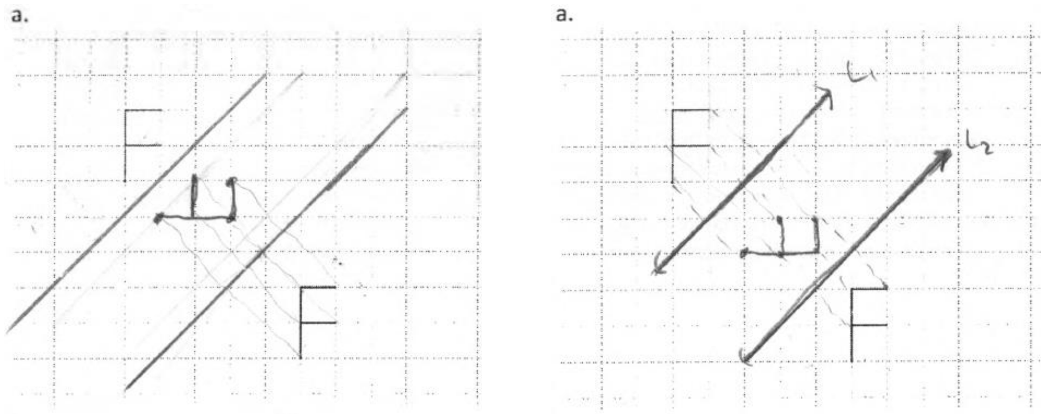
Examination of prospective teachers' written responses showed that they were able to generalize the ideas that emerged during pair work after the exploration of the GeoGebra file. That is, as mentioned before, they found that the composition of translations are equal to a translation in Task 3E and understood that the sum of vectors yields the translation vector of the composition of translations in Task 3F. Here, they explored many different compositions and then discovered that the ideas they developed in pair work were always valid. As seen in Table 4.31, all of them stated that "composition of translations always equals to a single translation" and "the translation vector is equal to the sum of each translation vector".

4.4.2. Performing Composition of Two Reflections

The aim of Task 4 of Activity 3 is to have prospective teachers recognize that the composition of two reflections corresponds to a single translation when the reflection lines are parallel to each other and to a single rotation when the reflection lines intersect with each other. Particularly, translation and rotation tasks included in Task 1A and 1C were presented again to prospective teachers in Task 4A and 4B. As explained before, Task 1 asked participants to identify the relevant type of transformation. This time in Task 4, it was expressed that the translation in Task 1A and the rotation in Task 1B were obtained by two successive reflections, and the prospective teachers were asked to identify these two reflections.

In Task 4A, a translation can be obtained by two successive reflections in which two reflection lines are parallel to each other. In Task 4B, a rotation can be obtained by two successive reflections in which the two reflection lines intersect with each other. Meanwhile, the intersection point of the reflection lines yields the center of rotation and the angle measure between these two lines is half of the measurement of the angle of rotation. Finally, Task 4C asks prospective teachers to compare the answers they gave for Task 4A and 4B.

The examination of audio recordings, activity sheets and field notes showed that all pairs were able to obtain translation by using composition of two reflections whose reflection lines are parallel in the course of pairwise discussions. To illustrate, PT5 & PT6's, and PT7 & PT8's answers to Task 4A are presented in Figure 4.20.



(a) (b)

Figure 4.20. PT5 & PT6's (a) and PT7 & PT8's (b) response to Task 4A of Activity 3

As seen in Figure 4.20, PT5 & PT6 and PT7 & PT8 reflected F along a line and they re-reflected the obtained F' along another line which is parallel to the initial line. In both of the examples, the locations and the slopes of the parallel lines were constructed correctly by the pairs and the distances between the parallel lines were equal to the half of the length of the translation vectors. The rest of the pairs gave answers similar to that of PT5 & PT6 or PT7 & PT8.

In Task 4B, prospective teachers were able to obtain the rotation by using composition of two reflections whose reflection lines intersect on the center of rotation. To illustrate, PT1 & PT2's and PT11 & PT12's answers to Task 4B are presented in Figure 4.21.

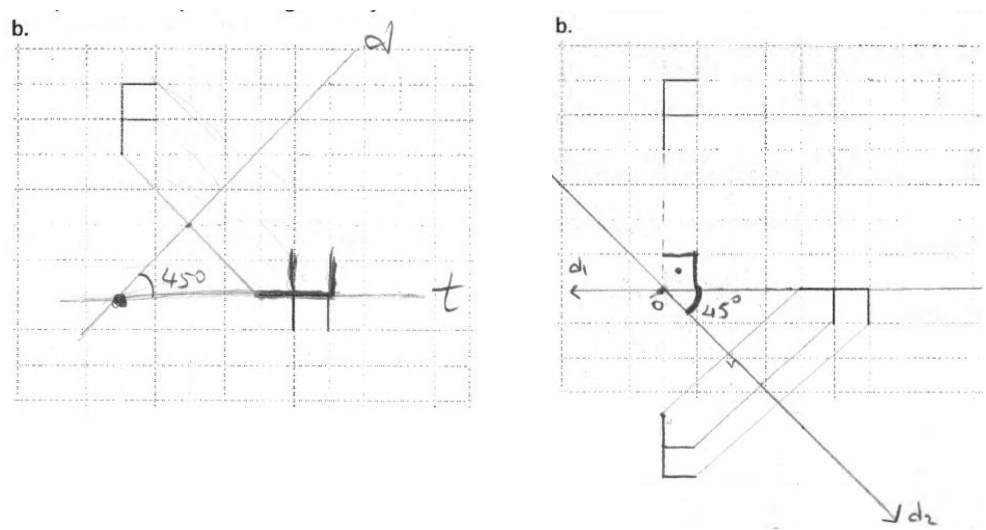


Figure 4.21. PT1 & PT2's (Left) and PT11 & PT12's (Right) response to Task 4B of Activity 3

As seen in Figure 4.21, PT1 & PT2 and PT11 & PT12 reflected F along a line and they re-reflected the obtained F' along another line intersecting with the initial line. In both of the examples, the location of intersection point and the slopes of the lines were constructed correctly. In more detail, the angle of rotation was 90 degrees and the angle measure between the lines were equal to 45 degrees which is half of the angle measure of rotation. Besides, the intersection point of the two lines coincides with the center of rotation.

In Task 4C, prospective teachers were asked to compare the responses generated in Task 4A with that of Task 4B. Examples of their written responses are given in Table 4.32.

Table 4.32. Examples of prospective teachers' responses to Task 4C of Activity 3

Pairs	Responses to Task 4C
PT3 & PT4	Ötelene ve dönme dönüşümleri elde etmek için yansıma kullandığımızda, öteleme için iki paralel yansıma doğrusu, dönme içinse dönme noktasında kesişen iki doğru elde ederiz. [For translation we need to use two reflections with parallel reflection lines. For rotation we need two reflections whose reflection lines intersect at the center of rotation.]
PT5 & PT6	Bir cismin paralel doğrulardan yansıması sonucu öteleme oluşur. 2 doğrunun kesişmesi sonucu dönme merkezi oluşur [Reflecting an object along parallel lines yields a translation. Center of rotation occurs when two lines intersect.]
PT15 & PT16	a şıkkında birbirine paralel iki doğruya göre F harfinin yansımasını alarak son şeklimizi elde ettik. b şıkkında aralarında 45° lik açı olan d_1 ve d_2 doğrularına göre F harfinin yansımasını alarak son şeklimizi elde ettik. [In alternative a, we obtained our final figure by reflecting F along two lines that are parallel to each other. In alternative b, we obtained our final figure by reflecting F along two lines whose intersection forms an angle with a measure of 45°.]

To summarize, it can be said that prospective teachers were able to perform composition of reflections alternative to a single translation by using parallel reflection lines and to single rotation by using intersecting lines. Besides, they were able to notice that the intersection point coincides with the center of rotation and the angle measure between intersecting lines is equal to half of the angle measure of rotation.

In whole class discussion of Task 4A, prospective teachers studied on the GeoGebra file that was prepared by the researcher in advance. The aim of this GoeGebra file was to help prospective teachers make generalizations about composition of two reflections in which reflection lines are parallel. The corresponding GeoGebra screenshot is presented in Figure 4.22.

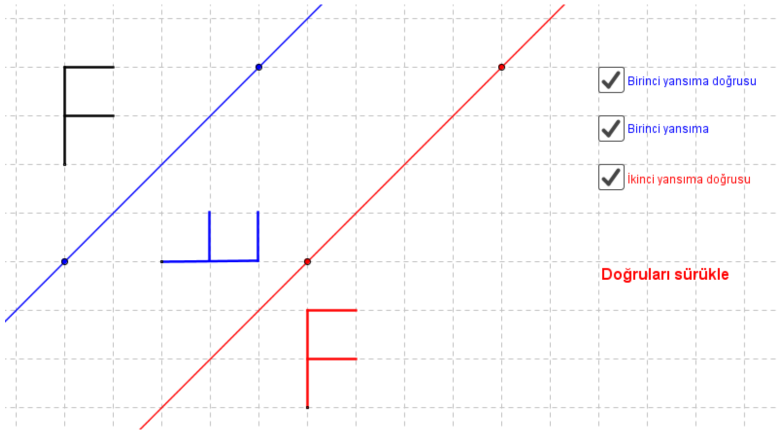


Figure 4.22 A. GeoGebra screenshot of Task 4A of Activity 3

While prospective teachers were studying composition of reflections via GeoGebra file, I asked them to drag the reflection lines and to examine the composition of two reflections. I constructed the reflection lines in a way that they are always parallel to each other. In the GeoGebra file, prospective teachers realized that the composition of two reflections along the two parallel lines always corresponded to a single translation by dragging the lines while keeping them parallel.

In whole class discussion of Task 4B, prospective teachers were asked to examine another GeoGebra file and the corresponding GeoGebra screenshot is presented in Figure 4.23. The aim of this GoeGebra file was to help prospective teachers make generalizations about composition of two reflections with intersecting reflection lines.

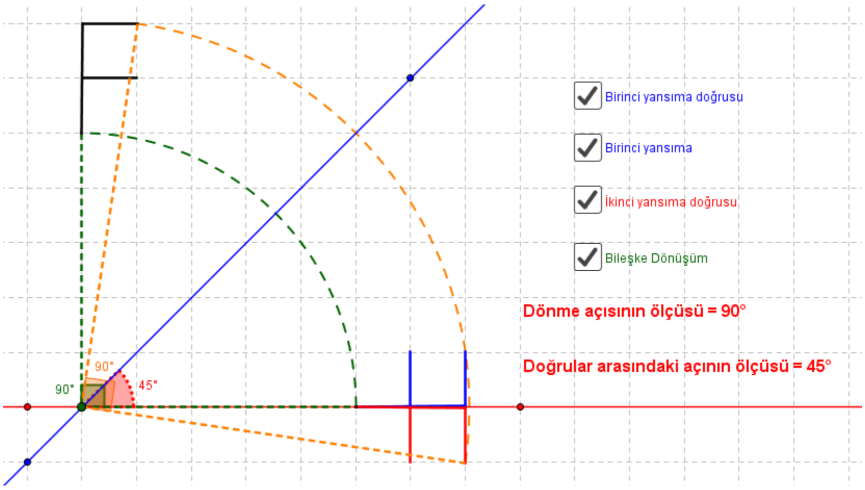


Figure 4.23 GeoGebra screenshot for Task 4B of Activity 3

During exploration of compositions of reflections, I asked prospective teachers to form different situations by changing the two lines and the intersection point and to observe the consequences. Another example (PT11 & PT12’s response) examined by them are presented in Figure 4.24.

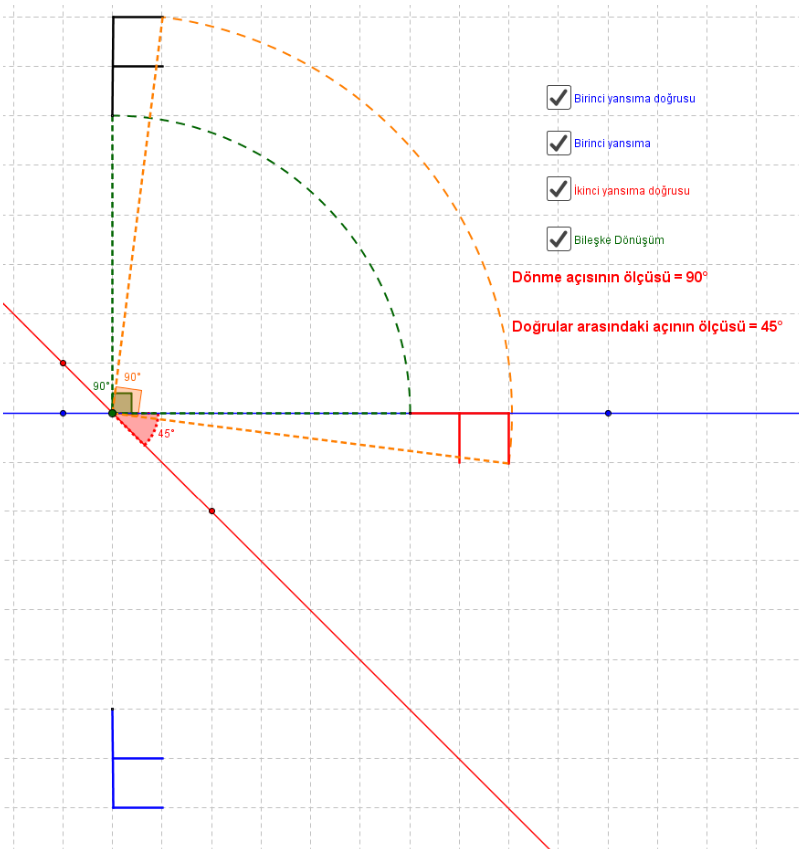


Figure 4.24. An example from the whole class discussion of Task 4B of Activity 3

I asked prospective teachers to summarize the ideas they developed during the exploration of compositions of reflections in two GeoGebra files. During the exploration of the GeoGebra file, prospective teachers realized that the composition of two reflections along the two intersecting lines always corresponded to a single rotation. Some of their written ideas are presented in Table 4.33.

Table 4.33. Examples of prospective teachers' ideas developed during the exploration of GeoGebra

Pairs	Ideas developed about composition of two reflections
PT1 & PT2	İki yansımanın bileşkesi dönmedir Yansıma doğrularının kesiştiği nokta dönme merkezidir. Yansıma doğruları birbirine paralelse öteleme dönüşümü görülür. [The composition of two reflections is a rotation. The intersection point of reflection lines is center of rotation. If reflection lines are parallel to each other, then there will be a translation.]
PT5 & PT6	Bir cismin paralel 2 doğruyla yansımasının bileşkesi ötelenmesini verir Bir cismin bir noktada kesişen 2 doğruya göre yansıması o noktaya göre döndürülmesini verir. Döndürülme açısı kesişen doğrular arasındaki açının 2 katıdır. [The composition of two reflections whose lines are parallel yields a translation. Reflecting an object along two intersecting lines yields a rotation in which the center of rotation is the intersection point. The intersection point of reflection lines is center of rotation. Angle of rotation is equal to the twofold of the angle between the two lines.]
PT11 & PT12	Yansıma doğruları paralel ise yansımaların bileşkesi ötelemedir. Yansıma doğruları α açısı ile kesişiyorsa yansımaların bileşkesi dönme dönüşümüdür. Yansıma doğruları arasındaki α açısının iki katıda dönme açısını verir. [If reflection lines are parallel, then the composition of reflections is a rotation. If the reflection lines intersect with an angle α, then the composition of reflection is a rotation. Twofold of the angle α yields the angle of rotation.]
PT13 & PT14	Şekle 2 kere yansıma yapıldığında yansıma yapılan doğrular birbirine paralel ise öteleme, kesişiyorsa dönme dönüşümü uygulanır (kesişen doğrular arası açının 2 katı kadar dönme yapılır) [The composition of two reflections is equal to a trnaslation if the reflection lines are parallel to each other and to a rotation if the reflection lines intersect (Angle of rotation is equal to the twofold of the angle between the two lines).]

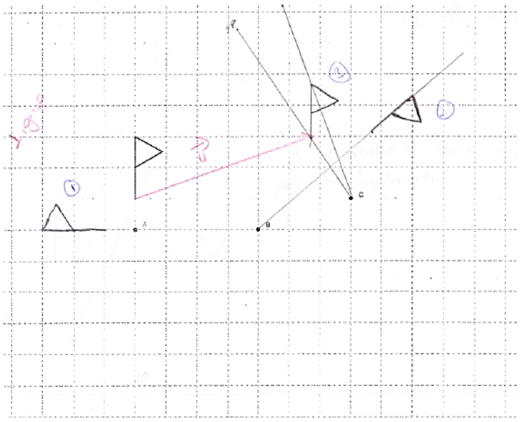
It was seen that prospective teachers could make necessary generalizations about composition of two reflections after further exploration of the ideas they gained in pairwise discussions through GeoGebra file. That is, they were able to recognize that the composition of two reflections corresponded always either to a translation or a rotation.

4.4.3. Performing Compositions of Rotations

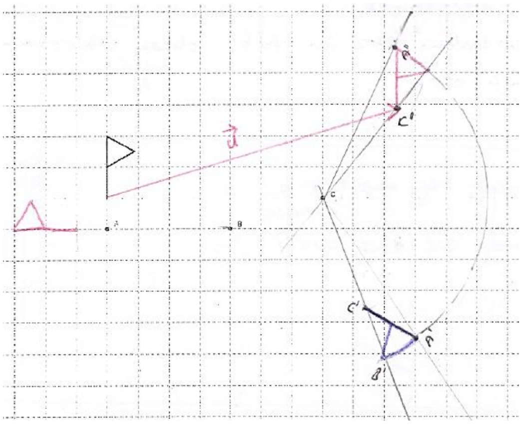
Task 5 of Activity 3 includes two sub-tasks as Task 5A and Task 5B and aims at having prospective teachers recognize the key point related to composition of rotations. That is, prospective teachers need to make an inference that the composition of two or more rotations corresponds to a single translation if the sum of measures of each angle rotation is equal to 360° or to a multiple of 360° and corresponds to a single rotation in other cases (i.e., if the sum of measures of each angle of rotation is not equal to 360° or to a multiple of 360°).

Task 5A includes a flag and the points A, B, and C. Besides, this sub-task includes four different compositions of rotation tasks in order for each pair to perform rotations with different angle measures. For instance, the first group had 90°, 220°, and 50° angle of rotations while the second group had 90°, 150°, and 120° angle of rotations. In each task, the composition equals to a single translation however the translation vector of the tasks differs from each other. By selecting different angle measures which all add up to 360°, I aimed to emphasize the critical role of 360° in compositions of rotations.

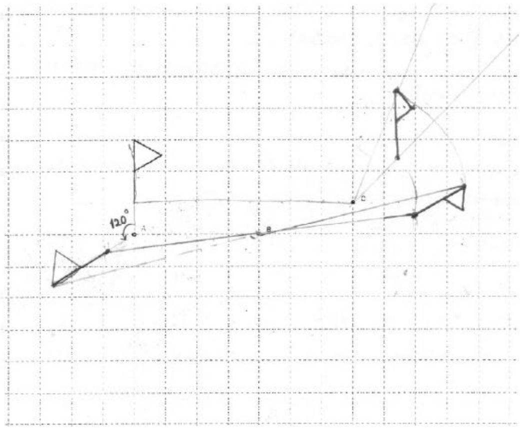
The examination of prospective teachers' activity sheets showed that all pairs were able to perform composition of rotations correctly. Four pairs' answers to Task 5A are presented in Figure 4.25 as illustrative examples.



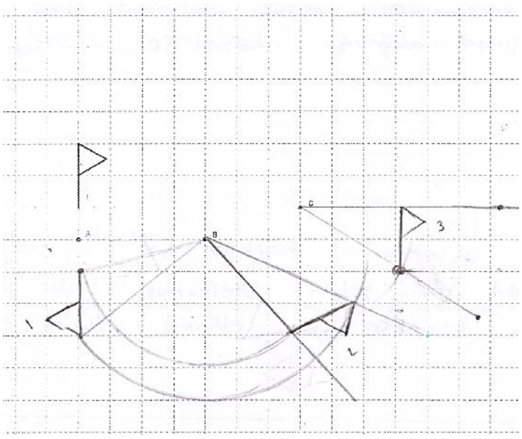
PT13 & PT14 (first task)



PT15 & PT16 (second task)



PT11 & PT12 (third task)



PT9 & PT10 (fourth task)

Figure 4.25. PT1 & PT2’s, PT15 & PT16’s, PT11 & PT12’s and PT9 & PT10’s response to Task 5A of Activity 3

As shown in Figure 4.25, the pairs were able to perform composition of rotations correctly by using their parameters.

In Task 5B, prospective teachers were expected to explain the composition of rotations with a single transformation. As can be seen, all pairs’ composition of rotations correspond to a single translation. Since the angles of rotations are different from each other, the translation vectors identified are different from each other as well. Examination of activity sheets and audio recordings showed that all prospective teachers were able to identify these translations in Task 5B. Some of their responses to Task 5B are given in Table 4.34.

Table 4.34. Examples of prospective teachers' responses to Task 5B of Activity 3

Pairs	Responses to Task 5B
PT1 & PT2	u = (6,2) vektörü kadar ötelenmiştir. Üç döndürme dönüşümünün bileşkesi bir öteleme dönüşümüdür. [It was translated by vector u=(6,2). Composition of three rotations is a translation.]
PT5 & PT6	Öteleme dönüşümüyle yapılabilir $\vec{u}$=(9,3) br. ötelenmiştir. [It can be explained by a translation. It was translated by the vector u=(9,3).]
PT11 & PT12	u(8.5, 1.5) ötelenmiştir, (birim) [It was translated by u=(8.5, 1.5).]
PT13 & PT14	son görüntüye bakarsak öteleme durumdur. [If we have a look at the final image, it is a translation.]
PT15 & PT16	$\vec{v}$ kadar ötelenmiştir. [It was translated by the vector v.]

In short, all pairs were able to perform composition of three different rotations and to identify the resulting single transformation, namely the translation, in pair work. Besides, before the exploration of GeoGebra file, they started to reason that 360° has a critical role since the sum of the angle measures of the three successive rotations is equal to 360° in all four tasks and the compositions of these rotations are equal to a translation.

After completing Task 5A and Task 5B in pair work, the prospective teachers discussed their answers in whole class discussion. To do so, the GeoGebra file prepared by the researcher for Task 5 was examined. The corresponding GeoGebra screenshot is presented in Figure 4.26.

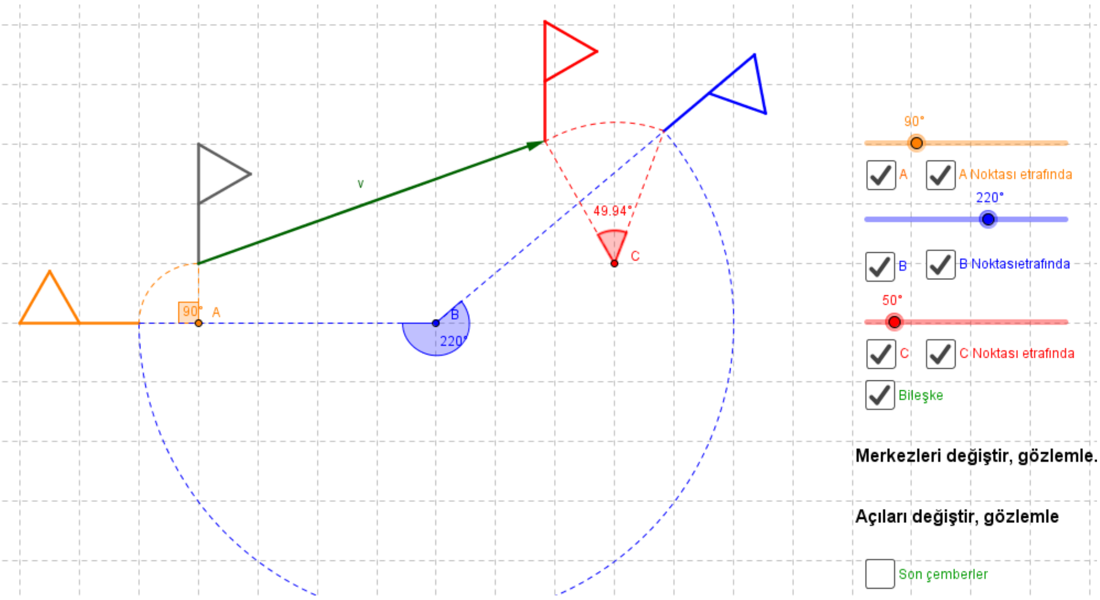


Figure 4.26 GeoGebra screenshot for Task 5 of Activity 3

The GeoGebra file given in Figure 4.26 was examined both in pair work and in whole class discussion. Whole class discussion proceeded by one prospective teacher's exploration of the file on the smart board. The discussion focused on two main points. First, all angles of rotations were kept the same and only the centers of rotations were changed. An example in which the third center of rotation changed was presented in Figure 4.27. Second, while the centers of three rotations and the first and the second angle of rotations were kept the same, the third angle of rotation was changed. An example illustrating this main point is given in Figure 4.28.

To explore the first main point, PT13 dragged the points A, B, and C (i.e., the center of rotations) on the smart board without changing their angle of rotation. The prospective teachers could notice that the composition of rotations always yielded a translation however the translation vector changed in each alteration of the center of rotation. For instance, when PT13 dragged point C, the composition of rotations changed as follows:

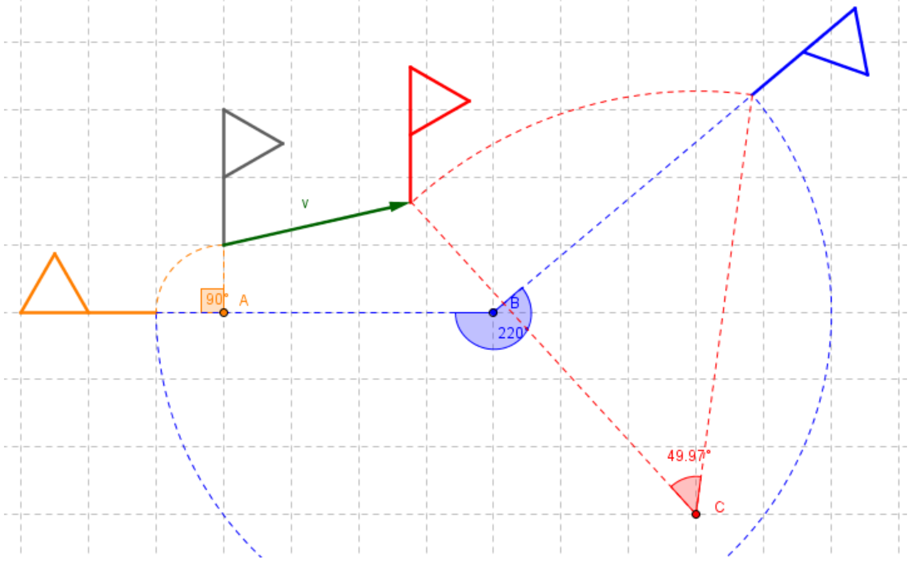


Figure 4.27. The composition of rotations in Task 5 when point C is dragged four units down

The prospective teachers saw that the composition of rotations was again a translation but it was different from the translation obtained in Figure 4.26 as seen in Figure 4.27.

To explore the second main point, while all three centers of rotations and the angles of the first and the second rotation were kept the same, the third angle of rotation (around point C) was changed. Meanwhile, the prospective teachers observed the change in the composition of rotations and realized that it was no more a translation. However, initially, they could not decide whether the composition of rotations might be explained by a single transformation and determine the type of this single transformation. The researcher gave a clue that the third angle of rotation can be set to 230° to have them easily determine the composition and asked them to state the type of transformation that the composition of rotations have. PT13 deactivated the buttons for the first and the second rotation and clicked the 'son çemberler' button. Thereafter, the prospective teachers could notice that the composition of three rotations yielded a single 180° rotation as seen in Figure 4.28.

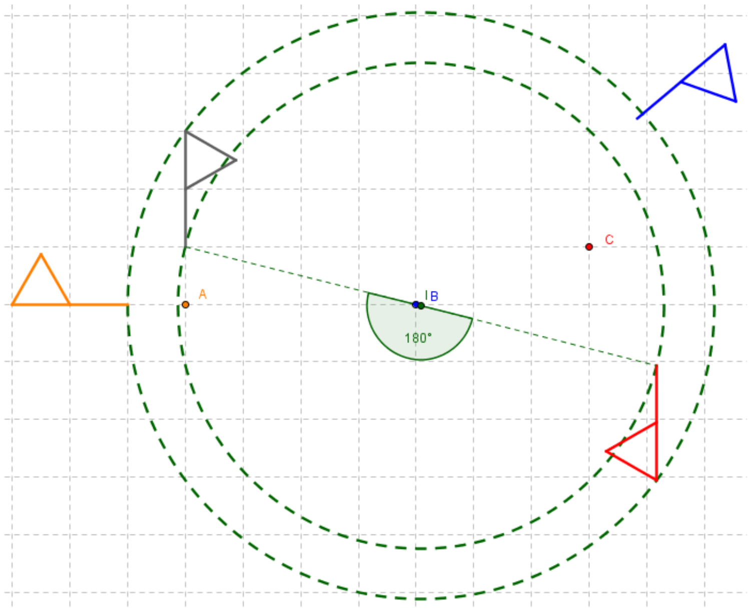


Figure 4.28. The composition of rotations with angle measurement 180°

The three angles of rotations given in Figure 4.28 were 90° (the center of rotation was point A), 220° (the center of rotation was point B) and 230° (the center of rotation was point C) respectively and thus the sum of their angle measures was 540°. The difference between 540° and 360° yielded the angle of resulting rotation. Namely, it was equal to 180°. Then, I let prospective teachers spend some time on GeoGebra file in pair work.

The prospective teachers examined various examples by changing all angles of rotations and centers of rotations and made an inference that the composition of rotations yielded a rotation when the sum of angles of rotations was not equal to 360°. By means of a number of explorations, they were able to notice that the difference between 360° and the sum of angles of rotations yielded the angle of the resultant rotation. Prospective teachers' written responses related to the ideas they developed during the exploration of composition of rotations are presented in Table 4.35.

Table 4.35. Examples of prospective teachers' ideas developed during the exploration of GeoGebra file for Task 5

Pairs	Ideas developed about composition of rotations
PT1 & PT2	Döndürdüğümüz açılar toplamı 360° ise öteleme dönüşümü görülür. 360° den fazla veya eksikse dönme dönüşümü olur. Dönme dönüşümünün bileşkesi döndürme merkezlerinden bağımsız olarak ya tek öteleme dönüşümüne yada tek dönme dönüşümüne eşittir. [If the sum of angle of rotations is 360°, then it is translation. If it is less or more than 360°, then it is a rotation. The composition of rotations is equal to a single translation or rotation regardless of center of rotations.]
PT5 & PT6	Bir cisme uygulanan dönmelerin dönme açılarının toplamı 360°'yi veriyorsa cisme öteleme dönüşümü uygulanmış olur. Ama 360°'den fazla veriyorsa 360°'den fazla olan kısmı dönme açısını ifade eder. [If the sum of angle of rotations is 360°, then a translation is applied on the object. However, if it is less or more than 360°, then the leftover angle measure corresponds to the angle of rotation.]
PT9 & PT10	Dönme açılarının toplamı 360° olursa son şekil ilk şeklin ötelenmiş hali olur. 360° den küçükse yine dönme olmuş olur [If the sum of angle of rotations is equal to 360°, the final figure is the translated form of the initial figure. If it is less than 360°, it is a rotation, again.]
PT13 & PT14	Şekle yapılan dönme açıları toplamı 360° ye eşit ise öteleme, değil ise dönme dönüşümü yapılır [If the sum of angle of rotations is equal to 360°, then it is a translation. If it is not equal to 360°, then it is a rotation.]
PT15 & PT16	Dönme açılarının toplamı 360° eşitse bileşke bir öteleme dönüşümüdür. Dönmelerin merkezleri değiştirilme sonucun öteleme dönüşümü olmasını değiştirmez Dönme açılarının toplamı 360° den farklı olursa bileşkenin sonucu dönme dönüşümü olur. [If the sum of angle of rotations is equal to 360°, the composition is a translation. Changing the center of rotations, does not change the composition. Namely, it will again be a translation. If the sum of angle of rotations is different from 360°, the composition will be a rotation.]

It can be seen that prospective teachers could generalize the ideas they gained in pairwise discussions after exploring the researcher prepared GeoGebra file. They were able to understand that composition of rotations corresponded always either to a translation or a rotation. Besides, they noticed that the change in centers of rotations did not change the type of composition. Meanwhile, they noticed that the difference between the sum of angle of rotations and 360° yielded the angle of resulting rotation. Finally, prospective teachers started to make connections among different transformations.

According to Flanagan (2001), an object understanding of transformations might involve reasoning about the composition of two or more transformations and thinking about the properties that would be preserved by the composite transformations. When prospective teachers' understanding of compositions of translations were viewed through the lens of APOS theory, it can be said that prospective teachers began to reason transformations as "object" understanding.

Thus far, findings related to compositions of transformations involving same type of geometric transformations (i.e., translation-translation, reflection-reflection, and rotation-rotation) were presented. In the next section, findings related to composition of different type of geometric transformations (i.e., reflection-translation, reflection-rotation, and rotation-translation) are presented.

4.4.4. Performing Composition of a Reflection and a Translation

The aim of this part was to have prospective teachers perform several different compositions of a reflection and a translation. Namely, prospective teachers were expected to notice that the composition of a reflection and a translation corresponded to a single reflection when the reflection line and the translation vector were perpendicular to each other and otherwise to a glide reflection. For this purpose, Task 1 and Task 4 each of with two sub-tasks were administered to the prospective teachers. In each sub-task, the letter F, a line, and a vector were given on a grid and prospective teachers were first asked to reflect the letter F along the given line and then translate the resultant F by using the given vector. Later, they were asked to find a single transformation alternative to the composition of a reflection and a translation.

4.4.4.1. Composition of a reflection and a translation when the translation vector is perpendicular to the reflection line

In Task 1 of Activity 4, prospective teachers were expected to perform compositions of a reflection and a translation when the reflection lines were perpendicular to translation vectors. Therefore, the compositions of these transformations were equal to a single reflection. The examination of prospective teachers' responses to Task 1A and 1B showed that all pairs were able to perform compositions of reflection and translation correctly and explained it with a single reflection alternative to the compositions. To illustrate, PT3 & PT4's response to Task 1A is presented in Figure 4.29.

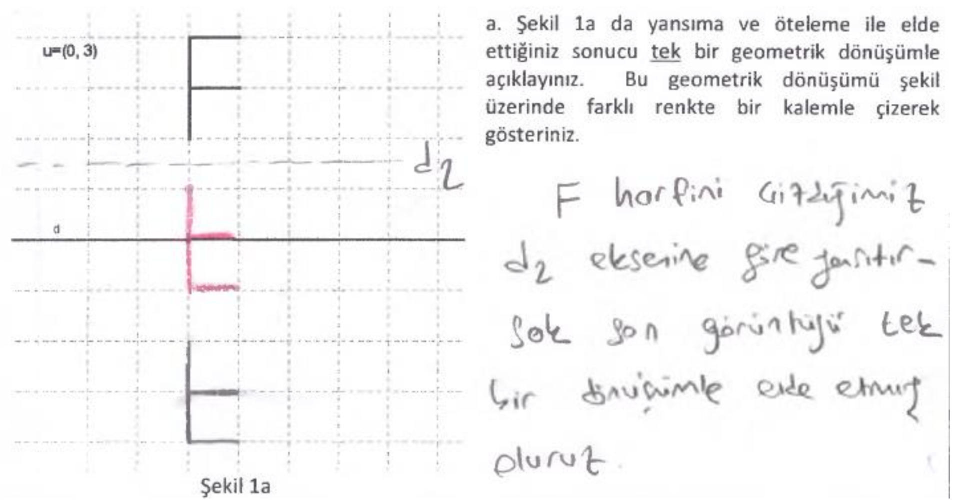


Figure 4.29. PT3 & PT4's response to Task 1A of Activity 4

As seen in Figure 4.29, PT3 & PT4 reflected F along the given line, obtained the F', translated F' by using the given vector and obtained F''. They stated that "if we reflect F only along d_2, we obtain F''". In short, PT13 & PT14 could find a single reflection alternative to the composition of a reflection and a translation whose reflection line and translation vector were perpendicular to each other.

Similar to Task 1A, in Task 1B, all pairs were able to perform reflection and translation correctly and find a reflection which was equal to the composition of these transformations. To illustrate, PT11 & PT12's response to Task 1B is presented in Figure 4.30.

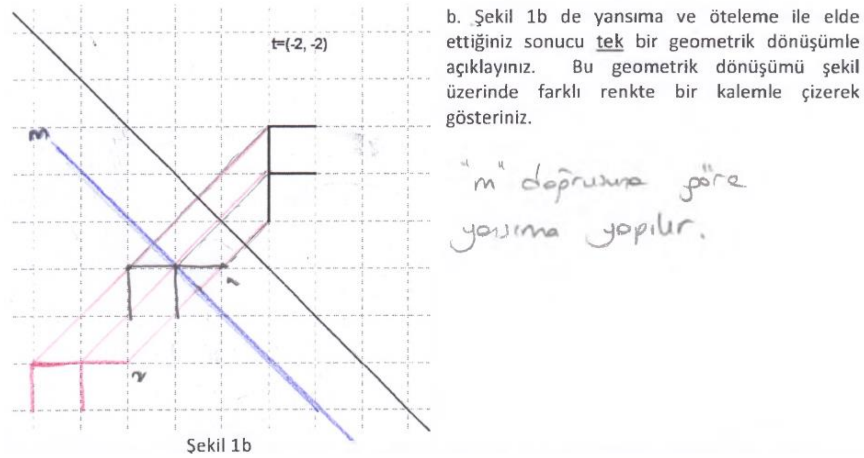


Figure 4.30. PT11 & PT12's response to Task 1B of Activity 4

As seen in Figure 4.30, PT11 & PT12 reflected F along the given line, obtained the F' (labelled with 1), translated F' by using the given vector and obtained F'' (labelled with 2). They could find a single reflection which is equal to the composition of a reflection and a translation and determine the parameter of this reflection (i.e., the reflection line m). PT11 & PT12 stated that "a reflection along line m was applied".

