ON SOME CLASSES OF SEMI-DISCRETE DARBOUX INTEGRABLE EQUATIONS

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

ERGÜN BİLEN

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY
IN
MATHEMATICS

DECEMBER 2017

Approval of the thesis:

ON SOME CLASSES OF SEMI-DISCRETE DARBOUX INTEGRABLE EQUATIONS

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ABSTRACT

ON SOME CLASSES OF SEMI-DISCRETE DARBOUX INTEGRABLE EQUATIONS

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December 2017, 71 pages

In this thesis we consider Darboux integrable semi-discrete hyperbolic equations of the form

$$t_{1x} = f(t, t_1, t_x), \qquad \frac{\partial f}{\partial t_x} \neq 0.$$

We use the notion of characteristic Lie ring for a classification problem based on dimensions of characteristic x- and n-rings.
Let $A = (a_{ij})_{N\times N}$ be a $N \times N$ matrix. We also consider semi-discrete hyperbolic equations of exponential type

$$u^i_{1,x} - u^i_x = e^{\sum a^+_{ij} u^j_1 + \sum a^-_{ij} u^j} \quad i, j = 1, 2, \ldots, N.$$

We find the conditions on a_{ij}'s so that the above equation is Darboux integrable when $N = 2$.

Keywords: Darboux integrability, Characteristic Lie ring, Hyperbolic equations

ÖZ

YARI AYRIK DARBOUX İNTEGRALLENEBİLİR DENKLEMLERİN BAZI ALTSINIFLARI ÜZERİNE

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Aralık 2017 , 71 sayfa

Bu tezde,

$$t_{1x} = f(t, t_1, t_x), \qquad \frac{\partial f}{\partial t_x} \neq 0,$$

formuna sahip Darboux integrallenebilir yarı ayrık hiperbolik denklemleri ele aldık. Karakteristik Lie halkası kavramını kullanarak karakteristik x- ve n- halkalarının boyutuna bağlı sınıflandırma problemi üzerine çalıştık. Bunun yanında $A = (a_{ij})_{N\times N}$ $N \times N$ boyutuna sahip bir A matrisi için,

$$u^i_{1,x} - u^i_x = e^{\sum a^+_{ij} u^j_1 + \sum a^-_{ij} u^j} \quad i, j = 1, 2, \dots, N$$

formuna sahip yarı ayrık üssel hiperbolik denklemleri de ele alıyoruz. $N = 2$ durumu için yukarıdaki diferansiyel denklemi Darboux integrallenebilir yapacak a_{ij} değerlerini elde ediyoruz.

Anahtar Kelimeler: Darboux integrallenebilir, Karakteristik Lie Halkasi, Hiperbolik Denklemler

To my family

ACKNOWLEDGMENTS

My supervisor Assoc. Prof. Dr. Kostyantyn Zheltukhin helped me to a great extent, encouraged me to the very end and made this thesis possible. I take great pleasure in expressing my sincere gratitude to Kostyantyn Zheltukhin.

My deepest gratitude further goes to my family for being with me in any situation, their encouragement, endless love and trust.

I have been receiving a scholarship about for five years from "Yurt İçi Doktora Burs Programı" of TÜBİTAK. I would like to thank to TÜBİTAK.

I would like to thank all my close friends whose names only a special chapter could encompass.

Finally, with deepest feelings I would like to thank my wife Duygu, without whom this thesis could not exist.

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CHAPTER 1

INTRODUCTION

Throughout the history mathematicians have been interested in methods to solve partial differential equations. Only a few class of nonlinear equations have known methods of solutions. This reveals the following questions, is there any criteria to determine if an equation is solvable and how do we solve this equations? To answer this question we can study properties of equations solved by a given method and try to find all equations satisfying particular properties. Thus we call an equation integrable if it satisfies some specially chosen properties. Since it is convenient to use different properties to describe different classes of equations there is no universal definition of integrability in the literature. For more about the integrable equations, see [1], [8], [9] and [13].

The problem of finding all equations of certain type, is known as a "classification problem", such classification problems are very important in the area of integrable systems because we want to know all equations solved by a particular method. Many mathematicians become interested in classification problem since discovery of the method of inverse scattering transform by Gardner, Green, Kruskal and Miura in 1967 (see [22]). They used inverse scattering transform to solve Koteveg-de Vries equation. Later it was shown that it was possible to use this method for other nonlinear equations including nonlinear Shrodinger equation (see [40]) and sine-Gordon equations (see [27]).

Many approaches were introduced attempting to classify equations solved by inverse scattering transform. In 1977 Ablowitz and Segur established a relation between integrable partial differential equations which are solved by using inverse scattering transform and the equations that satisfy Painleve test (see [2] and also [6], [39], [29], [21] and [30]). In 1979 Mikhailov, Sokolov and Shabat introduced the notion of generalized symmetry which is very useful to classify evolution equations

$$u_t = f(x, u, u_x, \ldots, u_{nx}). \tag{1.1}$$

It was observed that equations solved by inverse scattering transform have infinite hierarchy of generalized symmetries. For many classes of the evolution equations complete classifications were obtained by using symmetry approach (see [11], [12], [24], [33], [34], [37], [38] , [49] and [50]).

Unfortunately symmetry approach does not work well for hyperbolic equations

$$u_{xy} = f(x, y, u, u_x, u_y). \tag{1.2}$$

In 1915, Darboux suggested definition of integrable hyperbolic equation (1.2) based on the notion of x- and y-integrals (see [7]).
The functions $G = G(x, y, u, u_y, u_{yy}, \ldots, u_{ny})$ and $F = F(x, y, u, u_x, u_{xx}, \ldots, u_{mx})$ are called x-integral and y-integrals of equation (1.2) if $D_xG = 0$ and $D_yF = 0$ for all solutions of equation (1.2), respectively. We call equation (1.2) is Darboux integrable if it admits a nontrivial $x-$ and y-integrals. Indeed, if such integrals exist for an equation (1.2), then the equation can be transformed to a couple of ordinary differential equations

$$F(x, y, u, u_x, u_{xx}, \ldots, u_{mx}) = \alpha(x),$$
$$G(x, y, u, u_y, u_{yy}, \ldots, u_{ny}) = \beta(y).$$

The above ordinary differential equations in principle can be solved for u thus providing solution of equation (1.2) (see [4],[5] and [10]).

In 1981, Shabat and Yamilov suggested a method for determining Darboux integrability of hyperbolic equations (1.2). They introduced the notion of characteristic Lie ring for hyperbolic equations (see [23] and [32]). Using this approach, they classify equations of the form

$$u^i_{xy} = e^{\sum a_{ij}u^j} \quad i,j = 1,2,\ldots,N, \tag{1.3}$$

(see [32]). It was proved that the system (1.3) is Darboux integrable if and only if the matrix $A = (a_{ij})_{N\times N}$ is a Cartan matrix of a semi-simple Lie algebra. This method was also successfully apllied to classify other classes of hyperbolic equations (see [20], [25], [35], [36], [51] and [52]).

In 1999, Adler and Startsev defined Darboux integrability for discrete hyperbolic equations (see [3]) and in 2005, Habibullin extended definition of characteristic Lie ring to discrete hyperbolic equations (see [15]). Later in 2007, Habibullin and Pekcan introduced the the similar definitions for semi-discrete hyperbolic equations

$$t_{1x} = f(t,t_1,t_x), \qquad \frac{\partial f}{\partial t_x} \neq 0, \tag{1.4}$$

where unknown $t = t(n,x)$ is a function of two independent variables; a discrete variable n and a continuous variable x (see [17]).
To define Darboux integrability for equation (1.4), which is the main subject of our study, we introduce the shift operator D, setting $D^j t(n,x) = t(n+j,x) = t_j(n,x)$ and derivative operator D_x, setting $D_x^j t(n,x) = \frac{\partial^j}{\partial x^j} t(n,x) = t_{[j]}(n,x)$.
The functions $F = F(x,n,t,t_1,t_{-1},t_2,t_{-2},\ldots)$ and $I = I(x,n,t,t_x,t_{xx},\ldots)$ are x-integral and n-integrals of the chain (1.4) if $D_xF = 0$ and $DI = I$ for all solutions of (1.4). For more details on the Darboux integrability of equations (1.4), see [14], [16], [18] and [45].

As we mentioned above there are many classification results for continuous hyperbolic equations. But there is no comparable result for discrete and semi-discrete equations. A restricted classification was done for equations of the form $t_{1,x} = t_x + d(t,t_1)$ (see [46]). Some results were obtained toward classification of semi-discrete hyperbolic equation with $x-$ and n-ring of small dimension (see [17], [28], [43] and [44]). For instance it was proven that the only chain with 3 dimensional x-ring and 2 dimensional n-ring is $t_{1x} = t_x + t_1 - t$ (see [28]).

In 2011, Habibullin, Zheltukhin and Yangubaeva considered discretized form of the equations (1.3) which are called as semi-discrete hyperbolic equations of exponential type (see [41]). They proposed following discretization that supposedly preserves integrability

$$u_{1,x}^{i} - u_{x}^{i} = e^{\sum a_{ij}^{+} u_{1}^{j} + \sum a_{ij}^{-} u^{j}} \quad i, j = 1, 2, \ldots, N, \tag{1.5}$$

where the matrix $A = (a_{ij})_{N\times N}$ is decomposed into a sum of two triangular matrices $A = A_{+} + A_{-}$ with $A_{+} = \{a_{ij}^{+}\}$ being upper triangular and $A_{-} = \{a_{ij}^{-}\}$ being lower triangular matrices such that all diagonal entries of A_{+} are equal unity. It was proven for $N = 2$ that if A is a Cartan matrix, then the system (1.5) is Darboux integrable. Also it is hypothesized that the system (1.5) is Darboux integrable if and only if A is the Cartan matrix of a semi-simple Lie algebra.

We note that discretization hyperbolic equations have many applications in physics including Toda field equations on discrete space-time, Laplace sequence in discrete geometry, Q-system, Stokes phenomena in 1D Schrodinger problem and so on. For more on this subject, see a rewiev paper [26].

This thesis is organized as follows. In Chapter 2, we give the necessary definitions for Darboux integrability and characteristic Lie rings. In Chapter 3, we prove that when $N = 2$ and if the system (1.5) is Darboux integrable then A must be a Cartan matrix (see also [42]). Finally in Chapter 4, we make a classification of equations (1.4) which has four dimensional characteristic x-ring and two dimensional characteristic n-ring (see also[48]).

CHAPTER 2

DARBOUX INTEGRABILITY AND CHARACTERISTIC LIE RINGS

2.1 Darboux Integrability and Characteristic Lie Rings for Continuous Case

The partial differential equations of the form

$$u_{xy} = f(x, y, u, u_x, u_y). \tag{2.1}$$

belong to the class of hyperbolic type differential equations. We introduce notations $u_{ix} = \dfrac{\partial^i}{\partial x^i}$ and $u_{iy} = \dfrac{\partial^i}{\partial y^i}$. We define Darboux integrability of equation (2.1) in terms of x- and y-integrals.

Definition 1 *[7] Let $n, m < \infty$.*
• *A function $G = G(x, y, u, u_y, u_{yy}, \ldots, u_{ny})$ depending for x, y, u and derivatives of u with respect to y is x-integral of* (2.1) *if $D_xG = 0$ on all solutions of* (2.1).
• *A function $F = F(x, y, u, u_x, u_{xx}, \ldots, u_{mx})$ depending for x, y, u and derivatives of u with respect to x is y-integral of* (2.1) *if $D_yF = 0$ on all solutions of* (2.1).
• *We call* (2.1) *Darboux integrable if it admits a nontrivial x-integral and a nontrivial y-integral.*

Here D_x and D_y are operators of differentiation with respect to x and y. In above equalities the variables $u, u_x, u_y, \ldots$ are assumed to be independent. An x-integral cannot depend on $u_x, u_{xx}, \ldots$. Indeed, suppose G depends on x-derivatives and u_{kx} is the highest derivative that is $G = G(x, y, u, u_y, u_{yy}, \ldots, u_{ny}, u_x, \ldots, u_{kx})$. Then,

$$D_xG = G_x + G_y + G_u u_x + \cdots + G_{u_x} u_{xx} + \cdots + G_{u_{kx}} u_{(k+1)x} = 0.$$

The variables $u, u_x, \dots, u_{(k+1)x}$ are independent. Note that above expression does not contain the variable $u_{(k+1)x}$ except the last term. Hence $G_{u_{kx}}$ should be zero, that is G does not depend on u_{kx}.

From the above definitons it follows that if (2.1) is Darboux integrable, then it can be transformed to a couple of ordinary differential equations

$$F(x, y, u, u_x, u_{xx}, \dots, u_{mx}) = \alpha(x),$$
$$G(x, y, u, u_y, u_{yy}, \dots, u_{ny}) = \beta(y).$$

Note that $D_x G$ depends on u_x.

Example 2 *[19] Liouville equation $u_{xy} = e^u$ is Darboux integrable and the corresponding x-integral and y-integral are given as,*

$$G = u_{yy} - \frac{1}{2}u_y^2, \qquad F = u_{xx} - \frac{1}{2}u_x^2.$$

Example 3 *[19] The equation $u_{xy} = e^u\sqrt{u_y^2 - 4}$ is Darboux integrable and the corresponding x-integral and y-integral are given as,*

$$G = \frac{u_{yy} - u_y^2 + 4}{\sqrt{u_y^2 - 4}}, \qquad F = u_{xx} - \frac{1}{2}u_x^2 - \frac{1}{2}e^{2u}.$$

It is convenient to define Darboux integrability in terms of characteristic ring. Using chain rule and the equation (2.1) we can write,

$$D_x G = (\frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + f\frac{\partial}{\partial u_y} + D_y(f)\frac{\partial}{\partial u_{yy}} + \dots)G = 0$$

We define a vector field,

$$X_1 = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + f\frac{\partial}{\partial u_y} + D_y(f)\frac{\partial}{\partial u_{yy}} + \dots,$$

so that $X_1 G = 0$. We also have

$$\frac{\partial}{\partial u_x}(G) = 0.$$

since G cannot depend on x-derivatives of u. We define $X_2 = \frac{\partial}{\partial u_x}$. Then we have two conditions $X_1 G = 0$ and $X_2 G = 0$. Note that G will be in the kernel of commutators of the vector fields X_1 and X_2. Therefore, any vector field in the Lie ring generated by X_1 and X_2 annulates G.

Definition 4 *[32] A Lie ring generated by the vector fields X_1 and X_2 is called characteristic Lie ring L_x of the equation* (2.1) *in the direction of x. Characteristic Lie ring of the equation* (2.1) *in the direction of y denoted by L_y is defined similarly.*

Theorem 5 *[23] The equation* (2.1) *admits x- and y-integrals if and only if characteristic x- and y-rings are finite dimensional, respectively.*

Example 6 *[13] Liouville equation, $u_{xy} = e^u$ has 3 dimensional characteristic Lie ring L_x with the operators X_0, X_1 and X_2 with a multiplication table,*

L_x	X_1	X_2	X_3
X_1	*0*	$-X_3$	$u_x X_3 - X_1$
X_2	X_3	*0*	*0*
X_3	$X_1 - u_x X_3$	*0*	*0*

where $X_1 = D_x$, $X_2 = \dfrac{\partial}{\partial u_x}$ and $X_3 = [X_2, X_1] = \dfrac{\partial}{\partial u}$.

The notion of Darboux integrability is easily extended to the systems of hyperbolic type,

$$u^i_{xy} = f^i(x, y, u, u_x, u_y), \quad i = 1, 2, \dots, n. \tag{2.2}$$

We introduce notations $u^i_{jx} = \dfrac{\partial^j u^i}{\partial x^j}$ and $u^i_{jy} = \dfrac{\partial^j u^i}{\partial y^j}$. Similar to Definition 1, definition of Darboux integrability for such systems is given as follows.

Definition 7 *Let $r, s < \infty$*

- *A function $G = G(x, y, u^1, \dots, u^n, u^1_y, \dots, u^n_y, \dots, u^1_{ry}, \dots, u^n_{ry})$ depending on derivatives of u^i with respect to y is x-integral of* (2.2) *if $D_x G = 0$ for all solutions of* (2.2).
- *A function $F = F(x, y, u^1, \dots, u^n, u^1_x, \dots, u^n_x, \dots, u^1_{sx}, \dots, u^n_{sx})$ depending on derivatives of u^i with respect to x is y-integral of* (2.2) *if $D_y F = 0$ for all solutions of* (2.2).
- *We call the system* (2.2) *is Darboux integrable if it admits n functionally independent nontrivial x-integrals and n functionally independent nontrivial y-integrals.*

From above definitions it follows that if the system (2.2) is integrable then it can be transformed to a system of ordinary differential equations,

$$F^i(x, y, u^1, \ldots, u^n, u^1_x, \ldots, u^n_x, \ldots, u^1_{sx}, \ldots, u^n_{sx}) = \alpha^i(x), \quad i = 1, 2, \ldots, n \quad (2.3)$$

$$G^j(x, y, u^1, \ldots, u^n, u^1_y, \ldots, u^n_y, \ldots, u^1_{ry}, \ldots, u^n_{ry}) = \beta^j(y), \quad j = 1, 2, \ldots, n \quad (2.4)$$

where α_i, β_j are arbitrary functions.

Example 8 *[41] The following system,*

$$u_{xy} = e^{2u-v}$$
$$v_{xy} = e^{-u+2v}$$

is Darboux integrable with x-integrals

$$G_1 = u_{yy} + v_{yy} - u_y^2 + u_y v_{xy} - v_y^2$$
$$G_2 = v_{yyy} + u_y(v_{yy} - 2u_{yy}) + u_y^2 v_y - u_y v_y^2$$

and y-integrals

$$F_1 = u_{xx} + v_{xx} - u_x^2 + u_x v_x - v_x^2$$
$$F_2 = v_{xxx} + u_x(v_{xx} - 2u_{xx}) + u_x^2 v_x - u_x v_x^2.$$

The characteristic rings for a system are defined in the same way as for equations. By using chain rule we can write

$$\begin{aligned} D_x G \;=\; & \Big(\frac{\partial}{\partial x} + u^1_x \frac{\partial}{\partial u^1} + \cdots + u^n_x \frac{\partial}{\partial u^n} + f^1 \frac{\partial}{\partial u^1_y} + \ldots \\ & + f^n \frac{\partial}{\partial u^n_y} + D_y(f^1) \frac{\partial}{\partial u^1_{yy}} + \ldots \Big) G = 0. \end{aligned}$$

We define a vector field,

$$\begin{aligned} X_1 = \quad & \frac{\partial}{\partial x} + u^1_x \frac{\partial}{\partial u^1} + \cdots + u^n_x \frac{\partial}{\partial u^n} + f^1 \frac{\partial}{\partial u^1_y} + \ldots \\ & + f^n \frac{\partial}{\partial u^n_y} + D_y(f^1) \frac{\partial}{\partial u^1_{yy}} + \ldots \quad . \end{aligned}$$

An x-integral for the system (2.2) does not depend on x derivatives of u^i, $i = 1, \ldots, n$. Hence we define vector fields $X_2 = \frac{\partial}{\partial u^1_x}$, ..., $X_{n+1} = \frac{\partial}{\partial u^n_x}$. Then we have $(n+1)$ conditions, that is $X_i G = 0$, $i = 1, \ldots, (n+1)$. Note that G will be in the kernel of the commutators of the vector fields X_i where $i = 1, \ldots, (n+1)$. Therefore, any vector field in the Lie ring composed by X_i's for $i = 1, \ldots, (n+1)$ annulates G.

Definition 9 *The Lie ring defined above is called characteristic Lie ring L_x of the equation* (2.2) *in the direction of x. The characteristic Lie ring L_y of the equation* (2.2) *in the direction of y can be defined similarly.*

Theorem 10 *(see [32]) A system of equations* (2.2) *admits a non-trivial x-integral and a non-trivial y-integral if and only if its characteristic x-ring L_x and characteristic y-ring L_y are of finite dimension, respectively.*

It can be concluded that the system (2.2) is Darboux integrable if and only if characteristic rings L_x and L_y are finite dimensional.

