# METHODS FOR SOURCE LOCALIZATION FROM TIME DIFFERENCE OF ARRIVAL MEASUREMENTS AND THEIR PERFORMANCE IMPROVEMENT IN ILL-CONDITIONED CASES

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NUSRET ÇELENK

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**METHODS FOR SOURCE LOCALIZATION FROM TIME DIFFERENCE OF ARRIVAL MEASUREMENTS AND THEIR PERFORMANCE IMPROVEMENT IN ILL-CONDITIONED CASES**

submitted by **NUSRET ÇELENK** in partial fulfillment of the requirements for the degree of **Master of Science in Electrical and Electronics Engineering Department, Middle East Technical University** by,

Prof. Dr. Halil Kalıpçılar
Dean, Graduate School of **Natural and Applied Sciences** ______________

Prof. Dr. Tolga Çiloğlu
Head of Department, **Electrical and Electronics Eng.** ______________

Prof. Dr. Çağatay Candan
Supervisor, **Electrical and Electronics Eng. Dept., METU** ______________

**Examining Committee Members:**

Prof. Dr. Tolga Çiloğlu
Electrical and Electronics Engineering Dept., METU ______________

Prof. Dr. Çağatay Candan
Electrical and Electronics Engineering Dept., METU ______________

Assoc. Prof. Dr. Süleyman Serdar Kozat
Electrical and Electronics Engineering Dept., Bilkent Uni. ______________

Assoc. Prof. Dr. Umut Orguner
Electrical and Electronics Engineering Dept., METU ______________

Assist. Prof. Dr. Gökhan Muzaffer Güvensen
Electrical and Electronics Engineering Dept., METU ______________

**Date:** ______________

**I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.**

Name, Last Name: NUSRET ÇELENK

Signature :

# ABSTRACT

## METHODS FOR SOURCE LOCALIZATION FROM TIME DIFFERENCE OF ARRIVAL MEASUREMENTS AND THEIR PERFORMANCE IMPROVEMENT IN ILL-CONDITIONED CASES

Çelenk, Nusret
M.S., Department of Electrical and Electronics Engineering
Supervisor : Prof. Dr. Çağatay Candan

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Estimating the location of a radiating source is crucial in a wide variety of fields, such as military applications, navigation systems and geophysics. In such fields, power and cost limitations, requirement of remaining undetected or the nature of the problem may necessitate passive localization techniques. One of such techniques is time difference of arrival (TDOA) based source localization.

In this thesis, TDOA based closed-form source localization methods are studied. An extensive overview of the available methods is given. Some modifications are made on the existing methods to solve some ambiguities, reduce computational cost and increase estimation performance. Moreover, sensor and source distribution scenarios causing the problem to be ill-conditioned are analyzed. A new method robust to ill-conditioned cases, namely efficient constrained weighted least squares with coordinate separation (ECWLS-CS) is proposed. Existing methods and the proposed one are implemented in the same framework and compared under certain scenarios.

Keywords: time difference of arrivals, source localization, hyperbolic navigation, ill-conditioned

# ÖZ

## VARIŞ ZAMAN FARKI ÖLÇÜMLERİ İLE KAYNAK KONUMLANDIRMA YÖNTEMLERİ VE KÖTÜ KOŞULLANMIŞ DURUMLARDA PERFORMANS İYİLEŞTİRMESİ

Çelenk, Nusret
Yüksek Lisans, Elektrik ve Elektronik Mühendisliği Bölümü
Tez Yöneticisi : Prof. Dr. Çağatay Candan

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Yayın yapan bir kaynağın konum kestirimi askeri uygulamalar, navigasyon sistemleri ve jeofizik gibi birçok alanda önemli bir yere sahiptir. Bahsi geçen alanlarda; güç ve maliyet limitleri, tespit edilmeme isteği ya da problemin doğası pasif konumlandırma tekniklerinin kullanılmasını gerektirebilmektedir. Bu tekniklerden biri varış zaman farkı (VZF) temelli konum kestirimidir.

Bu tezde, VZF temelli kapalı formdaki konum belirleme teknikleri incelenmiştir. Mevcut yöntemler kapsamlı bir şekilde gözden geçirilmiş; yöntemlerde gözlenen bazı belirsizliklerin giderilmesi, hesaplama karmaşıklığının düşürülmesi ve kestirim performansının artırılmasına yönelik değişiklikler önerilmiştir. Ayrıca, problemin kötü koşullanmış hale gelmesine sebep olan kaynak ve algılayıcı yerleşim senaryoları analiz edilmiştir. Kötü koşullanmış durumlara dayanıklı yeni bir kestirim yöntemi, koordinat ayrımı ile verimli kısıtlı ağırlıklı en küçük kareler (VKAEKK-KA) adıyla önerilmiştir. Önerilen yöntem ile literatürde hâlihazırda bulunan yöntemlerin başarım seviyeleri aynı benzetim ortamında ve belirli senaryolarda karşılaştırılmıştır.

Anahtar Kelimeler: varış zaman farkları , kaynak konumlandırma, hiperbolik navigasyon, kötü-koşullanmış

*To my dear family*

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| | |
|---|---|
| 2D | Two-Dimensional |
| 3D | Three-Dimensional |
| AOA | Angle Of Arrival |
| CLS | Constrained Least Squares |
| CWLS | Constrained Weighted Least Squares |
| CWLS-RDS | Constrained Weighted Least Squares with Range Difference Separation |
| CRLB | Cramér–Rao Lower Bound |
| ECWLS | Efficient Constrained Weighted Least Squares |
| ECWLS-CS | Efficient Constrained Weighted Least Squares with Coordinate Separation |
| ECWLS-RDS | Efficient Constrained Weighted Least Squares with Range Difference Separation |
| ECWLS-RDSH | Efficient Constrained Weighted Least Squares with Range Difference Separation for Hyperbolic Distribution |
| eLORAN | enhanced LOng RAngle Navigation |
| FDOA | Frequency Difference Of Arrival |
| GM | Global Minimizer |
| GMS | Global Minimizer for Single constraint |
| GPS | Global Positioning System |
| GTRS | Generalized Trust Region Subproblems |
| LM | Local Minimizer |
| LS | Least Squares |
| LS-SC | Least Squares for Suboptimum Cost function |

| | |
|---|---|
| ML | Maximum Likelihood |
| MLE | Maximum Likelihood Estimator |
| MSE | Mean Square Error |
| MUSIC | MUltiple SIgnal Classification |
| MVU | Minimum Variance Unbiased |
| PDF | Probability Density Function |
| RSP | Root Selection Procedure |
| SI | Spherical Interpolation |
| SM | Subspace Minimization |
| SNR | Signal to Noise Ratio |
| SONAR | SOund Navigation And Ranging |
| SX | Spherical intersection |
| TDOA | Time Difference Of Arrivals |
| TRS | Trust Region Subproblems |
| TSWLS | Two Stage Weighted Least Squares |
| ULS | Unconstrained Least Squares |
| UWLS | Unconstrained Weighted Least Squares |

# CHAPTER 1

# INTRODUCTION

Estimating the location of a radiating source is crucial in a wide variety of fields. In military applications, location estimation of an intruder is one of the most important parts of a successful defence system and a possible counterattack. In navigation systems (GPS, eLORAN, etc.), the problem of self-positioning is equivalent to that of radiating source localization. Performance of the guidance and autonomous driving systems in the land, sea and air vehicles strongly depend on the accuracy of the self-positioning. In geophysics, estimating the location of seismic events is one of the fundamental problems. In addition to these fields; speaker localization for speech enhancement, mobile station positioning in emergency, sound source positioning in search and rescue operations are some of the other areas in which radiating source localization problem is encountered.

In military and seismic applications, cooperation of the source with the locator system is impossible because of the nature of the problem. Although it is possible in navigation systems, cooperation is generally limited since it is quite costly and impractical in some cases. Besides, remaining undetected is essential in warfare, which necessitates the use of passive sensors. Since the systems using active sensors need more power and costly equipments, passive systems may also be preferred in other application areas. All of these limitations result in a need for the localization techniques which use only passive sensors that do not require cooperation with the emitter.

In the literature, a variety of source localization methods exists that meet the restrictions mentioned. These methods use one or more properties of the signal coming from the source; such as arrival time, direction, frequency and power. In one of these methods, namely time difference of arrival (TDOA) based source localization, differences of the time instants at which the signal arrives to the sensors are used to estimate the source location.

TDOA based source localization techniques are composed of two steps. In the first step, TDOA values are estimated using the outputs of stationary omnidirectional sensors. Then in the second step, the source position is estimated from the obtained TDOA values and the sensor positions. Extended versions of the subspace methods such as multiple signal classification (MUSIC) could also be used to estimate the source position [1]. However, since these methods have been originally developed for the angle of arrival estimation of far field sources emitting narrow-band signals, wide-band applications generally needs significant computational resources. On the other hand, bandwidth of the source signal affects only the first step of the TDOA based source localization techniques, which results in a barely noticeable change in the computational cost [2].

## 1.1 Literature Review

In the literature, there are various studies on TDOA based source localization algorithms. In [3], it has been shown that because of the nonlinearity of the problem, closed-form maximum likelihood estimator (MLE) does not exist. Then, an iterative method which uses Taylor series expansion has been proposed to obtain the ML estimate. However, this method needs an initial source location estimate. Moreover, convergence is not guaranteed [4].

Suboptimal closed-form methods have received a considerable interest in the literature concerning TDOA based source localization, since such methods are desirable for real-time systems having low computational capability. Additionally, such methods are also required to obtain an initial estimate for the iterative ML methods. Spherical interpolation [5]; subspace minimization [6]; spherical intersection [7], [8]; two stage weighted least squares [9]; constrained weighted least squares [10], [11], [12] are some of the closed-form TDOA based source localization methods, which we will examine in the following chapters.

There are also hybrid methods which use angle of arrival (AOA) [13], [14] or frequency difference of arrival (FDOA) measurements [15], [16], [17] in addition to the TDOA.

In [18], [19] and [20]; sensor position errors have also been taken into account in the source localization.

Works focusing on the use of TDOA based source localization in non-line-of-sight environments, such as mobile station positioning in a wireless network, also exist [21], [22].

In [23], a closed-form method focused on the estimation of far field sources using the triangulation of TDOA hyperbole asymptotes is available.

There are also studies which apply semidefinite programming to the TDOA based source localization problem [24], [25].

Studies on bias reduction of certain available methods are given in [26] and [27].

## 1.2 Scope and Contributions of the Thesis

In this thesis, TDOA based closed-form source localization methods have been studied. An extensive overview of the available methods has been given with the comments on their shortcomings and advantageous points. Besides, sensors and source placement scenarios resulting in an ill-conditioned problem, and the methods resistant to such scenarios have been investigated. Additionally, some modifications have been made on the existing methods to solve some ambiguities, reduce computational cost and increase estimation performance. Modified version of constrained weighted least squares (CWLS) method proposed in Section 2.3.2.5, in particular, could be considered as a new algorithm, in part. Moreover, a novel algorithm, efficient constrained weighted least squares with coordinate separation (ECWLS-CS), has been proposed; which circumvent the ill-conditioning problem emerging in the available methods when the sensor distribution is linear or hyperbolic in 2D plane (and planar or hyperboloidal in 3D). Existing methods and the proposed one have been implemented in the same framework and compared under certain source and sensor distribution scenarios.

### 1.3 Organization of the Thesis

Organization of the thesis is as follows:

In Chapter 2, firstly TDOA based source localization problem is clearly stated along with the necessary formulation. Then TDOA estimation is briefly mentioned. Finally, the estimation techniques available in the literature for TDOA based source localization problem are explained in detail. Modifications made on the existing methods to solve some ambiguities, reduce computational cost and increase estimation performance are also provided.

In Chapter 3, sensor and source distribution scenarios causing the methods given in Chapter 2 to face ill-conditioned problem, and the methods resistant to such scenarios are investigated. A new method, ECWLS-CS, which is resistant to most of the ill-conditioned cases is proposed.

In Chapter 4, MATLAB® simulation results are provided. Performances of the proposed and the available methods are compared under certain scenarios.

In Chapter 5, a summary and conclusions drawn from the study are provided.

# CHAPTER 2

# TDOA BASED SOURCE LOCALIZATION

## 2.1 Problem Statement

One of the common ways of solving the problem of source localization is to use the differences of the times at which the source signal arrives to the sensors. As stated in the introduction chapter, this method is composed of two steps. In the first step, signal arrival times of the sensors relative to that of a chosen reference sensor are estimated using the sensor outputs. These estimated values are named as time difference of arrival (TDOA) measurements. In the second step, the obtained TDOA values are used to estimate the position of the source.

In the following section, the second step of the TDOA based source localization problem is formulated, assuming the first step is completed and TDOA values are obtained. To ease the exposition, the formulation and the remaining parts of the thesis consider the two-dimensional localization problem. Extending to three-dimensional space causes nothing more than an increase in matrix dimensions and polynomial degrees, and, in general, it is straightforward.

### 2.1.1 Formulation of TDOA Based Source Localization Problem

Consider a passive sensor array composed of N elements. Sensors are distributed in a two-dimensional plane as shown in Figure 2.1. Let $\boldsymbol{s} = [x\ y]^{\mathrm{T}}$ be the radiating source position and $\boldsymbol{s_i} = [x_i\ y_i]^{\mathrm{T}}$, $i = 1, 2, ..., N$ be the position of the $i^{th}$ sensor. Note that $\boldsymbol{s_i}$ are the known coordinates and $\boldsymbol{s}$ is the unknown to be estimated. Let $d_i$ denote the time of arrival (TOA) of the $i^{th}$ sensor, i.e., time it takes the signal travel from the

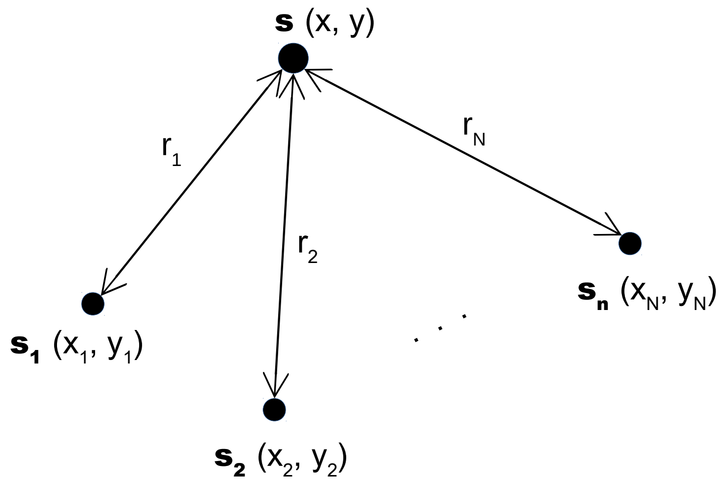


Figure 2.1: Sensor Placement

source to the $i^{th}$ sensor. Then, without loss of generality, if the $1^{st}$ sensor is chosen as the reference, TDOA between the $i^{th}$ and the $1^{st}$ sensor is computed as follows:

$$d_{i,1} = d_i - d_1, \qquad i = 2, 3, ..., N. \tag{2.1}$$

Let $r_i$ represent the distance between the source and the $i^{th}$ sensor, i.e.,

$$r_i = \|\boldsymbol{s} - \boldsymbol{s_i}\|, \qquad i = 1, 2, ..., N \tag{2.2}$$

where $\|.\|$ represents the Euclidean norm. Similar to the TDOA computation, range differences are found from

$$r_{i,1} = r_i - r_1, \qquad i = 2, 3, ..., N. \tag{2.3}$$

If the measurement and calibration noises are neglected, then the following equation set could be written:

$$r_{i,1} = c\, d_{i,1}, \qquad i = 2, 3, ..., N, \tag{2.4}$$

where $c$ is the signal propagation speed of the medium. In (2.4), $d_{i,1}$ are the TDOA values estimated previously using the sensor outputs; and $r_{i,1}$ are the true range differences, which are the nonlinear functions of the unknown source location $\boldsymbol{s}$.

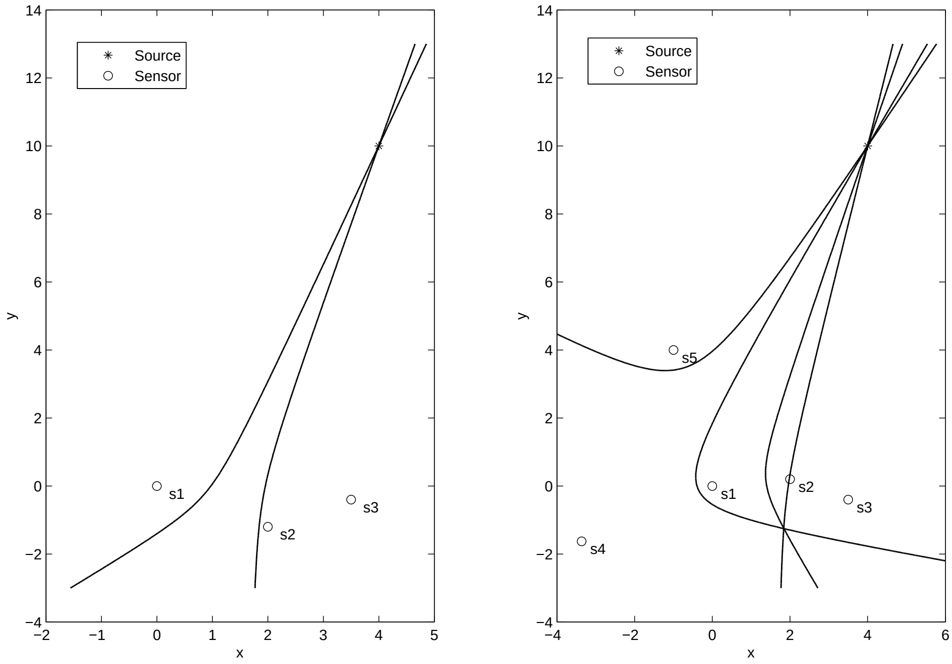


Figure 2.2: TDOA Hyperbolas for Two Different Placement Scenarios

Equation 2.4 represents a set of the single-branch hyperbolas with the focal points $s_i$ and the reference sensor $s_1$. In Figure 2.2, these hyperbola branches are drawn for two different source and sensor placement scenarios,without adding any noise to the true sensor positions and TDOA values.

In the placement scenario illustrated in the left hand side of Figure 2.2, it can be seen that three sensors are enough to locate the radiating source (four sensors in a 3D space). However, in some cases such as the one given in the right hand side of the Figure 2.2 , single-branch hyperbolas may intersect at two points. Therefore, even though there is no need to have a higher estimation accuracy resulting from the larger number of sensors, more than three sensors may be needed to locate the source without ambiguity. On the other hand, if the estimation method used has the ability to give both intersection points, then even a rough estimate obtained from another method based on other properties of the signal may be enough to solve the ambiguity.

As previously mentioned, TDOA based source localization is composed of two main steps:

1. Estimation of the TDOA values using the sensor outputs
2. Estimation of the source location using the obtained TDOA values

The scope of this thesis is the second step. Therefore, TDOA estimation methods are not examined. Nevertheless, for the sake of completeness, some common TDOA estimators are briefly mentioned in Section 2.2.

In Section 2.3, firstly maximum likelihood (ML) estimator for the TDOA based source localization problem is explained. After showing that the ML estimator could be solved only iteratively, closed-form suboptimum estimators in the literature are investigated in detail. Advantages, weak points and computational costs of these closed-form methods are explained.

## 2.2 TDOA Estimation

In TDOA based source localization, performance of the estimation result is strongly dependent on the accuracy of the estimated TDOA values. Therefore, considering the physical environment, sensor properties, characteristics of the radiating signal, etc.; most convenient TDOA estimation technique should be chosen. Since this topic is outside the scope of the thesis, only a brief explanation of some common TDOA estimation methods are given in this section.

One of the simple methods for TDOA estimation is leading edge detection [28]. In this method, the time instant at which the sensor output exceeds a certain threshold value is taken as the signal arrival time. Then, by subtracting the arrival time of the reference sensor from that of the others, TDOA values are obtained.

Another method for TDOA estimation is cross-correlation [29]. In this method, the reference sensor output and that of other sensors are cross-correlated. The time index of the peak point of a cross-correlation result gives the displacement of the source signal in the corresponding sensor output relative to the reference sensor, i.e., the TDOA value of the corresponding sensor. If the source signal and noise spectra are known or could be estimated, then a prefiltering before the cross-correlation could be applied to emphasize the signal at the frequencies where the SNR is high [29].

### 2.3 Source Location Estimation from TDOA Values

After the TDOA values are obtained, the final step of localization is to estimate the source position using these values. In this section, estimators which could be used in the second step, particularly the closed-form ones, are examined in detail.

Assume TDOA estimation results are in the form of

$$d_{i,1} = d^0_{i,1} + n_{i,1}, \qquad i = 2, 3, ..., N, \tag{2.5}$$

where $d^0_{i,1}$ represent the noise-free TDOA values and $n_{i,1}$ represent zero mean additive Gaussian noises. If TDOA vector is $\boldsymbol{d} = [d_{2,1}\ d_{3,1}\ ...\ d_{N,1}]^{\mathrm{T}}$, true range difference vector is $\boldsymbol{r} = [r_{2,1}\ r_{3,1}\ ...\ r_{N,1}]^{\mathrm{T}}$, signal propogation speed is $c$, noise vector is $\boldsymbol{n} = [n_{2,1}\ n_{3,1}\ ...\ n_{N,1}]^{\mathrm{T}}$ and the covariance matrix of $\boldsymbol{n}$ is $\boldsymbol{Q}$, then the PDF of $\boldsymbol{d}$ could be written as

$$p(\boldsymbol{d};\boldsymbol{s}) = \frac{1}{(2\pi)^{(N-1)/2}|\boldsymbol{Q}|^{1/2}} \exp\{-\frac{1}{2}(\boldsymbol{d}-\boldsymbol{r}/c)^{\mathrm{T}}\boldsymbol{Q}^{-1}(\boldsymbol{d}-\boldsymbol{r}/c)\}. \tag{2.6}$$

In (2.6), to emphasize that the PDF of $\boldsymbol{d}$ is related to the source position $\boldsymbol{s}$, since $\boldsymbol{r}$ is a nonlinear function of $\boldsymbol{s}$, the notation $p(\boldsymbol{d};\boldsymbol{s})$ is used rather than $p(\boldsymbol{d})$.

Since the natural error criterion, minimum mean square error (MSE), generally results in estimators which are not realizable, which is the result of the bias term depending on the unknown parameter to be estimated; constraining the estimator to be unbiased and trying to find the one whose variance attains the Cramér–Rao Lower Bound (CRLB) is an appropriate starting point [30]. To find out if there exists such an estimator, the following theorem stated in [30] could be used:

**Theorem 1.** *Let $\boldsymbol{\theta}$ be the unknown vector to be estimated, $\boldsymbol{x}$ be the observation vector, and $p(\boldsymbol{x};\boldsymbol{\theta})$ denote the probability density function (PDF) of $\boldsymbol{x}$. Then, if the regularity condition*

$$\mathrm{E}\left[\frac{\partial\ \ln\ p(\boldsymbol{x};\boldsymbol{\theta})}{\partial\boldsymbol{\theta}}\right] = \mathbf{0} \tag{2.7}$$

*holds, then an unbiased estimator whose variance is equal to the CRLB could be found if and only if*

$$\frac{\partial \ \ln \ p(\boldsymbol{x};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \boldsymbol{I}(\boldsymbol{\theta})(\boldsymbol{g}(\boldsymbol{x}) - \boldsymbol{\theta}) \tag{2.8}$$

*where $\boldsymbol{I}$ and $\boldsymbol{g}$ are some functions. In (2.8), $\boldsymbol{g}(\boldsymbol{x})$ is the minimum variance unbiased (MVU) estimator and $[\boldsymbol{I}^{-1}(\boldsymbol{\theta})]_{ii}$ is the variance of the $i^{th}$ element of the estimator, where $[\boldsymbol{I}^{-1}(\boldsymbol{\theta})]_{ii}$ denotes the $(i,i)^{th}$ element of the inverse of $\boldsymbol{I}(\boldsymbol{\theta})$, which is the Fisher information matrix.*

In Appendix A, CRLB for TDOA based source localization problem is obtained. It is also shown that the regularity condition given in (2.7) is met. Furthermore, $\partial \, ln \, p(\boldsymbol{x};\boldsymbol{\theta})/\partial\boldsymbol{\theta}$ for this problem is calculated as

$$\frac{\partial \ \ln \ p(\boldsymbol{d};\boldsymbol{s})}{\partial \boldsymbol{s}} = \frac{1}{c}\frac{\partial \boldsymbol{r}^{\mathrm{T}}}{\partial \boldsymbol{s}}\boldsymbol{Q}^{-1}(\boldsymbol{d} - \boldsymbol{r}/c). \tag{2.9}$$

Since the true range difference vector $\boldsymbol{r}$ is a nonlinear function of the source position vector $\boldsymbol{s}$, (2.9) could not be put in the form of (2.8). Therefore, it can be said that an unbiased estimator attaining the CRLB does not exist for the TDOA based source localization problem.

### 2.3.1 Maximum Likelihood Estimator

In this section, the most well known approach for obtaining practical estimators [30], the maximum likelihood estimator (MLE), is formulated for TDOA based source localization problem.

MLE for the TDOA based source localization problem is the $\boldsymbol{s}$ vector maximizing $p(\boldsymbol{d};\boldsymbol{s})$ given in (2.6), for a TDOA set. Since the natural logarithm is a monotonically increasing function, finding the $\boldsymbol{s}$ vector which maximizes the natural logarithm of (2.6) gives the same result:

$$\hat{\boldsymbol{s}}_{\mathrm{MLE}} = \arg \max_{\boldsymbol{s}} \{p(\boldsymbol{d};\boldsymbol{s})\} = \arg \max_{\boldsymbol{s}} \{\ln(p(\boldsymbol{d};\boldsymbol{s}))\}. \tag{2.10}$$

Taking the natural logarithm of (2.6) gives

$$\ln(p(\boldsymbol{d};\boldsymbol{s})) = -\ln((2\pi)^{(N-1)/2}|\boldsymbol{Q}|^{1/2}) \ -\frac{1}{2}(\boldsymbol{d} - \boldsymbol{r}/c)^{\mathrm{T}}\boldsymbol{Q}^{-1}(\boldsymbol{d} - \boldsymbol{r}/c)\}. \tag{2.11}$$

Since the first term of the right hand side of (2.11) does not depend on $\boldsymbol{s}$:

$$\hat{\boldsymbol{s}}_{\text{MLE}} = \arg\min_{\boldsymbol{s}} \{(\boldsymbol{d} - \boldsymbol{r}/c)^{\text{T}} \boldsymbol{Q}^{-1} (\boldsymbol{d} - \boldsymbol{r}/c)\}. \tag{2.12}$$

Note that even if the Gaussian assumption could not be made about the additive noise in (2.5), the estimator criteria in (2.12) is still meaningful. In such case, the obtained estimator is regarded as the least squares estimator and $\boldsymbol{Q}^{-1}$ becomes the weighting matrix [3].

To find the $\boldsymbol{s}$ vector minimizing (2.12), taking the partial derivative of (2.11) with respect to $\boldsymbol{s}$ and equating the result to zero gives

$$\frac{\partial \ \ln \ p(\boldsymbol{d}; \boldsymbol{s})}{\partial \boldsymbol{s}} = \frac{1}{c} \frac{\partial \boldsymbol{r}^{\text{T}}}{\partial \boldsymbol{s}} \boldsymbol{Q}^{-1} (\boldsymbol{d} - \boldsymbol{r}/c) = \boldsymbol{0}_{\text{2x1}}. \tag{2.13}$$

Since $\boldsymbol{r}$ is a nonlinear function of $\boldsymbol{s}$, which can be seen explicitly in (2.16), there is no closed form solution for MLE. Hence numerical methods should be employed.

In [4] and [3], an iterative method based on the Taylor series expansion is proposed to obtain the ML estimate for the problems in which the observations are nonlinear functions of the unknowns to be estimated. This method is explained for the TDOA based source localization problem as follows:

Under the assumption given in (2.5), to show the relation between the source position $\boldsymbol{s}$ and the TDOA vector $\boldsymbol{d}$ explicitly

$$\boldsymbol{d} = \boldsymbol{r}(\boldsymbol{s})/c + \boldsymbol{n} \tag{2.14}$$

where

$$\boldsymbol{r}(\boldsymbol{s}) = \begin{bmatrix} \|\boldsymbol{s} - \boldsymbol{s_2}\| - \|\boldsymbol{s} - \boldsymbol{s_1}\| \\ \|\boldsymbol{s} - \boldsymbol{s_3}\| - \|\boldsymbol{s} - \boldsymbol{s_1}\| \\ \vdots \\ \|\boldsymbol{s} - \boldsymbol{s_N}\| - \|\boldsymbol{s} - \boldsymbol{s_1}\| \end{bmatrix} \tag{2.15}$$

$$= \begin{bmatrix} \left(\sqrt{(x - x_2)^2 + (y - y_2)^2} - \sqrt{(x - x_1)^2 + (y - y_1)^2}\right) \\ \left(\sqrt{(x - x_3)^2 + (y - y_3)^2} - \sqrt{(x - x_1)^2 + (y - y_1)^2}\right) \\ \vdots \\ \left(\sqrt{(x - x_N)^2 + (y - y_N)^2} - \sqrt{(x - x_1)^2 + (y - y_1)^2}\right) \end{bmatrix} \tag{2.16}$$

could be written.