In whole class discussion of Task 1, prospective teachers studied on two GeoGebra files that were prepared by the researcher in advance. The aim of these GoeGebra files was to help prospective teachers make generalizations about composition of a reflection and a translation in which reflection lines and the translation vector were parallel. The corresponding GeoGebra screenshots are presented in Figure 4.31 and Figure 4.32.

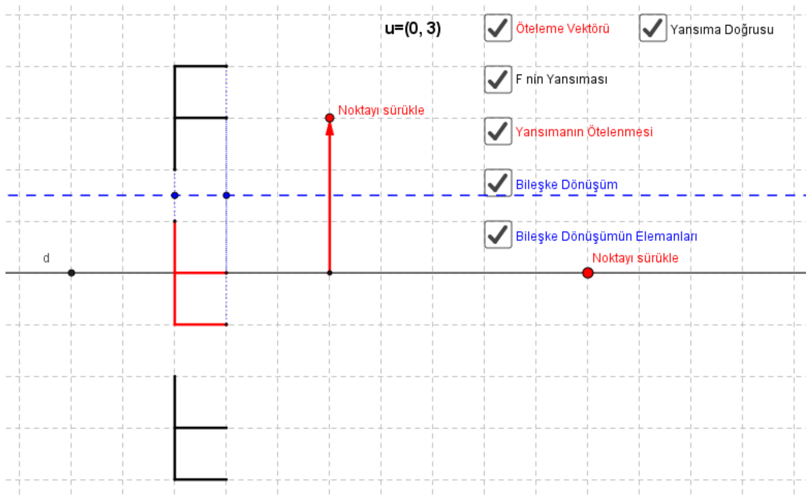


Figure 4.31. A. GeoGebra screenshot of Task 1A of Activity 4

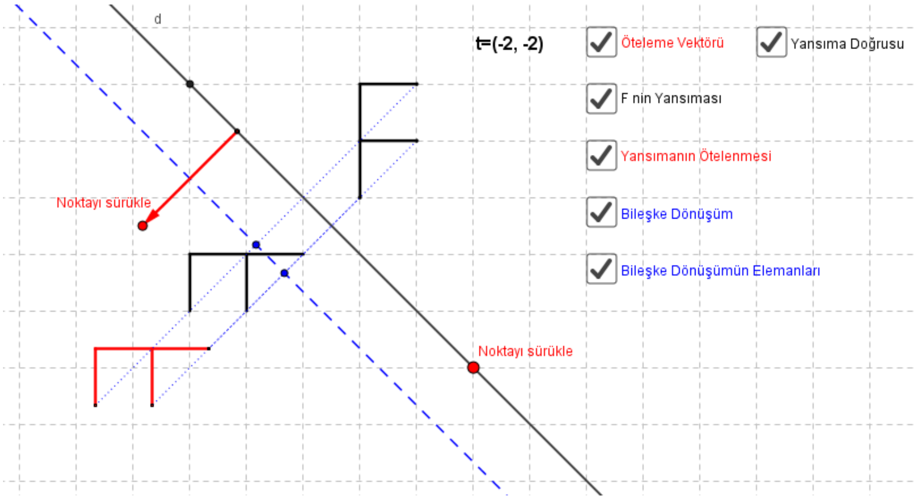


Figure 4.32. GeoGebra screenshot of Task 1B of Activity 4

Prospective teachers examined GeoGebra files related to Task 1A and Task 1B respectively. When the files were opened, the buttons were inactive and the objects given in these tasks were on the screen. The prospective teachers examined the files both in pair work and in whole class discussions. While prospective teachers were studying compositions via GeoGebra files, I asked them to change the reflection line and the magnitude of the vector. Before coming to the classroom, I constructed the reflection line and the translation vector in a way that they were always perpendicular to each other. The prospective teachers examined several examples of compositions of a reflection and a translation when their parameters were perpendicular and arrived at a conclusion. I asked prospective teachers to write the ideas they developed during the exploration of composition of a reflection and a translation. Their written responses are presented in Table 4.36.

Table 4.36. Prospective teachers' ideas developed during the exploration of GeoGebra file for Task 1 of Activity 4

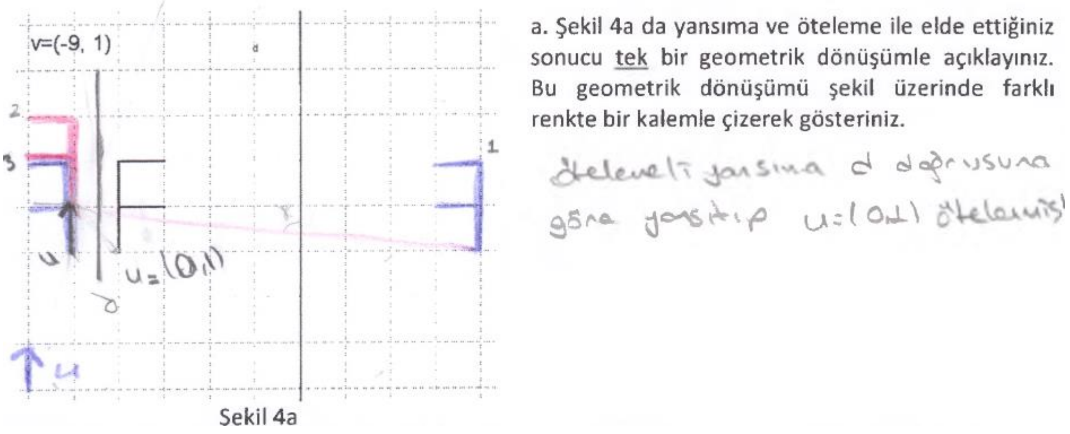
Pairs	Ideas developed about composition of a reflection and a translation
PT1 & PT2	Öteleme yapılan vektör ile ilk yansıma doğruyla dik ise Öteleme ve yansıma dönüşümlerinin bileşkesi tek bir yansıma dönüşümü olur. [If the translation vector and the reflection line are perpendicular, then the composition of a translation and a reflection is a reflection.]
PT3 & PT4	Yansıma ve öteleme ile elde ettiğimiz sonuç tek bir yansıma ile elde ettiğimiz zaman iki yansıma doğrusu birbirine paralel olmaktadır. [The two reflection lines are parallel to each other when the composition of a reflection and a translation is equal to a single reflection.]
PT5 & PT6	Öteleme vektörü ile yansıma doğrusu birbirine dik olduğu durumlarda bileşke dönüşüm yansıma olur. [The composition is a reflection when the translation vector is perpendicular to the reflection line.]
PT7 & PT8	Yansıma doğrusu ile öteleme vektörü birbirine dik ise öteleme ve yansımayla elde edilen şekil ilk şeklin yansıması olur. [If the reflection line and the translation vector are perpendicular to each other, then the composition of a reflection and a translation is equal to a reflection.]
PT9 & PT10	Yansıma doğrusuyla öteleme vektörü birbirine dik olduğunda son şekil tek bir dönüşümle açıklanır Yani yansımayla ötelemenin bileşkesi yine yansıma olur [When the reflection line and the translation vector are perpendicular, then the final figure can be explained by a single transformation. Namely, the composition of a reflection and translation is again a translation.]
PT11 & PT12	Yansıma ve öteleme dönüşümünün bileşkesi yine yansıma dönüşümüdür. Fakat yeni oluşan yansıma doğrusu ilk yansıma doğrusuna paraleldir. [The composition of a reflection and a translation is again a reflection. However, the new reflection line is parallel to the initial reflection line.]
PT15 & PT16	Yansıma doğrusu ile öteleme vektörü birbirine dikse sonuç başka bir doğruya göre yansımadır. [If the translation vector and the reflection line are perpendicular to each other, then the composition of is a reflection along another line.]

It can be seen that prospective teachers could generalize the ideas they gained in pairwise discussions after exploring the researcher prepared GeoGebra files. By dragging the reflection line and the translation vector while keeping them perpendicular in the GeoGebra files, they generalized the idea that composition of a reflection and a translation corresponded to a single reflection when the translation vector is perpendicular to the reflection line.

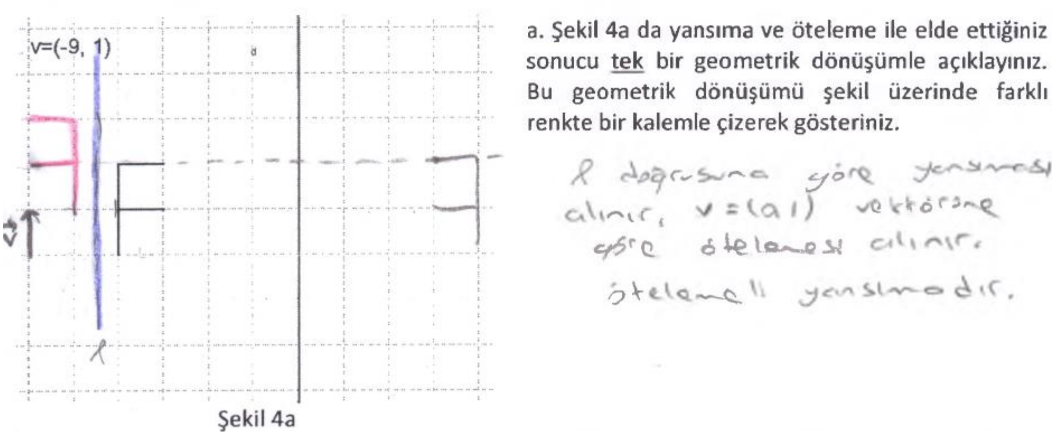
4.4.4.2. Composition of a reflection and a translation when the translation vector is not perpendicular to the reflection line

In Task 4 of Activity 4, prospective teachers were expected to perform compositions of a reflection and a translation as in Task 1. However, in Task 4 there were no perpendicularities between the reflection lines and the translation vectors. Therefore, the compositions of these transformations were equal to a glide reflection. Task 4 included two sub-tasks as Task 4A and 4B. The examination of prospective teachers' responses to this tasks showed that all pairs were able to perform compositions of a reflection and a translation correctly. Beside, except for PT9 & PT10 all pairs were able to find a glide reflection alternative to the compositions. PT9 & PT10 could perform compositions correctly however they had difficulty in explaining the composition with one glide reflection. As an example, PT1 & PT2's and PT7 & PT8's responses to Task 4A are presented in Figure 4.33.

As seen in Figure 4.33, both pairs performed reflection and translation successively and identified their composition as a glide reflection. Both pairs stated that "it was a glide reflection". In PT1 & PT2's and PT7 & PT8's responses, the reflection lines were labeled as *d* and *l* respectively. Besides, the translation vector which was parallel to the reflection line was identified as *u* by the first pair and as *v* by the second pair. In both responses, the slope and the location of the reflection line were correct and the translation vector was equal to (0, 1).



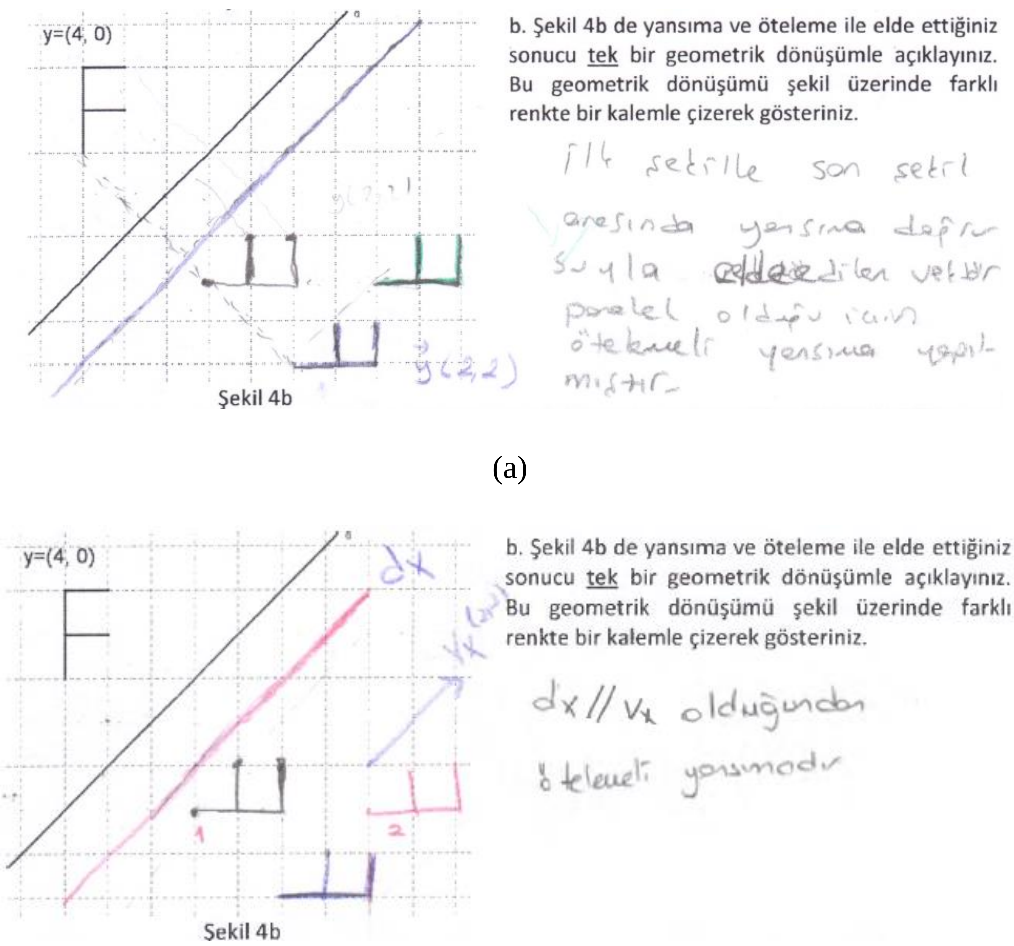
(a)



(b)

Figure 4.33. PT1 & PT2’s (a) and PT7 & PT8’s (b) response to Task 4A

Similar to Task 4A, all pairs were able to perform the composition of a reflection and a translation correctly in Task 4B. Excluding PT9 & PT10, all pairs could find a glide reflection which was equal to the composition of these transformations. To illustrate, PT5 & P6’s and PT13 & PT14’s responses to Task 4B are presented in Figure 4.34.



(a)

(b)

Figure 4.34. PT5 & PT6’s (a) and PT13 & PT14’s (b) response to Task 4B

Both PT5 & PT6 and PT13 & PT14 could perform the composition of a reflection and a translation. Both pairs stated that “it was a glide reflection”. In PT13 & PT14’s response, the reflection of F was labelled with 1 and the translation of this reflected F (F') was labelled with 2. Both pairs could identify the composition as a glide reflection. In PT5 & PT6’s and PT13 & PT14’s responses, the reflection lines were correctly located two units right to the reflection line and the translation vector was identified as (2, 2). Both pairs stated that since the reflection line and the translation vector were parallel, the composition was a glide reflection.

In whole class discussion of Task 4, prospective teachers studied on two GeoGebra files that were prepared by the researcher in advance. The aim of these GeoGebra files were to help prospective teachers make generalizations about composition of a reflection and a translation in which the reflection line is not perpendicular to the translation vector. The corresponding GeoGebra screenshots are presented in Figure 4.35 and Figure 4.36.

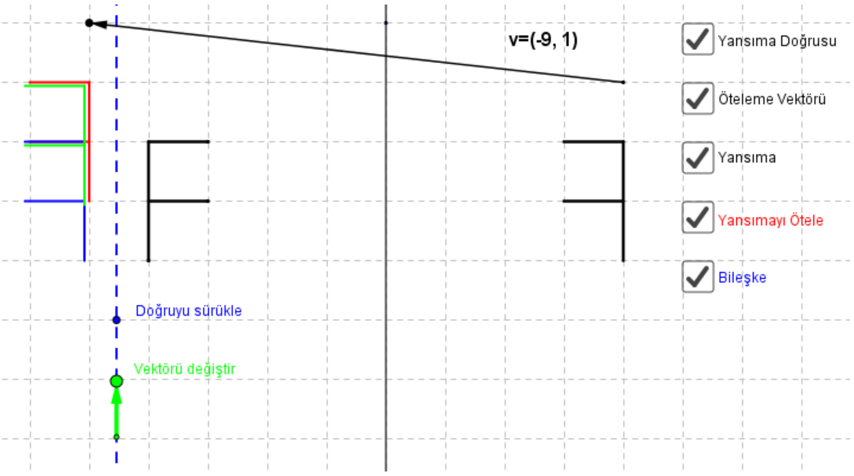


Figure 4.35. A. GeoGebra screenshot of Task 4A

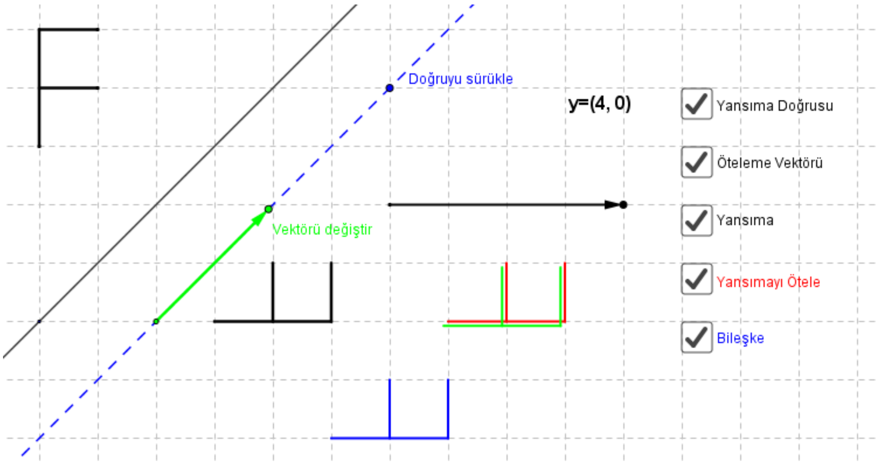


Figure 4.36. GeoGebra screenshot of Task 4B

Prospective teachers examined GeoGebra files related to Task 4A and Task 4B respectively. When the files were opened, the buttons were inactive and the objects given in these tasks were on the screen. The prospective teachers examined the files both in pair work and in whole class discussions. While prospective teachers were studying compositions via GeoGebra files, I asked them to generate other examples of compositions of a reflection and a translation by changing the reflection line and translation vector and to identify a single transformation alternative to them. They dynamically examined the glide reflection and its effect on the objects. Then, I asked prospective teachers to write down the ideas they developed during the exploration of composition of a reflection and a translation. Their written responses are presented in Table 4.37.

Table 4.37. Examples of prospective teachers' ideas developed during the exploration of GeoGebra file for Task 4 of Activity 4

Pairs	Ideas developed about composition of a reflection and a translation
PT1 & PT2	yansıma doğrusuyla öteleme vektörü dik olmadığından ötelemeli yansımadır. [It is a glide reflection since the reflection line and the translation vector are not perpendicular to each other.]
PT5 & PT6	Dik olmadığı durumlarda öteleme ve yansıma dönüşümlerinin bileşkesi ötelemeli yansıma olur. [In cases where there is not a perpendicularity, the composition of a translation and a reflection is a glide reflection.
PT7 & PT8	Yansıma doğrusu ile öteleme vektörü dik olduğundan yansıma, eğer dik değilse ötelemeli yansıma olur. [It is a reflection since the reflection line and the translation vector are perpendicular to each other. If there is no perpendicularity, then it is glide reflection.]
PT11 & PT12	Yeni yansıma doğrumuzu ilk yansıma doğrusuna paralel olmak şartıyla istediğimiz gibi çizebiliriz. Daha sonra da öteleme vektörünü buluruz. [We can construct the new reflection line in a way that it is perpendicular to the initial reflection line. Then we identify the translation vector.]
PT13 & PT14	Dik ise yansıma, dik olmadığı durumlarda ötelemeli yansıma dönüşümüdür. [It is a reflection when there is perpendicularity and a glide reflection when there is no perpendicularity.]
PT15 & PT16	Öteleme vektörü ile yansıma doğrusu dikse bileşke yine bir yansımadır. Ama dik değilse ötelemeli yansımadır. [The composition is a reflection when the reflection line and the translation vector are perpendicular to each other. But it is glide reflection when they are not perpendicular to each other.]

As seen in Table 4.37, all pairs were able to generalize the ideas they gained in pairwise discussions during the exploration of GeoGebra files. They could combine the ideas gained in Task 1 and Task 4 of Activity 4 and gained a complete understanding of compositions of a reflection and a translation. That is, prospective teachers gained the idea that the composition of a reflection and a translation corresponded to a single reflection when the reflection line and the translation vector were perpendicular to each other and otherwise to a glide reflection.

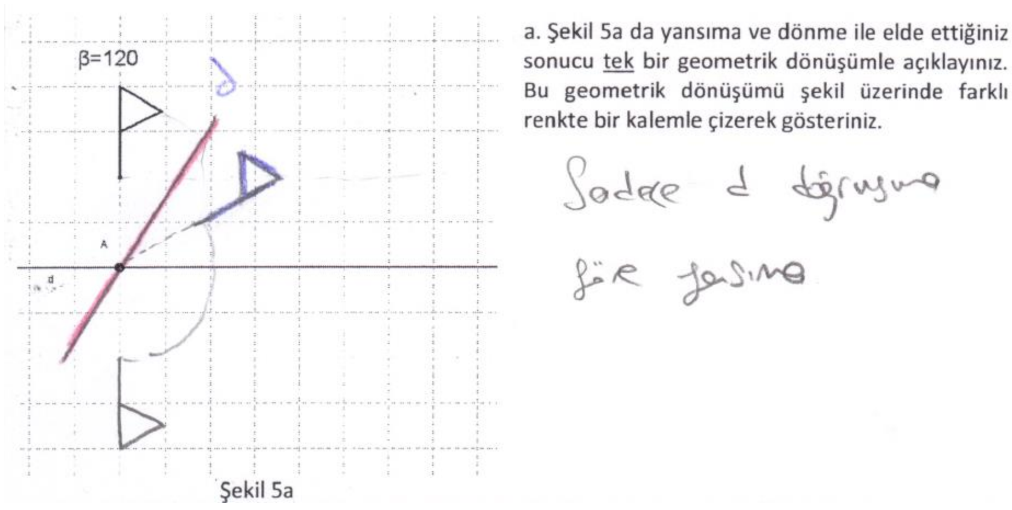
4.4.5. Performing Compositions of a Reflection and a Rotation

The aim of Task 5 and Task 6 of Activity 4 is to have prospective teachers recognize that the composition of a reflection and a rotation corresponds to a single reflection when the center of rotation is on the reflection line and otherwise to a glide reflection. For this purpose, Task 5 and Task 6, each with two sub-tasks, were administered to the prospective teachers. In each sub-task, a flag, a line, and a point were given on a grid and prospective teachers were asked to reflect the flag along the given line and then rotate the resultant flag around the given point. Then, they were asked to find a single transformation alternative to the composition of a reflection and a rotation.

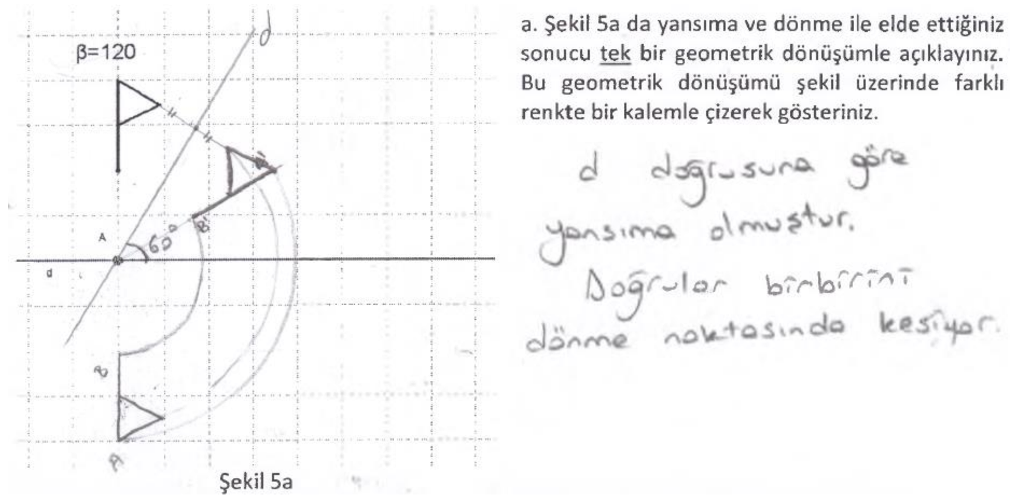
4.4.5.1. Compositions of a reflection and a rotation when the center of rotation is on the reflection line

In Task 5 of Activity 4, prospective teachers performed reflection and rotation respectively and the center of rotation was on the reflection line. Therefore, the composition of these transformations was equal to a single reflection. The examination of prospective teachers' responses to Task 5A and 5B showed that all pairs were able to perform composition of a reflection and a rotation correctly and find a single reflection alternative to the two transformations. To illustrate, PT3 & PT4's and PT15 & PT16's responses to Task 5A are presented in Figure 4.37.

As seen in Figure 4.37, both pairs reflected the flag and rotated the resulting flag 120 degrees around point A correctly. PT3 & PT4 stated that "it was a reflection along line *d*". They were able to identify the composition as a single reflection and labeled the reflection line as *d*. Besides, PT15 & PT16 indicated that "reflection lines intersect at the center of rotation". They identified the angle measure between these lines as 60° which is equal to the half of the angle of rotation.



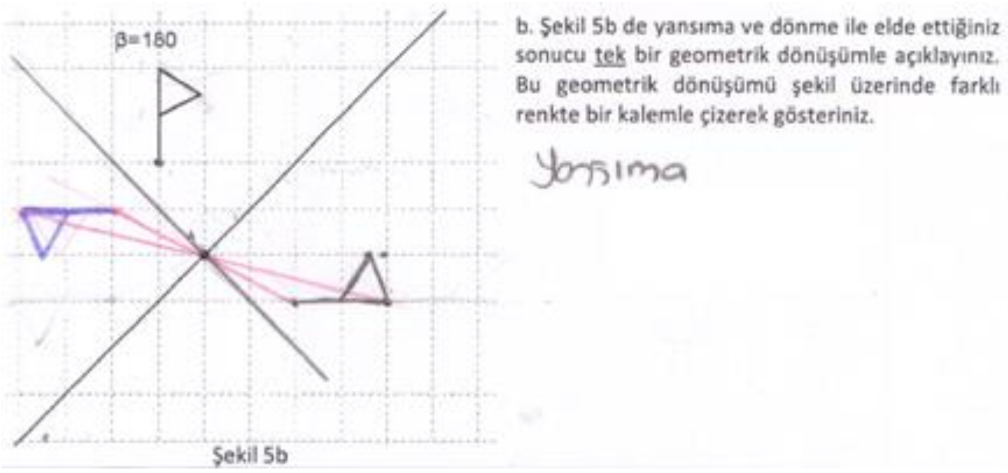
(a)



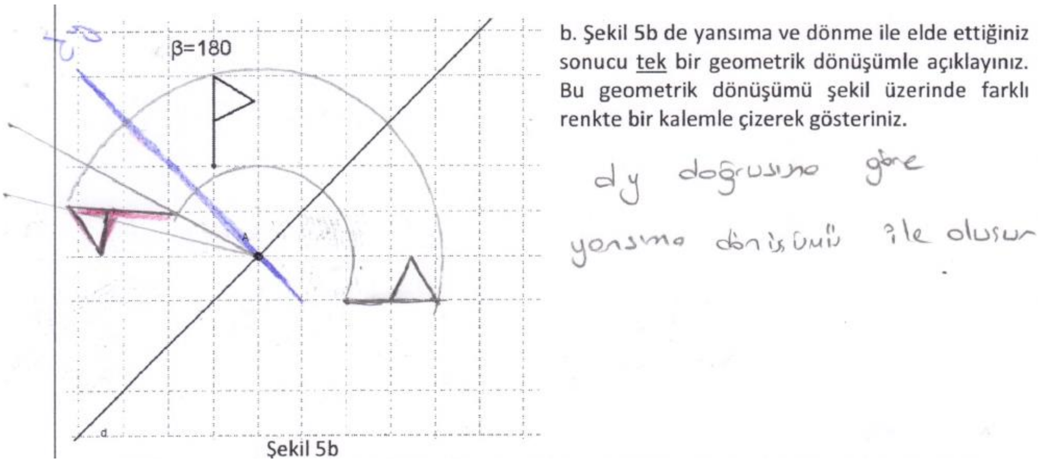
(b)

Figure 4.37. PT3 & PT4's (a) and PT15 & PT16's (b) response to Task 5A of Activity 4

Similar to Task 5A, all pairs were able to perform compositions of a reflection and a rotation correctly in Task 5B and find a single reflection alternative to the two transformations. To illustrate, PT9 & PT10's and PT13 & PT14's responses to Task 5B are presented in Figure 4.38.



(a)



(b)

Figure 4.38. PT9 & PT10's (a) and PT13 & PT14's (b) response to Task 5B of Activity 4

As in Task 5A, both pairs were able to identify the composition as a single reflection and they drew a reflection line. PT13 & PT14 stated that "it was a reflection along line d_y". In whole class discussion of Task 5, prospective teachers studied on two GeoGebra files that were prepared by the researcher in advance. The aim of these GeoGebra files was to help prospective teachers make generalizations about the composition of a reflection and a rotation when the center of rotation is on the reflection line. The corresponding GeoGebra screenshots are presented in Figure 4.39 and Figure 4.40.

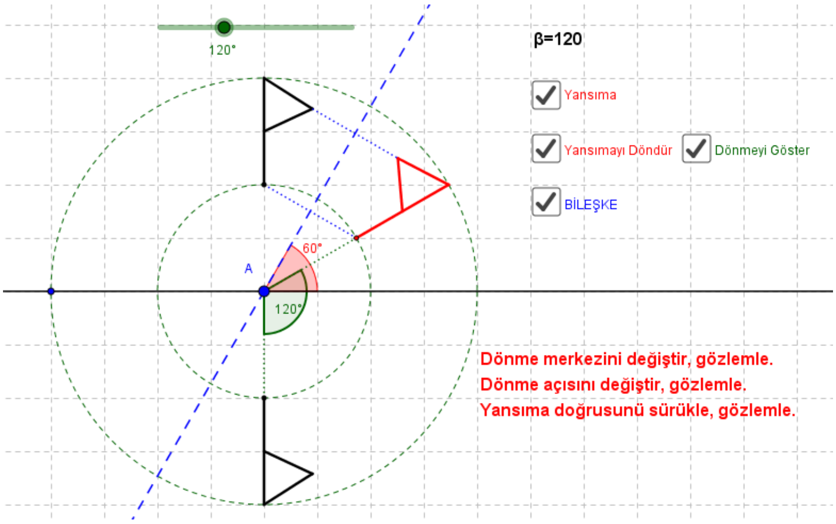


Figure 4.39. A. GeoGebra screenshot of Task 5A of Activity 4

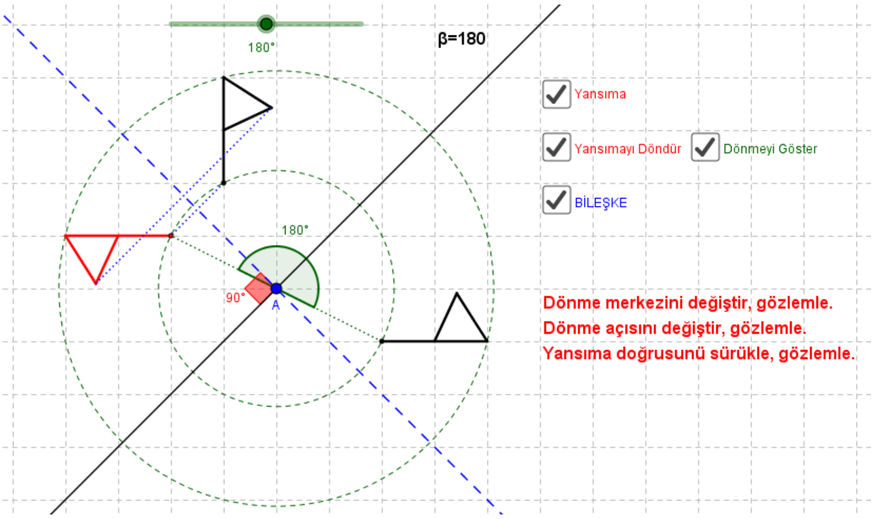


Figure 4.40. A. GeoGebra screenshot of Task 5B of Activity 4

Prospective teachers examined GeoGebra files related to Task 5A and Task 5B respectively. When the files were opened, the buttons were inactive and the objects given in this task were on the screen. The prospective teachers examined the files both in pairs and in whole class discussions. While prospective teachers were studying compositions via GeoGebra files, I asked them to generate other examples of compositions of a reflection and a rotation by changing the reflection line, by changing the center of rotation located on the reflection line and by changing the angle of rotation. They examined several examples of compositions of a reflection and a rotation and tried to make generalizations. Then, I asked prospective teachers to write down the ideas they developed during the exploration of composition of a reflection and a rotation. Their written responses are presented in Table 4.38.

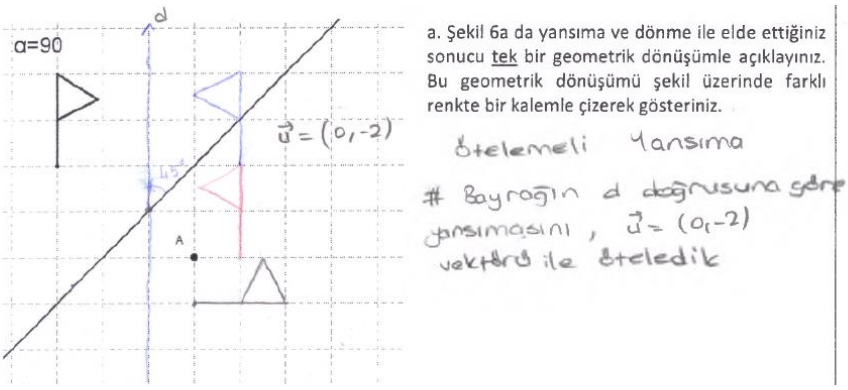
Table 4.38. Examples of prospective teachers' ideas developed during the exploration of GeoGebra file for Task 5 of Activity 4

Pairs	Ideas developed about composition of a reflection and a translation
PT1 & PT2	Yansıma ve dönme ile elde ettiğimiz şekil ilk şeklin yansıma doğrusuna göre dönme açısının yarısı kadar açı yapacak şekilde yansımasıdır. Bu durum dönme noktasının yansıma doğrusunun üstünde bulunmasıyla gerçekleşir. İkinci yansıma doğrusu dönme merkezinden geçer. [The figure obtained by reflection and rotation is a reflection whose reflection line intersects the first reflection line with an angle measure which is half of the angle of rotation. This occurs when the center of rotation is on the reflection line. The second reflection line passes through the center of rotation.]
PT3 & PT4	Sadece yansıma ile elde edebiliriz. Ancak bileşke dönüşümünde yansıma doğrusu, dönme merkezinden geçmek zorunda. [We obtain only by reflection. But, the reflection line must pass through the center of rotation.]
PT5 & PT6	Dönme noktası yansıma doğrusu üzerindeyse bileşke dönüşümdeki yansıma doğrusu da dönme noktasından geçer. Bu iki doğru arasındaki açı dönme açısının yarısıdır. [The second reflection line passes through the center of rotation when the center of rotation is on the first reflection line. The angle between these two reflection lines is half of the angle of rotation.]
PT11 & PT12	Yansıma ile açıklanır. Doğrular dönme merkezinde kesişiyor. (Bileşke dönüşümü) [It can be explained by reflection. Lines intersect at the center of rotation.]
PT13 & PT14	Yansıma ve dönme dönüşümü yapıldığında bileşke dönüşümü yansıma olur. Bu yansıma doğrusu ile önceki yansıma doğrusu dönme noktasında kesişirler. [The composition of a reflection and a rotation is a reflection. The first reflection line and the second reflection line intersect at the center of rotation.]
PT15 & PT16	Oluşan yansıma doğrusu ile ilk yansıma doğrusu arasındaki açı dönme açısının yarısı kadardır. Doğrular birbirini dönme noktasında keser. [The measure of the angle between the first reflection line and the second reflection line is half of the angle of rotation. Lines intersect at the center of rotation.]

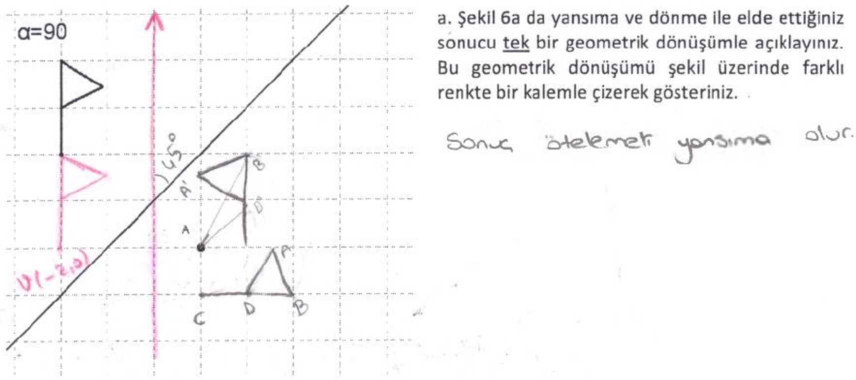
It can be seen that prospective teachers could generalize the ideas they gained in pairwise discussions after exploring the researcher prepared GeoGebra files. They generalized the idea that composition of a reflection and a rotation corresponded to a single reflection when the center of rotation was on the reflection line.

4.4.5.2. Compositions of a reflection and a rotation when the center of rotation is not on the reflection line

In Task 6 of Activity 4, prospective teachers performed a reflection and a rotation respectively and the center of rotation was not on the reflection line. Therefore, the composition of these transformations was equal to a single glide reflection. The examination of prospective teachers' responses to Task 6A showed that all pairs were able to perform the composition of a reflection and a rotation correctly. Except for PT9 & PT10, all pairs were able to find a single glide reflection alternative to each composition. To illustrate, PT11 & PT12's and PT15 & PT16's responses to Task 6A are presented in Figure 4.41.



