MULTISCALE VOLATILITY ANALYSIS VIA MALLIAVIN CALCULUS

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ABSTRACT

MULTISCALE VOLATILITY ANALYSIS VIA MALLIAVIN CALCULUS

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In this thesis, we study multifractal stochastic processes and stability properties of stochastic processes with the aim of analyzing the multiscale characteristics of dynamic risk premiums present in financial asset prices. Multifractal processes are first defined to model the statistical properties of turbulent flows and characterized by the scale-invariance property, which implies volatility clustering, long-range dependency and multiplicative instead of additive behavior. The multifractal characterization of a dataset can be obtained, also, via the multifractal spectrum, the singularity spectrum and the generalized dimensions. The complex dynamics of financial markets resembling chaos recently gave rise to the development of multifractal models in finance. In the present study we aim to relate the multifractal behaviour of markets to the existence of multiscale risk premiums. We employ Malliavin calculus techniques to analyze the dynamics of the instantaneous risk premiums by estimating the price-volatility feedback effect rate, which is defined as the expansion rate of the rescaled variation resulting from the perturbation of the stochastic process. Throughout our study, we discover that the price-volatility feedback effect rate is the local Lyapunov exponent of the perturbation resulting in the change of measure. The fundamental indicator of chaotic dynamics is generally accepted to be the sensitive dependency to initial conditions, which can be measured via the Lyapunov exponents. The local Lyapunov exponents (LLE) characterize the finite-time behaviour of the expansion rates. We analyze the dimensional properties of the price-volatility feedback effect rate to show the existence of multiscale risk premiums in financial return series. The generalized dimensions constitutes the basis of our study as they allow for the analysis of perturbations of multifractal processes and LLEs.

To bring the multifractal framework and Malliavin calculus techniques together, we first perform multifractal analysis of the empirical datasets. Then, we estimate the instantaneous volatilities and the price-volatility feedback effect rate series of the datasets using the recently defined Fourier series method. Additionally, analyze the multifractal characteristics of the instantaneous volatilities, while the usual multifractal analysis assumes multifractality of absolute returns. To demonstrate the existence of multiscale risk premiums, we perform dimensional analysis of both the return and the estimated instantaneous price-volatility feedback effect rate series. We conclude with the observation that the generalized dimensions spectrums of both series coincide, which suggests that the existence of scale-dependent non-linear type of behavior of the risk premiums in financial asset prices.

Keywords : Multifractal processes, Malliavin calculus, Lyapunov exponents, volatility modelling, the price-volatility feedback effect rate

ÖZ

MALLİAVİN KALKÜLÜS İLE ÇOK ÖLÇEKLİ OYNAKLIK MODELLEMESİ

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Bu tezde, dinamik risk primlerinin çoklu-ölçekli analizinin yapılması amacıyla çoklu-fraktal süreçler ve stokastik süreçlerin stabilite özellikleri çalışılmıştır. Çoklu-fraktal süreçler ilk olarak türbülant akışların istatistiksel özelliklerinin modellenmesi amacıyla tanımlanmıştır ve oynaklık birikimi, uzun-dönem bağlılık ve toplamsal yerine çarpımsal davranışı ima eden ölçek-değişmezliği ile karakterize edilir. Bir veri setinin çoklu-fraktal karakterizasyonu, aynı zamanda, çoklu-fraktal tayfı, tekillik tayfı ve genellenmiş boyutlar ile de elde edilebilir. Finansal piyasaların karmaşık dinamikleri kaotik davranışı andırmaktadır ve bu benzerlik çoklu-fraktal finansal modellerin oluşturulmasını sağlamıştır. Bu tezde, finansal piyasaların çoklu-fraktal yapısını çoklu-ölçekli risk primlerinin varlığı ile ilişkilendirilmesi amaçlanmıştır. Anlık risk primlerinin ölçülmesi için Malliavin kalkülüs teknikleri kullanılarak fiyat-oynaklık geribesleme etkisi ölçülmüştür. Fiyat-oynaklık geribesleme etkisi, yeniden ölçeklenmiş varyasyonun genişleme oranı olarak tanımlanmıştır ve çalışmamız dahilinde bu etkinin, ölçü değişimine yol açan pertürbasyonun yerel Lyapunov üsteli olduğu gözlemlenmiştir. Kaotik davranışı tanımlayan temel karakteristik başlangıç durumuna hassas bağlılıktır ve varlığı Lyapunov üstelleri ile ölçülebilir. Çalışmamızda, fiyat-oynaklık geribesleme etkisinin boyutsal analizi yapılarak çoklu-ölçekli risk primlerinin varlığı gösterilmiştir. Bu bağlamda, genellenmiş boyutlar çalışmanın temel unsuru olarak ön plana çıkmaktadır.

Çalışmada ilk olarak empirik verinin çoklu-fraktal analizi yapılmış, sonrasında Fourier serisi tekniği ile anlık oynaklık ve fiyat-oynaklık geribesleme serileri tahmin edilmiştir.

Literatüre empirik katkı olarak, Fourier tekniği ile tahmin edilen anlık oynaklık serilerinin çoklu-fraktal analizi yapılmıştır. Çoklu ölçekli risk primlerinin varlığını göstermek amacı ile, getiri ve fiyat-oynaklık geribesleme serilerinin boyutsal analizi yapılmış ve elde edilen genellenmiş boyut tayflarının birbirine benzer davranışı gözlemlenmiştir. Bu sonuç, çoklu-ölçekli doğrusal olmayan risk primlerinin varlığına işaret etmektedir.

Anahtar Kelimeler: Multifraktal süreçler, Malliavin kalkülüs, oynaklık modelleme, Lyapunov üstelleri, fiyat-oynaklık geribesleme etkisi oranı

To My Family

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CHAPTER 1

INTRODUCTION

1.1 Motivation and Literature Review

The seemingly random, mainly due to unpredictability of the fundamental drivers, behaviour of financial markets inspired the employment of various stochastic processes of various levels of complexity for modeling financial data. It is accepted by many authors that the dawn of financial modeling is the work of Báchelier in early twentieth century [3]. Then, approximately 70 years later, comes the celebrated Black&Scholes&Merton (BSM) formula (see [10] and [45]), which utilizes the techniques of statistical physics to model the dynamics of price processes. It is now widely known that BSM formula is built on the similar arguments as the work of Báchelier, in the sense that both works employ Brownian motion as the stochastic process that is the source of randomness. As the financial markets grow and and become more and more complex during the recent decades, more advanced financial models have been developed mostly motivated by the inclusion of non-Gauissianity and non-linear features to better model complexity in financial market dynamics.

The observed behaviour of financial markets are far from being stable, mostly referred as being ”‘chaotic”’. The notion of chaos had emerged from the study of turbulent flows where multiplicative processes are frequently employed, started with the pioneering work of Kolmogorov [29]. Multiplicative processes are built in an iterative fashion through various scales, creating a feature called ”‘scale-invariance”’. Scale-invariance is in fact a generalization of self-similarity property, first used in the area of fractal geometry and later extended to stochastic processes. Fractal geometry, mainly developed by Mandelbrot based on the work of Gaston Julia [28] on dynamical systems. As the fractal geometry is shown to be a useful tool for the study of turbulence and chaos, the link between multiplicative processes and fractals was also brought into the light, and a new family of stochastic processes have emerged: ”‘Multifractals”’. The term Multifractal was first coined by Parisi and Frisch, in their work on turbulence [48]. Mandelbrot has attempted to define Multifractal processes via fractal sets on which the turbulence is concentrated. Parisi and Frisch, however, proposed that there should be a *duality* between the Hausdorff dimension of the fractal sets and the order of exteme realizations observed in turbulence. They then defined a Legendre duality between the scaling of the moments of a Multifractal process and codimension of its singularities of certin orders. Multifractal models were built in discrete time and scale in the beginning, until the pioneering work of Lovejoy and Schertzer where they defined stochastic integrals that generate Multifractal processes. Multifractals was widely used in modelling of weather phenomena such as rainfall and clouds (see [51], [60], [34]) also in geophysics [52], physiology [24] and in finance [41], [54], [5], [11], [12]. In [39], Mandelbrot et. al. developed the Multifractal Model of Asset Returns (MMAR), by defining a Multifractal process via a time-changed Brownian motion, where the time-change is performed by a Multifractal random measure. Later, Bacry and Muzy have defined the Multifractal Random Walk (MRW) in a similar fashion to Mandelbrot. They have defined the Multifractal process as an Itô integral of a Multifractal measure with respect to a Brownian motion. Multifractal models in finance developed with the aim of capturing the empirical characteristic of the financial asset prices such as volatility clustering, long-range autocorrelations and scale-invariance.

As the models aim to capture complex and intermittent behavior observed in empirical studies, investors prefer stability when they make investment decisions. The increasing frequency of financial crisis have motivated regulators and authorities to first define, and then ensure stability. Market stability is considered as the ability of the market to absorb fluctuations up to a certain degree. During the 2008 crisis, markets witnessed a specific type of instability caused by a sudden drain of liquidity in the market, resulted in large price moves in general assumed to be belonging to the tails of the probability distributions assumed modeling the market dynamics. It can be argued that the lack of liquidity had induced significant changes in investors' perception of risk, which can be observed by dynamical estimation of the market price of risk, which is the agent that serves as the generator of the feedback loop between price and volatility.

Stability characteristics of random dynamical systems is investigated via the Lyapunov exponents [36]. Lyapunov exponents indicate whether the underlying process features sensitive dependency to initial conditions, which is the distinguishing property of chaotic systems. Furthermore, the local, i.e. time-dependent, Lyapunov exponents are used to characterize the intermittency observed in the underlying system [8], [61]. Lyapunov exponents of Multifractals have been studied by Lorenz [33] and based on his work Aurell et. al. derived a duality relation for local Lyapunov exponents in a similar way to the construction of the singularity spectrum for Multifractals [2]. The Lyapunov spectrum is the counterpart of the singularity spectrum in Multifractal stochastic framework.

Although there is a huge literature on Multifractals and Lyapunov exponents, only a very small portion of them involve both concepts. On the other hand, there is only a small number of studies featuring the price-volatility feedback effect rate to our knowledge [6], [46]. Therefore, the novelty of this study is the attempt to establish the relation between Multifractals and the feedback effect rate through the local Lyapunov exponents, by the observation that the feedback rate itself is a local Lyapunov exponent. This relation is difficult to detect using the standard statistical methods so we employ Multifractal analysis to capture the dimensional properties of Multifractals and the feedback effect rate associated to Multifractal process.

1.1.1 The Aim of the Thesis

The aim of this thesis is to contribute to the existing literature in both theoretical and empirical terms:

- Empirical contributions: In the present study, we have employed Multifractal analysis techniques to investigate the scale-invariance properties of absolute returns of BIST30 index, USD/TRY and EUR/USD exchange rates. We have also estimated the instantaneous volatilities and feedback effect rate series via the Fourier method developed by Barucci et. al. in [6]. The obtained feedback effect rate values are interpreted in terms of their ability to indicate market stability. Then, we analyzed the scale-invariance properties of both the instantaneous volatilities and the feedback effect rate series to report whether the two are related in terms of dimensional properties. The dimensional analysis of volatilities, returns and the feedback effect rate may reveal the change of measure dynamics of the underlying asset price. This feature of the price volatility feedback effect rate constitutes the basis for the possibility of a multiscale market price of risk present in the markets.
- Theoretical contributions: Our theoretical contribution is based on our observation that the feedback effect rate itself is in fact the local Lyapunov exponent of the Girsanov factor resulting from a perturbation of the underlying stochastic process. This observation allows us to investigate consistently the Multifractal properties of the feedback effect rate to compute approximately the local Lyapunov spectrum of the measure transforming perturbation.

1.1.2 Plan of the Thesis

In Chapter 1, we start by briefly summarizing the building blocks of Multifractal processes and then proceed to Multifractal framework. We beriefly summarized some examples of Multifractal models and parameter estimation techniques applied to absolute returns of several datasets as a proxy of the instantaneous volatility to detect Multifractal scaling in volatility series. Results are compared to the results of unifractal examples to distinguish Multifractality from unifractality. In Chapter 2, we study the methods to identify the stability characteristics of stochastic processes and report the relation between the feedback effect rate and the local Lyapunov exponents in the context of stochastic stability. To include the feedback effect rate to Multifractal framework, we employ the Lyapunov exponents within the context of Multifractal characterizations of the underlying stochastic processes such as the singularity spectrum, generalized dimensions spectrum and the Lyapunov spectrum. The theoretical background on Lyapunov exponents allows us to build this relation by using the link between the Lyapunov spectrum to singularity spectrum of Multifractal processes. In Chapter 3, we summarize our results and conclude by discussing future research directions.

CHAPTER 2

MULTIFRACTAL PROCESSES

2.1 Multifractal Processes in Finance

Multifractal processes was first developed to model turbulent flows. The motivation behind using multifractal processes to model financial asset prices are due to empirical findings which are commonly observed in financial markets which are considered to be of similar characteristics to that of turbulent flows (see [11], [42], [39]). These stylized facts have now become prerequisites for newly developed financial models. Let us briefly mention some of those below:

- *Long-range autocorrelations, or long-memory, of the return amplitudes*: This observation is in contradiction with the efficient market hypotheses as it implies that the amplitude of the returns of the past observations can be used to predict the amplitudes of the future returns. It is clear that this feature gave rise to arbitrage opportunities where a portfolio of return amplitudes can be constructed in such a way to create risk-free returns.
- *Scale dependent shape of the distribution of returns*: The Gaussianity assumption for the random behaviour of the returns was made in the midst of the twentieth century, where at the time the empirical studies were mainly carried out at the daily time scale. As the data analysis at finer time scales become available, it is observed that at smaller time scales, the distribution of returns are highly non-Gaussian, whereas at larger time scales, for instance daily or weekly, the distributions converge to Gaussian.
- *Scale invariance*: The scale invariance property, also called **multifractal scaling**, refers to non-linear behaviour of the rate of growth for the moments of financial returns along different time-scales. We want to emphasize that the self-similarity property implies linear rate of growth for all finite moments at any time-scale considered.

Let us first state that the findings listed above does not coincide with the properties of some of the most popular stochastic models employed in financial modeling. To build up models that exhibit long-range correlations one can consider the fractional Brownian motions as the source of randomness. However, the effect of time-scale on the

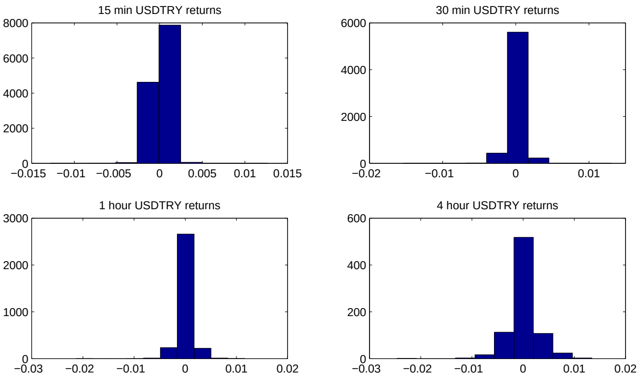

Figure 2.1: Histograms of USD/TRY exchange rate at various time scales

shape of the distribution of returns can not be obtained via using infinitely divisible, self-similar nor stable processes. Similarly, for these processes, the rate of the change of moments is linear with changing time-scale. The name **multifractal scaling** refers to the definitive property of a new family of stochastic processes: Multifractals. Multifractal stochastic processes, first constructed in the attempt to model the dynamics of *chaotic* systems, such as *turbulent flows* [48], [41], fits into the picture: financial markets are thought to be at a level of complexity probably never seen before. One can now trade at time scales ranging from milliseconds to months. Therefore, the scale dependent distributional behaviour is of fundamental importance for any trading or hedging activity.

The effect of changing time scale on the distributional characteristics of the process informs on the behaviour of financial markets. Let us start by considering a very small time-scale, *i.e.* high-frequency data samples, and the characteristics of the distribution at this scale. Naturally, at small time-scales, one witnesses much less extreme amplitudes of return series, which is the result of the restriction of time window to very small sizes that makes it impossible for a large number of buy and sell orders to occur. Putting aside the automated-trading phenomena, it is almost impossible to build up an investment or trading strategy based on the information obtained only by analyzing at such a small time scale. Therefore, it is natural to think that the tails of the return distribution of the returns at smaller time-scales would exhibit different characteristics than the tails of the return distribution at larger time scales. We can therefore assume that the volatility at smaller time scales would be smaller compared to volatility at larger time scales. However, employing unifractal processes, it is impossible to distinguish between the characteristics of the distribution at very small time scales and very large ones. Let us also discuss the pricing of risk in markets with the increasing frequency of the occurrence of financial crisis. On the long term, a volatile market is perceived as an unstable market, where investors would require higher returns or even avoid participating in the market. Higher volatility at larger time-scales, therefore, can be assumed to be an undesirable property and that is probably why central banks and financial regulators aim to suppress extreme volatility and prevent longer periods with high volatility to occur. Some examples can be used to verify these assertions. The tails of the large time-scale samples, mostly, corresponds to financial crisis or market bubbles. The increasing frequency of the occurrence of extreme realizations forces the authorities to establish tighter regulations to restore stability in the markets. It is clear that the time scale does effect the distributional properties of financial prices and acknowledgement of this fact lead to researchers to employ multifractal models to model financial asset prices, or more precisely, price fluctuations.

In what follows, we will first briefly summarize the main properties of unifractal, or self-similar, stochastic processes and how multifractal processes are built on unifractal processes using multiplicative cascades.

2.1.1 Unifractal Stochastic Processes

In financial modeling, randomness is mostly observed as the main source of risk, and therefore practitioners frequently attempted to model the price volatility. The need for predictability is generally predominates the fact that the financial markets are *almost* impossible to predict. The linear growth of variance rule is one of the reasons that financial institutions prefer linear additive models to measure the risk. The famous "square root of time to maturity" rule is known to be an oversimplification of the risk in financial markets. Nevertheless, it offers people predictability and probably that is the reason it is still being used in financial practices. It in fact is the result of the self-similarity of the Gaussian random variables! Let us state the definition of the self-similarity below [50]:

Definition 2.1. Let $\{S(t),\, t \geq 0\}$ be a self-similar stochastic process. Then for each $\lambda > 0$, there exists $\beta > 0$ such that

$$S(\lambda t) \overset{d}{=} \beta S(t). \tag{2.1}$$

The distribution of a self-similar stochastic process at various time scales can be computed using the relation between λ and β which is characterized by a single exponent H, namely the self-similarity, or Hurst, exponent of $S(t)$. The following theorem sets the basic relation between λ and β, where the role of the self-similarity exponent H is explained [31]:

Theorem 2.1. *If $\{S(t),\, t \geq 0\}$ is nontrivial, stochastically continuous at $t = 0$ and self-similar, then there exists a unique exponent $H \geq 0$ such that β can be expressed as $\beta = \lambda^H$. Moreover, $H > 0$ if and only if $S(0) = 0$ a.s..*

The self-similarity exponent is denoted by the letter H and also called the Hurst exponent in regard to H. E. Hurst, who has first discovered the long-range dependence in hydrological time series data of Nile River [23]. In general, we have $0 < H < 1$. Some examples of self-similar processes include Brownian motion $\{B(t),\, t > 0\}$, in which case we have $H = \frac{1}{2}$:

$$B(\lambda t) \overset{d}{=} \lambda^{\frac{1}{2}} B(t), \tag{2.2}$$

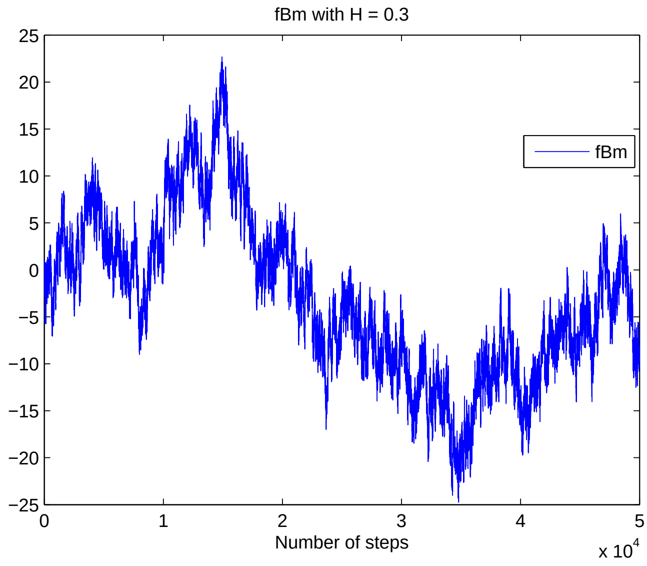

Figure 2.2: Simulation of fBm with Hurst parameter of 0.3

where it is said that the Brownian motion is "$\frac{1}{2}$-self-similar". Among the continuum of processes with self similarity exponent $0 < H < 1$, the Brownian motion is a member of continuous self-similar stochastic processes, namely the fractional Brownian motions. Brownian motion has stationary independent increments. It turns out that the independent increment property is a specific feature of Brownian motion which is the result of its $\frac{1}{2}$ self-similarity. The following theorem in [50] explains the effect of H on the auto-covariance structure of increments of a stochastic process:

Theorem 2.2. *Let $\{S(t),\, t \geq 0\}$ be nontrivial and H self-similar with stationary increments and suppose $\mathbb{E}[|S(1)|^2] < \infty$. Then*

$$\mathbb{E}[S(t)S(s)] = \frac{1}{2}\{t^{2H} + s^{2H} - |t-s|^{2H}\}\mathbb{E}[|S(1)|^2]. \tag{2.3}$$

Proof. We refer to [50] for the proof of the theorem. □

Let us observe that for $H = \frac{1}{2}$, we obtain:

$$\mathbb{E}[S(t)S(s)] = \frac{1}{2}(t + s - |t-s|) = min\{t, s\},$$

which is the covariance structure of the increments of a standard Brownian motion. However, when $H \neq \frac{1}{2}$, the covariation structure implies the dependency of increments. Furthermore, the fractional Brownian motion with self-similarity (or Hurst) exponent H is classified as persistent, $H > \frac{1}{2}$ and anti-persistent $H < \frac{1}{2}$ in reference to positive and negative correlations between subsequent increments, respectively. These features of fractional Brownian motions have extensively used in modeling of natural

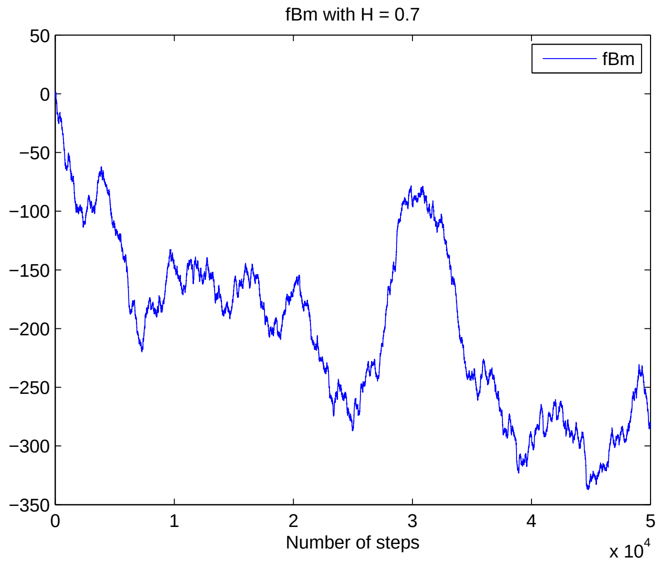

Figure 2.3: Simulation of fBm with Hurst parameter of 0.7

phenomena, spot electricity prices, financial prices and physiological observations (see [24, 41, 52] and the references therein).