Assuming that an initial estimate of the source position ($\hat{\boldsymbol{s}}_0$) is available, Taylor series expansion around this initial guess could be written for the terms less than second order as follows:

$$\boldsymbol{r}(\boldsymbol{s}) \simeq \boldsymbol{r}(\hat{\boldsymbol{s}}_0) + \boldsymbol{G}_{\hat{\boldsymbol{s}}_0}\,(\boldsymbol{s} - \hat{\boldsymbol{s}}_0), \tag{2.17}$$

where

$$\boldsymbol{G}_{\hat{\boldsymbol{s}}_0} \triangleq \left.\frac{\partial \boldsymbol{r}(\boldsymbol{s})}{\partial \boldsymbol{s}}\right|_{\boldsymbol{s}=\hat{\boldsymbol{s}}_0} = \begin{bmatrix} \left.\partial r_{2,1}/\partial x\right|_{\boldsymbol{s}=\hat{\boldsymbol{s}}_0} & \left.\partial r_{2,1}/\partial y\right|_{\boldsymbol{s}=\hat{\boldsymbol{s}}_0} \\ \left.\partial r_{3,1}/\partial x\right|_{\boldsymbol{s}=\hat{\boldsymbol{s}}_0} & \left.\partial r_{3,1}/\partial y\right|_{\boldsymbol{s}=\hat{\boldsymbol{s}}_0} \\ \vdots & \vdots \\ \left.\partial r_{N,1}/\partial x\right|_{\boldsymbol{s}=\hat{\boldsymbol{s}}_0} & \left.\partial r_{N,1}/\partial y\right|_{\boldsymbol{s}=\hat{\boldsymbol{s}}_0} \end{bmatrix}. \tag{2.18}$$

Putting (2.17) into the MLE cost function given in (2.12) results in

$$C_{\text{MLE}}(\boldsymbol{s}) = (\boldsymbol{d}'_{\hat{\boldsymbol{s}}_0} - \boldsymbol{G}_{\hat{\boldsymbol{s}}_0}\boldsymbol{s}/c)^{\text{T}}\boldsymbol{Q}^{-1}(\boldsymbol{d}'_{\hat{\boldsymbol{s}}_0} - \boldsymbol{G}_{\hat{\boldsymbol{s}}_0}\boldsymbol{s}/c), \tag{2.19}$$

$$\boldsymbol{d}'_{\hat{\boldsymbol{s}}_0} = \boldsymbol{d} - \boldsymbol{r}(\hat{\boldsymbol{s}}_0)/c + \boldsymbol{G}_{\hat{\boldsymbol{s}}_0}\hat{\boldsymbol{s}}_0/c.$$

Taking the gradient of (2.19) and equating zero

$$\nabla_{\boldsymbol{s}} C_{\text{MLE}}(\boldsymbol{s})\Big|_{\boldsymbol{s}=\hat{\boldsymbol{s}}_1} = 2\boldsymbol{G}^{\text{T}}_{\hat{\boldsymbol{s}}_0}\boldsymbol{Q}^{-1}\boldsymbol{G}^{\text{T}}_{\hat{\boldsymbol{s}}_0}\hat{\boldsymbol{s}}_1/c^2 - 2\boldsymbol{G}^{\text{T}}_{\hat{\boldsymbol{s}}_0}\boldsymbol{Q}^{-1}\boldsymbol{d}'_{\hat{\boldsymbol{s}}_0}/c = \boldsymbol{0}_{2\text{x}1}, \tag{2.20}$$

solving (2.20) for $\hat{\boldsymbol{s}}_1$ gives

$$\hat{\boldsymbol{s}}_1 = c(\boldsymbol{G}^{\text{T}}_{\hat{\boldsymbol{s}}_0}\boldsymbol{Q}^{-1}\boldsymbol{G}^{\text{T}}_{\hat{\boldsymbol{s}}_0})^{-1}\boldsymbol{G}^{\text{T}}_{\hat{\boldsymbol{s}}_0}\boldsymbol{Q}^{-1}\boldsymbol{d}'_{\hat{\boldsymbol{s}}_0}. \tag{2.21}$$

In general form

$$\hat{\boldsymbol{s}}_{k+1} = c(\boldsymbol{G}^{\text{T}}_{\hat{\boldsymbol{s}}_k}\boldsymbol{Q}^{-1}\boldsymbol{G}^{\text{T}}_{\hat{\boldsymbol{s}}_k})^{-1}\boldsymbol{G}^{\text{T}}_{\hat{\boldsymbol{s}}_k}\boldsymbol{Q}^{-1}\boldsymbol{d}'_{\hat{\boldsymbol{s}}_k}, \quad k = 0, 1, 2, ... \tag{2.22a}$$

$$\boldsymbol{d}'_{\hat{\boldsymbol{s}}_k} = \boldsymbol{d} - \boldsymbol{r}(\hat{\boldsymbol{s}}_k)/c + \boldsymbol{G}_{\hat{\boldsymbol{s}}_k}\hat{\boldsymbol{s}}_k/c, \tag{2.22b}$$

$$\boldsymbol{G}_{\hat{\boldsymbol{s}}_k} = \left.\frac{\partial \boldsymbol{r}(\boldsymbol{s})}{\partial \boldsymbol{s}}\right|_{\boldsymbol{s}=\hat{\boldsymbol{s}}_k}. \tag{2.22c}$$

Iterating (2.22) until $\hat{\boldsymbol{s}}$ converges gives the source location estimate.

**Comments**

In the literature [9], [31], [32]; iterative MLE procedures, such as the Taylor series method, are regarded as the most successful estimation techniques for the TDOA based source localization problem. However, they have some disadvantages:

- An initial estimate of the source position is required. Performance of the result is dependent on the accuracy of this estimation.

- Convergence is not guaranteed.

- Since it is an iterative method, computational complexity is generally high compared to the closed-form methods.

### 2.3.2 Least Squares Estimators for Squared TDOA Measurements

Systems having low computational capacity and the requirement of real-time computation necessitate a closed-form location estimator. Furthermore, such an estimator is also required to obtain an initial estimate for the iterative methods such as the one mentioned in Section 2.3.1. In this section, closed form estimators for TDOA based source localization problem are investigated in detail. These estimators are least squares (LS) based and for a suboptimal cost function in the ML sense.

#### 2.3.2.1 Unconstrained Least Squares

Without losing generality, just to make the exposition easier, let the origin of the coordinate system be the location of the reference sensor, i.e.,

$$\boldsymbol{s}_1 = [0\ 0]^{\mathrm{T}}. \tag{2.23}$$

Then, combining (2.2), (2.3) and (2.4), the following equation could be written:

$$\begin{aligned} \|\boldsymbol{s}_i - \boldsymbol{s}\| &= cd_{i,1} + \|\boldsymbol{s}_1 - \boldsymbol{s}\| \\ &= cd_{i,1} + \|\boldsymbol{s}\|, \qquad i = 2, 3, ..., N. \end{aligned} \tag{2.24}$$

Squaring both sides of (2.24)

$$(\|\boldsymbol{s}_i - \boldsymbol{s}\|)^2 = (cd_{i,1} + \|\boldsymbol{s}\|)^2, \qquad i = 2, 3, ..., N, \tag{2.25}$$

and then expanding the brackets gives

$$\|\boldsymbol{s}_i\|^2 - 2\boldsymbol{s}_i^{\mathrm{T}}\boldsymbol{s} + \|\boldsymbol{s}\|^2 = c^2 d_{i,1}^2 + 2cd_{i,1}\|\boldsymbol{s}\| + \|\boldsymbol{s}\|^2, \qquad i = 2, 3, ..., N. \tag{2.26}$$

Finally, rearranging the terms results in

$$2\boldsymbol{s}_i^{\mathrm{T}}\boldsymbol{s} + 2cd_{i,1}r_1 = \|\boldsymbol{s}_i\|^2 - c^2 d_{i,1}^2, \qquad i = 2, 3, ..., N. \tag{2.27}$$

Equation set (2.27) could be put into matrix form as follows:

$$\boldsymbol{A\theta} = \boldsymbol{b} \tag{2.28a}$$

where

$$\boldsymbol{A} = \begin{bmatrix} x_2 & y_2 & cd_{2,1} \\ x_3 & y_3 & cd_{3,1} \\ \vdots & \vdots & \vdots \\ x_N & y_N & cd_{N,1} \end{bmatrix} \tag{2.28b}$$

$$\boldsymbol{\theta} = [x \quad y \quad r_1]^{\mathrm{T}} \tag{2.28c}$$

$$\boldsymbol{b} = 0.5 \begin{bmatrix} x_2^2 + y_2^2 - c^2 d_{2,1}^2 \\ x_3^2 + y_3^2 - c^2 d_{3,1}^2 \\ \vdots \\ x_N^2 + y_N^2 - c^2 d_{N,1}^2 \end{bmatrix} \tag{2.28d}$$

In (2.28), $\boldsymbol{A}$ and $\boldsymbol{b}$ can be calculated using the sensor positions and noisy TDOA measurements. $\boldsymbol{\theta}$ is the unknown array to be estimated to obtain the source position $\boldsymbol{s}$. Because of the synchronization and measurement errors, equation (2.28) holds approximately. Therefore, a least squares criteria based on this equation could be used to obtain an estimate of the source position:

$$\hat{\boldsymbol{s}}_{\text{LS-SC}} = \arg\min_{\boldsymbol{s}} \{(\boldsymbol{A\theta} - \boldsymbol{b})^{\mathrm{T}}(\boldsymbol{A\theta} - \boldsymbol{b})\}, \tag{2.29}$$

where LS-SC (LS for suboptimum cost function) stands for the emphasis on the cost function $(\boldsymbol{A\theta} - \boldsymbol{b})^{\mathrm{T}}(\boldsymbol{A\theta} - \boldsymbol{b})$, which is suboptimum in the maximum likelihood sense [11].

Note that the third element of $\boldsymbol{\theta}$ is nonlinearly related to the first two elements. This relation can be seen explicitly by writing (2.2) for $i = 1$:

$$r_1 = \|\boldsymbol{s} - \boldsymbol{s_i}\| = \sqrt{x^2 + y^2}, \tag{2.30}$$

or equivalently

$$\boldsymbol{\theta}^{\mathrm{T}}\mathbf{C}\boldsymbol{\theta} = 0 \tag{2.31a}$$

and

$$\theta_3 \geq 0, \tag{2.31b}$$

where

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

and $\theta_3$ represents the $3^{th}$ element of $\boldsymbol{\theta}$. Throughout the thesis, (2.31a) and (2.31b) will be called as "equality constraint" and "inequality constraint", respectively.

The nonlinear relation among the elements of $\boldsymbol{\theta}$ makes the equation set (2.28) non-linear. However, if the relation is ignored and $r_1$ is considered as an independent variable, then (2.28) becomes a linear equation. In this case, LS estimate of $\boldsymbol{\theta}$ could be found as

$$\hat{\boldsymbol{\theta}}_{\mathrm{ULS}} = (\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A})^{-1}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{b}, \tag{2.32}$$

and the source location estimate is the first two elements of the result:

$$\hat{\boldsymbol{s}}_{\mathrm{ULS}} = [\hat{\theta}_{\mathrm{ULS\text{-}1}} \;\; \hat{\theta}_{\mathrm{ULS\text{-}2}}]^{\mathrm{T}}, \tag{2.33}$$

where $\hat{\theta}_{\mathrm{ULS}-i}$ represents the $i^{th}$ element of $\hat{\boldsymbol{\theta}}_{\mathrm{ULS}}$. Since the constraints in (2.31) are ignored, the estimate is named as ULS (Unconstrained LS). Although the cost function is the same with the LS-SC, SC is dropped just to have a short abbreviation. Note that, to have a nonsingular $(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A})$ matrix, at least 4 sensors are required in a two dimensional space.

**Spherical Interpolation**

In [5], to find the source location in closed-from, a method named as Spherical Interpolation (SI) is proposed. In this method, $r_1$ in (2.28) is assumed to be known and the source coordinates $x$ and $y$ are computed in terms of $r_1$ as follows:

Rewriting (2.28a) as

$$\begin{aligned} \boldsymbol{A}_{\text{(N-1)x3}}\ \boldsymbol{\theta}_{\text{3x1}} &= \boldsymbol{b}_{\text{(N-1)x1}} \\ [\boldsymbol{S}_{\text{(N-1)x2}}\ \ c\boldsymbol{d}_{\text{(N-1)x1}}]\ [\boldsymbol{s}_{\text{2x1}}^{\text{T}}\ r_1]^{\text{T}} &= \boldsymbol{b}_{\text{(N-1)x1}} \\ \boldsymbol{S}_{\text{(N-1)x2}}\ \boldsymbol{s}_{\text{2x1}} + r_1 c\boldsymbol{d}_{\text{(N-1)x1}} &= \boldsymbol{b}_{\text{(N-1)x1}}, \end{aligned} \tag{2.34}$$

where $\boldsymbol{S}$ is the matrix composed of the first two columns of $\boldsymbol{A}$, and $c\boldsymbol{d}$ is the third column of $\boldsymbol{A}$. Then, assuming $r_1$ is known, LS estimate of the source location $\boldsymbol{s}$ could be found as

$$\hat{\boldsymbol{s}}_{\text{SI}-r_1} = (\boldsymbol{S}^{\text{T}}\boldsymbol{S})^{-1}\boldsymbol{S}^{\text{T}}(\boldsymbol{b} - r_1 c\boldsymbol{d}). \tag{2.35}$$

Then, if (2.35) is inserted back to (2.34)

$$\begin{aligned} &\boldsymbol{S}(\boldsymbol{S}^{\text{T}}\boldsymbol{S})^{-1}\boldsymbol{S}^{\text{T}}(\boldsymbol{b} - r_1 c\boldsymbol{d}) + r_1 c\boldsymbol{d} = \boldsymbol{b} \\ &r_1 c(\boldsymbol{I} - \boldsymbol{S}(\boldsymbol{S}^{\text{T}}\boldsymbol{S})^{-1}\boldsymbol{S}^{\text{T}})\boldsymbol{d} = (\boldsymbol{I} - \boldsymbol{S}(\boldsymbol{S}^{\text{T}}\boldsymbol{S})^{-1}\boldsymbol{S}^{\text{T}})\boldsymbol{b}. \end{aligned} \tag{2.36}$$

Define

$$\boldsymbol{P} \triangleq \boldsymbol{I} - \boldsymbol{S}(\boldsymbol{S}^{\text{T}}\boldsymbol{S})^{-1}\boldsymbol{S}^{\text{T}}. \tag{2.37}$$

Then LS estimate of $r_1$ becomes

$$\hat{r}_1 = \frac{\boldsymbol{d}^{\text{T}}\boldsymbol{P}^{\text{T}}\boldsymbol{P}\boldsymbol{b}}{c\boldsymbol{d}^{\text{T}}\boldsymbol{P}^{\text{T}}\boldsymbol{P}\boldsymbol{d}} \tag{2.38}$$

Inserting (2.38) into (2.35) gives the source location estimate:

$$\hat{\boldsymbol{s}}_{\text{SI}} = (\boldsymbol{S}^{\text{T}}\boldsymbol{S})^{-1}\boldsymbol{S}^{\text{T}}(\boldsymbol{I} - \frac{\boldsymbol{d}\boldsymbol{d}^{\text{T}}\boldsymbol{P}^{\text{T}}\boldsymbol{P}}{\boldsymbol{d}^{\text{T}}\boldsymbol{P}^{\text{T}}\boldsymbol{P}\boldsymbol{d}})\boldsymbol{b}. \tag{2.39}$$

Since it solves the same equation set ignoring the constraints, this two stage cumbersome process is mathematically equivalent to (2.32). Therefore, spherical interpolation (SI) and unconstrained least squares methods (ULS) are identical [33].

### Subspace Minimization

In [6], an alternative method for the TDOA based source localization problem, namely subspace minimization (SM), is proposed. In this method, (2.34) is multiplied by a projection matrix whose null-space contains $\boldsymbol{d}$, i.e.,

$$\boldsymbol{M}\boldsymbol{d} = \boldsymbol{0}. \tag{2.40}$$

Such an $\boldsymbol{M}$ could be obtained as

$$\boldsymbol{M}_k = (\boldsymbol{I} - \boldsymbol{Z}^k)\boldsymbol{D}, \quad k = 1, 2, ... \tag{2.41}$$

where

$$\boldsymbol{D} = \begin{bmatrix} d_{2,1} & & & & \\ & d_{3,1} & & \mathbf{0} & \\ & & \ddots & & \\ & \mathbf{0} & & d_{(\text{N-1}),1} & \\ & & & & d_{\text{N},1} \end{bmatrix}^{-1}_{(\text{N-1})\text{x}(\text{N-1})},$$

$$\boldsymbol{Z} = \begin{bmatrix} 0 & 1 & 0 & \dots & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & \vdots \\ \vdots & \dots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \dots & \dots & 0 & 1 & 0 \\ 0 & \dots & \dots & \dots & 0 & 1 \\ 1 & 0 & \dots & \dots & \dots & 0 \end{bmatrix}_{(\text{N-1})\text{x}(\text{N-1})}.$$

For the sake of simplicity, let $\boldsymbol{M}_1$ be the projection matrix. Then, multiplying (2.34) by $\boldsymbol{M}_1$ results

$$\begin{aligned} \boldsymbol{M}_1(\boldsymbol{S}\boldsymbol{s} + r_1 c\boldsymbol{d}) &= \boldsymbol{M}_1\boldsymbol{b} \\ \boldsymbol{M}_1\boldsymbol{S}\boldsymbol{s} &= \boldsymbol{M}_1\boldsymbol{b}. \end{aligned} \tag{2.42}$$

Finally, LS estimate of (2.42) could be found as

$$\hat{\boldsymbol{s}}_{\text{SM}} = (\boldsymbol{S}^{\text{T}}\boldsymbol{M}_1^{\text{T}}\boldsymbol{M}_1\boldsymbol{S})^{-1}\boldsymbol{S}^{\text{T}}\boldsymbol{M}_1^{\text{T}}\boldsymbol{M}_1^{\text{T}}\boldsymbol{M}_1\boldsymbol{b}. \tag{2.43}$$

In general,

$$\hat{\boldsymbol{s}}_{\text{SM}} = (\boldsymbol{S}^{\text{T}}\boldsymbol{M}_k^{\text{T}}\boldsymbol{M}_k\boldsymbol{S})^{-1}\boldsymbol{S}^{\text{T}}\boldsymbol{M}_k^{\text{T}}\boldsymbol{M}_k^{\text{T}}\boldsymbol{M}_k\boldsymbol{b}. \tag{2.44}$$

Since $\boldsymbol{M}_k$ is singular with $N - 2$ rank, number of rows of $\boldsymbol{S}$ should be higher than that of columns to have a nonsingular $\boldsymbol{M}_k\boldsymbol{S}$. This implies that $N - 1 > n$, where $n$ is the dimension of the space. Therefore, in a two dimensional space, at least 4 sensors are required to use SM method.

Recall that Spherical Interpolation method is the LS solution of

$$\boldsymbol{S}\boldsymbol{s} = (\boldsymbol{I} - \frac{\boldsymbol{d}\boldsymbol{d}^{\text{T}}\boldsymbol{P}^{\text{T}}\boldsymbol{P}}{\boldsymbol{d}^{\text{T}}\boldsymbol{P}^{\text{T}}\boldsymbol{P}\boldsymbol{d}})\boldsymbol{b}. \tag{2.45}$$

Multiplying (2.45) by $\boldsymbol{M}_k$ results

$$\begin{aligned}\boldsymbol{M}_k\boldsymbol{S}\boldsymbol{s} &= \boldsymbol{M}_k\boldsymbol{b} - \boldsymbol{M}_k\boldsymbol{d}\frac{\boldsymbol{d}^{\mathrm{T}}\boldsymbol{P}^{\mathrm{T}}\boldsymbol{P}}{\boldsymbol{d}^{\mathrm{T}}\boldsymbol{P}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{d}}\boldsymbol{b} \\ &= \boldsymbol{M}_k\boldsymbol{b} \qquad (2.46)\end{aligned}$$

which is the same equation as (2.42). This shows that subspace minimization method is equivalent to spherical interpolation [6].

#### Comments

ULS, and equivalently SI and SM, give a closed form solution which is quite valuable for computationally limited systems and iterative methods requiring an initial guess. However, there are important points that should not be ignored:

- ULS method uses a cost function which is not optimum in the ML sense.
- ULS method ignores the constraints among the elements of $\boldsymbol{\theta}$, which is given in (2.31). This may even results in a negative $\hat{\theta}_3$, which is the range between the source and the reference sensor. However, it will be shown in Section 2.3.2.4 that the estimator obtained by considering the constraints becomes approximately equal to the ULS estimator when the noise level is small.

As mentioned earlier, SI needs a two stage computationally cumbersome process. SM is less cumbersome than SI; however it still needs unnecessarily complicated calculations and gives the same result as ULS [33]. Because of this, it is seen in the literature that among these three equivalent closed-form methods, ULS is usually chosen as the base to work on for improvements [9], [11], [10], [34].

#### 2.3.2.2 Spherical Intersection

Schau and Robinson have proposed an alternative TDOA based closed form source localization method, namely spherical intersection (SX), in [7]. The proposed method is for $n+1$ sensors, where $n$ is the dimension of space. In [8], SX has been generalized for $N \geq n+1$ sensors with a weighting matrix, and the conditions for a solution to exist are stated. In this section, we will represent the SX method as well as its generalized version, and then examine whether SX gives the global minimum of the problem (2.29) with the constraints (2.31). At the end, we will address the reasons of weak performance of this method.

Recall that in SI method, $\boldsymbol{A}$ was divided into a sensor position matrix and TDOA vector as (2.34):

$$\begin{aligned} \boldsymbol{A}_{\text{(N-1)x3}}\ \boldsymbol{\theta}_{\text{3x1}} &= \boldsymbol{b}_{\text{(N-1)x1}} \\ [\boldsymbol{S}_{\text{(N-1)x2}}\ \ c\boldsymbol{d}_{\text{(N-1)x1}}]\ [\boldsymbol{s}_{\text{2x1}}^{\text{T}}\ r_1]^{\text{T}} &= \boldsymbol{b}_{\text{(N-1)x1}} \\ \boldsymbol{S}_{\text{(N-1)x2}}\ \boldsymbol{s}_{\text{2x1}} + r_1 c\boldsymbol{d}_{\text{(N-1)x1}} &= \boldsymbol{b}_{\text{(N-1)x1}}, \end{aligned}$$

with the cost function

$$\text{C}_{\text{SI}}(\boldsymbol{s}, r_1) = (\boldsymbol{b} - \boldsymbol{S}\boldsymbol{s} - r_1 c\boldsymbol{d})^{\text{T}}(\boldsymbol{b} - \boldsymbol{S}\boldsymbol{s} - r_1 c\boldsymbol{d}), \tag{2.47}$$

and then a source location estimate was obtained as (2.35) by assuming $r_1$ is known. In SX method, same approach is used for $N = 3$:

$$\hat{\boldsymbol{s}}_{\text{SX}} = \hat{\boldsymbol{s}}_{\text{SI}-r_1} = \boldsymbol{S}^{-1}(\boldsymbol{b} - r_1 c\boldsymbol{d}). \tag{2.48}$$

However, rather than inserting (2.48) back in (2.34), this time (2.48) is inserted in the quadratic equality constraint (2.31a) as follows:

$$\begin{aligned} &\boldsymbol{\theta}^{\text{T}}\mathbf{C}\boldsymbol{\theta} = \boldsymbol{s}^{\text{T}}\boldsymbol{s} - r_1^2 = 0 \\ &(\boldsymbol{b} - r_1 c\boldsymbol{d})^{\text{T}}\boldsymbol{S}^{-\text{T}}\boldsymbol{S}^{-1}(\boldsymbol{b} - r_1 c\boldsymbol{d}) - r_1^2 = 0 \end{aligned} \tag{2.49}$$

Rearranging terms, (2.49) becomes [7]

$$a_{\text{SX}} r_1^2 + b_{\text{SX}} r_1 + c_{\text{SX}} = 0, \tag{2.50}$$

where

$$a_{\text{SX}} = 1 - c^2\boldsymbol{d}^{\text{T}}\boldsymbol{S}^{-\text{T}}\boldsymbol{S}^{-1}\boldsymbol{d}, \tag{2.51a}$$

$$b_{\text{SX}} = 2c\boldsymbol{d}^{\text{T}}\boldsymbol{S}^{-\text{T}}\boldsymbol{S}^{-1}\boldsymbol{b}, \tag{2.51b}$$

$$c_{\text{SX}} = -\boldsymbol{b}^{\text{T}}\boldsymbol{S}^{-\text{T}}\boldsymbol{S}^{-1}\boldsymbol{b}. \tag{2.51c}$$

In [8], SX method has been generalized to utilize additional sensors and a weighting matrix is inserted to the minimization problem. When $N > 3$, since $\boldsymbol{S}$ is no more a square matrix, $\boldsymbol{S}^{-1}$ in (2.48) is replaced by the pseudo-inverse $(\boldsymbol{S}^{\mathrm{T}}\boldsymbol{S})^{-1}\boldsymbol{S}^{\mathrm{T}}$ to obtain generalized SX (GSX) estimate:

$$\hat{\boldsymbol{s}}_{\text{GSX}} = \hat{\boldsymbol{s}}_{\text{SI}-r_1} = (\boldsymbol{S}^{\mathrm{T}}\boldsymbol{S})^{-1}\boldsymbol{S}^{\mathrm{T}}(\boldsymbol{b} - r_1 c\boldsymbol{d}). \tag{2.52}$$

and weighting matrix $\boldsymbol{W}$ is inserted in to obtain generalized weighted SX (GWSX) cost function

$$\mathrm{C}_{\text{GWSX}}(\boldsymbol{s}, r_1) = (\boldsymbol{b} - \boldsymbol{S}\boldsymbol{s} - r_1 c\boldsymbol{d})^{\mathrm{T}}\boldsymbol{W}(\boldsymbol{b} - \boldsymbol{S}\boldsymbol{s} - r_1 c\boldsymbol{d}), \tag{2.53}$$

which results in [8]

$$\hat{\boldsymbol{s}}_{\text{GWSX}} = \hat{\boldsymbol{s}}_{\text{SI}-r_1} = (\boldsymbol{S}^{\mathrm{T}}\boldsymbol{W}\boldsymbol{S})^{-1}\boldsymbol{S}^{\mathrm{T}}\boldsymbol{W}(\boldsymbol{b} - r_1 c\boldsymbol{d}). \tag{2.54}$$

Inserting (2.54) in (2.31a) and then rearranging terms gives [8]

$$a_{\text{GWSX}}\, r_1^2 + b_{\text{GWSX}}\, r_1 + c_{\text{GWSX}} = 0, \tag{2.55}$$

where

$$a_{\text{GWSX}} = l_1^2 + l_2^2 + l_3^2 - 1, \tag{2.56a}$$

$$b_{\text{GWSX}} = -2(k_1 l_1 + k_2 l_2 + k_3 l_3), \tag{2.56b}$$

$$c_{\text{GWSX}} = (k_1^2 + k_2^2 + k_3^2), \tag{2.56c}$$

$$(\boldsymbol{S}^{\mathrm{T}}\boldsymbol{W}\boldsymbol{S})^{-1}\boldsymbol{S}^{\mathrm{T}}\boldsymbol{W}\boldsymbol{b} = [k_1\ k_2\ k_3]^{\mathrm{T}}, \tag{2.56d}$$

$$(\boldsymbol{S}^{\mathrm{T}}\boldsymbol{W}\boldsymbol{S})^{-1}\boldsymbol{S}^{\mathrm{T}}\boldsymbol{W}c\boldsymbol{d} = [l_1\ l_2\ l_3]^{\mathrm{T}}. \tag{2.56e}$$

To find the SX source location estimate, $r_1$ is firstly obtained by solving (2.50). Then the obtained value is substituted into (2.48). For the GWSX estimate, aforementioned equations should be replaced by (2.55) and (2.54), respectively.

Since (2.50) and (2.55) are second order equations, it is possible to end up with two positive roots, two negative roots or even imaginary roots. For the first case, [7] has stated that the $r_1$ candidates are generally far from each other, which allows to eliminate one of them by considering the region of interest. However, it is possible to have an interest region including both candidates. As an alternative approach, the one resulting smaller cost function could be chosen as the estimate. Moreover, for the latter case, there is no solution proposed in either of [7] and [8]. Therefore, it could be stated that SX method does not guarantee a solution to the TDOA based source localization problem.

Apart from the solution existence problem, it should also be questioned that whether it is the global minimizer of the problem (2.29) with the constraints (2.31), when an estimate is obtained from the SX method. At the first step of the SX method, the set of position vectors which minimizes the cost function (2.53) for any $r_1 \in \mathbb{R}$ are assigned as the set of possible solutions. Then in this set, the one satisfying the constraints (2.31) is chosen as the source location estimate. However, it is quite possible that the global minimizer is not in the obtained set in the SX method, i.e.,

$$\begin{aligned} &\exists(\boldsymbol{S}, \boldsymbol{d}) \quad s.t. \\ &\quad \arg\min_{\boldsymbol{s}} \{\|\boldsymbol{b} - \boldsymbol{S}\boldsymbol{s} - r_1 c\boldsymbol{d}\|^2 : \ \boldsymbol{s}^{\mathrm{T}}\boldsymbol{s} = r_1^2 \,,\, r_1 \geq 0\} \\ &\notin \left\{ \ \boldsymbol{s} \in \mathbb{R}^2 \ : (\exists r_1' \in \mathbb{R})\big[\boldsymbol{s} = \arg\min_{\boldsymbol{s}} \left\{\|\boldsymbol{b} - \boldsymbol{S}\boldsymbol{s} - r_1' c\boldsymbol{d}\|^2\right\}\big] \ \right\}. \end{aligned} \tag{2.57}$$

Therefore, SX estimate is not the global minimizer, i.e.,

$$\begin{gathered} (\boldsymbol{s}_{\mathrm{GM}}, r_{1-\mathrm{GM}}) = \arg\min_{(\boldsymbol{s}, r_1)} \{\|\boldsymbol{b} - \boldsymbol{S}\boldsymbol{s} - r_1 c\boldsymbol{d}\|^2 : \boldsymbol{s} \in S_c\} \\ \neq \\ (\hat{\boldsymbol{s}}_{\mathrm{SX}}, \hat{r}_{1-\mathrm{SX}}) = \arg\min_{(\boldsymbol{s}, r_1)} \{\|\boldsymbol{b} - \boldsymbol{S}\boldsymbol{s} - r_1 c\boldsymbol{d}\|^2 : \boldsymbol{s} \in S_c \cap S_{sx}\} \end{gathered}$$

where $\hat{\boldsymbol{s}}_{\mathrm{GM}}$ is the global minimizer of (2.29) with the constraints (2.31),

$$\begin{aligned} &r_{1-\mathrm{GM}} = \sqrt{\boldsymbol{s}_{\mathrm{GM}}^{\mathrm{T}} \boldsymbol{s}_{\mathrm{GM}}}, \\ &S_c = \{(\boldsymbol{s}, r_1) \in (\mathbb{R}^2, \mathbb{R}) : \boldsymbol{s}^{\mathrm{T}}\boldsymbol{s} = r_1^2 \,,\, r_1 \geq 0\}, \\ &S_{sx} = \left\{(\boldsymbol{s}, r_1) \in (\mathbb{R}^2, \mathbb{R}) : \boldsymbol{s} = \arg\min_{\boldsymbol{s}} \{\|\boldsymbol{b} - \boldsymbol{S}\boldsymbol{s} - r_1 c\boldsymbol{d}\|^2\}\right\}. \end{aligned}$$

**Comments**

SX method starts with the same equation set used in the SI method, and then takes into account the equality constraint; which is ignored in SI, and equivalently in ULS methods. Because of this, at first glance, it could be seen as superior to the ULS. However, as we have explained in detail, SX does not guarantee to give a solution. Moreover, even if the method gives an estimate, it is quite possible that the obtained estimate is not the global minimum of the cost function. On the other hand, as we will show in Section 2.3.2.4, the LS estimator obtained by considering the constraints becomes approximately equal to the ULS estimator when the noise level is small. In [5], although the problems of the SX method has not been mentioned, some simulation results comparing the performance of SX and SI methods have been presented. The given results show that SI gives a better performance.

