LAGRANGIAN PERTURBATIONS OF LAGRANGIAN NODAL SPHERES IN THE COMPLEX PLANE

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ABSTRACT

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Classification of monotone Lagrangian tori in $\mathbb{C}^2$ up to Hamiltonian isotopy and rescaling is still an open problem and the only classes of such tori that are currently known are Clifford and Chekanov tori. In this thesis, we analyze how these two classes of tori can be obtained by Lagrangian perturbations of a Lagrangian nodal sphere in $\mathbb{C}^2$.

Keywords: Chekanov torus, Clifford torus, Whitney immersion, Lagrangian nodal spheres, Lagrangian perturbations

ÖZ

KARMAŞIK DÜZLEMDEKİ LAGRANJİYEN BOĞUMSAL KÜRELERİN LAGRANJİYEN TEDİRGEMELERİ

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Karmaşık $\mathbb{C}^2$ düzlemindeki tekdüze Lagranjiyen simitlerin Hamiltoniyen izotopi ve yeniden ölçeklendirme altında sınıflandırılması hâlâ açık bir problemdir. Clifford ve Chekanov simitleri bu sınıflandırılma altında bilinen tek simitlerdir. Bu tezde, Clifford ve Chekanov simitlerinin karmaşık $\mathbb{C}^2$ düzlemindeki bazı Lagranjiyen boğumsal kürelerin Lagranjiyen tedirgemeleri ile elde edilmesi incelenecektir.

Anahtar Kelimeler: Chekanov simidi, Clifford simidi, Whitney daldırması, Lagranjiyen boğumsal küreler, Lagranjiyen tedirgemeler

To my family: Mahmut, Melek, Sevim and Zelal

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$\mathbb{S}^n$	Unit n-sphere in $\mathbb{R}^{n+1}$.
$\mathbb{S}^n_r$	Round n-sphere in $\mathbb{R}^{n+1}$ with radius r.
$\mathbb{S}^n_+$	Upper hemisphere of unit n-sphere in $\mathbb{R}^{n+1}$.
$\mathbb{S}^n_-$	Lower hemisphere of unit n-sphere in $\mathbb{R}^{n+1}$
$\mathbb{D}^n_{\leq r}$	Round closed n-disc in $\mathbb{R}^n$ with radius r.
$\mathbb{D}^n_{<r}$	Round open n-disc in $\mathbb{R}^n$ with radius r.
$\Lambda(n)$	The Lagrangian Grassmannian of $\mathbb{R}^{2n}$.
T_pM	The tangent space of the smooth manifold M at point p.
TM	The total space of the tangent bundle of the smooth manifold M.
T^*_pM	The cotangent space of the smooth manifold M at point p.
T^*M	The total space of the cotangent bundle of the smooth manifold M.
$\iota_X\omega$	Contraction of ω with respect to X.
$F : (M, L) \to (N, K)$	A smooth map $F : M \to N$ such that $F(L) \subseteq K$ where $L \subseteq M$ and $K \subseteq N$.

CHAPTER 1

INTRODUCTION

1.1 The Subject of the Thesis

It is well-known and trivial fact that a closed orientable Lagrangian submanifold in $\mathbb{C}^n$ must have zero Euler characteristic and in particular orientable closed connected Lagrangian surfaces in $\mathbb{C}^2$ are tori. All Lagrangian tori in $\mathbb{C}^2$ are Lagrangian isotopic but not Hamiltonian isotopic [21]. In addition to Hamiltonian isotopy if we allow rescaling and consider the only monotone Lagrangian tori in $\mathbb{C}^2$, there exist two known classes of such tori. One class consists of Clifford tori and the other one consists of Chekanov tori. It is still unknown if there are other classes besides these two.

Problem. ***(Chekanov)*** *Is it true that a monotone Lagrangian torus in $\mathbb{C}^2$ is either a Clifford torus or a Chekanov torus?*

A key observation motivating our research is existence of two essentially different local models of Lagrangian perturbations of a Lagrangian nodal singularity. Existence of such perturbations shows that a Lagrangian nodal sphere in $\mathbb{C}^2$ must have only one self-intersection point. Furthermore, for a Lagrangian nodal sphere in $\mathbb{C}^2$, one of such perturbations gives a Clifford torus and another gives a Chekanov torus

Our aim in this research is to give explicit descriptions of how the Clifford and Chekanov tori can be obtained by Lagrangian perturbations of certain Lagrangian nodal spheres in $\mathbb{C}^2$.

1.2 History and Motivation

Exactness is a very effective condition for a Lagrangian submanifold. However, M. Gromov proved that there is no closed exact Lagrangian submanifold in $\mathbb{C}^n$. For Lagrangian submanifolds, the concept of monotonicity is a generalization of exactness and it is introduced by Y.-G. Oh in [17]. Monotone Lagrangian submanifolds play important role in the recent technologies developed in symplectic topoloy, such as, Floer homology, pearl homology, symplectic quasi-states, Fukaya categories etc. By using examples of monotone Lagrangian submanifolds one can test and refine these tools. Monotonicity is preserved by a Hamiltonian isotopy.

A trivial example of a monotone Lagrangian torus in $\mathbb{C}^n$ is a monotone split Lagrangian torus which is the product of n circles of same radius, $\mathbb{S}^1_r \times ... \times \mathbb{S}^1_r$. A Lagrangian torus is called a Clifford torus if it is Hamiltonian isotopic to a monotone split Lagrangian torus.

A monotone Lagrangian is torus exotic if it is not Hamiltonian isotopic to a Clifford torus. Y. V. Chekanov constructed the first examples of exotic monotone Lagrangian tori in $\mathbb{C}^n$ [7]. Y. Eliashberg and L. Polterovich provided another interpretation of a Chekanov torus in $\mathbb{C}^2$ [11]. By a generalization of this interpretation of a Chekanov torus in $\mathbb{C}^2$, new examples of monotone Lagrangian tori in $\mathbb{C}^n$, $\mathbb{CP}^n$ and $\times_n \mathbb{S}^2$ were provided by Y. V. Chekanov and F. Schlenk [8]. The number of monotone Lagrangian tori in the work of Chekanov and Schlenk is increasing with the dimension but it is finite for any dimension. D. Auroux showed that there exist infinitely many monotone Lagrangian tori in $\mathbb{C}^3$, which are pairwise non-Hamiltonian isotopic to any rescaling of one another [6].

R. Vianna showed that there exist infinitely many exotic monotone Lagrangian tori which are pairwise non-Hamiltonian isotopic in $\mathbb{CP}^2$, $\mathbb{S}^2 \times \mathbb{S}^2$ and the del Pezzo surfaces $\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$ for $k = 3, 4, 5, 6, 7, 8$ [24,26]. The tori constructed by Vianna in $\mathbb{CP}^2$ yield monotone Lagrangian tori in $\mathbb{C}^2$ [16, 25]. However, it is not known whether these tori give new classes of monotone Lagrangian tori in $\mathbb{C}^2$ or not. Hence the problem of Chekanov explained in section 1.1 still remains open.

1.3 The Goals of the Thesis

Some Lagrangian nodal spheres arise in the construction of Chekanov torus in $\mathbb{C}^2$ given by Eliashberg and Polterovich in [11]. We prove that there exist Lagrangian perturbations of these Lagrangian nodal spheres which are Clifford and Chekanov tori following [11].

In the paper [14], F. Lalonde and J. C. Sikorav introduced a method of smoothing nodal singularities of immersed Lagrangian surfaces. L. Polterovich in [19] and M. Audin in [5] provided generalizations of this smoothing for any dimension. This smoothing procedure is generally called Polterovich surgery. We provide a description of the Polterovich surgery which is a mixture of the descriptions in [19] and [18].

The Whitney immersion is a Lagrangian immersion of an n-sphere into $\mathbb{C}^n$ and it gives a Lagrangian nodal sphere in $\mathbb{C}^n$. Chekanov remarked in [7] that the tori obtained by applying Polterovich surgery to the Whitney immersion in $\mathbb{C}^2$ are Chekanov and Clifford tori. One of our main goals is to prove the following theorem.

Theorem A. *The two Lagrangian tori obtained by Polterovich surgeries of the Lagrangian nodal sphere given by the Whitney immersion in $\mathbb{C}^2$ are Clifford and Chekanov tori.*

M.-L. Yau considered the following integrable Hamiltonian system on $\mathbb{R}^4$:

$$G : \mathbb{R}^4 \to \mathbb{R}, \quad G(q_1, p_1, q_2, p_2) = (p_1^2 + p_2^2) - (q_1^2 + q_2^2) + (q_1^2 + q_2^2)^2 \tag{1.1}$$

$$H : \mathbb{R}^4 \to \mathbb{R}, \quad H(q_1, p_1, q_2, p_2) = p_2 q_1 - p_1 q_2 \,. \tag{1.2}$$

where its momentum map is given by $F : \mathbb{R}^4 \to \mathbb{R}^2$ defined by $F = (G, H)$. Yau described the fibers of the momentum map $F : \mathbb{R}^4 \to \mathbb{R}^2$ in [28] without proof. Another main goal of this thesis is to find the range of $F : \mathbb{R}^4 \to \mathbb{R}^2$, provide a proof for the classification of its fibers and hence prove the following theorem.

Theorem B. *The fiber $F^{-1}(0,0)$ is a Lagrangian nodal sphere and the Lagrangian perturbations of $F^{-1}(0,0)$ given by $F^{-1}(\epsilon, 0)$ are Clifford tori if $\epsilon > 0$ and Chekanov tori if $-\frac{1}{4} < \epsilon < 0$.*

1.4 The Structure of the Thesis

In section 2.1, we introduce Maslov index following [3]. In sections 2.2 and 2.3, we give some background material in symplectic topology. In sections 2.4 and 2.5, we discuss concepts of exact and monotone Lagrangian submanifolds in a symplectic manifold. In section 2.6, we introduce Chekanov suspension.

In section 3.1, we provide a description of a Chekanov torus in $\mathbb{C}^2$ and prove that it is monotone by using Chekanov suspension. In section 3.2, we provide another example of monotone torus which is described by Eliashberg and Polterovich in [11]. We show that this torus is exotic by counting holomorphic discs with Maslov index 2. In section 3.3, we show that the monotone torus given by Eliashberg and Polterovich is a Chekanov torus following [12]. In section 3.4, we show that the two kinds of Lagrangian perturbations of Lagrangian nodal spheres introduced in section 3.2 are Clifford and Chekanov tori.

In section 4.1, we introduce Whitney immersion and show that its image is a Lagrangian nodal sphere in $\mathbb{C}^n$ by using method of generating functions. In section 4.2, we give a description of the Polterovich surgery. In section 4.3, we show that Polterovich surgery procedure yields two Lagrangian perturbations of the Lagrangian nodal sphere which is the image of the Whitney immersion in $\mathbb{C}^2$ are Clifford and Chekanov tori.

In section 5.1, we discuss an integrable Hamiltonian system given by Yau in her paper [28]. In section 5.2 , we find the range and classify the fibers of the integrable Hamiltonian system described in the section 5.1.

CHAPTER 2

PRELIMINARIES

2.1 Lagrangian Grassmannian and Maslov Index

Recall, a *symplectic vector space* is a pair (V, Ω) where V is an m-dimensional real vector space and $\Omega : V \times V \to \mathbb{R}$ is a non-degenerate skew-symmetric bilinear form which is called a *symplectic form.* For a symplectic vector space (V, Ω), the dimension of V has to be even ($m = 2n$) and there exists an ordered basis $\{e_1, f_1, ..., e_n, f_n\}$ such that $\Omega(e_i, e_j) = \Omega(f_i, f_j) = 0$, $\Omega(e_i, f_j) = \delta_{ij}$ for all $i, j = 1, ..., n$.

The matrix

$$J_0 = \mathrm{diag}\Big(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, ..., \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \Big) \in \mathbb{R}^{2n \times 2n} \tag{2.1}$$

defines a symplectic form Ω_0 on the vector space $\mathbb{R}^{2n}$ with respect to the standard basis $\{e_1, f_1, ..., e_n, f_n\}$ of $\mathbb{R}^{2n}$, in other words, we have $\Omega_0(u, v) = u^T J_0 v$ for all $u, v \in \mathbb{R}^{2n}$.

A linear isomorphism $\Phi : (V_1, \Omega_1) \to (V_2, \Omega_2)$ of symplectic vector spaces is called a *linear symplectomorphism* if $\Phi^*(\Omega_2) = \Omega_1$. Any $2n$-dimensional symplectic vector space (V, Ω) is symplectomorphic to $(\mathbb{R}^{2n}, \Omega_0)$. For a symplectic vector space (V, Ω), the group of symplectomorphisms of (V, Ω) i.e. $\{\Phi : V \to V, \,|\, \Phi^*(\Omega) = \Omega\}$ is denoted by $\mathrm{Sp}(V, \Omega)$. An element of the group $\mathrm{Sp}(\mathbb{R}^{2n}, \Omega_0)$ can be identified with a $2n \times 2n$ real matrix, set of these matrices is called *the symplectic group* and it is denoted by $\mathrm{Sp}(2n)$. We have

$$\mathrm{Sp}(2n) \cap \mathrm{O}(2n) = \mathrm{Sp}(2n) \cap \mathrm{GL}(n, \mathbb{C}) = \mathrm{O}(2n) \cap \mathrm{GL}(n, \mathbb{C}) = \mathrm{U}(n)\,. \tag{2.2}$$

A *linear Lagrangian subspace* of a symplectic vector space (V, Ω) is a linear subspace L of (V, Ω) such that $\Omega|_L \equiv 0$. Set of all linear Lagrangian subspaces of $(\mathbb{R}^{2n}, \Omega_0)$ is called *Lagrangian Grassmannian* and it is denoted it by $\Lambda(n)$.

Lemma 2.1.1 ([16]). $\mathrm{U}(n)$ *and thus* $\mathrm{Sp}(2n)$ *act transitively on* $\Lambda(n)$. *The stabilizer of the action of* $\mathrm{U}(n)$ *on* $\Lambda(n)$ *is* $\mathrm{O}(n)$. $\square$

Lemma 2.1.2 ([10]). $\mathrm{Sp}(2n)$ *acts transitively on pairs of transverse Lagrangian subspaces of* $(\mathbb{R}^{2n}, \Omega_0)$. $\square$

Lemma 2.1.1 shows that $\Lambda(n)$ is a manifold which is diffeomorphic to $\mathrm{U}(n)/\mathrm{O}(n)$ whose dimension $n(n+1)/2$.

Theorem 2.1.3 ([3]). *The map* $\det_{\mathbb{C}}^2 : \mathrm{U}(n) \to \mathbb{S}^1$ *induces an isomoprhism of fundamendal groups* $\pi_1(\Lambda(n))$ *and* $\pi_1(\mathbb{S}^1)$.

Proof. The map $\det_{\mathbb{C}}^2 : \mathrm{U}(n) \to \mathbb{S}^1$ induces a map $\bar{d} : \mathrm{U}(n)/\mathrm{O}(n) \to \mathbb{S}^1$ since $\mathrm{O}(n) \subset Ker(\det_{\mathbb{C}}^2)$. Map $\bar{d}$ is a fibration with fibers $\mathrm{SU}(n)/\mathrm{SO}(n)$. The exact sequence induced by the fibration $\bar{d}$ shows that $\pi_1(\Lambda(n)) \simeq \pi_1(\mathrm{U}(n)/\mathrm{O}(n)) \simeq \pi_1(\mathbb{S}^1)$. $\square$

If $\gamma : \mathbb{S}^1 \to \Lambda(n)$ is a loop of Lagrangian subspaces of $(\mathbb{R}^{2n}, \Omega_0)$, then degree of the map $\bar{d} \circ \gamma : \mathbb{S}^1 \to \mathbb{S}^1$ (where $\bar{d}$ is defined in the proof of Theorem 2.1.3) is called *Maslov index* of the loop γ and we denote it by $\mu(\gamma)$. It follows immediately from the definition that Maslov index induces a homomorphism $\mu : \pi_1(\Lambda(n)) \to \mathbb{Z}$. In some sense, Maslov index measures how much linear Lagrangian spaces rotates along a loop in $\Lambda(n)$.

2.2 Symplectic Manifolds and Lagrangian Submanifolds

A *symplectic manifold* is a pair (M, ω) where M is an $2n-$dimensional smooth manifold and ω is closed non-degenerate ($w^n \neq 0$ pointwisely) 2-form. A diffeomorphism $\phi : (M_1, \omega_1) \to (M_2, \omega_2)$ of symplectic manifolds is called *symplectomorhism* if it preserves the sypmlectic structure, $\phi^* \omega_2 = \omega_1$.

Example 2.2.1.

