$u\tau$-CONVERGENCE IN LOCALLY SOLID VECTOR LATTICES

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

YOUSEF A. M. DABBOORASAD

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY
IN
MATHEMATICS

MARCH 14, 2018

Approval of the thesis:

$u\tau$-CONVERGENCE IN LOCALLY SOLID VECTOR LATTICES

submitted by **YOUSEF A. M. DABBOORASAD** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy in Mathematics Department, Middle East Technical University** by,

Prof. Dr. Halil Kalıpçılar
Dean, Graduate School of **Natural and Applied Sciences**

Prof. Dr. Yıldıray Ozan
Head of Department, **Mathematics**

Prof. Dr. Eduard Emel'yanov
Supervisor, **Department of Mathematics, METU**

Examining Committee Members:

Prof. Dr. Murat Yurdakul
Department of Mathematics, METU

Prof. Dr. Eduard Emel'yanov
Department of Mathematics, METU

Prof. Dr. Bahri Turan
Department of Mathematics, Gazi University

Prof. Dr. Birol Altın
Department of Mathematics, Gazi University

Assoc. Prof. Dr. Kostyantyn Zheltukhin
Department of Mathematics, METU

Date:

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last Name: YOUSEF A. M. DABBOORASAD

Signature :

ABSTRACT

$u\tau$-CONVERGENCE IN LOCALLY SOLID VECTOR LATTICES

Dabboorasad, Yousef A. M.
Ph.D., Department of Mathematics
Supervisor : Prof. Dr. Eduard Emel'yanov

March 14, 2018, 60 pages

We say that a net (x_α) in a locally solid vector lattice (X, τ) is $u\tau$-convergent to a vector $x \in X$ if $|x_\alpha - x| \wedge w \xrightarrow{\tau} 0$ for all $w \in X_+$. The aim of the thesis is to study general properties of $u\tau$-convergence, which generalizes unbounded norm convergence. Besides, general investigation of $u\tau$-convergence, we carry out detailed investigation of its very important case, so-called "unbounded m-convergence" (um-convergence, for short) in multi-normed vector lattices. Unlike "unbounded order convergence", we showed that the $u\tau$-convergence is topological and the corresponding topology serves as a generalization of the unbounded norm topology.

Keywords: Vector Lattice, Locally solid Vector Lattice, $u\tau$-Convergence, uo-Convergence, un-Convergence, um-Convergence, Lebesgue property, Levi property.

ÖZ

YEREL KATI VECTÖR ÖRGÜSÜNDE $u\tau$-YAKINSAKLIK

Dabboorasad, Yousef A. M.
Doktora, Matematik Bölümü
Tez Yöneticisi : Prof. Dr. Eduard Emel'yanov

Mart 14, 2018 , 60 sayfa

x_α yerel katı vectör örgüsü (X,τ) da bir net olsun ; Eğer her $w \in X_+$ için $|x_\alpha - x| \wedge w \xrightarrow{\tau} 0$ oluyorsa, bu durumda x_α neti $x \in X$ vektörüne sınırsız τ-yakınsaktır diyeceğiz. Bu tezin amacı sınırsız norm yakınsamanın bir genellemesi olan sınırsız τ-yakınsaklığın (kısaca, $u\tau$-yakınsaklığın) genel özelliklerini calışacağız. Ayrıca, multi normlu vectör örgülermde $u\tau$-yakınsamanın önemli çeşiti olan sınırsız m-yakınsaklık" veya (kısaca um-yakınsaklık) çalışılmıştır. Sınırsız sıra yakınsaklığının aksine, $u\tau$-yakınsaklığı ve um-yakınsaklığı topolojik olduğu ve bunlara karşılık gelen topolojilerin sınırsız norm topolojinin genellemelerine karşılık geldiği gösterilmiştir.

Anahtar Kelimeler: Yöney örgüsü, yerel som yöney örgüsü, $u\tau$-Yakınsama, uo-Yakınsama, un-Yakınsama, um-Yakınsama,Lebesgue özelliği, Levi özelliği.

To my father, mother, wife, sons, to all my family and all people who are reading this work

ACKNOWLEDGMENTS

First of all, thanks to Allah who guided me to this step in my life. I would like to thank my supervisor Professor Eduard Emel'yanov, for his constant support, guidance and friendship. I would like to deeply thank my best friend Dr. Mohammad Marabeh for his support, time and guidance. He also motivated and influenced me highly in scientific context. It was a great honor to work with them. Our cooperation influenced me academically and scientifically.

There are a lot of people that were with me in these years. They defined me, they made me who I am, they are the true owners of this work. It is not possible to write down why each of them is important to me and to this work, because it will take more space than the work itself. I am very grateful to all those people, they changed me deeply: my vision towards life, happiness and friendship. I am very lucky to have them all. So I'll just give names of some of them; my father Atef Dabboorasad and my mother Asmaa Asaad, the real persons behind all my successes in life, my wife Maha Alattar, the one who never hesitate to support me in all kinds of support, my sons Mohammed, Abdelrahman and Anas who where the source of my motivation to work and success, to my teachers from the elemntary school to the PhD level, especially My mother, the great mathematics teacher, Mr. Hisham Hasouna (may his soul rest in peace), Dr Ahmed Al Mabhouh, Prof. Asa'ad Asa'ad, Prof. Hisham Mahdi and Prof. Eduard Emel'yanov. Finally again the distinct person Mohammad Marabeh, whose friendship is the best thing I got in Turkey, and his help was a real reason for me to reach this step in my scientific life. Thank you all for supporting and believing in me all the way throughout my academic life.

Lastly, this work is supported by YTB Scholarship Programme for PhD and by a special Scholarship from the President Recep Tayyip Erdogan, sincerest thanks to them all.

TABLE OF CONTENTS

ABSTRACT v

ÖZ vi

ACKNOWLEDGMENTS viii

TABLE OF CONTENTS ix

CHAPTERS

1 INTRODUCTION 1

2 PRELIMINARIES 5

3 UNBOUNDED τ-CONVERGENCE IN LOCALLY SOLID VECTOR LATTICES 11

3.1 $u\tau$-Topology 13

3.2 $u\tau$-Convergence in sublattices 18

3.3 Unbounded relatively uniform convergence 19

3.4 Topological orthogonal systems and metrizabililty 22

3.5 $u\tau$-Completeness 30

4 UNBOUNDED m-TOPOLOGY IN MULTI-NORMED VECTOR LATTICES 33

4.1 Multi-normed vector lattices 33

4.2 um-Topology 36

4.3 Metrizabililty of um-topology 37

4.4 um-Completeness 40

4.5 um-Compact sets 47

REFERENCES 53

CURRICULUM VITAE 59

CHAPTER 1

INTRODUCTION

The subject of “unbounded convergence” has attracted many researchers [57, 53, 31, 30, 21, 18, 61, 36, 8, 41, 35, 28, 52]. It is well-investigated in Banach lattices [30, 31, 33, 36, 58, 61]. In this thesis, we study unbounded convergence in locally solid vector lattices. Results in this thesis extend previous works [18, 30, 36, 61].

Many types of "unbounded convergences" were defined in vector lattices, normed lattices, locally solid vector lattices and in lattice-normed vector lattices; see, e.g. [7, 8, 10, 16, 17, 18, 23, 31, 38, 54, 57, 61]. Using those unbounded convergences, several related topologies were introduced; see, e.g. [15, 16, 34, 35, 36, 37, 51, 52, 61]. Some new classes of operators were defined and investigated using unbounded convergences; see, e.g. [6, 9, 12, 13, 24, 25, 29, 44, 47, 62]. Furthermore, unbounded convergences has been used in the study of Brezis-Lieb lemma, risk measures, Kolomos properties and universal completion for vector lattices ; see, e.g. [11, 19, 21, 28, 29, 30, 32, 41, 43].

A net $(x_\alpha)_{\alpha\in A}$ in a vector lattice X is said to be *order convergent* (or *o-convergent*) to a vector $x \in X$ if there is another net (y_β), possibly over a different index set, such that $y_\beta \downarrow 0$ and, for every β, there exists α_β satisfying $|x_\alpha - x| \leqslant y_\beta$ whenever $\alpha \geqslant \alpha_\beta$. In this case we write, $x_\alpha \xrightarrow{o} x$. A net (x_α) in a vector lattice X is *unbounded order convergent* to a vector $x \in X$ if $|x_\alpha - x| \wedge u \xrightarrow{o} 0$ for all $u \in X_+$, in this case we say that the net (x_α) uo-converges to x and we write $x_\alpha \xrightarrow{uo} x$. H. Nakano (1948) was the first who defined uo-convergence in [45], but he called it “*individual convergence*”. He extended the individual ergodic theorem, which is known also as Birkhoff’s ergodic theorem, to KB-spaces. Later, R. DeMarr (1964) proposed the name "unbounded order convergence" in [17]. He defined the uo-convergence in ordered vector spaces and mainly showed that any locally convex space E can be embedded in a particular ordered vector space X so that topological convergence in E is equivalent to uo-convergence in X. In 1977, A. Wickstead investigated the relation between weak and uo-convergences in Banach lattices in [57]. Two characterizations of uo-convergence in order (Dedekind) complete vector lattices having weak units were established in [38] by S. Kaplan (1997/98). In [20], they studied stability of order convergence in vector lattices and some types of order ideals in vector lattices. Order convergence of nets was studied in][2, 55].

Recently, in [31], N. Gao and F. Xanthos studied uo-convergent and uo-Cauchy nets in Banach lattices and used them to characterize Banach lattices with the positive Schur property and KB-spaces. Moreover, they applied uo-Cauchy sequences to extend Doob's submartingale convergence theorem to a measure-free setting. Next, N. Gao (2014) studied unbounded order convergence in dual spaces of Banach lattices; see [27]. Quite recently, N. Gao, V. Troitsky, and F. Xanthos (2017) examined more properties of uo-convergence in [30]. They proved the stability of the uo-convergence under passing to and from regular sublattices. Using that fact, several results in [31, 27] were generalized. In addition, they studied the convergence of Cesàro means in Banach lattices using the uo-convergence. As a result, they obtained an intrinsic version of Komlós' Theorem in Banach lattices and developed a new and unified approach to study Banach-Saks properties and Banach-Saks operators in Banach lattices based on uo-convergence.

Moreover, E. Emelyanov and M. Marabeh (2016) derived two measure-free versions of Brezis-Lieb lemma in vector lattices using uo-convergence in [21]. In 2017, H. Li and Z. Chen showed in [41] that every norm bounded positive increasing net in an order continuous Banach lattice is uo-Cauchy and that every uo-Cauchy net in an order continuous Banach lattice has a uo-limit in the universal completion.

Regarding applications, unbounded order convergence has been applied in finance. For instance, N. Gao and F. Xanthos have exploited uo-convergence to derive a w^*-representation theorem of proper convex increasing functionals on particular dual Banach lattices in [32]. Extending this work, representation theorems of convex functionals and risk measures was established using unbounded order continuous dual of a Banach lattice in [28].

Let X be a normed lattice, then a net (x_α) in X is *unbounded norm convergent* to a vector $x \in X$ (or x_α un-convergent to x) if $|x_\alpha - x| \wedge u \xrightarrow{\|\cdot\|} 0$ for all $u \in X_+$. In this case, we write $x_\alpha \xrightarrow{un} x$. In 2004, V. Troitsky defined the unbounded norm convergence in [53]. He called it the "*d-convergence*", and studied the relation between the d-convergence and measure of non-compactness.

Later, in 2016, Y. Deng, M. O'Brien, and V. Troitsky introduced the name "unbounded norm convergence" in [18]. They studied basic properties of un-convergence and investigated its relation with uo- and weak convergences. Finally, they showed that un-convergence is topological.

The "*unbounded norm topology*" (or *un-topology*) in Banach lattices was deeply investigated in [36], by M. Kandić, M. Marabeh, and V. Troitsky (2017). They showed that the un-topology and the norm topology agree iff the Banach lattice has a strong unit. The un-topology is metrizable iff the Banach lattice has a quasi-interior point. The un-topology in an order continuous Banach lattice is locally convex iff it is atomic. An order continuous Banach lattice X is a KB-space iff its closed unit ball B_X is un-complete. For a Banach lattice X, B_X is un-compact iff X is an atomic KB-space. Also, they studied un-compact operators and the relationship between un-convergence and weak*-convergence.

The concept of unbounded norm convergence has been generalized in [35] by M.

Kandić, H. Li, and V. Troitsky (2017) as follows: let X be a normed lattice and Y a vector lattice such that X is an order dense ideal in Y, then a net (y_α) *un-converges to $y \in Y$ with respect to X* if $|y_\alpha - y| \wedge x \xrightarrow{\|\cdot\|} 0$ for every $x \in X_+$. They extended several known results about un-convergence and un-topology to this new setting.

At the same time, O. Zabeti (2017) introduced and studied the *unbounded absolute weak convergence* (or uaw-convergence). A net (x_α) in a Banach lattice X uaw-converges to $x \in X$ if $|x_\alpha - x| \wedge u \xrightarrow{w} 0$ for all $u \in X_+$; [61]. Zabeti investigated the relations of uaw-convergence with other convergnces. Moreover, he obtained a characterization of order continuous and reflexive Banach lattices in terms of uaw-convergence.

After that, Mitchell A. Taylor in [52, 51] investigated unbounded convergence and minimal topologies in locally solid vector lattices. In particular, he prove that a Banach lattice is boundedly uo-complete iff it is monotonically complete. In addition, he studied completeness-type properties of minimal topologies; which are exactly the Hausdorff locally solid topologies in which uo-convergence implies topological convergence. Together with Marko Kandić, they proved in [34] that a minimal topology is metrizable iff X has the countable sup property and a countable order basis. Moreover, they proved relations between minimal topologies and uo-convergence that generalize classical relations between convergence almost everywhere and convergence in measure.

The structure of this thesis is as follows. In **Chapter 2** we provide basic notions and results form vector lattice theory that are needed throughout this thesis.

Chapter 3 consists of five sections. We study general properties of unbounded τ-convergence (shortly, $u\tau$-convergence). For a net (x_α) in a locally solid vector lattice (X, τ); we say that (x_α) is unbounded τ-convergent to a vector $x \in X$ if $|x_\alpha - x| \wedge w \xrightarrow{\tau} 0$ for all $w \in X_+$. The $u\tau$-convergence generalizes unbounded norm convergence and unbounded absolute weak convergence in normed lattices that have been investigated recently [18, 36, 61]. Besides, we introduce $u\tau$-topology and study briefly metrizabililty and completeness of this topology.

Finally, in **Chapter 4** we carry out a detailed investigation of its very important case, the so-called "unbounded m-convergence" (um-convergence, for short) in multi-normed vector lattices [15]. If $\mathcal{M} = \{m_\lambda\}_{\lambda \in \Lambda}$ is a separating family of lattice semi-norms on a vector lattice X, then the pair $(X, \mathcal{M})$ is called a *multi-normed vector lattice (or* MNVL). We write $x_\alpha \xrightarrow{\mathrm{m}} x$ if $m_\lambda(x_\alpha - x) \to 0$ for all $\lambda \in \Lambda$. A net (x_α) in an MNVL $X = (X, \mathcal{M})$ is said to be unbounded m-convergent (or um-convergent) to x if $|x_\alpha - x| \wedge u \xrightarrow{\mathrm{m}} 0$ for all $u \in X_+$. The um-convergence generalizes un-convergence [18, 36] and uaw-convergence [61], and specializes up-convergence [8] and $u\tau$-convergence [16]. The um-convergence is always topological, whose corresponding topology is called *unbounded m-topology (or um*-topology). We show that, for an m-complete metrizable MNVL $(X, \mathcal{M})$, the um-topology is metrizable if and only if X has a countable topological orthogonal system. In terms of um-completeness, we present a characterization of MNVLs possessing both Lebesgue's and Levi's properties. Then, we characterize MNVLs possessing simultaneously the

σ-Lebesgue and σ-Levi properties in terms of sequential um-completeness. Finally, we prove that every m-bounded and um-closed set is um-compact if and only if the space is atomic and has Lebesgue's and Levi's properties.

The results of Chapters 3, and 4 appear in the preprint [16] and the article [15].

CHAPTER 2

PRELIMINARIES

For the convenience of the reader, we present in this chapter the general background needed in this thesis.

Let "$\leq$" be an order relation on a real vector space X. Then X is called an *ordered vector space*, if it satisfies the following conditions: (i) $x \leq y$ implies $x + z \leq y + z$ for all $z \in X$; and (ii) $x \leq y$ implies $\lambda x \leq \lambda y$ for all $\lambda \in \mathbb{R}_+$.

For an ordered vector space X we let $X_+ := \{x \in X : x \geq 0\}$. The subset X_+ is called the *positive cone* of X. For each x and y in an ordered vector space X we let $x \vee y := \sup\{x, y\}$ and $x \wedge y := \inf\{x, y\}$. If $x \in X_+$ and $x \neq 0$, then we write $x > 0$.

An ordered vector space X is said to be a *vector lattice* (or a *Riesz space*) if for each pair of vectors $x, y \in X$ the $x \vee y$ and $x \wedge y$ both exist in X. Let X be a vector lattice and $x \in X$ then $x^+ := x \vee 0$, $x^- := (-x) \vee 0$ and $|x| := (-x) \vee x$ are the *positive part*, *negative part* and *absolute value* of x, respectively. Two elements x and y of a vector lattice X are *disjoint* written as $x \perp y$ if $|x| \wedge |y| = 0$. For a nonempty set A of X then its *disjoint complement* A^d is defined by $A^d := \{x \in X : x \perp a \text{ for all } a \in A\}$. A subset S of a vector lattice X is *bounded from above* (respectively, *bounded from below*) if there is $x \in X$ with $s \leq x$ (respectively, $x \leq s$) for all $s \in S$. If $a, b \in X$, then the subset $[a, b] := \{x \in X : a \leq x \leq b\}$ is called an *order interval* in X. A subset S of X is said to be *order bounded* if it is bounded from above and below or equivalently there is $u \in X_+$ so that $S \subseteq [-u, u]$. If a net (x_α) in X is increasing and $x = \sup_\alpha x_\alpha$, then we write $x_\alpha \uparrow x$. The notation $x_\alpha \downarrow x$ means the net (x_α) in X is decreasing and $x = \inf_\alpha x_\alpha$. A vector lattice X is said to be *Archimedean* if $\frac{1}{n}x \downarrow 0$ holds for each $x \in X_+$. Throughout this thesis, all vector lattices are assumed to be Archimedean.

A vector lattice X is called *order complete* or *Dedekind complete* if every order bounded from above subset has a supremum, equivalently if $0 \leq x_\alpha \uparrow \leq u$ then there is $x \in X$ such that $x_\alpha \uparrow x$.

A vector subspace Y of a vector lattice X is said to be a $sublattice$ of X if for each y_1 and y_2 in Y we have $y_1 \vee y_2 \in Y$. A sublattice Y of X is *order dense* in X if for each $x > 0$ there is $0 < y \in Y$ with $0 < y \leq x$ and Y is said to be *majorizing* in X if for each $x \in X_+$ there exists $y \in Y$ such that $x \leq y$.

A linear operator $T : X \to Y$ between vector lattices is called *lattice homomorphism* if $|Tx| = T|x|$ for all $x \in X$. A one-to-one lattice homomorphism is referred as a *lattice isomorphism*. Two vector lattices X and Y are said to be *lattice isomorphic* when there is a lattice isomorphism from X onto Y.

If X is a vector lattice, then there is a (unique up to lattice isomorphism) order complete vector lattice X^δ that contains X as a majorizing order dense sublattice. We refer to X^δ as the *order (or Dedekind) completion* of X.

A subset A of X is said to be *solid* if for $x \in X$ and $a \in A$ such that $|x| \leq |a|$ it follows that $x \in A$. A solid vector subspace of a vector lattice is referred as *ideal*. Let A be a nonempty subset of X then I_A the *ideal generated by* A is the smallest ideal in X that contains A. This ideal is given by

$$I_A := \{x \in X : \exists a_1, \dots, a_n \in A \text{ and } \lambda \in \mathbb{R}_+ \text{with } |x| \leq \lambda \sum_{j=1}^{n} |a_j|\}.$$

For $x_0 \in X$ then I_{x_0} the ideal generated by x_0 is referred as a *principal ideal*. This ideal has the form $I_{x_0} := \{x \in X : \exists \lambda \in \mathbb{R}_+ \text{with } |x| \leq \lambda |x_0|\}$.

For a net (x_α) in a vector lattice X, we write $x_\alpha \xrightarrow{o} x$, if x_α *converges* to x *in order*. This means that there is a net (y_β), possibly over a different index set, such that $y_\beta \downarrow 0$ and, for every β, there exists α_β satisfying $|x_\alpha - x| \leqslant y_\beta$ whenever $\alpha \geqslant \alpha_\beta$. It follows that an order convergent net has an order bounded tail, whereas an order convergent sequence is order bounded. For a net (x_α) in a vector lattice X and $x \in X$ we have $|x_\alpha - x| \xrightarrow{o} 0$ if and only if $x_\alpha \xrightarrow{o} x$. For an order bounded net (x_α) in an order complete vector lattice we have, $x_\alpha \xrightarrow{o} x$ if and only if $\inf_\alpha \sup_{\beta \geq \alpha} |x_\beta - x| = 0$. A net (x_α) is *unbounded order convergence* to a vector $x \in X$ if $|x_\alpha - x| \wedge u \xrightarrow{o} 0$ for every $u \in X_+$. We write $x_\alpha \xrightarrow{uo} x$ and say that x_α *uo-converges* to x. The *unbounded order convergent* was introduced in [45] under the name *individual convergence*, where the name *unbounded order convergence* was first proposed by DeMarr (1964) [17]. Clearly, order convergence implies uo-convergence and they coincide for order bounded nets. The uo-convergence is an abstraction of a.e.-convergence in L_p-spaces for $1 \leq p < \infty$, [30, 31]. For a measure space (Ω, Σ, μ) and for a sequence f_n in $L_p(\mu)$ $(0 \leq p \leq \infty)$, $f_n \xrightarrow{uo} 0$ if and only if $f_n \to 0$ almost everywhere (cf. [30, Remark 3.4]). It is well known that almost everywhere convergence is not topological in general [46]. Therefore, the uo-convergence might not be topological. Quite recently, it has been shown that order convergence is never topological in infinite dimensional vector lattices [14].

Suppose that X is a vector lattice. By [30, Corollary 3.6], every disjoint sequence in X is uo-null. Recall that a sublattice Y of X is *regular* if the inclusion map preserves suprema and infima of arbitrary subsets. It was shown in [30, Theorem 3.2] that uo-convergence is stable under passing to and from regular sublattices. That is, if (y_α) is a net in a regular sublattice Y of X, then $y_\alpha \xrightarrow{uo} 0$ in Y if and only if $y_\alpha \xrightarrow{uo} 0$ in X (in fact, this property characterizes regular sublattices).

A net $(x_\alpha)_{\alpha \in A}$ in X is said to be *order Cauchy* (or *o-Cauchy*) if the double net $(x_\alpha - x_{\alpha'})_{(\alpha,\alpha') \in A \times A}$ is order convergent to 0. A linear operator $T : X \to Y$ between vector lattices is said to be *order continuous* if $x_\alpha \xrightarrow{o} 0$ in X implies $Tx_\alpha \xrightarrow{o} 0$ in Y. Order convergence is the same in a vector lattice and in its order completion, see [30, Corollary 2.9].

A subset A of X is called *order closed* if for any net (a_α) in A such that $a_\alpha \xrightarrow{o} x$ it follows that $x \in A$. An order closed ideal is a *band*. For $x_0 \in X$ the *principal band* generated by x_0 is the smallest band that includes x_0. We denote this band by B_{x_0} and it is described as $B_{x_0} := \{x \in X : |x| \wedge n|x_0| \uparrow |x|\}$. A band B in a vector lattice X is said to be a *projection band* if $X = B \oplus B^d$. If B is a projection band, then each $x \in X$ can be written uniquely as $x = x_1 + x_2$ where $x_1 \in B$ and $x_2 \in B^d$. The projection $P_B : X \to X$ defined by $P_B(x) := x_1$ is called the *band projection* corresponding to the band projection B. If P is a band projection then it is a lattice homomorphism and $0 \leq P \leq I$; i.e., $0 \leq Px \leq x$ for all $x \in X_+$. So band projections are order continuous.

A vector lattice X equipped with a norm $\|\cdot\|$ is said to be a *normed lattice* if $|x| \leq |y|$ in X implies $\|x\| \leq \|y\|$. If a normed lattice is norm complete, then it is called a *Banach lattice*. A normed lattice $(X, \|\cdot\|)$ is called *order continuous* if $x_\alpha \downarrow 0$ in X implies $\|x_\alpha\| \downarrow 0$ or equivalently $x_\alpha \xrightarrow{o} 0$ in X implies $\|x_\alpha\| \to 0$. A normed lattice $(X, \|\cdot\|)$ is called a *KB-space* if for $0 \leq x_\alpha \uparrow$ and $\sup_\alpha \|x_\alpha\| < \infty$ we get the net (x_α) is norm convergent. Clearly, if the norm is order continuous, then uo-convergence implies un-convergence.

Let X be a vector lattice. An element $0 \neq e \in X_+$ is called a *strong unit* if $I_e = X$, where I_e denotes the ideal generated by e (equivalently, for every $x \geqslant 0$, there exists $n \in \mathbb{N}$ such that $x \leqslant ne$), and $0 \neq e \in X_+$ is called a *weak unit* if $B_e = X$, (equivalently, $x \wedge ne \uparrow x$ for every $x \in X_+$). Here B_e denotes the band generated by e.

Recall that a vector lattice V is a *locally solid vector lattice* if it is Hausdorff topological vector space possessing a zero base of solid neighborhoods. If (X, τ) is a locally solid vector lattice, then $0 \neq e \in X_+$ is called a *quasi-interior point*, if the principal ideal I_e is τ-dense in X, that is $\overline{I_e}^\tau = X$. [49, Def. II.6.1]. If X is a normed lattice. Then it can be shown that $0 < e \in X$ is a quasi-interior point if and only if for every $x \in X_+$ we have $\|x - x \wedge ne\| \to 0$ as $n \to \infty$. It is known that in a normed lattice

$$\text{strong unit} \Rightarrow \text{quasi-interior point} \Rightarrow \text{weak unit.}$$

An element $a > 0$ in a vector lattice X is called an *atom* whenever for every $x \in [0, a]$ there is some real $\lambda \geq 0$ such that $x = \lambda a$. It is known that B_a the band generated by a is a projection band and $B_a = I_a = span\{a\}$, where I_a is the ideal generated by a. A vector lattice X is called *atomic* if the band generated by its atoms is X. For any $x > 0$ there is an atom a such that $a \leq x$. For any atom a, let P_a be the band projection corresponding to B_a. Then $P_a(x) = f_a(x)a$ where f_a is the biorthogonal functional corresponding to a. Since band projections are lattice homomorphisms and are order continuous, so is f_a for any atom a.

Finally we characterize order convergence in atomic order complete vector lattices, and for the convenience of the reader we provide the following technical lemma.