#### 2.3.2.3 Two Stage Weighted Least Squares

Chan and Ho proposed an improved version of the LS based source localization method in [9]. In this method, a weighting matrix is added to the cost function in (2.29) to approximate the maximum likelihood estimator (2.12). Then, similar to the ULS method, an UWLS (unconstrained weighted LS) estimator is obtained by ignoring the constraints among the elements of $\boldsymbol{\theta}$, which is given in (2.31). Finally, a second LS computation is done to improve the UWLS estimation result by considering the previously ignored constraints. Because of this two stage process, the method given in [9] is named as TSWLS (two stage weighted least squares). Details of obtaining the weighting matrix and the two step LS computation are as follows:

For the equation set given in (2.28), error vector could be defined as

$$\boldsymbol{e} \triangleq \boldsymbol{b} - \boldsymbol{A\theta}. \tag{2.58}$$

Considering the noise assumption given in (2.5); $e_i$, the $i^{th}$ element of $\boldsymbol{e}$, is written as

$$\begin{aligned} e_i &= 0.5(x_i^2 + y_i^2 - c^2 d_{i,1}^2) - x_i x - y_i y - r_1 c d_{i,1}, \qquad i = 2, 3, ..., N. \\ &= 0.5(x_i^2 + y_i^2 - c^2 (d_{i,1}^0 + n_{i,1})^2) - x_i x - y_i y - c r_1 (d_{i,1}^0 + n_{i,1}). \end{aligned} \tag{2.59}$$

Expanding the brackets gives

$$\begin{aligned} e_i = \; & 0.5(x_i^2 + y_i^2) - 0.5c^2 (d_{i,1}^0)^2 - c^2 d_{i,1}^0 n_{i,1} - 0.5c^2 n_{i,1}^2 \\ & - x_i x - y_i y - c r_1 d_{i,1}^0 - c r_1 n_{i,1}. \end{aligned} \tag{2.60}$$

Since $d_{i,1} = d_i - d_1$ and $cd_i^0 = r_i$, equation (2.60) could be written as

$$\begin{aligned} e_i = &\ 0.5(x_i^2 + y_i^2) - 0.5c^2(d_{i,1}^0)^2 - x_i x - y_i y - cr_1 d_{i,1}^0 \\ &- cr_i n_{i,1} - 0.5c^2 n_{i,1}^2 \\ = &\ 0.5((x_i - x)^2 + (y_i - y)^2) - 0.5r_i^2 - cr_i n_{i,1} - 0.5c^2 n_{i,1}^2 \\ = &\ - cr_i n_{i,1} - 0.5c^2 n_{i,1}^2 \end{aligned} \tag{2.61}$$

Then, covariance matrix of the error vector becomes

$$\boldsymbol{\Psi} \triangleq \text{cov}(\boldsymbol{e}) = \text{cov}(c\boldsymbol{R}\boldsymbol{n} + 0.5c^2\boldsymbol{n} \odot \boldsymbol{n}) \tag{2.62}$$

where $\boldsymbol{n} = [n_{2,1}, n_{3,1}, ..., n_{N,1}]^{\text{T}}$, $\boldsymbol{R} \triangleq diag\{r_2, r_3, ..., r_N\}$ and $\odot$ is the Hadamard product.

If $r_i \gg cn_{i,1}$ for $i = 2, 3, ..., N$, which is usually the case in practice [9], the second term of (2.62) could be ignored. Then, the covariance matrix of the error vector becomes

$$\boldsymbol{\Psi} \approx \text{cov}(c\boldsymbol{R}\boldsymbol{n}) = c^2\boldsymbol{R}\boldsymbol{Q}\boldsymbol{R}. \tag{2.63}$$

Inserting $\boldsymbol{\Psi}$ into the LS-SC cost function (2.29) as a weighting matrix results in a cost function as follows:

$$C_{\text{WLS}} = (\boldsymbol{b} - \boldsymbol{A}\boldsymbol{\theta})^{\text{T}}\boldsymbol{\Psi}^{-1}(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{\theta}). \tag{2.64}$$

Note that, $\boldsymbol{e}$ could be written as

$$\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{A}\boldsymbol{\theta} \ \approx \ c\boldsymbol{R}\boldsymbol{n} = c\boldsymbol{R}(\boldsymbol{d} - \boldsymbol{r}/c). \tag{2.65}$$

Inserting (2.63) and (2.65) into (2.64) gives

$$\begin{aligned} C_{\text{WLS}} &\approx [c\boldsymbol{R}(\boldsymbol{d} - \boldsymbol{r}/c)]^{\text{T}}(c^2\boldsymbol{R}\boldsymbol{Q}\boldsymbol{R})^{-1}[c\boldsymbol{R}(\boldsymbol{d} - \boldsymbol{r}/c)] \\ &\approx (\boldsymbol{d} - \boldsymbol{r}/c)^{\text{T}}\boldsymbol{Q}^{-1}(\boldsymbol{d} - \boldsymbol{r}/c), \end{aligned} \tag{2.66}$$

which is the same cost function obtained for the MLE in (2.12). This means that a source position estimate minimizing (2.64) is an approximation of the MLE. It is an approximation since:

1. Noise-squared term of the equation (2.62) is ignored,

2. $\boldsymbol{R}$ is composed of the true range values between the source and the sensors, therefore it is not available in practice. An approximate $\boldsymbol{R}$ matrix have to be used in the calculation of the weighting matrix $\boldsymbol{\Psi}^{-1}$.

Using the same approach as in the ULS method, a closed form estimator could be obtained by ignoring the relation among the elements of $\boldsymbol{\theta}$ as follows:

$$\begin{aligned}\hat{\boldsymbol{\theta}}_{\text{UWLS}} &= \arg\min_{\boldsymbol{\theta}} \{(\boldsymbol{A\theta} - \boldsymbol{b})^{\text{T}}\boldsymbol{\Psi}^{-1}(\boldsymbol{A\theta} - \boldsymbol{b})\}. \\ &= (\boldsymbol{A}^{\text{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A})^{-1}\boldsymbol{A}^{\text{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{b}, \end{aligned} \tag{2.67}$$

and the source location estimate is the first two elements of the result:

$$\hat{\boldsymbol{s}}_{\text{UWLS}} = [\hat{\theta}_{\text{UWLS-1}} \;\; \hat{\theta}_{\text{UWLS-2}}]^{\text{T}}, \tag{2.68}$$

where $\hat{\theta}_{\text{UWLS}-i}$ represents the $i^{th}$ element of $\hat{\boldsymbol{\theta}}_{\text{UWLS}}$. Since the constraints in (2.31) are ignored, the estimate is named as UWLS (unconstrained WLS).

Note that, to have a nonsingular $(\boldsymbol{A}^{\text{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A})$ matrix, at least 4 sensors are required in a two dimensional space. In the case of 4 sensors, $\boldsymbol{A}$ becomes a 3x3 full rank matrix (except some special cases such as zero valued TDOAs or linear sensor distribution). Then the inverse of $\boldsymbol{A}$ exists, i.e., a location estimate equating the cost function (2.64) to zero could be obtained. In such case, the result is not affected by the weighting and it is identical to the ULS estimate. Therefore, at least 5 sensors are needed to see the effect of the weighting matrix.

In practice, $\boldsymbol{\Psi}$ could not be known exactly, since $\boldsymbol{R}$ includes the true range values between the sensors and the source. In [9], a method to obtain $\boldsymbol{\Psi}$ approximately is proposed. In this approach, by assuming that the source is far from the area in which the sensors are distributed, $r_i,\ i = 2, 3, ..., N$, values are considered nearly equal to $r_1$, and therefore

$$\boldsymbol{R} \approx r_1 \boldsymbol{I}, \tag{2.69}$$

where $\boldsymbol{I}$ is $(N-1)$x$(N-1)$ identity matrix. Then the weighting matrix could be calculated as

$$\boldsymbol{\Psi} \approx r_1^2 c^2 \boldsymbol{Q}. \tag{2.70}$$

Finally, (2.67) becomes

$$
\begin{aligned}
v &= (\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A})^{-1}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{b} \\
&\approx (\boldsymbol{A}^{\mathrm{T}}(r_1^2c^2\boldsymbol{Q})^{-1}\boldsymbol{A})^{-1}\boldsymbol{A}^{\mathrm{T}}(r_1^2c^2\boldsymbol{Q})^{-1}\boldsymbol{b} \\
&\approx (\boldsymbol{A}^{\mathrm{T}}\boldsymbol{Q}^{-1}\boldsymbol{A})^{-1}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{Q}^{-1}\boldsymbol{b}.
\end{aligned} \tag{2.71}
$$

If the source is close to the sensors, (2.71) is used to obtain an initial estimate of the source position to calculate $\boldsymbol{R}$ approximately. Then, using (2.63) and (2.67) UWLS estimate of the source location could be obtained [9].

If $\hat{\boldsymbol{\theta}}_{\mathrm{UWLS}}$ is written as the sum of the true source location and range vector $\boldsymbol{\theta}$ and a noise term, i.e.,

$$
\hat{\boldsymbol{\theta}}_{\mathrm{UWLS}} = \boldsymbol{\theta} + \boldsymbol{e}_{\mathrm{UWLS}}, \tag{2.72}
$$

then multiplying both side of (2.67) with $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A}$ and inserting (2.72)

$$
\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A}(\boldsymbol{\theta} + \boldsymbol{e}_{\mathrm{UWLS}}) = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{b} \tag{2.73}
$$

is obtained. Expanding the brackets and rearranging the terms gives

$$
\begin{aligned}
\boldsymbol{e}_{\mathrm{UWLS}} &= (\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A})^{-1}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{\theta}) \\
&= (\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A})^{-1}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{e}.
\end{aligned} \tag{2.74}
$$

Remember that $\boldsymbol{\Psi} \triangleq \mathrm{cov}(\boldsymbol{e})$. If the noise squared terms of $\boldsymbol{e}$ in (2.62) are ignored, $\mathrm{E}[\boldsymbol{e}]$ becomes zero vector. Then the expected value and covariance of $\hat{\boldsymbol{\theta}}_{\mathrm{UWLS}}$ could be approximated as

$$
\mathrm{E}[\hat{\boldsymbol{\theta}}_{\mathrm{UWLS}}] = \boldsymbol{\theta} + \mathrm{E}[\boldsymbol{e}_{\mathrm{UWLS}}] \approx \boldsymbol{\theta}, \tag{2.75}
$$

$$
\begin{aligned}
\mathrm{cov}(\hat{\boldsymbol{\theta}}_{\mathrm{UWLS}}) &\approx \mathrm{E}[\boldsymbol{e}_{\mathrm{UWLS}}\, \boldsymbol{e}_{\mathrm{UWLS}}^{\mathrm{T}}] \\
&\approx (\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A})^{-1}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{\Psi}\boldsymbol{\Psi}^{-1}\boldsymbol{A}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A})^{-1} \\
&\approx (\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A})^{-1}.
\end{aligned} \tag{2.76}
$$

Equations (2.75) and (2.76) show that when the bias resulting from the noise squared terms are negligible, $\hat{\boldsymbol{\theta}}_{\mathrm{UWLS}}$ could be considered as a random vector whose expected value is the true value. Then $\hat{\boldsymbol{\theta}}_{\mathrm{UWLS}}$ could be written as

$$
\hat{\boldsymbol{\theta}}_{\mathrm{UWLS}} \triangleq \begin{bmatrix} \theta_{\text{UWLS-1}} \\ \theta_{\text{UWLS-2}} \\ \theta_{\text{UWLS-3}} \end{bmatrix} = \begin{bmatrix} x + e_x \\ y + e_y \\ r_1 + e_r \end{bmatrix}, \tag{2.77}
$$

where $e_i,\ i = 1,2,3$ are the estimation errors. Considering the previously ignored constraint (2.31a), an equation set could be written as follows [9]:

$$\boldsymbol{e}' = \boldsymbol{b}' - \boldsymbol{A}'\boldsymbol{s}' \tag{2.78}$$

where

$$\boldsymbol{b}' = \begin{bmatrix} (\theta_{\text{UWLS-1}})^2 \\ (\theta_{\text{UWLS-2}})^2 \\ (\theta_{\text{UWLS-3}})^2 \end{bmatrix}, \qquad \boldsymbol{A}' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \qquad \boldsymbol{s}' = \begin{bmatrix} x^2 \\ y^2 \end{bmatrix}.$$

From (2.77) and (2.78), $\boldsymbol{e}'$ is obtained as

$$\boldsymbol{e}' = \begin{bmatrix} 2xe_x + e_x^2 \\ 2ye_y + e_y^2 \\ 2r_1e_r \end{bmatrix} \approx \begin{bmatrix} 2xe_x \\ 2ye_y \\ 2r_1e_r \end{bmatrix}, \tag{2.79}$$

and

$$\boldsymbol{\Psi}' \triangleq \text{cov}(\boldsymbol{e}') = 4\boldsymbol{R}'\text{cov}(\hat{\boldsymbol{\theta}}_{\text{UWLS}})\boldsymbol{R}' \tag{2.80}$$

where $\boldsymbol{R}' = \text{diag}(x, y, r_1)$. Then, the WLS estimate of $\boldsymbol{s}'$ is obtained as

$$\begin{aligned} \hat{\boldsymbol{s}'} &= \arg\min_{\boldsymbol{s}'} \{(\boldsymbol{A}'\boldsymbol{s}' - \boldsymbol{b}')^{\text{T}}\boldsymbol{\Psi}'^{-1}(\boldsymbol{A}'\boldsymbol{s}' - \boldsymbol{b}')\}. \\ &= (\boldsymbol{A}'^{\text{T}}\boldsymbol{\Psi}'^{-1}\boldsymbol{A}')^{-1}\boldsymbol{A}'^{\text{T}}\boldsymbol{\Psi}'^{-1}\boldsymbol{b}'. \end{aligned} \tag{2.81}$$

$\boldsymbol{R}$ and $\boldsymbol{R}'$ could be estimated using $\hat{\boldsymbol{\theta}}_{\text{UWLS}}$, hence an approximate $\boldsymbol{\Psi}'$ is obtained. Then, $\boldsymbol{s}'$ is estimated as

$$\hat{\boldsymbol{s}'} \approx (\boldsymbol{A}'^{\text{T}}\boldsymbol{R}'^{-1}\boldsymbol{A}^{\text{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A}\boldsymbol{R}'^{-1}\boldsymbol{A}')^{-1}(\boldsymbol{A}'^{\text{T}}\boldsymbol{R}'^{-1}\boldsymbol{A}^{\text{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A}\boldsymbol{R}'^{-1})\boldsymbol{b}'. \tag{2.82}$$

If the source is far from the sensor array, then the covariance of $\hat{\boldsymbol{\theta}}_{\text{UWLS}}$ becomes

$$\text{cov}(\hat{\boldsymbol{\theta}}_{\text{UWLS}}) \approx c^2r_1^2(\boldsymbol{A}^{\text{T}}\boldsymbol{Q}^{-1}\boldsymbol{A})^{-1} \tag{2.83}$$

and (2.81) becomes

$$\hat{\boldsymbol{s}'} \approx (\boldsymbol{A}'^{\text{T}}\boldsymbol{R}'^{-1}\boldsymbol{A}^{\text{T}}\boldsymbol{Q}^{-1}\boldsymbol{A}\boldsymbol{R}'^{-1}\boldsymbol{A}')^{-1}(\boldsymbol{A}'^{\text{T}}\boldsymbol{R}'^{-1}\boldsymbol{A}^{\text{T}}\boldsymbol{Q}^{-1}\boldsymbol{A}\boldsymbol{R}'^{-1})\boldsymbol{b}'. \tag{2.84}$$

Finally, TSWLS position estimate is obtained as

$$\hat{\boldsymbol{s}}_{\text{TSWLS}} = \begin{bmatrix} \pm\sqrt{s_1'} \\ \pm\sqrt{s_2'} \end{bmatrix} \tag{2.85}$$

where $s'_1$ and $s'_2$ are the first and the second elements of $\hat{\boldsymbol{s}'}$, respectively. Because of the sign ambiguity, there are four different source position candidate. In [9], choosing the one which is in the search area is proposed. However, it is possible that more than one solution candidate lies in the search area. We propose an alternative method to solve the sign ambiguity. Among the four TSWLS estimate candidate found in (2.85), the one closest to the UWLS estimate is chosen as the final estimate. In other words,

$$\hat{\boldsymbol{s}}_{\text{TSWLS}} = \underset{\hat{\boldsymbol{s}}_{\text{TSWLS}-i}}{\arg\min} \{\|\hat{\boldsymbol{s}}_{\text{TSWLS}-i} - \hat{\boldsymbol{s}}_{\text{UWLS}}\|\}, \quad i = 1, 2, 3, 4, \tag{2.86}$$

where

$$\begin{aligned}
\hat{\boldsymbol{s}}_{\text{TSWLS}-1} &\triangleq [+\sqrt{s'_1} \quad + \sqrt{s'_2}]^{\text{T}}, \\
\hat{\boldsymbol{s}}_{\text{TSWLS}-2} &\triangleq [-\sqrt{s'_1} \quad + \sqrt{s'_2}]^{\text{T}}, \\
\hat{\boldsymbol{s}}_{\text{TSWLS}-3} &\triangleq [+\sqrt{s'_1} \quad - \sqrt{s'_2}]^{\text{T}}, \\
\hat{\boldsymbol{s}}_{\text{TSWLS}-4} &\triangleq [-\sqrt{s'_1} \quad - \sqrt{s'_2}]^{\text{T}}, \\
\hat{\boldsymbol{s}}_{\text{UWLS}} &\triangleq [\theta_{UWLS-1} \;\; \theta_{UWLS-2}]^{\text{T}}.
\end{aligned}$$

Depending on the noise level and the source position, it is possible to have imaginary roots in (2.85). In these cases, as an empirical method, it is proposed in [9] to set the imaginary part zero.

In summary, TSWLS source localization method steps are as follows:

1. $\boldsymbol{A}$ and $\boldsymbol{b}$ are obtained using the sensor locations and TDOA values.
2. Initial source position estimate is calculated using (2.71).
3. If the initial estimate shows that source is far from the sensor array, it is jumped to step 7.
4. Using the initial estimate, $\boldsymbol{R}$ is found approximately.
5. $\boldsymbol{\Psi}$ is calculated using $\boldsymbol{R}$ and the TDOA covariance matrix $\boldsymbol{Q}$.
6. Initial source position estimate is updated with (2.67).
7. $\boldsymbol{R}'$ is approximated using the initial estimate.
8. $\hat{\boldsymbol{s}'}$ in (2.82) is calculated.
9. TSWLS source location estimate is obtained using (2.85) and (2.86).

### Comments

TSWLS source localization method [9] is based on ULS with two main improvements. The first one is the use of covariance matrix of TDOA values. Since the TDOA value for each sensor is obtained with respect to a common reference sensor, there is a nonzero correlation between TDOA values. Utilizing this relation is expected to improve the estimation result. The second improvement is to take into account the ignored constraint among the elements of $\boldsymbol{\theta}$ via a second LS. However, there are important points that should not be ignored:

- To see the affect of weighting in the estimation result, more than 4 sensors (in general, more than N+2 for N-dimensional space) should be used.
- Note that in the derivation of TSWLS, noise squared terms are ignored in (2.63) and (2.79). Moreover, an estimate of the true range matrix $\boldsymbol{R}$ is used to obtain the weighting matrix $\boldsymbol{\Psi}$. Therefore, under low SNR condition, TSWLS estimate may even worse than ULS.
- There is a sign ambiguity in the TSWLS result, as seen in (2.85). To solve the ambiguity, we have proposed a solution in (2.86).

#### 2.3.2.4 Constrained Least Squares

In section 2.3.2.1, nonlinear equation set given in (2.28) is linearized by considering the source range $r_1$ as a new variable, despite the fact that it is a function of the source coordinates $[x\ y]^{\mathrm{T}}$, which is given in (2.30). In section 2.3.2.3, this ignored relation is used via a second LS to improve the estimate. To take into account the relation from the beginning, it is proposed in [10] to impose the relation (2.30) on the minimization problem via Lagrange multiplier technique:

Our problem is to minimize the cost function

$$(\boldsymbol{A\theta} - \boldsymbol{b})^{\mathrm{T}}(\boldsymbol{A\theta} - \boldsymbol{b}) \tag{2.87}$$

subject to the constraint

$$\boldsymbol{\theta}^{\mathrm{T}}\mathbf{C}\boldsymbol{\theta} = 0, \tag{2.88}$$

where $\mathbf{C} = diag(1, 1, -1)$. Therefore, corresponding Lagrange function becomes

$$\mathcal{L}(\boldsymbol{\theta}, \lambda) = (\boldsymbol{A\theta} - \boldsymbol{b})^{\mathrm{T}}(\boldsymbol{A\theta} - \boldsymbol{b}) \ + \ \lambda\boldsymbol{\theta}^{\mathrm{T}}\mathbf{C}\boldsymbol{\theta}, \tag{2.89}$$

where $\lambda$ is the Lagrange multiplier. Expanding (2.89) results in

$$\begin{aligned}\mathcal{L}(\boldsymbol{\theta}, \lambda) &= (\boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}} - \boldsymbol{b}^{\mathrm{T}})(\boldsymbol{A\theta} - \boldsymbol{b}) \ + \ \lambda\boldsymbol{\theta}^{\mathrm{T}}\mathbf{C}\boldsymbol{\theta} \\ &= \boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A\theta} - 2\boldsymbol{b}^{\mathrm{T}}\boldsymbol{A\theta} + \boldsymbol{b}^{\mathrm{T}}\boldsymbol{b} + \lambda\boldsymbol{\theta}^{\mathrm{T}}\mathbf{C}\boldsymbol{\theta} \\ &= \boldsymbol{\theta}^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} + \lambda\mathbf{C})\boldsymbol{\theta} - 2\boldsymbol{b}^{\mathrm{T}}\boldsymbol{A\theta} + \boldsymbol{b}^{\mathrm{T}}\boldsymbol{b}.\end{aligned} \tag{2.90}$$

In order to obtain the $\boldsymbol{\theta}$ value minimizing (2.90), gradient of the Lagrange function is taken with respect to $\boldsymbol{\theta}$ and then the result is equated to zero:

$$\frac{\partial\mathcal{L}(\boldsymbol{\theta}, \lambda)}{\partial\boldsymbol{\theta}} = 2(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} + \lambda\mathbf{C})\boldsymbol{\theta} - 2\boldsymbol{A}^{\mathrm{T}}\boldsymbol{b} = 0. \tag{2.91}$$

If (2.91) is solved for $\boldsymbol{\theta}$, constrained least squares (CLS) estimate is obtained as

$$\begin{aligned}\hat{\boldsymbol{\theta}}_{\mathrm{CLS}} &\triangleq \arg\min_{\boldsymbol{\theta}} \{\mathcal{L}(\boldsymbol{\theta}, \lambda)\} \\ &= (\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} + \lambda\mathbf{C})^{-1}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{b}.\end{aligned} \tag{2.92}$$

Once $\lambda$ is obtained, CLS estimate of the source location could be obtained using (2.92). To find $\lambda$, substituting (2.92) into (2.88) results in

$$\boldsymbol{b}^{\mathrm{T}}\boldsymbol{A}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} + \lambda\mathbf{C})^{-1}\mathbf{C}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} + \lambda\mathbf{C})^{-1}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{b} = 0. \tag{2.93}$$

Note that $\mathbf{C}^{-1} = \mathbf{C}$, therefore

$$\begin{aligned}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} + \lambda\mathbf{C})^{-1} &= \mathbf{C}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\mathbf{C} + \lambda\mathbf{C}\mathbf{C})^{-1} \\ &= \mathbf{C}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\mathbf{C} + \lambda\boldsymbol{I})^{-1}.\end{aligned} \tag{2.94}$$

Hence (2.93) could be rewritten as

$$\boldsymbol{b}^{\mathrm{T}}\boldsymbol{A}\mathbf{C}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\mathbf{C} + \lambda\boldsymbol{I})^{-1}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\mathbf{C} + \lambda\boldsymbol{I})^{-1}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{b} = 0. \tag{2.95}$$

Let $\boldsymbol{D}$ represent a 3x3 diagonal matrix whose diagonal elements are the eigenvalues of $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\mathbf{C}$, i.e., $\boldsymbol{D} \triangleq diag(\zeta_1, \zeta_2, \zeta_3)$, where $\zeta_i$, $i = 1, 2, 3$ is the $i^{\mathrm{th}}$ eigenvalue of $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\mathbf{C}$ (ordered decreasingly); and $\boldsymbol{S}$ represent a 3x3 matrix whose columns are the corresponding eigenvectors of $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\mathbf{C}$. Then, by using eigendecomposition, $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\mathbf{C}$ could be diagonalized as

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\mathbf{C} = \boldsymbol{S}\boldsymbol{D}\boldsymbol{S}^{-1}. \tag{2.96}$$

Inserting (2.96) into (2.95) results in

$$\boldsymbol{g}^{\mathrm{T}}(\boldsymbol{D}+\lambda\boldsymbol{I})^{-2}\boldsymbol{h}=0 \tag{2.97}$$

where

$$\boldsymbol{g}=\boldsymbol{S}^{\mathrm{T}}\mathbf{C}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{b}=[\alpha_1\ \ \alpha_2\ \ \alpha_3]^{\mathrm{T}}$$
$$\boldsymbol{h}=\boldsymbol{S}^{-1}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{b}=[\beta_1\ \ \beta_2\ \ \beta_3]^{\mathrm{T}}.$$

Rewriting (2.97) in terms of the elements of the vectors gives

$$\sum_{i=1}^{3}\frac{\alpha_i\beta_i}{(\lambda+\zeta_i)^2}=0. \tag{2.98}$$

Multiplying (2.98) with $\prod_{i=1}^{3}(\lambda+\zeta_i)^2$ gives a fourth degree polynomial of $\lambda$ as below:

$$\sum_{i=1}^{3}\alpha_i\beta_i\prod_{j=1,j\neq i}^{3}(\lambda+\zeta_j)^2=0. \tag{2.99}$$

Since there are four roots of (2.99) corresponding to local extrema; a method to find the $\lambda$ value for global minimum, $\lambda_{\text{GM}}$, is needed. In [10], it has been proposed to find the Lagrange multiplier using secant method around $\lambda=0$ by claiming that the polynomial is smooth near $\lambda=0$ and $\lambda$ is expected to be small. However, there is no guarantee that the found root is $\lambda_{\text{GM}}$. It is even possible to end up with an estimate $\hat{\boldsymbol{\theta}}$ having negative third element. In [34] and [35], choosing the root which makes the cost function (2.87) minimum is proposed. However, finding all real roots of (2.99), since the Lagrange multiplier of a real optimization problem is real [34]; obtaining the source location estimates; and then calculating the cost functions for the estimates satisfying the constraint $\hat{\theta}_3\geq 0$ is a computationally expensive process. Moreover, in a three dimensional space, (2.99) becomes a sixth degree polynomial of $\lambda$ and therefore iterative methods have to be used to find the roots [33], which makes the process even more complicated.

In [11], it has been shown that $\lambda_{\text{GMS}}$, the Lagrange multiplier of the minimization problem (2.87) with the single constraint (2.88), lies in a certain interval. Even though the second constraint, $\theta_3\geq 0$, is not taken into account; the $\lambda$ value lying in the defined interval generally corresponds to the global minimum, i.e. $\lambda_{\text{GMS}}=\lambda_{\text{GM}}$, unless the noise level is high [11]. Additionally, it has been proven in [36] that (2.99) is strictly decreasing in the defined interval, which makes it possible to find $\lambda_{\text{GMS}}$ using a simple bisection algorithm. $\lambda_{\text{GMS}}$ finding approach proposed in [11] is explained in detail as follows:

Recall our minimization problem:

$$\min_{\boldsymbol{\theta}}\{\|\boldsymbol{A\theta} - \boldsymbol{b}\|^2 : \ \boldsymbol{\theta}^{\mathrm{T}}\mathbf{C}\boldsymbol{\theta} = 0, \ \theta_3 \geq 0\}. \tag{2.100}$$

Note that (2.100) is a non-convex quadratic function minimization problem with two quadratic constraints (since a linear constraint is a member of general quadratic constraints). For such problems, there is no known way to obtain a general characterization of the global minimizer efficiently [11]. However, if the second constraint, $\theta_3 \geq 0$, is ignored; then the problem becomes the minimization of a quadratic function subject to a single quadratic constraint. Such problems are called as generalized trust region subproblems (GTRS) [11]. GTRS problems have necessary and sufficient optimality conditions, which make it possible to establish efficient closed form solution techniques [11], [36].