(a)

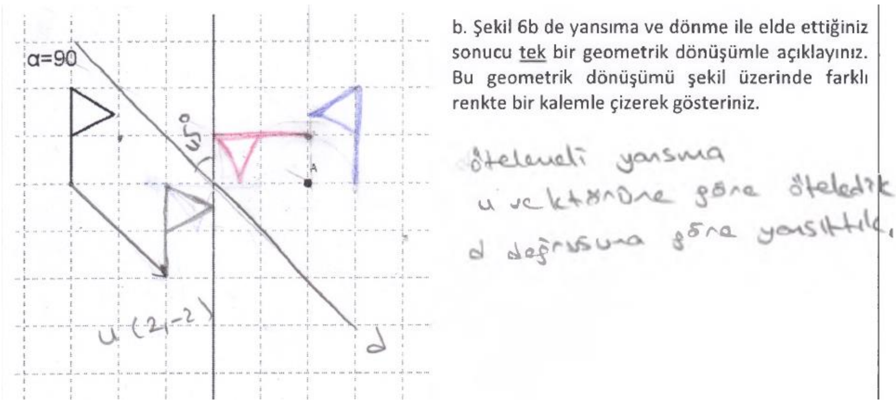


(b)

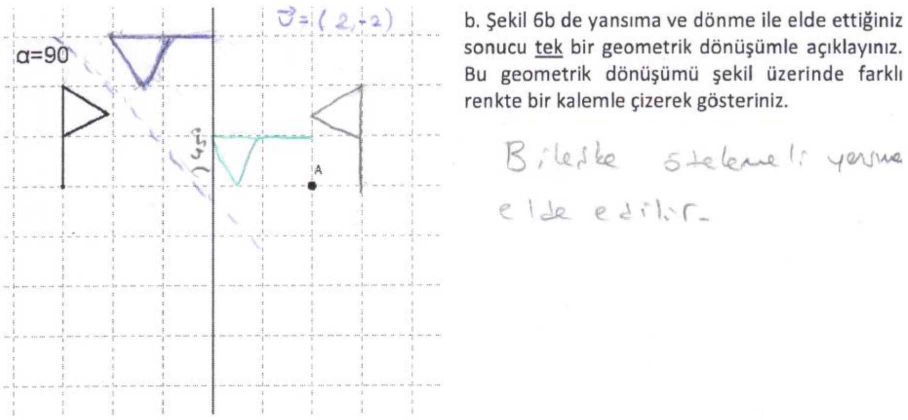
Figure 4.41. PT11 & PT12's (a) and PT15 & PT16's (b) response to Task 6A of Activity 4

Both pairs performed a reflection and a rotation respectively and identified their composition as a glide reflection. Both pairs stated that "it was a glide reflection". In PT11 & PT12's response, the reflection lines were labeled with *d* and the translation vector which was parallel to the reflection line was identified as *u*. In both responses, the slope and the location of the reflection line were determined correctly and the translation vector was equal to *u* = (-2, 0). However, PT11 & PT12 reflected the flag first and then translated the resulting flag while PT15 & PT16 first translated the flag and then reflected the resulting flag. The angle measure between two reflection lines were identified as 45 degrees by both pairs.

Similar to Task 6A, prospective teachers were able to perform the composition of a reflection and a rotation correctly. Except for PT9 & PT10, all pairs were able to find a single glide reflection alternative to each composition in Task 6B. To illustrate, PT1 & PT2's and PT5 & PT6's responses to this sub-task are presented in Figure 4.42.



(a)



(b)

Figure 4.42. PT1 & PT2's (a) and PT5 & PT6's (b) response to Task 6B of Activity 4

As in Task 6A, both pairs were able to identify the composition as a single glide reflection in Task 6B. Both pairs stated that "composition is a glide reflection. In both responses, the slope and the location of the reflection line were determined correctly and the translation vector was equal to u = (2, -2). While, PT1 & PT2 first performed a translation and then a reflection, PT5 & PT6 first performed a reflection and then a translation. In each response, the angle measure between the two reflection lines was identified as 45°.

In whole class discussion of Task 6, prospective teachers studied on two GeoGebra files that were prepared by the researcher in advance. The aim of these GeoGebra files were to help prospective teachers make generalizations about composition of a reflection and a rotation when the center of rotation was not on the reflection line. The corresponding GeoGebra screenshots are presented in Figure 4.43 and Figure 4.44.

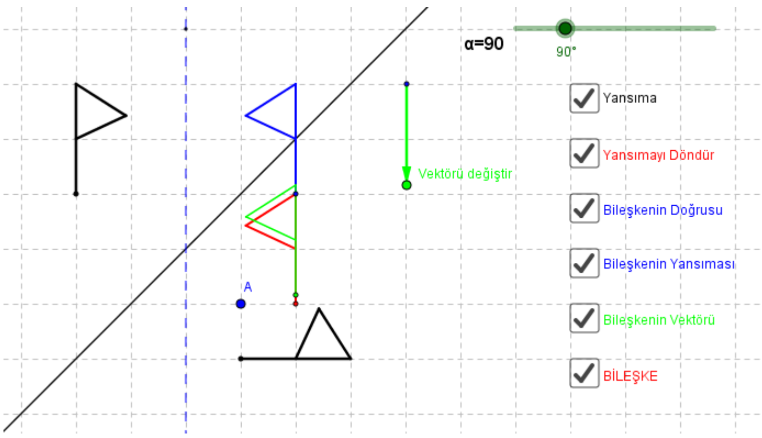


Figure 4.43. A. GeoGebra screenshot of Task 6A of Activity 4

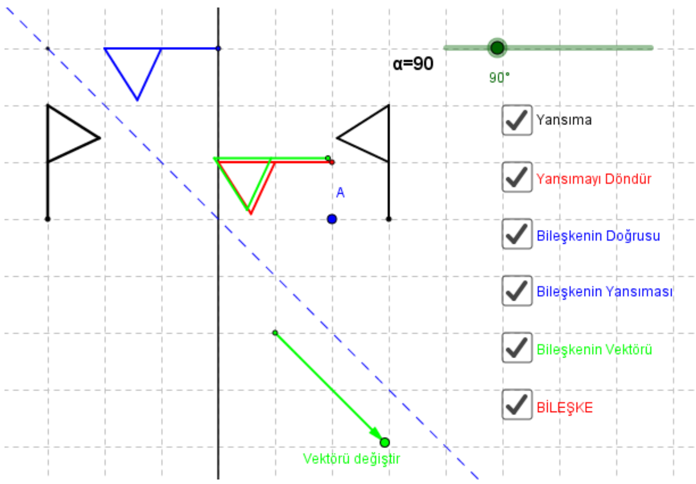


Figure 4.44. GeoGebra screenshot of Task 6B of Activity 4

Prospective teachers examined GeoGebra files related to Task 6A and Task 6B respectively. When the files were opened, the buttons were inactive and the objects given in this task were on the screen. The prospective teachers examined the files both in pairs and in whole class discussions. While prospective teachers were studying compositions via GeoGebra files, I asked them to generate other examples of compositions of a reflection and a rotation by changing the reflection line, by changing the center of rotation which was not located on the reflection line and by changing the angle of rotation. They examined several examples of compositions of a reflection and a rotation and tried to make generalizations. Then, I asked prospective teachers to write down the ideas they developed during the exploration of composition of a reflection and a rotation. Their written responses are presented in Table 4.39.

As seen in Table 4.39, during the exploration of GeoGebra files, all pairs were able to generalize the ideas they gained in pairwise discussions of Task 6. Besides, they could combine the ideas gained in Task 5 and Task 6 of Activity 4 and gained a complete understanding of the composition of a reflection and a rotation. That is, prospective teachers gained the idea that the composition of a reflection and a rotation corresponded to a single reflection when the center of rotation was on the reflection line and otherwise to a glide reflection.

Table 4.39. Examples of prospective teachers' ideas developed during the exploration of GeoGebra file for Task 6 of Activity 4

Pairs	Ideas developed about composition of a reflection and a translation
PT5 & PT6	Dönme noktesi yansıma doğrusu üzerinde değilse bileşke ötelemeli yansımadır. Ayrıca yansıma yaptığımız birinci doğruyla ötelemeli yansımanın yansıma doğrusu arasındaki açı döndürmenin yarısıdır. [If the center of rotation is not on the reflection line, then the composition is a glide reflection. Besides, the measure of the angle between the first reflection line and the reflection line of the glide reflection rotation is half of the angle of rotation.]
PT7 & PT8	Dönme merkezinin yansıma doğrusu üzerinde değilse oluşan şekli ötelemeli yansımadan elde edilir. [The figure can be obtained by a glide reflection if the center of rotation is not on the reflection line.]
PT9 & PT10	Dönme merkezi yansıma doğrusu üzerinde değilse bileşke şekil ötelemeli yansımadır 90° döndürünce iki yansıma doğrusu arasındaki açı 45° olur [The composition is a glide reflection if the center of rotation is not on the reflection line. When it is rotated 90°, the measure of the angle between reflection lines is 45°.]
PT11 & PT12	Nokta doğru üzerinde değilse ötelemeli yansıma oldu dönmenin yarısı kadar doğrular arasında açı oluyor [It is glide reflection when the point is not on the line. The measure of the angle between reflection lines is half of the angle of rotation.]
PT13 & PT14	Yansıma ve dönme dönüşümü yapıldığında merkez doğrunun dışında ise bileşke dönüşüm ötelemeli yansımadır [The composition is a glide reflection if the center of rotation is not on the reflection line.]
PT15 & PT16	Dönme noktası yansıma doğrusu üzerinde değilse sonuçtaki bileşke ötelemeli yansımadır. İlk yansıma doğrusu ile son yansıma doğrusu arasındaki açı da dönme açısının yarısı kadardır. [The composition is a glide reflection when the center of rotation is not on the reflection line. The measure of the angle between the initial reflection line and the final reflection line is half of the angle of rotation.]

4.4.6. Performing Composition of a Rotation and a Translation

The aim of Task 7 of Activity 4 was to have prospective teachers recognize that the composition of a rotation and a translation corresponds to a single rotation. For this purpose, In Task 7, a flag, a point, an angle and a vector were given on a grid. Prospective teachers were asked to rotate the flag 120 degrees around the given point and then translate the resultant flag by using the given vector. Finally, they were asked to find a single transformation alternative to the composition of a rotation and a translation. The examination of prospective teachers' responses to Task 7 showed that all pairs were able to perform compositions of a rotation and a translation correctly. Except for PT3 & PT4 and PT7 & PT8, all pairs could find a single rotation alternative to this composition. To illustrate, PT1 & PT2's, PT5 & PT6's and PT13 & PT14's responses to Task 7 are presented in Figure 4.45.

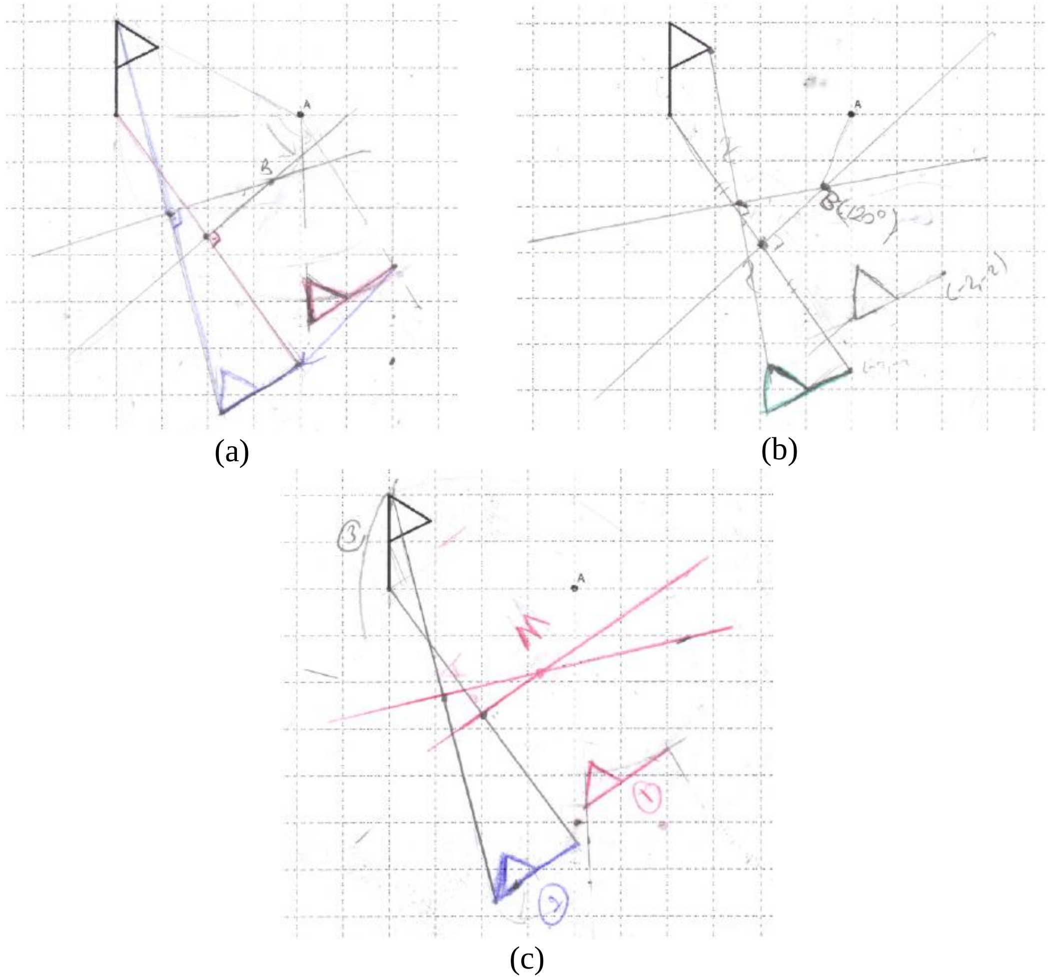


Figure 4.45. PT1 & PT2's (a), PT5 & PT6's (b), PT13 & PT14's (c) response to Task 7 of Activity 4

As seen in Figure 4.45, pairs used the perpendicular bisector to find the center of rotation. First they determined two corresponding points on the first and the final flag. Then, constructed perpendicular bisectors of line segments connecting two corresponding points. In their responses to Task 7, pairs stated that the intersection of two perpendicular bisectors is the center of rotation and that the angle of the rotation is equal to the initial angle of rotation.

In whole class discussion of Task 7, prospective teachers studied on a GeoGebra file that was prepared by the researcher in advance. The aim of this GeoGebra file is to help prospective teachers make generalizations about composition of a rotation and a translation. The corresponding GeoGebra screenshots are presented in Figure 4.47, Figure 4.48, and Figure 4.49.

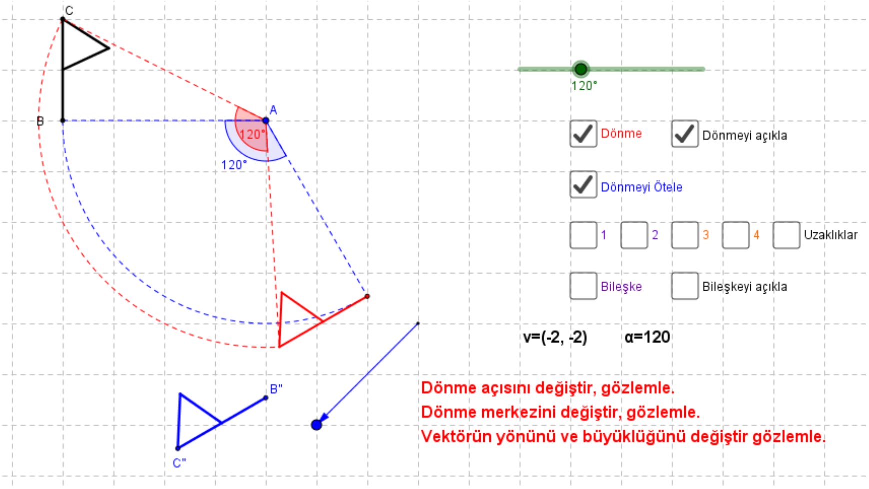


Figure 4.46. A. The first GeoGebra screenshot of Task 7 of Activity 4

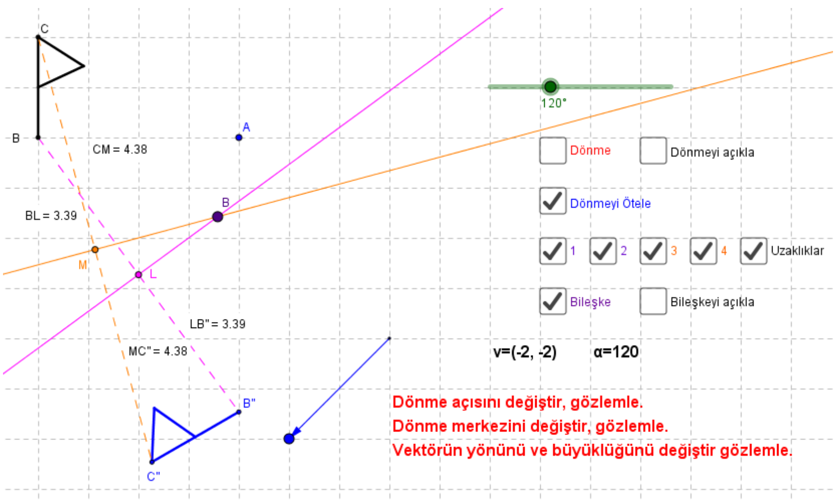


Figure 4.47. The second GeoGebra screenshot of Task 7 of Activity 4

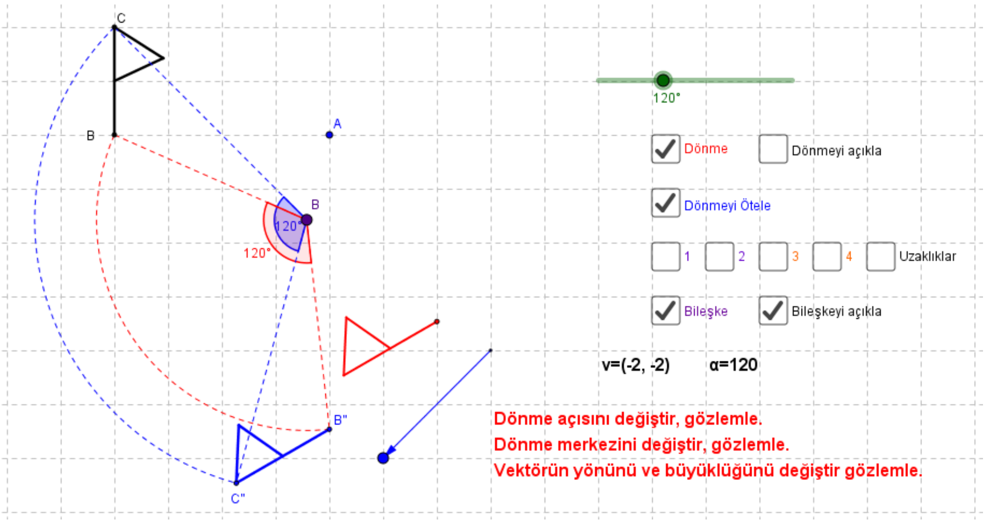


Figure 4.48. The third GeoGebra screenshot of Task 7 of Activity 4

Prospective teachers examined GeoGebra files related to Task 7. When the file was opened, the buttons were inactive and the objects given in this task were on the screen. The prospective teachers examined the files both in pairs and in whole class discussions. While prospective teachers were studying compositions via GeoGebra files, I asked them to generate other examples of composition of a rotation and a translation by changing the angle and the center of rotation and by changing the translation vector. They examined several examples of composition of a rotation and a translation and tried to make generalizations. Finally, I asked prospective teachers to write down the ideas they developed during the exploration of composition of a rotation and a translation. Their written responses are presented in Table 40.

Table 4.40. Examples of prospective teachers' ideas developed during the exploration of GeoGebra file for Task 7 of Activity 4

Pairs	Ideas developed about composition of a reflection and a translation
PT3 & PT4	Sonuç tek bir dönme ile elde edilebilir. Dönme açısı değişmez ancak öteleme yaptığımız için dönme merkezi değişir. [The composition is a rotation. The angle of rotation does not change. However, the center of rotation changes since we applied a translation.]
PT7 & PT8	Elde edilen şekli tek dönüşüm kullanılarak elde edebiliriz. Dönme dönüşümü yapılabilir. Dönme yapılırken açı değişmez sadece merkez değişir. [The composition can be explained by a single translation. Rotation can be applied. The angle of rotation does not change, only the center of rotation changes.]
PT9 & PT10	İki orta dikmenin kesiştiği yer dönme merkezimiz olur. [The intersection point of the two perpendicular bisectors is the center of rotation.]
PT13 & PT14	Öteleme ve dönme dönüşümleri yapıldığında bileşke yine dönmedir. Açı değişmez merkez değişir. 2. ve 3. şeklin orta noktalarını çizeriz ve bunların dikmelerinin kesişimleri merkezi olur. [The composition of a translation and a rotation is again a rotation. The angle of rotation does not change. The center of rotation changes.]
PT15 & PT16	Dönme ile öteleme dönüşümünün bileşkesi yine bir dönme dönüşümüdür. Dönmenin merkezini son oluşan şekil ile ilk oluşan şekil arasında belirli iki nokta seçilir. Onların arasındaki uzaklıkların orta noktalarına orta dikmeler çizilir. Ve bu dikmelerin kesişim noktası dönme noktasını belirler. Son dönmenin açısı ilk dönme açısı ile aynıdır. [The composition of a rotation and a translation is a rotation. We identify the center of rotation by selecting two points and constructing perpendicular bisectors. The intersection point of these perpendicular bisectors is the center of rotation. The final angle of rotation is the initial angle of rotation.]

As seen in Table 4.40, all pairs stated that the composition of a rotation and a translation is equal to a rotation. Other ideas gained by the pairs were that the center of the rotation is the intersection point of the two perpendicular bisectors and angle of rotation is same.

At the end of the Activity 4, prospective teachers made connections among different transformations. Making connections among different transformations is an indicator of object understanding (Flanagan, 2001). Based on the data obtained from pairwise and whole class discussions and activity sheets, it was understood that prospective teachers started to gain an object understanding of transformations. Because, object understanding of transformations involves reasoning about the composition of two or more transformations and thinking about the properties that would be preserved by the composite transformations.

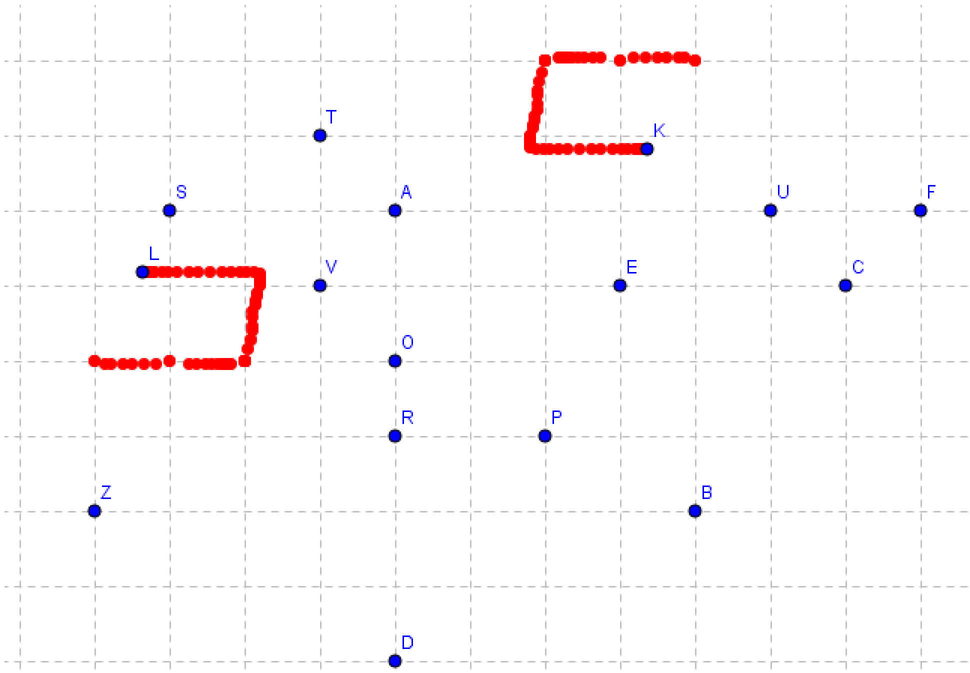
4.5. Exploring Functional Dependency of Points in the Plane

In this study, it was hypothesized that understanding functional dependency of points in the plane is critical for understanding geometric transformations as functions from R^2 to R^2 (i.e., functions defined on the plane). Through Activity 5, the researcher aimed at having prospective teachers gain ideas about properties of relations defined on the plane. These ideas are necessary (but not sufficient) for understanding geometric transformations as functions from $R^2 \rightarrow R^2$. In more detail, participants have already been familiar with algebraic functions prior to this activity and they already know that these functions include single real numbers as inputs and outputs. By the help of this activity, participants were expected to extend their knowledge of algebraic functions to functions defined on the plane. Namely, these functions take points in the plane as inputs and produce points in the plane as outputs. Participants were expected to classify the points in the plane according to their behaviors and labeled them as independent and dependent points. Moreover, prospective teachers were expected to distinguish functions from non-functions. That is, by dragging the independent point, and considering the definition of function concept, they were asked to decide whether the relation on the plane was a function or not. Another idea that the participants were expected to gain was expressing variables of functions on the plane with ordered pairs of real numbers. Finally, by observing the relative directions and relative speeds of dependent and independent points, participants were asked to determine the coordinates of the points and then the rule of the relations.

In this part, findings related to prospective teachers' exploration of functional dependency of points in the plane are presented. The data came from four tasks included in Activity 5. In these tasks, prospective teachers were expected to classify points in the plane as dependent and independent, observe the relative speed and direction of dependent and independent points, distinguish functions from non-functions and finally express rules of functions respectively. Findings related to prospective teachers' classification of dependent and independent points in the plane are presented below.

4.5.1. Classifying Dependent and Independent Points in the Plane

Thinking that points might be variables is important because it is a significant step in understanding the domain of transformations as all points in the plane. Therefore, Task 1 of Activity 5 involved classifying points in the plane as dependent and independent points. In Task 1 of Activity 5 prospective teachers were presented a GeoGebra file including several points in the grid. They were asked to observe the behaviors of the given points by dragging. Consequently, they were expected to explore and describe the relationships among the given points. The corresponding GeoGebra screenshot is presented in Figure 4.49.



Note: The traces of Point K and Point L were activated only for this figure for the purpose of explaining the nature of this task to the readers.

Figure 4.49. GeoGebra screenshot for Task 1 of Activity 5

As seen in Figure 4.49, there were 16 points in the grid. If some of the points are dragged, a change in some other points is observed as well. However, it is not possible for other points to drag. For instance, when Point K is dragged, the movement of Point L in relation to the change in the position of Point K is observed. However, there is an attempt to drag Point L, it is seen that this point cannot be dragged.

The related points were S and R; O and D; E and C; P, U, and F; A and Z; T and V; K and L. On the other hand, the dragging of Point B did not have any effect on any of the other points. Examination of prospective teachers' activity sheets and audio recordings of pairwise discussions showed that all prospective teachers could determine all relations among the points by dragging them correctly in Task 1A of Activity 5. Although this task had an important role in having prospective teachers gain gradually the ideas planned by the researcher for this activity, it was fairly simple to respond. Therefore, it was seen sufficient for one prospective teacher to show the relations among the points on the smart board. Consequently, PT16 showed them on the board during whole class discussion.

In Task 1B of Activity 5, the prospective teachers were asked to classify the given 16 points into two by considering their general characteristics. The prospective teachers were given some time to discuss this with their partners. In whole class discussion, the following dialogue took place.

Researcher: How did you group the points, and how did you label them?

PT11: Some points are movable whereas some others are immovable. For instance, Points R, C, and D do not move when we drag. Namely, they seem independent. Points S, E, and O are dependent on R, C, and D.

Researcher: Do you have any other idea?

PT15: Some points affect other points so we can group the points as affecting points and affected points.

Researcher: Ok, what else?

PT1: We think that some points are dependent on some other points. So, the whole points can be grouped into two as independent and dependent points.

Researcher: Well, your friends took into consideration points' movability when grouping the points and suggested three different labels. Which one would you prefer?

Students: Independent and dependent points!

Researcher: Then, which points are independent and dependent points?
PT4: S, O, E, P, A, T, and K are independent points, and R, D, C, U, F, Z, V, and L are dependent points.
Researcher: Well done! Let's move on to the next question.

In the above given dialogue, prospective teachers PT11 suggested considering movability of the points, PT15 suggested considering the effects of points on each other and PT11 suggested considering dependability of the points to each other. As understood by the above given dialogue, participants decided to label these groups as dependent and independent points at the end of the discussion. That is, by means of whole class discussion, the prospective teachers agreed upon using 'dependent-independent' terminology.

To summarize, the prospective teachers examined the behavior of points by dragging and classified them according their behaviors. As mentioned before, thinking that points might be variables is important because it is a significant step in understanding the domain of transformations as all points in the plane. According to APOS Theory, process conception requires understanding the domain of transformations as all points in the plane. Thus, due to gaining one of the many ideas necessary for the process conception, it can be said that prospective teachers were able to progress towards this conception.

Next, the prospective teachers were guided by the researcher to observe the relative speeds and directions of dependent and independent points. Related findings are presented below.

4.5.2. Observing the Relative Speeds and Directions of Dependent and Independent Points

In Task 2 of Activity 5, prospective teachers were expected to observe the relative directions and relative speeds of independent and dependent points by dragging. All pairs dragged the given points, noted down the relative directions and the speeds of the points and finally filled in the table given in Task 2 in pair work. For instance, PT15 & 16's table is presented in Figure 4.50.

İlgili Noktalar	Noktaların Birbirlerine göre Yönleri ile ilgili Gözlemler	Noktaların Birbirlerine göre Hızları ile ilgili Gözlemler
S → R	S sadece sağa-sola, R 'de yukarı-aşağı gidiyor. S sağa ve R yukarı S sola ve R aşağı	Hızları eşit
P → U F	Yönleri aynı Aşağı -yukarı ve sağa ve sola gidiyor	Hızları eşit
K → L	Yönleri zıt. Doğrultuları aynı	Hızları eşit
A → Z	Sağa ve sola aynı zamanda yukarı aşağı gidiyor.	Hızları eşit
T → V	T sağa gidiyor V yukarı T sola " V aşağı T yukarı " V sola T aşağı " V sağa	Hızları eşit
O → D	Yönleri aynı Sağa -sola ve aşağı -yukarı gidiyor.	D üç birim (3V hızı) O bir birim (V hızı)
E → C	Yönleri aynı Aşağı -yukarı ve sağa -sola gidiyor.	E bir birim C iki birim sağ -solda farklı yukarı aşağı doğrultuda hızları aynı
B	Tek başına	

Figure 4.50. PT15 & PT16’s response to Task 2 of Activity 5

All prospective teachers took notes similar to the one presented in Figure 4.50 about the behavior of independent and dependent points (variables). In essence, by having prospective teachers observe the behaviors of points, the researcher aimed at having them connect relative speeds and directions to the coordinates of independent and dependent points (variables). By making these connections, the prospective teachers were expected to understand functional relationships among independent and dependent points and then find the rules of the functions in Task 4 of Activity 5. The researcher felt that it was compulsory for participants to evaluate whether any given relation is a function or not since exploring functions from a geometric perspective is a novel idea for prospective teachers. The researcher suggested that it might be relevant for a learner to relate the function concept with a geometric perspective in order for him/her to understand geometric transformations as functions. Therefore, the next part is related to distinguishing functions from non-functions.

4.5.3. Distinguishing Functions from Non-Functions

Task 3 of Activity 7 asked prospective teachers to examine the relations between points on the screen in terms of being a function or not. Before moving on to Task 3 of Activity 5, the researcher asked prospective teachers to define verbally what ‘function’ was. The following dialogue took place among the researcher and the participants during discussion.

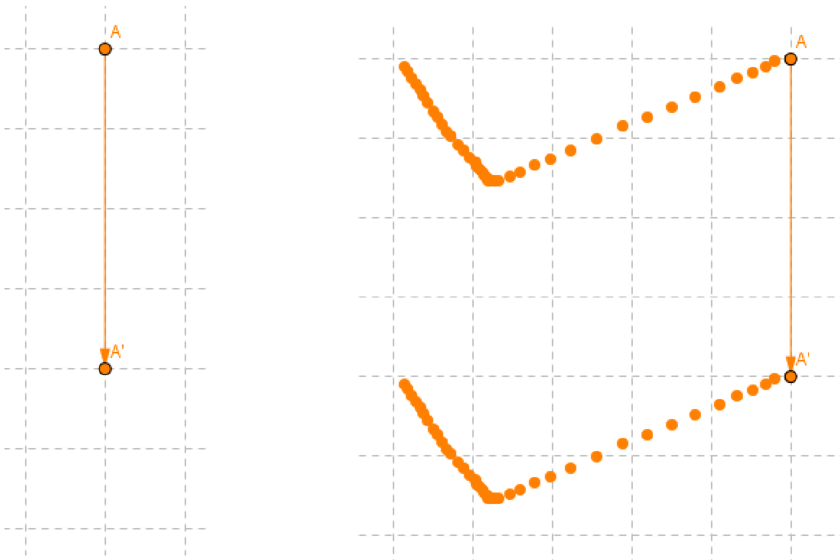
> Researcher: What is a function?
> PT9: We have two sets as a domain and range. The relations between these two sets are called a function.
> Researcher: Do these relations have other properties?
> PT14: There must not be any unmapped element in the domain.
> Students: An element in the domain must not match with two different elements.
> Researcher: Well, ok then, please consider these points when examining the relations in Task 3.

In Task 3 of Activity 5, prospective teachers were presented four GeoGebra files. They explored these files by dragging the independent points in pair work and whole class discussions. In each of these files there were two relations. Thus, prospective teachers examined eight relations in total and determined whether the given relations were a function or not. Examination of prospective teacher’s activity sheets and audio recordings of pair works showed that all pairs could correctly identify the functions from non-functions and they accepted the Relations AA', EE', DD', GG' HH' as functions. Prospective sample responses are presented in Table 4.41.

Table 4.41. Prospective teachers' sample explanations to the functions in Task 3 of Activity 5

Pairs	Prospective teachers' sample explanations to the functions		
PT5 & PT6	A → A'	Fonksiyon	A'nın sürüklendiği her nokta için bir A' değeri olduğu için
PT11 & PT12	A → A'	+	A noktasını hareket ettirdiğimizde A' noktası tek bir tanedir.
PT1 & PT2	D→D'	Fonksiyondur.	Tanım kümesindeki iki eleman görüntü kümesinde bir elemana gidebilir
PT7 & PT8	D→D'	Fonksiyondur	Tanım kümesindeki iki tane eleman değer kümesindeki bir elemana gidebilir. Bu fonksiyon olmasını etkilemez
PT3 & PT4	E→E'	Fonk	Ne yaparsak yapalım E'nin karşısında yalnız 1 E' geliyor.
PT9 & PT10	E→E'	Fonksiyon	Tanım kümesindeki her eleman değer kümesinden bir elemana gitmiştir.
PT7 & PT8	G→G'	Fonksiyon	Tanım kümesindeki her eleman görüntü kümesinde bir elemana karşılık gelir.
PT13 & PT14	G→G'	fonksiyondur.	G yer değiştirdiğinde G' ile eşleşiyor
PT11 & PT12	H→H'	+	Sabit fonksiyondur. Tanım kümesindeki H elemanını istediğimiz kadar değiştirsekte, değer kümesindeki H' elemanının yeri değişmiyor, sabit kalıyor
PT15 & PT16	H→H'	fonksiyon	H ye verilen her değer için (H') sadece bir noktaya gidiyor. Bu da sabit fonksiyondur.

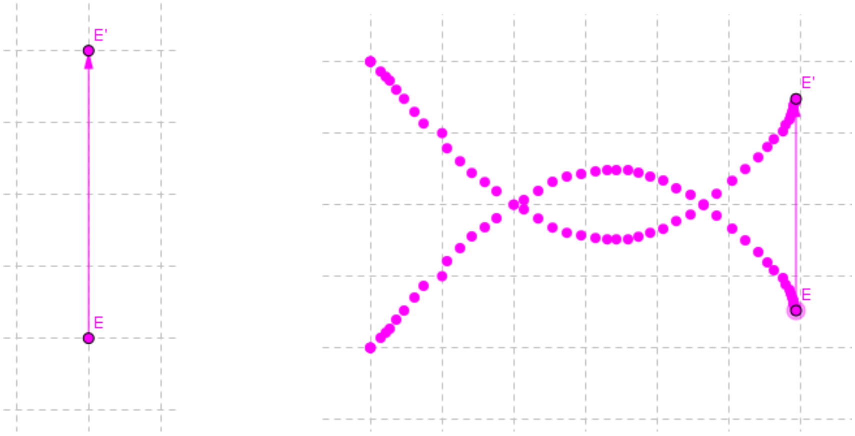
In the following paragraphs the eight relations examined by prospective teachers were explained. First, functions and then non-functions were explained. The GeoGebra screenshots for Relation AA', EE', DD', GG', and HH' are presented in Figure 4.51, Figure 4.52, Figure 4.53, Figure 4.54, and Figure 4.55 respectively. The traces appearing in the following screenshots were activated only for the purpose of explaining the nature of the relations to the readers. It is important to note that the trace tool was not activated while prospective teachers were examining the relations.



(a) (b)

Figure 4.51. Initial GeoGebra screenshot (a) and the screenshot for dragged independent point A (b) for Task 3 of Activity 5

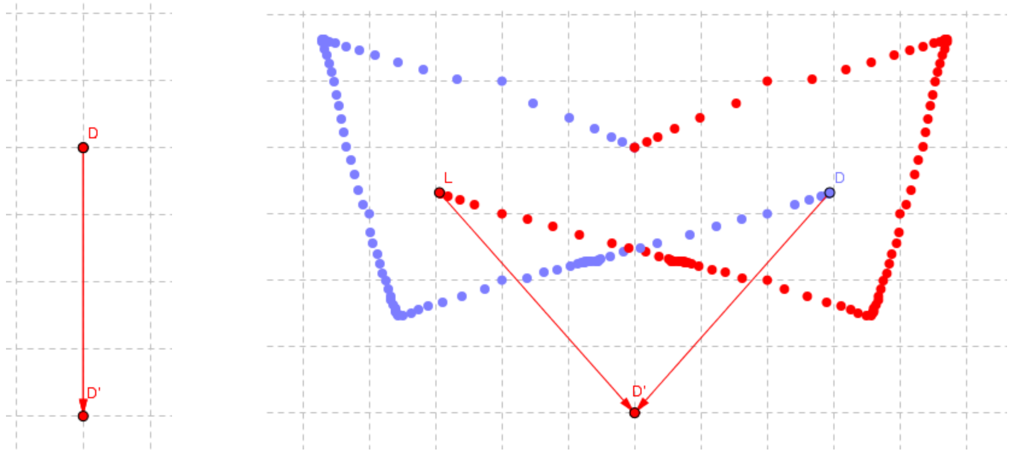
Relation AA' presented in Figure 4.51 satisfied all properties of a function. When Point A, the independent variable, was dragged on the screen, a single point A' always corresponded to it. Therefore, all prospective teachers accepted Relation AA' as function. As an example, see PT5 & PT6's and PT11 & PT12's responses in Table 4.41.



