EXOTIC 4-MANIFOLDS AND HYPERELLIPTIC LEFSCHETZ FIBRATIONS

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ABSTRACT

EXOTIC 4-MANIFOLDS AND HYPERELLIPTIC LEFSCHETZ FIBRATIONS

Altunöz, Tülin
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In this thesis, we explicitly construct genus-3 Lefschetz fibrations over $\mathbb{S}^2$ whose total space is $\mathbb{T}^2 \times \mathbb{S}^2 \# 6\overline{\mathbb{CP}}^2$ using the monodromy of Matsumoto's genus-2 Lefschetz fibration over $\mathbb{S}^2$. We also present exotic minimal symplectic 4-manifolds $3\mathbb{CP}^2 \# k\overline{\mathbb{CP}}^2$ for $k = 13, \dots, 19$ by twisted fiber summing of our monodromy or the genus-3 version of generalized Matsumoto's fibration constructed by Korkmaz or by applying lantern substitutions to these twisted fiber sums. In addition, we generalize our construction of genus-3 Lefschetz fibration to genus-$3k$ Lefschetz fibrations over $\mathbb{S}^2$ using the generalized Matsumoto's genus-$2k$ Lefschetz fibration over $\mathbb{S}^2$ constructed by Korkmaz and independently by Cadavid. Using the generalized version of our monodromy, we derive exotic 4-manifolds via Luttinger surgery and twisted fiber sum. Secondly, we prove that the minimal number of singular fibers in a hyperelliptic Lefschetz fibration over a sphere is $2g + 4$ for even $g \geq 4$, and also, we find a lower bound for odd $g \geq 5$ when the fibration is holomorphic. In addition, we discuss the number of singular fibers of a hyperelliptic Lefschetz fibration over a sphere which does not carry a complex structure.

Keywords: Lefschetz Fibrations, Hyperelliptic Lefschetz fibrations, Exotic 4-manifolds, Mapping Class Groups.

ÖZ

EGZOTİK 4-MANİFOLDLAR VE HİPERELİPTİK LEFSCHETZ LİF DEMETLERİ

Altunöz, Tülin
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Şubat 2018 , 91 sayfa

Bu tezde, Matsumoto'nun cinsi 2 olan $\mathbb{S}^2$ üzerindeki Lefschetz liflemesini kullanarak cinsi 3 ve total uzayı $\mathbb{T}^2 \times \mathbb{S}^2 \# 6\overline{\mathbb{CP}}^2$ olan $\mathbb{S}^2$ üzerinde Lefschetz liflemeleri ürettik. Ayrıca, elde ettiğimiz Lefschetz liflemelerine ya da Korkmaz ve bağımsız olarak Cadavid'ın elde ettiği cinsi 3 olan genelleştirilmiş Matsumoto Lefschetz liflemesine lif toplamını ve bu lif toplamlarına lantern değişimini uygulayarak $k = 13, \dots, 19$ için egzotik minimal simplektik $3\mathbb{CP}^2 \# k\overline{\mathbb{CP}}^2$ manifoldlarını elde ettik. Bunlara ek olarak, elde ettiğimiz cinsi 3 olan $\mathbb{S}^2$ üzerindeki Lefschetz liflemesini genelleştirerek cinsi $3k$ olan $\mathbb{S}^2$ üzerindeki Lefschetz liflemeleri elde ettik. Bu Lefschetz liflemelerini kullanarak, Luttinger operasyonu ve lif toplamı aracılığıyla egzotik 4-manifoldlar elde ettik. İkinci olarak, küre üzerindeki holomorfik hiperelliptik Lefschetz liflemelerinin tekil liflerinin minimal sayılarının $g \geq 4$ ve çift olmak üzere $2g + 4$ olduğunu, $g \geq 5$ ve tek olmak üzere $2g + 6$ dan büyük ya da eşit olduğunu ispatladık. Ek olarak, total uzayı kompleks yapı taşımayan küre üzerindeki hiperelliptik Lefschetz liflemelerinin singüler liflerini araştırdık.

Anahtar Kelimeler: Lefschetz Liflemeleri, Hipereliptik Lefschetz Liflemeleri, Egzotik 4-manifoldlar, Gönderim Sınıfı Grupları.

To my family

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TABLE OF CONTENTS

LIST OF FIGURES

FIGURES

CHAPTER 1

INTRODUCTION

There is a close relationship between objects in 4-dimensional topology and algebra by virtue of the pioneering works of Donaldson and Gompf. By the remarkable work of Donaldson, it was shown that every closed symplectic 4-manifold has a structure of a Lefschetz pencil which, after blowing up at its base points, yields a Lefschetz fibration [23]. Conversely, Gompf [38] proved that the total space of a genus-g Lefschetz fibration admits a symplectic structure if $g \geq 2$. This relation between symplectic 4-manifolds and Lefschetz fibrations provides a way to understand any symplectic 4-manifold via a positive factorization of its monodromy, if it exists. Given a genus-g Lefschetz fibration $f : X \to \mathbb{S}^2$, one can associate to it the identity word $W = 1$ in the mapping class group of a closed orientable genus-g surface. Conversely, one can construct a genus-g Lefschetz fibration over $\mathbb{S}^2$ corresponding to a given monodromy consisting of right handed Dehn twist factorization $t_{\alpha_1} t_{\alpha_2} \cdots t_{\alpha_n} = 1$ in the mapping class group of the regular fiber.

Proving the existence of minimal symplectic structures on 4-manifolds and constructing such manifolds in the homeomorphism classes of simply connected 4-manifolds with very small topology, such as rational surfaces with $b_2^+ = 1, 3$ have been an interesting topic that has used several construction techniques such as rational blowdowns, knot surgery, fiber sums and Luttinger surgeries. (e.g. [1, 6, 11, 22, 32, 33, 34, 36, 37, 47, 59, 60, 65].) Recently [7, 4, 5, 9, 28, 29, 27], some authors have applied some relations in the mapping class group, such as lantern relation or Luttinger surgery, to construct Lefchetz fibrations with $b_2^+ = 1, 3$. For instance, in [15, 2, 7], genus-2 Lefschetz fibrations are studied and exotic genus-2 Lefschetz fibrations with

$b_2^+ \leq 3$ are obtained via several constructions and especially their monodromies. Also, Akhmedov and Monden constructed some higher genus fibrations via lantern and daisy substitutions [3]. We would like to specify that the aim of this study is not only to construct exotic smooth structures on very small 4-manifolds with $b_2^+ = 3$ but also to use the twisted fiber sum operation and lantern substitution corresponding to the symplectic rational blowdown surgery along a -4 sphere [28] to study smooth structures on various 4-manifolds using the monodromies of Lefschetz fibrations with small numbers of singular fibers.

In this thesis, we construct a relation $W = 1$ in the mapping class group of a closed orientable genus-3 surface, denoted by Mod_3, using Matsumoto's well known relation [55], via the construction technique given by Baykur and Korkmaz in [14], (see [13] for more examples of this technique), and we obtain $\mathbb{T}^2 \times \mathbb{S}^2 \# 6\overline{\mathbb{CP}}^2$ admitting genus-3 Lefschetz fibration over $\mathbb{S}^2$. We apply lantern substitutions to the twisted fiber sums of genus-3 Lefschetz fibrations over $\mathbb{S}^2$ with monodromy $W = 1$, to get minimal genus-3 Lefschetz fibrations whose total spaces are homeomorphic but not diffeomorphic to $3\mathbb{CP}^2 \# p\overline{\mathbb{CP}}^2$ for $p = 13, 14, 15$. We also construct simply connected genus-3 Lefchetz fibrations via fiber sums of the genus-3 Lefschetz fibrations corresponding to $W = 1$ and Korkmaz's fibration for $g = 3$ [43], which is also constructed independently by Cadavid [19] and later with a different proof [20], and use lantern substitution to the twisted fiber sums to get exotic minimal symplectic 4-manifolds in the homeomorphism classes of $3\mathbb{CP}^2 \# q\overline{\mathbb{CP}}^2$ for $q = 16, \ldots, 19$. Moreover, we generalize our relation $W = 1$ in Mod_3 to the relation $W_k = 1$ in Mod_{3k}, the corresponding total space of which is diffeomorphic to $\Sigma_k \times \mathbb{S}^2 \# 6\overline{\mathbb{CP}}^2$. Using this Lefschetz fibration structure, we produce exotic copies of $(4k-1)\mathbb{CP}^2 \# (4k+5)\overline{\mathbb{CP}}^2$ for any positive integer k via Luttinger surgery and finally we construct minimal exotic copies of $(4k^2 - 2k + 1)\mathbb{CP}^2 \# (4k^2 + 4k + 7)\overline{\mathbb{CP}}^2$ admitting Lefschetz fibration structure for any integer $k > 0$ via twisted fiber sum.

In Chapter 2, we give a review of background information about Lefschetz fibrations, symplectic 4-manifolds, classification of simply connected 4-manifolds, Luttinger surgery and Seiberg-Witten invariants.

In Chapter 3, we construct a factorization W of t_δ in the mapping class group of genus-3 surface with one boundary component denoted by Mod_3^1. Let X denote the genus-3 Lefschetz fibration with the monodromy W. We prove

Theorem 1.0.1. *The* 4*-manifold* X *is diffeomorphic to* $\mathbb{T}^2 \times \mathbb{S}^2 \# 6\overline{\mathbb{CP}}^2$.

By twisted fiber summing and applying Lantern relation to the twisted fiber sum of X, we construct Lefschetz fibrations (X_1, f_1), (X_2, f_2) and (X_3, f_3) and we prove

Theorem 1.0.2. *For* $i = 1, 2, 3$*, the genus-*3 *Lefschetz fibration* $f_i : X_i \to \mathbb{S}^2$ *is minimal and has*

(i) $e(X_i) = 21 - i$,

(ii) $c_1^2(X_i) = 3 + i$,

(iii) $\pi_1(X_i) = 1$.

In Chapter 4, using simply connected genus-3 Lefschetz fibrations constructed in Chapter 3, and Matsumoto's genus-3 Lefchetz fibration, we derive exotic copies of $3\mathbb{CP}^2 \# k\overline{\mathbb{CP}}^2$, for $k = 13, \ldots, 19$. Moreover, we generalize the factorization $W = t_\delta$ in Mod_3^1 to the factorization $W_k = t_\delta$ in Mod_{3k}^1. Let $X(k)$ denote the genus-$3k$ Lefschetz fibration with the monodromy W_k. We prove

Theorem 1.0.3. *The* 4*-manifold* $X(k)$ *is diffeomorphic to* $\Sigma_k \times \mathbb{S}^2 \# 6\overline{\mathbb{CP}}^2$ *for* k *any non-negative integer.*

Using twisted fiber sum of the genus-$3k$ Lefcshetz fibration $X(k)$, we prove

Theorem 1.0.4. *There exist new minimal symplectic exotic copies of* $(4k^2 - 2k + 1)\mathbb{CP}^2 \# (4k^2 + 4k + 7)\overline{\mathbb{CP}}^2$ *admitting genus-*$3k$ *Lefschetz fibration structure for each integer* $k \geq 1$.

Moreover, using Luttinger surgery we contruct smaller exotic manifolds. We prove

Theorem 1.0.5. *There exist new smooth exotic copies of* $(4k - 1)\mathbb{CP}^2 \# (4k + 5)\overline{\mathbb{CP}}^2$.

In Chapter 5, after giving some background information about hyperelliptic Lefschetz fibrations, the number of singular fibers in Lefschetz fibrations and classification of complex surfaces, we prove

Theorem 1.0.6. *Let N_g^h be the minimal number of singular fibers in a genus-g hyperelliptic Lefschetz fibration over $\mathbb{S}^2$. Then*

1. $N_4^h = 12$,
2. $N_5^h \geq 15$,
3. $N_6^h = 16$,
4. $N_7^h \geq 17$,
5. $N_8^h \in \{19, 20\}$,
6. $N_9^h \geq 24$,
7. $N_{10}^h \in \{23, 24\}$.

For hyperelliptic holomorphic Lefschetz fibrations, let M_g^h be the minimal number of singular fibers. We prove

Theorem 1.0.7. *Let g be grater than 3 and even. Then $M_g^h = 2g + 4$.*

Theorem 1.0.8. *Let g be grater than 6 and odd. Then $M_g^h \geq 2g + 6$.*

.

CHAPTER 2

LEFSCHETZ FIBRATIONS

2.1 Preliminaries

In this section, we first state some preliminary definitions and recall some useful results concerning mapping class groups, lantern relations, Lefschetz fibrations [31, 38, 53]. Then we give some details on Matsumoto's well known fibration [55] and generalized Matsumoto's fibration to higher genus orientable surfaces [43]. Also, we give the Endo and Nagami's method to compute signatures Lefschetz fibrations [30].

2.1.1 Mapping class groups

Let Σ_g^n denote a compact connected oriented surface of genus g with n boundary components, $\mathrm{Diff}^+(\Sigma_g^n)$ denote the group of all orientation preserving self-diffeomorphisms of Σ_g^n that fixes all points on the boundary and let $\mathrm{Diff}_0^+(\Sigma_g^n)$ denote the subgroup of $\mathrm{Diff}^+(\Sigma_g^n)$ consisting of orientation preserving self-diffeomorphisms of Σ_g^n which are isotopic to the identity. *The mapping class group* Mod_g^n is defined to be the group of isotopy classes of orientation preserving self-diffeomorphisms of Σ_g^n fixing all points on the boundary, i.e.,

$$\mathrm{Mod}_g^n = \mathrm{Diff}^+(\Sigma_g^n)/\mathrm{Diff}_0^+(\Sigma_g^n)$$

We denote Mod_g^0 and Σ_g^0 by Mod_g and Σ_g, respectively.

Definition 2.1.1. *Let a be a simple closed curve on an oriented surface Σ_g^n. A right (or positive) Dehn twist about a is the diffeomorphism t_a obtained by cutting Σ_g^n along a and gluing it back after rotating one of the sides by 360 degrees to the right.*

Throughout this thesis, for any two mapping classes, the multiplication fg means that g is applied first and then f.

Lemma 2.1.2. *[31] Let f be an orientation preserving self-diffeomorphism of Σ_g^n and a and b be two simple closed curves on Σ_g^n. Then*

1. $ft_af = t_{f(a)}$,
2. *if a and b are disjoint, then t_a and t_b commute,*
3. *if a intersects b transversely at a single point, then the corresponding Dehn twists t_a and t_b satisfy the braid relation $t_at_bt_a = t_bt_at_b$.*

2.1.2 Some relations in the mapping class group

In the following we review some relations in the mapping class group.

2.1.2.1 Even chain relation

A *chain* of length $2h$ is an ordered $2h$-tuple $(e_1, e_2, \ldots, e_{2h})$ of simple closed curves on a genus-g surface Σ_g if

(i) for each $i = 1, 2, \ldots, 2h-1$, the simple closed curves e_i and e_{i+1} intersect tranversely at one point,

(ii) $e_i \cap e_j = \varnothing$ when $|i-j| > 1$.

Now, consider the even chain $(c_1, c_2, \ldots, c_{2h})$. A tubular neighborhood of

$$(c_1 \cup c_2 \cup \cdots \cup c_{2h})$$

is a genus-h surface with one boundary component d. Then the relation

$$(t_{c_1}t_{c_2}\cdots t_{c_{2h}})^{4h+2} = t_d \tag{2.1}$$

is called *even chain relation* in Mod_g (cf. Figure 2.1). A curve bounding a subsurface of genus h (such as d) is called a *separating curve of type h*.

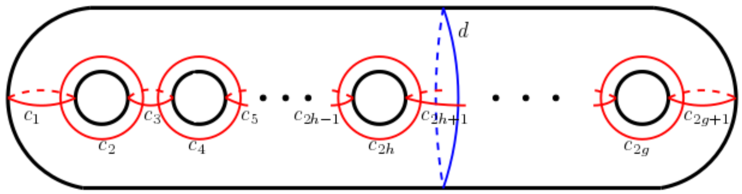

Figure 2.1: The curves in the even chain relation and the hyperelliptic relation

2.1.2.2 Hyperelliptic relation

Definition 2.1.3. *A hyperelliptic involution on a closed orientable surface Σ_g is (the isotopy class of) a self diffeomorphism of order two which has exactly $2g + 2$ fixed points.*

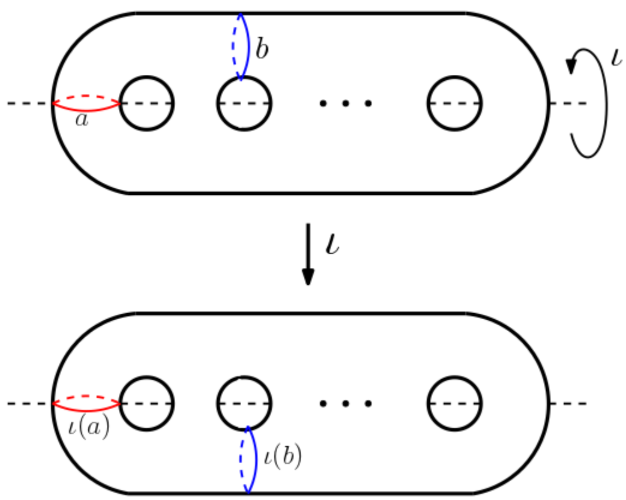

Figure 2.2: Hyperelliptic involution ι

Let us embed the surface Σ_g in $\mathbb{R}^3$ as in Figure 2.2, so that it is invariant under the rotation ι by π about the y-axis, which we take as the hyperelliptic involution.

The hyperelliptic involution ι can be written as

$$\iota = t_{c_1} \cdots t_{c_{2g}} t_{c_{2g+1}}^2 t_{c_{2g}} \cdots t_{c_1}$$

so that

$$(t_{c_1} \cdots t_{c_{2g}} t_{c_{2g+1}}^2 t_{c_{2g}} \cdots t_{c_1})^2 = 1, \tag{2.2}$$

which is called *hyperelliptic relation* in Mod_g. It is easy to see that the simple closed curves $(c_1, c_2, \ldots, c_{2g+1})$ form the longest chain on Σ_g as in Figure 2.1.

2.1.2.3 Lantern relation

Let us record the lantern relation which was proved by Dehn and reproved by Johnson.

Lemma 2.1.4. *[31, 41] Let $\delta_1, \delta_2, \delta_3$ and δ_4 be the boundary curves of Σ_0^4 and x_1, x_2 and x_3 be the simple closed curves as shown in Figure2.3. Then the following relation holds in* Mod_0^4.

$$t_{\delta_1}t_{\delta_2}t_{\delta_3}t_{\delta_4} = t_{x_1}t_{x_2}t_{x_3}.$$

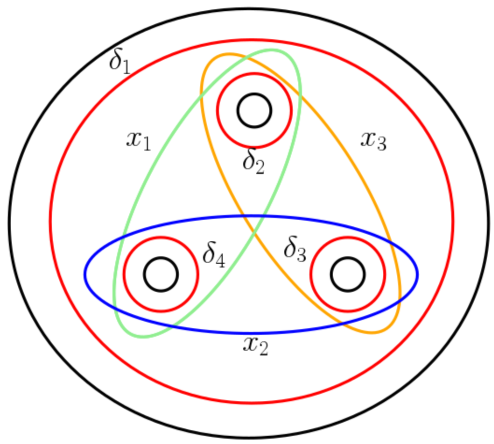

Figure 2.3: Lantern relation

2.1.3 Lefschetz fibrations and monodromy representations

We start with a review of some basic definitions and properties of Lefschetz fibrations.

Let M be a compact oriented smooth 4-manifold. A smooth surjective map $f : M \to \mathbb{S}^2$ is a *Lefschetz fibration* of genus g if it has finitely many critical points and can be written as $f(z_1, z_2) = z_1^2 + z_2^2$ with respect to some local complex coordinates around each critical point. The genus of a regular fiber F is called *the genus of the fibration*. We assume that all the critical points lie in the distinct fibers, called *singular fibers*, which can be achieved after a small perturbation. Each singular fiber is obtained by shrinking a simple closed curve, called *vanishing cycle*, to a point in the regular fiber. If the vanishing cycle is nonseparating (resp. separating), then the singular fiber is called *irreducible* (resp. *reducible*). In this work, we also assume that all Lefschetz fibrations are nontrivial, i.e. there exists at least one singular fiber and fibrations are relatively minimal, i.e. no fiber contains a (-1)-sphere, otherwise one can blow it down without changing the rest of fibration.

Lefschetz fibrations can be described combinatorially via their monodromies. The monodromy of a Lefschetz fibration $f : M \to \mathbb{S}^2$ is given by a positive factorization $t_{\alpha_1} t_{\alpha_2} \cdots t_{\alpha_n} = 1$ in Mod_g where α_i's are the vanishing cycles of the singular fibers. Conversely, for given a positive factorization $t_{a_1} t_{a_2} \cdots t_{a_k} = 1$ in Mod_g, one can construct a genus-g Lefschetz fibration over $\mathbb{S}^2$ by attaching 2-handles along vanishing cycles a_i in a Σ_g fiber in $\Sigma_g \times D^2$ with -1 framing, and then close it up by a fiber preserving map to get a fibration over $\mathbb{S}^2$. Such a fibration is uniquely determined up to isomorphisms, which are orientation preserving self-diffeomorphisms of the total spaces and $\mathbb{S}^2$ making the fibrations commute. The relation $t_{a_1} t_{a_2} \cdots t_{a_k} = 1$ in Mod_g is uniquely determined up to *Hurwitz moves* (exchanging subwords $t_{a_i} t_{a_{i+1}} = t_{a_{i+1}} t_{t_{a_{i+1}}(a_i)}$ and *global conjugations* (changing each t_{a_i} with $t_{\varphi(a_i)}$ for some $\varphi \in \mathrm{Mod}_g$) if $g \geq 2$. A map $\sigma : \mathbb{S}^2 \to M$ is called a *section* of a Lefschetz fibration $f : M \to \mathbb{S}^2$ if $f \circ \sigma = id_{\mathbb{S}^2}$. If there exists a lift of a positive relation $t_{\alpha_1} t_{\alpha_2} \cdots t_{\alpha_n} = 1$ in Mod_g to Mod_g^k such that $t_{\tilde{\alpha_1}} t_{\tilde{\alpha_2}} \cdots t_{\tilde{\alpha_n}} = t_{\delta_1}^{m_1} t_{\delta_2}^{m_2} \cdots t_{\delta_k}^{m_k}$ where m_i's are integers and δ_i's are boundary curves then the Lefschetz fibration $f : M \to \mathbb{S}^2$ admits k disjoint sections $S_1, \ldots, S_k$, where S_j is of self-intersection $-m_j$ and vice versa [16].

For $i = 1, 2$, let $f_i : M_i \to \mathbb{S}^2$ be a genus-g Lefschetz fibration with a regular fiber F_i and monodromy factorization $W_i = 1$. Let r be an orientation-reversing self-diffeomorphism of S^1 and $\phi : F_2 \to F_1$ be an orientation-preserving diffeomorphism. We remove a fibred neighborhood of F_i from M_i and glue the resulting manifolds along their boundaries using the orientation reversing diffeomorphism $r \times \phi$. Then the resulting 4-manifold is a genus-g Lefschetz fibration over $\mathbb{S}^2$ with monodromy factorization $W_1 W_2^{\phi}$, which is called a *twisted fiber sum* of Lefschetz fibrations f_1 and f_2. Moreover if , for $i = 1, 2$, the Lefschetz fibration $f_i : M_i \to \mathbb{S}^2$ admits a section with self-intersection m_i, then the twisted fiber sum of f_1 and f_2 admits a section with self-intersection $m_1 + m_2$. Here the notation W^{ϕ} denotes the conjugated word of W, i.e., $W^{\phi} = t_{\phi(\alpha_1)} t_{\phi(\alpha_2)} \cdots t_{\phi(\alpha_n)}$, if $W = t_{\alpha_1} t_{\alpha_2} \cdots t_{\alpha_n}$.

2.1.4 Generalized Matsumoto's relation

Let $B_0, B_1, \dots, B_g, a, b$ and C be simple closed curves on Σ_g as shown in Figure 2.4 and W_g be the following word:

$$W_g = \begin{cases} (t_{B_0} t_{B_1} \dots t_{B_g} t_C)^2 & \text{if} \quad g = 2k, \\ (t_{B_0} t_{B_1} \dots t_{B_g} t_a^2 t_b^2)^2 & \text{if} \quad g = 2k+1. \end{cases}$$

The word W_g represents the identity in the mapping class group Mod_g, which was shown by Matsumoto in [55] for $g = 2$, and by Korkmaz [43] and Cadavid independently [19] for $g \geq 3$. Stipsicz and Ozbagci showed that W_g is equal to the Dehn twist t_δ when there is one boundary component [56], where δ is the boundary of the genus-g surface. When there are two boundary components δ_1 and δ_2, there is also a lifting of W_g that equals to the product of $t_{\delta_1} t_{\delta_2}$ obtained by Korkmaz [45]. Recently, Hamada gave a maximal set of disjoint (-1) sections of W_g. One of the liftings that he constructed is $W_g = t_{\delta_1} t_{\delta_2}$, where the curves δ_1 and δ_1 are as depicted in Figure 2.5 [39]. Let M_g be the total space of the Lefschetz fibration over $\mathbb{S}^2$ with the monodromy factorization W_g. It is known that M_g is diffeomorphic to $\Sigma_k \times \mathbb{S}^2 \# 4\overline{\mathbb{CP}}^2$ (resp. $\Sigma_k \times \mathbb{S}^2 \# 8\overline{\mathbb{CP}}^2$) when $g = 2k$ (resp. $g = 2k+1$) [43, 19].