Lemma 1. *Let X and Y be vector lattices. If $T : X \to Y$ is an order continuous lattice homomorphism and A a subset of X such that* $\sup A$ *exists in X, then* $T(\sup A) = \sup T(A)$.

Proof. Note that $\{a_1 \vee \cdots \vee a_n : n \in \mathbb{N}, a_1, \ldots, a_n \in A\} \uparrow \sup A$. So $T\big(\{a_1 \vee \cdots \vee a_n : n \in \mathbb{N}, a_1, \ldots, a_n \in A\}\big) \uparrow T(\sup A)$. Furthermore, $T\big(\{a_1 \vee \cdots \vee a_n : n \in \mathbb{N}, a_1, \ldots, a_n \in A\}\big) = \{T(a_1 \vee \cdots \vee a_n) : n \in \mathbb{N}, a_1, \ldots, a_n \in A\} = \{Ta_1 \vee \cdots \vee Ta_n : n \in \mathbb{N}, a_1, \ldots, a_n \in A\} \uparrow \sup T(A)$. Hence $T(\sup A) = \sup T(A)$. $\square$

Lemma 2. *If X is an atomic order complete vector lattice and (x_α) is an order bounded net such that $f_a(x_\alpha) \to 0$ for any atom a, then $x_\alpha \xrightarrow{o} 0$.*

Proof. Suppose the contrary, then $\inf_\alpha \sup_{\beta \geq \alpha} |x_\beta| > 0$, so there is an atom a such that $a \leq \inf_\alpha \sup_{\beta \geq \alpha} |x_\beta|$. Hence $a \leq \sup_{\beta \geq \alpha} |x_\beta|$ for any α.

Let f_a be the biorthogonal functional corresponding to a, then it follows from Lemma 1 that $1 = f_a(a) \leq f_a\big(\sup_{\beta \geq \alpha} |x_\beta|\big) = \sup_{\beta \geq \alpha} |f_a(x_\beta)|$ for each α. Thus

$\limsup_\alpha |f_a(x_\alpha)| \geq 1$ which is a contradiction. $\square$

Lemma 3. *[30, Corollary 2.9] For any net (x_α) in a vector lattice X, $x_\alpha \xrightarrow{o} 0$ in X if and only if $x_\alpha \xrightarrow{o} 0$ in X^δ.*

Combining Lemmas 2 and 3 we obtain the following result.

Proposition 1. *If X is an atomic vector lattice and (x_α) is an order bounded net such that $f_a(x_\alpha) \to 0$ for any atom a, then $x_\alpha \xrightarrow{o} 0$.*

For a net (x_α) in a normed lattice $(X, \|\cdot\|)$, we write $x_\alpha \xrightarrow{\|\cdot\|} x$ if x_α converges to x in norm. We say that x_α *unbounded norm converges* to $x \in X$ (or x_α *un-converges* to x) if $|x_\alpha - x| \wedge u \xrightarrow{\|\cdot\|} 0$ for every $u \in X_+$. We write $x_\alpha \xrightarrow{un} x$. The *$un$-convergence* was introduced in [53] under the name *d-convergence* and studied in [18] and [36]. Clearly, norm convergence implies un-convergence. The converse need not be true.

Example 1. *Consider the sequence (e_n) of standard unit vectors in c_0. Let $u = (u_1, u_2, \ldots)$ be an element in $(c_0)_+$. Let $0 < \varepsilon < 1$ then there is $n_\varepsilon \in \mathbb{N}$ such that $u_n < \varepsilon$ for all $n \geq n_\varepsilon$. Thus for $n \geq n_\varepsilon$, $\|ne_n \wedge u\|_\infty = u_n < \varepsilon$. Hence $ne_n \xrightarrow{un} 0$. The sequence (ne_n) is not norm bounded, and so it can not be norm convergent.*

For order bounded nets, un-convergence and norm convergence coincide. If the norm of a normed lattice is order continuous then uo-convergence implies un-convergence.

Proposition 2. *[18, Lemma 2.11] Let X be a normed lattice with a quasi-interior point e. Then for any net (x_α) in X, $x_\alpha \xrightarrow{un} 0$ if and only if $\big\||x_\alpha| \wedge e\big\| \to 0$.*

Let Y be a sublattice of a Banach lattice X. Clearly, if (y_α) is a net in Y and $y_\alpha \xrightarrow{un} 0$ in X, then $y_\alpha \xrightarrow{un} 0$ in Y. The converse need not be true.

Example 2. *Let (e_n) be the sequence of standard unit vectors in c_0. Then $e_n \xrightarrow{un} 0$ in c_0, but this does not hold in ℓ_∞. Indeed, let $u = (1, 1, 1, \ldots)$ then $e_n \wedge u = e_n$ and $\|e_n\|_\infty = 1 \not\to 0$.*

Theorem 1. *[36, Theorem 4.3] Let Y be a sublattice of a normed lattice X and (y_α) a net in Y such that $y_\alpha \xrightarrow{un} 0$ in Y. The following statements hold.*

1. *If Y is majorizing in X, then $y_\alpha \xrightarrow{un} 0$ in X.*
2. *If Y is norm dense in X, then $y_\alpha \xrightarrow{un} 0$ in X.*
3. *Y is a projection band in X, then $y_\alpha \xrightarrow{un} 0$ in X.*

Since every Archimedean vector lattice X is majorizing in its order completion X^δ, we have the following result.

Corollary 1. *[36, Corollary 4.4] If X is a normed lattice and $x_\alpha \xrightarrow{un} x$ in X, then $x_\alpha \xrightarrow{un} x$ in the order completion X^δ of X.*

Corollary 2. *[36, Corollary 4.5] If X is a KB-space and $x_\alpha \xrightarrow{un} 0$ in X, then $x_\alpha \xrightarrow{un} 0$ in X^{**}.*

Example 2 shows that the assumption that X is a KB-space cannot be removed.

Corollary 3. *[36, Corollary 4.6] Let Y be a sublattice of an order continuous Banach lattice X. If $y_\alpha \xrightarrow{un} 0$ in Y then $y_\alpha \xrightarrow{un} 0$ in X.*

While uo-convergence need not be given by a topology, it was observed in [18] that un-convergence is topological. For every $\varepsilon > 0$ and non-zero $u \in X_+$, put

$$V_{\varepsilon,u} = \big\{x \in X : \big\||x| \wedge u\big\| < \varepsilon\big\}.$$

The collection of all sets of this form is a base of zero neighborhoods for a topology, and the convergence in this topology agrees with un-convergence. This topology is referred as *un-topology* and it was investigated in [36].

Recall that for a net (x_α), $x_\alpha \xrightarrow{w} 0$ if and only if $f(x_\alpha) \to 0$ for all $f \in X^*$, where "w" refers to weak convergence, and X^* is the *topological dual* of X (the space of all real valued continuous functionals on X).

A net (x_α) is *unbounded absolute weak convergent* to $x \in X$ (or x_α *uaw-converges* to x) if $|x_\alpha - x| \wedge u \xrightarrow{w} 0$ for all $u \in X_+$. We write $x_\alpha \xrightarrow{uaw} x$. Absolute weak convergence implies uaw-convergence. The notions of uaw-convergence and uaw-topology were introduced in [61].

Let X be a Banach lattice. If $x_\alpha \xrightarrow{|\sigma|(X,X^*)} 0$, then $x_\alpha \xrightarrow{\text{uaw}} 0$, where $|\sigma|(X, X^*)$ denotes the absolute weak topology on X. It was pointed out in [61, Example 3] that the converse need not be true. For order bounded nets uaw-convergence and absolute weak convergence are equivalent.

As in the case of un-convergence the following result illustrates that uaw-convergence can only be evaluated at a quasi-interior point.

Proposition 3. *[61, Lemma 6] Let X be a Banach lattice with a quasi-interior point e. Then for any net (x_α) in X, $x_\alpha \xrightarrow{\text{uaw}} 0$ if and only if $|x_\alpha| \wedge e \xrightarrow{w} 0$.*

Similar to the situation in Corollary 3 uaw-convergence on atomic order continuous Banach lattices can transfer from a sublattice to the whole space.

Proposition 4. *[61, Proposition 16] Suppose X is an order continuous Banach lattice and Y is a sublattice of X. If $y_\alpha \xrightarrow{\text{uaw}} 0$ in Y then $y_\alpha \xrightarrow{\text{uaw}} 0$ in X.*

Next result shows that uo-, un- and uaw-convergences all agree on atomic order continuous Banach lattices.

Proposition 5. *[61, Corollary 14] Suppose X is an order continuous Banach lattice. Then uo-convergence un-convergence and uaw-convergence are agree if and only if X is atomic.*
Thus if X is an atomic order continuous Banach lattice, (x_α) is a net in X and f_a is the biorthogonal functional corresponding to an atom $a \in X$. Then $x_\alpha \xrightarrow{\text{uo}} 0$ if and only if $x_\alpha \xrightarrow{\text{un}} 0$ if and only if $x_\alpha \xrightarrow{\text{uaw}} 0$ if and only if $f_a(x_\alpha) \to 0$ for any atom $a \in X$.

CHAPTER 3

UNBOUNDED τ-CONVERGENCE IN LOCALLY SOLID VECTOR LATTICES

Recall that a *topological vector space* is a vector space assigned with a topology in which the vector operations are continuous. If X is a vector lattice, and τ is a linear topology on X that has a base at zero consisting of solid sets, then the pair (X, τ) is called a *locally solid vector lattice*. It should be noted that all topologies considered throughout this thesis are assumed to be Hausdorff. It follows from [3, Theorem 2.28] that a linear topology τ on a vector lattice X is locally solid if and only if it is generated by a family $\{\rho_j\}_{j\in J}$ of Riesz pseudonorms, where a Riesz pseudonorm ρ is a real-valued function defined on a vector lattice X satisfying the following properties:

1. $\rho(x) \geq 0$ for all $x \in X$.
2. $\rho(x+y) \leq \rho(x) + \rho(y)$ for all $x, y \in X$.
3. $\rho(\lambda x) \to 0$ as $\lambda \to 0$ for each $x \in X$.
4. If $|x| \leq |y|$ then $\rho(x) \leq \rho(y)$.

Moreover, if a family of Riesz pseudonorms generates a locally solid topology τ on a vector lattice X, then $x_\alpha \xrightarrow{\tau} x$ in X if and only if $\rho_j(x_\alpha - x) \to 0$ in $\mathbb{R}$ for each $j \in J$. Since X is Hausdorff, the family $\{\rho_j\}_{j\in J}$ of Riesz pseudonorms is separating; i.e., if $\rho_j(x) = 0$ for all $j \in J$, then $x = 0$.

A subset A in a topological vector space (X, τ) is called *topologically bounded* (or simply τ-*bounded*) if, for every τ-neighborhood V of zero, there exists some $\lambda > 0$ such that $A \subseteq \lambda V$. If ρ is a Riesz pseudonorm on a vector lattice X and $x \in X$, then $\frac{1}{n}\rho(x) \leq \rho(\frac{1}{n}x)$ for all $n \in \mathbb{N}$. Indeed, if $n \in \mathbb{N}$ then $\rho(x) = \rho(n\frac{1}{n}x) \leq n\rho(\frac{1}{n}x)$. The following standard fact is included for the sake of completeness.

Proposition 6. *Let (X, τ) be a locally solid vector lattice with a family of Riesz pseudonorms $\{\rho_j\}_{j\in J}$ that generates the topology τ. If a subset A of X is τ-bounded then $\rho_j(A)$ is bounded in $\mathbb{R}$ for any $j \in J$.*

Proof. Let $A \subseteq X$ be τ-bounded and $j \in J$. Put $V := \{x \in X : \rho_j(x) < 1\}$. Clearly, V is a neighborhood of zero in X. Since A is τ-bounded, there is $\lambda > 0$ satisfying

$A \subseteq \lambda V$. Thus $\rho_j(\frac{1}{\lambda}a) \leq 1$ for all $a \in A$. There exists $n \in \mathbb{N}$ with $n > \lambda$. Now, $\frac{1}{n}\rho_j(a) \leq \rho_j(\frac{1}{n}a) \leq \rho_j(\frac{1}{\lambda}a) \leq 1$ for all $a \in A$. Hence, $\sup_{a \in A} \rho_j(a) \leq n < \infty$. □

Next, we discuss the converse of the above proposition.

Let $\{\rho_j\}_{j \in J}$ be a family of Riesz pseudonorms for a locally solid vector lattice (X, τ). For $j \in J$, let $\tilde{\rho}_j := \frac{\rho_j}{1+\rho_j}$. Then $\tilde{\rho}_j$ is a Riesz pseudonorm on X. Moreover, the family $(\tilde{\rho}_j)_{j \in J}$ generates the topology τ on X. Clearly, $\tilde{\rho}_j(A) \leq 1$ for any subset A of X, but still we might have a subset that is not τ-bounded.

Recall that a locally solid vector lattice (X, τ) is said to have the *Lebesgue property* if $x_\alpha \downarrow 0$ in X implies $x_\alpha \xrightarrow{\tau} 0$; or equivalently $x_\alpha \xrightarrow{o} 0$ implies $x_\alpha \xrightarrow{\tau} 0$; and (X, τ) is said to have the *σ-Lebesgue property* if $x_n \downarrow 0$ in X implies $x_n \xrightarrow{\tau} 0$; and (X, τ) is said to have the *pre-Lebesgue* property if $0 \leq x_n \uparrow \leq x$ implies only that (x_n) is τ-Cauchy. Finally, (X, τ) is said to have the *Levi property* if (x_α) is τ-bounded net, with $0 \leq x_\alpha \uparrow$, implies that (x_α) has the supremum in X; and (X, τ) is said to have the *σ-Levi property* if $0 \leq x_n \uparrow$ and (x_n) is τ-bounded, then (x_n) has supremum in X, see [3, Defintion 3.16].

Let X be a vector lattice, and take $0 \neq u \in X_+$. Then a net (x_α) in X is said to be *u-uniform convergent* to a vector $x \in X$ if, for each $\varepsilon > 0$, there exists some α_ε such that $|x_\alpha - x| \leq \varepsilon u$ holds for all $\alpha \geqslant \alpha_\varepsilon$; and (x_α) is said to be *u-uniform Cauchy* if, for each $\varepsilon > 0$, there exists some α_ε such that, for all $\alpha, \alpha' \geqslant \alpha_\varepsilon$, we have $|x_\alpha - x_{\alpha'}| \leq \varepsilon u$. A vector lattice X is said to be *u-uniform complete* if every u-uniform Cauchy sequence in X is u-uniform convergent; and X is said to be *uniform complete* if X is u-uniform complete for each $0 \neq u \in X_+$.

It should be noted that, in a u-uniform complete vector lattice, each u-uniform Cauchy net is u-uniform convergent. Indeed, suppose that (x_α) is a u-uniform Cauchy net in a vector lattice X. Then, for each $n \in \mathbb{N}$, there is α_n such that $|x_\alpha - x_{\alpha'}| \leq \frac{1}{n}u$ for all $\alpha, \alpha' \geq \alpha_n$. We select a strictly increasing sequence α_n. Then, it is clear that the sequence (x_{α_n}) is u-uniform Cauchy and so there is $x \in X$ such that (x_{α_n}) u-uniform converges to x. Let $n_0 \in \mathbb{N}$. Then, for all $\alpha \geqslant \alpha_{n_0}$, we get $|x_\alpha - x_{\alpha_{n_0}}| \leq \frac{1}{n_0}u$, and for all $n \geqslant n_0$, $|x_{\alpha_n} - x_{\alpha_{n_0}}| \leq \frac{1}{n_0}u$. As $n \to \infty$, $|x - x_{\alpha_{n_0}}| \leq \frac{1}{n_0}u$. For $\alpha \geqslant \alpha_{n_0}$, $|x - x_\alpha| \leq \frac{2}{n_0}u$.

Lemma 4. *[42, Theorem 42.2] The vector lattice X is uniform complete if and only if, for every $u \in X_+$, any monotone u-uniform Cauchy sequence has an u-uniform limit.*

Recall that a Banach lattice X is called an *AM-space* if $\|x \vee y\| = \max\{\|x\|, \|y\|\}$ for all $x, y \in X$ with $x \wedge y = 0$.

We prove that any sequentially complete locally solid vector lattice is uniform complete. First we provide the following fact.

Lemma 5. *Let (X, τ) be a sequentially complete locally solid vector lattice and $(\rho_j)_{j \in J}$ be a family of Riesz pseudonorms that generates τ. Given $j \in J$ and $u \in X$. Then, for all $\varepsilon > 0$, there is $\delta > 0$ such that $\rho_j(\delta u) < \varepsilon$.*

Proof. Given $j \in J$ and $u \in X$. If there exists $\varepsilon_0 > 0$ such that $\rho_j(\delta u) \geq \varepsilon_0$ for all $\delta > 0$, then we have, in particular, $\rho_j(\frac{1}{n}u) \geq \varepsilon_0$ for all $n \in \mathbb{N}$. It follows from [3, Defintion 2.27(3)] that $\lim_{n\to\infty} \rho_j(\frac{1}{n}u) = 0$ and so $\varepsilon_0 \leq 0$, a contradiction. $\square$

Proposition 7. *Let (X, τ) be a sequentially complete locally solid vector lattice. Then X is uniform complete.*

Proof. Let $(\rho_j)_{j\in J}$ be a family of Riesz pseudonorms that generates τ. Let $0 \neq u \in X_+$ and (x_n) be an increasing sequence which is u-uniform Cauchy. We show that X is uniform complete. Given $j \in J$ and $\varepsilon > 0$, then, by Lemma 5, there is $\delta > 0$ such that $\rho_j(\delta u) < \varepsilon$. Since (x_n) is u-uniform Cauchy, there is $n_\delta \in \mathbb{N}$ satisfying $|x_n - x_m| \leq \delta u$ for all $n, m \geq n_\delta$. Then $\rho_j(|x_n - x_m|) \leq \rho_j(\delta u) < \varepsilon$ for all $n, m \geq n_\delta$. Thus, (x_n) is τ-Cauchy and, since (X, τ) is sequentially complete, (x_n) is τ-convergent, so there is $x \in X$ such that $x_n \xrightarrow{\tau} x$. Since (x_n) is increasing, $x_n \uparrow x$. It remains to show that (x_n) u-converges to x. Take $\varepsilon > 0$. Since (x_n) is u-uniform Cauchy, there is $n_\varepsilon \in \mathbb{N}$ satisfying

$$|x_n - x_m| \leq \varepsilon u, \text{ for all } n, m \geqslant n_\varepsilon. \tag{3.0.1}$$

Letting $m \to \infty$ in (3.0.1), we get $|x_n - x| \leq \varepsilon u$ for all $n \geqslant n_\varepsilon$. $\square$

Let (X, τ) be a sequentially complete locally solid vector lattice. By Proposition 7, it is uniform complete. So, for each $0 \neq u \in X_+$, let I_u be the ideal generated by u and $\|\cdot\|_u$ be the norm on I_u given by

$$\|x\|_u = \inf\{r > 0 : |x| \leq ru\}.$$

Then, by [5, Theorem 2.58], the pair $(I_u, \|\cdot\|_u)$ is a Banach lattice. Now Theorem 3.4 in [1] implies that $(I_u, \|\cdot\|_u)$ is an AM-space with a strong unit u, and then, by [1, Theorem 3.6], it is lattice isometric to $C(K)$ for some compact Hausdorff space K in such a way, that the strong unit u is identified with the constant function $\mathbb{1}$ on K.

3.1 $u\tau$-Topology

In this section we introduce the $u\tau$-topology in an analogous manner to the un-topology [36] and uaw-topology [61]. First we define the $u\tau$-convergence.

Definition 1. *Suppose (X, τ) is a locally solid vector lattice. Let (x_α) be a net in X. We say that (x_α) is unbounded τ-convergent to $x \in X$ if, for any $w \in X_+$, we have $|x_\alpha - x| \wedge w \xrightarrow{\tau} 0$. In this case, we write $x_\alpha \xrightarrow{u\tau} x$ and say that x_α* $u\tau$-converges to x.

Obviously, if $x_\alpha \xrightarrow{\tau} x$ then $x_\alpha \xrightarrow{u\tau} x$. The converse holds if the net (x_α) is order bounded. Note also that $u\tau$-convergence respects linear and lattice operations. It is clear that $u\tau$-convergence is a generalization of un-convergence [18, 36] and, of uaw-convergence [61].

Theorem 2. *The $u\tau$-convergence is topological.*

Proof. Let $\mathscr{N}_0$ be the collection of all sets of the form

$$V_{\varepsilon,w,j} = \{x \in X : \rho_j(|x| \wedge w) < \varepsilon\},$$

where, $j \in J, 0 \neq w \in X_+$, and $\varepsilon > 0$. We claim that $\mathscr{N}_0$ is a base of neighborhoods of zero for some topology.

For that note that every set in $\mathscr{N}_0$ contains zero since $\rho_j(|0 \wedge w|) = \rho_j(0) = 0 < \varepsilon$ for all $0 \neq w \in X_+$, $j \in J$, and $\varepsilon > 0$.

1. Now let $V_{\varepsilon_1,w_1,j_1}$, and $V_{\varepsilon_2,w_2,j_2} \in \mathscr{N}_0$, put $\varepsilon = \varepsilon_1 \wedge \varepsilon_2$, $w = w_1 \vee w_2$, $\rho_j \geq \rho_{j_1}$, and $\rho_j \geq \rho_{j_2}$. For $x \in V_{\varepsilon,w,j}$, $\rho_j(|x| \wedge w) < \varepsilon$, but $|x| \wedge w_1 \leq |x| \wedge w$ implies that $\rho_{j_1}(|x| \wedge w_1) \leq \rho_j(|x| \wedge w_1) \leq \rho_j(|x| \wedge w) < \varepsilon \leq \varepsilon_1$, similarly $\rho_{j_2}(|x| \wedge w_2) < \varepsilon_2$, that is $x \in V_{\varepsilon_1,w_1,j_1} \cap V_{\varepsilon_2,w_2,j_2}$, and hence $V_{\varepsilon,w,j} \subseteq V_{\varepsilon_1,w_1,j_1} \cap V_{\varepsilon_2,w_2,j_2}$ which means that the intersection of ant two sets in $\mathscr{N}_0$ contains another set from $\mathscr{N}_0$.

2. Let $x_1 + x_2 \in V_{\varepsilon,w,j} + V_{\varepsilon,w,j}$, then $\rho_j(|x_1| \wedge w) < \varepsilon$, and $\rho_j(|x_2| \wedge w) < \varepsilon$, so $\rho_j(|x_1 + x_2| \wedge w) \leq \rho_j(|x_1| \wedge w + |x_2| \wedge w) \leq \rho_j(|x_1| \wedge w) + \rho_j(|x_2| \wedge w) < 2\varepsilon$, that is $x_1 + x_2 \in V_{2\varepsilon,w,j} \in \mathscr{N}_0$, and hence for any $W \in \mathscr{N}_0$ there exists $V \in \mathscr{N}_0$ such that $V + V \subseteq W$.

3. Let $\alpha \in \mathbb{R}$ such that $|\alpha| \leq 1$, and $W = V_{\varepsilon,w,j} \in \mathscr{N}_0$, then for any $x \in \alpha W = \alpha V_{\varepsilon,w,j}$, $x = \alpha t$ for some $t \in V_{\varepsilon,w,j}$, with $|\alpha|\,|t| \leq |t|$ because $|\alpha| \leq 1$, and $\rho_j(|t| \wedge w) < \varepsilon$ which implies that $\rho_j(|x| \wedge w) = \rho_j(|\alpha|\,|t| \wedge w) \leq \rho_j(|t| \wedge w) < \varepsilon$, hence $x \in W$, and so $\alpha W \subseteq W$.

4. Let $x \in X$, and $W = V_{\varepsilon,w,j} \in \mathscr{N}_0$, if $\rho_j(|x|) = 0$, then take $\alpha = 1$ to get that $x \in \alpha W$. If $\rho_j(|x|) \neq 0$, take $\alpha = \frac{2\rho_j(|x|)}{\varepsilon}$ to get that $\rho_j(\frac{1}{\alpha}|x| \wedge w) \leq \rho_j(\frac{1}{\alpha}|x|) = \frac{1}{\alpha}\rho_j(|x|) = \frac{\varepsilon}{2\rho_j(|x|)}\rho_j(|x|) = \frac{\varepsilon}{2} < \varepsilon$, so $\frac{1}{\alpha}x \in W$, that is $x \in \alpha W$, and hence W is absorbing.

Now let $W = V_{\varepsilon,w,j} \in \mathscr{N}_0$, and let $y \in W$. Put $\delta = \varepsilon - \rho_j(|y| \wedge w) > 0$ since $y \in W$, for $x \in V_{\delta,w,j}$, we have $\rho_j(|y + x| \wedge w) \leq \rho_j(|y| \wedge w + |x| \wedge w) \leq \rho_j(|y| \wedge w) + \rho_j(|x| \wedge w) < \rho_j(|y| \wedge w) + \delta = \varepsilon$, hence $y + x \in V_{\varepsilon,w,j}$, and thus $y + V_{\delta,w,j} \subseteq V_{\varepsilon,w,j}$. Therefore, by [39, Theorem 5.1] $\mathscr{N}_0$ is a base of neighborhoods of zero for some linear topology, call it τ.

Moreover, we show that this topology is Hausdorff. Indeed, suppose that $0 \neq x \in \bigcap\{V_{\varepsilon,w,j} : V_{\varepsilon,w,j} \in \mathscr{N}_0\}$, then $\rho_j(|x| \wedge w) < \varepsilon$ for all $j \in J, 0 \neq w \in X_+$, and $\varepsilon > 0$. In particular for $w = |x|$, we have $\rho_j(|x| \wedge |x|) < \varepsilon$, for all $j \in J$, and $\varepsilon > 0$; i.e., for all $j \in J$, $\rho_j(|x|) < \varepsilon$ for all $\varepsilon > 0$, hence $\rho_j(|x|) = 0$, for all $j \in J$, but $(\rho_j)_{j \in J}$ is a separating family of seminorms, then $|x| = 0$, that is $x = 0$ which is a contradiction.

Finally we show that $x_\alpha \xrightarrow{u\tau} 0$ if and only if $x_\alpha \to 0$ in the topology defined above. First suppose that a net (x_α) in X $u\tau$-converges to 0. Let $V_{\varepsilon_0,w_0,j_0} \in \mathscr{N}_0$. Since $x_\alpha \xrightarrow{u\tau} 0$, for any $0 \neq w \in X_+$, $\rho_j(|x_\alpha| \wedge w) \to 0$ in $\mathbb{R}$ for all $j \in J$. In particular,

$\rho_{j_0}(|x_\alpha| \wedge w_0) \to 0$, and so for $\varepsilon_0 > 0$, there exists α_0 such that $\rho_{j_0}(|x_\alpha| \wedge w_0) < \varepsilon_0$ for all $\alpha \geqslant \alpha_0$. Thus $x_\alpha \in V_{\varepsilon_0, w_0, j_0}$ for all $\alpha \geqslant \alpha_0$. On the other hand, suppose that $x_\alpha \to 0$ in the topology defined above. Let $w \in X_+$, take $j \in J$ and $\varepsilon > 0$, then $V_{\varepsilon,w,j} \in \mathscr{N}_0$, and thus, there exist α_0 such that $x_\alpha \in V_{\varepsilon,w,j}$ for all $\alpha \geqslant \alpha_0$. That is $\rho_j(|x_\alpha| \wedge w) < \varepsilon$ for all $\alpha \geqslant \alpha_0$. Thus $\rho_j(|x_\alpha| \wedge w) \to 0$. Therefore $x_\alpha \xrightarrow{u\tau} 0$.