(a) (b)

Figure 4.52. Initial GeoGebra screenshot (a) and the screenshot for dragged independent point E (b) for Task 3 of Activity 5

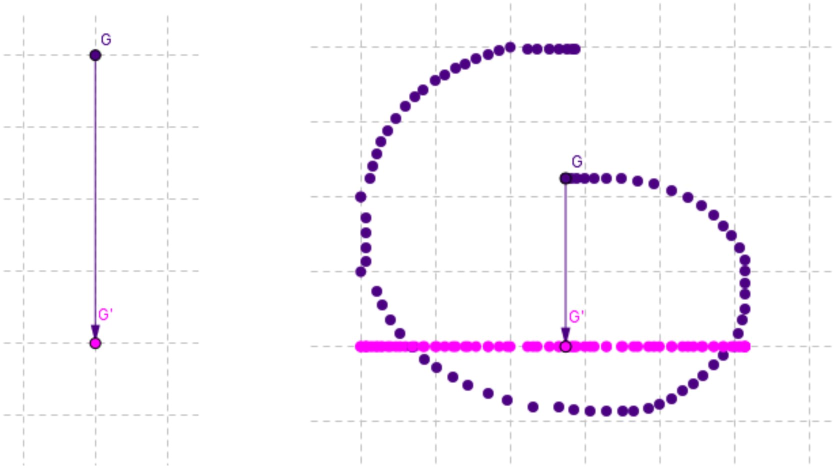
Relation EE' presented in Figure 4.52 satisfied all properties of a function. It is important to note that the impact of the independent variable of Relation AA' was different from that of Relation EE' on the dependent variable. Although the prospective teachers were not guided to relate these functions with geometric transformations at this moment, the Relation AA' was a translation while the Relation EE' was a reflection along a horizontal line. All prospective teachers accepted Relation EE' as a function. As an example, see PT3 & PT4's and PT9 & PT10's responses in Table 4.41.



(a) (b)

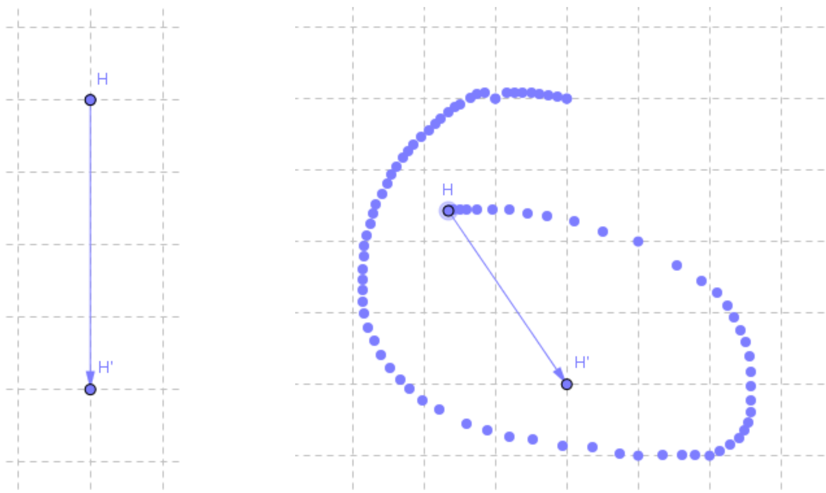
Figure 53. Initial GeoGebra screenshot (a) and the screenshot for dragged independent point D (b) for Task 3 of Activity 5

The prospective teachers accepted the Relation DD' as a function. They explained that both Point L and Point D matched with Point D' and added that this was not a problem for being a function. As an example, see PT1 & PT2's and PT7 & PT8's responses in Table 4.41.



(a) (b)

Figure 4.54. Initial GeoGebra screenshot (a) and the screenshot for dragged independent point G (b) for Task 3 of Activity 5



(a) (b)

Figure 55. Initial GeoGebra screenshot (a) and the screenshot for dragged independent point H (b) for Task 3 of Activity 5

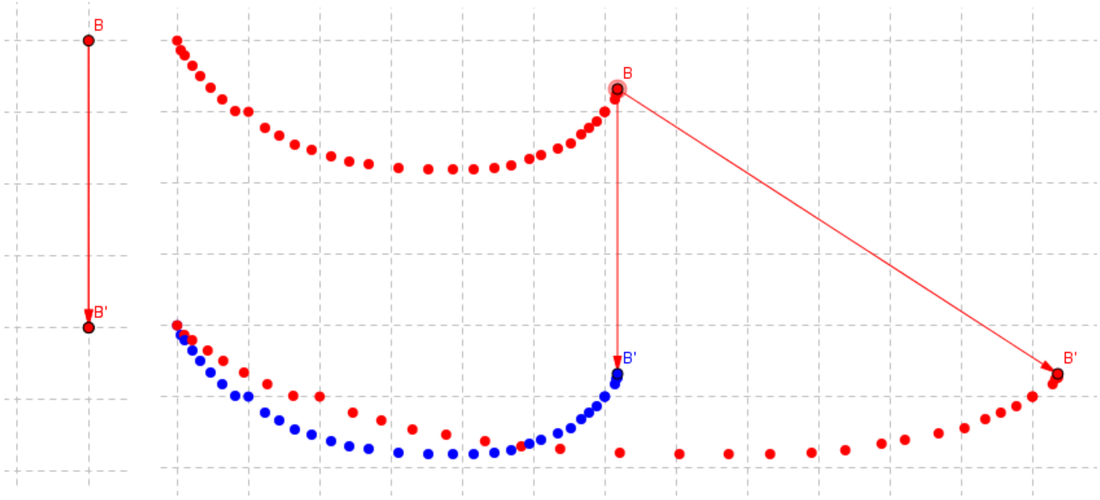
The Relation GG' and the Relation HH' presented in Figure 4.54 and Figure 4.55 satisfied all properties of a function. Thus, they were functions. Except for the impact of its independent variable on the dependent variable, Relation GG' was analogues to Relation AA' and Relation EE'. In Relation HH', the Point H could be dragged in all directions on the screen. However, it did not have any impact on Point H' since Point

H' remained invariant despite all draggings. The prospective teachers pointed out that this situation was not an obstacle for Relation HH' to be a function. They even stated that Relation HH' was an example for a constant function. See PT7 & PT8's and PT13 & PT14's responses for Relation GG' and see PT11 & PT12's and PT15 & PT16's responses for Relation HH' in Table 4.41. Examinations of prospective teacher's activity sheets and audio recordings of pair works showed that all pairs could correctly identify the relations that were non-functions. That is, they did not accept the Relations BB' and FF' as functions. Prospective sample responses are presented in Table 4.42.

Table 4.42. Prospective teachers' sample explanations for Relation BB' and FF' in Task 3 of Activity 5

Pairs	Prospective teachers' sample explanations for Relation BB' and FF'		
PT3 & PT4	B→ B'	Fonk. değil	B'nin 2 tane B' şeklinde görüntüsü var. Bu fonk.'a aykırı bir durum.
PT9 & PT10	B→ B'	Fonksiyon değil	Tanım kümesinden 1 eleman değer kümesinden 2 elemanla eşleşiyor.
PT11 & PT12	B→ B'	—	B yi hareket ettirdiğimizde 2 farklı B' noktasına götürdüğü için fonksiyon değil. (iki farklı görüntü)
PT5 & PT6	F→F'	Fonksiyon Değil	F'nın bir değerinde 2 tane görüntüsü olduğu için
PT13 & PT14	F→F'	fonksiyon değildir	F yer değiştiğinde 2 (F') ile eşleşiyor.
PT15 & PT16	F→F'	fonksiyon değil	Tanım kümesindeki bir eleman görüntü kümesinde iki elemana gittiği için fonksiyon olmaz.

In the following paragraphs the two relations which were not accepted as functions by the prospective teachers are explained. The GeoGebra screenshots for Relation BB' and FF' are presented in Figure 4.56 and Figure 4.57 respectively.



(a) (b)

Figure 4.56. Initial GeoGebra screenshot (a) and the screenshot for dragged independent point BB' (b) for Task 3 of Activity 5

The Relation BB' presented in Figure 4.56 does not satisfy all properties of a function. When Point B, the independent variable, was dragged on the screen, two points (two B's) corresponded to it. Therefore, none of the prospective teachers accepted Relation BB' as a function. As an example, see PT3 & PT4, PT9 & PT10, and PT11 & PT12's responses in Table 4.42.

Relation FF' presented in Figure 4.57 does not satisfy all properties of a function. As in Relation BB', the Point F matched with two points (two F's). That is, when Point F, the independent variable, was dragged on the screen, two points (two F's) corresponded to it. Therefore, the Relation FF' was not accepted as a function by the prospective teachers. As an example, see PT5 & PT6, PT13 & PT14, and PT15 & PT16's responses in Table 4.42.

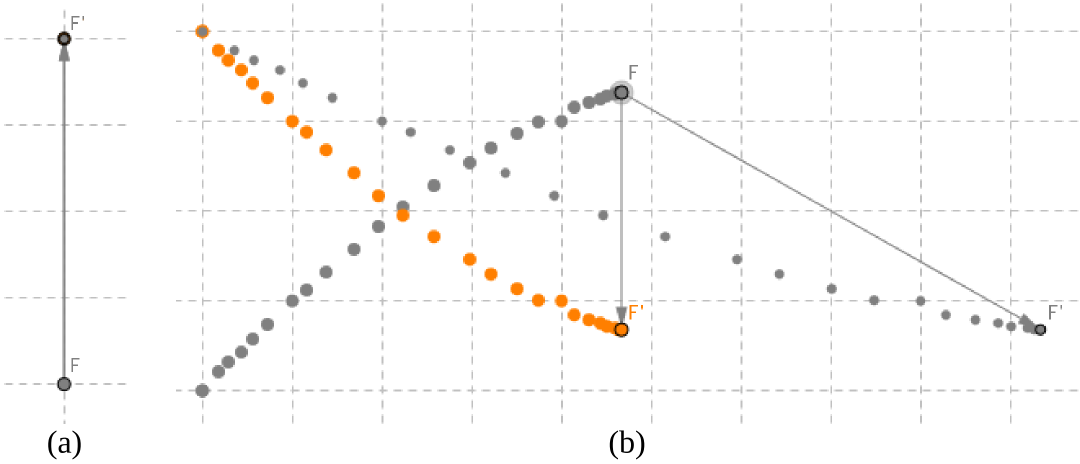


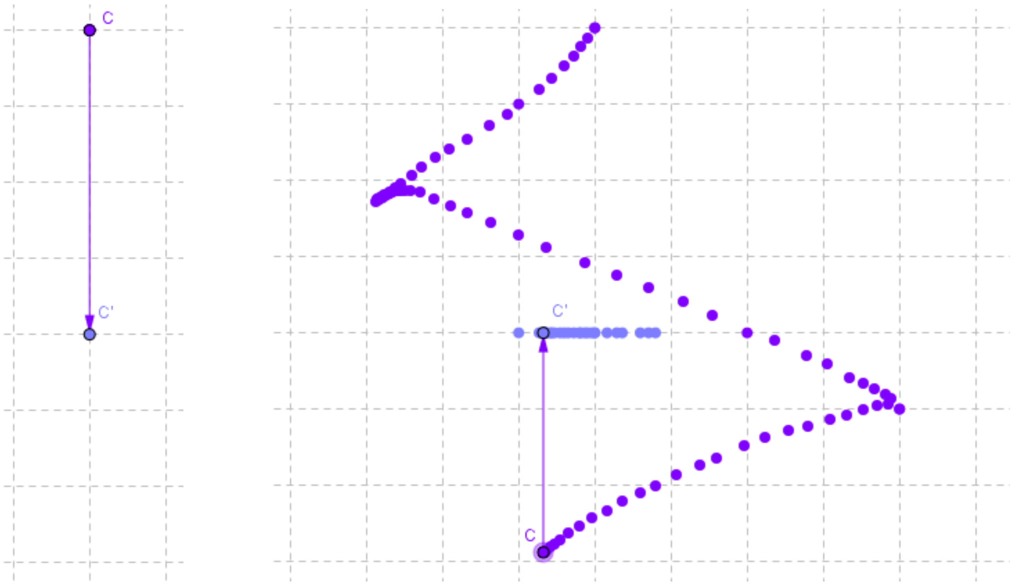
Figure 4.57. Initial GeoGebra screenshot (a) and the screenshot for dragged independent point FF' (b) for Task 3 of Activity 5

Examination of prospective teacher's activity sheets and audio recordings of pair works showed that except for PT1 & PT2 and PT9 &PT10, all pairs stated that the Relation CC' was not a function. PT1 & PT2 and PT9 &PT10 stated that the Relation CC' was a function that is defined on a specific region. Besides, Although PT3 & PT4 and PT7 & PT8 did not accepted the Relation CC' as a function, they explained that this relation could be accepted as a function when it was defined on a specific region. PT5 & PT6, PT11 & PT12, PT13 & PT14, and PT15 & PT16 explained that point C was not defined for some region or that point C was unmapped for some region. All prospective teachers' responses are presented in Table 4.43.

Table 4.43. Prospective teachers’ explanations for Relation CC' in Task 3 of Activity 5

Pairs	Prospective teachers’ sample explanations for Relation CC'		
PT1 & PT2	$C \rightarrow C'$	Fonksiyondur.	Fakat tanımlı old. aralıkta fonksiyondur.
PT3 & PT4	$C \rightarrow C'$	Fonk. Değil.	Belli aralıklarda fonk, bölgelerinde değil yani tanım kümesini sınırlamamız gerekiyor.
PT5 & PT6	$C \rightarrow C'$	Fonksiyon değil	Tanımsız noktalar olduğu için
PT7 & PT8	$C \rightarrow C'$	Fonksiyon değil	Tanım kümesinde her elemanın değer kümesinde bir elemana karşılık gelmesi lazım. Belli aralık tanımlanırsa fonksiyon denilebilir.
PT9 & PT10	$C \rightarrow C'$	Fonksiyon	Belli bir tanım aralığı için
PT11 & PT12	$C \rightarrow C'$	—	C'nin bazı değerlerinde görüntüsü olmadığı için fonk. değildir.
PT13 & PT14	$C \rightarrow C'$	fonksiyon değildir.	Çünkü bazı noktalarda C noktasının görüntüsü olmuyor.
PT15 & PT16	$C \rightarrow C'$	fonksiyon Değil	C tanım kümesinde boşta kalan eleman vardır.

As seen in Table 4.43, PT1 & PT2, PT3 & PT4, PT7 & PT8, and PT9 & PT10 stated that the Relation CC' could be a function when it was defined on a specific region. Other pairs stated that this relation was not a function. The screenshot presented in Figure 4.58 explains the nature of the Relation CC'.



(a) (b)

Figure 4.58. Initial GeoGebra screenshot (a) and the screenshot for dragged independent point CC' (b) for Task 3 of Activity 5

When all points in the plane (GeoGebra screen in this case) were considered as a domain, CC' was a not function. However, when a limited region on the screen was accepted as a domain, CC' turned into a function. On GeoGebra screen, when Point C was dragged in all directions, Point C' appeared in some regions and disappeared in some other regions. Actually, by designing Relation CC' in this way, the researcher aimed at having prospective teachers ponder on the concept of a domain through a geometric perspective. As it was expected, some of the pairs (PT1 & PT2, PT3 & PT4, PT7 & PT8, and PT9 & PT10) noticed the difference between the Relation CC' and other relations.

The prospective teachers discussed each relation in Task 3 of Activity 5 during whole class discussions. For all relations excluding Relation CC', the prospective teachers were in consensus with each other in terms of being a function or not. During whole class discussion of Relation CC', some participants argued that Relation CC' was a function while others stated that it was not. As mentioned before, four of the pairs explained that this relation could be a function when it was defined on a specific region (see Table 4.43). The dialogue that took place during the discussion of Relation CC' is presented as follows.

Researcher: Some of you (PT5 & PT6, PT11 & PT12, PT13 & PT14, and PT15 & PT16) argue that Relation CC' is not a function.

PT13, would you please tell me why Relation CC' is not a function?

PT13: In order for it to be a function, all elements in the domain should match. When we change the location of Point C in our screen, in some region the point C' disappeared. This means, the range becomes undefined.

PT14: Actually, I changed my mind, the Relation CC' turns into a function in some limited regions.

Researcher: PT14, would you please elaborate on it?

PT14: If we specify an interval, the Relation CC' becomes a function.

Researcher: Can you tell us more about this interval?

PT14: I mean, the domain. If we specify the elements of the domain into a specific interval, the Relation CC' becomes a function. Let's say this point is 1 and this is 3 (pointing to the borders of the region that Point C' appeared when Point C was dragged horizontally). Besides, Point C' always appears when we drag Point C vertically in this interval. We can say that the Relation CC' is a function in the [1, 3] interval on x-axis.

In the above given dialogue, PT13 explained why they thought that Relation CC' was not a function. Her partner, PT14, stated that she changed her mind and explained that the Relation CC' turns into a function when the domain was considered as a specific region. Researcher asked the remaining three pairs (PT5 & PT6, PT11 & PT12, and PT15 & PT16) whether they changed their mind. They stated that they started to think that it was a function. The discussion about the domain of the Relation CC' continued in Activity 6.

Briefly, Task 3 of Activity 5 was related to identifying whether a given relation was a function or not. After prospective teachers became familiar with this type of tasks, the researcher guided them towards expressing rules of functions. Thus, the researcher presented them Task 4 of Activity 5.

4.5.4. Expressing Rules of Relations

In Task 1 of Activity 5, the points were presented to participants on a grid. In Task 4 of Activity 5, the same points were used but this time they were located on a coordinate system. The corresponding GeoGebra screenshot is presented in Figure 4.59.

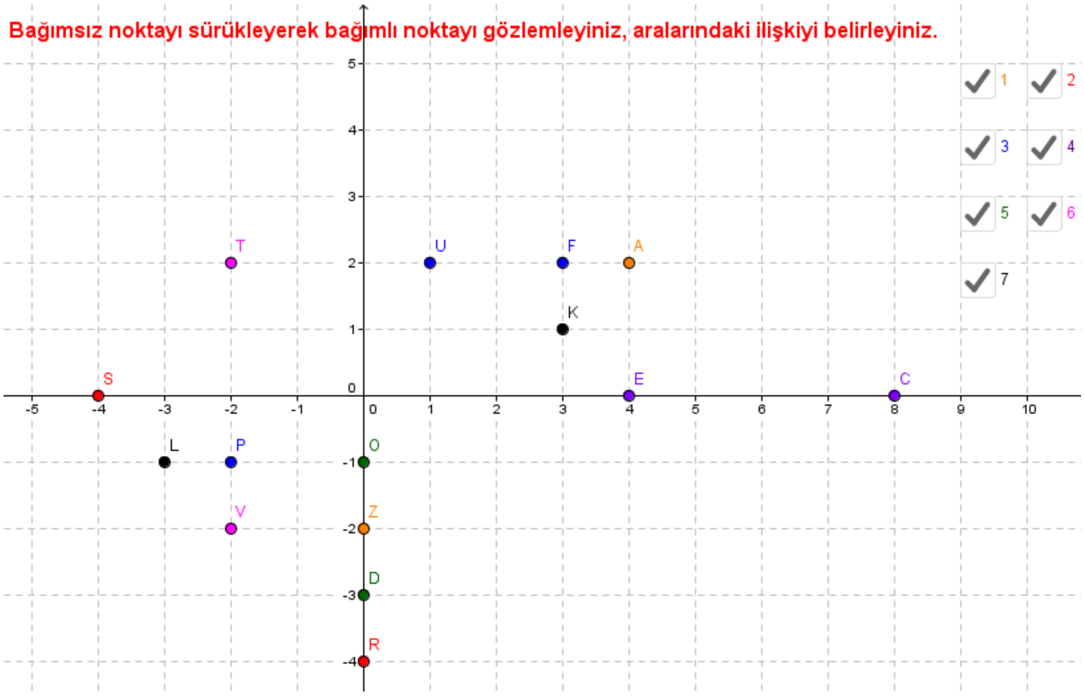
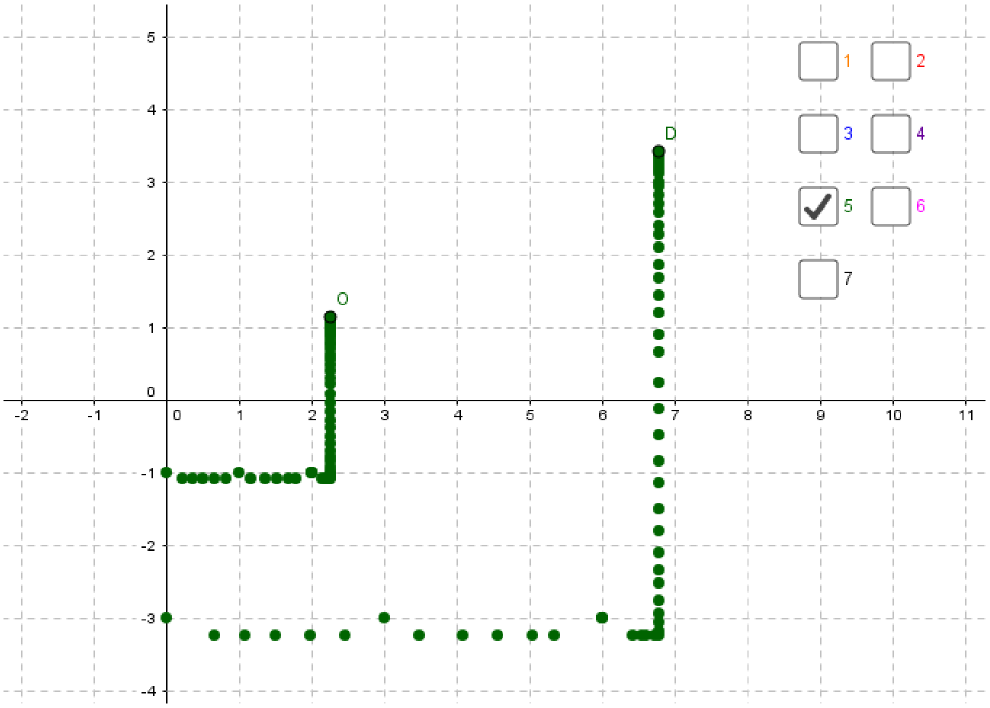


Figure 4.59. GeoGebra screenshot for Task 4 of Activity 5

To examine each relation separately, the corresponding buttons were clicked in an order. For instance, clicking only Button 5 made the Relation OD become apparent. Other points did not appear on the screen. By this way, the prospective teachers had an opportunity to examine each relation in detail. While prospective teachers were examining the relations, the trace tool was not activated. The corresponding screenshot in presented in Figure 4.60. As seen in Figure 60, point D was three times faster than point O. Therefore, the rule of the Relation OD was (x, y) → (3x, 3y).



Note: The traces of Point O and Point D were activated only for this figure for the purpose of explaining the nature of this task to the readers.

Figure 4.60. GeoGebra screenshot for Relation OD in Task 4 of Activity 5

In Task 2 of Activity 5, prospective teachers observed the relative directions and speeds of related points such as point O and point D. Here, they were expected to determine the coordinates of points for each relation and write them as ordered pairs. By writing the relationship between an independent and dependent point, prospective teachers were asked to find the rules of relations (i.e., the relations which are functions or not functions). The relations and their rules included in Task 4 of Activity 5 are presented in Table 4.44.

Table 4.44. The relations and their rules included in Task 4 of Activity 5

Relation	Rule	Relation	Rule
A →Z	(x, y) → (x - 4, y - 4)	E→C	(x, y) → (2x, y)
S→R	(x, 0) → (0, x)	O→D	(x, y) → (3x, 3y)
P→U	(x, y) → (x+3, y+3)	T →V	(x, y) → (-y, x)
P→F	(x, y) → (x+5, y+5)	K →L	(x, y) → (-x, -y)

Examination of prospective teachers' activity sheets regarding Task 4 of Activity 5 showed that all prospective teachers were able to express the rules of all relations correctly. Prospective teachers' sample responses to this task are presented in Table 4.45.

Table 4.45. Prospective teachers' sample responses to Task 4 of Activity 5

Pairs	Relation	Rule	Explanation
PT3 & PT4	A→Z	$(4,2) \to A$ $(0,-2) \to Z$	$x' \to (x-4)$, $y' \to (y-4)$ } Eşit. $(x,y) \to (x-4, y-4)$ Hız ve yön aynı eklenme çıkarma etkiliyor?
PT13 & PT14	S→R	$S(x,0)$ $R(0,x)$	hız aynı Doğrultuları değişiyor
PT1 & PT2	P→U	$(x,y) \to (x+3, y+3)$	Hızları aynıdır katsayı yoktur. Noktalar arasındaki uzaklık her zaman eşittir.
PT1 & PT2	P→F	$(x,y) \to (x+5, y+3)$	Hızları aynıdır kat sayı yoktur.
PT7 & PT8	E→C	$(x,y) \to (2x,y)$	aşağı-yukarı hareket ederken aynı hızdalar sağa-sola hareket ederken C'nin hızı E'nin hızının 2 katı olur. x değerinin 2 kat olduğundan
PT5 & PT6	O→D	$(x,y) \to (3x,3y)$	Yönleri aynı D, O'nun 3 katı hızla
PT11 & PT12	T→V	$(x,y) \to (-y,x)$	Hızları eşit, yön olarak orijine göre 90° dönme.
PT15 & PT16	K→L	$(x,y) \to (-x,-y)$	Hızları eşit yönleri zıt Orijine göre simetriktir.

As seen in Table 4.45, prospective teachers could explain the rules of relations. Besides, although it was not asked, some pairs (e.g., PT11 & PT12 and PT15 & PT16) related these relations to geometric transformations. For instance, PT11 & PT12 explained that Relation TV was an example of a rotation and PT15 & PT16 explained that Relation KL was an example of a symmetry.

In Activity 5, all prospective teachers gained some experiences about functions defined on the plane. In this activity, prospective teachers explored many functions defined on the plane including functions which are examples of geometric transformations and other function examples. Although prospective teachers were not given information about the type of functions being explored, they also gained important ideas related to understanding geometric transformations as functions from $R^2 \rightarrow R^2$.

The researcher deliberately delayed informing prospective teachers about whether the function being explored was a geometric transformation or not. Thus, in Activity 5, prospective teachers did not have the chance to recognize that geometric transformations are in fact specific functions defined on the plane. Besides, they were not expected to relate geometric transformations to functions in this activity. The researcher postponed covering these ideas until Activity 6. Because, the researcher planned to teach the domain concept at the beginning of Activity 6 and then have prospective teachers explore geometric transformations as R^2 to R^2 functions.

4.6. Exploring Geometric Transformations as $R^2 \rightarrow R^2$ Functions

Towards the end of Activity 5, prospective teachers began to explain relations via geometric transformations as they developed fluency with the rules of functions. That is, although they were not asked to explain the relationship between behaviors of points and coordinates by means of geometric transformations, the researcher observed such notes in their activity sheets. At the end of Activity 6, prospective teachers are expected to be able to relate these relations with geometric transformations and then to define geometric transformations as functions. Besides, in this activity, prospective teachers were asked to identify the domain and range and the elements of these two sets.

In this part, findings related to prospective teachers' exploration of transformations as R^2 to R^2 functions are presented. The data came from both pairwise and whole class discussions in Activity 6. First, the findings related to prospective teachers' exploration of domain and range are presented below.

4.6.1. Recognizing Variables of Transformations as Points in the Plane and the Domain and Range of Transformations as Plane

The two relations, the Relation AA' and the Relation CC' examined in Activity 5, were re-presented to the prospective teachers at the beginning of Activity 6 (see Figure 4.51 and Figure 4.58). While the Relation AA' was defined on R^2, the Relation CC' was defined on a specific region. In Activity 5, prospective teachers only examined whether these two relations were a function or not. In Activity 6, prospective teachers were asked to determine the domain and range of Function AA' and Function CC' and their elements. Examination of prospective teachers' pairwise discussion records showed that there were three types of understanding about the domain and range of Relation AA' and their elements. Five prospective teachers held the first type of understanding about domain and range. Namely, PT3, PT5, PT6, PT7, and PT8 conceived domain as Set A, range as Set A', the elements of Set A as x_i where x_i are real numbers, and the elements of Set A' as y_i where y_i are real numbers. Besides, prospective teachers holding this conception were aware that y was a function of x. To give an example, during pairwise discussion PT3 uttered her idea as follows: "Let's say that Set A includes x's. Because y belongs to Set A' ". PT4 interrupted PT3 and rejected her idea by saying "y also belongs to Set A. x' and y' belong to Set A' ". Because, PT4 was holding a different conception. To be more precise, including PT4, five prospective teachers held the second type of understanding about domain and range. PT4, PT11, PT12, PT15, and PT16 conceived the domain as Set A, range as Set A', the elements of Set A as ordered pairs (x, y), and the elements of Set A' as ordered pairs (x', y'). Although, they expressed that both x and y were from real number set and the elements of domain and range are ordered pairs, they never mentioned explicitly that the set of all ordered pairs comprised plane namely the R^2. For instance, PT4 explained his idea during pairwise discussion also as follows:

> PT4: Let's think of Point A(x, y). If each x is an element of R, then x' must also be the element of R. Uhm…what about y? (Silence) Ok, we can write the same thing for y. In this case, if each y is an element of R, then each $y' = y$-4 must also be the element of R. Aren't I?

Another pair, PT11 & PT12, expressed similar statements as follows:

PT12: Domain is Set A, and range is Set A'. The domain comprises ordered pairs (x, y) as elements.
PT11: Uhm, domain is the set of x and y
PT12: However, x and y's represent a point at the same time.
PT11: It doesn't matter. We take x and y as point A. We can change x and y. Thus, this is our domain. Likewise, our range is x' and y' and they also change.
PT12: Where do we select A(x, y) from?
PT11: From whole real numbers.
PT12: Okay… It is a point as well.

Similar to PT11 & PT12, PT15 & PT16 indicated domain and range as the set of all ordered pairs. The following dialogue took place among PT15 and PT16 during pairwise discussion:

PT16: The set containing x values is our domain.
PT15: Saying that domain comprises x values seems a bit strange to me.
PT16: Well, I mean the set of ordered pairs.
PT15: Ok, let's write it then. We select ordered pairs A(x, y) from a set of ordered pairs.
PT16: The x and y values we select from Set A correspond to the x' and y' values in Set A'.

Finally, PT1, PT2, PT9, PT10, PT13, and PT14 formed another group who held a conception different from the two aforementioned ones. The prospective teachers holding this conception had the most accurate knowledge from a mathematical standpoint. Namely, these prospective teachers were able to explain domain and range as plane (i.e., R^2) and the points (x, y) as elements of this plane. For instance, the dialogue that took place between PT1 and PT2 is given below:

PT1: In Relation AA', the domain is R^2, the range is R^2. Here, R^2 means plane and the elements are (x, y). Do you agree?
PT2: R^2 …, you said, x is an element of R. You are right.
PT1: x is an element of R, y is an element of R, and then (x, y) must be an element of R^2.
PT2: Yes.
PT1: Consider R^2 as the Cartesian product of two number lines.
PT2: Right, R^2 means R x R.

Another pair, PT9 & PT10, expressed similar statements as follows:

PT10: Let's consider Relation CC'. It appeared in some regions and disappeared in some other regions. So, Relation AA' is different from Relation CC'. It is defined everywhere between negative infinity and positive infinity. Do you agree?

PT9: Actually, you said in R^2, right?

PT10: Yes, it includes all points in the plane. All points in the plane form Set A.

A similar discussion took place between PT13 and PT14. However, they used the term "coordinate system" in place of the "plane".

After the completion of pairwise discussions, the class proceeded to whole class discussion. However, just before the beginning of whole class discussion, the researcher presented the participants a dynagraph example to have them clear up their ideas about domain and range. The dynagraph file was taken from The Geometer's Sketchpad resource center. The corresponding dynagraph screenshot is presented in Figure 4.61.

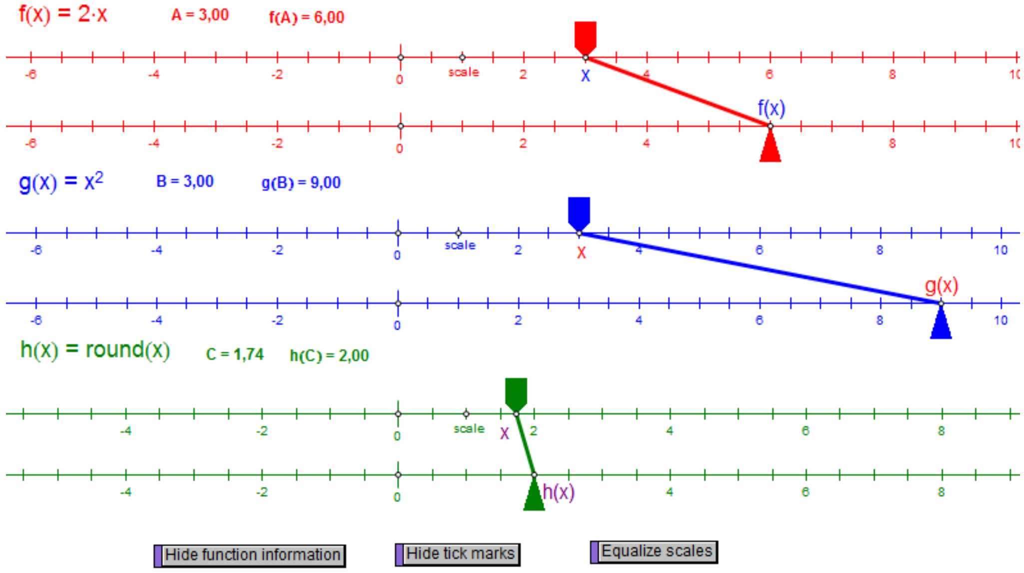


Figure 4.61. A dynagraph example presented to participants in Activity 6

Prospective teachers examined functions defined from R to R such as $y = 2x$ through the file given in Figure 4.61. By doing so, the researcher aimed at helping prospective teachers shift from R to R^2 when considering the domain of functions represented with $f(x, y)$. After the presentation and exploration of the dynagraph example, the whole class discussion began. The dialogue that took place among the researcher and the participants during the whole class discussion is presented below:

Researcher: You have just examined the $f(x) = 2x$ function. You said that elements of this function was selected from the real number set. Now, let's move on to Relation AA' which is also a function. Where do we select Point A from? To which set does Point A belong?
PT14: From the Coordinate system.
PT1: It is an element of R^2.
PT12: The element of plane
Researcher: You all gave similar answers. But, why?
PT14: Due to *x* and *y*.
PT9: Due to owning an *x*- and *y*- axes simultaneously.

In the aforementioned dialogue, PT1, PT9, PT12, and PT14 expressed that domain of Relation AA' is a plane (i.e., R^2). In whole class discussion, these prospective teachers explained that Point A can take different values on the plane and that Point A' can take different values depending on the rule of Relation AA'. Meanwhile, they expressed that the different values taken by Point A' comprised range. Therefore, they gained the idea that inputs are variables which is another indication of the process conception. According to APOS theory, understanding the domain as all points in the plane corresponds to a process conception. This idea can be regarded as the most important one for holding process conception. Thus, due to gaining these ideas necessary for the process conception, it can be said that PT1, PT2, PT9, PT10, PT12, PT13, and PT14 were able to progress towards this conception.

Next, the researcher asked the class to ponder on the graph of functions defined from R^2 to R^2 such as Relation AA'. The class was given some time to think on this issue. After a while, PT15 explained her idea as follows:

PT15: We drew in this way (showing her paper). Actually, we limited our graph to the four values of Relation AA'. However, generally speaking, it covers all the plane, thus I cannot say something. That is, we cannot say that this is the graph of the function. These are examples for some values. Normally, it covers the whole plane.

All prospective teachers had similar ideas with PT15. As an example PT15 & PT16's drawing is presented below.

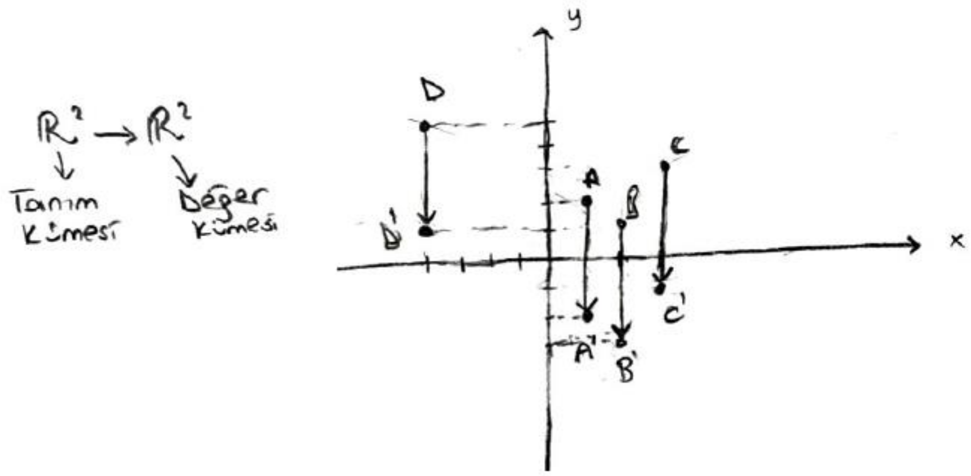


Figure 4.62. PT15 & PT16’s drawing for Function AA' in Activity 6

As understood from PT15 & PT16’s drawing for Function AA' and PT15’s explanations in the whole class discussion, this pair considered all points in the plane as a domain and they explained the rule of this function by demonstrating the correspondence between points (i.e., variables) for four values in their graphs. Another prospective teacher, PT4, explained his idea as follows:

> PT4: Let me explain in this way. Let this be a plane (he grasped an A4 paper and positioned it parallel to his desk). Let this be another plane (pointing to his desk). I think that there are some things that go from one plane (pointing to A4 paper) to another (Pointing to the desk). I could explain in this way. Let this be our defined plane (pointing to the paper he grasped). From this plane to the one below (from A4 paper to the desk). Here *x*s are constant, it does not matter how they go but it will be either in this way (He moved the paper back and forth) or this way. But, I think it will be between the paper and desk. There are two distinct Coordinate planes.