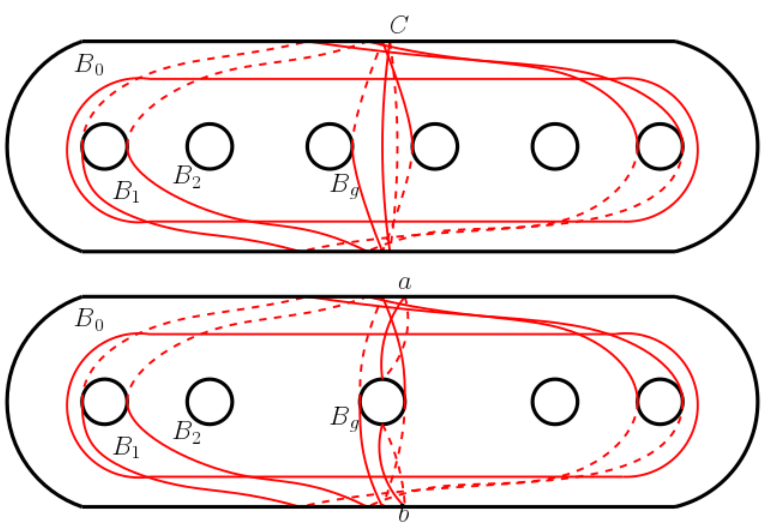

Figure 2.4: The curves B_i' s, B_0', a, b and C on Σ_g

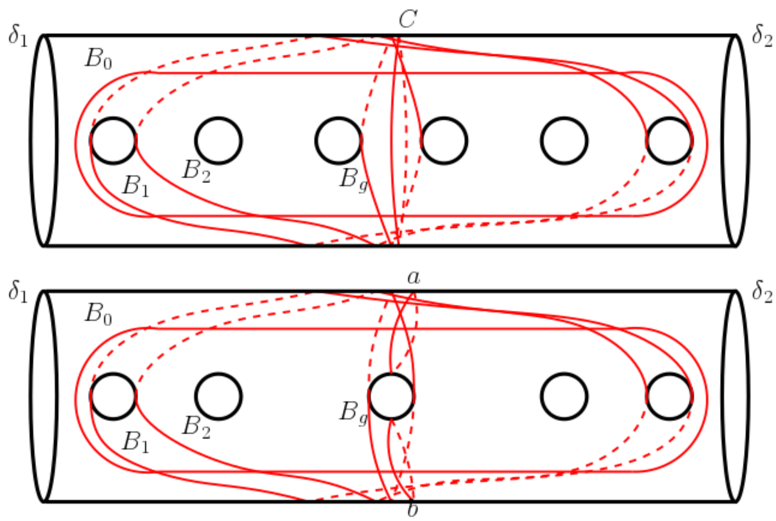

Figure 2.5: The curves B_i' s, B_0', a, b and C on Σ_g^2

2.1.5 Signature of a relation

In [30], Endo and Nagami discovered a useful method to calculate the signature of a Lefschetz fibration over $\mathbb{S}^2$ by introducing the notion of the signature of a relation in a mapping class group. This method allows one to determine the signature of a Lefschetz fibration over $\mathbb{S}^2$ as the sum of signatures of basic relations in its monodromy. They also explicitly compute the signature of some known relations. Let us recall the definition of the signature of a relation and some results that we will need later.

Let $\mathcal{F}$ be the free group generated by all isotopy classes of simple closed curves on Σ_g. There is a natural homomorphism $\varrho : \mathcal{F} \to \mathrm{Mod}_g$ mapping a simple closed curve a on Σ_g to the right-handed Dehn twist t_a. Since Mod_g is generated by Dehn twists [21, 50], the homomorphism ϱ is surjective. We call an element of $Ker \varrho$ a *relator*. A relator ρ is of the form $\rho = c_1^{\epsilon_1} c_2^{\epsilon_2} \cdots c_n^{\epsilon_n}$ where c_i' s are simple closed curves on Σ_g and $\epsilon_i = \pm 1$ for $i = 1, \ldots, n$. The word ρ is said to be a *positive relator* if $\epsilon_i = +1$ for $i = 1, \ldots, n$. For instance,

$$L = x_1 x_2 x_3 \delta_1^{-1} \delta_2^{-1} \delta_3^{-1} \delta_4^{-1}$$

is a relator of Mod_0^4 coming from the Lantern relation (2.1.2.3), which we call the *lantern relator*. The words

$$(B_0 B_1 \cdots B_g C)^2 \text{ if } g \text{ is even,}$$

and

$$(B_0 B_1 \cdots B_g a^2 b^2)^2 \text{ if } g \text{ is odd}$$

are also relators in Mod_g. There is an explicit homomorphism $c_g : Ker\varrho \to \mathbb{Z}$ inducing the evaluation map $H_2(\mathrm{Mod}_g) \to \mathbb{Z}$ for the cohomology class of τ_g, where $\tau_g : \mathrm{Mod}_g \times \mathrm{Mod}_g \to \mathbb{Z}$ is the Meyer's signature cocycle. For a relator $\rho \in Ker\varrho$, *the signature* of ρ is given by

$$I_g(\rho) := -c_g(\rho) - s(\rho),$$

where $s(\rho)$ is the sum of the exponents of Dehn twists about separating simple closed curves appearing in the word ρ. Endo and Nagami also extend this definition for any element of the free group $\mathcal{F}$.

Definition 2.1.5. *Let $\rho = W_1^{-1}W_2$ and $\xi = UW_1V$ be relators such that U, V, W_1 and W_2 are positive words in $\mathcal{F}$. Then we can obtain a new positive relator $\xi' = \xi V^{-1}\rho V = UW_2V$. This operation is called ρ-substitution to ξ. When ρ is a lantern relator then we say that ξ' is obtained by applying the lantern substitution to ξ.*

For the proofs of the following lemma and theorem, we refer the reader to [30].

Lemma 2.1.6. *The signature I_g satisfies the following:*

(i) $I_g(a) = -1$, where a is the isotopy class of a separating curve .

(ii) $I_g(L) = +1$, where L is a lantern relator.

(iii) $I_g((B_0B_1 \cdots B_gC)^2) = -4$ if g is even,
$I_g((B_0B_1 \cdots B_g a^2 b^2)^2) = -8$ if g is odd.

Theorem 2.1.7. *Let $f : X \to \mathbb{S}^2$ be a genus-g Lefschetz fibration with the monodromy $t_{c_1}t_{c_2} \cdots t_{c_n}$, so that $c_1c_2 \cdots c_n \in Ker(\varrho)$ a positive relator. Then the signature $\sigma(X)$ of the 4-manifold X is equal to the signature of $c_1c_2 \cdots c_n$, i.e.,*

$$\sigma(X) = I_g(c_1c_2 \cdots c_n).$$

2.1.6 Minimality of symplectic fiber sums

In this subsection, we give the definition of symplectic fiber sum operation and some useful theorems to determine minimality of fiber sums.

Definition 2.1.8. *A symplectic 4-manifold is called minimal if it does not contain any symplectically embedded 2-sphere with self-intersection -1.*

Definition 2.1.9. *Let X_i be a closed, oriented, smooth manifold of dimension 4 containing a smoothly embedded surface Σ of genus $g \geq 1$ such that the surface Σ has zero self-intersection in X_i and represents a homology class of infinite order for each $i = 1, 2$. The generalized fiber sum $X_1 \#_\varphi X_2$ along closed embedded genus-g surfaces Σ is defined as $(X_1 \setminus \upsilon\Sigma) \cup_\varphi (X_2 \setminus \upsilon\Sigma)$, where $\upsilon\Sigma \cong \Sigma \times D^2$ in both X_1 and X_2 denotes a tubular neighbourhood of the surface Σ and the gluing map φ is an orientation-reversing and fiber preserving self-diffeomorphism of $S^1 \times \Sigma$.*

For a symplectic manifold X_i and embedded symplectic submanifold in X_i for each $i = 1, 2$, Gompf showed that the resulting manifold $X_1 \#_\varphi X_2$ admits a symplectic structure [37].

Let $e(X)$ be the Euler characteristic of a manifold X. Some topological invariants of $X_1 \#_\varphi X_2$ can be computed using the following lemma.

Lemma 2.1.10. *Let $X_1 \#_\varphi X_2$ be the fiber sum of X_1 and X_2 along closed embedded surface Σ of genus g ($g \geq 1$) determined by φ. Then*

(i) $e(X_1 \#_\varphi X_2) = e(X_1) + e(X_2) - 2e(\Sigma)$,

(ii) $\sigma(X_1 \#_\varphi X_2) = \sigma(X_1) + \sigma(X_2)$.

One can describe the minimality of a symplectic fiber sum using the following theorem:

Theorem 2.1.11. *[68, 24] Let (X, ω_X) and (Y, ω_Y) be two symplectic 4-manifolds containing an embedded symplectic surface S of genus $g \geq 0$ and M be the symplectic fiber sum $X \#_S Y$.*

1. *The 4-manifold M is not minimal if*

(a) $X \setminus S_X$ or $Y \setminus S_Y$ *contains an embedded symplectic* (-1)*-sphere, where* $S_X \subset X$ *and* $S_Y \subset Y$ *are copies of the surface* S*,*

or

(b) $X\#_S Y = Z\#_{S_{\mathbb{CP}^2}}\mathbb{CP}^2$ *with* $S_{\mathbb{CP}^2}$ *an embedded* $+4$*-sphere in class* $[S_{\mathbb{CP}^2}] = 2[H] \in H_2(\mathbb{CP}^2;\mathbb{Z})$ *and* Z *has at least two disjoint exceptional spheres* E_i *so that each* E_i *meets the submanifold* $S_Z \subset Z$ *transversely and positively in a single point with* $[E_i]\cdot[S_X] = 1$.

2. *If* $X\#_S Y = Z\#_{S_B} B$ *where* B *is a* $\mathbb{S}^2$*-bundle over a surface of genus-*g *and* S_B *is a section of this bundle then* M *is minimal if and only if* Z *is minimal.*

3. M *is minimal in all other cases.*

We will use the following proposition which is a simple corollary of Theorem 2.1.11 on symplectic sums to verify that our Lefschetz fibrations are minimal symplectic 4-manifolds, (see also [15]).

Proposition 2.1.12. *Let* (X, f) *be a Lefschetz fibration associated to a factorization* $W = W_1^{\phi}W_2$ *in* Mod_g*, where* ϕ *is any mapping class and* W_1*,* W_2 *are positive factorizations in* Mod_g*. Then the* 4*-manifold* X *is minimal.*

Definition 2.1.13. *For a* 4*-manifold* X*, the topological blow-up of* X *at one point is diffeomorphic to the* 4*-manifold* $X\#\overline{\mathbb{CP}^2}$*. Here* $\overline{\mathbb{CP}^2}$ *is* $\mathbb{CP}^2$ *with the reversed orientation.*

The reverse process of the topological blow-up operation is called topological blow-down.

Note that every symplectic 4-manifold can be made minimal by blowing down a maximal collection of symplectically embedded (-1)-spheres.

Definition 2.1.14. *Symplectic manifolds that blow-down to an* $\mathbb{S}^2$*-bundle over a Riemann surface of genus-*$g \geq 0$ *are called ruled surfaces.*

Definition 2.1.15. *Symplectic manifolds that blow-down to* $\mathbb{CP}^2$ *or* $\mathbb{S}^2 \times \mathbb{S}^2$ *are called rational surfaces.*

The minimal model of a ruled surface is not unique. It is known that there are exactly two minimal models of ruled surfaces of genus g, the trivial bundle $\Sigma_g \times \mathbb{S}^2$ and the nontrivial bundle $\Sigma_g \ltimes \mathbb{S}^2$ [12]. Li and Liu proved the following two theorems about symplectic structures of ruled surfaces.

Theorem 2.1.16. *[48] The symplectic structure of an $\mathbb{S}^2$-bundle over a genus-g Riemann surface is unique up to symplectic deformation and diffeomorphism.*

Theorem 2.1.17. *[48] The symplectic structures of blow ups of geometrically ruled surfaces are unique.*

Therefore, for $k > 0$, $\Sigma_g \times \mathbb{S}^2 \# k\overline{\mathbb{C}P^2}$ and $\Sigma_g \ltimes \mathbb{S}^2 \# k\overline{\mathbb{C}P^2}$ are symplectomorphic.

Theorem 2.1.18. *([67], [51]) Let X be a minimal symplectic 4-manifold which is not a ruled surface. Then $c_1^2(X) \geq 0$, where $c_1^2(X) = 3\sigma(X) + 2e(X)$.*

Let (X, ω) be a symplectic 4-manifold. Then the set ε_X is defined to be the set of all $E \in H^2(X; \mathbb{Z})$ such that E is the Poincaré dual of the homology classes which can be represented by a smoothly embedded (-1)-sphere in (X, ω) and $E \cdot [\omega] > 0$. When the 4-manifold X is neither rational nor ruled, we may assume that $E \cdot [\omega] > 0$, because one can change the orientation of the smoothly embedded (-1)-sphere that represents the class E if necessary to get $E \cdot [\omega] > 0$. Then we have the following theorem.

Theorem 2.1.19. *[62] Let (X, f) be a relatively minimal Lefschetz fibration of genus-g over $\mathbb{S}^2$ where $g \geq 2$. Suppose that the set ε_X is non-empty and $\varepsilon_X = \{E_1, E_2, \ldots, E_n\}$. If X is neither rational nor ruled, then*

(i) $n \leq 2g - 2$.

(ii) $\left(\sum_{i=1}^{n} E_i \right) \cdot F \leq 2g - 2$.

(iii) $1 \leq E_i \cdot F \leq 2g - 2$, *for any i with $1 < i < n$.*

Theorem 2.1.20. *[64] Let $g \geq 2$ and (X, f) be a genus-g Lefschetz fibration over a sphere with $b_2^+(X) = 1$. Then either $e(X) \geq 0$ or X is the blow-up of a ruled surface.*

2.1.7 Classification of simply connected 4-Manifolds

In this subsection, we will state Freedman's remarkable theorem which determines the homeomorphism type of a simply-connected closed 4-manifold. His theorem is based on the intersection form of 4-manifolds.

For a closed oriented 4-manifold X , let $a, b \in H_2(X;\mathbb{Z})$ be homology classes and let the cohomology classes $\alpha, \beta \in H^2(X;\mathbb{Z})$ be their Poincaré duals. The intersection form Q_X of X is defined by $Q_X(a,b) = \langle a \cup b, [X] \rangle$ where $\cup : H^2(X;\mathbb{Z}) \times H^2(X;\mathbb{Z}) \to H^4(X;\mathbb{Z})$ the cup product of cohomology groups. It is known that the intersection form Q_X is symmetric and unimodular. The *rank* of the form Q_X is defined to be the dimension of $H_2(X;\mathbb{Z})\big/$Torsion, which is the group obtained by dividing out the torsion part of $H_2(X;\mathbb{Z})$. The form is called *even* if $Q_X(\alpha,\alpha) \equiv 0$ mod2 for all $\alpha \in H_2(X;\mathbb{Z})$. Otherwise the form Q_X is called *odd*. The *signature* $\sigma(X)$ of X is defined to be the signature of the diagonalizable (extended) form $Q_{\mathbb{R}}$, which is given by $b_2^+ - b_2^-$, where b_2^+ and b_2^- denote the number of positive and negative eigenvalues associated to the form $Q_{\mathbb{R}}$, respectively.

Theorem 2.1.21. *[35] Given a unimodular bilinear symmetric form Q, there exists a simply-connected, closed topolological 4-manifold X such that the intersection form Q_X of X is isomorphic to Q. If Q is even, there exists unique homeomorphism class with this property. If Q is odd, there exist two different homeomorphism classes of 4-manifolds with this property. At most one of these classes can be represented by a 4-manifold with a smooth structure.*

2.1.8 Luttinger surgery

Luttinger surgery is a technique used to produce symplectic 4-manifolds using known symplectic 4-manifolds. We will use this technique in Chapter 4 to construct some exotic 4-manifolds.

Let (X, ω) be a symplectic 4-manifold. Given any Lagrangian torus T in X, a neighbourhood $\nu(T)$ of T can be identified (symplectomorphically) with a neighbourhood of the zero section of the cotangent bundle of T with its standard symplectic structure. This identification is called *Lagrangian push off* or *Lagrangian framing*.

Let γ be a co-oriented simple closed loop in T. One can identify T with $\mathbb{R}^2/\mathbb{Z}^2$ so that γ and its co-orientation agrees with the first coordinate x_1 and the second coordinate axis x_2 with the standard orientation, respectively. The symplectic form is given by $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$, where (x_1, x_2) denotes the corresponding coordinates on T and (y_1, y_2) denotes the dual coordinates on the cotangent bundle. Let r be a positive real number in such a way that the neighbourhood $\nu(T)$ of T contains the set $U_r = \mathbb{R}^2/\mathbb{Z}^2 \times [-r, r] \times [-r, r]$ under the identification. Take a smooth function $\chi : [-r, r] \to [0, 1]$ satisfying

- $\chi(t) = 0$ if $t \leq -\frac{r}{3}$,
- $\chi(t) = 1$ if $t \geq \frac{r}{3}$,
- $\int_{-r}^{r} t\chi'(t)\, dt = 0$.

Then for any k, define $\phi_k : U_r - U_{r/2} \to U_r - U_{r/2}$ by

- $\phi_k(x_1, x_2, y_1, y_2) = x_1 + k\chi(y_1), x_2, y_1, y_2)$, if $y_2 \geq \dfrac{r}{2}$,
- $\phi_k = id$, otherwise.

Then define the manifold $X(T, \gamma, k)$ is to be obtained by removing the neighbourhood $U_{r/2}$ and gluing U_r using the self-diffeomorphism ϕ_k of $U_r - U_{r/2}$ to identify their boundaries.

The surgery described above is equivalent to the surgery operation introduced by Luttinger [52] (see also [25]).

Disregarding the symplectic structure, in the topological set up one can describe this construction as a certain type of Dehn surgery along a Lagrangian torus as follows:

Cut out a neighbourhood $T \times D^2$ of T and glue back it by identifying the boundaries $T \times \mathbb{S}^1$ with a diffeomorphism so that it acts trivially on $H_1(T; \mathbb{Z})$ and maps $[\mu]$ to $[\mu] + k[\gamma]$ where $[\mu]$ is the homology class of the meridian.

Under a natural framing of the normal bundle to T along γ, one can push away the loop γ in a canonical way. This provides us to define homotopy class of γ in $\pi_1(X -$

T). After performing this surgery, the fundamental group $\pi_1(X(T,\gamma,k))$ of the 4-manifold $X(T,\gamma,k)$ is computed as follows and also some topological invariants of the resulting manifolds satisfies the following:

Lemma 2.1.22.

(1) $\pi_1(X(T,\gamma,k)) = \pi_1(X-T)\big/N(\mu\gamma^k)$*, where* $N(\mu\gamma^k)$ *is the normal closure of the group generated by* $\mu\gamma^k$*,*

(2) $e(X(T,\gamma,k)) = e(X)$*,*

(3) $\sigma(X(T,\gamma,k)) = \sigma(X)$*.*

It is known that this surgery operation is symplectically well-defined [10, 52]. The above construction is obtained by $1/k$-Dehn surgery along a Lagrangian torus T in a symplectic 4-manifold X. One can also perform p/q-Dehn surgery along a Lagrangian torus T. In this case, after cutting out a neighbourhood $T \times D^2$ of T, a diffeomophism ϕ satisfying $\phi([\mu]) = p[\mu] + q[\gamma]$ is used when gluing it back to identify the boundaries. The fundamental group of the resulting manifold is isomorphic to

$$\pi_1(X(T,\gamma,p/q)) = \pi_1(X-T)\big/N(\mu^p\gamma^q)$$

where $N(\mu^p\gamma^q)$ is the normal closure of $\mu^p\gamma^q$ in the group $\pi_1(X-T)$. When $p \neq \pm 1$, the 4-manifold $X(T,\gamma,p/q)$ generally does not admit a symplectic structure.

2.1.9 Seiberg-Witten invariants

In this subsection we review the basics of Seiberg-Witten invariants which are a diffeomorphism type invariant for compact smooth 4-manifolds.

For a smooth closed 4-manifold M with $b_2^+ > 1$, the Seiberg-Witten invariant of the manifold M is an integer valued function from the set of $spin^c$ structures on M [69]. If $H_1(M;\mathbb{Z})$ has no 2-torsion (in particular if M is a simply-connected 4-manifold) then there is a one to one correspondence between the $spin^c$ structures on M and characteristic classes of elements of $H^2(M;\mathbb{Z})$ (i.e. their Poincare duals reduce mod2 to the second Stiefel-Whitney class ω_2 of M). Under this identification, each $spin^c$

structure s on M corresponds to a bundle of positive spinors W_s^+ on M. Hence one can view the Seiberg-Witten invariant as a function

$$SW_M : \{k \in H^2(M;\mathbb{Z}) | k \equiv \omega_2 (mod2)\} \to \mathbb{Z}$$

If $SW_M(\beta) \neq 0$ for $\beta \in H^2(M;\mathbb{Z})$ then β is said to be *basic class* of M. It is known that the set of basic classes of a 4-manifold is finite. If β is a basic class, the Seiberg-Witten invariant of $-\beta$ is given as follows:

$$SW_M(-\beta) = (-1)^{(e(M)+\sigma(M))/4} SW_M(\beta).$$

Hence, one can conclude that $-\beta$ is also a basic class of M. Now, we will state some useful theorems about Seiberg-Witten invariants.

Theorem 2.1.23. *[66] Let (M, ω) be a compact, oriented, closed, symplectic 4-manifold with $b_2^+ \geq 2$. Then $SW_M(c_1(X)) = \pm 1$ where $c_1(X)$ is the canonical class of the symplectic structure of M.*

Theorem 2.1.24. *(Connected Sum)(cf.[61]) Let M_i be a compact oriented smooth 4-manifold with $b_2^+(M_i) \geq 1$ for $i = 1, 2$. Then all Seiberg-Witten invariants of $M_1 \# M_2$ are zero.*

CHAPTER 3

CONSTRUCTIONS OF GENUS-3 LEFSCHETZ FIBRATIONS

3.1 Construction of a genus-3 Lefschetz fibration from Matsumoto's genus-2 Lefschetz fibration

In this section, we explicitly construct a positive factorization for a genus-3 Lefschetz fibration whose total space is diffeomophic to $\mathbb{T}^2 \times \mathbb{S}^2 \# 6\overline{\mathbb{CP}}^2$. The technique we used to produce the genus-3 Lefchetz fibration over $\mathbb{S}^2$ comes from the idea of the construction of the smallest hyperelliptic genus-3 Lefchetz fibration produced by Korkmaz and Baykur [14]. This technique is also used in [13] and [40].

Consider the genus-3 surface Σ_3^1 represented in Figure 3.1. The lifting of W_2 constructed by Hamada given in Section 2.1.4 and the embeddings of the genus-2 surfaces Σ_2^1 and Σ_2^1 in Σ_3^1 give rise to the following identities in Mod_3^1:

$$(t_{B_0}t_{B_1}t_{B_2}t_C)^2 = (t_C t_{B_0}t_{B_1}t_{B_2})^2 = t_{C'}, \tag{3.1}$$

$$(t_{B_0'}t_{B_1'}t_{B_2'}t_{C'})^2 = t_C t_\delta, \tag{3.2}$$

where the curves B_i, B_i', C and C' are as shown in Figure 3.1 and δ is a curve parallel to the boundary component of Σ_3^1. The first identity (3.1) comes from the commutativity of the Dehn twists along disjoint curves C and C' and the second identity (3.2) is obtained by capping off one boundary component in Hamada's lifting given in Section 2.1.4. Using the fact that t_C and $t_{C'}$ commute, we get the following relation in Mod_3^1:

$$t_{B_0}t_{B_1}t_{B_2}t_C t_{B_0}t_{B_1}t_{B_2}t_{B_0'}t_{B_1'}t_{B_2'}t_{C'}t_{B_0'}t_{B_1'}t_{B_2'}t_{C'}t_C t_C^{-1}t_{C'}^{-1} = t_\delta.$$

Finally, we get the following identity in Mod_3^1 consisting of the product of positive

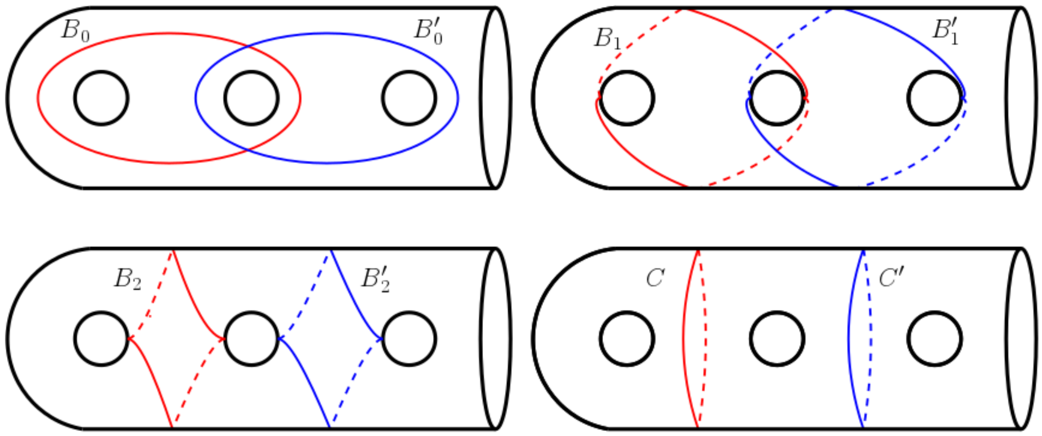

Figure 3.1: The curves B_i, B_i', C, C'.

Dehn twists along 12 nonseparating curves B_i, B_i' and 2 separating simple closed curves C and C':

$$t_{B_0}t_{B_1}t_{B_2}t_C t_{B_0}t_{B_1}t_{B_2}t_{B_0'}t_{B_1'}t_{B_2'}t_{C'}t_{B_0'}t_{B_1'}t_{B_2'} = t_\delta. \tag{3.3}$$

Let W be the positive factorization of t_δ in the equation 3.3 and let X denote the smooth 4-manifold admitting the genus-3 Lefschetz fibration over $\mathbb{S}^2$, with a section of self-intersection -1, whose global monodromy is W.

Theorem 3.1.1. *The* $4-$*manifold* X *is diffeomorphic to* $\mathbb{T}^2 \times \mathbb{S}^2 \# 6\overline{\mathbb{CP}}^2$.

Proof. We first compute the fundamental group $\pi_1(X)$. Since the Lefschetz fibration (X, f) with monodromy W has a section, by the theory of Lefschetz fibrations [38], $\pi_1(X)$ is isomorphic to the quotient of $\pi_1(\Sigma_3)$ by the normal subgroup generated by vanishing cycles of (X, f).

Using the generators a_i, b_i of $\pi_1(\Sigma_3)$ shown in the Figure 3.2, we get the following relations coming from the vanishing cycles:

$$B_0 \quad = \quad b_1 b_2 = 1, \tag{3.4}$$
$$B_1 \quad = \quad a_2^{-1}[a_3, b_3]b_2^{-1}b_1^{-1}a_1^{-1} = 1, \tag{3.5}$$
$$B_2 \quad = \quad a_2^{-1}[a_1, b_1^{-1}]a_1^{-1} = 1, \tag{3.6}$$
$$B_0' \quad = \quad b_2 b_3 = 1, \tag{3.7}$$
$$B_1' \quad = \quad a_3^{-1}b_3^{-1}b_2^{-1}a_2^{-1} = 1, \tag{3.8}$$

$$B'_2 = b_3 a_3^{-1} b_3^{-1} a_2^{-1} = 1, \tag{3.9}$$

$$C = [a_1, b_1] = 1, \tag{3.10}$$

$$C' = [a_3, b_3] = 1. \tag{3.11}$$

In addition, $\pi_1(X)$ has the relation

$$b_3^{-1} b_2^{-1} b_1^{-1} (a_1 b_1 a_1^{-1})(a_2 b_2 a_2^{-1})(a_3 b_3 a_3^{-1}) = 1. \tag{3.12}$$

Thus, $\pi_1(X)$ admits a presentation with generators $a_1, a_2, a_3, b_1, b_2, b_3$ and the relations $(3.4) - (3.12)$.