□

The linear Hausdorff topology in the proof of Theorem 2 will be referred as *$u\tau$-topology*.

Clearly, if $x_\alpha \xrightarrow{\tau} 0$, then $x_\alpha \xrightarrow{u\tau} 0$, and so the τ-topology, in general, is finer than $u\tau$-topology. On the contrary to Theorem 2.3 in [36], example 5 in chapter 4 provides a locally solid vector lattice which has a strong unit, yet the τ-topology and $u\tau$-topology do not agree.

It is known that the topology of any linear topological space can be derived from a unique translation-invariant uniformity, i.e., any linear topological space is uniformisable (cf. [50, Theorem 1.4]). It follows from [22, Theorem 8.1.20] that any linear topological space is completely regular. In particular, the unbounded τ-convergence is completely regular.

Remark 1. *Let (X, τ) be a locally solid vector lattice, where τ is generated by a family $(\rho_j)_{j\in J}$ of Riesz pseudonorms. For all $j \in J, 0 \neq w \in X_+$, and $\varepsilon > 0$, $V_{\varepsilon,w,j}$ is solid.*

Proof. Let $y \in V_{\varepsilon,w,j}$, and let $|x| \leq |y|$, then $|x| \wedge w \leq |y| \wedge w$, and so $\rho_j(|x| \wedge w) \leq \rho_j(|y| \wedge w) < \varepsilon$. Hence $x \in V_{\varepsilon,w,j}$. □

The next result should be compared with [36, Lemma 2.1].

Proposition 8. *Let (X, τ) be a sequentially complete locally solid vector lattice, where τ is generated by a family $(\rho_j)_{j\in J}$ of Riesz pseudonorms. Let $\varepsilon > 0$, $j \in J$, and $0 \neq w \in X_+$. Then either $V_{\varepsilon,w,j}$ is contained in $[-w, w]$, or it contains a non-trivial ideal.*

Proof. Suppose that $V_{\varepsilon,w,j}$ is not contained in $[-w, w]$. Then there exists $x \in V_{\varepsilon,w,j}$ such that $x \notin [-w, w]$. Replacing x with $|x|$, we may assume $x > 0$. Since $x \notin [-w, w]$, $y = (x - w)^+ > 0$. Now, letting $z = x \vee w$, we have that the ideal I_z generated by z, is lattice and norm isomorphic to $C(K)$ for some compact and Hausdorff space K, where z corresponds to the constant function $\mathbb{1}$. Also x, y, and w in I_z correspond to $x(t)$, $y(t)$, and $w(t)$ in $C(K)$ respectively.

Our aim is to show that for all $\alpha \geq 0$ and $t \in K$, we have

$$(\alpha y)(t) \wedge w(t) \leq x(t) \wedge w(t).$$

For this, note that $y(t) = (x - w)^+(t) = (x - w)(t) \vee 0$.

Let $t \in K$ be arbitrary.

- Case (1): If $(x-w)(t) > 0$, then $x(t) \wedge w(t) = w(t) \geq (\alpha y)(t) \wedge w(t)$ for all $\alpha \geq 0$, as desired.

- Case (2): If $(x-w)(t) < 0$, then $(\alpha y)(t) \wedge w(t) \leq (\alpha y)(t) = \alpha(x-w)(t) \vee 0 = 0 \leq x(t) \wedge w(t)$, as desired.

Hence, for all $\alpha \geq 0$ and $t \in K$, we have $(\alpha w)(t) \wedge w(t) \leq x(t) \wedge w(t)$ and so $(\alpha y) \wedge w \leq x \wedge w$ for all $\alpha \geq 0$. Note, that $\alpha y, w, x \in X_+$. Thus $\rho_j(|\alpha y| \wedge w) \leq \rho_j(|x| \wedge w) < \varepsilon$, so $\alpha y \in V_{\varepsilon,w,j}$ and, since $V_{\varepsilon,w,j}$ is solid, $I_z \subseteq V_{\varepsilon,w,j}$. $\square$

Note that the sequential completeness in Proposition 8 can be removed, as we see later in Theorem 5.

Theorem 3. *[3, Theorem 2.8 and 2.40] Let (X,τ) be a Hausdorff locally solid vector lattice. Then there is a unique (up to isomorphism) Hausdorff topological vector space $(\widehat{X},\widehat{\tau})$ having the following properties:*

1. *The topological vector space $(\widehat{X},\widehat{\tau})$ is $\widehat{\tau}$-complete.*

2. *The $\widehat{\tau}$-closure of X_+ is a cone of $\widehat{X}$ and $(\widehat{X},\widehat{\tau})$ equipped with this cone is a Hausdorff locally solid vector lattice containing X as a vector sublattice.*

3. *The topology $\widehat{\tau}$ induces τ in X.*

4. *The vector sublattice X is $\widehat{\tau}$-dense in $\widehat{X}$.*

5. *The $\widehat{\tau}$-closure of a solid subset of X is a solid subset of $\widehat{X}$. In particuler, if $\mathcal{N}$ is a base of zero for (X,τ) consisting of solid sets, then $\{\overline{V}^{\widehat{\tau}} : V \in \mathcal{N}\}$ is also a base of zero for $(\widehat{X},\widehat{\tau})$ consisting of solid sets.*

The Hausdorff locally solid vector lattice $(\widehat{X},\widehat{\tau})$ in Theorem 3 is the *topological completion* of (X,τ).

In the following theorem we gather some properties of $(\widehat{X},\widehat{\tau})$. Recall that

Theorem 4. *Let $(\widehat{X},\widehat{\tau})$ be the topological completion of a Lebesgue Hausdorff locally solid vector lattice (X,τ). Then the following statements hold:*

1. *$(\widehat{X},\widehat{\tau})$ satisfies Lebesgue property.*

2. *$\widehat{X}$ is Dedekind complete.*

3. *X is order dense in $\widehat{X}$, and so X is regular in $\widehat{X}$.*

4. *If X^δ is the Dedekind completion of X, then $X \subseteq X^\delta \subseteq \widehat{X}$ and both X and X^δ are regular vector sublattices of $\widehat{X}$.*

Proof. (1) It follows from [3, Theorem 3.23] that (X, τ) satisfies pre-Lebesgue property. Now, [3, Theorem 3.26] implies that $(\widehat{X}, \widehat{\tau})$ satisfies Lebesgue property.

(2) Since $(\widehat{X}, \widehat{\tau})$ satisfies Lebesgue property, it follows from [3, Theorem 3.24] that $\widehat{X}$ is Dedekind complete.

(3) Since $(\widehat{X}, \widehat{\tau})$ satisfies Lebesgue property, it satisfies Faton property; see e.g., [3, Lemma 4.2] . Thus, X is order dense in $\widehat{X}$ by [3, Theorem 4.31].

(4) Since $X \subseteq \widehat{X}$, $X^\delta \subseteq (\widehat{X})^\delta = \widehat{X}$ as $\widehat{X}$ is Dedekind complete. So, $X \subseteq X^\delta \subseteq \widehat{X}$. Since X is regular in $\widehat{X}$, it follows from [30, Theorem 2.10] that X^δ is regular in $\widehat{X}$. Also, since X is regular in X^δ and X^δ is regular in $\widehat{X}$, we get X is regular in $\widehat{X}$.
Again suppose that (X, τ) is Lebesgue Hausdorff locally solid vector lattice. Then by [3, Theorem 4.12] there is a unique Lebesgue Hausdorff locally solid topology τ^δ on X^δ that induces τ on X.
Also, since X^δ is a vector sublattice of $\widehat{X}$, we can equip X^δ with the relative topology induces by $\widehat{\tau}$. Since $(\widehat{X}, \widehat{\tau})$ is a Lebesgue Hausdorff locally solid space, so is $(X^\delta, \widehat{\tau})$. Now [3, Theorem 4.12] implies that $\widehat{\tau} = \tau^\delta$ on X^δ. □

Theorem 5. *Let (X, τ) be a locally solid vector lattice, where τ is generated by a family $(\rho_j)_{j\in J}$ of Riesz pseudonorms. Let $\varepsilon > 0$, $j \in J$, and $0 \neq w \in X_+$. Then either $V_{\varepsilon,w,j}$ is contained in $[-w, w]$ or $V_{\varepsilon,w,j}$ contains a non-trivial ideal.*

Proof. Take $\varepsilon > 0$, $j \in J$, and $0 \neq w \in X_+$. Let $(\widehat{X}, \widehat{\tau})$ be the topological completion of (X, τ). In view of Theorem 3, $(\widehat{X}, \widehat{\tau})$ is also a locally solid vector lattice. It follows from the proof of Proposition 22F in [26] that if $\widehat{\rho}_j$ is the continuous extension of ρ_j to $\widehat{X}$, then $\widehat{\rho}_j$ is also a Riesz pseudonorm and $\widehat{\tau}$ is generated by $(\widehat{\rho}_j)_{j\in J}$. In particular, $(\widehat{X}, \widehat{\tau})$ is a sequentially complete locally vector lattice. Let $\widehat{V}_{\varepsilon,w,j} = \{\widehat{x} \in \widehat{X} \colon \widehat{\rho}_j(|\widehat{x}| \wedge w) < \varepsilon\}$. Then $V_{\varepsilon,w,j} = X \cap \widehat{V}_{\varepsilon,w,j}$. By Proposition 8, either $\widehat{V}_{\varepsilon,w,j}$ is a subset of $[-w, w]_{\widehat{X}}$ in $\widehat{X}$ or $\widehat{V}_{\varepsilon,w,j}$ contains a non-trivial ideal of $\widehat{X}$. If $\widehat{V}_{\varepsilon,w,j} \subseteq [-w, w]_{\widehat{X}}$, then

$$V_{\varepsilon,w,j} = X \cap \widehat{V}_{\varepsilon,w,j} \subseteq X \cap [-w, w]_{\widehat{X}} = [-w, w] \subseteq X.$$

If $\widehat{V}_{\varepsilon,w,j}$ contains a non-trivial ideal, then $\widehat{V}_{\varepsilon,w,j} \nsubseteq [-w, w]_{\widehat{X}}$. By solidity, we can take $0 < \widehat{x} \in \widehat{V}_{\varepsilon,w,j}$ such that $\widehat{x} \notin [-w, w]_{\widehat{X}}$, that is, $(\widehat{x} - w)^+ > 0$. Now take a net $(x_\alpha) \subset X$ such that $x_\alpha \xrightarrow{\tau} \widehat{x}$. Then $x_\alpha^+ \xrightarrow{\tau} \widehat{x}^+ = \widehat{x}$, and $(x_\alpha^+ - w)^+ \xrightarrow{\tau} (\widehat{x} - w)^+$. Since $\widehat{V}_{\varepsilon,w,j}$ is an open set containing $\widehat{x}$, we may take $x := x_\alpha^+ \in \widehat{V}_{\varepsilon,w,j} \cap X$ such that $y := (x - w)^+ > 0$. By the same argument in Proposition 8 to $(\widehat{X}, \widehat{\tau})$, we get $(\alpha y) \wedge w \leq x \wedge w$ for all $\alpha \in \mathbb{R}_+$. Since $x \in \widehat{V}_{\varepsilon,w,j}$, $\alpha y \in \widehat{V}_{\varepsilon,w,j}$ for all $\alpha \in \mathbb{R}_+$. But $\alpha y \in X_+$ for all $\alpha \in \mathbb{R}_+$ and, since $V_{\varepsilon,w,j} = X \cap \widehat{V}_{\varepsilon,w,j}$, we get $\alpha y \in V_{\varepsilon,w,j}$ for all $\alpha \in \mathbb{R}_+$. Since $V_{\varepsilon,w,j}$ is solid, we conclude that the principal ideal I_y taken in X is a subset of $V_{\varepsilon,w,j}$. □

Lemma 6. *Let (X, τ) be a locally solid vector lattice, where τ is generated by a family $(\rho_j)_{j\in J}$ of Riesz pseudonorms. If $V_{\varepsilon,w,j}$ is contained in $[-w, w]$, then w is a strong unit.*

Proof. Suppose $V_{\varepsilon,w,j} \subseteq [-w,w]$. Since $V_{\varepsilon,w,j}$ is absorbing, for any $x \in X_+$, there exist $\alpha > 0$ such that $\alpha x \in V_{\varepsilon,w,j}$, and so $\alpha x \in [-w,w]$, or $x \leq \frac{1}{\alpha}w$. Thus w is a strong unit, as desired. □

3.2 $u\tau$-Convergence in sublattices

Let Y be a sublattice of a locally solid vector lattice (X,τ). If (y_α) is a net in Y then $y_\alpha \xrightarrow{u\tau} 0$ in Y means: $|y_\alpha| \wedge y \xrightarrow{\tau} 0$ for all $y \in Y_+$. Clearly, $y_\alpha \xrightarrow{u\tau} 0$ in X implies $y_\alpha \xrightarrow{u\tau} 0$ in Y. The converse does not hold in general. For example, the sequence (e_n) of standard unit vectors is un-null in c_0, but not in ℓ_∞. In this section, we study when the $u\tau$-convergence passes from a sublattice to the whole space.

The following theorem extends [36, Theorem 4.3] to locally solid vector lattices.

Theorem 6. *Let (X,τ) be a locally solid vector lattice and Y be a sublattice of X. If (y_α) is a net in Y and $y_\alpha \xrightarrow{u\tau} 0$ in Y, then $y_\alpha \xrightarrow{u\tau} 0$ in X in each of the following cases*:

1. *Y is majorizing in X;*
2. *Y is τ-dense in X;*
3. *Y is a projection band in X.*

Proof. 1. It is obvious to see that.

2. Let $u \in X_+$. Fix $\varepsilon > 0$ and take $j \in J$. Since Y is τ-dense in X, there is $v \in Y_+$ such that $\rho_j(u-v) < \varepsilon$. But $y_\alpha \xrightarrow{u\tau} 0$ in Y and so, in particular, $\rho_j(|y_\alpha| \wedge v) \to 0$. So there is α_0 such that $\rho_j(|y_\alpha| \wedge v) < \varepsilon$ for all $\alpha \geqslant \alpha_0$. It follows from $u \leq v + |u-v|$, that $|y_\alpha| \wedge u \leq |y_\alpha| \wedge v + |u-v|$, and so $\rho_j(|y_\alpha| \wedge u) \leq \rho_j(|y_\alpha| \wedge v) + \rho_j(u-v) < 2\varepsilon$. Thus, $\rho_j(|y_\alpha| \wedge u) \to 0$ in $\mathbb{R}$. Since $j \in J$ was chosen arbitrary, we conclude that $y_\alpha \xrightarrow{u\tau} 0$ in X.
3. Let $u \in X_+$. Then $u = v + w$, where $v \in Y_+$ and $w \in Y_+^d$. Now $|y_\alpha| \wedge u = |y_\alpha| \wedge v + |y_\alpha| \wedge w = |y_\alpha| \wedge v \xrightarrow{\tau} 0$ in X, since $y_\alpha \in Y$

□

Corollary 4. *If (X,τ) is a locally solid vector lattice and $x_\alpha \xrightarrow{u\tau} 0$ in X, then $x_\alpha \xrightarrow{u\tau} 0$ in the Dedekind completion X^δ of X.*

Corollary 5. *If (X,τ) is a locally solid vector lattice and $x_\alpha \xrightarrow{u\tau} 0$ in X, then $x_\alpha \xrightarrow{u\tau} 0$ in the topological completion $\widehat{X}$ of X.*

The next result generalizes Corollary 4.6 in [36] and Proposition 16 in [61].

Theorem 7. *Let (X,τ) be a Dedekind complete locally solid vector lattice that has the Lebesgue property, and Y be a sublattice of X. If $y_\alpha \xrightarrow{u\tau} 0$ in Y, then $y_\alpha \xrightarrow{u\tau} 0$ in X.*

Proof. Suppose $y_\alpha \xrightarrow{u\tau} 0$ in Y. By Theorem 6(1), $y_\alpha \xrightarrow{u\tau} 0$ in the ideal $I(Y)$ generated by Y in X. By Theorem 6(2), $y_\alpha \xrightarrow{u\tau} 0$ in the closure $\overline{\{I(Y)\}}^\tau$ of $I(Y)$. It follows from [3, Theorem 3.7] that $\overline{\{I(Y)\}}^\tau$ is a band in X. Since X is Dedekind complete, $\overline{\{I(Y)\}}^\tau$ is a projection band in X. Then $y_\alpha \xrightarrow{u\tau} 0$ in X, in view of Theorem 6(3). $\square$

Suppose that (X, τ) is a locally solid vector lattice possessing the Lebesgue property. Then, in view of Theorem 4 part (1), its topological completion $(\widehat{X}, \widehat{\tau})$ possesses the Lebesgue property as well. Hence, by [3, Theorem 3.24], $\widehat{X}$ is Dedekind complete. It follows from [3, Theorem 2.41] that X is regular in $\widehat{X}$, so that $X^\delta \subseteq \widehat{X}$ by [30, Theorem 2.10]. Now, Theorem 7 assures that, given a net (z_α) in X^δ, if $z_\alpha \xrightarrow{u\tau} 0$ in X^δ then $z_\alpha \xrightarrow{u\tau} 0$ in $\widehat{X}$.

Proposition 9. *Every band in a locally solid vector lattice is $u\tau$-closed.*

Proof. Let B be a band in X. Suppose (x_α) is a net in B such that $x_\alpha \xrightarrow{u\tau} x$. Let $z \in B^d$, then $|x_\alpha| \wedge |z| \xrightarrow{\tau} |x| \wedge |z|$. But $|x_\alpha| \wedge |z| = 0$ for all α and so $|x| \wedge |z| = 0$. So $x \in B^{dd} = B$. $\square$

3.3 Unbounded relatively uniform convergence

In this section we discuss unbounded relatively uniform convergence. Recall that a net (x_α) in a vector lattice X is said to be *relatively uniform convergent* to $x \in X$ if, there is $u \in X_+$ such that for any $n \in \mathbb{N}$, there exists α_n satisfying $|x_\alpha - x| \leq \frac{1}{n}u$ for $\alpha \geqslant \alpha_n$. In this case we write $x_\alpha \xrightarrow{ru} x$ and the vector $u \in X_+$ is called *regulator*, see [56, Defintion III.11.1]. Moreover, in a locally solid vector lattice (X, τ), $x_\alpha \xrightarrow{ru} 0$ implies that $x_\alpha \xrightarrow{\tau} 0$. Indeed, let V be a solid neighborhood at zero. Since $x_\alpha \xrightarrow{ru} 0$, there is $u \in X_+$ such that, for a given $\varepsilon > 0$, there is α_ε satisfying $|x_\alpha| \leq \varepsilon u$ for all $\alpha \geq \alpha_\varepsilon$. Since V is absorbing, there is $c \geq 1$ such that $\frac{1}{c}u \in V$. There is some α_0 such that $|x_\alpha| \leq \frac{1}{c}u$ for all $\alpha \geq \alpha_0$. Since V is solid and $|x_\alpha| \leq \frac{1}{c}u$ for all $\alpha \geq \alpha_0$, $x_\alpha \in V$ for all $\alpha \geq \alpha_0$. That is $x_\alpha \xrightarrow{\tau} 0$.

The following result might be considered as an ru-version of Theorem 1 in [14].

Theorem 8. *Let X be a vector lattice. Then the following conditions are equivalent.*

(1) *There exists a linear topology τ on X such that, for any net (x_α) in X: $x_\alpha \xrightarrow{ru} 0$ if and only if $x_\alpha \xrightarrow{\tau} 0$.*

(2) *There exists a norm $\|\cdot\|$ on X such that, for any net (x_α) in X: $x_\alpha \xrightarrow{ru} 0$ if and only if $\|x_\alpha\| \to 0$.*

(3) *X has a strong order unit.*

Proof. $(1) \Rightarrow (3)$ It follows from [14, Lemma 1].

$(3) \Rightarrow (2)$ Let $e \in X$ be a strong order unit. Then $x_\alpha \xrightarrow{ru} 0$ if and only if $\|x_\alpha\|_e \to 0$, where $\|x\|_e := \inf\{r : |x| \leqslant re\}$.

$(2) \Rightarrow (1)$ It is trivial. $\square$

Let X be a vector lattice. A net (x_α) in X is said to be *unbounded relatively uniform convergent* to $x \in X$ if $|x_\alpha - x| \wedge w \xrightarrow{ru} 0$ for all $w \in X_+$. In this case, we write $x_\alpha \xrightarrow{uru} x$. Clearly, if $x_\alpha \xrightarrow{uru} 0$ in a locally solid vector lattice (X, τ), then $x_\alpha \xrightarrow{u\tau} 0$.

In general, uru-convergence is also not topological. Indeed, consider the vector lattice $L_1[0,1]$. It satisfies the diagonal property for order convergence by [42, Theorem 71.8]. Now, by combining Theorems 16.3, 16.9, and 68.8 in [42] we get that for any sequence f_n in $L_1[0,1]$ $f_n \xrightarrow{o} 0$ if and only if $f_n \xrightarrow{ru} 0$. In particular, $f_n \xrightarrow{uo} 0$ if and only if $f_n \xrightarrow{uru} 0$. But the uo-convergence in $L_1[0,1]$ is equivalent to $a.e.$-convergence which is not topological, see [46].

However, in some vector lattices the uru-convergence could be topological. For example, if X is a vector lattice with a strong unit e, It follows from Theorem 8, that ru-convergence is equivalent to the norm convergence $\|\cdot\|_e$, where $\|x\|_e := \inf\{\lambda > 0 : |x| \leq \lambda e\}$, $x \in X$. Thus uru-convergence in X is topological.

Consider the vector lattice c_{00} of eventually zero sequences. It is well known that in c_{00}: $x_\alpha \xrightarrow{ru} 0$ if and only if $x_\alpha \xrightarrow{o} 0$. For the sake of completeness we include a proof of this fact. Clearly, $x_\alpha \xrightarrow{ru} 0 \Rightarrow x_\alpha \xrightarrow{o} 0$. For the converse, suppose $x_\alpha \xrightarrow{o} 0$ in c_{00}. Then there is a net $y_\beta \downarrow 0$ in c_{00} such that, for any β, there is α_β satisfying $|x_\alpha| \leq y_\beta$ for all $\alpha \geq \alpha_\beta$. Let (e_n) denote the sequence of standard unit vectors in c_{00}. Fix β_0. Then $y_{\beta_0} = c_1^{\beta_0} e_{k_1} + \cdots + c_n^{\beta_0} e_{k_n}$, $c_i^{\beta_0} \in \mathbb{R}, i = 1, \ldots, n$. Since y_β is decreasing, $y_\beta \leq y_{\beta_0}$ for all $\beta \geq \beta_0$. So, $y_\beta = c_1^\beta e_{k_1} + \cdots + c_n^\beta e_{k_n}$ for all $\beta \geq \beta_0, c_i^\beta \in \mathbb{R}, i = 1, \ldots, n$. Since $y_\beta \downarrow 0$, $\lim_\beta c_i^\beta = 0$ for all $i = 1, \ldots, n$. Let $u = e_{k_1} + \cdots + e_{k_n}$. Take $\varepsilon > 0$. Then, there is $\beta_\varepsilon \geq \beta_0$ such that $c_i^\beta < \varepsilon$ for all $\beta \geq \beta_\varepsilon$ for $i = 1, \ldots, n$. Consider y_{β_ε} then there is α_ε such that $|x_\alpha| \leq y_{\beta_\varepsilon}$ for all $\alpha \geq \beta_\varepsilon$. But $y_{\beta_\varepsilon} = c_1^{\beta_\varepsilon} e_{k_1} + \cdots + c_n^{\beta_\varepsilon} e_{k_n} \leq \varepsilon u$. So, $|x_\alpha| \leq \varepsilon u$ for all $\alpha \geq \alpha_\varepsilon$. That is $x_\alpha \xrightarrow{ru} 0$. Thus, the uru-convergence in c_{00} coincides with the uo-convergence which is pointwise convergence and, therefore, is topological.

Proposition 10. *Let X be a Lebesgue and complete metrizable locally solid vector lattice. Then $x_\alpha \xrightarrow{ru} 0$ if and only if $x_\alpha \xrightarrow{o} 0$.*

Proof. The necessity is obvious. Let d be the metric that induces the Lebesgue locally solid topology on X. For the sufficiency assume that $x_\alpha \xrightarrow{o} 0$. Then there exists $y_\beta \downarrow 0$ such that for any β there is α_β with $|x_\alpha| \leqslant y_\beta$ as $\alpha \geqslant \alpha_\beta$. Since $d(y_\beta, 0) \to 0$, there exists an increasing sequence $(\beta_k)_k$ of indeces with $d(k y_{\beta_k}, 0) \leqslant \frac{1}{2^k}$. Let $s_n =$

$\sum_{k=1}^{n} ky_{\beta_k}$. We show the sequence (s_n) is Cauchy. For $n > m$,

$$d(s_n, s_m) = d(s_n - s_m, 0) = d\Big(\sum_{k=m+1}^{n} ky_{\beta_k}, 0\Big) \leq \sum_{k=m+1}^{n} d\big(ky_{\beta_k}, 0\big)$$
$$\leq \sum_{k=m+1}^{n} \frac{1}{2^k} \to 0, \text{ as } n, m \to \infty.$$

Since X is complete, the sequence (s_n) converges to some $u \in X_+$. That is, $u := \sum\limits_{k=1}^{\infty} ky_{\beta_k}$. Then

$$k|x_\alpha| \leqslant ky_{\beta_k} \leqslant u \quad (\forall \alpha \geqslant \alpha_{\beta_k})$$

which means that $x_\alpha \xrightarrow{ru} 0$. □

Let $X = \mathbb{R}^\Omega$ be the vector lattice of all real-valued functions on a set Ω.