The $\frac{1}{2}$-self-similarity of Brownian motion is a consequence of the Central Limit Theorem (CLT): The square root of time, or number of observations, rule for the growth of the standard deviation of the sum of the sequence, or realizations of the stochastic process $B(t)$. Therefore, for self-similar process with $H \neq \frac{1}{2}$, one can obtain the following form of the CLT [27]:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{X_i - a(n)}{b(n)} = L_\alpha, \tag{2.4}$$

where $b(n) = n^{1/\alpha}$ and $0 \leq \alpha \leq 2$ is called the **stability** or Lévy index of the variable L_α. For self-similar processes, one can see that $H = \frac{1}{\alpha}$. Equation (2.4) is in fact the defining equation of stable stochastic processes. For details on stable processes, we refer to Appendix A.

Self-similarity property is a simplification of a wide range of possible distributional behavior of stochastic processes, and therefore it is a very restrictive one. It can be argued that self-similarity holds for large class of processes for a predefined interval of time scales. However, the limiting behavior may differ significantly which results in misinterpretation of distributional properties of a stochastic process for very large or very small time scales. To properly capture the multi-scale distributional characteristics of stochastic processes, the notion of self-similarity was generalized to allow for non-linear scaling of the moments. To better capture the complex structure of the scaling, the following generalization of the self-similarity is considered:

$$\mathbb{E}[|X(t,\Delta t)|^q] = c(q)(\Delta t)^{\zeta(q)}. \tag{2.5}$$

where the *scaling exponent* $\zeta(q)$ depends not only on the order of moment q, but also on the time scale at which the moment is estimated. For existence and non-degeneracy purposes, $\zeta(q)$ is assumed to be a convex function of q. For unifractal self-similar processes it is a linear function of q. To obtain a non-linear $\zeta(q)$, unifractal processes are organized in a specific hierarchy. In the next section, we summarize the beautiful *multifractal framework*.

2.2 A Hierarchical Structure of Unifractal Processes: Multifractals and Scale-Invariance

In recent studies, self-similarity have shown to be an oversimplification of the behaviour of financial markets [11], [39], [5]. In these studies, the growth rate of variance with time scale was found to be behaving in a non-linear way for datasets of many different markets. This property is called the *scale invariance*. The idea of scale-invariance in distributions of random processes dates back to Kolmogorov [29], where his proposition of the existence scaling in turbulent flows inspired a series of studies of scaling in natural sciences, especially physics, meteorology, geophysics and economics. The earliest example for a scale invariant equation is the Navier-Stokes equation (see [59]). Scale invariance is, in a very general manner, expressed by Equation (2.5), which was very familiar with mathematicians and physicists for almost half a century. However, with the emergence of a new paradigm called "chaos", the unifractal, or uniscaling, modelling suddenly became obsolete. The observations of chaotic systems suggests a new type behaviour, namely, *multiscaling*. Multiscaling manifests itself as the non-linearity of the scaling function $\zeta(q)$ as a function of time scale at which the process is observed. As the researchers agreed upon the evident results from various analysis, the next debate was about how one can built a multiscaling model that demonstrates a similar behaviour to that of the chaotic systems in terms of its finite dimensional distributions. A natural candidate was found to be the so-called "multiplicative cascades", which has its origins in the pioneering work of Richardson on the weather prediction ([49]). Multiplicative cascades can be used to build processes with non-linear scaling, which makes it the ideal candidate as a technique for building multifractal models.

Now let us state the definition of multifractal stochastic processes ([39]):

Definition 2.2. A stochastic process $\{X(t),\, t \in \mathcal{T}\}$ is called multifractal if it has stationary increments and satisfies:

$$\mathbb{E}[|X(t)|^q] = c(q)t^{\zeta(q)}, \quad \text{for all} \quad t \in \mathcal{T}, \quad q \in \mathcal{Q}, \tag{2.6}$$

where $\mathcal{T}$ and $\mathcal{Q}$ are intervals on the real line, $\zeta(q)$ and $c(q)$ are functions with domain $\mathcal{Q}$. Moreover, we assume that $\mathcal{T}$ and $\mathcal{Q}$ have positive lengths, and that $0 \in \mathcal{T}$, $[0,1] \subseteq \mathcal{Q}$.

It can be seen that Equations (2.6) and (2.5) are equivalent, and a process satisfying these equations are called *scale invariant*. In stochastic analysis terminology, scale-invariance implies that the distribution of the process is invariant under change of scale. For financial modelling, one mostly deals with the scale in terms of time. Therefore, without loss of generality, in what follows, by saying scale we mean time scale. So when it is said that the Brownian motion is scale invariant, it is in fact equivalent to saying that when when the (time)scale is changed, its distribution is still Gaussian with mean equal to zero and variance equal to (time)scale considered. Therefore, Equation (2.6) is satisfied by scale invariant processes, which can be *unifractal* or *multifractal*. The distinction between unifractality and multifractality comes into open when one analyses the form of the scaling function $\zeta(q)$: A linear function of q corresponds to a unifractal process, which is uniscaling, whereas in case of a non-linear $\zeta(q)$, the process is a multifractal. The case for self-similar processes can directly be seen since, for a self-similar process $\{S(t),\, t \in \mathcal{T}\}$ with self-similarity exponent H we have $S(t) \overset{d}{=} t^H S(1)$, which leads to the following form of Equation (2.6):

$$\mathbb{E}[|S(t)|^q] = t^{Hq}\,\mathbb{E}[|S(1)|^q],$$

which implies $\zeta(q) = Hq$ and $c(q) = \mathbb{E}[|S(1)|^q]$.

To obtain the general properties of the scaling function $\zeta(q)$, we first set $q = 0$ to conclude that for all scaling functions we have $\zeta(0) = 0$. Let us note at this point that the specification of $\zeta(q)$ is slightly varies from the original definition of Mandelbrot in [39], where he defines $\zeta_M(q) = \zeta(q)+1$, where the subscript M stands for Mandelbrot. Holding the different definitions in mind, we will explain the reason behind our choice of the scaling function later. Another property of $\zeta(q)$ (and $K_M(q)$) is that it is a strictly concave function of q.

Unifractality can be seen as a direct consequence of self-similarity whereas to obtain multifractal dynamics, a more flexible approach is needed. Suppose that instead of a constant self-similarity exponent, the distributional equivalence between large-scale and small-scale increments is defined via a random variable $R(\cdot)$ with a distribution depending only on the *scale ratio* $\lambda > 0$:

$$X(\lambda t) \overset{d}{=} R(\lambda) X(t), \tag{2.7}$$

where $\{X(t),\, t \geq 0\}$ and $R(\cdot)$ are independent. If we assume that $\{X(t), t \geq 0\}$ is stationary, then Equation (2.7) can further be extended to local scaling:

$$X(t + \lambda\Delta t) - X(t) \overset{d}{=} R(\lambda)[X(t + \Delta t) - X(t)], \tag{2.8}$$

for all $\lambda > 0$ and the distribution of $R(\lambda)$ does not depend on t. One can see that self-similar processes correspond to the deterministic case $R(\lambda) = \lambda^H$. The scale invariance property can be rewritten in a more suitable form by defining $H(\lambda) = \log_\lambda R(\lambda)$:

$$X(\lambda t) \overset{d}{=} \lambda^{H(\lambda)} X(t), \tag{2.9}$$

where $H(\lambda)$ is a random function of λ. This equation can be used to obtain some important features of scale invariance and multiscaling processes. Let us assume that $\frac{\lambda_2}{\lambda_1} = \frac{\lambda_3}{\lambda_2}$, with $\lambda_1, \lambda_2, \lambda_3 > 0$ are constants. Then, the following holds:

$$\frac{X(\lambda_2 t)}{X(\lambda_1 t)} \stackrel{d}{=} \frac{X(\lambda_3 t)}{X(\lambda_2 t)}, \tag{2.10}$$

as both sides of the equality in distribution is equal in distribution to the random variable $R(\frac{\lambda_2}{\lambda_1})$. It is also possible to obtain a very important feature of the random variable $R(\cdot)$ by iterating Equation (2.7) as follows:

$$\begin{aligned} X(\lambda_1\lambda_2 t) &\stackrel{d}{=} R(\lambda_1\lambda_2)X(t) \\ &\stackrel{d}{=} R(\lambda_1)X(\lambda_2 t) \\ &\stackrel{d}{=} R(\lambda_1)R(\lambda_2)X(t), \end{aligned}$$

where λ_1 and λ_2 are positive constants, R_1 and R_2 are independent and identically distributed (i.i.d) random variables which have the same distribution as R. Multifractal framework offers flexibility to properly model how a change of scale effects the distribution of the underlying process $X(t)$ through its scale dependent moments. It is well-known that characteristic functions, and equivalently the moment generating functions, are unique for a specified probability distribution. Scale invariance suggests that the scaling function $\zeta(q)$ has a specific shape along the q axis at any scale considered. Therefore, when we consider multifractal processes, we are dealing with a collection of probability distributions each corresponding to a scale. The scaling function $\zeta(q)$ allows for the parsimonious approach to model this complex behaviour. Recall the empirical features of financial prices, where it is a common observation that the shape of the distribution is highly non-Gaussian at smaller scales converging to quasi-Gaussian distribution as one considers larger scales. Multifractals constitute a natural candidate for modelling this specific type of behaviour.

In financial modelling, one usually deals with processes, and for multifractal processes, at a predefined time t, there are (at least) finitely many time-scales for which we can compute the distribution of the process. An important question is that how to build multifractal models from scratch? An important method the multiscale behaviour can be analysed is the *multiplicative cascades*, which is the subject of the next section.

2.2.1 Discrete Multiplicative Cascades and Multiplicative Measures

A multiplicative cascade is an iterative procedure where at each iteration, the time scale is reduced by a predetermined ratio called the scale ratio. Multiplicative cascades was first introduced by Richardson for weather prediction [49]. Multiplicative cascades can be used to generate multiscaling models that properly mimic the behaviour of complex dynamical systems. In order to build multifractal models that demonstrate multiscaling, one starts with building multifractal measures and then extend multifractality from stochastic measures to stochastic processes [39].

In a more formal manner, multifractality is defined first for measures and then extended to processes. The definition involves also *fractal* sets such as the Cantor set, the Koch curve and Peano curves (see for instance [40]).

The following example in [56], which features one of the first fractal sets defined in real analysis and named after its creator, explains how a multiplicative cascade is built: the Cantor set [13].

Example 2.1 (The Cantor set**).** Let us consider the closed interval $[0, 1]$, σ-algebra $\mathcal{B}[0, 1]$ and associate the probability measure $\mathbb{P}$ the uniform measure, which assigns a probability to every interval $[a, b]$ equal to its length $b - a$. The Cantor set is constructed by the iterative procedure that first divides an interval into 3 equal parts and then removes the interval in the middle, i.e. if $C_0 = [0, 1]$ then $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, where $\mathbb{P}(C_1) = \frac{2}{3}$. The second iteration results in

$$C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1], \tag{2.11}$$

where one can see that $\mathbb{P}(C_2) = \frac{4}{9}$. At the k-th iteration, the set C_k contains 2^k intervals, with assigned probability of $\frac{1}{3^k}$, and the whole set C_k has therefore probability $(\frac{2}{3})^k$. The Cantor set C is defined as

$$C = \cap_{k=1}^{\infty} C_k.$$

A straightforward computation shows that

$$\mathbb{P}(C) = \lim_{k\to\infty} \mathbb{P}(C_k) = \lim_{k\to\infty} (\frac{2}{3})^k = 0.$$

Based on the sets C_k, one can construct a random variable using, for instance, a binomial random variable Y_n with $\mathbb{P}(Y_n = 1) = \mathbb{P}(Y_n = 0) = \frac{1}{2}$, $n = 1, 2, \ldots$:

$$Y = \sum_{n=1}^{\infty} \frac{2Y_n}{3^n}.$$

After the first n coin tosses, the random variable Y takes values in the set C_n which shows that Y can only take values in the Cantor set $C = \cap_{n=1}^{\infty} C_n$. Y has very interesting properties: it does not have a density nor a probability density function. Its cumulative distribution function is a *singularly continuous* function, i.e. it is a non-constant continuous function with a derivative equal to zero almost everywhere.

The Cantor set is a *fractal*, a set with a fractional dimension, defined on the real line. Fractal sets were defined in the beginning of the twentieth century. The iteration procedure and the choice of the probability measure is later generalized to construct multiplicative measures through which we can obtain multifractal processes.

In a similar manner to the construction of the Cantor set, multiplicative measures are constructed via an iterative procedure. Recall that, the Cantor function is defined by assigning a uniform probability measure to each interval equal to the length of the interval. In the context of multifractal processes we consider infinitely divisible distributions to assign probability to each interval in the cascading process.

Suppose there are two real numbers $g_0, g_1 > 0$ with $g_0 + g_1 = 1$. At the initial step $i = 0$ of the cascade, we consider the uniform probability measure ς_0 on the compact interval $[0, 1]$. In the first step $i = 1$, the measure ς_1 is obtained via assigning the *weight* g_0 to subinterval $[0, 1/2]$ and g_1 to subinterval $[1/2, 1]$. Note that the subintervals are created by dividing the initial interval $[0, 1]$ to two equal halves and therefore *the scale ratio is equal to* $1/2$. Similarly, in the second step $i = 2$, we now have 4 subintervals: $[0, 1/4]$, $[1/4, 1/2]$, $[1/2, 3/4]$ and $[3/4, 1]$ and the interval that was a subinterval in the previous step, namely $[0, 1/2]$ and $[1/2, 1]$ are treated in the same way the unit interval $[0, 1]$ was treated in the first step, i.e. the measure ς_2 assigns the weights to subintervals as follows:

$$\begin{aligned}
\varsigma_2[0, 1/4] &= g_0 g_0, \quad \varsigma_2[1/4, 1/2] = g_0 g_1, \\
\varsigma_2[1/2, 3/4] &= g_1 g_0, \quad \varsigma_2[3/4, 1] = g_1 g_1,
\end{aligned}$$

where the weights g_is are independent at each step. The *binomial* measure is defined as the limit of the sequence of measures ς_k.

The binomial measure cascade is the *simplest multifractal* example. It has some important properties. For instance, considering the dyadic interval $[t, t + 2^{-k}]$ with $t = 0.\eta_1\eta_2 \ldots \eta_k$ in the counting base of 2, one can compute the measure of a dyadic interval according to

$$\varsigma[t, t + 2^{-k}] = g_0^{k\nu_0} g_1^{k\nu_1},$$

where ν_0 and ν_1 denote the relative frequencies of 0's and 1's in the binary representation of t, respectively. One can also proceed to compute the measure at a *smaller* (or *finer*) scale from a *larger* according to the following principle:

$$\varsigma_i[0.\eta_1\eta_2 \ldots \eta_k] = g_i(t)\varsigma_i[0.\eta_1\eta_2 \ldots \eta_{k-1}]. \tag{2.12}$$

The properties of multifractal random measures are nontrivial: Multifractal random measures are continuous but singular probability measures, i.e. they have no density and no point mass, which are the features of the Cantor set.

Preservation of the mass at each step with the constraint $g_0 + g_1 = 1$ has been named the *microcanonical* property. Multifractal measures can be built via multiplicative cascade procedure with not only two but a larger number of weights g_m, which results in *multinomial* measures. In the microcanonical setting, we assume $\sum g_m = 1$, with $0 \leq m \leq b - 1$. For the binomial case we have $0 \leq m \leq 1$.

The generalization of the binomial and multinomial measures can be obtained with allowing the non-negative weights g_m to be general random variables, instead of discrete ones. As a result, the class of *multiplicative measures* are obtained. Again, in the first step of the multiplicative cascade, the unit interval $[0, 1]$ is subdivided into b-adic cells that have length $1/b$, where for every m the random weight g_m is assigned to mth cell. Iterating this procedure, we obtain the measure of an interval of length $\Delta t = b^{-1}$:

$$g(\Delta t) = g(\eta_1)g(\eta 1, \eta_2)\ldots g(\eta_1, \ldots, \eta_i),$$

and therefore for the q-th moment, we have

$$g(\Delta t)^q = g(\eta_1)^q g(\eta_1, \eta_2)^q \ldots g(\eta_1, \ldots, \eta_i)^q, \quad \forall q \geq 0. \tag{2.13}$$

Taking expectation of the both sides of Equation (2.13), we obtain the intended scaling rule:

$$\mathbb{E}[g(\Delta t)^q] = (\mathbb{E}[g^q])^i, \tag{2.14}$$

as the weights $g(\eta_i),\ i = 1, 2, \ldots, n$ are independent.

To relax the restriction of microcanonical conservation imposed on the multifractal measure, it can be required that the measure conserves the *expected* mass, *i.e.* $\mathbb{E}[\sum g_m] = 1$. In this case, the measure is called *canonical*. Canonical multifractal measures play an important role in the attempts to build universal multifractal processes and multifractal stochastic equations. For canonical multifractals, Equation 2.13 takes the following form:

$$g(\Delta t)^q = \Omega(\eta_1, \ldots, \eta_i)g(\eta_1)^q g(\eta 1, \eta_2)^q \ldots g(\eta_1, \ldots, \eta_i)^q, \quad \forall q \geq 0, \tag{2.15}$$

where Ω denotes the total mass of the multifractal measure. In this setting, $\Omega(\eta_1, \ldots, \eta_i) \stackrel{d}{=} \Omega$, and therefore the multifractal measure satisfies Equation (2.14), which is the desired multifractal scaling property.

2.2.2 Multifractal Random Fields and Change of Measure

The analysis of scaling properties of random fields involves a very well-known concept in financial mathematics and modeling, as well as stochastic analysis, of course: change of measure. Change of measure is in fact one of the key ingredients of BSM model, which makes it possible to define the option pricing problem in a conformable setting to the fundamental theorem of asset pricing (see for instance [15]). Now let us turn back to scaling random fields to discover the role of the change of measure in the construction of multifractal fields.

Now let us start by considering a random measure $\mu(A)$, defined over the σ-field of subsets $A \in \mathbb{R}^D$, where D denotes the dimension of the underlying space. The average density at scale L is defined as [44]:

$$F_L(r) = L^{-D}\mu(C), \tag{2.16}$$

where C is a D-dimensional cube of side length L centered at r and $F_L(r)$ is a random field. To analyze the density μ at different scales l and L, one can consider the relative density $a_{l;L}$, which is the well-known Radon-Nikodym derivative process:

$$a_{l;L} = F_l(r)/F_L(r), \tag{2.17}$$

where $L > l$ and $a_{l;L}$ satisfies:

$$a_{l;L} = a_{l;g}a_{g;L}, \tag{2.18}$$

where $l < g < L$ and $V_l < V_g < V_L$ and we assume that $a_{l;L}$ is a random variable, the distribution of which depends only on the ratio l/L and is independent of the positions of volume center r provided that $V_l \in V_L$. For the moments of $a_{l;L}$ we write:

$$\mathbb{E}[a^q_{l;L}] = \mathbb{E}[a^q_{l;g}]\mathbb{E}[a^q_{g;L}], \tag{2.19}$$

and since $\mathbb{E}[a^q_{l;L}]$ depends only on the ratio l/L, we obtain the following general form

$$\mathbb{E}[a^q_{l;L}] = (l/L)^{-K(q)}, \tag{2.20}$$

where by definition we see that $K(0) = K(1) = 0$.

Now let us introduce the logarithmic ratio

$$\begin{aligned} \kappa_{l;L} &= -\ln\left(F_l l^D/F_L L^D\right) \\ &= -\ln\left(F_l/F_L\right)\left(l/L\right) \\ &= -\ln\left(a_{l;L}\right)\left(l/L\right) \end{aligned} \tag{2.21}$$

where we can see that $\kappa_{l;L}$ is a non-negative random variable that is depending only on the ratio l/L. Let us observe that one can define a sequence of n cubes (or intervals in 1-dimensional case) of side length ρ_n lying between V_l and V_L such that

$$1/\rho_1 = \rho_1/\rho_2 = \cdots = \rho_n/L = (l/L)^{1/n}. \tag{2.22}$$

The representation of $\kappa_{l;L}$ in terms of $a_{l;L}$ allows us to write:

$$\kappa_{l;L} = \kappa_{l;\rho_1} + \kappa_{\rho_1;\rho_2} + \cdots + \kappa_{\rho_n;L}, \tag{2.23}$$

which implies that $\kappa_{l;L}$ is an infinitely divisible random variable.

One can see that in multifractal random fields, change of scale implies change of measure through the Radon-Nikodym derivative process $a_{l;L}$. Therefore, each change of scale from L to l corresponds to a change of measure. In fact, the multifractal scaling is observed in $a_{l;L}$. As the scale ratio between two intervals increase, the number of times the measure changed also increases. In the small time scale limit, it can be assumed that there are almost infinite number of measure changes.

2.2.2.1 Generator of the cascade

Let us recall that the random variable $R(\lambda)$ is responsible for the scale dependent behaviour of the multifractal process $X(t)$. An equivalent approach to multifractal analysis is using multifractal random measures of the previous section. Writing $R(\lambda)$ for $a_{l;L}$ we can write:

$$R(\lambda_1\lambda_2) \stackrel{d}{=} R(\lambda_1)R(\lambda_2), \tag{2.24}$$

which implies that $R(\cdot)$ is a *multiplicative* process. Moving to logarithmic coordinates we obtain a familiar distributional property of $R(\lambda)$:

$$\log R(\lambda_1\lambda_2) \stackrel{d}{=} \log R(\lambda_1) + \log R(\lambda_2), \tag{2.25}$$

i.e. $\log R(\cdot)$ is infinitely divisible and therefore $R(\cdot)$ is log-infinitely-divisible. This result is also obtained for self-similar random measures.

Remark 2.1. The random weights $R(\cdot)$ determining the scale-dependent behaviour of a multifractal process $X(t)$ is log-infinitely divisible.

This important remark restricts the class of stochastic processes to be chosen to determine the scale-dependent behavior of multifractal processes. Members of the family of infinitely divisible processes ranges from the Brownian motion to Poisson processes, and the multifractal processes are named after the distribution of $R(\cdot)$, with examples such as the log-Normal multifractal and the log-Poisson multifractal.

We can write via scale invariance:

$$X(t) \stackrel{d}{=} R(t)X(1), \tag{2.26}$$

and rewrite Equation (2.6) as:

$$\begin{aligned} \mathbb{E}[|X(t)|^q] &= \mathbb{E}[|R(t)X(1)|^q] \\ &= \mathbb{E}[R(t)]\mathbb{E}[|X(1)|^q] \\ &= c(q)t^{K(q)}, \end{aligned}$$

where $c(q) = \mathbb{E}[|X(1)|^q]$ is a constant and $R(\cdot)$ satisfies

$$\mathbb{E}[R(\lambda)] = \lambda^{K(q)}. \tag{2.27}$$

The log-infinite divisibility of $R(\cdot)$ allows for relating the scaling function $\zeta(q)$ and the moment generating function of the infinitely divisible distribution of $W(\lambda) = \ln(R(\lambda))$ since:

$$\begin{aligned} \mathbb{E}[R(\lambda)^q] &= \mathbb{E}[\exp(qW(\lambda))] && (2.28)\\ &= \lambda^{K(q)}. && (2.29) \end{aligned}$$

Let us denote the moment generating function of the infinitely divisible distribution by $\xi(q)$. Then, we have the following equality that defines $K(q)$:

$$K(q) = qD - \xi(q), \tag{2.30}$$

where D is the dimension of the underlying space. The conservation property implies $\xi(1) = D$ [44].

What is the distribution of $X(t)$? The multifractal process is named after the distribution of $W(\lambda)$, the generator of the cascade. In fact, the choice of the infinitely divisible distribution fully characterizes the distribution of the multifractal process $X(t)$ since it determines the form of the scaling functions $\zeta(q)$ and $K(q)$.

Using Equation (2.26) on a multiplicative cascade requires the scale ratio λ to be set equal to a constant greater than one. The usual choice is $\lambda = 2$. Holding in mind that we are only considering models that satisfy canonical conservation property, *i.e.* $\mathbb{E}[R(\lambda)] = 1$, let us proceed to the construction process.