As understood from PT4’s explanations, he put the paper parallel to his desk and regarded them as two parallel planes. He stated that “I think that there are some things that go from one plane to another”. By moving the paper back and forth, he tried to demonstrate that the points in one plane do not go to the same points in the other plane. Actually, he excluded the identity function but it seems that he tried to explain that there was a rule between these related points. Thus, it can be suggested that PT4 gained the idea that the points in the plane map to the points in the plane according to a specific rule. In other words, he started to consider the domain as all points in the plane.

It can be concluded that PT1, PT2, PT4, PT9, PT10, PT12, PT13, PT14, PT15, and PT16 considered the domain as a plane, variables as points on the plane (i.e., ordered pairs), geometric transformations as mappings of points from plane to plane according to a specific rule.

4.6.2. Defining Geometric Transformations as $R^2 \to R^2$ Functions

After PT4's explanation, the researcher presented another GeoGebra file which included two functions. The corresponding screenshot is presented in Figure 4.63.

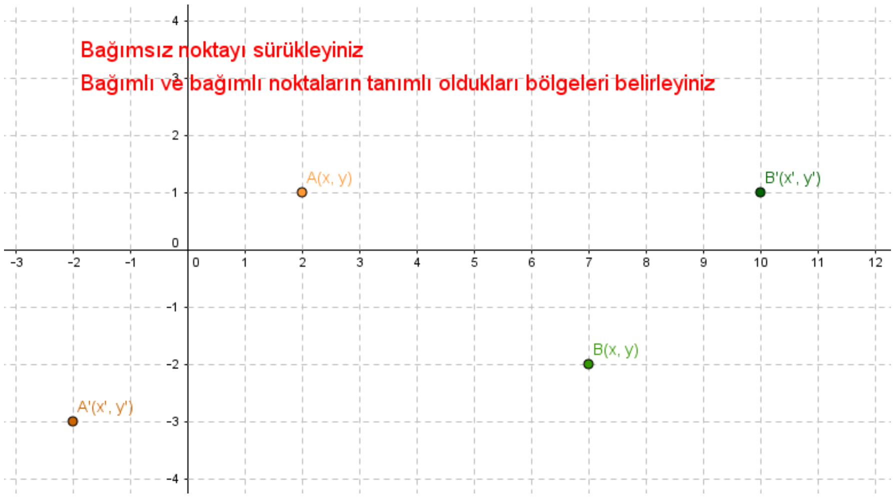


Figure 4.63. GeoGebra screenshot for Relation AA' and BB' in Activity 6

Relation AA' and Relation BB' were defined on a specific region. Prospective teachers were asked to identify this region and to express it mathematically. Besides, they were asked to determine the rules of two functions. They were given some time to examine the behavior of two relations by dragging and to discuss this with their partners.

In whole class discussion, all pairs stated that the rule of Relation AA' was (x, y) → (x-4, y-4) and the rule of Relation BB' was (x, y) → (x+3, y+3). Besides, they expressed that the Relation AA' was defined from 2 to 4 on the x-axis and from 1 to 3 on the y-axis. When the researcher asked prospective teachers to express this mathematically, the following dialogue took place among them.

Researcher: You determined two intervals in which the Relation AA' was defined. How do you write this mathematically?
PT13: We write this as a closed interval.
PT2: It is a square and we can write this as a Cartesian product.
Researcher: PT2, could you please write it on the board?

PT2 wrote the following on the board: [2, 4] × [1, 3] → [-2, 0] × [-1, -3]. Then researcher opened the trace tool and showed the regions that the relations were defined on the smart board. When the trace tool was activated and Point A and Point B were dragged, the following situation occurred on the smart board (see Figure 4.64).

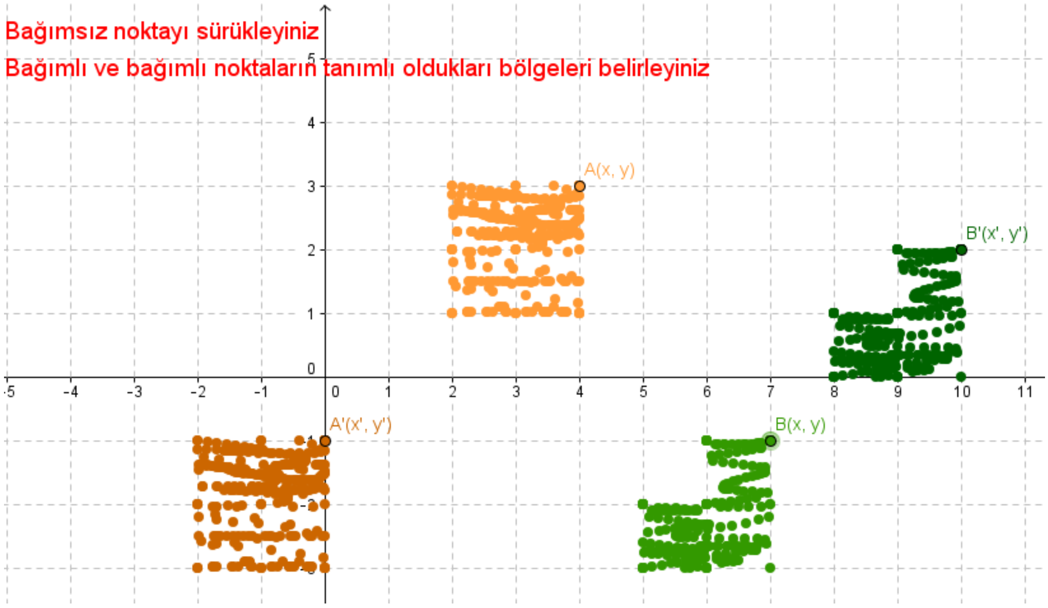


Figure 4.64. GeoGebra screenshot for the Relation AA' and the Relation BB' in Activity 6 when the trace tool was activated

There were two square shaped regions and two ladder shaped regions on the smart board. The Relation AA' was defined on square shaped regions and Relation BB' were defined on ladder shaped regions. The prospective teachers were asked to examine these traces for some time. Then in whole class discussion, the prospective teachers expressed that the trace tool made the relations clearer. Besides, they mentioned about one to one correspondence between points. That is, they indicated that each point in the domain corresponded to a point in the range.

While the Relation AA' and the Relation BB' were on the smart board (see Figure 4.64), the researcher asked prospective teachers to relate them to the other concepts they know.

PT1: I remembered the translation.
Students: Yes, they are translations.
Researcher: Why?
PT16: Because the appearance did not change.
PT14: Their sizes did not change.
PT11: Only the locations changed.
Researcher: Actually, we were examining functions that map points on one region to other points on another region. Now you said that these are translations. Well, might there be a relationship between translations and functions?
PT13: Yes. If *x* and *y* change as positive and negative… For instance, if the coordinates change as +1, +2, -1, and -2, it is a translation. I mean, if the rule of the relation tells us to add to or subtract from x and y, it is a translation.
Researcher: Do you agree with PT13? What do you think?
Students: Yes!
PT2: I think it is a translation for all conditions.
Researcher: How? Give me another example.
PT2: … (No response)
PT11: It is also a translation for the rule that tells us to turn x into 3x.
Researcher: What do all of you think?
PT16: No! The area changes.
PT13: The area might become larger since the speed increases.
Researcher: What happens when the area becomes larger?
PT13 & PT15 & PT16: It will no more be a translation.
PT4: I think, the relation should not include coefficient of x and y coordinates. However, it can include addition and subtraction.
Researcher: Ok then, briefly can we say that translations are functions?
PT1: I think yes. When we were presented a function, we performed the rule. The result was a translation. Thus, translations are also functions.
PT16: I think yes. Since the relation among points denotes a function, we translate the points here.
PT8: There are two functions on the board. Translation was performed for both of them.
PT7: Namely, x became x+3 and it increased 3 units. This means translate that point 3 units right.
PT4: It is a function. We can determine the magnitude in this way: the distance between the points is equal to the translation vector.

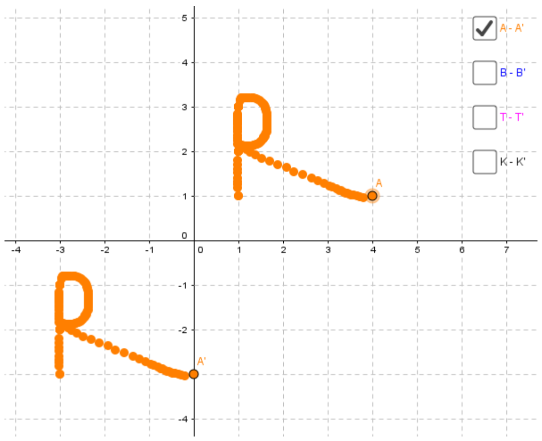
Prospective teachers expressed that the Relation AA' and the Relation BB' were examples of translations. They stated that the regions they defined were same in size and shape and the only difference was their locations. They reached a consensus that in order for a relation to be a translation, the rule of the relation should not include coefficients. Since some of the participants (e.g., PT 13, PT15, and PT16) were aware that coefficients in the rule made the range larger than the domain. They indicated that when the rule of a relation included coefficients, the sizes differed and then the relation was no more a translation. At the end, by the help of PT4 and PT7, prospective teachers were able to relate the rule of the relation to the concept of translation vector.

Shortly, up to now, PT1, PT4, PT7, PT8, PT11, PT13, PT14, and PT16 showed evidence that they conceived translation as functions defined on the plane. After prospective teachers gained the idea that translations were functions, the researcher requested them to express their ideas about reflections and rotations. The following dialogue took place among the researcher and the participants.

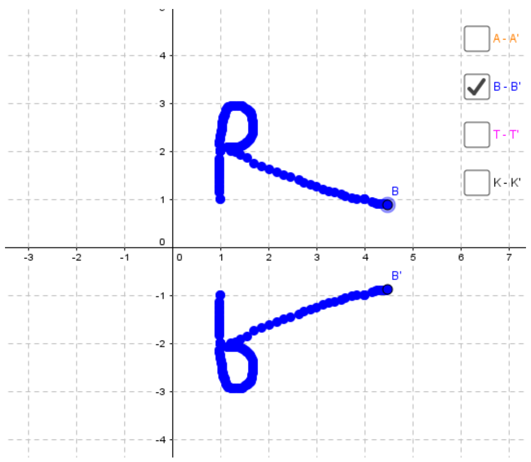
> Researcher: Is this valid for only translation? What about other transformations? Reflection? Rotation?
>
> PT13: This has to do with the rule of function. If the rule of function tells us to multiply x by (-1) and remain y constant, this function will denote a reflection. If the rule of function also tells us to change the order of x and y (i.e., (y, -x)), this will be a rotation. I could imagine in this way.
>
> PT4: Yes, that is true. I absolutely agree with PT13.

In above given dialogue, PT13 explained that for a relation to be a geometric transformation or not depends on the rule of the corresponding relation. She gave vertical reflection and 90 degrees rotation examples whose rules are $(x, y) \rightarrow (-x, y)$ and $(x, y) \rightarrow (-y, x)$ respectively.

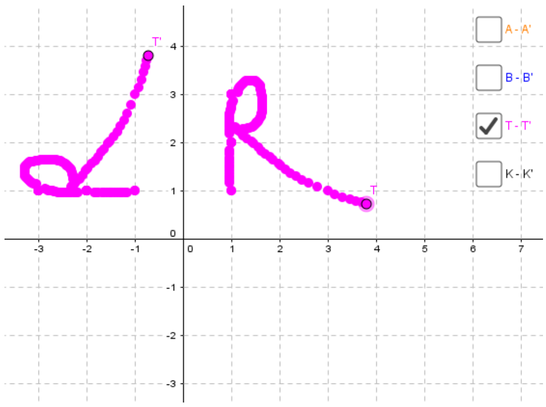
The researcher re-presented some of the relations in the GeoGebra file that were presented the previous week. During that time, the prospective teachers accepted that these relations were all functions from R^2 to R^2. However, there was no discussion whether they were transformations or not. The GeoGebra file included translation, reflection along x-axis, 90 degrees rotation around the origin, and 180 degrees rotation around the origin (see Figure 4.65). The researcher asked the participants to explore the relations by dragging their independent points when the Trace tool was activated. Some of the examples that were examined by the prospective teachers during whole class discussion are presented in Figure 4.65.



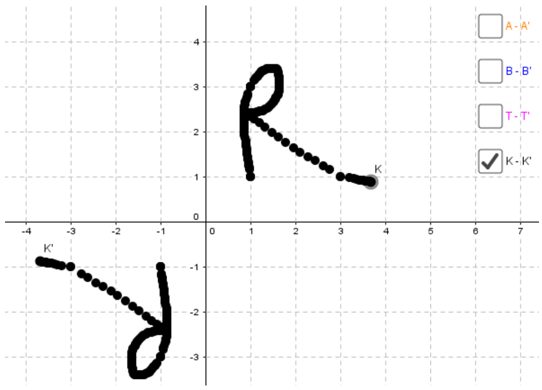
(a)



(b)



(c)



(d)

Figure 4.65. Relation AA', Relation BB', Relation KK', and Relation TT'

As mentioned before, PT1, PT4, PT7, PT8, PT11, PT13, PT14, and PT16 explained that Relation AA' (see Figure 4.64 and Figure 4.65a) was a translation with vector *v* = (-4, -4). They expressed this here as well. Besides, in whole class discussion, PT13 stated that Relation TT' was the rotation that she had explained before. Because, PT13 gave a 90 degrees rotation example whose rule is (x, y)→(-y, x) previously. All prospective teachers stated in whole class discussion that Relation BB' and Relation KK' were a reflection along *x*-axis and 180 degrees rotation around the origin respectively. These are only four of the examples that prospective teachers examined in whole class discussion. It can be concluded that towards the end of the Activity 6, PT1, PT4, PT7, PT8, PT11, PT13, PT14, and PT16 could conceive geometric transformations as functions from R^2 to R^2.

To summarize, prospective teachers recognized that some of the relations they examined were functions defined from R^2 to R^2 the previous week. This week, they recognized that the domain of these functions was plane. Besides, this week, they re-examined the functions defined from R^2 to R^2 by activating the Trace tool, and they noticed that the some of these functions corresponded to a specific geometric transformation. Hence, they came to reason geometric transformations as functions defined from R^2 to R^2 and by this way, their understanding of geometric transformations involved all ideas required for the process conception of geometric transformations.

Thereafter, the prospective teachers examined the properties of these specific functions till the end of the instructional unit. For instance, they discussed bijectivity (one to one and onto property) of geometric transformations in the following part. Understanding one to one and onto property of geometric transformations was an indicator of the object conception of geometric transformations.

4.7. Exploring Functional Properties of Transformations

At the end of Activity 6, all prospective teachers started to reason geometric transformations as functions from R^2 to R^2. In Activity 7, prospective teachers were expected to reason other properties of geometric transformations as functions. These properties were points that remain fixed after transformation, one-to-one and onto property, identity functions, and inverse functions.

4.7.1. Determining Points That Remain Fixed After Transformation

The researcher explained that fixed points meant "points that mapped to themselves after transformation" and asked prospective teachers to ponder on points that remain fixed after translation, reflection, and rotation. Each type of transformation was discussed separately. The participants were allocated some time for pairwise discussions. Examination of prospective teachers' pairwise discussion records showed that six out of eight pairs were able to determine the points that remain fixed after transformations correctly. One of the pairs (PT13 & PT14) provided a completely incorrect answer. This was because they misunderstood the question and determined identity functions instead of fixed points. Besides, another pair (PT15 & PT16) expressed a response that was only true for some specific cases

(These issues were presented in whole class discussion). The dialogue that took place among PT3 and PT4 is presented below.

> PT4: Well, here the points on the axis remain fixed (implying reflection). In translation… Suppose that we translated this point. Is there a point that remain fixed here?
> PT3: Perhaps, no.
> PT4: There is not any fixed point indeed. What about rotation? ... I think it may be the center of rotation.
> PT3: Yes, you are right.

PT1 & PT2 discussed these ideas in a similar way. Their dialogue is presented below.

> PT1: When we perform reflection, the points on the reflection line remain fixed. The reflection of this point along this line is itself, isn't it? Think about it!
> PT2: Yes, I agree with you.
> PT1: Well then, in reflection the points on the reflection line remain fixed. What about translation? ... You translate a point but it remains fixed after translation. I do not think there is not such a probability.
> PT2: I wonder if it could be points on the translation vector.
> PT1: I do not think so. It seems to me that all points change in translation.
> PT2: Right, you take all the points away. Thus, all points change.
> PT1: All right, what about rotation?
> PT2: In rotation, the center remains fixed.
> PT1: Yes, you are right.

As can be understood from both of the discussions, prospective teachers could provide correct answers for translation, reflection, and rotation. The prospective teachers consolidated their ideas in whole class discussion. The following dialogue took place among them in whole class discussion.

> PT15: We argue that origin remains fixed after reflection. In reflection along x-axis, along y-axis, and along x=y line, the origin always maps to itself.
> Researcher: For the three specific cases you mentioned, your answer is true. However, I want you to give an answer that satisfies all cases. For instance, think about reflection along x = 1 line.
> PT15: … (No response)
> PT2: Teacher, we decided it to be points on the reflection line for reflection.
> Researcher: Is there anyone else who think in this way?
> Students: We! ...We! … (All prospective teachers except for PT13 and PT14 agreed with PT2).

Researcher: PT13 and PT14, what do you think?
PT13: We considered the letter P. We thought that whichever we reflect, we cannot obtain the letter P itself.
Researcher: Well… Actually, you are right but that is not what I asked. This has to do with a concept (identity function of transformations) which we will discuss later. Well, what do you think about the points on the reflection line?
PT13: If we think from that standpoint, it is true.
Researcher: Well, what do the whole class think about rotation?
PT8: It is the center of rotation.
PT16: The center
PT1: The center of rotation
PT13: From this point of view, it will be the center of rotation. Angle of rotation is not important.
Researcher: What about translation?
Students: No points! None of the points remain fixed after translation.

After whole class discussion, the prospective teachers explained that the points on the reflection line remain fixed after reflection, the center of rotation remains fixed after rotation regardless of the angle of rotation, and that none of the points in the plane remain fixed after translation. Thus, by the help of whole class discussion PT13 & PT14 and PT15 & PT16 realized that they misunderstood the question. Besides, as mentioned at the beginning, other pairs reached correct answers during pairwise discussion. Therefore, all prospective teachers were able to notice the points that mapped to themselves after each transformation.

Determination of points that remain fixed after translation, reflection, and rotation is significant since this conception is an indicator of the idea that all points in the plane map to other points in the plane when a transformation is applied. When a participant can differentiate points that remain fixed from points that remain unfixed, this indicates that s/he is aware of the fact that all points in the plane map to other points in the plane. The idea that all points map to other points in the plane is crucial for understanding geometric transformations as functions in the plane and consequently for process conception. All prospective teachers' noticing of fixed points after transformations also supported the finding that they had a process conception.

4.7.2. Determining Identity Functions of Transformations

The researcher asked prospective teachers to think over the identity functions of translation, reflection, and rotation. They were asked to find a translation, a rotation and a reflection that maps all points in the plane to themselves. Prospective teachers' discussion and exploration of fixed points after the transformations might have helped them in finding the identity functions of each transformations. Because, they easily found the identity function of each type transformations in whole class discussion. For instance, PT13 stated that "when we select the translation vector as a zero vector, all points in the plane map to themselves". Besides, PT1 stated that "When the angle of rotation is zero or is a multiple of 360 degrees, all points in the plane map to themselves. The center of rotation is not important". When prospective teachers were asked to state the identity function of reflection, none of them gave an answer. They all agreed that none of the reflections could map all points in the plane to themselves.

Determination of the identity function of translation, reflection, and rotation is important since this conception is another indicator of the idea that all points in the plane map to other points in the plane when a transformation is performed. In brief, by having prospective teachers determine fixed points and identity function of geometric transformations, she made sure that they had understood geometric transformations as functions defined on the plane hereafter. By this way, the participants understood all ideas that had to do with the process conception according to APOS theory.

4.7.3. Determining Inverse Functions of Transformations

The researcher asked prospective teachers to ponder on inverse function of translation, reflection, and rotation. The pairs were allocated some time for discussion. Examination of prospective teachers' pairwise discussions showed that all pairs excluding PT15 & PT16 were able to find inverse function of a translation, a reflection, and a rotation. For instance, the dialogue that took place between PT9 and PT10 is presented below.

PT10: The inverse function of a translation is the translation in opposite direction.
PT9: Yes, actually we must use the inverse of translation vector.
PT10: Ok. In a rotation, the inverse function is rotation in opposite direction.
PT9: Right. In reflection, if we re-reflect the image along the same reflection line, we obtain the pre-image.

PT10: Yes, if we reflect this point we get this one, then if the re-reflect we get the initial one.

The dialogue that took place among PT11 and PT12 is given below.

PT12: We must put a minus sign in front of the translation vector. That is, if our translation vector is *u*, then the translation vector of the inverse function is –*u*.
PT11: Ok, I see.
PT12: In rotation, the center of rotation remains the same while angle of rotation changes.
P11: In that case, the angle of rotation must be -α.
PT12: Negative alpha (-α) is equal to 2π-α.
PT11: -α or 2π-α, it makes no difference. In reflection, we must reflect again along the same line.
PT12: Then the reflection line does not change.
P11: Yes, it remains the same.

The dialogue that took place among PT13 and PT14 is given below.

PT14: Now, let (3, 5) be our point. Let's translate this point with (1, 1). We obtain point (4, 6). To get back to point (3, 5), we must translate the point (4, 6) with (-1, -1). Namely, if the translation vector is *v*, the inverse function is –*v*. That is it.
PT13: Then, it is -360° in rotation.
PT14: Yes, if the angle of rotation is 360°, then the inverse is -360°. Shortly, we change α to –α.
PT13: In reflection, the same reflection line is the inverse function. If we reflect the image along the same reflection line we obtain the pre-image.
PT14: We reflect the image along line *l* again. Uhm, I understood.

After pairwise discussion, the whole class discussion was carried out. The dialogue that took place among the researcher and the class in the whole class discussion is provided below.

Researcher: Let's start with translation. What did you think?
PT11: We take the inverse of the translation vector.
Researcher: Others?
PT1, PT4, PT7, PT 9, and PT13: We had the same answer.
Researcher: You all said that the translation vector of the inverse function is the inverse of the translation vector. Well, what about reflection?
PT14: We reflect along the same line.
PT11: We reflect again.
PT9: The resulting figure must be reflected along the same line.
Researcher: You said the inverse function of a reflection is equal to itself, right?

Students: Yes!
Researcher: Well, what about rotation?
PT11: It must be -α.
PT9: Same rotation must be done in counterclockwise direction.
PT14: If the angle of rotation is α, the angle of the inverse function must be -α. The centers are the same.
PT12: We can also say 2π-α.

The pairwise and whole class discussions showed that prospective teachers could develop ideas about inverse functions of geometric transformations.

4.7.4. Recognizing Transformations as One to One and Onto Functions

The researcher asked the participants to explain if there is something special for geometric transformations. The following dialogue took place during whole class discussion.

Researcher: Is there a property that makes these transformations more specific functions? A property that makes them more specific?
Students: … (Silence)
Researcher: Think about the properties of functions.
PT13: Might they be one-to-one functions?
Researcher: How did you decide on this property?
PT13: Because all matched with different points.
Research: Well, do these transformations have any other property?
PT13: Actually, they are also onto functions
Researcher: Why do you think so?
PT13: Because, there is not any element that remains unmatched in the range. I mean, all elements of range match with an element from domain.
PT16: We can also see this geometrically, when we examine each point on "R" separately. Moreover, if we drag this point everywhere, corresponding points becomes apparent immediately. Thus, geometric transformations are one-to-one and onto functions from R^2 to R^2.

To make PT16's explanation more clear to the other participants, the researcher presented on the smart board the Relation AA'. While the coordinates of Point A and Point A' were visible and the Trace tool was activated for the vector, Point A and Point A', the researcher dragged the Point A (independent point) rapidly on the screen. The resulting image is presented in Figure 4.66.

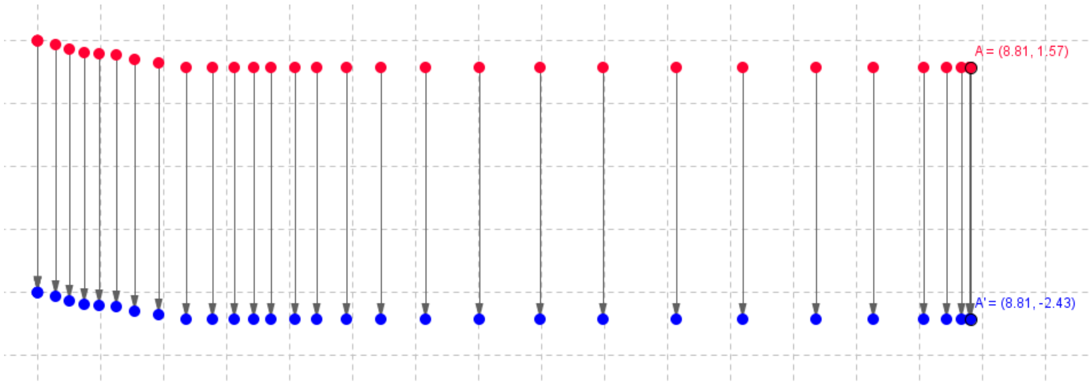


Figure 4.66. Traces of the Relation AA'

After the researcher showed this relation on the smart board, PT13 and PT16 stated that it was just what they tried to explain. Other prospective teachers expressed that geometric transformations are one-to-one and onto functions as well. However, until this point we only had an evidence that support PT13 and PT16's conception of geometric transformations as one to one and onto functions. Other prospective teachers' indication of their agreement with PT13 and PT16 was not regarded sufficient for the justification of conceiving geometric transformations as one to one and onto functions. Therefore, it can be said that PT13 and PT16 could proceed towards object conception of geometric transformations at the end of Activity 7. Because, according to APOS Theory, object understanding of transformations involves reasoning about transformations as one-to-one and onto functions that map points in the plane to points in the plane.

4.8. Prospective Middle School Mathematics Teachers' Knowledge and Understanding of Geometric Transformations after the Implementation of the Instructional Unit

At the end of the instructional unit, all prospective teachers were administered the Transformation Geometry Questionnaire as a post-test. After post-test, follow-up interviews were conducted with each participant. The results of the TGQ are presented in two sections. First, quantitative results regarding prospective teachers' pre-existing knowledge of geometric transformations are presented based on post-test data. Next, qualitative results regarding interpretation of prospective teachers' understanding of geometric transformations through APOS theory are presented based on pre-test and follow-up interview data.

4.8.1. Prospective Middle School Mathematics Teachers' Knowledge of Geometric Transformations after the Implementation of the Instructional Unit

Descriptive results regarding prospective middle school mathematics teachers' knowledge of geometric transformations after the implementation of the instructional unit are presented in Table 4.46.

Table 4.46. Descriptive results regarding participants' knowledge of geometric transformations

		Post-Test Scores	
Task #	Max	*M (%)*	*SD*
1	24	20.25 (84.38)	2.77
2	18	17.50 (97.22)	0.73
3	3	3.00 (100.00)	0.00
4	3	3.00 (100.00)	0.00
5	6	5.88 (98.00)	0.34
6	9	7.88 (87.56)	1.09
7	4	3.25 (81.25)	0.86
8	4	3.63 (90.75)	0.72
9	18	15.38 (85.44)	2.39
10	11	10.31 (93.73)	1.54
Total	100	90.06 (90.06)	4.45

As seen in Table 4.46, TGQ includes ten tasks and maximum score of each task differs from each other. For that reason, in addition to mean scores, the percentages for each task was calculated. As seen in Table 4.46, prospective teachers' transformation geometry performances for each task ranged between 81.25 and 100 out of 100 points in post-test. Prospective teachers displayed higher performance in Task 3 and Task 4, lower performance in Task 7.

When prospective teachers' pre-test and post-test performances are compared, it can be seen that prospective teachers' mean score increased to 90.06 from 32.44 after the implementation of the instructional unit. The mean scores for each task increased to a considerable extent as well. More detailed information related to each component of TGQ is presented in Table 4.47.

Table 4.47. Descriptive results for each component of TGQ

			Post-Test Scores	
Components of TGT	Related Tasks	Points	*M %*	*SD*
Define geometric transformations	1	24	20.25(84.38)	2.77
Identify rotation	2A, 2F	6	5.56(92.67)	0.73
Identify reflection	2B, 2E	6	5.93(98.83)	0.25
Identify translation	2C	3	3.00(100.00)	0.00
Identify glide reflection	2D	3	3.00(100.00)	0.00
Perform reflection	3, 6A, 8	10	8.78(87.80)	1.15
Perform translation	4, 6B	6	5.88(98.00)	0.34
Perform rotation	5, 6C, 7	13	11.88(91.38)	1.40
Relate geometric transformations with functions	9	18	15.38(85.44)	2.39
Perform compositions of transformations	10	11	10.31(93.73)	1.54

Prospective teachers' transformation geometry performances for each component ranged between 84.38 and 100.00 out of 100 points in post-test. The analysis of post-test data showed that participants' mean scores increased considerably for each component of the TGT after the implementation of the instructional unit. Prospective teachers' performance in the composition of transformations task (93.73%) and in relating geometric transformations and functions task (85.44%) increased to a considerable extent.

4.8.2. Interpretation of Prospective Middle School Mathematics Teachers' Understanding of Geometric Transformations through the Lens of APOS Theory

In this part, results regarding analysis of participants' understanding of geometric transformations through APOS theory are presented. The post-test and follow-up interview data showed that there appeared three clusters of prospective teachers in terms of understanding geometric transformations. In the first cluster, there were eight participants holding a complete object conception. In the second cluster, there were seven participants holding a conception between process and object. Finally, in the third cluster there were one participant holding a conception between action and process. These three different levels of understandings are described in more detail in the following paragraphs. Based on post-test and follow-up interview data, prospective teachers' ideas related to geometric transformations are summarized in Table 4.48.

Based on the interview and post-test data, it was found out that all prospective teachers were able to use parameters to define each type of geometric transformation and to reason about properties that remain invariant under each of them. However, PT5 was in the first cluster and he defined geometric transformations from a motion perspective. Although he used parameters to define transformations, he still defined them as a motion. For instance, the definition "PT5: Translation is the movement of an object along with a vector which has a specified direction and magnitude" includes the term "translation vector" and thus indicates a defined motion. To give another example, "PT5: Turning an object around a point with a specific angle." indicates a defined motion as well. In short, although PT5 reasoned about parameters and the effects of changing them on geometric transformations, and reasoned about properties that remain invariant under a transformation, he did not present any evidence to support the idea that transformations are applied to all points in the plane. He did not provide an evidence that indicates full mapping understanding of geometric transformations and thus he was not able to complete process conception of geometric transformations. Therefore, his understandings of geometric transformations fell within a continuum among an action conception and a process conception.

Table 4.48. Summary of prospective teachers’ ideas related to geometric transformations based on pre-test and follow-up interview data

		Prospective Teachers															
	Ideas related to geometric transformations	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Reflection	Reflecting an object along the a line					x											
	Function	x	x	x	x		x	x	x	x	x	x	x	x	x	x	x
	Defined from R^2 to R^2	x	x	x	x		x	x	x	x	x	x	x	x	x	x	x
	Requires a line	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
	One to one and onto			x	x			x		x				x	x	x	x
Rotation	Turning an object around a point with a specific angle					x											
	Function	x	x	x	x		x	x	x	x	x	x	x	x	x	x	x
	Defined from R^2 to R^2	x	x	x	x		x	x	x	x	x	x	x	x	x	x	x
	Requires a center and an angle	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
	One to one and onto			x	x			x		x				x	x	x	x
Translation	Movement of an object along with a vector which has a specified direction and magnitude					x											
	Function	x	x	x	x		x	x	x	x	x	x	x	x	x	x	x
	Defined from R^2 to R^2	x	x	x	x		x	x	x	x	x	x	x	x	x	x	x
	Requires a translation vector	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
	One to one and onto			x	x			x		x				x	x	x	x

PT1, PT2, PT6, PT8, PT10, PT11, and PT12 were in the second cluster, and these participants could define geometric transformations as functions from R^2 to R^2. Besides, they were able to understand that transformations were applied to all points in the plane. Namely, these participants were able to gain a mapping understanding of transformations and consequently to hold a process conception of transformations. Besides, they were able to reason about the composition of two or more transformations and reason about the properties that were preserved under composition of transformations. However, post-test and interview data did not present any evidence to suggest that they thought transformations as one-to-one and onto functions. Therefore, it cannot be stated that these participants had a complete object conception of transformations. Their understandings of geometric transformations fell within a continuum among a process conception and an object conception.

PT3, PT4, PT7, PT9, PT13, PT14, PT15, and PT16 were in the third cluster and post-test and interview data showed that these participants gained a complete object understanding. In addition to the ideas gained by the participants in the second cluster, these participants were able to define geometric transformations as one-to-one and onto functions from R^2 to R^2. Therefore, it can be concluded that these participants had a complete object conception of transformations.

CHAPTER 5

DISCUSSION, CONCLUSION AND IMPLICATIONS

The purpose of this study was to explore prospective middle school mathematics teachers' evolving understanding of geometric transformations through an instructional unit. Through this purpose, findings related to analysis of prospective middle school mathematics teachers' understanding of geometric transformations before, during and after the implementation of the instructional unit were reported in the results chapter. Prospective teachers' developing understanding of geometric transformations was documented as they gained experienced with the following ideas: identifying geometric transformations, performing single and compositions of transformations, exploring functional dependency of points on the plane, $R^2 \rightarrow R^2$ functions, and functional properties of transformations.

In this chapter, first, the results related to the prospective teachers' knowledge and understanding of geometric transformations were discussed in line with the existing body of literature. Next, the prospective teachers' understanding related to geometric transformations as functions were discussed. Finally, implications and limitations of this study and suggestions for future research were presented.

5.1. Development in Prospective Middle School Mathematics Teachers' Knowledge and Understanding of Geometric Transformations

In this study, prospective middle school mathematics teachers' knowledge of geometric transformations were measured through the Transformation Geometry Questionnaire (TGQ). This questionnaire included tasks involving defining and identifying geometric transformations, performing single and composite geometric transformations, and relating geometric transformations with functions. Results revealed that the prospective teachers had limited knowledge of geometric transformations. Namely, only about one third of their solutions for the tasks were correct. In the following sections, the findings related to each component of TGQ are summarized and discussed in line with the related research studies.

5.1.1. Development in Prospective Teachers' Definitions of Geometric Transformations

Findings showed that prospective teachers could respond to about one third of the definition tasks correctly in pre-test. Namely, before the implementation of instructional unit, they had difficulty in defining geometric transformations with all parameters of geometric transformations, and consequently in noticing the relationships between a parameter, a pre-image point and the corresponding image point. This result is consistent with the findings of earlier studies. Past research has shown that both students and prospective teachers had difficulty in defining transformations with their all parameters (e.g., Hollebrands, 2004; Thaqi et al., 2011; Yanık, 2011, 2014). For instance, Thaqi and others (2011) found out that prospective elementary teachers mentioned only the angle of rotation but not the center of rotation when defining rotation. Similarly, high school students' definition of rotation included the angle of rotation explicitly while the center of the rotation seemed to be implicit (Hollebrands, 2004). Furthermore, Yanık (2011, 2014) reported that neither middle school students nor prospective middle school mathematics teachers mentioned the magnitude or the direction of the translation vector when defining translation. Parameters are important since they are parts of the definitions of geometric transformations (Hollebrands, 2003).

During and after the implementation of the instructional unit, the prospective teachers showed development in defining geometric transformations. Most of the prospective teachers became competent in defining geometric transformations with their all parameters at the end of the first phase of the instructional unit. Similarly, prospective teachers' mean score was 20.25 over 24 (84%) in defining geometric transformations in the post-test. This development in defining geometric transformations can be attributed to the design of identification tasks that involved numerous finite figures and frieze patterns whose parameters exemplified all possible variations (e.g., center of rotations which were inside and outside the given figures, vertical, horizontal, and inclined reflection lines which passed inside and outside the given figures, etc.) and the instructor's/researcher's attempts to have all prospective teachers focus on all parameters during the identification of these geometric transformations. Furthermore, the prospective teachers' development in defining geometric transformations can be explained by the researcher's directing of the whole class discussions in a way that focus on the parameters and the relations among parameter(s), pre-image points, and corresponding image points throughout the instructional unit.

5.1.2. Development in Prospective Teachers' Identification of Geometric Transformations

The analysis of data revealed that prospective teachers had difficulty in identifying geometric transformations. Figures and their transformed images were presented to participants in the identification of geometric transformation tasks included in TGQ. Findings revealed that prospective teachers could solve about half of the identification of geometric transformation tasks correctly in pre-test. Namely, before the implementation of the instructional unit, they had difficulty in identifying geometric transformations and defining these transformations with their all parameters. This result is consistent with the findings of earlier studies. Past research has shown that both students and prospective teachers had difficulty in identifying transformations (e.g., Harper, 2003; Hollebrands, 2004; Thaqi et al., 2011). For instance, Harper (2003) found out that prospective elementary teachers had difficulty identifying translations, reflections and rotations. She presented a figure and its rotated image to four participants and it was revealed that none of them could explain the transformation by using only one rotation. The prospective elementary teachers tried to explain the given rotation by a combination of simple steps without identifying any specific center of rotation and angle of rotation. Similarly, when a figure and its reflected image were presented to prospective elementary teachers, they had difficulty in finding a single reflection line.