2.2 Darboux Integrability and Characteristic Lie Rings for Semi-Discrete Case

We consider semi-discrete chains of the given form,

$$t_{1x} = f(t, t_1, t_x), \qquad \frac{\partial f}{\partial t_x} \neq 0 \tag{2.5}$$

where unknown $t = t(n, x)$ is a function of two independent variables; a discrete variable n and a continuous variable x [3]. Recall that we defined $t_k = D^k t$ and $t_{[k]} = D_x^k t$. The variables $(t_k)_{k=-\infty}^{\infty}$ and $(t_{[k]})_{k=1}^{\infty}$ are considered as independent variables. Darboux integrability for semi-discrete hyperbolic type equations defined as follows.

Definition 11 *[3]*

• *A function $F = F(x, n, t, t_1, t_{-1}, t_2, t_{-2}, \dots)$ depending on shifts of t is x-integral of the chain* (2.5) *if $D_x F = 0$ for all solutions of* (2.5).

• *A function $I = I(x, n, t, t_x, t_{xx}, \dots)$ depending on x-derivatives of t is n-integral of the chain* (2.5) *if $DI = I$ for all solutions of* (2.5) *where,*

$$DI = I(x, n+1, t_1, f, f_x, f_{xx}, \dots) = I(x, n, t, t_x, t_{xx}, \dots). \tag{2.6}$$

In other words, the function I is in the kernel of the operator $(D - \mathbb{I})I = 0$.

• *We call the chain* (2.5) *Darboux integrable if it admits a nontrivial x-integral $F(x, n, t, t_1, t_{-1}, t_2, t_{-2}, \dots)$ and a nontrivial n-integral $I(x, n, t, t_x, t_{xx}, \dots)$.*

Remark 12

1. *In the same way as in continuous case we can show that an x-integral cannot*

depend on x-derivatives of t and an n-integral cannot depend on shifts of t.
2. *Note that D_xI depend on t_x and DI depends on shifts of t.*

It follows that if the chain (2.5) admits a nontrivial x-integral and a nontrivial n-integral then by above definitions the following identities,

$$F(x,n,t,t_1,t_{-1},t_2,t_{-2},\dots) = \alpha(n)$$
$$I(x,n,t,t_x,t_{xx},\dots) = \beta(x)$$

must be satisfied for arbitrary $\alpha(n)$ and $\beta(x)$. This means that the chain (2.5) can be reduced to a pair of equations; one differential equation and one difference equation.

Example 13 *[3] As an example, we consider coupled Riccati equation $t_{1x} = t_x + t_1^2 - t^2$. It is Darboux integrable and the corresponding n-integral and x-integral are given as,*

$$I = t_x - t^2, \qquad F = \frac{(t-t_2)(t_1-t_3)}{(t-t_3)(t_1-t_2)}.$$

Example 14 *[3] The chain $t_{1x} = \dfrac{t_1+t}{t_x}$ is Darboux integrable and the corresponding n-integral and x-integral are given as,*

$$I = \frac{(t_{xx}-1)^2}{t_x^2}, \qquad F = \frac{(t_3-t_1)(t_2-t)}{(t_2+t_1)}.$$

By shifting (2.5) backward it is possible to write as

$$t_{-1x} = g(t,t_{-1},t_x) \tag{2.7}$$

for some appropriate function g. This can be done owing the condition $\frac{\partial f}{\partial t_x} \neq 0$ we supposed for (2.5).

The characteristic x-ring for the systems is defined in the same way as in continuous case. We consider vector fields $D_0 = D_x$ and $D_1 = \frac{\partial}{\partial t_x}$. Then we have two conditions $D_0F = 0$ and $D_1F = 0$. Note that F will be in the kernel of commutators of the vector fields D_0 and D_1. Therefore, any vector field in the Lie ring composed by D_0 and D_1 annulates F.

Definition 15 *Lie ring generated by the vector fields D_0 and D_1 is called characteristic Lie ring L_x of the equation* (2.5) *in the direction of x.*

Theorem 16 *(See [32]) Semi discrete chain* (2.5) *admits a non-trivial x-integral if and only if its characteristic x-ring L_x is of finite dimension.*

Example 17 *[47] For the chain,*

$$t_{1x} = t_x + \sqrt{e^{2t_1} + e^{t_1+t} + e^{2t}},$$

characteristic Lie ring L_x is 3 dimensional with the operators D_0, D_1 and D_2 with a multiplication table,

L_x	D_0	D_1	D_2
D_0	0	$-D_2$	$t_xD_2 - D_0$
D_1	D_2	0	0
D_2	$D_0 - t_xD_2$	0	0

where $D_0 = D_x$, $D_1 = \dfrac{\partial}{\partial t_x}$ and $D_2 = [D_1, D_0] = \displaystyle\sum_{k=-\infty}^{\infty} \frac{\partial}{\partial t_k}$.

The notion of Darboux integrability is easily extended to systems of semi-discrete type,

$$t^i_{1x} = f^i(t^1, \ldots, t^m, t^1_1, \ldots, t^m_1, t^1_x, \ldots, t^m_x), \qquad \frac{\partial f^i}{\partial t_x} \neq 0 \quad i = 1, \ldots, m \tag{2.8}$$

where unknowns $t^i = t^i(n, x)$ are functions of two independent variables; a discrete variable n and a continuous variable x. We define $D^j t^i(n, x) = t^i_j(n, x) = t^i(n+j, x)$ and $D^j_x t(n, x) = t^i_{[j]}(n, x) = \frac{\partial^j}{\partial x^j} t^i(n, x)$. The variables $(t^i_k)^{\infty}_{k=-\infty}$ and $(t^i_{[k]})^{\infty}_{k=1}$ are considered as independent variables.

Definition 18

• *A function $F = F(x, n, t^1, \ldots, t^m, t^1_1, \ldots, t^m_1, t^1_{-1}, \ldots, t^m_{-1}, t^1_2, \ldots)$ depending on shifts of t^i's is x-integral of the chain* (2.8) *if $D_xF = 0$ for all solutions of* (2.8)

• *A function $I = I(x, n, t^1, \ldots, t^m, t^1_x, \ldots, t^m_x, t^1_{xx}, \ldots)$ depending on x-derivatives of $t_i(n, x)$ is n-integral of the chain* (2.8) *if $DI = I$ for all solutions of* (2.8)*, where*

$$DI = I(x, n+1, t^1_1, \ldots, t^m_1, f^1, \ldots, f^m, f^1_x, \ldots, f^n_x, f^1_{xx} \ldots).$$

In other words, the function I is in the kernel of the difference operator $(D - \mathbb{I})I = 0$.

• *We call the chain* (2.8) *is Darboux integrable if it admits m functionally independent nontrivial x-integrals and m functionally independent nontrivial n-integrals.*

Note that if the chain (2.8) is Darboux integrable, then the following identities,

$$F^i(x,n,t^1,\dots,t^m,t_1^1,\dots,t_1^m,t_{-1}^1,\dots,t_{-1}^m,t_2^1,\dots) = \alpha^i(n) \quad i=1,...,m$$
$$I^j(x,n,t^1,\dots,t^m,t_x^1,\dots,t_x^m,t_{xx}^1,\dots) = \beta^j(x) \qquad j=1,...,m$$

should be satisfied for arbitrary functions $\alpha^i(n)$ and $\beta^j(x)$.

Example 19 *[41] For the system*

$$u_{1x} - u_x = e^{u+u_1-v_1}$$
$$v_{1x} - v_x = e^{-u+v+v_1}$$

x-integrals are

$$F_1 = e^{-v+v_1} + e^{-u+u_1+v_1-v_2} + e^{u_1-u_2},$$
$$F_2 = e^{-u+u_1} + e^{u_1-u_2-v_1+v_2} + e^{v_2-v_3},$$

and n-integrals are given as

$$I_1 = u_{xx} + v_{xx} - u_x^2 + u_x v_x - v_x^2,$$
$$I_2 = u_{3x} + u_x(v_{xx} - 2u_{xx}) + u_x^2 v_x - u_x v_x^2.$$

In the same way as in the continuous case, we define vector fields,

$$D_0 = D_x$$
$$D_i = \frac{\partial}{\partial t_x^i}, \quad i = 1,\dots,m.$$

Then any vector field in the Lie ring generated by D_i's annulates F.

Definition 20 *Lie ring generated by the vector fields D_i's $i = 0,1,\dots,m$ is called characteristic Lie ring L_x of the equation* (2.8) *in the direction of x.*

Example 21 *[41] For the equation,*

$$u_{1x} - u_x = e^{u+u_1-v_1}$$
$$v_{1x} - v_x = e^{-u+v+v_1}$$

characteristic Lie-ring L_x consist of the operators $\frac{\partial}{\partial x}$, Y_1, Y_2, A, B, P_1, P_2 and $T_1 = [P_1, P_2]$ where,

$$Y_1 = \frac{\partial}{\partial u_x}, \qquad Y_2 = \frac{\partial}{\partial v_x}, \qquad A = \sum_{j=-\infty}^{\infty} \frac{\partial}{\partial u_j}, \qquad B = \sum_{j=-\infty}^{\infty} \frac{\partial}{\partial v_j},$$

$$P_1 = \sum_{k=1}^{\infty}(\sum_{i=1}^{k} e^{u_{i-1}+u_i-v_i})\frac{\partial}{\partial u_k} - \sum_{k=1}^{\infty}(\sum_{i=1}^{k} e^{u_{-i}+u_{-i+1}-v_{-i+1}})\frac{\partial}{\partial u_{-k}},$$
$$P_2 = \sum_{k=1}^{\infty}(\sum_{i=1}^{k} e^{-u_{i-1}+v_{i-1}+v_i})\frac{\partial}{\partial v_k} - \sum_{k=1}^{\infty}(\sum_{i=1}^{k} e^{-u_{-i}+v_{-i}+v_{-i+1}})\frac{\partial}{\partial v_{-k}}.$$

One can also define the characteristic n-ring such that a nontrivial n-integral exists if and only if the characteristic n-ring is finite dimensional. Since we are not using characteristic n-ring in our work we will not give its definition here. The definition is rather technical and can be found in [17].

CHAPTER 3

ON DARBOUX INTEGRABLE SEMI-DISCRETE HYPERBOLIC SYSTEMS OF EXPONENTIAL TYPE

3.1 Semi-Discrete Hyperbolic Systems of Exponential Type

Let $A = (a_{ij})_{N\times N}$ be an $N \times N$ matrix and consider hyperbolic systems of exponential type,

$$r^i_{xy} = e^{\sum a_{ij} r^j} \quad i, j = 1, 2, \ldots, N. \tag{3.1}$$

The Darboux integrability of such systems is described by the following theorem.

Theorem 22 *(see [32]) The system* (3.1) *is Darboux integrable if and only if $A = (a_{ij})_{N\times N}$ is the Cartan matrix of a semi-simple Lie algebra.*

Note that a square matrix $A = (a_{ij})_{N\times N}$ is a Cartan matrix if it satifies the followings,(See [31])

1) $a_{ij} \in \{-3, -2, -1, 0, 2\}$,

2) $a_{ii} = 2$,

3) $a_{ij} \leq 0$ when $i \neq j$,

4) $a_{ij} = 0 \Longrightarrow a_{ji} = 0$,

5) There exists a diagonal matrix D such that DAD^{-1} gives a symmetric and positive definite quadratic form.

Note that Cartan matrices in two dimension has the form,

$$A = \begin{pmatrix} 2 & -1 \\ -c & 2 \end{pmatrix}, \quad \text{where} \quad c = 1, 2, 3. \tag{3.2}$$

We want to consider discretization of (3.1) which preserves integrability. Following [41], we decompose A into a sum of two triangular matrices $A = A_+ + A_-$ with $A_+ = \{a^+_{ij}\}$ being upper triangular and $A_- = \{a^-_{ij}\}$ being lower triangular matrices such that all diagonal entries of A_+ are equal unity. For example when $N = 2$, this decomposition is given by,

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad A_- = \begin{pmatrix} a_{11}-1 & 0 \\ a_{21} & a_{22}-1 \end{pmatrix}, \quad A_+ = \begin{pmatrix} 1 & a_{12} \\ 0 & 1. \end{pmatrix}$$

Now we consider discretized form of (3.1). Let $r^i(n,x)$, $i = 1, 2, \ldots, N$ depend on a continuous variable x and discrete variable n. We consider semi-discrete chains of the form (See [41])

$$r^i_{1,x} - r^i_x = e^{\sum a^+_{ij} r^j_1 + \sum a^-_{ij} r^j} \quad i, j = 1, 2, \ldots, N. \tag{3.3}$$

For example if we put $r^1 = u$, $r^2 = v$, then for Cartan matrix (3.1), the system (3.3) becomes

$$\begin{aligned} u_{1x} - u_x &= e^{u+u_1-v_1} \\ v_{1x} - v_x &= e^{-cu+v+v_1} \end{aligned} \qquad c = 1, 2, 3. \tag{3.4}$$

For Darboux integrability of system (3.3) we have the following conjecture.

Conjecture 23 *(see [41]) System* (3.3) *is Darboux integrable if and only if A is a Cartan matrix .*

In [41] it was proven that for 2×2 Cartan matrix we have Darboux integrable system. Also for the corresponding system (3.4), characteristic x-ring, x-integrals and n-integrals are fully described.

In this chapter, we classify all $A = (a_{ij})_{2\times 2}$ matrixes such that the system (3.3) is Darboux integrable. In the end, we will show (3.3) is Darboux integrable only if A is the Cartan matrix. That will complete the proof of Conjecture 23 for the case $N = 2$. Our proof is based on characteristic x-ring. Considering characteristic x-ring we also reproduce results of [41].

Theorem 24 *The system of hyperbolic equations of exponential type,*

$$\begin{aligned} u_{1x} - u_x &= e^{(a_{11}-1)u+u_1+a_{12}v_1} \\ v_{1x} - v_x &= e^{a_{21}u+(a_{22}-1)v+v_1} \end{aligned} \tag{3.5}$$

is Darboux integrable if and only if the matrix $A = (a_{ij})_{2\times 2}$ is the Cartan matrix.

The proof of the main theorem will be given later. In the next section we consider the characteristic x-ring to prove this theorem.

3.2 Characteristic x-ring

Consider an arbitrary matrix $A = (a_{ij})_{2x2}$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad a_{12}, a_{21} \neq 0$$

Setting $r^1 = u$, $r^2 = v$, the system (3.3) takes form,

$$\begin{aligned} u_{1x} - u_x &= e^{(a_{11}-1)u + u_1 + a_{12}v_1} \\ v_{1x} - v_x &= e^{a_{21}u + (a_{22}-1)v + v_1}. \end{aligned} \tag{3.6}$$

Conditions $a_{12}, a_{21} \neq 0$ are needed because if $a_{12} = 0$ or $a_{21} = 0$, then (3.6) is reduced to a set of ordinary differential equations. We assume (3.6) is Darboux integrable and prove that this is possible only if A is the Cartan matrix. The proof will be based on by constructing x-ring of the system. Let us construct x-ring of the system. Following Definition 15 the characteristic x-ring is generated by vector fields

$$\begin{aligned} Z &= \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + v_x\frac{\partial}{\partial v} + u_{1x}\frac{\partial}{\partial u_1} + v_{1x}\frac{\partial}{\partial v_1} + u_{(-1)x}\frac{\partial}{\partial u_{-1}} + v_{(-1)x}\frac{\partial}{\partial v_{-1}} + \dots \\ &= \frac{\partial}{\partial x} + \sum_{k=-\infty}^{\infty}\left(u_{kx}\frac{\partial}{\partial u_k} + v_{kx}\frac{\partial}{\partial v_k}\right) \end{aligned}$$

and

$$Y_1 = \frac{\partial}{\partial u_x}, \qquad Y_2 = \frac{\partial}{\partial v_x}.$$

For convenience we define,

$$\begin{aligned} M_i &= e^{(a_{11}-1)u_{i-1} + u_i + a_{12}v_i}, \qquad M_{-i} = e^{(a_{11}-1)u_{-i} + u_{-i+1} + a_{12}v_{-i+1}}, \\ N_i &= e^{(a_{22}-1)v_{i-1} + v_i + a_{21}u_{i-1}}, \quad N_{-i} = e^{(a_{22}-1)v_{-i} + v_{-i+1} + a_{21}u_{-i}}. \end{aligned}$$

Note that, M_i, M_{-i}, N_i and N_{-i} are never zero and we have

$$u_{kx} = u_x + \sum_{i=1}^{k} M_i, \quad u_{(-k)x} = u_x - \sum_{i=1}^{k} M_{-i}, \tag{3.7}$$

$$v_{kx} = v_x + \sum_{i=1}^{k} N_i, \quad v_{(-k)x} = v_x - \sum_{i=1}^{k} N_{-i}. \tag{3.8}$$

We prove the identities (3.7) and (3.8) by using induction. Firstly note that if we shift (3.6) backward we get,

$$\begin{aligned} u_{(-1)x} &= u_x - e^{(a_{11}-1)u_{-1}+u+a_{12}v} = u_x - M_{-1}, \\ v_{(-1)x} &= v_x - e^{a_{21}u_{-1}+(a_{22}-1)v_{-1}+v} = v_x - N_{-1}. \end{aligned} \tag{3.9}$$

We will prove the identities in (3.7). The other identities be can proved similarly. For $k = 1$, the identities in (3.7) follows from (3.6) and (3.9). Now we suppose (3.7) are true for $k = n$. That is

$$u_{nx} = u_x + \sum_{i=1}^{n} M_i, \quad u_{(-n)x} = u_x - \sum_{i=1}^{n} M_{-i}. \tag{3.10}$$

Then we find

$$u_{(n+1)x} = u_{1x} + \sum_{i=2}^{(n+1)} M_i = u_x + M_1 + \sum_{i=2}^{(n+1)} M_i = u_x + \sum_{i=1}^{(n+1)} M_i$$

and

$$u_{-(n+1),x} = u_{(-1)x} - \sum_{i=2}^{(n+1)} M_{-i} = u_x + M_{-1} + \sum_{i=2}^{(n+1)} M_{-i} = u_x - \sum_{i=1}^{(n+1)} M_{-i}.$$

Hence the identities (3.7) are true for $k = n + 1$, this completes the proof. $\square$

Consider the commutators of Z, Y_1 and Y_2. They have the following form,

$$A = [Y_1, Z] = \sum_{j=-\infty}^{\infty} \frac{\partial}{\partial u_j}, \quad B = [Y_2, Z] = \sum_{j=-\infty}^{\infty} \frac{\partial}{\partial v_j}.$$

Next we define $A^* = [A, Z]$ and $B^* = [B, Z]$. Then

$$\begin{aligned} A^* = a_{11} \sum_{k=1}^{\infty} \left(\sum_{i=1}^{k} M_i \right) \frac{\partial}{\partial u_k} - a_{11} \sum_{k=1}^{\infty} \left(\sum_{i=1}^{k} M_{-i} \right) \frac{\partial}{\partial u_{-k}} + \\ + a_{21} \sum_{k=1}^{\infty} \left(\sum_{i=1}^{k} N_i \right) \frac{\partial}{\partial v_k} - a_{21} \sum_{k=1}^{\infty} \left(\sum_{i=1}^{k} N_{-i} \right) \frac{\partial}{\partial v_{-k}} \end{aligned}$$

and

$$\begin{aligned} B^* = a_{12} \sum_{k=1}^{\infty} \left(\sum_{i=1}^{k} M_i \right) \frac{\partial}{\partial u_k} - a_{12} \sum_{k=1}^{\infty} \left(\sum_{i=1}^{k} M_{-i} \right) \frac{\partial}{\partial u_{-k}} + \\ + a_{22} \sum_{k=1}^{\infty} (N_i) \frac{\partial}{\partial v_k} - a_{22} \sum_{k=1}^{\infty} \left(\sum_{i=1}^{k} N_{-i} \right) \frac{\partial}{\partial v_{-k}}. \end{aligned}$$

For convenience instead of A^* and B^* we define the vector fields,

$$P_1 = \frac{1}{a_{22}a_{11} - a_{21}a_{12}}(a_{22}A^* - a_{21}B^*), \qquad P_2 = \frac{1}{a_{22}a_{11} - a_{21}a_{12}}(-a_{12}A^* + a_{11}B^*).$$

They have the form,

$$P_1 = \sum_{k=1}^{\infty}\Big(\sum_{i=1}^{k} M_i\Big)\frac{\partial}{\partial u_k} - \sum_{k=1}^{\infty}\Big(\sum_{i=1}^{k} M_{-i}\Big)\frac{\partial}{\partial u_{-k}},$$
$$P_2 = \sum_{k=1}^{\infty}\Big(\sum_{i=1}^{k} N_i\Big)\frac{\partial}{\partial v_k} - \sum_{k=1}^{\infty}\Big(\sum_{i=1}^{k} N_{-i}\Big)\frac{\partial}{\partial v_{-k}}.$$

Hence the vector fields,

$$\frac{\partial}{\partial x}, \quad Y_1, \quad Y_2, \quad A, \quad B, \quad P_1 \quad and \quad P_2$$

belong to the characteristic x-ring of the system (3.6). Here note that the vector field Z can be written as linear combination of the above vector fields. Moreover, we have the following commutator table for this vector fields.