The relations (3.4) and (3.7) give $b_1 = b_2^{-1} = b_3$. From the relations (3.6), (3.7), (3.8) and (3.10), we obtain $a_1 = a_2^{-1} = a_3$. We conclude that $\pi_1(X)$ is a free abelian group of rank 2 generated by a_1 and b_1.

We next show that the signature $\sigma(X)$ of X is -6 using Endo and Nagami's method given in Section 2.1.5. (Alternatively, the signature of X can be calculated using Ozbagci's algorithm [57]).

Consider the relator $(B_0 B_1 B_2 C)^2 (B'_0 B'_1 B'_2 C')^2 C^{-1} C'^{-1}$ associated to the relation

$$W = t_{B_0} t_{B_1} t_{B_2} t_C t_{B_0} t_{B_1} t_{B_2} t_{B'_0} t_{B'_1} t_{B'_2} t_{C'} t_{B'_0} t_{B'_1} t_{B'_2} = 1$$

in Mod_3 obtained by the factorization of t_δ in Mod_3^1 by capping off the boundary component. It follows from Theorem 2.1.7, the additivity of the signature I_g and Lemma 2.1.6 that we have,

$$\begin{aligned} \sigma(X) &= I_g\big((B_0 B_1 B_2 C)^2 (B'_0 B'_1 B'_2 C')^2 C^{-1} (C')^{-1}\big), \\ &= I_g\big((B_0 B_1 B_2 C)^2\big) + I_g\big((B'_0 B'_1 B'_2 C')^2\big) - I_g(C) - I_g(C'), \\ &= -4 - 4 - (-1) - (-1) = -6. \end{aligned}$$

Other topological invariants of X we need are computed as follows:

$$\begin{aligned} e(X) &= e(\mathbb{S}^2) e(\Sigma_3) + \#\text{singular fibers} = 2(-4) + 14 = 6, \\ c_1^2(X) &= 2e(X) + 3\sigma(X) = -6. \end{aligned}$$

We will now prove that X is a ruled surface. Suppose that X is neither rational nor ruled. Let $\widetilde{X}$ be the minimal model of X, so $X \cong \widetilde{X} \# k\overline{\mathbb{CP}}^2$ for some non-negative

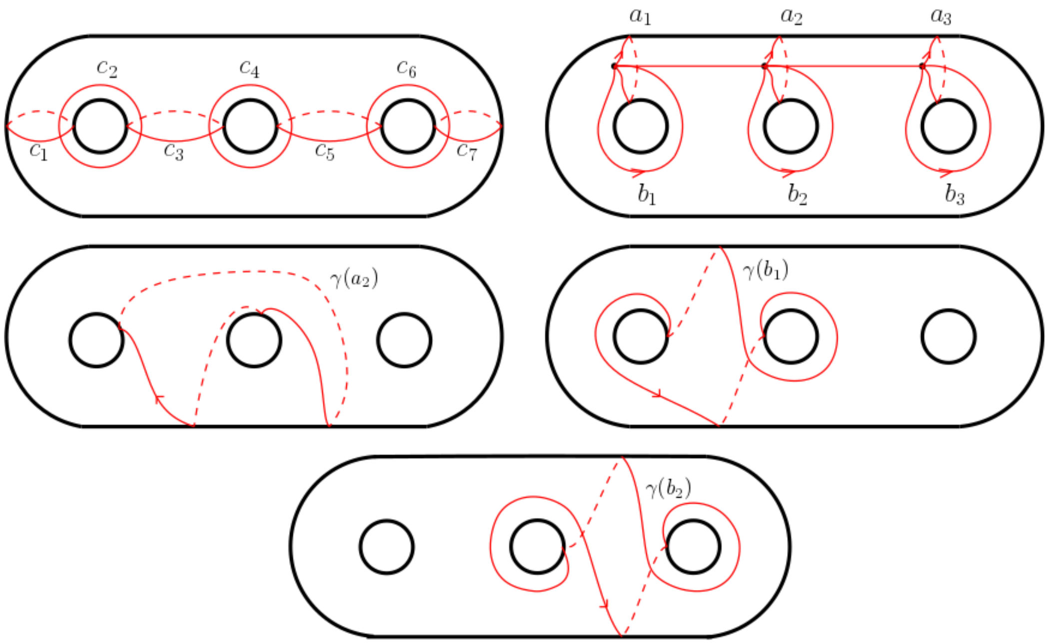

Figure 3.2: The curves c_i's, $\gamma(a_2), \gamma(b_1)$ and $\gamma(b_2)$ and the generators of $\pi_1(\Sigma_3)$.

integer k. It is easy to see that

$$c_1^2(\widetilde{X}) = c_1^2(X) + k = -6 + k.$$

Since $\widetilde{X}$ is a minimal symplectic 4-manifold that is neither rational nor ruled, $c_1^2(\widetilde{X}) \geq 0$ by Theorem 2.1.18, so that we have $k \geq 6$. Moreover, since X has $k \geq 6$ disjoint exceptional spheres, it follows from Theorem 2.1.19 that $k \leq 2g - 2 = 4$, which is a contradiction.

Therefore, X is either a rational or a ruled surface. Since $b_1(X) = 2$, we conclude that X is diffeomorphic to (a blow up of) a ruled surface with invariants $(b_2^+, b_2^-, b_1) = (1, 7, 2)$. So, X is diffeomorphic to $\mathbb{T}^2 \times \mathbb{S}^2 \# 6\overline{\mathbb{CP}}^2$. $\square$

3.2 Construction the genus-3 Lefschetz fibrations X_1, X_2 and X_3

Recall the factorization

$$W = t_{B_0} t_{B_1} t_{B_2} t_C t_{B_0} t_{B_1} t_{B_2} t_{B_0'} t_{B_1'} t_{B_2'} t_{C'} t_{B_0'} t_{B_1'} t_{B_2'}$$

of t_δ in the mapping class group of Σ_3^1. We may rewrite W as

$$W = V t_{B_2'}^2 t_{B_2}^2,$$

where

$$V = t_{t_{B_2}^{-2}(B_0)} t_{t_{B_2}^{-2}(B_1)} t_{t_{B_2}^{-1}(C)} t_{t_{B_2}^{-1}(B_0)} t_{t_{B_2}^{-1}(B_1)} t_{B_0'} t_{B_1'} t_{t'_{B_2}(C')} t_{t_{B_2'}(B_0')} t_{t_{B_2'}(B_1')}.$$

Let

$$\alpha = t_{c_4} t_a t_{B_2'} t_{c_4} t_{c_2} t_{c_1} t_{B_2} t_{c_2},$$

and

$$\beta = t_{b_1}^7 t_{c_4} t_b t_{B_2} t_{c_4} t_{c_6} t_{c_7} t_{B_2'} t_{c_6},$$

where the curves c_i are as in Figure 3.2, and the curves a and b are as in Figure 2.5 for $g = 3$. It is easy to see that $\alpha(B_2') = a$, $\alpha(B_2') = c_1$, $\beta(B_2') = b$ and $\beta(B_2) = c_7$.

The conjugations of W with α and β give the factorizations

$$t_\delta = W^\alpha = V^\alpha t_{\alpha(B_2')}^2 t_{\alpha(B_2)}^2 = V^\alpha t_a^2 t_{c_1}^2$$

and

$$t_\delta = W^\beta = V^\beta t_{\beta(B_2')}^2 t_{\beta(B_2)}^2 = V^\beta t_b^2 t_{c_7}^2 = t_b^2 t_{c_7}^2 V^\beta.$$

It follows that

$$t_\delta^2 = W^\alpha W^\beta = V^\alpha t_a^2 t_{c_1}^2 t_b^2 t_{c_7}^2 V^\beta = V^\beta V^\alpha t_a^2 t_b^2 t_{c_1}^2 t_{c_7}^2. \tag{3.13}$$

We see that the curves $\{c_1, c_1, a, b\}$ bound a sphere with four boundary components, which allows us to use the lantern substitution explained in Section 2.1.2.3. Using the lantern relation $t_a t_b t_{c_1}^2 = t_{c_3} t_C t_{B_2}$ we get the identity

$$V^\beta V^\alpha t_{c_3} t_C t_{B_2} t_a t_b t_{c_7}^2 = t_\delta^2 \tag{3.14}$$

in Mod_3^1. Moreover, the curves $\{c_7, c_7, a, b\}$ bound a sphere with four boundary components. By applying the lantern substitution $t_a t_b t_{c_7}^2 = t_{c_5} t_{C'} t_{B_2'}$, we get

$$V^\beta V^\alpha (t_{c_3} t_C t_{B_2})(t_{c_7} t_{C'} t_{B_2'}) = t_\delta^2. \tag{3.15}$$

For later use, up to conjugation and the inversion, we write the vanishing cycles of X_1, X_2 and X_3 in the fundamental group of $\pi_1(\Sigma_3)$ in the generating set $\{a_1, a_2, a_3, b_1, b_2, b_3\}$.

Let

$$\begin{array}{ll}
\mathrm{U}_1 = \alpha(t_{B_2}^{-2}(B_0)) & \mathrm{U}'_1 = \beta(t_{B_2}^{-2}(B_0)) \\
\mathrm{U}_2 = \alpha(t_{B_2}^{-2}(B_1)) & \mathrm{U}'_2 = \beta(t_{B_2}^{-2}(B_1)) \\
\mathrm{U}_3 = \alpha(t_{B_2}^{-1}(C)) & \mathrm{U}'_3 = \beta(t_{B_2}^{-1}(C)) \\
\mathrm{U}_4 = \alpha(t_{B_2}^{-1}(B_0)) & \mathrm{U}'_4 = \beta(t_{B_2}^{-1}(B_0)) \\
\mathrm{U}_5 = \alpha(t_{B_2}^{-1}(B_1)) & \mathrm{U}'_5 = \beta(t_{B_2}^{-1}(B_1)) \\
\mathrm{U}_6 = \alpha(B'_0) & \mathrm{U}'_6 = \beta(B'_0) \\
\mathrm{U}_7 = \alpha(B'_1) & \mathrm{U}'_7 = \beta(B'_1) \\
\mathrm{U}_8 = \alpha(t_{B'_2}(C')) & \mathrm{U}'_8 = \beta(t_{B'_2}(C')) \\
\mathrm{U}_9 = \alpha(t_{B'_2}(B'_0)) & \mathrm{U}'_9 = \beta(t_{B'_2}(B'_0)) \\
\mathrm{U}_{10} = \alpha(t_{B'_2}(B'_1)) & \mathrm{U}'_{10} = \beta(t_{B'_2}(B'_1))
\end{array}$$

so that (3.13), (3.14) and (3.15) are given, respectively, as

$$t_\delta^2 = t_{\mathrm{U}'_1} t_{\mathrm{U}'_2} \cdots t_{\mathrm{U}'_{10}} t_{\mathrm{U}_1} t_{\mathrm{U}_2} \cdots t_{\mathrm{U}_{10}} t_a^2 t_b^2 t_{c_1}^2 t_{c_7}^2. \tag{3.16}$$

$$t_\delta^2 = t_{\mathrm{U}'_1} t_{\mathrm{U}'_2} \cdots t_{\mathrm{U}'_{10}} t_{\mathrm{U}_1} t_{\mathrm{U}_2} \cdots t_{\mathrm{U}_{10}} (t_{c_3} t_C t_{B_2}) t_a t_b t_{c_7}^2. \tag{3.17}$$

$$t_\delta^2 = t_{\mathrm{U}'_1} t_{\mathrm{U}'_2} \cdots t_{\mathrm{U}'_{10}} t_{\mathrm{U}_1} t_{\mathrm{U}_2} \cdots t_{\mathrm{U}_{10}} (t_{c_3} t_C t_{B_2})(t_{c_5} t_{C'} t_{B'_2}). \tag{3.18}$$

Let $(X_1, f_1), (X_2, f_2)$ and (X_3, f_3) be the genus-3 Lefschetz fibrations with the monodromies (3.16), (3.17) and (3.18) respectively.

The vanishing cycles U_i are shown in Figure 3.3. One may find that

$$\begin{aligned}
\mathrm{U}_1 = {} & [b_1^{-1}, a_1] b_2 a_2^{-1} b_3 a_3^{-1} b_3^{-1} b_2^{-1} a_1 b_1^{-1} a_1^2 b_2 b_3 a_3 b_3^{-1} (a_2 b_2^{-1})^2 a_1 b_1^{-1} {a_1}^2 \\
& [b_1^{-1}, a_1] b_2 a_2^{-1} b_3 a_3^{-1} b_3^{-1} b_2^{-1} a_1,
\end{aligned} \tag{3.19}$$

$$\mathrm{U}_2 = [b_1^{-1}, a_1] b_2 a_2^{-1} b_3 a_3^{-1} b_3^{-1} b_2^{-1} a_1 b_1^{-1} a_1^2 [b_1^{-1}, a_1] a_2 b_2^{-1} a_1 b_1^{-1} a_1^2, \tag{3.20}$$

$$\mathrm{U}_3 = [b_1^{-1}, a_1] b_2 a_2^{-1} b_3 a_3^{-1} b_3^{-1} b_2^{-1} a_1^{-1} b_1 a_1^{-1} b_2 b_3 a_3 b_3^{-1} a_2 b_2^{-1} a_1 b_1^{-1} a_1, \tag{3.21}$$

$$\begin{aligned}
\mathrm{U}_4 = {} & [b_1^{-1}, a_1] b_2 a_2^{-1} b_3 a_3^{-1} b_3^{-1} b_2^{-1} a_1 b_1^{-1} a_1 b_2 b_3 a_3 b_3^{-1} (a_2 b_2^{-1})^2 a_1 b_1^{-1} a_1 \\
& [b_1^{-1}, a_1] b_2 a_2^{-1} b_3 a_3^{-1} b_3^{-1} b_2^{-1} a_1,
\end{aligned} \tag{3.22}$$

$$\mathrm{U}_5 = [b_1^{-1}, a_1] a_2 b_2^{-1} a_1 b_1^{-1} a_1 [b_1^{-1}, a_1] b_2 a_2^{-1} b_3 a_3^{-1} b_3^{-1} b_2^{-1} a_1 b_1^{-1} a_1, \tag{3.23}$$

$$\mathrm{U}_6 = b_2a_2^{-1}b_3a_3^{-1}b_3^{-1}b_2a_2^{-1}b_3^{-1}[a_1, b_1^{-1}]b_1a_1^{-1}, \tag{3.24}$$

$$\mathrm{U}_7 = b_2^2a_2^{-1}a_3^{-1}b_3^{-1}[a_1, b_1^{-1}]b_1a_1^{-1}, \tag{3.25}$$

$$\mathrm{U}_8 = b_3a_3b_3^{-1}a_2b_2a_2^{-1}a_3^{-1}a_2b_2^{-1}a_2^{-1}, \tag{3.26}$$

$$\mathrm{U}_9 = b_2a_2^{-1}b_3a_3^{-1}b_3^{-1}a_2b_2a_2^{-1}b_3^{-1}a_2[a_1, b_1^{-1}]b_1a_1^{-1}, \tag{3.27}$$

$$\mathrm{U}_{10} = [b_1^{-1}, a_1]a_2^{-1}b_3a_3a_2b_2^{-1}a_2^{-1}b_2^{-1}a_1b_1^{-1}. \tag{3.28}$$

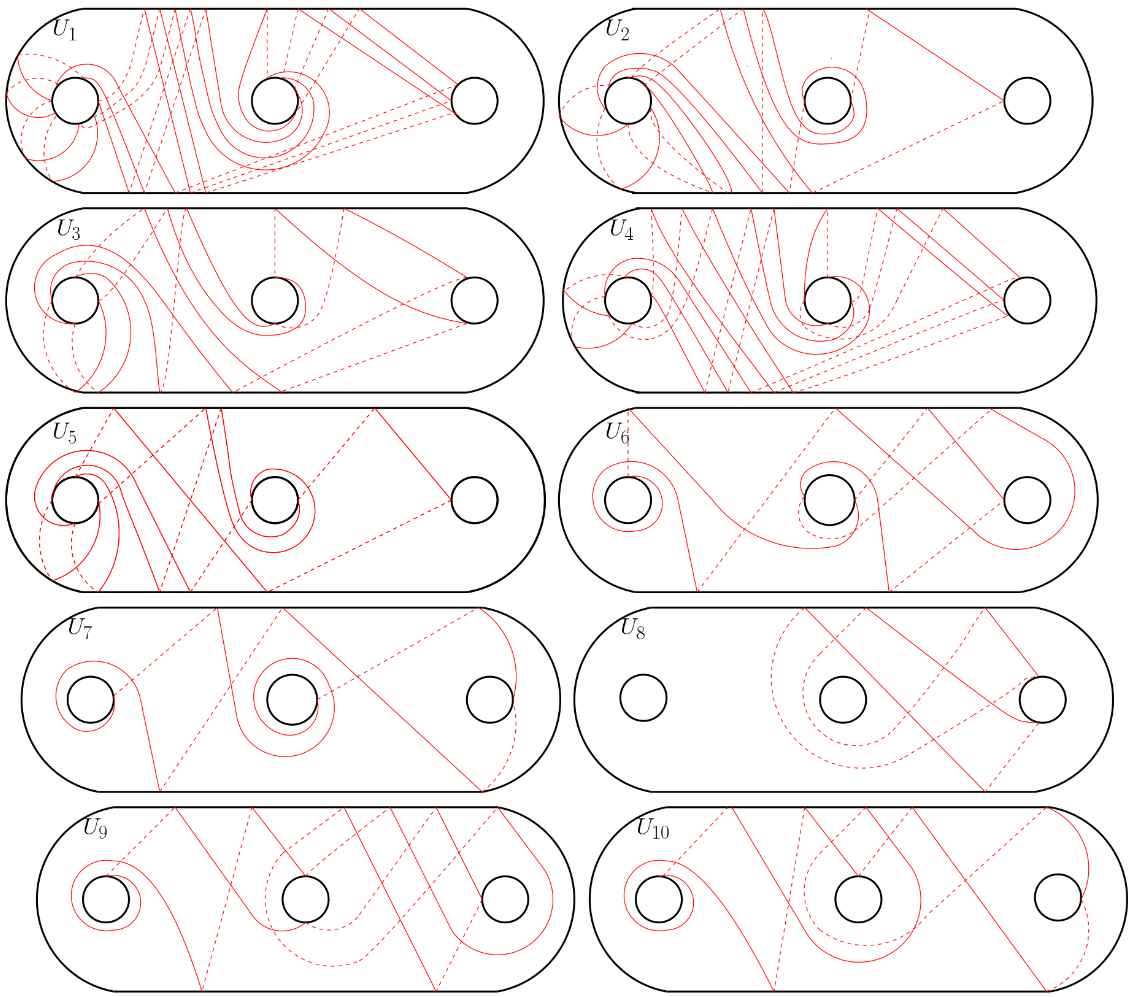

Figure 3.3: The curves U_i's,

We now prove that for each $i = 1, 2, 3$, the fundamental group $\pi_1(X_i)$ of the 4-manifolds X_i is trivial.

Lemma 3.2.1. *The* 4 *manifold* X_1 *is simply connected.*

Proof. The monodromy of (X_1, f_1) is given in (3.16). Since this Lefschetz fibration has a section, $\pi_1(X_1)$ has a presentation with generators $a_1, a_2, a_3, b_1, b_2, b_3$ and with the defining relations

$$b_3^{-1}b_2^{-1}b_1^{-1}(a_1b_1a_1^{-1})(a_2b_2a_2^{-1})(a_3b_3a_3^{-1}) = 1,$$
$$U_i' = U_i = a = b = c_1 = c_7 = 1, \ \ (i = 1, 2, \dots, 10).$$

Note that $a = a_2$, $c_1 = a_1$ and $c_7 = a_3$.

The relations $\mathrm{U}_1 = \mathrm{U}_2 = \mathrm{U}_4 = \mathrm{U}_5 = 1$, then gives

$$b_1^{-1}b_2^{-1}b_1^{-1} = 1. \tag{3.29}$$

Similarly, the relations $\mathrm{U}_6 = \mathrm{U}_7 = 1$ and $\mathrm{U}_9 = \mathrm{U}_{10} = 1$ yield

$$b_3 = b_1b_2b_2. \tag{3.30}$$

By the relation (3.29), we get

$$b_2 = b_1^{-2},$$

also, using the identity (3.30), we obtain

$$b_3 = b_1b_2^2 = b_1(b_1^{-2})^2.$$

Therefore b_2 and b_3 can be written in terms of b_1. It follows that $\pi_1(X_1)$ is abelian and is isomorphic to a quotient of the free abelian group $\mathbb{Z}$, generated by b_1. Therefore $\pi_1(X_1)$ is isomorphic to $H_1(X_1; \mathbb{Z})$.

Let us now determine the homology class of the vanishing cycle $\mathrm{U}_1' = \beta(t_{B_2}^{-2}(B_0))$. One can easily determine the effect of the Dehn twist t_a on the homology class of the curve b as to be:

$$[t_a^n(b)] = [b] + ni(a, b)[a] \in H_1(\Sigma_3, \mathbb{Z}); \tag{3.31}$$

where $i(a, b)$ is the algebraic intersection number of the oriented simple closed curves a and b.

In order to find the homology class of $t_{B_2^{-2}}(B_0)$, use $B_0 = b_1 + b_2$ and $B_2 = a_1 + a_2$

in $H_1(\Sigma_3;\mathbb{Z})$ and the formula (3.31). Hence, applying the Dehn twist $t_{B_2}^{-2}$ to the curve B_0, we get

$$\begin{aligned} t_{B_2^{-2}}(B_0) &= B_0 - 2\big(i(B_2,B_0)B_2\big) = B_0 - 4B_2 \\ &= -4a_1 - 4a_2 + b_1 + b_2. \end{aligned} \tag{3.32}$$

It is enough to determine the effect of the diffeomorphism β on the homology generators a_1, a_2, b_1 and b_2 of $H_1(\Sigma_3;\mathbb{Z})$ in order to find the homology class of U_1'.

Let $\beta = t_{b_1}^7\gamma$ where $\gamma = t_{c_4}t_bt_{B_2}t_{c_4}t_{c_6}t_{c_7}t_{B_2'}t_{c_6}$. Using $\gamma(a_1) = a_1$ and the curves $\gamma(a_2), \gamma(b_1)$ and $\gamma(b_2)$ in Figure 3.2, we obtain

$$\gamma(a_1) = a_1, \tag{3.33}$$

$$\gamma(a_2) = -a_1 - a_2, \tag{3.34}$$

$$\gamma(b_1) = a_1 + a_2 + b_1 - b_2, \tag{3.35}$$

$$\gamma(b_2) = a_2 + a_3 - b_2 - b_3. \tag{3.36}$$

It follows from formula (3.31) that

$$\beta(a_1) = a_1 - 7b_1, \tag{3.37}$$

$$\beta(a_2) = -a_1 - a_2 + 7b_1, \tag{3.38}$$

$$\beta(b_1) = a_1 + a_2 - 6b_1 - b_2, \tag{3.39}$$

$$\beta(b_2) = a_2 + a_3 - b_2 - b_3. \tag{3.40}$$

Therefore the identity (3.32) gives rise to

$$\mathrm{U}_1' = \beta(t_{B_2^{-2}}(B_0)) = \beta(-4a_1 - 4a_2 + b_1 + b_2)$$

and using the identities (3.37)-(3.40)

$$\mathrm{U}_1' = a_1 + 6a_2 + a_3 - 6b_1 - 2b_2 - b_3. \tag{3.41}$$

Combining the identities $a_1 = a_2 = a_3 = 0$, (3.41), (3.29) and (3.30), we have the following relations in $H_1(X_1;\mathbb{Z})$:

$$-6b_1 - 2b_2 - b_3 = 0 \tag{3.42}$$

$$2b_1 + b_2 = 0 \tag{3.43}$$

$$b_1 + 2b_2 - b_3 = 0 \tag{3.44}$$

This completes our claim that $H_1(X_1;\mathbb{Z}) = 0$ since the above relations (3.42)-(3.44) in $H_1(X_1;\mathbb{Z})$ imply that $b_1 = b_2 = b_3 = 0$. $\square$

Lemma 3.2.2. *The* 4*-manifold* X_2 *is simply connected.*

Proof. The monodromy of (X_2, f_2) is (3.17). Since this Lefschetz fibration has a section, $\pi_1(X_2)$ has a presentation with generators $a_1, a_2, a_3, b_1, b_2, b_3$ and with defining relations

$$b_3^{-1}b_2^{-1}b_1^{-1}(a_1b_1a_1^{-1})(a_2b_2a_2^{-1})(a_3b_3a_3^{-1}) = 1,$$

$$U_i' = U_i = c_3 = C = B_2 = a = b = c_7 = 1, \;\; i = 1, 2, \ldots, 10.$$

Note that $a = a_2$, $c_7 = a_3$ and

$$c_3 = a_1a_2^{-1}, \tag{3.45}$$

$$C = [a_1, b_1], \tag{3.46}$$

$$B_2 = [b_1^{-1}, a_1]a_2a_1. \tag{3.47}$$

It follows from identity (3.45) and the relation $a = a_2 = 1$ in $\pi_1(X_2)$ that $a_1 = 1$. Therefore, $\pi_1(X_2)$ has the same relations as $\pi_1(X_1)$. We conclude that $\pi_1(X_2) = 1$ by the proof of Lemma 3.2.1. $\square$

Lemma 3.2.3. *The* 4*-manifold* X_3 *is simply connected.*

Proof. The monodromy of (X_3, f_3) is (3.18). Since this Lefschetz fibration has a section, $\pi_1(X_3)$ has a presentation with generators $a_1, a_2, a_3, b_1, b_2, b_3$ and with the defining relations

$$b_3^{-1}b_2^{-1}b_1^{-1}(a_1b_1a_1^{-1})(a_2b_2a_2^{-1})(a_3b_3a_3^{-1}) = 1,$$

$$U_i' = U_i = c_3 = C = B_2 = c_5 = C' = B_2' = 1, \;\; i = 1, 2, \ldots, 10.$$

In addition to the identities (3.45), (3.46) and (3.47), we have

$$c_5 = a_2a_3^{-1}, \tag{3.48}$$

$$C' = [a_3, b_3], \tag{3.49}$$

$$B_2' = [b_3, a_3]a_3a_2. \tag{3.50}$$

Using the relations

$$b_3^{-1}b_2^{-1}b_1^{-1}(a_1b_1a_1^{-1})(a_2b_2a_2^{-1})(a_3b_3a_3^{-1}) = 1,$$

$$C = [a_1, b_1] = 1, C' = [a_3, b_3] = 1,$$

then $[a_2, b_2] = 1$ holds in $\pi_1(X_3)$. By the identities (3.45) and (3.48), we have

$$a_1 = a_2 = a_3. \tag{3.51}$$

Also, by the relations (3.47) and (3.50) we get the equations

$$a_2a_1 = a_3a_2 = 1, \tag{3.52}$$

It follows from (3.51), (3.52), $[a_1, b_1] = [a_2, b_2] = [a_3, b_3] = 1$ and the presentations (3.27) and (3.28) of U_9 and U_{10}, respectively that $\mathrm{U}_9 = \mathrm{U}_{10} = 1$ in $\pi_1(X_3)$ give

$$\mathrm{U}_9 = b_2^2b_3^{-1}b_1 = 1 \tag{3.53}$$

$$\mathrm{U}_{10} = a_2^{-1}b_3b_2^{-2}b_1^{-1} = 1. \tag{3.54}$$

The equations (3.53) and (3.54) yield $a_2 = 1$. Hence $a_1 = a_2 = a_3 = 1$ and $\pi_1(X_3)$ has the same relations as $\pi_1(X_1)$. This implies that $\pi_1(X_3) = 1$ by the proof of Lemma 3.2.1. □

Theorem 3.2.4. *For $i = 1, 2, 3$ the genus-3 Lefschetz fibration $f_i : X_i \to \mathbb{S}^2$ is minimal and has*

(i) $e(X_i) = 21 - i$,

(ii) $c_1^2(X_i) = 3 + i$,

(iii) $\pi_1(X_i) = 1$.