Trust region methods are a class of iterative solution approaches used to solve the unconstrained optimization problems. In such methods, a new trial step is obtained by solving the following subproblem

$$\min_{\boldsymbol{x}\in\mathbb{R}^{\mathrm{n}}}\{m_k(\boldsymbol{x})\} \tag{2.101a}$$

$$s.t. \quad \|\boldsymbol{Dx}\| \leq \Delta_k, \tag{2.101b}$$

where $m_k(\boldsymbol{x})$ is a quadratic model function which is an approximation of the objective function, and (2.101b) is the trust region in which the model function is valid [37]. The minimization problem (2.101) is called as trust region subproblem (TRS). In [36], a generalized version of TRS (GTRS) is considered by extending (2.101) as

$$\min_{\boldsymbol{x}\in\mathbb{R}^{\mathrm{n}}}\{f(\boldsymbol{x})\} \tag{2.102a}$$

$$s.t. \quad c(\boldsymbol{x}) \leq 0, \tag{2.102b}$$

where both $f(\boldsymbol{x})$ and $c(\boldsymbol{x})$ are $\mathbb{R}^{\mathrm{n}} \to \mathbb{R}$ quadratic functions. In [36], global minimizer of (2.102) is characterized by first considering the equality constraint

$$c(\boldsymbol{x}) = 0. \tag{2.103}$$

Recall the TDOA based source localization problem:

$$\min_{\boldsymbol{\theta}}\{(\boldsymbol{A\theta}-\boldsymbol{b})^{\mathrm{T}}(\boldsymbol{A\theta}-\boldsymbol{b})\} \tag{2.104a}$$

$$s.t. \quad \boldsymbol{\theta}^{\mathrm{T}}\mathbf{C}\boldsymbol{\theta}=0 \quad and \quad \theta_3 \geq 0. \tag{2.104b}$$

Ignoring the second constraint $\theta_3 \geq 0$, (2.104) could be considered as a GTRS with the equality constraint, where

$$f(\boldsymbol{\theta})=(\boldsymbol{A\theta}-\boldsymbol{b})^{\mathrm{T}}(\boldsymbol{A\theta}-\boldsymbol{b}), \tag{2.105}$$

$$c(\boldsymbol{\theta})=\boldsymbol{\theta}^{\mathrm{T}}\mathbf{C}\boldsymbol{\theta}. \tag{2.106}$$

Therefore, in [11], the results of [36] related to GTRS has been taken as the base to develop a closed form efficient solution algorithm. One of the most valuable theorems given in [36] and its significance for our source localization problem are stated below:

**Theorem 2.** *For a problem in the form of*

$$\min_{\boldsymbol{x}\in\mathbb{R}^{\mathrm{n}}}\{f(\boldsymbol{x}):c(\boldsymbol{x})=0\}, \tag{2.107}$$

*where $f(\boldsymbol{x})$ and $c(\boldsymbol{x})$ are $\mathbb{R}^{\mathrm{n}}\rightarrow\mathbb{R}$ quadratic functions defined on $\mathbb{R}^{\mathrm{n}}$; assume that*

$$min\{c(\boldsymbol{x}):\, x\in\mathbb{R}^{\mathrm{n}}\} \;<\; 0 \;<\; max\{c(\boldsymbol{x}):\, x\in\mathbb{R}^{\mathrm{n}}\} \tag{2.108}$$

*and the Hessian of $c(\boldsymbol{x})$ satisfies*

$$\nabla^2_{\boldsymbol{xx}}c(\boldsymbol{x})\neq 0. \tag{2.109}$$

*Then, a vector $\boldsymbol{x}_{GMS}$ is a global minimizer of (2.107) if and only if the constraint is satisfied, i.e.,*

$$c(\boldsymbol{x}_{GMS})=0 \tag{2.110}$$

*and there exist a $\lambda_{GMS}\in\mathbb{R}$ such that*

$$\nabla_{\boldsymbol{x}}f(\boldsymbol{x}_{GMS})+\lambda_{GMS}\nabla_{\boldsymbol{x}}c(\boldsymbol{x}_{GMS})=\mathbf{0} \tag{2.111}$$

*and*

$$\nabla^2_{\boldsymbol{xx}}f(\boldsymbol{x}_{GMS})+\lambda_{GMS}\nabla^2_{\boldsymbol{xx}}c(\boldsymbol{x}_{GMS})\geq 0. \tag{2.112}$$

A detailed proof of Theorem 2 could be seen in [36]. To examine whether this theorem is applicable to (2.104), taking the Hessian of (2.106) results in

$$\nabla^2_{\boldsymbol{\theta\theta}}c(\boldsymbol{\theta})=\nabla^2_{\boldsymbol{\theta\theta}}(\boldsymbol{\theta}^{\mathrm{T}}\mathbf{C}\boldsymbol{\theta})=\mathbf{C}. \tag{2.113}$$

Moreover, since $\mathbf{C} = diag\{1, 1, -1\}$ is indefinite, both (2.108) and (2.109) are satisfied. Therefore, a vector $\boldsymbol{\theta}_{\text{GMS}}$ is a global minimizer if and only if

$$c(\boldsymbol{\theta}_{\text{GMS}}) = \boldsymbol{\theta}_{\text{GMS}}^{\text{T}}\mathbf{C}\boldsymbol{\theta}_{\text{GMS}} = 0 \tag{2.114}$$

and there is a $\lambda_{\text{GMS}} \in \mathbb{R}$ satisfying

$$\nabla_{\boldsymbol{\theta}} f(\boldsymbol{\theta}_{\text{GMS}}) + \lambda_{\text{GMS}} \nabla_{\boldsymbol{\theta}} c(\boldsymbol{\theta}_{\text{GMS}}) = 2(\boldsymbol{A}^{\text{T}}\boldsymbol{A}\boldsymbol{\theta}_{\text{GMS}} - \boldsymbol{A}^{\text{T}}\boldsymbol{b} + \lambda_{\text{GMS}}\mathbf{C}\boldsymbol{\theta}_{\text{GMS}}) = \mathbf{0} \tag{2.115}$$

and

$$\nabla^2_{\boldsymbol{\theta\theta}} f(\boldsymbol{\theta}_{\text{GMS}}) + \lambda_{\text{GMS}} \nabla^2_{\boldsymbol{\theta\theta}} c(\boldsymbol{\theta}_{\text{GMS}}) = 2(\boldsymbol{A}^{\text{T}}\boldsymbol{A} + \lambda_{\text{GMS}}\mathbf{C}) \geq 0. \tag{2.116}$$

From (2.115), $\boldsymbol{\theta}_{\text{GMS}}$ could be obtained as

$$\boldsymbol{\theta}_{\text{GMS}} = (\boldsymbol{A}^{\text{T}}\boldsymbol{A} + \lambda\mathbf{C})^{-1}\boldsymbol{A}^{\text{T}}\boldsymbol{b}. \tag{2.117}$$

Note that the equations (2.114), (2.115) and (2.117) have been already obtained by applying the Lagrange multiplier method and given as (2.91), (2.88) and (2.92), respectively. Additionally, it has been derived using these equations a four degree polynomial whose one of the roots gives $\lambda_{\text{GMS}}$. What the Theorem (2) gives us additional about $\lambda_{\text{GMS}}$ is that $\boldsymbol{A}^{\text{T}}\boldsymbol{A} + \lambda_{\text{GMS}}\mathbf{C}$ should be positive semidefinite.

Note that

$$\boldsymbol{A}^{\text{T}}\boldsymbol{A} + \lambda\mathbf{C} = (\boldsymbol{A}^{\text{T}}\boldsymbol{A})^{1/2}(\boldsymbol{I} + \lambda(\boldsymbol{A}^{\text{T}}\boldsymbol{A})^{-1/2}\boldsymbol{C}(\boldsymbol{A}^{\text{T}}\boldsymbol{A})^{-1/2})(\boldsymbol{A}^{\text{T}}\boldsymbol{A})^{1/2}. \tag{2.118}$$

So, the numbers of the eigenvalues with the same signs are equal for $\boldsymbol{A}^{\text{T}}\boldsymbol{A} + \lambda\mathbf{C}$ and $\boldsymbol{I} + \lambda(\boldsymbol{A}^{\text{T}}\boldsymbol{A})^{-1/2}\boldsymbol{C}(\boldsymbol{A}^{\text{T}}\boldsymbol{A})^{-1/2}$.

If the generalized eigenvalues of the matrix pair $(\boldsymbol{C}, \boldsymbol{A}^{\text{T}}\boldsymbol{A})$ are defined as

$$\lambda_i(\boldsymbol{C}, \boldsymbol{A}^{\text{T}}\boldsymbol{A}) \triangleq \lambda_i[(\boldsymbol{A}^{\text{T}}\boldsymbol{A})^{-1/2}\boldsymbol{C}(\boldsymbol{A}^{\text{T}}\boldsymbol{A})],\ i = 0, 1, 2 \tag{2.119}$$

with the order

$$\lambda_0(\boldsymbol{C}, \boldsymbol{A}^{\text{T}}\boldsymbol{A}) \leq \lambda_1(\boldsymbol{C}, \boldsymbol{A}^{\text{T}}\boldsymbol{A}) \leq \lambda_2(\boldsymbol{C}, \boldsymbol{A}^{\text{T}}\boldsymbol{A}), \tag{2.120}$$

and

$$\gamma_i \triangleq -1/\lambda_i(\boldsymbol{C}, \boldsymbol{A}^{\text{T}}\boldsymbol{A}), \quad i = 0, 1, 2, \tag{2.121}$$

then the eigenvalues of $\boldsymbol{I} + \lambda(\boldsymbol{A}^{\text{T}}\boldsymbol{A})^{-1/2}\boldsymbol{C}(\boldsymbol{A}^{\text{T}}\boldsymbol{A})^{-1/2}$ could be written as

$$\lambda_i(\boldsymbol{I} + \lambda(\boldsymbol{A}^{\text{T}}\boldsymbol{A})^{-1/2}\boldsymbol{C}(\boldsymbol{A}^{\text{T}}\boldsymbol{A})^{-1/2}) = 1 - \lambda/\gamma_i, \quad i = 0, 1, 2. \tag{2.122}$$

Note that $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$ is positive definite and $\boldsymbol{C}$ has two positive eigenvalues and one negative eigenvalue,therefore

$$\gamma_2 \leq \gamma_1 < 0 < \gamma_0. \tag{2.123}$$

Let define four interval as

$$I_a = (\gamma_0, \infty),\ I_b = (\gamma_1, \gamma_0),\ I_c = (\gamma_2, \gamma_1),\ I_d = (-\infty, \gamma_2). \tag{2.124}$$

- If $\lambda \in I_a$, then $1 - \lambda/\gamma_i > 0$ for $i = 1, 2$ and $1 - \lambda/\gamma_0 < 0$. Hence $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} + \lambda\mathbf{C}$ has two positive eigenvalues and one negative eigenvalue.
- If $\lambda \in I_b$, then $1 - \lambda/\gamma_i > 0$ for $i = 0, 1, 2$. Therefore $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} + \lambda\mathbf{C}$ is positive definite.
- If $\lambda \in I_c$, then $1 - \lambda/\gamma_i > 0$ for $i = 0, 2$ and $1 - \lambda/\gamma_1 < 0$. Hence $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} + \lambda\mathbf{C}$ has two positive eigenvalues and one negative eigenvalue.
- If $\lambda \in I_d$, then $1 - \lambda/\gamma_0 > 0$ and $1 - \lambda/\gamma_i < 0$ for $i = 1, 2$. Hence $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} + \lambda\mathbf{C}$ has one positive eigenvalue and two negative eigenvalues.

From Theorem 2, Lagrange multiplier corresponding the the global minimizer, i.e. $\lambda_{\text{GMS}}$, should satisfy (2.116), which is equivalent to the equation set

$$1 - \lambda_{\text{GMS}}/\gamma_i > 0, \quad i = 0, 1, 2. \tag{2.125}$$

For (2.125) to hold true, $\lambda_{\text{GMS}}$ should lie in $I_b = I_{\text{GMS}}$ [11].

Note that the inequalities in (2.125) are strict. This removes the case in which $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} + \lambda_{\text{GMS}}\mathbf{C} = 0$. Since the equality case is considered as the "hard-case" in GTRS, and to face such a case in TDOA based source localization problems is extremely unlikely [11]; it is assumed that $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} + \lambda_{\text{GMS}}\mathbf{C}$ is positive definite for simplicity.

Using Theorem 2, the interval in which $\lambda_{\text{GMS}}$ lies is obtained. The following theorem given in [36] will make it easer to obtain $\lambda_{\text{GMS}}$ in this interval:

**Theorem 3.** *Let $f(\boldsymbol{x})$ and $c(\boldsymbol{x})$ are $\mathbb{R}^{\mathrm{n}} \to \mathbb{R}$ quadratic functions defined on $\mathbb{R}^{\mathrm{n}}$, and assume $I_{GMS} \triangleq \{\lambda : \boldsymbol{A}^{T}\boldsymbol{A} + \lambda\boldsymbol{C}\ \ positive\ \ definite\} \neq \varnothing$. If the solution of*

$$\nabla_{\boldsymbol{x}} f[(\boldsymbol{x}(\lambda)] + \lambda \nabla_{\boldsymbol{x}} c[\boldsymbol{x}(\lambda)] = \mathbf{0}$$

*is denoted by* $\boldsymbol{x}(\lambda) \in \mathbb{R}^n$*, then* $c[\boldsymbol{x}(\lambda)]$ *is a strictly decreasing function on* $I_{GMS}$ *unless* $\boldsymbol{x}(\lambda)$ *is constant on the same interval with*

$$\nabla_{\boldsymbol{x}} f[(\boldsymbol{x}(\lambda)] = \mathbf{0}, \quad \nabla_{\boldsymbol{x}} c[\boldsymbol{x}(\lambda)] = \mathbf{0}, \quad \lambda \in I_{GMS}.$$

A complete proof of Theorem 3 is available in [36]. For the TDOA based source localization problem, this theorem could be proven as follows:

From (2.92) and (2.106)

$$\begin{aligned}\frac{d}{d\lambda}\boldsymbol{\theta}(\lambda) &= \frac{d}{d\lambda}[(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} + \lambda\mathbf{C})^{-1}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{b}] \\ &= -(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} + \lambda\mathbf{C})^{-1}\boldsymbol{C}\boldsymbol{\theta}(\lambda) \\ &= -\frac{1}{2}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} + \lambda\mathbf{C})^{-1}\nabla_{\boldsymbol{\theta}} c[\boldsymbol{\theta}(\lambda)]. \end{aligned} \tag{2.126}$$

is obtained. Putting (2.126) into the following equation

$$\frac{d}{d\lambda}c[\boldsymbol{\theta}(\lambda)] = \nabla_{\boldsymbol{\theta}} c[\boldsymbol{\theta}(\lambda)]^{\mathrm{T}}\frac{d}{d\lambda}\boldsymbol{\theta}(\lambda), \tag{2.127}$$

it is obtained that

$$\frac{d}{d\lambda}c[\boldsymbol{\theta}(\lambda)] = -\frac{d}{d\lambda}\boldsymbol{\theta}(\lambda)^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} + \lambda\mathbf{C})\frac{d}{d\lambda}\boldsymbol{\theta}(\lambda). \tag{2.128}$$

Since $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} + \lambda\mathbf{C}$ is positive definite in $I_{\mathrm{GMS}} = (\gamma_1, \gamma_0)$, (2.128) is negative in the same interval. This shows that $c[\boldsymbol{\theta}(\lambda)]$ is a strictly decreasing function in $I_{\mathrm{GMS}}$. Hence, $\lambda_{\mathrm{GMS}}$ could be easily found by applying a bisection algorithm to (2.98) in the interval $I_{\mathrm{GMS}}$.

To show that $\lambda_{\mathrm{GMS}}$ is close to zero in high SNR, (2.117) could be written as [35]

$$[(\boldsymbol{A}^0 + \varDelta\boldsymbol{A})^{\mathrm{T}}(\boldsymbol{A}^0 + \varDelta\boldsymbol{A}) + \lambda\boldsymbol{C}](\boldsymbol{\theta}^0 + \varDelta\boldsymbol{\theta}) = (\boldsymbol{A}^0 + \varDelta\boldsymbol{A})^{\mathrm{T}}(\boldsymbol{b}^0 + \varDelta\boldsymbol{b}), \tag{2.129}$$

where $\boldsymbol{A} = \boldsymbol{A}^0 + \varDelta\boldsymbol{A}$, $\boldsymbol{\theta} = \boldsymbol{\theta}^0 + \varDelta\boldsymbol{\theta}$, $\boldsymbol{b} = \boldsymbol{b}^0 + \varDelta\boldsymbol{b}$, $\{*\}^0$ is the noise-free term and $\varDelta\{*\}$ is the noise. Assuming sufficiently low noise conditions, the error terms other than the first-order ones could be ignored. Then (2.129) becomes

$$\begin{aligned}&\lambda\boldsymbol{C}(\boldsymbol{\theta}^0 + \varDelta\boldsymbol{\theta}) \\ &\approx \boldsymbol{A}^{0^{\mathrm{T}}}\varDelta\boldsymbol{b} + \varDelta\boldsymbol{A}^{\mathrm{T}}\boldsymbol{b}^0 - \boldsymbol{A}^{0^{\mathrm{T}}}\varDelta\boldsymbol{A}\boldsymbol{\theta}^0 - \varDelta\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}^0\boldsymbol{\theta}^0 - \boldsymbol{A}^{0^{\mathrm{T}}}\boldsymbol{A}^0\varDelta\boldsymbol{\theta}. \end{aligned} \tag{2.130}$$

Since $\mathrm{E}[\boldsymbol{A}] = \boldsymbol{A}^0$, $\mathrm{E}[\boldsymbol{b}] \approx \boldsymbol{b}^0$ and $\mathrm{E}[\boldsymbol{\theta}] = \boldsymbol{\theta}^0$ [34], taking the expectation of (2.130) results in

$$\mathrm{E}[\lambda] = 0. \tag{2.131}$$

Therefore, under low noise conditions, it could be expected that the Lagrange multiplier is close to zero.

Note that $\lambda_{\text{GMS}}$ is the Lagrange multiplier corresponding to the global minimizer of the problem (2.100) as long as the second constraint, $\theta_3 \geq 0$, is ignored. Therefore, it is possible to end up with a negative $\theta_3(\lambda_{\text{GMS}})$, especially under low SNR [11].

The minimization problem given in (2.100) has also been considered by taking into account all constraints in [11], with the help of the following proposition [38]:

**Proposition 1** (Karush-Kuhn-Tucker Necessary Conditions)**.** *Let $\boldsymbol{x}_{LM}$ is a local minimum of*

$$\min_{\boldsymbol{x}\in\mathrm{I\!R}^{\mathrm{n}}} \{f(\boldsymbol{x}):\ c_i(\boldsymbol{x}) = 0,\ d_j(\boldsymbol{x}) \leq 0,\ i = 1, 2, ..., k,\ j = 1, 2, ..., l\} \tag{2.132}$$

*where $f(\boldsymbol{x})$, $c_i(\boldsymbol{x})$ and $d_j(\boldsymbol{x})$ are $\mathrm{I\!R}^{\mathrm{n}} \rightarrow \mathrm{I\!R}$ continuously differentiable functions, and assume that the gradients of the active constraints among $c_i(\boldsymbol{x})$ and $d_j(\boldsymbol{x})$ are independent at $\boldsymbol{x} = \boldsymbol{x}_{LM}$. Then, there are unique Lagrange multipliers vectorized as*

$$\boldsymbol{\lambda}_{LM} = [\lambda_{LM-1}, \lambda_{LM-2}, ..., \lambda_{LM-k}]^T,$$
$$\boldsymbol{\mu}_{LM} = [\mu_{LM-1}, \lambda_{LM-2}, ..., \mu_{LM-l}]^T,$$

*which satisfy*

$$\nabla_{\boldsymbol{x}} L(\boldsymbol{x}_{LM}, \boldsymbol{\lambda}_{LM}, \boldsymbol{\mu}_{LM}) = 0, \tag{2.133}$$
$$\mu_{LM-j} \geq 0, \quad j = 1, 2, ..., l, \tag{2.134}$$
$$\mu_{LM-j} = 0, \quad \forall j \notin P(\boldsymbol{x}_{LM}), \tag{2.135}$$

*where $P(\boldsymbol{x}_{LM})$ is the set of active constraints among $d_j(\boldsymbol{x})\, j = 1, 2, ..., l$, i.e. the ones which satisfy $d_j(\boldsymbol{x}_{LM}) = 0$. Moreover, if $f(\boldsymbol{x})$, $c_i(\boldsymbol{x})$ and $d_j(\boldsymbol{x})$ are twice continuously differentiable, then*

$$\boldsymbol{w}^T \nabla^2_{\boldsymbol{x}\boldsymbol{x}} L(\boldsymbol{x}_{LM}, \boldsymbol{\lambda}_{LM}, \boldsymbol{\mu}_{LM}) \boldsymbol{w} \geq 0, \tag{2.136}$$

*for any* $\boldsymbol{w} \in \mathbb{R}^{\mathrm{n}}$ *satisfying*

$$\nabla c_i(\boldsymbol{x}_{LM})^T \boldsymbol{w} = 0, \quad i = 1, 2, ..., k, \tag{2.137}$$

$$\nabla d_i(\boldsymbol{x}_{LM})^T \boldsymbol{w} = 0, \quad \forall j \in P(\boldsymbol{x}_{LM}). \tag{2.138}$$

For TDOA based source localization problem:

$$f(\boldsymbol{\theta}) = (\boldsymbol{A}\boldsymbol{\theta} - \boldsymbol{b})^{\mathrm{T}}(\boldsymbol{A}\boldsymbol{\theta} - \boldsymbol{b}), \tag{2.139}$$

$$c_1(\boldsymbol{\theta}) = c(\boldsymbol{\theta}) = \boldsymbol{\theta}^{\mathrm{T}}\mathbf{C}\boldsymbol{\theta}, \quad k = 1, \tag{2.140}$$

$$d_1(\boldsymbol{\theta}) = d(\boldsymbol{\theta}) = -\theta_3, \quad l = 1. \tag{2.141}$$

For the nonzero solution, i.e. $\boldsymbol{\theta}_{GM} \neq \mathbf{0}$, $d(\boldsymbol{\theta}_{GM})$ is not active. Additionally, both constraints are twice continuously differentiable. Therefore

$$\nabla f(\boldsymbol{\theta}_{GM}) + \lambda_{\mathrm{GM}} \nabla c(\boldsymbol{\theta}_{GM}) + \mu_{GM} \nabla d(\boldsymbol{\theta}_{GM}) = 0, \tag{2.142}$$

$$c(\boldsymbol{\theta}_{GM}) = 0, \quad d(\boldsymbol{\theta}_{GM}) < 0, \tag{2.143}$$

$$\mu_{GM} = 0. \tag{2.144}$$

From (2.142) and (2.144), $\boldsymbol{\theta}_{GM}$ is obtained as

$$\boldsymbol{\theta}_{GM} = (\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} + \lambda_{\mathrm{GM}}\mathbf{C})^{-1}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{b}. \tag{2.145}$$

Furthermore,

$$\boldsymbol{w}^{\mathrm{T}} \nabla^2_{\boldsymbol{\theta}\boldsymbol{\theta}} L(\boldsymbol{\theta}_{GM}, \boldsymbol{\lambda}_{GM}, \boldsymbol{\mu}_{GM}) \boldsymbol{w} \;=\; \boldsymbol{w}^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} + \lambda_{\mathrm{GM}}\mathbf{C})\boldsymbol{w} \;\geq\; 0, \tag{2.146}$$

for all $\boldsymbol{w} \in \mathbb{R}^3$ such that

$$\nabla c(\boldsymbol{\theta}_{GM})^{\mathrm{T}} \boldsymbol{w} = 2\boldsymbol{\theta}_{GM}^{\mathrm{T}}\mathbf{C}\boldsymbol{w} = 0. \tag{2.147}$$

In addition, using Courant-Fischer-Weyl min-max theorem, the second smallest eigenvalue of $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} + \lambda\mathbf{C}$ could be obtained as

$$\lambda_{SS}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} + \lambda_{\mathrm{GM}}\mathbf{C}) = \max_{\boldsymbol{v} \neq 0} \min_{\boldsymbol{w}^{\mathrm{T}}\boldsymbol{v}=0} \frac{\boldsymbol{w}^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} + \lambda_{\mathrm{GM}}\mathbf{C})\boldsymbol{w}}{\boldsymbol{w}^{\mathrm{T}}\boldsymbol{w}}, \tag{2.148}$$

where $\boldsymbol{v} \in \mathbb{R}^3$. Using (2.146) and (2.147), the following inequality could be written:

$$\begin{aligned} \lambda_{SS}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} + \lambda_{\mathrm{GM}}\mathbf{C}) &= \max_{\boldsymbol{v} \neq 0} \min_{\boldsymbol{w}^{\mathrm{T}}\boldsymbol{v}=0} \frac{\boldsymbol{w}^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} + \lambda_{\mathrm{GM}}\mathbf{C})\boldsymbol{w}}{\boldsymbol{w}^{\mathrm{T}}\boldsymbol{w}}, \\ &\geq \min_{\boldsymbol{\theta}_{GM}^{\mathrm{T}}\mathbf{C}\boldsymbol{w}=0} \frac{\boldsymbol{w}^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} + \lambda_{\mathrm{GM}}\mathbf{C})\boldsymbol{w}}{\boldsymbol{w}^{\mathrm{T}}\boldsymbol{w}} \geq 0, \end{aligned} \tag{2.149}$$

which implies that at most one of the eigenvalues of $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} + \lambda_{\mathrm{GM}}\mathbf{C}$ is negative.

Recall that $\mathbb{R}$ space was divided into four interval in (2.124) and then signs of the eigenvalues of $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} + \lambda\mathbf{C}$ were examined in these intervals. It was stated that the matrix is positive definite in the interval $I_b$ and it has one negative eigenvalue in $(I_a \cup I_c)$. As a result, it could be said that $\lambda_{\mathrm{GM}} \in (I_a \cup I_b \cup I_c)$. Additionally, since $I_b = I_{\mathrm{GMS}}$, if $\lambda_{\mathrm{GM}} \in I_b$ then $\lambda_{\mathrm{GM}} = \lambda_{\mathrm{GMS}}$.

Using the obtained results, an algorithm to find the global optimal solution of the problem stated in (2.104) could defined as follows [11]:

1. Using the TDOA values and the sensor positions, obtain $\boldsymbol{A}$ and $\boldsymbol{b}$ given in (2.28).
2. In the polynomial given in (2.99), calculate unknowns except $\lambda$. Then find $\lambda_{\mathrm{GMS}}$ in the interval $I_b$. Since the polynomial is strictly decreasing in this interval, a simple bisection method could be utilized.
3. Calculate $\boldsymbol{\theta}_{\mathrm{GMS}}$ using (2.117) and $\lambda_{\mathrm{GMS}}$. If $\theta_{\text{GMS-3}}$ is not negative, then $\boldsymbol{\theta}_{\mathrm{GM}} = \boldsymbol{\theta}_{\mathrm{GMS}}$ and the first two elements of $\boldsymbol{\theta}_{\mathrm{GM}}$, i.e. $[\theta_{\text{GM-1}}\ \theta_{\text{GM-2}}]^{\mathrm{T}}$, is the source coordinate estimate. There is no need to continue with the following steps.
4. If $\theta_{\text{GMS-3}}$ is negative, find all real roots $\lambda_{\text{LM-1}}, \lambda_{\text{LM-2}}, ..., \lambda_{\mathrm{LM}-t},\ t \leq 3$ of (2.99) in the interval $(I_a \cup I_c)$, and then calculate the corresponding $\boldsymbol{\theta}$ estimates, which could be represented as $\boldsymbol{\theta}(\lambda_{\mathrm{LM}-i})$, $i = 1, 2, ..., t$.
5. Calculate the objective function (2.87) for $\boldsymbol{\theta}(\lambda_{\mathrm{LM}-i})$ values satisfying $[0\ 0\ 1]\,\hat{\boldsymbol{\theta}}(\lambda_{\mathrm{LM}-i}) \geq 0$ and for $\boldsymbol{\theta}_0 = [0\ 0\ 0]^{\mathrm{T}}$. The one with the smallest objective function is $\boldsymbol{\theta}_{\mathrm{GM}}$.
6. Source location estimate $\hat{\boldsymbol{s}}_{\mathrm{CLS}}$ is the first two elements of $\boldsymbol{\theta}_{\mathrm{GM}}$.

**Comments**

CLS is a ULS based source localization method with a significant improvement. In this method, constraints (2.104b) ignored in ULS are taken into account, while solving the minimization problem (2.104a). The equality constraint is added to the minimization problem via Lagrange multiplier method, and the inequality constraint is taken into consideration while choosing the global solution among the candidates.

Finding the Lagrange multiplier corresponding to the global minimizer is a computationally cumbersome process. However; Bech, Stoica and Li made a valuable analysis in [11] about the interval in which the Lagrange multiplier lies. Their results reduce the computational cost significantly, especially when the noise level is low. Moreover, by taking into account the all zero solution, they have proposed a procedure which guarantees to obtain the global minimum of the unweighted cost function with both constraints. We will call the CLS method using the Lagrange multiplier finding approach given in [11] as efficient CLS (ECLS).

Unlike the TSWLS method, CLS considers the constraints from the beginning of the minimization process, therefore it could be considered as a better approach. However, it should be noted that the correlation among the TDOA values is not taken into consideration in CLS, while it is utilized in TSWLS.

Note that the only difference between the ULS estimate (2.32) and the CLS estimate (2.92) is the term $\lambda\mathbf{C}$. Since it has shown in (2.131) that the expected value of $\lambda$ is zero when the noise level is small, it could be said that the ULS method approximates the global minimum of the TDOA based source localization problem under high SNR.

#### 2.3.2.5 Constrained Weighted Least Squares

In order to increase the estimation performance of the CLS method, correlation among the TDOA estimates could be exploited via a weighting matrix, which is the same approach used in the TSWLS method. Such a method has been proposed in [39] as a combination of CLS [10] and TSWLS [9]. Throughout the remaining parts, the proposed method will be called as CWLS (constrained weighted least squares).