Proposition 11. *In the vector lattice $X = \mathbb{R}^\Omega$, the following conditions are equivalent*:

(1) *for any net (f_α) in X: $f_\alpha \xrightarrow{o} 0$ if and only if $f_\alpha \xrightarrow{ru} 0$;*

(2) *Ω is countable.*

Proof. $(1) \Rightarrow (2)$ Suppose $f_\alpha \xrightarrow{o} 0 \Leftrightarrow f_\alpha \xrightarrow{ru} 0$ for any net (f_α) in $X = \mathbb{R}^\Omega$. Our aim is to show that Ω is countable. Assume, in contrary, that Ω is uncountable. Let $\mathcal{F}(\Omega)$ be the collection of all finite subsets of Ω. For each $\alpha \in \mathcal{F}(\Omega)$, put $f_\alpha = \mathcal{X}_\alpha$, the characteristic function on α. Clearly, $f_\alpha \uparrow \mathbb{1}$, where $\mathbb{1}$ denotes the constant function one on Ω. Then $\mathbb{1} - f_\alpha \downarrow 0$ or $\mathbb{1} - f_\alpha \xrightarrow{o} 0$ in $\mathbb{R}^\Omega$. So, there is $0 \leq g \in \mathbb{R}^\Omega$ such that, for any $\varepsilon > 0$, there exists α_ε satisfying $\mathbb{1} - f_\alpha \leq \varepsilon g$ for all $\alpha \geqslant \alpha_\varepsilon$. Let $n \in \mathbb{N}$. Then there is a finite set $\alpha_n \subseteq \Omega$ such that $\mathbb{1} - f_{\alpha_n} \leq \frac{1}{n} g$. Consequently, $g(x) \geqslant n$ for all $x \in \Omega \setminus \alpha_n$. Let $S = \cup_{n=1}^{\infty} \alpha_n$. Then S is countable and $\Omega \setminus S \neq \emptyset$. Moreover, for each $x \in \Omega \setminus S$, we have $g(x) \geqslant n$ for all $n \in \mathbb{N}$, which is impossible.

$(2) \Rightarrow (1)$ Suppose that Ω is countable. So, we may assume that $X = s$, the space of all sequences. Since, from $x_\alpha \xrightarrow{ru} 0$ always follows that $x_\alpha \xrightarrow{o} 0$, it is enough to show that if $x_\alpha \xrightarrow{o} 0$ then $x_\alpha \xrightarrow{ru} 0$. To see this, let $(x_\alpha^n)_n = x_\alpha \xrightarrow{o} 0$. Then, the net (x_α) is eventually bounded, say $|x_\alpha| \leqslant u = (u_n)_n \in s$. Take $w := (nu_n)_n \in s$. We show that $x_\alpha \xrightarrow{ru} 0$ with the regulator w. Let $k \in \mathbb{N}$. Since $x_\alpha \xrightarrow{o} 0$, for each $n \in \mathbb{N}$, $x_\alpha^n \to 0$ in $\mathbb{R}$. Hence, there is α_k such that $k|x_\alpha^1| < w_1$, $k|x_\alpha^2| < w_2$, $\cdots$, $k|x_\alpha^{k-1}| < w_{k-1}$ for all $\alpha \geqslant \alpha_k$. Note that for $n \geqslant k$, $k|x_\alpha^n| < w_n$. Therefore, $k|x_\alpha| < w$ for all $\alpha \geqslant \alpha_k$. □

It follows from Proposition 11 that, for countable Ω, the uru-convergence in $\mathbb{R}^\Omega$ coincides with the uo-convergence (which is pointwise) and therefore is topological. We do not know, whether or not the countability of Ω is necessary for the property that uru-convergence is topological in $\mathbb{R}^\Omega$.

3.4 Topological orthogonal systems and metrizabililty

A collection $\{e_\gamma\}_{\gamma\in\Gamma}$ of positive vectors in a vector lattice X is called an *orthogonal system* if $e_\gamma \wedge e_{\gamma'} = 0$ for all $\gamma \neq \gamma'$. If, moreover, $x \wedge e_\gamma = 0$ for all $\gamma \in \Gamma$ implies $x = 0$, then $\{e_\gamma\}_{\gamma\in\Gamma}$ is called a *maximal orthogonal system*. It follows from Zorn's Lemma that every vector lattice containing at least one non-zero element has a maximal orthogonal system. Motivated by Definition III.5.1 in [49], we introduce the following notion.

Definition 2. *Let (X,τ) be a locally solid vector lattice. An orthogonal system $Q = \{e_\gamma\}_{\gamma\in\Gamma}$ of non-zero elements in X_+ is said to be a* topological orthogonal system *if the ideal I_Q generated by Q is τ-dense in X.*

Lemma 7. *If $Q = \{e_\gamma\}_{\gamma\in\Gamma}$ is a topological orthogonal system in a locally solid vector lattice (X,τ), then Q is a maximal orthogonal system in X.*

Proof. Assume $x \wedge e_\gamma = 0$ for all $\gamma \in \Gamma$. By the assumption, there is a net (x_α) in the ideal I_Q such that $x_\alpha \xrightarrow{\tau} x$. Without lost of generality, we may assume $0 \leq x_\alpha \leq x$ for all α. Since $x_\alpha \in I_Q$, there are $0 < \mu_\alpha \in \mathbb{R}$ and $\gamma_1,\ \gamma_2, \ldots,\ \gamma_n$, such that $0 \leq x_\alpha \leq \mu_\alpha(e_{\gamma_1}+e_{\gamma_2}+\cdots+e_{\gamma_n})$. So $0 \leq x_\alpha = x_\alpha \wedge x \leq [\mu_\alpha(e_{\gamma_1}+e_{\gamma_2}+\cdots+e_{\gamma_n})]\wedge x$ $= [\mu_\alpha e_{\gamma_1}] \wedge x + \cdots + [\mu_\alpha e_{\gamma_n}] \wedge x = 0$. Hence $x_\alpha = 0$ for all α, and so $x = 0$. $\square$

We recall the following construction from [49, page 169]. Let X be a vector lattice and $Q = \{e_\gamma\}_{\gamma\in\Gamma}$ be a maximal orthogonal system of X. Let $\mathcal{F}(\Gamma)$ denote the collection of all finite subsets of Γ ordered by inclusion. For each $(n,H) \in \mathbb{N}\times\mathcal{F}(\Gamma)$ and $x \in X_+$, define

$$x_{n,H} := \sum_{\gamma\in H} x \wedge n e_\gamma.$$

Clearly $\{x_{n,H} : (n,H) \in \mathbb{N}\times\mathcal{F}(\Gamma)\}$ is directed upward, and by Theorem 6.5 in [60] it follows that

$$x_{n,H} \leq x \quad \text{for all} \quad (n,H) \in \mathbb{N}\times\mathcal{F}(\Gamma). \tag{3.4.1}$$

Moreover, Proposition II.1.9 in [49] implies $x_{n,H} \uparrow x$.

Theorem 9. *Let $Q = \{e_\gamma\}_{\gamma\in\Gamma}$ be an orthogonal system of a locally solid vector lattice (X,τ). Then Q is a topological orthogonal system if and only if we have $x_{n,H} \xrightarrow{\tau} x$ over $(n,H) \in \mathbb{N}\times\mathcal{F}(\Gamma)$ for each $x \in X_+$.*

Proof. For the backward implication take $x \in X_+$. Since

$$x_{n,H} = \sum_{\gamma\in H} x \wedge n e_\gamma \leq n \sum_{\gamma\in H} e_\gamma,$$

it follows that $x_{n,H} \in I_Q$ for each $(n,H) \in \mathbb{N}\times\mathcal{F}(\Gamma)$. Also, we have, by assumption, $x_{n,H} \xrightarrow{\tau} x$. Thus, $x \in \overline{I}_Q^{\tau}$, i.e., Q is a topological orthogonal system of X.

For the forward implication, note that Q is a maximal orthogonal system, by Lemma 7. Let $x \in X_+$, and $j \in J$. Take $\varepsilon > 0$, let $V_{\varepsilon,x,i} := \{z \in X : \rho_j(z-x) < \varepsilon\}$. Then

$V_{\varepsilon,x,j}$ is a neighborhood of x in the τ-topology. Since I_Q is dense in X with respect to the τ-topology, there is $x_\varepsilon \in I_Q$ such that $\rho_j(x_\varepsilon - x) < \varepsilon$.

Note that

$$\begin{aligned} |x_\varepsilon^+ \wedge x - x| &= |x_\varepsilon^+ \wedge x - x \wedge x| \\ &\leq |x_\varepsilon^+ - x| \text{ by Theorem 1.9(2) in [4]} \\ &= |x_\varepsilon^+ - x^+| \\ &\leq |x_\varepsilon - x| \text{ again by Theorem 1.9(2) in [4].} \end{aligned}$$

Since $x_\varepsilon \in I_Q$ which is an ideal, $x_\varepsilon^+ \wedge x \in I_Q$. Thus without lost of generality, we can assume that there is $x_\varepsilon \in I_Q$ with $0 \leq x_\varepsilon \leq x$ such that $\rho_j(x_\varepsilon - x) < \varepsilon$. Now, $x_\varepsilon \in I_Q$ implies that there are $H_\varepsilon \in \mathscr{F}(\Gamma)$ and $n_\varepsilon \in \mathbb{N}$ such that

$$x_\varepsilon \leq n_\varepsilon \sum_{\gamma \in H_\varepsilon} e_\gamma. \tag{3.4.2}$$

Let

$$w := x \wedge \sum_{\gamma \in H_\varepsilon} n_\varepsilon e_\gamma. \tag{3.4.3}$$

It follows from $0 \leq w \leq \sum\limits_{\gamma \in H_\varepsilon} n_\varepsilon e_\gamma$ and the Riesz decomposition property, that, for each $\gamma \in H_\varepsilon$, there exists y_γ with

$$0 \leq y_\gamma \leq n_\varepsilon e_\gamma \tag{3.4.4}$$

such that

$$w = \sum_{\gamma \in H_\varepsilon} y_\gamma. \tag{3.4.5}$$

From (3.4.3) and (3.4.5), we have

$$y_\gamma \leq x \quad (\forall \gamma \in H_\varepsilon). \tag{3.4.6}$$

Also, (3.4.4) and (3.4.6) imply that $y_\gamma \leq n_\varepsilon e_\gamma \wedge x$. Now

$$w = \sum_{\gamma \in H_\varepsilon} y_\gamma \leq \sum_{\gamma \in H_\varepsilon} x \wedge n_\varepsilon e_\gamma = x_{n_\varepsilon, H_\varepsilon}. \tag{3.4.7}$$

But, from (3.4.2) and (3.4.3), we get

$$0 \leq x_\varepsilon \leq w. \tag{3.4.8}$$

Thus, it follows from (3.4.7), (3.4.8), and (3.4.1), that $0 \leq x_\varepsilon \leq x_{n_\varepsilon, H_\varepsilon} \leq x$. Hence, $0 \leq x - x_{n_\varepsilon, H_\varepsilon} \leq x - x_\varepsilon$ and so $\rho_j(x - x_{n,H}) \leq \rho_j(x - x_{n_\varepsilon, H_\varepsilon}) \leq \rho_j(x - x_\varepsilon)$ for each $(n, H) \geq (n_\varepsilon, H_\varepsilon)$. Therefore $x_{n,H} \xrightarrow{\tau} x$. $\square$

Corollary 6. *Let (X, τ) be a locally solid vector lattice. The following statements are equivalent:*

1. $e \in X_+$ is a quasi-interior point;

2. for each $x \in X_+,\ x - x \wedge ne \xrightarrow{\tau} 0$ as $n \to \infty$.

Moreover, if (X, τ) possesses the σ-Lebesgue property, then every weak unit in X is a quasi-interior point.

Proof. The first part is obvious, for the second part, let $x \in X^+$, and let e be a weak unit. Then $x \wedge ne \uparrow x$. So, by the σ-Lebesgue property, we get $x - x \wedge ne \xrightarrow{\tau} 0$ as $n \to \infty$. □

Proposition 12. *Let $e \in X_+$. Then e is a quasi-interior point in (X, τ) if and only if e is a quasi-interior point in the topological completion $(\widehat{X}, \widehat{\tau})$.*

Proof. For the forward implication let $\widehat{x} \in \widehat{X}_+$. Our aim is to show that $\widehat{x} - \widehat{x} \wedge ne \xrightarrow{\tau} 0$ in $\widehat{X}$ as $n \to \infty$. By Theorem 4, part (2), $\widehat{X}_+ = \overline{X_+}^{\widehat{\tau}}$. So, there is a net (x_α) in X_+ such that $x_\alpha \xrightarrow{\widehat{\tau}} \widehat{x}$ in $\widehat{X}$. Let $j \in J$ and $\varepsilon > 0$. Since $\widehat{\rho}_j(x_\alpha - \widehat{x}) \to 0$, there is α_ε satisfying

$$\widehat{\rho}_j(x_{\alpha_\varepsilon} - \widehat{x}) < \varepsilon. \tag{3.4.9}$$

Since e is a quasi-interior point in X and $x_{\alpha_\varepsilon} \in X_+$, we have $x_{\alpha_\varepsilon} - x_{\alpha_\varepsilon} \wedge ne \xrightarrow{\tau} 0$ in X as $n \to \infty$. Thus, there is $n_\varepsilon \in \mathbb{N}$ such that

$$\widehat{\rho}_j(x_{\alpha_\varepsilon} - ne \wedge x_{\alpha_\varepsilon}) = \rho_j(x_{\alpha_\varepsilon} - ne \wedge x_{\alpha_\varepsilon}) < \varepsilon \quad (\forall n \geqslant n_\varepsilon). \tag{3.4.10}$$

Now, $0 \leq \widehat{x} - \widehat{x} \wedge ne = \widehat{x} - x_{\alpha_\varepsilon} + x_{\alpha_\varepsilon} - ne \wedge x_{\alpha_\varepsilon} + ne \wedge x_{\alpha_\varepsilon} - \widehat{x} \wedge ne$. So $\widehat{\rho}_j(\widehat{x} - \widehat{x} \wedge ne) \leq \widehat{\rho}_j(\widehat{x} - x_{\alpha_\varepsilon}) + \widehat{\rho}_j(x_{\alpha_\varepsilon} - ne \wedge x_{\alpha_\varepsilon}) + \widehat{\rho}_j(ne \wedge x_{\alpha_\varepsilon} - \widehat{x} \wedge ne)$. For $n \geqslant n_\varepsilon$, we have, by (3.4.9), (3.4.10), and [4, Theorem 1.9(2)], that

$$\widehat{\rho}_j(\widehat{x} - \widehat{x} \wedge ne) \leq \varepsilon + \varepsilon + \widehat{\rho}_j(ne \wedge x_{\alpha_\varepsilon} - \widehat{x} \wedge ne) \leq \varepsilon + \varepsilon + \widehat{\rho}_j(x_{\alpha_\varepsilon} - \widehat{x}) \leq 3\varepsilon.$$

Therefore, e is a quasi-interior point in $\widehat{X}$.

The backward implication follows trivially from Corollary 6. □

Another way to see the forward implication of Proposition 12, suppose that e is a quasi-interior point of X, then the closure of I_e in the τ-topology is X. By Theorem 3(iii), $\widehat{\tau}$ induces τ in X, so the closure of I_e with respect to $\widehat{\tau}$ in X is X itself. But $\overline{I_e}^{\widehat{\tau}}$ in X is subset of $\overline{I_e}^{\widehat{\tau}}$ in $\widehat{X}$, so $X \subseteq \overline{I_e}^{\widehat{\tau}}$ which implies by Theorem 3 (iv) that $\widehat{X} = \overline{X}^{\widehat{\tau}} \subseteq \overline{I_e}^{\widehat{\tau}}$. Hence $\widehat{X} = \overline{I_e}^{\widehat{\tau}}$. Therefore e is a quasi-interior point of $\widehat{X}$.

Theorem 10. *Let (X, τ) be a locally solid vector lattice, and $Q = \{e_\gamma\}_{\gamma \in \Gamma}$ be a topological orthogonal system of (X, τ). Then $x_\alpha \xrightarrow{u\tau} 0$ if and only if $|x_\alpha| \wedge e_\gamma \xrightarrow{\tau} 0$ for every $\gamma \in \Gamma$.*

Proof. The forward implication is trivial. For the backward implication, assume $|x_\alpha| \wedge e_\gamma \xrightarrow{\tau} 0$ for every $\gamma \in \Gamma$. Let $u \in X_+$, $j \in J$. Fix $\varepsilon > 0$. We have

$$\begin{aligned}
|x_\alpha| \wedge u &= |x_\alpha| \wedge (u - u_{n,H} + u_{n,H}) \\
&\leq |x_\alpha| \wedge (u - u_{n,H}) + |x_\alpha| \wedge u_{n,H} \\
&\leq (u - u_{n,H}) + |x_\alpha| \wedge \sum_{\gamma \in H} u \wedge ne_\gamma \\
&\leq (u - u_{n,H}) + |x_\alpha| \wedge \sum_{\gamma \in H} ne_\gamma \\
&\leq (u - u_{n,H}) + n\big(|x_\alpha| \wedge \sum_{\gamma \in H} e_\gamma\big) \\
&= (u - u_{n,H}) + n \sum_{\gamma \in H} |x_\alpha| \wedge e_\gamma,
\end{aligned}$$

where the last equality is provided by Theorem 6.5 in [60].

Now, Theorem 9 assures that $u_{n,H} \xrightarrow{\tau} u$, and so, there exists $(n_\varepsilon, H_\varepsilon) \in \mathbb{N} \times \mathscr{F}(\Gamma)$ such that

$$\rho_j(u - u_{n_\varepsilon, H_\varepsilon}) < \varepsilon. \tag{3.4.11}$$

Thus, $|x_\alpha| \wedge u \leq u - u_{n_\varepsilon, H_\varepsilon} + \sum\limits_{\gamma \in H_\varepsilon} n_\varepsilon(e_\gamma \wedge |x_\alpha|)$. But, by the assumption, $e_\gamma \wedge |x_\alpha| \xrightarrow{\tau} 0$ for all $\gamma \in \Gamma$, and so $n_\varepsilon(e_\gamma \wedge |x_\alpha|) \xrightarrow{\tau} 0$. Hence, there is $\alpha_{\varepsilon, H_\varepsilon}$ such that

$$\rho_j\big(n_\varepsilon(e_\gamma \wedge |x_\alpha|)\big) < \frac{\varepsilon}{|H_\varepsilon|} \quad (\forall \alpha \geq \alpha_{\varepsilon, H_\varepsilon},\ \forall \gamma \in H_\varepsilon). \tag{3.4.12}$$

Here $|H_\varepsilon|$ denotes the cardinality of H_ε. For $\alpha \geq \alpha_{\varepsilon, H_\varepsilon}$, we have

$$\begin{aligned}
\rho_j(|x_\alpha| \wedge u) &\leq \rho_j(u - u_{n_\varepsilon, H_\varepsilon}) + \rho_j\big(n_\varepsilon \sum_{\gamma \in H_\varepsilon} |x_\alpha| \wedge e_\gamma\big) \\
&\leq \varepsilon + \sum_{\gamma \in H_\varepsilon} \rho_j\big(n_\varepsilon(e_\gamma \wedge |x_\alpha|)\big) < \varepsilon + \sum_{\gamma \in H_\varepsilon} \frac{\varepsilon}{|H_\varepsilon|} = 2\varepsilon,
\end{aligned}$$

where the second inequality follows from (3.4.11) and the third one from (3.4.12). Therefore, $\rho_j(|x_\alpha| \wedge u) \to 0$, and so $x_\alpha \xrightarrow{u\tau} 0$. $\square$

Corollary 7. *Let (X, τ) be a locally solid vector lattice, and $e \in X_+$ be a quasi-interior point. Then $x_\alpha \xrightarrow{u\tau} 0$ if and only if $|x_\alpha| \wedge e \xrightarrow{\tau} 0$.*

Proof. The forward implication is trivial. For the backward implication assume $|x_\alpha| \wedge e \xrightarrow{\tau} 0$. Let $u \in X_+$, and fix $\varepsilon > 0$. Note that for all $k \in \mathbb{N}$,

$$|x_\alpha| \wedge u \leq |x_\alpha| \wedge (u - u \wedge ke + u \wedge ke) \leq |x_\alpha| \wedge (u - u \wedge ke) + |x_\alpha| \wedge (u \wedge ke)$$

$$\leq (u - u \wedge ke) + k|x_\alpha| \wedge (ku \wedge ke) = (u - u \wedge ke) + k\left[|x_\alpha| \wedge (u \wedge e)\right].$$

Hence $|x_\alpha| \wedge u \leq (u - u \wedge ke) + k\,(|x_\alpha| \wedge e)$. Thus for all $j \in J$,

$$\rho_j(|x_\alpha| \wedge u) \leq \rho_j\,(u - u \wedge ke) + k\rho_j\,(|x_\alpha| \wedge e)$$

for all α and for all $k \in \mathbb{N}$. Since e is a quasi-interior point, and $u \in X_+$, for the fixed ε, and for all $j \in J$, there exist $k_{\varepsilon,j} \in \mathbb{N}$ such that $\rho_j\left(u - u \wedge k_{\varepsilon,j}e\right) < \frac{\varepsilon}{2}$.

Furthermore. it follows from $x_\alpha \wedge e \xrightarrow{\tau} 0$, that for the fixed ε, and for all $j \in J$, there exists $\alpha_{j,\varepsilon}$, such that $\rho_j\left(|x_\alpha| \wedge e\right) < \frac{\varepsilon}{2k_{\varepsilon,j}}$, that is $k_{\varepsilon,j}\rho_j\left(|x_\alpha| \wedge e\right) < \frac{\varepsilon}{2}$. Thus for the fixed ε, and for all $j \in J$, there exists $\alpha_{j,\varepsilon}$, and $k_{\varepsilon,j} \in \mathbb{N}$, such that $\alpha \geq \alpha_{j,\varepsilon}$ implies that $\rho_j\left(|x_\alpha| \wedge u\right) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Therefore $x_\alpha \xrightarrow{u\tau} 0$ as desired. □

Theorem 11. *Let (X, τ) be a sequentially complete locally solid vector lattice, where τ is generated by a family $(\rho_j)_{j\in J}$ of Riesz pseudonorms. Let $e \in X_+$. The following are equivalent:*

1. *e is a quasi-interior point;*
2. *for every net (x_α) in X_+, if $x_\alpha \wedge e \xrightarrow{\tau} 0$ then $x_\alpha \xrightarrow{u\tau} 0$;*
3. *for every sequence (x_n) in X_+, if $x_n \wedge e \xrightarrow{\tau} 0$ then $x_n \xrightarrow{u\tau} 0$.*

Proof. (1)⇒(2) It follows from Corollary 7.

(2)⇒(3) is trivial.

(3)⇒(1).

Suppose (3). Fix $x \in X_+$. We need to show that $x - (x \wedge ne) \xrightarrow{\tau} 0$ or, equivalently by [4, Theorem 1.7(1)] $(x - ne)^+ \xrightarrow{\tau} 0$ as a sequence of n. Put $w = x \vee e$. The ideal I_w is lattice and norm isomorphic (as a vector lattice) to $C(K)$ for some compact Hausdorff space K, with w corresponding to $\mathbb{1}$. Since $x, e \in I_w$, we may consider x and e as elements of $C(K)$. Note that $x \vee e = \mathbb{1}$ implies that x and e never vanish simultaneously.

For each $n \in \mathbb{N}$, we define

$$F_n = \big\{t \in K \ :\ x(t) \geqslant ne(t)\big\} \text{ and } O_n = \big\{t \in K \ :\ x(t) > ne(t)\big\}.$$

Observe that $O_n \subseteq F_n$, and O_n is open in K, because for any $t \in O_n$, $(x - ne)(t) > 0$, that is O_n is the inverse image of $(0, \infty)$.

And F_n is closed, because for any $t \in F_n$, $(x - ne)(t) \geqslant 0$, that is F_n is the inverse image of $[0, \infty)$.

Claim 1: $F_{n+1} \subseteq O_n$. Indeed, let $t \in F_{n+1}$. Then $x(t) \geqslant (n+1)e(t)$. If $e(t) > 0$ then $x(t) > ne(t)$, so that $t \in O_n$. If $e(t) = 0$ then $x(t) \geqslant 0$, but x and e never vanish simultaneously, so $x(t) > 0$, and hence $t \in O_n$.

By Urysohn's Lemma, we find $f_n \in C(K)$ such that $0 \leqslant f_n \leqslant x$, f_n agrees with x on F_{n+1} and vanishes outside of O_n. We can also view f_n as an element of X.

Claim 2: $n(f_n \wedge e) \leqslant x$. Let $t \in K$. If $t \in O_n$ then $n(f_n \wedge e)(t) \leqslant ne(t) < x(t)$. If $t \notin O_n$ then $f_n(t) = 0$, so that the inequality is satisfied trivially.

Claim 3: $\big(x-(n+1)e\big)^+ \leqslant f_n$. Again, let $t\in K$. If $t\in F_{n+1}$ then $\big(x-(n+1)e\big)^+ \leqslant x(t)=f_n(t)$. If $t\notin F_{n+1}$ then $x(t)<(n+1)e(t)$, so that $\big(x-(n+1)e\big)^+(t)=0$ and the inequality is satisfied trivially.

Now, Claim 2 yields $f_n\wedge e\leqslant \frac{1}{n}x$, but $f_n\wedge e\geqslant 0$, so $0\leqslant f_n\wedge e\leqslant \frac{1}{n}x$, and so for all $j\in J$, we have $0\leq \rho_j(f_n\wedge e)\leq \frac{1}{n}\rho_j(x)$, and as $n\to\infty$, we get that $\rho_j(f_n\wedge e)\to 0$, that is $f_n\wedge e\xrightarrow{\tau} 0$. By assumption, this yields $f_n\xrightarrow{u\tau} 0$. Since $0\leqslant f_n\leqslant x$ for every n, the sequence (f_n) is order bounded, so take $w=x$, to get that $f_n\wedge x\xrightarrow{\tau} 0$, therefore $f_n\xrightarrow{\tau} 0$. Now Claim 3 yields $\big(x-(n+1)e\big)^+\xrightarrow{\tau} 0$, which concludes the proof. □

Recall that a topological vector space is metrizable if and only if it has a countable neighborhood base at zero, [3, Theorem 2.1]. In particular, a locally solid vector lattice (X,τ) is metrizable if and only if its topology τ is generated by a countable family $(\rho_k)_{k\in\mathbb{N}}$ of Riesz pseudonorms because there is a one to one corresponding between Riesz pseudonorms and neighborhood base at zero, as follows:
Let $\varepsilon_n=\frac{1}{n},\ n\in\mathbb{N}$;

$$V_{n,k}=\{x\in X:\rho_k(x)<\frac{1}{n}\}$$

Lemma 8. *Suppose that $\rho: X\times X\longrightarrow[0,\infty]$ is a semimetric, then $d: X\times X\longrightarrow[0,\infty]$ defined by $d(x,y):=\frac{\rho(x,y)}{1+\rho(x,y)}$ is also a semimetric. In particuler, if ρ is a metric, then d is a metric as well.*

Proof. Clearly $d(x,y)\geq 0$ for all $x,y\in X$ and $d(x,y)=d(y,x)$. We prove the triangle inequality. That is for all x,y,z we have $d(x,y)\leq d(x,z)+d(z,y)$. Let $f(t)=\frac{t}{1+t}$ for $t\in[0,\infty)$, then $f'(t)=\frac{1}{(1+t)^2}>0$. Thus, f is an increasing function over $[0,\infty)$. That is, if $t\leq s$ then $\frac{t}{1+t}\leq\frac{s}{1+s}$. We know that ρ satisfies triangle inequality. So, $\rho(x,y)\leq\rho(x,z)+\rho(z,y)$. Then we get

$$\begin{aligned}\frac{\rho(x,y)}{1+\rho(x,y)}&\leq\frac{\rho(x,z)+\rho(z,y)}{1+\rho(x,z)+\rho(z,y)}\\&=\frac{\rho(x,z)}{1+\rho(x,z)+\rho(z,y)}+\frac{\rho(z,y)}{1+\rho(x,z)+\rho(z,y)}\\&\leq\frac{\rho(x,z)}{1+\rho(x,z)}+\frac{\rho(z,y)}{1+\rho(z,y)}.\end{aligned}$$

Thus $d(x,y)\leq(x,z)+d(z,y)$. □

Lemma 9. *Let (x_α) be a net in $\mathbb{R}$. Then, $x_\alpha\to x$ in $\mathbb{R}$ if and only if $\frac{|x_\alpha-x|}{1+|x_\alpha-x|}\to 0$ in $\mathbb{R}$.*

Proof. $(\Rightarrow)$ Trivial.
$(\Leftarrow)$ Suppose $\frac{|x_\alpha-x|}{1+|x_\alpha-x|}\to 0$ in $\mathbb{R}$. Our aim is to show that $x_\alpha\to x$ in $\mathbb{R}$. Given $\varepsilon>0$. Take $\delta=\frac{\varepsilon}{1+\varepsilon}$. Note $0<\delta<1$. Since $\frac{|x_\alpha-x|}{1+|x_\alpha-x|}\to 0$ in $\mathbb{R}$, there is α_0 such that $\frac{|x_\alpha-x|}{1+|x_\alpha-x|}<\delta$ for all $\alpha\geqslant\alpha_0$. or $\frac{|x_\alpha-x|}{1+|x_\alpha-x|}<\frac{\varepsilon}{1+\varepsilon}$ for all $\alpha\geqslant\alpha_0$, so

$(1+\varepsilon)|x_\alpha - x| < \varepsilon + \varepsilon|x_\alpha - x|$, that is $|x_\alpha - x| < \varepsilon$ for all $\alpha \geqslant \alpha_0$. Thus, $x_\alpha \to x$ in $\mathbb{R}$. $\square$

The following result gives a sufficient condition for the metrizabililty of $u\tau$-topology.