Proof. For each $i = 1, 2, 3$ the Euler characteristic $e(X_i)$ of X_i is given by

$$\begin{aligned} e(X_i) &= e(\mathbb{S}^2)e(\Sigma_3) + \#\text{singular fibers} \\ &= 2(-4) + 29 - i = 21 - i, \end{aligned}$$

and the signature $\sigma(X_i)$ is

$$\sigma(X_i) = \sigma(X_1) + \sigma(X_1) + i - 1 = -13 + i,$$

by using Novikov additivity and the fact that Lantern substitution increases the signature by 1 (Lemma2.1.6 ii). The topological invariant $c_1^2(X_i)$ is as follows:

$$c_1^2(X_i) = 2e(X_i) + 3\sigma(X_i) = 3 + i.$$

By Proposition 2.1.12, each X_i is minimal. Also, one can explain the minimality of X_i by considering a lantern substitution as a rational blowdown surgery along a -4 sphere [32]. The rational blowdown surgery along a -4 sphere can be obtained by the symplectic sum operation. So, $X_2 = X_1 \#_{S,V_{\mathbb{CP}^2}} \mathbb{CP}^2$ where S is symplectic -4 sphere in X_1 and $V_{\mathbb{CP}^2}$ which is an embedded $+4$ sphere in $\mathbb{CP}^2$ in class of $[V_{\mathbb{CP}^2}] = 2[H] \in H_2(\mathbb{CP}^2, \mathbb{Z})$. Since X_1 is minimal by Theorem 2.1.11 (iii) then again it follows from the Theorem 2.1.11 X_2 is minimal. Similarly, since X_3 can be viewed as a symplectic sum of minimal X_2 and $\mathbb{CP}^2$, we get X_3 is minimal using the same argument.

Finally, for $i = 1, 2, 3$, the fundamental group $\pi_1(X_i)$ of each X_i is trivial by Lemma 3.2.1, Lemma 3.2.2 and Lemma 3.2.3.

□

CHAPTER 4

CONSTRUCTION OF EXOTIC 4-MANIFOLDS

4.1 Exotic fibered 4-manifolds with $b_2^+ = 3$

In this section, we will construct minimal symplectic 4-manifolds admitting Lefschetz fibration of genus 3 over $\mathbb{S}^2$ which are homeomorphic but not diffeomorphic to $3\mathbb{CP}^2\#k\overline{\mathbb{CP}}^2$ for $k = 13, \ldots, 19$ using the Lefschetz fibration prescribed by the factorization W and generalized the Matsumoto Lefschetz fibration for genus 3.

Consider the generalized Matsumoto's Lefschetz fibration for genus 3, the total space M_3 corresponding to the word $W_3 = (t_{\beta_0} t_{\beta_1} t_{\beta_2} t_{\beta_3} t_a^2 t_b^2)^2$ is diffeomorphic to $\mathbb{T}^2 \times \mathbb{S}^2\#8\overline{\mathbb{CP}}^2$. Here we denote the vanishing cycles of M_3 by β_i's instead of B_i's as shown in Figure 2.5 to distinguish them from the vanishing cycles of X. To determine the relations in $\pi_1(M_3)$, consider the following identification of the fundamental group of M_3 using the existence of sections of Matsumoto's fibrations :

$$\pi_1(M_3) = \pi_1(\Sigma_3)/\langle \beta_0, \beta_1, \beta_2, \beta_3, a, b \rangle,$$

$$\begin{aligned}
\beta_0 &= b_1 b_2 b_3 = 1, && (4.1)\\
\beta_1 &= b_1 b_2 b_3 a_3 a_1 = 1, && (4.2)\\
\beta_2 &= b_2 b_3 a_3 b_3^{-1} a_1 = 1, && (4.3)\\
\beta_3 &= a_2 b_2 [b_3, a_3] a_2 = 1, && (4.4)\\
a &= a_2 = 1, && (4.5)\\
b &= [a_1, b_1^{-1}] a_2^{-1} = 1. && (4.6)
\end{aligned}$$

The equations (4.1) and (4.2) yield the relation $a_3a_1 = 1$. Since

$$b_3^{-1}b_2^{-1}b_1^{-1}(a_1b_1a_1^{-1})(a_2b_2a_2^{-1})(a_3b_3a_3^{-1}) = 1,$$

$a_2 = 1$ and the relation (4.6) give $[b_3, a_3] = 1$. Using these relations and the relation (4.4) , we get $b_2 = 1$. We have the relations

$$a_3a_1 = 1 \text{ and } b_1b_3 = 1$$

in $\pi_1(M_3)$. Therefore $\pi_1(M_3)$ is the free abelian group of rank 2.

We first present the minimal symplectic genus-3 Lefschetz fibrations with $(b_2^+, b_2^-) = (3, 19)$ and $(3, 18)$. To obtain such Lefschetz fibrations, consider the following positive factorization:

$$W_3W_3^{\phi} = (t_{\beta_0}t_{\beta_1}t_{\beta_2}t_{\beta_3}t_a^2t_b^2)^2(t_{\phi(\beta_0)}t_{\phi(\beta_1)}t_{\phi(\beta_2)}t_{\phi(\beta_3)}t_{\phi(a)}^2t_{\phi(b)}^2)^2 = t_\delta^2, \tag{4.7}$$

where the diffeomorphism $\phi = t_{b_3^{-1}}t_{\beta_0}t_{a_1}$.

Let us rewrite the positive factorization W_3. In [43], it is shown that the product of positive Dehn twists $t_{\beta_0}t_{\beta_1}t_{\beta_2}t_{\beta_3}t_a^2t_b^2$ is the vertical involution ι of the genus-3 surface with two fixed points. Hence, it preserves the curve β_0, then we have

$$t_{\beta_0}t_{\beta_1}t_{\beta_2}t_{\beta_3}t_a^2t_b^2(\beta_0) = \beta_0.$$

By applying Lemma 2.1.2, we get the following identity of the factorization W_3:

$$\begin{aligned} W_3 &= t_{\beta_0}t_{\beta_1}t_{\beta_2}t_{\beta_3}t_a^2t_b^2t_{\beta_0}t_{\beta_1}t_{\beta_2}t_{\beta_3}t_a^2t_b^2 \\ &= t_{\beta_0}t_{\beta_0}t_{\beta_1}t_{\beta_2}t_{\beta_3}t_a^2t_b^2t_{\beta_1}t_{\beta_2}t_{\beta_3}t_a^2t_b^2 \\ &= t_{\beta_0}^2(t_{\beta_1}t_{\beta_2}t_{\beta_3}t_a^2t_b^2)^2 = t_\delta. \end{aligned}$$

It is easy to see that $\phi(\beta_0) = a_1$. It follows that

$$W_3W_3^{\phi} = T_1t_at_bt_{a_1}^2T_2 = t_\delta^2, \tag{4.8}$$

where

$$T_1 = t_{\beta_0}t_{\beta_1}t_{\beta_2}t_{\beta_3}t_a^2t_b^2t_{\beta_0}t_{\beta_1}t_{\beta_2}t_{\beta_3}t_at_b$$

and

$$T_2 = (t_{\phi(\beta_1)}t_{\phi(\beta_2)}t_{\phi(\beta_3)}t_{\phi(a)}^2t_{\phi(b)}^2)^2.$$

Since the curves $\{a, b, c_1, c_1\}$ bound a sphere with four holes, we can use the lantern relation $t_a t_b t_{c_1}^2 = t_{c_3} t_C t_{B_2}$ to get the identity

$$W_3 W_3^{\phi} = T_1 t_{c_3} t_C t_{B_2} T_2 = t_{\delta}^2. \tag{4.9}$$

Let M_{19} and M_{18} be the genus-3 Lefschetz fibrations with the monodromies (4.7) and (4.9), respectively. We now prove that $\pi_1(M_{19}) = \pi_1(M_{18}) = 1$.

Lemma 4.1.1. *The fundamental group $\pi_1(M_{19})$ of M_{19} is trivial.*

Proof. Since the Lefschetz fibration M_{19} has a section, $\pi_1(M_{19})$ has a presentation with the generators a_j and b_j, $(j = 1, 2, 3)$ and with the relations

$$b_3^{-1} b_2^{-1} b_1^{-1} (a_1 b_1 a_1^{-1})(a_2 b_2 a_2^{-1})(a_3 b_3 a_3^{-1}) = 1,$$

$$\beta_i = a = b = \phi(\beta_i) = \phi(a) = \phi(b) = 1, \ \ (i = 0, 1, 2, 3).$$

Since the monodromy of (M_{19}, f_{19}) contains the factorization W_3, the relations coming from the factorization W_3 make $\pi_1(M_{19})$ a quotient of the free abelian group of rank 2. So, we can find additional relations coming from the conjugated W_3^{ϕ} using the action of ϕ on the generators of first homology group $H_1(\Sigma_3; \mathbb{Z})$. Using the relations (4.1)-(4.6) in $\pi_1(M_3)$, in addition to the relations $a_2 = b_2 = 0$, the following abelianized relations hold in $H_1(M_{19}; \mathbb{Z})$:

$$a_1 + a_3 \ \ = 0, \tag{4.10}$$

$$b_1 + b_3 \ \ = 0. \tag{4.11}$$

as can be seen in Figure 4.1 the effect of the diffeomorphism ϕ on the generators a_1 and a_3 of $H_1(\Sigma_3; \mathbb{Z})$ are as follows :

$$\phi(a_1) = a_1 - b_1 - b_2 - b_3, \tag{4.12}$$

$$\phi(a_3) = a_3 - b_1 - b_2 - 2b_3, \tag{4.13}$$

From the fact that $\phi(\beta_1)$ and $\phi(\beta_0) = a_1$ are the vanishing cycles in the Lefschetz fibration, and using the identities (4.2) and (4.3), we get the relation

$$\phi(\beta_1) = \phi(\beta_0 + a_1 + a_3) = \phi(a_1 + a_3) = 0.$$

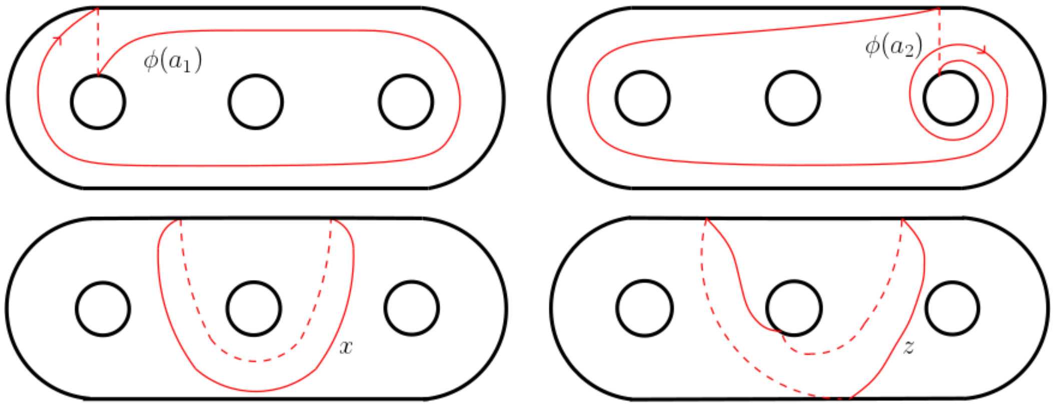

Figure 4.1: The curves $\phi(a_1)$, $\phi(a_2)$, x and z .

Hence, by the identities (4.12) and (4.13), the relation

$$\phi(a_1) + \phi(a_3) = a_1 + a_3 - 2b_1 - 2b_2 - 3b_3 = 0$$

holds in $H_1(\Sigma_3; \mathbb{Z})$, which implies that $b_3 = 0$ by the relations (4.10) and (4.11). Also, using $\phi(\beta_0) = a_1 = 0$ and the relations (4.10) and (4.11), we conclude that $H_1(\Sigma_3; \mathbb{Z}) = 0$. This proves that the 4-manifold M_{19} is simply connected. □

Lemma 4.1.2. *The fundamental group $\pi_1(M_{18})$ of M_{18} is trivial.*

Proof. The monodromy of the Lefschetz fibration M_{18} is given as (4.9). Hence $\pi_1(M_{18})$ has a presentation with generators a_j and b_j, $(j = 1, 2, 3)$ and, for each $i = 0, 1, 2, 3$ and $k = 1, 2, 3$, the relations

$$b_3^{-1} b_2^{-1} b_1^{-1} (a_1 b_1 a_1^{-1})(a_2 b_2 a_2^{-1})(a_3 b_3 a_3^{-1}) = 1,$$
$$\beta_i = a = b = c_3 = C = B_2 = \phi(\beta_k) = \phi(a) = \phi(b) = 1.$$

The relations coming from the factorization W_3 hold in $\pi_1(M_{18})$. Since c_3 is a vanishing cycle of M_{18} , the relation $c_3 = a_1 a_2^{-1} = 1$ is satisfied in $\pi_1(M_{18})$. It follows from the relation $a = a_2 = 1$ that $a_1 = 1$. Moreover the relation $\phi(\beta_1) = 1$, $\pi_1(M_{18})$ has the same presentation as $\pi_1(M_{19})$. By the proof of Lemma 4.1.1, $\pi_1(M_{18}) = 1$. □

Theorem 4.1.3. *The* 4*-manifolds* M_{18} *and* M_{19} *are exotic copies of the manifolds* $3\mathbb{CP}^2 \# 18\overline{\mathbb{CP}}^2$ *and* $3\mathbb{CP}^2 \# 19\overline{\mathbb{CP}}^2$*, respectively.*

Proof. The 4-manifolds M_{18} and M_{19} have the following topological invariants:

The Euler characteristics of M_{19} and M_{18} are given by

$$\begin{aligned} e(M_{19}) &= e(\mathbb{S}^2)e(\Sigma_3) + \#\text{singular fibers} \\ &= 2(-4) + 32 = 24, \\ e(M_{18}) &= e(\mathbb{S}^2)e(\Sigma_3) + \#\text{singular fibers} \\ &= 2(-4) + 31 = 23, \end{aligned}$$

and the signatures of M_{19} and M_{18} are given by

$$\begin{aligned} \sigma(M_{19}) &= \sigma(M_3) + \sigma(M_3) = -16, \\ \sigma(M_{18}) &= \sigma(M_{19}) + 1 = -15, \end{aligned}$$

using Novikov additivity and the fact that lantern substitution increases the signature by 1 (Lemma2.1.6 ii).

By Lemma 4.1.1 and 4.1.2, M_{19} and M_{18} are simply connected. Hence the identities

$$\begin{aligned} e(M_{19}) &= 24 = 2 - 2b_1(M_{19}) + b_2(M_{19}) \\ &= 2 + b_2^+(M_{19}) + b_2^-(M_{19}), \end{aligned}$$

and

$$\sigma(M_{19}) = -16 = b_2^+(M_{19}) - b_2^-(M_{19})$$

imply that $(b_2^+(M_{19}), b_2^-(M_{19})) = (3, 19)$. Using Theorem 2.1.21 , we conclude that M_{19} and M_{18} are homeomorphic to $3\mathbb{CP}^2\#19\overline{\mathbb{CP}}^2$ and $3\mathbb{CP}^2\#18\overline{\mathbb{CP}}^2$, respectively. It follows from Theorem 2.1.11 that M_{18} and M_{19} are minimal by similar arguments in the proof of Theorem 3.2.4 i.e., it cannot contain a smoothly embedded -1 sphere. But the manifolds $3\mathbb{CP}^2\#19\overline{\mathbb{CP}}^2$ and $3\mathbb{CP}^2\#18\overline{\mathbb{CP}}^2$ contain smoothly embedded -1 spheres, the exceptional spheres. Hence M_{19} and M_{18} cannot be diffeomorphic to $3\mathbb{CP}^2\#19\overline{\mathbb{CP}}^2$ and $3\mathbb{CP}^2\#18\overline{\mathbb{CP}}^2$, respectively. □

We now present the minimal symplectic genus-3 Lefschetz fibrations with $(b_2^+, b_2^-) = (3, 17)$ and $(3, 16)$. To obtain such Lefschetz fibrations, consider the following identity of the factorization W in (3.3):

$$W = T_3 t_C t_{C'} = t_\delta,$$

where $T_3 = t_{t_C^{-1}(B_0)} t_{t_C^{-1}(B_1)} t_{t_C^{-1}(B_2)} t_{B_0} t_{B_1} t_{B_2} t_{B_0'} t_{B_1'} t_{B_2'} t_{t_{C'}(B_0')} t_{t_{C'}(B_1')} t_{t_C'(B_2')}$.

Recall that the factorization

$$W_3 = T_1 t_a t_b = t_\delta,$$

where $T_1 = t_{\beta_0} t_{\beta_1} t_{\beta_2} t_{\beta_3} t_a^2 t_b^2 t_{\beta_0} t_{\beta_1} t_{\beta_2} t_{\beta_3} t_a t_b$.

Thus we have

$$W_3 W = T_1 t_a t_b t_C t_{C'} T_3 = t_\delta^2. \tag{4.14}$$

Since the curves $\{a, b, C, C'\}$ bound a sphere with four holes, we have the identity $t_a^2 t_C t_{C'} = t_x t_b t_z$. Therefore we get the following factorization

$$T_1 t_x t_b t_z T_3 = t_\delta^2, \tag{4.15}$$

where the Dehn twist curves x and z are depicted in Figure 4.1. Let M_{17} and M_{16} be the genus-3 Lefschetz fibration with the monodromy (4.14) and (4.15), respectively.

Lemma 4.1.4. *The fundamental group of M_{17} is trivial.*

Proof. The monodromy of the Lefschetz fibration M_{17} is (4.14). Hence, $\pi_1(M_{17})$ has a presentation with the generators a_j and b_j, $(j = 1, 2, 3)$ and for each $i = 0, 1, 2, 3$ and $k = 0, 1, 2$ with the relations

$$b_3^{-1} b_2^{-1} b_1^{-1} (a_1 b_1 a_1^{-1})(a_2 b_2 a_2^{-1})(a_3 b_3 a_3^{-1}) = 1,$$
$$\beta_i = a = b = B_k = B'_k = C = C' = 1.$$

Therefore the relations (4.1)-(4.6) and (3.4)-(3.11) hold in $\pi_1(M_{17})$. These relations immediately imply that all generators are trivial in $\pi_1(M_{17})$. □

Lemma 4.1.5. *The fundamental group $\pi_1(M_{16})$ of M_{16} is trivial.*

Proof. The monodromy of the Lefschetz fibration (M_{16}, f_{16}) is (4.15). Hence, $\pi_1(M_{16})$ has a presentation with the generators a_j and b_j, $(j = 1, 2, 3)$ and for each $i = 0, 1, 2, 3$ and $k = 0, 1, 2$ with the relations

$$b_3^{-1} b_2^{-1} b_1^{-1} (a_1 b_1 a_1^{-1})(a_2 b_2 a_2^{-1})(a_3 b_3 a_3^{-1}) = 1,$$
$$\beta_i = a = b = B_k = B'_k = C = C' = 1.$$

Thus, the relations (4.1)-(4.6) and (3.4)-(3.9) hold in $\pi_1(M_{16})$ which gives rise to $\pi_1(M_{16}) = 1$. □

Theorem 4.1.6. *The 4-manifolds M_{17} and M_{16} are exotic copies of the manifolds $3\mathbb{CP}^2\#17\overline{\mathbb{CP}}^2$ and $3\mathbb{CP}^2\#16\overline{\mathbb{CP}}^2$, respectively.*

Proof. The manifolds M_{17} and M_{16} have the following topological invariants:

The Euler characteristic $e(M_{17})$ of M_{17} and $e(M_{16})$ of M_{16} are

$$\begin{aligned} e(M_{17}) &= e(\mathbb{S}^2)e(\Sigma_3) + \#\text{singular fibers} \\ &= 2(-4) + 30 = 22, \\ e(M_{16}) &= e(\mathbb{S}^2)e(\Sigma_3) + \#\text{singular fibers} \\ &= 2(-4) + 29 = 21 \end{aligned}$$

and the signature $\sigma(M_{19})$ of M_{19} and $\sigma(M_{18})$ of M_{18} are

$$\begin{aligned} \sigma(M_{17}) &= \sigma(M_3) + \sigma(X) = -14, \\ \sigma(M_{16}) &= \sigma(M_{17}) + 1 = -13, \end{aligned}$$

using the Novikov additivity and Lemma 2.1.6 ii.

It follows from Lemma 4.1.4 and Lemma 4.1.5 that M_{17} and M_{16} are simply connected. Thus, the identities

$$\begin{aligned} e(M_{17}) &= 22 = 2 - 2b_1(M_{17}) + b_2(M_{17}) \\ &= 2 + b_2^+(M_{17}) + b_2^-(M_{17}), \end{aligned}$$

and

$$\sigma(M_{17}) = -14 = b_2^+(M_{17}) - b_2^-(M_{17})$$

imply that $(b_2^+(M_{17}), b_2^-(M_{17})) = (3, 17)$. Similarly, we obtain that $(b_2^+(M_{16}), b_2^-(M_{16})) = (3, 16)$. Using Theorem 2.1.21 , we see that M_{17} and M_{16} are homeomorphic to $3\mathbb{CP}^2\#17\overline{\mathbb{CP}}^2$ and $3\mathbb{CP}^2\#16\overline{\mathbb{CP}}^2$, respectively. It is shown that the manifolds M_{17} and M_{16} are minimal in a similar way in the proof of Theorem 4.1.3. Therefore, M_{17} and M_{16} cannot be diffeomorphic to $3\mathbb{CP}^2\#17\overline{\mathbb{CP}}^2$ and $3\mathbb{CP}^2\#16\overline{\mathbb{CP}}^2$, respectively. □

Now, let us consider the minimal genus-3 Lefschetz fibrations (X_i, f_i) $(i = 1, 2, 3)$ constructed in Subsection 3.2.

Theorem 4.1.7. *The* 4*-manifolds* X_1*,* X_2*, and* X_3 *are exotic copies of* $3\mathbb{CP}^2\#15\overline{\mathbb{CP}}^2$*,* $3\mathbb{CP}^2\#14\overline{\mathbb{CP}}^2$ *and* $3\mathbb{CP}^2\#13\overline{\mathbb{CP}}^2$*, respectively.*

Proof. For each $i = 1, 2, 3$, the manifold X_i is minimal, simply connected and has the following invariants:

$$\begin{aligned} e(X_i) &= = 21 - i = 2 - 2b_1(X_i) + b_2(X_i) \\ &= 2 + b_2^+(X_i) + b_2^-(X_i), \end{aligned}$$

and the signature $\sigma(X_i)$ is

$$\sigma(X_i) = -13 + i = b_2^+(X_i) + b_2^-(X_i).$$

Therefore, $(b_2^+(X_i), b_2^-(X_i)) = (3, 16 - i)$. It follows from Theorem 2.1.21 , we see that X_i is homeomorphic to $3\mathbb{CP}^2\#(16-i)\overline{\mathbb{CP}}^2$. Since each X_i is minimal, X_i can not be diffeomorphic to $3\mathbb{CP}^2\#(16-i)\overline{\mathbb{CP}}^2$. This finishes our proof. □

4.2 Constructions of genus-$3k$ Lefschetz fibrations and some exotic 4-manifolds

In this section, we generalize our construction of genus-3 Lefschetz fibration over $\mathbb{S}^2$ to the construction of genus-$3k$ Lefschetz fibration over $\mathbb{S}^2$ with total space is diffeomorphic to $\Sigma_k \times \mathbb{S}^2\#6\overline{\mathbb{CP}}^2$ using generalized Matsumoto's genus-$2k$ fibration. Moreover, we give some fibered and nonfibered examples of exotic structures using our generalized construction via twisted fiber sum or Luttinger surgery.