The multifractal cascades can be built starting from a predetermined largest scale S down to a smallest scale s. The scale considered when analysing the multifractal process is called the resolution. The multifractal process at the smallest scale is called "bare", in analogy to the energy flux in turbulence. There is also the "dressed" process, corresponding to integrated bare process. The detailed discussion on the distinction between dressed and bare processes is not in the scope of this study. However, one must keep in mind that in empirical studies, the incrasing scale is used to obtain the dressed process for parameter estimation procedures. In the cascading process, at each iteration, the scale is reduced to the subsequent scale by the scale ratio λ: $s_i = s_{i-1}/\lambda$. Denoting the resolution by r, we have $S = r \times \lambda^n$. As the scale decreases, the number of intervals increase by λ: at the n-th iteration, the scale is $s_n = L/\lambda^n$ and the number of intervals is λ^n, where the size of an interval is fixed as r.

Inspired by Richardson's cascade setting, the cascade is constructed with random weights. Suppose $t \in [0, T]$, with the largest scale $S = T$. At the n-th iteration of the cascade, the realization of the multifractal process $X(\cdot)$ at t is given by:

$$X(t) = \prod_{i=1}^{n} R_i(t), \tag{2.31}$$

where $W_i(t)$ stands for the i-th step weight corresponding to time t. These weights are independent of each other, helping to satisfy:

$$\mathbb{E}[X(t)^q] = \prod_{i=1}^{n} \mathbb{E}[(R_i(t))^q] = \mathbb{E}[R^q]^n, \tag{2.32}$$

with $R \overset{d}{=} R_i(t), \quad i = 1, \cdots, n$. If one considers a total number of cascade steps N, the *total scale ratio* Λ is defined as

$$\Lambda = \frac{S}{r} = \lambda^N.$$

Now let us write $X_\Lambda(t)$ to denote the multifractal process at resolution r. The scale invariance property suggests that

$$\mathbb{E}[X_\Lambda(t)] = \Lambda^{\zeta(q)},$$

where the scaling function reads $\zeta(q) = \log_\Lambda \mathbb{E}[R^q]$, and we see that the result is independent of the time point t.

The multiplicative cascade setting restricts the possible choices of the scale ratios by determining the rate of increase of the scale via a constant scale ratio. Investigation of the relation between the dynamics of the process X_λ at different cascade steps we can write:

$$X_{\lambda^{m+n}}(t) = T_n(X_{\lambda^m}(t))X_{\lambda^n}(t), \tag{2.33}$$

where $X_{\lambda^m}(t)$ and $X_{\lambda^n}(t)$ are independent m-th and n-th step realizations of the process at cascade, respectively and we have, $T_n[X_{\lambda^m}(t)] = X_{\lambda^m}(t)(\lambda^{-n}t)$, where we can rewrite Equation (2.33) as

$$X_{\lambda^{m+n}}(t) = X_{\lambda^m}(t)(\lambda^{-n}t)X_{\lambda^n}(t), \tag{2.34}$$

which allows for the computation of $m+n$-th step realization using m-th and n-th step realizations. Let us switch to the generator setting $G(\lambda) = \ln(X(\lambda))$, where we dropped the time point t for simplicity. For the multifractal process X_λ to satisfy the canonical conservation property, the following normalization can be employed

$$\exp G' = \frac{\exp G}{\mathbb{E}[\exp G]}, \tag{2.35}$$

where we compute $\mathbb{E}[\Gamma'] = 1$. This normalization is in fact the well-known *Esscher transform*, which is in accordance with the observation that a change of scale implies a change of measure for multiplicative multifractals.

The random variable G satisfies the *additive group property*:

$$G_{\lambda^{m+n}} = T_n(G_{\lambda^m}) + G_{\lambda^n}.$$

At the smallest scale of the cascade,that is the *homogeneity scale*, the process converges to the limiting behaviour, which can be characterized by its scaling function $\zeta(q)$. The choice of the distribution of the generator G determines the form of $\zeta(q)$. Let us demonstrate this property in the following example [55]:

Example 2.2 (The log-Normal multifractal)**.** This class of multifractals corresponds to the choice of a Gaussian random variable as the generator of the cascade. Holding in mind the conservative property, we require the scaling function to satisfy $K(0) = K(1) = 0$. For simplicity, we assume a scale ratio $\lambda = 2$. The log-Normal generator L is given by

$$L = \exp(X), \quad X \sim N(a, b^2).$$

The stability property of Gaussian random variables allows us to write:

$$X \stackrel{d}{=} a + bZ, \quad Z \sim N(0, 1).$$

The logarithmic transformation implies the moment generating function of X is equivalent to the q-th order moment of L:

$$\begin{aligned} \mathbb{E}[\exp(qX)] &= \mathbb{E}[L^q] \\ &= \exp(qa + \frac{1}{2}b^2q^2). \end{aligned} \tag{2.36}$$

The conservation property implies the following relation between the parameters a and b, the mean and the variance of the distribution respectively:

$$a + \frac{1}{2}b^2 = 0 \quad \text{or} \quad a = -\frac{1}{2}b^2,$$

and defining $\kappa = K(2) = \log_2(\mathbb{E}[L^2])$, we can rewrite Equation (2.36) as

$$L = \exp(\sqrt{\kappa \log 2}Z - \frac{\kappa}{2}\log 2).$$

Then, one can show that the log-Normal generator results in the following form of the scaling function:

$$\zeta(q) = \frac{\kappa}{2}(q^2 - q).$$

The log-Normal multifractal was first proposed by Kolmogorov and Obukhov as the first example of a multiscaling process [29], [47]. It perfectly demonstrates how the choice of the infinitely divisible distribution, the Gaussian distribution in this case, determines the form of the scaling function $\zeta(q)$ of the multifractal process $X(t)$.

This observation in fact points out to a stronger result obtained for infinitely divisible random processes, which is stated in the following theorem [17]:

Theorem 2.3. *The class of infinitely divisible distributions coincides with the class of limit distributions of compound Poisson distributions.*

We will not repeat the proof of Theorem (2.3) and refer to [17] for the proof. However, we report some of the important results in Appendix A.

It is therefore possible to specify the distribution of the log-infinitely divisible multifractals via specifying the canonical Lévy measure M and therefore to compute the scaling function $\zeta(q)$ as soon as the canonical Lévy measure is specified. The probability distribution of a multifractal process is, of course, scale dependent. In fact, it can be interpreted that the probability distribution of a multifractal process is a collection of probability distributions along different time-scales. This interpretation is a very useful one for financial applications since many empirical studies report that the shape of the distribution of financial returns are highly non-Gaussian for small time-scales and converges to a quasi-Gaussian distribution as the time-scale increases. The resulting collection of scale-dependent distributions manifests itself in the scaling function $\zeta(q)$: unlike unifractal processes, the scaling function of a multifractal process is not a vector of values for each order of moments but a matrix of values, where the second dimension is the time scale. This matrix contains all the necessary information on the distribution of the multifractal process.

Suppose that one wants to investigate the characteristics of the extreme realizations of the process at each time-scale considered. These realizations correspond to the tails of the probability distribution. Since the scaling function $\zeta(q)$ contains all the necessary information on the distribution, one can also identify how the tails of the distribution behaves, at each scale. It turns out that the existence of a multifractal process is possible when there is a *dual* relationship between the order of the extreme realizations and their probabilities. This duality has in fact led to the definition of the multifractal processes, and the term multifractal to be coined by Parisi and Frisch in their pioneering study [48].

2.3 Singularities and Codimensions: Knitting Unifractals in Hierarchy

Multifractal processes used frequently to model the statistical dynamics of chaotic systems such as cloud formations, atmospheric wind, turbulent fluids, rainfall fields, human heart beat and financial markets. In early studies, the multiplicative cascades and their statistical properties were considered. As the main purpose is to build multifractal processes, the focus is first on the notion of dimension. The key relation relation upon which the multifractal framework was built is the one between what is called a *singularity*, its order, and the fractal dimension of the set it is observed. We proceed with some important definitions.

2.3.1 Dimension and Singularity

The key point in the definition of multifractals is the intrinsic dual relationship between dimension and singularity of a function, or a stochastic process. This duality was first proposed by Parisi and Frisch in [48]. Their idea is that in order for multifractal behavior to be possible, the magnitude of large observations and the dimension of the sets that support those observations must have a specific type of dependence. Let us start with the definition of *singularity* [48]:

Definition 2.3. A process $\nu(\cdot)$ is said to have a singularity of order $h > 0$ at the point x if

$$\overline{\lim}_{x \to y} |\nu(x) - \nu(y)| / |x - y|^h \neq 0. \tag{2.37}$$

The early multifractal models were built on the assumption that their singularities are concentrated on fractal sets. Let us denote by $S(h)$ the set of points for which the process has a singularity of order h. The notion of singularity is closely related to *local Hölder exponents* [58]:

Definition 2.4. A function f is h-Hölder continuous at point t_0 iff there exists a polynomial P of degree $h^{'} < h$ such that

$$|f(t) - P(t - t_0)| \leq C_{t_0} |t - t_0|^h \tag{2.38}$$

in a neighborhood of t_0, where C_{t_0} is a constant. Let $C^h(t_0)$ denote the space of real-valued functions that satisfy Equation (2.38) at t_0. A function f is said to have local Hölder exponent h_f if for $h < h_f$, $f \in C^h(t_0)$ and for $h > h_f$, $f \notin C^h(t_0)$.

Connecting two previous definitions, it is observed that when the function f has a singularity of order h, which is the case on the set $S(h)$, it is said that f is not an Hölder function of order h. Therefore one can define two subspaces of the underlying space where the singularities of order h occurs and where the Hölder continuity of order h are observed.

The Hölder exponent of Lévy processes are path dependent. However, there is a famous exception: the fractional Brownian motion with self-similarity exponent H (including the standard Brownian motion corresponding to $H = \frac{1}{2}$) has local exponent $h_{fBm} = \frac{1}{H}$ almost surely almost everywhere, *i.e* for almost all sample paths. However, we do not have similar results obtained for sample paths of Lévy processes or even for α-stable Lévy motion. This difficulty is overcome by defining the *singularity spectrum* of a stochastic process. The definition of singularity spectrum defined based on the Hausdorff-Besicovitch dimension. We proceed to the definition of the Hausdorff measure [57]:

Definition 2.5. For any set $E \in \mathbb{R}^d$, we define the *exterior α-dimensional* Hausdorff measure of E by

$$m_\alpha^*(E) = \lim_{\delta \to 0} \inf \Big\{ \sum_k (diam F_k)^\alpha : E \subset \bigcup_{k=1}^{\infty} F_k, \quad diam F_k \leq \delta \quad \text{all} \quad k \Big\}, \tag{2.39}$$

where $diamS$ denotes the diameter of the set S, that is, $diamS = \sup\{|x-y| : x, y \in S\}$. The exterior Hausdorff measure considers for each $\delta > 0$, a union of arbitrary sets F_k with diameter less than δ and take the infimum of the sum $\sum_k (diamF_k)^\alpha$. An important feature of the exterior Hausdorff measure is that, the measure of a set scales according to its dimension. To see this, consider a set F scaled by r, where r is a positive constant, then $(diamF)^\alpha$ scales by λ^α, i.e.

$$m_\alpha(\lambda E) = \lambda^\alpha m_\alpha(E), \quad \forall \lambda > 0. \tag{2.40}$$

The scaling property of the Hausdorff measure resembles self-similarity of random measures and one can see the relation between the exponent α in Equation (2.40) and the self-similarity exponent H. To better understand this relation, let us state the definition of the Hausdorff-Besicovitch dimension of a stochastic process [57]:

Definition 2.6. Hausdorff dimension Given a Borel subset E of $\mathbb{R}^d$, there exists a unique α such that

$$m_\beta = \begin{cases} \infty & \text{if} \quad \beta < \alpha, \\ 0 & \text{if} \quad \alpha < \beta, \end{cases}$$

where α is the *Hausdorff dimension* of E, given by

$$\alpha = \sup\left\{\beta : m_\beta(E) = \infty\right\} = \inf\left\{\beta : m_\beta(E) = 0\right\}. \tag{2.41}$$

The Hausdorff dimension is a very fundamental concept for *fractals*. Fractals are defined to be objects with fractional dimension. The Cantor set has a Hausdorff dimension of $\alpha = \log 2/\log 3$. The famous example of a fractal set in real world had been discovered by Richardson in his study to measure the length of the coast of Britain, where he computed that the coast has a fractional dimension of 1.5, and therefore it is a fractal! Fractals simultaneously defined in complex analysis by Gastón Julia in his study of dynamical systems in [28]. However, an important point that was emphasized by Richardson on the applicability of fractals in natural sciences is that empirical studies of fractals requires advanced computational power and it was Mandelbrot who first attempted to construct fractal sets using computers [40]. In fact Julia has drawn the Julia set by hand. The fractals have drawn a lot of attention since then. Their distinctive property, the self-similarity, has been used in modeling the geometry of natural objects. As we have seen in the previous chapter, self-similarity property was extended to stochastic processes, of course, in distributional sense. In his pioneering studies, Kolmogorov suggested a self-similar stochastic process to model turbulent flows. In fact, the model he constructed corresponds to a fractional Brownian motion with $H = \frac{1}{3}$, which is a unifractal. The uniscaling structure of unifractals was later criticized to be an underestimation of the level of complexity observed in turbulence. The idea of multifractal, or multiple, scaling later proposed to properly reflect the characteristic features of turbulence.

The characterization of the behaviour of singularities in view of the Hausdorff dimensions of the sets on which they are observed, brings the question of whether there is a dependency between these two. This question is answered by Parisi and Frisch in [48] where the authors conjectured that if these type of singularities exist, then in case the scaling function is non-linear, as in the case for multifractals, the Hausdorff dimension of the set $S(h)$, denoted by $d(h)$, has a "nontrivial dependence on h: different kind of singularities are associated with sets having different Hausdorff dimensions". This is where the term *multifractal* is first coined: a hierarchical structure of fractal sets with different values of Hausdorff dimensions, on which the singularities h are concentrated. The counterexample of unifractals corresponds to self-similar processes with linear scaling functions $\zeta(q)$. For multifractals, the mixture of linear scaling exponents result in a non-linear scaling function. Furthermore, the degree of multifractality of a stochastic process can be measured by measuring how much its scaling function deviates from linearity. Now let us sketch briefly how Parisi and Frisch have made the connection between the scaling function $\zeta(q)$ and the singularity spectrum, which they had first demonstrated in their β model.

The fractal dimension of the cascade is responsible for the multiscaling characterization of the process. Recall that fractal sets have fractional Hausdorff dimension, whereas regular sets have integer Hausdorff dimension which is equal to its Euclidean dimension. This distinction leads to homogeneous and "intermittent" cascades. Homogeneous cascades are sets with integer Hausdorff dimension that demonstrate regular and predictable behaviour. In the case of a cascade with a fractional Hausdorff dimension, the resulting structure is a fractal which demonstrates multiple scaling.

Let us briefly summarize the β model of Parisi and Frisch. To mimic the aggregation properties of turbulent flows, their idea was to construct the cascade from larger scale to smaller scales in a way that at each iteration, the random weights are either dead ,equal to zero, or alive, equal to a predetermined constant. Suppose a multiplicative random variable W_m with the following binomial distribution:

$$\begin{aligned} \mathbb{P}(W_m = \lambda^c) &= \lambda^{-c} && (2.42) \\ \mathbb{P}(W_m = 0) &= 1 - \lambda^{-c}, && (2.43) \end{aligned}$$

where we can see that to satisfy the canonical conservation property $\mathbb{E}[W_m] = 1$, a duality between singularities and their probability of occurence has been formed: The probability of occurence of a singularity, $\mathbb{P}(W_m = \lambda^c)$, is proportional to its order λ^{-c}. This duality constitutes the basis of the multifractal framework.

Suppose that we have at the n-th iteration of the cascade of the β model. Since the cascade steps are independent of each other, the probability that a weight is alive after n iterations is

$$\mathbb{P}(W_{m,n} = \lambda^c | W_{m,1} = W_{m,2} = \cdots = W_{m,n-1} = \lambda^c) = (\lambda^{-c})^n, \tag{2.44}$$

where $W_{m,n}$ denotes the multiplicative weight at the n-th iteration of the cascade. Equation (2.44) implies a power-law behaviour for the proability of staying alive for the weight W_m through n iteration of the cascading process. The asymptotic exponent

c is called the (fractal) *codimension* of the process. Let us consider a d-dimensional cascade, where at each iteration, the number of "pixels" increases by λ^d. Then, the average number of active weights is computed as

$$\#(active) = \#(pixels)\mathbb{P}(\epsilon_{m,n} = \lambda^c) = \lambda^d \lambda^{-c} = \lambda^D; \quad D = d - c,$$

where the exponent D is the difference between the dimension of the space and the codimension of W_m. It can be interpreted that the active weights is concentrated on a volume of λ^{-c} of the total volume of λ^d. As we progress through cascading, D goes to the (fractal) Hausdorff dimension of the set of active points whenever $c \leq d$, since the empty space where there is no alive weight determines the complexity of the fractal set that is built by the multiplicative cascade. In case d is an integer, the fractality of the set of non-zero points directly implies the fractality of the set of zero points as both would have fractional dimensions. Let us compute the scaling function $K_\beta(q)$ of the β model:

$$\mathbb{E}[W_m^q] = \lambda^{qc}\lambda^{-q} = \lambda^{q(c-1)}, \tag{2.45}$$

which concludes $K_\beta(q) = q(c-1)$ and therefore the β model is a unifractal.

The β model simply captured the connection between the dimension and the order of singularities. However, the resulting process was not a multifractal but a unifractal self-similar process, which are taught to be too simplistic to reflect the *intermittency* observed in chaotic dynamical systems. To obtain multifractality, Schertzer and Lovejoy improved the β model and developed the α model in a similar fashion to the β model, by multiplicative cascading with dead and alive weights (see [35]). The idea behind the α model was to complicate the setting by allowing the alive weights at eacht step to increase or decrease in magnitude according to predefined exponents.

Let us start with the unit interval $[0, 1]$ and at iteration 0, the weight $W_0 = 1$. Consider as before a scale ratio λ, which is an integer denoting the number of subintervals generated by the cascade at each iteration. The cascade proceeds to smaller scales by multiplying i.i.d. random weights W_m with $\mathbb{E}[W_m] = 1$, at each iteration. Let $x \in [0, 1]$, then, the value at x at the n-th iteration of the cascade is computed as

$$W_n(x) = \prod_{j=1}^{n} W_{m,j}(x) \tag{2.46}$$

and the smallest scale, or the resolution, is now $1/\lambda^n$. The canonical conservation together with the i.i.d. assumption for $W_{m,j}, j = 1, \ldots, n$ implies $\mathbb{E}[W_n(x)] = 1, \forall x \in [0, 1]$. The multiplicative cascade upon which the α model is built is a binomial cascade:

$$\begin{aligned}\mathbb{P}(W_m = \lambda^{h^+}) &= \lambda^{-c} \\ \mathbb{P}(W_m = \lambda^{h^-}) &= 1 - \lambda^{-c},\end{aligned}$$

where $h^+ > 0$ corresponds to a *boost*, i.e. $\epsilon_m > 1$ and $h^- < 0$ corresponds to a *decrease* in the magnitude of the weights. We can see that when we want to compute the expectation of the α model at any step, the canonical conservation property moves in, and the relationship between $h^{\pm}$ and the codimension c comes into view:

$$\mathbb{E}[W_m] = \lambda^{-c}\lambda^{h^+} + (1 - \lambda^{-c})\lambda^{h^-} = 1,$$

which also implies that among the three parameters of the α model, (h^+, h^-, c), only two of them can be decided freely. It is possible to recover β model as $h^+ \to c$, which corresponds to $h^- \to -\infty$. The improvement of the α model on the β model is that in the latter, the part of the space is reserved for zero realizations due to occurence of dead weights whereas in the former there exist realizations, even if very small, resulting in a more continuous structure of the process. The α model, therefore, as an example of a multifractal model, allows for the construction of stochastic models with intermediate behaviour between jump models and continuous models [39].

Both β and α models assume that the probability of a realization is inversely related to its order of singularity. Now consider the multifractal process ϵ_Λ defined as the limit of the multiplicative cascade with weights W_{λ^i} at cascade step i:

$$\epsilon_\lambda = \prod_{i=1}^{N} W_{\lambda^i}, \tag{2.47}$$

for which the following form of probability density in terms of the singularities is assumed:

$$\mathbb{P}(\epsilon_\lambda \geq \lambda^h) = p(h)\lambda^{-c(h)}, \quad \frac{dc}{d\gamma} > 0, \tag{2.48}$$

where $p(h)$ is a normalization factor and $c(h)$ is the *codimension function* of the process. This approach is similar to that of scaling function $\zeta(q)$ of a multifractal. Recall that to obtain multifractality, the consant scaling exponent assumption relaxed so that we have a scaling function. The codimension function $c(h)$ is the corresponding relaxation of the codimensions.

In a more general way, one can define $c(h)$ instead of a single codimension constant c as follows:

$$\mathbb{P}(\epsilon_\lambda \geq \lambda^h) \cong \frac{\#(\text{singularities with orders} > h)}{\#(\text{pixels})} \cong \lambda^{-c(h)}. \tag{2.49}$$

The singularity of multifractal processes constructed via multiplicative cascades offers an opportunity to define a universal class [53]. We have seen that the idea of a multifractal process is developed on the observation that the singularities of various orders exist on fractal sets of various Hausdorff dimensions. This idea is formally stated in terms of a *codimension function* related to the scaling function $\zeta(q)$ via a Legendre transform. Let us first observe that for a multifractal cascade ϵ_l, the singularities can be defined by a power of the scale, or resolution λ:

$$\epsilon_\lambda \geq \lambda^{-h},$$

which implies that the rate of divergence of ϵ_λ is greater of equal to the rate of divergence of λ^{-h}. Borrowing the terminology of thermodynamics, recall that at each iteration, the "generator", which we have called the mother breaks up to λ^d "offsprings", which we have called the daugter, where $\lambda > 1$ is the scale ratio for one iteration of the cascade and d is the dimension of the space on which we construct the cascade. Let us denote the resolution at n-th step by l_n, implying $\lambda_0 = l_{n+1}/l_n$. Furthermore, let us restrict the development of the cascade by introducing L, the largest scale to be considered in our construction. At the n-th step, the total scale ratio is $\lambda^n = \Lambda$. As we have mentioned in previous sections, the logarithm of ϵ_λ is a member of the class of infinitely divisible random variables, and therefore switching from ϵ_λ to $\Gamma_\lambda = \ln(\epsilon\lambda)$ will be helpful in the analysis of multifractal cascades.