As in the case of the TSWLS method, the weighting matrix includes unknown range vector, which requires the estimation procedure to be repeated once more. However, unlike TSWLS, CWLS needs significant computational cost, since it requires to find the Lagrange multiplier for the global minimum. Doubling such computations are not desirable for the real-time systems with limited resources.

In this section, we will explain in detail the CWLS method which was firstly proposed in [39] and then a slightly changed version of which was restated in [34]. Moreover, we will show that it is possible to apply the Lagrange multiplier finding approach proposed in [11] for CLS to the CWLS method. Then, using this approach, we will propose a modified version of the algorithm given in [34], namely efficient CWLS (ECWLS), which has significantly lower computational cost.

Recall that in Section 2.3.2.3, it has been shown that for the error defined as

$$\boldsymbol{e} \triangleq \boldsymbol{b} - \boldsymbol{A\theta}, \tag{2.150}$$

covariance matrix of the error vector is approximately

$$\boldsymbol{\Psi} \triangleq \text{cov}(\boldsymbol{e}) \approx \text{cov}(c\boldsymbol{Rn}) = c^2\boldsymbol{RQR}, \tag{2.151}$$

where $\boldsymbol{n} = [n_{2,1}, n_{3,1}, ..., n_{N,1}]^{\mathrm{T}}$, $\boldsymbol{R} \triangleq diag\{r_2, r_3, ..., r_N\}$ and $\boldsymbol{Q}$ is the covariance matrix of $\boldsymbol{n}$, or equivalently the TDOA vector.

Inserting $\boldsymbol{\Psi}$ into the LS-SC cost function (2.29) as a weighting matrix results in

$$C_{\text{WLS}} = (\boldsymbol{b} - \boldsymbol{A\theta})^{\mathrm{T}}\boldsymbol{\Psi}^{-1}(\boldsymbol{b} - \boldsymbol{A\theta}), \tag{2.152}$$

and the minimization problem could be defined as

$$\min_{\boldsymbol{\theta}}\{(\boldsymbol{b} - \boldsymbol{A\theta})^{\mathrm{T}}\boldsymbol{\Psi}^{-1}(\boldsymbol{b} - \boldsymbol{A\theta}) : \; \boldsymbol{\theta}^{\mathrm{T}}\mathbf{C}\boldsymbol{\theta} = 0, \; \theta_3 \geq 0\}. \tag{2.153}$$

Ignoring the inequality constraint, corresponding Lagrange function becomes

$$\mathcal{L}_{\text{W}}(\boldsymbol{\theta}, \lambda_{\text{W}}) = (\boldsymbol{A\theta} - \boldsymbol{b})^{\mathrm{T}}\boldsymbol{\Psi}^{-1}(\boldsymbol{A\theta} - \boldsymbol{b}) \; + \; \lambda_{\text{W}}\boldsymbol{\theta}^{\mathrm{T}}\mathbf{C}\boldsymbol{\theta}. \tag{2.154}$$

Recall that in (2.66), we have shown that (2.152) is approximately the same cost function obtained for the MLE in (2.12).

Equation (2.154) could be expanded as

$$\begin{aligned}
\mathcal{L}_{\text{W}}(\boldsymbol{\theta}, \lambda_{\text{W}}) &= (\boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}} - \boldsymbol{b}^{\mathrm{T}})\boldsymbol{\Psi}^{-1}(\boldsymbol{A\theta} - \boldsymbol{b}) \; + \; \lambda_{\text{W}}\boldsymbol{\theta}^{\mathrm{T}}\mathbf{C}\boldsymbol{\theta} \\
&= \boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A\theta} - 2\boldsymbol{b}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A\theta} + \boldsymbol{b}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{b} + \lambda_{\text{W}}\boldsymbol{\theta}^{\mathrm{T}}\mathbf{C}\boldsymbol{\theta} \\
&= \boldsymbol{\theta}^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A} + \lambda_{\text{W}}\mathbf{C})\boldsymbol{\theta} - 2\boldsymbol{b}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A\theta} + \boldsymbol{b}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{b}.
\end{aligned} \tag{2.155}$$

To find the $\boldsymbol{\theta}$ value minimizing (2.155), gradient of the Lagrange function is taken with respect to $\boldsymbol{\theta}$ and then the result is equated to zero:

$$\frac{\partial\mathcal{L}_{\text{W}}(\boldsymbol{\theta}, \lambda_{\text{W}})}{\partial\boldsymbol{\theta}} = 2(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A} + \lambda_{\text{W}}\mathbf{C})\boldsymbol{\theta} - 2\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{b} = 0. \tag{2.156}$$

Solving (2.156) for $\boldsymbol{\theta}$ gives [39]

$$\hat{\boldsymbol{\theta}}_{\text{CWLS}} \triangleq \arg\min_{\boldsymbol{\theta}} \{\mathcal{L}_{\text{W}}(\boldsymbol{\theta}, \lambda_{\text{W}})\}$$
$$= (\boldsymbol{A}^{\text{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A} + \lambda_{\text{W}}\mathbf{C})^{-1}\boldsymbol{A}^{\text{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{b}. \tag{2.157}$$

After obtaining $\lambda_{\text{W}}$, CWLS estimate of the source location could be obtained via (2.157). To find $\lambda_{\text{W}}$, substituting (2.157) into the quadratic equality constraint (2.88) results in

$$\boldsymbol{b}^{\text{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A}(\boldsymbol{A}^{\text{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A} + \lambda_{\text{W}}\mathbf{C})^{-1}\mathbf{C}(\boldsymbol{A}^{\text{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A} + \lambda_{\text{W}}\mathbf{C})^{-1}\boldsymbol{A}^{\text{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{b} = 0. \tag{2.158}$$

Since $\mathbf{C}^{-1} = \mathbf{C}$

$$(\boldsymbol{A}^{\text{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A} + \lambda_{\text{W}}\mathbf{C})^{-1} = \mathbf{C}(\boldsymbol{A}^{\text{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A}\mathbf{C} + \lambda_{\text{W}}\mathbf{C}\mathbf{C})^{-1}$$
$$= \mathbf{C}(\boldsymbol{A}^{\text{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A}\mathbf{C} + \lambda_{\text{W}}\boldsymbol{I})^{-1}. \tag{2.159}$$

Hence (2.158) could be rewritten as

$$\boldsymbol{b}^{\text{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A}\mathbf{C}(\boldsymbol{A}^{\text{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A}\mathbf{C} + \lambda_{\text{W}}\boldsymbol{I})^{-1}(\boldsymbol{A}^{\text{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A}\mathbf{C} + \lambda_{\text{W}}\boldsymbol{I})^{-1}\boldsymbol{A}^{\text{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{b} = 0. \tag{2.160}$$

Let $\boldsymbol{D}_{\text{W}}$ represent a 3x3 diagonal matrix whose diagonal elements are the eigenvalues of $\boldsymbol{A}^{\text{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A}\mathbf{C}$, i.e., $\boldsymbol{D}_{\text{W}} \triangleq diag(\zeta_1', \zeta_2', \zeta_3')$, where $\zeta_i'$, $i = 1, 2, 3$ is the $i^{\text{th}}$ eigenvalue of $\boldsymbol{A}^{\text{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A}\mathbf{C}$ (ordered decreasingly); and $\boldsymbol{S}_{\text{W}}$ represent a 3x3 matrix whose columns are the corresponding eigenvectors of $\boldsymbol{A}^{\text{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A}\mathbf{C}$. Then, by using eigendecomposition, $\boldsymbol{A}^{\text{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A}\mathbf{C}$ could be diagonalized as

$$\boldsymbol{A}^{\text{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A}\mathbf{C} = \boldsymbol{S}_{\text{W}}\boldsymbol{D}_{\text{W}}\boldsymbol{S}_{\text{W}}^{-1}. \tag{2.161}$$

Inserting (2.161) into (2.160) gives

$$\boldsymbol{g}_{\text{W}}^{\text{T}}(\boldsymbol{D}_{\text{W}} + \lambda_{\text{W}}\boldsymbol{I})^{-2}\boldsymbol{h}_{\text{W}} = 0 \tag{2.162}$$

where

$$\boldsymbol{g}_{\text{W}} = \boldsymbol{S}_{\text{W}}^{\text{T}}\mathbf{C}\boldsymbol{A}^{\text{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{b} = [\alpha_1' \;\; \alpha_2' \;\; \alpha_3']^{\text{T}}$$
$$\boldsymbol{h}_{\text{W}} = \boldsymbol{S}_{\text{W}}^{-1}\boldsymbol{A}^{\text{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{b} = [\beta_1' \;\; \beta_2' \;\; \beta_3']^{\text{T}}.$$

Rewriting (2.162) in terms of the elements of the vectors gives

$$\sum_{i=1}^{3} \frac{\alpha_i'\beta_i'}{(\lambda_{\text{W}} + \zeta_i')^2} = 0. \tag{2.163}$$

Multiplying (2.163) with $\prod_{i=1}^{3} (\lambda_{\text{W}} + \zeta_i^{'})^2$ results in a fourth degree polynomial of $\lambda_{\text{W}}$ as below:

$$\sum_{i=1}^{3} \alpha_i^{'} \beta_i^{'} \prod_{j=1, j \neq i}^{3} (\lambda_{\text{W}} + \zeta_j^{'})^2 = 0. \tag{2.164}$$

H. C. So and S. P. Hui has proposed in [39] the following steps to find the CWLS estimate:

1. Set $\boldsymbol{\Psi} = \boldsymbol{I}$.

2. Using Newton–Raphson method with the initial guess $\lambda_0 = 0$, find the root of (2.164) closest to zero. Then use (2.157) to obtain the corresponding $\hat{\boldsymbol{\theta}}$.

3. Calculate $\hat{\boldsymbol{R}}$ using the obtained $\hat{\boldsymbol{\theta}}$. Then update $\boldsymbol{\Psi}$ using (2.151).

4. Repeat Step 2 to obtain the final CWLS estimate.

It could be expected that the $\lambda_{\text{W}}$ value corresponding to the global minimum, $\lambda_{\text{W-GM}}$, is close to zero, particularly when the noise level is low. However, it has not been guaranteed. It is possible to end up with a local extremum, even the one having a negative third element. Therefore, the proposed algorithm in [39] does not guarantee to find the global minimizer of the cost function (2.152) with the constraints (2.104b).

In [34], a slightly changed version of the CWLS algorithm in [39] has been restated as below:

1. Set $\boldsymbol{\Psi} = \boldsymbol{I}$.

2. Find the all roots of (2.164).

3. Using the real valued roots of (2.164) and the equation (2.157), find the corresponding $\hat{\boldsymbol{\theta}}$ candidates. Then choose the one making the cost function (2.152) minimum.

4. Calculate $\hat{\boldsymbol{R}}$ using the chosen $\hat{\boldsymbol{\theta}}$. Then update $\boldsymbol{\Psi}$ using (2.151).

5. Repeat steps (2), (3) and (4) until $\hat{\boldsymbol{\theta}}$ converges.

Inequality constraint, i.e. $\theta_3 \geq 0$, has been ignored in the proposed algorithm; however, it could be easily taken into account in the step 3. Then, since the CWLS algorithm given in [34] examines all real roots of (2.164), it could be said that the proposed algorithm finds the global minimizer of the cost function (2.152) with the constraints (2.104b), assuming the weighting matrix $\boldsymbol{\Psi}$ is constant. Moreover, by iterating the procedure, we can increase the accuracy of the weighting matrix and hence the estimate. However, finding all roots of (2.164), obtaining the source location estimates, and then calculating the cost functions for the estimates satisfying the constraint $\hat{\theta}_3 \geq 0$ for each iteration increase computational cost significantly.

To decrease the computational complexity, we propose to apply the Lagrange multiplier finding approach proposed in [11] for CLS to the CWLS method. Firstly, we will show that it is applicable to the CWLS. Then, using this approach, we will propose a computationally more efficient version of the algorithm given in [34], namely ECWLS.

Recall Theorem 2, which has been used in [11] to obtain an efficient CLS algorithm. Since the constraint function $c(\boldsymbol{\theta}) = \boldsymbol{\theta}^{\mathrm{T}}\mathbf{C}\boldsymbol{\theta}$ remains the same for the CWLS minimization problem, this theorem is still applicable. Therefore, it follows from Theorem 2 that a vector $\boldsymbol{\theta}_{\text{W-GMS}}$ is a global minimizer of the minimization problem whose objective function is (2.152) and single constraint is (2.88), if and only if

$$c(\boldsymbol{\theta}_{\text{W-GMS}}) = \boldsymbol{\theta}_{\text{W-GMS}}^{\mathrm{T}}\mathbf{C}\boldsymbol{\theta}_{\text{W-GMS}} = 0 \tag{2.165}$$

and there is a $\lambda_{\text{W-GMS}} \in \mathbb{R}$ satisfying

$$\begin{aligned}&\nabla_{\boldsymbol{\theta}} f_{\mathrm{W}}(\boldsymbol{\theta}_{\text{W-GMS}}) + \lambda_{\text{W-GMS}}\nabla_{\boldsymbol{\theta}} c(\boldsymbol{\theta}_{\text{W-GMS}})\\&= 2(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A}\boldsymbol{\theta}_{\text{W-GMS}} - \boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{b} + \lambda_{\text{W-GMS}}\mathbf{C}\boldsymbol{\theta}_{\text{W-GMS}}) = \mathbf{0}\end{aligned} \tag{2.166}$$

and

$$\begin{aligned}&\nabla^2_{\boldsymbol{\theta\theta}} f_{\mathrm{W}}(\boldsymbol{\theta}_{\text{W-GMS}}) + \lambda_{\text{W-GMS}}\nabla^2_{\boldsymbol{\theta\theta}} c(\boldsymbol{\theta}_{\text{W-GMS}})\\&= 2(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A} + \lambda_{\text{W-GMS}}\mathbf{C}) \geq 0.\end{aligned} \tag{2.167}$$

Equation (2.166) gives (2.157), and the inequality (2.167) gives the requirement of positive semidefiniteness of $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A} + \lambda_{\text{W-GMS}}\mathbf{C}$, which will be exploited to obtain an interval in which $\lambda_{\text{W-GMS}}$ lies.

Note that

$$
\begin{aligned}
&\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A}+\lambda_{\mathrm{W}}\mathbf{C}\\
&=(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A})^{1/2}(\boldsymbol{I}+\lambda_{\mathrm{W}}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A})^{-1/2}\boldsymbol{C}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A})^{-1/2})(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A})^{1/2}.
\end{aligned} \tag{2.168}
$$

Therefore, the numbers of zero, positive and negative eigenvalues of $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A}+\lambda_{\mathrm{W}}\mathbf{C}$ are same as the ones of $\boldsymbol{I}+\lambda_{\mathrm{W}}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A})^{-1/2}\boldsymbol{C}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A})^{-1/2}$.

Let define the generalized eigenvalues of the matrix pair $(\boldsymbol{C},\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A})$ as

$$
\lambda_i(\boldsymbol{C},\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A})\triangleq\lambda_i[(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A})^{-1/2}\boldsymbol{C}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A})],\ i=0,1,2 \tag{2.169}
$$

with the following order

$$
\lambda_0(\boldsymbol{C},\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A})\leq\lambda_1(\boldsymbol{C},\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A})\leq\lambda_2(\boldsymbol{C},\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A}). \tag{2.170}
$$

If

$$
\gamma_i'\triangleq-1/\lambda_i(\boldsymbol{C},\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A}),\quad i=0,1,2, \tag{2.171}
$$

then the eigenvalues of $\boldsymbol{I}+\lambda_{\mathrm{W}}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A})^{-1/2}\boldsymbol{C}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A})^{-1/2}$ could be written as

$$
\lambda_i(\boldsymbol{I}+\lambda_{\mathrm{W}}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A})^{-1/2}\boldsymbol{C}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A})^{-1/2})=1-\lambda_{\mathrm{W}}/\gamma_i',\quad i=0,1,2. \tag{2.172}
$$

Note that $\boldsymbol{\Psi}$ is a covariance matrix and therefore it is positive semidefinite. If the approximation given in (2.151) is used to obtain an approximate $\boldsymbol{\Psi}$, the obtained matrix is still positive semidefinite since $\boldsymbol{Q}$ is a covariance matrix and $\boldsymbol{R}$ is a diagonal matrix. Hence $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A}$ is positive semidefinite. Additionally, $\boldsymbol{C}$ has two positive eigenvalues and one negative eigenvalue,therefore

$$
\gamma_2'\leq\gamma_1'<0<\gamma_0'. \tag{2.173}
$$

Let define four interval as

$$
I_a'=(\gamma_0',\infty),\ I_b'=(\gamma_1',\gamma_0'),\ I_c'=(\gamma_2',\gamma_1'),\ I_d'=(-\infty,\gamma_2'). \tag{2.174}
$$

- If $\lambda\in I_a'$, then $1-\lambda/\gamma_i'>0$ for $i=1,2$ and $1-\lambda/\gamma_0'<0$. Therefore $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A}+\lambda_{\mathrm{W}}\mathbf{C}$ has two positive eigenvalues and one negative eigenvalue.
- If $\lambda\in I_b'$, then $1-\lambda/\gamma_i'>0$ for $i=0,1,2$. Therefore $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A}+\lambda_{\mathrm{W}}\mathbf{C}$ is positive definite.

- If $\lambda \in I'_c$, then $1 - \lambda/\gamma'_i > 0$ for $i = 0, 2$ and $1 - \lambda/\gamma'_1 < 0$. Therefore $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A} + \lambda_{\mathrm{W}}\mathbf{C}$ has two positive eigenvalues and one negative eigenvalue.
- If $\lambda_{\mathrm{W}} \in I'_d$, then $1 - \lambda/\gamma'_0 > 0$ and $1 - \lambda/\gamma'_i < 0$ for $i = 1, 2$. Therefore $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A} + \lambda_{\mathrm{W}}\mathbf{C}$ has one positive eigenvalue and two negative eigenvalues.

From Theorem 2, Lagrange multiplier corresponding the global minimizer, i.e. $\lambda_{\text{W-GMS}}$, should satisfy (2.167), which is equivalent to the equation set

$$1 - \lambda_{\text{W-GMS}}/\gamma'_i > 0, \quad i = 0, 1, 2. \tag{2.175}$$

For $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A}$ to be positive definite, $\lambda_{\text{W-GMS}}$ should lie in $I'_b = I_{\text{W-GMS}}$.

Moreover, from (2.157) and (2.106)

$$\begin{aligned}\frac{d}{d\lambda_{\mathrm{W}}}\boldsymbol{\theta}(\lambda_{\mathrm{W}}) &= \frac{d}{d\lambda_{\mathrm{W}}}[(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A} + \lambda_{\mathrm{W}}\mathbf{C})^{-1}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{b}] \\ &= -(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A} + \lambda_{\mathrm{W}}\mathbf{C})^{-1}\boldsymbol{C}\boldsymbol{\theta}(\lambda_{\mathrm{W}}) \\ &= -\frac{1}{2}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A} + \lambda_{\mathrm{W}}\mathbf{C})^{-1}\nabla_{\boldsymbol{\theta}}c[\boldsymbol{\theta}(\lambda_{\mathrm{W}})].\end{aligned} \tag{2.176}$$

is obtained. Inserting (2.176) into the following equation

$$\frac{d}{d\lambda_{\mathrm{W}}}c[\boldsymbol{\theta}(\lambda_{\mathrm{W}})] = \nabla_{\boldsymbol{\theta}}c[\boldsymbol{\theta}(\lambda_{\mathrm{W}})]^{\mathrm{T}}\frac{d}{d\lambda_{\mathrm{W}}}\boldsymbol{\theta}(\lambda_{\mathrm{W}}), \tag{2.177}$$

results in

$$\frac{d}{d\lambda_{\mathrm{W}}}c[\boldsymbol{\theta}(\lambda_{\mathrm{W}})] = -\frac{d}{d\lambda_{\mathrm{W}}}\boldsymbol{\theta}(\lambda_{\mathrm{W}})^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A} + \lambda_{\mathrm{W}}\mathbf{C})\frac{d}{d\lambda_{\mathrm{W}}}\boldsymbol{\theta}(\lambda_{\mathrm{W}}). \tag{2.178}$$

Since $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A} + \lambda_{\mathrm{W}}\mathbf{C}$ is positive definite in $I_{\text{W-GMS}} = (\gamma'_1, \gamma'_0)$, (2.178) is negative in the same interval. This implies that $c[\boldsymbol{\theta}(\lambda_{\mathrm{W}})]$ is a strictly decreasing function in $I_{\text{W-GMS}}$. Therefore, $\lambda_{\text{W-GMS}}$ could be easily found by applying a bisection algorithm to (2.164) in the interval $I_{\text{W-GMS}}$.

Note that $\lambda_{\text{W-GMS}}$ is the Lagrange multiplier corresponding to the global minimizer of the problem (2.153) as long as the second constraint, $\theta_3 \geq 0$, is ignored. Therefore, it is possible to end up with a negative $\theta_3(\lambda_{\text{W-GMS}})$.

Similar to the CLS minimization problem, (2.153) could also been considered by taking into account all constraints, with the help of Proposition 1, since the weighted cost function (2.152) is twice continuously differentiable. the Proposition 1 gives that if $\boldsymbol{\theta}_{\text{W-GM}}$ is the global minimizer of (2.153), then

$$\nabla f_{\text{W}}(\boldsymbol{\theta}_{\text{W-GM}}) + \lambda_{\text{W-GM}} \nabla c(\boldsymbol{\theta}_{\text{W-GM}}) + \mu_{\text{W-GM}} \nabla d(\boldsymbol{\theta}_{\text{W-GM}}) = 0, \tag{2.179}$$

$$c(\boldsymbol{\theta}_{\text{W-GM}}) = 0, \quad d(\boldsymbol{\theta}_{\text{W-GM}}) < 0, \tag{2.180}$$

$$\mu_{\text{W-GM}} = 0, \tag{2.181}$$

for the nonzero solution, i.e. $\boldsymbol{\theta}_{\text{W-GM}} \neq \mathbf{0}$. $f_{\text{W}}(\boldsymbol{\theta})$ represents the cost function (2.152). Constraint functions $c(\boldsymbol{\theta})$ and $d(\boldsymbol{\theta})$ have been defined in (2.140) and (2.141), respectively.

From (2.179) and (2.181), $\boldsymbol{\theta}_{\text{W-GM}}$ is obtained as

$$\boldsymbol{\theta}_{\text{W-GM}} = (\boldsymbol{A}^{\text{T}} \boldsymbol{\Psi}^{-1} \boldsymbol{A} + \lambda_{\text{W-GM}} \mathbf{C})^{-1} \boldsymbol{A}^{\text{T}} \boldsymbol{\Psi}^{-1} \boldsymbol{b}. \tag{2.182}$$

Furthermore,

$$\boldsymbol{w}^{\text{T}} \nabla^2_{\boldsymbol{\theta\theta}} L(\boldsymbol{\theta}_{\text{W-GM}}, \boldsymbol{\lambda}_{\text{W-GM}}, \boldsymbol{\mu}_{\text{W-GM}}) \boldsymbol{w} \;=\; \boldsymbol{w}^{\text{T}} (\boldsymbol{A}^{\text{T}} \boldsymbol{A} + \lambda_{\text{W-GM}} \mathbf{C}) \boldsymbol{w} \;\geq\; 0, \tag{2.183}$$

for all $\boldsymbol{w} \in \mathbb{R}^3$ such that

$$\nabla c(\boldsymbol{\theta}_{\text{W-GM}})^{\text{T}} \boldsymbol{w} = 2\boldsymbol{\theta}_{\text{W-GM}}^{\text{T}} \mathbf{C} \boldsymbol{w} = 0. \tag{2.184}$$

In addition, using Courant-Fischer-Weyl min-max theorem, the second smallest eigenvalue of $\boldsymbol{A}^{\text{T}} \boldsymbol{\Psi}^{-1} \boldsymbol{A} + \lambda \mathbf{C}$ could be obtained as

$$\lambda_{\text{SS}}(\boldsymbol{A}^{\text{T}} \boldsymbol{\Psi}^{-1} \boldsymbol{A} + \lambda_{\text{W-GM}} \mathbf{C}) = \max_{\boldsymbol{v} \neq 0} \min_{\boldsymbol{w}^{\text{T}} \boldsymbol{v} = 0} \frac{\boldsymbol{w}^{\text{T}} (\boldsymbol{A}^{\text{T}} \boldsymbol{\Psi}^{-1} \boldsymbol{A} + \lambda_{\text{W-GM}} \mathbf{C}) \boldsymbol{w}}{\boldsymbol{w}^{\text{T}} \boldsymbol{w}}, \tag{2.185}$$

where $\boldsymbol{v} \in \mathbb{R}^3$. Using (2.183) and (2.184), the following inequality could be written:

$$\begin{aligned} \lambda_{\text{SS}}(\boldsymbol{A}^{\text{T}} \boldsymbol{\Psi}^{-1} \boldsymbol{A} + \lambda_{\text{W-GM}} \mathbf{C}) &= \max_{\boldsymbol{v} \neq 0} \min_{\boldsymbol{w}^{\text{T}} \boldsymbol{v} = 0} \frac{\boldsymbol{w}^{\text{T}} (\boldsymbol{A}^{\text{T}} \boldsymbol{\Psi}^{-1} \boldsymbol{A} + \lambda_{\text{W-GM}} \mathbf{C}) \boldsymbol{w}}{\boldsymbol{w}^{\text{T}} \boldsymbol{w}}, \\ &\geq \min_{\boldsymbol{\theta}_{\text{W-GM}}^{\text{T}} \mathbf{C} \boldsymbol{w} = 0} \frac{\boldsymbol{w}^{\text{T}} (\boldsymbol{A}^{\text{T}} \boldsymbol{\Psi}^{-1} \boldsymbol{A} + \lambda_{\text{W-GM}} \mathbf{C}) \boldsymbol{w}}{\boldsymbol{w}^{\text{T}} \boldsymbol{w}} \geq 0, \end{aligned} \tag{2.186}$$

which implies that at most one of the eigenvalues of $\boldsymbol{A}^{\text{T}} \boldsymbol{\Psi}^{-1} \boldsymbol{A} + \lambda_{\text{W-GM}} \mathbf{C}$ is negative.

Recall that $\mathbb{R}$ space was divided into four interval in (2.174) and then signs of the eigenvalues of $\boldsymbol{A}^{\text{T}} \boldsymbol{\Psi}^{-1} \boldsymbol{A} + \lambda_{\text{W}} \mathbf{C}$ were examined in these intervals. It was stated that the matrix is positive definite in the interval $I'_b$ and it has one negative eigenvalue in $(I'_a \cup I'_c)$. As a result, it could be said that $\lambda_{\text{W-GM}} \in (I'_a \cup I'_b \cup I'_c)$. Additionally, since $I'_b = I_{\text{W-GMS}}$, if $\lambda_{\text{W-GM}} \in I'_b$ then $\lambda_{\text{W-GM}} = \lambda_{\text{W-GMS}}$.

Using the obtained results, we propose an algorithm, namely ECWLS, for the problem stated in (2.153) as follows:

1. Using the TDOA values and the sensor positions, obtain $\boldsymbol{A}$ and $\boldsymbol{b}$ given in (2.28).

2. Set $\boldsymbol{R} = I$, i.e. $\boldsymbol{\Psi} = \boldsymbol{Q}$.

3. In the polynomial given in (2.164), calculate unknowns except $\lambda_{\text{W}}$. Then find $\lambda_{\text{W-GMS}}$ in the interval $I_b'$. Since the polynomial is strictly decreasing in this interval, a simple bisection method could be utilized.

4. Calculate $\boldsymbol{\theta}_{\text{W-GMS}}$ using (2.157) and $\lambda_{\text{W-GMS}}$. If $\theta_{\text{W-GMS-3}}$ is not negative, then $\boldsymbol{\theta}_{\text{W-GM}} = \boldsymbol{\theta}_{\text{W-GMS}}$ and the first two elements of $\boldsymbol{\theta}_{\text{W-GM}}$, i.e. $\boldsymbol{s}_{\text{W-GM}} = [\theta_{\text{W-GM-1}}\ \theta_{\text{W-GM-2}}]^{\text{T}}$, is the current source coordinate estimate. Jump to step 7. Otherwise, continue with the following step.

5. If $\theta_{\text{W-GMS-3}}$ is negative, find all real roots $\lambda_{\text{W-LM-1}}, \lambda_{\text{W-LM-2}}, ..., \lambda_{\text{W-LM-t}},\ t \leq 3$ of (2.164) in the interval $(I_a' \cup I_c')$, and then calculate the corresponding $\boldsymbol{\theta}$ estimates, which could be represented as $\boldsymbol{\theta}(\lambda_{\text{W-LM-i}})$, $i = 1, 2, ..., t$.

6. Calculate the objective function (2.152) for $\boldsymbol{\theta}(\lambda_{\text{W-LM-i}})$ values satisfying $[0\ \ 0\ \ 1]\,\hat{\boldsymbol{\theta}}(\lambda_{\text{W-LM-i}}) \geq 0$ and for $\boldsymbol{\theta}_0 = [0\ \ 0\ \ 0]^{\text{T}}$. The one with the smallest objective function is $\boldsymbol{\theta}_{\text{W-GM}}$, and the first two elements of $\boldsymbol{\theta}_{\text{W-GM}}$, i.e. $\boldsymbol{s}_{\text{W-GM}} = [\theta_{\text{W-GM-1}}\ \theta_{\text{W-GM-2}}]^{\text{T}}$, is the current source coordinate estimate..

7. Using $\boldsymbol{s}_{\text{W-GM}}$, update $\boldsymbol{R}$ and hence $\boldsymbol{\Psi}$. Repeat steps 3, 4, 5 and 6 until $\boldsymbol{s}_{\text{W-GM}}$ converges. The resultant estimate is $\hat{\boldsymbol{s}}_{\text{ECWLS}}$.

Simulation results suggest that single iteration is generally enough unless the sensor geometry is poor. However, even in such cases, it is seen that the estimate converges in three iterations. Therefore, for a given sensor distribution scenario, a constant number of iteration steps could be safely defined.