$[.,.]$	$\frac{\partial}{\partial x}$	Y_1	Y_2	A	B	P_1	P_2
$\frac{\partial}{\partial x}$	0	0	0	0	0	0	0
Y_1	0	0	0	0	0	0	0
Y_2	0	0	0	0	0	0	0
A	0	0	0	0	0	$a_{11}P_1$	$a_{21}P_2$
B	0	0	0	0	0	$a_{12}P_1$	$a_{22}P_2$

We can conclude from the above table that except the commutators of P_1 and P_2 the other terms do not produce any new vector field. Thus the characteristic x-ring is finite dimensional if and only if the ring generated by P_1, P_2 is finite dimensional.

Let $\mathcal{A}$ be the ring generated by P_1, P_2. To describe the structure of $\mathcal{A}$ we define a sequence of linear spaces $\mathbb{X}_n$ as follows,

$$\mathbb{X}_0 = Lin\big\{P_1, P_2\big\},$$
$$\mathbb{X}_1 = Lin\big\{[P_1, P_2]\big\},$$
$$\mathbb{X}_2 = Lin\big\{[P_{t_{2_0}}, [P_{t_{2_1}}, P_{t_{2_2}}]]\big\}, \quad t_{2_0}, t_{2_1}, t_{2_2} \in \{1, 2\},$$
$$\vdots$$
$$\mathbb{X}_n = Lin\big\{[P_{t_{n_0}}, [P_{t_{n_1}}, [....[P_{t_{n_{n-1}}}, P_{t_{n_n}}]]]]\big\}, \quad t_{n_0}, t_{n_1}, ..., t_{n_n} \in \{1, 2\},$$

Lemma 25 *The intersection of the linear space $\mathbb{X}_i$ and $\mathbb{X}_j$ is empty that is $\mathbb{X}_i \bigcap \mathbb{X}_j = 0$ when $i \neq j$.*

Proof: Consider the vector fields $X_i \in \mathbb{X}_i$ for $i = 1, 2, \ldots$. Then the vector fields $X_1, X_2, \ldots, X_k$ are

$$
\begin{aligned}
X_1 &= \sum_{k=-\infty}^{\infty}\left(\sum_{i,j=-\infty}^{\infty}\sum^{\infty} A_{1ij}M_iN_j\right)\frac{\partial}{\partial u_k} + \sum_{k=-\infty}^{\infty}\left(\sum_{i,j=-\infty}^{\infty}\sum^{\infty} B_{1ij}M_iN_j\right)\frac{\partial}{\partial v_k},\\
X_2 &= \sum_{k=-\infty}^{\infty}\left(\sum_{i,j=-\infty}^{\infty}\sum^{\infty} A_{2ij}M_i^{r_2}N_j^{R_2}\right)\frac{\partial}{\partial u_k} + \sum_{k=-\infty}^{\infty}\left(\sum_{i,j=-\infty}^{\infty}\sum^{\infty} B_{2ij}M_i^{r_2}N_j^{R_2}\right)\frac{\partial}{\partial v_k},\\
&\vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots\\
X_k &= \sum_{k=-\infty}^{\infty}\left(\sum_{i,j=-\infty}^{\infty}\sum^{\infty} A_{kij}M_i^{r_k}N_j^{s_k}\right)\frac{\partial}{\partial u_k} + \sum_{k=-\infty}^{\infty}\left(\sum_{i,j=-\infty}^{\infty}\sum^{\infty} B_{kij}M_i^{r_k}N_j^{s_k}\right)\frac{\partial}{\partial v_k},
\end{aligned}
$$

where A_{kij}'s and B_{kij}'s are constants and $1 \leq r_k, s_k \leq k, \quad r_k + s_k = k + 1$ for $k = 1, 2, 3, \ldots$.

Suppose contrary, $\mathbb{X}_m \bigcap \mathbb{X}_n \neq 0$ for some $m \neq n$, then there exist a vector field $C \in \mathbb{X}_m \bigcap \mathbb{X}_m$ such that $C = c_m X_m = c_n X_n$ and $c_m, c_n \neq 0$ for some $X_m \in \mathbb{X}_m$ and $X_n \in \mathbb{X}_n$. Then the coefficients of $\dfrac{\partial}{\partial u_k}$ and $\dfrac{\partial}{\partial v_k}$ must be same for $c_m X_m$ and $c_n X_n$. We must have,

$$
\sum_{i,j=-\infty}^{\infty}\sum^{\infty} c_m A_{mij} M_i^{r_m} N_j^{s_m} = \sum_{i,j=-\infty}^{\infty}\sum^{\infty} c_n A_{nij} M_i^{r_n} N_j^{s_n},
$$

$$
\sum_{i,j=-\infty}^{\infty}\sum^{\infty} c_m B_{mij} M_i^{r_m} N_j^{s_m} = \sum_{i,j=-\infty}^{\infty}\sum^{\infty} c_n B_{nij} M_i^{r_n} N_j^{s_n},
$$

where

$$
\begin{aligned}
M_i^{r_m} N_j^{s_m} &= e^{r_m(u_i + a_{12}v_i + (a_{11}-1)u_{i-1}) + s_m(v_j + a_{21}u_{j-1} + (a_{22}-1)v_{j-1})},\\
M_i^{r_n} N_j^{s_n} &= e^{r_n(u_i + a_{12}v_i + (a_{11}-1)u_{i-1}) + s_n(v_j + a_{21}u_{j-1} + (a_{22}-1)v_{j-1})},
\end{aligned}
$$

are polynomials of exponential powers. Since both of $r_m = r_n$ and $s_m = s_n$ cannot hold at the same time, the polynomials of exponential powers are not same. Hence we must have $c_m = c_n = 0$. This means that the vector field $C = 0$ and this is a contradiction with our assumption. Hence the intersection of the linear space $\mathbb{X}_i$ and $\mathbb{X}_j$ is empty. $\square$

Lemma 25 shows that $\mathcal{A}$ is a direct sum of $\mathbb{X}_n$'s.

$$\mathcal{A} = \mathbb{X}_0 \oplus \mathbb{X}_1 \oplus \mathbb{X}_2 + \dots .$$

Lemma 26 *If* $\dim\mathcal{A} < \infty$ *then the linear space* $\mathbb{X}_n = 0$ *for some* $n < \infty$*.*

Proof: Suppose contrary, let $\dim\mathcal{A} = p < \infty$ for some $p \in \mathbb{N}$ but $\mathcal{A} = \sum_{n=1}^{\infty} \mathbb{X}_n$. By Lemma 25, for arbitrary $m \in \mathbb{Z}^+$, we can find vector fields $X_i \in \mathbb{X}_i$ for $i = 1, 2, \dots, m$. Hence we can find m linearly independent vector fields in $\mathcal{A}$. Then m must be less than $dim\mathcal{A}$. But since m is arbitrary this is not possible and that is a contradiction with our assumption. Therefore, we must have $\mathbb{X}_n = 0$ for some $n < \infty$. $\square$

To prove our claim we will need the condition $\mathbb{X}_n = 0$ for some n. In this case, all vector fields in $\mathbb{X}_n$ must be zero. To ensure this we must have a condition to prove a vector field is zero. Thus, the following lemma is very useful to show that characteristic x-ring is of finite dimension.

Lemma 27 *(See [28]) Suppose that the vector field*

$$X = \sum_{k=1}^{\infty}\Big(\alpha_k \frac{\partial}{\partial u_k} + \alpha_{-k}\frac{\partial}{\partial u_{-k}}\Big) + \sum_{k=1}^{\infty}\Big(\beta_k \frac{\partial}{\partial v_k} + \beta_{-k}\frac{\partial}{\partial v_{-k}}\Big)$$

satisfies the equality $DXD^{-1} = hX$*, where* h *is a function depending on shifts and derivatives of variables* u *and* v *then* $X = 0$*.*

The next lemma shows how the transformation $D(,)D^{-1}$ acts on the vector fields A, B, P_1 and P_2.

Lemma 28 *We have the following identities,*

$$\begin{aligned} &DAD^{-1} = A, && DBD^{-1} = B, \\ &DP_1D^{-1} = P_1 - M_1A, && DP_2D^{-1} = P_2 - N_1B. \end{aligned} \tag{3.11}$$

Proof: It is easy to check that $DAD^{-1} = A$ and $DBD^{-1} = B$. Let

$$DP_1D^{-1} = \sum_{k=1}^{\infty}\Big(\alpha_k\frac{\partial}{\partial u_k} + \alpha_{-k}\frac{\partial}{\partial u_{-k}}\Big)$$

then

$$\alpha_k = DP_1D^{-1}(u_k) = DP_1(u_{k-1}) = D\Big(\sum_{k=1}^{k-1} M_i\Big) = \sum_{k=1}^{k-1} M_{i+1}. \qquad k = 1, 2, \dots \quad .$$

Hence we have

$$\alpha_k = \Big(\sum_{k=1}^{k} M_i\Big) - M_1.$$

In the same way, we can find the equality for α_{-k}, $k = 1, 2 \dots$. Thus, we have $DP_1D^{-1} = P_1 - M_1A$. The formula for DP_2D^{-1} follows similarly. $\square$

Let us we introduce the sequences of the commutators;

$$\begin{aligned}
&\mathbb{X}_1: \quad T_1 = [P_1, P_2],\\
&\mathbb{X}_2: \quad T_2 = [P_1, T_1], \quad R_2 = [P_2, T_1],\\
&\mathbb{X}_3: \quad T_3 = [P_1, T_2], \quad K_3 = [P_2, T_2], \quad H_3 = [P_1, R_2], \quad R_3 = [P_2, R_2].
\end{aligned}$$

In general we define

$$\begin{aligned}
\mathbb{X}_n: \qquad & T_n = [P_1, T_{n-1}], \qquad K_n = [P_2, K_{n-1}],\\
& H_n = [P_1, H_{n-1}], \quad R_n = [P_2, R_{n-1}] \quad n = 4, 5, \dots \quad ,
\end{aligned}$$

We also define

$$\begin{aligned}
\mathbb{X}_n: \qquad & W_n = [P_2, W_{n-1}], \quad n \geq 4, 5 \dots \quad \text{and} \quad W_3 = T_3,\\
& V_n = [P_2, V_{n-1}], \quad n \geq 5, 6 \dots \quad \text{and} \quad V_4 = T_4,
\end{aligned}$$

and

$$Q_{ij} = [T_i, T_j] \quad i, j = 1, 2, 3, \dots \quad ,$$

We have two lemmas which are very useful for our calculations.

Lemma 29 *For the vector fields in x-ring we have the following identities,*

a) $A(M_1) = a_{11}M_1, \qquad A(N_1) = a_{21}N_1.$

b) $B(M_1) = a_{12}M_1, \qquad B(N_1) = a_{22}N_1.$

c) $P_1(M_1) = M_1^2, \qquad P_1(N_1) = 0.$

d) $P_2(M_1) = a_{12}M_1N_1, \qquad P_2(N_1) = N_1^2.$

e) $T_1(M_1) = -a_{12}M_1^2N_1, \quad T_i(M_1) = 0 \quad i \geq 2, \qquad T_i(N_1) = 0, \quad i \geq 1.$

f) $R_2(M_1) = -a_{12}(a_{12}+1)M_1^2N_1^2 \quad i \geq 2 \qquad R_i(N_1) = 0, \quad i \geq 1.$

g) $K_i(M_1) = 0, \qquad K_i(N_1) = 0, \quad i \geq 3.$

Proof: We prove (e) and (f), the others follow similarly. We start with (e),

$$\begin{aligned} T_1(M_1) &= [P_1, P_2](M_1) = P_1P_2M_1 - P_2P_1M_1 \\ &= P_1(a_{12}M_1N_1) - P_2(M_1^2) = -a_{12}M_1^2N_1. \end{aligned}$$

$$\begin{aligned} T_1(N_1) &= [P_1, P_2](N_1) = P_1P_2N_1 - P_2P_1N_1 \\ &= P_1(N_1^2) - P_2(0) = 0. \end{aligned}$$

$$\begin{aligned} T_2(M_1) &= [P_1, T_1](M_1) = P_1T_1M_1 - T_1P_1M_1 \\ &= P_1(-a_{12}M_1^2N_1) - T_1(M_1^2) = 0. \end{aligned}$$

when $i \geq 2$, using induction we suppose $T_k(M_1) = 0$ and $T_k(N_1) = 0$. Then,

$$T_{k+1}(M_1) = [P_1, T_k](M_1) = P_1T_k(M_1) - T_kP_1(M_1) = T_k(M_1^2) = 0.$$

$$T_{k+1}(N_1) = [P_1, T_k](N_1) = P_1T_k(N_1) - T_kP_1(N_1) = 0.$$

Hence they are true for $n = k + 1$. That completes the proof of part (e). Now we prove (f), We have

$$\begin{aligned} R_2(M_1) &= [P_2, T_1](M_1) = P_2T_1M_1 - T_1P_2M_1 \\ &= P_2(-a_{12}M_1^2N_1) - T_1(a_{12}M_1N_1) = -a_{12}(a_{12}+1)M_1^2N_1^2. \end{aligned}$$

$$\begin{aligned} R_2(N_1) &= [P_2, T_1](N_1) = P_2T_1N_1 - T_1P_2N_1 \\ &= -T_1(N_1^2) = 0. \end{aligned}$$

when $i \geq 2$, using induction we suppose $R_k(N_1) = 0$. Then,

$$R_{k+1}(N_1) \;=\; [P_2, R_k](N_1) = P_2 R_k(N_1) - R_k P_2(N_1) = -R_k(N_1^2) = 0.$$

That means $R_i(N_1) = 0$. Hence it is true for $i \geq 2$. $\square$

Lemma 30 *We have the following identities;*

a) $[A, T_n] = (na_{11} + a_{21})T_n,$

b) $[B, T_n] = (na_{12} + a_{22})T_n,$

c) $[B, R_n] = (a_{12} + na_{22})R_n,$

d) $[A, R_n] = (a_{11} + na_{21})R_n,$

e) $[B, K_n] = (a_{12} + na_{22})K_n,$

f) $[B, W_n] = (3a_{12} + (n-2)a_{22})W_n,$

g) $[B, V_n] = (4a_{12} + (n-3)a_{22})V_n.$

Proof: We prove (a) and (b), the others follow similarly. We use induction. For $n = 1$ we have

$$\begin{aligned} [A, T_1] = [A, [P_1, P_2]] &= -[P_1, [P_2, A]] - [P_2, [A, P_1]] \\ &= -[P_1, -a_{21}P_2]] - [P_2, a_{11}P_1] = (a_{11} + a_{21})T_1. \end{aligned}$$

Hence (a) is true for $n = 1$. Now we suppose (a) is true for $n = k$. That is, we suppose $[A, T_k] = (ka_{11} + a_{21})T_k$. Then

$$\begin{aligned} [A, T_{k+1}] = [A, [P_1, T_k]] &= -[P_1, [T_k, A]] - [T_k, [A, P_1]] \\ &= -[P_1, -(ka_{11} + a_{21})T_k]] - [T_k, a_{11}P_1] \\ &= ((k+1)a_{11} + a_{21})T_{k+1}. \end{aligned}$$

That means (a) is true for all n.

Now we prove(b). We use induction again. For $n = 1$ we have

$$\begin{aligned} [B, T_1] = [B, [P_1, P_2]] &= -[P_1, [P_2, B]] - [P_2, [B, P_1]] \\ &= -[P_1, -a_{22}P_2]] - [P_2, a_{12}P_1] = (a_{12} + a_{22})T_1. \end{aligned}$$

Hence (b) is true for $n = 1$. Now we suppose it is true for $n = k$. That is, we suppose $[B, T_k] = (ka_{12} + a_{22})T_k$. Then,

$$\begin{aligned} [B, T_{k+1}] = [B, [P_1, T_k]] &= -[P_1, [T_k, B]] - [T_k, [B, P_1]] \\ &= -[P_1, -(ka_{12} + a_{22})T_k]] - [T_k, a_{12}P_1] \\ &= ((k+1)a_{12} + a_{22})T_{k+1}. \end{aligned}$$

That means (b) is true for all n. □

3.3 Proof of Main Theorem

We suppose system (3.5) is Darboux integrable. Then from definition of Darboux integrability, system (3.5) must admit two functionally independent nontrivial x-integrals and two functionally independent nontrivial n-integrals. Firstly, we consider x-integrals. Since system (3.5) has a nontrivial x-integral, Theorem 16 implies characteristic x-ring of the system is finite dimensional, that is $dim\mathcal{A} < \infty$. Then Lemma 26 implies the linear space $\mathbb{X}_n = 0$ for some $n < \infty$. Hence we will study all the cases such that the linear space $\mathbb{X}_n = 0$ and $\mathbb{X}_k \neq 0$ for $k = 1, 2, .., n-1$. Therefore we will find all cases of the system (3.5) that are Darboux integrable. We will also find the dimension and bases of characteristic x-ring for each Darboux integrable system. Firstly we will consider the cases $\mathbb{X}_1$, $\mathbb{X}_2$ and $\mathbb{X}_3$ are equal to zero, then we will prove some general formulas to generalize our result to $\mathbb{X}_n$ for all $n \in \mathbb{N}$. We start with the case $\mathbb{X}_1 = 0$.

3.3.1 The Case $\mathbb{X}_1$ = 0

The only vector field in $\mathbb{X}_1$ is

$$T_1 = [P_1, P_2].$$

We will show $\mathbb{X}_1$ is never zero because the vector field T_1 is never zero.