- ***i.*** *$\mathbb{R}^{2n}$ is a symplectic manifold with the sypmlectic form $\omega_0 = \sum_{i=1}^{n} dq_i \wedge dp_i$ where $(q_1, p_1 ..., q_n, p_n)$ are coordinates of $\mathbb{R}^{2n}$. If we only consider linear subspaces of $\mathbb{R}^{2n}$ then ω_0 is same as Ω_0 under the identifications $dq_i = e_i$ and $dp_i = f_i$ for $i = 1, ..., n$.*
- ***ii.*** *If we identify coordinates $(z_1, ..., z_n)$ of $\mathbb{C}^n$ with coordinates above of $\mathbb{R}^{2n}$ by $z_j = q_j + ip_j$ then ω_0 above can be written as $\omega_0 = \frac{i}{2}\sum_{j=1}^{n} dz_j \wedge d\bar{z}_j$.*
- ***iii.*** *Cotangent bundle T^*X of an arbitrary smooth n-dimensional manifold X has a canonical symplectic structure. Let $(\mathcal{U}, q_1, ..., q_n)$ be a coordinate chart for X and $(T^*\mathcal{U}, q_1, p_1, ..., q_n, p_n)$ be associated coordinate chart for T^*X, then canonical symplectic structure is $\omega_{can} = \sum_{i=1}^{n} dq_i \wedge dp_i$. We see that $(T^*\mathbb{R}^n, \omega_{can})$ is symplectomorhic to $(\mathbb{R}^{2n}, \omega_0)$.*

A diffeomorphism $F : X \to Y$ induces a symplectomorphism $F_\# : T^*X \to T^*Y$ which is defined by $F_\#(q, p) = (\tilde{q}, \tilde{p})$ where $\tilde{q} = f(q)$ and $p = dF_q^*(\tilde{p})$ for any $(q, p) \in T^*X$, $(\tilde{q}, \tilde{p}) \in T^*Y$ [9]. Similarly, if $F : X \to Y$ is a smooth embedding and X, Y are smooth manifolds of the same dimension then $F_\# : T^*X \to T^*Y$ is a symplectic embbedding of cotangent bundles.

Any $2n$-dimensional symplectic manifold (M, ω) is locally symplectomorphic to $(\mathbb{R}^{2n}, \omega_0)$. This fact is known as *Darboux theorem*. A *Lagrangian submanifold* of symplectic manifold (M, ω) is an n-dimensional submanifold L such that $\omega|_L \equiv 0$.

Example 2.2.2.

- ***i.*** *Zero section of the T^*X is a Lagrangian submanifold of (T^*X, ω_{can}).*
- ***ii.*** *The graph $\Gamma_{df} = \{(x, df_x) \in T^*X \mid x \in X\}$ of the differential of a smooth function $f : X \to \mathbb{R}$ is a Lagrangian submanifold of (T^*X, ω_{can}).*

The function $f : X \to \mathbb{R}$ in the Example 2.2.2-(ii) is called a *generating function* for the Lagrangian Γ_{df}.

Let L be a Lagrangian in a symplectic manifold (M, ω). Then *the symplectic area* *class* of L is defined by

$$\omega_L : \pi_2(M, L) \to \mathbb{R},\ [u] \mapsto \int_{\mathbb{D}^2} u^*\omega$$

where $u : (\mathbb{D}^2, \partial\mathbb{D}^2) \to (M, L)$ is a smooth map representing a homotopy class $[u] \in \pi_2(M, L)$.

There are interesting obstructions to the Lagrangian embeddings n-manifolds into $2n$-symplectic manifolds in contrast to Whintney embedding theorem. The following proposition is just one example, for more examples reader may consult the section 9.2 of [15].

Proposition 2.2.3. *Among all closed orientable connected surfaces, only torus admits a Lagrangian embedding into* $(\mathbb{C}^2, \omega_0)$*.*

Proof. Normal bundle and tangent bundle of an orientable Lagrangian are isomorphic. Self-intersection number of an orientable Lagrangian L is $(-1)^{\frac{n(n-1)}{2}}\chi(L)$ where $\chi(L)$ is Euler characteristic of L. An orientable Lagrangian in $(\mathbb{C}^n, \omega_0)$ has self-intersection number zero. This implies an orientable Lagrangian L in $(\mathbb{C}^n, \omega_0)$ must have zero Euler characteristic. □

2.3 Hamiltonian, Lagrangian and Exact Lagrangian Isotopies

Let (M, ω) be a symplectic manifold and $H : M \to \mathbb{R}$ be a smooth function. Consider the equation $\iota_{X_H}\omega = dH$. This equation has a unique vector field solution X_H by nondegeneracy of ω. (Note that if H is time dependent then X_H will be time-dependent as well.) H is called *Hamiltonian funciton* and X_H is called the *Hamiltonian vector field* corresponding to H and the isotopy $\{\Phi_H^t : M \to M\}_{t\in\mathbb{R}}$ generated by X_H is called the *Hamiltonian flow* of X_H. Using Cartan magic formula and Lie derivative of the sypmlectic structure ω with respect to the vector field X_H, we can prove that a Hamiltonian flow $\{\Phi_H^t : M \to M\}_{t\in\mathbb{R}}$ preserves the symplectic structure, $(\Phi_H^t)^*\omega = \omega$.

Two Lagrangians L_0 and L_1 are said to be *Hamiltonian isotopic* if there exists a time-dependent Hamiltonian $H : M \times [0,1] \to \mathbb{R}$ such that $L_1 = \Phi_H^1(L_0)$ where $\{\Phi_H^t\}_{t\in\mathbb{R}}$ is the Hamiltonian flow induced by H.

Example 2.3.1. *We know that* $\mathrm{U}(2)$ *is path connected. Hence we can find a smooth path* $\gamma : [0,1] \to \mathrm{U}(2)$ *such that*

$$\gamma(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \gamma(1) = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{bmatrix}.$$

Consider the isotopy $\Upsilon : \mathbb{C}^2 \times [0,1] \to \mathbb{C}^2$ *given by*

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \underbrace{\begin{bmatrix} \gamma_{11}(t) & \gamma_{12}(t) \\ \gamma_{21}(t) & \gamma_{22}(t) \end{bmatrix}}_{=\gamma(t)} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.$$

For each $t \in [0,1]$ *the diffeomorphism* $\Upsilon^t : \mathbb{C}^2 \to \mathbb{C}^2$ *is a symplectomorphism since the matrix* $\gamma(t) \in \mathrm{U}(2) \subseteq \mathrm{Sp}(4)$*. By the famous Cartan magic formula and the definition of Lie derivative we have* $d\iota_{X_t}\omega = 0$*, where* X_t *is the time-dependent vector field generated by the isotopy* $\Upsilon : \mathbb{C}^2 \times [0,1] \to \mathbb{C}^2$*. Then the isotopy* $\Upsilon : \mathbb{C}^2 \times [0,1] \to \mathbb{C}^2$ *is a Hamiltonian isotopy, since every closed form is exact in* $\mathbb{C}^2$*.*

Proposition 2.3.2 ([1, 16])**.** *For any two embedded discs* $u_1, u_2 : \mathbb{D}^2 \to \mathbb{C}$ *bounding the same area there exists a Hamiltonian isotopy* $\Phi : \mathbb{C} \times [0,1] \to \mathbb{C}$ *such that* $\Phi^1(u_1(\mathbb{D}^2)) = u_2(\mathbb{D}^2)$ *and* $\Phi^t(0) = 0$ *for all* $t \in [0,1]$ *where* $\Phi^t = \Phi(\cdot, t)$*. Moreover, if* $u_1(\mathbb{D}^2)$ *and* $u_2(\mathbb{D}^2)$ *are symmetric with respect to origin of* $\mathbb{C}$ *then* $\Phi^t(u_1(\mathbb{D}^2))$ *is symmetric with respect to origin of* $\mathbb{C}$ *for all* $t \in [0,1]$*.* □

A smooth isotopy $\Phi : L\times[0,1] \to M$ of L is called a *Lagrangian isotopy* if each step of the isotopy is a Lagrangian in M. For a Lagrangian isotopy $\Phi : L \times [0,1] \to M$ we can find a family of one-forms $\{\alpha_t\}_{t\in[0,1]}$ on L such that $\Phi^*\omega = \alpha_t \wedge dt$ and $\iota_{\frac{\partial}{\partial t}}\alpha_t = 0$ since $\Phi^*\omega$ vanishes on L_t for all $t \in [0,1]$. Furthermore, α_t is closed for all $t \in [0,1]$ since $0 = d\Phi^*\omega = d\alpha_t \wedge dt$. A Lagrangian isotopy $\Phi : L \times [0,1] \to M$ is an *exact Lagrangian isotopy* if α_t is exact for all $t \in [0,1]$.

Proposition 2.3.3 ([18], [20]). *Let L be a closed manifold and $\Phi : L \times [0,1] \to M$ be a Lagrangian isotopy. Then the following are equivalent for Φ :*

i. *It is an exact Lagrangian isotopy.*

ii. *It can be extended to a Hamiltonian isotopy of M.*

iii. *It preserves the symplectic area class, in other words, for any smooth map $u : (\mathbb{D}^2, \partial\mathbb{D}^2) \to (M, L)$ representing a homotopy class $[u] \in \pi_2(M, L)$ we have*

$$\int_{\mathbb{D}^2} u^*\omega = \int_{\mathbb{D}^2} (\Phi^t(u))^*\omega.$$

□

2.4 Exact Lagrangian Submanifolds

A symplectic manifold (M, ω) is an *exact symplectic manifold* if the symplectic form ω is exact. A Lagrangian L in an exact symplectic manifold $(M, d\alpha)$ is an *exact Lagrangian* if $i_L^*\alpha$ is exact for the inclusion map $i_L : L \to M$. An immersion $i_N : N \to (M, \omega = d\alpha)$ is *Lagrangian* if $i_N^*\omega = 0$ and *exact Lagrangian* if $i_N^*\alpha$ is exact.

Let L be an exact Lagrangian in $(M, \omega = d\alpha)$ and $u : (\mathbb{D}^2, \partial\mathbb{D}^2) \to (M, L)$ be a smooth map. Then

$$\int_{\mathbb{D}^2} u^*\omega = \int_{\mathbb{D}^2} u^*d\alpha = \int_{\mathbb{D}^2} du^*\alpha = \int_{\mathbb{S}^1} u^*\alpha = \int_{u(\mathbb{S}^1)} i^*\alpha = \int_{u(\mathbb{S}^1)} df = 0\,. \quad (2.3)$$

Proposition 2.4.1 ([13]). *A closed exact Lagrangian L in $(\mathbb{C}^n, \omega_0)$ must bound a holomorphic discs with non-zero area.* □

The Proposition 2.4.1 and 2.3 implies that there does not exist a closed exact Lagrangian in $(\mathbb{C}^n, \omega_0)$. Since $\pi_1(\mathbb{S}^n)$ is trivial for $n \geq 2$, any one-form on $\mathbb{S}^n$ is exact. This shows that $\mathbb{S}^n$ does not admit a Lagrangian embedding into the symplectic manifold $(\mathbb{C}^n, \omega_0)$ for $n \geq 2$.

In the paper [27], H. Whitney described a general method to construct immersions of n-manifolds into $2n$-Euclidean space with transeversal, double and isolated self-intersection points. An immersion which has such self-intersection points is a *generic immersion* [19]. A *Lagrangian nodal sphere* in $\mathbb{C}^2$ is an immersed Lagrangian sphere which has only double transversal self-intersection points. [23].

2.5 Monotone Lagrangian Submanifolds

Let L be a Lagrangian in a symplectic manifold (M, ω) and $u : (\mathbb{D}^2, \partial\mathbb{D}^2) \to (M, L)$ be a map representing a homotopy class $[u] \in \pi_2(M, L)$. Then there exists a unique trivialization (up to homotopy) of the bundle $u^*TM \simeq \mathbb{D}^2 \times \mathbb{C}^n$ as a symplectic vector bundle [17]. Then *Maslov class* $\mu_L([u])$ of the disc $[u] \in \pi_2(M, L)$ is defined as Maslov index of $\partial\mathbb{D}^2$ after this trivialization. It follows from the definition that Maslov class induces a homomorphism $\mu_L : \pi_2(M, L) \to \mathbb{Z}$. Maslov class remain invariant under a Lagrangian isotopy of L [11].

A Lagrangian L in a symplectic manifold (M, ω) is called a *monotone Lagrangian* if there exist a real number $\kappa_L > 0$ such that for any homotopy class $[u] \in \pi_2(M, L)$ we have $\omega_L([u]) = \kappa_L \mu_L([u])$.

Monotonicity of Lagrangian submanifolds is preserved under Hamiltonian isotopies since both Maslov class and symplectic area are preserved under Hamiltonian isotopies.

Example 2.5.1. *Let $T = \mathbb{S}^1_{r_1} \times ... \times \mathbb{S}^1_{r_n}$ be a Lagrangian split torus in $(\mathbb{C}^n, \omega_0)$. Lagrangian torus T is monotone if and only if we have $r_1 = ... = r_n$. Because the basic generators of the $\pi_2(\mathbb{C}^n, T) \simeq \mathbb{Z}^n$ has Maslov class 2 and symplectic area πr_k^2 for all $k = 1, ..., n$. As a result, the split monotone Lagrangian torus $\mathbb{S}^1_r \times ... \times \mathbb{S}^1_r$ has monotonicity constant $\frac{\pi r^2}{2}$.*

A *Clifford torus* in $(\mathbb{C}^n, \omega_0)$ is a torus Hamiltonian isotopic to a split Lagrangian torus $\mathbb{S}^1_r \times ... \times \mathbb{S}^1_r$ for some $r > 0$ and it will be denoted by $\mathbb{T}^n_{Cl}$.

Proposition 2.5.2 ([13]). *If $\{\Phi^t : \mathbb{C}^2 \to \mathbb{C}^2\}_{t\in[0,1]}$ is a Hamiltonian isotopy, then* *for each point on the torus $\Phi^t(\mathbb{T}^2_{Cl})$ there exist at least two holomorphic discs whose boundaries are homologous to the cycles $\Phi^t(e_1)$ and $\Phi^t(e_2)$ where e_1 and e_2 are standard generators of the $H_1(\mathbb{T}^2_{Cl};\mathbb{Z})$.* □

2.6 The Chekanov Suspension

Consider the map

$$\vartheta_n : \mathbb{R}^n \times \mathbb{S}^1 \to \mathbb{R}^{n+1}, \quad (t_1, ..., t_n, \theta) \mapsto (e^{t_1}\cos\theta, e^{t_1}\sin\theta, t_2, ..., t_n)\,.$$

The map ϑ_n is a smooth embedding of $\mathbb{R}^n \times \mathbb{S}^1$ into $\mathbb{R}^{n+1}$ and image of ϑ_n is diffeomorphic to $(\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^{n-1}$. The map ϑ_n induces a symplectic embedding $\Theta_n : T^*(\mathbb{R}^n \times \mathbb{S}^1) \to T^*\mathbb{R}^{n+1}$ of the cotangent bundles where $\Theta_n = (\vartheta_n)_\#$. Since $T^*(\mathbb{R}^n \times \mathbb{S}^1)$ and $T^*\mathbb{R}^{n+1}$ are symplectomorphic to $\mathbb{R}^{2n} \times T^*\mathbb{S}^1$ and $\mathbb{R}^{2n+2}$ respectively, we can think of Θ_n as a symplectic embedding of $\mathbb{R}^{2n} \times T^*\mathbb{S}^1$ into $\mathbb{R}^{2n+2}$.

Let L be an arbitrary Lagrangian submanifold of $\mathbb{R}^{2n}$. Consider the Lagrangian submanifold $S_a = \{(\theta, \tau) \in T^*\mathbb{S}^1 \simeq \mathbb{S}^1 \times \mathbb{R} \mid \tau = a\}$ of $T^*\mathbb{S}^1$. The submanifold $L \times S_a$ is a Lagrangian in $\mathbb{R}^{2n} \times T^*\mathbb{S}^1$ since product of Lagrangian submanifolds is Lagrangian in product of symplectic manifolds. Symplectic embedding Θ_n gives a Lagrangian $\Theta_n(L \times S_a)$ in $\mathbb{R}^{2n+2}$. The Lagrangian $\Theta_n(L \times S_a)$ is called *Chekanov suspension of L at the level a* and it will be denoted as $\mathcal{C}_a(L)$.

For any circle S_a, the image $\Theta_n(S_a)$ will bound a disc with symplectic area $2\pi a$ and Maslov index of the circle $\Theta_n(S_a)$ will be zero since S_a does not bound a disc. So these show that if L is monotone Lagrangian in $\mathbb{R}^{2n}$ then $\mathcal{C}_0(L)$ is a monotone Lagrangian in $\mathbb{R}^{2n+2}$.