The canonical conservation property is defined based on the assumption that the expectation, the first moment, of ϵ_λ is finite. However, the singularities of higher orders may cause the divergence of moments. To compute the bounds for the convergence of moments, i.e. $\zeta(q) < \infty$, the "trace moments" introduced by Schertzer and Lovejoy ([35]) of ϵ_λ on a D dimensional set A_λ, where the subscript λ indicates that the set A is measured at the same resolution as ϵ_λ:

$$\mathbb{E}[\int_{A_\lambda} \epsilon_\lambda^q d^{qD} x] = \mathbb{E}[\sum_{A_\lambda} \epsilon_\lambda^q \lambda^{-qD}], \tag{2.50}$$

which is bounded since

$$\mathbb{E}[\sum_{A_\lambda} \epsilon_\lambda^q \lambda^{-qD}] \geq \#(\text{singularities with orders} > h)\lambda^{qh}\lambda^{-qD} = \lambda^{(qh-c(h))-(h-1)D}. \tag{2.51}$$

The scaling function $K_\epsilon(q)$ of the multifractal variable ϵ_λ is defined by

$$\mathbb{E}[\epsilon_\lambda^q] = \mathbb{E}[\epsilon_1^h]\lambda^{\zeta(q)} = \exp(\zeta(q)\ln(\lambda))\mathbb{E}[\epsilon_1]. \tag{2.52}$$

The scale invariance property establishes a duality between the order of moments and the order of singularities which creates a hierarchical order of singularities and the fractal sets in the multiplicative cascade. Now let us focus on the connection between the codimension function $c(h)$ and the scaling function $K(q)$. We can specify the following form of scaling tail probabilities for the multifractal process:

$$\mathbb{P}(\epsilon_\lambda \geq \lambda^h) \cong \lambda^{-c(h)}; \quad \frac{dc}{dh} > 0, \tag{2.53}$$

and the probability density of singularities can be computed as

$$\mathbb{P}(h) = \frac{d\mathbb{P}(\epsilon_\lambda \geq \lambda^h)}{dh} = c^{'}(h)(\ln(\lambda)\lambda^{-c(h)}) \cong \lambda^{-c(h)}, \tag{2.54}$$

where $c^{'}(h)\ln(\lambda)$ is a slowly varying function at infinity. Combining Equation 2.52 and Equation 2.54, one can write:

$$\mathbb{E}[\epsilon_\lambda^h] = \int \epsilon_\lambda^q d\mathbb{P} \sim \int dh\lambda^{-c(h)}\lambda^{qh}, \tag{2.55}$$

where the change of variables $\epsilon_\lambda = \lambda^h$ is used to obtain the right-hand side. The motivation behind this transformation is that we focus our attention on the realizations of ϵ_λ that are greater than or equal to singularities λ^h. In view of Equation (2.55), we obtain

$$\mathbb{E}[\epsilon_\lambda^q] = \lambda^{K(q)} = \exp(K(q)\ln(\lambda)) \sim \int_{-\infty}^{\infty} dh \exp(\ln(\lambda)(qh - c(h))), \tag{2.56}$$

which is of similar form to Equation 2.64 and for $\ln(\lambda) >> 1$, the largest contribution to the integral in the right-hand side of Equation (2.56) comes from integrand with the maximum value of the exponent:

$$K(q) = \max_h(qh - c(h)), \tag{2.57}$$

which is called a Legendre transform. The Legendre transform has a very special property that inverse of the Legendre transform is again a Legendre transform, which allows to obtain:

$$c(h) = \max_q(qh - K(q)). \tag{2.58}$$

2.3.1.1 Mandelbrot's Approach

Mandelbrot reports similar results with a slight change of perspective: instead of the scale ratio λ getting larger, he considers the time scale δt getting smaller to its limit value of zero. The codimension function takes another name in this context: the singularity spectrum. Similarly, the singularity spectrum brings out the scale dependent

Hölder continuity characteristics of the multifractal process. This, also, defines a specific hierarchy of Hölder exponents. We first restate the definition of the *singularity spectrum* [39]:

Definition 2.7. Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function and for each $\alpha > 0$ define the set of points at which f has local Hölder exponent h:

$$\Omega(\alpha) = \{t, h_f(t) = \alpha\}. \tag{2.59}$$

The singularity spectrum of f is the function $D : \mathbb{R}^+ \to \mathbb{R}$ which associates to each $\alpha > 0$ the Hausdorff dimension of $\Omega(\alpha)$:

$$D_f(\alpha) = dim_H \Omega(\alpha). \tag{2.60}$$

Put into words, the singularity spectrum of a function, or a stochastic process in our case, is the set of Hausdorff dimensions of points with a specific local Hölder exponent.

We have mentioned that the singularity spectrum of stochastic processes may be path-dependent and therefore the estimation of singularity spectrum of a process is of no special importance. However, Jaffard ([26]) showed that for a large class of Lévy processes, the singularity spectrum is the same for almost all sample paths In fact, the singularity spectrum of a Lévy process can be characterized by the Blumenthal-Getoor index defined as

$$\xi = \inf\Big\{\gamma > 0, \int_{|x|\le 1} x^\gamma \nu(dx) < \infty\Big\},$$

where $\nu(dx)$ is the Lévy measure of the process. An important result is that, for α-stable stochastic processes, the Blumenthal-Getoor index is equal to the index of stability α. This result provides important information on the multiplicative cascades with α-stable weigts and their singularity spectrum.

More formally, the following proposition in [26] for the singularity spectrum of Lévy processes:

Proposition 2.4. *Let $\{X(t), t \ge 0\}$ be a Lévy process with Lévy triplet (σ^2, ν, b) and Blumenthal-Getoor index ξ.*

- *If $2 > \xi > 0$ and $\sigma = 0$ then for almost every sample path*

  $$dim\Omega(\alpha) = \xi\alpha \quad \textit{for} \quad \alpha \le \frac{1}{\xi}$$

  and $\Omega(\alpha) = \emptyset$ for $\alpha > 1/\xi$.

- *If $2 > \xi > 0$ and $\sigma \neq 0$ then for almost every sample path*

  $$dim\Omega(\alpha) = \begin{cases} \xi\alpha & \textit{if} \quad \alpha < \frac{1}{2}, \\ 1 & \textit{if} \quad \alpha = \frac{1}{2}, \end{cases}$$

  and $\Omega(\alpha) = \emptyset$ for $\alpha > 1/2$.

- *If $\xi = 0$ then for each $\alpha > 0$ with probability 1, $dim\Omega(\alpha) = 0$.*

Proof. For the proof of the proposition, we refer to [26]. $\square$

Proposition 2.4 states that for Lévy processes, the singularity spectrum reduces to a single number. This better explains the scale-independency of the moments of self-similar processes: Regardless of the time-scale considered, Lévy processes behave according to a single value of singularity, i.e. the set of points where a Lévy process has a local Hölder exponent α has the Hausdorff dimension equal to a multiple of the Hölder exponent and the Blumenthal-Getoor index ξ.

This result is of practical importance, since the local Hölder exponents of Lévy processes may differ between sample paths, one can obtain information on the singularities, that is, the *roughness*, of the process by estimating its singularity spectrum. The interpretation of the singularity spectrum requires some insight on the Hausdorff dimension. We have stated that fractals have fractional dimension and the singularity spectrum of Lévy processes involves this case whenever the value of the Hausdorff dimension, which is given as $\xi\alpha$, is a fraction. This feature is in accordance with the log-linear behaviour observed for the scaling function $\zeta(q)$ of the self-similar processes.

Mandelbrot emphasizes there are more than one interpretation of the singularity spectrum, for which he prefers the term *multifractal spectrum*. First consider a discretization of time of length $\Delta t = b^{-k}$, where b is a positive constant, and define

$$\alpha_k(t) = \frac{\ln X(t, \Delta t)}{\ln(\Delta t)}, \tag{2.61}$$

where $\alpha_k(t)$ are called the *coarse* Hölder exponents. To obtain the limit as $\Delta t \to 0$, we increase k and in the limit, the frequency histogram of coarse Hölder exponents converges to the frequency distribution of local Hölder exponents. Mandelbrot states that the singularity, or multifractal, spectrum can be interpreted as the following equivalent alternatives:

1. The limit of a renormalized histogram of Hölder exponents,

2. The Hausdorff (fractal) dimension of the set of instants with Hölder exponent α,

3. The limit of $k^{-1} \log_b \mathbb{P}(\alpha_k > \alpha) + 1$.

An implication of this definition is that the number of intervals with Hölder exponent α, N_α, behaves as

$$N_\alpha(\Delta t) \sim (\Delta t)^{-D_f(\alpha)}.$$

which can be obtained by the scaling property of the Hausdorff measure given by Equation (2.40).

Now let us consider a finite interval $[0, T]$ with N discrete points and define the estimator $\hat{\zeta}(q)$ of the scaling function $\zeta(q)$ of the process $X(\cdot)$ as follows:

$$\hat{\zeta}(q) = \ln\Big(\sum_{i=0}^{N-1} |X(i\Delta t, \Delta t)|^q\Big) / \ln(\Delta t), \tag{2.62}$$

and it is known that for a given sample, one can estimate the local Hölder exponent by the relation

$$|X(t, \Delta t)| \sim (\Delta t)^{q\alpha(t)}, \tag{2.63}$$

where $\alpha(t)$ denotes the local Hölder exponent at t. The singularity spectrum $D_f(\alpha)$, implies that the distribution of α is of the form $c(\alpha)(\Delta t)^{-D_f(\alpha)}$, where $c(\alpha)$ is a constant. As we have obtained the form of the distribution of α, using Equation 2.63, we can write

$$\sum_{i=0}^{N-1} |X(i\Delta t, \Delta t)|^q \sim \int c(\alpha)(\Delta t)^{q\alpha - D_f(\alpha)} d\alpha. \tag{2.64}$$

In the limit $\Delta t \to 0$, the main contributor to the integral above is the following term:

$$(\Delta t)^{\min_\alpha (q\alpha - D_f(\alpha)}, \tag{2.65}$$

which shows that in the limiting case we have

$$\hat{\zeta}(q) \to \min_\alpha (q\alpha - D_f(\alpha)), \quad \text{as} \quad \Delta t \to 0, \tag{2.66}$$

where $\min_\alpha (q\alpha - D_f(\alpha))$ is the *Legendre transform* of $D_f(\alpha)$ ([14]), which has an important property that it also satisfies

$$\hat{D}_f(\alpha) = \min_\alpha (q\alpha - \hat{\zeta}(q)), \tag{2.67}$$

which provides an estimator for the singularity, or multifractal, spectrum of the process $X(t)$.

2.3.2 Universal Multifractals

Until now we have seen that multiscaling property of multifractal processes, denoted by ϵ, can be obtained by employing multiplicative cascades with random weights (multipliers), which are called generators and given by the relation $\epsilon = \exp(\Gamma)$. Furthermore, we have showed that the scale invariance property implies the log-infinite divisibility of the random variable that characterizes the scale dependent behaviour of the cascading process. If we further restrict our attention to α-stable generators, it has been shown that we can reach a universal characterization for multifractals. The generators that result in universal multifractal processes should be chosen according to the following requirements listed below:

1. Since $\ln(\epsilon) \geq 0$, it is required that $\Gamma \geq 1$.

2. Denoting by $K(q)$ the scaling exponent of the limiting process ϵ, we can write
$$K_\lambda(q) \approx \ln(\lambda) K(q),$$

3. For some orders of moments $q > 0$ to converge, it is necessary that the right tail of the probability distribution of the generator Γ_λ must decay faster than exponentially. This requirement is particularly important and restrictive since it describes the distribution of an asymmetric α-stable random variable: $S_\alpha(\sigma, -1, \mu)$ where we have assumed the lowest value of -1 for the skewness parameter β. By setting $\beta = -1$, we ensure that all moments of ϵ is finite ([50]).

4. The canonical conservation property must be satisfied: $K(1) = 0$.

The debate of the existence of universal multifractals was a very active one especially around 1980's ([53], [22]). Finally, Schertzer and Lovejoy have developed the universal multifractals based on extremely asymmetric, $\beta = -1$, α-stable random generators. They have also obtained the form of the scaling function $\zeta(q)$ as

$$K(q) = \begin{cases} \frac{C_1}{\alpha-1}(q^\alpha - q) & \text{if} \quad \alpha \in [0,2] \setminus \{1\}, \\ C_1 q \ln(q) & \text{if} \quad \alpha = 1, \end{cases} \tag{2.68}$$

where α is the stability index of the α-stable distribution and $C_1 = K^{'}(1)$. This representation is of particular importance in terms of applications as one can estimate the shape of the empirical scaling function of, for instance, return amplitudes, i.e. the absolute returns, and calibrate the parameters α and C_1 to the sample under consideration. Furthermore, using Equation 2.57, we obtain the codimension function of the universal multifractals as

$$c(h) = \begin{cases} C_1\left(\frac{h}{C_1\alpha'} + \frac{1}{\alpha'}\right)(q^\alpha - q) & \text{if} \quad \alpha \in [0,2] \setminus \{1\}, \\ C_1 \exp\left(\frac{h}{C_1} - 1\right) & \text{if} \quad \alpha = 1, \end{cases} \tag{2.69}$$

where $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$.

Universal multifractals are named after the generator process employed in multiplicative cascade procedure. Some of the important examples are given as follows [4]:

1. **The log-Lévy multifractal** In this case, one considers a stable generator with stability index $1 < \alpha < 2$ and we have almost surely discontinuous paths. This case corresponds to the canonical Lévy measure $M(dx) \sim x^{2-\alpha}$, and the resulting scaling function is

$$K_L = \sigma^\alpha |q|^\alpha.$$

2. **The log-Normal multifractal** We have already mentioned the log-Normal multifractal in Example 2.2. It is the first multiscaling process proposed by Kolmogorov and Obukhov ([29, 47]) and the scaling function is defined as:

$$K_{LN} = \frac{\kappa}{2}(q^2 - q),$$

where $\kappa = K(2) = \log_2 \mathbb{E}[W]$ and W is the log-Normal generator of the cascade.

2.3.3 Order of Singularities vs. Fractal Dimension

We have seen that Equations (2.58) and (2.67) completely specifies the scaling functions $K(q)$ and $\zeta(q)$, where the former denotes the scaling function of the multifractal random measure and the latter denotes the scaling function of the multifractal process. The relation between two functions, in the context of log-infinitely divisible random measures is given as:

$$\zeta(q) = qH - K(q),$$

with $H = \zeta(1)$, which is the exponent of the mean. Comparing Equations (2.58) and (2.67), one can see the inverse-type relationship between codimensions $c(h)$ and singularity spectrum $D_f(\alpha)$, which reveals one of the fundamental implications of scale invariance: the minimum deviation of the Hausdorff dimension of the process from linearity is equivalent to, or implies, a maximum deviation of the codimension of the process from linearity. This duality results in the hierarchical structure of multifractals, where the singularities of highest orders occur on fractal sets with the smallest fractal dimensions. This observation brings into question the distinction between rare events and extreme events.

It is argued by Schertzer and Lovejoy that the singularity spectrum was developed for deterministic chaos and the codimension formalism expressed in Equations (2.55) through (2.57) is more general and necessary for stochastic processes.

Now let us get back to the investigation of the convergence of the scaling function $\zeta(q)$. By Equations 2.51 and 2.57, we can see that the moments diverge when

$$\zeta(q) = qh - c(h) \geq (h - 1)D, \tag{2.70}$$

and we can define the critical order of singularities $K_D(q) \equiv \zeta(q) - (q-1)D$ and therefore the divergence occurs whenever $K_D(q) \geq 0$. From this point of view, the critical codimensions $C(q)$ are defined as

$$C(q)(q-1) = \zeta(q). \tag{2.71}$$

The canonical conservation requires $K(1) = 0$ and by definition $K(0) = 0$, which are the properties of the scaling function $\zeta(q)$ we have mentioned before and it can be shown that $\zeta(q)$ is convex, that is $K^{''}(q) > 0$. An important value is the "codimension of the mean" $C_1 = C(1) = K^{'}(1)$.

Going back to the β model of Parisi and Frisch, we recall that it constitutes an example of unifractal processes. The scaling function for the β model is computed as

$$K_\beta(q) = C_1(q-1), \tag{2.72}$$

which is, as expected, a linear function of q, and this observation lead to a suggestion to measure the degree of multifractality of a process by measuring the deviation of its scaling function from linearity. For instance, the $K_\beta(q)$ of the β model. It is known that linear scaling functions points out to unifractal, or self-similar, processes. One can see that the β model is self-similar with the scaling exponent $H = C_1$. Recall that to construct multifractality, the constant scaling exponents are replaced by functions $H(q)$. Similarly, in this context, one has the scaling exponent function $C(q)$. A local, in q, measure of multifractality is suggested as follows:

$$\rho = K^{''}(q)/K^{'}(q), \tag{2.73}$$

which is in fact a measure of the non-linearity of $\zeta(q)$. In case of unifractal processes, we have $\rho = 0$.

We have seen in 2.4 that for Lévy processes, this dimension can be computed by its Blumenthal-Getoor index, which is the index of stability α for α-stable processes. Note that the case $c > d$, which results in $D < 0$, leads to the so called "latent" dimensions, which is out of the scope of our study.

The investigation of the process in terms of its singularities and the shape of the singularity spectrum, and its *Legendre dual*, scaling function led to the construction of the class "universal multifractals" by Schertzer and Lovejoy ([60]). Universal multifractals are particularly important for the definition of the stochastic equations that are the solutions of multifractal processes. In the next subsection, we will briefly explain the approach to universality through multiplicative cascades.

2.3.4 Generalized Dimensions

The Hausdorff dimension is a member of the set of generalized dimensions, defined as the generalization of the box-counting dimension. The generalized dimensions are used to construct the generalized dimension spectrum, which is also a characterization of the stochastic processes and dynamical systems (see [19] for a brief discussion). The generalized dimensions are also important for the study of multifractal models and the concept of codimension.

Let us assume that we assign a probability p_i to each nonempty cell i. A trivial choice of assignment would be to use n_i/N where n_i is the number of points in the ith cell and N is the total number of points considered. When the total number of nonempty cells is n, we write

$$D(q) = \frac{1}{q-1} \lim_{\epsilon \to 0} \Big(\frac{\log \sum_{i=1}^{n} p_i^q}{\log \epsilon} \Big), \quad q \in \mathbb{R}. \tag{2.74}$$

Generalized dimensions for each q contains specific information on the underlying process. For $q = 1$, we obtain the information dimension:

$$D_I = \lim_{\epsilon \to 0} \Big(\frac{\log \sum_{i=1}^{n} p_i^q}{\log \epsilon} \Big), \tag{2.75}$$

and it is equal to the pointwise dimension α in general:

$$p(l) \sim l^{\alpha}, \quad l \to 0,$$

where $p(l)$ denotes the measure of a neighborhood of size l. $D(q)$ is called the correlation dimension, and so on.

The spectrum of generalized dimensions characterizes the multifractal properties of a process. Let us define a function $f(\alpha)$ as the dimension of points with a pointwise dimension α, which is called *the singularity spectrum* in the multifractal framework. The generalized dimensions have the following property:

$$D(q) = \frac{1}{q-1} \big[q\alpha - f(\alpha) \big], \tag{2.76}$$

and by computing the derivative we obtain $\alpha = \frac{d}{dq}\big((q-1)D(q)\big)$ and $f(\alpha) = (1 - q)D(q) + q\alpha$.

Defined by Grassberger and Procaccia, the generalized dimensions emphasizes the theoretical background of multifractals which extends to fractal geometry [20, 21]. It is the dimensional properties that distinguishes multifractals and generalized dimensions establishes the link between entropy and the fractal dimension of the process, and of critical importance to our study.

We have seen that the scaling exponents $\zeta(q)$ hold critical information about the multiscale characteristics of the process. However, one may need to obtain further information such as the predictability properties of a multifractal process, which is of fundamental importance since the characteristic feature of chaotic dynamics is the sensitive dependency to initial conditions. In the multifractal framework, this feature is explained by the long-term scale dependent structure of autocorrelation function. However, this explanation is in terms of average deviations and causes loss of critical local information. A natural candidate for the analysis of predictability of multifractals is the local Lyapunov exponents. Inspired by the study of Lorenz [33], Aurell et al. have shown that one can define a multifractal characterization of a dynamical system using the generalized dimensions of its local Lyapunov exponents. In fact, this characterization constitutes the basis of our study. We will see in Chapter 3 that using Malliavin calculus techniques, it is possible to measure the local Lyapunov exponents resulting from a perturbation of a stochastic process. Then, by estimating the generalized dimensions of these exponents, we will show that the multifractal behaviour of the process manifests itself in the generalized dimensions of the local Lyapunov exponents. Similar to the multifractal spectrum, the generalized dimension spectrum of a multifractal process exhibit non-linear behavior and high variability. In case of a unifractal process, the generalized dimensions will be of similar magnitude, and the spectrum is a constant value along different orders of dimension. However, for multifractal processes, the dimensions $D(q)$ will show high variability, as we see for the scaling functions $\zeta(q)$, and as q gets larger, we see a sharper decrease than linear order. We will estimate the generalized dimensions of empirical data to show the listed features in the following sections.

2.4 Examples of Multifractal Processes

Multifractal processes are built upon unifractal processes with the employment of the method of multiplicative cascades. The choice of the unifractal process in fact determines the scale-invariance properties of the constructed multifractal process. Several multifractal models are built according to this feature. We will mention some of the most famous examples.

2.4.1 The Multifractal Random Walk

In financial modelling, multifractality is considered to exist in the volatility series of financial prices. This approach is first employed by Mandelbrot et al. in [39] by considering a time-change based on a multifractal random measure, which is called the "trading time". An equivalent approximation by Bacry and Muzy is suggested to employ a multifractal process as the volatility coefficient of an Itô integral with respect to a Brownian motion $\{B(t), t \in [0, T]\}$ ([4]). Let us consider the following process $\{P_\lambda(t), t \in [0, T], \lambda \in [\lambda_0, \Lambda]\}$ as a model for financial prices:

$$P_\lambda(t) = \int_0^t \exp(\frac{1}{2}\gamma_\lambda(s))dB(s), \tag{2.77}$$

where we assume that $\gamma_\lambda(s) \perp W(s), \forall s \in [0, T]$. The Multifractal Random Walk (MRW) $P(t)$ is the limiting process:

$$P(t) = \lim_{\lambda \to 0^+} P_\lambda(t). \tag{2.78}$$

The resulting MRW is in fact equal in distribution to Mandelbrot's MMAR model, that is, a Brownian motion subordinated with a multifractal trading time $M(t)$. The MRW exhibits the following scaling:

$$\mathbb{E}[|P(t)|^q] = \sigma^q \frac{2^{q/2}\Gamma(\frac{q+1}{2})}{\Gamma(\frac{1}{2})} K_{q/2} t^{K_q}, \tag{2.79}$$

where $K_q = T^{\zeta(q)}\mathbb{E}[M([0,T])^q]$, $K_q = q/2 - \psi(q/2)$ and $\psi(q/2)$ is the scaling function of the canonical measure $M(dx)$ of the multifractal process $\epsilon_\lambda(t)$.

2.4.2 Continuous Multifractals

Let us consider a variable scale ratio $1 \le \lambda \le \Lambda$, where Λ is the fixed largest scale ratio. Introduce $R = \log \Lambda$ and $r = \log \lambda$. The elementary scale ratio is now $\lambda_1 = \lambda,^{1/n} = \exp R/n$. The discrete cascade corresponds to introducing a stochastic kernel M, and intervals A_p and B_p such that (recall that $\epsilon(x) = \prod_{i=1}^{n-1} W_{i,x}$)

$$\Gamma(x) = \log \epsilon(x) = \sum_{p=0}^{n-1} M(A_p, B_p(x)), \tag{2.80}$$

where $M(A, B)$ is a random variable depending only on $m(A)$, resulting in $\psi_{M(A,B)}(q) = m(A)\psi_0(q)$. The intervals A_p and B_p, responsible for the cascading behaviour are given by:

$$A_p = [\frac{pR}{n}, \frac{(p+1)R}{n}], \text{and} \quad B_p(x) = [x - \frac{K}{2}\exp(pR/n), x + \frac{K}{2}\exp(pR/n)],$$

where $K = L/\Lambda$ is the resolution and $\lambda_1^p = \exp(pR/n)$.

The densification of the cascade actually corresponds to $n \to \infty$, transforms Equation (2.80) into a stochastic integral and one can show that:

$$\epsilon_\Lambda(x) = \Lambda^{-c} \exp(\int_1^\Lambda M(\frac{cd\lambda}{\lambda}, D_\lambda I_0)(x)), \tag{2.81}$$

where $c > 0$ is a parameter, $I_0(x)$ is the interval of the length K centered in x, and D_λ is the dilatation operator of factor λ.

The stochastic integral generates a multifractal field as can be seen via scaling of moments as $\mathbb{E}[\epsilon_\Lambda^q] = \Lambda^{\zeta(q)}$ with $\zeta(q) = c(\psi(q) - q)$. Moreover, the two-points statistics can also be recovered.