Lemma 31 *We have the following identity for the vector field* T_1

$$DT_1D^{-1} = T_1 + a_{12}N_1P_1 - a_{21}M_1P_2 + a_{21}M_1N_1B. \tag{3.12}$$

Proof: Using (3.11) we can write,

$$\begin{aligned} DT_1D^{-1} &= D[P_1, P_2]D^{-1} = [DP_1D^{-1}, DP_2D^{-1}] = [P_1 - M_1A, P_2 - N_1B] \\ &= [P_1, P_2] - [P_1, N_1B] - [M_1A, P_2] + [M_1A, N_1B] \\ &= T_1 + a_{12}N_1P_1 - a_{21}M_1P_2 + a_{21}M_1N_1B. \end{aligned}$$

$\square$

Proposition 32 *For the system* (3.5)*, the linear space* $\mathbb{X}_1 = 0$ *is not possible.*

Proof: By Lemma 27, $T_1 = 0$ if and only if $DT_1D^{-1} = T_1$ and using Lemma 31 this is possible if and only if $a_{12} = a_{21} = 0$, which is a contradiction with our assumption. Hence linear space $\mathbb{X}_1 = 0$ is not possible. $\square$

3.3.2 The Case $\mathbb{X}_2$ = 0

All possible vector fields in $\mathbb{X}_2$ are given as follows,

$$T_2 = [P_1, T_1], \qquad R_2 = [P_2, T_1].$$

Lemma 33 *We have the following identities for the vector fields* T_2 *and* R_2 *in* $\mathbb{X}_2$*,*

$$DT_2D^{-1} = T_2 + \alpha^2_{T_1} M_1 T_1 + \alpha^2_{P_2} M_1^2 P_2 + \alpha^2_{P_1} M_1 N_1 P_1 + \alpha^2_B M_1^2 N_1 B \tag{3.13}$$

where

$$\alpha^2_{T_1} = -(2a_{21} + a_{11}), \qquad \alpha^2_{P_1} = -a_{12}(a_{11} + 2a_{21}),$$
$$\alpha^2_{P_2} = a_{21}(a_{11} + a_{21} - 1), \quad \alpha^2_B = a_{21}(1 - a_{11} - a_{21}),$$

and

$$DR_2D^{-1} = R_2 + \gamma^2_{T_1} N_1 T_1 + \gamma^2_{P_2} N_1^2 P_2, \tag{3.14}$$

where

$$\gamma^2_{T_1} = -(2a_{12} + a_{22}), \qquad \gamma^2_{P_2} = a_{12}(1 - a_{22} - a_{12}).$$

Proof: We can write

$$\begin{aligned} DT_2D^{-1} &= D[P_1,T_1]D^{-1} = [DP_1D^{-1}, DT_1D^{-1}] \\ &= [P_1 - M_1A, T_1 + a_{12}N_1P_1 - a_{21}M_1P_2 + a_{21}M_1N_1B] \\ &= [P_1,T_1] + a_{12}[P_1,N_1P_1] - a_{21}[P_1,M_1P_2] + a_{21}[P_1,M_1N_1B] \\ &\quad -[M_1A,T_1] - a_{12}[M_1A,N_1P_1] + a_{21}[M_1A,M_1P_2] - a_{21}[M_1A,M_1N_1B]. \end{aligned}$$

Using the identities in Lemma 29 and Lemma 30 we get the desired result for T_2. We also have

$$\begin{aligned} DR_2D^{-1} &= D[P_2,T_1]D^{-1} = [DP_1D^{-1}, DT_1D^{-1}] \\ &= [P_2 - N_1B, T_1 + a_{12}N_1P_1 - a_{21}M_1P_2 + a_{21}M_1N_1B] \\ &= [P_2,T_1] + a_{12}[P_2,N_1P_1] - a_{21}[P_2,M_1P_2] + a_{21}[P_2,M_1N_1B] \\ &\quad -[N_1B,T_1] - a_{12}[N_1B,N_1P_1] + a_{21}[N_1B,M_1P_2] - a_{21}[N_1B,M_1N_1B]. \end{aligned}$$

In the same way, using the identities in Lemma 29 and Lemma 30 we get the desired result for R_2. □

Proposition 34 *For the system* (3.5) *the linear space* $\mathbb{X}_2 = 0$ *if and only if* a_{ij}*'s satisfy,*

$$\begin{aligned} a_{11} &= 2, \qquad a_{12} = -1 \\ a_{21} &= -1, \quad\; a_{22} = 2 \end{aligned}$$

in which case $\mathcal{A}$*-ring of the system* (3.5) *is* 3 *dimensional with bases* P_1*,* P_2 *and* T_1*.*

Proof: Since $T_2, R_2 \in \mathbb{X}_2$, they must be zero. Firstly, $DT_2D^{-1} = T_2$ must be satisfied. From the coefficient of T_1 and P_2 in (3.13) the followings must be satisfied

$$\begin{aligned} \alpha^2_{T_1} &= -(2a_{21} + a_{11}) = 0, \\ \alpha^2_{P_2} &= a_{21}(a_{11} + a_{21} - 1) = 0. \end{aligned}$$

They imply

$$a_{11} = 2, \quad a_{21} = -1. \tag{3.15}$$

Under conditions (3.15), $DT_2D^{-1} = T_2$ is satisfied and Lemma 27 implies $T_2 = 0$. In addition to that we must also have $DR_2D^{-1} = R_2$, from the coefficient of T_1 and P_2 on the right hand side of (3.14) we have

$$\gamma^2_{T_1} = -(2a_{12} + a_{22}) = 0,$$
$$\gamma^2_{P_2} = a_{21}(1 - a_{22} - a_{12}) = 1.$$

which imply

$$a_{12} = -1, \quad a_{22} = 2 \tag{3.16}$$

Under the conditions (3.16), $DR_2D^{-1} = R_2$ is satisfied and Lemma 27 implies $R_2 = 0$. This means that all vector fields in $\mathbb{X}_2$ are zero. Hence $\mathbb{X}_2 = 0$.
Under conditions (3.15) and (3.16) the nonzero vectors fields in $\mathbb{X}_0$ and $\mathbb{X}_1$ are P_1, P_2 and T_1. Hence, the ring $\mathcal{A}$ of (3.5) is 3-dimensional with bases P_1, P_2 and T_1 and we have the following 8 dimensional characteristic x-ring.

$[.,.]$	$\frac{\partial}{\partial x}$	Y_1	Y_2	A	B	P_1	P_2	T_1
$\frac{\partial}{\partial x}$	0	0	0	0	0	0	0	0
Y_1	0	0	0	0	0	0	0	0
Y_2	0	0	0	0	0	0	0	0
A	0	0	0	0	0	$2P_1$	$-P_2$	T_1
B	0	0	0	0	0	$-P_1$	$2P_2$	T_1
P_1	0	0	0	$-2P_1$	P_1	0	T_1	0
P_2	0	0	0	P_2	$-2P_2$	$-T_1$	0	0
T_1	0	0	0	$-T_1$	$-T_1$	0	0	0

□

Note that we obtained

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \tag{3.17}$$

which is a Cartan matrix. In this case the system (3.5) reduces to

$$\begin{aligned} u_{1x} - u_x &= e^{u+u_1-v_1}, \\ v_{1x} - v_x &= e^{-u+v+v_1}. \end{aligned} \tag{3.18}$$

For this system our results reproduce the results of [41]. Also corresponding x-integrals are [41],

$$F_1 = e^{-v+v_1} + e^{-u+u_1+v_1-v_2} + e^{u_1-u_2},$$
$$F_2 = e^{-u+u_1} + e^{u_1-u_2-v_1+v_2} + e^{v_2-v_3},$$

and n-integrals are given as follows

$$I_1 = u_{xx} + v_{xx} - u_x^2 + u_x v_x - v_x^2,$$
$$I_2 = u_{3x} + u_x(v_{xx} - 2u_{xx}) + u_x^2 v_x - u_x v_x^2.$$

3.3.3 The Case $\mathbb{X}_3 = 0$

All possible vector fields in $\mathbb{X}_3$ are given as follows

$$[P_1, T_2] = T_3, \qquad [P_2, T_2] = K_3, \tag{3.19}$$
$$[P_1, R_2] = H_3, \qquad [P_2, R_2] = R_3. \tag{3.20}$$

Note that the relation $H_3 = K_3$ is satisfied by the Jacobi identity.

Lemma 35 *We have the following identities for the vector fields T_3 and R_3 in $\mathbb{X}_3$*

$$\begin{aligned} DT_3D^{-1} &= T_3 + \alpha^3_{T_2} M_1 T_2 + \alpha^3_{T_1} M_1^2 T_1 + \alpha^3_{P_1} M_1^2 N_1 P_1 \\ &\quad + \alpha^3_{P_2} M_1^3 P_2 + \alpha^3_B M_1^3 N_1 B, \end{aligned} \tag{3.21}$$

where

$$\alpha^3_{T_2} = -3(a_{11} + a_{21}),$$
$$\alpha^3_{T_1} = a_{11}(2a_{11} + 6a_{21} - 1) + a_{21}(3a_{21} - 3),$$
$$\alpha^3_{P_1} = a_{21}a_{12}(6a_{11} + 3a_{21} - 3) + a_{11}a_{12}(2a_{11} - 1),$$
$$\alpha^3_{P_2} = -a_{21}(1 - a_{11} - a_{21})(2 - 2a_{11} - a_{21}),$$
$$\alpha^3_B = a_{21}(1 - a_{11} - a_{21})(2 - 2a_{11} - a_{21}),$$

and

$$DR_3D^{-1} = R_3 + \gamma^3_{R_2} N_1 R_2 + \gamma^3_{T_1} N_1^2 T_1 + \gamma^3_{P_2} N_1^3 P_2 + \gamma^3_B N_1^4 B, \tag{3.22}$$

where

$$\gamma^3_{R_2} = -(3a_{12} + 3a_{22}),$$
$$\gamma^3_{T_1} = (a_{12} + 2a_{22} - 1)(2a_{12} + a_{22}),$$
$$\gamma^3_{P_2} = a_{12}(2 - 3a_{22})(1 - a_{12} - a_{22}),$$
$$\gamma^3_{B} = a_{12}(1 - a_{12} - a_{22}).$$

Proof: We can write

$$\begin{aligned}
DT_3D^{-1} &= D[P_1, T_2]D^{-1} = [DP_1D^{-1}, DT_2D^{-1}] \\
&= [P_1 - M_1A, T_2 + \alpha^2_{T_1}M_1T_1 + \alpha^2_{P_2}M_1^2P_2 + \alpha^2_{P_2}M_1N_1P_1 + \alpha^2_B M_1^2N_1B] \\
&= [P_1, T_2] + \alpha^2_{T_1}[P_1, M_1T_1] + \alpha^2_{P_2}[P_1, M_1^2P_2] + \alpha^2_{P_2}[P_1, M_1N_1P_1] \\
&\quad +\alpha^2_B[P_1, M_1^2N_1B] - [M_1A, T_2] - \alpha^2_{T_1}[M_1A, M_1T_1] \\
&\quad -\alpha^2_{P_2}[M_1A, M_1^2P_2] - \alpha^2_{P_2}[M_1A, M_1N_1P_1] - \alpha^2_B[M_1A, M_1^2N_1B].
\end{aligned}$$

Using the identities in Lemma 29 and Lemma 30 we get the desired result for T_3. We also have

$$\begin{aligned}
DR_3D^{-1} &= D[P_2, R_2]D^{-1} = [DP_2D^{-1}, DR_2D^{-1}] \\
&= [P_2 - N_1B, R_2 - (2a_{12} + a_{22})N_1T_1 + a_{12}(1 - a_{22} - a_{12})N_1^2P_2] \\
&= [P_2, R_2] - (2a_{12} + a_{22})[P_2, N_1T_1] + a_{12}(1 - a_{22} - a_{12})[P_2, N_1^2P_2] \\
&\quad -[N_1B, R_2] + (2a_{12} + a_{22})[P_2, N_1T_1] - a_{12}(1 - a_{22} - a_{12})[N_1B, N_1^2P_2].
\end{aligned}$$

In the same way, using the identities in Lemma 29 and Lemma 30 we get the desired result for R_3. □

Lemma 36 *If the vector field R_3 satisfies $R_3 = 0$, then $R_2 = 0$*

Proof: We suppose $R_2 \neq 0$ and $DR_3D^{-1} = R_3$ must be satisfied. Then from the coefficients of R_2, T_1 and B on the right hand side of (3.22) the followings must be satisfied,

$$\gamma^3_{R_2} = -(3a_{12} + 3a_{22}) = 0, \tag{3.23}$$
$$\gamma^3_{T_1} = (a_{12} + 2a_{22} - 1)(2a_{12} + a_{22}) = 0, \tag{3.24}$$
$$\gamma^3_{B} = a_{12}(1 - a_{12} - a_{22}) = 0. \tag{3.25}$$

Equalities (3.23), (3.24) and (3.25) are satisfied if and only if $a_{12} = 0$ and $a_{22} = 0$ which is not possible. Hence we find $R_3 = 0$ if and only if $R_2 = 0$. $\square$

Proposition 37 *For the system* (3.5)*, the linear space* $\mathbb{X}_3 = 0$ *if and only if* a_{ij}*'s satisfy,*

$$
\begin{aligned}
a_{11} &= 2, \qquad a_{12} = -1, \\
a_{21} &= -2, \quad\; a_{22} = 2.
\end{aligned}
$$

in which case $\mathcal{A}$*-ring of the system* (3.5) *is* 4 *dimensional with bases* P_1*,* P_2*,* T_1 *and* T_2*.*

Proof: Assume that $\mathbb{X}_3 = 0$. Since $T_3, R_3 \in \mathbb{X}_3$ they must be zero. Since $T_3 = [P_1, T_2]$ assuming $T_2 \neq 0$, $DT_3D^{-1} = T_3$ must be satisfied. From the coefficient of T_2 and T_1 on the right hand side of (3.21) we have

$$\alpha^3_{T_2} = -3(a_{11} + a_{21}) = 0, \tag{3.26}$$

$$\alpha^3_{T_1} = a_{11}(2a_{11} + 6a_{21} - 1) + a_{21}(3a_{21} - 3) = 0. \tag{3.27}$$

Solving (3.26) and (3.27) we find

$$a_{11} = 2, \quad a_{21} = -2. \tag{3.28}$$

Under the conditions (3.28), $DT_3D^{-1} = T_3$ is satisfied and Lemma 27 implies $T_3 = 0$.

Also, using Lemma 36, $R_3 = 0$ implies $R_2 = 0$, Then from the proof of Proposition 34, we find

$$a_{12} = -1, \quad a_{22} = 2. \tag{3.29}$$

Under the conditions (3.28) and (3.29), since $R_2 = 0$ and $H_3 = K_3$ the vector fields in (3.19) and (3.20) are zero. Then we get $\mathbb{X}_3 = 0$ and the nonzero vector fields in $\mathbb{X}_i$, for $i = 0, 1, 2$ are P_1, P_2, T_1 and T_2. Therefore, $\mathcal{A}$ ring of the system (3.5) is 4-dimensional with bases P_1, P_2, T_1 and T_2 and we have the following 9 dimensional characteristic x-ring.

$[.,.]$	$\frac{\partial}{\partial x}$	Y_1	Y_2	A	B	P_1	P_2	T_1	T_2
$\frac{\partial}{\partial x}$	0	0	0	0	0	0	0	0	0
Y_1	0	0	0	0	0	0	0	0	0
Y_2	0	0	0	0	0	0	0	0	0
A	0	0	0	0	0	$2P_1$	$-2P_2$	0	$2T_2$
B	0	0	0	0	0	$-P_1$	$2P_2$	T_1	0
P_1	0	0	0	$-2P_1$	P_1	0	T_1	T_2	0
P_2	0	0	0	$2P_2$	$-2P_2$	$-T_1$	0	0	0
T_1	0	0	0	0	$-T_1$	$-T_2$	0	0	0
T_2	0	0	0	$-2T_2$	0	0	0	0	0

□

Note that we obtained

$$A = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}, \tag{3.30}$$

which is a Cartan matrix. In this case the system (3.5) reduces to

$$\begin{aligned} u_{1x} - u_x &= e^{u+u_1-v_1}, \\ v_{1x} - v_x &= e^{-2u+v+v_1}. \end{aligned} \tag{3.31}$$

For this system our results reproduce the results of [41]. Also corresponding x-integrals are [41],

$$\begin{aligned} F_1 &= e^{-u+u_1} + e^{-u_1+u_2+v_2-v_3} + e^{u_1-u_2-v_1+v_2} + e^{v_2-v_3}, \\ F_2 &= e^{-v+v_1} + e^{-2u+2u_1+v_1-v_2} + 2e^{-u+2u_1-u_2} + e^{2u_1-2u_2-v_1+v_2} + e^{v_2-v_3}, \end{aligned}$$

and n-integrals are given as

$$\begin{aligned} I_1 &= 2u_{xx} + v_{xx} - 2u_x^2 + 2u_xv_x - v_{xx}^2, \\ I_2 &= u_{[4]} + u_x(v_{[3]} - 2u_{[3]}) + u_{xx}(4u_xv_x - 2u_x^2 - v_x^2) \\ &\quad + u_{xx}(v_{xx} - u_{xx}) + v_{xx}u_x(u_x - 2v_x) + u_x^4 + u_x^2v_x^2 - 2u_x^2v_x. \end{aligned}$$

Before considering the next case, we will prove some general formulas for the vector fields T_n and R_n in $\mathbb{X}_n$.

3.3.4 Important Formulas For General Case

Let us find how the transformation $D(,)D^{-1}$ acts on the introduced sequences of vector fields.

Lemma 38 *We have the following identities for the vector fields T_n and R_n.*

$$DT_nD^{-1} = T_n + \alpha^n_{T_{n-1}} M_1 T_{n-1} + \alpha^n_{T_{n-2}} M_1^2 T_{n-2} + \dots \quad , \tag{3.32}$$

$$DR_nD^{-1} = R_n + \gamma^n_{R_{n-1}} N_1 R_{n-1} + \gamma^n_{R_{n-2}} N_1^2 R_{n-2} + \dots \quad . \tag{3.33}$$

Proof: We use induction. Equations (3.13) and (3.22) imply the above formulas for $n = 2, 3$. Now we suppose they are true for $n = k$ and we check the case $n = k+1$,

$$\begin{aligned} DT_{k+1}D^{-1} &= [DP_1D^{-1}, DT_kD^{-1}] \\ &= [P_1 - M_1A, T_k + \alpha^k_{T_{k-1}} M_1 T_{k-1} + \alpha^k_{T_{k-2}} M_1^2 T_{k-2} + \dots] \\ &= T_{k+1} + \alpha^{k+1}_{T_k} M_1 T_k + \alpha^{k+1}_{T_{k-1}} M_1^2 T_{k-1} + \dots + \alpha^{k+1}_{T_i} M_1^{k-i+1} T_i + \dots, \end{aligned}$$

where

$$\alpha^{k+1}_{T_k} = \alpha^k_{T_{k-1}} - (ka_{11} + a_{21}), \tag{3.34}$$

$$\alpha^{k+1}_{T_{k-1}} = \alpha^k_{T_{k-2}} + (1 - ka_{11} - a_{21})\alpha^k_{T_{k-1}}, \tag{3.35}$$

$$\alpha^{k+1}_{T_{k-2}} = \alpha^k_{T_{k-3}} + (2 - ka_{11} - a_{21})\alpha^k_{T_{k-2}}, \tag{3.36}$$

and

$$\begin{aligned} DR_{k+1}D^{-1} &= [DP_2D^{-1}, DR_kD^{-1}] \\ &= [P_2 - N_1B, R_k + \gamma^k_{R_{k-1}} N_1 R_{k-1} + \gamma^k_{R_{k-2}} N_1^2 R_{k-2} + \dots] \\ &= R_{k+1} + \gamma^{k+1}_{R_k} N_1 R_k + \gamma^{k+1}_{R_{k-1}} N_1^2 R_{k-1} + \dots, \end{aligned}$$

where

$$\gamma^{k+1}_{R_k} = \gamma^k_{R_{k-1}} - (ka_{22} + a_{12}), \tag{3.37}$$

$$\gamma^{k+1}_{R_{k-1}} = \gamma^k_{R_{k-2}} + (1 - ka_{22} - a_{12})\gamma^k_{R_{k-1}}, \tag{3.38}$$

$$\gamma^{k+1}_{R_{k-2}} = \gamma^k_{R_{k-3}} + (2 - ka_{22} - a_{12})\gamma^k_{R_{k-2}}. \tag{3.39}$$

Hence (3.32) and (3.33) are true for $n = k + 1$. That means (3.32) is true for $n \geq 2$ and (3.33) is true for $n \geq 3$. $\square$

Lemma 39 *We have the following formulas for the coefficients of the terms of DT_nD^{-1} and DR_nD^{-1} in* (3.32) *and* (3.33)*,*

$$\alpha^n_{T_{n-1}} = -\frac{n(n-1)}{2}a_{11} - na_{21}, \qquad n \geq 2$$

$$\begin{aligned}\alpha^n_{T_{n-2}} = \frac{1}{24}n\Big(&-8a_{11} + 12na_{11} - 4n^2a_{11} - 2a_{11}^2 + 9na_{11}^2 - 10n^2a_{11}^2\\ &+ 3n^3a_{11}^2 + 12a_{21} - 12na_{21} + 12a_{11}a_{21} - 24na_{11}a_{21}\\ &+ 12n^2a_{11}a_{21} - 12a_{21}^2 + 12na_{21}^2\Big), \qquad n \geq 3\end{aligned}$$

$$\begin{aligned}\alpha^n_{T_{n-3}} = \Big(&-\frac{1}{48}n(n^2-3n+2)(n(n^2-4n+3)a_{11}^3 + 2a_{11}^2(n^2(3a_{21}-2)\\ &- 7n(a_{21}-1) + 2a_{21} - 3) + 4a_{11}(3na_{21}^2 - 5na_{21} + n - 3a_{21}^2 + 7a_{21} - 3)\\ &+ 8a_{21}(a_{21}^2 - 3a_{21} + 2))\Big), \qquad n \geq 4\end{aligned}$$

$$\gamma^n_{R_{n-1}} = -\frac{n(n-1)}{2}a_{22} - na_{12}, \qquad n \geq 4$$

$$\begin{aligned}\gamma^n_{R_{n-2}} = \frac{1}{24}(-2+n)\Big(&3n^3a_{22}^2 - 4n^2a_{22}(1 + a_{22} - 3a_{12}) + 12a_{12}(-1 + a_{22} + a_{12})\\ &+ n(4a_{22} + a_{22}^2 + 12(-1 + a_{12})a_{12})\Big), \qquad n \geq 5\end{aligned}$$

$$\begin{aligned}\gamma^n_{R_{n-3}} = -\frac{1}{48}(-3+n)(-2+n)\Big(&n^4a_{22}^3 - 2n^3a_{22}^2(2 + a_{22} - 3a_{12})\\ &+ 8a_{12}(4 - 5a_{22} + a_{22}^2 - 6a_{12} + 3a_{22}a_{12} + 2a_{12}^2)\\ &+ n^2a_{22}(4 + 6a_{22} + a_{22}^2 - 20a_{12} - 2a_{22}a_{12} + 12a_{12}^2) \qquad n \geq 6\\ &+ 2n\Big(a_{22}^2(-1 + 6a_{12}) + 4a_{12}(2 - 3a_{12} + a_{12}^2) + a_{22}(-2 - 6a_{12} + 6a_{12}^2)\Big)\Big).\end{aligned}$$

Proof: Solving the recursive equations (3.34), (3.35), (3.36), (3.37), (3.38) and (3.39) we get the desired formulas. □

The following two lemmas are very important for our calculations.