4.2.0.1 Construction genus-$3k$ Lefschetz fibrations from generalized Matsomoto's genus-$2k$ fibrations

We have the following two identities in Mod_{2k}^2 using the liftings of generalized Matsumoto's fibration for even g given by Hamada as explained in 2.1.4:

$$\begin{aligned} (t_{B_0}t_{B_1}t_{B_2}\ldots t_{B_{2k}}t_C)^2 &= (t_Ct_{B_0}t_{B_1}t_{B_2}\cdots t_{B_{2k}})^2 = t_{C'} \\ (t_{B_0'}t_{B_1'}t_{B_2'}\cdots t_{B_{2k}'}t_{C'})^2 &= t_Ct_\delta, \end{aligned}$$

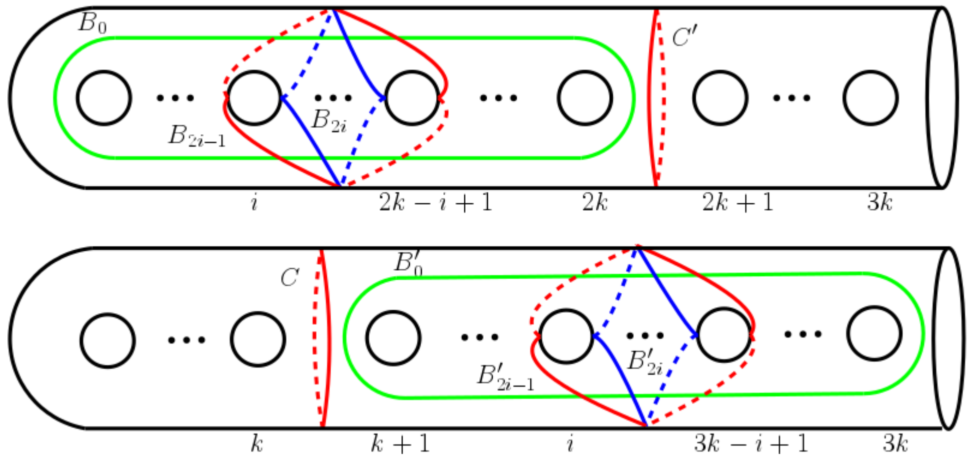

Figure 4.2: The curves for the monodromy W_k

Here the curves B_i, B'_i, C and C' are as shown in Figure 4.2 and the curve δ is the curve that is parallel to the boundary component of Σ^1_{3k}. The first identity holds by the commutativity of the separating curves C and C' and the lifting of the factorization to Mod^1_{2k}, which is easily obtained by capping off the boundary component δ_1. Embedding these curves into Σ^1_{3k}, and again using the fact that t_C and $t_{C'}$ commute, we obtain the following relation in Mod^1_{3k}.

$$t_{B_0}t_{B_1}\cdots t_{B_{2k}}t_Ct_{B_0}t_{B_1}\cdots t_{B_{2k}}t_{B'_0}t_{B'_1}\cdots t_{B'_{2k}}t_{C'}t_{B'_0}t_{B'_1}\cdots t_{B'_{2k}}t_{C'}t_Ct_C^{-1}t_{C'}^{-1}$$
$$= t_{B_0}t_{B_1}\cdots t_{B_{2k}}t_Ct_{B_0}t_{B_1}\cdots t_{B_{2k}}t_{B'_0}t_{B'_1}\cdots t_{B'_{2k}}t_{C'}t_{B'_0}t_{B'_1}\cdots t_{B'_{2k}} = t_\delta.$$

Let us denote the above relation in Mod^1_{3k} by W_k . Note that W_k is a product of $8k+6$ positive Dehn twists, two of which are about separating simple closed curves. Let $X(k) \to \mathbb{S}^2$ be the genus-$3k$ Lefschetz fibration corresponding to the word W_k. By applying the technique of Endo and Nagami explained in Subsection (2.1.5) to compute the signature $\sigma(X(k))$ of $X(k)$ and the Euler characteristic formula for the Lefshetz fibrations, we get the topological invariants of $X(k)$ as follows:

$$\begin{aligned}
e(X(k)) &= e(\mathbb{S}^2)e(\Sigma_{3k}) + \#\text{singular fibers} \\
&= 2(2-6k) + 8k + 6 = -4k + 10, \\
\sigma(X(k)) &= I_{3k}((B_0B_1\cdots B_{2k}C)^2(B'_0B'_1\cdots B'_{2k}C')^2C^{-1}{C'}^{-1}) \\
&= I_{3k}((B_0B_1\cdots B_{2k}C)^2) + I_{3k}((B'_0B'_1\cdots B'_{2k}C')^2) - I_{3k}(C) - I_{3k}(C') \\
&= -4 - 4 - (-1) - (-1) = -6, \\
c_1^2(X(k)) &= 3\sigma(X(k)) + 2e(X(k)) = -8k + 2, \\
\chi_h(X(k)) &= (e(X(k)) + \sigma(X(k)))/4 = -k + 1.
\end{aligned}$$

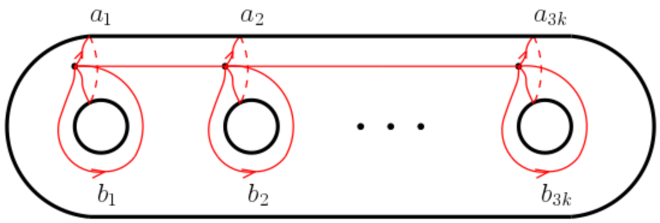

Figure 4.3: The generators of $\pi_1(\Sigma_{3k})$

Lemma 4.2.1. *For each* $1 \leq i \leq k$ *and* $1 \leq j \leq k-1$ *, the fundamental group* $\pi_1(X(k))$ *of* $X(k)$ *has the following relations:*

$$a_i a_{2k-i+1} = 1, \;\; a_{k+i} a_{3k-i+1} = 1, \tag{4.16}$$

$$b_{j+1} b_{j+2} \cdots b_{2k-j} = [a_{2k+1-j}, b_{2k+1-j}][a_{2k+2-j}, b_{2k+2-j}] \cdots [a_{2k}, b_{2k}], \tag{4.17}$$

$$b_{k+j+1} b_{k+j+2} \cdots b_{3k-j} = [a_{3k+1-j}, b_{3k+1-j}][a_{3k+2-j}, b_{3k+2-j}] \cdots [a_{3k}, b_{3k}]. \tag{4.18}$$

Proof. Let a_i and b_i be the generators of $\pi_1(\Sigma_{3k})$ for $i = 1, \ldots, 3k$ as in Figure 4.3. Since the genus-k Lefschetz fibration $X(k) \to \mathbb{S}^2$ admits a section, $\pi_1(X(k))$ is isomorphic to the quotient of $\pi_1(\Sigma_{3k})$ by the normal closure of vanishing cycles. The fundamental group $\pi_1(X(k))$ has the following relations up to conjugation:

$$\begin{aligned}
B_0 &= b_1 b_2 \cdots b_{2k} = 1, \\
B_{2i-1} &= a_i b_i b_{i+1} \cdots b_{2k+1-i} c_{2k+1-i} a_{2k+1-i} = 1, \;\; 1 \leq i \leq k, \\
B_{2i} &= a_i b_{i+1} b_{i+2} \cdots b_{2k-i} c_{2k-i} a_{2k+1-i} = 1, \;\; 1 \leq i \leq k-1, \\
B_{2k} &= a_k c_k a_{k+1} = 1, \\
B'_0 &= b_{k+1} b_{k+2} \cdots b_{3k} = 1, \\
B'_{2i-1} &= a_{k+i} b_{k+i} b_{k+i+1} \cdots b_{3k+1-i} c_{3k+1-i} a_{3k+1-i} = 1, \;\; 1 \leq i \leq k, \\
B'_{2i} &= a_{k+i} b_{k+i+1} b_{k+i+2} \cdots b_{3k-i} c_{3k-i} a_{3k+1-i} = 1, \;\; 1 \leq i \leq k-1, \\
B'_{2k} &= a_{2k} c_{2k} a_{2k+1} = 1, \\
C &= c_k = 1, \\
D &= c_{2k} = 1, \\
c_{3k} &= [a_1, b_1][a_2, b_2] \cdots [a_{3k}, b_{3k}] = 1,
\end{aligned}$$

where $c_j = [a_1, b_1][a_2, b_2] \cdots [a_j, b_j]$ for $1 \leq j \leq k$. First consider the relations

$$\begin{aligned}
B_1 &= a_1 b_1 b_2 \cdots b_{2k} c_{2k} a_{2k} = 1, \\
B_0 &= b_1 b_2 \cdots b_{2k} = 1,
\end{aligned}$$

and the relation $c_{2k} = D = 1$ then one can easily get $a_1 a_{2k} = 1$. Now consider

$$B_2 = a_1 b_2 b_3 \cdots b_{2k-1} c_{2k-1} a_{2k} = 1.$$

Using the equation $a_1 a_{2k} = 1$, we get

$$b_2 b_3 \cdots b_{2k-1} c_{2k-1} = 1.$$

Using the relation

$$B_3 = a_2 b_2 \cdots b_{2k-1} c_{2k-1} a_{2k-1} = 1$$

we have $a_2 a_{2k-1} = 1$. Inductively, one can obtain the relations $a_i a_{2k-i+1} = 1$ for $1 \leq i \leq k$.

To get the relations $a_{k+i} a_{3k-i+1} = 1$ for $1 \leq i \leq k$, combine the equations

$$\begin{aligned} B'_1 &= a_{k+1} b_{k+1} b_{k+2} \cdots b_{3k} c_{3k} a_{3k} = 1, \\ B'_0 &= b_{k+1} b_{k+2} \cdots b_{3k} = 1, \end{aligned}$$

and $c_{3k} = 1$, we have $a_{k+1} a_{3k} = 1$. Then, consider the relation

$$B'_2 = a_{k+1} b_{k+2} b_{k+3} \cdots b_{3k-1} c_{3k-1} a_{3k} = 1$$

together with the equation $a_{k+1} a_{3k} = 1$, we obtain

$$b_{k+2} b_{k+3} \cdots b_{3k-1} c_{3k-1} = 1.$$

Then, by inserting it into

$$B'_3 = a_{k+2} b_{k+2} b_{k+2} \cdots b_{3k} c_{3k-1} a_{3k-1} = 1$$

we have $a_{k+2} a_{3k-1} = 1$. Continuing in this way, we conclude that $a_{k+i} a_{3k-i+1} = 1$ for $1 \leq i \leq k$.

We next show that $b_{i+1} b_{i+2} \cdots b_{2k-i} = [a_{2k+1-i}, b_{2k+1-i}][a_{2k+2-i}, b_{2k+2-i}] \cdots [a_{2k}, b_{2k}]$ for $1 \leq i \leq k-1$. Since we have the equations

$$B_{2i} = a_i b_{i+1} b_{i+2} \cdots b_{2k-i} c_{2k-i} a_{2k+1-i} = 1$$

and $a_i a_{2k-i+1} = 1$ then it follows that

$$b_{i+1} b_{i+2} \cdots b_{2k-i} = c_{2k-i}^{-1}.$$

This actually gives the required relation by the definition of c_{2k-i}.

The last equation (4.18) comes from the relations

$$a_{k+i}a_{3k-i+1} = 1$$

and

$$B'_{2i} = a_{k+i}b_{k+i+1}b_{k+i+2}\cdots b_{3k-i}c_{3k-i}a_{3k+1-i} = 1$$

for $1 \leq i \leq k-1$ in a similar way. $\square$

Corollary 4.2.2. *The first homology group $H_1(X(k);\mathbb{Z})$ of $X(k)$, is isomorphic to $\mathbb{Z}^{2k}$.*

Proof. Observe that a_{k+i} for all $i = 1,\ldots,2k$ can be written in terms of a_j for $j = 1,\ldots,k$ by the relation (4.16) in the Lemma 4.2.1. The abelianization of the relation (4.17) gives rise to the equation $b_{j+1} = -b_{2k-j}$ for each $j = 1,\ldots,k-1$. Furthermore, it follows from the abelianization of the relation (4.18) that $b_{k+j+1} = -b_{3k-j}$ for each $j = 1,\ldots,k-1$. One can easily observe that b_{k+i} for all $i = 1,\ldots,2k$ can be written also in terms of b_j for $j = 1,\ldots,k$. This finishes the proof. $\square$

Theorem 4.2.3. *Let $X(k) \to \mathbb{S}^2$ be the genus-$3k$ Lefschetz fibration constructed above. The 4-manifold $X(k)$ is diffeomorphic to $\Sigma_k \times \mathbb{S}^2 \# 6\overline{\mathbb{CP}}^2$ for all k non-negative integers.*

Proof. Using $H_1(X(k);\mathbb{Z}) \cong \mathbb{Z}^{2k}$ and some other topological invariants that we obtained above, one can easily compute that $b_1(X(k)) = 2k$ and $b_2^+(X(k)) = 1$. When $k = 1$, we showed that the total space X of the Lefschetz fibration with the factorization $W = W_1$ is diffeomorphic to $\mathbb{T}^2 \times \mathbb{S}^2 \# 6\overline{\mathbb{CP}}^2$ in Theorem 3.1.1.

When $k = 2$, we will show that $X(2)$ is diffeomorphic to a blow-up of a ruled surface. Assuming that $X(2)$ is not diffeomorphic to (a blow-up of) a ruled surface, then $X(2) \cong \widetilde{X(2)} \# m\overline{\mathbb{CP}}^2$ where $\widetilde{X(2)}$ is the minimal model of $X(2)$ and m is some non-negative integer. It is easily computed that $c_1^2(\widetilde{X(2)}) = c_1^2(X(2)) + m = -14 + m$. By Theorem 2.1.18, since the minimal 4-manifold $X(2)$ is neither rational nor ruled then $c_1^2(\widetilde{X(2)}) = -14 + m \geq 0$, which implies that $m \geq 14$. On the other hand, it follows from Theorem 2.1.19 that $m \leq 2g - 2 = 10$, where g is the genus of the

Lefschetz fibration $X(2) \to \mathbb{S}^2$, which is 6. since $X(2)$ is not rational nor ruled then it admits m disjoint exceptional spheres. This yields a contradiction.

Lastly, when $k > 2$ since $e(X(k)) < 0$, $X(k)$ is diffeomorphic to a blow-up of a ruled surface by Theorem 2.1.20. Thus using the signature and Euler characteristic of $X(k)$, we can deduce that $X(k)$ is diffeomorphic to $\Sigma_k \times \mathbb{S}^2 \# 6\overline{\mathbb{CP}}^2$. Therefore, the fundamental group $\pi_1(X(k))$ of $X(k)$ that has the representation in Lemma(4.2.1) is isomophic to the surface group $\pi_1(\Sigma_k)$. □

4.2.0.2 Construction of fibered exotic $(4k^2 - 2k + 1)\mathbb{CP}^2 \# (4k^2 + 4k + 7)\overline{\mathbb{CP}}^2$, using genus-$3k$ fibration on $\Sigma_k \times \mathbb{S}^2 \# 6\overline{\mathbb{CP}}^2$

To produce exotic 4-manifolds $(4k^2-2k+1)\mathbb{CP}^2\#(4k^2+4k+7)\overline{\mathbb{CP}}^2$ which carry the Lefschetz fibration structure, we will perform sufficiently many twisted fiber sums of the genus-$3k$ Lefschetz fibration $X(k)$ to get a simply-connected 4-manifold.

Theorem 4.2.4. *There exist new minimal symplectic exotic copies of* $(4k^2 - 2k + 1)\mathbb{CP}^2\#(4k^2+4k+7)\overline{\mathbb{CP}}^2$ *admitting genus-*$3k$ *Lefschetz fibration structure for each integer* $k \geq 1$.

Proof. We start with the Lefschetz fibration $X(k) = \Sigma_k \times \mathbb{S}^2 \# 6\overline{\mathbb{CP}}^2$, we can choose a diffeomorphism in such a way that when we perform twisted fiber sum of W_k, the word induced by conjugating W_k with this diffeomorphism kills some generators of the fundamental group $\pi_1(X_k)$. Consider the disjoint vanishing cycles B_{2k} and B'_{2k}, one can find the diffeomorphisms $f_i(B_{2k}) = a_i$, $f_i(B'_{2k}) = b_{i+1}$ for each $i = 1, \ldots, k-1$ and $f_k(B_{2k}) = a_k$ and $f_i(B'_{2k}) = b_1$ by the classification of surfaces where a_i, b_i' s are the generators of $\pi_1(X_k)$ as in Figure 4.3 and the curves B_{2k} and B'_{2k} are depicted in Figure 4.2. We obtain the monodromy factorization $W_k W_k^{f_1} \ldots W_k^{f_k}$ by conjugating such diffeomorphisms. Let X_k^{k+1} be the corresponding total space of the Lefschetz fibration to the monodromy factorication $W_k W_k^{f_1} \cdots W_k^{f_k}$. Using the theory of Lefschetz fibrations and the existence of a section, the fundamental group $\pi_1(X_k^{k+1})$ of X_k^{k+1} is a quotient of $\pi_1(X_k)$ that is the surface group with generators a_i, b_i for $i = 1, \ldots, k$. The conjugated words $W_i^{f_i}$ induce the additional relations containing $f_i(B_{2k}) = a_i = 1$, $f_i(B'_{2k}) = b_{i+1} = 1$ for each $i = 1, \ldots, k-1$,

$f_k(B_{2k}) = a_k = 1$ and $f_i(B'_{2k}) = b_1 = 1$. So, the additional relations induced by conjugated monodromy factorizations kill all generators a_i, b_i for $i = 1, \ldots, k$, which makes $\pi_1(X_k^{k+1})$ is trivial. The other topological invariants can be computed using the fiber sum computations as follows:

$$\begin{aligned}
e(X_k^{k+1}) &= e(\mathbb{S}^2)e(\Sigma_{3k}) + \#\text{singular fibers} \\
&= 2(2-6k) + (k+1)(8k+6) = 8k^2 + 2k + 10, \\
\sigma(X_k^{k+1}) &= (k+1)\sigma(X(k)) = (k+1)(-6) = -6k - 6, \\
c_1^2(X_k^{k+1}) &= 3\sigma(X_k^{k+1}) + 2e(X_k^{k+1}) = 16k^2 - 14k - 16, \\
\chi_h(X_k^{k+1}) &= (\sigma(X_k^{k+1}) + e(X_k^{k+1}))/4 = 2k^2 - k + 1.
\end{aligned}$$

Using Theorem 2.1.21, X_k^{k+1} is homeomorphic to $(4k^2 - 2k + 1)\mathbb{CP}^2\#(4k^2 + 4k + 7)\overline{\mathbb{CP}}^2$ for any integer $k > 0$. Theorem (2.1.11) (or Proposition 2.1.12) implies the minimality of X_k^{k+1}, so they are not diffeomorphic to $(4k^2 - 2k + 1)\mathbb{CP}^2\#(4k^2 + 4k + 7)\overline{\mathbb{CP}}^2$ for any integer $k > 0$. $\square$

Remark 4.2.5. *Further fibered minimal exotic examples can be constructed using other genus-$3k$ Lefschetz fibrations over $\mathbb{S}^2$ and performing lantern substitutions.*

4.2.0.3 Construction of exotic, not fibered, $(4k-1)\mathbb{CP}^2\#(4k+5)\overline{\mathbb{CP}}^2$ using genus-$3k$ fibration on $\Sigma_k \times \mathbb{S}^2\#6\overline{\mathbb{CP}}^2$

To construct exotic copies of $(4k-1)\mathbb{CP}^2\#(4k+5)\overline{\mathbb{CP}}^2$ for any positive integer k, we will use the following family of symplectic building block. It is obtained from $\Sigma_{3k} \times \mathbb{T}^2$ by performing a sequence of torus surgeries. Also, our computations are similar to the some computations in [8]. Let us denote this construction by $Y_k(1/p, m/q)$ which is smooth 4-manifold obtained by performing the following $6k$-torus surgeries on $\Sigma_{3k} \times \mathbb{T}^2$ for fixed integers $m, k \geq 1$ and $p, q \geq 0$:

$$\begin{aligned}
&(\beta'_1 \times c'', \beta_{2k}, -1), (\alpha''_{3k} \times d', d', m/q), \\
&(\beta'_2 \times c'', \beta_{2k+1}, -1), (\alpha'_1 \times c', \alpha'_1, -1), \\
&(\beta'_3 \times c'', \beta_{2k+2}, -1), (\alpha'_2 \times c', \alpha'_2, -1), \\
&\ldots, \ldots
\end{aligned}$$

$$(\beta'_k \times c'', \beta_{3k-1}, -1), (\alpha'_{k-1} \times c', \alpha'_{k-1}, -1),$$
$$(\beta'_{k+1} \times c'', \beta_1, -1), (\alpha'_k \times c', \alpha'_k, -1),$$
$$(\beta'_{k+2} \times c'', \beta_2, -1), (\alpha'_{k+1} \times c', \alpha'_{k+1}, -1),$$
$$\dots, \dots$$
$$(\beta'_{2k} \times c'', \beta_k, -1), (\alpha'_{2k-1} \times c', \alpha'_{2k-1}, -1),$$
$$(\beta'_{2k+1} \times c'', \beta_{k+1}, -1), (\alpha'_{2k} \times c', \alpha'_{2k}, -1),$$
$$(\beta'_{2k+2} \times c'', \beta_{k+2}, -1), (\alpha'_{2k+1} \times c', \alpha'_{2k+1}, -1),$$
$$\dots, \dots$$
$$(\beta'_{3k-1} \times c'', \beta_{2k-1}, -1), (\alpha'_{3k-2} \times c', \alpha'_{3k-2}, -1),$$
$$(\alpha'_{3k} \times c', c', 1/p), (\alpha'_{3k-1} \times c', \alpha'_{3k-1}, -1),$$

where α_i, β_i, $i = 1, 2 \dots 3k$ and c, d are the generators of $\pi_1(\Sigma_{3k})$ and $\pi_1(\mathbb{T}^2)$, respectively. When we set $m = 1$, the above torus surgeries are Luttinger surgeries and in this case the Luttinger surgery preserves the minimality and can be performed symplectically as explained in Subsection 2.1.22.

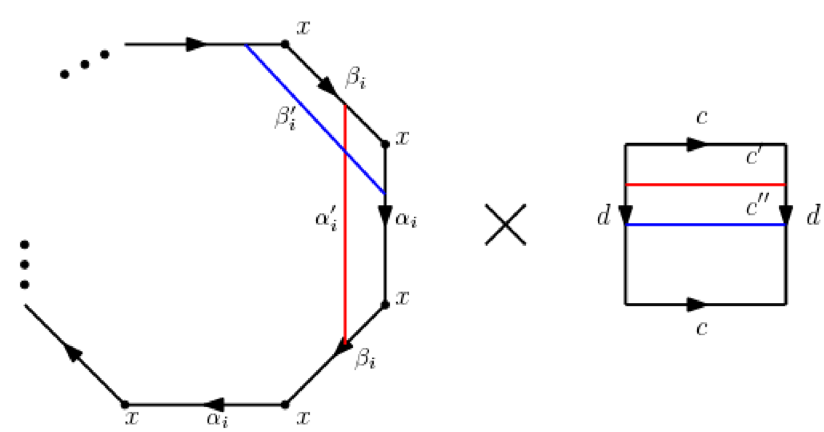

Figure 4.4: Lagrangian tori $\beta'_i \times c''$ and $\alpha'_i \times c'$

The fundamental group of the resulting manifold $Y_k(1/p, m/q)$ is generated by α_i, β_i, $i = 1, 2 \dots 3k$ and c, d and it has the following relations:

$$[\alpha_1^{-1}, d] = \beta_{2k}, [\alpha_2^{-1}, d] = \beta_{2k+1}, \cdots, [\alpha_k^{-1}, d] = \beta_{3k-1},$$

$$[\alpha_{k+1}^{-1}, d] = \beta_1, [\alpha_{k+2}^{-1}, d] = \beta_2, \cdots, [\alpha_{3k-1}^{-1}, d] = \beta_{2k-1},$$

$$[c^{-1}, \beta_{3k}]^{-m} = d^q, [\beta_1^{-1}, d^{-1}] = a_1, [\beta_2^{-1}, d^{-1}] = \alpha_2,$$

$$\cdots,\cdots$$

$$[\beta_{3k-1}^{-1}, d^{-1}] = \alpha_{3k-1}, [d^{-1}, \beta_{3k}^{-1}] = c^p, [\beta_j, c] = 1,$$

$$[\alpha_{3k}, d] = 1, [\alpha_j, c] = 1, [\alpha_{3k}, c] = 1,$$

$$[\alpha_1, \beta_1][\alpha_1, \beta_1]\cdots[\alpha_{3k}, \beta_{3k}] = 1, [c, d] = 1. \tag{4.19}$$

where $1 \leq j \leq 3k-1$.

Theorem 4.2.6. *There exist new smooth exotic copies of* $(4k-1)\mathbb{CP}^2\#(4k+5)\overline{\mathbb{CP}}^2$.