CHAPTER 3

THE CHEKANOV TORI IN $\mathbb{C}^2$

3.1 Description of a Chekanov Torus via Chekanov Suspension

Proposition 3.1.1 ([7])**.** *The following subset*

$$\mathbb{T}^2_{Ch} = \{((e^t + ise^{-t})\cos\theta, (e^t + ise^{-t})\sin\theta) \in \mathbb{C}^2 \mid s^2 + t^2 = 2r^2, \theta \in [0, 2\pi]\}$$

of $\mathbb{C}^2$ *is a monotone Lagrangian torus.*

Proof. Let L be a circle of radius $\sqrt{2}r > 0$ in $\mathbb{R}^2$ centered at the origin. Monotonicity constant of L is $\kappa_L = \pi r^2$. Then Chekanov suspension $\mathcal{C}_0(L)$ is a monotone Lagrangian in $\mathbb{R}^4 \simeq \mathbb{C}^2$, which is diffeomorphic to torus and its monotonicity constant is πr^2 as well. To find $\mathcal{C}_0(L)$ explicitly :

Consider the smooth embedding

$$\vartheta_1 : \mathbb{R} \times \mathbb{S}^1 \to \mathbb{R}^2, \quad (t, \theta) \mapsto (e^t \cos\theta, e^t \sin\theta) .$$

For the map ϑ_1, the dual of the differential $d\vartheta_1{}^*_{(t,\theta)} : T^*_{(t,\theta)}(\mathbb{R} \times \mathbb{S}^1) \to T^*_{\vartheta_1(t,\theta)}\mathbb{R}^2$ at the point (t, θ) is given by the matrix

$$A_{(t,\theta)} = \begin{bmatrix} e^t \cos\theta & e^t \sin\theta \\ -e^t \sin\theta & e^t \cos\theta \end{bmatrix}$$

for the bases $\{dt, d\theta\}$ and $\{dq_1, dq_2\}$. Then the symplectic embedding

$$\Theta_1 : T^*\mathbb{R} \times T^*\mathbb{S}^1 \to T^*\mathbb{R}^2 \simeq \mathbb{R}^4 , \quad (t, s, \theta, \tau) \mapsto (q_1, p_1, q_2, p_2)$$

is given by $(q_1, q_2) = \vartheta_1(t, \theta)$ and $[p_1\ p_2]^T = A^{-1}_{(t,\theta)}[s\ \tau]^T$ i.e. we have explicitly $\Theta_1(t, s, \theta, \tau) = (e^t \cos\theta, se^{-t}\cos\theta - \tau e^{-t}\sin\theta, e^t \sin\theta, se^{-t}\sin\theta + \tau e^{-t}\cos\theta)$.

The Chekanov suspension $\mathcal{C}_0(L)$ of L at level $\tau = 0$ is the following subset of $\mathbb{R}^4$:

$$\mathcal{C}_0(L) = \{(e^t \cos\theta, e^t \sin\theta, se^{-t}\cos\theta, se^{-t}\sin\theta) \in \mathbb{R}^4 \mid s^2+t^2 = 2r^2, \theta \in [0, 2\pi]\} .$$

If we identify $\mathbb{C}^2$ with $\mathbb{R}^4$ via the identification explained in example 2.2.1 , then we have

$$\mathcal{C}_0(L) = \{((e^t + ise^{-t})\cos\theta, (e^t + ise^{-t})\sin\theta) \in \mathbb{C}^2 \mid s^2 + t^2 = 2r^2, \theta \in [0, 2\pi]\}.$$

□

A *Chekanov torus* in $\mathbb{C}^2$ is a monotone Lagrangian torus that is Hamiltonian isotopic to a rescaling of the torus $\mathbb{T}^2_{Ch}$ in Proposition 3.1.1 .

3.2 Description of a Chekanov Torus via Conics in $\mathbb{C}^2$

Consider the map $G : \mathbb{C}^2 \to \mathbb{C}$ given by $G(z_1, z_2) = z_1 z_2$ and the Hamiltonian function $H : \mathbb{C}^2 \to \mathbb{R}$ given by $H(z_1, z_2) = \frac{1}{2}(|z_2|^2 - |z_1|^2)$. The fiber $G^{-1}(z_0)$ is topologically a cylinder if $z_0 \neq 0$ and a cone if $z_0 = 0$.

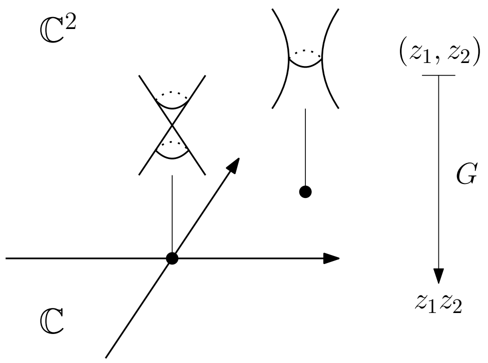

Figure 3.1: Topological model of the fibers of $G : \mathbb{C}^2 \to \mathbb{C}$.

Let $u : \mathbb{D}^2 \to \mathbb{C}$ be an embedded disc with the area $A_u > 0$ and the boundary $\gamma : \mathbb{S}^1 \to \mathbb{C}$. As a remark, instead of $\gamma(e^{is})$ we write $\gamma(s)$ for short. Now consider the following subset of $\mathbb{C}^2$:

$$L^\gamma_\beta = \{(z_1, z_2) \in \mathbb{C}^2 \mid H(z_1, z_2) = \beta,\ \beta \in \mathbb{R}\} \cap G^{-1}(\gamma(\mathbb{S}^1)) \tag{3.1}$$

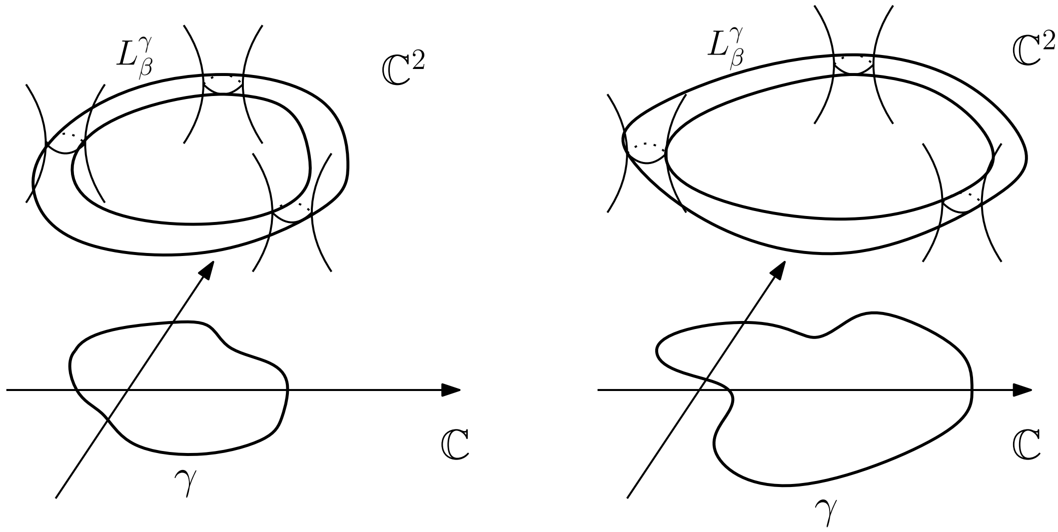

Figure 3.2: L_β^γ when β is not zero or γ does not pass through origin.

Proposition 3.2.1 ([11]). *L_β^γ is a Lagrangian torus if β is not zero or $\gamma(\mathbb{S}^1)$ does not pass through origin.*

Sketch of proof. Let $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$ and $\gamma = r_\gamma e^{i\theta_\gamma}$ be polar coordinates representations of z_1, z_2 and γ respectively. If $z_1, z_2 \in L_\beta^\gamma$, then we have the equalites $\gamma = r_\gamma e^{i\theta_\gamma} = z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$ and $2\beta = |z_2|^2 - |z_1|^2 = r_2^2 - r_1^2$. Then we have $r_\gamma^2 = r_1^2 r_2^2 = 2\beta r_1^2 + r_1^4$. As a result, we have

$$r_1 = \sqrt{-\beta + \sqrt{\beta^2 + r_\gamma^2}}, \quad r_2 = \sqrt{\beta + \sqrt{\beta^2 + r_\gamma^2}} \quad \text{and} \quad e^{i\theta_2} = e^{-i\theta_1} e^{i\theta_\gamma}\,.$$

If we let $g(r_\gamma) = \sqrt{-\beta + \sqrt{\beta^2 + r_\gamma^2}}$ and $h(r_\gamma) = \sqrt{\beta + \sqrt{\beta^2 + r_\gamma^2}}$, then the subset L_β^γ of $\mathbb{C}^2$ is given by

$$L_\beta^\gamma = \{(\pm g(r_{\gamma(s)}) e^{i\theta}, \pm h(r_{\gamma(s)}) e^{i\theta_{\gamma(s)}} e^{-i\theta}) \mid \gamma(s) = r_{\gamma(s)} e^{i\theta_{\gamma(s)}},\ s, \in [0, 2\pi], \theta \in [0, \pi]\}.$$

If $\beta \neq 0$ or γ does not pass through origin, thenL_β^γ is a torus. On L_β^γ we have

$$dz_1 = \pm\, e^{i\theta_{\gamma(s)}/2} e^{i\theta} \Big(\big(\partial_s g(r_{\gamma(s)}) + \frac{i}{2}(\partial_s \theta_{\gamma(s)})\big) ds + i g(r_{\gamma(s)}) d\theta \Big) \tag{3.2}$$

$$dz_2 = \pm\, e^{i\theta_{\gamma(s)}/2} e^{-i\theta} \Big(\big(\partial_s h(r_{\gamma(s)}) + \frac{i}{2}(\partial_s \theta_{\gamma(s)})\big) ds - i h(r_{\gamma(s)}) d\theta \Big) \tag{3.3}$$

then

$$\omega_0 = \frac{i}{2}\big(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2\big) = \Big(\big(\partial_s h(r_{\gamma(s)})\big) h(r_{\gamma(s)}) - \big(\partial_s g(r_{\gamma(s)})\big) g(r_{\gamma(s)}) \Big) ds \wedge d\theta.$$

This shows that $\omega_0|_{L^\gamma_\beta} \equiv 0$ since we have

$$\partial_s h(r_{\gamma(s)})\big)h(r_{\gamma(s)}) = \frac{r_{\gamma(s)}\partial_s r_{\gamma(s)}}{\sqrt{\beta^2 + r^2_{\gamma(s)}}} = \partial_s g(r_{\gamma(s)})\big)g(r_{\gamma(s)})\,.$$

□

Proposition 3.2.2 ([11])**.** *The subset L^γ_0 of $\mathbb{C}^2$ is a Lagrangian nodal sphere if $\gamma(\mathbb{S}^1)$ passes through origin.*

Sketch of proof. The set

$$L^\gamma_\beta = \{(\pm g(r_{\gamma(s)})e^{i\theta}, \pm h(r_{\gamma(s)})e^{i\theta_{\gamma(s)}}e^{-i\theta}) \mid \gamma(s) = r_{\gamma(s)}e^{i\theta_{\gamma(s)}},\ s \in [0, 2\pi], \theta \in [0, \pi]\}.$$

in the proof of Proposition 3.2.1 becomes

$$L^\gamma_0 = \{(\pm\sqrt{r_{\gamma(s)}}e^{i\theta}, \pm\sqrt{r_{\gamma(s)}}e^{i\theta_{\gamma(s)}}e^{-i\theta}) \mid \gamma(s) = r_{\gamma(s)}e^{i\theta_{\gamma(s)}},\ s \in [0, 2\pi], \theta \in [0, \pi]\}.$$

Since γ passes through origin, $\sqrt{r_{\gamma(s_0)}} = 0$ for some $s_0 \in [0, 2\pi]$. Hence the set L^γ_0 is an immersed Lagrangian sphere. Transversality of intersection follows from 3.2 and 3.3. □

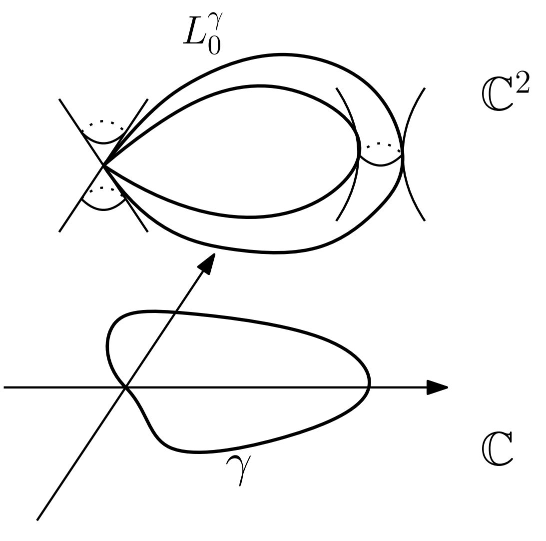

Figure 3.3: The subset L^γ_0 if γ passes through origin.

Proposition 3.2.3 ([11])**.** *The Lagrangian torus L^γ_0 is an exotic monotone torus in $\mathbb{C}^2$ if the disc $u : \mathbb{D}^2 \to \mathbb{C}$ does not meet the set $\Delta = \{a + ib \in \mathbb{C} \mid a \leq 0, b = 0\}$ through the origin of $\mathbb{C}$.*

Proof. If $\beta = 0$ then $H(z_1, z_2) = 0$ i.e. $|z_1| = |z_2|$. So we have $z_1 = z_2 e^{i2\theta}$ for some $\theta \in [0, \pi]$. Then for a point $(z_1, z_2) \in L_0^\gamma$ we have $\gamma(s) = G(z_1, z_2) = z_2^2 e^{i2\theta}$ for some $s \in [0, 2\pi]$. Then $(z_1, z_2) = \pm(\sqrt{\gamma(s)} e^{i\theta}, \sqrt{\gamma(s)} e^{-i\theta})$; as a result, the Lagrangian torus L_0^γ is given by the subset

$$L_0^\gamma = \{(\sqrt{\gamma(s)} e^{i\theta}, \sqrt{\gamma(s)} e^{-i\theta}) \in \mathbb{C}^2 \mid s, \theta \in [0, 2\pi]\}.$$

The two generators of $\pi_2(\mathbb{C}^2, L_0^\gamma) \simeq \mathbb{Z}^2$ have the following representative discs

$$\begin{aligned} u_1 &: \mathbb{D}^2 \to \mathbb{C}^2, \qquad w \mapsto (\sqrt{u(w)}, \sqrt{u(w)}) \\ u_2 &: \mathbb{D}^2 \to \mathbb{C}^2, \qquad \rho e^{i\theta} \mapsto (\rho e^{i\theta}, \rho e^{-i\theta}) \end{aligned} \quad .$$

First disc u_1 lies on the diagonal of the $\mathbb{C}^2$ and it's boundary $\sigma_1 : \mathbb{S}^1 \to \mathbb{C}^2$ is given by $s \mapsto (\sqrt{\gamma(s)}, \sqrt{\gamma(s)})$. Maslov class of u_1 is 2 since it is an embedded disc lying on a plane. It follows immediately from the definition of u_1 that by changing A_u we can adjust the symplectic area of u_1 as we want. Let's choose A_u so that u_1 has symplectic area $2\pi r^2$. (Note that, in this case each of the projections of the disc u_1 to the first and second coordinates of $\mathbb{C}^2$ has symplectic area πr^2 .)

For the disc u_2, its boundary $\sigma_2 : \mathbb{S}^1 \to \mathbb{C}^2$ induces the loop $\widetilde{\sigma}_2 : \mathbb{S}^1 \to \mathrm{U}(2)/\mathrm{O}(2)$ given by $\theta \mapsto \mathrm{diag}(e^{i\theta}, e^{-i\theta})$. We have $\det^2_{\mathbb{C}}(\mathrm{diag}(e^{i\theta}, e^{-i\theta})) = 1$. So Maslov class of u_2 is 0 . Symplectic area of u_1 is 0 since we have

$$\sigma_2^*(z_1 d\bar{z}_1 + z_2 d\bar{z}_2) = e^{i\theta} de^{-i\theta} + e^{-i\theta} de^{i\theta} = e^{i\theta}(-ie^{-i\theta})d\theta + e^{-i\theta}(ie^{i\theta})d\theta = 0\,.$$

This proves that L_0^γ is a monotone Lagrangian with monotonicity constant πr^2 .