Lemma 40 *If characteristic x-ring $\mathcal{A}$ is finite dimensional then the vector field $R_2 = 0$ and this is equivalent to $a_{12} = -1$ and $a_{22} = 2$.*

Proof: Lemma 26 implies that the linear space $\mathbb{X}_n = 0$ for some $n < \infty$. We show that $\mathbb{X}_n = 0$ will imply the desired results. We use induction, because of Proposition 32, $\mathbb{X}_1$ cannot be zero. So we start from $\mathbb{X}_2 = 0$. Since $R_2 \in \mathbb{X}_2$, it should be zero. Then from the proof of Proposition 34 we get

$$a_{12} = -1, \quad a_{22} = 2. \tag{3.40}$$

The conditions (3.40) imply $DR_2D^{-1} = R_2$ and Lemma 27 implies $R_2 = 0$.
Suppose now $\mathbb{X}_3 = 0$. Since $R_3 \in \mathbb{X}_3$ it must be zero. Then Lemma 36 implies $R_2 = 0$ and this implies $a_{12} = -1$, $a_{22} = 2$.
Now suppose it is true for $n = k$. That is, $\mathbb{X}_k = 0$ implies $R_2 = 0$ and $a_{12} = -1$, $a_{22} = 2$. Let $n = k + 1$ and suppose $\mathbb{X}_{k+1} = 0$. Then since $R_{k+1} \in \mathbb{X}_{k+1}$, R_{k+1} must be zero, which is equivalent to $DR_{k+1}D^{-1} = R_{k+1}$. Using (3.37) we need $\gamma^{k+1}_{R_k} = 0$, $\gamma^{k+1}_{R_{k-1}} = 0$ and $\gamma^{k+1}_{R_{k-2}} = 0$ and solving them together we find $a_{12} = a_{22} = 0$, which is a contradiction. Since $R_{k+1} = [P_2, R_k]$ we must have $R_k = 0$. Hence by induction we get the desired result. □

Lemma 41 *The vector field $T_n = 0$ with the condition $T_{n-1} \neq 0$ implies the following conditions on a_{ij}'s,*

$$a_{11} = 2, \quad a_{21} = 1 - n, \quad \textit{for} \quad n \geq 2. \tag{3.41}$$

Proof: When $n = 2$, the proof of Proposition 34 implies (3.41). When $n \geq 3$, $DT_nD^{-1} = T_n$ must be satisfied. Hence we set the coefficient of T_{n-1} and T_{n-2} equal to zero in (3.32), that is $\alpha^n_{T_{n-1}} = 0$ and $\alpha^n_{T_{n-2}} = 0$. Solving them we get the desired conditions for a_{11} and a_{21}. □

3.3.5 The Case $\mathbb{X}_4$ = 0

All possible vector fields in $\mathbb{X}_4$ are given as follows,

$$[P_i, T_3] \quad i = 1, 2 \ , \qquad [P_i, R_3] \quad i = 1, 2 \ , \qquad [P_i, K_3] \quad i = 1, 2 \ .$$

Lemma 42 *We have the following identity for the vector field Q_{12} in $\mathbb{X}_4$*

$$\begin{aligned} DQ_{12}D^{-1} \;\; &= \;\; Q_{12} + \theta^{12}_{T_3} N_1 T_3 + \theta^{12}_{K_3} M_1 K_3 + \theta^{12}_{T_2} M_1 N_1 T_2 \\ &\quad + \theta^{12}_{R_2} M_1^2 R_2 + \theta^{12}_{T_1} M_1^2 N_1 T_1 + \ldots \quad , \end{aligned} \tag{3.42}$$

where

$$\begin{aligned} \theta^{12}_{T_3} &= a_{12}, & \theta^{12}_{K_3} &= -a_{21}, \\ \theta^{12}_{T_2} &= a_{21}(2_{12} + a_{22}), & \theta^{12}_{R_2} &= a_{21}(a_{21} + 1), \\ \theta^{12}_{T_1} &= a_{21}(a_{11} + a_{21})(a_{12} + a_{22}). \end{aligned}$$

Proof: We can write

$$\begin{aligned} DQ_{12}D^{-1} \;\; &= \;\; D[T_1, T_2]D^{-1} = [DT_1D^{-1}, DT_2D^{-1}] \\ &= \;\; \Big[T_1 + a_{12}N_1P_1 - a_{21}M_1P_2 + a_{21}M_1N_1B, \\ &\qquad T_2 + \alpha^2_{T_1} M_1 T_1 + \alpha^2_{P_2} M_1^2 P_2 + \alpha^2_{P_1} M_1 N_1 P_1 + \alpha^2_B M_1^2 N_1 B\Big]. \end{aligned}$$

Using the identities in Lemma 29 and Lemma 30 we get the desired result for Q_{12}. □

Lemma 43 *When $\mathbb{X}_4 = 0$, the vector field $Q_{12} = 0$ implies $T_3 = 0$.*

Proof: Firstly, Lemma 40 implies $R_2 = 0$ and by the case $\mathbb{X}_3 = 0$ we have $K_3 = 0$. Since $Q_{12} = [T_1, T_2]$, we suppose $T_2 \neq 0$. From the coefficients of T_3 on the right hand side of (3.42) we require

$$\theta^{12}_{T_3} = a_{12} = 0,$$

but this is not possible and we find $T_3 = 0$. □

Proposition 44 *For the system* (3.5) *the linear space* $\mathbb{X}_4 = 0$ *but* $\mathbb{X}_k \neq 0$ *for* $k = 1, 2, 3$ *is not possible.*

Proof: Suppose contrary, $\mathbb{X}_4 = 0$ and $\mathbb{X}_k \neq 0$ for $k = 1, 2, 3$. We know $R_4, Q_{12} \in \mathbb{X}_4$. Then Lemma 40 implies $R_2 = 0$ and Lemma 43 implies $T_3 = 0$. But by Proposition 37, this two together imply $\mathbb{X}_3 = 0$ and this is a contradiction with our assumption. Hence $\mathbb{X}_4 = 0$ but $\mathbb{X}_k \neq 0$ for $k = 1, 2, 3$ is not possible. $\square$

3.3.6 The Case $\mathbb{X}_5$ = 0

All possible vector fields in $\mathbb{X}_5$ are given as follows,

$$[P_i, [P_j, T_3]] \quad i, j = 1, 2,$$
$$[P_i, [P_j, R_3]] \quad i, j = 1, 2,$$
$$[P_i, [P_j, K_3]] \quad i, j = 1, 2.$$

Lemma 45 *We have the following identity for the vector fields* Q_{13} *and* V_5 *in* $\mathbb{X}_5$*,*

$$\begin{aligned} DQ_{13}D^{-1} &= Q_{13} + \theta^{13}_{T_4} N_1 T_4 + \theta^{13}_{W_4} M_1 W_4 + \theta^{13}_{Q_{12}} M_1 Q_{12} + \theta^{13}_{T_3} M_1 N_1 T_3 + \theta^{13}_{K_3} M_1^2 K_3 \\ &\quad + \theta^{13}_{T_2} M_1^2 N_1 T_2 + \theta^{13}_{R_2} M_1^3 R_2 + \theta^{13}_{T_1} M_1^3 N_1 T_1 + \theta^{13}_{P_1} M_1^3 N_1^2 P_1, \end{aligned} \tag{3.43}$$

where

$$\begin{aligned}
&\theta^{13}_{T_4} = a_{12}, && \theta^{13}_{W_4} = -a_{21}, \\
&\theta^{13}_{Q_{12}} = -3(a_{11} + a_{21}), && \theta^{13}_{T_3} = a_{21}a_{22} - 3a_{11}a_{12}, \\
&\theta^{13}_{K_3} = 3a_{21}(a_{11} + a_{21}), && \theta^{13}_{T_2} = -3a_{21}(a_{11} + a_{21})(a_{22} + 2a_{12}), \\
&\theta^{13}_{R_2} = -a_{21}(1 + a_{21})(-2 + 3a_{11} + 2a_{21}), \\
&\theta^{13}_{T_1} = a_{21}(1 + a_{21})(-2 + 3a_{11} + 2a_{21})(a_{22} + 2a_{12}), \\
&\theta^{13}_{P_1} = a_{21}(1 + a_{21})(-2 + 3a_{11} + 2a_{21})a_{12}(-1 + a_{22} + a_{12}).
\end{aligned}$$

and

$$\begin{aligned} DV_5D^{-1} &= V_5 + \varphi^5_{W_4} M_1 W_4 + \varphi^5_{T_4} N_1 T_4 + \varphi^5_{T_3} M_1 N_1 T_3 + \varphi^5_{K_3} M_1^2 K_3 \\ &\quad + \varphi^5_{T_2} M_1^2 N_1 T_2 + \varphi^5_{R_2} M_1^3 R_2 + \varphi^5_{T_1} M_1^3 N_1 T_1 + \dots \quad , \end{aligned} \tag{3.44}$$

where

$$\begin{array}{ll}
\varphi^5_{W_4} = \alpha^4_{T_3}, & \varphi^5_{T_2} = -(2a_{12}+a_{22})\alpha^4_{T_2}, \\
\varphi^5_{T_4} = -(4a_{12}+a_{22}), & \varphi^5_{R_2} = \alpha^4_{T_1}, \\
\varphi^5_{T_3} = -(3a_{12}+a_{22})\alpha^4_{T_3}, & \varphi^5_{T_1} = -(a_{12}+a_{22})\alpha^4_{T_1} - \alpha^4_{P_1}, \\
\varphi^5_{K_3} = \alpha^4_{T_2}. &
\end{array}$$

Proof: We can write

$$\begin{aligned}
DQ_{13}D^{-1} &= D[T_1,T_3]D^{-1} = [DT_1D^{-1}, DT_3D^{-1}] \\
&= \Big[T_1 + a_{12}N_1P_1 - a_{21}M_1P_2 + a_{21}M_1N_1B,\ T_3 + \alpha^3_{T_2}M_1T_2 \\
&\qquad + \alpha^3_{T_1}M_1^2T_1 + \alpha^3_{P_1}M_1^2N_1P_1 + \alpha^3_{P_2}M_1^3P_2 + \alpha^3_B M_1^3N_1B\Big].
\end{aligned}$$

Using the identities in Lemma 29 and Lemma 30 we get the desired result for Q_{13}. For DV_5D^{-1} we can write

$$\begin{aligned}
DV_5D^{-1} &= D[P_2,T_4]D^{-1} = [DP_2D^{-1}, DT_4D^{-1}] \\
&= \Big[P_2 - N_1B,\ T_4 + \alpha^4_{T_3}M_1T_3 \\
&\qquad + \alpha^4_{T_2}M_1^2T_2 + \alpha^4_{T_1}M_1^3T_1 + \alpha^4_{P_1}M_1^3N_1P_1 + \dots\Big].
\end{aligned}$$

In the same way, using the identities in Lemma 29 and Lemma 30 we get the desired result for V_5. □

Lemma 46 *When $\mathbb{X}_5 = 0$, the vector field $V_5 = 0$ implies,*
a) $T_4 = 0$,
b) $Q_{13} = 0$.

Proof: a) Firstly, Lemma 40 implies $R_2 = 0$, $K_3 = 0$ and $a_{12} = -1, a_{22} = 2$. Then, DV_5D^{-1} takes the form,

$$\begin{aligned}
DV_5D^{-1} &= V_5 + \alpha^4_{T_3}M_1W_4 - (4a_{12}+a_{22})N_1T_4 \\
&\qquad - (3a_{12}+a_{22})\alpha^4_{T_3}M_1N_1T_3 + \dots \quad .
\end{aligned} \tag{3.45}$$

Since $V_5 = [P_2, T_4]$, we suppose $T_4 \neq 0$ and we must have $DV_5D^{-1} = V_5$. Then, since $K_3 = 0$ the coefficients of T_3 on the right hand side of (3.45) must be zero, since $3a_{12}+a_{22}\neq 0$ we get $\alpha^4_{T_3}=0$. Then, DV_5D^{-1} takes the form,

$$DV_5D^{-1} \;=\; V_5-(4a_{12}+a_{22})N_1T_4+\dots \quad . \tag{3.46}$$

Then since the coefficient of T_4 is not equal to zero, we get $T_4=0$.

b) Using Lemma 40 and (3.43), $DQ_{13}D^{-1}$ reduces to

$$DQ_{13}D^{-1}=Q_{13}-(2a_{21}+3a_{11})M_1Q_{12}+(2a_{21}+3a_{11})M_1N_1T_3.$$

By part (a), $T_4=0$ and Lemma 41 implies $a_{11}=2, a_{21}=-3$. Then $DQ_{13}D^{-1}=Q_{13}$ is satisfied and we get the desired result. $\square$

Lemma 47 *When $\mathbb{X}_5=0$, we have the following identities ,*

$a)$ $W_4=-Q_{12}$,

$b)$ $W_5=0$,

$c)$ $[P_1,W_4]=0$.

Proof: Since $\mathbb{X}_5=0$, Lemma 40 implies $R_2=0$ and $K_3=0$.

a) The vector field $[P_1,K_3]$ is in $\mathbb{X}_4$ and we have

$$\begin{aligned} 0=[P_1,K_3]=[P_1,[P_2,T_2]]] &= -[P_2,[T_2,P_1]]-[T_2,[P_1,P_2]] \\ &= [P_2,T_3]+[T_1,T_2]=W_4+Q_{12}. \end{aligned} \tag{3.47}$$

Hence we find $W_4=-Q_{12}$.

b) The vector field $W_5=[P_2,W_4]$ is in $\mathbb{X}_5$ and by part (a)

$$\begin{aligned} W_5=[P_2,W_4]=[P_2,-Q_{12}]=-[P_2,[T_1,T_2]] &= [T_1,[T_2,P_2]]+[T_2,[P_2,T_1]] \\ &= -[T_1,K_3]+[T_2,R_2]=0. \end{aligned}$$

c) Using Lemma 46 and part (a) we can write

$$\begin{aligned} 0=-Q_{13}=-[T_1,T_3]=-[T_1,[P_1,T_2]] &= [P_1,[T_2,T_1]]+[T_2,[T_1,P_1]] \\ &= -[P_1,Q_{12}]-[T_2,T_2]=-[P_1,Q_{12}] \\ &= [P_1,W_4]. \end{aligned}$$

Hence we are done. $\square$

Proposition 48 *For the system* (3.5)*, the linear space* $\mathbb{X}_5 = 0$ *if and only if* a_{ij}*'s satisfy,*

$$
\begin{aligned}
a_{11} &= 2, \quad a_{12} = -1, \\
a_{21} &= -3, \quad a_{22} = 2,
\end{aligned}
\tag{3.48}
$$

in which case $\mathcal{A}$*-ring of the system* (3.5) *is* 6 *dimensional with bases* P_1*,* P_2*,* T_1*,* T_2*,* T_3 *and* W_4*.*

Proof: Since $R_5, V_5 \in \mathbb{X}_5$, they must be zero. Lemma 40 and Lemma 46 imply (3.48). It follows that the vector fields $[P_i, [P_j, R_3]],\ i, j = 1, 2$ and $[P_i, [P_j, K_3]]$, $i, j = 1, 2$ are all zero. Now we check $[P_i, [P_j, T_3]]$, for $i, j = 1, 2$. Since $T_4 = 0$ we need to check only the vector fields $[P_1, W_4]$ and $W_5 = [P_2, W_4]$. By Lemma 47, they are also zero. That means all the vector fields in $\mathbb{X}_5$ are zero. That is $\mathbb{X}_5 = 0$. The only vector field in $\mathbb{X}_4$ which is nonzero is W_4. Hence W_4 is also a bases vector field for the characteristic x-ring of system (3.6) in that case.
The nonzero vector fields in $\mathbb{X}_i$, for $i = 0, 1, 2, 3$ are P_1, P_2, T_1, T_2 and T_3. Hence they form a bases for the characteristic x-ring of the system (3.6). Hence, the ring $\mathcal{A}$ of the system (3.6) is 6-dimensional with bases $\{P_1, P_2, T_1, T_2, T_3, W_4\}$ and we have the following 11 dimensional characteristic x-ring.