The participants of the current study might have experienced difficulty in identifying geometric transformations due to their aforementioned difficulties in defining geometric transformations. Furthermore, participants' difficulty in identifying transformations can also be explained by their difficulty in understanding relationships among the parameter(s), a pre-image point and the corresponding image point in geometric transformations. Hollebrands (2004) explained that while solving geometric transformations tasks, students did not carefully consider the relationships among a pre-image, a parameter(s), and an image. In particular, she pointed out that when solving rotation tasks, students did not consider the property that corresponding pre-image and image points are equidistant from the center of rotation. Similarly, Ramful and others (2015) reported that learners were not able to carefully consider that reflection line must be the perpendicular bisector of the segments formed by joining corresponding pre-image and image points.

During and after the implementation of the instructional unit, prospective teachers showed development in identifying geometric transformations. Most of the prospective teachers became competent in identifying geometric transformations and defining these transformations with parameters at the end of the first phase of the instructional unit. More specifically, all the prospective teachers could define the translation with its translation vector in the last identification of translation task (i.e., in Task 1A of Activity 3. Similarly, when all identification of reflection tasks were considered, it was found that most of the prospective teachers could define the reflections with their all parameters (i.e., all the reflection lines that the given figure has). Most of the prospective teachers could identify the rotations with at least one of their parameters at the end of the first phase of the instructional unit as well. In the post-test, prospective teachers showed development in identifying geometric transformations as well. Prospective teachers' development in identification of geometric transformations can be attributed to the fact that they gained experience with identification of all possible variations of parameters. After the prospective teachers experienced identification of geometric transformations tasks in the first phase, they were required to perform two or more geometric transformations successively and to identify a single transformation as a composition of these transformations. Therefore, the prospective teachers continued to identify geometric transformations until the implementation of the post-test and consequently they developed themselves in terms of identifying geometric transformations with their all parameters. It is important to note that they especially developed themselves in identifying both parameters of a rotation.

5.1.3. Development in Prospective Teachers' Performing of Geometric Transformations

In performing geometric transformation tasks included in the TGQ, the prospective teachers were presented figures and parameters and they were asked to perform single and compositions of geometric transformations. Findings revealed that the prospective teachers had difficulty in performing geometric transformations in pre-test. This result is consistent with the findings of earlier studies. For instance, middle and high school students (e.g., Hollebrands, 2004; Yanık, 2014) and prospective teachers (e.g., Harper, 2003; Yanık, 2011) had difficulty in performing translations. In more detail, the participants in these studies failed to use the translation vector when they were asked to perform translation for a given figure and a translation vector. They misinterpreted translation vectors and used them as either a reflection line, a direction indicator or a location on which the given figure must be placed. Similarly, Hollebrands (2003) found out that although high school students seemed to relate rotation with turning, they used centers of rotations differently. In performing a rotation task, half of the students rotated the figure around different points rather than the given point as the center of rotation. Furthermore, these high school students in Hollebrands' (2003) study and the high school students in Ramful and others' (2015) study had difficulty in reflecting a figure along an inclined line of reflection.

Prospective teachers' difficulty in considering the role of parameters might have led to difficulties not only in identifying geometric transformations but also in performing them. Namely, while performing geometric transformations they might have not considered the relationships among a pre-image, a parameter(s), and an image. Besides, prospective teachers' might have used intuitive strategies in performing geometric transformations and this might have led to difficulties as well. Ramful and others (2015) ascribed students' erroneous conception of reflection related to inclined reflection lines to their intuitive understanding of perpendicularity as being opposite. Furthermore, they found out that this intuitive understanding helped the participants perform reflections in which the reflection lines were vertical or horizontal, but it led to student errors in reflections with inclined lines.

During and after the implementation of the instructional unit, the prospective teachers' performing of geometric transformations developed. All the prospective teachers could successfully perform single and composition of geometric transformations included in the second phase of the instructional unit. Similarly, in post-test, prospective teachers showed development in performing geometric transformations. Prospective teachers' development in performing geometric transformations can be attributed to the design and implementation of identification activities in the first phase of the instructional unit and to the design and implementation of the performing geometric transformation activities in the second phase of the instructional unit. For instance, prospective teachers' development in performing geometric transformations can be explained by the researcher's directing of the whole class discussions in a way that focus on the parameters, the role of parameter(s) in performing geometric transformations, and invariant properties of geometric transformations throughout the instructional unit. In these phases, the activities were designed in a way that required each pair of prospective teachers to respond to each performing geometric transformation task first and then participate in whole class discussions related to these performing tasks. Participating in whole class discussions which focused on the aforementioned ideas before moving on to another performing task, might have helped participants to develop their understandings in performing geometric transformations. In other words, shared understandings emanated from the discussion of the initial task were used by the prospective teachers in responding to the second task. Similarly, the shared understandings from the first and second task were used in the solution of the third task. This cycle continued in this way until the last task. Therefore, the prospective teachers might have reached a rich understanding about performing of geometric transformations as they widened and accumulated their understandings after each whole class discussion.

Prospective teachers' development in performing geometric transformations can also be attributed to the researcher's designing of pre-constructed GeoGebra files related to performing single and compositions of geometric transformations. Namely, in the study, the prospective teachers first performed geometric transformations on paper and then they explored geometric transformations dynamically through corresponding

GeoGebra files. Through these files, the prospective teachers had an opportunity to explore the relations among parameters, pre-images, and images and to explore the properties of geometric transformations and to make generalizations related to compositions of geometric transformations.

5.1.4. Development in Prospective Teachers' Understanding of Geometric Transformations

This study revealed that the prospective middle school mathematics teachers held motion conception of geometric transformations prior to the implementation of the instructional sequence. Namely, they considered geometric transformations as physical movements. This result is consistent with the findings of earlier studies. Research studies revealed that learners from different ranges of ages (ranging from elementary to undergraduate students) mainly held motion conception of transformations (e.g., Edwards, 2003; Glass, 2001; Hollebrands, 2003, 2004; Portnoy et al., 2006; Thaqi et al., 2011; Yanık; 2011, 2014). For instance, Yanık (2011) examined prospective middle school mathematics teachers' pre-conceptions of geometric translations and found out that prospective teachers had motion conception of translations. Similarly, Hollebrands (2004) examined high school students' prior understanding of rotation, translation and reflection and she found that the students conceived these transformations as motions or actions that were applied to a figure. Moreover, Yanık (2014) explored the nature of middle school students' understanding of geometric translations and he found that the students conceived geometric translations as either translational motion or both translational and rotational motion. Learners holding a motion conception of transformations view the plane as an invisible background, they think that geometric figures sit on this plane, and consider transformations as physical movements of geometric figures on the top of the plane (Edwards, 2003). There are some possible reasons for learners' understanding of geometric transformations in this way. Edwards (2003) attributes this way of thinking to learners' embodied experience. To be more precise, she explained that in real life, people experience that objects given on a table are not embedded subsets of this table but rather they sit on it. Based on this idea, she attributed the source of learners' motion conception to natural understanding of motion experienced by themselves in their daily lives. Actually, these experiences are conceptions that are useful outside the formal mathematics context. Therefore, it is difficult for learners to conceive geometric transformations as mappings of the plane unless they reconceptualize that the transformed objects are subsets of the plane rather than entities sitting on it (table in the aforementioned example). Similarly, Yanık (2006) stated that that learners do not necessarily or naturally interpret transformation situations as mapping of the plane onto itself. He further explained that conceptualizing geometric transformations as motion towards mapping depends on a variety of factors and therefore processing from motion conception to mapping conception is a complex process which requires a long term careful designed instruction.

During and after the implementation of the instructional unit, the prospective teachers' understanding of geometric transformations developed. All the prospective teachers displayed progress in motion-mapping continuum. Prospective teachers' developments became evident in whole class discussions in the fourth and fifth phase of the instructional unit. For instance, ten prospective teachers (PT1, PT2, PT4, PT9, PT10, PT12, PT13, PT14, PT15, and PT16) started to consider the domain as a plane, variables as points on the plane (i.e., ordered pairs), and geometric transformations as mappings of points from plane to plane according to a specific rule.

The findings of the post-test and follow-up interviews showed that eight out of sixteen prospective teachers conceived geometric transformations as one to one and onto functions defined on the plane at the end of the study. This means that they gained complete mapping conception of geometric transformations. Edwards' (2003) notion of mapping conception of geometric transformations corresponds to Arnon et al.'s (2013) notion of object conception in APOS Theory. Meanwhile, the characteristics of learners with motion conception correspond to the characteristics of learners with action conception. Thus, it can be said that these eight prospective teachers gained complete object conception of geometric transformations at the end of the study. Seven prospective teachers reached an understanding that fell within a continuum between a process conception and an object conception at the end of the study. Namely, they conceived geometric transformations as functions defined on the plane but they did not explicitly state that these functions were one to one and onto functions. Therefore, their understanding was not regarded as a complete object conception. It is important to note that these fifteen prospective teachers were holding action conception of geometric transformations at the beginning of the study. One prospective teacher (PT5) could not reach a complete process conception of geometric transformations at the end of the study. His understanding of geometric transformations fell within a continuum between action and process conception. Although PT5 considered parameters and the effects of changing them on geometric transformations, and considered properties that remain invariant under a transformation, there was not enough evidence to support the idea that he conceived transformations are applied to all points in the plane and that he had a full mapping understanding of transformations. Thus, he was not able reach a complete process conception of transformations at the end of the study. It is important to note that although PT5's final understanding of geometric transformations fell between action and process conceptions, his development might be regarded as remarkable since his understanding was below the action conception at the beginning of the study.

Prospective teachers' development in understanding geometric transformations can be attributed to the design of the instructional unit in terms of many aspects. First, in this study, a hypothetical learning trajectory (HLT) was developed. In designing the hypothetical learning trajectory, the critical ideas for understanding geometric transformations as functions suggested by Hollebrands (2003) and Yanık (2006) were took into consideration. These ideas were explained in detail in the literature chapter and in the following part. In designing the HLT, shortcomings of the previous studies were carefully evaluated and these shortcomings were tried to be eliminated. To do so, the researcher of the current study conjectured that understanding functional dependency of points in the plane would help prospective teachers understand geometric transformations as functions. By integrating the ideas coming from the literature with the new ideas and teaching them in an integrated way in a seven-week instructional sequence might be the most important reason for prospective teachers' development.

Prospective teachers' development in understanding geometric transformations might also related to the use of a dynamic geometry software, namely GeoGebra.

Dynamic geometry environments' positive effect was expressed by several researchers (e.g., Güven, 2012; Harper, 2003; Tatar et al., 2014). In this study, the prospective teachers explored geometric transformations both in a static and a dynamic environment. Especially in the first two phases, the prospective teachers responded to geometric transformation tasks first in a paper and pencil environment and then they explored these tasks in a dynamic environment through corresponding pre-constructed GeoGebra files. By the help of these files, they were able to make generalizations and were able to understand the connections within and among geometric transformations. Studying both in a static and a dynamic geometry environment and exploring the intended ideas through pre-constructed files might be other reasons for prospective teachers' development in understanding geometric transformations.

Prospective teachers' development in understanding of domain as plane might be related to the use of Trace tool in GeoGebra. In the last two phases of the instructional unit, the researcher used the Trace tool to have prospective teachers explore the domain of functions defined on the plane and relate geometric transformations with functions. Therefore, the use of Trace tool seemed to have an important role in understanding the domain, the most critical idea, as a pre-requisite for understanding geometric transformations as functions. In other words, using the Trace tool apart from reflect, rotate, and translate tools might have particularly influenced prospective teachers' development in understanding geometric transformations.

5.2. Ideas Involved in Understanding Geometric Transformations as Functions

Findings of the study revealed that prospective teachers had difficulty in identifying geometric transformations with their all parameters and they had limited knowledge of the relationships among pre-images, images, and parameters before the implementation of the instructional unit. However, understanding parameters and their effects, and the relationships among pre-images, images, and parameters were critical factors for supporting the development of understanding of transformations as functions (Hollebrands, 2003). Similarly, understanding parameters is critical in terms of facilitating one's transition from an undefined motion of a single object to a defined motion of a single object (Yanık, 2006). Therefore, the initial activities of the instructional unit were devoted to the parameters, the effects of parameters, and relationships among parameters, images and pre-images. Namely, in the first phase of the instructional unit, the prospective teachers were required to examine many finite figures and frieze patterns whose parameters exemplified all possible variations (e.g., center of rotations which were inside and outside the given figures, vertical, horizontal, and inclined reflection lines which passed inside and outside the given figures, etc.). Besides, they also had to participate in whole class discussions which focus on parameters and the relations among parameter(s), pre-image points, and corresponding image points. Focusing on the parameters, by considering the previous studies, was a well-suited decision. Because, the findings revealed that most of the prospective teachers became competent in identifying geometric transformations with their all parameters at the end of the first phase of the instructional unit and they could respond to almost all of the identification tasks in the post-test correctly. As was the case in Yanık's (2006) study, the participants of the current study moved away from undefined motion conception of geometric transformations and thus they showed progress within the motion-mapping continuum by focusing on parameters as well.

Findings of the study revealed that participants conceived the domain of geometric transformations as single objects rather than the whole plane before the implementation of the instructional unit. However, in order to understand geometric transformations as functions, one should know that both domain and range of geometric transformations are the whole plane (Hollebrands, 2003). Moreover, Yanık (2006) stated that understanding domain is critical in terms of facilitating one's transition from a defined motion of a single object to a defined motion of all points in the plane. Furthermore, he emphasized that while understanding of parameters and domain is a necessary step for understanding transformations as mapping, understanding the concept of plane plays a major role in complete understanding of geometric transformations as functions. By considering these conclusions drawn by the previous studies, the forth phase of the instructional unit was devoted to understanding the domain of geometric transformations as plane. In this phase, domain of geometric transformations were introduced to the participants by the help of Drag and Trace tools. Namely, participants sampled the entire plane by dragging an independent point and observed the continuous variation of the independent and dependent points by the help of the Trace tool. At the end of the study, all prospective teachers except for one conceived geometric transformations as functions and conceived the plane as the domain of geometric transformations. This means that, as indicated by Yanık (2006), understanding domain is critical in terms of facilitating participants' transition from understanding domain as a single object to understanding domain of all points in the plane.

Although Yanık (2006) and Hollebrands (2003) explained several critical ideas required for understanding geometric transformations as functions, these ideas did not suffice to reach a complete mapping understanding (i.e., object conception) of geometric transformations for their participants except for one participant in Hollebrands' (2003) study. For instance, Yanık (2006) stated that by the end of the study, although two of the four prospective teachers reached process understanding of transformations, these participants were still operating from the motion conception, their conceptions of domain and plane were incomplete, and they did not quite reach the stage attributed as mapping of the plane onto itself. He added that although there was a considerable amount of data gathered to understand how students progressed from action to process conception of transformations, the picture was still incomplete of what would be a more complete understanding of object conception of transformations and how might students come to understand this notion (Yanık, 2006).

Since learners are most often exposed to functions in a numerical setting, they have difficulty in understanding that functions are relationships between variables (Carlson, 1998). More precisely, learners are often introduced to functional relationships between variables merely in terms of discrete variables and consequently they have difficulty understanding the multifaceted idea of variable (Trigueros & Ursini, 1999). Understanding geometric transformations involves understanding variables as points in the plane and the relationship between these points as functions defined in the plane. By considering these studies, the researcher of the current study conjectured that learners' (participants of previous studies) difficulty in understanding geometric transformations would stem from their difficulty in learning functions in a nonnumeric setting. Thus, the researcher of the current study hypothesized that understanding functions defined on the plane in which variables are ordered pairs of real numbers would help prospective teachers understand geometric transformations as functions and implemented these ideas through designing an activity, namely Activity 5.

The newly added ideas involve understanding functional dependency between points in the plane. These are understanding dependent and independent variables of a function defined on the plane as points in the plane, the notion of function as a relationship between these variables, and understanding the domain of these functions as plane. Since the prospective teachers are familiar with algebraic functions that include single real numbers as inputs and outputs, this understanding involves extending their knowledge of algebraic functions to functions defined on the plane in which variables are ordered pairs of real numbers.

Thinking that points might be variables is important because it is a significant step in understanding the domain of geometric transformations as all points in the plane. Steketee (2012) explained that while dragging the independent variable, learners could explore a variety of values of independent variable easily and could observe the behavior of both variables. Therefore, continuous variation is a natural concept since the action of dragging is continuous. In the third phase of the instructional sequence participants examined the points given on pre-constructed GeoGebra files. Findings showed that participants could notice the dependency between points. They could classify the points as independent and dependent points by noticing their effects on each other. Gaining the idea that points may be variables contributed to gaining the idea that functions may be defined on the plane and thus understanding geometric transformations as functions defined on the plane.

Thinking special relations between the points as functions is important in understanding geometric transformations as functions. Actually, geometric transformations are only some specific examples of functions defined on the plane. Hazzan and Goldenberg (1997) devised a dynamic geometry context with no algebra, graphs or numbers. They pointed to the role of dynamic geometry environment in grasping the notion of function as a relationship between points. In this study, participants explored the function concept on its own in the dynamic geometry environment without algebra. In the third phase of the instructional sequence participants examined many relations between points on the GeoGebra screen in terms of being a function or not. Findings showed that participants were able to extend their knowledge of algebraic functions to functions defined on the plane and thus they were able to distinguish functions from non-function relations. Gaining this new understanding in the third phase helped prospective teachers conceive geometric transformations as some special examples of functions defined on the plane in the fourth phase of the instructional unit.

Exploring the domain of functions defined on the plane or a specific region is significant in understanding the domain of geometric transformations as plane. As mentioned before, geometric transformations are only some specific examples of functions defined on the plane. Steketee and Scher (2011) spoke to the use of Drag and Trace tools in helping learners understand the domain and range of functions as plane. Therefore, in the fourth phase of the instructional unit, participants explored the domain of functions by the help of the Trace tool. Some of these functions were defined on the plane while some others were defined on specific regions. Findings showed that participants were able to conceive the domain as the whole plane or a specific region if necessary during the fourth phase of the instructional unit. At the end of the fourth phase, they noticed that the some of the functions they accepted as functions defined on the plane corresponded to a specific geometric transformation by the help of Trace tool. Hence, they came to reason geometric transformations as functions defined from R^2 to R^2 and by this way, their understanding of geometric transformations involved all ideas required for the process conception of geometric transformations.

In conclusion, findings of the current study revealed that all the prospective teachers showed progress in motion-mapping continuum (i.e., action-object conception). Specifically, the findings showed that eight out of sixteen prospective teachers gained a complete object conception of geometric transformations and seven prospective teachers reached an understanding that fell within a continuum between a process conception and an object conception at the end of the study. It is important to note that all prospective teachers were at or below the level of action conception prior to the implementation of the instructional unit. Thus, adding the aforementioned new ideas to the pre-existing ideas explained by Hollebrands (2003) and Yanık (2006) and designing an instructional unit in which these ideas were in accord with each other was a well-suited decision for participants development in understanding geometric transformations.

5.3. Evaluation of Instructional Sequence

The instructional sequence covered the critical ideas required for understanding geometric transformations as functions. The first activity was related to identifying reflection and rotation with their parameters in finite figures. The center of rotation was on the figure itself and the line of reflection was passing through the figure. This activity worked well in revealing and developing participants' identification of rotations and reflections. The difficulty level of the tasks included in Activity 1 was appropriate. Besides, the content of the activity was appropriate in terms of being covered in one course session.

Activity Two was related to identifying reflection, rotation, and translation with their parameters in frieze patterns. Different form the first activity, it involved identification of translation tasks, the center of rotation was outside the figure and the line of reflection was not passing through the figure. This activity worked well in revealing and developing participants' identification geometric transformations. The difficulty level of tasks and time allocated for these tasks were appropriate. Findings revealed that tasks included in the first two activities helped participants develop in identification of geometric transformations.

Activity Three was related mainly to performing single and composition of geometric transformations. This activity worked well in revealing and developing participants' performing of compositions of geometric transformations and identification of a single geometric transformation alternative to each of these compositions. The difficulty level of tasks included in this activity was appropriate. However, the content of this activity required more time to be covered. Namely, during the implementation of Activity 3, the prospective teachers expressed that after they performed geometric transformations on activity sheets, the remaining time for exploring a task in the corresponding GeoGebra file was limited. Since tasks included in Activity 3 involved performing geometric transformations both in paper and pencil environment and dynamic geometry environment, more time was required. Therefore, the time allocated for these might be increased in the further iterations.

Activity Four was also related to performing composition of geometric transformations. In general, this activity worked well. The difficulty level of tasks included in this activity was appropriate. However, as in the case of Activity 3, the time allocated to completing all tasks included in this activity was not enough. Therefore, the content of Activity 3 and Activity 4 can be covered in three activities. Namely, in the further iteration, a new activity can be added between Activity 4 and Activity 5. This can be done in this way: the last task of Activity 3 can be moved to the beginning of Activity 4. Similarly, the last two tasks of Activity 4 can be moved to a new activity. In the remaining time period in the new activity, compositions of geometric transformations can be reviewed. Namely, in this study, participants performed all possible combinations of geometric transformation types. However, due to time limitation, there was no opportunity for the participants to discuss that all possible combinations of compositions of geometric transformations also yield a geometric transformation. Doing this, might have helped prospective teachers see compositions of geometric transformations in a broader perspective.

Activity Five was related to having prospective teachers explore functional ideas in a dynamic geometry environment. This activity worked well in revealing and developing participants understanding of geometric transformations as functions. The difficulty level of tasks and time allocated for these tasks were appropriate. The pre-constructed GeoGebra files, the key element of this activity, worked well. Findings revealed that this activity helped participants gain a deeper understanding and more abstract thinking of functions which would pave the way for a mapping conception of geometric transformations.

Activity Six was related to exploring the domain of functions and Activity Seven was related to characteristics of geometric transformations, namely identity function, inverse function, fixed points, and one-to-one and onto property. These activities worked well. Whole class discussions and pair discussions helped participants integrate their knowledge and understanding of geometric transformations obtained through the first four activities with the ones obtained through Activity Five.

5.4. Implications

Based on the findings of the current study and the related literature on geometric transformations, some possible implications for prospective teachers, in-service teachers, mathematics education researchers, mathematics teacher educators, textbook authors, and curriculum developers are presented below.

Current study was carried out in order to contribute to the literature about learners' knowledge and understanding of geometric transformations in general and development of their understanding as they participated in an instructional unit related to geometric transformations. Namely, this study is expected to reveal and refine big ideas which help prospective middle school mathematics teachers proceed from motion to mapping understanding of geometric transformations.

In this study, prospective teachers' development in geometric transformations was traced by the help of a seven-week instructional unit. This unit included activities, all related tools, and pre-constructed GeoGebra files and they were used in a classroom teaching experiment. Mathematics teacher education programs in Turkey do not include compulsory geometry courses which provide prospective teachers systematic training for understanding geometric transformations as functions and functional properties of geometric transformations. Thus, courses involving the aforementioned seven-week instructional unit and its components might play an important role for these programs in designing various courses that help prospective teachers develop mapping conception of geometric transformations.

In this study, growths in prospective teachers' understanding of geometric transformations were explored and consequently the big ideas which were found to be critical for this development were revealed. The critical ideas that supported prospective teachers' development in understanding geometric transformations as functions (i.e., mapping understanding of geometric transformations) were understanding parameters of geometric transformations, understanding the relationships among parameter(s), a pre-image point, and a corresponding image point, understanding properties of geometric transformations, understanding the variables of functions defined on the plane, and understanding the domain of functions defined on the plane, understanding the inverses of geometric transformations, and understanding the fixed and the identity mappings of each family of transformations. These ideas might guide mathematics education researchers in conducting research on prospective teachers' developing understanding of geometric transformations. Namely, researchers might use these ideas in their own research contexts and revise them or incorporate new ideas into them if necessary. By this way, ideas proposed in this study as critical for understanding geometric transformations will be ascertained and refined. Furthermore, mathematics teacher educators can use these ideas in designing courses related to geometric transformations. Mathematics teacher educators are suggested to design these courses by focusing on mapping conception as a core approach and consequently use the aforementioned ideas in having prospective teachers gain this conception.

In this study, a dynamic geometry environment, GeoGebra, was used to enhance prospective teachers' knowledge and understanding of geometric transformations. Most of the tasks included in the activities were presented to prospective teachers first in a paper-pencil and then in a dynamic geometry environment and thus they had the opportunity to explore transformations in both environments. The GeoGebra files pre-constructed by the researcher provided support for their development of geometric transformations. Thus, tasks involving use of technology and their implementations in this study might provide some ideas for mathematics education researchers and mathematics teacher educators about how dynamic geometry software can be used in improving prospective teachers' knowledge and understanding of geometric transformations.

In this study, the researcher considered that understanding functions defined on the plane was a prerequisite for understanding geometric transformations as functions. Thus, the participants of this study explored the concept of functions defined on the plane through GeoGebra. More specifically, they explored variables and domain of functions defined on the plane through Trace and Drag tools of GeoGebra. For instance, by the help of the Trace tool the prospective teachers were provided with an opportunity to see the domain of functions defined from R^2 to R^2 as plane and variables as point in the plane. The relevant sections of the instructional unit might provide a basis for those who want to teach functions in a geometric context. Besides, functions are often exposed to learners in a numerical context. Thus, the current study might help learners raise awareness that functions are defined not only numerically (functions from R to R) but also geometrically (functions from R^2 to R^2). High school mathematics teachers might use the aforementioned relevant sections if they want to teach functions through a geometric perspective. Moreover, mathematics educators and researchers with a focus on extending prospective teachers' knowledge of functions (i.e., from $R \rightarrow R$ and $R^2 \rightarrow R$ to $R^2 \rightarrow R^2$), might take into account the design and findings of this study.

In this study, prospective teachers' understanding of geometric transformations was analyzed based on Arnon et al.'s (2013) APOS theory. More precisely, APOS Theory framework applied to geometric transformations by Hollebrands (2003) was used in data analysis. Research conducted thus far showed that students could reach a process conception utmost (e.g., Hollebrands, 2003; Yanık, 2006) However, in this study half of the participants were able to reach an object conception. Therefore, the data analyzed in this study might shed some light on the ideas that are indicators of prospective teachers' object understanding of geometric transformations. Besides, classroom teaching experiment was conducted to explore prospective teachers' understanding of geometric transformations. There is a lack of studies that analyze prospective teachers' understanding of geometric transformations while they were participating in pair discussions and more importantly in whole class discussions which involved ideas related to object conception of geometric transformations. Thus, data analysis carried out by using APOS theory in this study may contribute to mathematics education researchers in analyzing learners' understanding of geometric transformations.

5.5. Suggestions

In this study it was revealed that participating prospective middle school mathematics teachers had limited knowledge of geometric transformations prior to the implementation of the seven-week instructional unit. However, they were able to improve their knowledge and understanding of geometric transformations after being exposed to the instructional unit. Based on this finding, it can be suggested that pre-service mathematics teacher education programs should include courses that help prospective teachers extend their knowledge and understanding of geometric transformations. These courses might be designed by considering the activities included in the seven-week instructional unit. Providing prospective teachers such courses is important for several reasons. First, as they will start teaching in the near future, they need to have a sound body of knowledge about geometric transformations before entering into the classrooms. Second, if they begin their teaching career with some difficulties about geometric transformations, it is most likely that these difficulties would be conveyed to their own students by themselves. Similarly, the content of existing methods of mathematics teaching courses should pay considerable attention to the teaching of geometric transformations in a way that enable prospective teachers to reach a mapping conception and move away from motion conception.

Yanık (2014) found out that textbooks and classroom instruction seemed to cause fundamental student misconceptions about geometric transformations. More specifically, some of the textbooks and teacher generated real life examples might interfere with students' understanding of geometric transformations (e.g., a rolling ball and movement of a car) (Yanık, 2014). Similarly, at the beginning of this study, the prospective teachers held motion conception of transformations which could be regarded as a naïve conception that might hinder formation of mapping conception. Thus, textbook authors should prepare textbooks that include carefully selected transformation geometry examples. Namely, textbook authors should provide examples with careful explanations that can avoid unexpected student conceptions about geometric transformations and can help students move away from motion conception and gain a mapping conception of geometric transformations. Furthermore, in-service mathematics teachers' understanding of geometric transformations might be improved by using the instructional unit developed in this study. In-service teachers might develop awareness about factors associated with effective transformation geometry instruction and organize their classroom instruction accordingly.

In Turkey, the learning objectives regarding school mathematics topics are specified in national curriculum documents. They provide us a picture of what is to be taught and learned in the classrooms. Thus, curriculum developers should notice the importance of understanding geometric transformations as functions for learning both geometric transformations and functions and prepare learning objectives that explicitly emphasize the teaching of ideas that are found to be critical in this study for gaining mapping understanding of geometric transformations.

Thus far, some suggestions for mathematics teacher education program developers, national curriculum developers, and textbook writers were proposed. Below, some suggestions for future research studies are offered.

The current study was conducted with sixteen senior and junior prospective middle school mathematics teachers enrolled in state university in the inner region of Turkey. This study might be replicated by using a different group of prospective middle school mathematics teachers to test whether the instructional unit leads to a similar development in participants' understanding of geometric transformations. Besides, prospective secondary school mathematics teachers would be selected as participants and it might be explored how their understanding of geometric transformations improve and it might be revealed whether their development differs from prospective middle school mathematics and if yes in what ways.

Moreover, in-service middle or secondary school mathematics teachers might be selected as participants and it might be explored whether they move along the motion-mapping continuum in a way similar to prospective middle or secondary school teachers. Studying with in-service teachers is also significant in terms of examining whether their teaching experiences have any effect on their understanding of geometric transformations. In other words, it should be explored whether the instructional unit developed in this study would have the similar influences on prospective teachers, novice teachers and experienced teachers (especially the ones with at least 20 years of mathematics teaching).

In this study, understanding the parameters of geometric transformations, understanding relationships among parameter(s), a pre-image point and a corresponding image point, understanding properties of geometric transformations, understanding the variables of functions defined on the plane, understanding the domain of functions defined on the plane, understanding the inverses of geometric transformations, and understanding the fixed and the identity mappings of each family of transformations were critical ideas for prospective teachers' gaining of mapping conception. Other research studies might investigate whether there are other ideas that are critical for developing mapping conception of prospective teachers' geometric transformations. Similarly, the ideas that are critical for in-service mathematics teachers' mapping conception of geometric transformations might be explored.

In this study, some activities and tasks were developed in order to keep track of prospective teachers' developing understanding of geometric transformations. Other researchers might design different activities with different tasks and examples which focus on the same critical ideas mentioned above. More explicitly, in the current study half of the participants could gain an object conception and less than half of them could gain a process conception. Thus, researchers might examine whether a higher rate of participants can display a well-developed understanding of geometric transformations with different activities focusing the same abovementioned ideas.

In this study, the Transformation Geometry Questionnaire was used to assess relatively a small number of prospective mathematics teachers' pre-existing knowledge of geometric transformations. Other researchers might use this questionnaire for a broader sample of learners and focus on their pre-existing knowledge of geometric transformations by conducting quantitative research studies. In particular, researchers might focus on learners' possible difficulties, errors, or misconceptions about geometric transformations by using this questionnaire and might provide a more general picture about learners' understanding of geometric transformations.

The current study included an activity to have participants explore functional ideas in a dynamic geometry environment and these ideas were considered essential for prospective teachers to understand that geometric transformations can also be viewed as functions. The findings revealed that exploring functions on the plane helped prospective teachers in understanding geometric transformations as functions. Further studies can be conducted to find out whether there is a reflexive relationship between understanding functions and understanding geometric transformations. In other words, researchers might examine whether understanding geometric transformations helps learners in understanding functions.

Steketee (2012) argued that formal teaching of functions should begin with geometric transformations in a dynamic geometry environment. Although his study was not an empirical research, he hypothesized that teaching functions in this way might help students in developing stronger and clearer ideas related to functions. Based on this idea, it can be suggested that future studies might concentrate on whether it is more effective to use geometric transformations instead of using numeric functions.

5.6. Limitations

The limitations that should be taken into account while interpreting the findings of this study are explained as follows.

The participants of this study were senior and junior prospective school mathematics teachers enrolled in a state university in the inner region of Turkey. In Turkey, students are placed to the programs they wish to study based on their university examination scores. Naturally, some universities are more frequently preferred than the other ones. Consequently, more popular universities accept students with higher examination scores and prospective middle school mathematics teachers enrolled in each university have similar examination scores. Prospective teachers who participated in this study had similar university entrance examination scores and thus similar backgrounds in this respect. Thus, the findings of this study shed light only on participating prospective middle school mathematics teachers’ knowledge and understanding of geometric transformations. Prospective teachers who are enrolled in universities that accept students with higher or lower university entrance examination scores might have different backgrounds. Therefore, the findings of this study should be evaluated by considering specified classrooms and university contexts.

In this study, the researcher was also an instructor. During the implementation of the instructional unit, the researcher guided all of the whole class discussions upon completion of each transformation geometry task in pair work. Prospective teachers generated mathematical ideas under the guidance of the researcher. Therefore, the researcher might have had some effect on prospective teachers' mathematical practices and ideas that emerged during whole class discussions.

In this study, four male and twelve female prospective teachers took part in the study. Uneven distribution of participants' gender might be another limitation for this study. Moreover, the data of this study regarding prospective teachers' pre-existing knowledge of geometric transformations is limited to the Transformation Geometry Questionnaire developed by the researcher.

Finally, participants' willingness to respond to all tasks included in the activities, their willingness to participate in pairwise and whole class discussions, and their willingness to respond to the questions included in the follow-up interviews were also a limitation for this study since their understanding of geometric transformations were determined based on the explanations obtained through the aforementioned procedures.

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APPENDIX A

TRANSFORMATION GEOMETRY QUESTIONNAIRE

DÖNÜŞÜM GEOMETRİSİ KAVRAMLARI TESTİ

1. Aşağıdaki kavramlarla ilgili istenilen bilgileri yazınız.

A. Yansıma Dönüşümü

1.A.1. Yansıma dönüşümü nedir? Tanımlayınız.

1.A.2. Yansıma dönüşümüne bir örnek veriniz ve örneğinizi açıklayınız.

1.A.3. Yansıma dönüşümünün özelliklerini yazınız.

B. Öteleme Dönüşümü

1.B.1. Öteleme dönüşümü nedir? Tanımlayınız.

1.B.2. Öteleme dönüşümüne bir örnek veriniz ve örneğinizi açıklayınız.

1.B.3. Öteleme dönüşümünün özelliklerini yazınız.

C. Dönme Dönüşümü

1.C.1. Dönme dönüşümü nedir? Tanımlayınız.

1.C.2. Dönme dönüşümüne bir örnek veriniz ve örneğinizi açıklayınız.

1.C.3. Dönme dönüşümünün özelliklerini yazınız.

D. Ötelemeli Yansıma Dönüşümü

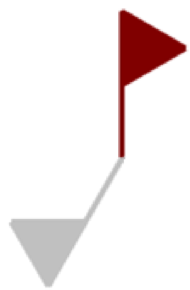
1.D.1. Ötelemeli yansıma dönüşümü nedir? Tanımlayınız.

1.D.2. Ötelemeli yansıma dönüşümüne bir örnek veriniz ve örneğinizi açıklayınız.

1.D.3. Ötelemeli yansıma dönüşümünün özelliklerini yazınız.

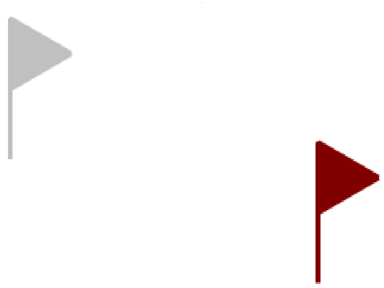
2. Aşağıda koyu renkli bayraklar ve bu bayraklara geometrik dönüşümlerin uygulanmasıyla elde edilen görüntüleri verilmiştir. Bu görüntülerin her birinin koyu renkli bayrağa hangi dönüşümün ya da dönüşümlerin uygulanmasıyla elde edilebileceğini yanlarında verilen boşlukta ayrıntılı bir şekilde açıklayınız.