The above approximation for the causal cascades can also be used with the assumption that the position is time, and the past does not depend on the future. For this purpose, the interval is modified as $B_p(t) = [t - K\exp(pR/n), t]$. This gives the following causal stochastic evolution law for continuous multifractals:

$$\epsilon_\Lambda(t) = \Lambda^{-c}\exp(\int_1^\Lambda M(\frac{cd\lambda}{\lambda}, [t - K\lambda, t]). \tag{2.82}$$

Finally, let us consider an important family, log-Stable multifractals, including the log-normal case. Stable laws are infinitely divisible; $M(kA) \overset{d}{=} k^{1/\alpha}M(A)$ for $k > 0$ constant, and $0 < \alpha \leq 2$ is the Lévy index. This result in $\psi_0(q) = q^\alpha$; when $\alpha < 2$, the second Laplace characteristic function is defined for positive moments only for asymmetric laws for which hyperbolic pdf corresponds to negative fluctuations, $\mathbb{P}(-X > x) \approx x^{-\alpha}$, whereas positive fluctuations have an exponential decay. Then, by splitting Equation (2.80) into two integrals, corresponding to backward and forward domains, and introducing the change of variables $u = x - \frac{K}{2}\lambda$ and $v = x + \frac{K}{2}\lambda$ respectively, one obtains, with the Lévy measue $L_\alpha(du) = M(du, [u, x])$ a stable stochastic integral:

$$\epsilon_\Lambda(x) = \Lambda^{-c}\exp(\int_{A(x)} |u - x|^{-1/\alpha} dL_\alpha(cu)) \tag{2.83}$$

where $A(x) = [x - X/2, x - K/2] \cup [x + K/2, x + X/2]$ and $\Lambda = X/K$.This equation corresponds to the exponential of a fractional integration (over a limited domain) of order $(1 - \frac{1}{\alpha})$ of a Lévy-stable noise. When the position is in time, we obtain with a fixed scale ratio $\Lambda = \frac{T}{K}$:

$$\epsilon_\Lambda(x) = \Lambda^{-c}\exp(\int_{t-T}^{t-K} (u - x)^{-1/\alpha} dL_\alpha(cu)) \tag{2.84}$$

where $L_\alpha(cu) \overset{d}{=} c^{1/\alpha}L_\alpha(u)$.

2.4.2.1 Continuous Universal Multifractals

One important point is that we have considered mainly discrete-in-scale multiplicative cascades, and yet have not presented any results regarding the continuous time limit of multiplicative cascades with infinitely divisible, or more specifically, α stable generators. Let us recall that we have defined $\epsilon_\lambda = \exp(\Gamma_\lambda)$, where $\lambda = L/l$ indicates the scale ratio under consideration, where L is the largest scale and l is the time scale (resolution) at scale ratio λ. We write

$$\frac{\partial \epsilon_\lambda}{\partial \lambda} = \gamma_\lambda \epsilon_\lambda, \quad \gamma_\lambda = \frac{\partial \Gamma_\lambda}{\partial \lambda}, \tag{2.85}$$

where γ_λ is the infinitesimal generator of the cascade. One question arises is how one can define a continuous time multifractal process. It turns out that this can be done in a formal way as follows [51, 52]:

$$\epsilon \equiv \lim_{\lambda \to \infty} \epsilon_\lambda = \exp(\Gamma), \tag{2.86}$$

where we also define $\Gamma = \lim_{\lambda \to \infty} \Gamma_\lambda$. By taking $\lambda \to \infty$, we take $l \to 0$ and obtain a continuous in scale (and therefore in time) multifractal process. Our fundamental concern is, as expected, to build a continuous model that demonstrates multiple scaling which implies logarithmic divergence of the moments of ϵ_λ as $\lambda \to 0$:

$$\mathbb{E}[\epsilon_\lambda^q] \sim \lambda^{\zeta(q)} \Rightarrow \mathbb{E}[\exp(\Gamma_\lambda)^q] \sim \exp(\zeta(q) \ln(\lambda)).$$

To stay within the borders of the universal multifractals, one chooses a fractionally integrated, of order h, α-stable Lévy process γ_λ as the generator of the cascade:

$$\Gamma_\lambda(x) = g_\lambda * \gamma_\lambda(x).$$

where $g_\lambda * \gamma_\lambda(x) \sim |x|^{-h}$ and restricting the domain of integration to the interval $\mathbb{D}_\lambda : \{|x^{'}| \in [L/\lambda, L]\}$, we write

$$\Gamma_\lambda(x) = \int_{\mathbb{D}_\lambda} dx^{'} |x^{'}|^{-h} \gamma_\lambda(x - x^{'}), \tag{2.87}$$

we obtain the following form for the generator:

$$\Gamma_\lambda(x) = (\int_{\mathbb{D}_\lambda} dx^{'} |x^{'}|^{-\alpha h})^{1/\alpha} \gamma + \gamma_0, \tag{2.88}$$

which results in the following scaling function

$$\mathbb{E}[\Gamma_\lambda(x)] = \exp((q^\alpha \int_{\frac{L}{\lambda}}^{L} dx^{'} |x^{'}|^{-\alpha h} + q\gamma_0), \tag{2.89}$$

where γ_0 is a recentering parameter to ensure the canonical conservation of the mass is satisfied in the multiplicative cascade. For obtaining the desired logarithmic divergence of scaling function in λ, that is

$$\int_{L/\lambda}^{L} dx|x|^{-\alpha h} \sim \ln(\lambda),$$

it is required that

$$|x|^{-\alpha h} \sim |x|^{-d},$$

where d is the dimension of the embedding space, which gives $h = \frac{d}{\alpha}$. This result shows that the index of stability, α, also determines the order of fractional integration of the α-stable Lévy process, to obtain universal continuous multifractals.

2.5 Empirical Analysis Procedures

Multifractal processes exhibit fundamental properties of the financial markets and furthermore it is possible to characterize the distribution of financial prices using only a small number of parameters. As usual, to be able to employ multifractal processes, it is necessary to check whether the underlying sample exhibits the characteristic features of multifractals. Recall that the main features consist of long-range correlations in return amplitudes, multifractal scaling and scale-invariance. The analysis of the data in terms of scaling function, codimension functions and moment generating function can be used to identify if it is appropriate to employ multifractals in the modeling process. Let us briefly summarize some of the analysis techniques available for multifractal analysis.

2.5.1 Structure Functions

This is the most frequently used method since it is built solely on the definition of scale-invariance. Multifractal processes are processes that satisfy multiple scaling, a property that requires a specific form of the scaling function $\zeta(q)$. It is therefore natural to compute the empirical moments at various time-scales and examine its form to detect the non-linear scaling of a multifractal process. In fact, we have already reported the estimator of the scaling function based on the structure functions in Equation (2.62):

$$\hat{K}(q) = \ln\Big(\sum_{i=0}^{N-1} |X(i\Delta t, \Delta t)|^q\Big) / \ln(\Delta t). \tag{2.90}$$

Computation of the empirical scaling function $\zeta(q)$ reveals the scaling characteristics of the process. A linear structure in logarithmic coordinates suggests unifractality while non-linearity is a sign of multifractality [39].

2.5.2 Detrended Fluctuation Analysis

An important consideration in the analysis of structure functions is local trends, seasonalities and/or non-stationarities present in the data. A method developed to overcome these possible issues is the *multifractal detrended fluctuation analysis* (MFDFA) (see [24]). The MDFA algorithm detrends the subsamples at different scale ratios λ_i, and then explores the scale dependent moment structure via cumulative series. MFDFA is used to estimate the scaling function and the singularity spectrum via estimating the "generalized Hurst exponent" $H(q)$, which is related to the scaling function $\zeta(q)$ as:

$$\zeta(q) = qH(q) - 1.$$

Using the scaling function $\zeta(q)$, then, as we have seen, it is possible to compute the singularity spectrum via a Legendre transform:

$$f(\alpha) = q\alpha - \zeta(q),$$

where $\alpha = K^{'}(q)$. Furthermore, we can estimate the generalized dimensions:

$$D(q) = qf(\alpha) - \zeta(q), \tag{2.91}$$

which is of essence for our study as we will see in the next chapter where we will estimate the generalized dimensions of the price-volatility feedback effect rate series.

2.5.3 Double Trace Moments

The Double Trace Moments (DTM) of technique, defined by Lavallee in ([32]) aims to detect the scaling behaviour of the multifractal process by first taking its ηth power at the scale ratio $\lambda \leq \Lambda$ and define:

$$\epsilon^{\eta}_{\lambda,\Lambda} = \frac{(\epsilon_{\lambda})^{\eta}}{\mathbb{E}[(\epsilon_{\lambda})^{\eta}]}\mathbb{E}[(\epsilon_{\Lambda})^{\eta}],$$

which has the following scaling structure:

$$\mathbb{E}[\epsilon^{\eta}_{\lambda,\Lambda}] \approx \lambda^{K(q,\eta)},$$

where $K(q,\eta)$ is called the *double trace moment scaling exponent* and related to the usual scaling function with the following equality:

$$K(q,\eta) = K(q\eta) - qK(\eta),$$

Table 2.1: Sample statistics for datasets

Dataset	Mean	Median	St. dev.	Skewness	Kurtosis
BIST30	$7.1473e-06$	0	0.0029	-1.3	102.9986
USD/TRY	0	0	0.0007	-0.3085	38.0840
EUR/USD	0	0	0.0011	-0.1813	33.1317

and $K(q,1) = \zeta(q)$.

The DTM method is especially useful to obtain parameter estimates for the universal multifractal processes:

$$K(q,\eta) = \eta^{\alpha}\zeta(q).$$

Therefore, one can estimate α by plotting $K(q,\eta)$ against η in logarithmic coordinates. The linear fit gives the estimate of α.

2.6 Empirical Results

In this study, we employed multifractal analysis to three datasets: 5 minute observations of BIST30 index between $4.1.2007$ and $29.4.2010$, 15 minute observations of USD/TRY between $4.1.2016$ and $22.1.2016$ and 1 hour observations of EUR/USD exchange rate between $12.6.2013$ and $17.12.2013$. The BIST30 dataset includes the critical 2008 period, where the global crises have caused turbulence in financial markets worldwide. The USD/TRY and EUR/USD exchange rates are chosen two compare their multifractal characteristics here first and their stability properties in Chapter 3. The obtained results suggest that all three datasets feature multifractality via bursts of volatility, non-linear structure of the scaling exponent and local discontinuities.

Let us first summarize some statistical properties of the datasets used:

The datasets we use in our analysis share some common features listed below:

- Near zero mean and median,
- Low volatility over the whole sample,
- Negative skewness and excess kurtosis.

Excess kurtosis is one of the stylized facts of financial returns. However, we are interested in the change in the statistics with the changing time scale. As usual, we expect the return series converge to a Gaussian distribution as the time scale increases.

Table 2.2: Time scale and statistics for USD/TRY returns

Time scale	Mean	Median	St. dev.	Skewness	Kurtosis
15 min	0	0	0.0007	−0.3085	38.0840
30 min	0	0	0.001	−0.2391	27.2431
60 min	0	0	0.0014	−0.757	27.2913
120 min	0	0	0.002	−0.8972	19.9106

We see that for USD/TRY sample, as the time scale increases, the volatility increases whereas the kurtosis decreases. The skewness of the sample also increases in magnitude towards the more negative values. Since we are not interested in the precise time scale at which the sample is quasi-Gaussian but how the sample moments change as the time scale changes, we proceed to multifractal analysis of the datasets.

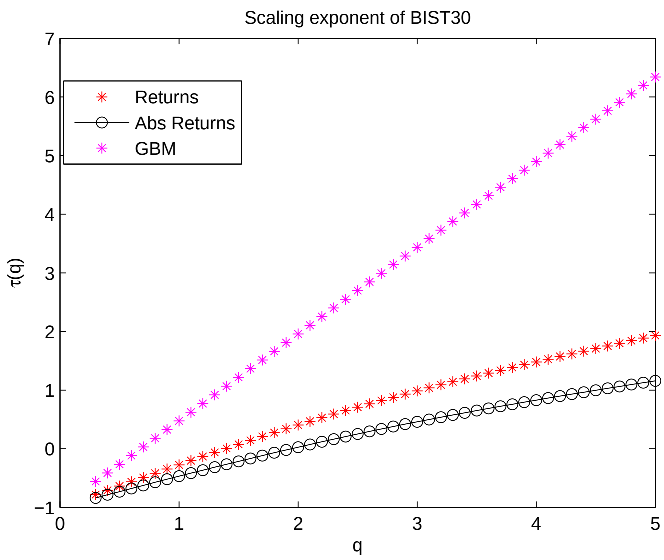

Figure 2.4: Comparison of scaling functions for BIST30 Index

We quantitatively measure the multifractality in datasets via the non-linearity of the scaling functions, the generalized dimensions and the generalized Hurst exponents, which are estimated using the Multifractal DFA method [24].

Singularity spectrums of all three datasets also support our observation that multifractal scaling is present in the return series of EUR/USD, BIST30 and USD/TRY. The spectrums vary significantly when compared to the spectrum of a Brownian motion.

We proceed with the estimation of generalized Hurst exponents $H(q)$ for BIST30 returns and a standard Brownian motion to check whether the desired multifractal scaling could be obtained. The generalized Hurst exponents of the return series emphasizes the multifractal nature of the three datasets analyzed in Figure 2.7.

We interpret the estimated scaling exponents as the degree of multifractality varies between datasets. The scaling exponent of USD/TRY is almost linear and it is the closest

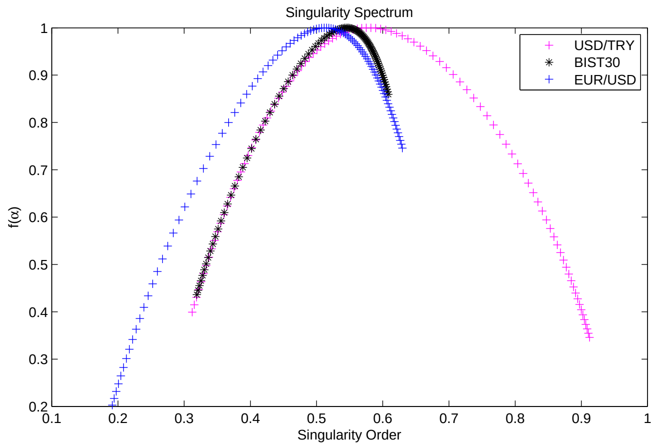

Figure 2.5: Comparison of singularity spectrums of return series

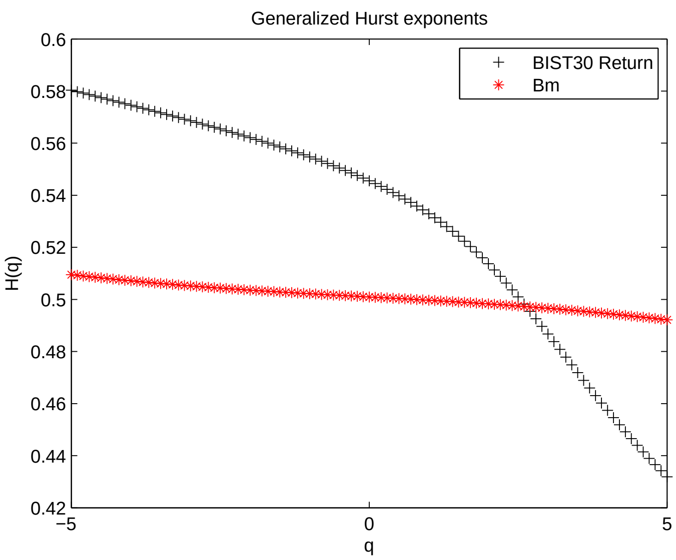

Figure 2.6: Generalized Hurst exponents of BIST30 return series

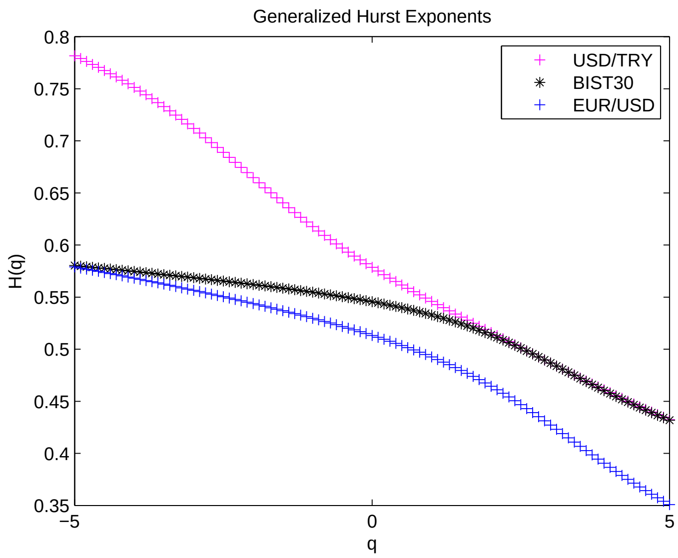

Figure 2.7: Comparison of the generalized Hurst exponents of return series

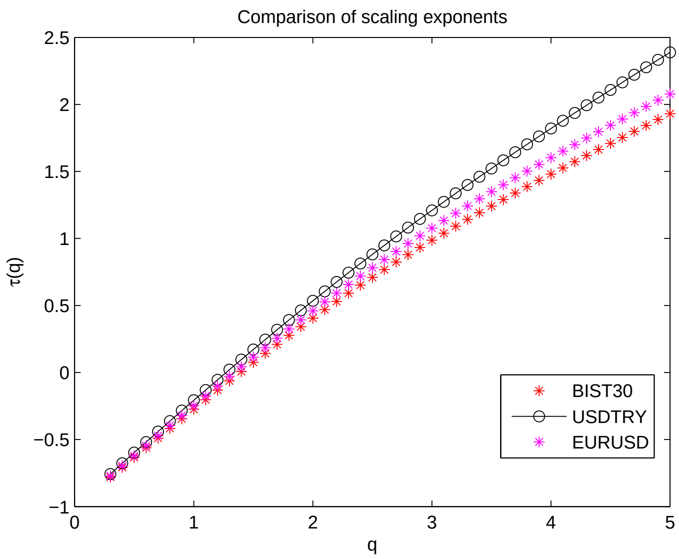

Figure 2.8: Comparison of the scaling exponents of absolute return series

to unifractal behavior among the three datasets. We can also see that its singularity spectrum is symmetrical with the most probable singularity of around 0.5 in Figure 2.5. For BIST30 and EUR/USD datasets, however, the multifractal behavior is easier to see with non-linear scaling functions, asyymetric and skewed singularity spectrums.

2.7 Conclusion of Empirical Results

Our analysis suggest that although of different levels of significance, all three datasets exhibit multifractal features. The generalized Hurst exponents $H(q)$ differ significantly from the Brownian motion value of 0.5. We see weak signs of multifractality in USD/TRY return series. This may be a result of the particular time period chosen or the relatively low number of observations in the data sample. The multifractal behavior of BIST30 and EUR/USD series can be seen via the generalized Hurst exponents, scaling exponents and the singularity spectrums. One can interpret that for large time scales, the volatility of USD/TRY would exceed the volatility of BIST30 and EUR/USD returns. The variability of USD/TRY series do not seem to differ with changing time scales.

By looking at the value of $H = H(1)$, the usual Hurst exponent for the three datasets, we see the anti-persistent nature of EUR/USD returns whereas the returns of both USD/TRY and BIST30 series are persistent. This may be a result of the very high trade volumes and orders for EUR/USD exchange rate, which may cause very fast reversion to mean and low volatility. Similarly, the effect of positive or negative return observations do not easily vanish for USD/TRY and BIST30 series. The observed persistence may be used to build trading strategies.

CHAPTER 3

STABILITY PROPERTIES OF STOCHASTIC PROCESSES

It is the usual assumptions in financial modeling practice that the investment decisions are based on two main drivers of asset prices: the risk and the return of a financial asset. While it is much more straightforward to define the return on an asset, e.g. the change in its value through time, the risk of an asset is even difficult to define. The usual practice is to use the volatility as the fundamental measure of risk. However, as the markets evolve to a more complex structure and markets witnessed new type of crises, practitioners and investors invented new risk measures.

The latest big financial crisis in 2008 have added a new type of risk the investors did not taking seriously before: The liquidity risk. A sudden drain of liquidity in the market induces large price movements as the investors with large asset portfolios would consider liquidating some of their holdings to reduce the risk of falling prices. However, liquidity shocks result in widening bid/ask spreads and the more investors trying to sell their assets paradoxically put pressure on prices. After 2008, maintaining the stability in markets have become one of the main targets of central banks and financial authorities.

The stability of financial markets can be, in some context, defined as a market's ability to absorb "small" price fluctuations [6]. Conversely, instability refers to an easily altered path of asset prices, where one considers the fluctuations as perturbations of the price process. Therefore, the notion of stability is closely related to the reactions of a market to perturbations. The behavior of a (stochastic) process under perturbation can be analyzed using dynamical systems approach where the stability of the process is measured via *Lyapunov exponents*.

Lyapunov exponents are introduced to measure the stability of dynamical systems. It is later extended to semi-martingales by X. Mao [43] and Arnold [1]. The stability behaviour of both Itô and Stratonovich type stochastic differential equations are extensively studied. Furthermore, Lyapunov exponents of multifractal processes are recently studied and the so-called *generalized Lyapunov exponents* are shown to be related to the entropy of the process [2, 61].

In a more recent study, Barucci et al. have analyzed the stability of stochastic processes, Itô processes specifically, by employing Malliavin calculus techniques and proposed a stability index, the so-called *price-volatility feedback effect rate* [6]. In this study, we show that the price-volatility feedback effect rate is the Lyapunov exponent of the *Girsanov factor* for the change of measure induced by infinitesimal perturbations of a stochastic process through time. This claim is in line with the idea that the price-volatility feedback effect rate is a stability index, as Lyapunov exponents are used to decide whether a process is stochastically stable or not.

In the following section we briefly summarize the definition of, and some important results regarding Lyapunov exponents in the context of stochastic processes.

3.1 Lyapunov Exponents and Stochastic Stability

Lyapunov exponents are roughly the exponential rate of change of a process. It is introduced as an indicator of the stability in the sense we will briefly explain below. As the theoretical background of Lyapunov exponents is mostly beyond the scope of this study, we briefly summarize some of the results mentioned in [1]. Let us consider an SDE of the form:

$$dX(t) = a(t, X(t))dt + b(t, X(t))dW(t), \tag{3.1}$$

where $\{W(t), t \in [0, T]\}$ is a Brownian motion, $a(\cdot)$ and $b(\cdot)$ are $\mathbb{R}$ valued functions which satisfy the existence and uniqueness conditions and $X(0) = x_0 \in \mathbb{R}$. Let us further assume that $a(t, 0) = b(t, 0) = X(t, 0) = 0$. Then it is said that SDE in Equation (3.1) admits the trivial solution $x(t, 0) \equiv 0$ Let us state the definition of stochastic stability ([43]):

Definition 3.1. The trivial solution of Equation (3.1) is said to be *stochastically stable* or *stable in probability* if for every pair of $\epsilon \in (0, 1)$ and $r > 0$, there exists a $\delta = \delta(\epsilon, r) > 0$ such that

$$\mathbb{P}(\{|X(t; x_0)| < r, \forall\, t \geq 0\}), \tag{3.2}$$

whenever $|x_0| < \delta$. Otherwise, it is said to be *stochastically unstable*.

Stochastic stability can be detected via the Lyapunov exponents, more precisely the sign of the Lyapunov exponents. The stability of a process can be defined as its insensitivity to changes in the initial conditions. The sensitivity to initial conditions is also a subject of financial modeling where the option Greek Delta measures the sensitivity of the option price with respect to changes in the initial condition. Let us mention the definition of the Lyapunov exponent of a SDE in an informal manner [43]:

$$\Lambda(x_0) = \lim_{t \to \infty} \frac{1}{t} \log |\frac{X(t, x_0)}{x_0}|, \tag{3.3}$$

where the quantity $\Lambda(x_0)$ is called the Lyapunov exponent of $X(t, x_0)$. A negative $\Lambda(x_0)$ value indicates stochastic stability whereas a positive one implies instability.

More precisely, a positive Lyapunov exponent implies sensitive dependence on the initial conditions, a feature of chaotic dynamical systems. One can see from Equation (3.3) that the Lyapunov exponents are time averages of the local quantity $\log|X(t, x_0)|$.