$[.,.]$	$\frac{\partial}{\partial x}$	Y_1	Y_2	A	B	P_1	P_2	T_1	T_2	T_3	W_4
$\frac{\partial}{\partial x}$	0	0	0	0	0	0	0	0	0	0	0
Y_1	0	0	0	0	0	0	0	0	0	0	0
Y_2	0	0	0	0	0	0	0	0	0	0	0
A	0	0	0	0	0	$2P_1$	$-3P_2$	$-T_1$	T_2	$3T_3$	$3W_4$
B	0	0	0	0	0	$-P_1$	$2P_2$	T_1	0	$-T_3$	$-W_4$
P_1	0	0	0	$-2P_1$	P_1	0	T_1	T_2	T_3	0	0
P_2	0	0	0	$-2P_2$	$-2P_2$	$-T_1$	0	0	0	W_4	0
T_1	0	0	0	T_1	$-T_1$	$-T_2$	0	0	W_4	0	0
T_2	0	0	0	$-T_2$	0	$-T_3$	0	$-W_4$	0	0	0
T_3	0	0	0	$-3T_3$	T_3	0	$-W_4$	0	0	0	0
W_4	0	0	0	$-3W_4$	W_4	0	$-W_4$	0	0	0	0

□

Note that we obtained

$$A = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}, \tag{3.49}$$

which is a Cartan matrix. In this case the system (3.5) reduces to,

$$\begin{aligned} u_{1x} - u_x &= e^{u+u_1-v_1} \\ v_{1x} - v_x &= e^{-3u+v+v_1}. \end{aligned} \tag{3.50}$$

For this system our results reproduce the results of [41]. Also corresponding x-integrals are [41]

$$\begin{aligned} F_1 &= e^{-u+u_1} + e^{-2u_1+2u_2+v_2-v_3} + 2e^{-u_1+2u_2-u_3} + e^{-u_2+u_3+v_3-v_4} \\ &\quad + e^{2u_2-2u_3-v_2+v_3} + e^{u_1-u_2-v_1+v_2} + e^{u_3-u_4}, \\ F_2 &= e^{-v+v_1} + 3e^{-2u+3u_1+v_1-v_2} + 3e^{-u_1+3u_2-2u_3} + 3e^{-u+u_1+u_2-u_3} + e^{-3u+3u_1+v_1-v_2} \\ &\quad + 3e^{-u+3u_1-2u_2-v_1+v_2} + 3e^{u_1-u_3-v_1+v_2} + e^{3u_1-3u_2-2v_1+2v_2} + e^{-3u_1+3u_2+2v_2-2v_3} \\ &\quad + 3e^{-u+u_2+v_2-v_3} + 3e^{-2u_1+3u_2-u_3+v_2-v_3} + 2e^{-v_1+2v_2-v_3} + e^{3u_2-3u_3-v_2+v_3} + e^{v_3-v_4}, \end{aligned}$$

and n-integrals are given as

$$\begin{aligned} I_1 &= u_{xx} + \frac{1}{3}v_{xx} - u_x^2 + u_x v_x - \frac{1}{3}v_x^2, \\ I_2 &= u_{[6]} - 2u_{[5]}u_x + v_{[5]}u_x + 32u_{[4]}u_x^2 - 30u_{[4]}u_x v_x + 11u_{[4]}v_x^2 - 40u_{[4]}u_{xx} \\ &\quad + 11u_{[4]}v_{xx} + 14v_{[4]}u_x^2 - 15v_{[4]}u_x v_x + \frac{13}{3}v_{[4]}v_x^2 - 10v_{[4]}u_{xx} - \frac{13}{3}v_{[4]}v_{xx} \\ &\quad + 19u_{[3]}^2 + \frac{13}{6}v_{[3]}^2 + 16u_{[3]}v_{[3]} - 36u_{[3]}u_{xx}u_x + 18u_{[3]}u_{xx}v_x + 80u_{[3]}v_{xx}u_x \\ &\quad + -45u_{[3]}v_{xx}v_x - 52v_{[3]}u_{xx}u_x + 33v_{[3]}u_{xx}v_x - 5v_{[3]}v_{xx}u_x - 64u_{[3]}u_x^3 \\ &\quad + 102u_{[3]}u_x^2 v_x - 2u_{[3]}u_x v_x^2 + 13u_{[3]}v_x^3 + 32v_{[3]}u_x^3 - 58v_{[3]}u_x^2 v_x + 38v_{[3]}u_x v_x^2 \\ &\quad - \frac{26}{3}v_{[3]}v_x^3 + 66u_{xx}^3 + \frac{26}{3}v_{xx}^3 - 35u_{xx}^2 v_{xx} - 5u_{xx}v_{xx}^2 + 30u_{xx}^2 u_x^2 - 18u_{xx}^2 u_x v_x \\ &\quad - \frac{11}{2}u_{xx}^2 v_x^2 - 34u_{xx}v_{xx}u_x^2 + 32u_{xx}v_{xx}u_x v_x - 2u_{xx}v_{xx}v_x^2 - 2v_{xx}u_x v_x + 6u_x u_x^4 \\ &\quad - 24u_{xx}u_x^3 v_x + 25u_{xx}u_x^2 v_x^2 - 9u_{xx}u_x v_x^3 + u_{xx}v_x^4 - v_{xx}u_x^4 + 8v_{xx}u_x^3 v_x - 8v_{xx}u_x^2 v_x^2 \\ &\quad + 2v_{xx}u_x v_x^3 - 2u_x^6 + 6u_x^5 v_x - \frac{13}{2}u_x^4 v_x^2 + 3u_x^3 v_x^3 - \frac{1}{2}u_x^2 v_x^4. \end{aligned}$$

3.3.7 The Case $\mathbb{X}_6 = 0$

All possible vector fields in $\mathbb{X}_5$ are given as follows,

$$\begin{aligned}
&[P_i, [P_j, [P_k, T_3]]] \quad i, j = 1, 2, \\
&[P_i, [P_j, [P_k, R_3]]] \quad i, j = 1, 2, \\
&[P_i, [P_j, [P_k, K_3]]] \quad i, j = 1, 2.
\end{aligned}$$

Lemma 49 *We have the following identity for the vector field $V_6 \in \mathbb{X}_6$,*

$$\begin{aligned}
DV_6D^{-1} &= V_6 - (8a_{12} + 3a_{22})N_1V_5 + \alpha^4_{T_3}M_1W_5 \\
&\qquad -(4a_{12} + a_{22})\alpha^5_{T_4}N_1^2T_4 + \dots \quad ,
\end{aligned} \tag{3.51}$$

Proof: We have

$$\begin{aligned}
DV_6D^{-1} &= D[P_2, V_5]D^{-1} = [DP_2D^{-1}, DV_5D^{-1}] \\
&= \Big[P_2 - N_1B, V_5 + \varphi^5_{W_4}M_1W_4 + \varphi^5_{T_4}N_1T_4 + \dots\Big] \\
&= [P_2, V_5] + \varphi^5_{W_4}[P_2, W_4] + \varphi^5_{T_4}[P_2, T_4] \\
&\quad -[N_1B, V_5] - \varphi^5_{W_4}[N_1B, W_4] - \varphi^5_{T_4}[N_1B, T_4] + \dots \quad ,
\end{aligned}$$

Using the identities in Lemma 29 and Lemma 30 we get the desired result for V_6. □

Lemma 50 *When $\mathbb{X}_6 = 0$, the vector field $T_4 = 0$.*

Proof: Firstly, note that using Lemma 40 and the proof of Lemma 47 we get $W_5 = 0$. Since $V_6 \in \mathbb{X}_6$, it must be zero. Using $W_5 = 0$ and Lemma 40, DV_6D^{-1} takes the form,

$$DV_6D^{-1} = V_6 - (8a_{12} + 3a_{22})N_1V_5 - (4a_{12} + a_{22})\alpha^5_{T_4}N_1^2T_4 + \dots . \tag{3.52}$$

Then by Lemma 27, $DV_6D^{-1} = V_6$ must be satisfied. But since $a_{12} = -1$ and $a_{22} = 2$, the coefficient of V_5 is not zero so V_5 must be zero. Then using Lemma 46 we find $T_4 = 0$. □

Proposition 51 *For the system* (3.5)*, the linear space* $\mathbb{X}_6 = 0$ *but* $\mathbb{X}_k \neq 0$ *for* $k = 1, \dots, 5$ *is not possible.*

Proof: Since $R_6, V_6 \in \mathbb{X}_6$, they must be zero. Using Lemma 40 and Lemma 50 we get $R_2 = 0$ and $T_4 = 0$. But as in proof of Proposition 48 this two together imply $\mathbb{X}_5 = 0$ and this is a contradiction with our assumption. Hence $\mathbb{X}_6 = 0$ but $\mathbb{X}_k \neq 0$ for $k = 1, \dots, 5$ is not possible. $\square$

3.3.8 The Case $\mathbb{X}_n$ = 0 $\quad n \geq 7$

Lemma 52 *The vector field* V_n *in* $\mathbb{X}_n$ *has the form*

$$DV_nD^{-1} = V_n + \varphi^n_{V_{n-1}} N_1 V_{n-1} + \varphi^n_{W_{n-1}} M_1 W_{n-1}, \tag{3.53}$$

where

$$\varphi^n_{V_{n-1}} = \frac{1}{2}[(42 - 8n)a_{22} + (7n - n^2 - 22)a_{12}]. \tag{3.54}$$

Proof: We use induction. Using (3.51) (3.53) is true for $n = 6$. Now we suppose it is true for $n = k$ and we check the case $n = k + 1$,

$$\begin{aligned} DV_{k+1}D^{-1} &= [DP_2D^{-1}, DV_kD^{-1}] \\ &= \left[P_2 - N_1B, V_k + \varphi^n_{V_{k-1}} N_1 V_{k-1} + \varphi^k_{W_{k-1}} M_1 W_{k-1} + \dots\right] \\ &= V_{k+1} + \varphi^{k+1}_{V_k} N_1 V_k + \varphi^{k+1}_{W_k} M_1 W_k + \dots \quad , \end{aligned}$$

where

$$\begin{aligned} \varphi^{k+1}_{V_k} &= \varphi^k_{V_{k-1}} - (4a_{12} + (k-3)a_{22}), \\ \varphi^{k+1}_{W_k} &= \varphi^k_{W_{k-1}}, \\ \varphi^{k+1}_{V_{k-1}} &= \varphi^k_{V_{k-2}} - (4a_{12} + (k-3)a_{22} - 1)\varphi^k_{V_{k-1}}, \\ \varphi^{k+1}_{W_{k-1}} &= \varphi^k_{W_{k-2}} - (3a_{12} + (k-3)a_{22})\varphi^k_{W_{k-1}}, \\ \varphi^{k+1}_{K_{k-1}} &= \varphi^k_{K_{k-2}}. \end{aligned}$$

Hence (3.53) is true for $n = k + 1$. That means it is true for all $n \geq 7$. $\square$

Lemma 53 *When $n \geq 7$ if $\mathbb{X}_n = 0$, then we have*
a) $W_i = 0$, $i \geq 5$,
b) $T_4 = 0$.

Proof: Using Lemma 40, we get $R_2 = 0$ and $K_3 = 0$.
a) Lemma 47 implies $W_5 = 0$. Hence $W_i = 0$, when $i \geq 5$.
b) Since $V_n \in \mathbb{X}_6$, it must be zero. Using (3.53) with part (a) and (b), DV_nD^{-1} takes the form

$$DV_nD^{-1} = V_n + \varphi^n_{V_{n-1}} N_1 V_{n-1} + \varphi^n_{V_{n-2}} N_1^2 V_{n-2} + \dots \quad . \tag{3.55}$$

Then we use Lemma 27, so $DV_nD^{-1} = V_n$ must be satisfied. We suppose $V_{n-1} \neq 0$, but for $a_{12} = -1$ and $a_{22} = 2$ the coefficient of V_{n-1} on the right hand side of (3.55)

$$\varphi^n_{V_{n-1}} = \frac{1}{2}(n^2 - 23n + 106) \tag{3.56}$$

is nonzero for $n \in \mathbb{N}$. Hence we must have $V_{n-1} = 0$. Continuing in that way we get $V_5 = 0$ and using Lemma 46 we get $T_4 = 0$. $\square$

Proposition 54 , *For the system* (3.5)*, when $n \geq 7$ the linear space $\mathbb{X}_n = 0$ but $\mathbb{X}_k \neq 0$, $k = 1, 2, ..., (n-1)$ is not possible.*

Proof: If $\mathbb{X}_n = 0$, Lemma 40 imply $R_2 = 0$ and from Lemma 53 we get $T_4 = 0$. But in that case from the proof of Proposition 48, it follows that $\mathbb{X}_5 = 0$ and this is a contradiction with our assumption.
Hence for the system (3.6) when $n \geq 7$, $\mathbb{X}_n = 0$ but $\mathbb{X}_k \neq 0$, $k = 1, 2, ..., (n-1)$ is not possible. $\square$

As a result, Proposition 32, Proposition 34, Proposition 37, Proposition 44, Proposition 48, Proposition 51 and Proposition 54 imply that, if characteristic x-ring is finite dimensional, then A should be a Cartan matrix in the form (3.2). In [41], when A is a Cartan matrix in the form (3.2) two functionally independent nontrivial n-integrals of the corresponding system were constructed. Hence we can conclude that if the system (3.5) is Darboux integrable, then A must be a Cartan matrix and that completes the proof of Theorem 24.

CHAPTER 4

DARBOUX INTEGRABLE EQUATIONS WITH SMALL DIMENSION OF CHARACTERISTIC RINGS

In this chapter, we consider some classification problems for the chains of the form

$$t_{1x} = f(x, t, t_1, t_x). \tag{4.1}$$

In particular we consider chains with x- and n-ring of small dimension.

4.1 Characteristic x-ring

For semi-discrete chains of type (4.1), x-ring is generated as described in Chapter 2 by the vector fields,

$$\begin{aligned} D_0 &= \frac{\partial}{\partial x} + t_x\frac{\partial}{\partial t} + f\frac{\partial}{\partial t_1} + g\frac{\partial}{\partial t_{-1}} + D(f)\frac{\partial}{\partial t_2} + D^{-1}(g)\frac{\partial}{\partial t_{-2}} + \dots \\ D_1 &= \frac{\partial}{\partial t_x}. \end{aligned}$$

where $g = t_{-1,x}$. To be able to construct characteristic x-ring, we consider the commutators of D_0 and D_1. Let us define sequences of vector fields

$$C_1 = [D_0, D_1] \quad \text{and} \quad C_n = [X, C_{n-1}] \quad n = 2, 3, \dots,$$

and

$$Z_1 = [D_0, C_1] \quad \text{and} \quad Z_n = [D_0, Z_{n-1}] \quad n = 2, 3, \dots .$$

This vector fields have form [28], [43]

$$\begin{aligned} C_1 &= \frac{\partial}{\partial t} + f_{t_x}\frac{\partial}{\partial t_1} + g_{t_x}\frac{\partial}{\partial t_{-1}} + (f_1)_{t_x}\frac{\partial}{\partial t_2} + (g_{-1})_{t_x}\frac{\partial}{\partial t_{-2}}\dots \\ C_2 &= f_{t_xt_x}\frac{\partial}{\partial t_1} + g_{t_xt_x}\frac{\partial}{\partial t_{-1}} + (f_1)_{t_xt_x}\frac{\partial}{\partial t_2} + (g_{-1})_{t_xt_x}\frac{\partial}{\partial t_{-2}}\dots \\ Z_1 &= (f_{t_{xx}} + t_x f_{t_xt} + f f_{t_xt_1} - f_t - f_{t_x}f_{t_1})\frac{\partial}{\partial t_1} + \\ &\quad + (g_{t_{xx}} + t_x g_{t_xt} + g g_{t_xt_1} - g_t - g_{t_x}g_{t_1})\frac{\partial}{\partial t_{-1}} + \dots \quad . \end{aligned}$$

In general we have

$$C_n = \sum_{i=1}^{\infty}\Big((D_1)^n(f_{i-1})\frac{\partial}{\partial t_i} + (D_1)^n(g_{-i+1})\frac{\partial}{\partial t_{-i}}\Big),$$

where $f_0 := f$ and $g_0 := g$. In what follows we use some commutator relations between vector fields.

Lemma 55 *(See [28]) The vector field D_0 satisfies a commutation relation*

$$DD_0 = \frac{1}{f_{t_x}}D_0D. \tag{4.2}$$

Assumption that characteristic x-ring has a small dimension puts restrictions on the function f. Such restrictions were found in [28] and [43].

Lemma 56 *(See [28]) For the chain* (4.1)*, dimension of the characteristic x-ring is three if and only if the function f satisfies*

$$f_{t_xt_x} = 0 \tag{4.3}$$

and

$$-\frac{t_x f_{t_xt}}{f_{t_x}^2} - \frac{f f_{t_xt_1}}{f_{t_x}^2} + \frac{f_t}{f_{t_x}^2} + \frac{f_{t_1}}{f_{t_x}} = 0. \tag{4.4}$$

In this case, characteristic x-ring is generated by the vector fields D_0, D_1 and C_1, which form the following commutator table

L_x	D_0	D_1	C_1
D_0	*0*	C_1	0
D_1	$-C_1$	*0*	*0*
C_1	0	*0*	*0*

When dimension of the characteristic x-ring is 4, we have to consider two different cases, $f_{t_xt_x} \neq 0$ and $f_{t_xt_x} = 0$.

Lemma 57 *(See [43]) Suppose $f_{t_xt_x} \neq 0$ then the chain* (4.1) *has characteristic x-ring of dimension four if and only if the following conditions hold*

$$D\left(\frac{f_{t_xt_xt_x}}{f_{t_xt_x}}\right) = \frac{f_{t_xt_xt_x}f_{t_x} - 3f_{t_xt_x}^2}{f_{t_xt_x}f_{t_x}^2}, \tag{4.5}$$

$$\begin{aligned} &D\left(\frac{f_{xt_x} + t_xf_{t_xt} + ff_{t_xt_1} - f_t - f_{t_x}f_{t_1}}{f_{t_xt_x}}\right) = \\ &\frac{f_{xt_x} + t_xf_{t_xt} + ff_{t_xt_1} - f_t - f_{t_x}f_{t_1}}{f_{t_xt_x}}f_{t_x} - (f_x + t_xf_t + f_{t_1}.f). \end{aligned} \tag{4.6}$$

The characteristic x-ring is generated by the vector fields D_0, D_1, C_1, C_2 and we have the following commutator table:

L_x	D_0	D_1	C_1	C_2
D_0	0	C_1	C_2	μC_2
D_1	$-C_1$	0	λC_2	ρC_2
C_1	$-C_2$	$-\lambda C_2$	0	ηC_2
C_2	$-\mu C_2$	$-\rho C_2$	ηC_2	0

where $\rho = \lambda\mu + D_0(\lambda)$ and $\eta = D_0(\rho) - D_1(\mu)$.

Lemma 58 *(See [43]) Supose $f_{t_xt_x} = 0$ then the chain* (4.1) *has the characteristic x-ring of dimension four if and only if the following condition hold*

$$D\left(\frac{K(m)}{m} - m + \frac{f_t}{f_{t_x}}\right) = \frac{K(m)}{m} + m - f_{t_1}, \tag{4.7}$$

where K is the following vector field

$$K = \frac{\partial}{\partial x} + t_x\frac{\partial}{\partial t} + f\frac{\partial}{\partial t_1} + \dots,$$

and $m = \dfrac{-(f_{xt_x} + t_xf_{t_xt} + ff_{t_xt_1}) + f_t + f_{t_x}f_{t_1}}{f_{t_x}}$. In this case, characteristic x-ring is generated by the vector fields D_0, D_1, C_1, Z_1 and we have the following commutator table

L_x	D_0	D_1	C_1	Z_1
D_0	0	C_1	0	0
D_1	$-C_1$	0	Z_1	αZ_1
C_1	0	$-Z_1$	0	$D_0(\alpha)Z_1$
Z_1	0	αZ_1	$-D_0(\alpha)Z_1$	0

where $Z_2 = \alpha Z_1$.

When characteristic x-ring has dimension 4, one can obtain further restrictions on $f(x, t, t_1, t_x)$.

Lemma 59 *(See [43]) Let the chain* (4.1) *have four dimensional characteristic x-ring. Then f has the following form*

$$f = M(x, t, t_x)A(x, t, t_1) + t_x B(x, t, t_1) + C(x, t, t_1), \tag{4.8}$$

where M, A, B and C are some functions.

Lemma 60 *(See [43]) Let the chain* (4.1) *have four dimensional characteristic x-ring and $f_{t_x t_x} \neq 0$. Then*

$$Df = -H_1(x, t, t_1, t_2)t_x + H_2(x, t, t_1, t_2)f + H_3(x, t, t_1, t_2), \tag{4.9}$$

where H_1, H_2 and H_3 are some functions.

4.2 Characteristic n-ring

For semi-discrete chains of type (4.1) the characteristic n-ring is generated, as described in Chapter 2, by the vector fields,

$$\begin{aligned} &Y_0 = \frac{\partial}{\partial t_1}, \qquad Y_{-0} = \frac{\partial}{\partial t_{-1}}, \\ &Y_j = D^{-j}\frac{\partial}{\partial t_1}D^j, \quad j = 1, 2, \ldots \quad , \\ &Y_{-j} = D^j\frac{\partial}{\partial t_{-1}}D^{-j}, \quad j = 1, 2, \ldots \quad , \\ &X_j = \frac{\partial}{\partial t_{-j}}, \quad j = 1, 2, \ldots \quad . \end{aligned}$$

When dimension of characteristic n-ring is 2 the restriction on the function f are obtained in [28]. We have the following condition.

Lemma 61 *(See [28]) The dimension of the characteristic n-ring is 2 if and only if f satisfied the following condition*

$$D\left(\frac{f_t}{f_{t_x}}\right) = -f_{t_1}. \tag{4.10}$$

In this case, the characteristic n-ring is generated by the vector fields X_1 and Y_1 and this vector fields commute.

4.3 Classification of Semi-Discrete Hyperbolic Type Equations Admitting Characteristic x-rings and n-rings of Small Dimensions

The case of three dimensional characteristic x-ring and two dimensional characteristic n-ring is considered in [28] and it is found that the chain (4.1) must have the form,

$$t_{1x} = t_x + t_1 - t.$$

In this section we want to classify all chains (4.1) such that characteristic x-ring is four dimensional and characteristic n-ring is two dimensional. To make this classification, we use Lemma 57, Lemma 58 and Lemma 61. Firstly we remark that using equality (4.10) in Lemma 61 we get

$$\frac{\partial}{\partial t_1}\left(\frac{f_t}{f_{t_x}}\right) = 0 \tag{4.11}$$

since f_{t_1} does not depend on t_2. In the same way, using the identities (4.6) and (4.7) we find

$$\frac{\partial}{\partial t_1} m = 0, \qquad \frac{\partial}{\partial t_1}\tilde{m} = 0. \tag{4.12}$$

where $\tilde{m} = \frac{f_{xt_x} + t_x f_{t_x t} + f f_{t_x t_1} - f_t - f_{t_x} f_{t_1}}{f_{t_x t_x}}$.