A

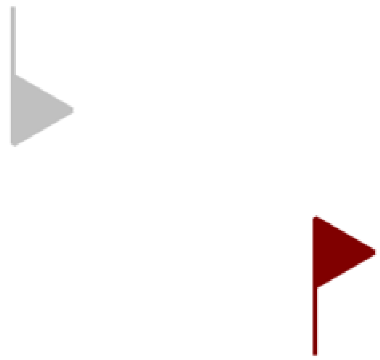


B

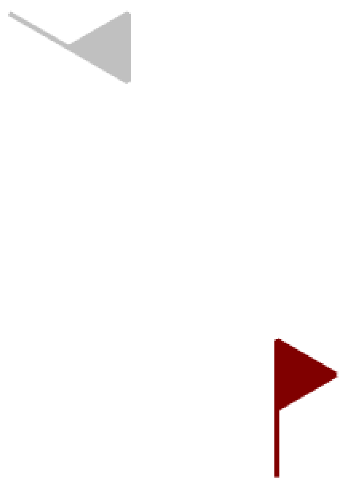
C



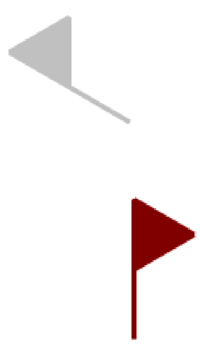
D



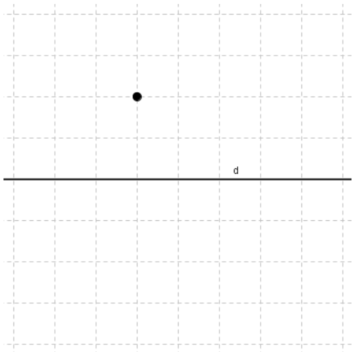
E



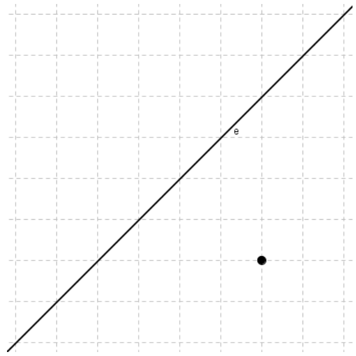
F



3. Aşağıdaki noktaları verilen doğrulara göre yansıtınız.



A



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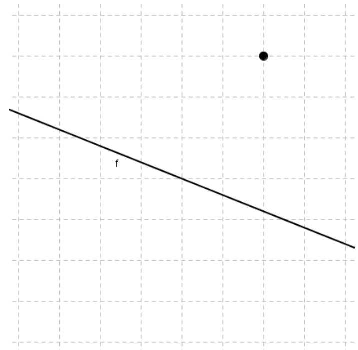
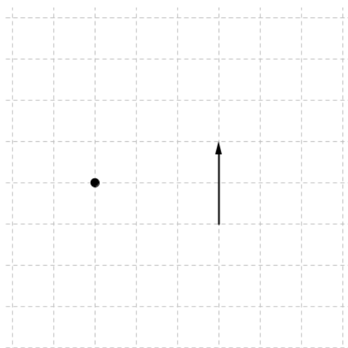
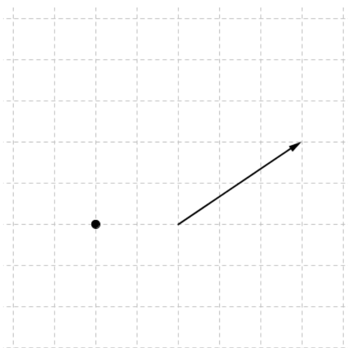


C

4. Aşağıdaki noktaları verilen vektörler ile öteleyiniz.



A



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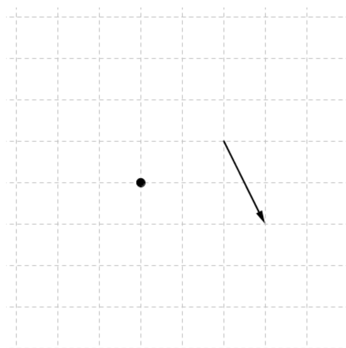
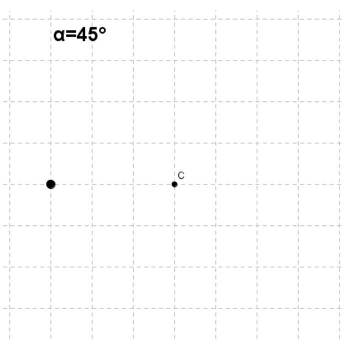
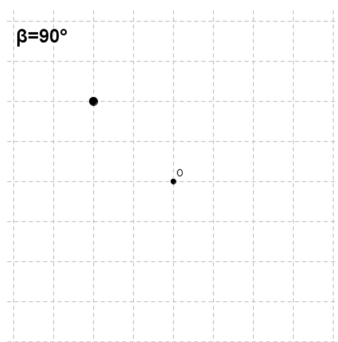


C

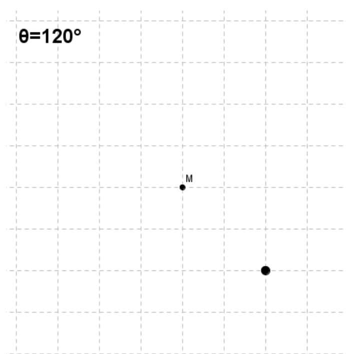
5. Aşağıda A şıkkında verilen noktayı C noktası etrafında α açısı kadar, B şıkkında verilen noktayı O noktası etrafında β açısı kadar ve C şıkkında verilen noktayı M noktası etrafında θ açısı kadar döndürünüz.



A



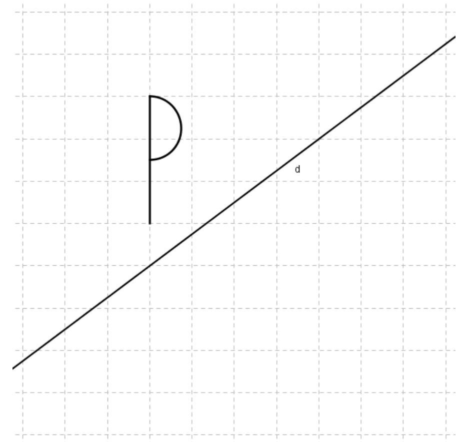
B



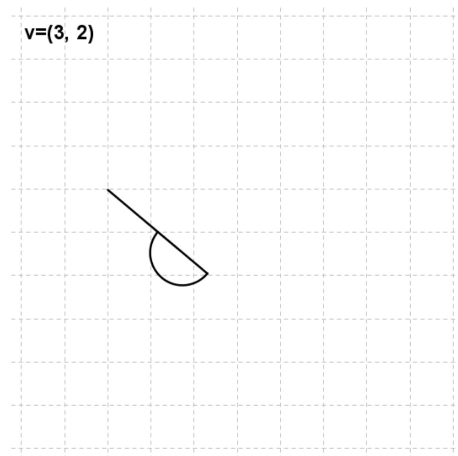
C

6. P harfine aşağıdaki dönüşümleri uygulayınız.

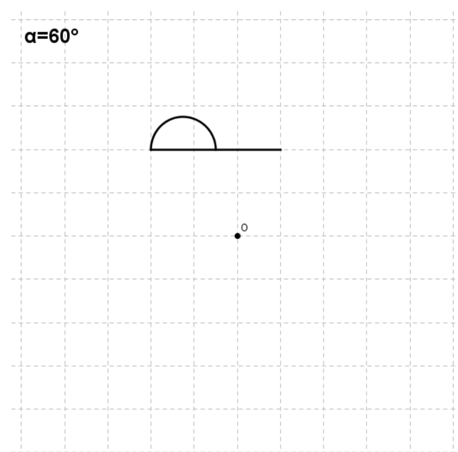
a. P harfinin d doğrusuna göre yansımasını çiziniz ve çiziminizi açıklayınız.



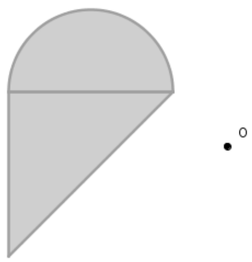
b. P harfinin v vektörü ile ötelenmesi sonucunda oluşan görüntüsünü çiziniz ve çiziminizi açıklayınız.



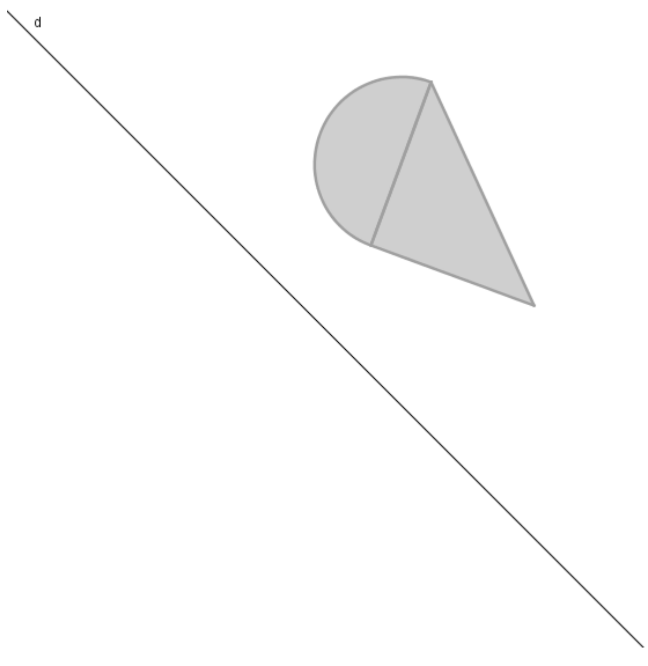
c. P harfinin O noktası etrafında pozitif yönde α açısı kadar döndürülmesi sonucunda oluşan görüntüsünü çiziniz ve çiziminizi açıklayınız.



7. Aşağıda verilen şeklin O noktası etrafında pozitif yönde 140° döndürüldüğünde oluşan görüntüsünü pergel, cetvel ve açıölçer yardımıyla çiziniz. Çiziminizin aşamalarını açıklayınız.



8. Aşağıda verilen şeklin d doğrusuna göre yansımasını pergel, cetvel ve açıölçer yardımıyla çiziniz. Çiziminizin aşamalarını açıklayınız.



9. Geometrik dönüşümler (yansıma, öteleme ve dönme dönüşümleri) bir fonksiyon belirtir mi? Cevabınız Evet ise;

a. Nedenini açıklayınız.

b. Bu fonksiyonun tanım kümesini, değer kümesini ve bu kümelerin elemanlarını belirleyiniz.

10. Geometrik dönüşümlerin bileşkesi her zaman bir geometrik dönüşüm müdür? Cevabınızı örnekler yardımıyla açıklayınız.

APPENDIX B

SCORING RUBRIC FOR TRANSFORMATION GEOMETRY QUESTIONNAIRE

Rubric for Question 1		
Sub-questions	Score	Explanation
1A1, 1B1, 1C1 and 1D1	0	Blank or irrelevant answer
	1	Defining transformations as motions on the plane (i.e., action level)
	2	Defining transformations as functions defined from plane to plane (i.e., process level)
	3	Defining transformations as one to one and onto functions defined from plane to plane (i.e., object level)
1A2, 1B2, 1C2 and 1D2	0	Blank or irrelevant answer
	1	Transformation example is correct (only example is given, its explanation is not provided)
	2	Transformation example and its explanation are both correct.
1A3, 1B3, 1C3 and 1D3	0	Blank or irrelevant answer
	0,5	One of the characteristics of transformation is explained correctly
	1	Two of the characteristics of transformation is explained correctly

Rubric for Question 2		
Sub-questions	Score	Explanation
A and F	0	Blank or irrelevant answer
	1	Rotation is identified but none of the parameters are explained.
	2	Rotation and one of its parameters (angle or center of rotation) are identified correctly.
	3	Rotation and both parameters are identified correctly.
B and E	0	Blank or irrelevant answer
	1	Reflection is identified but the reflection line (parameter) is not drawn.
	2	Reflection and reflection line are identified but the reflection line is not drawn precisely (e.g., ignoring equal distances to the reflection line or mid-points).
	3	Reflection and reflection line are identified precisely. The distances from related points (A and A') to the reflection line are equal.
C	0	Blank or irrelevant answer
	1	Translation is identified but the translation vector (parameter) is not explained.
	2	Translation is identified and either the direction or the magnitude of the translation vector is explained.
	3	Translation is identified and both the direction and the magnitude of the translation vector are explained.
D	0	Blank or irrelevant answer
	1	Glide reflection is identified but none of the parameters are explained.
	2	Glide reflection is identified but either the reflection line, the direction or the magnitude of the translation vector is not explained.
	3	Glide reflection and parameters of both reflection and translation are identified. That is, reflection line, the direction and the magnitude of the translation vector are identified precisely.

Rubric for Question 3, Question 4, and Question 5		
Sub-questions	Score	Explanation
3A, 3B, and 3C	0	Blank answer or given point is not reflected correctly.
	1	Given point is reflected correctly.
4A, 4B, and 4C	0	Blank answer or given point is not translated correctly.
	1	Given point is translated correctly.
5A, 5B, and 5C	0	Blank answer or given point is not rotated correctly.
	1	Given point is not rotated precisely. That is, either the angle of rotation or the distance from the center is not correctly determined.
	2	Given point is rotated correctly. That is, the angle of rotation and the distance from the center is correctly determined.

Rubric for Question 6		
Sub-questions	Score	Explanation
6A	0	Blank or irrelevant answer.
	1	Only one point on the line segment is reflected correctly but the reflected figure is not correct.
	2	Two points on the line segment are reflected correctly but the reflected figure is not entirely correct.
	3	All components of the figure are reflected correctly. That is, both the line segment and the semi-circle are reflected correctly.
6B	0	Blank or irrelevant answer.
	1	Only one point on the line segment is translated correctly but the translated figure is not correct.
	2	Two points on the line segment are translated correctly but the translated figure is not entirely correct.
	3	All components of the figure are translated correctly. That is, both the line segment and the semi-circle are translated correctly.
6C	0	Blank or irrelevant answer.
	1	Only one point on the line segment is rotated correctly but the rotated figure is not correct.
	2	Two points on the line segment are rotated correctly but the rotated figure is not entirely correct.
	3	All components of the figure are rotated correctly. That is, both the line segment and the semi-circle are rotated correctly.

Rubric for Question 7 and Question 8		
Questions	Score	Explanation
7	0	Blank or irrelevant answer.
	1	Only one vertex of the triangle is rotated correctly but the rotated figure is not correct.
	2	Two points on the line segment are rotated correctly but the reflected figure is not entirely correct.
	3	Three vertices of the triangle or two vertices of the triangle and another point on the semi-circle are rotated correctly but the rotated figure is not entirely correct.
	4	All components of the figure are rotated correctly. That is, both the triangle and the semi-circle are rotated correctly.
8	0	Blank or irrelevant answer.
	1	Only one vertex of the triangle is reflected correctly but the reflected figure is not correct.
	2	Two points on the line segment are reflected correctly but the reflected figure is not entirely correct.
	3	Three vertices of the triangle or two vertices of the triangle and another point on the semi-circle are reflected correctly but the reflected figure is not entirely correct.
	4	All components of the figure are reflected correctly. That is, both the triangle and the semi-circle are reflected correctly.

Rubric for Question 9		
Sub-questions	Score	Explanation
9A	0	Blank or irrelevant answer.
	1	Saying yes without explanation
	5	Saying yes and explaining transformations as relations from one set to another set
	7	Explaining transformations as relations from one set to another set and explaining that every element of domain are mapped
	9	Explaining transformations as relations from one set to another set and explaining that each element of domain is mapped only one element in the range
9B	0	Blank or irrelevant answer.
	3	Explaining only domain as plane or explaining only range as plane
	6	Explaining domain and range as plane
	9	Explaining the elements of domain and range as points in the plane

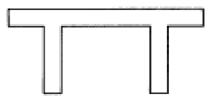
Rubric for Question 10		
Question	Score	Explanation
10	0	Blank or irrelevant answer
	1	Saying yes without explanation
	5	Composition of geometric transformations example is correct
	11	Composition of geometric transformations example and its explanation are both correct

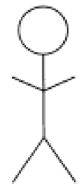
APPENDIX C

ACTIVITIES

ETKİNLİK 1

1. Aşağıdaki şekiller geometrik dönüşümler uygulanarak elde dilmiştir.



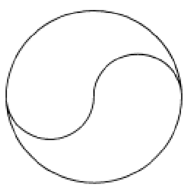


a. Her iki şekilde ortak olan geometrik dönüşümleri belirleyiniz.

b. Belirlediğiniz geometrik dönüşümleri birinci şekil için açıklayınız.

c. Belirlediğiniz geometrik dönüşümleri ikinci şekil için açıklayınız.

2. Aşağıdaki şekiller geometrik dönüşümler uygulanarak elde dilmiştir.

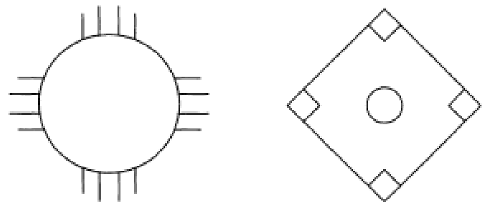


a. Her iki şekilde ortak olan geometrik dönüşümleri belirleyiniz.

b. Belirlediğiniz geometrik dönüşümleri birinci şekil için açıklayınız.

c. Belirlediğiniz geometrik dönüşümleri ikinci şekil için açıklayınız.

3. Aşağıdaki şekiller geometrik dönüşümler uygulanarak elde dilmiştir.

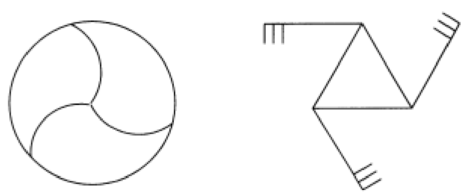


a. Her iki şekilde ortak olan geometrik dönüşümleri belirleyiniz.

b. Belirlediğiniz geometrik dönüşümleri birinci şekil için açıklayınız.

c. Belirlediğiniz geometrik dönüşümleri ikinci şekil için açıklayınız.

4. Aşağıdaki şekiller geometrik dönüşümler uygulanarak elde dilmiştir.

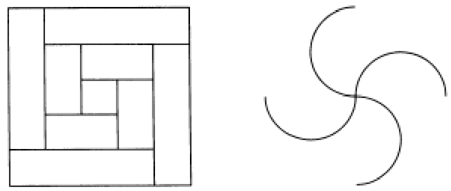


a. Her iki şekilde ortak olan geometrik dönüşümleri belirleyiniz.

b. Belirlediğiniz geometrik dönüşümleri birinci şekil için açıklayınız.

c. Belirlediğiniz geometrik dönüşümleri ikinci şekil için açıklayınız.

5. Aşağıdaki şekiller geometrik dönüşümler uygulanarak elde dilmiştir.



a. Her iki şekilde ortak olan geometrik dönüşümleri belirleyiniz.

b. Belirlediğiniz geometrik dönüşümleri birinci şekil için açıklayınız.

c. Belirlediğiniz geometrik dönüşümleri ikinci şekil için açıklayınız.

6. Kutudan rastgele bir kâğıt çekiniz. Size çıkan şekli kimseye göstermeden inceleyiniz ve bu şeklin sahip olduğu özellikleri belirten matematiksel ifade/ler yazınız. Daha sonra yazdığınız ifadeyi ikinci kutuya atınız. İkinci kutu tamamlandığında içinden rastgele bir kâğıt çekip arkadaşınızın yazmış olduğu ifadeden onun ilk kutudan seçmiş olduğu şekli tahmin etmeye çalışınız (İkinci kutudan kendi yazdığınız ifadeyi çekti iseniz bir başkası ile değişiniz).

7. Bu derste incelenen şekilleri ve geometrik dönüşümleri anımsayınız. Benzer şekilde siz de geometrik dönüşümleri kullanarak yeni iki şekil oluşturunuz. Şeklinizi geometrik dönüşümler ile matematiksel olarak açıklayınız.

a. Sadece dönme dönüşümü kullanarak elde edilebilecek bir şekil oluşturunuz.

b. Dönmeyi uyguladığınız birimi çiziniz.

c. Şekli elde etmek için bu birime uyguladığınızı dönmeyi açıklayınız.

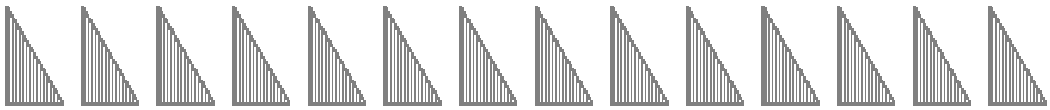
d. Hem dönme hem yansıma dönüşümü kullanarak elde edilebilecek bir şekil oluşturunuz.

e. Dönüşümleri uyguladığınız birimi belirleyiniz.

f. Şekli elde etmek için bu birime uyguladığınız dönüşümleri açıklayınız.

ETKİNLİK 2

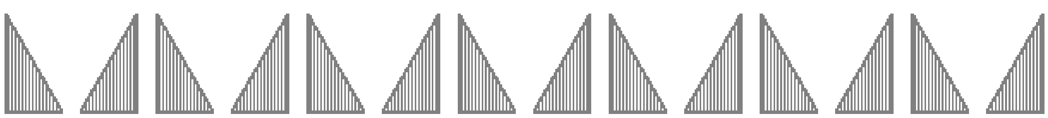
1. Bir dik üçgene geometrik dönüşümlerin uygulanmasıyla oluşturulan şerit süslemeyi inceleyiniz.



a. Bu süslemede kullanılan geometrik dönüşümü/dönüşümleri belirleyiniz.

b. Bu süslemenin nasıl oluşturulabileceğini belirlediğiniz dönüşümlerle <u>aşama aşama</u> açıklayınız.

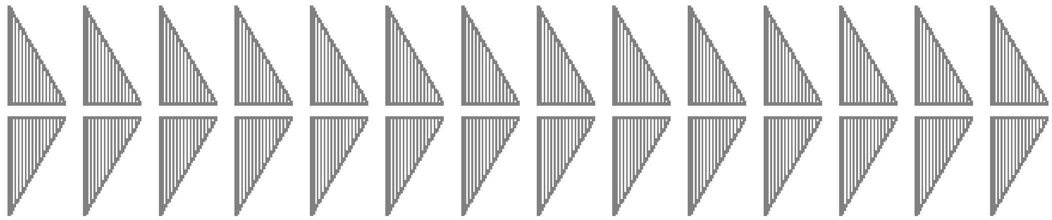
2. Bir dik üçgene geometrik dönüşümlerin uygulanmasıyla oluşturulan şerit süslemeyi inceleyiniz.



a. Bu süslemede kullanılan geometrik dönüşümü/dönüşümleri belirleyiniz.

b. Bu süslemenin nasıl oluşturulabileceğini belirlediğiniz dönüşümlerle <u>aşama aşama</u> açıklayınız.

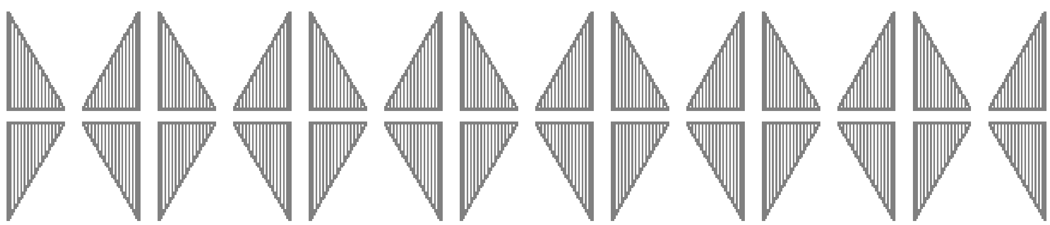
3. Bir dik üçgene geometrik dönüşümlerin uygulanmasıyla oluşturulan şerit süslemeyi inceleyiniz.



a. Bu süslemede kullanılan geometrik dönüşümü/dönüşümleri belirleyiniz.

b. Bu süslemenin nasıl oluşturulabileceğini belirlediğiniz dönüşümlerle aşama aşama açıklayınız.

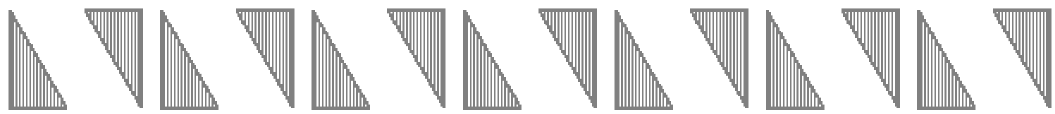
4. Bir dik üçgene geometrik dönüşümlerin uygulanmasıyla oluşturulan şerit süslemeyi inceleyiniz.



a. Bu süslemede kullanılan geometrik dönüşümü/dönüşümleri belirleyiniz.

b. Bu süslemenin nasıl oluşturulabileceğini belirlediğiniz dönüşümlerle aşama aşama açıklayınız.

5. Bir dik üçgene geometrik dönüşümlerin uygulanmasıyla oluşturulan şerit süslemeyi inceleyiniz.



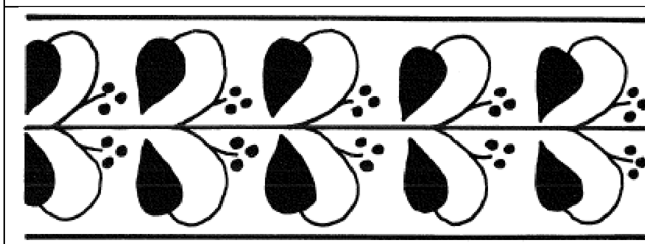
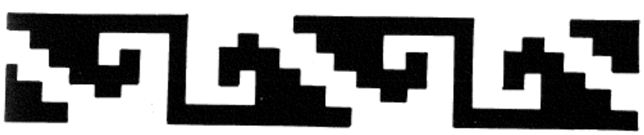
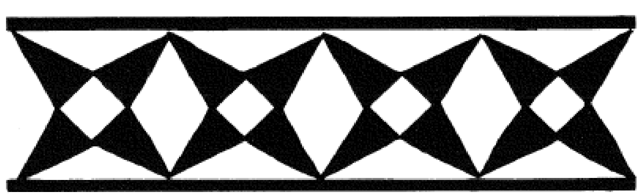
a. Bu süslemede kullanılan geometrik dönüşümü/dönüşümleri belirleyiniz.

b. Bu süslemenin nasıl oluşturulabileceğini belirlediğiniz dönüşümlerle aşama aşama açıklayınız.

6. Aşağıda verilen şerit süslemeleri inceleyiniz.

a. Aşağıdaki her bir şerit süslemeyi 1 – 5 sorulardaki şerit süslemelerle (üçgenlere geometrik dönüşümlerin uygulanmasıyla oluşturulan süslemeler) karşılaştırınız ve <u>aynı dönüşümlerin</u> uygulanmasıyla elde edilen şerit süsleme ile eşleştiriniz.

b. Bu dönüşümlerin uygulandığı motifleri verilen boşluğa çizerek gösteriniz.

Şerit Süslemeler	**Süsleme ile ilgili Açıklama**
	………………………… motifine ……sorudaki dönüşümlerin uygulanmasıyla oluşmuştur.
	………………………… motifine ……sorudaki dönüşümlerin uygulanmasıyla oluşmuştur.
	………………………… motifine ……sorudaki dönüşümlerin uygulanmasıyla oluşmuştur.
	………………………… motifine ……sorudaki dönüşümlerin uygulanmasıyla oluşmuştur.
	………………………… motifine ……sorudaki dönüşümlerin uygulanmasıyla oluşmuştur.

7.Kutudan rastgele bir kâğıt çekiniz. Grubunuza kutudan çıkan süslemeyi inceleyiniz. Süslemenizin 1 – 5 sorulardan hangisindeki dönüşümlerin uygulanmasıyla elde edildiğine karar veriniz ve kâğıdınızın arkasına yazınız.

8.Bu soruda sizden aşağıdaki aşamaları takip ederek bir şerit süsleme oluşturmanız istenmektedir.
a. Ç, F, G, J, L, P, R harflerinden birisini seçiniz.

b. Bu harfe geometrik dönüşümler uygulayarak şerit süsleme elde ediniz.

c. Şerit süslemenizi nasıl oluşturduğunuzu geometrik dönüşümlerle açıklayınız.

9.Bu soruda sizden aşağıdaki aşamaları takip ederek bu derste incelemiş olduğunuz şerit süslemelere benzer bir şerit süsleme oluşturmanız beklenmektedir.

a. Simetri özelliğine sahip olmayan bir model çiziniz.

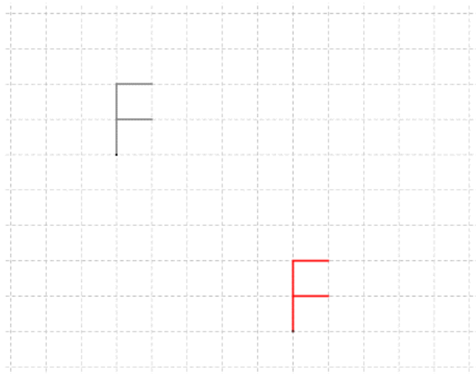
b. Bu modele geometrik dönüşümler uygulayarak şerit süsleme elde ediniz.

c. Şerit süslemenizi nasıl oluşturduğunuzu geometrik dönüşümlerle açıklayınız.

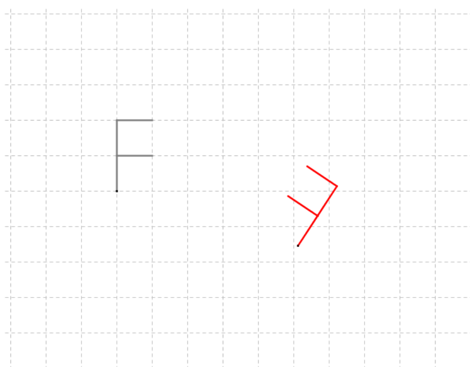
ETKİNLİK 3

1.Aşağıda verilen şekilleri grup arkadaşınızla birlikte inceleyiniz. Her bir şıkta F harfine hangi geometrik dönüşüm veya dönüşümlerin uygulanmış olabileceğini belirleyiniz. Belirlediğiniz dönüşümlerin nasıl gerçekleştiğini açıklayınız.

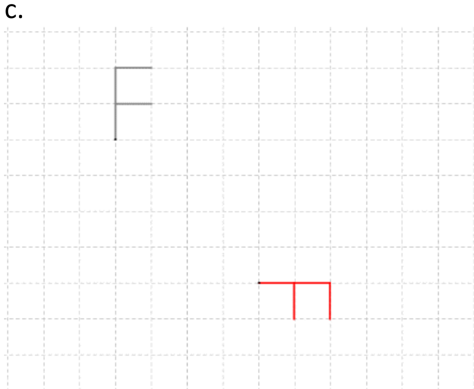
a.



b.

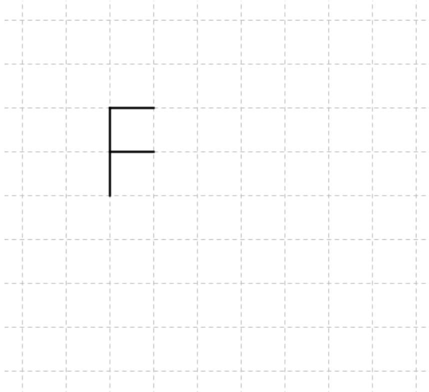


c.

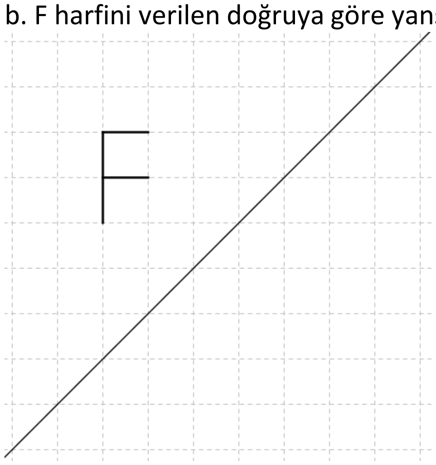


2. F harfine aşağıdaki dönüşümleri uygulayınız.

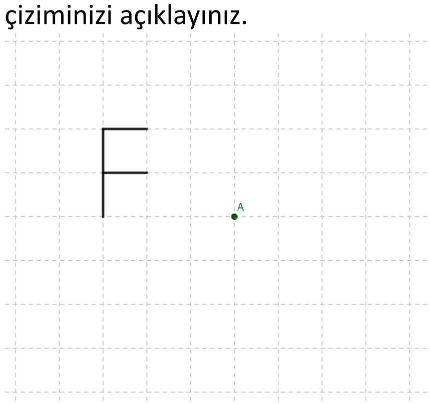
a. F harfinin u=(3, 1) vektörü ile ötelenmesi sonucunda oluşan görüntüsünü çiziniz ve çiziminiz açıklayınız.



b. F harfini verilen doğruya göre yansımasını çiziniz ve çiziminizi açıklayınız.



c. F harfini A noktası etrafında 120° döndürülmesi sonucunda oluşan görüntüsünü çiziniz ve çiziminizi açıklayınız.

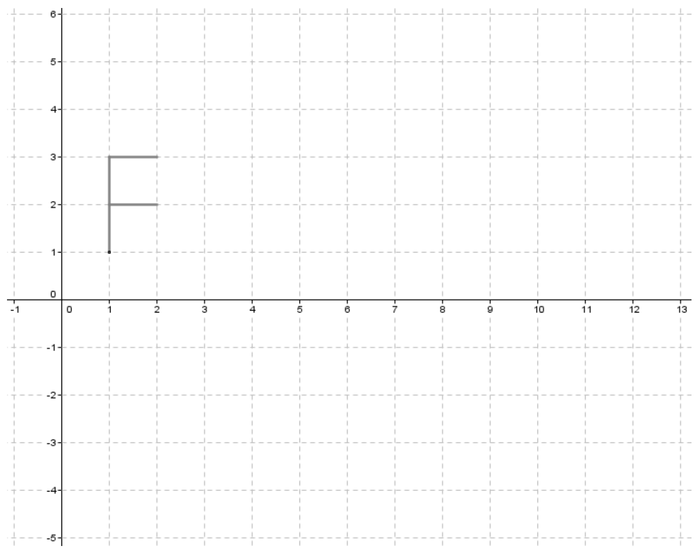


3. Aşağıda koordinat düzleminde bir F harfi verilmiştir. F harfini a, b ve c şıklarında istendiği gibi aşama aşama öteleyiniz. Her bir şıkta farklı renk kalem kullanınız.

a. F harfini $\vec{v} = (3, 2)$ ile öteleyiniz, oluşan görüntüyü $\vec{w} = (4, -3)$ile öteleyiniz.

b. F harfini $\vec{d} = (2, -4)$ile öteleyiniz, oluşan görüntüyü$\vec{e} = (3, -1)$ ile öteleyiniz, son olarak oluşan görüntüyü $\vec{\imath} = (2, 4)$ile öteleyiniz.

c. F harfini $\vec{\jmath} = (9, 2)$ ile öteleyiniz, oluşan görüntüyü $\vec{k} = (-4, 1)$ ile öteleyiniz, daha sonra oluşan görüntüyü $\vec{p} = (5, -3)$ ile öteleyiniz, son olarak oluşan görüntüyü $\vec{q} = (-3, -1)$ ile öteleyiniz.



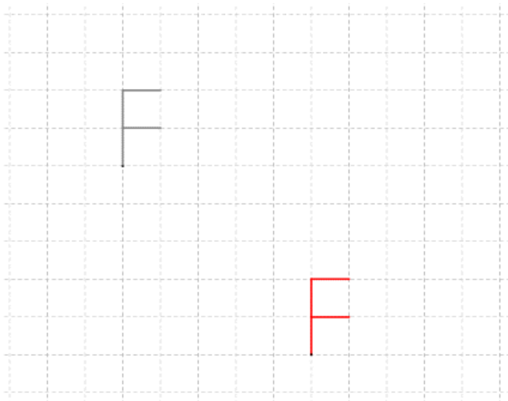
d. a, b ve c şıklarında elde ettiğiniz sonuçları karşılaştırınız ve gözleminizi aşağıya yazınız.

e. Her bir şık için F in ilk ve son görüntüsünü inceleyiniz ve tek bir dönüşüm ile açıklayınız.

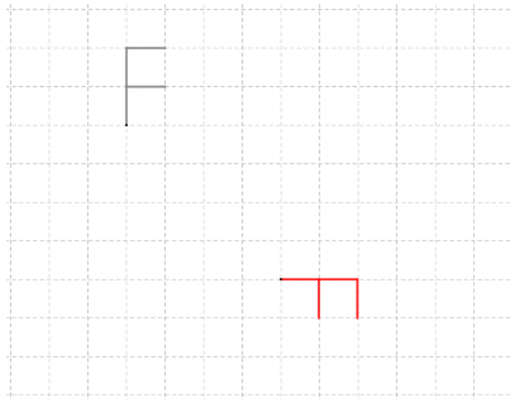
f.F harfini a şıkkında iki, b şıkkında üç, c şıkkında dört kez verilen vektörler ile ötelediniz ve F in son görüntülerini elde ettiniz. Bu dönüşümleri çizerek gerçekleştirmeden F in son görüntüsünü belirlemek mümkün müdür? Mümkünse nasıl olduğunu açıklayınız.

4. Birinci soruda incelemiş olduğunuz şekillerden ikisi aşağıda tekrar verilmiştir. Her şıkta F harfine iki defa yansıma dönüşümü uygulandığı bilinmektedir. Bu yansımaların yansıma doğrularını çiziniz.

a.



b.



c. a ve b şıklarında elde ettiğiniz sonuçları karşılaştırınız.

5. Aşağıdaki şekilde bir bayrak ve A, B ve C noktaları verilmiştir.

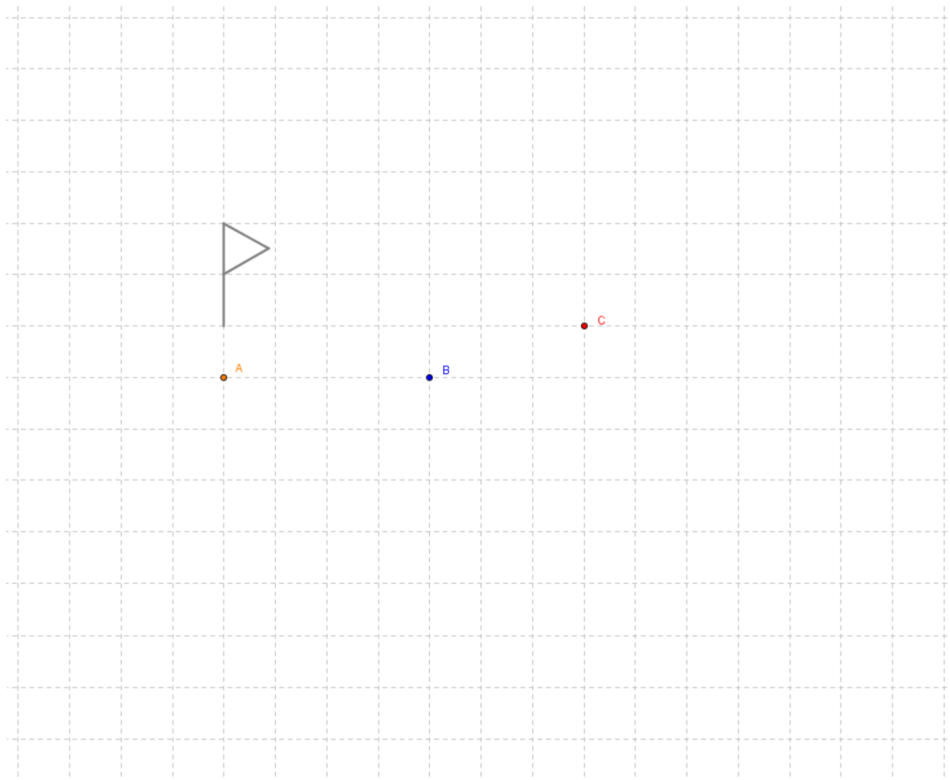
a. Şekil üzerinde kendi grubunuzun karşısında verilen üç döndürmeyi her biri için farklı renk kalem kullanarak gerçekleştiriniz. Grubunuzu işaretleyerek belirtiniz.

1. Grup: Bayrağı A noktası etrafında 90°, oluşan görüntüyü B noktası etrafında 220°, oluşan görüntüyü de C noktası etrafında 50° döndürünüz.

2. Grup: Bayrağı A noktası etrafında 90°, oluşan görüntüyü B noktası etrafında 150°, oluşan görüntüyü de C noktası etrafında 120° döndürünüz.

3. Grup: Bayrağı A noktası etrafında 120°, oluşan görüntüyü B noktası etrafında 180°, oluşan görüntüyü de C noktası etrafında 60° döndürünüz.