In a similar fashion, local Lyapunov exponents are defined based on the expansion rate of the perturbation of a dynamical system. Let us assume an initial perturbation of $\omega(0) = \omega_0$ of the process and define the local Lyapunov exponents (LLE) as follows [8, 61]:

$$\Lambda(t, x_0) = \frac{1}{t} \log|\frac{\omega(t)}{\omega_0}|, \tag{3.4}$$

with $d\omega(t)/dt = \Lambda(t, x_0)\omega(0)$. The idea is that in chaotic systems, the initial perturbation ω_0 will expand exponentially in accordance with sensitive dependency to initial conditions.

A similar argument is used by Barucci and coworkers to define the price-volatility feedback effect rate [6]. In the next section, we will show that the price-volatility feedback effect rate is in fact the LLE of the so-called Girsanov factor that is responsible for the change of measure induced by a perturbation of ω_0 of the underlying process. The critical feature of the LLEs is that they fluctuate according to a probability distribution $\mathbb{P}(\Lambda, t)$. Therefore, one can define moments of various orders of LLEs, which are called generalized Lyapunov exponents (GLE) [61]:

$$L(q) = \frac{1}{q} \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}[|\frac{\omega(t)}{\omega(0)}|^q], \tag{3.5}$$

and for $t >> 1$, the following form of the distribution is obtained [8, 61]:

$$\mathbb{P}(\Lambda, t) \sim \exp[-tf(\Lambda)], \tag{3.6}$$

where $f(\Lambda)$ denotes the entropy function and, just previously done in the multifractal framework, one can show that it is related to the GLEs via the Legendre transform:

$$f(\Lambda) = \max_q (q\lambda - qL(q)). \tag{3.7}$$

By comparing Equation (3.7) to Equation (2.58), one can see that $f(\lambda) \sim c(h)$ and $K(q) \sim qL(q)$. This relation implies that the Lyapunov exponents are of similar order to singularities of a multifractal process, which is straightforward to see since both quantities are responsible for the expansion of the underlying process. A more important result is obtained by Aurell et al. in [2], where the authors analyzed the predictability problem for multifractals.

Suppose that an infinitesimal perturbation of initial size δ grows to a threshold θ after the so-called *predictability time* denoted by $T(\delta, \theta)$. The “finite-size” Lyapunov exponent (FSLE) is then defined as:

$$\Lambda(\delta, \theta) = \langle [\frac{1}{T(\delta, \theta)}] \rangle \ln(\frac{\delta}{\theta}), \tag{3.8}$$

where $\langle\rangle$ denotes statistical average. Then, it is shown that the FSLE has the scaling exponent

$$\chi(q) = \min_q (1 + \frac{2 - D(q)}{q}),$$

which is "a new invariant of the multifractal approach to turbulence". This result has motivated us to investigate the dimensional properties of the LLE by estimating the generalized dimensions.

3.2 The Price-Volatility Feedback Effect Rate

The price-volatility feedback effect rate is developed as a stability, or liquidity, index for financial markets in [6] and later applied to BIST30 index to analyze the (market) stability characteristics of the index in [46]. The market stability refers to the ability of the market to absorb relatively small price fluctuations. This definition is in fact closely related to the stability concept introduced by Lyapunov.

To properly capture the motivation behind the price-volatility feedback effect rate, we first mention the pioneering study of Fournié et al. [18]. In their study, authors have employ Bismut-Elworthy-Li formula [9], [16] to suggest a new method to estimate the Greeks. The Greek Rho, the sensitivity of the option price to the changes in the mean rate of return, or the drift, parameter, can be used to investigate the stability properties of a stochastic process with respect to random perturbations at random times. Let us briefly summarize how the connection between the change of measure and the stability is established via Malliavin calculus techniques.

3.2.1 Change of Measure and Parametric Sensitivities

In financial practices, one seldomly thinks in marginal terms. The alchemy of financial markets is the relative value, or the sensitivity, of the value of asset classes with respect to each other, as the obvious opportunities are easily captured, it is the hidden patterns that makes the difference. In turn, a multi-correlated, dense structure of financial markets have evolved. It is still uncertain what drives the prices most of the time as the reasoning generally lags behind the movement. However, one can interpret changes in the prices as the changes in the underlying measure of the investors through which they measure the price of risk and manage investment decisions accordingly.

Fournié et al. suggested that the changes in the price levels of a financial asset results from the changes in the underlying measure. Consider a functional $\phi(\cdot)$ of the price process of a financial asset. The price of this payoff functional is computed via conditional expectations with respect to two equivalent probability distributions [18]:

$$\begin{aligned} \text{Change in price} &= \mathbb{E}^{\mathbb{Q}_0}[\phi] - \mathbb{E}^{\mathbb{Q}}[\phi] & (3.9)\\ &= \mathbb{E}^{\mathbb{Q}_0}[\phi \times \upsilon], & (3.10)\end{aligned}$$

where the Malliavin weight υ is defined as follows:

$$\upsilon = \frac{d\mathbb{Q} - d\mathbb{Q}_0}{d\mathbb{Q}_0}. \tag{3.11}$$

When one considers a parametrized family $\mathbb{Q}_\nu$ of distributions with parameter set $\nu = \{\nu_i\}, i = 1, \ldots, n$, we have the following result:

$$\frac{\partial}{\partial \nu_i}\mathbb{E}^{\mathbb{Q}_0}[\phi] = \mathbb{E}^{\mathbb{Q}_0}[\phi \times \upsilon_i], \tag{3.12}$$

with $Z = \frac{d\mathbb{Q}}{d\mathbb{Q}_0}$ and $\upsilon_i = \frac{\partial Z}{\partial \nu_i}$, which states that υ_i is the logarithmic derivative of $\mathbb{Q}$ at $\mathbb{Q}_0$ in the ν_i direction.

The Malliavin weight is defined based on the change of measure arguments: it is the logarithmic derivative of the Radon-Nikodym derivative process Z in a specific direction. In general, to compute the Greeks, Equation (3.12) is employed with $\nu_i = 1$, i.e. a unit change along the axis. Now consider the Greek Rho, the sensitivity of the price of the contingent claim ϕ with respect to the changes in the mean rate of return, or the drift, parameter.

3.2.2 Perturbation and the Feedback Effect Rate

Assume that the function ϕ bay be such that it can depend on the whole history of the process $\{X(t), 0 \in [0,T]\}$ satisfying $\mathbb{E}[\phi(X(\cdot))^2] < \infty$. Consider the reference path as in Equation (3.1) and the perturbed path, or process, as follows:

$$dX^\epsilon(t) = [a(t, X(t)) + \epsilon\gamma(t, X^\epsilon(t))]dt + b(t, X(t))dW(t), \tag{3.13}$$

where $\epsilon \in \mathbb{R}$ is a small parameter and $\gamma : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ is bounded. The infinitesimal random perturbation of $X(t)$ induces a change of measure through the Radon-Nikodym derivative process $\frac{d\mathbb{Q}_0}{d\mathbb{Q}}$:

$$Z^\epsilon(T) = \exp\big(-\epsilon\int_0^T \frac{\gamma(X(t))}{b(X(t))}dW(t) - \frac{\epsilon^2}{2}\int_0^T (\frac{\gamma(X(t))}{b(X(t))})^2 dt\big), \tag{3.14}$$

and its logarithmic derivative at $\mathbb{Q}_0$ in the γ direction results in the Malliavin weight for the Greek Rho (see [18]):

$$\int_0^T \frac{\gamma(X(t))}{b(X(t))} dW(t). \tag{3.15}$$

Barucci et al. coins the term *the Girsanov factor* for the ratio $\frac{\gamma(X(t))}{b(X(t))}$ and investigates its dynamics for a special choice of perturbation by choosing $\gamma(X(t))$ as the first variation process and define the price-volatility feedback effect rate.

The causality behind volatility feedback effect is the time-varying risk premium on the underlying asset. In [6], this causality is used to relate volatility feedback effect to Girsanov transformation and Malliavin calculus notions. We start by defining the *first variation* of $X(t)$, $Y(t) = \frac{\partial}{\partial x} X^x(t),\ t \in [0,T]$:

$$dY(t) = \mu^{'}(X^x(t))Y(t)dt + \sigma^{'}(X^x(t))Y(t)dW(t), \quad Y(0) = 1. \tag{3.16}$$

Now suppose that we choose $\gamma(X(t)) := Y(t)$ and analyze the change of measure induced by an infinitesimal random perturbation by $\epsilon Y(t)$, where ϵ is a small real parameter. We have seen in Equation (3.14) that this perturbation defines a new Brownian motion $\{\tilde{B}(t), t \in [0,T]\}$ under the new measure $\mathbb{Q}_0$:

$$d\tilde{B}(t) = \epsilon \frac{Y(t)}{\sigma(S(t))} dt + dB(t), \tag{3.17}$$

where $z(t) = \frac{Y(t)}{\sigma(S(t))}$ is called the *rescaled variation*. In line with the dynamical systems framework, one can think of $B(t)$ as the reference sample path and $\tilde{B}(t)$ as the perturbed path, and Equation (3.17) shows that the perturbation of the process $X(t)$ by $\epsilon Y(t)$ results in the change of the measure by the rescaled variation $z(t)$. Volatility feedback effect rate is defined as the change in the rescaled variaton of a stochastic process through time and in this context it allows us to analyze how the measure is affected by perturbations at different times. The following theorem, which is first proposed in [6] plays an important role in our study:

Theorem 3.1. *The rescaled variation is a differentiable function with respect to* t*, with its logarithmic derivative* $\lambda(t)$ *being called the feedback effect rate. Thus we have*

$$Z(t) = \exp\left(\int_s^t \lambda(\tau)d\tau\right)Z(s), \quad s \leq t, \tag{3.18}$$

where $t \in [0,T]$.

Proof. We closely follow the proof in [6]. Let us first obtain the differential equations that define $\sigma(\cdot)$ and $\frac{1}{\sigma(\cdot)}$. Applying Itô formula to $f(x) = \sigma(x)$ and $g(x) = \frac{1}{\sigma(x)}$

respectively, which satisfy the usual regularity assumptions, we can write

$$\begin{aligned} d\sigma(X(t)) &= \sigma^{'}(X(t))(\mu(X(t))dt + \sigma(X(t))dB(t)) + \frac{1}{2}\sigma^{''}(X(t))\sigma^2(X(t))dt, \\ d(\frac{1}{\sigma(X(t))}) &= -\frac{\sigma^{'}(X(t))}{\sigma^2(X(t))}\mu(X(t))dt - \frac{\sigma^{'}(X(t))}{\sigma(X(t))}dB(t) - \frac{1}{2}\sigma^{''}(X(t))dt \\ &\quad + \frac{(\sigma^{'}(X(t)))^2}{\sigma(X(t))}dt. \end{aligned}$$

Then we obtain the differential equation for $Z(\cdot)$ as follows:

$$\begin{aligned} dZ(t) &= d(\frac{Y(t)}{\sigma(X(t))}) \\ &= Y(t)d(\frac{1}{\sigma(X(t)}) + \frac{1}{\sigma(X(t)}dY(t) + \left\langle dY(t), d(\frac{1}{\sigma(X(t))}) \right\rangle \\ &= \frac{Y(t)}{\sigma(X(t))}(-\frac{\sigma^{'}(X(t))}{\sigma}(X(t))\mu(X(t))dt - \sigma^{'}(X(t))dB(t) \\ &\quad -\frac{1}{2}\sigma(X(t))\sigma^{''}(X(t))dt + (\sigma^{'})^2dt + \frac{Y(t)}{\sigma(X(t))}(\sigma^{'}(X(t))dB(t) \\ &\quad +\mu^{'}(X(t)dt)) - \frac{Y(t)}{\sigma(X(t))}(\sigma^{'})^2dt, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the quadratic covariation. The logarithmic derivative of $Z(t)$ can be expressed as follows:

$$\frac{dZ(t)}{Z(t)} = \left[\mu^{'}(X(t)) - \frac{\sigma^{'}(X(t))}{\sigma(X(t))}\mu(X(t)) - \frac{1}{2}\sigma(X(t))\sigma^{''}(X(t))\right] dt.$$

Let us define $\frac{dZ(t)}{Z(t)} = \lambda(t)dt$. Integrating both sides from s to t yields

$$\ln Z(t) - \ln Z(s) = \int_s^t \lambda(\tau)d\tau,$$

which gives

$$Z(t) = \exp\,(\int_s^t \lambda(\tau)d\tau)Z(s),$$

where $\lambda(t)$ is defined as

$$\lambda(t) = -\frac{1}{2}\left[-2\mu^{'}(X(t)) + 2\mu(X(t))\frac{\sigma^{'}(X(t))}{\sigma(X(t))} + \sigma(X(t))\sigma^{''}(X(t))\right]. \qquad (3.19)$$

□

Theorem (3.1) states the basis of our study: the feedback effect rate is the LLE of the rescaled variation process, which is responsible for the measure transformation resulting from a perturbation of the stochastic process $X(t)$. This claim is supported in [38] since the authors stated that the volatility feedback effect rate can be seen as the appreciation rate of the rescaled variation and that while large positive values of λ indicates market instability, negative feedback effect rate values imply market stability in the sense that the market oscillates around an equilibrium state. We have already seen in Definition (3.1) that, negative Lyapunov exponents indicate stochastic stability of the underlying SDE by means of its reactions under random perturbations. The following proposition explains the relation between negativity of the feedback effect rate and the duration of the effect of perturbations on the process, which results in what the authors uses the term *remote memory*, which should not be confused with the long-memory observed in fractional Brownian motions in case $H > 1/2$.

Proposition 3.2. *Assume that $\mu = 0$. Furthermore assume that there exists $\delta > 0$ such that the price-volatility feedback effect rate associated to price process defined by Equation (3.24) satisfies*

$$\lambda(t) < -\delta, \quad \forall t \in [0, T].$$

Then, the market has no remote memory (that is $Z(t) \to 0$ as $t \to +\infty$). More precisely, we have the estimate

$$|Z(t)| \leq \exp(-\delta(t - t_0))|Z(t_0)|, \quad \forall t \in (t_0, T].$$

Proof. See [38] for the proof of this proposition. □

The feedback effect rate, the LLE of the rescaled variation, is not a constant quantity and oscillates with the transition of the underlying stochastic process between stability and instability in the sense that the increase and decrease in the sensitivity of the underlying process to random perturbations. This feature of fluctuating feedback rates can be compared to features of multifractal processes that can be observed as volatility clustering, intermittency and the decreasing dimension of the sets upon which large singularities are observed, i.e. the definitive duality of multifractals. Since one can consider multifractals as random measures as well as random processes, a continuous change of measure is observed through both time and scale. Since the feedback effect rate corresponds to LLEs of the rescaled variation, dimensional properties would coincide with that of a multifractal process. We will in fact display this feature of the feedback effect rate in our empirical analysis.

Let us briefly summarize how the feedback effect rate is computed in the next subsection.

3.2.2.1 Quadratic Variation and Covariations

To compute volatility feedback effect rate $\lambda(.)$, we need to know the analytic expressions of $\sigma(\cdot)$ and $\mu(\cdot)$ in advance, which is generally not the case in applications. One can overcome this difficulty by using non-parametric methods for estimation of high-frequency volatility and covariance series. For this purpose, Malliavin and Mancino developed a non-parametric method based on Fourier analysis to compute time series volatility for semimartingales in [37]. In what follows, $\langle \cdot, \cdot \rangle$ denotes quadratic covariation. Let us first give the representations of the quantities needed as follows

$$\langle dX(t), dX(t) \rangle = Adt, \langle dX(t), dA(t) \rangle = Bdt, \langle dB(t), dX(t) \rangle = Cdt.$$

So, the instantaneous quadratic variation and covariatons are defined as functions of time. Furthermore, the following theorem in [6] states that volatility feedback effect rate $\lambda(\cdot)$ can be expressed as a function of A, B and C. Note that, in this setting, the variance of the log return series is equal to A, the quadratic covariation between log return and log return variance series is equal to B. Hence, C is the quadratic covariation betweeen B and the log return series.

Theorem 3.3. *The volatility feedback effect rate function $\lambda(\cdot)$ can be expressed as*

$$\lambda(t) = \frac{3B^2}{8A^3} - \frac{C}{4A^2} + \mu^{'}(X(t)) - \mu(X(t))\frac{B}{2A^2}. \tag{3.20}$$

Proof. We first observe that $Adt = \sigma^2(X(t))dt$. To compute B, which is defined as the quadratic covariation between return and return variance, we will first compute the differential form of A, which can be expressed as:

$$d(\sigma^2(X(t))) = 2\sigma(X(t))\sigma^{'}(X(t))dX(t) + \frac{1}{2}2\sigma^{'}(X(t))(\sigma^{'}(X(t)\sigma(X(t)))^2.$$

Then,

$$Bdt = \langle dX, dA \rangle = \left\langle dX, 2\sigma^{'}\sigma dX \right\rangle \quad \text{and} \quad \sigma(X(t))\sigma^{'}(X(t)) = \frac{B}{2A}.$$

Using these equations, we can write

$$\begin{aligned} \left\langle dX, 2d(\sigma(X(t))\sigma^{'}(X(t))) \right\rangle &= \left\langle dX, 2d(\frac{B}{2A}) \right\rangle \\ &= 2\left[\sigma^{''}(X(t))\sigma(X(t)) + (\sigma^{'}(X(t)))^2)\sigma^2(X(t)\right] dt \\ &= \frac{A\langle dX, dB \rangle - B\langle dX, dB \rangle}{A^2}. \end{aligned} \tag{3.21}$$

Substituting Equation (3.21) in Equation (3.19), we obtain the following representation of λ :

$$\lambda(t) = \frac{3B^2}{8A^3} - \frac{C}{4A^2} + \mu^{'}(X(t)) - \mu(X(t))\frac{B}{2A^2}, \quad t \in [0, T].$$

□ □

3.2.3 Estimating Volatility

The Fourier series method to estimate volatility was first proposed in [37]. The method requires computation of Fourier coefficients of the series. After computing the coefficients, it is possible to reconstruct the series using Fourier-Féjer inversion formula. The first step in applying the method is scaling the original sampling interval to $[0, 2\pi]$. Moreover, the series that we will compute the coefficients must be detrended in such a way that we will have $X(0) = X(2\pi)$. The Fourier coefficients of the series are then computed using the following equations:

$$\begin{gathered} a_0(dX) = \frac{1}{2\pi}\int_0^{2\pi} dX(t), \quad a_k(dX) = \frac{1}{\pi}\int_0^{2\pi} \cos(kt)dX(t), \\ b_k(dX) = \frac{1}{\pi}\int_0^{2\pi} \sin(kt)dX(t), \quad t \in [0, 2\pi]. \end{gathered} \tag{3.22}$$

Applying integration by parts and previous-tick interpolation scheme, in order to avoid any bias in the computation of volatilities (see [7]), the integral equation for $a_k(dX)$ in equation (3.22) can be approximated by:

$$a_k(dX) = \frac{X(2\pi) - X(0)}{\pi} + \int_0^{2\pi} \sin(kt)X(t)dt.$$

Previous-tick interpolation assumes $X(t) = X(t_i)$ on $[t_i, t_{i+1}]$, this assumption leads to the following approximation:

$$\frac{k}{\pi}\int_{t_i}^{t_{i+1}} X(t)dt = X(t_i)\frac{1}{\pi}[\cos(kt_i) - \cos(kt_{i+1})].$$

Then, we can compute the Fourier coefficients of X by the following equation:

$$a_k(dX) = \frac{X(2\pi) - X(0)}{\pi} + \sum_{i=1}^{N} X(t_i)\frac{1}{\pi}[\cos(kt_i) - \cos(kt_{i+1})].$$

The modified coefficients defined below are used in the computation of volatility series in order to guarantee its positivity (see [38] for details):

$$a_k^* = \begin{cases} a_k(dp) & \text{for } k > 0, \\ a_{-k}(dp) & \text{for } k < 0\,, \end{cases} \qquad b_k^* = \begin{cases} b_k(dp) & \text{for } k > 0, \\ -b_{-k}(dp) & \text{for } k < 0\,, \end{cases}$$

with $a_0^* = b_0^* = 0$. The Fourier coefficients of the volatility series are represented in terms of a_k^* and b_k^* are as follows:

$$\begin{aligned} a_k(A) &= \lim_{N\to\infty} \frac{1}{2N+1}\sum_{s=-N}^{N-k} [a_s^*(dX)a_{s+k}^*(dX) + b_s^*(dX)b_{s+k}^*(dX)], \\ b_k(A) &= \lim_{N\to\infty} \frac{1}{2N+1}\sum_{s=-N}^{N-k} [a_s^*(dX)b_{s+k}^*(dX) - b_s^*(dX)a_{s+k}^*(dX)]. \end{aligned}$$

Finally, the volatility series are reconstructed using the Fourier-Féjer inversion formula given by

$$A(t_i) = \lim_{N\to\infty} \sum_{k=0}^{N} (1 - \frac{k}{N})[a_k(A)\cos(kt_i) + b_k(A)\sin(kt_i)].$$

We iterated Fourier series method three times to compute A, B and C. The sensitivity and the resolution of the quantity decreases at each iteration and in order to avoid estimation errors in these computations, a smaller number of coefficients in the Fourier-Féjer inversion formula is used. It is shown in [7] that Fourier series method results in an unbiased estimator of volatility. The coefficients of A are used to obtain the coefficients of B, and then the coefficients of B are used to obtain the coefficients of C [6]. After all three quantities computed, the volatility feedback effect rate can be computed using Equation (3.20).

3.2.4 Instantaneous Volatilities and the Feedback Effect Rate

To investigate market stability, or more specifically the stability of a chosen asset price process, we employ the Fourier method to estimate the instantaneous feedback effect rate values. Since the feedback rate is suggested as a measure of market stability in terms of market liquidity, we have chosen arguably the most liquid exchange rate in the world, the EUR/USD exchange rate, and two relatively illiquid datasets, BIST30 index and USD/TRY exchange rate.

In what follows, assume that the stock price $S(\cdot)$ follows a diffusion process in the form given below:

$$dS(t) = \tilde{\mu}(S(t))S(t)dt + \tilde{\sigma}(S(t))S(t)dB(t); \quad S(0) = s \in \mathbb{R}^+, \quad t \in [0,T]. \tag{3.23}$$

where $\tilde{\mu}(\cdot)$ and $\tilde{\sigma}(\cdot)$ are deterministic functions that are continuously differentiable and satisfy the usual assumptions [38]. We further assume that volatilities are functions of price levels. Hence, the dynamics of the logarithmic price process $X(t) = \log(S(t))$ has the following form:

$$X(t) = \log(S(0)) + \int_0^t [\tilde{\mu}(S(s)) - \frac{1}{2}\tilde{\sigma}^2(S(s))]ds + \int_0^t \tilde{\sigma}(S(s))dB(s). \tag{3.24}$$

For simplicity, let us define $\mu(x) := \tilde{\mu}(\exp(x)) - \frac{1}{2}\tilde{\sigma}^2(\exp(x))$ and $\sigma(x) = \tilde{\sigma}(\exp(x))$ to obtain the following stochastic differential equation (SDE) for $X(t)$:

$$dX(t) = \mu(X(t))dt + \sigma(X(t))dB(t), \quad X(0) = \log(s) = x.$$

We first estimate the instantaneous volatilities of the return series where we observe characteristic features of financial volatility such as clustering and extreme observations.

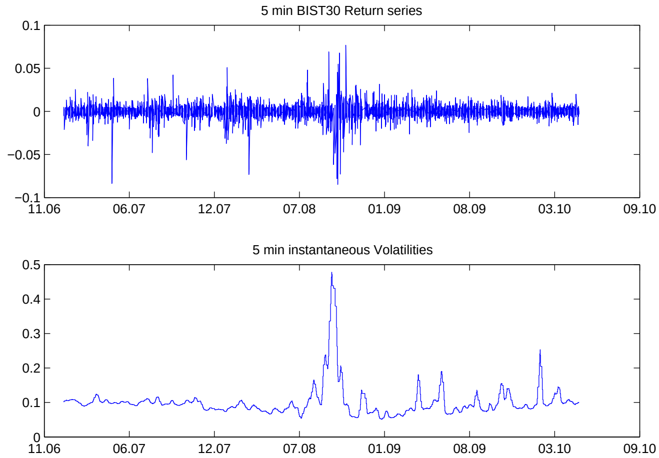

Figure 3.1: BIST30 returns and instantaneous volatility estimations

The BIST30 sample includes the observations during the 2008 crisis and we observe large volatility estimations around July and August 2008. An important point is that before July 2008, volatility levels are very low, without any significant clustering effect. However, after the extreme volatility observed around September, we also observe large positive feedback effect rate values and after that date, the volatility levels increase and on the other hand, feedback estimations decrease to before crisis levels and we observe a significant upward movement in the index maintained until March 2010, where the higher volatility are coupled with negative or small feedback values result in the stable behavior of the market.