Note that if the chain (4.1) has four dimensional characteristic x-ring then by Lemma 59 the function f has form

$$f = A(x,t,t_1)M(x,t,t_x) + B(x,t,t_1)t_x + C(x,t,t_1).$$

For simplicity we assume that the function M depends only on t_x and f does not depend on x explicitly. That is we study equations of the form

$$t_{1x} = A(t,t_1)M(t_x) + B(t,t_1)t_x + C(t,t_1). \tag{4.13}$$

We have to consider two cases: $f_{t_x t_x} = 0$ and $f_{t_x t_x} \neq 0$.

4.3.1 f is Linear With Respect to t_x $\quad (f_{t_x t_x} = 0)$

Since f is linear in t_x we have

$$f(t, t_1, t_x) = A(t, t_1)t_x + B(t, t_1) \tag{4.14}$$

and equation (4.1) becomes

$$t_{1x} = A(t, t_1)t_x + B(t, t_1). \tag{4.15}$$

Lemma 62 *For the chain* (4.15) *the dimension of the characteristic n-ring is two if and only if function f has the form*

$$f(t, t_1, t_x) = \frac{c_1\gamma(t)}{\gamma(t_1)}t_x + \frac{c_2}{\gamma(t_1)} - \frac{c_1\gamma(t)\sigma(t)}{\gamma(t_1)} + \sigma(t_1), \tag{4.16}$$

where γ and σ are some functions of the given variables and c_1, c_2 are some constants.

Proof: If characteristic n-ring is two dimensional, then f must satisfy the condition (4.10) in Lemma 61

$$D\left(\frac{f_t}{f_{t_x}}\right) = -f_{t_1}. \tag{4.17}$$

Then using (4.11) we get that

$$\frac{f_t}{f_{t_x}} = \frac{A_t}{A}t_x + \frac{B_t}{A}. \tag{4.18}$$

which does not depend on t_1. Thus $\frac{A_t}{A}t_x$ and $\frac{B_t}{A}$ does not depend on t_1. Hence $\frac{A_t}{A} = \alpha(t)$ for some function $\alpha(t)$ and we find,

$$A(t, t_1) = \gamma(t)\varphi(t_1)$$

for some functions γ and φ. Then,

$$\frac{B_t(t, t_1)}{A(t, t_1)} = \frac{B_t(t, t_1)}{\gamma(t)\varphi(t_1)} = \tilde{\alpha}(t)$$

and we find

$$B(t, t_1) = l(t)\varphi(t_1) + \sigma(t_1).$$

Then (4.15) and (4.18) reduce to

$$f(t, t_1, t_x) = \gamma(t)\varphi(t_1)t_x + l(t)\varphi(t_1) + \sigma(t_1)$$

and

$$\frac{f_t}{f_{t_x}} = \frac{\gamma'(t)}{\gamma(t)}t_x + \frac{l'(t)}{\gamma(t)}. \tag{4.19}$$

Using (4.19) and (4.17) we obtain

$$\frac{\gamma'(t_1)}{\gamma(t_1)}(\gamma(t)\varphi(t_1)t_x + l(t)\varphi(t_1) + \sigma(t_1)) + \frac{l'(t_1)}{\gamma(t_1)} = \gamma(t)\varphi'(t_1)t_x + l(t)\varphi'(t_1) + \sigma'(t_1). \tag{4.20}$$

By comparing the coefficients of t_x in (4.20) we get

$$\frac{\gamma'(t_1)\gamma(t)\varphi(t_1)}{\gamma(t_1)} = -\gamma(t)\varphi'(t_1) \quad \Longleftrightarrow \quad \frac{\gamma'(t_1)}{\gamma(t)} + \frac{\varphi'(t_1)}{\varphi(t)} = 0.$$

Then we find,

$$\varphi(t_1) = \frac{c_1}{\gamma(t_1)} \qquad \text{where } c_1 \text{ is a constant.} \tag{4.21}$$

Using (4.21) in (4.20) we obtain

$$\frac{\gamma'(t_1)}{\gamma(t_1)}\sigma(t_1) + \frac{l'(t_1)}{\gamma(t_1)} = -\sigma'(t_1) \quad \Longleftrightarrow \quad \gamma'(t_1)\sigma(t_1) + \gamma(t_1)\sigma'(t_1) + l'(t_1) = 0. \tag{4.22}$$

Solving (4.22), we find

$$l(t) = -\gamma(t)\sigma(t) + \tilde{c_2}, \quad \text{where } \tilde{c_2} \text{ is a constant.} \tag{4.23}$$

Substituting φ and l into (4.19) we get equation (4.16) and condition (4.10) is satisfied for this f.

$$t_{1_x} = c_1\frac{\gamma(t)}{\gamma(t_1)}t_x - c_1\frac{\gamma(t)}{\gamma(t_1)}\sigma(t) + c_2\frac{1}{\gamma(t_1)} + \sigma(t_1), \quad c_1, c_2 : \text{constants} \tag{4.24}$$

where γ and σ are some functions of one variable. □

Let us introduce a new variable τ

$$\tau = L(t),$$

such that $L'(t) = \gamma(t)$. Then (4.24) becomes

$$\tau_{1_x} = c_1\tau_x + c_2 - c_1Q(\tau) + Q(\tau_1) \tag{4.25}$$

where Q is function of one variable. For the above equation, it can be shown that (4.10) is satisfied. Hence for the chain (4.25) dimension of characteristic n-ring is still remains two.

For the chain (4.25) characteristic x-ring must be four dimensional. Therefore, we check the condition (4.7) and we have the following lemma.

Lemma 63 *The equation* (4.25) *has four dimensional characteristic x-ring if and only if*

$$Q(\tau) = A_1\tau^2 + A_2\tau \quad or \quad Q(\tau) = A_1e^{\alpha\tau} + A_2e^{-\alpha\tau} \tag{4.26}$$

for some constants A_1, A_2 and α.

Proof: In this case we take the function $f = c_1\tau_x + c_2 - c_1Q(\tau) + Q(\tau_1)$ and we check (4.7) for f. We get

$$\frac{c_1\tau_x + c_2 - c_1Q(\tau) + Q(\tau_1)}{Q'(\tau_2) - Q'(\tau_1)}(c_1Q''(\tau_2) - Q''(\tau_1)) +$$
$$+ \frac{Q''(\tau_2)(c_2 - c_1Q(\tau_1) + Q(\tau_2))}{Q'(\tau_2) - Q'(\tau_1)} - Q'(\tau_2) + Q'(\tau_1) =$$
$$\frac{c_1Q''(\tau_1) - Q''(\tau)}{Q'(\tau_1) - Q'(\tau)}\tau_x + \frac{Q''(\tau_1)(c_2 - c_1Q(\tau) + Q(\tau_1))}{Q'(\tau_1) - Q'(\tau)} + Q'(\tau_1) - Q'(\tau).$$

By comparing coefficients of τ_x in the above equality we get

$$c_1D\left(\frac{c_1Q''(\tau_1) - Q''(\tau)}{Q'(\tau_1) - Q'(\tau)}\right) = \frac{c_1Q''(\tau_1) - Q''(\tau)}{Q'(\tau_1) - Q'(\tau)}.$$

Let $F(t, t_1) = \dfrac{c_1Q''(t_1) - Q''(t)}{Q'(t_1) - Q'(t)}$, then (4.27) reduces to,

$$c_1D(F(t, t_1)) = F(t, t_1). \tag{4.27}$$

Since right hand side of (4.27) does not depend on t_2, left hand side of (4.27) also does not depend on t_2. That means F does not depend on t_1. Hence we get, $c_1D(F(t)) = F(t)$. Since F does not depend on t_1, $D(F(t))$ does not depend on t_1. Then, F does not depend on t. Hence $c_1 = 1$ or $F = 0$. When $F = 0$, $c_1Q''(\tau_1) - Q''(\tau) = 0$. Again this implies $c_1 = 1$. Therefore, the system (4.25) reduces to,

$$\tau_{1x} = \tau_x + d(\tau, \tau_1). \tag{4.28}$$

in which case complete list of Darboux integrable chains is given at [46] (also, their x and n characteristic rings). By checking the list of Darboux integrable equations of this form at [46] we conclude that Q should be in the form of (4.26). $\square$

Returning the variable t and using each form of Q given by (4.26), we have the following theorem.

Theorem 64 *Let f be a linear function with respect to t_x. The equation* (4.15) *has four dimensional characteristic x-ring and two dimensional characteristic n-ring if and only if*

$$f = \frac{\gamma(t)}{\gamma(t_1)} t_x - \frac{\gamma(t)}{\gamma(t_1)} \sigma(t) + \sigma(t_1),$$

where functions γ and σ satisfy one of the following relations

$$(\gamma(t)\sigma(t))' = \gamma(t)\sqrt{B_1 + B_2 \gamma(t)\sigma(t)} \tag{4.29}$$

or

$$(\gamma(t)\sigma(t))' = \gamma(t)\sqrt{B_1 + B_2 (\gamma(t)\sigma(t))^2} \tag{4.30}$$

with B_1, B_2 being arbitrary constants.

Proof: Definition of τ implies $Q(\tau) = \sigma(t)\gamma(t)$ and Lemma 63 implies Q has the following forms

$$Q(\tau) = A_1\tau^2 + A_2\tau \quad \text{or} \quad Q(\tau) = A_1 e^{\alpha\tau} + A_2 e^{-\alpha\tau}.$$

In the first case, $Q'(\tau) = 2A_1\tau + A_2$ which implies,

$$(Q'(\tau))^2 = 4A_1(A_1\tau^2 + A_2\tau) + A_2^2 = 4A_1 Q(\tau) + A_2^2.$$

Since $(Q'(\tau))^2 = ((\gamma(t)\sigma(t))')^2$ we get (4.29). Equality (4.30) follows similarly. $\square$

4.3.2 f is Nonlinear With Respect to t_x $\quad$ ($f_{t_x t_x} \neq 0$)

We suppose that f is nonlinear in t_x and

$$f(t, t_1, t_x) = A(t, t_1)M(t_x) + B(t, t_1)t_x + C(t, t_1) \tag{4.31}$$

so equation (4.1) becomes

$$t_{1x} = A(t, t_1)M(t_x) + B(t, t_1)t_x + C(t, t_1). \tag{4.32}$$

Using that characteristic n-ring is two dimensional, we have the following lemma.

Lemma 65 *Let equation* (4.32) *has characteristic n-ring of dimension two and ($M_{t_x} \neq constant$) be a non-linear function, then M satisfies*

$$M' = -\frac{\alpha_2 M + \alpha_4 t_x + \alpha_6}{\alpha_1 M + \alpha_3 t_x + \alpha_5}. \tag{4.33}$$

where $\alpha_1 M + \alpha_3 t_x + \alpha_5 \neq 0$.

Proof: If characteristic n-ring is two dimensional, then f must satisfy the condition (4.11). Using (4.31) for f we obtain

$$\left(\frac{A_t M + t_x B_t + C_t}{AM' + B}\right)_{t_1} =$$
$$\frac{(A_{tt_1} M + t_x B_{tt_1} + C_{tt_1})(AM' + B) - (A_{t_1} M' + B_{t_1})(A_t M + t_x B_t + C_t)}{(AM' + B)^2} = 0.$$

Rearranging terms in above expression we obtain

$$M'(\alpha_1 M + \alpha_3 t_x + \alpha_5) = -(\alpha_2 M + \alpha_4 t_x + \alpha_6), \tag{4.34}$$

where the constants α_i, $i = 1, 2, \ldots, 5$ are given by,

$$\alpha_1 = A_{tt_1} A - A_{t_1} A_t,$$
$$\alpha_2 = A_{tt_1} B - B_{t_1} A_t,$$
$$\alpha_3 = B_{tt_1} A - B_t A_{t_1},$$
$$\alpha_4 = B_{tt_1} B - B_{t_1} B_t,$$
$$\alpha_5 = C_{tt_1} A - A_{t_1} C_t.$$

If $\alpha_1 M + \alpha_3 t_x + \alpha_5 = 0$, then M is linear in t_x, which implies we are in the case $f_{t_x t_x} = 0$. Hence we assume $\alpha_1 M + \alpha_3 t_x + \alpha_5 \neq 0$ and we obtain the result,

$$M' = -\frac{\alpha_2 M + \alpha_4 t_x + \alpha_6}{\alpha_1 M + \alpha_3 t_x + \alpha_5}. \tag{4.35}$$

$\square$

Using (4.35) we can write M' in terms of M. The next lemma enables us to write DM in terms of M.

Lemma 66 *If characteristic x-ring is four dimensional and $f_{t_x t_x} \neq 0$, then M satisfies*

$$DM = Q_1(t, t_1, t_2) M + Q_2(t, t_1, t_2) t_x + Q_3(t, t_1, t_2), \tag{4.36}$$

for some functions Q_1, Q_2 and Q_3.

Proof: Since we assume that the dimension of characteristic x-ring is four and $f_{t_x t_x} \neq 0$, by Lemma 60 we have

$$Df = -H_1(t, t_1, t_2)t_x + H_2(t, t_1, t_2)f + H_3(t, t_1, t_2),$$

where H_1, H_2 and H_3 are some functions. Thus,

$$D(AM + Bt_x + C) = -H_1 t_x + H_2(AM + Bt_x + \widetilde{C}). \tag{4.37}$$

We can write (4.37) in the form,

$$DM = Q_1(t, t_1, t_2)M + Q_2(t, t_1, t_2)t_x + Q_3(t, t_1, t_2). \tag{4.38}$$

where Q_1, Q_2 and Q_3 are some functions. □

The expressions (4.35) and (4.38) enable us to prove the following lemma.

Lemma 67 *Let equation* (4.32) *have characteristic* n*-ring of dimension two then either* $M = \dfrac{1}{t_x + P}$ *or* $M = \sqrt{t_x^2 + Pt_x + Q}$ *or* $M = t_x^2$.

Proof: Note that by (4.2) we have the identity $DD_0 = \dfrac{1}{f_{t_x}} D_0 D$, where $D_0 = \dfrac{\partial}{\partial t_x}$. Therefore

$$\begin{aligned}
DD_0 M &= \frac{1}{f_{t_x}} X D_0 M, \\
DM' &= \frac{1}{AM' + B}\frac{\partial}{\partial t_x} M, \\
-D\left(\frac{\alpha_2 M + \alpha_4 y + \alpha_6}{\alpha_1 M + \alpha_3 y + \alpha_5}\right) &= \frac{1}{AM' + B}(Q_1 M + Q_2 t_x + Q_3).
\end{aligned} \tag{4.39}$$

We use (4.35) and (4.38) and obtain

$$-\frac{\tilde{\alpha_2}(Q_1 M + Q_2 t_x + Q_3) + \tilde{\alpha_4}(AM + Bt_x + C) + \tilde{\alpha_6}}{\tilde{\alpha_1}(Q_1 M + Q_2 t_x + Q_3) + \tilde{\alpha_3}(AM + Bt_x + C) + \tilde{\alpha_5}} = \frac{Q_2 - Q_1 \dfrac{\alpha_2 M + \alpha_4 t_x + \alpha_6}{\alpha_1 M + \alpha_3 t_x + \alpha_5}}{B - A\dfrac{\alpha_2 M + \alpha_4 t_x + \alpha_6}{\alpha_1 M + \alpha_3 t_x + \alpha_5}}$$

where $D\alpha_k = \tilde{\alpha_k}$. Then

$$\frac{(\tilde{\alpha_2} Q_1 + \tilde{\alpha_4} A)M + (\tilde{\alpha_2} Q_2 + \tilde{\alpha_4} B)t_x + (\tilde{\alpha_2} Q_3 + \tilde{\alpha_4} C + \tilde{\alpha_6})}{(\tilde{\alpha_1} Q_1 + \tilde{\alpha_3} A)M + (\tilde{\alpha_1} Q_2 + \tilde{\alpha_3} B)t_x + (\tilde{\alpha_1} Q_3 + \tilde{\alpha_3} C + \tilde{\alpha_5})} = \frac{(Q_1\alpha_2 - Q_2\alpha_1)M + (Q_1\alpha_4 - Q_2\alpha_3)t_x + (Q_1\alpha_6 - Q_2\alpha_5)}{(A\alpha_2 - B\alpha_1)M + (A\alpha_4 - B\alpha_3)t_x + (A\alpha_6 - B\alpha_5)}.$$

The above equality can be reduced to

$$R_1 M^2 - (R_2 t_x + R_3)M + (R_4 t_x^2 + R_5 t_x + R_6) = 0, \tag{4.40}$$

where R_i, $i = 1, 2, \ldots, 6$ are some functions. In the case $R_1 = 0$, from (4.40) we find that

$$M = \frac{R_4 t_x^2 + R_5 t_x + R_6}{R_2 t_x + R_3}. \tag{4.41}$$

In the case $R_1 \neq 0$ we find

$$M = \frac{(R_2 t_x + R_3) \pm \sqrt{(R_2 t_x + R_3)^2 - 4R_1(R_4 t_x^2 + R_5 t_x + R_6)}}{2R_1}. \tag{4.42}$$

Substituting M given by (4.41) into $f = AM + Bt_x + C$ we can redefine M, B, C so that

$$M = t_x^2 \quad \text{or} \quad M = \frac{1}{t_x + P}.$$

For M given by (4.42) we can also redefine M, B, C so that $M = \sqrt{t_x^2 + Pt_x + Q}$.
□

Now we consider each case of M obtained in the above lemma separately.

4.3.2.1 Case 1 : $M = t_x^2$

Lemma 68 *The equation* (4.32) *cannot have four dimensional characteristic x-ring if $M = t_x^2$.*

Proof: In this case f takes the form

$$f = A(t, t_1)t_x^2 + B(t, t_1)t_x + C(t, t_1).$$

It can be checked that the condition (4.5) is not satisfied. Hence equation (4.32) cannot have 4 dimensional characteristic x-ring if $M = t_x^2$. □

4.3.2.2 Case 2 : $M = \dfrac{1}{t_x + P}$

In this case we have the following lemma.