Proposition 13. *Let (X,τ) be a complete metrizable locally solid vector lattice. If X has a countable topological orthogonal system, then the $u\tau$-topology is metrizable.*

Proof. First note that, since (X,τ) is metrizable, τ is generated by a countable family $(\rho_k)_{k\in\mathbb{N}}$ of Riesz pseudonorms.

Now suppose $(e_n)_{n\in\mathbb{N}}$ to be a topological orthogonal system. For each $n \in \mathbb{N}$, put $d_n(x,y) := \sum\limits_{k=1}^{\infty} \frac{1}{2^k} \frac{\rho_k(|x-y|\wedge e_n)}{1+\rho_k(|x-y|\wedge e_n)}$. Note that each d_n is a semimetric by Lemma 8, and $d_n(x,y) \leq 1$ for all $x,y \in X$. If $d_n(x,y) = 0$, then $\rho_k(|x-y|\wedge e_n) = 0$ for all $k \in \mathbb{N}$, so $(|x-y|\wedge e_n) = 0$. For $x,y \in X$, let $d(x,y) := \sum\limits_{n=1}^{\infty} \frac{1}{2^n} d_n(x,y)$. Clearly, $d(x,y)$ is nonnegative. Also d satisfies the triangle inequality, Indeed

$$\begin{aligned} d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} d_n(x,y) &\leq \sum_{n=1}^{\infty} \frac{1}{2^n} \left(d_n(x,z) + d_n(z,y)\right) \\ &= \sum_{n=1}^{\infty} \frac{1}{2^n} d_n(x,z) + \sum_{n=1}^{\infty} \frac{1}{2^n} d_n(z,y) \\ &= d(x,z) + d(z,y)\,. \end{aligned}$$

It is easy to see that $d(x,y) = d(y,x)$ for all $x,y \in X$. Now $d(x,y) = 0$ if and only if $d_n(x,y) = 0$ for all $n \in \mathbb{N}$ if and only if $\rho_k(|x-y|\wedge e_n) = 0$ for all $k \in \mathbb{N}$ if and only if $(|x-y|\wedge e_n) = 0$ for all $n \in \mathbb{N}$ if and only if $|x-y| = 0$ if and only if $x = y$. Thus (X,d) is a metric space.

It remains to show that d generates the $u\tau$-topology. Suppose that $(x_\alpha)_{\alpha\in A}$ is a net in X such that $x_\alpha \xrightarrow{u\tau} 0$. Then by Theorem 10 we have $|x_\alpha| \wedge e_n \xrightarrow{\tau} 0$ over α for each $n \in \mathbb{N}$. Thus, for each $k \in \mathbb{N}$, $\rho_k(|x_\alpha| \wedge e_n) \to 0$ over α and this holds also for each $n \in \mathbb{N}$. Given $n \in \mathbb{N}$. Then,

$$\rho_k(|x_\alpha| \wedge e_n) \to 0 \text{ over } \alpha \text{ for each } k \in \mathbb{N} \tag{3.4.13}$$

Our aim is to show that $x_\alpha \xrightarrow{\mathrm{d_n}} 0$ where $d_n(x,y) = \sum\limits_{k=1}^{\infty} \frac{1}{2^k} \frac{\rho_k(|x-y|\wedge e_n)}{1+\rho_k(|x-y|\wedge e_n)}$. Given $\varepsilon > 0$. Then there is $k_0 \in \mathbb{N}$ such that

$$\sum_{k=k_0}^{\infty} \frac{1}{2^k} < \frac{\varepsilon}{2} \tag{3.4.14}$$

For $k = 1, \cdots, k_0 - 1$, there is α_0 such that

$$\rho_1(|x_\alpha| \wedge e_n) + \cdots + \rho_{k_0-1}(|x_\alpha| \wedge e_n) < \frac{\varepsilon}{2} \text{ for all } \alpha \geqslant \alpha_0 \tag{3.4.15}$$

For $\alpha \geqslant \alpha_0$,

$$\begin{aligned} d_n\left(x_\alpha, 0\right) &= \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\rho_k\left(\left|x_\alpha\right| \wedge e_n\right)}{1+\rho_k\left(\left|x_\alpha\right| \wedge e_n\right)} \\ &= \sum_{k=1}^{k_0-1} \frac{1}{2^k} \frac{\rho_k\left(\left|x_\alpha\right| \wedge e_n\right)}{1+\rho_k\left(\left|x_\alpha\right| \wedge e_n\right)} + \sum_{k=k_0}^{\infty} \frac{1}{2^k} \frac{\rho_k\left(\left|x_\alpha\right| \wedge e_n\right)}{1+\rho_k\left(\left|x_\alpha\right| \wedge e_n\right)} \end{aligned}$$

In the first sum note that $\frac{1}{2^k} \frac{1}{1+\rho_k(|x_\alpha| \wedge e_n)} \leq 1$ and in the second sum $\frac{\rho_k(|x_\alpha| \wedge e_n)}{1+\rho_k(|x_\alpha| \wedge e_n)} \leq 1$. So for $\alpha \geqslant \alpha_0$,

$$\begin{aligned} d_n\left(x_\alpha, 0\right) &\leq \sum_{k=1}^{k_0-1} \rho_k\left(\left|x_\alpha\right| \wedge e_n\right) + \sum_{k=k_0}^{\infty} \frac{1}{2^k} \\ \text{By 3.4.14 and 3.4.15, } &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Hence, we have proved that for $n \in \mathbb{N}$, $d_n\left(x_\alpha, 0\right) \to 0$ over α. Note that $d\left(x, y\right) = \sum\limits_{n=1}^{\infty} \frac{1}{2^n} d_n\left(x, y\right)$. Given $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that

$$\sum_{n=n_0}^{\infty} \frac{1}{2^n} < \frac{\varepsilon}{2}. \tag{3.4.16}$$

Also, there is α_ε such that

$$d_1\left(x_\alpha, 0\right) + \cdots + d_{n_0-1}\left(x_\alpha, 0\right) < \frac{\varepsilon}{2} \text{ for all } \alpha \geqslant \alpha_\varepsilon. \tag{3.4.17}$$

Therefore, for all $\alpha \geqslant \alpha_\varepsilon$,

$$\begin{aligned} d\left(x_\alpha, 0\right) &= \sum_{n=1}^{\infty} \frac{1}{2^n} d_n\left(x_\alpha, 0\right) \\ &= \sum_{n=1}^{n_0-1} \frac{1}{2^n} d_n\left(x_\alpha, 0\right) + \sum_{n=n_0}^{\infty} \frac{1}{2^n} d_n\left(x_\alpha, 0\right) \\ &\leq \sum_{n=1}^{n_0-1} d_n\left(x_\alpha, 0\right) + \sum_{n=n_0}^{\infty} \frac{1}{2^n} \\ \text{By 3.4.16 and 3.4.17, } &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

So far we have shown that if $x_\alpha \xrightarrow{u\tau} 0$ then $x_\alpha \xrightarrow{d} 0$. Conversely, suppose that $x_\alpha \xrightarrow{d} 0$, i.e. $d\left(x_\alpha, 0\right) \to 0$ over α. But $d\left(x_\alpha, 0\right) = \sum\limits_{n=1}^{\infty} \frac{1}{2^n} d_n\left(x_\alpha, 0\right)$. Note that $\frac{1}{2^n} d_n\left(x_\alpha, 0\right) \leq d\left(x_\alpha, 0\right) \to 0$ over α, so $d_n\left(x_\alpha, 0\right) \to 0$ over α for all $n \in \mathbb{N}$. Note that $d_n\left(x_\alpha, 0\right) = \sum\limits_{k=1}^{\infty} \frac{1}{2^k} \frac{\rho_k(|x_\alpha| \wedge e_n)}{1+\rho_k(|x_\alpha| \wedge e_n)}$, and $\frac{1}{2^k} \frac{\rho_k(|x_\alpha| \wedge e_n)}{1+\rho_k(|x_\alpha| \wedge e_n)} \leq d_n\left(x_\alpha, 0\right) \to 0$ over α, then by Lemma 9 $\rho_k\left(\left|x_\alpha\right| \wedge e_n\right) \to 0$ over α for all $k \in \mathbb{N}$, and so for all $n \in \mathbb{N}$. It follows that $|x_\alpha| \wedge e_n \xrightarrow{\tau} 0$ for all $n \in \mathbb{N}$. Again by Theorem 10 we have $x_\alpha \xrightarrow{u\tau} 0$. $\square$

Recall that a topological space X is called *submetrizable* if its topology is finer that some metric topology on X.

Proposition 14. *Let (X, τ) be a metrizable locally solid vector lattice. If X has a weak unit, then the $u\tau$-topology is submetrizable.*

Proof. Note that, since (X, τ) is metrizable, τ is generated by a countable family $(\rho_k)_{k\in\mathbb{N}}$ of Riesz pseudonorms.

Suppose that $e \in X_+$ is a weak unit. Put $d(x, y) := \sum\limits_{k=1}^{\infty} \frac{1}{2^k} \frac{\rho_k(|x-y|\wedge e)}{1+\rho_k(|x-y|\wedge e)}$. Note that $d(x, y) = 0$ if and only if $\rho_k(|x-y| \wedge e) = 0$ for all $k \in \mathbb{N}$ if and only if $|x-y| \wedge e = 0$ and, since e is a weak unit, $x = y$. By the same argument used in the proof of Proposition 13, it can be shown that d satisfies the triangle inequality. Assume $x_\alpha \xrightarrow{u\tau} x$. Then, $\rho_k(|x-y| \wedge e) \to 0$ for all $k \in \mathbb{N}$. Now, we show shown that $x_\alpha \xrightarrow{d} x$. Fix $\varepsilon > 0$. There is $k_0 \in \mathbb{N}$ such that $\sum\limits_{k=k_0}^{\infty} \frac{1}{2^k} < \frac{\varepsilon}{2}$. Since $\rho_k(|x-y| \wedge e) \to 0$ for all $k \in \mathbb{N}$, there is α_0 such that $\sum\limits_{k=1}^{k_0-1} \frac{1}{2^k} \frac{\rho_k(|x-y|\wedge e)}{1+\rho_k(|x-y|\wedge e)} < \frac{\varepsilon}{2}$ for all $\alpha \geq \alpha_0$. Thus, for all $\alpha \geq \alpha_0$,

$$\begin{aligned} d(x_\alpha, x) &= \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\rho_k(|x_\alpha - x| \wedge e)}{1+\rho_k(|x_\alpha - x| \wedge e)} \\ &\leq \sum_{k=1}^{k_0-1} \frac{1}{2^k} \frac{\rho_k(|x_\alpha - x| \wedge e)}{1+\rho_k(|x_\alpha - x| \wedge e)} + \sum_{k=k_0}^{\infty} \frac{1}{2^k} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus, $x_\alpha \xrightarrow{d} x$.

Therefore, the $u\tau$-topology is finer than the metric topology generated by d, and hence $u\tau$-topology is submetrizable. □

The converse of Proposition 13 holds for a particular case as shown in Proposition 21. Where the converse of Proposition 14 in general, does not hold, see [34, Example 2.1].

3.5 $u\tau$-Completeness

A subset A of a locally solid vector lattice (X, τ) is said to be (*sequentially*) $u\tau$-*complete* if, it is (sequentially) complete in the $u\tau$-topology. In this section, we relate sequential $u\tau$-completeness of subsets of X with the Lebesgue and Levi properties. First, we remind the following theorem.

Theorem 12. *[59, Theorem 1] If (X, τ) is a locally solid vector lattice, then the following statements are equivalent*:

1. (X, τ) *has the Lebesgue and Levi properties*;
2. X *is* τ*-complete, and* c_0 *is not lattice embeddable in* (X, τ).

Recall that two locally solid vector lattices (X_1, τ_1) and (X_2, τ_2) are said to be *isomorphic*, if there exists a lattice isomorphism from X_1 onto X_2 that is also a homeomorphism; in other words, if there exists a mapping from X_1 onto X_2 that preserves the algebraic, the lattice, and the topological structures. [3, Page 52].

A locally solid vector lattice (X_1, τ_1) is said to be *lattice embeddable* into another locally solid vector lattice (X_2, τ_2) if there exists a sublattice Y_2 of X_2 such that (X_1, τ_1) and (Y_2, τ_2) are isomorphic.

Note that (X, τ) can have the Lebesgue and Levi properties and simultaneously contains c_0 as a sublattice, but not as a lattice embeddable copy. The following example illustrates this.

Example 3. *Let* s *denote the vector lattice of all sequences in* $\mathbb{R}$ *with coordinatewise ordering. Clearly,* c_0 *is a sublattice of* s*. For* $j \in \mathbb{N}$*, define the Riesz pseudonorm* ρ_j *on* s *as follows:*

$$\rho_j((x_n)_{n\in\mathbb{N}}) := |x_j|.$$

Let $\mathcal{R} := \{\rho_j : j \in \mathbb{N}\}$*. Then* $\mathcal{R}$ *generates a locally solid topology* τ *on* s*. We show that* (s, τ) *has the Lebesgue and Levi properties. Let* $0 \leq x^\alpha \uparrow$ *be a* τ*-bounded net in* s*. For each* α*,* $x^\alpha = (x_n^\alpha)_{n\in\mathbb{N}}$*. The condition* $0 \leq x^\alpha \uparrow$ *implies that, for each* $j \in \mathbb{N}$*,* $(x_j^\alpha)_\alpha$ *is an increasing net in* $\mathbb{R}_+$*. Note that the* ρ_j*'s here are Riesz seminorms, so the* τ*-boundedness of the net* (x^α) *assures that, for each* j*, the net* $(x_j^\alpha)_\alpha$ *is bounded in* $\mathbb{R}$*. Thus, by the monotone convergence theorem in* $\mathbb{R}$*, we have for each* j*,* $0 \leq x_j^\alpha \uparrow x_j$ *for some* $x_j \in \mathbb{R}$*. Define* $x := (x_j)_{j\in\mathbb{N}} \in s$*, then* $x^\alpha \uparrow x$*. Now, suppose* $x^\alpha \downarrow 0$ *in* s*. Then, for each* $j \in \mathbb{N}$*, the sequence* $(x_j^\alpha)_\alpha$ *decreases to zero in* $\mathbb{R}$*. That is* $\rho_j(x^\alpha) = x_j^\alpha \to 0$ *in* $\mathbb{R}$ *for each* $j \in \mathbb{N}$*. Hence,* $x^\alpha \xrightarrow{\tau} 0$*. Therefore,* (s, τ) *possesses the Lebesgue and Levi properties. Although* c_0 *is a sublattice of* s*, but* $(c_0, \|\cdot\|_\infty)$ *is not lattice embeddable in* (s, τ)*. To see this, let* $\Phi : (c_0, \|\cdot\|_\infty) \to (s, \tau)$ *be a lattice embedding. Let* (e_n) *be the standard basis in* c_0*. Then* (Φe_n) *is a disjoint sequence in* (s, τ)*, which is easily seen to converge to* 0 *in* (s, τ)*. It follows that* $e_n \to 0$ *in* $(c_0, \|\cdot\|_\infty)$*, which is absurd.*

Proposition 15. *Let* (X, τ) *be a complete locally solid vector lattice that has the Lebesgue property. If every* τ*-bounded* $u\tau$*-Cauchy sequence is* $u\tau$*-convergent in* X*, then* (X, τ) *also has the Levi property.*

Proof. Suppose X does not possess the Levi property. Then, by Theorem 12, c_0 is lattice embeddable in (X, τ). So there is a map $\Phi : (c_0, \|\cdot\|_\infty) \to (X, \tau)$ which is a lattice embedding. Let $s_n = \sum_{k=1}^n e_k$, where e_k's denote the standard unit vectors in c_0. It follows from [36, Lemma 6.1] that (s_n) is un-Cauchy in $(c_0, \|\cdot\|_\infty)$. Thus (Φs_n) is $u\tau$-Cauchy in $(\Phi c_0, \tau)$. Now [3, Theorem 3.24] assures that X is Dedekind complete and hence (Φs_n) is $u\tau$-Cauchy in (X, τ) by Theorem 7. Suppose $\Phi s_n \xrightarrow{u\tau} x$ in X. Since $0 \leq \Phi s_n \uparrow$ and (X, τ) has the Lebesgue property, it follows by a similar argument to [36, Lemma 1.2(i)] that $x = \sup_n \Phi s_n$, so that $\Phi s_n \to x$ in (X, τ) due to the Lebesgue property again. This implies (Φs_n) is Cauchy in (X, τ), so that (s_n) is Cauchy in $(c_0, \|\cdot\|_\infty)$, which is absurd. $\square$

Theorem 13. *[59, Theorem 1′] If (X, τ) is a Dedekind complete locally solid vector lattice, then the following statements are equivalent*:

1. *(X, τ) has the σ-Lebesgue and σ-Levi properties*;
2. *X is τ-sequentially complete, and c_0 is not lattice embeddable in (X, τ).*

Using the proof of Proposition 15 and Theorem 13, one can easily prove the following result.

Proposition 16. *Let X be a Dedekind complete vector lattice equipped with a sequentially complete locally solid topology τ. If (X, τ) has the Lebesgue property and every τ-bounded $u\tau$-Cauchy sequence is $u\tau$-convergent in X, then (X, τ) also has the σ-Levi property.*

As it was observed in [36, page 271 before Example 6.5], the Lebesgue property can not be removed from Propositions 15 and 16.

Clearly, every finite dimensional locally solid vector lattice (X, τ) is $u\tau$-complete. On the contrary of [36, Proposition 6.2], we provide an example of a τ-complete locally solid vector lattice (X, τ) possessing the Lebesgue property such that it is $u\tau$-complete and $\dim X = \infty$.

Example 4. *Let $X = s$ and $\mathcal{R} = (\rho_j)_{j \in \mathbb{N}}$ such that $\rho_j((x_n)) := |x_j|$, where $(x_n) \in s$.*

First, we show that $(X, \mathcal{R})$ is τ-complete. Let (x^α) be a τ-Cauchy net in $(X, \mathcal{R})$, then $x^\alpha = (x_n^\alpha)_{n \in \mathbb{N}}$ and $x^\alpha - x^\beta \xrightarrow{\tau} 0$ over α, β. For $j \in \mathbb{N}$, we have $\rho_j(x^\alpha - x^\beta) \to 0$ in $\mathbb{R}$ over α, β. That is, for $j \in \mathbb{N}$, $|x_j^\alpha - x_j^\beta| \to 0$ in $\mathbb{R}$ over α, β. Thus, for each $j \in \mathbb{N}$, the net $(x_j^\alpha)_\alpha$ is Cauchy in $\mathbb{R}$ and so, there is $x_j \in \mathbb{R}$ such that $x_j^\alpha \to x_j$ over α. Take $x := (x_j)_{j \in \mathbb{N}} \in s$. Since, for each $j \in \mathbb{N}$, $x_j^\alpha \to x_j$ over α in $\mathbb{R}$, it follows that $\rho_j(x^\alpha - x) \to 0$ in $\mathbb{R}$. Hence, $x^\alpha \xrightarrow{\tau} x$. Therefore, $(X, \mathcal{R})$ is τ-complete.

Second, we show that $(X, \mathcal{R})$ has the Lebesgue property. Assume $x^\alpha \downarrow 0$, our aim is to show that $x^\alpha \xrightarrow{\tau} 0$. We know that $x^\alpha = (x_n^\alpha)_{n \in \mathbb{N}}$. For each $j \in \mathbb{N}$, $x^\alpha \downarrow 0$ implies that $x_j^\alpha \downarrow 0$ in $\mathbb{R}$. That is $\rho_j(x^\alpha) \downarrow 0$ in $\mathbb{R}$. Thus, $x^\alpha \xrightarrow{\tau} 0$.

Finally, we show that $(X, \mathcal{R})$ is $u\tau$-complete. Suppose (x^α) is $u\tau$-Cauchy net. Then, for each $u \in X_+$, we have $|x^\alpha - x^\beta| \wedge u \xrightarrow{\tau} 0$. Now, $u = u_n$ and, $x^\alpha = x_n^\alpha$. Let $j \in \mathbb{N}$, then $\rho_j(|x^\alpha - x^\beta| \wedge u) \to 0$ in $\mathbb{R}$ over α, β if and only if $|x_j^\alpha - x_j^\beta| \wedge u_j \to 0$ in $\mathbb{R}$ if and only if $|x_j^\alpha - x_j^\beta| \to 0$ in $\mathbb{R}$ over α, β.
Thus, $(x_j^\alpha)_\alpha$ is Cauchy in $\mathbb{R}$ and so there is $x_j \in \mathbb{R}$ such that $x_j^\alpha \to x_j$ in $\mathbb{R}$ over α. Let $x = (x_j)_{j \in \mathbb{N}} \in s$, then, clearly, $x^\alpha \xrightarrow{u\tau} x$.

CHAPTER 4

UNBOUNDED m-TOPOLOGY IN MULTI-NORMED VECTOR LATTICES

Unbounded convergences have attracted many researchers (see for instance [31, 27, 30, 21, 18, 61, 36, 8, 41, 37, 35, 29, 28, 52, 16]. Unbounded convergences are well-investigated in vector and normed lattices (cf. [18, 30, 36, 53, 57]). In this chapter, we also extend several previous results from [18, 30, 36, 53, 57, 61] to multi-normed setting. This work is a continuation of Chapter 3, in which unbounded topological convergence was studied in locally solid vector lattices.

Let (X, τ) be a locally solid vector lattice, if τ has base at zero consisting of convex-solid sets, then (X, τ) is called a *locally convex-solid vector lattice*. It is known that a linear topology τ on X is locally convex-solid if and only if there exists a family $\mathcal{M} = \{m_\lambda\}_{\lambda\in\Lambda}$ of lattice seminorms that generates τ (cf. [3, Theorem 2.25]). Moreover, for such $\mathcal{M}$, $x_\alpha \xrightarrow{\tau} x$ if and only if $m_\lambda(x_\alpha - x) \underset{\alpha}{\to} 0$ in $\mathbb{R}$ for each $m_\lambda \in \mathcal{M}$. Since τ is Hausdorff, the family $\mathcal{M}$ is separating.

Recall that subset A in a topological vector space (X, τ) is called τ-*bounded* if, for every τ-neighborhood V of zero, there exists $\lambda > 0$ such that $A \subseteq \lambda V$. In the case when the topology τ is generated by a family $\{m_\lambda\}_{\lambda\in\Lambda}$ of seminorms, a subset A of X is τ-bounded if and only if $\sup_{a\in A} m_\lambda(a) < \infty$ for all $\lambda \in \Lambda$.

4.1 Multi-normed vector lattices

Let (X, τ) be a locally convex-solid vector lattice with an upward directed family $\mathcal{M} = \{m_\lambda\}_{\lambda\in\Lambda}$ of lattice seminorms generating τ. Throughout this chapter, the pair $(X, \mathcal{M})$ will be referred as a *multi-normed vector lattice* (*MNVL*). Also, τ-convergence, τ-Cauchy, τ-complete, etc. will be denoted by m-convergence, m-Cauchy, m-complete, etc.

Let X be a vector space, E be a vector lattice, and $p : X \to E_+$ be a vector norm (i.e. $p(x) = 0 \Leftrightarrow x = 0$, $p(\lambda x) = |\lambda| p(x)$ for all $\lambda \in \mathbb{R}$, $x \in X$, and $p(x+y) \le p(x)+p(y)$ for all $x, y \in X$), then (X, p, E) is called a *lattice-normed space*, abbreviated as *LNS*, see [40]. If X is a vector lattice, and the vector norm p is monotone (i.e. $|x| \le |y| \Rightarrow p(x) \le p(y)$), then the triple (X, p, E) is called a lattice-normed vector lattice, abbreviated as *LNVL* (cf. [8, 9]).

Given an LNS (X, p, E). Recall that a net (x_α) in X is said to be *p-convergent* to x (see [8]) if $p(x_\alpha - x) \xrightarrow{o} 0$ in E. In this case, we write $x_\alpha \xrightarrow{p} x$. A subset A of X is called *p-bounded* if there exists $e \in E$ such that $p(a) \leq e$ for all $a \in A$.

Proposition 17. *Every MNVL induces an LNVL. Moreover, for arbitrary nets, p-convergence in the induced LNVL implies m-convergence, and they coincide in the case of p-bounded nets.*

Proof. Let $(X, \mathcal{M})$ be an MNVL, then there is a separating family $\{m_\lambda\}_{\lambda \in \Lambda}$ of lattice seminorms on X. Let $E = \mathbb{R}^\Lambda$ be the vector lattice of all real-valued functions on Λ, and define $p : x \mapsto p_x$ from X into E_+ such that $p_x[\lambda] := m_\lambda(x)$.

We show that p is a vector norm on X.

- If $x = 0$, then $p_0[\lambda] = m_\lambda(0) = 0$, so $p_0[\lambda] = 0$ for all $\lambda \in \Lambda$. So $p_0 = 0$. Assume, $p_x = 0$, then $p_x[\lambda] = 0$ for all $\lambda \in \Lambda$, or $m_\lambda(x) = 0$ for all $\lambda \in \Lambda$. Since $(m_\lambda)_{\lambda \in \Lambda}$ is a separating family of lattice seminorms on X, we have $x = 0$. Therefore, $p_x = 0$ if and only if $x = 0$.
- For $r \in \mathbb{R}$, we show $p_{rx} = |r|p_x$. Indeed, $p_{rx}[\lambda] = m_\lambda(rx) = |r|m_\lambda(x) = |r|p_x$. Next we show triangle inequality. For all $x \in X$. Let $\lambda \in \Lambda$, then $p_{(x+y)}[\lambda] = m_\lambda(x + y) \leq m_\lambda(x) + m_\lambda(y) = p_x[\lambda] + p_y[\lambda] = (p_x + p_y)[\lambda]$. Thus, $p_{(x+y)} \leq p_x + p_y$.