The torus L_0^γ is the boundary of the solid torus foliated by the holomorphic discs

$$g^\theta : \mathbb{D}^2 \to \mathbb{C}^2, \qquad w \mapsto (\sqrt{u(w)} e^{i\theta}, \sqrt{u(w)} e^{-i\theta}), \qquad \theta \in [0, 2\pi) \ .$$

If we let $(\sqrt{u(w_1)} e^{i\theta_1}, \sqrt{u(w_1)} e^{-i\theta_1}) = (\sqrt{u(w_2)} e^{i\theta_2}, \sqrt{u(w_2)} e^{-i\theta_2})$ then we have $\sqrt{u(w_1)} = \sqrt{u(w_2)} e^{i\theta_2 - \theta_1}$ and $\sqrt{u(w_1)} = \sqrt{u(w_2)} e^{i\theta_1 - \theta_2}$. This implies either we have $\theta_1 - \theta_2 = 0$ or $\theta_1 - \theta_2 = \pi$ since u does not pass through origin. So we have either $\sqrt{u(w_1)} = \sqrt{u(w_2)}$ or $\sqrt{u(w_1)} = -\sqrt{u(w_2)}$. The latter is not possible because u is embedded and square root operation confines the image of the disc $\sqrt{u}$ to only half of the $\mathbb{C}$. So we have $\theta_1 = \theta_2$ and $u(w_1) = u(w_2)$ (hence $w_1 = w_2$ since u is embedded). This proves the holomorphic discs $g^\theta : \mathbb{D}^2 \to \mathbb{C}^2$ are disjoint.

Now we will prove that a holomorphic disc $g : \mathbb{D}^2 \to \mathbb{C}^2$ in $\mathbb{C}^2$ with the boundary lying on L_0^γ and symplectic area $2\pi r^2$ must be one of the discs $g^\theta : \mathbb{D}^2 \to \mathbb{C}^2$ of the holomoprhic foliation above. Consider the map $F = G \circ g \circ u^{-1} : u(\mathbb{D}^2) \to \mathbb{C}$. This map is holomorhic since each of the maps in the composition of F is holomorphic. Observe that for any point $\gamma(s)$ on the boundary of $u(\mathbb{D}^2)$ we have a point e^{it} on the boundary of $\mathbb{D}^2$ and this point is mapped to a point $(\sqrt{\gamma(s_t)}e^{i\theta_t}, \sqrt{\gamma(s_t)}e^{-i\theta_t})$ on L_0^γ. So we have $F = G \circ g \circ u^{-1}(\gamma(s)) = G(\sqrt{\gamma(s_t)}e^{i\theta_t}, \sqrt{\gamma(s_t)}e^{-i\theta_t}) = \gamma(s_t)$ i.e. F maps the boundary of $u(\mathbb{D}^2)$ to itself. Since $\int\limits_{\partial\mathbb{D}^2} g^*\lambda_0 = 2\pi r^2$, it can be shown that $F|_{\partial u(\mathbb{D}^2)=\gamma}$ has a degree greater than equal to 1. For the time being assume its degree is 1. Then F is a bi-holomorphic mapping of $u(\mathbb{D}^2)$ to itself. By a conformal change of variable we can have $F(w) = w$ or equivalently $G \circ g(w) = u(w)$ for $w \in \mathbb{D}^2$. If $g(w) = (g_1(w), g_2(w))$ then we have

$$G \circ g(w) = g_1(w)g_2(w) = u(w)\,.$$

Since $u(w)$ is non-zero we can define a function

$$\Phi(w) = \frac{g_1(w)}{g_2(w)} \quad \text{for} \quad w \in \mathbb{D}^2$$

which is a non-zero holomorphic function. Then by maximum modulus principle Φ is a constant. Now observe that, when $w \in \partial\mathbb{D}^2$ we have $|g_1(w)| = |g_1(w)|$ since $g(w)$ lies on L_0^γ. This implies Φ is equal to a constant $e^{i\theta_0}$ for some $e^{i\theta_0} \in [0, 2\pi)$. This says that $g_1(w) = g_1(w)e^{i\theta_0/2}$ which is equivalent to $g(w) = (\sqrt{u(w)}e^{i\theta}, \sqrt{u(w)}e^{-i\theta})$ where $\theta = \theta_0/2 \mod \pi$. This shows that $g = g^\theta$ for some $\theta \in [0, 2\pi)$. If the degree of $F|_{\partial u(\mathbb{D}^2)=\gamma}$ is greater than 1, with a similar approach we will find that $g : \mathbb{D}^2 \to \mathbb{C}^2$ will be multiple cover of one of the embedded discs $g^\theta : \mathbb{D}^2 \to \mathbb{C}^2$. However, since the area of the image of $g : \mathbb{D}^2 \to \mathbb{C}^2$ and $g^\theta : \mathbb{D}^2 \to \mathbb{C}^2$ are $2\pi r^2$, $g : \mathbb{D}^2 \to \mathbb{C}^2$ is not a multiple cover.

Theorem 2.5.2 of Gromov tells us that if $\mathbb{T}^2_{Cl}$ and L_0^γ were Hamiltonian isotopic then there should be two holomorphic discs corresponding the two generators of $\pi_2(\mathbb{C}^2, \mathbb{T}^2_{Cl})$. However, we proved that there is only one holomorphic disc of area $2\pi r^2$ passing through each point of L_0^γ. As a result, L_0^γ is exotic. □

Remark 3.2.4. *Proposition 3.2.3 remains true when we replace Δ by origin of $\mathbb{C}$,* *because $u : \mathbb{D}^2 \to \mathbb{C}$ will miss a branch cut of $\mathbb{C}$ in this case and by a conformal change of variable this branch cut can be mapped to Δ.*

By the Proposition 3.2.3, we obtained an infinite family of exotic monotone Lagrangian tori. Any two members of this family are Hamiltonian isotopic to each other up to rescaling. We will denote a member of this family by $\mathbb{T}^2_{EP}$. Equivalently, we can define $\mathbb{T}^2_{EP}$ as

$$\mathbb{T}^2_{EP} = \{(\sigma(s)e^{i\theta}, \sigma(s)e^{-i\theta}) \in \mathbb{C}^2 \mid s,\ \theta \in [0, 2\pi]\} \tag{3.4}$$

where $\sigma : \mathbb{S}^1 \to \mathbb{C}$ is an embedded circle bounding area πr^2 and whose image lies in the right half-plane $\{z \in \mathbb{C} \mid Re(z) > 0\}$.

3.3 Chekanov Tori in $\mathbb{C}^2$ are Exotic

Proposition 3.3.1 ([11, 12]). *The exotic torus $\mathbb{T}^2_{EP}$ is a Chekanov torus in $\mathbb{C}^2$.*

Proof. Without loss of generality let the $r = 1$ in the definition of $\mathbb{T}^2_{Ch}$. Then $\mathbb{T}^2_{Ch} = \{((e^t + ise^{-t})\cos\theta, (e^t + ise^{-t})\sin\theta) \in \mathbb{C}^2 \mid s^2 + t^2 = 2,\ \theta \in [0, 2\pi]\}$. If we let $(t, s) = (\sqrt{2}\sin\vartheta, \sqrt{2}\cos\vartheta)$ then we get $e^t + ise^{-t} = e^{\sqrt{2}\cos\vartheta} + i\sqrt{2}\sin\vartheta e^{-\sqrt{2}\cos\vartheta}$ which gives the embedded circle $\sigma : \mathbb{S}^1_{\sqrt{2}} \to \mathbb{C}$, $\sigma(\vartheta) = e^{\sqrt{2}\cos\vartheta} + i\sqrt{2}\sin\vartheta e^{-\sqrt{2}\cos\vartheta}$.

Consider $f : \mathbb{R}^2 \to \mathbb{C}$ given by $(x, y) \mapsto e^x + iye^{-x}$. The map f is a symplectomorphism since

$$dz \wedge d\bar{z} = (e^x dx + ie^{-x}dy - iye^{-x}dy) \wedge (e^x dx - ie^{-x}dy + iye^{-x}dx) = -2i dx \wedge dy.$$

Then embedded circle $\sigma : \mathbb{S}^1_{\sqrt{2}} \to \mathbb{C}$ bounds a disc of area 2π. Consider the Hamiltonian isotopy $\Upsilon : \mathbb{C}^2 \times [0, 1] \to \mathbb{C}^2$ given in Example 2.3.1. We have

$$\Upsilon^1(\mathbb{T}^2_{Ch}) = \{\frac{1}{\sqrt{2}}(\sigma(\vartheta)e^{i\theta}, \sigma(\vartheta)e^{-i\theta}) \in \mathbb{C}^2 \mid \vartheta,\ \theta \in [0, 2\pi]\}$$

which is $\mathbb{T}^2_{EP}$, see 3.4. This proves that $\mathbb{T}^2_{EP}$ is Hamiltonian isotopic to $\mathbb{T}^2_{Ch}$, in other words, $\mathbb{T}^2_{EP}$ is a Chekanov torus in $\mathbb{C}^2$. □

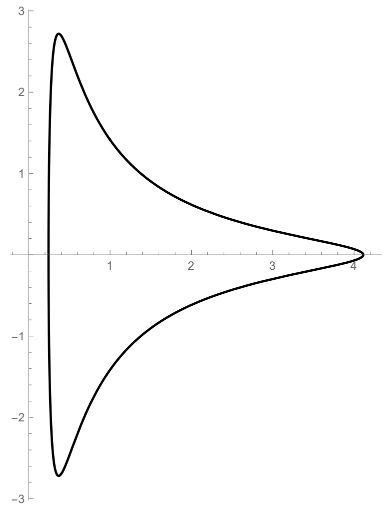

Figure 3.4: Trace of $\sigma : \mathbb{S}^1_{\sqrt{2}} \to \mathbb{C}$

3.4 Lagrangian Perturbations of a Lagrangian Nodal Sphere

Proposition 3.4.1 ([11]). *The subset L_0^γ of $\mathbb{C}^2$ is a Clifford torus if interior of the disc bounded by $\gamma(\mathbb{S}^1)$ contains the origin.*

Proof. If $\beta = 0$ then $H(z_1, z_2) = 0$ i.e. $|z_1| = |z_2|$. So we have $z_1 = z_2 e^{i2\theta}$ for some $\theta \in [0, \pi]$. Then for a point $(z_1, z_2) \in L_0^\gamma$ we have $\gamma(s) = G(z_1, z_2) = z_2^2 e^{i2\theta}$ for some $s \in [0, 2\pi]$. Then $(z_1, z_2) = \pm(\sqrt{\gamma(s)}e^{i\theta}, \sqrt{\gamma(s)}e^{-i\theta})$; as a result, the Lagrangian torus L_0^γ is given by the union of the subsets

$$\{\pm(\sqrt{\gamma(s)}e^{i\theta}, \sqrt{\gamma(s)}e^{-i\theta}) \in \mathbb{C}^2 \mid s \in [0, 2\pi], \theta \in [0, \pi]\}\,.$$

The union of curves $\sqrt{\gamma} \cup -\sqrt{\gamma}$ is a embedded circle, interior of the disc bounded by this embedded circle contains origin.

Let $\sigma : \mathbb{S}^1 \to \mathbb{C}$ be an embedded circle whose image is $\sqrt{\gamma} \cup -\sqrt{\gamma}$. Then by Proposition 2.3.2, there exists a Hamiltonian isotopy $\{\Phi^t : \mathbb{C} \to \mathbb{C}\}_{t\in[0,1]}$ which takes $\sigma(\mathbb{S}^1)$ to a round circle of radius r_0 centered at the origin of $\mathbb{C}$ and $\Phi^t(0) = 0$ for all $t \in [0, 1]$. Consider the map

$$\Psi : L_0^\gamma \times [0, 1] \to \mathbb{C}^2, \quad (\sigma(s)e^{i\theta}, \sigma(s)e^{-i\theta}, t) \mapsto (\Phi^t(\sigma(s))e^{i\theta}, \Phi^t(\sigma(s))e^{-i\theta})\,. \tag{3.5}$$

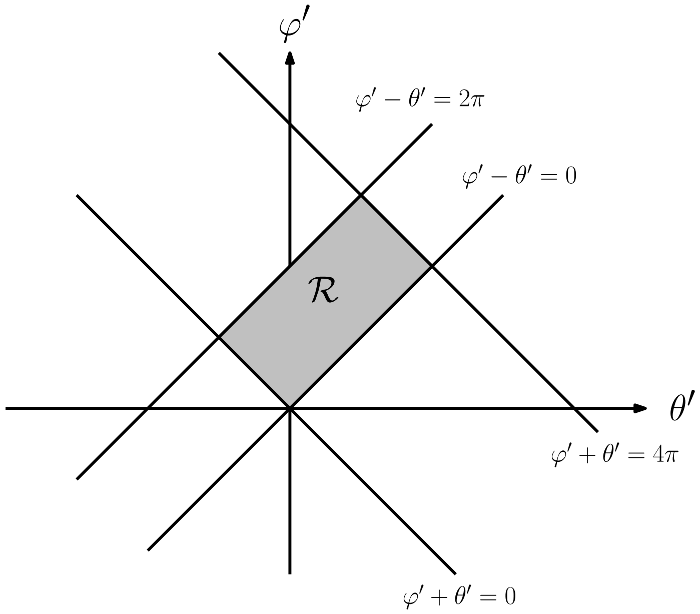

Figure 3.5: The region $\mathcal{R}$ which has area $4\pi^2$.

The subset $\Psi^t(L_0^\gamma)$ of $\mathbb{C}^2$ is a torus since $\Phi^t(\sigma(\mathbb{S}^1))$ does not pass through the origin of $\mathbb{C}$ for all $t \in [0,1]$. Then the map $\Psi : L_0^\gamma \times [0,1] \to \mathbb{C}^2$ is a Lagrangian isotopy since $\Psi^t(L_0^\gamma)$ is a Lagrangian for all $t \in [0,1]$. The following two embedded circles

$$\beta_1^t : \mathbb{S}^1 \to \mathbb{C}^2, \qquad e^{is} \mapsto (\Phi^t(\sigma(s)), \Phi^t(\sigma(s)))$$
$$\beta_2^t : \mathbb{S}^1 \to \mathbb{C}^2, \qquad e^{i\theta} \mapsto (\Phi^t(\sigma(0))e^{i\theta}, \Phi^t(\sigma(0))e^{-i\theta})$$

lie on the $\Psi^t(L_0^\gamma)$ and bound two embedded discs which correspond to the generators of $\pi_2(\mathbb{C}^2, \Psi^t(L_0^\gamma))$ for all $t \in [0,1]$. The discs bounded by $\beta_1^t : \mathbb{S}^1 \to \mathbb{C}^2$ has the same area for all $t \in [0,1]$ since $\{\Phi^t : \mathbb{C} \to \mathbb{C}\}_{t\in[0,1]}$ is a Hamiltonian isotopy. The discs bounded by $\beta_2^t : \mathbb{S}^1 \to \mathbb{C}^2$ has zero area since the one-form $z_1 d\bar{z}_1 + z_2 d\bar{z}_2$ vanishes on $\beta_2^t(\mathbb{S}^1)$. Hence by Proposition 2.3.3, the Lagrangian isotopy $\Psi : L_0^\gamma \times [0,1] \to \mathbb{C}^2$ is an exact Lagrangian isotopy and it can be extended to a Hamiltonian isotopy of $\mathbb{C}^2$. In other words, L_0^γ is Hamiltonian isotopic to

$$\begin{aligned}\tilde{L} =& \{(r_0 e^{i\varphi} e^{i\theta}, r_0 e^{i\varphi} e^{-i\theta}) \mid \varphi \in [0,2\pi], \theta \in [0,\pi]\} \\ =& \{(r_0 e^{i(\varphi+\theta)}, r_0 e^{i(\varphi-\theta)}) \mid \varphi \in [0,2\pi], \theta \in [0,\pi]\}.\end{aligned}$$

If we let $\varphi' = \varphi + \theta$ and $\theta' = \varphi - \theta$ then we have

$$\tilde{L} = \{(r_0 e^{i\varphi'}, r_0 e^{i\theta'}) \mid (\varphi', \theta') \in \mathcal{R} \subset \mathbb{R}^2\}$$

where $\mathcal{R} \subset \mathbb{R}^2$ given in the Figure 3.5. As a result we have

$$\tilde{L} = \{(r_0 e^{i\varphi'}, r_0 e^{i\theta'}) \mid \varphi, \theta \in [0, 2\pi]\}$$

which is a Clifford torus. □

Theorem 3.4.2 ([11])**.** *There exist Lagrangian perturbations of Lagrangian nodal spheres L_0^γ in Propostion 3.2.2 which are Clifford and Chekanov tori in* $(\mathbb{C}^2, \omega_0)$.