Volatility series of EUR/USD observations between January 2016 and January 2018 is very low. During the almost two year period we observe very low variability and the exchange rate oscillates between 1.2 and 1.3 levels. When we analyze the joint behavior of volatility and the feedback effect rate we observe that during periods with higher volatility, the feedback effect is negative, resulting in the observed stability of the exchange rate.

The feedback series of USD/TRY and BIST30 index takes both positive and negative values with large magnitudes. The large positive feedback observations point out to instability in the sense that the market is not liquid enough to handle small fluctuations, which effectively implies that prices are easily effected by marginal buy and sell transactions of significant levels.

3.3 Generalized Dimensions of the Feedback Effect Rate

Feedback is a characteristic feature of chaotic dynamical systems. However, the majority of multifractal models does not feature feedback. The sensitive dependency to initial conditions point out to chaotic behavior of the underlying process or system, which is assumed to manifest itself in the autocorrelation structure of multifractal

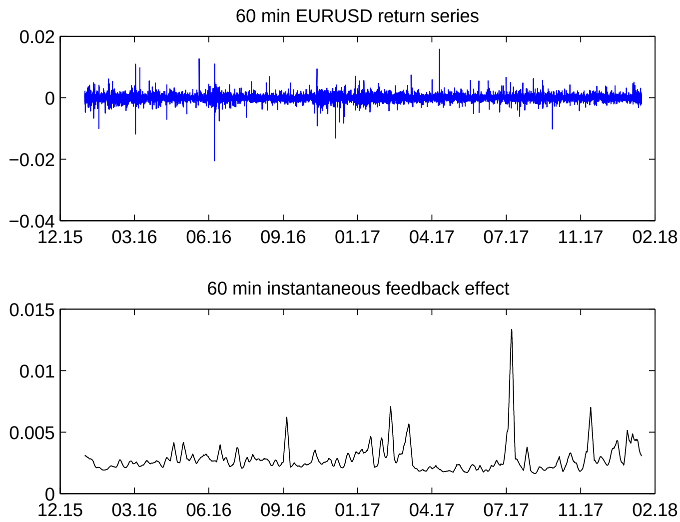

Figure 3.2: EUR/USD returns and instantaneous volatility estimations

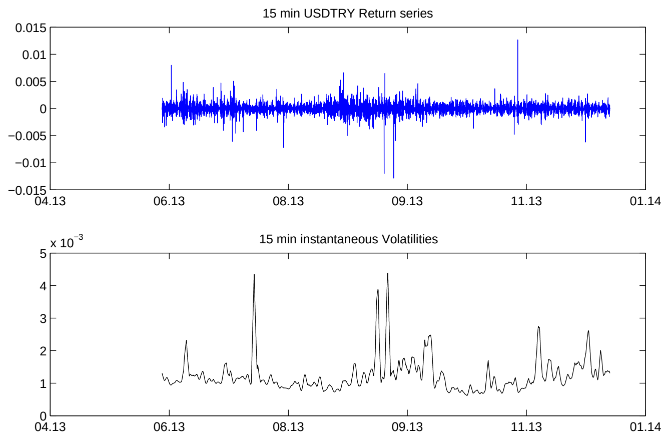

Figure 3.3: USD/TRY returns and instantaneous volatility estimations

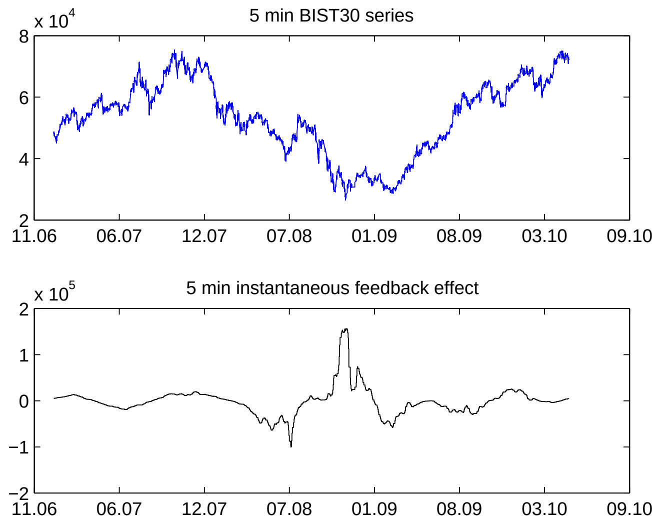

Figure 3.4: BIST30 observations and instantaneous feedback estimations

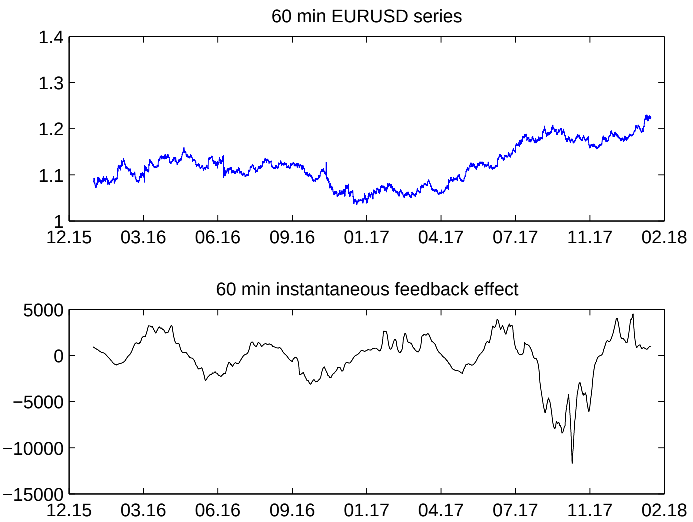

Figure 3.5: EUR/USD observations and instantaneous feedback estimations

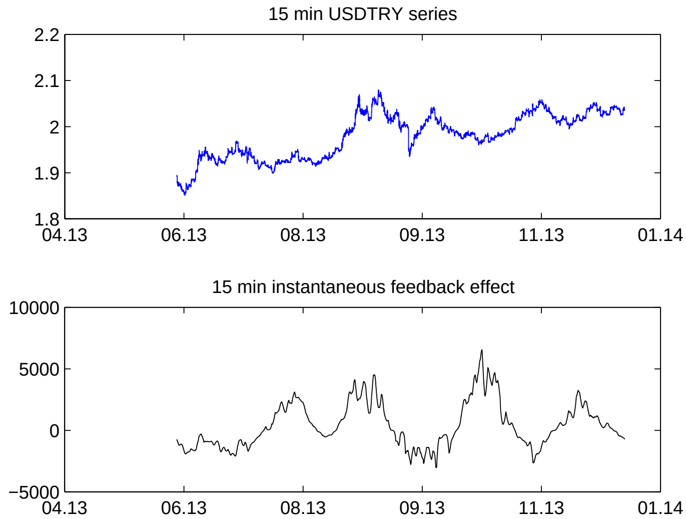

Figure 3.6: USD/TRY observations and instantaneous feedback estimations

models.

We have seen that the price volatility feedback effect rate is in fact the local Lyapunov exponent of the Girsanov factor that causes the change of measure. A critical assumption that allows for the computation of the feedback effect rate is the non-specified form of the volatility function $\sigma(\cdot)$. The dynamics of the stock price process $S(t)$ assumed to behave according to an Itô SDE. However, using the function $\sigma(\cdot)$, we can compute the feedback effect rate and investigate the stability and ergodicity properties of $S(t)$ [6].

We have also seen that continuous multifractal processes can be defined via fractional integration. Here, we do not specify the form of $\sigma(\cdot)$ but we assume that the SDE in Equation (3.23) represent the dynamics of a multifractal at a specific scale, at least in the sense of the dressed properties.

Based on this assumption and the result of Aurell et al. stated in Equation (3.8), we proceed to analyze the generalized dimensions of the feedback effect rate and compare with the generalized dimensions of the underlying return process.

The generalized dimensions of the price-volatility feedback effect rate, or the local Lyapunov exponent of the rescaled variation, is of similar characteristics for the positive values of h. The negative dimensions, which are also called latent dimensions, are out of the scope of our study and presented only for the interested reader. For the discussion on the negative dimensions we refer to [12] and the references therein.

The similar characteristics of the generalized dimensions of both the return series and the instantaneous price-volatility feedback effect rate series support our claim that the risk premiums have multiscale characteristics.

We observe that the generalized dimensions of the return series and the feedback effect

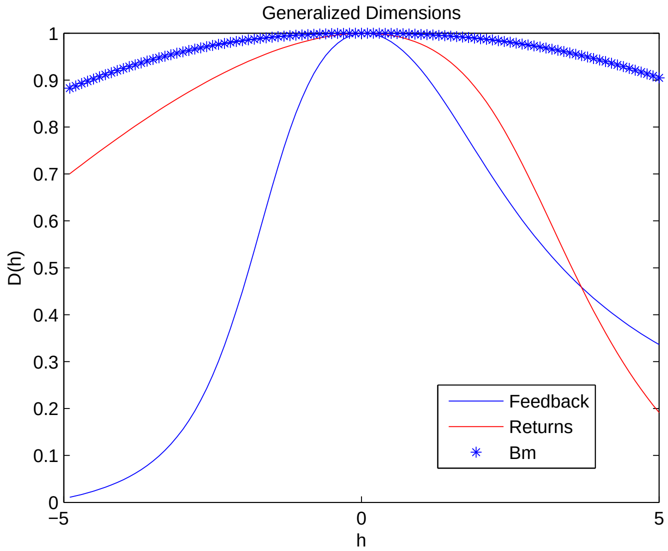

Figure 3.7: Comparison of the generalized dimensions of EUR/USD return series and the feedback effect

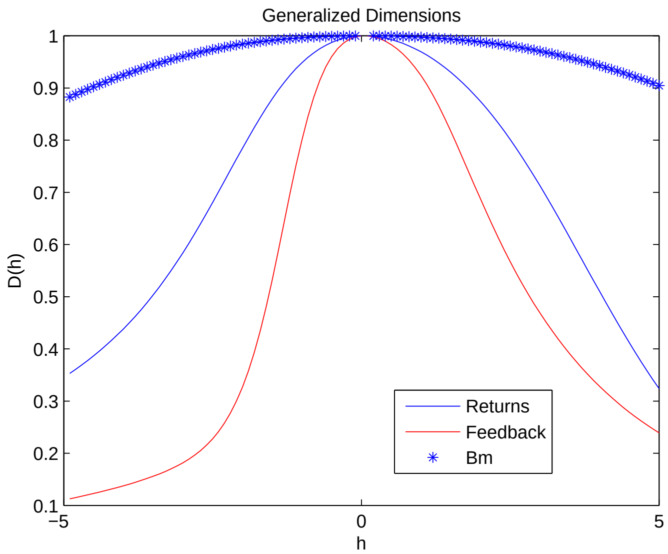

Figure 3.8: Comparison of the generalized dimensions of USD/TRY return series and the feedback effect

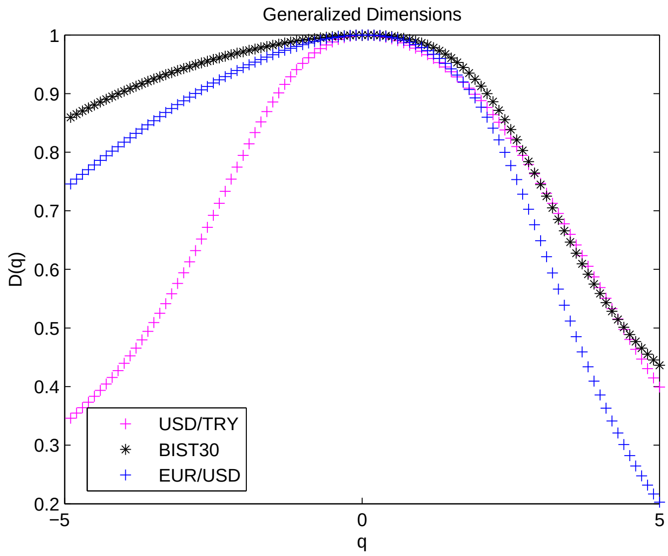

Figure 3.9: Comparison of the generalized dimensions of BIST30, EUR/USD and USD/TRY return series

rate series coincide in general. Our analysis support our claim that the feedback effect rate scales according to the generalized dimensions, which is in accordance with the result obtained by Aurell et al. for local Lyapunov exponents [2]. Since the feedback effect is a measure of the time-varying risk premiums in asset prices, we interpret the observed characteristics of the generalized dimensions of the feedback effect rate as the evidence of the existence of multiscale risk premiums in asset prices. This interpretation is valid for all three datasets we have analyzed, of various levels.

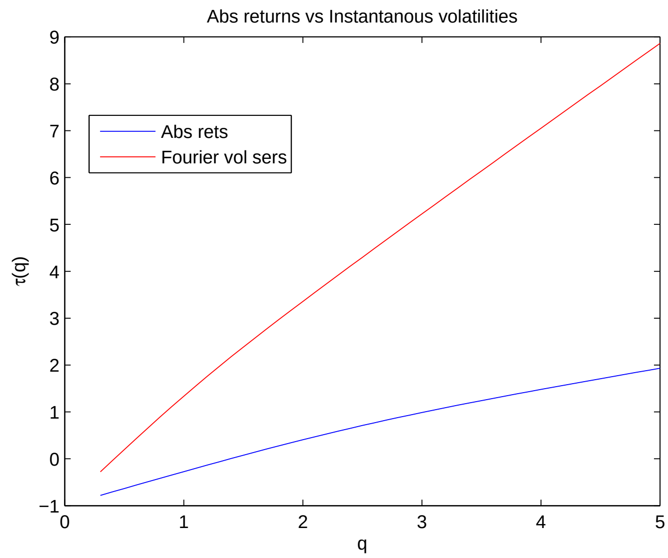

Figure 3.10: The scaling exponents of the absolute returns and the Fourier volatilities of BIST30

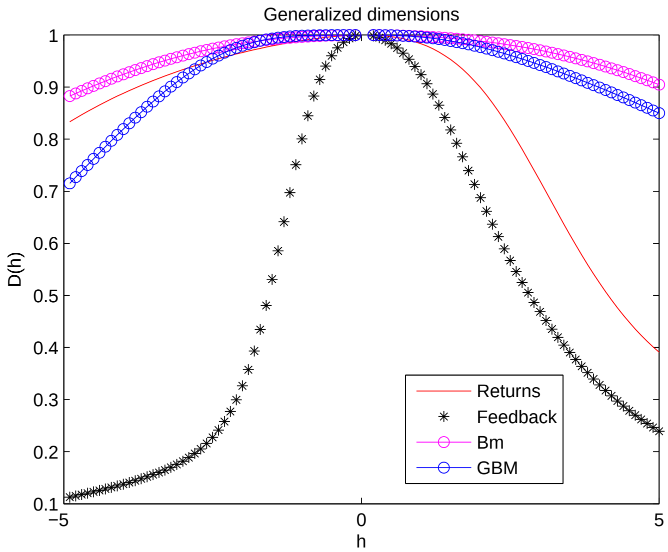

Figure 3.11: Comparison of the generalized dimensions of BIST30 return series and the feedback effect

This result have two main implications for financial modelling practice:

1. Market risk premium required by investors for the compensation of risk depends on the time scale, or the investment horizon, in a non-linear way: This result would not surprise the practitioners since the risk premium has an upper bound, i.e. the risk premium of a risky asset can not always increase with increasing time scale. This is in accordance with the flattening end of the scaling function $\zeta(q)$. The volatility does not always increase linearly with the time scale. The rate of increase decreases after a certain time window.

2. Market risk premium depends also on moments of higher order: The risk can not only be measured with the volatility, the second order moment, of the return series. Investors also take into account the higher moments such as skewness and kurtosis into consideration. The dimensional analysis of the feedback effect reveal the multiscale nature of time varying risk premiums present in asset prices.

CHAPTER 4

CONCLUSION

In this thesis, we study the multifractal processes and stability properties of stochastic processes with the aim of analyzing the multiscale characteristics of the time-varying risk premiums present in financial asset prices. We employ Malliavin calculus techniques to analyze the behavior of dynamic risk premiums by estimating the price-volatility feedback effect rate. Throughout our study, we have discovered that the price-volatility feedback effect rate is the local Lyapunov exponent of the perturbation resulting in the change of measure. Our aim is to prove the existence of multiscale risk premiums via dimensional analysis of the feedback effect rate.

For this purpose, we started with the investigation of the multifractal processes and their properties, with a focus on the dimensional characteristics. Even though multifractal processes are defined in terms of their scale dependent behavior, the theoretical framework was in fact built on the duality between the dimension of the fractal sets that support extreme observations. The detailed conceptualization includes singularities, Hölder exponents, Hausdorff dimensions and the Legendre transforms that knits the singular values to their supporting fractal sets in a unique way that defines multifractals. The resulting connection can be viewed in terms of both the scaling exponent and the singularity spectrum of the multifractal process. This connection also characterizes the generalized dimensions of the underlying process, which provides us with the chance to use the Lyapunov exponents in the study of multifractals. The Lyapunov exponents are defined in order to analyze the stability properties of dynamical systems and stochastic processes and play a similar role to codimensions in the multifractal framework.

Another approach to stability analysis of financial markets using Malliavin calculus techniques resulted in the concept of the price-volatility feedback effect rate. The aim of measuring the feedback effect rate is to identify how random perturbations enforce changes in the underlying measure of the underlying stochastic process. In our study, we discovered that the feedback effect rate is in fact the local Lyapunov exponent of the perturbation of the underlying probability measure. This observation suggests applying multifractal analysis to feedback effect rate series to analyze dimensional properties. Our analysis shows that the generalized dimensions of the return series and the estimated instantaneous price-volatility feedback effect rate series exhibit similar characteristics, a result which we interpret as the proof of existence of multiscale risk premium in financial asset prices.

We mainly employed the Multifractal Detrended Fluctuation Analysis (MFDFA) to estimate multiscale distributional characteristics of three datasets: a market index, BIST30, the USD/TRY exchange rate, which can be seen as an example of a relatively less liquid exchange rate and the EUR/USD exchange rate, which is arguably the most liquid exchange rate in the world, to compare their multifractal and feebdack effect rate characteristics.

Our empirical study is multilayered: We first estimate instantaneous volatility series using Fourier series method of Malliavin and Mancino and then employ multifractal analysis to estimated volatility series. We also perform multifractal analysis of absolute return series and compare the multifractal characteristics of volatilities and absolute returns. Then, we estimate the price-volatility feedback effect rate series to analyze the stability dynamics of the price processes.

Finally, we analyze the dimensional properties of the estimated price-volatility feedback effect rate series and based on the result obtained for the predictability of multifractals we compare the generalized dimensions of the return series with the price-volatility feedback effect rate series, to obtain the desired result that the generalized dimensions spectrums of both series coincide, which shows the existence of multiscale risk premiums in the analyzed datasets.

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APPENDIX A

DISTRIBUTIONAL PROPERTIES OF ADDITIVE PROCESSES

A.1 Additivity of Stochastic Processes and Infinite Divisibility

Additivity is the fundamental property that is needed to build up stochastic models if one aims to employ stochastic integration. Consider an i.i.d. sequence of random variables $\{\Theta_i\}$ where we may explicitly know the form of the underlying distribution or not. A basic question arises when one attempts to characterize the distribution of the sum of the sequence: "Can the distribution of the sum be explained in terms of the distribution of the sequence?"

Stochastic processes are classified according to the properties of their distributions under arithmetic operations. The application of these operations to stochastic processes is in fact non-trivial; For instance, to perform addition to a sequence of random variables, the sequence must be realizations of an "additive" stochastic process:

Definition A.1. A stochastic process $\{\Theta_t, t \in [0, T]\}$ is called an **additive** process if the following two conditions are satisfied:

- $X(t) = 0, \quad$ *a.s.*
- For any choice of $t_0 \leq t_1 < \cdots < t_n < T$, $X(t_i) - X(t_{i-1})$, $i = 1, 2, \cdots, n$, are independent.

Investigation of the distributional characteristics of additive processes is of fundamental importance for statistical modelling and therefore financial modelling. The behaviour of sums and averages of financial returns holds very critical information for applicational purposes such as portfolio optimization, building trading strategies, pricing options and financial derivatives. The relation between the distribution of the sums of sequences and the underlying distribution of the sequence used to classify stochastic processes. In this regard, let us consider the additive process and their distributions. The **Fundamental Construction Theorem** (FCT) (see [25]), stated below, establishes the connection between additive processes and sequences of distributions of increments:

Theorem A.1 (Fundamental Construction Theorem)**.** *Assume that* $\{X(t), t \in [0, T]\}$ *is an additive stochastic process with* $X(t) - X(s) \sim \phi_{st}, 0 < s < t < T$*. Then, if a* *family* $\{\phi_{st}, 0 < s < t < T\}$ *of probability distributions, satisfies*

$$\phi_{su} = \phi_{st} * \phi_{tu}, s < t < u, \tag{A.1}$$

where $*$ *denotes convolution operator, then an additive process* $X_t, t \in [0, T]$ *can be constructed on a suitable probability space in such a way that the distribution of* $X(t) - X(s)$ *is given by* ϕ_{st}.

In the FCT, it is stated that the distribution of the increment of the process $\{X(t), t \in [0, T]\}$ can be defined as the convolution of the distributions of the "sub-increments" which can also be interpreted as the increments at different time-scales $s - u = (s - t) + (t - u)$: The distribution of the large time-scale increment $X(s) - X(u)$, can be obtained via convolution of the distributions of the small time-scale increments $X(s) - X(t)$ and $X(t) - X(u)$. In the early financial models, this relation is assumed to be linear and the distributions does not depend on the time-scale of the increment. This assumption leads to the definition of the notion of self-similarity for stochastic processes, which we will explain in the following sections.

Now consider a special family of stochastic processes, where each realization of the process is defined to be an additive process. This property is called **infinite divisibility** and the members of this family, infinitely divisible processes, include the Brownian motion, the fractional Brownian motion, Poisson process, *i.e.* Lévy processes are infinitely divisible.

Let us first recall the definition of infinite divisibility ([17]):

Definition A.2. A stochastic process $\Theta_t, t \geq 0$, is said to have an infinitely divisible distribution if for each $t \geq 0$ and $n = 1, 2, \ldots$, there exist a sequence of i.i.d. random variables $\Theta_{1,t}, \ldots, \Theta_{n,t}$ such that

$$\Theta_t \stackrel{d}{=} \Theta_{1,t} + \cdots + \Theta_{n,t}.$$

We have already seen that the additivity property is defined based on a sequence of distributions and their convolutions, which implies the existence of the sequence of distributions $\phi_{ij,t}$ satisfying:

$$\phi_{ij,t} = \phi_{ik,t} * \phi_{kj,t},$$

with $\Theta_{i,t} - \Theta_{j,t} \sim \phi_{ij,t}$.