Now we show that p is monotone. Assume that $|x| \leq |y|$, then for $\lambda \in \Lambda$, $p_x[\lambda] = m_\lambda(x) \leq m_\lambda(y) = p_y[\lambda]$, hence p is monotone. Therefore (X, p, E) is an LNVL.

Let (x_α) be a net in X. If $x_\alpha \xrightarrow{p} 0$, then $p_{x_\alpha} \xrightarrow{o} 0$ in $\mathbb{R}^\Lambda$, and so $p_{x_\alpha}[\lambda] \to 0$ or $m_\lambda(x_\alpha) \to 0$ for all $\lambda \in \Lambda$. Hence $x_\alpha \xrightarrow{m} 0$.

Finally, assume a net (x_α) to be p-bounded. If $x_\alpha \xrightarrow{m} 0$, then $m_\lambda(x_\alpha) \to 0$ or $p_{x_\alpha}[\lambda] \to 0$ for each $\lambda \in \Lambda$. Since (x_α) is p-bounded, $p_{x_\alpha} \xrightarrow{o} 0$ in $\mathbb{R}^\Lambda$. That is $x_\alpha \xrightarrow{p} 0$. $\square$

The following proposition characterizes quasi-interior points, and should be compared with [4, Theorem 4.85].

Proposition 18. *Let $(X, \mathcal{M})$ be an MNVL, then the following statements are equivalent*:

1. *$e \in X_+$ is a quasi-interior point;*
2. *for all $x \in X_+$, $x - x \wedge ne \xrightarrow{m} 0$ as $n \to \infty$;*
3. *e is strictly positive on X^*, i.e., $0 < f \in X^*$ implies $f(e) > 0$, where X^* denotes the topological dual of X.*

Proof. (1)⇒(2) Suppose that e is a quasi-interior point of X, then $\overline{I_e}^m = X$. Let $x \in X_+$. Then $x \in \overline{I_e}^m$, so there exists a net (x_α) in I_e that m-converges to x. But $x_\alpha \xrightarrow{m} x$ implies $|x_\alpha| \xrightarrow{m} |x| = x$. Moreover, $x_\alpha \wedge x \xrightarrow{m} x \wedge x = x$, and $x_\alpha \wedge x \leq x_\alpha$ implies that $x_\alpha \wedge x \in I$, because I_e is an ideal. So we can assume also that $x_\alpha \leq x$. Hence, for any $x \in X_+$, there is a net $0 \leq x_\alpha \in I_e$ and $x_\alpha \leq x$. Then $0 \leq x_\alpha \wedge ne \leq x \wedge ne \leq x$ for all $n \in \mathbb{N}$. Now, take $\lambda \in \Lambda$, and let $\varepsilon > 0$, then there is α_ε such that $m_\lambda(x - x_{\alpha_\varepsilon}) < \varepsilon$. But $0 \leq x_{\alpha_\varepsilon} \in I_e$, so $0 \leq x_{\alpha_\varepsilon} \leq k_\varepsilon e$ for some $k_\varepsilon \in \mathbb{N}$. Since $0 \leq x_{\alpha_\varepsilon} = x_{\alpha_\varepsilon} \wedge k_\varepsilon e \leq x \wedge k_\varepsilon e \leq x$, we get $m_\lambda(x - x \wedge ne) \leq m_\lambda(x - x \wedge k_\varepsilon e) \leq m_\lambda(x - x_\alpha \wedge k_\varepsilon e) = m_\lambda(x - x_{\alpha_\varepsilon}) < \varepsilon$ for all $n \geq k_\varepsilon$. Hence $m_\lambda(x - x \wedge ne) \to 0$ as $n \to \infty$. Since $\lambda \in \Lambda$ was chosen arbitrary, we get $x - x \wedge ne \xrightarrow{m} 0$.

(2)⇒(3) Let $0 < f \in X^*$ and assume in contrary that $f(e) = 0$. Now let $x \in X_+$, then $0 \leq x \wedge ne \leq ne$ for all $n \in \mathbb{N}$. Since $0 < f \in X^*$, $f(x \wedge ne) \leq f(ne) = nf(e) = 0$. So, $f(x \wedge ne) = 0$ for all $n \in \mathbb{N}$. Since $x \wedge ne \xrightarrow{m} x$ and $f \in X^*$, by continuity of f, we have $f(x \wedge ne) \to f(x)$ as $n \to \infty$, i.e., $f(x) = 0$ for all $x \in X_+$. and so $f \equiv 0$ which is a contradiction.

(3)⇒(1) If I_e is not dense in X with respect to m-topology, then by Hahn-Banach Theorem [48, Theorem 3.5] there is a non-zero $f \in X^*$ such that $f(x) = 0$ for every $x \in I_e$. Since $f = f^+ - f^-$ and $f \neq 0$, either $f^+ \neq 0$ or $f^- \neq 0$. Assume without lose of generality that $f^+ > 0$. Now Riesz-Kantorovich formula implies that

$$\begin{aligned} f^+(e) &= \sup\{f(x) : x \in X \text{ and } 0 \leq x \leq e\} \\ &= \sup\{f(x) : x \in I_e \text{ and } 0 \leq x \leq e\} = 0 \end{aligned}$$

which is a contradiction. Thus, $\overline{I_e}^m = X$, that is e is a quasi-interior point of X^+.

□

It should be noted that in the proof of (1) ⇒ (2) of Proposition 18 we can select an increasing bounded from above net (x_α) in I_e^+ such that $x_\alpha \xrightarrow{m} x$. Indeed, if $x \in \overline{I_e}^m$, then we know that there is a net $(x_\alpha)_{\alpha \in A}$ in I_e^+ such that $0 \leq x_\alpha \leq x$ for all $\alpha \in A$. Let $\mathcal{F}(A)$ denote the collection of all finite subsets of A. Clearly, $\mathcal{F}(A)$ is directed upward. For each $\Delta \in \mathcal{F}(A)$ let $y_\Delta : = \sup\limits_{\alpha \in \Delta} x_\alpha$. Then $y_\Delta \uparrow$ and $y_\Delta \leq x$ for all $\Delta \in \mathcal{F}(A)$. We claim that $y_\Delta \xrightarrow{m} x$. Let $\lambda \in \Lambda$. Given $\varepsilon > 0$, since $x_\alpha \xrightarrow{m} x$, there is α_ε satisfying $m_\lambda(x - x_\alpha) < \varepsilon$ for all $\alpha \geqslant \alpha_\varepsilon$. Let $\Delta_\varepsilon = \{\alpha_\varepsilon\}$. For $\Delta \supseteq \Delta\varepsilon$, we have $y_\Delta \geqslant x_{\alpha_\varepsilon}$ or $-y_\Delta \leq -x_{\alpha_\varepsilon}$ and so $0 \leq x - y_\Delta \leq x - x_{\alpha_\varepsilon}$. Hence, $m_\lambda(x - y_\Delta) \leq m_\lambda(x - x_{\alpha_\varepsilon}) < \varepsilon$ for all $\Delta \supseteq \Delta_\varepsilon$. Therefore, $0 \leq y_\Delta \uparrow$ in I_e and $y_\Delta \xrightarrow{m} x$.

More generally we have,

Proposition 19. *Let (X, p, E) be an LNVL and I be an ideal in X. For $x \in X_+$, if there is a net (x_α) in I satisfying $x_\alpha \xrightarrow{p} x$, then there is a net $0 \leq y_\beta$ in I with $y_\beta \uparrow$ and $y_\beta \xrightarrow{p} x$.*

Proof. Suppose that $x \in X_+$ and there exists a net $(x_\alpha) \in I$ with $x_\alpha \xrightarrow{p} x$, then by the same argument used in the proof of (1) ⇒ (2) of Proposition 18, we may consider

$x_\alpha \in I^+$ with $x_\alpha \leq x$ or $x_\alpha \in [0, x]$ for all α. Let $B = [0, x] \cap I$, then B is directed upward, and the net $(y_b) = (b)$ for all $b \in B$ is increasing in I with $0 \leq y_b$. In particular $y_{x_\alpha} = x_\alpha$ for all α. For $b \geqslant x_\alpha$, we have $0 \leq x - y_b = x - b \leq x - x_\alpha = x - y_{x_\alpha}$, and so $p(y_b - x) \leq p(x_\alpha - x)$ as $b \geq x_\alpha$. Now by assumption $x - x_\alpha \xrightarrow{p} 0$ as $\alpha \to \infty$, i.e., $p(x_\alpha - x) \xrightarrow{o} 0$ in E, then there is a net $e_\gamma \downarrow 0$ in E, such that for all γ, there exist α_γ satisfying $p(x_\alpha - x) \leq e_\gamma$ for all $\alpha \geqslant \alpha_\gamma$. In particular $p(x_{\alpha_\gamma} - x) \leq e_\gamma$. We want to show that $p(y_b - x) \xrightarrow{o} 0$. For that consider the net (e_γ) as above, then $e_\gamma \downarrow 0$ in E, and for all γ, take $b_\gamma = x_{\alpha_\gamma}$. Then for all $b \geqslant b_\gamma = x_{\alpha_\gamma}$, we have $p(b - x) \leq p(b_\gamma - x) = p(x_{\alpha_\gamma} - x) \leq e_\gamma$. Therefore $p(y_b - x) \xrightarrow{o} 0$. □

Corollary 8. *Let $(X, \mathcal{M})$ be an $MNVL$, and let I be an ideal in $(X, \mathcal{M})$ with $\overline{I}^m = X$. Then for every $x \in X_+$, there exist a net $(y_\beta) \in I$ such that $0 \leq y_\beta \uparrow \leq x$ and $y_\beta \xrightarrow{m} x$.*

Proof. Suppose that I is an ideal in $(X, \mathcal{M})$ with $\overline{I}^m = X$, then for every $x \in X_+$, there is a net $(x_\alpha) \in I$ such that $x_\alpha \xrightarrow{m} x$, and by the same argument used in the proof of (1) $\Rightarrow$ (2) of Proposition 18, we may assume that $x_\alpha \in I_+$ with $x_\alpha \leq x$. Now by Proposition 17, $(X, \mathcal{M})$ induces an LNVL (X, p, E) with $E = \mathbb{R}^\Lambda$, and $p : X \to E_+$, such that $x \mapsto p_x$, where $p_x \colon \Lambda \to \mathbb{R}$ and $p_x[\lambda] := m_\lambda(x)$. Then for all $\lambda \in \Lambda$, $p_{x_\alpha}[\lambda] = m_\lambda(x_\alpha) \leq m_\lambda(x) = p_x[\lambda]$, so $p(x_\alpha) \leq p(x)$. Hence $x_\alpha \in I_+$ is p-bounded. But $x_\alpha \xrightarrow{m} x$, then by Proposition 17 $x_\alpha \xrightarrow{p} x$, hence by Proposition 19, there exist a net $(y_b) \in I$ such that $y_b \uparrow$ and $y_b \xrightarrow{p} x$. Again by Proposition 17 $y_b \xrightarrow{m} x$ as desired. □

It follows from Theorem 6.63 (ii) and (iv) [3] that an MNVL satisfies the KB-property if and only if it has the Lebesgue and Levi properties.

4.2 um-Topology

In this section we introduce the um-topology in a analogous manner to the un-topology [36] and uaw-topology [61]. First we define the um-convergence.

Definition 3. *Let $(X, \mathcal{M})$ be an MNVL, then a net (x_α) is said to be* unbounded m-convergent *to x, if $|x_\alpha - x| \wedge u \xrightarrow{m} 0$ for all $u \in X_+$. In this case, we say (x_α)* um-converges *to x and write $x_\alpha \xrightarrow{um} x$.*

Clearly, that um-convergence is a generalization of un-convergence. The following result generalizes [36, Corollary 4.5].

Proposition 20. *If $(X, \mathcal{M})$ is an MNVL possessing the Lebesgue and Levi properties, and $x_\alpha \xrightarrow{um} 0$ in X, then $x_\alpha \xrightarrow{um} 0$ in X^{**}.*

Proof. It follows from Theorem 6.63 of [3] that $(X, \mathcal{M})$ is m-complete and X is a band in X^{**}. Now, [3, Theorem 2.22] shows that X^{**} is Dedekind complete, and so

X is a projection band in X^{**}. The conclusion follows now from Theorem 6, part 3. $\square$

In a similar way as in Theorem 2, one can show that $\mathcal{N}_0$, the collection of all sets of the form

$$V_{\varepsilon,u,\lambda} = \{x \in X : m_\lambda(|x| \wedge u) < \varepsilon\},$$

where $\varepsilon > 0,\ 0 \neq u \in X_+$, and $\lambda \in \Lambda$, forms a neighborhood base at zero for some Hausdorff locally solid topology τ such that, for any net (x_α) in X: $x_\alpha \xrightarrow{um} 0$ if and only if $x_\alpha \xrightarrow{\tau} 0$. Thus, the um-convergence is topological, and we will refer to its topology as the *um-topology*.

Clearly, if $x_\alpha \xrightarrow{m} 0$, then $x_\alpha \xrightarrow{um} 0$, and so the m-topology, in general, is finer than um-topology. On the contrary to Theorem 2.3 in [36], the following example provides an MNVL which has a strong unit, yet the m-topology and um-topology do not agree.

Example 5. *Let $X = C[0,1]$. Let $\mathcal{A} := \{[a,b] \subseteq [0,1] : a < b\}$. For $[a,b] \in \mathcal{A}$ and $f \in X$, let $m_{[a,b]}(f) := \frac{1}{b-a}\int_a^b |f(t)|dt$. Then $\mathcal{M} = \{m_{[a,b]} : [a,b] \in \mathcal{A}\}$ is a separating family of lattice seminorms on X. Thus, $(X,\mathcal{M})$ is an MNVL. For each $2 \leq n \in \mathbb{N}$, let*

$$f_n = \begin{cases} n & \text{if } x \in [0, \frac{1}{n}], \\ n^2(1-n)x + n^2 & \text{if } x \in [\frac{1}{n}, \frac{1}{n-1}], \\ 0 & \text{if } x \in [\frac{1}{n-1}, 1]. \end{cases}$$

So we have

$$f_n \wedge \mathbb{1} = \begin{cases} 1 & \text{if } x \in [0, \frac{n+1}{n^2}], \\ n^2(1-n)x + n^2 & \text{if } x \in [\frac{n+1}{n^2}, \frac{1}{n-1}], \\ 0 & \text{if } x \in [\frac{1}{n-1}, 1]. \end{cases}$$

Now, let $0 < b \leq 1$, then there is $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0-1} < b$. So, for $n \geq n_0$, we have $\frac{1}{n-1} < b$, and so we get $m_{[0,b]}(f_n) = \frac{1}{b}(1 + \frac{1}{n-1}) \to \frac{1}{b} \neq 0$ as $n \to \infty$. Thus, $f_n \not\xrightarrow{m} 0$. On the other hand, if $[a,b] \in \mathcal{A}$ then there is $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0-1} < b$ so, for $n \geq (n_0-1)$, we have $m_{[a,b]}(f_n \wedge \mathbb{1}) = \frac{1}{b-a}(\frac{n+1}{n^2} + \frac{1}{2n^2(n-1)}) \to 0$ as $n \to \infty$. Since $\mathbb{1}$ is a strong unit in X, by Corollary 6, $f_n \xrightarrow{um} 0$.

4.3 Metrizabililty of um-topology

The main result in this section is Proposition 21, which shows that the um-topology is metrizable if and only if the space has a countable topological orthogonal system.

It is well known (cf. [3, Theorem 2.1]) that a topological vector space is metrizable if and only if it has a countable neighborhood base at zero. Furthermore, an MNVL $(X,\mathcal{M})$ is metrizable if and only if the m-topology is generated by a countable family of lattice seminorms, see [56, Theorem VII.8.2].

Notice that, in an MNVL $(X, \mathcal{M})$ with countable $\mathcal{M} = \{m_k\}_{k\in\mathbb{N}}$, an equivalent translation-invariant metric $\rho_{\mathcal{M}}$ can be constructed by the formula

$$\rho_{\mathcal{M}}(x,y) = \sum_{k=1}^{\infty} \frac{m_k(x-y)}{2^k(m_k(x-y)+1)} \quad (x, y \in X). \tag{4.3.1}$$

Since the function $t \to \frac{t}{t+1}$ is increasing on $[0, \infty)$, $|x| \leqslant |y|$ in X implies that $\rho_{\mathcal{M}}(x, 0) \leqslant \rho_{\mathcal{M}}(y, 0)$.

A series $\sum_{i=1}^{\infty} x_i$ in a multi-normed space $(X, \mathcal{M})$ is called *absolutely m-convergent* if $\sum_{i=1}^{\infty} m_\lambda(x_i) < \infty$ for all $\lambda \in \Lambda$; and the series is *m-convergent*, if the sequence $s_n := \sum_{i=1}^{n} x_i$ of partial sums is m-convergent.

Lemma 10. *A metrizable multi-normed space $(X, \mathcal{M})$ is m-complete if and only if every absolutely m-convergent series in X is m-convergent.*

Proof. $(\Rightarrow)$ Let $(X, \mathcal{M})$ be sequentially m-complete, with $\mathcal{M} = (m_k)_{k\in\mathbb{N}}$. If the series $\sum\limits_{i=1}^{\infty} x_i$ is an absolutely convergent in $(X, \mathcal{M})$, then for each $k \in \mathbb{N}$, $\sum\limits_{i=1}^{\infty} m_k(x_i) < \infty$. Given $\varepsilon > 0$, there exists N_ε such that $\sum\limits_{n=N_\varepsilon}^{\infty} m_k(x_i) < \varepsilon$. Let $S_n = \sum\limits_{i=1}^{n} x_i$ the sequence of partial sums of the series $\sum\limits_{i=1}^{\infty} x_i$, then for $n \geq m \geq N_\varepsilon$ we have

$$\begin{aligned} m_k(S_n - S_m) &= m_k\left(\sum_{i=m}^{n} x_i\right) \\ &\leq \sum_{i=m}^{n} m_k(x_i) \\ &\leq \sum_{i=N_\varepsilon}^{\infty} m_k(x_i) < \varepsilon. \end{aligned}$$

But $k \in \mathbb{N}$ is arbitrary, so the sequence $(S_n)_{n\in\mathbb{N}}$ is m-Cauchy and by sequentially m-completeness of $(X, \mathcal{M})$, $(S_n)_{n\in\mathbb{N}}$ m-converges to an element say $x \in X$.
$(\Leftarrow)$ Let (x_n) be an m-Cauchy sequence in X. For $k = 1$, $m_1(x_n - x_m) \longrightarrow 0$ as $n, m \longrightarrow \infty$.
For each $i \in \mathbb{N}$, there exist $n_i \in \mathbb{N}$ such that $m_1(x_n - x_m) < 2^{-i}$ for all $n, m > n_i$, and we may choose that $n_i's$ so that $n_{i+1} > n_i$. Then $(x_{n_i})_{i=1}^{\infty}$ is a subsequence of (x_n). Letting $y_1 = x_{n_1}$, and $y_i = x_{n_i} - x_{n_{i-1}}$ for $i \geq 2$ we obtain a series $\sum\limits_{i=1}^{\infty} y_i$ whose i^{th} partial sum is x_{n_i}. But $m_1\left(x_{n_i} - x_{n_{i-1}}\right) < 2^{-(i-1)}$, so we have $m_1(y_i) \leq 2^{-i+1}$ for $i \geq 2$. Thus

$$\sum_{i=1}^{\infty} m_1(y_i) \leq m_1(y_1) + \sum_{i=2}^{\infty} 2^{-i+1} = m_1(y_1) + 1. \tag{4.3.2}$$

Hence, the sequence (y_i) which is a subsequence of (x_i) satisfies the condition in (4.3.2). We repeat the same argument above for $k = 2$ to produce a subsequence (z_i)

of (y_i) which satisfies that

$$\sum_{i=1}^{\infty} m_2\left(z_i\right) \leq m_2\left(z_1\right) + 1 < \infty.$$

So by this diagonal argument we obtain a common subsequence $\left(x_{n_j}\right)$ of (x_n) such that for each $k \in \mathbb{N}$, $\sum\limits_{j=1}^{\infty} m_k\left(y_j\right) < \infty$ where $y_1 = x_{n_1}$ and for $j \geq 2$, $y_j = x_{n_j} - x_{n_{j-1}}$. Thus, $\sum\limits_{j=1}^{\infty} y_j$ is absolutely convergent series. By hypothesis it follows that the series $\sum\limits_{j=1}^{\infty} y_j$ is convergent. That is the sequence $(S_\ell)_{\ell\in\mathbb{N}}$ of partial sums of $\sum\limits_{j=1}^{\infty} y_j$ is m-convergent in X. That is $S_\ell = \sum\limits_{j=1}^{\ell} y_j = x_{n_j}$, i.e. $\left(x_{n_j}\right)$ is m-convergent. Therefore, we have an m-Cauchy sequence (x_n) and an m-convergent subsequence $\left(x_{n_j}\right)$ which implies that (x_n) is m-convergent. $\square$

The following result extends [36, Theorem 3.2].

Proposition 21. *Let $(X, \mathcal{M})$ be a metrizable m-complete MNVL. Then the following conditions are equivalent*:

(i) *X has a countable topological orthogonal system*;

(ii) *the um-topology is metrizable*;

(iii) *X has a quasi interior point.*

Proof. Since $(X, \mathcal{M})$ is metrizable, we may suppose that $\mathcal{M} = \{m_k\}_{k\in\mathbb{N}}$ is countable and directed.

$(i) \Rightarrow (ii)$ It follows directly from Proposition 13 . Notice also that a metric d_{um} of the um-topology can be constructed by the following formula:

$$d(x, y) = \sum_{k,n=1}^{\infty} \frac{1}{2^{k+n}} \cdot \frac{m_k(|x-y| \wedge e_n)}{1 + m_k(|x-y| \wedge e_n)}, \tag{4.3.3}$$

where $\{e_n\}_{n\in\mathbb{N}}$ is a countable topological orthogonal system for X.

$(ii) \Rightarrow (iii)$ Assume that the um-topology is generated by a metric d_{um} on X. For each $n \in \mathbb{N}$, let $B_{um}(0, \frac{1}{n}) = \{x \in X : d_{um}(x, 0) < \frac{1}{n}\}$. Since the um-topology is metrizable, for each $n \in \mathbb{N}$, there are $k_n \in \mathbb{N}, 0 < u_n \in X_+$, and $\varepsilon_n > 0$ such that $V_{\varepsilon_n,u_n,k_n} \subseteq B_{um}(0, \frac{1}{n})$, where

$$V_{\varepsilon,u_n,k} = \{x \in X : m_k(|x| \wedge u_n) < \varepsilon\}.$$

Notice that $\{V_{\varepsilon,u_n,k}\}_{\varepsilon>0,n,k\in\mathbb{N}}$ is a base at zero of the um-topology on X.

Let $B_m(0, 1) = \{x \in X : d_m(x, 0) < 1\}$, where d_m is the metric generating the m-topology. There is a zero neighborhood V in the m-topology such that $V \subseteq B_m(0, 1)$.

Since V is absorbing, for every $n \in \mathbb{N}$, there is $c_n \geq 1$ such that $\frac{1}{c_n} u_n \in V$. Thus $\frac{1}{c_n} u_n \in V \subseteq B_m(0,1)$ for each $n \in \mathbb{N}$. Hence, the sequence $\frac{1}{c_n} u_n$ is d_m-bounded and so it is bounded with respect to the multi-norm $\mathcal{M} = \{m_k\}_{k \in \mathbb{N}}$. Let

$$e := \sum_{n=1}^{\infty} \frac{u_n}{2^n c_n}. \tag{4.3.4}$$

We verify the absolute convergence of the above series. Fix $k \in \mathbb{N}$. Since the sequence $\frac{u_n}{c_n}$ is bounded with respect to $\mathcal{M}$, there exists $r_k \in \mathbb{R}_+$ such that $m_k(\frac{u_n}{c_n}) \leq r_k < \infty$ for all $n \in \mathbb{N}$. Hence,

$$\sum_{n=1}^{\infty} m_k\left(\frac{u_n}{2^n c_n}\right) = \sum_{n=1}^{\infty} \frac{1}{2^n} m_k\left(\frac{u_n}{c_n}\right) \leq r_k \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty.$$

Thus, the series $\sum_{n=1}^{\infty} \dfrac{u_n}{2^n c_n}$ is absolutely m-convergent. Since X is m-complete, Lemma 10 assures that the series $\sum_{n=1}^{\infty} \dfrac{u_n}{2^n c_n}$ is m-convergent to some $e \in X$.

Now, we show that e is a quasi-interior point in X. Let (x_α) be a net in X_+ such that $x_\alpha \wedge e \xrightarrow{m} 0$. Our aim is to show that $x_\alpha \xrightarrow{um} 0$. Since

$$x_\alpha \wedge u_n \leq 2^n c_n x_\alpha \wedge 2^n c_n e = 2^n c_n (x_\alpha \wedge e) \xrightarrow{m} 0 \quad (\alpha \to \infty),$$

we have $x_\alpha \wedge u_n \xrightarrow{m} 0$ for all $n \in \mathbb{N}$. In particular, $m_{k_n}(x_\alpha \wedge u_n) \to 0$. Thus, there exists α_n such that $m_{k_n}(x_\alpha \wedge u_n) < \varepsilon_n$ for all $\alpha \geq \alpha_n$. That is $x_\alpha \in V_{\varepsilon_n, u_n, k_n}$ for all $\alpha \geq \alpha_n$, which implies $x_\alpha \in B_{um}(0, \frac{1}{n})$. Therefore, $x_\alpha \xrightarrow{\mathrm{d_{um}}} 0$ and so $x_\alpha \xrightarrow{um} 0$. Hence, Corollary 6 implies that e is a quasi interior point

$(iii) \Rightarrow (i)$ It is trivial. □

Similar to [36, Proposition 3.3], we have the following result.

Proposition 22. *Let $(X, \mathcal{M})$ be an m-complete metrizable MNVL. The um-topology is stronger than a metric topology if and only if X has a weak unit.*

Proof. The sufficiency follows from 14.

For the necessity, suppose that the um-topology is stronger than the topology generated by a metric d. Let e be as in (4.3.4) above. Assume $x \wedge e = 0$. Since $e \geq \frac{u_n}{2^n c_n}$ for all $n \in \mathbb{N}$, we get $x \wedge \frac{u_n}{2^n c_n} = 0$, and hence $x \wedge u_n = 0$ for all n. Then $x \in V_{\varepsilon_n, u_n, k_n}$ for all n, and $x \in B(0, \frac{1}{n}) = \{x \in X : d(x, 0) < \frac{1}{n}\}$ for each $n \in \mathbb{N}$. So $x = 0$, which means that e is a weak unit. □

4.4 um-Completeness

A subset A of an MNVL $(X, \mathcal{M})$ is said to be (*sequentially*) *um-complete* if, it is (sequentially) complete in the um-topology. In this section, we characterize um-complete subsets of X in terms of the Lebesgue and Levi properties.