Proof. For a Lagrangian nodal sphere L_0^γ, the curve γ passes through origin. The curve γ can be deformed around origin in a small neighborhood to an embedded circle γ_δ in two ways. In one case, γ_δ does not pass through origin anymore and the disc bounded by γ_δ contains the origin. In the other case, γ_δ does not pass through origin anymore and the disc bounded by γ_δ does not contain the origin. Hence, proof follows by Proposition 3.2.3 and Proposition 3.4.1. □

Theorem 3.4.3 ([11])**.** *A Clifford torus and a Chekanov torus is Lagrangian isotopic.*

Proof. Let L_0^γ be a Clifford torus and $L_0^{\tilde{\gamma}}$ be a Chekanov torus where $\gamma, \tilde{\gamma} : \mathbb{S}^1 \to \mathbb{C}$ two embedded circles. Let $\{\gamma_t : \mathbb{S}^1 \to \mathbb{C}\}_{t \in [0,1]}$ be a smooth isotopy of embedded circles so that $\gamma_0 = \gamma$ and $\gamma_1 = \tilde{\gamma}$. Then the family $\{L_t^{\gamma_t}\}_{t \in [0,1]}$ induces a Lagrangian isotopy between the Clifford torus L_0^γ and the Chekanov torus $L_0^{\tilde{\gamma}}$. □

CHAPTER 4

THE WHITNEY IMMERSION AND ITS POLTEROVICH SURGERY

4.1 The Whitney Immersion of an n-sphere in $\mathbb{C}^n$

Consider the functions $g_{\pm} : [-1,1] \to \mathbb{R}$ given by $g_{\pm}(x) = \pm\frac{1}{3}(1-x^2)^{\frac{3}{2}}$. Differentials of these functions are $dg_{\pm} = g'_{\pm}(x)dx = \mp x(1-x^2)^{\frac{1}{2}}dx$. We will denote the graphs of differentials dg_+, dg_- by W^1_+, W^1_- respectively.

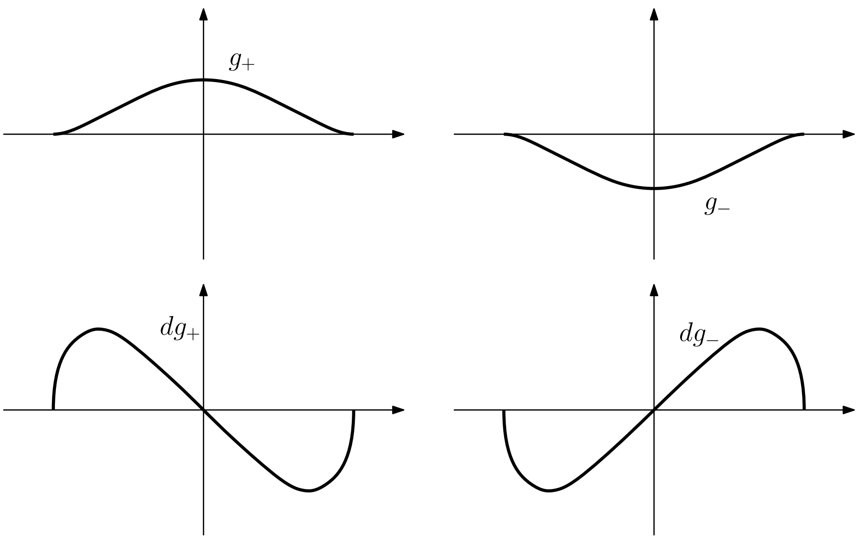

Figure 4.1: Graphs of $g_{\pm}$ and the differentials $dg_{\pm}$.

If we consider W^1_+ and W^1_- as a subset of $\mathbb{C}$, then we have

$$\begin{aligned} W^1_{\pm} =& \left\{x + i\, g'_{\pm}(x) \in \mathbb{C} \middle| x \in [-1,1]\right\} = \left\{x \mp ix(1-x^2)^{\frac{1}{2}} \in \mathbb{C} \middle| x \in [-1,1]\right\} \\ =& \left\{x + ixy \in \mathbb{C} \middle| (x,y) \in \mathbb{S}^1_{\pm}\right\}. \end{aligned}$$

The immersed circle $W^1 = W^1_+ \cup W^1_-$ is given by the set $\left\{x+ixy \in \mathbb{C} \middle| (x,y) \in \mathbb{S}^1\right\}$. It is the image of the immersion $\mathcal{W}_1 : \mathbb{S}^1 \to \mathbb{C}$ given by $(x,y) \mapsto x + ixy$.

In coordinates $(x, y) = (\sin\varphi, \cos\varphi)$ we have

$$\mathcal{W}_1(\sin\varphi, \cos\varphi) = \sin\varphi + \frac{i}{2}\sin 2\varphi \quad \text{where} \quad \varphi \in [0, 2\pi]\,. \tag{4.1}$$

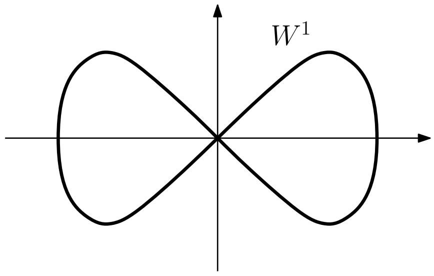

Figure 4.2: Immersed circle W^1

Higher dimensional analog of $\mathcal{W}_1$ is $\mathcal{W}_n : \mathbb{S}^n \to \mathbb{C}^n$ which is given by

$$\mathcal{W}_n(x_1, ..., x_n, x_{n+1}) = (x_1 + ix_1x_{n+1}, ..., x_n + ix_nx_{n+1}). \tag{4.2}$$

The immersion $\mathcal{W}_n : \mathbb{S}^n_1 \to \mathbb{C}^n$ is called *Whitney immersion* and its image will be denoted by W^n.

Proposition 4.1.1. *The immersed submanifold W^n is a Lagrangian nodal sphere in $(\mathbb{C}^n, \omega_0)$.*

Proof. Using functions $g_\pm$ define the following functions

$$f_\pm : \mathbb{D}^n_{\leq 1} \to \mathbb{R}\,, \qquad (x_1, ..., x_n) \mapsto g_\pm(\|(x_1, ..., x_n)\|)$$

where their differentials are given by

$$df_\pm = \mp \sum_{i=1}^{n} x_i(1 - \|(x_1, ..., x_n)\|^2)^{\frac{1}{2}} dx_i\,.$$

By method of generating functions the graphs $W^n_\pm$ of differentials $df_\pm$ are Lagrangians in contangent bundles $T^*\mathbb{D}^n_{\leq 1} \simeq \mathbb{D}^n_{\leq 1} \times \mathbb{R}^n \subseteq \mathbb{R}^{2n} \simeq \mathbb{C}^n$. If denote $(x_1, ..., x_n)$ as $\bar{x}_n$ for short then we have

$$\begin{aligned} W^n_\pm =& \big\{(x_1 \pm ix_1x_{n+1}, ..., x_n \pm ix_nx_{n+1}) \in \mathbb{C}^n \mid \bar{x}_n \in \mathbb{D}^n_{\leq 1},\, x_{n+1} = (1 - \|\bar{x}_n\|^2)^{\frac{1}{2}}\big\} \\ =& \big\{(x_1 \pm ix_1x_{n+1}, ..., x_n \pm ix_nx_{n+1}) \in \mathbb{C}^n \mid \bar{x}_{n+1} \in \mathbb{S}^n_\pm \subseteq \mathbb{R}^{n+1}\big\}\,. \end{aligned}$$

Then union of the Lagrangians $W^n_{\pm}$ is

$$W^n_+ \cup W^n_- = \left\{(x_1 + ix_1x_{n+1}, ..., x_n + ix_nx_{n+1}) \in \mathbb{C}^n \middle| \, \bar{x}_{n+1} \in \mathbb{S}^n_1 \subseteq \mathbb{R}^{n+1}\right\} = W^n.$$

So W^n is Lagrangian since it is smooth gluing of two Lagrangians. There is a self-intersection point at the origin $0 \in \mathbb{C}^n$ and we have $\mathcal{W}_n^{-1}(0) = \{(0, ..., 0, \pm 1)\}$. Having only one-self intersection point follows from a straightforward computation.

The tangent spaces at the self-intersection point which is at the origin $0 \in \mathbb{C}^n$ are given by:

$$V_1 = \mathrm{Span}_{\mathbb{R}} \left\{ \begin{bmatrix} 1-i \\ \vdots \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ \vdots \\ 1-i \end{bmatrix} \right\}$$

$$V_2 = \mathrm{Span}_{\mathbb{R}} \left\{ \begin{bmatrix} 1+i \\ \vdots \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ \vdots \\ 1+i \end{bmatrix} \right\}$$

which proves the transversality at the self-intersection point. □

4.2 The Polterovich Surgery

Let V_1 and V_2 be Lagrangian linear subspaces of $(\mathbb{C}^n, \omega_0)$ which intersect transversally at the origin. A *Lagrangian handle* joining V_1 and V_2 is the image of a a Lagrangian embedding $H : \mathbb{S}^{n-1} \times \mathbb{R} \to \mathbb{C}^n$ satisfying the properties:

- **i.** There exist two n-discs D_1, D_2 such that each contains origin of $\mathbb{C}^n$ and lies in V_1, V_2 respectively.
- **ii.** $H(\mathbb{S}^{n-1} \times [c, +\infty)) = V_1 \setminus D_1$ and $H(\mathbb{S}^{n-1} \times (-\infty, -c]) = V_2 \setminus D_2$ for some $c > 0$.

A Lagrangian handle Γ joining V_1 and V_2 is *positive* (or $sgn(\Gamma) = 1$) if the inclusion maps $V_1 \setminus D_1 \hookrightarrow \Gamma$ and $V_2 \setminus D_2 \hookrightarrow \Gamma$) induce the same orientation and *negative* (or $sgn(\Gamma) = -1$) otherwise.

Proposition 4.2.1 ([19]). *Let V_1 and V_2 be Lagrangian linear subspaces of* $(\mathbb{C}^n, \omega_0)$

which intersect transversally at the origin. There exist two Lagrangian handles Γ and $\tilde{\Gamma}$ joining V_1 and V_2 such that $sgn(\Gamma) = (-1)^n sgn(\tilde{\Gamma})$.

Proof. Let $h_\epsilon : \mathbb{R}_+ \to \mathbb{R}_+$ be a smooth function such that $h_\epsilon(t) = 1$ if $t \leq \epsilon/2$, $h_\epsilon(t) = 0$ if $t \geq \epsilon$ and $h'_\epsilon(t) < 0$ if $\epsilon/2 < t < \epsilon$.

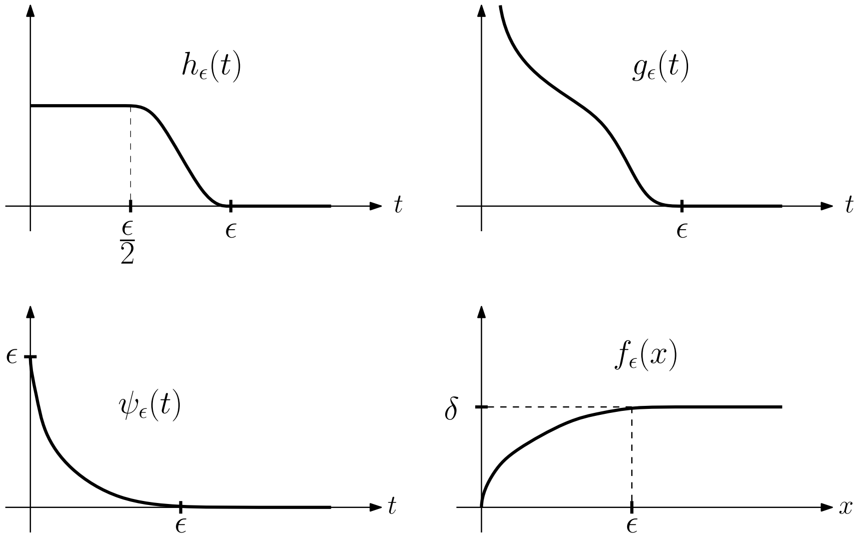

Figure 4.3: Functions $h_\epsilon(t)$, $g_\epsilon(t)$, $\psi_\epsilon(t)$ and $f_\epsilon(t)$.

Define $g_\epsilon : \mathbb{R}_+ \to \mathbb{R}_+$ by $g_\epsilon(t) = \dfrac{h(t)}{t}$. Observe that $g_\epsilon(t)$ is strictly decreasing when $t < \epsilon$ ($g'_\epsilon(t) =< 0$ if $t < \epsilon$) and $g_\epsilon(t) = 0$ if $t \geq \epsilon$.

Define $\psi_\epsilon(t) = h(t)(g|_{(0,\epsilon]})^{-1}(t)$ and using this function define $f_\epsilon(x) = \int_0^x \psi_\epsilon(t)dt$ for $x > 0$.

Then consider the following generating function

$$F_\epsilon^n : \mathbb{R}^n \setminus 0 \to \mathbb{R}, \quad (x_1, ..., x_n) \mapsto f_\epsilon(\|(x_1, ..., x_n)\|) \tag{4.3}$$

Graph of its differential $\Gamma_{dF_\epsilon^n}$ is a Lagrangian in $\mathbb{C}^n$ and it is smoothly tangent to $i\mathbb{R}^n$ along $i\mathbb{R}^n \cap \mathbb{S}_\epsilon^{2n-1}$. As a result, $\Gamma_{dF_\epsilon^n} \cup (i\mathbb{R}^n \setminus \mathbb{D}_{<\epsilon}^{2n})$ is a Lagrangian. The Lagrangian $\Gamma_{dF_\epsilon^n} \cup (i\mathbb{R}^n \setminus \mathbb{D}_{<\epsilon}^{2n})$ is diffeomorphic to $\mathbb{R}^n \setminus 0 \simeq \mathbb{S}^{n-1} \times \mathbb{R}$ since $\Gamma_{dF_\epsilon^n}$ is diffeomorphic to $\mathbb{R}^n \setminus 0 \simeq \mathbb{S}^{n-1} \times \mathbb{R}$. These show that $\Gamma_{dF_\epsilon^n} \cup (i\mathbb{R}^n \setminus \mathbb{D}_{<\epsilon}^{2n})$ is a Lagrangian handle joining $\mathbb{R}^n$ and $i\mathbb{R}^n$. This Lagrangian handle will be denoted by Γ_n. If we use the generating function $-F_\epsilon^n$ instead of F_ϵ^n, then we will obtain another

Lagrangian handle joining $\mathbb{R}^n$ and $i\mathbb{R}^n$. The Lagrangian handle obtained by $-F^n_\epsilon$ will be denoted by $\tilde{\Gamma}_n$. A quick observation shows that $sgn(\Gamma_n) = (-1)^n sgn(\tilde{\Gamma}_n)$.

By Lemma 2.1.2 we know that $\mathrm{Sp}(2n)$ acts transitively on pair of transverse linear Lagrangian subspaces of $\mathbb{C}^n$. Hence there exists a linear symplectomorphism Ψ of $\mathbb{C}^n$ which maps (V_1, V_2) to $(\mathbb{R}^n, i\mathbb{R}^n)$. Images of Lagrangian handles Γ_n and $\tilde{\Gamma}_n$ under this symplectomorphism give desired Lagrangian handles joining V_1 and V_2. □

Let N be an immersed Lagrangian submanifold of a $2n$ dimensional symplectic manifold (M, ω) where all self-intersection points of N are transversal, double and isolated. Let p be a self-intersection point of N, T_pM be tangent space of M at p and V_1, V_2 be two tangent spaces of N at the point p. By a version of Darboux theorem one can find a neighbourhood $\mathcal{U}$ of M around p and a symplectic embedding $I : \mathcal{U} \to \mathbb{C}^n$ satisfying:

i. $N \cap \mathcal{U} = D_1 \cup D_2$ where D_1 and D_2 are two n-discs such that $T_pD_1 = V_1$, $T_pD_2 = V_2$ and $D_1 \cap D_2 = \{p\}$,

ii. $I(D_1) \subset V_1$, $I(D_2) \subset V_2$ and $I(p) = 0$.

Attach a Lagrangian handle Γ joining V_1 and V_2 such that closure of $(\Gamma \setminus (V_1 \cup V_2))$ lies in $I(\mathcal{U})$ and glue $N \setminus \mathcal{U}$ with $I^{-1}(\Gamma)$. This procedure is called a *Polterovich surgery* of N at the self-intersection point p. The submanifold obtained after Polterovich surgery is an immersed Lagrangian submanifold of M if N has more than one self-intersection point, otherwise resulting submanifold is an embedded Lagrangian submanifold of M. If N is oriented immersed submanifold of M then the Lagrangian surgery is called *positive (negative)* if the Lagrangian handle Γ is positive (negative) with respect to the orientations of V_1 and V_2 induced from the orientation of N.

The Polterovich surgery procedure is the same when we replace immersed Lagrangian submanifold N of (M, ω) with a pair of transverse Lagrangian submanifolds L_1 and L_2 of (M, ω). In this case the Polterovich surgery is called *Lagrangian connect sum* of L_1 and L_2.