Proof. Consider the symplectic manifold $X(k)$ constructed above (Subsection 4.2.0.1) with a genus-$3k$ symplectic submanifold Σ_{3k}, a regular fiber coming from its genus-$3k$ Lefschetz fibration structure (Theorem 4.2.3) and the symplectic 4-manifold $Y_k(1,1)$ which we obtained by performing torus surgeries from $\Sigma_{3k}\times\mathbb{T}^2$ where $p=q=m=1$ above with the symplectic submanifold Σ'_{3k} that is a copy of $\Sigma_{3k}\times\{pt\}$ in $Y_k(1,1)$. Let $Z(k)$ be the 4-manifold obtained by symplectic fiber sum of $X(k)$ and $Y_k(1,1)$ along the surfaces Σ_{3k} and Σ'_{3k}. We need to find an orientation-reversing gluing diffeomorphism to perform symplectic fiber sum such that $Z(k)$ is simply-connected. Recall from the Lemma 4.2.1 a_i, b_i ($i=1,\ldots,3k$) are the generators of $\pi_1(X(k))\cong\pi_1(\Sigma_k\times\mathbb{S}^2\#6\overline{\mathbb{CP}}^2)\cong\pi_1(\Sigma_k)$ but the generators a_i, b_i ($k+1\leq i\leq 3k$) are nullhomotopic. $\pi_1(X(k)\setminus\upsilon\Sigma_{3k})$ is isomorphic to the fundamental group $\pi_1(X(k))$ since the genus-$3k$ Lefschetz fibration $X(k)\to\mathbb{S}^2$ admits a section and hence the normal circle to Σ_{3k}, denote it by λ is nullhomotopic in $\pi_1(X(k)\setminus\upsilon\Sigma_{3k})$. The generators of $\pi_1(Y_k(1,1))$ are α_i, β_i, c and d for $i=1,\ldots,3k$ and $\pi_1(Y_k(1,1))$ has the relations (4.19). Choose a base point p of $\pi_1(Y_k(1,1))$ on $\partial\upsilon\Sigma'_{3k}\cong\Sigma'_{3k}\times S^1$ in such a way that $\pi_1(Y_k(1,1)\setminus\Sigma'_{3k}, p)$ is normally generated by α_i, β_i, c and d for $i=1,\ldots,3k$. One can perform above tori surgeries such that $\Sigma'_{3k}\subset Y_k(1,1)$ is disjoint from all tori surgeries performed. Hence the relations in (4.19) still hold in $\pi_1(Y_k(1,1)\setminus\Sigma'_{3k}, p)$ except for $[c,d]=1$, which represents a meridian, denote it by λ', in $\pi_1(Y_k(1,1)\setminus\Sigma'_{3k}, p)$. Now, choose the gluing diffeomorphism $\varphi:\partial(\Sigma_{3k})\to\partial(\Sigma'_{3k})$ mapping the generators of π_1 as follows:

$$a_i \mapsto \alpha_i$$

$$b_i \mapsto \beta_i$$

$$\lambda \mapsto \lambda'$$

By Van Kampen's theorem, the fundamental group $\pi_1(Z(k))$ of the resulting 4-manifold $Z(k) = (X(k) \setminus \upsilon\Sigma_{3k} \cup_\varphi Y_k(1,1) \setminus \upsilon\Sigma'_{3k})$ is isomorphic to

$$\pi_1(Z(k)) \cong \frac{\pi_1(X(k) \setminus \upsilon\Sigma_{3k} * \pi_1(Y_k(1,1) \setminus \Sigma'_{3k})}{\langle a_i = \alpha_i, b_i = \beta_i, \lambda = \lambda' \rangle}.$$

One can conclude that $\pi_1(Z(k))$ admits a presentation with generators a_i, b_i ($i = 1, \ldots, 3k$), c and d and the relations (4.16), (4.17), (4.18) and relations (4.19) hold in $\pi_1(Z(k))$. Keep in mind $a_i = \alpha_i, b_i = \beta_i, \lambda = \lambda'$ and $[c, d] = \lambda'$, it is enough to prove that $c = d = 1$ in $\pi_1(Z(k))$ to show that $\pi_1(X(k))$ is trivial using the relations in (4.19). To do this, first consider the relations in (4.19)

$$[a_{3k}, d] = 1, [a_{k+1}^{-1}, d] = b_1$$

and the relation in (4.16)

$$a_{3k} = a_{k+1}^{-1},$$

which yields $b_1 = 1$. Next, since $[b_1^{-1}, d^{-1}] = a_1$ in (4.19) then $a_1 = 1$. Also, using the relations

$$a_1 a_{2k} = a_{2k} a_{2k+1} = 1,$$

we have $a_{2k} = a_{2k+1} = 1$ by relations (4.19). Using these equations, it can be obtained that

$$b_k = b_{k+1} = 1$$

since $[a_{2k}^{-1}, d] = b_k$ and $[a_{2k+1}^{-1}, d] = b_{k+1}$. Using the relation in (4.19) $[b_k^{-1}, d^{-1}] = a_k$ and $b_k = 1$, we get

$$a_k = 1.$$

Also, since $[a_k^{-1}, d] = b_{3k-1}$ in (4.19) and $a_k = 1$ then

$$b_{3k-1} = 1.$$

Similarly, the relations $[b_{3k-1}^{-1}, d^{-1}] = a_{3k-1}$ and $b_{3k-1} = 1$ give the equation

$$a_{3k-1} = 1.$$

Recall that we found $a_k = 1$, using equations $[a_k, a_{k+1}] = [a_{k+1}, a_{3k}] = 1$, we get

$$a_{3k} = 1.$$

Now, the relation (4.18) gives the relation

$$b_{k+2}b_{k+3}\cdots b_{3k-1} = [a_{3k-1}, b_{3k-1}][a_{3k}, b_{3k}]$$

and we have the relation

$$b_{k+1}b_{k+2}\cdots b_{3k} = 1$$

coming from vanishing cycle B_0' in Figure 4.2. So, they result in the relations

$$b_{k+2}b_{k+3}\cdots b_{3k-1} = b_{k+1}b_{k+2}\cdots b_{3k} = 1$$

using $a_{3k-1} = a_{3k} = 1$. Then we get

$$b_{k+1}b_{3k} = 1,$$

which impiles that $b_{3k} = 1$ using $b_{k+1} = 1$. Finally, we can obtain that

$$c = d = 1$$

by using the relations $[c^{-1}, b_{3k}] = d$ and $[d^{-1}, b_{3k}^{-1}] = c$. Therefore, the following relations

$$[b_i^{-1}, d^{-1}] = a_i, \quad i = 1, \dots, 3k-1$$

$$[a_i^{-1}, d] = b_{2k-1+i}, \quad i = 1, \dots, k$$

$$[a_i^{-1}, d] = b_{i-k}, \quad i = k+1, \dots, 3k-1$$

coming from the equations (4.19) prove that $\pi_1(Z(k)) = 1$.

Using the fact that Luttinger surgery preserves the Euler characteristic and the signature by Lemma 2.1.22, the topological invariants of $Z(k)$ are computed as follows:

$$\begin{aligned}
e(Z(k)) &= e(X(k)) + e(Y_k(1,1)) - 2e(\Sigma_{3k}) = 6 + 8k, \\
\sigma(Z(k)) &= \sigma(Z(k)) + \sigma(Y_k(1,1)) = -6, \\
c_1^2 &= 3\sigma(Z(k)) + 2e(Z(k)) = 6 + 16k, \\
\chi_h(Z(k)) &= (\sigma(Z(k)) + e(Z(k)))/4 = 2k.
\end{aligned}$$

Using Theorem 2.1.21, $Z(k)$ is homeomorphic to $(4k-1)\mathbb{CP}^2\#(4k+5)\overline{\mathbb{CP}}^2$ for positive integer k. Since $Z(k)$ is symplectic and $b_2^+(Z(k)) \geq 2$ then the Seiberg-Witten invariant of the canonical class of $Z(k)$ is ± 1 by Theorem 2.1.23 . However, the Seiberg-Witten invariant of the canonical class of $(4k-1)\mathbb{CP}^2\#(4k+5)\overline{\mathbb{CP}}^2$ is trivial by Theorem 2.1.24. Hence, we distinguish $Z(k)$ with $(4k-1)\mathbb{CP}^2\#(4k+5)\overline{\mathbb{CP}}^2$ up to diffeomorphism, since Seiberg-Witten invariant is a diffeomorphism invariant. Also, we replace 4-manifold $Y_k(1,1)$ with $Y_k(1,m)$ in our construction above, where the integer $m \neq 1$ to construct an infinitely many exotic copies of $(4k-1)\mathbb{CP}^2\#(4k+5)\overline{\mathbb{CP}}^2$. $\square$

CHAPTER 5

THE NUMBER OF SINGULAR FIBERS IN HYPERELLIPTIC LEFSCHETZ FIBRATIONS

This chapter is devoted to hyperelliptic Lefschetz fibrations over a sphere, which are a special kind of Lefschetz fibration. More precisely, we examine the minimal number of singular fibers in genus-g hyperelliptic Lefschetz fibrations over a sphere. We obtain some results about them when $g = 4, 5, 6, 7, 8, 9$ and 10. We next focus on the minimal number of singular fibers in genus-g holomorphic hyperelliptic Lefschetz fibrations over a sphere. In this case, we obtain the exact values of the minimal number of singular fibers in such Lefschetz fibrations for even genus $g \geq 4$ and improve a lower bound for them for odd genus $g \geq 5$. Since the total spaces of homolorphic Lefschetz fibrations carry complex structures, then we give a summary of the Enrique-Kodaira classification of complex surfaces, which is crucial in proving our results in holomorphic cases.

5.1 Preliminaries

In this chapter we focus on a special kind of Lefschetz fibrations called hyperelliptic Lefschetz fibrations. First, we present some definitons and properties related to them.

Definition 5.1.1. *The hyperelliptic mapping class group of a genus-g surface Σ_g, denoted by HMod_g, is the group of mapping classes of Σ_g that commute with the hyperelliptic involution ι (as defined in Subsection 2.1.2.2).*

Definition 5.1.2. *A genus-g Lefschetz fibration is said to be hyperelliptic if its vanishing cycles are invariant under some hyperelliptic involution.*

It follows that for a genus-g hyperelliptic Lefschetz fibration over $\mathbb{S}^2$ with global monodromy

$$t_{a_1}t_{a_2}\cdots t_{a_n} = 1,$$

there exists a mapping class $\phi \in \mathrm{Mod}_g$ such that $\phi t_{a_i}\phi^{-1} \in \mathrm{HMod}_g$ for all $i = 1, 2, \ldots, n$.

5.1.1 First homology group of the hyperelliptic mapping class group

We collect some useful facts about the first homology group of the hyperelliptic mapping class group.

Recall that for any group G, the first homology group with integral coefficients,

$$H_1(G;\mathbb{Z}) = G/[G,G],$$

is the abelianization of the group G, where $[G,G]$ is the commutator subgroup of G, which is the subgroup generated by all commutators $[a,b] = aba^{-1}b^{-1}$ for all $a, b \in G$. It is known that $H_1(\mathrm{Mod}_g;\mathbb{Z})$ is generated by the class of a Dehn twist about a nonseparating simple closed curve and also we have the following lemma:

Lemma 5.1.3. *For a closed orientable surface of genus $g \geq 1$, the first homology group $H_1(\mathrm{Mod}_g;\mathbb{Z})$ of the mapping class group Mod_g*

$$H_1(\mathrm{Mod}_g;\mathbb{Z}) = \begin{cases} \mathbb{Z}/12, & \text{if } g = 1, \\ \mathbb{Z}/10, & \text{if } g = 2, \\ 0, & \text{if } g \geq 3. \end{cases}$$

For further details about the homology groups of the mapping class group and the proof of the Lemma 5.1.3, see [44].

The following lemma can be proven from the presentation of hyperellipitic mapping class group [17].

Lemma 5.1.4. *For a closed orientable genus-g surface, the first homology group $H_1(\mathrm{HMod}_g;\mathbb{Z})$ of the hyperelliptic mapping class group HMod_g is*

$$H_1(\mathrm{HMod}_g;\mathbb{Z}) = \begin{cases} \mathbb{Z}/4(2g+1), & \text{if } g \text{ is odd}, \\ \mathbb{Z}/2(2g+1), & \text{if } g \text{ is even}. \end{cases}$$

All right Dehn twists about nonseparating hyperelliptic simple closed curves on Σ_g are nontrivial in the hyperelliptic mapping class group HMod_g of Σ_g and each of them maps to the same generator in $H_1(\mathrm{HMod}_g)$ under the quotient map $\mathrm{HMod}_g \to H_1(\mathrm{HMod}_g)$. Note that, by the even chain relation, a right handed Dehn twist about the separating simple closed curve of type h in HMod_g can be written as a product of $2h(4h+2)$ right Dehn twists about the nonseparating simple closed curves. The following lemma is from this observation.

Lemma 5.1.5. *Let n be the number of nonseparating and s_h be the number of separating vanishing cycles of type h in a factorization of the identity in HMod_g. Then*

$$n + \sum_{h=1}^{[g/2]} 2h(4h+2)s_h \equiv \begin{cases} 0 \pmod{4(2g+1)} & \text{if } g \text{ is odd,} \\ 0 \pmod{2(2g+1)} & \text{if } g \text{ is even.} \end{cases} \tag{5.1}$$

5.1.2 Signatures of hyperelliptic Lefschetz fibrations

Here we review the signatures of hyperelliptic Lefschetz fibrations.

Lemma 5.1.6. *[26, 54, 55] Let (X, f) be a genus-g hyperelliptic Lefschetz fibration over $\mathbb{S}^2$. Let n and $s = \sum_{h=1}^{[g/2]} s_h$ be the numbers of nonseparating and separating vanishing cycles of this fibration, respectively, where s_h denotes the number of separating vanishing cycles of type h. Then the signature of X is*

$$\sigma(X) = -\frac{g+1}{2g+1}n + \sum_{h=1}^{[g/2]} \left(\frac{4h(g-h)}{2g+1} - 1\right)s_h. \tag{5.2}$$

Lemma 5.1.7. *[57] For any 4-manifold X admitting a genus-g Lefschetz fibration over $\mathbb{S}^2$ or $\mathbb{D}^2$, $\sigma(X) \leq n - s$.*

Lemma 5.1.8. *[57] For any 4-manifold X admitting a genus-g hyperelliptic Lefschetz fibration over $\mathbb{S}^2$, $\sigma(X) \leq n - s - 4$.*

Lemma 5.1.9. *[19] For any 4-manifold X admitting a genus-g Lefschetz fibration over $\mathbb{S}^2$,*

$$\sigma(X) \leq n - s - 2(2g - b_1(X)).$$

It can be easily obtained that $\sigma(X) \leq n - s - 2$ using $b_1(X) \leq 2g - 1$ by the handlebody decomposition of Lefcshetz fibrations.

Recall that the topological invariant $c_1^2(X) = 3\sigma(X) + 2e(X)$ of a symplectic 4-manifold satisfies the following inequalities.

Theorem 5.1.10. *[63] Any relatively minimal nontrivial Lefschetz fibration X over $\mathbb{S}^2$ satisfies $c_1^2(X) \geq 4 - 4g$.*

Theorem 5.1.11. *[49] Let X be a relatively minimal Lefschetz fibration of genus g over a surface of genus-h. If X is not rational nor ruled then*

$$c_1^2(X) \geq 2(g-1)(h-1),$$

and it is sharp if $h = 0$.

Theorem 5.1.12. *[42] Let Σ_g and Σ_h be closed, oriented surfaces with $h \geq 1$ and let Σ_h be connected. If $f : \Sigma_g \to \Sigma_h$ be a continuous map of degree d, then*

$$d\,|e(\Sigma_h)| \leq |e(\Sigma_g)|.$$

Theorem 5.1.13. *[63] If (X, f) be a relatively genus-g Lefschetz fibration over $\mathbb{S}^2$ then the fiber sum $X \# X$ is minimal.*

Theorem 5.1.14. *[64] For any genus-g Lefschetz fibration $X \to \mathbb{S}^2$ with homologically essential fiber F, the homology class $[F]$ is primitive.*

5.1.3 The number of singular fibers in Lefschetz fibrations

In this section, we collect some results about the number singular fibers in Lefschetz fibrations.

Recall that for any genus-g Lefschetz fibration, n and s denote the number of nonseparating and separating vanishing cycles, respectively.

Lemma 5.1.15. *[49] If any Lefschetz fibration of genus g has $\sigma \geq -(n+s)+4$, it has*

(i) $b_1 \leq 2g - 2$,

(ii) $b_1 \leq (n+s) - 2$,

(iii) $b_2^+ \leq n - 3$,

(iv) $\sigma \leq n - s - 4$.

Theorem 5.1.16. *[64] Let $X \to \mathbb{S}^2$ be a nontrivial Lefschetz fibration of genus g with $b_2^+ = 1$.*

(1) If $g \geq 6$ is even, then it admits at least $2g+4$ singular fibers. (This lower bound is sharp.)

(2) If $g \geq 15$ is odd, then it admits at least $2g+10$ singular fibers. (This lower bound is sharp.)

(3) If $g \geq 9$ is odd, then it contains at least $2g+6$ singular fibers.

Theorem 5.1.17. *[18] Let $X \to \mathbb{S}^2$ be a nontrivial Lefschetz fibration of genus $g \geq 1$. Then $X \to \mathbb{S}^2$ has at least $\frac{1}{5}(8g-3)$ nonseparating vanishing cycles.*

Let N_g be the minimal number of singular fibers in a genus-g Lefschetz fibration over $\mathbb{S}^2$. Combining the results about N_g, we have the following theorem:

Theorem 5.1.18. *[46, 15, 70, 18, 19, 43] For the number N_g the following holds.*

(i) $N_1 = 12$.

(ii) $N_2 = 7$.

(iii) *(a)* $\frac{1}{5}(8g-3) \leq N_g \leq 2g+4$ *if* $g \geq 4$ *is even.*

(b) $\frac{1}{5}(8g-3) \leq N_g \leq 2g+10$ *if* $g \geq 3$ *is odd.*

Therefore, if $g \geq 3$ then the exact value of N_g has not been known yet.

If we restrict ourselves to hyperelliptic Lefschetz fibrations over a sphere then we have the following estimates for the minimal number of singular fibers, denote it by N_g^h :

Theorem 5.1.19. *[46, 15, 70, 14, 18, 19, 43] For the number N_g^h the following holds.*

(i) $N_1^h = 12$.

(ii) $N_2^h = 7$.

(iii) $N_3^h = 12$.

(iv) *(a)* $\frac{1}{5}(8g-3) \leq N_g^h \leq 2g+4$ *if* $g \geq 4$ *is even.*

(b) $\frac{1}{5}(8g-3) \leq N_g^h \leq 5g-3$ *if* $g \geq 5$ *is odd.*

Since all genus-g Lefschetz fibrations are hyperelliptic for $g = 1, 2$, $N_1^h = N_1 = 12$ and $N_2^h = N_2 = 7$. For even $g \geq 4$, the upper bound comes from the generalized Matsumoto's fibration explained in Subsection 2.1.4. However, the generalized Matsumoto's fibration is not hyperelliptic for odd $g \geq 3$. Recently, Korkmaz constructed a genus-g hyperelliptic Lefschetz fibration over a sphere with $5g-3$ singular fibers. Therefore, the upper bound for the number N_g^h is $5g-3$ for odd $g \geq 5$.

5.1.4 Classification of complex surfaces

In section 5.3, we examine the minimal number of singular fibers in a genus-g hyperelliptic holomorphic Lefschetz fibrations over $\mathbb{S}^2$. The classification of complex surfaces helps us to determine the total spaces of hyperelliptic holomorphic Lefschetz fibrations. For further information about the classification of complex surfaces see [12].

The Kodaira dimension κ takes four values $-\infty, 0, 1$ and 2 and so it divides complex surfaces into four classes. Starting from the coarse classification using Kodaira dimension, complex surfaces are divided into ten classes by Enriques-Kodaira classification.

Theorem 5.1.20. *Every minimal complex surface is in exactly one of the classes* $(1)-(10)$ *in the following table. The minimal model of the complex surfaces is unique, up to isomorphism, except for the complex surfaces with minimal models in the classes* (1) *and* (3).

The class of X	$\kappa(X)$	$b_1(X)$	$c_1^2(X)$	$e(X)$
(1) minimal rational surfaces	$-\infty$	0	8 or 9	4 or 3
(2) minimal surfaces of class VII	$-\infty$	1	≤ 0	≥ 0
(3) ruled surfaces of genus $g \geq 1$	$-\infty$	$2g$	$8(1-g)$	$4(1-g)$
(4) Enriques surfaces	0	0	0	12
(5) hyperelliptic surfaces	0	2	0	0
(6) Kodaira surfaces-$g \geq 1$	0	1 or 3	0	0
(7) K3-surfaces	0	0	0	24
(8) tori	0	4	0	0
(9) minimal properly elliptic surfaces	1		0	≥ 0
(10) minimal surfaces of general type	2	$\equiv 0 \mod 2$	> 0	> 0

Recall that the holomorphic Euler characteristic χ_h of a manifold X is given by

$$\chi_h = \frac{\sigma(X) + e(X)}{4} = \frac{c_1^2(X) + e(X)}{12}.$$

In the above table the minimal model of a complex surface with $\chi_h < 0$ is a minimal surface of class VII (the class (2) in the Table) or a ruled surface of genus $g \geq 1$ (the class (3) in the Table). It is known that closed symplectic 4-manifolds have $b_2^+ > 0$ ([38], p.390). Therefore, it follows from surfaces of class VII have $b_2 = 0$ that they do not carry a symplectic structure. One can conclude that the minimal model of a complex surface with $\chi_h < 0$ is a ruled surface of genus $g \geq 1$ if it admits a symplectic structure.

The minimal model of a ruled surface is an $\mathbb{S}^2$ bundle over a Riemann surface of genus $g \geq 0$. By Theorem 5.1.20, this minimal model is not unique. It is known that there are exactly two minimal models of ruled surfaces of genus g, the trivial bundle $\Sigma_g \times \mathbb{S}^2$ and the nontrivial bundle $\Sigma_g \ltimes \mathbb{S}^2$. It follows from Theorem 2.1.16 and Theorem 2.1.17 that for $k > 0$, $\Sigma_g \times \mathbb{S}^2 \# k\overline{\mathbb{C}P^2}$ and $\Sigma_g \ltimes \mathbb{S}^2 \# k\overline{\mathbb{C}P^2}$ are symplectomorphic. In a nutshell, it follows from the fact that the holomorphic Euler characteristic χ_h is invariant under blow ups that a non minimal symplectic complex surface with $\chi_h < 0$ is $\Sigma_g \times \mathbb{S}^2 \# k\overline{\mathbb{C}P^2}$ for some positive integers g and k.

5.2 The minimal number of singular fibers in hyperelliptic Lefschetz fibrations over a sphere

In this section, we determine the minimal number of singular fibers in some hyperelliptic Lefschetz fibrations over $\mathbb{S}^2$. Let N_g^h be denote the minimal number of singular fibers in a hyperelliptic Lefschetz fibration over $\mathbb{S}^2$.

Lemma 5.2.1. *The* 4*-manifold* $\Sigma_2 \times S^2 \# 3\overline{\mathbb{C}P^2}$ *does not admit any Lefschetz fibration of genus* 4 *over* $\mathbb{S}^2$.

Proof. Suppose that $\Sigma_2 \times \mathbb{S}^2 \# 3\overline{\mathbb{C}P^2}$ admits a genus-4 Lefschetz fibration and consider the homology class of a regular fiber F. We may write

$$[F] = a[U] + b[V] + \sum_{i=1}^{3} c_i[E_i] \in H_2(\Sigma_2 \times \mathbb{S}^2 \# 3\overline{\mathbb{C}P^2}; \mathbb{Z}),$$

where $[U]$, $[V]$ and $[E_i]$ denote the homology classes of the section and fiber of the ruling $\Sigma_2 \times S^2 \rightarrow \Sigma_2$ and E_i is the exceptional class of the ith blow-up such that $[U]^2 = [V]^2 = 0$, $[U] \cdot [V] = 1$, a,b and c_i are some integers.

The composition of the blowing down and the projection map $\Sigma_2 \times \mathbb{S}^2 \rightarrow \Sigma_2$ leads a degree-d map $F \rightarrow \Sigma_2$. The degree must be a. Moreover, since the fiber of $\mathbb{S}^2$-bundle $\Sigma_2 \times \mathbb{S}^2 \rightarrow \Sigma_2$ has pseudo-holomorphic representative [48] then the degree of the map $F \rightarrow \Sigma_2$ is positive by the positivity of intersection.

Consider a singular fiber F. Since the normalization of F has genus ≤ 3, by Theorem 5.1.12, for such a degree-d map yields to inequality

$$3 - 1 \geq g(F) - 1 \geq d(h-1) = a(2-1),$$

where $g(F)$ is the genus of the fibers F. Therefore, $0 < d = a \leq 2$.
Since $[F]^2 = 0$, we have

$$2ab = \sum_{i=1}^{3} c_i^2. \tag{5.3}$$

Since the symplectic structure on $\Sigma_2 \times \mathbb{S}^2 \# 3\overline{\mathbb{C}P^2}$ is unique up to deformations and diffeomorphism we can apply the adjunction formula

$$2g(F) - 2 = [F]^2 + [K] \cdot [F]$$

where $[K] = -2[U] + (2h-2)[V] + [E_1] + [E_2] + [E_3]$ is the canonical bundle where $h = g(\Sigma_2) = 2$. In this case the adjunction formula is

$$2g(F) - 2 = 2ah - 2a - 2b - \sum_{i=1}^{3} c_i. \tag{5.4}$$

Thus, for $g(F) = 4$, $h = 2$, we have

$$6 = 2a - 2b - \sum_{i=1}^{3} c_i. \tag{5.5}$$

If $a = 0$, then $c_i = 0$ by the identity (5.3) , which implies that $[F] = b[V]$. Also, by Theorem 5.1.14, $b = \pm 1$. However, $[F]$ cannot be represented by a smoothly embedded sphere [58].

For $a = 1$, by the identities (5.3) and (5.5) we have

$$\sum_{i=1}^{3} c_i^2 = 2b \text{ and } \sum_{i=1}^{3} c_i = -4 - 2b,$$

which leads to

$$\sum_{i=1}^{3} c_i^2 + \sum_{i=1}^{3} c_i = -4.$$

Hence $\sum_{i=1}^{3}(c_i + \frac{1}{2})^2 = -\frac{13}{4}$, which is not possible.

In the case $a = 2$, using the identities (5.3) and (5.5), we have the following equalities:

$$4b = \sum_{i=1}^{3} c_i^2 \text{ and } 2 = -2 - \sum_{i=1}^{3} c_i,$$

which gives

$$\sum_{i=1}^{3} c_i^2 + 2\sum_{i=1}^{3} c_i = -4.$$

Thus, the resulting equality is $\sum_{i=1}^{3}(c_i+1)^2 = -1$, which is a contradiction. Therefore, this shows that $\Sigma_2 \times \mathbb{S}^2 \# 3\overline{\mathbb{C}P^2}$ does not admit a genus-4 Lefschetz fibrations over $\mathbb{S}^2$. □

Lemma 5.2.2. *The 4-manifold $\Sigma_3 \times \mathbb{S}^2 \# 3\overline{\mathbb{C}P^2}$ does not admit any Lefschetz fibration of genus 7 over $\mathbb{S}^2$.*

Proof. Suppose that $\Sigma_3 \times \mathbb{S}^2 \# 3\overline{\mathbb{C}P^2}$ admits such a Lefschetz fibration and consider the homology class of a regular fiber F. We may write

$$[F] = a[U] + b[V] + \sum_{i=1}^{3} c_i[E_i] \in H_2(\Sigma_3 \times \mathbb{S}^2 \# 3\overline{\mathbb{C}P^2}; \mathbb{Z}),$$

where $[U]$, $[V]$ and $[E_i]$ denote the homology classes of the section and fiber of the ruling $\Sigma_3 \times S^2 \to \Sigma_3$ and e_i is the exceptional class of the ith blow-up such that $[U]^2 = [V]^2 = 0$, $[U] \cdot [V] = 1$, a,b and c_i are some integers.

The composition of blowing down and the projection map $\Sigma_3 \times \mathbb{S}^2 \to \Sigma_3$ give rise to a degree-d map $F \to \Sigma_3$. The degree d must be a. Also, since the fiber of any $\mathbb{S}^2$-bundle has pseudo-holomorphic representative [48] then the degree of the map $F \to \Sigma_2$ is positive by the positivity of intersection.

Let F be a singular fiber. Since the normalization of F has genus ≤ 6, Theorem 5.1.12 yields to the inequality

$$6 - 1 \geq g(F) - 1 \geq a(3 - 1).$$

This implies that $a \leq 2$.

Since $[F]^2 = 0$, we have

$$2ab = \sum_{i=1}^{3} c_i^2 \tag{5.6}$$

Also, applying the adjunction formula (5.4) for $g(F) = 7$ and $h = g(\Sigma_3) = 3$, we get

$$12 = 4a - 2b - \sum_{i=1}^{3} c_i. \tag{5.7}$$

If $a = 0$, then $c_i = 0$. In this case $[F] = b[V]$. It follows from Theorem 5.1.14 that $b = \pm 1$. But the homology class $[F]$ can not be represented by a smoothly embedded sphere [58].

For $a = 1$, the identities (5.6) and (5.7) imply that

$$\sum_{i=1}^{3} c_i^2 = 2b \text{ and } \sum_{i=1}^{3} c_i = -8 - 2b.$$

Therefore, we get

$$\sum_{i=1}^{3} c_i^2 + \sum_{i=1}^{3} c_i = -8.$$

This yields to $\sum_{i=1}^{3}(c_i + \frac{1}{2})^2 = -\frac{29}{4}$, which is a contradiction.