We begin with the following technical lemma.

Lemma 11. *Let $(X, \mathcal{M})$ be an MNVL, and $A \subseteq X$ be m-bounded, then $\overline{A}^{um}$ is m-bounded.*

Proof. Given $\lambda \in \Lambda$, then $M_\lambda = \sup_{a \in A} m_\lambda(a) < \infty$. Let $x \in \overline{A}^{um}$, then there is a net (a_α) in A such that $a_\alpha \xrightarrow{um} x$. So $m_\lambda(|a_\alpha - x| \wedge u) \to 0$ for any $u \in X_+$. In particular,

$$m_\lambda(|x|) = m_\lambda(|x| \wedge |x|) = m_\lambda(|x - a_\alpha + a_\alpha| \wedge |x|) \leq$$
$$m_\lambda(|x - a_\alpha| \wedge |x|) + \sup_{a \in A} m_\lambda(a) = m_\lambda(|x - a_\alpha| \wedge |x|) + M_\lambda.$$

Letting $\alpha \to \infty$, we get $m_\lambda(x) = m_\lambda(|x|) \leq M_\lambda < \infty$ for all $x \in \overline{A}^{um}$. □

The following theorem and its proof should be compared with [31, Theorem 4.7].

Theorem 14. *Let $(X, \mathcal{M})$ be an MNVL and let A be an m-bounded and um-closed subset in X. If X has the Lebesgue and Levi properties, then A is um-complete.*

Proof. Suppose that (x_α) is um-Cauchy in A, then, without lost of generality, we may assume that (x_α) consists of positive elements.
Case (1): If X has a weak unit e, then e is a quasi-interior point, by the Lebesgue property of X and Proposition 18. Note that, for each $k \in \mathbb{N}$,

$$|x_\alpha \wedge ke - x_\beta \wedge ke| \leq |x_\alpha - x_\beta| \wedge ke,$$

hence the net $(x_\alpha \wedge ke)_\alpha$ is m-Cauchy in X. Now, [3, Theorem 6.63] assures that X is m-complete, and so the net $(x_\alpha \wedge ke)_\alpha$ is m-convergent to some $y_k \in X$. Given $\lambda \in \Lambda$. Then

$$\begin{aligned} m_\lambda(y_k) &= m_\lambda(y_k - x_\alpha \wedge ke + x_\alpha \wedge ke) \\ &\leq m_\lambda(y_k - x_\alpha \wedge ke) + m_\lambda(x_\alpha) \\ &\leq m_\lambda(y_k - x_\alpha \wedge ke) + \sup_\alpha m_\lambda(x_\alpha). \end{aligned}$$

But $x_\alpha \wedge ke \xrightarrow{m} y_k$, so for all $\varepsilon > 0$, there exist α' such that $\alpha \geq \alpha'$ implies that $m_\lambda(y_k - x_\alpha \wedge ke) < \varepsilon$. Hence for all $\varepsilon > 0$, $m_\lambda(y_k) \leq \varepsilon + \sup_a m_\lambda(x_\alpha)$ that is $m_\lambda(y_k) \leq \sup_a m_\lambda(x_\alpha) < \infty$ by m-boundedness of A. Hence (y_k) is m-bounded in X. Note also that if $k_1 \leq k_2$, then $x_\alpha \wedge k_1 e \leq x_\alpha \wedge k_2 e$, and hence $y_{k_1} \leq y_{k_2}$ by montoncity of $m'_\lambda s$.
Thus (y_k) is m-bounded and increasing in X, but X has the Lebesgue and Levi properties, so by [3, Theorem 6.63], (y_k) is m-convergent to some $y \in X$.

It remains to show that y is the um-limit of (x_α). Given $\lambda \in \Lambda$. Note that, by Birkhoff's inequality,

$$|x_\alpha \wedge ke - x_\beta \wedge ke| \wedge e \leq |x_\alpha - x_\beta| \wedge e.$$

Thus

$$m_\lambda(|x_\alpha \wedge ke - x_\beta \wedge ke| \wedge e) \leq m_\lambda(|x_\alpha - x_\beta| \wedge e).$$

Taking limit over β, we get

$$m_\lambda(|x_\alpha \wedge ke - y_k| \wedge e) \leq \lim_\beta m_\lambda(|x_\alpha - x_\beta| \wedge e).$$

Now taking limit over k, we have

$$m_\lambda(|x_\alpha - y| \wedge e) \leq \lim_\beta m_\lambda(|x_\alpha - x_\beta| \wedge e).$$

Finally, as (x_α) is um-Cauchy, taking limit over α, yields

$$\lim_\alpha m_\lambda(|x_\alpha - y| \wedge e) \leq \lim_{\alpha,\beta} m_\lambda(|x_\alpha - x_\beta| \wedge e) = 0.$$

Thus, $x_\alpha \xrightarrow{um} y$ and, since A is um-closed, $y \in A$.

Case (2): If X has no weak unit. Let $\{e_\gamma\}_{\gamma\in\Gamma}$ be a maximal orthogonal system in X. Let Δ be the collection of all finite subsets of Γ. For each $\delta \in \Delta$, $\delta = \{\gamma_1, \gamma_2, \dots, \gamma_n\}$, consider the band B_δ generated by $\{e_{\gamma_1}, e_{\gamma_1}, \dots, e_{\gamma_n}\}$. It follows from [3, Theorem 3.24] that B_δ is a projection band. Then B_δ is an m-complete MNVL in its own right. Moreover, by Lemma 15, the m-topology restricted to B_δ possesses the Lebesgue and Levi properties. Note that B_δ has a weak unit, namely $e_{\gamma_1} + e_{\gamma_2} + \cdots + e_{\gamma_n}$. Let P_δ be the band projection corresponding to B_δ.
Claim 1: We want to show that for any $x \in X_+$, we have that $P_\delta x \uparrow x$. Now since $0 \leq P_\delta \leq I$, $P_\delta x \leq x$ for all $\delta \in A$. If $0 \leq z \leq x$ and $P_\delta x \leq z$ for all $\delta \in \Delta$, then $-P_\delta x \geq -z$ or $-z \leq -P_\delta x$ which implies that $0 \leq x - z \leq x - P_\delta x$. Note $x - P_\delta x \in B_\delta^d$ for all $\delta \in A$, since B_δ^d is an ideal, we get that $x - z \in B_\delta^d$. In particular, $x - z \in B_{e_\gamma}^d$ for all $\gamma \in \Gamma$, so $(x - z) \wedge e_\gamma = 0$ for all $\gamma \in \Gamma$, then by maximality we get that, $x - z = 0$, and so $x = z$.
For $\delta \in \Delta$, since (x_α) is um-Cauchy in X and P_δ is a band projection, and so lattice homomorphism, then $|P_{x_\alpha} - P_{x_\beta}| \wedge b = p|x_\alpha - x_\beta| \wedge b \leq |x_\alpha - x_\beta| \wedge b \xrightarrow{m} 0$, thus $|P_{x_\alpha} - P_{x_\beta}| \wedge b \xrightarrow{m} 0$, then $P_\delta x_\alpha$ is um-Cauchy in B_δ. Lemma 11 assures that $\overline{P_\delta(A)}^{um}$ is m-bounded in B_δ. Thus, by Case (1), there is $z_\delta \in B_\delta$ such that

$$P_\delta x_\alpha \xrightarrow{um} z_\delta \geq 0 \text{ in } B_\delta \quad (\alpha \to \infty).$$

Since B_δ is a projection band, we have

$$P_\delta x_\alpha \xrightarrow{um} z_\delta \geq 0 \text{ in } X \quad (\text{over } \alpha). \tag{4.4.1}$$

Note that, $0 \leq z_\delta \uparrow$, moreover, (z_δ) is m-bounded. Indeed, given $\lambda \in \Lambda$, then

$$\begin{aligned} m_\lambda(z_\delta) &= m_\lambda\left(|z_\delta| \wedge |z_\delta|\right) \\ &= m_\lambda\left(|z_\delta - P_\delta x_\alpha + P_\delta x_\alpha| \wedge z_\delta\right) \\ &\leq m_\lambda\left(|z_\delta - P_\delta x_\alpha| \wedge z_\delta\right) + m_\lambda\left(P_\delta x_\alpha \wedge z_\delta\right) \\ &\leq m_\lambda\left(|z_\delta - P_\delta x_\alpha| \wedge z_\delta\right) + m_\lambda\left(P_\delta x_\alpha\right) \\ &\leq m_\lambda\left(|z_\delta - P_\delta x_\alpha| \wedge z_\delta\right) + m_\lambda\left(x_\alpha\right) \\ &\leq m_\lambda\left(|z_\delta - P_\delta x_\alpha| \wedge z_\delta\right) + k_\lambda. \end{aligned}$$

Taking the limit over α, we get $m_\lambda\left(z_\delta\right) \leq k_\lambda < \infty$ where $m_\lambda\left(x_\alpha\right) \leq k_\lambda < \infty$ for all α. Thus, z_δ is m-bounded in X.

Since X has the Lebesgue and Levi properties, it follows from [3, Theorem 6.63], that there is $z \in X_+$ such that

$$z_\delta \xrightarrow{m} z, \text{ and so } z_\delta \uparrow z. \tag{4.4.2}$$

It follows also from (4.4.2) that $z_\delta \xrightarrow{um} z$.

Our aim is to show that $x_\alpha \xrightarrow{um} z$. Let $u \in X_+$, we verify $|x_\alpha - z| \wedge u \xrightarrow{m} 0$. Let B_u be the band generated by u and P_u be the corresponding band projection. As above, $(P_u x_\alpha)$ is um-Cauchy in B_u and so there is $0 \leq x_u \in B_u$ such that

$$P_u x_\alpha \xrightarrow{um} x_u \text{ over } \alpha, \text{ in } B_u$$

So,

$$P_u x_\alpha \xrightarrow{um} x_u \text{ in } X. \tag{4.4.3}$$

Note that $|x_\alpha - x_u| \wedge u \in B$ for all α. Hence,

$$\begin{aligned} |x_\alpha - x_u| \wedge u &= P_u\left(|x_\alpha - x_u| \wedge u\right) \\ &= |P_u x_\alpha - x_u| \wedge u \xrightarrow{m} 0 \text{ in } X \text{ by (4.4.3).} \end{aligned}$$

So,

$$|x_\alpha - x_u| \wedge u \xrightarrow{m} 0 \text{ over } \alpha \text{ in } X. \tag{4.4.4}$$

Given $\delta \in \Delta$;

$$\begin{aligned} |P_\delta x_\alpha - P_\delta x_u| \wedge u &= P_\delta\left(|x_\alpha - x_u|\right) \wedge u \\ &\leq |x_\alpha - x_u| \wedge u \xrightarrow{m} 0 \text{ over } \alpha \text{ in } X \text{ by (4.4.4).} \end{aligned}$$

Thus,

$$|P_\delta x_\alpha - P_\delta x_u| \wedge u \xrightarrow{m} 0 \text{ over } \alpha \text{ in } X. \tag{4.4.5}$$

But $P_\delta x_\alpha \xrightarrow{um} z_\delta$ in X by (4.4.1).
In particular,

$$|P_\delta x_\alpha - z_\delta| \wedge u \xrightarrow{m} 0 \text{ over } \alpha \text{ in } X. \tag{4.4.6}$$

Since
$$|z_\delta - P_\delta x_u| \wedge u \leq |z_\delta - P_\delta x_\alpha| \wedge u + |P_\delta x_\alpha - P_\delta x_u| \wedge u,$$
Taking limit over α we get from (4.4.5) and (4.4.6) that
$$|z_\delta - P_\delta x_u| \wedge u = 0. \tag{4.4.7}$$
Taking limit over δ in (4.4.7), it follows from (4.4.2) and Claim 1 that
$$|z - x_u| \wedge u = 0.$$
Note that $|z - x_u| \wedge u \in B_u$ and so
$$0 = |z - x_u| \wedge u = P_u\left(|z - x_u| \wedge u\right) = |P_u z - x_u| \wedge u.$$
Since u is a weak unit in B_u,
$$P_u z = x_u. \tag{4.4.8}$$
Now,
$$\begin{aligned} |x_\alpha - z| \wedge u &= P_u\left(|x_\alpha - z| \wedge u\right) \\ &= |P_u x_\alpha - P_u z| \wedge u \\ \text{by (4.4.8)} &= |P_u x_\alpha - x_u| \wedge u \xrightarrow{m} 0 \text{ over } \alpha \text{ by (4.4.3).} \end{aligned}$$
We get that
$$|x_\alpha - z| \wedge u \xrightarrow{m} 0$$
Since, $u \in X_+$ was arbitrary, we get $x_\alpha \xrightarrow{um} z$. Since (x_α) in A and A is um-closed, we get that $z \in A$ and so A is um-complete. □

Lemma 12. *Any monotone m-convergent net in an $MNVL$ $(X, \mathcal{M})$ o-converges to its m-limit.*

Proof. It is enough to show that if $(X, \mathcal{M}) \ni x_\alpha \uparrow$ and $x_\alpha \xrightarrow{m} u$, then $x_\alpha \uparrow x$. Fix arbitrary α. Then $x_\beta - x_\alpha \in X_+$ for all $\beta \geq \alpha$. So, taking the limit over β we get $x_\beta - x_\alpha \xrightarrow{m} x - x_\alpha$, hence $x - x_\alpha \geq 0$ and so $x \geq x_\alpha$. But α is arbitrary. Thus $x \geq x_\alpha$ for all α, that is x is an upper bound for (x_α). We show x is the least upper bound. Suppose that $y \geq x_\alpha$ for all α, then $y - x_\alpha \geq 0$ for all α, and since $y - x_\alpha \xrightarrow{m} y - x$ over α, $y - x \geq 0$ or $y \geq x$. Therefore $x_\alpha \uparrow x$. □

Lemma 13. *If (x_α) is an increasing net in an $MNVL$ $(X, \mathcal{M})$, and $x_\alpha \xrightarrow{um} x$, then $x_\alpha \uparrow x$ and $x_\alpha \xrightarrow{m} x$.*

Proof. Since lattice operations are um-continuous, the same argument in Lemma 12 applies here as well and we get that $x_\alpha \uparrow x$. Thus (x_α) is order bounded and so um-convergence agrees with m-convergence. □

Lemma 14. *Let $(X, \mathcal{M})$ be an $MNVL$ that has the pre-Lebesgue property. Let (x_n) be a positive disjoint sequence such that (x_n) is not m-null. Put $s_n := \sum\limits_{i=1}^{n} x_i$. then (s_n) is um-Cauchy but not um-convergent.*

Proof. The sequence (s_n) is monotone increasing, and since (x_n) is not m-null, we get that (s_n) does not m-converge, otherwise s_n and s_{n-1} also m-converge to some x, consequently $x_n = s_n - s_{n-1} \xrightarrow{m} 0$ which contradicts the hypothesis. Hence by Lemma 13 s_n is not um-convergent. To show that (s_n) is um-Cauchy, fix any $\varepsilon > 0$, and a non-zero $u \in X_+$. Since (x_i) is a positive disjoint sequence, we have $s_n \wedge u = \sum\limits_{i=1}^{n} (x_i \wedge u)$ by Theorem 6.5 in [60]. The sequence $(s_n \wedge u)$ is increasing and order bounded by u, hence is m-Cauchy by Theorem 3.22 in [3]. Fix $\lambda \in \Lambda$, we can find n_{ε_λ} such that $m_\lambda (s_m \wedge u - s_n \wedge u) < \varepsilon$ for all $m \geq n \geq n_{\varepsilon_\lambda}$. Observe that

$$\begin{aligned}
s_m \wedge u - s_n \wedge u &= \left(\sum_{i=1}^{m} x_i\right) \wedge u - \left(\sum_{i=1}^{n} x_i\right) \wedge u \\
&= \sum_{i=1}^{m} (x_i \wedge u) - \sum_{i=1}^{n} (x_i \wedge u) \\
&= \sum_{i=n+1}^{m} (x_i \wedge u) = \left(\sum_{i=n+1}^{m} x_i\right) \wedge u \\
&= (s_m - s_n) \wedge u = |s_m - s_n| \wedge u.
\end{aligned}$$

It follows that $m_\lambda (|s_m - s_n| \wedge u) < \varepsilon$ for all $m \geq n \geq n_{\varepsilon_\lambda}$. But λ was fixed arbitrary. Hence (s_n) is um-Cauchy. $\square$

Let $(X, \mathcal{M})$ be a finite dimensional m-complete $MNVL$, then by Theorem 5.4 in [4], it is um-complete.
On the contrary of [[36], Proposition 6.2] we provide an example of an m-complete $MNVL$ $(X, \mathcal{M})$ satisfying the Lebesgue property such that it is um-complete and $\dim X = \infty$.

Example 6. *Let $X = s$ and $\mathcal{M} = (m_j)_{j\in\mathbb{N}}$ such that $m_j((x_n)) := |x_j|$ where $(x_n) \in \ell_\infty$.*
First we show $(X, \mathcal{M})$ is m-complete. Let (x^α) be an m-Cauchy net in $(X, \mathcal{M})$, $x^\alpha = (x_n^\alpha)_{n\in\mathbb{N}}$, so, $x^\alpha - x^\beta \xrightarrow{m} 0$ over α, β. For $j \in \mathbb{N}$ we have $m_j\left(x^\alpha - x^\beta\right) \to 0$ in $\mathbb{R}$ over α, β. That is, for $j \in \mathbb{N}$, $|x_j^\alpha - x_j^\beta| \to 0$ in $\mathbb{R}$ over α, β. That is, for each $j \in \mathbb{N}$, the net (x_j^α) is Cauchy in $\mathbb{R}$ and so there is $x_j \in \mathbb{R}$ such that $x_j^\alpha \to x_j$ over α. Put $x := (x_j)_{j\in\mathbb{N}}$, then $x \in s$. Since for each $j \in \mathbb{N}$ $x_j^\alpha \to x_j$ over α in $\mathbb{R}$, this means that $m_j(x^\alpha - x) \to 0$ in $\mathbb{R}$. Hence, $x^\alpha \xrightarrow{m} x$. Therefore, $(X, \mathcal{M})$ is m-complete.

Second, $(X, \mathcal{M})$ has the Lebesgue property. Assume $x^\alpha \downarrow 0$, our aim is to show that $x^\alpha \xrightarrow{m} 0$. We know $x^\alpha = (x_n^\alpha)_{n\in\mathbb{N}}$. For each $j \in \mathbb{N}$; $x^\alpha \downarrow 0$, implies that $x_j^\alpha \downarrow 0$ in $\mathbb{R}$. That is $m_j(x^\alpha) \downarrow 0$ in $\mathbb{R}$. Thus, $x^\alpha \xrightarrow{m} 0$.

Finally, we show that $(X, \mathcal{M})$ is um-complete. Suppose (x^α) is um-Cauchy net. Then for each $u \in X_+$ we have $|x^\alpha - x^\beta| \wedge u \xrightarrow{m} 0$. Now, $u = (u_n)_{n\in\mathbb{N}}$, $x^\alpha = (x_n^\alpha)_{n\in\mathbb{N}}$. Let $j \in \mathbb{N}$ then $m_j\left(|x^\alpha - x^\beta| \wedge u\right) \to 0$ in $\mathbb{R}$ over α, β. if and only if $|x_j^\alpha - x_j^\beta| \wedge u_j \to 0$ in $\mathbb{R}$ if and only if $\Leftrightarrow |x_j^\alpha - x_j^\beta| \to 0$ in $\mathbb{R}$ over α, β.

Thus, (x_j^α) is Cauchy in $\mathbb{R}$ and so there is $x_j \in \mathbb{R}$ such that $x_j^\alpha \to x_j$ in $\mathbb{R}$ over α. Let $x = (x_j)_{j\in\mathbb{N}} \in s$, then clearly, $x^\alpha \xrightarrow{um} x$.

Lemma 15. *Let $(X, \mathcal{M})$ be an m-complete $MNVL$ which satisfies Lebesgue and Levi properties. Let B be a band in X. Then B is an m-complete $MNVL$ in its own right which in addition satisfies Lebesgue and Levi properties.*

Proof. Let (x_α) be an m-Cauchy net in B, then (x_α) is an m-Cauchy in X. Since X is m-complete, there is $x \in X$ such that $x_\alpha \xrightarrow{m} x$, but by [3, Theorem 2.21] B is m-closed and so $x \in B$. Thus, B is m-complete.
Assume $x_\alpha \downarrow 0$ in B. Since B is regular, see [30, Lemma 2.5], we have $x_\alpha \downarrow 0$ in X. But X satisfies Lebesgue property so $x_\alpha \xrightarrow{m} 0$, since (x_α) in B, $x_\alpha \xrightarrow{m} 0$ in B. Hence, B satisfies Lebesgue property.
Suppose $0 \leq x_\alpha \uparrow$ in B, then $0 \leq x_\alpha \uparrow$ in X. Since X has Levi property, there is $x \in X$ such that $0 \leq x_\alpha \uparrow x$ in X, i.e. $x_\alpha \xrightarrow{o} x$, but (x_α) in B and B is order closed. Hence, $x \in B$ and so $x \geq x_\alpha$ for all α. If $0 \leq z \leq x$ and $x_\alpha \uparrow z$ in B, then by regularity of B in X we have $x_\alpha \uparrow z$ in X, which implies $z = x$. □

Next theorem generalizes Theorem 6.4 in [36].

Theorem 15. *Let $(X, \mathcal{M})$ be an m-complete MNVL with the pre-Lebesgue property. Then X has the Lebesgue and Levi properties if and only if every m-bounded um-closed subset of X is um-complete.*

Proof. The necessity follows directly from Theorem 14.

For the sufficiency, first notice that, in an m-complete MNVL, the pre-Lebesgue and Lebesgue properties coincide [3, Theorem 3.24].

If X does not have the Levi property then, by [3, Theorem 6.63], there is a disjoint sequence (x_n) in X_+, which is not m-null, such that its sequence of partial sums $s_n = \sum_{j=1}^n x_j$ is m-bounded. Let $A = \overline{\{s_n : n \in \mathbb{N}\}}^{um}$. By Lemma 11, we have that A is m-bounded. By Lemma 14, the sequence (s_n) is um-Cauchy in X and so in A, in contrary with that the sequence $s_{n+1} - s_n = x_{n+1}$ is not m-null. □

Theorem 16. *Let $(X, \mathcal{M})$ be an m-complete metrizable MNVL, and let A be an m-bounded sequentially um-closed subset of X. If X has the σ-Lebesgue and σ-Levi properties then A is sequentially um-complete. Moreover, the converse holds if, in addition, X is Dedekind complete.*

Proof. Suppose $\mathcal{M} = \{m_k\}_{k\in\mathbb{N}}$. Let $0 \leq x_n$ be a um-Cauchy sequence in A. Let $e := \sum_{n=1}^\infty \frac{x_n}{2^n}$. For $k \in \mathbb{N}$,

$$\sum_{n=1}^\infty m_k\left(\frac{x_n}{2^n}\right) = \sum_{n=1}^\infty \frac{1}{2^n} m_k(x_n) \leq c_k \sum_{n=1}^\infty \frac{1}{2^n} < \infty,$$

where $m_k(a) \leq c_k < \infty$ for all $a \in A$. Since $\sum_{n=1}^\infty \frac{x_n}{2^n}$ is absolutely m-convergent, by Lemma 10, $\sum_{n=1}^\infty \frac{x_n}{2^n}$ is m-convergent in X. Note that, $x_n \leq 2^n e$, so $x_n \in B_e$ for all $n \in \mathbb{N}$. Since X has the Levi property, X is σ-order complete (see [3, Definition 3.16]). Thus B_e is a projection band. Also e is a weak unit in B_e. Then, by the same argument as in Theorem 14, we get that there is $x \in B_e$ such that $x_n \xrightarrow{um} x$ in B_e and so $x_n \xrightarrow{um} x$ in X. Since A is sequentially um-closed, we get $x \in A$. Thus A is sequentially um-complete.

The converse follows from Proposition 16. $\square$

4.5 um-Compact sets

A subset A of an MNVL $(X, \mathcal{M})$ is said to be (*sequentially*) um-*compact*, if it is (sequentially) compact in the um-topology. In this section, we characterize um-compact subsets of X in terms of the Lebesgue and Levi properties. We begin with the following result which shows that um-compactness can be "localized" under certain conditions.

Theorem 17. *Let $(X, \mathcal{M})$ be an MNVL possessing the Lebesgue property. Let $\{e_\gamma\}_{\gamma\in\Gamma}$ be a maximal orthogonal system. For each $\gamma \in \Gamma$, let B_γ be the band generated by e_γ, and P_γ be the corresponding band projection onto B_γ. Then $x_\alpha \xrightarrow{um} 0$ in X if and only if $P_\gamma x_\alpha \xrightarrow{um} 0$ in B_γ for all $\gamma \in \Gamma$.*

Proof. For the forward implication, we assume that $x_\alpha \xrightarrow{um} 0$ in X. Let $b \in (B_\gamma)_+$. Then

$$|P_\gamma x_\alpha| \wedge b = P_\gamma |x_\alpha| \wedge b \leq |x_\alpha| \wedge b \xrightarrow{m} 0,$$

that implies $P_\gamma x_\alpha \xrightarrow{um} 0$ in B_γ.