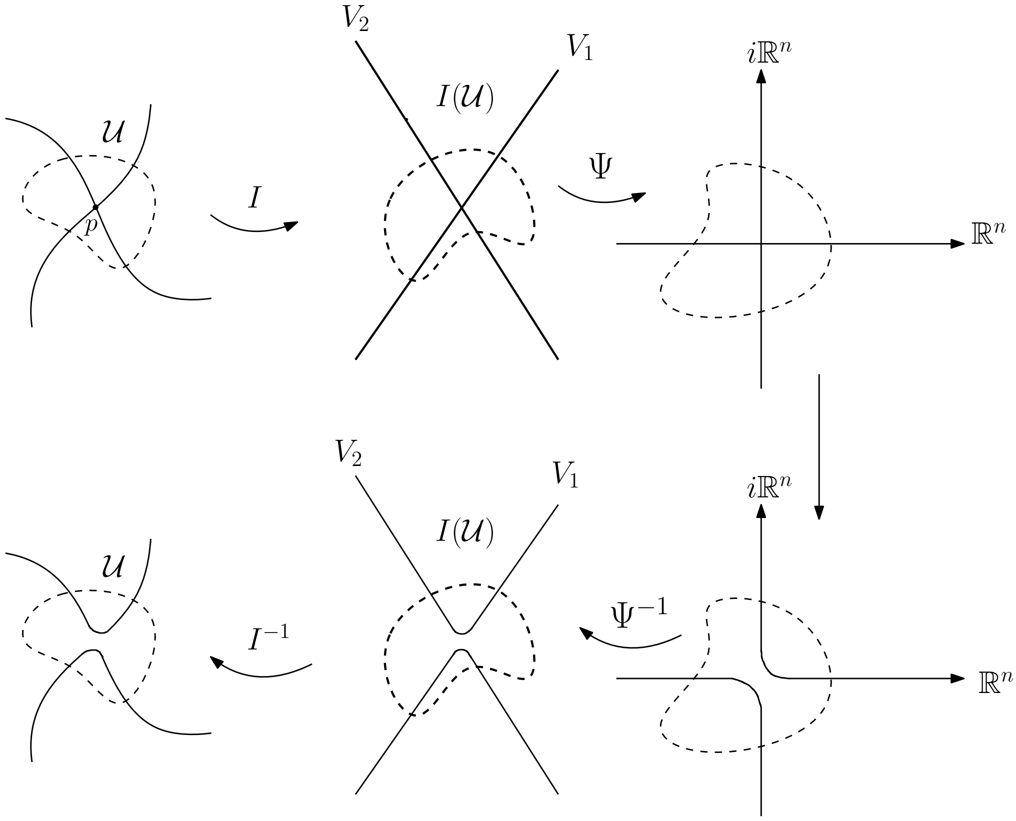

Figure 4.4: Model of the Polterovich Surgery

4.3 Polterovich Surgeries of a Lagrangian Nodal Sphere

Theorem 4.3.1 ([7,28]). *The two Lagrangian tori obtained by Polterovich surgeries of the Lagrangian nodal sphere W^2 given by the Whitney immersion in $\mathbb{C}^2$ are Clifford and Chekanov tori.*

Proof. Recall, $\mathcal{W}_2 : \mathbb{S}^2 \to \mathbb{C}^2$ is given by $(x_1, x_2, x_3) \mapsto (x_1 + ix_1x_3, x_2 + ix_2x_3)$. If we use spherical coordiantes $(x_1, x_2, x_3) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$ where $\varphi \in [0, \pi]$, $\theta \in [0, 2\pi]$ then we have

$$\mathcal{W}_2(x_1, x_2, x_3) = ((\sin\varphi + \frac{i}{2}\sin 2\varphi)\cos\theta, (\sin\varphi + \frac{i}{2}\sin 2\varphi)\sin\theta)\,.$$

If we let $\sigma(\varphi) = \sin\varphi + \frac{i}{2}\sin 2\varphi$ then the Lagrangian nodal sphere W^2 is given by

$$W^2 =\{(\sigma(\varphi)\cos\theta, \sigma(\varphi)\sin\theta) \mid \varphi \in [0, \pi],\ \theta \in [0, 2\pi]\} \tag{4.4}$$

$$=\{(\sigma(\varphi)\cos\theta, \sigma(\varphi)\sin\theta) \mid \varphi \in [0, 2\pi],\ \theta \in [0, \pi]\}\,. \tag{4.5}$$

From 4.1, we see that the image of the map $\sigma(\varphi) = \sin\varphi + \frac{i}{2}\sin 2\varphi$ is the immersed circle W^1 when $\varphi \in [0, 2\pi]$ and the right half of the immersed circle W^1 when $\varphi \in [0, \pi]$.

The tangent spaces of W^2 when $\varphi = 0$ and $\varphi = \pi$ are given by

$$V_1 = \mathrm{Span}_{\mathbb{R}}\left\{ \begin{bmatrix} 1-i \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1-i \end{bmatrix} \right\} \quad \text{and} \quad V_2 = \mathrm{Span}_{\mathbb{R}}\left\{ \begin{bmatrix} 1+i \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1+i \end{bmatrix} \right\}$$

respectively. There exists a neighborhood $\mathcal{U}$ of $0 \in \mathbb{C}^2$ and a symplectic embedding $I : \mathcal{U} \to \mathbb{C}^2$ such that I maps $\mathcal{U} \cap W^2 = D_1 \cup \mathbb{D}_2$ to $V_1 \cup V_2$ where D_1, D_2 are 2-discs. The matrix

$$A = \mathrm{diag}(e^{\frac{\pi i}{4}}, e^{\frac{\pi i}{4}}) \in \mathrm{Sp}(4)$$

gives a symplectomorphism which maps V_1 to $\mathbb{R}^2$ and V_2 to $i\mathbb{R}^2$. We perform Polterovich surgeries by attaching the Lagrangian handles Γ_2, $\tilde{\Gamma}_2$ joining $\mathbb{R}^2$ and $i\mathbb{R}^2$ which are described in the proof of Proposition 4.2.1.

The Lagrangian handles Γ_2, $\tilde{\Gamma}_2$ are described by the generating functions $\pm F^2_\epsilon$. The differentials of $\pm F^2_\epsilon$ are explicitly calculated as follows :

$$d(\pm F^2_\epsilon) = \pm df_\epsilon(\|(x_1, x_2)\|) = \pm\frac{\psi_\epsilon(\|(x_1, x_2)\|)}{(\|(x_1, x_2)\|)}(x_1 dx_1 + x_2 dx_2)$$

If we let $\bar{x} = (x_1, x_2)$ then we have

$$\Gamma_2 \cap \mathbb{D}^4_{\leq\epsilon} = \{(x_1 + i\frac{\psi_\epsilon(\|\bar{x}\|)}{(\|\bar{x}\|)}x_1, x_2 + i\frac{\psi_\epsilon(\|\bar{x}\|)}{(\|\bar{x}\|)}x_2) \mid \bar{x} \in \mathbb{R}^2 \cap \mathbb{D}^2_{\leq\epsilon}\} \tag{4.6}$$

$$\tilde{\Gamma}_2 \cap \mathbb{D}^4_{\leq\epsilon} = \{(x_1 - i\frac{\psi_\epsilon(\|\bar{x}\|)}{(\|\bar{x}\|)}x_1, x_2 - i\frac{\psi_\epsilon(\|\bar{x}\|)}{(\|\bar{x}\|)}x_2) \mid \bar{x} \in \mathbb{R}^2 \cap \mathbb{D}^2_{\leq\epsilon}\}\,. \tag{4.7}$$

In polar coordinates $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, the sets 4.6 and 4.7 become

$$\Gamma_2 \cap \mathbb{D}^4_{\leq\epsilon} = \{((r + i\psi_\epsilon(r))\cos\theta, (r + i\psi_\epsilon(r))\sin\theta) \mid r \in [0, \epsilon], \theta \in [0, 2\pi]\} \tag{4.8}$$

$$\tilde{\Gamma}_2 \cap \mathbb{D}^4_{\leq\epsilon} = \{((r - i\psi_\epsilon(r))\cos\theta, (r - i\psi_\epsilon(r))\sin\theta) \mid r \in [0, \epsilon], \theta \in [0, 2\pi]\}. \tag{4.9}$$

Comparing the formula 4.4 with the formulas 4.8 and 4.9 shows that two Lagrangian surgeries of W^2 obtained by Lagrangian handles Γ_2 and $\tilde{\Gamma}_2$ only affect the part $\sigma(\varphi)$ of the equations 5.1 and 5.2, in other words, two Lagrangian surgeries of W^2 are obtained from two Lagrangian surgeries of W^1 followed by a rotation.

The surgery obtained by Γ_1 on W^1 results in two disjoint embedded circles. Parametrize the piece lies in the right half plane by the embedded circle $\sigma_1 : \mathbb{S}^1 \to \mathbb{C}$, then $\sigma_1 \cup -\sigma_1$ gives the union of these two embedded circles. Then the surgery of W^2 obtained is

$$\begin{aligned} W_1^2 =& \{(((\sigma_1 \cup -\sigma_1)(\varphi))\cos\theta, ((\sigma_1 \cup -\sigma_1)(\varphi))\sin\theta) \mid \varphi \in [0, 2\pi], \theta \in [0, \pi]\} \\ =& \{(\sigma_1(\varphi)\cos\theta, \sigma_1(\varphi)\sin\theta) \mid \varphi \in [0, 2\pi], \theta \in [0, 2\pi]\} \end{aligned}$$

Consider the Hamiltonian isotopy $\Upsilon : \mathbb{C}^2 \times [0, 1] \to \mathbb{C}^2$ given in Example 2.3.1.

We have

$$\Upsilon^1({W_1}^2) = \{\frac{1}{\sqrt{2}}(\sigma_1(\varphi)e^{i\theta}, \sigma_1(\varphi)e^{-i\theta}) \mid \varphi \in [0, 2\pi], \theta \in [0, 2\pi]\}$$

which is a Chekanov Torus.

The surgery obtained by $\tilde{\Gamma}_1$ on W^1 results in an embedded circle whose interior contains the origin and let $\sigma_2 : \mathbb{S}^1 \to \mathbb{C}$ be a parametrization for this embedded circle. Then the surgery of W^2 obtained is

$$W_2^2 = \{(\sigma_2(\varphi)\cos\theta, \sigma_2(\varphi)\sin\theta) \mid \varphi \in [0, 2\pi], \theta \in [0, \pi]\}$$

Then there exists a Hamiltonian isotopy $\{\Phi^t : \mathbb{C} \to \mathbb{C}\}_{t\in[0,1]}$ which takes the image of $\sigma_2(\mathbb{S}^1)$ to the round circle given by $\tilde{\sigma}_2(\varphi) = r_{\sigma_2}e^{i\varphi}$ where $2\pi r_{\sigma_2}^2$ is the area of the embedded disc by $\sigma_2(\mathbb{S}^1)$, $\varphi \in [0, 2\pi]$ and $\Phi^t(0) = 0$ for all $t \in [0, 1]$. Consider the map $\Psi : W_2^2 \times [0, 1] \to \mathbb{C}^2$ given by

$$(\sigma_2(\varphi)\cos\theta, \sigma_2(\varphi)\sin\theta, t) \mapsto (\Phi^t(\sigma(\varphi))\cos\theta, \Phi^t(\sigma(\varphi))\sin\theta).$$

The subset $\Psi^t(W_2^2)$ of $\mathbb{C}^2$ is a torus since $\Phi^t(\sigma_2(\mathbb{S}^1))$ does not pass through the origin of $\mathbb{C}$ for all $t \in [0, 1]$. Then the map $\Psi : W_2^2 \times [0, 1] \to \mathbb{C}^2$ is a Lagrangian isotopy since $\Psi^t(W_2^2)$ is a Lagrangian for all $t \in [0, 1]$. The following two embedded circles

$$\begin{aligned} \beta_1^t : \mathbb{S}^1 \to \mathbb{C}^2, \qquad & e^{i\varphi} \mapsto (\Phi^t(\sigma(\varphi)), \Phi^t(\sigma(\varphi))) \\ \beta_2^t : \mathbb{S}^1 \to \mathbb{C}^2, \qquad & e^{i\theta} \mapsto (\Phi^t(\sigma(0))\cos\theta, \Phi^t(\sigma(0))\sin\theta) \end{aligned}$$

lie on the $\Psi^t(W_2^2)$ and bound two embedded discs which correspond to the generators of $\pi_2(\mathbb{C}^2, \Psi^t(W_2^2))$ for all $t \in [0, 1]$. The discs bounded by $\beta_1^t : \mathbb{S}^1 \to \mathbb{C}^2$

has the same area for all $t \in [0,1]$ since $\{\Phi^t : \mathbb{C} \to \mathbb{C}\}_{t\in[0,1]}$ is a Hamiltonian isotopy. The discs bounded by $\beta_2^t : \mathbb{S}^1 \to \mathbb{C}^2$ has zero area since the one-form $z_1 d\bar{z}_1 + z_2 d\bar{z}_2$ vanishes on $\beta_2^t(\mathbb{S}^1)$. Hence by Proposition 2.3.3, the Lagrangian isotopy $\Psi : W_2^2 \times [0,1] \to \mathbb{C}^2$ is an exact Lagrangian isotopy and it can be extended to a Hamiltonian isotopy of $\mathbb{C}^2$. In other words, W_2^2 is Hamiltonian isotopic to

$$\tilde{W}^2 = \{(r_{\sigma_2} e^{i\varphi} \cos\theta, r_{\sigma_2} e^{i\varphi} \sin\theta) \mid \varphi \in [0, 2\pi], \theta \in [0,\pi]\}\,.$$

By the Hamiltonian isotopy $\Upsilon : \mathbb{C}^2 \times [0,1] \to \mathbb{C}^2$ we have

$$\Upsilon^1(\tilde{W}^2) = \{\frac{r_{\sigma_2}}{\sqrt{2}}(e^{i(\varphi+\theta)}, e^{i(\varphi-\theta)}) \mid \varphi \in [0,2\pi], \theta \in [0,\pi]\}.$$

If we let $\varphi' = \varphi + \theta$ and $\theta' = \varphi - \theta$ then

$$\Upsilon^1(\tilde{W}^2) = \{\frac{r_{\sigma_2}}{\sqrt{2}}(e^{i\varphi'}, e^{i\theta'}) \mid (\varphi', \theta') \in \mathcal{R} \subset \mathbb{R}^2\}$$

where $\mathcal{R}$ as in Figure 3.5. As a result we have

$$\Upsilon^1(\tilde{W}^2) = \{\frac{r_{\sigma_2}}{\sqrt{2}}(e^{i\varphi'}, e^{i\theta'}) \mid \varphi' \in [0,2\pi], \theta' \in [0,2\pi]\}$$

which is a Clifford torus. □

Remark 4.3.2. *A Lagrangian nodal sphere in $\mathbb{C}^2$ can have only one self-intersection point. Otherwise, we could embed an orientable connected Lagrangian surface more of genus more than one.*

Remark 4.3.3. *The surgeries of A Lagrangian nodal sphere in $\mathbb{C}^2$ are always positive. Otherwise, by performing negative Polterovich surgery one could obtain Lagrangian embedding of a Klein bottle into $\mathbb{C}^2$ whose impossibility is proved in [22].*

CHAPTER 5

THE METHOD OF INTEGRABLE HAMILTONIAN SYSTEMS

5.1 An Example of Integrable Hamiltonian Systems

An *integrable Hamiltonian system* is a pair which consists of a $2n$-dimensional symplectic manifold (M, ω) and a set of real valued smooth functions $\{f_1, ..., f_n\}$ on M with properties:

i. The set differentials $\{df_1, ..., df_n\}$ are almost everywhere (except a set of zero measure) linearly independent on M,

ii. The set of Hamiltonian vector fields $\{X_{f_j} \mid \iota_{X_{f_j}}\omega = f_j,\ j = 1, ..., n\}$ satisfies the equality $\omega(X_{f_j}, X_{f_k})$ for all $j, k = 1, ..., n$.