In the case $a = 2$, the identities (5.6) and (5.7) result in

$$\sum_{i=1}^{3} c_i^2 = 4b \text{ and } \sum_{i=1}^{3} c_i = -4 - 2b$$

implying that $\sum_{i=1}^{3}(c_i + 1)^2 = -5$, which is a contradiction. This proves that the manifold $\Sigma_3 \times \mathbb{S}^2 \# 3\overline{\mathbb{C}P^2}$ does not admit a genus-7 Lefschetz fibrations over $\mathbb{S}^2$. □

Since the number N_g^h is known for $g \leq 3$ (Theorem 5.1.19), we will examine N_g^h for $g \geq 4$.

Theorem 5.2.3. *Let N_g^h be the minimal number of singular fibers in a hyperelliptic Lefschetz fibration over $\mathbb{S}^2$. Then*

1. $N_4^h = 12$,
2. $N_5^h \geq 15$,
3. $N_6^h = 16$,
4. $N_7^h \geq 17$,
5. $N_8^h \in \{19, 20\}$,
6. $N_9^h \geq 24$,
7. $N_{10}^h \in \{23, 24\}$.

Proof. The proof is divided into a series of lemmas, Lemma 5.2.4-Lemma 5.2.10. □

Lemma 5.2.4. $N_4^h = 12.$

Proof. Assume that $N_4^h < 12$, so that we have a hyperelliptic genus-4 Lefschetz fibration X. Let n and $s = s_1 + s_2$ denote the number of nonseparating and separating vanishing cycles. Hence $n + s < 12$.

The equation (5.1) leads to $n + 12s_1 + 4s_2 \equiv 0 \pmod{18}$, so that n is even. Moreover, we have $n \geq 6$ using Theorem 5.1.17. The signature and the Euler characteristic are given as

$$\sigma(X) = \frac{-5n + 3s_1 + 7s_2}{9}$$

and

$$e(X) = 4 - 4g + n + s = -12 + n + s_1 + s_2.$$

The possible values of (n, s_1, s_2) and $e(X), \sigma(X), c_1^2(X)$ are as follows:

	(n, s_1, s_2)	$e(X)$	$\sigma(X)$	$c_1^2(X)$
(1)	(6,1,0)	-5	-3	-19
(2)	(6,4,0)	-2	-2	-10
(3)	(6,0,3)	-3	-1	-9
(4)	(8,2,1)	-1	-3	-11

We now rule out all cases:

Case (1). In this case, $c_1^2(X) = -19 < 4 - 4g = -12$. This contradicts to Theorem 5.1.10.

Cases (2)-(4). In these cases, $c_1^2(X) < 2 - 2g = -6$. Theorem 5.1.11 implies that X is a blow up of a rational or ruled surface. Moreover, using the inequality in Lemma 5.1.9, one can conclude that X can not be simply-connected and so it is a blow up of a ruled surface. Thus we have $b_2^+(X) = 1$. The equalities

$$\begin{aligned} e(X) &= 2 - 2b_1(X) + b_2^+(X) + b_2^-(X) = 3 - 2b_1(X) + b_2^-(X) \\ \sigma(X) &= b_2^+(X) - b_2^-(X) = 1 - b_2^-(X) \end{aligned}$$

imply that $b_1(X) = 4$. But, then in the cases (2) and (3), the inequality in Lemma 5.1.9

$$\sigma(X) \leq n - s - 2(2g - b_1(X)) = n - s - 8$$

does not hold. In the case $(n, s_1, s_2) = (8, 2, 1)$, since $(b_1(X), b_2^+, b_2^-) = (4, 1, 4)$, X is diffeomorphic to $\Sigma_2 \times S^2 \# 3\overline{\mathbb{C}P^2}$, which is impossible by Lemma 5.2.1.

Since there is a hyperelliptic genus-4 Lefschetz fibration with 12 singular fibers by Theorem 5.1.19, $N_4^h = 12$. $\square$

Lemma 5.2.5. $N_5^h \geq 15$.

Proof. Suppose that $N_5^h < 15$ so that we have a hyperelliptic genus-5 Lefschetz fibration X. Let n and $s = s_1 + s_2$ be the number of nonseparating and separating vanishing cycles. Hence $n + s < 15$.

The equation (5.1) turns out

$$n + 12s_1 - 4s_2 \equiv 0 \pmod{44}$$

so that n is divided by 4. It follows from the Theorem 5.1.17 that $n \geq 8$. The signature and the Euler characteristic are computed as

$$\sigma(X) = \frac{-6n + 5s_1 + 13s_2}{11}$$

and

$$e(X) = 4 - 4g + n + s = -16 + n + s_1 + s_2.$$

Hence the possible values of (n, s_1, s_2) are as follows:

	(n, s_1, s_2)	$e(X)$	$\sigma(X)$	$c_1^2(X)$	$\chi_h(X)$
(1)	(8,2,0)	-6	-2	-18	-2
(2)	(8,3,0)	-5	-3	-19	-2
(3)	(8,1,5)	-2	2	2	0
(4)	(8,4,3)	-1	1	1	0

We now eliminate all cases:

Case (1), (2). In these cases, $c_1^2(X) < 4 - 4g = -16$. This is impossible by Theorem 5.1.10.

Case (3), (4). In these cases, $\sigma(X) > n - s - 4$. This contradicts to Lemma 5.1.8. Therefore, N_5^h can not be less than 15.

$\square$

Lemma 5.2.6. $N_6^h = 16.$

Proof. Suppose that $N_6^h < 16$ so that we have a hyperelliptic genus-6 Lefschetz fibration X. Let n and $s = s_1 + s_2 + s_3$ denote the number of nonseparating and separating vanishing cycles, respectively. Thus, $n + s < 16$. The equation (5.1) turns out to be

$$n + 12s_1 + 14s_2 + 6s_3 \equiv 0 \pmod{26},$$

so that n is even. The signature formula and the Euler characteristic computation give rise to

$$\sigma(X) = \frac{-7n + 7s_1 + 19s_2 + 23s_3}{13} \tag{5.8}$$

and

$$e(X) = 4 - 4g + n + s = -20 + n + s_1 + s_2 + s_3. \tag{5.9}$$

Therefore, the possible values of (n, s_1, s_2, s_3) and $e(X), \sigma(X), c_1^2(X)$ are as follows:

	(n, s_1, s_2, s_3)	$e(X)$	$\sigma(X)$	$c_1^2(X)$
(1)	(10,0,3,0)	-7	-1	-17
(2)	(10,3,0,1)	-6	-2	-18
(3)	(10,2,0,3)	-5	1	-7
(4)	(10,1,4,0)	-5	1	-7
(5)	(12,0,1,0)	-7	-5	-29
(6)	(12,1,2,0)	-5	-3	-19
(7)	(14,1,0,0)	-5	-7	-31

We now eliminate all cases:

Cases (5) and (7). In these cases,

$$c_1^2(X) < 4 - 4g = -22.$$

This contradicts to Theorem 5.1.10.

Cases (1), (2) and (6). In these cases, $c_1^2 < 2 - 2g = -10$. Hence, X is a blow up of rational or ruled surface by Theorem 5.1.11. Also, the inequality in Lemma 5.1.9 implies that X is not simply connected and so it is a blow up of a ruled surface. Thus, $b_2^+(X) = 1$. However, this contradicts to Theorem 5.1.16.

Cases (3) and (4). In these cases, we have the following identities:

$$\begin{aligned} \sigma(X) &= b_2^+(X) - b_2^-(X) = 1, & (5.10) \\ e(M) &= 2 - 2b_1(X) + b_2^+(X) + b_2^-(X) = -5. & (5.11) \end{aligned}$$

So, the equations (5.10) and (5.11) yield to

$$\begin{aligned} b_2^+(X) &= b_1(X) - 3, & (5.12) \\ b_2^-(X) &= b_1(X) - 4. & (5.13) \end{aligned}$$

Observe that M cannot be a rational surface because $b_1(M) > 0$ by the inequality in Lemma 5.1.9. Also, X is not a blow up of a ruled surface since ruled surfaces have $\sigma \leq 0$. Let $\widetilde{X}$ be the minimal model of X so that $X \cong \widetilde{X} \# k\overline{\mathbb{CP}}^2$ for some non-negative integer k. It follows from Theorem 2.1.18 that $c_1^2(\widetilde{X}) \geq 0$. Also, using the equation

$$c_1^2(\widetilde{X}) = c_1^2(X) + k = -7 + k,$$

we have $k \geq 7$. It is known that $b_2^-(X) \geq k \geq 7$. The identity (5.13) gives rise to $b_1(X) \geq 11$. Since $b_1(X) \leq 2g - 1 = 11$ by the theory of Lefcshetz fibrations, $b_1(X) = 11$. However, this contradicts with Lemma 5.1.15.

Hence N_6^h cannot be less than 16. Since there exists a genus-6 hyperelliptic Lefschetz fibration with 16 singular fiber by Theorem 5.1.19, $N_6^h = 16$. □

Lemma 5.2.7. $N_7^h \geq 17$.

Proof. Suppose that $N_7^h < 17$, so that we have a hyperelliptic Lefschetz fibration X. Let n and $s = s_1 + s_2 + s_3$ denote the number of nonseparating and separating vanishing cycles. Thus, $n + s < 17$.

The equation (5.1) yields to

$$n + 12s_1 - 20s_2 + 24s_3 \equiv 0 \pmod{60},$$

so that n is divided by 4. We get $n \geq 12$ by Theorem 5.1.17. The signature formula and the Euler characteristic are

$$\sigma(X) = \frac{-8n + 19s_1 + 25s_2 + 33s_3}{15}$$

and

$$e(X) = 4 - 4g + n + s = -24 + n + s_1 + s_2 + s_3.$$

The possible values of (n, s_1, s_2, s_3) and $e(X), \sigma(X), c_1^2(X)$ are as follows:

	(n, s_1, s_2, s_3)	$e(X)$	$\sigma(X)$	$c_1^2(X)$
(1)	(12,0,0,2)	-10	-2	-26
(2)	(12,2,0,1)	-9	-3	-27
(3)	(12,4,0,0)	-8	-4	-28

In all cases, the manifold X has $c_1^2(X) < 4 - 4g = -24$. This contradicts to Theorem 5.1.10. Therefore, $N_7^h \leq 17$.

□

Lemma 5.2.8. $N_8^h \in \{19, 20\}$.

Suppose that $N_8^h < 19$. so that there exists a hyperelliptic genus-4 Lefschetz fibration X. Let n and $s = s_1 + s_2 + s_3 + s_4$ be the number of nonseparating and separating vanishing cycles. Hence $n + s < 19$. For $g = 8$, the equation (5.1) turns out to be

$$n + 12s_1 + 6s_2 + 16s_3 + 8s_4 \equiv 0 \pmod{34},$$

so that n is even. Using Theorem 5.1.17, we have $n \geq 14$. The signature and the Euler characteristic of X are given as

$$\sigma(X) = \frac{-9n + 11s_1 + 31s_2 + 43s_3 + 47s_4}{17} \tag{5.14}$$

and

$$e(X) = 4 - 4g + n + s = -28 + n + s_1 + s_2 + s_3 + s_4. \tag{5.15}$$

Therefore, the possible values of (n, s_1, s_2, s_3, s_4) and $e(X), \sigma(X)$ and $c_1^2(X)$ are as follows:

	(n, s_1, s_2, s_3, s_4)	$e(X)$	$\sigma(X)$	$c_1^2(X)$
(1)	(14,1,0,0,1)	-12	-4	-36
(2)	(14,0,2,0,1)	-11	-1	-25
(3)	(14,0,1,3,0)	-10	2	-14
(4)	(14,4,1,0,0)	-9	-3	-27
(5)	(14,2,1,1,1)	-9	1	-15
(6)	(16,1,1,0,0)	-10	-6	-38
(7)	(16,0,3,0,0)	-9	-3	-27
(8)	(18,0,0,0,1)	-9	-7	-39

Now, we eliminate all cases:

Cases (1), (6), (8). In these cases, $c_1^2 < 4 - 4g = -28$. This contradicts to Theorem 5.1.10.

Cases (2), (4), (5), (7). In these cases $c_1^2 < 2 - 2g = -14$. Hence, the manifold X is a blow up of rational or ruled surface by Theorem 5.1.11. Also, since $b_1(X) > 0$ by Lemma 5.1.9, it can not be a rational surface. Hence, we have $b_2^+(X) = 1$. But this contradicts to Theorem 5.1.16.

Case (3). In this case, X is not rational nor ruled by Theorem 5.1.11. Let $\widetilde{X}$ be the minimal model of X. Then $X \cong \widetilde{X} \# k\overline{\mathbb{CP}}^2$ for some integer $k \geq 0$. The 4-manifold X has

$$\sigma(X) = b_2^+(X) - b_2^-(X) = 2 \tag{5.16}$$

and

$$e(X) = 2 - 2b_1(X) + b_2^+(X) + b_2^-(X) = -10. \tag{5.17}$$

So, the identities (5.16) and (5.17) result in

$$b_2^+(X) = b_1(X) - 5 \tag{5.18}$$

$$b_2^-(X) = b_1(X) - 7. \tag{5.19}$$

The minimal 4-manifold $\widetilde{X}$ must satisfy $c_1^2(\widetilde{X}) \geq 0$ by Theorem 2.1.18. We have the following identity:

$$c_1^2(\widetilde{X}) = c_1^2(X) + k = -14 + k \geq 0.$$

Then we get $k \geq 14$. It follows from

$$b_2^-(X) = b_1(X) - 7 \geq k \geq 14$$

that $b_1(X) \geq 21$. However, any genus-g Lefschetz fibration must satisfy $b_1 < 2g$. Therefore, there exist no such a Lefschetz fibration.

Since there exists a genus-8 hyperelliptic Lefschetz fibration with 20 singular fibers by Theorem 5.1.19, one can conclude that $N_8^h = 19$ or 20.

Lemma 5.2.9. $N_9^h \geq 24$.

Proof. Suppose that $N_9^h < 24$, so that we have a hyperelliptic Lefschetz fibration X. Let n and $s = s_1 + s_2 + s_3 + s_4$ denote the number of nonseparating and separating vanishing cycles. Hence, $n + s < 24$.

The equation (5.1) gives rlse to

$$n + 12s_1 + 40s_2 + 84s_3 + 144s_4 \equiv 0 \pmod{76},$$

so that n is divided by 4. We have $n \geq 16$ by Theorem 5.1.17. The signature formula and the Euler characteristic are

$$\sigma(X) = \frac{-10n + 13s_1 + 37s_2 + 53s_3 + 61s_4}{19}$$

and

$$e(X) = 4 - 4g + n + s = -32 + n + s_1 + s_2 + s_3 + s_4.$$

	(n, s_1, s_2, s_3, s_4)	$e(X)$	$\sigma(X)$	$c_1^2(X)$
(1)	(16,0,0,0,2)	-14	-2	-34
(2)	(16,1,1,1,0)	-13	-3	-35
(3)	(16,0,0,1,3)	-12	4	-12
(4)	(16,5,0,0,0)	-11	-5	-37
(5)	(16,1,1,2,1)	-11	3	-13
(6)	(16,0,3,2,0)	-11	3	-13
(7)	(16,3,1,0,2)	-10	2	-14
(8)	(16,3,0,3,0)	-10	2	-14
(9)	(16,2,3,0,1)	-10	2	-14

	(n, s_1, s_2, s_3, s_4)	$e(X)$	$\sigma(X)$	$c_1^2(X)$
(10)	(16,1,5,0,0)	-10	2	-14
(11)	(16,0,0,2,4)	-10	10	10
(12)	(16,5,0,1,1)	-9	1	-15
(13)	(16,4,2,1,0)	-9	1	-15
(14)	(16,2,0,0,5)	-9	9	9
(15)	(16,1,2,0,4)	-9	9	9
(16)	(16,1,1,3,2)	-9	9	9
(17)	(16,1,0,6,0)	-9	9	9
(18)	(16,0,4,0,3)	-9	9	9
(19)	(16,0,3,3,1)	-9	9	9
(20)	(20,0,1,2,0)	-9	-3	-27

Now, we eliminate all cases:

Cases (1), (2), (4). In these cases, $c_1^2 < 4 - 4g = -32$. This contradicts to Theorem 5.1.10.

Case (20). In this case $c_1^2 < 2 - 2g = -16$. Hence, the manifold X is a blow up of rational or ruled surface by Theorem 5.1.11. Also, since $b_1(X) > 0$ by Lemma 5.1.9, it is a ruled surface. Hence, we have $b_2^+(X) = 1$. But this contradicts to Theorem 5.1.16.

Cases (11), (14)-(19). In these cases, $\sigma(X) > n - s - 4$. However, this is a contradiction with Lemma 5.1.8.

Case (3). In this case, X is not rational nor ruled by Theorem 5.1.11. Let $\widetilde{X}$ be the minimal model of X. Then $X \cong \widetilde{X} \# k\overline{\mathbb{CP}}^2$ for some integer $k \geq 0$. The 4-manifold X has

$$\sigma(X) = b_2^+(X) - b_2^-(X) = 4 \tag{5.20}$$

and

$$e(X) = 2 - 2b_1(X) + b_2^+(X) + b_2^-(X) = -12. \tag{5.21}$$

So, the identities (5.20) and (5.21) result in

$$\begin{aligned} b_2^+(X) &= b_1(X) - 5 \quad (5.22) \\ b_2^-(X) &= b_1(X) - 9. \quad (5.23) \end{aligned}$$

The minimal 4-manifold $\widetilde{X}$ must satisfy $c_1^2(\widetilde{X}) \geq 0$ by Theorem 2.1.18. We have the following identity:

$$c_1^2(\widetilde{X}) = c_1^2(X) + k = -12 + k \geq 0.$$

Then we get $k \geq 12$. It follows from

$$b_2^-(X) = b_1(X) - 9 \geq k \geq 12$$

that $b_1(X) \geq 21$. However, any genus-g Lefschetz fibration must satisfy $b_1 < 2g$.

Cases (5), (6). In these cases, X is not rational nor ruled by Theorem 5.1.11. Let $\widetilde{X}$ be the minimal model of X. Then $X \cong \widetilde{X} \# k\overline{\mathbb{CP}}^2$ for some integer $k \geq 0$. The 4-manifold X has

$$\sigma(X) = b_2^+(X) - b_2^-(X) = 3 \quad (5.24)$$

and

$$e(X) = 2 - 2b_1(X) + b_2^+(X) + b_2^-(X) = -11. \quad (5.25)$$

So, the identities (5.24) and (5.25) result in

$$\begin{aligned} b_2^+(X) &= b_1(X) - 5 \quad (5.26) \\ b_2^-(X) &= b_1(X) - 8. \quad (5.27) \end{aligned}$$

The minimal 4-manifold $\widetilde{X}$ must satisfy $c_1^2(\widetilde{X}) \geq 0$ by Theorem 2.1.18. We have the following identity:

$$c_1^2(\widetilde{X}) = c_1^2(X) + k = -13 + k \geq 0.$$

Then we get $k \geq 13$. It follows from

$$b_2^-(X) = b_1(X) - 8 \geq k \geq 13$$

that $b_1(X) \geq 21$. However, any genus-g Lefschetz fibration must satisfy $b_1 < 2g$.

Cases (7)-(10). In these cases, X is not rational nor ruled by Theorem 5.1.11. Let $\widetilde{X}$ be the minimal model of X. Then $X \cong \widetilde{X} \# k\overline{\mathbb{CP}}^2$ for some integer $k \geq 0$. The 4-manifold X has

$$\sigma(X) \;=\; b_2^+(X) - b_2^-(X) = 2 \tag{5.28}$$

and

$$e(X) \;=\; 2 - 2b_1(X) + b_2^+(X) + b_2^-(X) = -10. \tag{5.29}$$

So, the identities (5.28) and (5.29) result in

$$b_2^+(X) \;=\; b_1(X) - 5 \tag{5.30}$$

$$b_2^-(X) \;=\; b_1(X) - 7. \tag{5.31}$$

The minimal 4-manifold $\widetilde{X}$ must satisfy $c_1^2(\widetilde{X}) \geq 0$ by Theorem 2.1.18. We have the following identity:

$$c_1^2(\widetilde{X}) = c_1^2(X) + k = -14 + k \geq 0.$$

Then we get $k \geq 14$. It follows from

$$b_2^-(X) = b_1(X) - 7 \geq k \geq 14$$

that $b_1(X) \geq 21$. This contradicts with the fact that any genus-g Lefschetz fibration has $b_1 < 2g$.

Cases (12), (13). In these cases, X is not rational nor ruled by Theorem 5.1.11. Let $\widetilde{X}$ be the minimal model of X. Then $X \cong \widetilde{X} \# k\overline{\mathbb{CP}}^2$ for some integer $k \geq 0$. The 4-manifold X has

$$\sigma(X) \;=\; b_2^+(X) - b_2^-(X) = 1 \tag{5.32}$$

and

$$e(X) \;=\; 2 - 2b_1(X) + b_2^+(X) + b_2^-(X) = -9. \tag{5.33}$$

So, the identities (5.32) and (5.33) result in

$$b_2^+(X) \;=\; b_1(X) - 5 \tag{5.34}$$

$$b_2^-(X) \;=\; b_1(X) - 6. \tag{5.35}$$

The minimal 4-manifold $\widetilde{X}$ must satisfy $c_1^2(\widetilde{X}) \geq 0$ by Theorem 2.1.18. We have the following identity:

$$c_1^2(\widetilde{X}) = c_1^2(X) + k = -15 + k \geq 0.$$

Then we get $k \geq 15$. It follows from

$$b_2^-(X) = b_1(X) - 6 \geq k \geq 15$$

that $b_1(X) \geq 21$. This contradicts with the fact that any genus-g Lefschetz fibration has $b_1 < 2g$. Thus, there exist no such a Lefschetz fibration. Therefore $N_9^h \geq 24$. $\square$

Lemma 5.2.10. $N_{10}^h \in \{23, 24\}$.

Proof. Assume that $N_{10}^h < 23$, so that we have a hyperelliptic genus-10 Lefschetz fibration X. Let n and $s = s_1 + s_2 + s_3 + s_4 + s_5$ be the number of nonseparating and separating vanishing cycles. Thus, $n + s < 23$.

The equation (5.1) gives rise to

$$n + 12s_1 - 2s_2 + 18s_4 + 10s_5 \equiv 0 \pmod{42}$$

so that n is even. Also, using Theorem 5.1.17, we have $n \geq 16$. The signature and the Euler characteristic are given as

$$\sigma(X) = \frac{-11n + 15s_1 + 43s_2 + 63s_3 + 75s_4 + 79s_5}{21}$$

and

$$e(X) = 4 - 4g + n + s = -36 + n + s_1 + s_2 + s_3 + s_4 + s_5.$$

Thus, the possible values of $(n, s_1, s_2, s_3, s_4, s_5)$ and $e(X), \sigma(X), c_1^2(X)$ are as follows:

	$(n, s_1, s_2, s_3, s_4, s_5)$	$e(X)$	$\sigma(X)$	$c_1^2(X)$
(1)	(16,0,1,0,1,1)	-17	1	-31
(2)	(16,1,2,0,1,0)	-16	0	-32
(3)	(16,0,1,1,1,1)	-16	4	-20
(4)	(16,1,2,1,1,0)	-15	3	-21
(5)	(16,1,0,0,2,2)	-15	7	-9
(6)	(16,0,2,0,0,3)	-15	7	-9
(7)	(16,0,1,2,1,1)	-15	7	-9
(8)	(16,4,0,0,0,2)	-14	2	-22
(9)	(16,2,1,0,2,1)	-14	6	-10
(10)	(16,1,3,0,0,2)	-14	6	-10
(11)	(16,1,2,2,1,0)	-14	6	-10
(12)	(16,1,0,1,2,2)	-14	10	2
(13)	(16,0,2,1,0,3)	-14	10	2
(14)	(16,0,2,0,4,0)	-14	10	2
(15)	(16,0,0,0,1,5)	-14	14	14
(16)	(16,5,1,0,0,1)	-13	1	-23
(17)	(16,4,0,1,0,2)	-13	5	-11
(18)	(16,3,2,0,2,0)	-13	5	-11
(19)	(16,2,4,0,0,1)	-13	5	-11
(20)	(16,2,1,1,2,1)	-13	9	1
(21)	(16,1,3,1,0,2)	-13	9	1
(22)	(16,1,2,3,1,0)	-13	9	1
(23)	(16,1,1,0,1,4)	-13	13	13
(24)	(16,1,0,2,2,2)	-13	13	13
(25)	(16,0,5,0,2,0)	-13	9	1
(26)	(16,0,2,2,0,3)	-13	13	13
(27)	(16,0,2,1,4,0)	-13	13	13
(28)	(16,0,1,4,1,1)	-13	13	13
(29)	(16,0,0,1,1,5)	-13	17	25
(30)	(16,0,0,0,5,2)	-13	17	25
(31)	(18,2,0,0,0,0)	-16	-8	-56
(32)	(18,2,0,1,0,0)	-15	-5	-45

	$(n, s_1, s_2, s_3, s_4, s_5)$	$e(X)$	$\sigma(X)$	$c_1^2(X)$
(33)	(18,2,0,2,0,0)	-14	-2	-34
(34)	(18,1,0,0,3,0)	-14	2	-22
(35)	(18,0,2,0,0,1)	-14	2	-22
(36)	(18,4,0,0,1,0)	-13	-3	-35
(37)	(18,2,0,3,0,0)	-13	1	-23
(38)	(18,1,3,0,1,0)	-13	1	-23
(39)	(18,1,0,1,3,0)	-13	5	-11
(40)	(18,0,2,1,1,1)	-13	5	-11
(41)	(20,1,0,0,0,1)	-14	-6	-46
(42)	(20,2,1,0,0,0)	-13	-7	-47
(43)	(20,1,0,1,0,1)	-13	-3	-35

We now rule out all cases:

Cases (31), (32), (41), (42). In these cases, $c_1^2(X) < 4 - 4g = -36$. This contradicts to Theorem 5.1.10.

Cases (1)-(4), (8), (16), (33)-(38), (43). In these cases, $c_1^2(X) < 2-2g = -18$. Thus, X is a blow up of rational or ruled surface by Theorem 5.1.11. Also, since $b_1(X) > 0$ by Lemma 5.1.9, it can not be a rational surface. Hence, we have $b_2^+(X) = 1$. But this contradicts to Theorem 5.1.16.

Cases (12)-(15), (20)-(30). In these cases, $\sigma(X) > n - s - 4$. However, this is a contradiction with Lemma 5.1.8.