For the backward implication, without lost of generality, we may assume that $x_\alpha \geq 0$ for all α. Let $u \in X_+$. Our aim is to show that $x_\alpha \wedge u \xrightarrow{m} 0$. It is known that $x_\alpha \wedge u = \sum_{\gamma\in\Gamma} P_\gamma(x_\alpha \wedge u)$. Let F be a finite subset of Γ. Then

$$x_\alpha \wedge u = \sum_{\gamma\in F} P_\gamma(x_\alpha \wedge u) + \sum_{\gamma\in\Gamma\setminus F} P_\gamma(x_\alpha \wedge u). \tag{4.5.1}$$

Note

$$\sum_{\gamma\in F} P_\gamma(x_\alpha \wedge u) = \sum_{\gamma\in F} P_\gamma x_\alpha \wedge P_\gamma u \xrightarrow{m} 0. \tag{4.5.2}$$

We have to control the second term in (4.5.1).

$$\sum_{\gamma\in\Gamma\setminus F} P_\gamma(x_\alpha \wedge u) \leq \frac{1}{n}\sum_{\gamma\in F} P_\gamma u + \sum_{\gamma\in\Gamma\setminus F} P_\gamma u, \tag{4.5.3}$$

where $n \in \mathbb{N}$. Let $\mathcal{F}(\Gamma)$ be the collection of all finite subsets of Γ. Let $\Delta = \mathcal{F}(\Gamma) \times \mathbb{N}$. For each $\delta = (F, n)$, put

$$y_\delta = \frac{1}{n}\sum_{\gamma\in F} P_\gamma u + \sum_{\gamma\in\Gamma\setminus F} P_\gamma u.$$

We show that (y_δ) is decreasing. Let $\delta_1 \leq \delta_2$ then $\delta_1 = (F_1, n_1), \delta_2 = (F_2, n_2)$. Then $\delta_1 \leq \delta_2$ if and only if $F_1 \subseteq F_2$ and $n_1 \leq n_2$. But $n_1 \leq n_2$ if and only if $\frac{1}{n_1} \geq \frac{1}{n_2}$. So,

$$\frac{1}{n_1} \sum_{\gamma \in F_1} P_\gamma u \geq \frac{1}{n_2} \sum_{\gamma \in F_1} P_\gamma u. \tag{4.5.4}$$

Note also

$$\frac{1}{n_2} \sum_{\gamma \in F_2} P_\gamma u = \frac{1}{n_2} \sum_{\gamma \in F_1} P_\gamma u + \frac{1}{n_2} \sum_{\gamma \in F_2 \setminus F_1} P_\gamma u. \tag{4.5.5}$$

Since $F_1 \subseteq F_2$, $\Gamma \setminus F_1 \supseteq \Gamma \setminus F_2$ and hence, $\sum_{\gamma \in \Gamma \setminus F_1} P_\gamma u \geq \sum_{\gamma \in \Gamma \setminus F_2} P_\gamma u$. Note, that

$$\sum_{\gamma \in \Gamma \setminus F_1} P_\gamma u = \sum_{\gamma \in F_2 \setminus F_1} P_\gamma u + \sum_{\gamma \in \Gamma \setminus F_2} P_\gamma u. \tag{4.5.6}$$

Now,

$$\sum_{\gamma \in F_2 \setminus F_1} P_\gamma u \geq \frac{1}{n_2} \sum_{\gamma \in F_2 \setminus F_1} P_\gamma u. \tag{4.5.7}$$

Combining (4.5.6) and (4.5.7), we get

$$\sum_{\gamma \in \Gamma \setminus F_1} P_\gamma u \geq \sum_{\gamma \in \Gamma \setminus F_2} P_\gamma u + \frac{1}{n_2} \sum_{\gamma \in F_2 \setminus F_1} P_\gamma u. \tag{4.5.8}$$

Adding (4.5.4) and (4.5.8), we get

$$\frac{1}{n_1} \sum_{\gamma \in F_1} P_\gamma u + \sum_{\gamma \in \Gamma \setminus F_1} P_\gamma u \geq \frac{1}{n_2} \sum_{\gamma \in F_1} P_\gamma u + \frac{1}{n_2} \sum_{\gamma \in F_2 \setminus F_1} P_\gamma u + \sum_{\gamma \in \Gamma \setminus F_2} P_\gamma u.$$

It follows from (4.5.5), that

$$\frac{1}{n_1} \sum_{\gamma \in F_1} P_\gamma u + \sum_{\gamma \in \Gamma \setminus F_1} P_\gamma u \geq \frac{1}{n_2} \sum_{\gamma \in F_2} P_\gamma u + \sum_{\gamma \in \Gamma \setminus F_2} P_\gamma u,$$

that is $y_{\delta_1} \geq y_{\delta_2}$. Next, we show $y_\delta \downarrow 0$. Assume $0 \leq x \leq y_\delta$ for all $\delta \in \Delta$. Let $\gamma_0 \in \Gamma$ be arbitrary and fix it. Let $F = \{\gamma_0\},\ n \in \mathbb{N}$, then

$$0 \leq x \leq \frac{1}{n} P_{\gamma_0} u + \sum_{\gamma \in \Gamma \setminus \{\gamma_0\}} P_\gamma u.$$

We apply P_{γ_0} for the expression above, so $0 \leq P_{\gamma_0} x \leq \frac{1}{n} P_{\gamma_0} u$ for all $n \in \mathbb{N}$, and so $P_{\gamma_0} x = 0$. Since $\gamma_0 \in \Gamma$ was chosen arbitrary, we get $P_{\gamma_0} x = 0$ for all $\gamma \in \Gamma$. Hence, $x = 0$ and so $y_\delta \downarrow 0$. Since $(X, \mathcal{M})$ has the Lebesgue property, we get $y_\delta \xrightarrow{m} 0$. Therefore, by (4.5.3),

$$\sum_{\gamma \in \Gamma \setminus F} P_\gamma (x_\alpha \wedge u) \leq y_\delta \xrightarrow{m} 0. \tag{4.5.9}$$

Hence (4.5.1), (4.5.2), and (4.5.9) imply $x_\alpha \wedge u \xrightarrow{m} 0$. $\square$

The next theorem and its proof should be compared with [36, Theorem 7.1].

Theorem 18. *Let $(X, \mathcal{M})$ be an MNVL possessing the Lebesgue and Levi properties. Let $\{e_\gamma\}_{\gamma\in\Gamma}$ be a maximal orthogonal system. Let A be a um-closed m-bounded subset of X. Then A is um-compact if and only if $P_\gamma(A)$ is um-compact in B_γ for each $\gamma \in \Gamma$, where B_γ is the band generated by e_γ and P_γ is the band projection corresponding to B_γ.*

Proof. Suppose A is um-compact. Since band projections are um-continuous, i.e. continuous with respect to um-topology and a continuous image of a compact set is compact, we conclude that $P_\gamma(A)$ is um-compact in B_γ for all γ.

For the converse, suppose that $P_\gamma(A)$ is um-compact in B_γ for every $\gamma \in \Gamma$. Let $H = \prod\limits_{\gamma\in\Gamma} B_\gamma$, the formal product of all the bands $B_\gamma, \gamma \in \Gamma$. That is, H consists of families $(x_\gamma)_{\gamma\in\Gamma}$ indexed by Γ, where $x_\gamma \in B_\gamma$. We equip H with the topology of coordinatewise um-convergence; this is the product of um-topologies on the bands that make up H. This makes H a topological vector space. Defined $\Phi : X \to H$ via $\Phi(x) = (P_\gamma x)_{\gamma\in\Gamma}$. Clearly, Φ is linear. But $\{e_\gamma : \gamma \in \Gamma\}$ is maximal orthogonal system, and so Φ is one-to-one. Then by Theorem 17, Φ is a homeomorphism from X equipped with um-topology onto its range in H.
Let K be a subset of H defined by $K = \prod\limits_{\gamma\in\Gamma} P_\gamma(A)$. By Tychonoff's Theorem, K is compact in H. It is easy to see that $\Phi(A) \subseteq K$. We claim that $\Phi(A)$ is closed in H. Indeed, suppose that $\Phi(x_\alpha) \to h$ in H for some net (x_α) in A. In particular, the net $(\Phi(x_\alpha))$ is Cauchy in H. Since Φ is a homeomorphism, the net (x_α) is um-Cauchy in A. Since (x_α) is m-bounded and X satisfies Lebesgue and Levi property, (x_α) um-converges to some $x \in X$ by Proposition 15. Since A is um-closed, we have $x \in A$. It follows that $h = \Phi(x)$, so $h \in \Phi(A)$. Being m-closed subset of a compact set, $\Phi(A)$ is its self compact. Since Φ is homeomorphism, we conclude A is um-compact.

□

If $(x_\alpha)_{\alpha\in A}$ is a net in a non-empty set X, then a net $(x_{\alpha_\beta})_{\beta\in B}$ is said to be *subnet* of $(x_\alpha)_{\alpha\in A}$ if there is a function $\varphi : B \to A$ satisfying:

1. For each $\beta \in B,\ x_{\alpha_\beta} = x_{\varphi(\beta)}$.
2. For each $\alpha_0 \in A$ there exists some $\beta_0 \in B$ such that if $\beta \geq \beta_0$ then $\varphi(\beta) \geq \alpha_0$.

See for example [4, Definition 2.15].

Lemma 16. *Let X be a vector lattice and $(x_\alpha)_{\alpha\in A}$ be an increasing net in X. If there is a subnet $(x_{\alpha_\beta})_{\beta\in B}$ such that $x_{\alpha_\beta} \uparrow x$, then $x_\alpha \uparrow x$.*

Proof. We know there is a function $\varphi : B \to A$ such that if $\alpha_0 \in A$ then there is $\beta_0 \in B$ satisfying $\varphi(\beta) \geq \alpha_0$ when $\beta \geq \beta_0$. Since $x_\alpha \uparrow$, $x_{\varphi(\beta)} \geq x_{\alpha_0}$ or $x_{\alpha_\beta} \geq x_{\alpha_0}$. Since $x_{\alpha_\beta} \uparrow x$, $x \geq x_{\alpha_0}$. But $\alpha_0 \in A$ was arbitrary, thus $x \geq x_\alpha$ for all $\alpha \in A$ and so x is an upper bound for $(x_\alpha)_{\alpha\in A}$. If $z \geq x_\alpha$ for all $\alpha \in A$, then in particular $z \geq x_{\alpha_\beta}$ for all $\beta \in B$ and since $x_{\alpha_\beta} \uparrow x$, $z \geq x$. Therefore, $x_\alpha \uparrow x$. □

The following theorem should be compared with [36, Theorem 7.5].

Theorem 19. *Let $(X, \mathcal{M})$ be an $MNVL$. The following are equivalent*:

1. *Any m-bounded and um-closed subset A of X is um-compact.*
2. *X is an atomic vector lattice and $(X, \mathcal{M})$ has the Lebesgue and Levi properties.*

Proof. (1) $\Rightarrow$ (2). Let $[a, b]$ be an order interval in X. For $x \in [a, b]$, we have $a \leq x \leq b$ and so $0 \leq x - a \leq b - a$. Consider the order interval $[0, b-a] \subseteq X_+$. Clearly, $[0, b-a]$ is m-bounded and um-closed in X. By (1), the order interval $[0, b-a]$ is um-compact. Let (x_α) be a net in $[0, b-a]$. Since $[0, b-a]$ is um-compact, there is a subset x_{α_β} such that $x_{\alpha_\beta} \xrightarrow{um} x$ in $[0, b-a]$. That is $|x_{\alpha_\beta} - x| \wedge u \xrightarrow{m} 0$ for all $u \in [0, b-a]$. Hence, $|x_{\alpha_\beta} - x| = |x_{\alpha_\beta} - x| \wedge (b-a) \xrightarrow{m} 0$. So, $x_{\alpha_\beta} \xrightarrow{m} x$ in $[0, b-a]$. Thus, $[0, b-a]$ is m-compact. Consider the following shift operator $T_a : X \to X$ given by $T_a(x) := x + a$. Clearly, T_a is continuous, and so $T_a([0, b-a]) = [a, b]$ is m-compact.
Since any order interval in X is m-compact, it follows from [3, Corollary 6.57] that X is atomic and has the Lebesgue property. It remains to show that X has the Levi property. Suppose $0 \leq x_\alpha \uparrow$ and is m-bounded. Let $A = \overline{\{x_\alpha\}}^{um}$. Then A is um-closed and, by Lemma 11, A is an m-bounded subset of X. Thus, A is um-compact and so, there are a subnet (x_{α_β}) and $x \in A$ such that $x_{\alpha_\beta} \xrightarrow{um} x$. Hence, by Lemma 13, $x_{\alpha_\beta} \uparrow x$, and so $x_\alpha \uparrow x$. Thus, X has the Levi property.

(2) $\Rightarrow$(1). Let A be an m-bounded and um-closed subset of X. We show that A is um-compact. Since X is atomic, there is a maximal orthogonal system $\{e_\gamma\}_{\gamma \in \Gamma}$ of atoms. For each $\gamma \in \Gamma$, let P_γ be the band projection corresponding to e_γ. Clearly, $P_\gamma(A)$ is m-bounded. Now, by the same argument as in the proof of Theorem 7.1 in [36], we get that $P_\gamma(A)$ is um-closed in $\prod_{\gamma \in \Gamma} B_\gamma$, and so it is um-closed in B_γ. But um-closedness implies m-closedness. So $P_\gamma(A)$ is m-bounded and m-closed in B_γ for all $\gamma \in \Gamma$. Since each e_γ is an atom in X, $B_\gamma = \operatorname{span}\{e_\gamma\}$ is a one-dimensional subspace. It follows from the Heine-Borel theorem that $P_\gamma(A)$ is m-compact in B_γ, and so it is um-compact in B_γ for all $\gamma \in \Gamma$. Therefore, Theorem 18 implies that A is um-compact in X. □

Lemma 17. *Let X be a topological space and $Y \subseteq X$. If $A \subseteq Y$ and A is compact in Y, then A is compact in X.*

Proof. The inclusion map $i : Y \hookrightarrow X$ is continuous (let O open set in X, then $i^{-1}(O) = O \cap Y$ which is open in Y.)
Since A is compact in Y, $i(A) = A$ is compact in X. □

Lemma 18. *Let X be a topological space. Let $S \subseteq Y \subseteq X$. If S is compact in X, then S is compact in Y.*

Proof. Let $(\mathcal{O}_\alpha)$ be an open cover for S in Y. Then for each α, there is G_α open in X such that $\mathcal{O}_\alpha = G_\alpha \cap Y$. Hence, $S \subseteq \bigcup\limits_\alpha \mathcal{O}_\alpha = \bigcup\limits_\alpha (G_\alpha \cap Y) \subseteq \bigcup\limits_\alpha G_\alpha$. Since S is compact in X, there is $\alpha_1, \ldots, \alpha_n$ such that $S \subseteq G_{\alpha_1} \cup \cdots \cup G_{\alpha_n}$, which implies

$$S = S \cap Y \subseteq (G_{\alpha_1} \cup \cdots \cup G_{\alpha_n}) \cap Y = \bigcup_{i=1}^{n} (G_{\alpha_i} \cap Y) = \bigcup_{i=1}^{n} \mathcal{O}_{\alpha_i}.$$

Thus, S is compact in Y. □

Lemma 19. *Let $(X, \mathcal{M})$ be a sequentially m-complete $MNVL$ that satisfies the Lebesgue property. Then X is σ-order (Dedekind) complete.*

Proof. Assume $0 \leq x_n \uparrow \leq u$. Since $(X, \mathcal{M})$ satisfies the Lebesgue property, by [3, Theorem 3.23], $(X, \mathcal{M})$ satisfies the pre-Lebesgue property. By Theorem 3.22 in [3] it follows that (x_n) is m-Cauchy. Since (x_n) is sequentially m-complete, $x_n \xrightarrow{m} x$ for some $x \in X$. Since $x_n \uparrow$, by Lemma 12 $x_n \uparrow x$. Thus, X is σ-order complete. □

In view of paragraph after [Definition 1.47,p.22] in [3] it follows that every σ-order complete vector lattice satisfies (PPP).

Proposition 23. *Let A be a subset of an m-complete metrizable MNVL $(X, \mathcal{M})$.*

1. *If X has a countable topological orthogonal system, then A is sequentially um-compact if and only if A is um-compact.*
2. *Suppose that A is m-bounded, and X has the Lebesgue property. If A is um-compact, then A is sequentially um-compact.*

Proof. (1). It follows immediately from Proposition 21.
(2). Let (x_n) be a sequence in A. Find $e \in X_+$ such that (x_n) is contained in B_e (e.g., take $e = \sum_{n=1}^{\infty} \frac{|x_n|}{2^n}$). Since A is um-compact, A is um-closed, but B_e is um-closed, so $A \cap B_e$ is um-closed, again A is um-compact, so $A \cap B_e$ is um-compact in A and hence by Lemma 17 $A \cap B_e$ is um-compact in X, and by Lemma 18 $A \cap B_e$ is um-compact in B_e. Now, since X is m-complete and has the Lebesgue property, by Lemma 15 B_e is also m-complete and has the Lebesgue property, so by Corollary 6, e is a quasi-interior point of B_e. Thus, by Proposition 21, the um-topology on B_e is metrizable, consequently, $A \cap B_e$ is sequentially um-compact in B_e. It follows that there is a subsequence (x_{n_k}) that um-converges in B_e to some $x \in A \cap B_e$. It follows from Lemma 19 that B_e is a projection band, then Theorem 6, part 3 implies $x_{n_k} \xrightarrow{um} x$ in X. Therefore, A is sequentially um-compact. □

REFERENCES

[1] Y. A. Abramovich and C. D. Aliprantis. *An invitation to operator theory*, volume 50 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2002.

[2] Y. A. Abramovich and G. Sirotkin. On order convergence of nets. *Positivity*, 9(3):287–292, 2005.

[3] C. D. Aliprantis and O. Burkinshaw. *Locally solid Riesz spaces with applications to economics*, volume 105 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2003.

[4] C. D. Aliprantis and O. Burkinshaw. *Positive operators*. Springer, Dordrecht, 2006. RJournal of the 1985 original.

[5] C. D. Aliprantis and R. Tourky. *Cones and duality*, volume 84 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2007.

[6] A. Aydın. Continuous operators with convergence in lattice-normed locally solid Riesz spaces. *arXiv:1711.10263*, 2017.

[7] A. Aydın. Unbounded p_τ-convergence in lattice-normed locally solid Riesz spaces. *arXiv:1711.00734*, 2017.

[8] A. Aydın, E.Yu. Emelyanov, N. Erkurşun Özcan, and M. A. A. Marabeh. Unbounded p-convergence in lattice-normed vector lattices. *arXiv:1609.05301v2*, 2016.

[9] A. Aydın, E.Yu. Emelyanov, N. Erkurşun Özcan, and M. A. A. Marabeh. Compact-like operators in lattice-normed spaces. *Indagationes Mathematicae*, 2017.

[10] A. Aydın, S.G. Gorokhova, and H. Gul. Nonstandard hulls of lattice-normed ordered vector spaces. *Turkish Journal of Mathematics*, 42(1):1–11, 2018.

[11] Y. Azouzi. Completeness for vector lattices. *arXiv:1710.03128*, 2017.

[12] A. Bahramnezhad and K. H. Azar. Strongly order continuous operators on Riesz spaces. *arXiv:1712.04275*, 2017.

[13] A. Bahramnezhad and K. H. Azar. Unbounded order continuous operators on Riesz spaces. *Positivity*, 2017.

[14] Y. A. Dabboorasad, E. Y. Emelyanov, and M. A. A. Marabeh. Order convergence in infinite-dimensional vector lattices is not topological. *arXiv:1705.09883*, 2017.

[15] Y. A. Dabboorasad, E. Y. Emelyanov, and M. A. A. Marabeh. um-topology in multi-normed vector lattices. *Positivity*, 2017.

[16] Y. A. Dabboorasad, E. Y. Emelyanov, and M. A. A. Marabeh. $u\tau$-convergence in locally solid vector lattices. *Positivity*, 2018.

[17] R. DeMarr. Partially ordered linear spaces and locally convex linear topological spaces. *Illinois J. Math.*, 8:601–606, 1964.

[18] Y. Deng, M. O'Brien, and V. G. Troitsky. Unbounded norm convergence in Banach lattices. *Positivity*, 21(3):963—974, 2017.

[19] E. Y. Emelyanov, N. Erkurşun Özcan, and S.G. Gorokhova. Komlos properties in Banach lattices. *arXiv:1710.02580*, 2017.

[20] E. Yu. Emelyanov and S.G. Gorokhova. On the notion of stability of order convergence in vector lattices. *Siberian Mathematical Journal*, 35(5):912–916, 1994.

[21] E. Yu. Emelyanov and M. A. A. Marabeh. Two measure-free versions of the Brezis-Lieb lemma. *Vladikavkaz. Mat. Zh.*, 18(1):21–25, 2016.

[22] R. Engelking. *General topology*. PWN—Polish Scientific Publishers, Warsaw, 1977. Translated from the Polish by the author, Monografie Matematyczne, Tom 60. [Mathematical Monographs, Vol. 60].

[23] Z. Ercan and M. Vural. Towards a theory of unbounded locally solid Riesz spaces. *arXiv:1708.05288*, 2017.

[24] N. Erkurşun Özcan, N. A. Gezer, and O. Zabeti. Spaces of $u\tau$-dunford-pettis and $u\tau$-compact operators on locally solid vector lattices. *arXiv:1710.11434*, 2017.

[25] N. Erkurşun Özcan, N. A. Gezer, and O. Zabeti. Unbounded absolute weak dunford-pettis and unbounded absolute weak compact operators. *arXiv:1708.03970*, 2017.

[26] D. H. Fremlin. *Topological Riesz spaces and measure theory*. Cambridge University Press, London-New York, 1974.

[27] N. Gao. Unbounded order convergence in dual spaces. *J. Math. Anal. Appl.*, 419(1):347–354, 2014.

[28] N. Gao, D. H. Leung, and F. Xanthos. Closedness of convex sets in orlicz spaces with applications to dual representation of risk measures. *arXiv:1610.08806*, 2016.

[29] N. Gao, D. H. Leung, and F. Xanthos. Duality for unbounded order convergence and applications. *Positivity*, 2017.

[30] N. Gao, V. G. Troitsky, and F. Xanthos. Uo-convergence and its applications to cesáro means in Banach lattices. *Israel J. Math.*, 220(2):649—-689, 2017.

[31] N. Gao and F. Xanthos. Unbounded order convergence and application to martingales without probability. *J. Math. Anal. Appl.*, 415(2):931–947, 2014.

[32] N. Gao and F. Xanthos. On the C-property and w^*-representations of risk measures. *Mathematical Finance*, 2017.

[33] S. G. Gorokhova. Intrinsic characterization of the space $c_0(A)$ in the class of Banach lattices. *Mat. Zametki*, 60(3):445–448, 1996.

[34] M. Kandić and M. A. Taylor. Metrizability of minimal and unbounded topologies. *arXiv:1709.05407*, 2017.

[35] M. Kandić, H. Li, and V. G. Troitsky. Unbounded norm topology beyond normed lattices. *Positivity*, 2017.

[36] M. Kandić, M. A. A. Marabeh, and V. G. Troitsky. Unbounded norm topology in Banach lattices. *J. Math. Anal. Appl.*, 451(1):259–279, 2017.

[37] M. Kandić and A. Vavpetič. Topological aspects of order in $C(x)$. *arXiv:1612.05410*, 2016.

[38] S. Kaplan. On unbounded order convergence. *Real Anal. Exchange*, 23(1):175–184, 1997/98.

[39] J. L. Kelley and I. Namioka. *Linear topological spaces*. With the collaboration of W. F. Donoghue, Jr., Kenneth R. Lucas, B. J. Pettis, Ebbe Thue Poulsen, G. Baley Price, Wendy Robertson, W. R. Scott, Kennan T. Smith. The University Series in Higher Mathematics. D. Van Nostrand Co., Inc., Princeton, N.J., 1963.

[40] A. G. Kusraev. Dominated operators, 2000.

[41] H. Li and Z. Chen. Some loose ends on unbounded order convergence. *Positivity*, 2017.

[42] W. A. J. Luxemburg and A. C. Zaanen. *Riesz spaces. Vol. I*. North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., New York, 1971. North-Holland Mathematical Library.

[43] M. Marabeh. Brezis-lieb lemma in convergence vector lattices. *Submitted*, 2017.

[44] O. V. Maslyuchenko, V. V. Mykhaylyuk, and M. M. Popov. A lattice approach to narrow operators. *Positivity*, 13(3):459–495, 2009.

[45] H. Nakano. Ergodic theorems in semi-ordered linear spaces. *Ann. of Math. (2)*, 49:538–556, 1948.

[46] E. T. Ordman. Classroom Notes: Convergence Almost Everywhere is Not Topological. *Amer. Math. Monthly*, 73(2):182–183, 1966.

[47] M. Pliev. Narrow operators on lattice-normed spaces. *Cent. Eur. J. Math.*, 9(6):1276–1287, 2011.

[48] W. Rudin. *Functional analysis*. International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, second edition, 1991.

[49] H. H. Schaefer. *Banach lattices and positive operators*. Springer-Verlag, New York-Heidelberg, 1974. Die Grundlehren der mathematischen Wissenschaften, Band 215.

[50] H. H. Schaefer and M. P. Wolff. *Topological vector spaces*, volume 3 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1999.

[51] M. A. Taylor. Completeness of unbounded convergences. *Proc. Amer. Math. Soc.*, 2017.

[52] M. A. Taylor. Unbounded topologies and uo-convergence in locally solid vector lattices. *arXiv:1706.01575*, 2017.

[53] V. G. Troitsky. Measures on non-compactness of operators on Banach lattices. *Positivity*, 8(2):165–178, 2004.

[54] Jan Harm van der Walt. The universal completion of c(x) and unbounded order convergence. *Journal of Mathematical Analysis and Applications*, 2017.

[55] B. Z. Vulih and O. S. Korsakova. Spaces in which convergence in norm coincides with order convergence. *Mat. Zametki*, 13:259–268, 1973.

[56] B. Z. Vulikh. *Introduction to the theory of partially ordered spaces*. Translated from the Russian by Leo F. Boron, with the editorial collaboration of Zaanen, A. C. and Kiyoshi Iséki. Wolters-Noordhoff Scientific Publications, Ltd., Groningen, 1967.

[57] A. W. Wickstead. Weak and unbounded order convergence in Banach lattices. *J. Austral. Math. Soc. Ser. A*, 24(3):312–319, 1977.

[58] A. Wirth. Order and norm convergence in Banach lattices. *Glasgow Math. J.*, 15:13, 1974.

[59] W. Wnuk. Locally solid Riesz spaces not containing c_0. *Bull. Polish Acad. Sci. Math.*, 36(1-2):51–56, 1988.

[60] A. C. Zaanen. *Introduction to operator theory in Riesz spaces*. Springer-Verlag, Berlin, 1997.

[61] O. Zabeti. Unbounded absolute weak convergence in Banach lattices. *Positivity*, 2017.

[62] O. Zabeti and A. Bahramnezhad. $u\tau$-continuous operators on locally solid vector lattices. *arXiv:1712.01737*, 2017.

CURRICULUM VITAE

PERSONAL INFORMATION

Surname, Name: Dabboorasad, Yousef
Nationality: Palestinian
Date and Place of Birth: 28.08.1983, Saudi Arabia
Marital Status: Married
Phone: +905318359619

EDUCATION

Degree	Institution	Year of Graduation
M.S.	Islamic University of Gaza, Mathematics	2010
B.S.	Islamic University of Gaza, Mathematics	2005
High School	Gifted Secondary School	2001

PROFESSIONAL EXPERIENCE

Year	Place	Enrollment
01/10/2010–25/10/2011	Islamic University of Gaza	Instructor of Mathematics
01/09/2005–01/10/2010	Islamic University of Gaza	Teaching assistant of Mathematics

PUBLICATIONS

1. **Y. Dabboorasad**, E. Y. Emelyanov, and M. A. A. Marabeh. um-Topology in multi-normed vector lattices, Positivity, (2017), https://doi.org/10.1007/s11117-017-0533-6.

2. **Y. Dabboorasad**, E. Y. Emelyanov, and M. A. A. Marabeh. $u\tau$-convergence in locally solid vector lattices, preprint, arXiv:1706.02006v3. [math.FA].

3. **Y. Dabboorasad**, E. Y. Emelyanov, and M. A. A. Marabeh. Order convergence in infinite-dimensional vector lattices is not topological, preprint, arXiv:1705.09883v1 [math.FA].

International Conference Publications

Your publications goes here. Do not try to use Bibtex, since Bibtex builds a single bibliography database for the document.