Proposition 5.1.1 ([2,4])**.** *Let $(M, \omega, \{f_1, ..., f_n\})$ be an integrable Hamiltonian system and $F : M \to \mathbb{R}^n$ be the smooth function given by $F = (f_1, ..., f_n)$. If $c \in \mathbb{R}^n$ is a regular value of $F : M \to \mathbb{R}^n$ then the fiber $F^{-1}(c)$ is a Lagrangian diffeomorphic to $\mathbb{T}^k \times \mathbb{R}^{n-k}$ and hence compact connected regular fibers are diffeomorphic to $\mathbb{T}^n$.*

Consider the following functions :

$$G : \mathbb{R}^4 \to \mathbb{R} \quad G(q_1, p_1, q_2, p_2) = (p_1^2 + p_2^2) - (q_1^2 + q_2^2) + (q_1^2 + q_2^2)^2\,, \tag{5.1}$$

$$H : \mathbb{R}^4 \to \mathbb{R} \quad H(q_1, p_1, q_2, p_2) = p_2 q_1 - p_1 q_2\,. \tag{5.2}$$

Proposition 5.1.2 ([28])**.** *The triple $(\mathbb{R}^4, \omega_0, \{G, H\})$ is an integrable system.*

Proof. The differentials of functions $G, H : \mathbb{R}^4 \to \mathbb{R}$ are given by

$$dG = 2q_1(2q_1^2 + 2q_2^2 - 1)dq_1 + 2q_2(2q_1^2 + 2q_2^2 - 1)dq_2 + 2p_1 dp_1 + 2p_2 dp_2$$
$$dH = p_2 dq_1 - p_1 dq_2 - q_2 dp_1 + q_1 dp_2\,.$$

The differentials dG and dH are linearly dependent if and only if one of the differentials dG and dH is 0 or $dH = \frac{\alpha}{2} dG$ for some non zero real number α.

The differential $dG = 0$ if and only if $(q_1, p_1, q_2, p_2) = (0, 0, 0, 0)$ or $2q_1^2 + 2q_2^2 = 1$, $(p_1, p_2) = (0, 0)$. The differential $dH = 0$ if and only if $(q_1, p_1, q_2, p_2) = (0, 0, 0, 0)$. Now let $dH = \frac{\alpha}{2} dG$ for some non zero real number α. Then we get the equations

$$p_2 = \alpha q_1(2q_1^2 + 2q_2^2 - 1),\ p_1 = -\alpha q_2(2q_1^2 + 2q_2^2 - 1),\ q_2 = -\alpha p_1,\ q_1 = \alpha p_2\,.$$

These equations have the following solutions:

i. $p_1 \neq 0,\ p_2 = \pm\frac{\sqrt{-2\alpha^4 p_1^2 + \alpha^2 + 1}}{\sqrt{2}\alpha^2},\ q_1 = \alpha p_2, q_2 = -\alpha p_1$
ii. $p_1 = 0,\ p_2 = \pm\frac{\sqrt{\alpha^2+1}}{\sqrt{2}\alpha^2},\ q_1 = \alpha p_2, q_2 = 0$
iii. $(q_1, p_1, q_2, p_2) = (0, 0, 0, 0)\,.$

Each of these cases yields either an immersed or an embedded submanifolds of $\mathbb{R}^4$ which has dimension≤ 3. Hence, the set of points where dG and dH are linearly dependent is measure zero in $\mathbb{R}^4$.

Let

$$X_H = A_H \partial_{q_1} + B_H \partial_{q_2} + C_H \partial_{p_1} + D_H \partial_{p_2}$$

$$X_G = A_G \partial_{q_1} + B_G \partial_{q_2} + C_G \partial_{p_1} + D_G \partial_{p_2}$$

be the vector fields such that $\iota_{X_G}\omega_0 = dG$ and $\iota_{X_H}\omega_0 = dH$. Then

$$\begin{aligned}\iota_{X_H}\omega_0 =&(dq_1 \wedge dp_1 + dq_2 \wedge dp_2)(X_H, \cdot)\\ =&dq_1(X_H)dp_1 - dp_1(X_H)dq_1 + dq_2(X_H)dp_2 - dp_2(X_H)dq_2\\ =&A_H dp_1 - C_H dq_1 + B_H dp_2 - D_H dq_2\end{aligned}$$

and similarly we have,

$$\iota_{X_G}\omega_0 = A_G dp_1 - C_G dq_1 + B_G dp_2 - D_G dq_2\,.$$

So

$$\begin{aligned}X_H =& - q_2 \partial_{q_1} + q_1 \partial_{q_2} - p_2 \partial_{p_1} + p_1 \partial_{p_2}\\ X_G =& 2p_1 \partial_{q_1} + 2p_2 \partial_{q_2} - 2q_1(2q_1^2 + 2q_2^2 - 1)\partial_{p_1} - 2q_2(2q_1^2 + 2q_2^2 - 1)\partial_{p_2}\,.\end{aligned}$$

As a result we have

$$\begin{aligned}\omega_0(X_H, X_G) =& dH(X_G) = -dG(X_H)\\ =& p_2 2p_1 - p_1 2p_2 + q_2 2q_1(2q_1^2 + 2q_2^2 - 1) - q_1 2q_2(2q_1^2 + 2q_2^2 - 1) = 0\end{aligned}$$

□

5.2 Lagrangian Perturbations through the Hamiltonian System

Let $F : \mathbb{R}^4 \to \mathbb{R}^2$ be the function defined by $F = (G, H)$ where $G, H : \mathbb{R}^2 \to \mathbb{R}$ are the functions given by the formulas 5.1 and 5.2. Define the following subset of $\mathbb{R}^2$:

$$\mathcal{B}_F = \{(a, \pm\frac{1}{3\sqrt{3}}(1 + 6a + \sqrt{1 + 3a})(1 + \sqrt{1 + 3a})) \in \mathbb{R}^2 \,|\, a \geq -\frac{1}{4}\}\,. \quad (5.3)$$

Theorem 5.2.1. *Range of the function $F : \mathbb{R}^4 \to \mathbb{R}^2$ is the region in $\mathbb{R}^2$ bounded by $\mathcal{B}_F$ and containing origin. The set of critical values of $F : \mathbb{R}^4 \to \mathbb{R}^2$ is the set $\mathcal{B}_F \cup \{(0, 0)\}$.*

Proof. If we use polar coordinates

$$p_1 = r_p \cos\theta_p \quad p_2 = r_p \sin\theta_p \qquad r_p \geq 0,\ \theta_p \in [0, 2\pi] \quad (5.4)$$

$$q_1 = r_q \cos\theta_q \quad q_2 = r_q \sin\theta_q \qquad r_q \geq 0,\ \theta_p \in [0, 2\pi] \quad (5.5)$$

then we have $G = r_p^2 - r_q^2 + r_q^4$ and $H = r_p r_q \sin(\theta_p - \theta_q)$. We see that H satisfies the inequality $-|r_p r_q| \leq H \leq |r_p r_q|$ and it takes every value in this interval and it achieves the boundary values when we have $\sin(\theta_p - \theta_q) = \pm 1$. If we let $G = a$, then we have $r_p^2 = r_q^2 - r_q^4 + a$ and as a result $H^2 = (r_q^2 - r_q^4 + a) r_q^2 \sin^2(\theta_p - \theta_q)$. If we maximize the function $h : \mathbb{R}_+ \to \mathbb{R}$ given by $h(r_q) = (r_q^2 - r_q^4 + a) r_q^2$ we see that it gets its maximum value $\frac{1}{27}(1 + 6a + \sqrt{1 + 3a})(1 + \sqrt{1 + 3a})$ at $r_q = \frac{\sqrt{1+\sqrt{1+3a}}}{\sqrt{3}}$. When $G = a$ and $r_q = \frac{\sqrt{1+\sqrt{1+3a}}}{\sqrt{3}}$, we have $r_p = \frac{1}{3}\sqrt{1 + 6a + \sqrt{1 + 3a}}$. Hence H has its maximum $M_a = \frac{1}{3\sqrt{3}}(1 + 6a + \sqrt{1 + 3a})(1 + \sqrt{1 + 3a})$ and minimum $-M_a = -\frac{1}{3\sqrt{3}}(1 + 6a + \sqrt{1 + 3a})(1 + \sqrt{1 + 3a})$ when $G = a$. This proves the first assertion of the theorem.

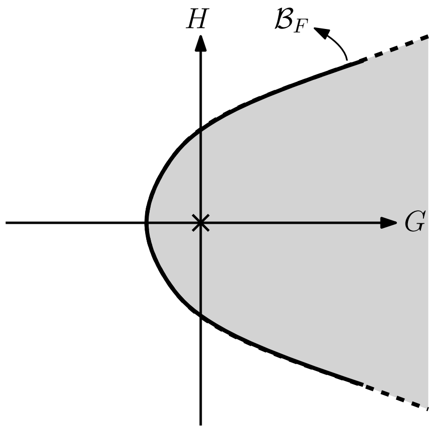

Figure 5.1: Sketch of range of the function $F : \mathbb{R}^4 \to \mathbb{R}^2$.

Critical points of F are the points where the differential dF has rank< 2. In the proof of Proposition 5.1.2 , we found the points where the differentials dG and dH are linearly dependent. These are exactly the critical points of F. When inserted in F we see that set of critical values of F is

$$\{(\frac{3+2\alpha^2-\alpha^4}{4\alpha^4}, \frac{\alpha^2+1}{2\alpha^3}) \in \mathbb{R}^2 \,|\, \alpha \in \mathbb{R} \setminus \{0\}\} \cup \{(0,0), (-\frac{1}{4}, 0)\}\,.$$

If we let $a = \frac{3+2\alpha^2-\alpha^4}{4\alpha^4}$ then we get $M_a = \frac{\alpha^2+1}{2\alpha^3}$ when $\alpha > 0$ and $M_a = -\frac{\alpha^2+1}{2\alpha^3}$ when $\alpha < 0$. If we let $a = -\frac{1}{4}$ then $M_{-\frac{1}{4}} = 0$. Hence the set of critical values of F is the set $\mathcal{B}_F \cup \{(0,0)\}$ which proves the second assertion of the theorem. □

Proposition 5.2.2 ([28])**.** *If $(a, b) \in \mathbb{R}^2$ lies in the image of F then the fibers $F^{-1}(a, b)$ is*

- ***i.*** *A circle if $(a, b) \in \mathcal{B}_F$,*
- ***ii.*** *A Chekanov torus if $-\frac{1}{4} < a < 0$ and $b = 0$,*
- ***iii.*** *The immersed Lagrangian sphere W^2 if $(a, b) = (0, 0)$,*
- ***iv.*** *A Clifford Torus if $a > 0$ and $b = 0$,*
- ***v.*** *A non-monotone Lagrangian torus if otherwise.*

Proposition 5.2.2 has two immediate important corollaries:

Theorem 5.2.3 ([28])**.** *There exist Lagrangian perturbations of the Lagrangian nodal sphere $W^2 = F^{-1}(0, 0)$, which are Clifford and Chekanov tori.*

Proof. If we perturb the Lagrangian nodal sphere $W^2 = F^{-1}(0,0)$ as $F^{-1}(\epsilon,0)$, then we get a Clifford torus when $\epsilon > 0$ and a Chekanov torus when $\epsilon < 0$ for sufficiently small ϵ. □

Theorem 5.2.4 ([28])**.** *Chekanov and Clifford tori are Lagrangian isotopic.*

Proof. Let $F^{-1}(a_1,0)$ be a Clifford torus and $F^{-1}(a_2,0)$ be a Chekanov torus. Let $\beta : [0,1] \to \mathbb{R}^2$ any path in the range of F connecting a_1 to a_2 and not passing through the origin. Then $\{F^{-1}(\beta(t))\}_{t\in[0,1]}$ induces a Lagrangian isotopy connecting the tori $F^{-1}(a_1,0)$ and $F^{-1}(a_2,0)$. □

Lemma 5.2.5. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function given by $f(q,p) = p^2 - q^2 + q^4$. The level set $f^{-1}(a)$ of the function $f : \mathbb{R}^2 \to \mathbb{R}$ is :*

- ***i.*** *Union of the two points $(\pm\frac{1}{\sqrt{2}}, 0)$ if $a = -\frac{1}{4}$,*
- ***ii.*** *Union of two embedded circles if $-\frac{1}{4} < a < 0$,*
- ***iii.*** *Immersed circle W^1 if $a = 0$,*
- ***iv.*** *An embedded circle containing origin if $a > 0$.*

Proof. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a function given by $g(\tilde{q},\tilde{p}) = \tilde{p}^2 - \tilde{q} + \tilde{q}^2$. Then we have $g(\tilde{q},\tilde{p}) = \tilde{p}^2 - (\tilde{q}-\frac{1}{2})^2 - \frac{1}{4}$ and $f(q,p) = g(q^2,p)$. Then a point (q_0,p_0) is in the level set $f^{-1}(a)$ if and only if the point (q_0^2,p_0) is in the set $g^{-1}(a) \cap \{(\tilde{q},\tilde{p}) \,|\, \tilde{q} \geq 0\} \subseteq \mathbb{R}^2$.

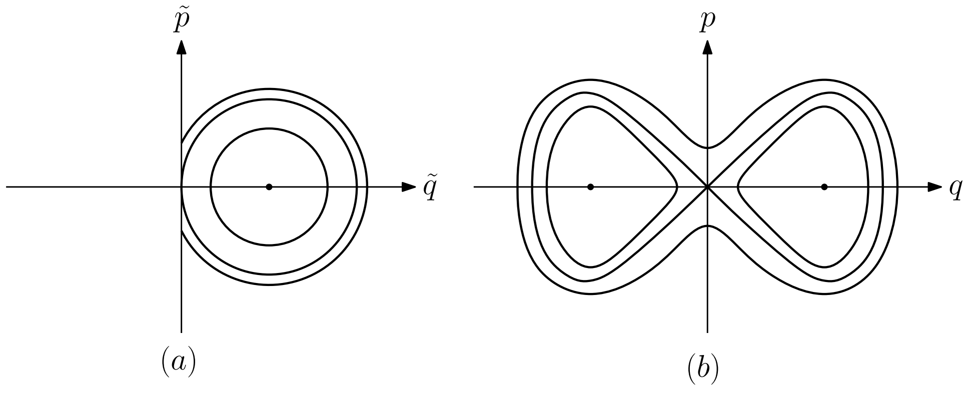

Figure 5.2: The sets $g^{-1}(a) \cap \{(\tilde{q},\tilde{p}) \,|\, \tilde{q} \geq 0\}$ and $f^{-1}(a)$ for different a values.

The values of the function $g : \mathbb{R}^2 \to \mathbb{R}$ are greater than equal to $-\frac{1}{4}$. If $a \in [-\frac{1}{4}, \infty)$, then the set $g^{-1}(a) \cap \{(\tilde{q},\tilde{p}) \,|\, \tilde{q} \geq 0\} \subseteq \mathbb{R}^2$ is

i. The point $(\frac{1}{2}, 0)$ if $a = -\frac{1}{4}$,
ii. The round circle centered at $(\frac{1}{2}, 0)$ of radius $\sqrt{a+\frac{1}{4}}$ if $-\frac{1}{4} < a < 0$,
iii. The round circle centered at $(\frac{1}{2}, 0)$ passing from $(0,0)$ if $a = 0$,
iv. The segment of the round circle centered at $(\frac{1}{2}, 0)$ of radius $\sqrt{a+\frac{1}{4}}$ if $a > 0$.

This proves the lemma except that the immersed circle in Lemma 5.2(iii) is W^1. If we let $a = 0$ and $q = \sin\varphi$, $\varphi \in [0, 2\pi]$, then we have $0 = p^2 - q^2 + q^4 = p^2 - \sin^2\varphi + \sin^4\varphi = p^2 - \frac{1}{4}\sin^2 2\varphi$ which is equivalent to $p = \pm\frac{1}{2}\sin 2\varphi$, $\varphi \in [0, 2\pi]$. Each case gives a parametrization of W^1 in $\mathbb{R}^2$, see 4.1 . □

Remark 5.2.6. *We have $G(q_1, p_1, q_2, p_2) = f(\sqrt{q_1^2 + q_2^2}, \sqrt{p_1^2 + p_2^2})$ or in polar coordinates 5.4 and 5.5 we have $G(q_1, p_1, q_2, p_2) = f(r_q, r_p)$ for the function $G : \mathbb{R}^4 \to \mathbb{R}$ given by 5.1.*

Proof of Proposition 5.2.2. In polar coordinates 5.4 and 5.5, we have $G = r_p^2 - r_q^2 + r_q^4$, $H = r_p r_q \sin(\theta_p - \theta_q)$. By the Remark 5.2.6 the fiber $F^{-1}(a, b)$ is given by

$$\{(r_q\cos\theta_q, r_p\cos\theta_p, r_q\sin\theta_q, r_p\sin\theta_p) \mid f(r_q, r_p) = a, r_p r_q \sin(\theta_p - \theta_q) = b\} \tag{5.6}$$

i. Let $(a, b) \in \mathcal{B}_F$. Then $b = \pm M_a$ where M_a is given in the proof of Theorem 5.2.1.
In the case $(a, b) = (-\frac{1}{4}, 0)$ we have $r_p = 0$, $r_q = \frac{1}{\sqrt{2}}$ and the fiber is $F^{-1}(-\frac{1}{4}, 0) = \{\frac{1}{\sqrt{2}}(\cos\theta_q, 0, \sin\theta_q, 0) \in \mathbb{R}^4 \mid \theta_q \in [0, 2\pi]\}$ is a circle.
In the case $(a, b) \neq (-\frac{1}{4}, 0)$, we have

$$(r_q, r_p) = (\frac{\sqrt{1+\sqrt{1+3a}}}{\sqrt{3}}, \frac{1}{3}\sqrt{1+6a+\sqrt{1+3a}})\,,\ \theta_p = \theta_q + \frac{\pi}{2} \text{ if } b = M_a,$$
$$(r_q, r_p) = (\frac{\sqrt{1+\sqrt{1+3a}}}{\sqrt{3}}, \frac{1}{3}\sqrt{1+6a+\sqrt{1+3a}})\,,\ \theta_p = \theta_q + \frac{\pi}{2} \text{ if } b = -M_a$$

for the fibers $F^{-1}(a, b)$ by proof of Theorem 5.2.1. As a result the fiber $F^{-1}(a, b)$ is again a circle.