**Comments**

Recall that in Section 2.3.2.4, we explained that the CLS method proposed in [11] is an exact solution which obtains the global optimum of the problem (2.104). In the CWLS method; the weighting matrix $\mathbf{\Psi}$ is imposed to the cost function of the problem to approximate the ML estimator. $\mathbf{\Psi}$ includes the unknown range matrix $\boldsymbol{R}$; therefore, an estimated $\boldsymbol{R}$ matrix is used in the algorithm. Because of this, CWLS method is not an exact but an approximate solution of (2.153). For this reason, it is expected that the performance of CWLS degrades faster than that of the CLS as the noise level increases.

# CHAPTER 3

# TDOA BASED SOURCE LOCALIZATION IN ILL-CONDITIONED CASES

In certain source and sensor placement scenarios, least squares based methods stated in Section 2.3.2 suffer from the rank deficient or ill-conditioned matrix problem; which cause that the estimate could not be obtained, or obtained with an increased MSE, respectively. In this chapter, such scenarios and the methods resistant to them will be investigated.

Recall the estimators given in Section 2.3.2:

$$\begin{aligned}\hat{\boldsymbol{\theta}}_{\text{ULS}} &= (\boldsymbol{A}^{\text{T}}\boldsymbol{A})^{-1}\boldsymbol{A}^{\text{T}}\boldsymbol{b},\\ \hat{\boldsymbol{\theta}}_{\text{UWLS}} &= (\boldsymbol{A}^{\text{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A})^{-1}\boldsymbol{A}^{\text{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{b},\\ \hat{\boldsymbol{\theta}}_{\text{CLS}} &= (\boldsymbol{A}^{\text{T}}\boldsymbol{A}+\lambda\mathbf{C})^{-1}\boldsymbol{A}^{\text{T}}\boldsymbol{b},\\ \hat{\boldsymbol{\theta}}_{\text{CWLS}} &= (\boldsymbol{A}^{\text{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{A}+\lambda_{\text{W}}\mathbf{C})^{-1}\boldsymbol{A}^{\text{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{b}.\end{aligned}$$

When the matrix $\boldsymbol{A}$ is rank deficient; estimators named as ULS and TSWLS, whose first step estimator is UWLS, could not be calculated. When it is ill-conditioned; rounding errors due to finite precision arithmetic become significant in the calculations of ULS and TSWLS. In addition, if the noise level of the TDOA measurements are low, expected values of the Lagrange multipliers $\lambda$ and $\lambda_{\text{W}}$ become zero, as given in (2.131). Then CLS and CWLS methods also suffer from the same problem.

As defined in (2.28b), $\boldsymbol{A}$ is composed of the sensor positions and TDOA values. Three different sensor and source distribution scenarios cause $\boldsymbol{A}$ to become ill-conditioned; namely circular, hyperbolic and linear distributions.

Firstly; we will explain a robust method proposed in [35], namely CWLS with range difference separation (CWLS-RDS). This method is suitable for circular sensor distributions with the source located near the center. Then, we will reduce the computational cost of this method by using the approach we have used for CWLS method.

Secondly; another distribution scenario, hyperbolic sensor distribution with the source near one of the foci, is investigated. After showing that such distributions cause ill-conditioned problem, we will propose a modified version of the CWLS-RDS method to overcome the problem.

Lastly; yet another scenario, linear sensor distribution, will be studied. Close-to-linear distributions cause the first two columns of $\boldsymbol{A}$ to become almost linearly dependent, resulting an ill-conditioned matrix problem. We will propose a new method to circumvent this issue. The proposed method is also applicable for the exact linear sensor distributions, where none of the methods given in Section 2.3.2 could be applied.

## 3.1 Constrained Weighted Least Squares with Range Difference Separation

When the sensor array is circular and the source is located near the array center, TDOA values become close to zero. Additionally, if the noise level of the TDOA values are low, then the matrix inverted to obtain the CWLS estimate becomes ill-conditioned. To overcome this problem; a new CWLS method, which we will call as CWLS with range difference separation (CWLS-RDS), is proposed in [35]. In this section, CWLS-RDS method will be explained in detail. Then, a computationally more efficient version of this method, namely efficient CWLS-RDS (ECWLS-RDS), will be presented by using the approach which has been used for CWLS method. At the end, some comments will be presented.

The main strategy of CWLS-RDS method is to separate $\boldsymbol{A}$ into a matrix containing the sensor positions and a vector composed of the TDOA values. In this way, the matrix to be inverted does not contain TDOA values. It should be noted that this is the same approach used in spherical interpolation method stated in Section 2.3.2.1.

To show explicitly, recall (2.34):

$$\begin{aligned} \boldsymbol{A}_{\text{(N-1)x3}}\ \boldsymbol{\theta}_{\text{3x1}} &= \boldsymbol{b}_{\text{(N-1)x1}} \\ [\boldsymbol{S}_{\text{(N-1)x2}}\ \ c\boldsymbol{d}_{\text{(N-1)x1}}]\ [\boldsymbol{s}_{\text{2x1}}^{\text{T}}\ r_1]^{\text{T}} &= \boldsymbol{b}_{\text{(N-1)x1}} \\ \boldsymbol{S}_{\text{(N-1)x2}}\ \boldsymbol{s}_{\text{2x1}} + r_1 c\boldsymbol{d}_{\text{(N-1)x1}} &= \boldsymbol{b}_{\text{(N-1)x1}}, \end{aligned} \tag{3.1}$$

where

$$\boldsymbol{A} = \begin{bmatrix} x_2 & y_2 & cd_{2,1} \\ x_3 & y_3 & cd_{3,1} \\ \vdots & \vdots & \vdots \\ x_N & y_N & cd_{N,1} \end{bmatrix},$$

$\boldsymbol{S}$ is the matrix composed of the first two columns of $\boldsymbol{A}$, and $c\boldsymbol{d}$ is the third column of $\boldsymbol{A}$. Then the cost function of CWLS (2.152) could be rewritten as

$$C_{\text{CWLS-RDS}}(\boldsymbol{s}, r_1) = (\boldsymbol{b} - \boldsymbol{S}\boldsymbol{s} - r_1 c\boldsymbol{d})^{\text{T}}\boldsymbol{\Psi}^{-1}(\boldsymbol{b} - \boldsymbol{S}\boldsymbol{s} - r_1 c\boldsymbol{d}), \tag{3.2}$$

and the CWLS minimization problem (2.153) is equivalent to

$$\min_{\boldsymbol{s}}\{(\boldsymbol{b} - \boldsymbol{S}\boldsymbol{s} - r_1 c\boldsymbol{d})^{\text{T}}\boldsymbol{\Psi}^{-1}(\boldsymbol{b} - \boldsymbol{S}\boldsymbol{s} - r_1 c\boldsymbol{d}) :\ \boldsymbol{s}^{\text{T}}\boldsymbol{s} = r_1^2,\ r_1 \geq 0\}. \tag{3.3}$$

Ignoring the inequality constraint, corresponding Lagrange function becomes

$$\mathcal{L}_{\text{W-RDS}}(\boldsymbol{s}, r_1, \lambda_{\text{W-RDS}}) = (\boldsymbol{b} - \boldsymbol{S}\boldsymbol{s} - r_1 c\boldsymbol{d})^{\text{T}}\boldsymbol{\Psi}^{-1}(\boldsymbol{b} - \boldsymbol{S}\boldsymbol{s} - r_1 c\boldsymbol{d})\ +\ \lambda_{\text{W-RDS}}(\boldsymbol{s}^{\text{T}}\boldsymbol{s} - r_1^2). \tag{3.4}$$

Differentiating (3.4) with respect to $\boldsymbol{s}$ and then equating the result to zero gives

$$\frac{\partial \mathcal{L}_{\text{W-RDS}}(\boldsymbol{s}, r_1, \lambda_{\text{W-RDS}})}{\partial \boldsymbol{s}} = 2\boldsymbol{S}^{\text{T}}\boldsymbol{\Psi}^{-1}(\boldsymbol{S}\boldsymbol{s} + r_1 c\boldsymbol{d} - \boldsymbol{b}) + 2\lambda_{\text{W-RDS}}\boldsymbol{s} = 0. \tag{3.5}$$

If (3.5) is solved for $\boldsymbol{s}$, CWLS-RDS estimate is obtained as

$$\begin{aligned} \hat{\boldsymbol{s}}_{\text{CWLS-RDS}} &\triangleq \arg\min_{\boldsymbol{s}}\ \{\mathcal{L}_{\text{W-RDS}}(\boldsymbol{s}, r_1, \lambda_{\text{W-RDS}})\} \\ &= (\boldsymbol{S}^{\text{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{S} + \lambda_{\text{W-RDS}}\boldsymbol{I})^{-1}(\boldsymbol{S}^{\text{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{b} - \boldsymbol{S}^{\text{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{d}cr_1). \end{aligned} \tag{3.6}$$

To obtain $\hat{\boldsymbol{s}}_{\text{CWLS-RDS}}$, $\lambda_{\text{W-RDS}}$ and $r_1$ are required.

Moreover, differentiating (3.4) with respect to $r_1$ and then equating the result to zero gives

$$r_1 = (\lambda_{\text{W-RDS}} - c^2\boldsymbol{d}^{\text{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{d})^{-1}(c\boldsymbol{d}^{\text{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{S}\boldsymbol{s} - c\boldsymbol{d}^{\text{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{b}). \tag{3.7}$$

Substituting (3.6) into the constraint $\boldsymbol{s}^{\mathrm{T}}\boldsymbol{s} = r_1^2$ gives

$$ar_1^2 + br_1 + c = 0, \tag{3.8}$$

where

$$\begin{aligned}
a &= c^2(\boldsymbol{S}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{d})^{\mathrm{T}}(\boldsymbol{S}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{S} + \lambda_{\text{W-RDS}}\boldsymbol{I})^{-2}\boldsymbol{S}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{d} - 1,\\
b &= -2c(\boldsymbol{S}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{b})^{\mathrm{T}}(\boldsymbol{S}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{S} + \lambda_{\text{W-RDS}}\boldsymbol{I})^{-2}\boldsymbol{S}\boldsymbol{\Psi}^{-1}\boldsymbol{d},\\
c &= (\boldsymbol{S}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{b})^{\mathrm{T}}(\boldsymbol{S}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{S} + \lambda_{\text{W-RDS}}\boldsymbol{I})^{-2}\boldsymbol{S}\boldsymbol{\Psi}^{-1}\boldsymbol{b}.
\end{aligned}$$

Once $\lambda_{\text{W-RDS}}$ is obtained, $r_1$ could be found using (3.8), and then $\hat{\boldsymbol{s}}_{\text{CWLS-RDS}}$ is obtained from (3.6). One way to obtain $\lambda_{\text{W-RDS}}$ is to express $r_1$ in terms of $\lambda_{\text{W-RDS}}$ using (3.6) and (3.7), then substituting the resultant equation in (3.8), which is a computationally complex process. However, if the minimization problems (3.3) and (2.153) are compared, it is seen that the objective functions and the constraints are the same, therefore they are equivalent [35]. Hence, $\lambda_{\text{W-RDS}}$ is equal to $\lambda_{\text{W}}$ and it could be obtained using (2.164).

After obtaining $\lambda_{\text{W-RDS}}$ values from (2.164), corresponding $r_1$ values should be found using (3.8). Since (3.8) is a second degree equation, a root selection procedure is needed. Such a procedure is stated in [35] as follows:

a. If the roots of (3.8) are real, then

   i if one of them is nonnegative and the other one is negative, choose the nonnegative one.

   ii if both of the roots are negative, set $r_1 = 0$.

   iii if both of the roots are positive, calculate the cost function using (3.6) and (3.2), respectively for both roots. Choose the one making the cost function minimum.

b. If the roots of (3.8) are complex, then

   i. if real part of the roots is positive, then assign this value to $r_1$.

   ii. if real part of the roots is negative, then $r_1 = 0$.

In [35], it has been stated that the cases apart from (a) are abnormal which occur under very high noise levels.

Note that the root selection procedure given in [35] takes into account the inequality constraint, $r_1 \geq 0$, while choosing the $r_1$ value corresponding to a $\lambda_{\text{W-RDS-GMS}}$ candidate. However, this constraint was ignored while the Lagrange multiplier method was being applied. Therefore, it is possible to have a negative $r_1$ value for a $\lambda_{\text{W-RDS-GMS}}$ candidate. In such cases, if one of the roots of (3.8) is positive and the other one is negative, then it does not cause an error to pair the aforementioned $\lambda_{\text{W-RDS-GMS}}$ candidate with the positive root, since the resulting estimate satisfies the equality constraint. In this case, the obtained pair is not an extremum. On the other hand; if both of the roots are negative, then the proposed root selection procedure sets $r_1 = 0$. This results in an estimate candidate which most probably does not satisfy the equality constraint. If the obtained estimate candidate makes the cost function smaller than the global minimizer of the problem with both of the constraints, i.e. (3.3), the proposed algorithm chooses the wrong candidate as the estimate. To overcome this problem, we propose the following updated root selection procedure:

a. If the roots of (3.8) are real, then

   i. if the magnitudes of the roots are equal, choose the positive one.

   ii. if the magnitudes of the roots are not equal, obtain the corresponding source position estimates using (3.6). Then calculate the resulting $r_1$ values from (3.7). The right $r_1$ root should match the resulting one obtained from (3.7); therefore, choose the matching one. Note that (3.7) has been ignored in the root selection procedure given in [35].

b. If the roots of (3.8) are complex, then assign the real part to $r_1$.

In [35], to find $\lambda_{\text{W-RDS-GM}}$, it has been proposed to consider all real roots of (2.164). However, as mentioned in Section 2.3.2.5; finding all real roots of (2.164), obtaining the source location estimates, and then calculating the cost functions for the estimates satisfying the constraint $\hat{r}_1 \geq 0$ is a computationally expensive process. Moreover, since the weighting matrix $\boldsymbol{\Psi}$ includes the range vector which is constructed from $\hat{\boldsymbol{s}}$; an iterative procedure is required to update $\boldsymbol{\Psi}$, which necessitates the repetition of this process. Additionally, in a three dimensional space, (2.164) becomes a sixth degree polynomial of $\lambda$ and therefore iterative methods have to be used to find the roots [33], which makes the process even more complicated. To reduce the computational burden, the Lagrange multiplier founding approach we have proposed in Section 2.3.2.5 could be also used in here, since $\lambda_{\text{W-RDS-GMS}} = \lambda_{\text{W-GMS}}$ and $\lambda_{\text{W-RDS-GM}} = \lambda_{\text{W-GM}}$.

If the global minimizer of (3.3), $\boldsymbol{s}_{\text{W-RDS-GMS}}$, is $[0\ \ 0]^{\text{T}}$; it could not be found by the CWLS-RDS algorithm unless $\boldsymbol{\theta} = [0\ 0\ 0]^{\text{T}}$ is an extremum of (3.3) with the inequality constraint is ignored, since $[0\ \ 0\ \ 0]^{\text{T}}$ is the boundary point in which the inequality constraint is active. Therefore, all zero solution should also be taken into account.

By imposing the Lagrange multiplier founding approach we have proposed in Section 2.3.2.5, changing the root selection procedure with the updated one, and taking into consideration the all zero solution; a complete and computationally more efficient version of the CWLS-RDS method, namely ECWLS-RDS, can be stated as follows:

1. Set $\boldsymbol{\Psi} = \boldsymbol{Q}$.
2. In the polynomial given in (2.164), calculate unknowns except $\lambda_{\text{W}}$. Then find $\lambda_{\text{W-GMS}}$ in the interval $I_b'$. Since the polynomial is strictly decreasing in this interval, a simple bisection method could be utilized.
3. Calculate the roots of (3.8) using $\lambda_{\text{W-GMS}}$, then use the updated root selection procedure. If the selected root is real nonnegative, then $\lambda_{\text{W-RDS-GM}} = \lambda_{\text{W-GMS}}$. Obtain the CWLS-RDS estimate from (3.6), and then jump to step 6. Otherwise, continue with the following step.
4. Find all real roots $\lambda_{\text{W-LM-1}}, \lambda_{\text{W-LM-2}}, ..., \lambda_{\text{W-LM-t}},\ t \leq 3$ of (2.164) in the interval $(I_a' \cup I_c')$, and then calculate $r_{1-i} = r_1(\lambda_{\text{W-LM-i}})$ and $\boldsymbol{s}_i = \boldsymbol{s}(r_1(\lambda_{\text{W-LM-i}}))$, $i = 1, 2, ..., t$.
5. Calculate the objective function (3.2) only for the sets $\{\lambda_{\text{W-LM-i}}, r_{1-i}, \boldsymbol{s}_i\},\ i = 1, 2, ..., t$ with nonnegative $r_1$ value and for $\boldsymbol{\theta}_0 = [0\ \ 0\ \ 0]^{\text{T}}$. The one with the smallest objective function is $\{\lambda_{\text{W-RDS-GM}}, r_{\text{1-GM}}, \boldsymbol{s}_{\text{GM}}\}$.
6. Using $\boldsymbol{s}_{\text{W-GM}}$, calculate the new range vector estimate $\hat{\boldsymbol{R}}$ and then update $\boldsymbol{\Psi}$ using (2.151). Repeat steps 2, 3,...6 until converges to obtain $\hat{\boldsymbol{s}}_{\text{CWLS-RDS}}$.

As mentioned at the end of Section 2.3.2.5, simulation results suggest that single iteration is generally enough unless the sensor geometry is poor. However, even in such cases, it is seen that the estimate converges in three iterations. Therefore, for a given sensor distribution scenario, a constant number of iteration steps could be safely defined.

In [35], the proposed algorithm starts with $\boldsymbol{\Psi} = \mathbf{I}$. In simulations, we see that although starting with $\boldsymbol{\Psi} = \boldsymbol{Q}$ gives a better initial estimate unless the noise level is quite high, there is no significant difference between the estimated values after first iteration.

In simulations, it is seen that when the sensor array is circular, source is placed to the array center and the noise level is extremely low; Lagrange multiplier corresponding to the global minimum may become complex valued due to the finite precision arithmetic. In such cases, Lagrange multiplier is close to zero, and the complex part is less than $5\%$ of the real part. It is seen that ignoring the complex part in such cases does not affect the general performance of the method.

**Comments**

In [35], some simulation results of the CWLS-RDS and the CWLS methods have been given for the circular placement scenarios. The results show that as variance of the Gaussian noise added to the TDOA values decreasing, MSE of the CWLS estimate gets much higher than the CRLB, while MSE of CWLS-RDS estimate attains it.

As mentioned in Section 2.3.2.5, the CWLS method given in [39] does not considers the inequality constraint in the selection procedure among the estimate candidates. This may reduce the estimation accuracy especially in the circular distribution. On the other hand, the modified CWLS method proposed in Section 2.3.2.5 and named as ECWLS considers the constraints. Besides, the CWLS-RDS method had originally aforementioned problems, which have been solved in the modified version proposed by us (ECWLS-RDS). In Section 4.3.1, simulation results of ECWLS and ECWLS-RDS methods will be compared to examine whether the relative performances are as given in [35].

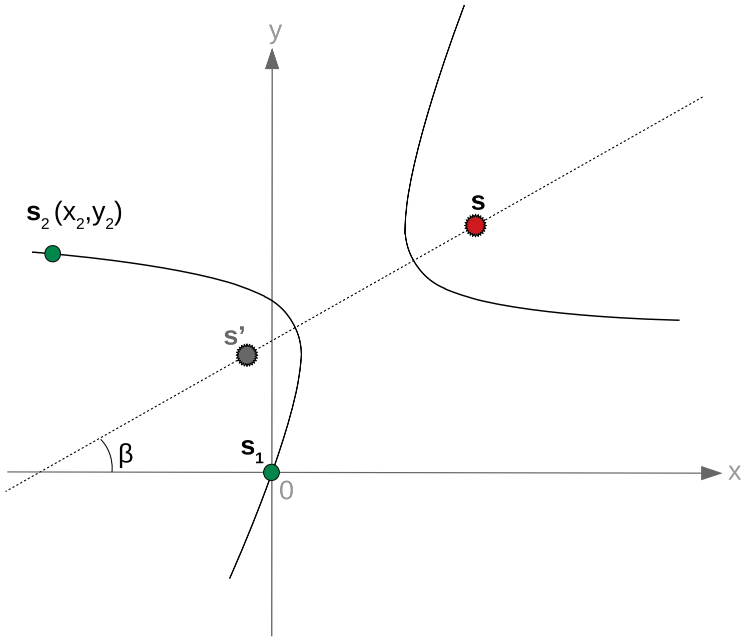


Figure 3.1: Hyperbolic Placement Scenario

## 3.2 Constrained Weighted Least Squares with Range Difference Separation for Hyperbolic Distribution

Consider a placement scenario in which the sensors are located on one of the branches of an hyperbola, and the source is located on one of the focal points of the same hyperbola. Such a placement scenario is illustrated for a two-element array in Figure 3.1. In the figure; $s_1$ and $s_2$ are the sensors, $s$ is the source, and $s'$ is the other focal point of the hyperbola where the source is not located. Similar to the previous sections, location of the reference sensor $s_1$ is taken as the origin of the coordinate system. To make the transverse axis of the hyperbola parallel to the x-axis, let the sensor array and the source are rotated around the origin. This process will make the exposition simpler. At the end, original coordinates can be obtained via an inverse rotation matrix. The result of the rotation process is given in Figure 3.2. Subscript "r" is used to indicate the rotated positions.

Considering Figure 3.2, the following equations can be written from Pythagorean

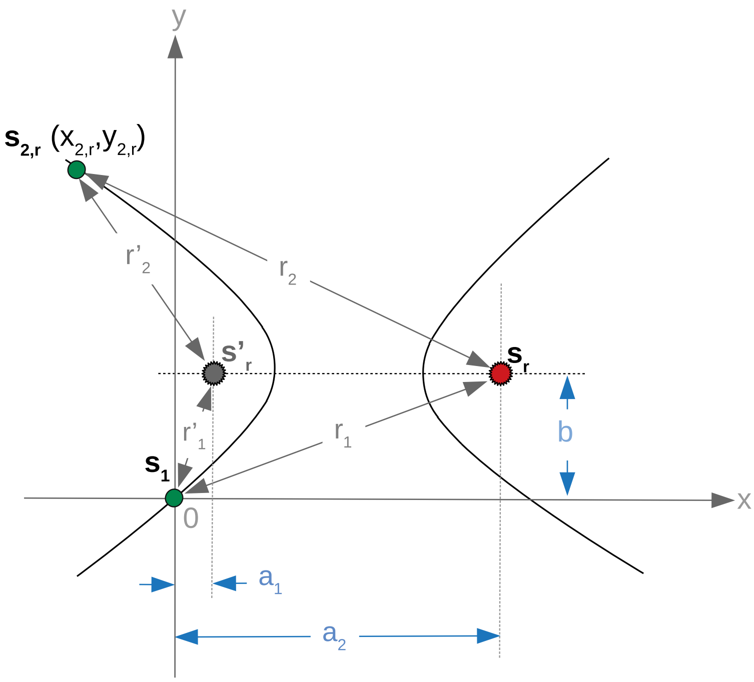


Figure 3.2: Rotated Hyperbolic Placement Scenario

theorem;

$$a_2^2 + b^2 = r_1^2, \tag{3.9}$$

$$a_1^2 + b^2 = {r'}_1^2, \tag{3.10}$$

$$(y_{2,r} - b)^2 + (x_{2,r} + a_2)^2 = r_2^2, \tag{3.11}$$

$$(y_{2,r} - b)^2 + (x_{2,r} + a_1)^2 = {r'}_2^2. \tag{3.12}$$

Subtracting (3.10) form (3.9), and (3.12) form (3.11) gives

$$r_1^2 - {r'}_1^2 = a_2^2 - a_1^2, \tag{3.13}$$

$$r_2^2 - {r'}_2^2 = a_2^2 - a_1^2 + 2x_{2,r}(a_2 - a_1). \tag{3.14}$$

Besides; since $\boldsymbol{s_1}$ and $\boldsymbol{s_{2,r}}$ are on the same hyperbola branch, and $\boldsymbol{s'_r}$ and $\boldsymbol{s_r}$ are the foci;

$$r_2 - {r'}_2 = r_1 - {r'}_1. \tag{3.15}$$

Therefore,

$$\begin{aligned} r_2^2 - {r'}_2^2 &= (r_1 - {r'}_1)(r_2 + {r'}_2) \\ &= (r_1 - {r'}_1)(r_1 + {r'}_1 + 2(r_2 - r_1)) \\ &= r_1^2 - {r'}_1^2 + 2(r_1 - {r'}_1)(r_2 - r_1). \end{aligned} \tag{3.16}$$

Inserting (3.13) and (3.16) into (3.14) results in

$$
\begin{aligned}
r_1^2 - {r'}_1^2 + 2(r_1 - {r'}_1)(r_2 - r_1) &= r_1^2 - {r'}_1^2 + 2x_{2,r}(a_2 - a_1) \\
(r_1 - {r'}_1)(r_2 - r_1) &= x_{2,r}(a_2 - a_1).
\end{aligned} \tag{3.17}
$$

From (3.17), range difference value is obtained as

$$
r_{2,1} = r_2 - r_1 = \frac{x_{2,r}(a_2 - a_1)}{(r_1 - {r'}_1)}. \tag{3.18}
$$

If the angle between the transverse axis of the hyperbola and the x-axis is $\beta$, then the rotation matrix becomes

$$
\boldsymbol{R} \triangleq \begin{bmatrix} \cos(\beta) & \sin(\beta) \\ -\sin(\beta) & \cos(\beta) \end{bmatrix}, \tag{3.19}
$$

and $x_{2,r}$ can be expressed as

$$
x_{2,r} = x_2 \cos(\beta) + y_2 \sin(\beta). \tag{3.20}
$$

Inserting (3.20) into (3.18) gives

$$
r_{2,1} = x_2 \left[ \frac{\cos(\beta)(a_2 - a_1)}{(r_1 - {r'}_1)} \right] + y_2 \left[ \frac{\sin(\beta)(a_2 - a_1)}{(r_1 - {r'}_1)} \right]. \tag{3.21}
$$

Note that (3.21) shows that the range difference value, $r_{2,1}$, is a linear combination of $\boldsymbol{s_2}$ coordinates. This means that if TDOA values are noiseless, i.e., range difference values are equal to the TDOA values times the propagation speed; then the third column of $\boldsymbol{A}$, which is defined in (2.28b), is a linear combination of the first two column. Therefore, hyperbolic source and sensors distributions result in an ill-conditioned $\boldsymbol{A}$ matrix under low noise conditions. Moreover, it has been shown in (2.131) that the expected value of the Lagrange multiplier is zero, i.e. $\mathrm{E}[\lambda] = 0$, for low noise conditions. Hence, the matrix to be inverted to obtain the CWLS estimate, $(\boldsymbol{A}^{\mathrm{T}} \boldsymbol{\Psi}^{-1} \boldsymbol{A} + \lambda_{\mathrm{W}} \mathbf{C})$, becomes ill-conditioned. To circumvent this issue, we propose a modified version of the ECWLS-RDS method given in Section 3.1.

Recall that in Section 3.1, the strategy was to separate $\boldsymbol{A}$ into a matrix containing the sensor positions and a vector composed of the TDOA values. In this way, the matrix to be inverted did not contain TDOA values, which are close to zero when the distribution is close to circular. In hyperbolic distribution, TDOA values are not close to zero, but they are close to a linear combination of the first two columns of $\boldsymbol{A}$.

Therefore, separating $\boldsymbol{A}$ should also solve the ill-conditioned problem resulting from the hyperbolic distributions.

Although both of the distribution scenarios cause the ill-conditioned problem which could be circumvented using the same matrix separation approach, it should not be ignored that there is a significant difference between them. In the exact circular sensor distribution with the source is located at the center, the single branch hyperbolas corresponding to the noiseless TDOA values intersect in a single point. This results in a cost function with a unique global minimum. When the inequality constraint is ignored, then the new cost function have two global minimum points; one is the true source coordinates and the true range, and the other one is the true source coordinates and the negative of the true range. On the other hand, in the exact hyperbolic distribution scenario, the single branch hyperbolas corresponding to the noiseless TDOA values intersect in two points, which are the foci of the hyperbola on which the sensors are distributed. Therefore, the original cost function, i.e., the cost function with both of the constraints are taken into account, have two global minimum points. As a result of this, applying ECWLS-RDS method in a hyperbolic distribution scenario may results in a source location estimate close to $\boldsymbol{s}'$, i.e., the other focal point of the hyperbola where the source is not located.

To solve the aforementioned problem, another property of the signal radiating from the source should be utilized. Received signal strength (RSS) is one of such properties which could be used. The RSS path loss model is [40]:

$$P_i = P_s - 10\mu \log_{10} r_i + n_{p,i}, \quad i = 1, 2, ..., N, \tag{3.22}$$

where $P_i$ is the average received signal power in dB of the $i$th sensor, $P_s$ is the unknown transmit power in dB of the source, $\mu$ is the known path loss factor of the medium, and $n_{p,i}$ is the additive noise term. Subtracting RSS of the reference sensor, $P_1$, from $P_i$ gives

$$P_i, 1 \triangleq P_i - P_1 = -10\mu \log_{10}\left(\frac{r_i}{r_1}\right) + n_{p,i,1}, \quad i = 2, ..., N. \tag{3.23}$$

For an $r_1$ estimate, $r_i, i = 2, 3, ..., N$ values could be estimated using the TDOA values. Then from (3.23), corresponding RSS differences could be estimated. Comparing the measured RSS values with the estimated ones, whether the obtained $r_1$ estimate corresponds to the right focal point or not could be seen. Using this approach, we propose a modified version of the ECWLS-RDS method, namely ECWLS-RDS for hyperbolic distributions (ECWLS-RDSH), as follows:

1. Set $\boldsymbol{\Psi} = \boldsymbol{Q}$.

2. In the polynomial given in (2.164), calculate unknowns except $\lambda_{\text{W}}$. Then find $\lambda_{\text{W-GMS}}$ in the interval $I'_b$. Since the polynomial is strictly decreasing in this interval, a simple bisection method could be utilized.