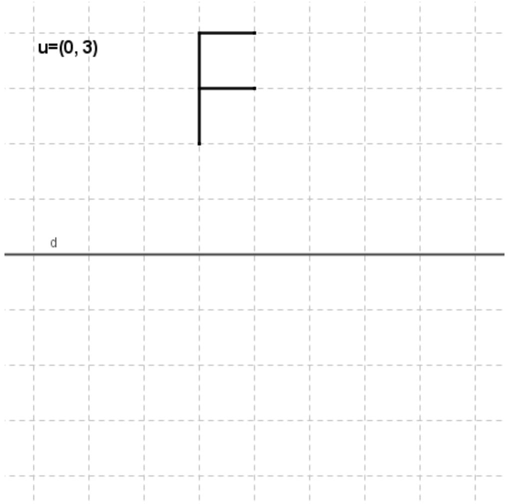
4. Grup: Bayrağı A noktası etrafında 180°, oluşan görüntüyü B noktası etrafında 120°, oluşan görüntüyü de C noktası etrafında 60° döndürünüz.



b. Bayrağın ilk ve son görüntülerini inceleyerek tek bir dönüşüm ile açıklayınız.

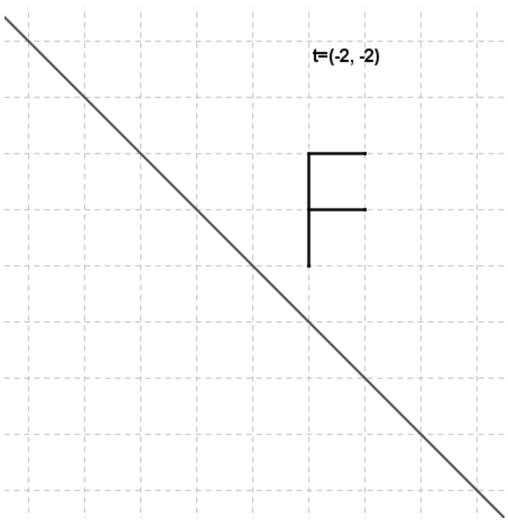
4. ETKİNLİK

1. Aşağıda her biri bir F harfi, bir doğru ve bir vektör içeren şekiller verilmiştir. Bir şekildeki F harfini önce verilen doğruya göre yansıtınız daha sonra elde ettiğiniz görüntüyü verilen vektör ile öteleyiniz.



Şekil 1a

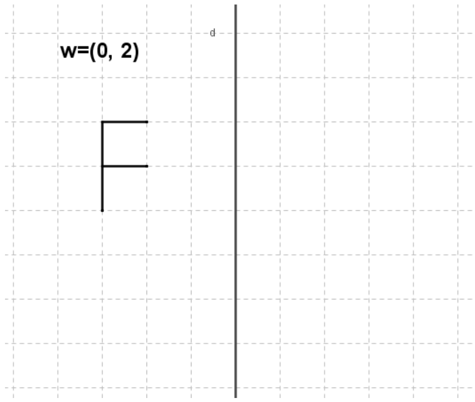
a. Şekil 1a da yansıma ve öteleme ile elde ettiğiniz sonucu <u>tek</u> bir geometrik dönüşümle açıklayınız. Bu geometrik dönüşümü şekil üzerinde farklı renkte bir kalemle çizerek gösteriniz.

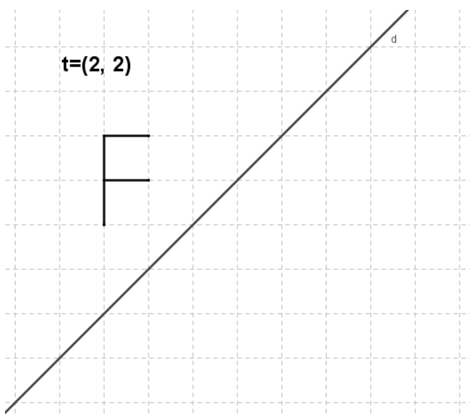


Şekil 1b

b. Şekil 1b de yansıma ve öteleme ile elde ettiğiniz sonucu <u>tek</u> bir geometrik dönüşümle açıklayınız. Bu geometrik dönüşümü şekil üzerinde farklı renkte bir kalemle çizerek gösteriniz.

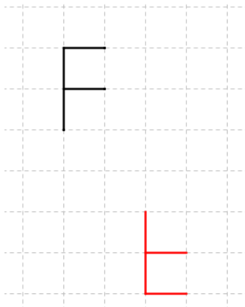
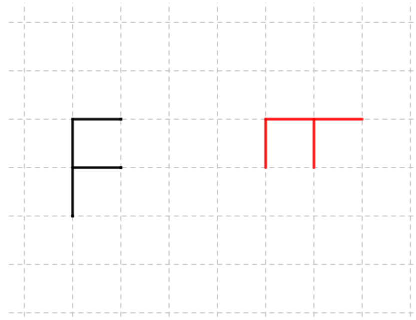
<u>Sonuç:</u>

<table>
<tr><td colspan="2">2. Aşağıda her biri bir F harfi, bir doğru ve bir vektör içeren şekiller verilmiştir. Bir şekildeki F harfini önce verilen doğruya göre yansıtınız daha sonra elde ettiğiniz görüntüyü verilen vektör ile öteleyiniz.</td></tr>
<tr><td>

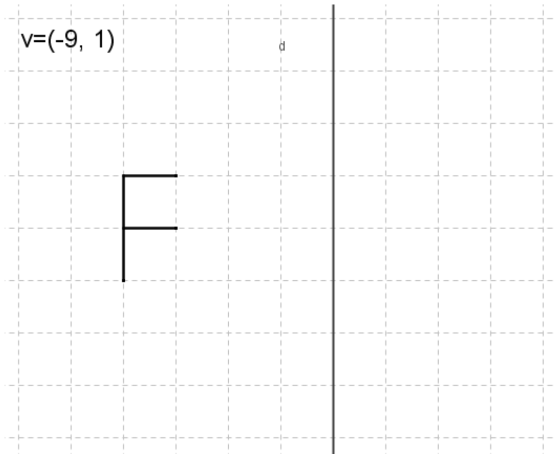
Şekil 2a</td><td>a. Şekil 2a da yansıma ve öteleme ile elde ettiğiniz sonucu <u>tek</u> bir geometrik dönüşümle açıklayınız. Bu geometrik dönüşümü şekil üzerinde farklı renkte bir kalemle çizerek gösteriniz.</td></tr>
<tr><td>

Şekil 2b</td><td>b. Şekil 2b de yansıma ve öteleme ile elde ettiğiniz sonucu <u>tek</u> bir geometrik dönüşümle açıklayınız. Bu geometrik dönüşümü şekil üzerinde farklı renkte bir kalemle çizerek gösteriniz.</td></tr>
</table>

<u>Sonuç:</u>

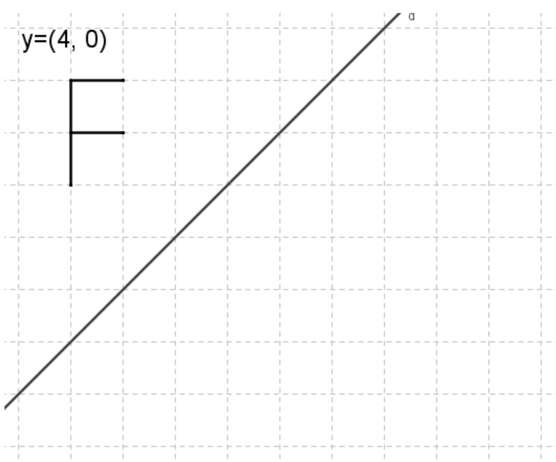
3. Aşağıda verilen şekilleri grup arkadaşınızla birlikte inceleyiniz. Her bir şıkta F harfine hangi geometrik dönüşüm veya dönüşümlerin uygulanmış olabileceğini belirleyiniz. Belirlediğiniz dönüşümlerin nasıl gerçekleştiğini açıklayınız	
a. 	b. 

4. Aşağıda her biri bir F harfi, bir doğru ve bir vektör içeren iki şekil verilmiştir. Bir şekildeki F harfini önce verilen doğruya göre yansıtınız daha sonra elde ettiğiniz görüntüyü verilen vektör ile öteleyiniz.



Şekil 4a

a. Şekil 4a da yansıma ve öteleme ile elde ettiğiniz sonucu tek bir geometrik dönüşümle açıklayınız. Bu geometrik dönüşümü şekil üzerinde farklı renkte bir kalemle çizerek gösteriniz.

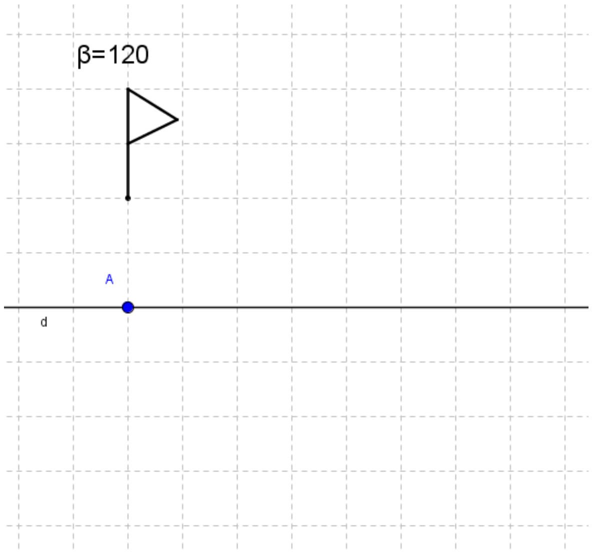


Şekil 4b

b. Şekil 4b de yansıma ve öteleme ile elde ettiğiniz sonucu tek bir geometrik dönüşümle açıklayınız. Bu geometrik dönüşümü şekil üzerinde farklı renkte bir kalemle çizerek gösteriniz.

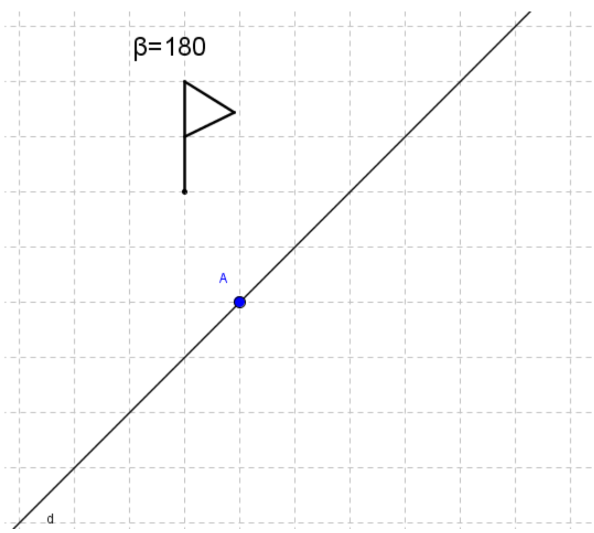
Sonuç:

5. Aşağıda her biri bir bayrak, bir doğru, bir nokta ve bir açı içeren şekiller verilmiştir. Bir şekildeki bayrağı önce verilen doğruya göre yansıtınız daha sonra elde ettiğiniz görüntüyü verilen nokta etrafında verilen açıda döndürünüz.



Şekil 5a

a. Şekil 5a da yansıma ve dönme ile elde ettiğiniz sonucu <u>tek</u> bir geometrik dönüşümle açıklayınız. Bu geometrik dönüşümü şekil üzerinde farklı renkte bir kalemle çizerek gösteriniz.

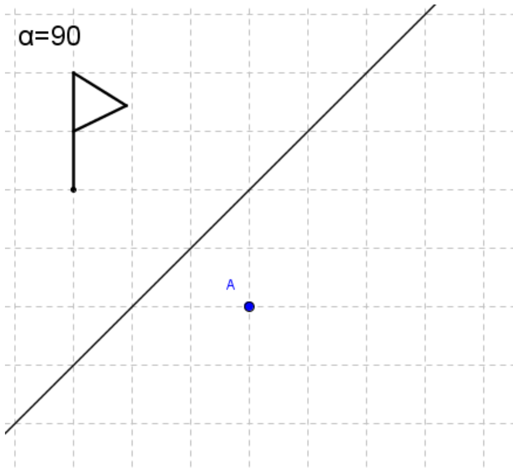


Şekil 5b

b. Şekil 5b de yansıma ve dönme ile elde ettiğiniz sonucu <u>tek</u> bir geometrik dönüşümle açıklayınız. Bu geometrik dönüşümü şekil üzerinde farklı renkte bir kalemle çizerek gösteriniz.

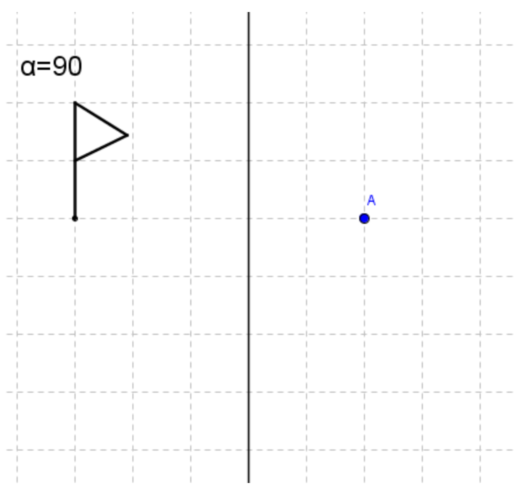
<u>Sonuç:</u>

6. Aşağıda her biri bir bayrak, bir doğru, bir nokta ve bir açı içeren iki şekil verilmiştir. Bir şekildeki bayrağı önce verilen doğruya göre yansıtınız daha sonra elde ettiğiniz görüntüyü verilen nokta etrafında verilen açıda döndürünüz.



Şekil 6a

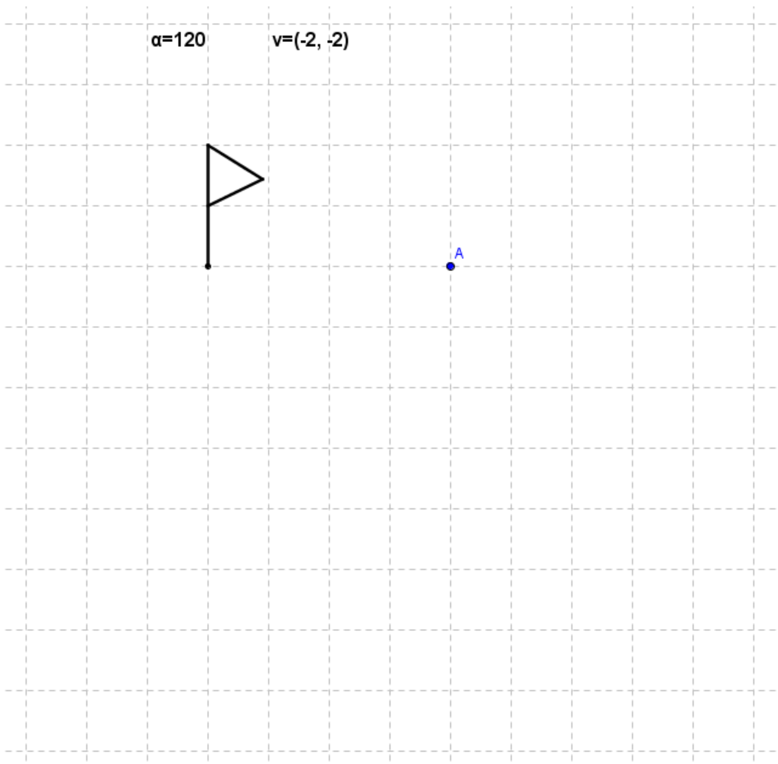
a. Şekil 6a da yansıma ve dönme ile elde ettiğiniz sonucu tek bir geometrik dönüşümle açıklayınız. Bu geometrik dönüşümü şekil üzerinde farklı renkte bir kalemle çizerek gösteriniz.



Şekil 6b

b. Şekil 6b de yansıma ve dönme ile elde ettiğiniz sonucu tek bir geometrik dönüşümle açıklayınız. Bu geometrik dönüşümü şekil üzerinde farklı renkte bir kalemle çizerek gösteriniz.

Sonuç:

7. Aşağıdaki şekilde bir bayrak, A noktası, α açısı ve v vektörü verilmiştir. Bayrağı önce A noktası etrafında α kadar döndürünüz. Daha sonra elde ettiğiniz görüntüyü v vektörü ile öteleyiniz. Elde ettiğiniz sonucu <u>tek</u> bir geometrik dönüşümle açıklayınız. Bu geometrik dönüşümü şekil üzerinde farklı renkte bir kalemle çizerek gösteriniz.


<u>Sonuç:</u>

5. ETKİNLİK

1. GeoGebra ekranındaki noktaları inceleyiniz.

a. Noktaları teker teker sürükleyiniz ve birbirleriyle ilgili olan noktaları belirleyip yazınız.

b. Genel özelliklerini dikkate alarak noktaları <u>iki</u> gruba ayırınız ve bu grupları isimlendiriniz. Grup isimlerini gerekçesiyle birlikte açıklayınız.

2. İncelediğiniz noktaların <u>birbirlerine göre</u> yönleri ve hızlarını inceleyerek tabloyu doldurunuz.

İlgili Noktalar	**Noktaların Birbirlerine göre Yönleri ile ilgili Gözlemler**	**Noktaların Birbirlerine göre Hızları ile ilgili Gözlemler**

3. GeoGebra sayfasında verilen noktalar arasındaki ilişkileri inceleyiniz. Her bir ilişkinin fonksiyon olup olmadığına karar veriniz ve nedenlerini yazınız.

İlişki	**Fonksiyon/ Fonksiyon Değil**	**Nedenleri**
A → A'		
B→ B'		
C→C'		
D→D'		
E→E'		
F→F'		
G→G'		
H→H'		

4. Birinci soruda incelediğiniz noktaların koordinatları arasındaki ilişkiyi belirleyiniz. Noktaların koordinatları ile noktaların birbirlerine göre hızlarını ve yönlerini ilişkilendiriniz.

Noktalar	**Koordinatlar**	**Hız ve Yönün Koordinatlar ile İlişkisi**
A $\rightarrow$ Z		
S $\rightarrow$ R		
P $\rightarrow$ U		
P $\rightarrow$ F		
E $\rightarrow$ C		
O $\rightarrow$ D		
T $\rightarrow$ V		
K $\rightarrow$ L		

Etkinlik 6 Tartışma Soruları

Bu etkinlikte öğretmenin soruları ve açıklamaları doğrultusunda sınıf olarak tartışma yapılır. Öğrencilere dağıtılacak bir doküman bulunmamaktadır. Tartışma soruları ile ilgili öğrencilerin çalışma kağıtları toplanır.

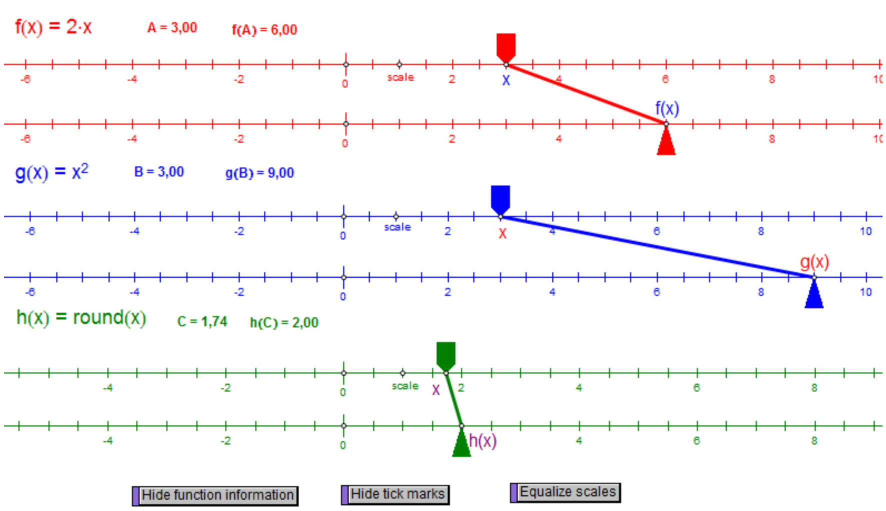
Fonksiyon Belirten İlişkilerde Tanım ve Değer Kümelerinin İncelenmesi

5. hafta_noktaların ilişkisi_3. Soru_2 dosyasını aç. C ve C' nü inceleyelim, fonksiyon belirtir mi? Geçen hafta buna fonksiyon değildir dedik. Çünkü Bazı bölgelerde C' vardı bazı bölgelerde ise C' yok oluyordu. Bu durum fonksiyon olma durumunu nasıl etkiler?

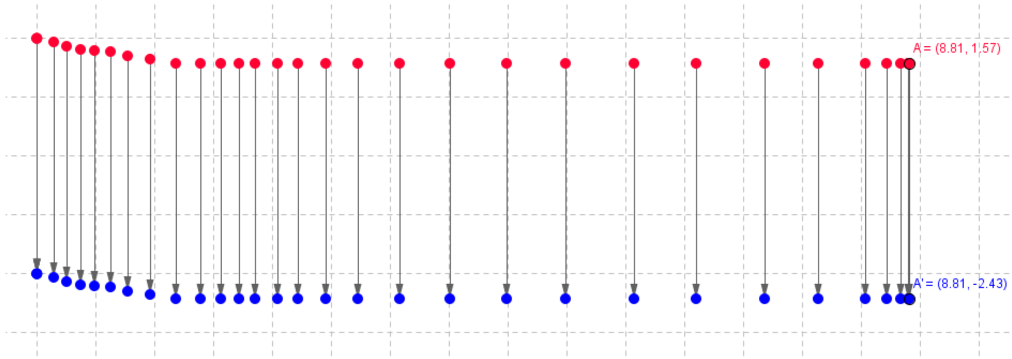
Yeniden *5. hafta_noktaların ilişkisi_3. Soru_1* dosyasına dönelim. Peki, fonksiyon ilişkisi belirtir dediğimiz A→A' ilişkisi nin **tanım** kümesi nedir? Tanım kümesinin elemanları nelerden oluşur? Bu soruları düşünmelerini isteyin, cevaplama şimdi değil. Düşünmeleri için süre ver.

Bunu daha iyi anlayabilmek için R→R bi örnek verelim. Şimdiye kadar aşina olduğumuz fonksiyonları düşünelim, örneğin f(x)=2x, bu fonksiyonun tanım kümesi nedir? Değer kümesi nedir? Tanım ve değer kümesinin elemanları nelerden oluşur?

R→R dediğimizde neyi kastediyoruz daha iyi anlaşılması için bu fonksiyonun grafiğini çizelim. Ayrıca bu fonksiyonun grafiğini **Geometr's Sketchpad** de **Dynagraph** olarak gösterelim. Sayı doğrusunun Reel sayılar kümesini ifade ettiğine değinerek bu grafikleri anlamlandırmaya çalışalım. Özetle, x de x' de reel sayı olduğundan R→R



Peki, A→A' yani (x, y)→ (x', y') olursa tanım ve değer kümeleri ne olur? A noktasının yani (x, y) nin alabileceği değerler nelerdir? A noktasını ekranda sürüklemelerini isteyelim, düşünmeleri için biraz süre verelim. Daha sonra akıllı tahtada A, A’ ve vektörün **izlerini açalım**, hızlı bir şekilde ekranda A yı sürükleyelim. Oluşan görüntüyü incelesinler, düşüncelerini soralım.



Tartışmanın sonunda öğrenciler A-A’ ilişkisinin $R^2 \rightarrow R^2$ tanımlı olduğunu fark etmeliler. Öğrencilere boş sayfa verelim ve <u>$R^2 \rightarrow R^2$ A-A’ ilişkisinin grafiğini çizmelerini isteyelim</u>. Süre ver.

Bir Bölgeden Başka Bir Bölgeye Tanımlı Fonksiyon

6. hafta_noktaların tanımlı oldukları bölgeler dosyasını açalım. Ekranda A(x, y), A’(x’, y’), B(x, y) ve B’(x’, y’) noktaları görünür durumda olacak. Gönüllü bir ö<u>ğrenciyi tahtaya çağıralım</u>, A noktasını sürüklesin ve sınıfla birlikte A’ noktasını gözlemlesinler.

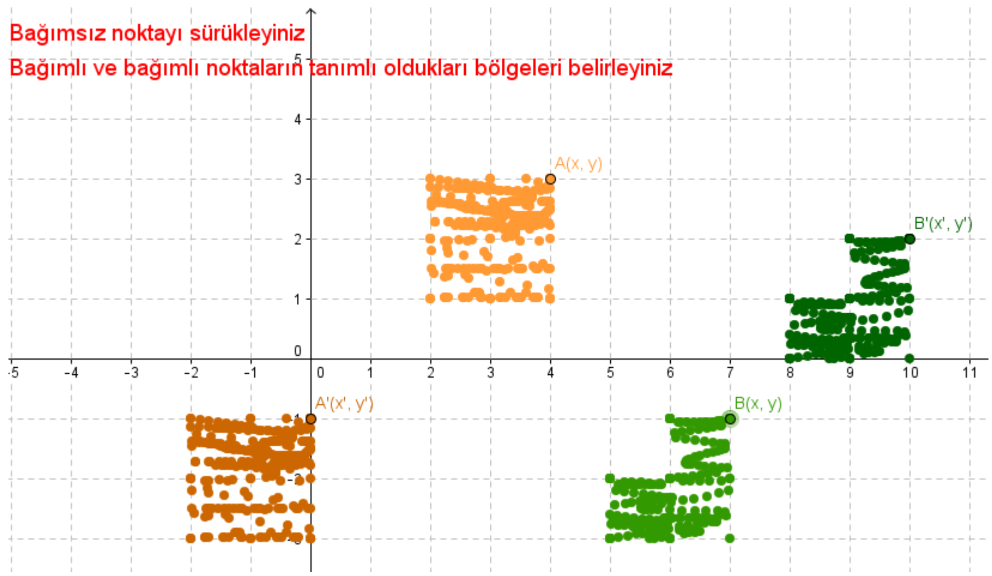
Sürükleme esnasında A ve A’ yok oluyor. Dolayısıyla $R^2 \rightarrow R^2$ tanımlı denilemiyor. Bu durumda <u>A→A’ ilişkisinin fonksiyon belirttiği noktaları nasıl sınırlandırırsınız?</u> Öğrenciler tanımlı olduğu bölgeye kendileri karar versin.

Daha sonra A ve A’ noktalarının **izlerini açalım** ve öğrenci sürüklemeye devam etsin. Ekranda iki kare oluşacak. Öğrenciler sadece bir **karesel bölgeden diğer karesel bölgeye** tanımlı olduğunu fark etmeliler. A→A’ sadece şu durumda bir fonksiyon belirtir: $[2,4]\times[1,3]\rightarrow[-2,0]\times[-1,-3]$. Aşağıdaki resimde sarı olan iki şekil.

A’nın oluşturduğu karenin her bir noktasından A’nün oluşturduğu karenin her bir noktasına bir **eşleşme** olduğunu fark etmeliler. Burası noktaların bire bir eşleşmesini gösterdiği için önemli.

B – B’ için gönüllü <u>başka bir öğrenciyi tahtaya çağıralım.</u> B noktasını sürüklesin ve sınıfla birlikte B’ noktasını gözlemlesinler. $R^2 \rightarrow R^2$ bir fonksiyon değil. Bu durumda B→B’ ilişkisinin fonksiyon belirttiği noktaları sınırlandırmaya çalışalım. Öğrenciler tanımlı olduğu bölgeye karar versin.

Daha sonra B ve B’ noktalarının **izlerini açalım** ve öğrenci sürüklemeye devam etsin. Ekranda iki şekil oluşacak. (aşağıdaki yeşil olan iki şekil). Açık yeşil şekilden koyu yeşil şekle birebir eşlemelerin olduğu görülmeli.



Öğrencilere ekrandaki görüntülerin ne ifade ettiğini soralım, geometrik dönüşümlerle ilişkilendirilmeli. Daha önceki haftalarda öğrendiğimiz geometrik dönüşümler birer fonksiyon olabilir mi? Öteleme? Yansıma? Dönme? Bu sorunun üzerinde düşünmeleri için zaman verelim.

Fonksiyondaki *bağımlı-bağımsız değişkenler*, fonksiyonun *kuralı*, *tanımlı oldukları bölgeler* açısından dönüşümleri incelesinler. Düşüncelerini bir kağıda yazsınlar.

Geometrik Dönüşümler Fonksiyon mudur?

5. hafta_noktaların ilişkisi_4. soru dosyasını açalım. Burada geçtiğimiz hafta incelenen ve kuralları belirlenen fonksiyonlar verilmiştir. A-A', B-B', T-T' ve K- K' (öteleme, yansıma, dönme, dönme) Burada noktalar arasındaki ilişkiler sırasıyla incelenecek.

Sırasıyla her bir ilişkiyi incelerken… Önce bağımsız nokta sürüklenerek ilişkinin kuralı belirlenecek, tahtaya yazılacak. Bu ilişkilerin sırasıyla $R^2 \rightarrow R^2$ olduğu görülmeli. Daha sonra geometrik dönüşümlerle ilişkilendirebilmek için izi aç seçeneğine tıklayalım. Her hangi bir harf ya da şekil çizelim, hangi dönüşüm olduğunu soralım. Dönüşümlerin $R^2 \rightarrow R^2$ bir fonksiyon olduğu sonucuna varılmalı. Tüm ilişkiler incelendikten sonra bu fonksiyonları özel yapan bir durum var mı? 1-1 ve örtenlik burada tartışılmalı.

Etkinlik 7 Tartışma Soruları

Bu etkinlikte öğretmenin soruları ve açıklamaları doğrultusunda sınıf olarak tartışma yapılır. Öğrencilere dağıtılacak bir doküman bulunmamaktadır. Tartışma soruları ile ilgili öğrencilerin çalışma kağıtları toplanır.

İki nokta arasındaki ilişki fonksiyon belirtir mi? Tanım ve değer kümelerinin incelenmesi

7.hafta_1, sonra *7.hafta_2*. en son da *7.hafta_3*. dosyalarını sırasıyla açalım. Gönüllü bir öğrenciyi tahtaya çağıralım. A(x, y) yi akıllı tahtada sürüklesin. A(x, y)→ A'(x', y') ilişkisi bir fonksiyon belirtir mi? Belirtirse tanım ve değer kümeleri nelerdir?

Aynı şekilde diğer iki noktayı da sürükleyerek değer ve tanım kümelerini belirleyelim. Ya da zaman kazanmak için her bir dosyadan birer ilişki gözlemlenebilir, örneğin A → A', D → D', G → G' gibi. Öğrencilere söz hakkı verelim, sebebiyle birlikte açıklasınlar.

Öteleme			Yansıma			Dönme		
A → A'	B → B'	C → C'	D → D'	E → E'	F → F'	G → G'	H → H'	J → J'
Her biri birer fonksiyon belirtir: Tanım Kümesi R^2 Değer Kümesi R^2								

Dönüşümler birer fonksiyon mu?

Daha sonra A ve A' nün Trace ini açalım. Farklı bir gönüllü öğrenci gelsin, akıllı tahtada A yı tekrar sürüklesin. A ve A' nün ekranda bıraktığı izleri inceleyelim. Trace açıkken öğrenciden belirli harfleri yazmasını isteyelim. P gibi simetrik olmayan harfler ya da nesneler. Burada öğrenciler oluşan görüntünün bir öteleme/yansıma/dönme dönüşümü olduğunu fark edeceklerdir.

Bu sayfadaki üç fonksiyon birer öteleme/yansıma/dönme dönüşümü olarak karşımıza çıktı. Peki, ilişkiye bir önceki bölümde fonksiyon belirtir demiştik, bu durumda dönüşümler birer fonksiyon mudur? Nedenleri ile açıklayınız. Diğer noktaların da aynı şekilde izlerini açalım ve inceleyelim. Parametreler bir sonraki bölümde ele alınacak.

Öteleme/Yansıma/Dönme fonksiyonlarının parametresi nedir?

Akıllı tahtada A, B ve C noktalarını aynı anda seçelim ve izlerle bir şekil oluşturalım. Peki, bu üç fonksiyon da aynı dönüşüm ise bunları birbirinden farklı yapan nedir? Burada parametrelerden bahsedilecek. Örneğin, öteleme dönüşümü bir fonksiyon ailesi ve parametrenin yani öteleme vektörünü farklılaştırarak çeşitli öteleme fonksiyonları üretebiliriz.

Diğer dönüşümler için de aynı süreç izlenecek. Yansıma dönüşümü bir fonksiyon ailesi ve parametrenin yani yansıma doğrusu farklılaştırarak çeşitli yansıma fonksiyonları üretebiliriz.

Matematiksel ifadeleri nasıldır?

Öteleme/yansıma/dönme fonksiyon aileleri için geçen hafta yazdığımız genel matematiksel ifadeyi parametreleriyle birlikte her bir fonksiyon için nasıl yazabiliriz? Geçen hafta F: $R^2 \rightarrow R^2$ F(x,y) → (x', y') yazmıştık. Parametreleriyle birlikte verilen bir matematiksel ifadede dönüşümün ne olduğunu belirlemek mümkün olmalı. F yerine her bir dönüşüm ailesine özgü harf verilebilir. Parametre nereye eklenmeli. Öteleme dışındakilerde (x', y') tam olarak kolayca belirlenemiyor, sebebi konuşulmalı.

Fonksiyon		Matematiksel İfade
Öteleme	A → A'	$T_{\vec{u}} : R^2 \rightarrow R^2 \ (x, y) \rightarrow (x+a, y+b)$
	B → B'	$T_{\vec{v}} : R^2 \rightarrow R^2 \ (x, y) \rightarrow (x+a, y+b)$
	C → C'	$T_{\vec{w}} : R^2 \rightarrow R^2 \ (x, y) \rightarrow (x+a, y+b)$
Yansıma	D → D'	$R_k : R^2 \rightarrow R^2 \ (x, y) \rightarrow (x', y')$
	E → E'	$R_l : R^2 \rightarrow R^2 \ (x, y) \rightarrow (x', y')$
	F → F'	$R_m : R^2 \rightarrow R^2 \ (x, y) \rightarrow (x', y')$
Dönme	G → G'	$R_{C,\alpha} : R^2 \rightarrow R^2 \ (x, y) \rightarrow (x', y')$
	H → H'	$R_{M,\beta} : R^2 \rightarrow R^2 \ (x, y) \rightarrow (x', y')$
	J → J'	$R_{O,\theta} : R^2 \rightarrow R^2 \ (x, y) \rightarrow (x', y')$

Öteleme/Yansıma/Dönme fonksiyon ailesinde sabit kalan noktalar

Bir öteleme/yansıma/dönme dönüşümünde kendisi ile eşleşen noktalar yani sabit kalan nokta var mıdır? Varsa hangi noktalardır?

Fonksiyon ailesi	Sabit kalan noktalar
Öteleme	Sabit kalan nokta yoktur, v=(0,0) ötelemesi dışında
Yansıma	Yansıma doğrusu üzerindeki noktalar sabit kalır
Dönme	Açı ne olursa olsun dönmenin merkezi olan nokta kendisi ile eşlenir.

Öteleme/Yansıma/Dönme fonksiyon ailesinde birim fonksiyon

Daha önceden aşına olduğumuz bir fonksiyonda bağımsız değişkenin yine kendisi ile eşleşmesine ne diyorduk? f(x)=x e ne diyorduk? Birim fonksiyon. Peki, tüm noktaları kendileri ile eşleştiren bir öteleme/yansıma/dönme fonksiyonu mümkün mü?

Fonksiyon ailesi	**Birim fonksiyon**
Öteleme	Öteleme vektörü v=(0,0) olması durumunda
Yansıma	Hiçbir durumda mümkün değil.
Dönme	Merkezi ne olursa olsun dönme açısı 0 olduğu durumda

Ötelemelerin/Yansımaların/Dönmelerin tersi ve birebir ve örtenlik

Ekrandaki üç fonksiyonu ayrı ayrı inceleyin, bu fonksiyonların tersi var mıdır? Bir fonksiyonun tersinin olması ne anlama gelir? Bu fonksiyonlar bire bir ve örten mi?

Fonksiyon ailesi	**Fonksiyon ailesinin tersi**
Öteleme	A dan B ye bir vektörün tersi B den A ya bir vektör olarak alınırsa
Yansıma	Tersi kendisine eşittir
Dönme	Dönme açısı eksi ile çarpılır

Bileşkelerin matematiksel ifadesi

Daha önceki haftalarda dönüşümlerin bileşkelerini aldık, dönüşümlerin bileşkesini matematiksel olarak nasıl ifade ederiz. Örneğin (x, önce l doğrusuna göre yansıma daha sonra elde edilen görüntünün v vektörü ile ötelenmesi bileşke fonksiyon olarak nasıl ifade edilir?

$$T_{\vec{v}}(R_l(x, y)) \rightarrow (x', y')$$

İki ötelemenin bileşkesinin yine bir ötelemeye eşit olduğunu hatırlayalım. (x, y) nin önce $\vec{u}$ vektörü ile ötelenmesi daha sonra oluşan görüntünün $\vec{v}$ vektörü ile ötelenmesinin bileşkesi $T_v(T_u(x, y)) = T_{u+v}(x,y)$

APPENDIX D

COURSE SCHEDULE

DATE	TOPIC
Feb, 19, 2015	Participants were informed about the study
Feb, 26, 2015	TGQ was administered as pre-test
Mar, 2 – Mar, 4, 2015	Follow-up interviews
Mar, 5, 2015	Activity 1
Mar, 12, 2015	Activity 2
Mar, 19, 2015	Activity 3
Mar, 26, 2015	Activity 4
Apr, 2, 2015	Activity 5
Apr, 9, 2015	Activity 6
Apr, 16, 2015	Activity 7
Apr, 23, 2015	National Holiday
Apr, 30, 2015	TGQ was administered as post-test
May, 4 – May, 6, 2015	Follow-up interviews

APPENDIX E

CURRICULUM VITAE

PERSONAL INFORMATION

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EDUCATION

Degree	Institution	Year of Graduation
PhD	METU Secondary Science and Mathematics Education	2017
MS	METU Elementary Science and Mathematics Education	2012
BS	METU Elementary Mathematics Teacher Education	2009

WORK EXPERIENCE

Years	Place	Enrollment
2009- Present	Aksaray University	Research Assistant

PUBLICATIONS

Avcu, R. & **Avcu, S**. (2015). Turkish adaptation of Utley geometry attitude scale: A validity and reliability study. *Eurasian Journal of Educational Research, 58,* 1-24.

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