Infinite divisibility property is frequently, and equivalently, defined via *characteristic functions* of the sequence of distributions $\phi_{i,t}, i = 1, \cdots, n$. Let us first recall the definition of the characteristic function of a distribution (or random variable):

Definition A.3. Let X be a random variable with probability distribution F. The characteristic function of F (or of X) is the function φ defined for real ζ by

$$\varphi(\zeta) \;=\; \int_{-\infty}^{\infty} \exp(i\zeta x) F(dx) = u(\zeta) + iv(\zeta), \tag{A.2}$$

where

$$u(\zeta) \;=\; \int_{-\infty}^{\infty} \cos(\zeta x)F(dx), \quad v(\zeta) = \int_{-\infty}^{\infty} \sin(\zeta x)F(dx), \tag{A.3}$$

Characteristic functions exhibit some important properties which allow for the analysis of random variables and their distributional properties. The following lemma lists the main properties:

Lemma A.2. *a) φ is continuous,*
b) $\varphi(0) = 1$ and $|\varphi(\zeta) \leq 1|$ for all ζ,
c) $aX + b$ has the characteristic function

$$\mathbb{E}[\exp(i\zeta(aX + b))] = \exp(ib\zeta)\varphi(a\zeta)$$

In particular, $\bar{\varphi} = u - iv$ is the characteristic function of $-X$.
d) u is even, v is odd. The characteristic function is real iff F is symmetric.
e) For all ζ, $0 \leq 1 - u(2\zeta) \leq 4(1 - u(\zeta))$.

The famous Lévy-Khinchin representation describes the form of the characteristic functions of infinitely divisible distributions, which is stated in the following theorem:

Theorem A.3. *Let F be an infinitely divisible distribution on $\mathbb{R}^d$. Its characteristic function can be represented as:*

$$\begin{aligned} \Phi_F(z) &= \exp(\Psi(z)), z \in \mathbb{R}^d, \\ \Psi(z) &= -\frac{1}{2}zAz + i\gamma z + \int_{\mathbb{R}^d} (\exp(izx - 1 - izx\mathbf{1}_{|x|\leq 1}))\nu(dx), \end{aligned}$$

where A is a symmetric positive $n \times n$ matrix, $\gamma \in \mathbb{R}^d$ and ν is a positive measure satisfying

$$\int_{|x|\leq 1} |x|^2 \nu(dx) < \infty, \quad \int_{|x|\geq 1} \nu(dx) < \infty,$$

and it is called the Lévy measure of the distribution F.

An important result can be verified for infinitely divisible characteristic functions:

Remark A.1. A characteristic function Φ is infinitely divisible *iff* for every $n \in \mathbb{N}$ there exists a characteristic function Φ_n satisfying

$$\Phi(\Theta) = (\Phi_n(\Theta))^n. \tag{A.4}$$

Infinite divisibility of a distribution (or a random process) does not conclude any restrictions on the shape of the distribution of the sum of the sequence. An example is the exponentially distributed random variables; the sum of i.i.d. exponentially distributed random variables has Gamma density. However, the sum of i.i.d. Gaussian random variables is again Gaussian. The distinction will be clear in the following sections where we introduce the **stable** distributions and random variables. The motivation is the search for a function $f(n)$ that would make the following relation possible:

$$\Theta_{1,t} + \cdots + \Theta_{n,t} \stackrel{d}{=} f(n)\Theta_{i,t}, \quad n = 1, 2, \ldots \tag{A.5}$$

where the function $f(n)$ is a deterministice function of n, which allows to express the distribution of the sequence in terms of the underlying distribution of the sample. The existence of this strong bond resulted in one of the most important theorems in statistical theory: **The Central Limit Theorem** (CLT). The CLT and its connection to *self-similar* random processes will be briefly explained in the next section.

A.2 The Roots of Self-Similarity: The Central Limit Theorem

The most important theorem that constitutes the basis for stochastic modelling is arguably the Central Limit Theorem (CLT). Consider any hypothetical statistical problem, where there are a large number of observations where one wants to characterize the statistics of the sample. It is possible to characterize the behaviour of a data sample with a large number of observations via the statement of the CLT. There are various alternative probability distributions and most of the time the most suitable choice is not apparent. However, one can switch her approach to the problem and investigate the behaviour of the sums and averages of the observations. A probability distribution is basically a rule that assigns each observation a probability depending on its magnitude. The integrability property of probability distributions result in distribution functions with a decrease in probability as the magnitude increases: the tails of the probability distribution corresponds to numbers with large magnitudes which are assigned with low probabilities of occurence. The CLT basically states that the sums of random variables can be characterized with a standard Gaussian distribution. We first state the definition where the notion of the central limit is expressed explicitly:

Definition A.4. A sequence $\{X_k, k \geq 1\}$ of real random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to have *the central limit property*, if there are sequences of constants a_n and b_n, $n \geq 1$, such that the sequence

$$Y_n = \frac{\sum_{k=1}^{n} X_k - a_n}{b_n} \tag{A.6}$$

converges in distribution to a standard Gaussian random variable, i.e.:

$$\lim_{n\to\infty} \mathbb{P}(Y_n \leq x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-\frac{u^2}{2})du, \quad x \in \mathbb{R}, \tag{A.7}$$

which can also be stated as $Y_n \sim \mathcal{N}(0,1)$, Gaussian distribution with $\mathbb{E}[Y_n] = 0$ and $Var(S_n) = 1$ where $Var(X_n) = \mathbb{E}[(X_n - \mu)^2]$.

The central limit property puts the Gaussian distribution to the very centre of statistical modelling. As a subdiscipline of statistical modelling, the central limit property is still the most widely used property in financial modelling practices. The Gaussianity assumption and the BSM model are still in use as it allows for a complete and closed-form characterization of many complex financial products. Let us give the *Lindeberg-Lévy* version of the CLT below ([30]):

Theorem A.4. *Let* $\{X_n, n \geq 1\}$ *be an independently identically distributed (i.i.d.) sequence in* $L^2(\Omega, \mathcal{F}, \mathbb{P})$*, the space of square integrable real random variables, and take* $\mathbb{E}[X_n] = \mu$ *and* $Var(X_n) = \sigma^2, \quad \forall n \geq 1$ *with* $\sigma > 0$*. Define*

$$S_n = \sum_{k=1}^{n} X_k, \quad \bar{S}_n = \frac{S_n}{n}, \quad \textit{and}$$
$$Y_n = \frac{S_n - n\mu}{\sigma\sqrt{n}} = \frac{\bar{S}_n - \mu}{\sigma/n}.$$

Then $\{X_n, n \geq 1\}$ *has the central limit property with* $a_n = n\mu$ *and* $b_n = \sigma\sqrt{n}$*.*

The CLT has various number of important implications on the behaviour of the sums of i.i.d. random variables. Most importantly, it allows for characterizing the sums of the random variables via only two statistics: the sample mean, or the centering parameter, μ and the sample variance σ^2. It is usually assumed that the centering parameter is trivial and can be shifted without distorting the shape of the distribution. This can be easily seen via the following example:

Example A.1. Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Then $X + c \sim \mathcal{N}(\mu + c, \sigma^2)$, where $c \in \mathbb{R}$ is a constant.

This feature is not specific to Gaussian random variables and holds also for various distributions, as we will see in the next sections. Another important feature of Gaussian distribution is that it can be characterized only by its first two moments, as we mentioned above, namely the mean and the variance. However, as we have seen that the role of the mean, or the centering or the location parameter, is not of practical importance and can easily be shifted along the real line (or space for higher dimensional variables), we focus our attention to the variance, or the scale parameter. We will avoid using the term scale parameter for the sample variance since it can cause confusion in the context of scale invariance and multifractals, which are the main topic of this study. The arguably most important implication of the CLT is that it defines the form of b_n as $\sigma\sqrt{n}$. For a sequence of i.i.d. random variables having the central limit property, one can say that the growth of the variance is proportional to the number of observations in the sample, which is equivalent to the proportionality of the standard deviation to the square root of the number of observations in the sample. The square root rule for the growth standard deviation, or the linear growth rule for the variance, has been generalized to define the self-similarity property of stochastic processes.

A.3 Stable Distributions and Processes

Stability property can be derived via a generalization of the CLT in terms of the form of the function $b(n)$ in Equation (2.4). The stable processes satisfy some deserved properties for modelling purposes such as the flexibility for modelling the tails of the distribution of observations. By choosing a stable random process as the source of randomness, one does not need to worry about the distribution of the data at any time-scale considered. When one is concerned about the time-scales of the sample, or its increments, stability property imposes that the shape of the distribution does not change with a change in time-scale. We will thorougly examine the behaviour of the financial returns to see whether this statement holds or not when we consider applications. The stability property is defined as follows:

Definition A.5. A random variable X has a stable distribution if and only if it has a *domain of attraction*, i.e., if there exist a sequence of $Y_1, Y_2, \ldots$ of i.i.d. random variables and sequences $\{d_n\}$ and $\{a_n\}$ of positive real numbers such that

$$\frac{Y_1 + Y_2 + \cdots + Y_n}{d_n} + a_n \xrightarrow{d} X. \tag{A.8}$$

where, in general, the form of d_n is

$$d_n = n^{1/\alpha} h(n) \tag{A.9}$$

with $h(x)$, $x \geq 0$ is a slowly varying function at infinity, i.e. $\lim_{x\to\infty} h(ux)/h(x) = 1$ for all $u > 0$ ([50], [17]). By saying that the stability property is a generalization of the CLT, it is meant that the the finite variance assumption has been relaxed in Definition (A.5). When Y_i's are i.i.d. random variables with finite variance, then X is Gaussian and the ordinary version of the CLT is obtained. Stable random variables are infinitely divisible, whereas the converse is not true. An important feature of stable processes is that the tails of the probability distribution obeys a power-law, which is also called Paretian, or scaling, tails:

$$\mathbb{P}(X > x) \sim Sx^{-\alpha}, \tag{A.10}$$

The most important parameter for a stable random variable is the "stability index" α. Stable random variables are also self-similar and the stability index α has a one-to-one correspondency with the self-similarity exponent H. Let us briefly summarize some of the properties of stable processes and mention some of the equivalent definitions of stability. Let us first give the definition of a stable distributed random variable ([50]):

Definition A.6. A random variable X is said to have a stable distribution if for any positive number a and b, there is a positive number c and a real number d such that

$$aX_1 + bX_2 \stackrel{d}{=} cX + d, \tag{A.11}$$

where X_1 and X_2 have the same distribution as X.

It can be seen that the above definition is a simplified restatement of Definition (A.5). In the following theorem, the role of the stability index α is expressed, which is the basis for motivation to employ stable processes in financial modelling:

Theorem A.5. *For any stable random variable X, there is a number $\alpha \in (0, 2]$ such that the number c in Equation (A.11) satisfies*

$$c^\alpha = a^\alpha + b^\alpha \tag{A.12}$$

The well-known Gaussian distributed random variable is the most famous member of the family of stable random variables. This is demonstrated in the following example:

Example A.2. If X is a Gaussian random variable with mean μ and variance σ^2, i.e. $X \sim N(\mu, \sigma^2)$, and X_1 and X_2 are equal in distribution to X. It is known that

$$aX_1 + bX_2 \sim N((a + b)\mu, (a^2 + b^2)\sigma^2) \tag{A.13}$$

which shows that Equation (A.12) holds with $c^2 = a^2 + b^2$ and

$$aX_1 + bX_2 \stackrel{d}{=} cX + d, \tag{A.14}$$

where $d = (a + b - c)\mu$.

The result obtained for the Gaussian case shows that a Gaussian random variable is stable with $\alpha = 2$. Recall that it is also $\frac{1}{2}$-self-similar. This observation is not trivial, it is in fact a general result for the stable random variables: a stable random variable with stability index α is $\frac{1}{\alpha}$-self-similar. We will mention this property in detail as we define the α-stable Lévy motion but first let us briefly mention some of the important properties of stable random variables. We begin with the parameters that determine the shape of the distribution of a stable random variable:

- the stability index $\alpha \in (0, 2]$,
- the location, or mean parameter $\mu \in (-\infty, \infty)$,
- the skewness parameter $\beta \in [-1, 1]$,
- the scale parameter $\sigma \in (0, \infty)$.

We will denote a stable random variable accordingly by $S_\alpha(\sigma, \beta, \mu)$, following [50]. The parameters α, β and μ are unique and when $\alpha = 2$ (Gaussian case), β is irrelevant, since Gaussian distribution is symmetric around its mean. In what follows, arithmetic properties of the stable random variables is summarized:

Definition A.7. Let X_1 and X_2 be two independent random variables with $X_i \sim S_\alpha(\sigma_i, \beta_i, \mu_i), \quad i = 1, 2$. Then $X_1 + X_2 \sim S_\alpha(\sigma, \beta, \mu)$ where

$$\sigma = (\sigma_1^\alpha + \sigma_2^\alpha)^{1/\alpha}, \quad \beta = \frac{\beta_1\sigma_1^\alpha + \beta_2\sigma_2^\alpha}{\sigma_1^\alpha + \sigma_2^\alpha}, \quad \mu = \mu_1 + \mu_2. \tag{A.15}$$

The skewness parameter β determines whether a stable random variable is symmetric about the location parameter μ. For instance, for any $0 < \alpha \leq 2$,

$$X \sim S_\alpha(\sigma, \beta, \mu) \Leftrightarrow -X \sim S_\alpha(\sigma, -\beta, \mu), \tag{A.16}$$

and a $S_\alpha(\sigma, \beta, \mu)$ is symmetric about μ iff $\beta = 0$.

We previously mentioned the scaling, or Paretian, tails of stable random variables. This property is more formally stated in the following way:

Remark A.2. Let $X \sim S_\alpha(\sigma, \beta, \mu)$ with $0 < \alpha < 2$. Then

$$\begin{cases} \lim_{x\to\infty} x^\alpha \mathbb{P}(X > x) = c_\alpha \frac{1+\beta}{2} \sigma^\alpha, \\ \lim_{x\to\infty} x^\alpha \mathbb{P}(X < -x) = c_\alpha \frac{1-\beta}{2} \sigma^\alpha, \end{cases} \tag{A.17}$$

where

$$c_\alpha = (\int_0^\infty \sin x dx)^{-1}.$$

The power-law behaviour of stable variables is used to obtain an important implication of the index α on the existence of moments of various orders. Since for $X \sim S_\alpha(\sigma, \beta, \mu)$ we have $\mathbb{E}[|X|^r] = \int_0^\infty \mathbb{P}(|X|^r > x)dx$, one can show that

$$\begin{cases} \mathbb{E}[|X|^q] < \infty & \text{if} \quad 0 < q < \alpha, \\ \mathbb{E}[|X|^q] = \infty & \text{if} \quad q \geq \alpha. \end{cases} \tag{A.18}$$

which implies α-stable random variables with $\alpha < 2$ have infinite second moments, which poses a huge problem in terms of applications to model financial returns as the volatility is defined via the second moment of the returns. In case $\alpha \leq 1$, we have infinite expectations.

The characteristic function of a stable random variable has specific form which is given in the following definition that is equivalent to Definitions A.5 and A.6:

Definition A.8. A random variable $X \sim S_\alpha(\sigma, \beta, \mu)$ has the characteristic function of the form:

$$\mathbb{E}[\exp(i\gamma X)] = \begin{cases} \exp\{-\sigma^\alpha |\gamma|^\alpha (1 - i\beta(sgn(\gamma)) \tan(\frac{\pi\alpha}{2}) + i\mu\gamma\} & \text{if } \alpha \neq 1, \\ \exp\{-\sigma|\gamma|(1 + i\beta\frac{2}{\pi}(sgn(\gamma)) \ln(\gamma)\} & \text{if } \alpha = 1. \end{cases} \tag{A.19}$$

where $sgn(\cdot)$ denotes the sign function.

A more familiar concept in the study of Lévy processes is the *Lévy-Khintchine representation*, which specifies the following form for the characteristic function of a stable random variable:

$$\mathbb{E}[\exp(i\gamma X)] = \begin{cases} \exp\{iK\gamma - \sigma^2\gamma^2\} & \text{if} \quad \alpha = 2, \\ \exp\{iK\gamma + P\int_0^\infty \psi(\gamma, x)\frac{dx}{x^{1+\alpha}} + Q\int_{-\infty}^0 \psi(\gamma, x)\frac{dx}{|x|^{1+\alpha}}\} & \text{if} \quad \alpha \leq 2. \end{cases} \tag{A.20}$$

where $M \in \mathbb{R}$, $\sigma \geq 0$ and P and Q are non-negative numbers and

$$\psi(\gamma, x) = \exp(i\gamma x) - 1 - \frac{i\gamma x}{1+x^2}.$$

Based on the representation given in Equation (A.20), the *Lévy measure* $L(dx)$ is defined as

$$L(dx) = \frac{P}{x^{1+\alpha}} \mathbf{1}_{(0,\infty)}(x)dx + \frac{Q}{|x|^{1+\alpha}} \mathbf{1}_{(-\infty,0)}(x)dx, \tag{A.21}$$

where $\mathbf{1}_A$ denotes the indicator function of set A.

Mandelbrot's idea to employ α-stable random variables in modelling of financial data is mainly built on the behaviour of their moments of various orders. This specific behaviour resulting from the power-law structure in the tails of the stable distributions, allows for more flexibility in terms of applications, as the fat-tailed distributions are frequently observed in analysis of financial data.

Now let us extend the stability property from random variables to random processes [27]:

Definition A.9. A stochastic process $\{X(t), t \in \mathbb{T}\}$, where $\mathbb{T}$ is an arbitrary set, is *stable* if all its finite dimensional distributions

$$X(t_1), X(t_2), \ldots, X(t_n), \quad t_1, t_2, \ldots, t_n \in \mathbb{T}, \quad n \geq 1$$

is stable. It is *symmetric stable* if all its finite-dimensional distributions are symmetric stable. Furthermore, it is symmetric stable iff all linear combinations

$$\sum_{i=1}^{n} a_i X(t_i), \quad n \geq 1, \quad t_1, t_2, \ldots, t_n \in \mathbb{T}, \quad a_1, a_2, \ldots, a_n \in \mathbb{R}$$

are symmetric stable.

The following example reveals some of the very important features of stable processes:

Definition A.10. A stochastic process $\{X(t), t \geq 0\}$ is called (standard) α-stable Lévy motion if

1. $X(0) = 0$,
2. X has independent increments: $X(t) - X(s) \perp X(s)$, $s < t$,
3. $X(t) - X(s) \sim S_\alpha((t-s)^{1/\alpha}, \beta, 0)$, for any $0 \leq s < t < \infty$.

A.4 Canonical Lévy measures of Infinitely Divisible Processes

Infinite divisibility is a very fundamental feature of the stochastic processes. It is directly related to the additivity of processes, which is a desired feature to define stochastic integrals. Infinite divisibility is equivalently defined for characteristic functions:

Theorem A.6. *A characteristic function W is infinitely divisible iff there exists a sequence (φ_n) of characteristic functions such that $(\varphi_n)^n \to W$.*
In this case W^t is characteristic function for every $t > 0$, and $W(\zeta) \neq 0$ for all ζ.

Theorem 2.3 allows for the specification of the most general form of infinitely divisible characteristic functions $W = \exp(\psi)$. It suffices to determine the general form of possible limits of sequences of characteristic functions $\exp(c_n(\zeta_n - 1))$ of the compound Poisson type, *i.e.* the possible limits of the characteristic functions of the form $W_n = \exp(\psi_n)$ with

$$\psi_n(\zeta) = c_n(\varphi_n(\zeta) - 1 - im_n\zeta), \tag{A.22}$$

and since W_n are infinitely divisible, its continuous limits are also infinitely divisible. Let us analyze the conditions under which there exists a continuous limit

$$\psi(\zeta) = \lim_{n\to\infty} \varphi_n(\zeta), \tag{A.23}$$

where it can be seen that φ_n is the characteristic function of a probability distribution F_n, the c_n are positive constants, and the centering constants m_n are real.

As it is always possible to recenter a distributon to zero, we will choose m_n accordingly when needed. The simplest such centering is obtained by the requirement that for $\zeta = 1$ the value of ψ_n be real. Let u_n and v_n denote the real and imaginery parts of φ_n, respectively. By Equation (A.3), this condition requires that

$$\beta_n = v_n(1) = \int_{-\infty}^{\infty} \sin x F_n(dx), \tag{A.24}$$

since

$$\psi(1) = \int_{-\infty}^{\infty} \exp(i\zeta x)F(dx) = u(1) + iv(1), \tag{A.25}$$

and with Equation A.24 for β_n, we obtain

$$\psi_n(1) = c_n(u(1) + iv(1) - 1 - iv(1)) = c_n(u(1) - 1), \tag{A.26}$$

which shows that centering is always possible. With it

$$\psi_n(\zeta) = c_n \int_{-\infty}^{\infty} [e^{i\zeta x} - 1 - i\zeta \sin x] F_n(dx). \tag{A.27}$$

Near the origin the integrand behaves like $-\frac{1}{2}\zeta^2 x^2$, which is the case with the normal distribution with zero mean and variance of ζ^2. The following lemma leads to the representation of infinitely divisible distributions in terms of canonical measures.

Lemma A.7. *Let $\{c_n\}$ and $\{\varphi_n\}$ be given. If there exist centering constants β_n such that ψ_n tends to a continouos limit ψ, then Equation A.27 will achieve the same goal.*

In what follows, it will be shown that with an arbitrarily chosen finite measure M, the integral in Equation A.28 defines an infinitely divisible characteristic function $\exp(\psi)$. Now let us define

$$\psi(\zeta) = \int_{-\infty}^{\infty} \frac{e^{i\zeta x} - 1 - i\zeta \sin x}{x^2} M(dx). \tag{A.28}$$

This integral is well-defined as the integrand is a bounded continuous function assuming at the origin the value $-\frac{1}{2}\zeta^2$. For the integral to be well-defined, it suffices that M attributes finite massses to finite intervals and that $M\{-\bar{x}, x\}$ increases sufficiently slowly for the integrals

$$M^+(x) = \int_x^{\infty} \frac{M(dy)}{y^2}, \qquad M^-(-x) = \int_{-\infty}^{-x} \frac{M(dy)}{y^2} \tag{A.29}$$

to converge for all $x > 0$. Measures defined by the densities $|x|^p dx$ with $0 < p < 1$ are typical examples. It will be proved that if the measure M has these properties, then Equation (A.28) defines an infinitely divisible characteristic function, and all such characteristic functions are obtained in this manner. The following definition introduces the special term for measure M:

Definition A.11. A measure M will be called canonical if it attributes finite masses to finite intervals and the integrals in Equation (A.29) converge for some (and therefore all) $x > 0$.

The following lemma provides the generalization for the study of infinitely divisible characteristic functions via canonical measures:

Lemma A.8. *If M is a canonical measure and ψ defined by Equation (A.28) then $\exp(\psi)$ is an infinitely divisible characteristic function.*

Proof of this lemma is especially important as two most widely used cases are considered:

Proof. a) Suppose that M is concentrated at the origin and attributes mass $m > 0$ to it. Then $\psi(\zeta) = -m\zeta^2/2$, and so $\exp(\psi)$ is a Gaussian characteristic function with variance ζ^2.

b) Suppose that M is concentrated on $|x| > \eta$ where $\eta > 0$. In this case, Equation A.28 may be rewritten in a simpler form. Indeed, $\frac{M(dx)}{x^2}$ now defines a finite measure with total mass $\mu = M^+(\eta) + M^-(-\eta)$. Accordingly, $(\frac{M(dx)}{x^2}/\mu = F(dx)$ defines a characteristic function φ, and obviously $\psi(\zeta) = \mu[\varphi(\zeta) - 1 - ib\zeta]$, where b is a real constant. Thus, in this case $\exp(\psi)$ is the characteristic function of the compound Poisson type, and hence infinitely divisible.

c) In the general case, let $m \geq 0$ be the mass attributed by M to the origin, and put

$$\psi_n(\zeta) = \int_{|x|>\eta} \frac{\exp(i\zeta x - 1 - i\zeta \sin x)}{x^2} M(dx). \tag{A.30}$$

Then

$$\psi(\zeta) = -\frac{m}{2}\zeta^2 + \lim_{\eta \to 0} \varphi_n(\zeta), \tag{A.31}$$

It has been seen that $\exp(\psi_n(\zeta))$ is the characteristic function of an infinitely divisible distribution U_η. If $m > 0$ the addition of $-m\zeta^2/2$ to $\psi_n(\zeta)$ corresponds to a convolution of U_η with a normal distribution. Thus, Equation (A.31) represents $\exp(\psi)$ as the limit of a sequence of infinitely divisible characteristic functions and hence $\exp(\psi)$ is infinitely divisible as asserted. $\square$

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