For the fibers $F^{-1}(a, 0)$ in (ii-iv), we have $(r_q, r_p) \in f^{-1}(a) \cap \{(q, p) \mid q, p \geq 0\}$ and θ_p is equal to θ_q or $\theta_q + \pi$.

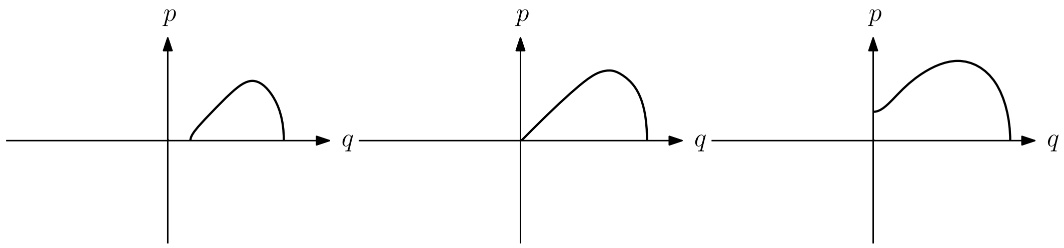

Figure 5.3: The sets $f^{-1}(a) \cap \{(q,p) \mid q, p \geq 0\}$.

By Lemma 5.2, the values (r_q, r_p) can take have graphs like in the Figure 5.3 where $a = 0$ case is one-fourth of the immersed circle W^1. If we parametrize these curves by $\gamma^a : [0,1] \to \mathbb{C}$ where $\gamma^a = (\gamma_1^a, \gamma_2^a)$, then the fibers $F^{-1}(a, 0)$ is the union of the following subsets of $\mathbb{R}^4$

$$\{(\gamma_1^a(s)\cos\theta_q, \pm\gamma_2^a(s)\cos\theta_q, \gamma_1(s)^a\sin\theta_q, \pm\gamma_2^a(s)\sin\theta_q) \mid s \in [0,1],\ \theta_q \in [0,2\pi]\}.$$

In $\mathbb{C}^2$, these sets are given by

$$\{((\gamma_1^a(s) \pm i\gamma_2^a(s))\cos\theta_q, (\gamma_1^a(s) \pm i\gamma_2^a(s))\sin\theta_q) \mid s \in [0,1],\ \theta_q \in [0,2\pi]\}.$$

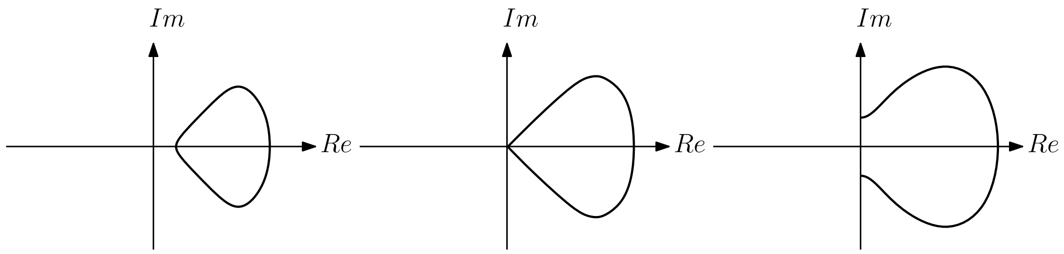

Figure 5.4: The sets $\{\gamma_1^a(s) \pm i\gamma_2^a(s) \mid s \in [0,1]\}$.

ii. If $-\frac{1}{4} < a < 0$ and $b = 0$ then union of the sets $\{\gamma_1^a(s) \pm i\gamma_2^a(s) \mid s \in [0,1]\}$ is an embedded circle which lies in the right half plane as in the Figure 5.4. If we parametrize this embedded circle by a map $\sigma^a : \mathbb{S}^1 \to \mathbb{C}$, then the fiber $F^{-1}(a, 0)$ is given by

$$\{(\sigma^a(\varphi)\cos\theta_q, \sigma^a(\varphi)\sin\theta_q) \mid \varphi, \theta_q \in [0,2\pi]\}$$

Consider the Hamiltonian isotopy $\Upsilon : \mathbb{C}^2 \times [0,1] \to \mathbb{C}^2$ given by Example

2.3.1. We have

$$\Upsilon^1(F^{-1}(a,0)) = \{\frac{1}{\sqrt{2}}(\sigma^a(\varphi)e^{i\theta}, \sigma^a(\varphi)e^{-i\theta}) \mid \varphi \in [0,2\pi], \theta \in [0,2\pi]\}$$

which is a Chekanov Torus.

iii. If $(a,b) = (0,0)$ then union of the sets $\{\gamma_1^a(s) \pm i\gamma_2^a(s) \mid s \in [0,1]\}$ is right half of the immersed circle W^1 as in the Figure 5.4. If we parametrize it by $\sigma : [0,1] \to \mathbb{C}$ where $\sigma(0) = \sigma(1) = 0 \in \mathbb{C}$, then the fiber $F^{-1}(0,0)$ is given by

$$F^{-1}(0,0) = \{(\sigma(s)\cos\theta_q, \sigma(s)\sin\theta_q) \mid s \in [0,1], \theta_q \in [0,2\pi]\} \quad (5.7)$$

$$= \{(\tilde{\sigma}(s)\cos\theta_q, \tilde{\sigma}(s)\sin\theta_q) \mid s \in [0,1], \theta_q \in [0,\pi]\} \quad (5.8)$$

where $\tilde{\sigma} : \mathbb{S}^1 \to \mathbb{C}$ is the concatenation of the maps $\sigma, -\sigma : [0,1] \to \mathbb{C}$. Then $\tilde{\sigma} : \mathbb{S}^1 \to \mathbb{C}$ traces the immersed circle W^1. Hence it can be parametrized by a map $\sigma^0 : \mathbb{S}^1 \to \mathbb{C}$ where $\sigma^0(\varphi) = \sin\varphi + \frac{i}{2}\sin 2\varphi$. Then $F^{-1}(0,0)$ is given by

$$\{((sin\varphi + \frac{i}{2}\sin 2\varphi)\cos\theta, (sin\varphi + \frac{i}{2}\sin 2\varphi)\sin\theta) \mid \varphi \in [0,2\pi], \theta \in [0,\pi]\}$$

which is Lagrangian nodal sphere W^2.

iv. If $a > 0$ and $b = 0$ then by similar arguments as in part (ii) of the proof, we see that the fiber $F^{-1}(a,0)$ is given by

$$\{(\sigma^a(s)\cos\theta_q, \sigma^a(s)\sin\theta_q) \mid s \in [0,1], \theta_q \in [0,2\pi]\} \quad (5.9)$$

where $\sigma^a : [0,1] \to \mathbb{C}$ is a parametrization of the right half of the embedded circle $f^{-1}(a)$ given in the Figure 5.4. The set given by equation 5.9 is equal to

$$\{(\tilde{\sigma}^a(\varphi)\cos\theta_q, \tilde{\sigma}^a(\varphi)\sin\theta_q) \mid \varphi \in [0,2\pi], \theta_q \in [0,\pi]\} \quad (5.10)$$

where $\tilde{\sigma}^a : \mathbb{S}^1 \to \mathbb{C}$ is a parametrization of the circle $\sigma^a \cup -\sigma^a$. There exists a Hamiltonian isotopy $\{\Phi^t : \mathbb{C} \to \mathbb{C}\}_{t\in[0,1]}$ which takes the image of $\tilde{\sigma}^a(\mathbb{S}^1)$ to the round circle given by $\tilde{\sigma}_2^a(\varphi) = r_a e^{i\varphi}$ where $\varphi \in [0,2\pi]$ and $\Phi^t(0) = 0$ for all $t \in [0,1]$. Consider the map $\Psi : F^{-1}(a,0) \times [0,1] \to \mathbb{C}^2$ given by $(\tilde{\sigma}^a\varphi)\cos\theta, \tilde{\sigma}^a(\varphi)\sin\theta, t) \mapsto (\Phi^t(\tilde{\sigma}^a\varphi))\cos\theta, \Phi^t(\tilde{\sigma}^a(\varphi))\sin\theta)$

The subset $\Psi^t(F^{-1}(a,0))$ of $\mathbb{C}^2$ is a torus since $\Phi^t(\tilde{\sigma}^a(\mathbb{S}^1))$ does not pass through the origin of $\mathbb{C}$ for all $t \in [0,1]$. The map $\Psi : F^{-1}(a,0) \times [0,1] \to \mathbb{C}^2$

is a Lagrangian isotopy since $\Psi^t(F^{-1}(a,0))$ is a Lagrangian for all $t \in [0,1]$. The following two embedded circles

$$\beta_1^t : \mathbb{S}^1 \to \mathbb{C}^2, \quad e^{i\varphi} \mapsto (\Phi^t(\tilde{\sigma}^a(\varphi)), \Phi^t(\tilde{\sigma}^a(\varphi)))$$
$$\beta_2^t : \mathbb{S}^1 \to \mathbb{C}^2, \quad e^{i\theta} \mapsto (\Phi^t(\tilde{\sigma}^a(0))\cos\theta, \Phi^t(\tilde{\sigma}^a(0))\sin\theta)$$

lie on the $\Psi^t(F^{-1}(a,0))$ and bound two embedded discs which correspond to the generators of $\pi_2(\mathbb{C}^2, \Psi^t(F^{-1}(a,0)))$ for all $t \in [0,1]$. The discs bounded by $\beta_1^t : \mathbb{S}^1 \to \mathbb{C}^2$ has the same area for all $t \in [0,1]$ since $\{\Phi^t : \mathbb{C} \to \mathbb{C}\}_{t\in[0,1]}$ is a Hamiltonian isotopy. The discs bounded by $\beta_2^t : \mathbb{S}^1 \to \mathbb{C}^2$ has zero area since the one-form $z_1 d\bar{z}_1 + z_2 d\bar{z}_2$ vanishes on $\beta_2^t(\mathbb{S}^1)$. Hence by Proposition 2.3.3, the Lagrangian isotopy $\Psi : F^{-1}(a,0) \times [0,1] \to \mathbb{C}^2$ is an exact Lagrangian isotopy and it can be extended to a Hamiltonian isotopy of $\mathbb{C}^2$. In other words, W_2^2 is Hamiltonian isotopic to

$$\mathbb{T}_a^2 = \{(r_a e^{i\varphi}\cos\theta, r_a e^{i\varphi}\sin\theta) \mid \varphi \in [0,2\pi], \theta \in [0,\pi]\}.$$

By the Hamiltonian isotopy $\Upsilon : \mathbb{C}^2 \times [0,1] \to \mathbb{C}^2$ given in the Example 2.3.1 we have

$$\Upsilon^1(\mathbb{T}_a^2) = \{\frac{r_a}{\sqrt{2}}(e^{i(\varphi+\theta)}, e^{i(\varphi-\theta)}) \mid \varphi \in [0,2\pi], \theta \in [0,\pi]\}.$$

If we let $\varphi' = \varphi + \theta$ and $\theta' = \varphi - \theta$ then

$$\Upsilon^1(\mathbb{T}_a^2) = \{\frac{r_a}{\sqrt{2}}(e^{i\varphi'}, e^{i\theta'}) \mid (\varphi', \theta') \in \mathcal{R} \subset \mathbb{R}^2\}$$

where $\mathcal{R}$ as in Figure 3.5. As a result we have

$$\Upsilon^1(\mathbb{T}_a^2) = \{\frac{r_a}{\sqrt{2}}(e^{i\varphi'}, e^{i\theta'}) \mid \varphi' \in [0,2\pi], \theta' \in [0,2\pi]\}$$

which is a Clifford torus.

v. Let (a,b) in the interior of the image of F and $(a,b) \neq (0,0)$. Without loss of generality assume $b > 0$. We have $r_p r_q \sin(\theta_p - \theta_q) = b$. This implies that $r_p r_q \geq b$. By Lemma 5.2, the values (r_q, r_p) can take lie on graphs like in the Figure 5.3 and satisfies $r_p r_q \geq b$. If we parametrize these curves by $\gamma^a : [0,1] \to \mathbb{C}$ where $\gamma^a = (\gamma_1^a, \gamma_2^a)$, then the points (q_1, p_1, q_2, p_2) on the fiber $F^{-1}(a,b)$ is given by

$$q_1 = \gamma_1^a(s)\cos\theta_q \tag{5.11}$$

$$p_1 = \gamma_2^a(s)\cos(\theta_q + \theta_0(s)) \text{ or } -\gamma_2^a(s)\cos(\theta_q - \theta_0(s)) \tag{5.12}$$

$$q_2 = \gamma_1^a(s)\sin\theta_q \tag{5.13}$$

$$p_2 = \gamma_2^a(s)\sin(\theta_q + \theta_0(s)) \text{ or } -\gamma_2^a(s)\sin(\theta_q - \theta_0(s)) \tag{5.14}$$

where $s \in [0,1]$, $\theta_q \in [0,2\pi]$ and $\theta_0(s) = \arcsin(\frac{b}{\gamma_1^a(s)\gamma_2^a(s)})$. If we identify $\mathbb{C}^2$ with $\mathbb{R}^4$ then we have

$$z_1 = \gamma_1^a(s)\cos\theta_q \pm i\gamma_2^a(s)\cos(\theta_q \pm \theta_0(s)) \tag{5.15}$$

$$z_2 = \gamma_1^a(s)\sin\theta_q \pm i\gamma_2^a(s)\sin(\theta_q \pm \theta_0(s)) \tag{5.16}$$

where $s \in [0,1]$, $\theta_q \in [0,2\pi]$ and $\theta_0(s) = \arcsin(\frac{b}{\gamma_1^a(s)\gamma_2^a(s)})$. This shows that the fiber $F^{-1}(a,b)$ is compact, since every function in the expressions 5.15 and 5.16 is smooth and the domain is compact. When $s = 0$, we have $\theta_0(0) = \frac{\pi}{2}$ since $\gamma_1^a(0)\gamma_2^a(0) = b$. So the expressions 5.15 and 5.16 become $z_1 = \gamma_1^a(0)$ and $z_2 = i\gamma_2^a(0)$. This shows that the fiber $F^{-1}(a,b)$ is connected since it is union of two connected spaces with non-empty intersection. Then by the Proposition 5.2.2 $F^{-1}(a,b)$ is a Lagrangian torus.
Consider the loop $\beta : [0,2\pi] \to \mathbb{C}^2$ on $F^{-1}(a,b)$ which is given by

$$z_1 = \gamma_1^a(0)\cos\theta_q - i\gamma_2^a(0)\sin\theta_q \text{ and } z_2 = \gamma_1^a(0)\sin\theta_q + i\gamma_2^a(0)\cos\theta_q$$

where $\theta_q \in [0,2\pi]$. The one-form $\frac{i}{2}(z_1 d\bar{z}_1 + z_2 d\bar{z}_2)$ is equal to the one-form $-\gamma_1^a(0)\gamma_2^a(0)d\theta_q = -bd\theta_q$ on the loop $\beta : [0,2\pi] \to \mathbb{C}^2$. This show that a disc bounded by β has area $2\pi b$. For each value of θ_q we have a unitary matrix A_{θ_q} associated to tangent plane of $F^{-1}(a,b)$ and given by

$$A_{\theta_q} = \operatorname{diag}(-\sin\theta_q - i\cos\theta_q, \cos\theta_q - i\sin\theta_q)\,. \tag{5.17}$$

$\det^2_{\mathbb{C}} A_{\theta_q} = 1$. Hence the map from $\mathbb{S}^1$ to $\mathbb{S}^1$ defined by A_{θ_q} is constant and its degree is zero. As a result, Maslov class of a disc bounded by $\beta : [0,2\pi] \to \mathbb{C}^2$ is zero. This proves that $F^{-1}(a,b)$ is non-monotone.

□

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