Lemma 69 *Let* $M = \dfrac{1}{t_x + P}$ *and equation* (4.32) *has four dimensional characteristic* x*-ring and two dimensional characteristic* n*-ring. Then chain* (4.32) *is either*

$$t_{1x} = \frac{c^* \eta(t)\eta(t_1)}{t_x} \quad \textit{or} \quad t_{1x} = \frac{c^* e^{c^{**}(t+t_1)}}{t_x + P} - P, \tag{4.43}$$

where c^* *and* c^{**} *are some constants.*

Proof: In this case f takes the form

$$f(t, t_1, t_x) = \frac{A(t, t_1)}{t_x + P} + B(t, t_1)t_x + C(t, t_1), \tag{4.44}$$

with

$$\begin{aligned} f_{t_x} &= \frac{B(t_x + P)^2 - A}{(t_x + P)^2}, \\ f_{t_x t_x} &= \frac{2A}{(t_x + P)^3}, \\ f_{t_x t_x t_x} &= -\frac{6A}{(t_x + P)^4}. \end{aligned}$$

By checking condition (4.5) for f we obtain

$$\frac{3(t_x + P)}{B(t_x + P)^2 + (C + P - BP)(t_x + P) + A} = \frac{3(t_x + P)(B(t_x + P)^2 + A)}{(B(t_x + P)^2 - A)^2}.$$

This equality reduces to

$$B(C + P - BP)(t_x + P)^3 + 4AB(t_x + P)^2 + A(C + P - BP)(t_x + P) = 0.$$

For this equality by checking the coefficients of $(t_x + P)^k$, k =1,2,3 we find

$$(t_x + P)^3 \;:\quad B(C + P - BP) = 0, \tag{4.45}$$

$$(t_x + P)^2 \;:\quad 4AB = 0, \tag{4.46}$$

$$(t_x + P) \;:\quad A(C + P - BP) = 0. \tag{4.47}$$

Firstly, note that $A(t, t_1) \neq 0$. Otherwise $f_{t_x t_x} = 0$. By solving (4.45), (4.46) and (4.47) together we find that

$$\begin{aligned} B &= 0, \\ C &= -P. \end{aligned}$$

Thus (4.44) takes the form

$$f(t, t_1, t_x) = \frac{A(t, t_1)}{t_x + P} - P. \tag{4.48}$$

Using the condition (4.10) we obtain

$$\frac{A_{t_1}(t_1,t_2)A(t,t_1)}{A(t_1,t_2)(t_x+P)} = \frac{A_{t_1}(t,t_1)}{t_x+P}$$

or

$$\frac{A_{t_1}(t_1,t_2)}{A(t_1,t_2)} = \frac{A_{t_1}(t,t_1)}{A(t,t_1)}. \tag{4.49}$$

Then using (4.49) $\dfrac{A_{t_1}(t_1,t_2)}{A(t_1,t_2)}$ does not depend on t_2, so we have $\dfrac{\partial}{\partial t_2}\dfrac{\partial}{\partial t_1}\ln A(t_1,t_2) = 0$. Also since $\dfrac{A_{t_1}(t,t_1)}{A(t,t_1)}$ does not depend on t we have $\dfrac{\partial}{\partial t}\dfrac{\partial}{\partial t_1}\ln A(t,t_1) = 0$. Hence we get

$$A(t,t_1) = \varphi(t)\eta(t_1).$$

Substituting A into (4.49) we obtain

$$\frac{\varphi'(t_1)}{\varphi(t_1)} = \frac{\eta'(t_1)}{\eta(t_1)} \Longrightarrow \frac{d}{dt_1}\left(\ln\frac{\varphi(t_1)}{\eta(t_1)}\right) = 0 \Longrightarrow \varphi(t_1) = c^*\eta(t_1).$$

Then we have

$$f(t,t_1,t_x) = \frac{c^*\eta(t)\eta(t_1)}{t_x+P} - P, \quad \text{where } P \text{ is a constant.}$$

Using condition (4.12)

$$\tilde{m} = \left(-\frac{P\eta'(t)}{2\eta(t)} + \frac{P\eta'(t_1)}{2\eta(t_1)}\right)(t_x+P) - \frac{(t_x+P)^2\eta'(t)}{2\eta(t)}$$

and it should not depend on t_1. This is possible in two cases

$$\begin{aligned} \text{Case 1} \quad &: \quad P = 0, \\ \text{Case 2} \quad &: \quad \eta'(t_1) = c^{**}\eta(t_1) \Longrightarrow \eta(t_1) = e^{c^{**}t_1}. \end{aligned}$$

For each of these cases condition (4.12) is satisfied and we have

$$\begin{aligned} \text{Case 1} \quad &: \quad f(t,t_1,t_x) = \frac{c^*\eta(t)\eta(t_1)}{t_x}, \\ \text{Case 2} \quad &: \quad f(t,t_1,t_x) = \frac{c^*e^{c^{**}(t+t_1)}}{t_x+P} - P, \qquad c^*, c^{**}: \ \text{constants.} \end{aligned}$$

It can be checked that the chains with such functions f have 2 dimensional characteristic n-ring and 4 dimensional characteristic x-ring. $\square$

4.3.2.3 Case 3 : $M = \sqrt{t_x^2 + Pt_x + Q}$

For this case we have the following lemma.

Lemma 70 *Let $M = \sqrt{t_x^2 + Pt_x + Q}$ and equation* (4.32) *has four dimensional characteristic x-ring and two dimensional characteristic n-ring. Then equation* (4.32) *takes form*

$$t_{1x} = \sqrt{B^2 - 1}\sqrt{t_x^2 + Pt_x + Q} + Bt_x + \frac{P}{2}(B - 1). \tag{4.50}$$

where B, Q, P are constants and $B^2 \neq 1$.

Proof: In this case f takes the form

$$f = A(t, t_1)\sqrt{t_x^2 + Pt_x + Q} + B(t, t_1)t_x + C(t, t_1). \tag{4.51}$$

Applying condition (4.5) to f we find,

$$-\frac{6BP + 12t_x + 12A\sqrt{t_x^2 + t_xP + Q}}{\left(AP + 2At_x + 2B\sqrt{t_x^2 + t_xP + Q}\right)^2} = \frac{-3P - 6C - 6Bt_x - 6A\sqrt{t_x^2 + Pt_x + Q}}{2Q + 2\left(C + Bt_x + A\sqrt{t_x^2 + Pt_x + Q}\right)\left(C + P + Bt_x + A\sqrt{t_x^2 + Pt_x + Q}\right)},$$

or

$$4\left(Q + \left(C + Bt_x + A\sqrt{t_x^2 + Pt_x + Q}\right)\left(C + P + Bt_x + A\sqrt{t_x^2 + Pt_x + Q}\right)\right) \times \times \left(BP + 2t_x + 2A\sqrt{t_x^2 + t_xP + Q}\right) = \left(AP + 2At_x + 2B\sqrt{t_x^2 + t_xP + Q}\right)^2\left(P + 2C + 2Bt_x + 2A\sqrt{t_x^2 + Pt_x + Q}\right)$$

Then by comparing coefficients of $\left(\sqrt{t_x^2 + Pt_x + Q}\right)^i \left(t_x\right)^j$, for $i, j = 0, 1, 2$, we find

$$AB(2C + P - BP) = 0, \tag{4.52}$$

$$A(-4C^2 - 4CP + A^2(P^2 - 4Q) + 4(B^2 - 1)Q) = 0, \tag{4.53}$$

$$(A^2 + B^2)(-2C + BP - P) = 0, \tag{4.54}$$

$$-2A^2P(2C + P) + 4B^3Q - -B(4C^2 + 4CP + 4Q + A^2(-3P^2 + 4Q)) = 0, \tag{4.55}$$

$$A^2(2C + P)(P^2 - 8Q) + +4B^2(2C + P)Q - 4BP(C^2 + CP + Q - A^2Q) = 0. \tag{4.56}$$

Equalities (4.52) and (4.54) are satisfied if

$$2C = PB - P. \tag{4.57}$$

Equality (4.53) is satisfied if

$$4C^2 + 4CP = A^2(P^2 - 4Q) - 4Q + 4B^2Q, \tag{4.58}$$

equalities (4.57) and (4.58) are satisfied in two cases

$$\begin{aligned}
&\text{Case 1} \quad : \qquad 2C = PB - P \ \text{ and } \ A^2 + 1 = B^2, \\
&\text{Case 2} \quad : \qquad 2C = PB - P \ \text{ and } \ P^2 = 4Q.
\end{aligned}$$

In the first case, M reduces to $M = t_x + \frac{P}{2}$ which is linear with respect to t_x. Therefore, we study only the Case 2. Thus we have

$$f = \sqrt{B(t,t_1)^2 - 1}\sqrt{t_x^2 + Pt_x + Q} + B(t,t_1)t_x + \frac{P}{2}(B(t,t_1) - 1), \tag{4.59}$$

where $B \neq \pm 1$. In the same way we check that condition (4.11) in the form

$$\left(D\frac{f_t}{f_{t_x}}\right)^2 - (f_{t_1})^2 = 0. \tag{4.60}$$

We find

$$\begin{aligned}
D\left(\frac{f_t}{f_{t_x}}\right) &= \frac{B_{t_1}(t_1,t_2)}{2\sqrt{B(t_1,t_2)^2 - 1}}\Bigg((P^2 + 4Q + 8Pt_x + 8t_x^2)B^2(t,t_1) - \\
&\quad -(P + 2t_x)^2 + 4(P + 2t_x)\sqrt{Q + t_x(P + t_x)}B(t,t_1)\sqrt{B^2(t,t_1)}\Bigg)^{\frac{1}{2}}, \\
f_{t_1} &= -\frac{1}{2}\left(P + 2t_x + \frac{2\sqrt{Q + t_x(P + t_x)}B(t,t_1)}{\sqrt{B^2(t,t_1) - 1}}\right)B_{t_1}(t,t_1).
\end{aligned}$$

Using the identities above, we check the coefficients of the terms,

$$\left(\sqrt{Q + t_x(P + t_x)}\right)^i \left(t_x\right)^j \quad \text{for } \ i,j = 0,1,2$$

in (4.60). Thus we obtain

$$(i,j)=(1,1) \Rightarrow \frac{2B(t,t_1)B_{t_1}^2(t,t_1)}{\sqrt{B^2(t,t_1)-1}}+ \\ +\frac{2B(t,t_1)\sqrt{B^2(t,t_1)-1}B_{t_1}(t_1,t_2)}{1-B^2(t_1,t_2)}=0; \tag{4.61}$$

$$(i,j)=(1,0) \Rightarrow \frac{P}{2}\Big(\frac{2B(t,t_1)B_{t_1}^2(t,t_1)}{\sqrt{B^2(t,t_1)-1}}+ \\ +\frac{2B(t,t_1)\sqrt{B^2(t,t_1)-1}B_{t_1}(t_1,t_2)}{1-B^2(t_1,t_2)}\Big)=0; \tag{4.62}$$

$$(i,j)=(0,2) \Rightarrow \frac{B^2(t,t_1)B_{t_1}^2(t,t_1)}{1-B^2(t,t_1)}-B_{t_1}^2(t,t_1)+ \\ +\frac{B_{t_1}^2(t_1,t_2)}{1-B^2(t_1,t_2)}-\frac{2B^2(t,t_1)B_{t_1}^2(t_1,t_2)}{1-B^2(t_1,t_2)}=0; \tag{4.63}$$

$$(i,j)=(0,1) \Rightarrow P\Big(\frac{B^2(t,t_1)B_{t_1}^2(t,t_1)}{1-B^2(t,t_1)}-B_{t_1}^2(t,t_1)+ \\ +\frac{B_{t_1}^2(t_1,t_2)}{1-B^2(t_1,t_2)}-\frac{2B^2(t,t_1)B_{t_1}^2(t_1,t_2)}{1-B^2(t_1,t_2)}\Big)=0; \tag{4.64}$$

$$(i,j)=(0,0) \Rightarrow \frac{B_{t_1}^2(t_1,t_2)}{4(1-B^2(t_1,t_2))}.\Big(P^2-P^2B^2(t,t_1)-4QB^2(t,t_1)\Big)+ \\ +B_{t_1}^2(t,t_1)\Big(\frac{QB^2(t,t_1)}{1-B^2(t,t_1)}-\frac{P^2}{4}\Big)=0. \tag{4.65}$$

Then (4.61) and (4.62) are satisfied if

$$\frac{B_{t_1}^2(t,t_1)}{B^2(t,t_1)-1}=\frac{B_{t_1}^2(t_1,t_2)}{B^2(t_1,t_2)-1}. \tag{4.66}$$

Equality (4.63) is equivalent to

$$-\Big(\frac{B_{t_1}^2(t,t_1)}{B^2(t,t_1)-1}-\frac{B_{t_1}^2(t_1,t_2)}{B^2(t,t_1)-1}\Big)+ \\ +2B^2(t,t_1)\Big(\frac{B_{t_1}^2(t,t_1)}{B^2(t,t_1)-1}-\frac{B_{t_1}^2(t_1,t_2)}{B^2(t_1,t_2)-1}\Big)=0. \tag{4.67}$$

So equality (4.63) is satisfied. Note that (4.64) is a multiple of (4.63), so (4.64) is also satisfied. Equality (4.65) is equivalent to

$$\frac{1}{4}\Big(\frac{B_{t_1}^2(t,t_1)}{B^2(t,t_1)-1}-\frac{B_{t_1}^2(t_1,t_2)}{B^2(t,t_1)-1}\Big)\Big(P^2-P^2B^2(t,t_1)-4QB^2(t,t_1)\Big)=0 \tag{4.68}$$

and it is satisfied by (4.66). Hence all equations above are satisfied provided we have equality (4.66). Equality (4.66) is equivalent to

$$\frac{B_{t_1}(t,t_1)}{A(t,t_1)}=\pm\frac{B_{t_1}(t_1,t_2)}{A(t_1,t_2)}.$$

Then $\dfrac{B_{t_1}(t,t_1)}{A(t,t_1)}$ does not depend on t and $\dfrac{B_t(t,t_1)}{A(t,t_1)}$ does not depend on t_1. Hence we find

$$B_{t_1}(t,t_1) = A(t,t_1)\varphi(t_1), \tag{4.69}$$

$$B_t(t,t_1) = \pm A(t,t_1)\varphi(t). \tag{4.70}$$

Now we check the condition (4.12). We will show that B is a constant function. We find

$$\tilde{m} = \mu_1 t_x(t_x^2 + Pt_x + Q) + \mu_2(t_x^2 + Pt_x + Q)^{\frac{3}{2}} + \mu_3(t_x^2 + Pt_x + Q),$$

where

$$\mu_1 = \frac{2PB(B_{t_1} + B_t)}{(P^2 - 4Q)(B^2 - 1)},$$

$$\mu_2 = \frac{2P\sqrt{B^2-1}(B_{t_1} + B_t)}{(P^2 - 4Q)(B^2 - 1)},$$

$$\mu_3 = \frac{(4Q - P^2 + P^2B)B_{t_1} + 4QBB_t}{(P^2 - 4Q)(B^2 - 1)}.$$

which does not depend on t_1. Hence, μ_1, μ_2 and μ_3 do not also depend on t_1. Using (4.69) μ_2 reduces to

$$\mu_2 = \frac{2P(\varphi(t_1) \pm \varphi(t))}{P^2 - 4Q}.$$

Since μ_2 does not depend on t_1 we have either $P = 0$ or φ is a constant function. Note that

$$\mu_1 = \frac{2PB(\varphi(t_1) \pm \varphi(t))}{(P^2 - 4Q)A},$$

and in both cases we find $\mu_1 = 0$ and $\mu_2 = 0$.

Firstly, assume $\phi = C$ to be constant. Using (4.69), μ_3 reduces to

$$\mu_3 = \frac{C(4Q - P^2) + C(P^2 \pm 4Q)B}{(P^2 - 4Q)\sqrt{B^2-1}}.$$

Derivative of μ_3 with respect to t_1 should be zero. Hence we get

$$0 = \frac{CB_{t_1}((P^2 \pm 4Q) + (4Q - P^2)B)}{(P^2 - 4Q)(B^2 - 1)^{\frac{3}{2}}},$$

which gives $B_{t_1} = 0$ or $((P^2 \pm 4Q) + (4Q - P^2)B) = 0$. Both cases imply that B is a constant.

Secondly, assume that $P = 0$ then using (4.69) again, μ_3 reduces to

$$\mu_3 = \frac{4Q(B_{t_1} + BB_t)}{-4Q(B^2 - 1)} = -\frac{\varphi(t_1) \pm B\varphi(t)}{\sqrt{B^2-1}}.$$

Derivative of μ_3 with respect to t_1 must be zero. Hence differentiating the above equality we have

$$0 = \frac{B_{t_1}(B\varphi(t_1) \pm \varphi(t))}{(B^2-1)^{\frac{3}{2}}},$$

which is possible if either $B_{t_1} = 0$ or $B = \pm\frac{\varphi(t)}{\varphi(t_1)}$. In the first case, if $B_{t_1} = 0$ then $B_t = 0$ by (4.69), so B is a constant. In the second case, if $B = \pm\frac{\phi(t)}{\phi(t_1)}$ then $\mu_3 = \pm\sqrt{\varphi^2(t) - \varphi^2(t_1)}$. Since μ_3 does not depend on t_1, ϕ must be a constant and hence B is a constant. As a result, in both cases we get B is a constant function and using (4.59) we get the statement of the lemma. □

Using these lemmas we have the following theorem.

Theorem 71 *Let f be a nonlinear function with respect to t_x. The equation* (4.32) *has four dimensional characteristic x-ring and two dimensional characteristic n-ring if and only if*

$$f = \frac{c_1\eta(t)\eta(t_1)}{t_x} \quad or \quad f = \frac{c_1 e^{c_2(t+t_1)}}{t_x + P} - P,$$

where c_1, c_2, P are arbitrary constants and η is an arbitrary function of one variable, or

$$f = \sqrt{B^2-1}\sqrt{t_x^2 + Pt_x + Q} + Bt_x + \frac{P}{2}(B-1),$$

where B, P and Q are arbitrary constants.

Proof: The proof of this lemma easily follows from Lemma 67, Lemma 68, Lemma 69 and Lemma 70. □

Examples

The functions f given in the Theorem 64 lead to the following examples.

Example 72 *The equation*

$$t_{1x} = \frac{\gamma(t)}{\gamma(t_1)} t_x - \frac{\gamma(t)}{\gamma(t_1)}\sigma(t) + \sigma(t_1),$$

where functions γ and σ satisfy the following relations

$$(\gamma(t)\sigma(t))' = \gamma(t)\sqrt{B_1 + B_2(\gamma(t)\sigma(t))}, \qquad B_1, B_2 \in \mathbb{R},$$

is Darboux integrable with x-integral $F = \frac{(L(t_3)-L(t_1))(L(t_2)-L(t))}{(L(t_3)-L(t_2))(L(t_2)-L(t_1))}$, where $L(t) = \int_0^t \gamma(\tau)d\tau$ and n-integral $I = \gamma(t)t_x - \sigma(t)$.

Example 73 *The equation*

$$t_{1x} = \frac{\gamma(t)}{\gamma(t_1)} t_x - \frac{\gamma(t)}{\gamma(t_1)} \sigma(t) + \sigma(t_1),$$

where functions γ and σ satisfy the following relations

$$(\gamma(t)\sigma(t))' = \gamma(t)\sqrt{B_1 + B_2(\gamma(t)\sigma(t))^2} \qquad B_1, B_2 \in \mathbb{R},$$

is Darboux integrable with x-integral $F = \frac{(e^{L(t)}-e^{L(t_2)})(e^{L(t_1)}-e^{L(t_3)})}{(e^{L(t)}-e^{L(t_3)})(e^{L(t_1)}-e^{L(t_2)})}$, where $L(t) = \int_0^t \gamma(\tau)d\tau$ and n-integral $I = \gamma(t)t_x - \sigma(t)$.

The functions f given in the Theorem 71 lead to the following examples

Example 74 *The equation*

$$t_{1x} = \frac{c_1\eta(t)\eta(t_1)}{t_x}$$

is Darboux integrable with x-integral $F = \int_0^{t_3} \eta^{-1}(\tau)d\tau - \int_0^{t_1} \eta^{-1}(\tau)d\tau$ and n-integral $I = \dfrac{t_x}{c_1\eta(t)} + \dfrac{\eta(t)}{t_x}$.

Example 75 *The equation*

$$t_{1x} = \frac{c_1 e^{c_2(t+t_1)}}{t_x + P} - P$$

is Darboux integrable with x-integral $F = e^{-c_2t_3+c_2Px} - e^{-c_2t_1+c_2Px}$ and n-integral $I = \dfrac{t_x + P}{c_1e^{c_2t}} + \dfrac{e^{c_2t}}{t_x + P}$.

Example 76 *The equation*

$$t_{1x} = \sqrt{B^2-1}\sqrt{t_x^2 + Pt_x + Q} + Bt_x + 0.5P(B-1)$$

is Darboux integrable with x-integral

$$F = (-8B^3 - 4B^2 + 4B - 1)t + (8B^3 - 2B + 1)t_1 + (-4B^2 + 2B - 1)t_2 + t_3$$

and n-integral

$$I = (B - \sqrt{B^2-1})^n(\sqrt{t_x^2 + Pt_x + Q} + t_x + 0.5P).$$

For each of these examples it can be checked that F is an x-integral and I is an n-integral by direct calculations.

CHAPTER 5

CONCLUSIONS AND FUTURE WORKS

In this thesis we studied Darboux integrable semi-discrete equations of the form

$$t_{1x} = f(t, t_1, t_x), \qquad \frac{\partial f}{\partial t_x} \neq 0. \tag{5.1}$$

In Chapter 3, we considered semi-discrete hyperbolic chains of exponential type

$$u^i_{1,x} - t^i_x = e^{\sum a^+_{ij} u^j_1 + \sum a^-_{ij} u^j} \quad i, j = 1, 2, \ldots, N \tag{5.2}$$

where $A = (a_{ij})_{N \times N}$ is a $N \times N$ matrix. For the equation (5.2) it was conjectured that the chain (5.2) is Darboux integrable if and only if A is the Cartan matrix (see [41]). We proved this conjecture for the case $N = 2$ but the other cases of N still remains as conjecture and needs to be proved.

In Chapter 4, we classify Darboux integrable equations (5.1) with four dimensional x-ring and two dimensional n-ring and some restrictions on the form of function f. For future studies we may consider classification of Darboux integrable equation (5.1) with no restrictions on dimension of characteristic rings and/or form of the function f.

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