3. Calculate the roots of (3.8) using $\lambda_{\text{W-GMS}}$, then use the updated root selection procedure (RSP) to obtain $r_1$ estimate. If the selected root is real nonnegative, then $\lambda_{\text{SW-GM}} = \lambda_{\text{W-GMS}}$. Calculate $r_i, i = 2, 3, ..., N$ estimates using the RSP result and the TDOA values. After that, estimate the RSS differences using (3.23). If the difference between the estimated values and measured RSS values are within a predefined "closeness" level, then obtain the ECWLS-RDS estimate from (3.6), and then jump to step 6. Otherwise, continue with the following step. If "closeness" between the estimated and the measured RSS difference values could not be defined mathematically, skip part 2 and 3.

4. Find all real roots $\lambda_{\text{W-LM-1}}, \lambda_{\text{W-LM-2}}, ..., \lambda_{\text{W-LM-t}},\ t \leq 3$ of (2.164) in the interval $(I'_a \cup I'_c)$, and then calculate $\hat{r}_{1-i} = \hat{r}_1(\lambda_{\text{W-LM-i}})$ and $\hat{\boldsymbol{s}}_i = \hat{\boldsymbol{s}}(r_1(\lambda_{\text{W-LM-i}}))$, $i = 1, 2, ..., t$.

5. Calculate the objective function (3.2) only for the sets $\{\lambda_{\text{W-LM-i}}, \hat{r}_{1-i}, \hat{\boldsymbol{s}}_i\},\ i = 1, 2, ..., t$ with nonnegative $r_1$ value and for $\boldsymbol{\theta}_0 = [0\ 0\ 0]^{\text{T}}$. Call the smallest objective function value as $f_1$, the second smallest one as $f_2$, and the third smallest one as $f_3$. If $|f_1 - f_2| << |f_2 - f_3|$, then using (3.23) obtain the RSS difference estimates for the sets corresponding to the $f_1$ and $f_2$. The one giving the closest RSS difference estimate to the measured ones is $\{\lambda_{\text{SW-GM}}, \hat{r}_{\text{1-GM}}, \hat{\boldsymbol{s}}_{\text{GM}}\}$. If $|f_1 - f_2|$ is not much less than $|f_2 - f_3|$, then one with the smallest objective function is $\{\lambda_{\text{SW-GM}}, \hat{r}_{\text{1-GM}}, \hat{\boldsymbol{s}}_{\text{GM}}\}$.

6. Using $\boldsymbol{s}_{\text{W-GM}}$, calculate the new range vector estimate $\hat{\boldsymbol{R}}$ and then update $\boldsymbol{\Psi}$ using (2.151). Repeat steps 2, 3,...6 until converges to obtain $\hat{s}_{\text{ECWLS-RDSH}}$.

As mentioned at the end of Section 2.3.2.5, simulation results suggest that single iteration is generally enough unless the sensor geometry is poor. However, even in such cases, it is seen that the estimate converges in three iterations. Therefore, for a given sensor distribution scenario, a constant number of iteration steps could be safely defined.

### Comments

In a circular sensor distribution with the source located at the center of the circle, noiseless TDOA values gives a single point for the source location. Reason of the ill-conditioning problem faced in the methods given in Section 2.3.2 is to ignore the inequality constraint. However, this is not the case in an hyperbolic distribution. When the sensors are located on one of the branches of an hyperbola, and the source is located on one of the focal points of the same hyperbola; then the noiseless TDOA values gives two possible points for the source location. Regardless of the method used, a decision between these two points could not be made without having additional information other than the TDOA values.

Note that to have an estimate from TDOA measurements, at least three sensors are required. Considering that three points in a plane uniquely define a circle, the ill-conditioned problem CWLS-RDS solves can be seen as a rare distribution case. On the other hand, to uniquely define a hyperbola, at least five points are required. Therefore, if three or four sensors are used, the required position of the source to have ill-conditioned problem is not a single point, but a certain area in the plane. This increases the probability of facing the ill-conditioned problem.

## 3.3 Constrained Weighted Least Squares with Coordinate Separation

When the sensor distribution is linear or close to linear, $\boldsymbol{A}$ and hence the matrices inverted in the calculation of the estimators given in Section 2.3.2 become rank deficient or ill-conditioned, respectively. To overcome this problem; we propose a new CWLS method, namely ECWLS with coordinate separation (ECWLS-CS), motivated by the approach used in [35] for circular sensor arrays. In this section, ECWLS-CS method is explained in detail with some comments at the end.

The main strategy of ECWLS-CS method is to separate the first two columns of $\boldsymbol{A}$, which are linearly dependent in the case of linear sensor array. Without losing generality, let the reference sensor be placed at $[0\ 0]^{\mathrm{T}}$ and the close-to-linear sensor array be placed along the x-axis. Then the second column of $\boldsymbol{A}$ becomes almost all zero vector. If we separate this column, (2.28a) becomes

$$\begin{aligned}\boldsymbol{A}_{\text{(N-1)x3}}\,\boldsymbol{\theta}_{\text{3x1}} &= \boldsymbol{b}_{\text{(N-1)x1}}\\ \boldsymbol{K}_{\text{(N-1)x2}}\,\boldsymbol{l}_{\text{2x1}} + y\boldsymbol{k}_{\text{(N-1)x1}} &= \boldsymbol{b}_{\text{(N-1)x1}},\end{aligned} \tag{3.24}$$

where

$$\boldsymbol{A} = \begin{bmatrix} x_2 & y_2 & cd_{2,1} \\ x_3 & y_3 & cd_{3,1} \\ \vdots & \vdots & \vdots \\ x_N & y_N & cd_{N,1} \end{bmatrix}, \qquad \boldsymbol{K} = \begin{bmatrix} x_2 & cd_{2,1} \\ x_3 & cd_{3,1} \\ \vdots & \vdots \\ x_N & cd_{N,1} \end{bmatrix},$$

$$\boldsymbol{k} = [y_2 \quad y_3 \dots y_N]^T, \quad \boldsymbol{\theta} = [x \quad y \quad r_1]^T, \quad \boldsymbol{l} = [x \quad r_1]^T.$$

Then the cost function of CWLS (2.152) could be rewritten as

$$C_{\text{ECWLS-CS}}(\boldsymbol{l}, y) = (\boldsymbol{b} - \boldsymbol{K}\boldsymbol{l} - y\boldsymbol{k})^{\mathrm{T}}\boldsymbol{\Psi}^{-1}(\boldsymbol{b} - \boldsymbol{K}\boldsymbol{l} - y\boldsymbol{k}), \tag{3.25}$$

and the CWLS minimization problem (2.153) is equivalent to

$$\min_{\boldsymbol{l},y}\{(\boldsymbol{b} - \boldsymbol{K}\boldsymbol{l} - y\boldsymbol{k})^{\mathrm{T}}\boldsymbol{\Psi}^{-1}(\boldsymbol{b} - \boldsymbol{K}\boldsymbol{l} - y\boldsymbol{k}) : \ \boldsymbol{l}^{\mathrm{T}}\mathbf{C}_2\boldsymbol{l} = -y^2, \ r_1 \geq 0\}, \tag{3.26}$$

where $\mathbf{C}_2 = \text{diag}(1, -1)$. Ignoring the inequality constraint, corresponding Lagrange function becomes

$$\mathcal{L}_{\text{W-CS}}(\boldsymbol{l}, y, \lambda_{\text{W-CS}}) = (\boldsymbol{b}-\boldsymbol{K}\boldsymbol{l}-y\boldsymbol{k})^{\mathrm{T}}\boldsymbol{\Psi}^{-1}(\boldsymbol{b}-\boldsymbol{K}\boldsymbol{l}-y\boldsymbol{k}) + \lambda_{\text{W-CS}}(\boldsymbol{l}^{\mathrm{T}}\mathbf{C}_2\boldsymbol{l}+y^2). \tag{3.27}$$

Differentiating (3.27) with respect to $\boldsymbol{l}$ and then equating the result to zero gives

$$\frac{\partial \mathcal{L}_{\text{W-CS}}(\boldsymbol{l}, y, \lambda_{\text{W-CS}})}{\partial \boldsymbol{l}} = 2\boldsymbol{K}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}(\boldsymbol{K}\boldsymbol{l} + y\boldsymbol{k} - \boldsymbol{b}) + 2\lambda_{\text{W-CS}}\mathbf{C}_2\boldsymbol{l} = 0. \tag{3.28}$$

If (3.28) is solved for $\boldsymbol{l}$, ECWLS-CS estimate is obtained as

$$\begin{aligned}\hat{\boldsymbol{l}}_{\text{ECWLS-CS}} &\triangleq \arg\min_{\boldsymbol{l}} \{\mathcal{L}_{\text{W-CS}}(\boldsymbol{l}, y, \lambda_{\text{W-CS}})\}\\ &= (\boldsymbol{K}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{K} + \lambda_{\text{W-CS}}\mathbf{C}_2)^{-1}(\boldsymbol{K}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{b} - \boldsymbol{K}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{k}y).\end{aligned} \tag{3.29}$$

To obtain $\hat{\boldsymbol{l}}_{\text{CWLS-CS}}$, $\lambda_{\text{W-CS}}$ and $y$ are required.

Moreover, differentiating (3.27) with respect to $y$ and then equating the result to zero gives

$$y = (\boldsymbol{k}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{k} + \lambda_{\text{W-CS}})^{-1}(\boldsymbol{k}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{b} - \boldsymbol{k}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{K}\boldsymbol{l}). \tag{3.30}$$

Substituting (3.29) into the constraint $\boldsymbol{l}^{\mathrm{T}}\mathbf{C}_2\boldsymbol{l} = -y^2$ gives

$$a_2 y^2 + a_1 y + a_0 = 0, \tag{3.31}$$

where

$$\begin{aligned}
a_2 &= (\boldsymbol{K}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{k})^{\mathrm{T}}\boldsymbol{T}(\boldsymbol{K}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{k}) + 1, \\
a_1 &= -2(\boldsymbol{K}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{b})^{\mathrm{T}}\boldsymbol{T}(\boldsymbol{K}\boldsymbol{\Psi}^{-1}\boldsymbol{k}), \\
a_0 &= (\boldsymbol{K}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{b})^{\mathrm{T}}\boldsymbol{T}(\boldsymbol{K}\boldsymbol{\Psi}^{-1}\boldsymbol{b}) \\
\boldsymbol{T} &\triangleq (\boldsymbol{K}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{K} + \lambda_{\text{W-CS}}\mathbf{C}_2)^{-1}\mathbf{C}_2(\boldsymbol{K}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{K} + \lambda_{\text{W-CS}}\mathbf{C}_2)^{-1}.
\end{aligned}$$

Once $\lambda_{\text{W-CS}}$ is obtained, $y$ could be found using (3.31), and then $\hat{\boldsymbol{l}}_{\text{ECWLS-CS}}$ is obtained from (3.29). One way to obtain $\lambda_{\text{W-CS}}$ is to express $y$ in terms of $\lambda_{\text{W-CS}}$ using (3.29) and (3.30), then substituting the resultant equation in (3.31), which is a computationally complex process. However, if the minimization problems (3.26) and (2.153) are compared, it is seen that the objective functions and the constraints are the same, therefore they are equivalent. Hence, $\lambda_{\text{W-CS}}$ is equal to $\lambda_{\text{W}}$ and it could be obtained using (2.164).

Note that when the sensor array is placed along the x-axis, second element of the gradient of the cost function (2.152) becomes zero. On the other hand, second element of the gradient of the equality constraint does not become zero, unless the source is located on the same line with the sensor array. Therefore, the Lagrange multiplier values corresponding to the point where the source is positioned and its symmetric point with respect to the sensor array are zero. Hence, among the methods given in Section 2.3.2, the ones using Lagrange multiplier method also face the rank deficient matrix problem, regardless of the noise level.

After obtaining $\lambda_{\text{W-CS-GM}}$ candidates from (2.164), corresponding $y$ values should be found using (3.31). Since (3.31) is a second degree equation, the following root selection procedure should be used:

a. If the roots of (3.31) are real, obtain corresponding $\hat{l}$ estimates using (3.29). Then calculate the resulting $y$ values from (3.30). The right $y$ root should match the resulting one obtained from (3.30); therefore, choose the matching one. Note that in the case of perfect linear array, $\boldsymbol{k}$ becomes all zero vector, which makes $a_1$ zero. In such cases, roots of (3.31) have the same magnitude but opposite signs. Therefore a priori information clarifying the side on which the source placed relative to the linear array axis, is required to choose the right root.

b. If the roots of (3.31) are complex, then take the real part as $y$.

Note that case (b) is abnormal, which occur when the noise level is quite high.

Since $\lambda_{\text{W-CS-GMS}} = \lambda_{\text{W-GMS}}$ and $\lambda_{\text{W-CS-GM}} = \lambda_{\text{W-GM}}$, the Lagrange multiplier founding approach we have proposed in Section 2.3.2.5 could be used in here, too. As a result, CWLS-CS method could be itemized as follows:

1. Set $\boldsymbol{\Psi} = \boldsymbol{Q}$.

2. In the polynomial given in (2.164), calculate unknowns except $\lambda_{\text{W}}$. Then find $\lambda_{\text{W-GMS}}$ in the interval $I_b'$. Since the polynomial is strictly decreasing in this interval, a simple bisection method could be utilized.

3. Calculate the roots of (3.31) using $\lambda_{\text{W-GMS}}$. If the roots are real, use the root selection procedure (RSP). Then, obtain $\hat{\boldsymbol{l}}$ from (3.29). If $\hat{l}_2 = \hat{r_1} \geq |\hat{l}_1| = |\hat{x}|$ , then $\lambda_{\text{W-CS-GM}} = \lambda_{\text{W-GMS}}$. Since the global minimizer of the problem is obtained, jump to step 6. If one of the following cases occurs, continue with the next step:

   - The roots are complex valued.
   - $\hat{r_1} < |\hat{x}|$.
   - There is a priori information clarifying the side on which the source placed, and the RSP result violates it.

4. Find all real roots $\lambda_{\text{W-LM-1}}, \lambda_{\text{W-LM-2}}, ..., \lambda_{\text{W-LM-t}},\ t \leq 3$ of (2.164) in the interval $(I_a' \cup I_c')$, and then calculate $\hat{y}_i = \hat{y}(\lambda_{\text{W-LM-i}})$ and $\hat{\boldsymbol{l}}_i = \hat{\boldsymbol{l}}(y(\lambda_{\text{W-LM-i}}))$, $i = 1, 2, ..., t$.

5. Calculate the objective function (3.25) only for the sets $\{\lambda_{\text{W-LM-i}}, \hat{y}_i, \hat{\boldsymbol{l}}_i\},\ i = 1, 2, ..., t$ satisfying $\hat{r}_1 \geq |\hat{x}|$ and for $\boldsymbol{\theta}_0 = [0\ 0\ 0]^{\text{T}}$. If a priori information about the source position is available, eliminate the sets incompatible with it. The one with the smallest objective function is $\{\lambda_{\text{W-CS-GM}}, y_{\text{W-CS-GM}}, \hat{\boldsymbol{l}}_{\text{W-CS-GM}}\}$.

6. Using $y_{\text{W-CS-GM}}$ and $\hat{l}_{\text{W-CS-GM-1}} = \hat{x}_{\text{W-CS-GM}}$, calculate the new range vector estimate $\hat{\boldsymbol{R}}$ and then update $\boldsymbol{\Psi}$ using (2.151). Repeat steps 2, 3,...6 until converges to obtain the source location estimate $\hat{\boldsymbol{s}}_{\text{ECWLS-CS}}$.

As mentioned at the end of Section 2.3.2.5, simulation results suggest that single iteration is generally enough unless the sensor geometry is poor. However, even in such cases, it is seen that the estimate converges in three iterations. Therefore, for a given sensor distribution scenario, a constant number of iteration steps could be safely defined.

**Comments**

Similar to the hyperbolic distribution, linear sensor distribution also causes the single branch hyperbolas corresponding to the noiseless TDOA values to intersect in two points. Therefore, some additional information is need. However, since these two points are symmetrical with respect to the sensor array axis; the RSS based approach used in ECWLS-RDSH, or the AOA method does not solve the direction ambiguity. As a result, if the linear sensor distribution is inevitable, the search area should be restricted to one of the sides of the sensor array.

Note that in the linear distribution, there is no requirement on the source position, unlike the circular and hyperbolic distributions.

Linear sensor distribution causes the methods given in Section 2.3.2 to face rank deficient matrix problem, regardless of the TDOA noise level. On the other hand, circular and hyperbolic distributions cause ill-conditioned problem only when the noise level is below a certain threshold.

Recall that it is shown in Section 3.2 that the hyperbolic distribution causes the third column of $\boldsymbol{A}$ to be a linear combination of the first two columns. In ECWLS-RDSH method, the third column is separated to circumvent the ill-conditioned problem. Another approach to circumvent the problem is to separate one of the first two columns of $\boldsymbol{A}$, which is done in ECWLS-CS. Therefore, ECWLS-CS method can also be used in hyperbolic distributions with the help of the RSS based approach used in ECWLS-RDSH.

# CHAPTER 4

# SIMULATION RESULTS

In this chapter, simulation results of the estimators given in Section 2.3.2 and Chapter 3 are presented. Firstly, performance of the LS based methods explained in Section 2.3.2 are analyzed and compared under the same framework. Then, performance of the each method given Chapter 3 are analyzed for the corresponding sensor and source placement scenarios. Unless otherwise stated, assumptions and the other details of the simulation process are as follows:

- Exact positions of the sensors are known.
- Multipath affect problem is not seen, or seen and solved in TDOA estimation step.
- For the sake of simplicity, simulations are made for 2D Space.
- Performance measure is RMSE. For $m$ Monte Carlo trials, RMSE of an estimator is calculated as

$$\text{RMSE} = \sqrt{\frac{\sum_{k=1}^{m} \|\hat{\boldsymbol{s}}(k) - \boldsymbol{s}\|^2}{m}}\ .$$

- TDOA vector $\boldsymbol{d}$ is constructed from the range vector $\boldsymbol{r}$ whose elements are corrupted by zero mean, independent and identically distributed Gaussian noise, i.e.,

$$\begin{aligned} d_{i,1} &= d_i - d_1, \qquad i = 2, 3, ..., N, \\ d_i &= r_i/c + n_i, \qquad i = 1, 2, ..., N, \end{aligned} \tag{4.1}$$

  where $d_{i,1}$ is the TDOA value of $i^{th}$ sensor, $d_i$ is the TOA value of the $i^{th}$ sensor, $r_i$ is the true range between the $i^{th}$ sensor and the source, $n_i$ is zero mean Gaussian random variable with variance $\sigma_n^2$, and $c$ is the propagation speed of the medium. As a result of the noise assumption, covariance matrix of $\boldsymbol{d}$ is taken as

$$\boldsymbol{Q} = \sigma_n^2 \begin{bmatrix} 2 & 1 & \dots & 1 \\ 1 & 2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \dots & 1 & 2 \end{bmatrix}_{\text{(N-1)x(N-1)}} .$$

- MATLAB® R2014a is used as simulation software.

### 4.1 Case 1: Random Distribution Scenario

In this simulation, the aim is to assess the relative performances of the methods given in Section 2.3.2 and Chapter 3. Details are as follows:

- Number of sensors: 5
- Number of Monte Carlo trials for each noise level: $4\text{x}10^6$
- Sensor and source positions: Randomly generated from a uniform distribution over [0, 10]x[0, 10] in each run
- 20$\log_{10}(c\,\sigma_n)$ dB range: Between -40dB and 10dB with 5dB steps
- For the methods using weighting matrix:
  - Initial value of $\boldsymbol{\Psi}$: $\boldsymbol{Q}$
  - Number of iterations: 3
- In order not to take into account ill-conditioned cases, only the distributions resulting in a noise free $\boldsymbol{A}$ matrix with a condition number less than $10^7$ are used.

Simulation results, which are given in Figure 4.1, shows that the weighting matrix $\boldsymbol{\Psi}$ does not have a significant effect on the ULS method. On the other hand, it decreases the RMSE of the ECLS. As the noise variance increases, the effect gradually diminish since the range vector estimate used to construct $\boldsymbol{\Psi}$ becomes more erroneous.

Besides, RMSE levels of TSWLS show that taking into account the constraints with a second LS causes significant performance improvement, although not as much as ECLS and ECWLS which consider the constraints throughout the estimation process. Lastly, it is seen that ECWLS, ECWLS-RDS and ECWLS-CS methods give the same performance when the problem is not ill-conditioned, as expected since their cost functions and the approaches in taking into account the constraints are identical.

In Table 4.1, intervals the Lagrange multiplier corresponding to the global minimizer is found are given as the percentage of the Monte Carlo trials of each noise level. In the table, "zero" stands for the all-zero solution. Note that as the noise level is increased, the percentage of $I_b$ reduces.

In Table 4.2, average computational time of each method is given. $t$ represents the average computational time of the ULS method. Note that the computational time of the methods using the Lagrange multiplier finding approach given in [11] are increased as the noise level increases. This is because the efficient Lagrange Multiplier finding method given in [11] is effective only when the Lagrange multiplier corresponding to the global minimizer lies in the interval $I_b$.

In section 2.3.2.2, it has been stated that the GSX method does not guarantee to give a solution. Therefore, simulation results of this method is not given in Figure 4.1. In Table 4.3, solution rate of the GSX method is given as a percentage of the Monte Carlo trials in each noise level.

## 4.2 Case 2: Passive Sonobuoy

In this simulation, the aim is to assess the performances of the methods given in Section 2.3.2 and Chapter 3 relative to the CRLB for a certain distribution scenario.

A sonobuoy is an expendable buoy having components to be a part of an active or passive SONAR system. It can be dropped from ships or aircrafts to use in underwater warfare or acoustic researches. The most inexpensive sonobuoy model is the one with a single omni-directional hydrophone. In the simulated scenario, it is assumed that an aircraft has ejected five passive omni-directional sonobuoys to an area where an

Table 4.1: $\lambda$ Intervals

| Method | Interval | $20\log_{10}(c\sigma_n)$ dB -40 dB | -30 dB | -20 dB | -10 dB | 0 dB | 10 dB |
|---|---|---|---|---|---|---|---|
| ECLS | $I_a$ | 0.04% | 0.16% | 0.96% | 6.52% | 22.66% | 26.66% |
| | $I_b$ | 99.96% | 99.84% | 99.00% | 93.00% | 73.82% | 52.34% |
| | $I_c$ | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% |
| | zero | 0.00% | 0.00% | 0.04% | 0.48% | 3.52% | 21.00% |
| ECWLS | $I_a$ | 0.04% | 0.21% | 1.03% | 6.97% | 24.40% | 19.85% |
| | $I_b$ | 99.96% | 99.79% | 98.98% | 92.71% | 71.70% | 52.56% |
| | $I_c$ | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% |
| | zero | 0.00% | 0.00% | 0.00% | 0.33% | 3.91% | 27.60% |
| ECWLS-RDS | $I_a$ | 0.04% | 0.22% | 0.92% | 6.45% | 21.68% | 17.49% |
| | $I_b$ | 99.96% | 99.79% | 98.97% | 92.61% | 71.63% | 52.56% |
| | $I_c$ | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% |
| | zero | 0.00% | 0.00% | 0.12% | 0.95% | 6.70% | 29.96% |
| ECWLS-CS | $I_a$ | 0.04% | 0.22% | 0.92% | 6.45% | 21.68% | 17.49% |
| | $I_b$ | 99.96% | 99.79% | 98.97% | 92.61% | 71.63% | 52.56% |
| | $I_c$ | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% |
| | zero | 0.00% | 0.00% | 0.12% | 0.95% | 6.70% | 29.96% |

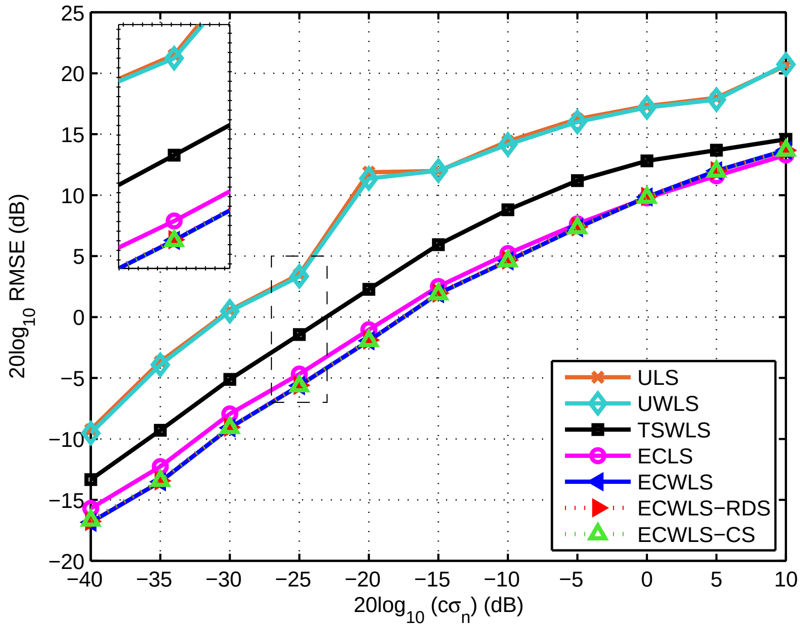


Figure 4.1: RMSE vs. $(c\sigma_n)$ for Randomly Distributed Source and Sensors

Table 4.2: Average Computational Time

| | $20\log_{10}(c\sigma_n)$ dB | | | | | |
|---|---|---|---|---|---|---|
| Method | -40 dB | -30 dB | -20 dB | -10 dB | 0 dB | 10 dB |
| ULS | 1 $t$ | 1 $t$ | 1 $t$ | 1 $t$ | 1 $t$ | 1 $t$ |
| UWLS | 2.62 $t$ | 2.45 $t$ | 2.43 $t$ | 2.49 $t$ | 2.58 $t$ | 2.46 $t$ |
| TSWLS | 3.10 $t$ | 3.05 $t$ | 2.90 $t$ | 2.92 $t$ | 2.99 $t$ | 2.90 $t$ |
| ECLS | 12.03 $t$ | 11.74 $t$ | 12.18 $t$ | 18.34 $t$ | 40.65 $t$ | 59.92 $t$ |
| ECWLS | 32.57 $t$ | 31.30 $t$ | 33.12 $t$ | 59.00 $t$ | 151.26 $t$ | 221.89 $t$ |
| ECWLS-RDS | 35.98 $t$ | 34.62 $t$ | 36.42 $t$ | 62.71 $t$ | 156.04 $t$ | 227.93 $t$ |
| ECWLS-CS | 34.84 $t$ | 33.45 $t$ | 35.31 $t$ | 61.39 $t$ | 154.20 $t$ | 226.26 $t$ |

Table 4.3: Solution Rate of GSX Method

| $20\log_{10}(c\sigma_n)$ dB | | | | | |
|---|---|---|---|---|---|
| -40 dB | -30 dB | -20 dB | -10 dB | 0 dB | 10 dB |
| 76.97% | 76.94% | 76.74% | 75.20% | 66.63% | 40.61% |

enemy submarine is located. Active SONAR signals or the propeller sound of the submarine is assumed to be detected by the sonobuoys. The scenario is illustrated in

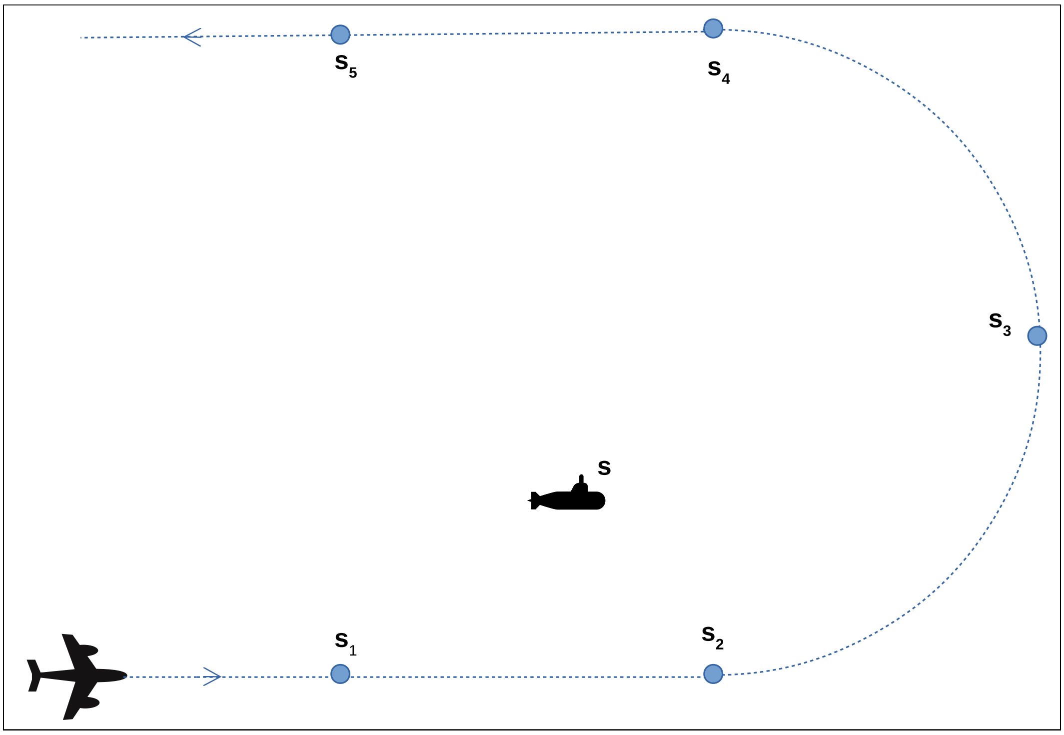


Figure 4.2: Sonobuoy Distribution Scenario

Figure 4.2.