Cases (17)-(19), (39), (40). In these cases, X is not rational nor ruled by Theorem 5.1.11. Let $\widetilde{X}$ be the minimal model of X. Then $X \cong \widetilde{X} \# k\overline{\mathbb{CP}}^2$ for some integer $k \geq 0$. The manifold X has

$$\sigma(X) = b_2^+(X) - b_2^-(X) = 5 \tag{5.36}$$

and

$$e(X) = 2 - 2b_1(X) + b_2^+(X) + b_2^-(X) = -13. \tag{5.37}$$

Thus, we have

$$b_2^+(X) = b_1(X) - 5 \tag{5.38}$$

$$b_2^-(X) = b_1(X) - 10. \tag{5.39}$$

Since the minimal 4-manifold $\widetilde{X}$ must satisfy $c_1^2(\widetilde{X}) \geq 0$ by Theorem 2.1.18. We have

$$c_1^2(\widetilde{X}) = c_1^2(X) + k = -11 + k \geq 0.$$

Then we conclude that $k \geq 11$. It follows from

$$b_2^-(X) = b_1(X) - 10 \geq k \geq 11$$

that $b_1(X) \geq 21$. However, any genus-g Lefschetz fibration must satisfy $b_1 < 2g$. Therefore, there exists no such a Lefschetz fibration.

Cases (5)-(7). In these cases, since X is not rational nor ruled by Theorem 5.1.11, then for some integer $k \geq 0$, $X \cong \widetilde{X} \# k\overline{\mathbb{CP}}^2$ where $\widetilde{X}$ is the minimal model of X. We have

$$\sigma(X) = b_2^+(X) - b_2^-(X) = 7 \tag{5.40}$$

$$e(X) = 2 - 2b_1(X) + b_2^+(X) + b_2^-(X) = -15. \tag{5.41}$$

So, we get the identities

$$b_2^+(X) = b_1(X) - 5 \tag{5.42}$$

$$b_2^-(X) = b_1(X) - 12. \tag{5.43}$$

The minimal 4-manifold $\widetilde{X}$ must satisfy $c_1^2(\widetilde{X}) \geq 0$ by Theorem 2.1.18. We have the following identity:

$$c_1^2(\widetilde{X}) = c_1^2(X) + k = -9 + k \geq 0.$$

Then we get $k \geq 9$. It follows from

$$b_2^-(X) = b_1(X) - 12 \geq k \geq 9$$

that $b_1(X) \geq 21$. However, this contradicts with the fact that any genus-g Lefschetz fibration must satisfy $b_1 < 2g$.

Cases (9)-(11). In these cases, it follows from Theorem 5.1.11 that X is not rational nor ruled. Thus, $X \cong \widetilde{X}\#k\overline{\mathbb{CP}}^2$ for some integer $k \geq 0$ where $\widetilde{X}$ is the minimal model of X. We get

$$\begin{aligned} \sigma(X) &= b_2^+(X) - b_2^-(X) = 6, && (5.44) \\ e(X) &= 2 - 2b_1(X) + b_2^+(X) + b_2^-(X) = -14. && (5.45) \end{aligned}$$

Consequently, we obtain the equations

$$\begin{aligned} b_2^+(X) &= b_1(X) - 5, && (5.46) \\ b_2^-(X) &= b_1(X) - 11. && (5.47) \end{aligned}$$

The minimal $\widetilde{X}$ has $c_1^2(\widetilde{X}) \geq 0$ by Theorem 2.1.18. We have the following identity:

$$c_1^2(\widetilde{X}) = c_1^2(X) + k = -10 + k \geq 0.$$

This implies that $k \geq 10$. It follows from

$$b_2^-(X) = b_1(X) - 11 \geq k \geq 10$$

that $b_1(X) \geq 21$. And so again we get a contradiction since b_1 can not be grater than 20 for any genus-10 Lefschetz fibration.

Therefore, the number N_{10}^h can not be less than 23. Since we have a genus-10 hyperelliptic Lefschetz fibration with 24 singular fibers by Theorem 5.1.19, N_{10}^h is either 23 or 24. $\square$

As long as genus-g increases, the number of possibilities of n and s increases, the number of irreducible and reducible fibers, respectively. Hence, it is hard to find the exact value of N_g^h. The odd case is more harder because of the upper bound of N_g^h which is $5g - 3$. However, one can improve the lower bound of N_g^h for small odd g as in the case of $g = 5$ and $g = 7$ in the Theorem 5.2.3. For general case we have the following:

Proposition 5.2.11. *Let $f : X \to \mathbb{S}^2$ be a genus-g hyperelliptic Lefschetz fibration with $n + s < 2g + 4$ and $g > 6$. Then the signature of X, $\sigma(X)$ is positive.*

Proof. Suppose that X admits a genus-g hyperelliptic Lefschetz fibration X with $n + s < 2g + 4$. Consider the 4-manifold $Y = X\#_f X$ obtained by fiber sum of X

with itself. The manifold Y admits a genus-g Lefschetz fibration over $\mathbb{S}^2$. It follows from Theorem 5.1.13 that Y is minimal. By Theorem 2.1.18 we have $c_1^2(Y) \geq 0$. Hence, we have the following inequality using Lemma 2.1.10:

$$\begin{aligned}
0 \leq c_1^2(Y) &= 3\sigma(Y) + 2e(Y) \\
&= 3(2\sigma(X)) + 2(4 - 4g + 2(n + s)) \\
&\leq 6\sigma(X) + 8 - 8g + 4(2g + 3) \\
&= 6\sigma(X) + 20.
\end{aligned}$$

This implies that $\sigma(X) \geq -3$. The manifold X is not a blow up of ruled surface by Theorem 5.1.16. Also, it cannot be a blow up of a rational surface since $b_1(X) > 0$ by the inequality in Lemma 5.1.9. So, it follows from Theorem 5.1.11 that $c_1^2(X) \geq 2 - 2g$. Therefore we get:

$$\begin{aligned}
2 - 2g \leq c_1^2(X) &= 3\sigma(X) + 2e(X) \\
&= 3\sigma(X) + 2(4 - 4g + n + s) \\
&\leq 3\sigma(X) + 2(4 - 4g + 2g + 3) \\
&= 3\sigma(X) + 14 - 4g.
\end{aligned}$$

The above inequality results in $\sigma(X) \geq \dfrac{2g-12}{3}$, which implies that $\sigma(X) > 0$ when $g > 6$. $\square$

Remark 5.2.12. *The above proposition implies that every hyperelliptic genus-g Lefschetz fibration with $n + s < 2g + 4$ has $b_1(X) > \dfrac{4g-19}{3}$ using the equation $\sigma(X) \leq n - s - 2(2g - b_1(X))$ by Lemma 5.1.9. However, there exists no known such a Lefschetz fibration.*

5.3 The number of singular fibers in a holomorphic hyperelliptic Lefschetz fibration over $\mathbb{S}^2$

In this section, we will focus on holomorphic Lefschetz fibrations and we will examine their minimal number of singular fibers using the classification of complex surfaces. Let M_g^h denote the minimal number of singular fibers in a holomorphic hyperelliptic Lefschetz fibration of genus g over $\mathbb{S}^2$.

Recall that n and s denote the number of nonsepataing and separating vanishing cycles, respectively.

Lemma 5.3.1. *Let $f : X \to \mathbb{S}^2$ be a genus-g holomorphic hyperelliptic Lefschetz fibration with $g \geq 6$ and even or $g \geq 9$ and odd. If $n + s < 2g + 4$ then $n \geq 2g + 2$.*

Proof. Suppose that there exist a holomorphic hyperelliptic Lefschetz fibration with $n < 2g + 2$.

Let us first consider $n < 2g$. Using the inequality $\sigma(X) \leq n - s - 4$ by Lemma 5.1.8 , we have

$$\begin{aligned}
\chi_h(X) &= \frac{e(X) + \sigma(X)}{4} \\
&\leq \frac{(4 - 4g + n + s) + (n - s - 4)}{4} \\
&= \frac{2n - 4g}{4} < 0.
\end{aligned}$$

Now, assume that $n = 2g$, which gives rise to $s \leq 3$. By the signature formula (5.2), we get

$$\begin{aligned}
\sigma(X) &= -\frac{g+1}{2g+1}n + \sum_{h=1}^{[g/2]}\left(\frac{4h(g-h)}{2g+1} - 1\right)s_h \\
&\leq -\frac{g+1}{2g+1}(2g) + 3\left(\frac{4(g/2)(g/2)}{2g+1} - 1\right) \\
&= \frac{g^2 - 8g - 3}{2g+1} \\
&< \frac{g}{2} - 3
\end{aligned}$$

and also, using $n + s \leq 2g + 3$ we have

$$\begin{aligned}
\chi_h(X) &= \frac{e(X) + \sigma(X)}{4} \\
&< \frac{4 - 4g + 2g + 3 + (g/2) - 3}{4} \\
&\leq \frac{-3(g/2) + 4}{4} < 0.
\end{aligned}$$

Hence, we conclude that $\chi_h(X) < 0$ if $n \leq 2g$. By classification of complex surfaces X is diffeomorphic to a blow up of a ruled surface which implies that $b_2^+ = 1$. However, this is a contradiction with Theorem 5.1.16. Therefore, $n > 2g$. Since the number n is even by equality (5.1), we get the required inequality.

□

Now, we are ready to prove one of the main theorems.

Theorem 5.3.2. *Let $g \geq 4$ and even. Then $M_g^h = 2g+4$.*

Proof. Suppose that we have a holomorphic hyperelliptic Lefschetz fibration X with $n+s < 2g+4$, $g \geq 6$ and even. Hence, $n \geq 2g+2$ by Lemma 5.3.1. The equality (5.1) implies that n is even and also $s = \sum_{h=1}^{[g/2]} s_h > 0$. Thus, $s = 1$.

The signature $\sigma(X)$ of X is computed using the signature formula (5.2) as follows:

$$\begin{aligned}
\sigma(X) &= -\frac{g+1}{2g+1}n + \sum_{h=1}^{[g/2]}(\frac{4h(g-h)}{2g+1} - 1)s_h \\
&\leq -\frac{g+1}{2g+1}(2g+2) + (\frac{4(g/2)(g/2)}{2g+1} - 1) \\
&= -\frac{g^2+6g+3}{2g+1} \\
&< -\frac{g}{2}.
\end{aligned}$$

Using $\sigma(X) < -\frac{g}{2}$, $n = 2g+2$ and $s = 1$, we get:

$$\begin{aligned}
\chi_h(X) &= \frac{e(X)+\sigma(X)}{4} \\
&< \frac{4-4g+2g+3-(g/2)}{4} \\
&\leq \frac{-5(g/2)+7}{4} < 0.
\end{aligned}$$

In this case, the classification of complex surfaces implies that X is a blow up of a ruled surface and hence $b_2^+ = 1$. However, this is impossible if $g \geq 6$ by Theorem 5.1.16.

Now, consider the remaining case, $g = 4$. It follows from Theorem 5.2.3 that minimal number of singular fibers in a genus-4 hyperelliptic Lefschetz fibration is 12. This completes the proof.

□

In the case of g odd and $g \geq 5$, we improve the lower bound of M_g^h. We prove the following theorem.

Theorem 5.3.3. *Let $g \geq 7$ and odd. Then $M_g^h \geq 2g+6$.*

Proof. Suppose that there exist a holomorphic hyperelliptic Lefschetz fibration X with $g \geq 5$, odd and $n+s < 2g+6$.

First consider the case $g \geq 9$. If $n < 2g$ then it can be shown that $\chi_h(X) < 0$ using the inequality $\sigma(X) \leq n-s-4$ as in the proof of Lemma 5.3.1. This implies that $b_2^+ = 1$ by the classification of complex surfaces. But, this gives a contradiction with Theorem 5.1.16. The odd case of the equation (5.1) leads to $n \equiv 0 \pmod 4$. We can conclude that $n \geq 2g+2$. The assumption $n+s < 2g+6$ gives rise to $n = 2g+2$ and $s \leq 3$. Therefore, the signature formula (5.2) implies the following inequality:

$$\begin{aligned}
\sigma(X) &= -\frac{g+1}{2g+1}n + \sum_{h=1}^{[g/2]}(\frac{4h(g-h)}{2g+1} - 1)s_h \\
&\leq -\frac{g+1}{2g+1}(2g+2) + 3(\frac{4(g/2)(g/2)}{2g+1} - 1) \\
&= \frac{g^2-10g-5}{2g+1} \\
&< \frac{g}{2} - 5.
\end{aligned}$$

Then, using the inequality $\sigma(X) < \frac{g}{2} - 5$, the holomorphic Euler characteristic $\chi_h(X)$ of X,

$$\begin{aligned}
\chi_h(X) &= \frac{e(X)+\sigma(X)}{4} = \frac{4-4g+n+s+\sigma(X)}{4} \\
&< \frac{4-4g+2g+5+(g/2)-5}{4} \\
&\leq \frac{-3g}{8} + 1 < 0.
\end{aligned}$$

Hence the classification of complex surfaces implies that X is a blow up of a ruled surface. In this case, $b_2^+(X) = 1$. However, this contradicts to Theorem 5.1.16.

Now consider the $g = 7$ case. By Lemma 5.2.7, the number of singular fibers must be grater than 16. Also , we know by the proof of Lemma 5.2.7 that

$$n \geq 12,$$

$$n \equiv 0 \pmod 4$$

and the equation

$$n + 12s_1 - 20s_2 + 24s_3 \equiv 0 \pmod{60}$$

must be satisfied where $s = s_1 + s_2 + s_3$. Hence the possible values of (n, s_1, s_2, s_3) are as follows:

	(n, s_1, s_2, s_3)	$e(X)$	$\sigma(X)$	$c_1^2(X)$	$\chi_h(X)$
(1)	(12,1,0,4)	-7	3	-5	-1
(2)	(12,0,3,2)	-7	3	-5	-1
(3)	(12,3,0,3)	-6	2	-12	-1
(4)	(12,2,3,1)	-5	1	-7	-1
(5)	(12,0,0,7)	-5	9	-5	1
(6)	(12,5,0,2)	-5	1	-7	-1
(7)	(12,4,3,0)	-5	1	-7	-1
(8)	(16,0,2,1)	-5	-3	-19	-2

Cases (1)-(4),(6)-(7). In these cases, $\chi_h(X) < 0$. Thus, X is a blow up of a ruled surface. However, $\sigma(X)$ must be nonnegative for such a manifold. Hence, we exclude these cases.

Case (5). In this case, the manifold X does not satisfy the inequality $\sigma(X) \leq n - s - 4$ by Lemma 5.1.8.

Case (8). In this case, since $\chi_h(X) < 0$, X is diffeomorphic to a blow up of a ruled surface. Hence $b_2^+ = 1$. We have

$$e(X) = -5 = 2 - 2b_1(X) + b_2^+(X) + b_2^-(X) = 3 - 2b_1(X) + b_2^-(X)$$

and

$$\sigma(X) = -3 = b_2^+(X) - b_2^-(X) = 1 + b_2^-(X).$$

Hence $(b_1(X), b_2^+(X), b_2^-(X)) = (6, 1, 4)$. Therefore, $X = \Sigma_3 \times S^2 \# 3\overline{\mathbb{C}P^2}$. But $\Sigma_3 \times S^2 \# 3\overline{\mathbb{C}P^2}$ does not admit a genus-7 Lefschetz fibration over $\mathbb{S}^2$ by Lemma 5.2.2. This finishes the proof. $\square$

REFERENCES

[1] A. Akhmedov, R.İ. Baykur, and D. Park. Constructing infinitely many smooth structures on small 4-manifolds. *J.Topol.*, 1(2):409–428, 2008.

[2] A. Akhmedov and N. Monden. Genus two Lefschetz fibrations with $b_2^+ = 1$ and $c_1^2 = 1$. *preprint;https://arxiv.org/abs/1509.01853*.

[3] A. Akhmedov and N. Monden. Lefschetz fibrations via monodromy substitutions. *preprint*.

[4] A. Akhmedov and N. Monden. Constructing Lefschetz fibrations via daisy substitution. *Kyoto J. Math.*, 56(3):501–529, 2016.

[5] A. Akhmedov and B. Ozbagci. Exotic Stein fillings with arbitrary fundamental group. *preprint;https://arxiv.org/abs/1212.1743*.

[6] A. Akhmedov and D. Park. Exotic smooth structures on small 4-manifolds with odd signatures. *Invent. Math.*, 181(3):577–603, 2010.

[7] A. Akhmedov and J-Y. Park. Lantern substitution and new symplectic 4-manifolds with ${b_2}^+ = 3$. *Math. Res. Lett.*, 21(1):1–17, 2014.

[8] A. Akhmedov and K. N. Saglam. New exotic 4-manifolds via Luttinger surgery on Lefschetz fibrations. *Internat. J. Math.*, 26(1):1550010–21, 2015.

[9] A. Akhmedov and W. Zhang. The fundamental group of symplectic 4-manifolds with $b_2^+ = 1$. *preprint;https://arxiv.org/abs/1506.08367*.

[10] D. Auroux, S. K. Donaldson, and L. Katzarkov. Luttinger surgery along Lagrangian tori and non-isotopy for singular symplectic plane curves. *Math. Ann*, 326(1):185–203, 2003.

[11] S. Baldridge and P. Kirk. A symplectic manifold homeomorphic but not diffeomorphic to $\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2$. *Geom. Topol.*, 12(2):919–940, 2008.

[12] W. Barth, C. Peters, and A. Van de Ven. *Compact complex surfaces*. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3).

[13] R.İ. Baykur. Small symplectic Calabi-Yau surfaces and exotic 4-manifolds via genus-3 pencils. *preprint; https://arxiv.org/abs/1511.05951*.

[14] R.İ. Baykur and M. Korkmaz. An interesting genus-3 Lefschetz fibration. *preprint*.

[15] R.İ. Baykur and M. Korkmaz. Small Lefschetz fibrations and exotic 4-manifolds. *Math. Ann.*, 367(3-4):1333–1361, 2017.

[16] R.İ. Baykur, M. Korkmaz, and N. Monden. Sections of surface bundles and Lefschetz fibrations. *Trans. Amer. Math. Soc.*, 365(11):5999–6016, 2013.

[17] J. S. Birman and Hilden H. M. On the mapping class groups of closed surfaces as covering spaces. *Ann. of Math. Studies, No. 66, Princeton Univ. Press*, 1971.

[18] V. Braungardt and D. Kotschick. Clustering of critical points in Lefschetz fibrations and the symplectic Szpiro inequality. *Trans. Amer. Math. Soc.*, 355(8):3217–3226, 2003.

[19] C. Cadavid. On a remarkable set of words in the mapping class group. *Thesis (Ph.D.), The University of Texas, Austin*, 1998.

[20] E. Dalyan, E. Medetoĝullari, and M. Pamuk. A note on the generalized matsumoto relation. *Turkish J. Math.*, 41(3):524–536, 2017.

[21] M. Dehn. *Papers on group theory and topology*. Springer-Verlag, New York, 1987. Translated from the German and with introductions and an appendix by John Stillwell, with an appendix by Otto Schreier.

[22] S. K. Donaldson. Irrationality and the h-cobordism conjecture. *J. Differential Geom.*, 26(1):141–168, 1987.

[23] S. K. Donaldson. Lefschetz pencils on symplectic manifolds. *J. Differential Geom.*, 53(2):205–236, 1999.

[24] J. Dorfmeister. Kodaira dimension of fiber sums along spheres. *Geom. Dedicata.*, 177:1–25, 2015.

[25] Y. Eliashberg and L. Polterovich. New applications of Luttinger's surgery. *Comment. Math. Helv.*, 69(4):512–522, 1994.

[26] H. Endo. Meyer's signature cocycle and hyperelliptic fibrations. *Math. Ann.*, 316(2):237–257, 2000.

[27] H. Endo and Y. Z. Gurtas. Positive dehn twist expression for a $\mathbb{Z}_3$ action on σ_g. *preprint; https://arxiv.org/abs/0808.0752*.

[28] H. Endo and Y. Z. Gurtas. Lantern relations and rational blowdowns. *Proc. Amer. Math. Soc.*, 138(3):1131–1142, 2010.

[29] H. Endo, T. E. Mark, and J. Van Horn-Morris. Monodromy substitutions and rational blowdowns. *J. Topol.*, 4(1):227–253, 2011.

[30] H. Endo and S. Nagami. Signature of relations in mapping class groups and non-holomorphic Lefschetz fibrations. *Trans. Amer. Math. Soc.*, 357(8):3179–3199, 2005.

[31] B. Farb and D. Margalit. *A primer on mapping class groups*. Princeton Mathematical Series,49, 2012.

[32] R. Fintushel and R. J. Stern. Rational blowdowns of smooth 4-manifolds. *J. Differential Geom.*, 46(2):181–235, 1997.

[33] R. Fintushel and R. J. Stern. Knots, links, and 4-manifolds. *Invent. Math.*, 134(2):363–400, 1998.

[34] R. Fintushel and R. J. Stern. Pinwheels and nullhomologous surgery on 4-manifolds with $b^+ = 1$. *Algebr. Geom. Topol.*, 11(3):1649–1699, 2011.

[35] M. H. Freedman. The topology of four-dimensional manifolds. *J. Differential Geom.*, 17(3):357–453, 1982.

[36] R. Friedman and J. W. Morgan. *Smooth four-manifolds and complex surfaces*. Springer-Verlag, Berlin, 27, 1994.

[37] R. E. Gompf. A new construction of symplectic manifolds. *Ann. of Math. (2)*, 142(3):527–595, 1995.

[38] R. E. Gompf and A. I. Stipsicz. *4-Manifolds and Kirby calculus*. Graduate Studies in Mathematics, 20, American Mathematical Society, Providence, RI, 1999.

[39] N. Hamada. Sections of the Matsumoto-Cadavid-Korkmaz Lefschetz fibration. *preprint; https://arxiv.org/abs/1610.08458*.

[40] N. Hamada and K. Hayano. Topology of holomorphic Lefschetz pencils on the four-torus. *Algebr. Geom. Topol.*, 18(3):1515–1572, 2018.

[41] D. L. Johnson. Homeomorphisms of a surface which act trivially on homology. *Proc. Amer. Math. Soc.*, 75(1):119–125, 1979.

[42] H. Kneser. Die kleinste bedeckungszahl innerhalb einer klasse von flächenabbildungen. *Math. Ann.*, 103(1):347–358, 1930.

[43] M. Korkmaz. Noncomplex smooth 4-manifolds with Lefschetz fibrations. *Internat. Math. Res. Notices*, (3):115–128, 2001.

[44] M. Korkmaz. Low-dimensional homology groups of mapping class groups: a survey. *Turkish J. Math.*, 26(1):101–114, 2002.

[45] M. Korkmaz. Lefschetz fibrations and an invariant of finitely presented groups. *Int. Math. Res. Not. IMRN*, (9):1547–1572, 2009.

[46] M. Korkmaz and B. Ozbagci. Minimal number of singular fibers in a Lefschetz fibration. *Proc. Amer. Math. Soc.*, 129(5):1545–1549, 2001.

[47] D. Kotschick. On manifolds homeomorphic to $\mathbf{C}\mathrm{P}^2\#8\overline{\mathbf{C}\mathrm{P}}^2$. *Invent. Math.*, 95(3):591–600, 1989.

[48] T-. J. Li and A. Liu. Symplectic structure on ruled surfaces and a generalized adjunction formula. *Math. Res. Lett.*, 2(4):453–471, 1995.

[49] T.-J. Li. Symplectic Parshin-Arakelov inequality. *Internat. Math. Res. Notices*, (18):941–954, 2000.

[50] W. B. R. Lickorish. A finite set of generators for the homeotopy group of a 2-manifold. *Proc. Cambridge Philos. Soc.*, 60:769–778, 1964.

[51] A. Liu. Some new applications of general wall crossing formula, Gompf's conjecture and its applications. *Math. Res. Lett.*, 3(5):569–585, 1996.

[52] K. M. Luttinger. Lagrangian tori in $\mathbb{R}^4$. *J. Differential Geom.*, 42(2):220–228, 1995.

[53] D. Margalit. A lantern lemma. *Algebr. Geom. Topol.*, 2:1179–1195, 2002.

[54] Y. Matsumoto. On 4-manifolds fibered by tori. II. *Proc. Japan Acad. Ser. A Math. Sci.*, 59(3):100–103, 1983.

[55] Y. Matsumoto. Lefschetz fibrations of genus two-a topological approach. *Topology and Teichmüller spaces*, pages 123–148, 1996.

[56] A. I. Ozbagci, B.and Stipsicz. Contact 3-manifolds with infinitely many stein fillings. *Proc. Amer. Math. Soc.*, 132(5):1549–1558, 2004.

[57] B. Ozbagci. Signatures of Lefschetz fibrations. *Pacific J. Math.*, 202(1):99–118, 2002.

[58] P. Ozsváth and Z. Szabó. The symplectic Thom conjecture. *Ann. of Math. (2)*, 151(1):93–124, 2000.

[59] B. D. Park. Exotic smooth structures on $3\mathbb{C}\mathbf{P}^2\#n\overline{\mathbb{C}\mathbf{P}}^2$. *Proc. Amer. Math. Soc.*, 128(10):3057–3065, 2000.

[60] J. Park. Simply connected symplectic 4-manifolds with $b_2^+ = 1$ and $c_1^2 = 2$. *Invent. Math.*, 159(3):657–667, 2005.

[61] D. Salamon. *Spin geometry and Seiberg–Witten invariants*. Citeseer, 1996.

[62] Y. Sato. 2-spheres of square -1 and the geography of genus-2 Lefschetz fibrations. *J.Math.Sci.Univ.Tokyo*, 15:461–491, 2008.

[63] A. I. Stipsicz. On the number of vanishing cycles in Lefschetz fibrations. *Math. Res. Lett.*, 6(3-4):449–456, 1999.

[64] A. I. Stipsicz. Singular fibers in lefschetz fibrations on manifolds with $b_2^+ = 1$. *Topology Appl.*, 117(1):9–21, 2002.

[65] A. I. Stipsicz and Z. Szabó. The smooth classification of elliptic surfaces with $b_2^+ > 1$. *Duke Math. J.*, 75(1):1–50, 1994.

[66] C. H. Taubes. The Seiberg-Witten invariants and symplectic forms. *Math. Res. Lett.*, 1(6):809–822, 1994.

[67] C. H. Taubes. SW$\rightarrow$ Gr: from the Seiberg-Witten equations to pseudo-holomorphic curves. *J. Amer. Math. Soc.*, 9(3):845–918, 1996.

[68] M. Usher. Minimality and symplectic sums. *Int. Math. Res. Not.*, (49857):17pp, 2006.

[69] E. Witten. Monopoles and four-manifolds. *Math. Res. Letters*, 1(6):769– 796, 1994.

[70] G. Xiao. *Surfaces fibrées en courbes de genre deux*. Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1985.

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