Details are as follows:

- Number of sensors: 5
- Number of Monte Carlo trials for each noise level: $10^4$
- Sensor positions are

$$s_1 = [0, 0]^{\mathrm{T}},$$
$$s_2 = [10, 0]^{\mathrm{T}},$$
$$s_3 = [20, 10]^{\mathrm{T}},$$
$$s_4 = [10, 20]^{\mathrm{T}},$$
$$s_4 = [0, 20]^{\mathrm{T}}.$$

- Source position is

$$s = [7, 5]^{\mathrm{T}}.$$

- $20\log_{10}(c\,\sigma_n)$ dB range: Between -20 dB and 70 dB with 5 dB steps

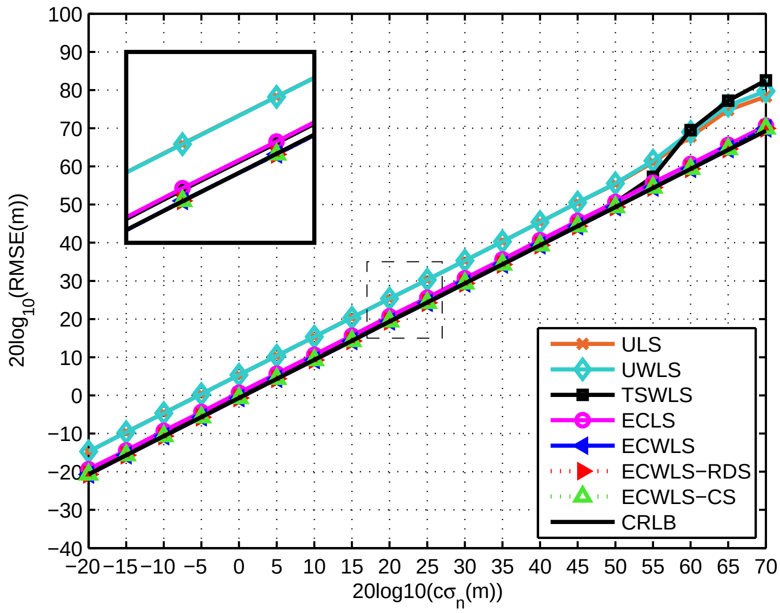


Figure 4.3: RMSE vs. $(c\sigma_n)$ for Passive Sonobuoy Scenario

- For the methods using weighting matrix:
  - Initial value of $\boldsymbol{\Psi}$: $\boldsymbol{Q}$
  - Number of iterations: 3
- Sensor and source positions are assumed to be in kilometers.

Simulation results are given Figure 4.3. Relative performances of the methods are similar to that of the random distribution scenario, except the TSWLS method. In this simulation, it is seen that TSWLS and ECLS methods give the same performance up to 55 dB noise standard deviation level. Besides, results show that ECWLS, ECWLS-RDS and ECWLS-CS attain the CRLB.

## 4.3 Case 3: Ill-Conditioned Distribution Scenarios

In Section 4.1, it is seen that the methods with the lowest RMSE levels among the compared ones are ECWLS, ECWLS-RDS and ECWLS-CS. Moreover, it is also seen that these methods attain the CRLB in a scenario given in Section 4.2. In this section, these methods are compared in the ill-conditioned cases analyzed in Section 3.

### 4.3.1 Circular Distribution Scenario

In Section 3.1, ill-conditioning problem resulting from the circular source and sensor placement has been examined. A method robust to such distributions, namely CWLS-RDS, is explained; then a modified version of CWLS-RDS, namely ECWLS-RDS, has been proposed. In this section, simulation results of CWLS-RDS and ECWLS-RDS methods are compared with the ones of the ECWLS and ECWLS-CS methods in two examples.

**Example 1:**

In [35], simulation results of CWLS-RDS and CWLS methods have been given for certain scenarios. Details of one of these scenarios is as follows:

- There are four sensors placed on a circle with the center $[5, 5]^{\mathrm{T}}$. Their positions are $[0, 0]^{\mathrm{T}}$, $[0, 10]^{\mathrm{T}}$, $[10, 0]^{\mathrm{T}}$ and $[10, 10]^{\mathrm{T}}$.
- Source position is $[5.1, 4.9]^{\mathrm{T}}$.
- Standard deviation of the noise term added to the true range values, i.e., $20\log_{10}(c\sigma_n)$ dB; is swept from -30 dB to 10 dB with steps of 2 dB.
- MSE values of the estimators for each noise level are computed using 1000 independent Monte Carlo trials.
- Initial value of $\mathbf{\Psi}$ is taken as identity matrix. 3 iterations are made to update $\mathbf{\Psi}$.

To assess the relative performances of ECWLS-RDS, ECWLS-CS and ECWLS; the parameters given above are used in our simulation without any change except the number of Monte Carlo trials, which is increased to $10^4$. Results are given in Figure 4.4. As seen from the figure, ECWLS method does not suffer from the ill-conditioning problem under the simulated noise levels. This result shows that the reason of the increase in MSE of CWLS method is not the ill-conditioning problem, but the disregard of the inequality constraint in the Lagrange multiplier selection procedure, as explained in Section 2.3.2.5. Moreover, it has been seen that the

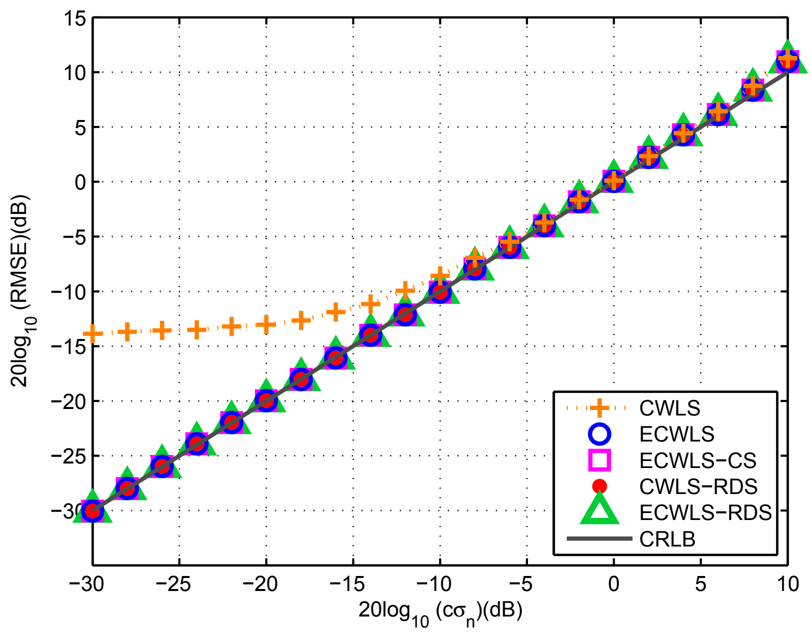


Figure 4.4: RMSE vs. $(c\sigma_n)$ for Circular Sensor Distribution, $\boldsymbol{s} = [5.1,\ 4.9]^{\mathrm{T}}$

CWLS-RDS and ECWLS-RDS methods give the same performance for the simulated scenario, which shows that the problems of the RSP of CWLS-RDS mentioned in Section 3.1 are not faced.

**Example 2:**

To see the effect of the ill-conditioning problem, the source is relocated to the sensor array center; $[5, 5]^{\mathrm{T}}$. Moreover, to simulate the systems having working precision lower than that of the MATLAB® numeric calculations, "vpa" function is used with 8 digits. Although "vpa" uses more digits than the specified one, and some functions such as "eig" does not support "vpa" and uses MATLAB® defaults; effect of reduced working precision is still observable in Figure 4.5.

Simulation results show that although the ill-conditioned problem is not seen in the ECWLS and ECWLS-CS methods for the source position and noise levels given in [35], it arises when the source is positioned to the array center and the noise variance and working precision decrease. The reason of that ECWLS-CS is slightly more robust than ECWLS is the RSP of ECWLS-CS and the reduced condition number of the matrix to be inverted [35]. On the other hand, ECWLS-RDS still attains CRLB

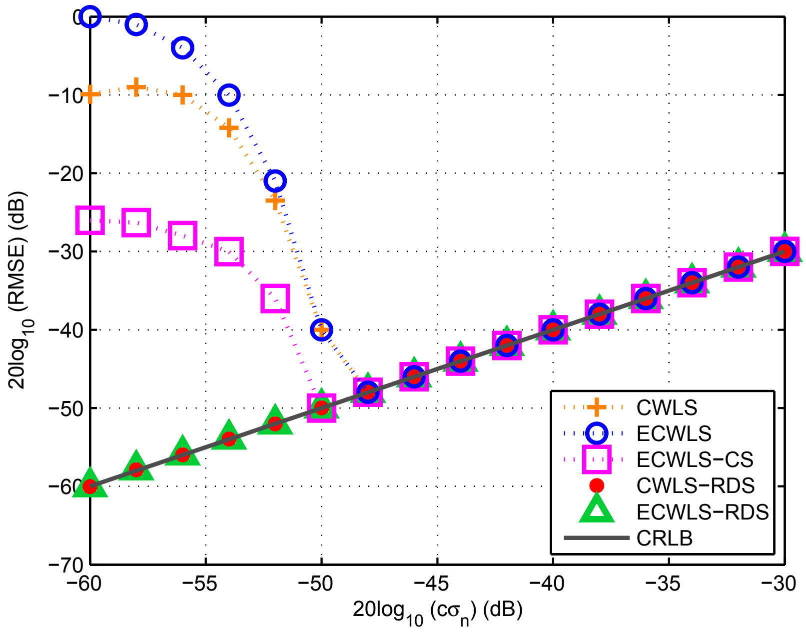


Figure 4.5: RMSE vs $(c\sigma_n)$ for Circular Sensor Distribution with a Reduced Computational Precision, $\boldsymbol{s} = [5,\ 5]^{\mathrm{T}}$

under the same circumstances. Moreover, it is seen in Section 4.1 that ECWLS-RDS, ECWLS and ECWLS-CS methods become equivalent when the distribution is not ill-conditioned.

### 4.3.2 Hyperbolic Distribution Scenario

In Section 3.2, ill-conditioning problem resulting from the hyperbolic source and sensor placement has been examined, and a modified version of the ECWLS-RDS method, namely ECWLS-RDS for hyperbolic distribution (ECWLS-RDSH), has been proposed to overcome the problem. In this section, simulation results of ECWLS, ECWLS-CS and ECWLS-RDSH methods are compared in a hyperbolic distribution scenario. To solve the ambiguity resulting from the hyperbolic distribution, RSS based approach used in ECWLS-RDSH is also applied to ECWLS and ECWLS-CS, and named as ECWLS-H and ECWLS-CSH, respectively. Simulation details are as follows:

- Number of sensors: 4

- Number of Monte Carlo trials for each noise level: $10^4$
- Four sensors are located on a single branch hyperbola with the following parameters:

$$
\begin{aligned}
\boldsymbol{f}_1 &= [-10, 50]^{\mathrm{T}}, \\
\boldsymbol{f}_2 &= [80, 100]^{\mathrm{T}}, \\
\|\boldsymbol{f}_2 - \boldsymbol{s}_i\| - \|\boldsymbol{f}_1 - \boldsymbol{s}_i\| &= 77, \quad i = 1, 2, 3, 4,
\end{aligned}
$$

  where $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ are the focal points. Sensor positions are

$$
\begin{aligned}
\boldsymbol{s}_1 &= [0, 0]^{\mathrm{T}}, \\
\boldsymbol{s}_2 &= [-3, -11.66]^{\mathrm{T}}, \\
\boldsymbol{s}_3 &= [-6, -22.45]^{\mathrm{T}}, \\
\boldsymbol{s}_4 &= [-9, -32.72]^{\mathrm{T}},
\end{aligned}
$$

  and source is located at $\boldsymbol{f}_2$:

$$\boldsymbol{s} = [80, 100]^{\mathrm{T}}.$$

- 20$\log_{10}(c\,\sigma_n)$ dB range: Between -75 dB and -40 dB with 5 dB steps
- For the methods using weighting matrix:
  - Initial value of $\boldsymbol{\Psi}$: $\boldsymbol{Q}$
  - Number of iterations: 3
- Sensor and source positions are assumed to be in meters.

Results are given Figure 4.6. It is seen that as the standard deviation of the additive noise is decreased, ill-conditioned problem causes an increase in the RMSE of ECWLS-H method, while ECWLS-RDSH and ECWLS-CSH are not affected.

Computational precision of the simulation is reduced from 16 digits to 8 digits using MATLAB® "vpa" function, whose details are explained in Example 2 of Section 4.3.1. Then, without changing any parameters, simulation is repeated. Results are given in Figure 4.7. As a result of the increased rounding errors, it is seen that ECWLS-H starts to suffer from ill-conditioned in a higher noise level, as expected.

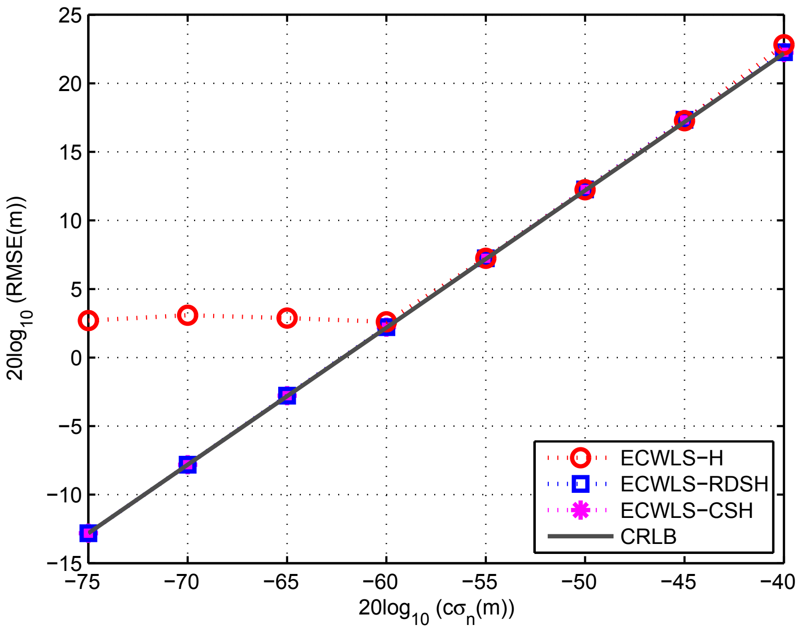


Figure 4.6: RMSE vs. $(c\sigma_n)$ for Hyperbolic Distribution

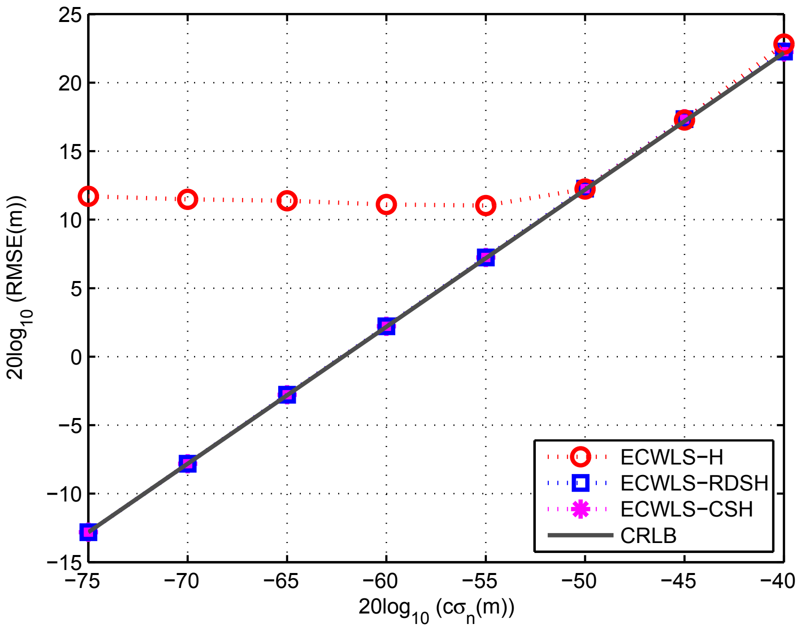


Figure 4.7: RMSE vs. $(c\sigma_n)$ for Hyperbolic Distribution with a Reduced Computational Precision

### 4.3.3 Linear Distribution Scenario

In Section 3.3, ill-conditioning problem resulting from the linear sensor distribution has been examined. A new method robust to such distributions, namely ECWLS-CS, has been proposed. In this section, simulation results of ECWLS-CS is compared with the CRLB.

Note that when the sensor distribution is linear, $\boldsymbol{A}$ becomes rank deficient regardless of the noise level, which is not the case in circular and hyperbolic distributions. Moreover, Lagrange multiplier corresponding to the global minimizer becomes zero, as explained in Section 3.3. Therefore, none of the methods given in Section 2.3.2 could be used in linear distribution scenarios.

Simulation details are as follows:

- Number of sensors: 4
- Number of Monte Carlo trials for each noise level: $10^4$
- Sensor positions: $[0, 0]^{\mathrm{T}}$, $[10, 0]^{\mathrm{T}}$, $[20, 0]^{\mathrm{T}}$ and $[30, 0]^{\mathrm{T}}$.
- Source position: $[15, 50]^{\mathrm{T}}$.
- It is assumed that a priori information clarifying the side on which the source is placed relative to the linear array axis is available
- 20log$_{10}(c\,\sigma_n)$ dB range: Between -30 dB and 10 dB with 2 dB steps.
- For the methods using weighting matrix:
  - Initial value of $\boldsymbol{\Psi}$: $\boldsymbol{Q}$
  - Number of iterations: 3
- Sensor and source positions are assumed to be in meters.

Simulation results are given in Figure 4.8. It is seen that ECWLS-CS method attains CRLB up to -15 dB noise standard deviation level. After this point, bias of the estimator becomes to increase, as can be seen in Figure 4.9.

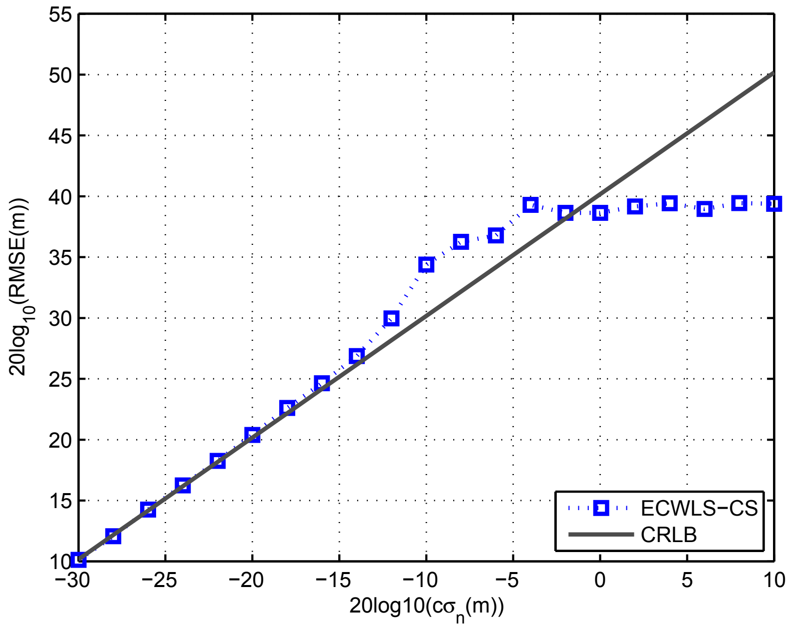


Figure 4.8: RMSE vs. $(c\sigma_n)$ for Linear Sensor Distribution, $\boldsymbol{s} = [15, 50]^{\mathrm{T}}$

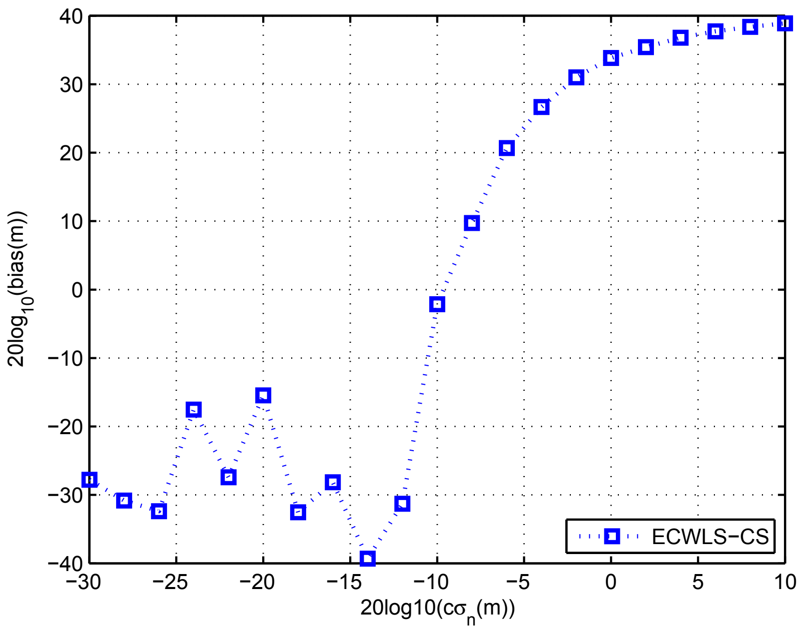


Figure 4.9: Bias vs. $(c\sigma_n)$ for Linear Sensor Distribution, $\boldsymbol{s} = [15, 50]^{\mathrm{T}}$

# CHAPTER 5

# CONCLUSIONS

In this thesis, TDOA based closed-form source localization methods have been studied. An extensive overview of the available methods has been given with the comments on their shortcomings and advantageous points. Besides, sensors and source placement scenarios resulting in an ill-conditioned problem, and the methods resistant to such scenarios have been investigated. Additionally, some modifications have been made on the existing methods to solve some ambiguities, reduce computational cost and increase estimation performance. Moreover, a novel algorithm, efficient constrained weighted least squares with coordinate separation (ECWLS-CS), has been proposed; which circumvent the ill-conditioning problem emerging in the available methods when the sensor distribution is linear or hyperbolic in 2D plane (and planar or hyperboloidal in 3D). Existing methods and the proposed one have been implemented in the same framework and compared under certain source and sensor distribution scenarios.

Simulation results in Section 4 show that ECWLS, and equivalently ECWLS-RDS and ECWLS-CS, give the best performance among the compared methods, in general. However, computational times given in Table 4.2 show that their computational costs are much higher than that of the ULS, UWLS and TSWLS methods, especially in the high noise levels. The reason of this difference is the Lagrange multiplier finding routine. In these methods, we utilize the efficient Lagrange multiplier finding approach given in [11]. However, although it significantly reduces computational time in low noise levels, the effect of this approach gradually diminish as the noise level increases. To reduce the computational cost further, number of iterations to update the weighting matrix could be reduced with the expense of a small performance decrease (see RSME levels of the CLS method in figures 4.1 and 4.3).

If the constrained methods; i.e. CLS, ECWLS, ECWLS-RDS and ECWLS-CS; are not feasible for an application because of their computational costs, TSWLS could be considered. In TSWLS, utilizing the equality constraint with a second LS results in a significant RSME decrease, according to the simulation results given in figures 4.1 and 4.3. However, performance improvement of this method over UWLS is highly dependent on the distribution scenario and the noise level. Furthermore, it should not be ignored that it is much more sensitive to the ill-conditioned distributions than the constrained methods.

In Chapter 3, we considered the ill-conditioned distributions; namely circular, hyperbolic and linear; in which the methods given in Section 2.3.2 suffer from the ill-conditioned matrix problem. Then we proposed two methods robust to such cases. The first one is ECWLS-RDS, which is a slightly changed version of the method given in[35]. This method is robust to circular and hyperbolic distributions. The second one is a new method, namely ECWLS-CS, which is robust to hyperbolic and linear distributions. In the circular distribution, ill-conditioned problem emerges only when the source is at the center of the circular sensor array. In the hyperbolic distribution, when the sensor number is less than five, the required position of the source to have the ill-conditioned problem is not a single point, but a certain area in the plane. In the linear distribution, there is no requirement on the source position. Therefore, when the sensor positions are not controllable, ECWLS-CS should be preferred over ECWLS-RDS to circumvent most of the possible ill-conditioned cases.

As a future work, the proposed method and the existing ones could be compared using real-world TDOA measurements. Furthermore, sensor position and synchronization errors in TDOA based source localization could also be studied. Finally, TDOA estimation methods could be investigated for a possible performance improvement.

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# APPENDIX A

# CRAMÉR–RAO LOWER BOUND FOR SOURCE LOCATION ESTIMATION FROM TDOA MEASUREMENTS

In [30], Cramér–Rao Lower Bound Theorem is stated as follows:

**Theorem 4.** *Let $\boldsymbol{\theta}$ be the unkonwn vector to be estimated, $\boldsymbol{x}$ be the observation vector, and $p(\boldsymbol{x};\boldsymbol{\theta})$ denote the probability density function (PDF) of $\boldsymbol{x}$. Then, if the regularity condition*

$$\mathrm{E}\left[\frac{\partial\, ln\, p(\boldsymbol{x};\boldsymbol{\theta})}{\partial\boldsymbol{\theta}}\right] = \mathbf{0} \tag{A.1}$$

*holds, then the variance of any unbiased estimator $\hat{\boldsymbol{\theta}}$ satisfy*

$$var(\hat{\theta}_i) \geq [\boldsymbol{I}^{-1}(\boldsymbol{\theta})]_{ii}, \tag{A.2}$$

*where $\hat{\theta}_i$ denotes the $i^{th}$ element of $\boldsymbol{\theta}$ and $[\boldsymbol{I}^{-1}(\boldsymbol{\theta})]_{ii}$ denotes the $(i,i)^{th}$ element of the inverse of the Fisher information matrix, which can be calculated as*

$$\boldsymbol{I}(\boldsymbol{\theta}) = -\mathrm{E}\left[\frac{\partial^2\, ln\, p(\boldsymbol{x};\boldsymbol{\theta})}{\partial\boldsymbol{\theta}^2}\right] = \mathrm{E}\left[\left(\frac{\partial\, ln\, p(\boldsymbol{x};\boldsymbol{\theta})}{\partial\boldsymbol{\theta}}\right)^2\right], \tag{A.3}$$

*where the expectations are taken with respect to $p(\boldsymbol{x};\boldsymbol{\theta})$, and the derivative is calculated for the true value of $\boldsymbol{\theta}$.*

In (2.6), the PDF of the TDOA values is given as

$$p(\boldsymbol{d};\boldsymbol{s}) = \frac{1}{(2\pi)^{(N-1)/2}|\boldsymbol{Q}|^{1/2}}\, exp\{-\frac{1}{2}(\boldsymbol{d}-\boldsymbol{r}/c)^{\mathrm{T}}\boldsymbol{Q}^{-1}(\boldsymbol{d}-\boldsymbol{r}/c)\}, \tag{A.4}$$

where $\boldsymbol{r}$ is the true range difference vector, $c$ is the signal propagation speed, $\boldsymbol{n}$ is the noise vector, $\boldsymbol{d} = \boldsymbol{r}/c + \boldsymbol{n}$ is the TDOA vector and $\boldsymbol{Q}$ is the covariance matrix of $\boldsymbol{n}$. Calculating the partial derivative of the natural logarithm of (A.4) with respect to $\boldsymbol{s}$, and then taking the expectation results in

$$\mathrm{E}\left[\frac{\partial\, ln\, p(\boldsymbol{d};\boldsymbol{s})}{\partial\boldsymbol{s}}\right] = \mathrm{E}\left[\frac{1}{c}\frac{\partial\boldsymbol{r}^{\mathrm{T}}}{\partial\boldsymbol{s}}\boldsymbol{Q}^{-1}(\boldsymbol{d}-\boldsymbol{r}/c)\right] = \mathbf{0}_{\text{2x1}}. \tag{A.5}$$

Equation (A.5) shows that the regularity condition given in Theorem 4 is satisfied for (2.6). Therefore, by applying Theorem 4, a lower bound for the variance of any unbiased TDOA based source localization estimator can be obtained. Calculating (A.3) for (2.6) gives

$$\begin{aligned} \boldsymbol{I}(\boldsymbol{s}) &= \mathrm{E}\left[\left(\frac{\partial\, ln\, p(\boldsymbol{d};\boldsymbol{s})}{\partial \boldsymbol{s}}\right)^2\right] \\ &= \frac{1}{c^2}\frac{\partial \boldsymbol{r}^{\mathrm{T}}}{\partial \boldsymbol{s}}\boldsymbol{Q}^{-1}\,\mathrm{E}[(\boldsymbol{d}-\boldsymbol{r}/c)(\boldsymbol{d}-\boldsymbol{r}/c)^{\mathrm{T}}]\,\boldsymbol{Q}^{-1}\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{s}} \\ &= \frac{1}{c^2}\left(\frac{\partial \boldsymbol{r}^{\mathrm{T}}}{\partial \boldsymbol{s}}\boldsymbol{Q}^{-1}\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{s}}\right)\Bigg|_{\boldsymbol{s}=\boldsymbol{s}^0}, \end{aligned} \tag{A.6}$$

where $\boldsymbol{s}^0$ is the true source location. $\partial \boldsymbol{r}^{\mathrm{T}}/\partial \boldsymbol{s}$ could be calculated as

$$\frac{\partial \boldsymbol{r}^{\mathrm{T}}}{\partial \boldsymbol{s}} = \begin{bmatrix} \left\{\frac{x-x_2}{r_2} - \frac{x-x_1}{r_1}\right\} & \cdots & \left\{\frac{x-x_N}{r_N} - \frac{x-x_1}{r_1}\right\} \\ \\ \left\{\frac{y-y_2}{r_2} - \frac{y-y_1}{r_1}\right\} & \cdots & \left\{\frac{y-y_N}{r_N} - \frac{y-y_1}{r_1}\right\} \end{bmatrix}_{2\mathrm{x}(N-1)}, \tag{A.7}$$

since

$$\frac{\partial\, r_{i,1}}{\partial \boldsymbol{s}} = \begin{bmatrix} \left\{\frac{\partial\left(\sqrt{(x-x_i)^2+(y-y_i)^2}-\sqrt{(x-x_1)^2+(y-y_1)^2}\right)}{\partial x}\right\} \\ \\ \left\{\frac{\partial\left(\sqrt{(x-x_i)^2+(y-y_i)^2}-\sqrt{(x-x_1)^2+(y-y_1)^2}\right)}{\partial y}\right\} \end{bmatrix} \tag{A.8}$$

$$= \begin{bmatrix} \left\{\frac{x-x_i}{r_i} - \frac{x-x_1}{r_1}\right\} \\ \\ \left\{\frac{y-y_i}{r_i} - \frac{y-y_1}{r_1}\right\} \end{bmatrix}. \tag{A.9}$$

Fisher information matrix $\boldsymbol{I}(\boldsymbol{s})$ in (A.6) has been also given in [9].